MA 6.101

Probability and Statistics

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RECAP

- Probability space, measurability, sigma algebra, $\mathcal{B}(\mathbb{R})$.
- Conditional probability, Bayes rule, law of total probability
- Independence, mutually exclusive event
- Conditional Independence
- Experiments involving counting

Motivation to random variables

Random variable

- Given a random experiment with associated $(\Omega, \mathcal{F}, \mathbb{P})$, it is sometimes difficult to deal directly with $\omega \in \Omega$. eg. rolling a dice ten times.
- Notice that each sample point $\omega \in \Omega$ is not a number but a sequence of numbers.
- Also, we may be interested in functions of these sample points rather than samples themselves. eg: Number of times 6 appears in the 10 rolls.
- In either case, it is often convenient to work in a new *simpler* probability space rather than the original space.
- Random variable is a device which precisely helps us make this mapping from $(\Omega, \mathcal{F}, \mathbb{P})$ to a 'simpler' $(\Omega', \mathcal{F}', P_X)$.
- P_X is called as an induced probability measure on Ω'.

Random variable as a measurable function

A random variable X is a function $X : \Omega \to \Omega'$ that transforms the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to $(\Omega', \mathcal{F}', P_X)$ and is '$(\mathcal{F}, \mathcal{F}')$-measurable'.

- The map $X : \Omega \to \Omega'$ implies $X(\omega) \in \Omega'$ for all $\omega \in \Omega$.
- A random variable could be non-injective and non-surjective.
- For event $B \in \mathcal{F}'$, the pre-image $X^{-1}(B)$ is defined as $X^{-1}(B) := \{\omega \in \Omega : X(\omega) \in B\}$

The '$(\mathcal{F}, \mathcal{F}')$-measurability' implies that for every $B \in \mathcal{F}'$, we have $X^{-1}(B) \in \mathcal{F}$.

Random variable as a measurable function

The '$(\mathcal{F}, \mathcal{F}')$-measurability' implies that for every $B \in \mathcal{F}'$, we have $X^{-1}(B) \in \mathcal{F}$.

- Since $X^{-1}(B) \in \mathcal{F}$, it can be measured using $\mathbb{P}$.
- What is $P_X(B)$?
- $P_X(B) := \mathbb{P}(X^{-1}(B))$ for all $B \in \mathcal{F}'$.
- $P_X(B)$ is therefore called as the induced probability measure.
- What if there is no $\omega \in \Omega$ such that $X(\omega) \in B$?

Random variables

- In general, the following convention is followed in most books:
  - Ω' will be the set of real numbers, denoted by $\mathbb{R}$.
  - $\mathcal{F}'$ as a result will be Borel σ-algebra, denoted by $\mathcal{B}(\mathbb{R})$.
  - Remember $\mathcal{B}(\mathbb{R})$?

Borel σ-algebra

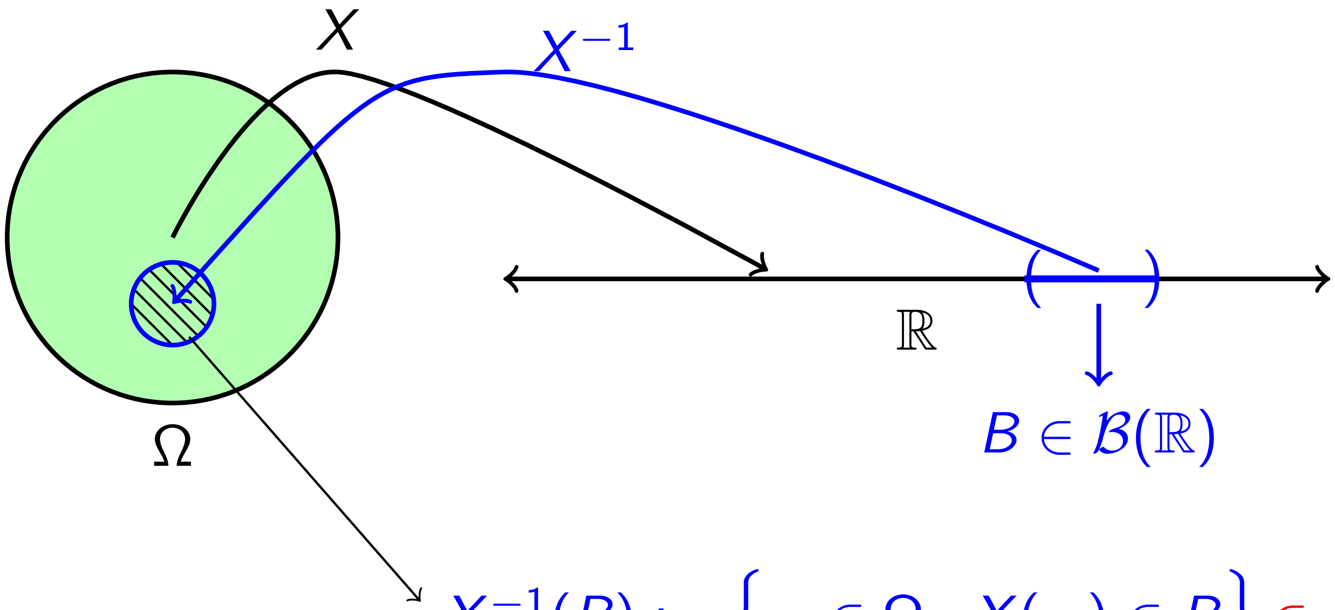
- Borel σ-algebra $\mathcal{B}(\mathbb{R})$:

  If $\Omega = \mathbb{R}$, then $\mathcal{B}(\mathbb{R})$ is the event set generated by open sets of the form (a, b) where $a \leq b$ and $a, b \in \mathbb{R}$.

- $\mathcal{B}(\mathbb{R})$ contains intervals of the form

$$[a, b]$$
$$[a, b)$$
$$(a, \infty)$$
$$[a, \infty)$$
$$(-\infty, b]$$
$$(-\infty, b)$$
$$\{a\}$$

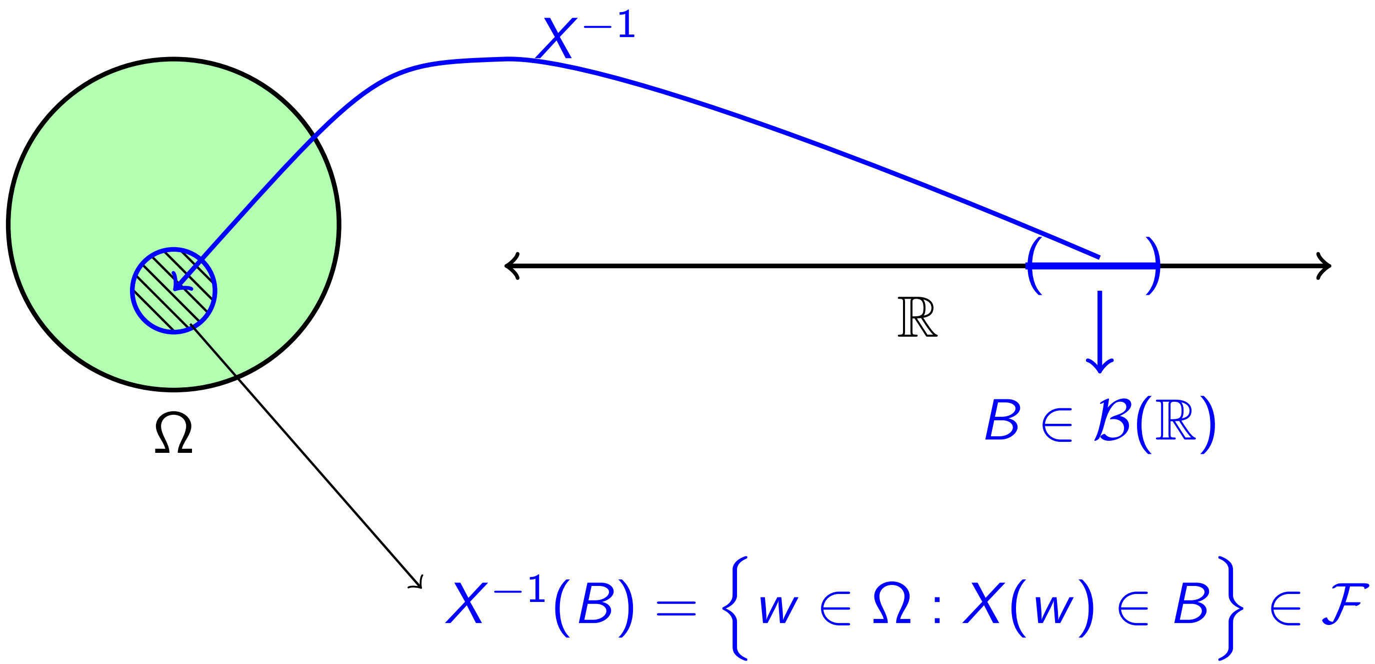
Random variables ($\Omega' = \mathbb{R}$)



$$X^{-1}(B) := \left\{ w \in \Omega : X(w) \in B \right\} \in \mathcal{F}$$

- $\Omega \xrightarrow{X} \mathbb{R}, \quad \mathcal{F} \xrightarrow{X} \mathcal{B}(\mathbb{R}),$ and $P(.) \xrightarrow{X} P_X(.)$
- Care must be taken such that the events you consider in the new event space $\mathcal{B}(\mathbb{R})$ are also valid events included in $\mathcal{F}$.
- $X^{-1}(B)$ is called as the preimage or the inverse image of B.

Definition of a random variables



A random variable X is a map $X : (\Omega, \mathcal{F}, P) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}), P_X)$ such that for each $B \in \mathcal{B}(\mathbb{R})$, the inverse image $X^{-1}(B) := \{w \in \Omega : X(w) \in B\}$ satisfies

$$X^{-1}(B) \in \mathcal{F} \text{ and}$$
$$P_X(B) = \Pr(w \in \Omega : X(w) \in B)$$

Random variable

- If Ω' is countable, then the random variable is called a discrete random variable.
- In this case it is convenient to use $\mathcal{F}'$ as power-set.
- If $\Omega' \subseteq \mathbb{R}$ or uncountable, then the random variable is a continuous random variable.
- In this case, $\mathcal{F}' = \mathcal{B}(\mathbb{R})$ and the definition is a bit tricky. We will deal with it later.
- You can also use $\Omega' = \mathbb{R}$ for a discrete random variable and survive! Lets not get into that.
- Notation: Random variables denoted by capital letters like X, Y, Z etc. anf their realizations by small letters x, y, z..

Discrete random variables

Example of rolling two dice

- Example of rolling two dice where we are interested in the sum of two dice.
- Suppose X = sum of two dice. Then we have

$$\begin{aligned}\Omega = \Big\{&(1,1),(1,2),\ldots,(1,6)\\ &(2,1),(2,2),\ldots,(2,6)\\ &\qquad\quad\vdots \qquad\qquad\qquad\qquad \xrightarrow{X} \qquad \Omega' = \big\{2,3,\ldots,12\big\}\\ &(6,1),(6,2),\ldots,(6,6)\Big\}\end{aligned}$$

- $\mathcal{F}$ and $\mathcal{F}'$ are power sets of Ω and Ω' respectively.
- Is X $(\mathcal{F},\mathcal{F}')$-measurable?

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- $X = 3$ is an event in $\mathcal{F}'$. What is its probability $P_X(3)$?
- $P_X(3) = \mathbb{P}(\{\omega \in \Omega : X(\omega) = 3\}) = \mathbb{P}(\{(1,2),(2,1)\})$.

In general for $x \in \Omega'$, we have $P_X(x) := \mathbb{P}(\{\omega \in \Omega : X(\omega) = x\})$.
Find $P_X(x)$ for all $x \in \Omega'$?