# MA 6.101

# Probability and Statistics

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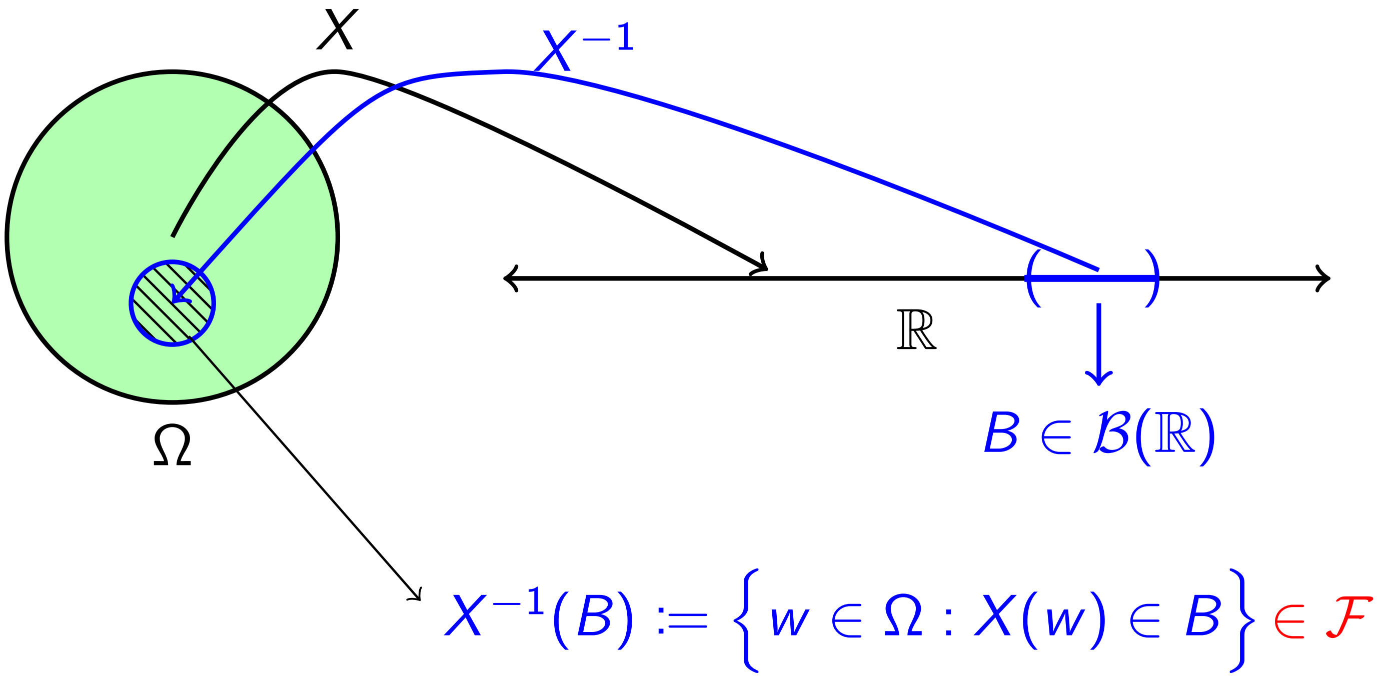
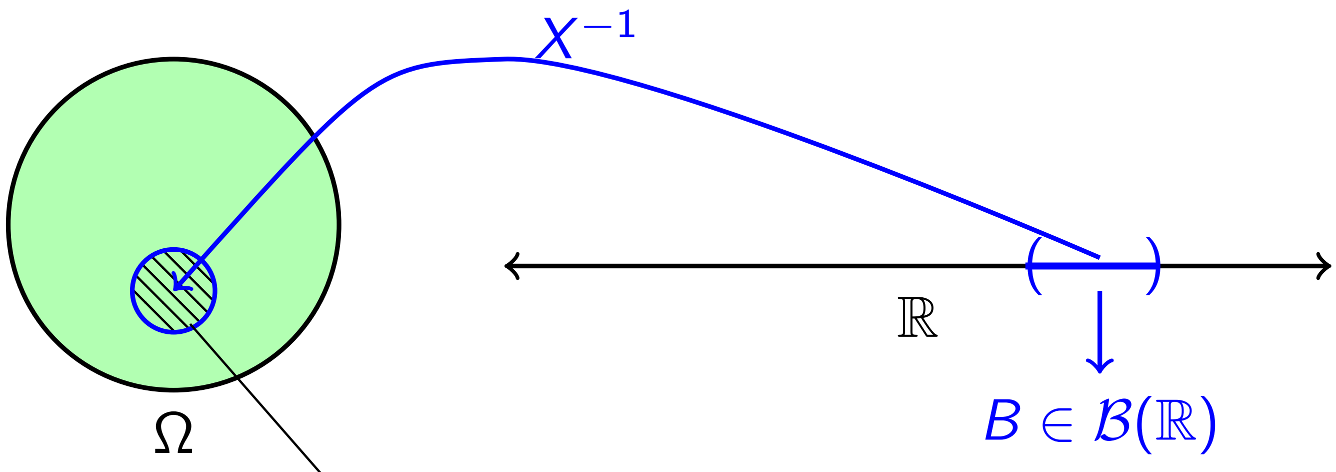
# Random variables ($\Omega' = \mathbb{R}$)




- $\Omega \xrightarrow{X} \mathbb{R}, \quad \mathcal{F} \xrightarrow{X} \mathcal{B}(\mathbb{R}),$ and $P(.) \xrightarrow{X} P_X(.)$

- $X^{-1}(B)$ is called as the preimage or the inverse image of $B$.

# Definition of a random variables




$$X^{-1}(B) = \Big\{ w \in \Omega : X(w) \in B \Big\} \in \mathcal{F}$$

A random variable $X$ is a map $X : (\Omega, \mathcal{F}, P) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}), P_X)$ such that for each $B \in \mathcal{B}(\mathbb{R})$, the inverse image $X^{-1}(B) := \{w \in \Omega : X(w) \in B\}$ satisfies

$$X^{-1}(B) \in \mathcal{F} \text{ and}$$
$$P_X(B) = \Pr(w \in \Omega : X(w) \in B)$$

# Induced measure $P_X$ and CDF

- The cumulative distribution function (CDF) $F_X(x)$ can be expressed using induced measure $P_X$.
- Since the domain of $P_X$ is $\mathcal{B}(\mathbb{R})$, we have seen that $\mathcal{B}(\mathbb{R})$ is made up of sets of the form $(-\infty, a]$ for $a \in \mathbb{R}$.
- $P_X((-\infty, x]) = \mathbb{P}\{w \in \Omega : X(w) \leq x\} := F_X(x)$.
- This is a general definition of CDF (applicable for both continuous or discrete).
- If $F_X(\cdot)$ is continuous (resp. piecewise continuous), then $X$ is continuous (resp. discrete) random variable.

# For a r.v. $X$, its CDF satisfies the following

- $F_X(\infty) = 1$ and $F_X(-\infty) = 0$ when $P(-\infty < X < \infty) = 1$.

- $F_X : \mathbb{R} \to [0, 1]$ is non-decreasing and right continuous.

- At point of discontinuity $x$ we have
  1. right hand limit $F_X(x+) := \lim_{\epsilon \downarrow 0} F_X(x + \epsilon)$
  2. left hand limit $F_X(x-) := \lim_{\epsilon \uparrow 0} F_X(x - \epsilon)$
  3. $F_X(x-) \neq F_X(x+)$.
  4. $F_X(x)$ could be set to either of the two. Which one?

- Right continuity mandates that at point of discontinuity, we have $F_X(x) = F_X(x+)$.

- By default, $F_X(x) = F_X(x+) = F_X(x-)$ if $F_X(x)$ is continuous at $x$.

# Right-continuity

$F_X : \mathbb{R} \to [0, 1]$ is non-decreasing and right continuous.

Proof

- Consider $a < b$ where $a$ and $b$ are arbitrary. We want to show that $F_X(a) \leq F_X(b)$.
- Define $A := \{\omega \in \Omega : X(\omega) \leq a\}$, $B := \{\omega \in \Omega : X(\omega) \leq b\}$.
- Easy to see that $A \subseteq B$ and hence $\mathbb{P}(A) \leq \mathbb{P}(B)$.
- $F_X(a) = P_X((-\infty, a]) = \mathbb{P}(A) \leq \mathbb{P}(B) = F_X(b)$.
- This proves the non-decreasing part.

# Right-continuity

$F_X : \mathbb{R} \to [0, 1]$ is non-decreasing and right continuous.

Proof for right-continuity

- We want to prove that $F_X(x) = F_X(x+)$.
- Consider a sequence of numbers $\{x_n\}$ decreasing to $x$. In this case, we have $F_X(x+) = \lim_{x_n \downarrow x} F_X(x_n)$.
- Define $A_n := \{\omega : X(\omega) \leq x_n\}$ and $A := \{\omega : X(\omega) \leq x\}$.
- Is $A_n \uparrow A$ or $A_n \downarrow A$ ? Clearly, $A_n \downarrow A$.
- From continuity of probability, $\lim_{n\to\infty} \mathbb{P}(A_n) = \mathbb{P}(A)$.
- This implies $\lim_{x_n \downarrow x} F_X(x_n) = F_X(x)$. $\square$
- You cannot prove the other way by considering $x_n \uparrow x$ because $\cup_n(-\infty, x_n] = (-\infty, x)$ and $P_X(-\infty, x) \neq F_X(x)$.

# Continuous random variables

- A random variable defined on $\mathbb{R}$ is discrete, if $F_X(\cdot)$ is piecewise constant.
- A random variable defined on $\mathbb{R}$ is continuous, if $F_X(\cdot)$ is a continuous function.
- Examples of Continuous random variables
  1. Pick a number uniformly from $[a, b]$.
  2. Time interval between successive customers entering DMart.
  3. Travel time from office to home.
  4. Level of water in a dam or pending workload on a server.

# Continuous random variables

- Associated with a continuous random variable is a probability density function (pdf) $f_X(x)$ for all $x \in \mathbb{R}$. Its unit is probability per unit length and is defined as

$$
\begin{aligned}
f_X(x) &:= \lim_{\Delta \to 0^+} \frac{P(x < X \leq x + \Delta)}{\Delta} \\
&= \lim_{\Delta \to 0^+} \frac{F_X(x + \Delta) - F_X(x)}{\Delta} \\
&= \frac{dF_X(x)}{dx}.
\end{aligned}
$$

- Alternatively, a random variable $X$ is continuous if there exists a non-negative real valued probability density function (PDF) $f_X(\cdot)$ such that $F_X(x) = \int_{u=-\infty}^{x} f_X(u)du$.

$$\frac{dF_X(x)}{dx} = f_X(x) \text{ or } P_X(x < X \leq x + h) \simeq f_X(x)h.$$

# Properties of pdf

- $P_X(\mathbb{R}) = \int_{u=-\infty}^{\infty} f_X(u)du = 1.$
- $P_X(a \leq X \leq b) = \int_a^b f_X(u)du.$ (Area under the curve)
- In general, $P_X(B) = \int_{u \in B} f_X(u)du.$
- $P_X(a \leq X \leq b) = P_X(a < X < b) = P_X(a \leq X < b) = P_X(a < X \leq b)$
- $P_X(X = a) = 0.$ (no mass at any point)

# Mean, Variance, Moments

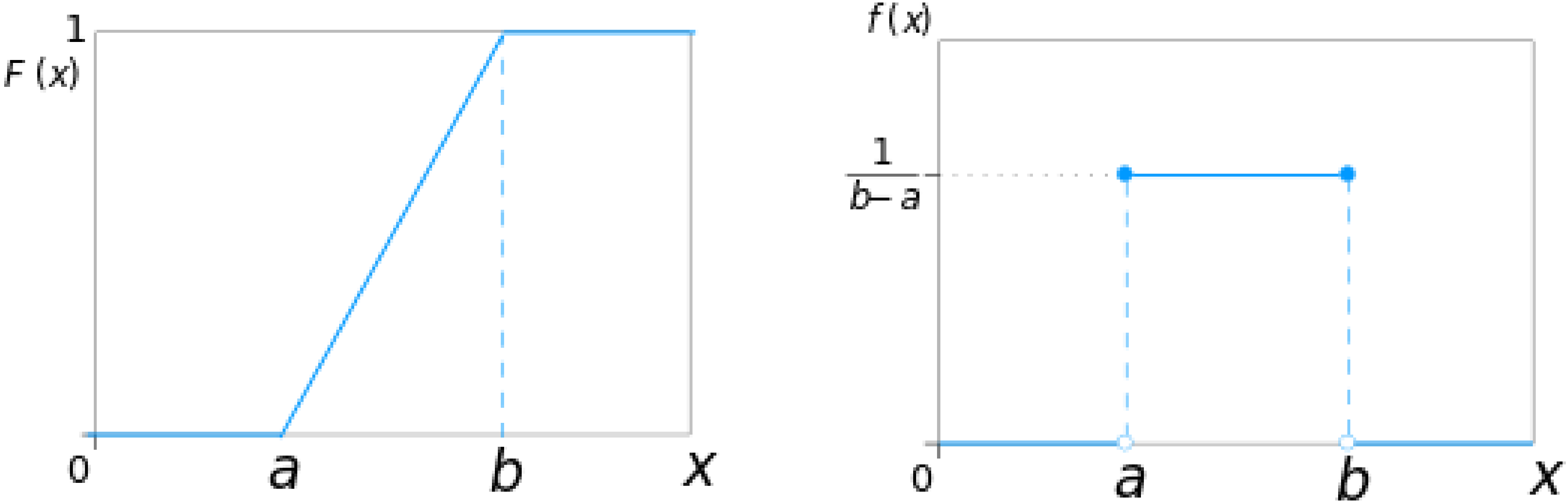
- $E[X] = \int_{-\infty}^{\infty} u f_X(u) du$
- $E[X^n] = \int_{-\infty}^{\infty} u^n f_X(u) du$
- $E[g(X)] = \int_{-\infty}^{\infty} g(u) f_X(u) du$
- $Var[X] = E[g(X)]$ where $g(x) = (x - E[X])^2$.
- For $Y = aX + b$, $E[Y] = aE[X] + b$.
- For $Y = aX + b$, $F_Y(y) = F_X(\frac{y-b}{a})$ when $a \geq 0$.
- For $Y = aX + b$ and $a < 0$, $F_Y(y) = 1 - F_X(\frac{y-b}{a})$.

# Standard Examples

# Uniform random variable ($U[a, b]$)

- This is a real valued r.v.
- Its pdf $f_X(x) = \frac{1}{b-a}$ for all $x \in [a, b]$.
- Its CDF is given by $F_X(x) = \begin{cases} 0 \text{ for } x < a. \\ \frac{x-a}{b-a} \text{ for } x \in [a, b] \\ 1 \text{ otherwise.} \end{cases}$
- HW: Verify $E[X] = \frac{a+b}{2}$ and $Var(X) = \frac{(b-a)^2}{12}$

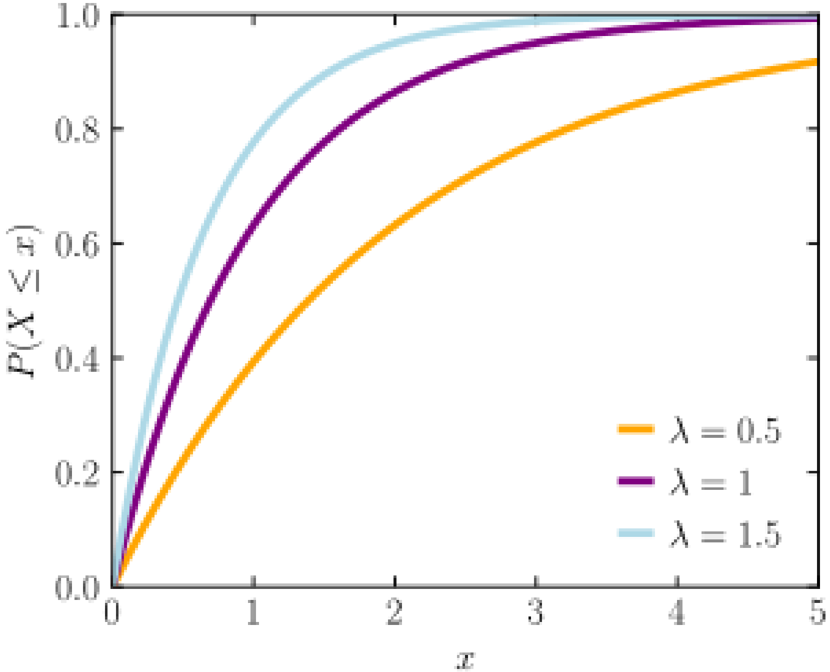
# $U[a, b]$



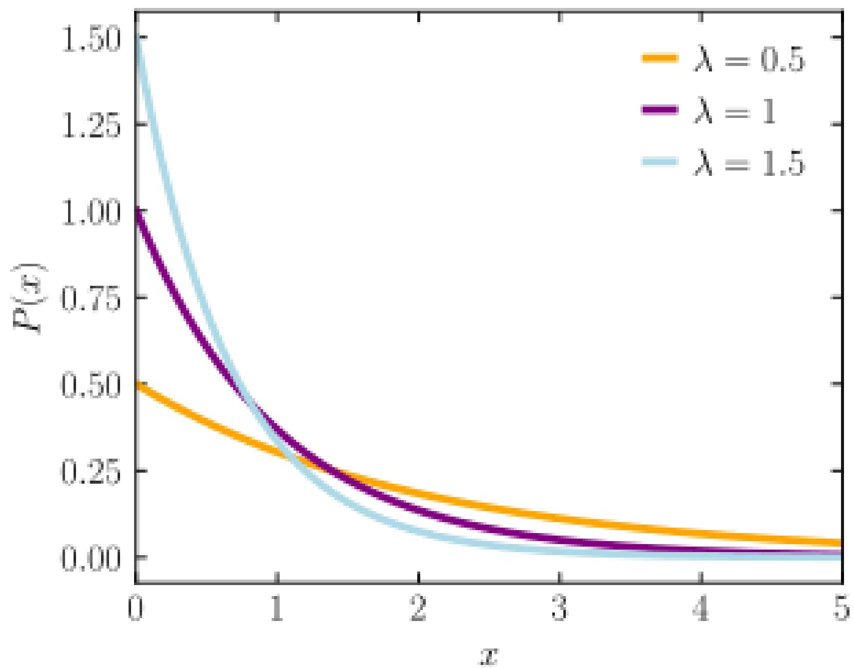
# Exponential random variable ($Exp(\lambda)$)

- This is a non-negative r.v. with parameter $\lambda$.
- Its pdf $f_X(x) = \lambda e^{-\lambda x}$ for $x \geq 0$.
- Its CDF is given by $F_X(x) = 1 - e^{-\lambda x}$ for $x \geq 0$.
- $E[X] = \frac{1}{\lambda}$ and $Var(X) = \frac{1}{\lambda^2}$
- $E[X^n] = \frac{n!}{\lambda^n}$

# $Exp(\lambda)$



1.0
0.8
0.6
0.4
0.2
0.0
0
1
2
3
4
5
$P(X \leq x)$
$x$
$\lambda = 0.5$
$\lambda = 1$
$\lambda = 1.5$

1.50
1.25
1.00
0.75
0.50
0.25
0.00
0
1
2
3
4
5
$P(x)$
$x$
$\lambda = 0.5$
$\lambda = 1$
$\lambda = 1.5$

# Significance of Exponential r.v.

- Building blocks for Continuous time Markov Chains.
- Demonstrate memory-less property (to be seen formally soon).
- $P(X > a + h | X > a) = \frac{e^{-\lambda(a+h)}}{e^{-\lambda(a)}} = e^{-\lambda(h)} = P(X > h)$.
- Used extensively in Queueing theory to model inter-arrival time and service time of jobs.