Example of rolling two dice

- Example of rolling two dice where we are interested in the sum of two dice.
- Suppose $X =$ sum of two dice. Then we have

$$\Omega = \Big\{(1,1),(1,2),\dots,(1,6) \\ (2,1),(2,2),\dots,(2,6) \\ \vdots \\ (6,1),(6,2),\dots,(6,6)\Big\} \xrightarrow{X} \Omega' = \big\{2,3,\dots,12\big\}$$

- $\{X = 3\}$ is an event in $\mathcal{F}'$. What is its probability $P_X(\{3\})$?
- $P_X(\{3\}) = \mathbb{P}(\{\omega \in \Omega : X(\omega) = 3\}) = \mathbb{P}(\{(1,2),(2,1)\})$.

In general for $x \in \Omega'$, $P_X(\{x\}) := \mathbb{P}(\{\omega \in \Omega : X(\omega) = x\})$.
Find $P_X(\{x\})$ for all $x \in \Omega'$?

Sum of two dice

- $\Omega' = \{2, 3, \ldots, 12\}$
- $\mathcal{F}' = \mathcal{P}(\Omega)$
- $P_X(\{x\}) = \begin{cases} \frac{x-1}{36} & \text{for } x \in \{2, 3, \ldots, 7\} \\ \frac{13-x}{36} & \text{for } x \in \{8, 9, \ldots, 12\}. \end{cases}$
- $Z =$Sum of 4 rolls ? Ω for 4 rolls is even complicated.
- This is where X is useful. $P(Z = 4) = P(X_1 = 2, X_2 = 2)$
- Here X_1 and X_2 are independent copies of random variable X.

> The function $P_X(\{x\})$ for $x \in \Omega'$ is called as a probability mass function (PMF) of random variable X. For ease of notation, we denote it by the function $p_X(x)$ on Ω'.

What is the PMF for a random variable corresponding to coin toss or roll a dice ?

PMF and CDF

- The function $p_X(x)$ for $x \in \Omega'$ is called as a probability mass function (PMF).

- Given a discrete random variable X and its induced measure P_X, one can obtain its PMF.

- Given a discrete random variable X and its PMF $p_X(x)$ for all $x \in \Omega'$, can you obtain the induced measure P_X? Yes! Why?

- The cumulative distribution function (CDF) $F_X(\cdot)$ is defined as $F_X(x_1) := \sum_{x \leq x_1} p_X(x) = \mathbb{P}\{\omega \in \Omega : X(\omega) \leq x_1\}$.

- What is the CDF for the random variable corresponding to the coin toss or dice experiment?

Expectation and Moments

- How do you define the arithmatic mean of a collection of numbers?
- The mean or expectation of a random variable X is denoted by $E[X]$ and is given by $E[X] = \sum_{x \in \Omega'} x p_X(x)$.
- What is $E[X]$ for the random variable X that corresponds to the outcome of coin toss or dice experiment?
- The n^{th} moment of a random variable X is denoted by $E[X^n]$ and is given by $E[X^n] = \sum_{x \in \Omega'} x^n p_X(x)$.
- For a function $g(\cdot)$ of a random variable X, its expectation is given by $E[g(X)] := \sum_{x \in \Omega'} g(x) p_X(x)$

What's in the name?

- Why call it random variable when it is a deterministic function of $\omega \in \Omega$?
- You are interested in outcomes of experiments which are random.
- You cannot say which $\omega \in \Omega$ is realized and hence cannot say apriori what value X will take.
- X is a variable because each time the experiment is performed, it can take different values $x' \in \Omega'$.
- There is no pattern in the values it can take, hence random.
- PMF goes one step ahead in capturing this randomness in X and assigns a probability to every value $x \in \Omega'$.

Consistency of the PMF

- PMF: $p_X(x) = \mathbb{P}(\{\omega \in \Omega : X(\omega) = x\})$ for $x \in \Omega'$.
- How do you check if p_X is legitimate PMF?
- $\sum_{x \in \Omega'} p_X(x) = 1$. Can you prove this?

$$\begin{aligned}
\sum_{x \in \Omega'} p_X(x) &= \sum_{x \in \Omega'} \mathbb{P}(\{\omega \in \Omega : X(\omega) = x\}) \\
&= \mathbb{P}(\cup_{x \in \Omega'} \{\omega \in \Omega : X(\omega) = x\}) \\
&= \mathbb{P}(\Omega) \quad \square
\end{aligned}$$

Linearity of Expectation

- Recall that $E[X] = \sum_{x \in \Omega'} x p_X(x)$.
- Functions of random variables are random variables.
- Furtermore, $E[g(X)] := \sum_{x \in \Omega'} g(x) p_X(x)$
- For $Y = aX + b$, what is $E[Y]$?

$$\begin{aligned} E[Y] &= \sum_{x \in \Omega'} (ax + b) p_X(x) \\ &= a \sum_{x \in \Omega'} x p_X(x) + b \\ &= aE[X] + b. \end{aligned}$$

- What is the PMF of Y?

PMF of Y where $Y = aX + b$.

- Suppose the range of X is $\Omega' = \{x_1, x_2, \ldots, x_n\}$. Then what is the range Ω'' of Y ?
- $\Omega''\{y_1, \ldots, y_n\}$ where $y_i = ax_i + b$ for $i \in \{1, 2, \ldots, n\}$.
- It is easy to see that, $p_Y(y_i) = p_X(x_i)$ for $i \in \{1, 2, \ldots, n\}$.

$$\begin{aligned}
E[Y] &= \sum_{y \in \Omega''} y p_Y(y) \\
&= \sum_{x \in \Omega'} (ax + b) p_y(ax + b) \\
&= \sum_{x \in \Omega'} (ax + b) P_x(x) \\
&= aE[X] + b.
\end{aligned}$$

- What if $Y = g(X)$ where the function $g(.)$ is many to one? What is the PMF of Y then ?