Science-1

Het Selarka
UG-2, ECE

$\boxed{1D}$ $\frac{dx}{dt} = \alpha x$

$\boxed{2D}$ $\frac{d\vec{R}}{dt} = \vec{J}_0 \vec{R}$

$$J(x,y) = \begin{pmatrix} \text{Partial der of } f(x,y) \text{ wrt } x & \text{Par der of } f(x,y) \text{ wrt } y \\ \text{Part der of } g \text{ wrt } x & \text{" " } g \text{ wrt } y \end{pmatrix}$$

$$\therefore J = \begin{pmatrix} \alpha - \beta y & -\beta x \\ \gamma y & \gamma x - \delta \end{pmatrix}$$

for

$f(x,y) = \alpha x - \beta y x \rightarrow$ Rate of Change of Prey Population

$g = \gamma x y - \delta y \rightarrow$ ROC of Predator Pop.

$\frac{dx}{dt}$ $\frac{dy}{dt}$

$$\therefore J(0,0) = \begin{pmatrix} \alpha & 0 \\ 0 & -\delta \end{pmatrix} \quad \begin{matrix} \nearrow \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \vec{v}_1 \\ \searrow -\delta \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow \vec{v}_2 \end{matrix}$$

By eigen vector thm,

Any vector can be written as sum of eigen vectors.

$$\therefore \vec{R}(0) = c_1(0)\vec{v}_1 + c_2(0)\vec{v}_2$$

$$\therefore \vec{R}(t) = c_1(t)\vec{v}_1 + c_2(t)\vec{v}_2$$

We know $\dfrac{dR}{dt} = \vec{J}_0 \vec{R}$

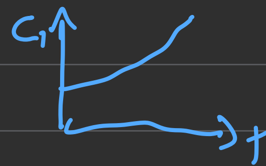
$$\therefore \frac{dc_1}{dt}\vec{v}_1 + \frac{dc_2}{dt}\vec{v}_2 = \vec{J}_0(c_1\vec{v}_1 + c_2\vec{v}_2)$$

$$- \vec{J}_0 \vec{v}_1 = \lambda_1 \vec{v}_1$$

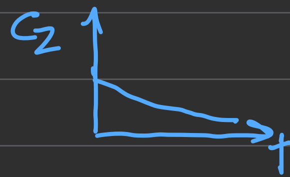
$$\vec{J}_0 \vec{v}_2 = \lambda_2 \vec{v}_2$$

$$\therefore \frac{dc_1}{dt}\vec{v}_1 + \frac{dc_2}{dt}\vec{v}_2 = \lambda_1 c_1 \vec{v}_1 + \lambda_2 c_2 \vec{v}_2$$

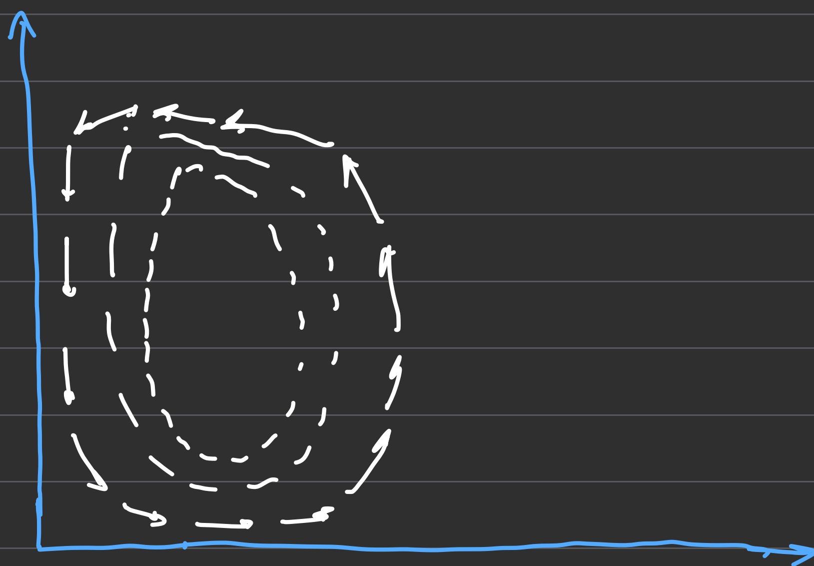
$$\therefore \; c_1(t) = c_1(0)\, e^{\lambda_1 t}$$



$$c_2(t) = c_2(0)\, e^{\lambda_2 t}$$



$\therefore$ Velocity Field for $x(t)$ and $y(t)$ sinusoid functions:



Ex: $\dfrac{d^2x}{dt^2} = f(x,t)$

Soln: Suppose $v(t) = \dfrac{dx}{dt}$

$\therefore \dfrac{dv}{dt} = f(x,t)$

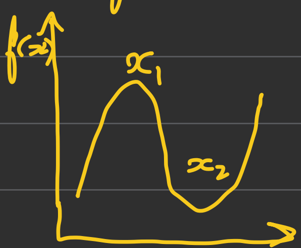
$\therefore$ If $\Phi = \begin{pmatrix} x \\ v \end{pmatrix}$ $\quad \dfrac{d\Phi}{dt} = \begin{pmatrix} v \\ f(x,t) \end{pmatrix}$

And then solve for J

→ HW Find info of 2nd steady state of $J \rightarrow \left(\dfrac{\delta}{\gamma}, \dfrac{\alpha}{\beta}\right)$

Note: Purpose of taking Jacobian matrix for steady states is to notice if the graphs of predator and prey are stable or unstable, by taking eigen values of J.

For exc:



$f'(x_1) = 0$ $\quad$ $f'(x_2) = 0$

$f''(x_1) > 0$ $\quad$ $f''(x_2) < 0$

Unstable system $\quad$ Stable

Now as $J = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}$

How the Jacobian Helps Analyze Stability

1. **Linearization Near Equilibrium:**
   At a given equilibrium (or fixed) point, the Jacobian matrix represents the first-order (linear) approximation of your system around that point. This means that the behavior of a nonlinear system near an equilibrium is (locally) similar to that of its linearization given by the Jacobian [2][9].

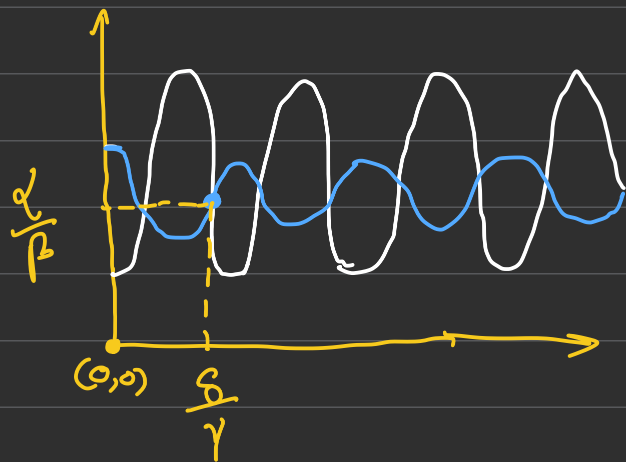
Key Rule:

- If all eigenvalues of the Jacobian evaluated at the equilibrium have negative real parts, the equilibrium is locally stable (trajectories move towards it).
- If any eigenvalue has a positive real part, the equilibrium is unstable (trajectories move away).
- If eigenvalues have zero real parts or are purely imaginary, the stability is inconclusive and may require deeper analysis (e.g., Lyapunov functions) [1][2][8][9].

] → oscillation of system instead of growth or decay

For multidimensional systems, this generalizes: **stability occurs when every eigenvalue has a negative real part.**

That's why the graphs of prey and predator continue to oscillate



Q- For $J\left(\frac{\delta}{\gamma}, \frac{\alpha}{\beta}\right)$

$$J(x,y) = \begin{pmatrix} \alpha - \beta y & -\beta x \\ \gamma y & \gamma x - \delta \end{pmatrix}$$

$$\therefore J = \begin{pmatrix} 0 & -\frac{\beta\delta}{\gamma} \\ \frac{\alpha\gamma}{\beta} & 0 \end{pmatrix}$$

$$= \underset{\lambda_1}{\underset{\downarrow}{\frac{\alpha^2}{\beta}}} \underset{v_1}{\underset{\downarrow}{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}} \quad \underset{\lambda_2}{\underset{\downarrow}{-\alpha\delta}} \underset{v_2}{\underset{\downarrow}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}}$$

$$\vec{R}_1 = C_1(t)\begin{pmatrix} 0 \\ 1 \end{pmatrix} + C_2(t)\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\frac{d\vec{R}_1}{dt} = \vec{J}\vec{R} = \frac{dc_1(t)}{dt}\vec{v}_1 + \frac{dc_2(t)}{dt}\vec{v}_2$$

$$= \vec{J}\left(C_1\vec{v}_1 + C_2\vec{v}_2\right)$$

$$= \lambda_1 C_1 \vec{v}_1 + \lambda_2 C_2 \vec{v}_2$$

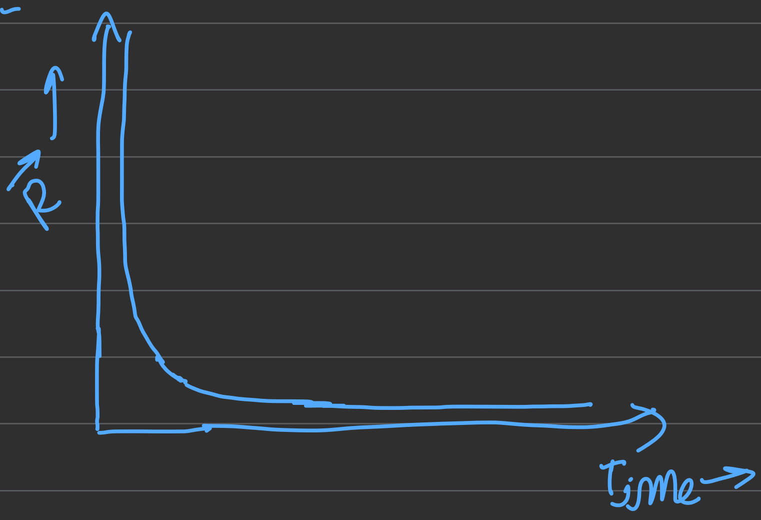
$$\therefore \frac{dc_1}{dt} = \frac{\alpha^2}{\beta} c_1 \qquad \frac{dc_2}{dt} = -\alpha\delta c_2$$

$$\therefore c_1 = c_1(0) e^{\frac{\alpha^2}{\beta} t} \quad \Big] - (1)$$

$$c_2 = c_2(0) e^{-\alpha\delta t} \quad \Big] - (2)$$

$$\therefore \vec{R} = c_1(0) e^{\frac{\alpha^2}{\beta} t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2(0) e^{-\alpha\delta t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$\therefore \vec{R}(t) =$



$$\nabla f(x,y,z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) \rightarrow \text{Gradient}$$

If $\vec{V} = (v_1, v_2, v_3)$

Then $\nabla \cdot \vec{V}(x, y, z) = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$

Gauss Law:

$$\nabla \vec{E}(x, y, z) = \frac{\rho}{\varepsilon_0}(x, y, z)$$

Force is conservative if $\vec{F}(\vec{r}) = -\nabla U(\vec{r})$

$$\therefore W = \int_{r_1}^{r_2} (-\nabla U)\, d\vec{r} = -(U_2 - U_1)$$