MA 6.101
Probability and Statistics

Tejas Bodas

Assistant Professor, IIIT Hyderabad

Topics

Last class we saw

- Conditioning
- Law of iterated Expectations

This class ..

- Sums of random variables & Convolutions
- Bayes Rule revisited
- Moment Generating functions

Sums of independent random variable

- Consider $Z = X + Y$. What is the pdf of Z when X and Y?
- What is $p_Z(z)$ or $f_Z(z)$?
- $p_Z(z) = \sum_{(x,y):x+y=z} p_{X,Y}(x,y)$
- $f_Z(z) = \int_{(x,y):x+y=z} f_{X,Y}(x,y)dxdy$.
- Integral of a surface over line.
- `https://en.wikipedia.org/wiki/Line_integral`
- Since X and Y are independent $p_{X,Y}(x,y) = p_X(x)p_Y(y)$ and $f_{X,Y}(x,y) = f_X(x)f_Y(y)$. This gives us

Convolution formula

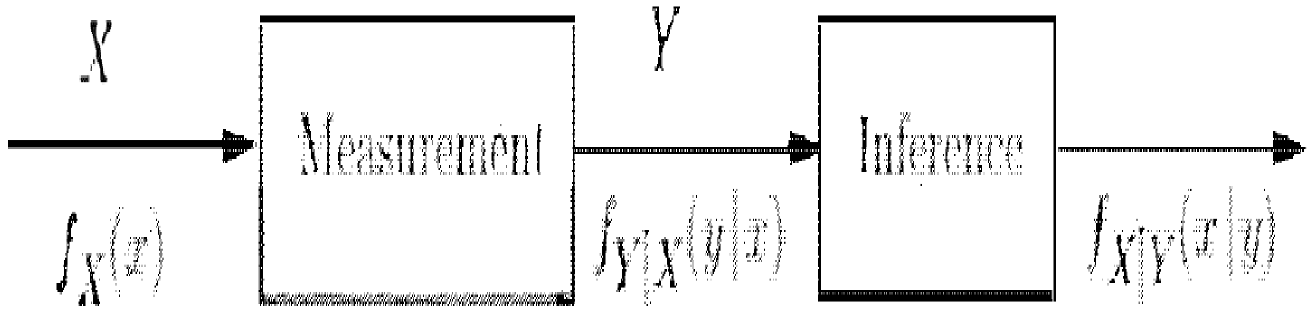
$$p_Z(z) = \sum_x p_X(x)p_Y(z-x)$$
$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x)f_Y(z-x)dx$$

HW: What if X and Y are not independent?

Examples

- EX1: Suppose X and Y are independent and $U[0, 1]$. Find the pdf and CDF of $Z = X + Y$.
- `https://en.m.wikipedia.org/wiki/File:Convolution_of_box_signal_with_itself2.gif`
- Ex2: Suppose X and Y are outcomes of independent roll of dice. Find the pmf of $Z = X + Y$.

Inference problem



- X is an unobservable random variable with a known distribution.
- We only observe measurements Y that takes values according to $f_{Y|X}(y|x)$.
- Objective is to draw inference about X having seen a realization of Y i.e., Obtain $f_{X|Y}(x|y)$ using only $f_X(x)$ and $f_{Y|X}(y|x)$, both of which are known.

Bayes Rule revisited

$$P(B/A) = \frac{P(A/B)P(B)}{P(A)}$$

For continuous random variables X and Y

$$f_{X|Y}(x|y) = \frac{f_{Y|X}(y|x)f_X(x)}{f_Y(y)} = \frac{f_{Y|X}(y|x)f_X(x)}{\int_{-\infty}^{\infty} f_{Y|X}(y|t)f_X(t)dt}$$

For discrete random variables X and Y

$$p_{X|Y}(x|y) = \frac{p_{Y|X}(y|x)p_X(x)}{p_Y(y)} = \frac{p_{Y|X}(y|x)p_X(x)}{\sum_i p_{Y|X}(y|i)p_X(i)}$$

Example 3.19(Bertsekas)

Lifetime of a Phillips bulb is assumed to be an exponential random variable Y with parameter Λ. Λ itself is a uniform random variable over $[1, 1.5]$. You test a bulb and see that it has a lifetime of y units. What can you say about randomness of Λ having observed $Y = y$.?

- What is $f_\Lambda(\lambda)$?
- What is $f_{Y|\Lambda}(y|\lambda)$?
- What is $f_Y(y)$?
- $f_{\Lambda|Y}(\lambda|y) = \frac{2\lambda e^{-\lambda y}}{\int_1^{1.5} 2te^{-ty}dt}$ for $\lambda \in [1, 1.5]$.

Bayes Rule revisited

For discrete N and continuous random variable Y

$$P(N = n|Y = y) = \frac{f_{Y|N}(y|n)p_N(n)}{f_Y(y)} = \frac{f_{Y|N}(y|n)p_N(n)}{\sum_i f_{Y|N}(y|i)p_N(i)}$$

Equivalently

$$f_{Y|N}(y|n) = \frac{P(N = n|Y = y)f_Y(y)}{p_N(n)} = \frac{P(N = n|Y = y)f_Y(y)}{\int_{-\infty}^{\infty} P(N = n|Y = t)f_Y(t)dt}$$

Example 3.20 (Bertsekas)

- Suppose $X = 1$ w.p. p and $X = -1$ w.p. $1 - p$. While transmitting this signal, it is corrupted by a Gaussian noise $N \sim \mathcal{N}(0, 1)$. We observe $Y = X + N$. Suppose you observe $Y = y$, then show that

$$P(X = 1|Y = y) = \frac{pe^y}{pe^y + (1-p)e^{-y}}$$

- Intuitively, this probability goes to zero as y decreases to $-\infty$ and increases to 1 as y increases to ∞.
- $P(X = 1|Y = y) = \frac{f_{Y|X}(y|1)p_X(1)}{f_Y(y)}$
- Here $f_Y(y) = f_{Y|X}(y|1)p_X(1) + f_{Y|X}(y|-1)p_X(1)$.
- Substitute values to obtain answer.