Recap

- Set theory 101
- $\mathbb{P}$ is a set-function.
- All sets in $\mathcal{P}(\Omega)$ need not be measurable.
- We restrict domain of $\mathbb{P}$ to sigma-algebra of measurable sets.
- $(\Omega, \mathcal{F}, \mathbb{P})$ is known as probability space.
- Formal definition of probability measure with its axioms.
- We looked at $\mathcal{B}([0,1])$ and $\mathcal{B}(\mathbb{R})$.

Consequences of the Probability Axioms

Definition

A probability measure $\mathbb{P}$ on the *measurable space* $(\Omega, \mathcal{F})$ is a function $\mathbb{P} : \mathcal{F} \to [0, 1]$ s.t.

1. $\mathbb{P}(\emptyset) = 0, \quad \mathbb{P}(\Omega) = 1$
2. For a disjoint collection of event sets $A_1, A_2, \ldots$ from $\mathcal{F}$ we have
$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$$
(countable additivity)

- The trio $(\Omega, \mathcal{F}, \mathbb{P})$ is called as a probability space.
- Identified the probability space in the coin, dice and experiment.

Consequences of the Probability Axioms

- $P(A^c) = 1 - P(A)$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
- If $A \subseteq B$, prove that $P(A) \leq P(B)$. ($A \subseteq B$ has the interpretation that Event A implies event B)
- $P(\cup_{i=1}^{\infty} B_i) \leq \sum_{i=1}^{\infty} P(B_i)$ (Boole's/Bonferroni's inequality). HW
- What is $P(A \cup B \cup C)$?
- State and prove the inclusion-exclusion principle for $P(\cup_{i=1}^{n} A_i)$

Impossible event v/s Zero prob. event

- In $U[0,1]$ what is $P(\omega = 0.5)$? $= 0$.
- Intuitive reasoning for this is that a point has zero length!
- If $P(\omega \in [a,b]) = b - a$ then $P([.5,.5]) = P(\{.5\}) = 0$.
- This is a zero probability event. In fact, every outcome of this experiment is a zero probability event.
- This implies that events of zero probability can happen and they are not impossible events.
- $P(\emptyset) = 0$, then is $\emptyset$ also possible ?No!
- What is $P(\omega \in [0,.25] \cap [.75,1])$?
- $P([0,.25] \cap [.75,1]) = P(\emptyset) = 0$ This event will never happen.

Impossible event v/s Zero prob. event

- Note that in the $U[0,1]$ experiment, $\Omega = \bigcup_{\omega \in \Omega} \{\omega\}$
- $P(\Omega) = P(\bigcup_{\omega \in \Omega} \{\omega\}) = \sum_{\omega \in \Omega} P(\{\omega\}) = 0.$
- What is the problem above ?
- Ω is an uncountable set and the probability set-function only has a countable additive property.
- $\bigcup_{\omega \in \Omega} \{\omega\}$ is an uncountable disjoint union!

Limits and Continuity

- How do we define limit of a sequence $\{a_1, a_2, \ldots, \}$?
- Notation: $\lim_{n\to\infty} a_n = L$.
- How do you define limit of a function at a point c?
- Notation: $\lim_{x\to c} f(x) = L$
- How do you define continuity of a function $f(x)$ at c ?
- When do you say a function is continuous ?
- (ϵ, δ)-definition of limits and continuity?

Limits and Continuity

Definition in terms of limits of sequences.

For a continuous function $f(\cdot)$, as $x \to c$, we have $f(x) \to f(c)$

For a continuous set-function S, as $A_n \to A$, we have $S(A_n) \to S(A)$

- Recall that $\mathbb{P}$ is a set-function. Is it continuous?
- We will see the proof shortly.

Sequence of sets

- Given $(\Omega, \mathcal{F})$, If $A_1 \subset A_2 \ldots$ is an increasing sequence of events defined on $\mathcal{F}$ and $\cup_{n=1}^{\infty} A_n = A \in \mathcal{F}$, then we say thats the sequence of sets A_n are increasing to A ($A_n \uparrow A$).

- Similarly when $A_1 \supset A_2 \ldots$ is a decreasing sequence of events and $\cap_{n=1}^{\infty} A_n = A$, then we have $A_n \downarrow A$.

- Alternative notation: For an increasing sequence of sets A_n we often write $\lim_{n \to \infty} A_n$ for $\cup_{n=1}^{\infty} A_n$ and for a decreasing sequence of sets A_n that $\lim_{n \to \infty} A_n = \cap_{n=1}^{\infty} A_n$.

Continuity of set-function $\mathbb{P}$

Lemma

For sequence of events of the type $A_n \uparrow A$ or $A_n \downarrow A$, we have $\lim_{n\to\infty} \mathbb{P}(A_n) = \mathbb{P}(A)$.

Proof

- Consider increasing sequence first. Similar arguments follow for decreasing seq.
- Define $F_n = A_n - A_{n-1}$
- $\cup_{n=1}^{\infty} A_n = \cup_{n=1}^{\infty} F_n$.
- $\mathbb{P}(A) = \mathbb{P}(\cup_{n=1}^{\infty} A_n) = \mathbb{P}(\cup_{n=1}^{\infty} F_n)$
- But $\mathbb{P}(\cup_{n=1}^{\infty} F_n) = \lim_{n\to\infty} \sum_{i=1}^{n} P(F_i) = \lim_{n\to\infty} \mathbb{P}(A_n)$. $\square$

Equivalently if $An \to \emptyset$, then $\mathbb{P}(A_n) \to 0$.