Recap

$p_{XY}(x, y) := \mathbb{P}\{\omega \in \Omega : X(w) = x \text{ and } Y(\omega) = y\}$.
$F_{XY}(x, y) := \mathbb{P}\{\omega \in \Omega : X(w) \leq x \text{ and } Y(\omega) \leq y\}$.

The marginal PMF's p_X and p_Y can be obtained from the joint PMF as follows:
$p_X(x) = \sum_y p_{XY}(x, y)$ and $p_Y(y) = \sum_x p_{XY}(x, y)$.

Two random variables, X and Y are independent if the following is true:
$p_{XY}(x, y) = p_X(x)p_Y(y)$, $F_{XY}(x, y) = F_X(x)F_Y(y)$ and $E[XY] = E[X]E[Y]$.

Consistency conditions

- $\sum_{x,y} p_X(x, y) = 1.$
- $F_{XY}(\infty, \infty) = 1.$
- $F_{XY}(-\infty, -\infty) = 0.$
- $F_{XY}(-\infty, \infty) = 0.$
- $F_{XY}(\infty, -\infty) = 0$
- $F_{XY}(x, \infty) = F_X(x)$ (marginal CDF)
- $F_{XY}(\infty, y) = F_Y(y)$ (marginal CDF)

Multiple continuous random variables

- Pick a number uniformly at random from a unit square centered at $(.5, .5)$.
- Random variables X and Y represent the respective x and y coordinate of the point chosen.
- $F_{X,Y}(x, y)$ denotes the probability that the point chosen lies below and to left of point (x, y).
- In this example, $F_{X,Y}(x, y) = xy$.
- Now visualize $F_{X,Y}(x + h, y) - F_{X,Y}(x, y)$. This is the probability that the point chosen lies in the thin strip below y and between x and $x + h$.

Multiple continuous random variables

- Visualize $F_{X,Y}(x+h,y) - F_{X,Y}(x,y)$. This is the probability that the point chosen lies in the thin strip below y and between x and $x+h$.
- $\frac{\partial F_{XY}(x,y)}{\partial x} = lim_{h \to 0} \frac{F_{X,Y}(x+h,y) - F_{X,Y}(x,y)}{h}$.
- This is the rate of change of the joint CDF $F_{XY}(x,y)$ in the x direction.

Multiple continuous random variables

- $\frac{\partial F_{XY}(x,y}{\partial y} = \lim_{h \to 0} \frac{F_{X,Y}(x,y+h) - F_{X,Y}(x,y)}{h}$ denotes the rate of change of the joint CDF in the y direction.
- $f_{X,Y}(x,y) := \frac{\partial^2 F_{XY}(x,y)}{\partial x \partial y}$ represents the joint probability density function.
- $f_{X,Y}(x,y)dxdy$ denotes the probability that (X,Y) are in a rectangle of area $dxdy$ around (x,y).
- In this example, $f_{X,Y}(x,y) = 1$.
- $F_{XY}(x,y) := \int_{-\infty}^{x} \int_{-\infty}^{y} f_{XY}(s,t)dsdt$.

Summary for Continuous random variable

- $f_{XY}(x, y)$ denotes the joint pdf for X and Y.
- $F_{XY}(x, y) := \int_{-\infty}^{x} \int_{-\infty}^{y} f_{XY}(s, t) ds dt$. $f_{X,Y}(x, y) := \frac{\partial^2 F_{XY}(x,y)}{\partial x \partial y}$.

> The marginal pdf's f_X and f_Y can be obtained from the joint PDF as follows:
> $f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y) dy$ and $f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y) dx$

> Two random variables, X and Y are independent if the following is true:
> $f_{XY}(x, y) = f_X(x) f_Y(y), F_{XY}(x, y) = F_X(x) F_Y(y)$ and $E[XY] = E[X]E[Y]$.

- Rules similar for more than 2 random variables.

Towards $E[g(X,Y)]$

- What about $E[aX + bY + c]$?

$$
\begin{aligned}
E[aX + bY + c] &= \sum_{x,y} (ax + by + c) p_{XY}(x,y) \\
&= a \sum_{xy} x p_{XY}(x,y) + b \sum_{xy} y p_{XY}(x,y) \\
&+ c \sum_{xy} p_{XY}(x,y) \\
&= aE[X] + bE[Y] + c.
\end{aligned}
$$

- What is $E[aX + bY + c]$ for the running example?

- Along similar lines, one would expect:

$$\boxed{E[g(X,Y)] = \sum_x \sum_y g(x,y) p_{XY}(x,y)}$$

Finding $p_Z(\cdot)$ where $Z = g(X, Y)$.

- Suppose $Z = g(X)$. Then what is $p_Z(z)$?
- $p_Z(z) = \sum_{\{x:g(x)=z\}} p_X(x)$.
- Now suppose $Z = g(X, Y)$ then we have

$$p_Z(z) = \sum_{\{x,y:g(x,y)=z\}} p_{XY}(x, y)$$

Does this need a proof? No if you are an unconscious statistician!

$E[g(X, Y)]$

- How do we define $E[g(X, Y)]$?
- One way is to define $Z = g(X, Y)$ and find $E[Z] = \sum_z z p_Z(z)$
- Recall $p_Z(z) = \sum_{\{x,y:g(x,y)=z\}} p_{XY}(x, y)$
- This gives us $E[Z] = \sum_z \sum_{\{x,y:g(x,y)=z\}} z p_{XY}(x, y)$.
- This is same as $E[g(X, Y)] = \sum_{\{x,y\}} g(x, y) p_{XY}(x, y)$.

$E[g(X, Y)] = \sum_{xy} g(x, y) p_{XY}(xy)$

$E[g(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f_{XY}(x, y) dxdy$ (for continuous r.v)

Example of function of Random variables

- Suppose X and Y are continuous independent random variables. Let $Y = max(X, Y)$ and $Z = min(X, Y)$ Find the CDF and pdf of Z.
- HW: When X and Y are exponential with parameters λ_1 and λ_2 then Z is also exponential with parameter $\lambda - 1 + \lambda_2$.

Covariance of X and Y

- $Cov(X, Y) = E[(X - E[X])(Y - E[Y])] = E[XY] - E[X]E[Y]$.
- When Covariance is zero, they are said to be uncorrelated.
- (Section 4.2 Bertsekas)