Discrete time Markov Chains (DTMC)

▶ A stochastic process $\{X_n, n \in \mathbb{Z}_+\}$ is a discrete time Markov chain if for any $n_1 < n_2 < \ldots < n_k < n$,

$$P(X_n = j | X_{n_1} = x_1, .., X_{n_k} = i) = P(X_n = j | X_{n_k} = i)$$

▶ This is called as the Markov property.

▶ $P(\text{next state} | \text{past states}, \text{present state}) = P(\text{next state} | \text{present state})$

▶ Why Chain? You can view the successive random variables as a chain of states being visited in a sequence and where the next state visited depends only on the current state.

▶ We will throughout assume that the state space $\mathcal{S}$ is countable.

Running example: Coin with memory!

▶ In a Markovian coin with memory, the outcome of the next toss depends on the current toss.

▶ $X_n = 1$ for heads and $X_n = -1$ otherwise. $\mathcal{S} = \{+1, -1\}$.

▶ Sticky coin : $P(X_{n+1} = 1|X_n = 1) = 0.9$ and $P(X_{n+1} = -1|X_n = -1) = 0.8$ for all n.

▶ Flippy Coin: $P(X_{n+1} = 1|X_n = 1) = 0.1$ while $P(X_{n+1} = -1|X_n = -1) = 0.3$ for all n.

▶ This can be represented by a transition diagram (see board)

▶ The one step transition probability matrix P for the two cases is $P_s = \begin{bmatrix} 0.9 & .1 \\ 0.2 & 0.8 \end{bmatrix}$ and $P_f = \begin{bmatrix} 0.1 & 0.9 \\ 0.7 & 0.3 \end{bmatrix}$

▶ The row corresponds to present state and the column corresponds to next state.

Running example: Dice with memory!

▶ In a markovian dice with memory, the outcome of the next roll depends on the current roll.

▶ $X_n = i$ for $i \in \mathcal{S}$ where $\mathcal{S} = \{1, \ldots, 6\}$.

▶ Example one-step transition probability matrix

$$P = \begin{bmatrix} 0.9 & .1 & 0 & 0 & 0 & 0 \\ 0 & .9 & .1 & 0 & 0 & 0 \\ 0 & 0 & 0.9 & 0.1 & 0 & 0 \\ 0 & 0 & 0 & 0.9 & 0.1 & 0 \\ 0 & 0 & 0 & 0 & 0.9 & 0.1 \\ 0.1 & 0 & 0 & 0 & 0 & 0.9 \end{bmatrix}$$

▶ State transition diagram on board

▶ Consider $S_n = \sum_{i=1}^n X_i$ and $\hat{\mu}_n = \frac{S_n}{n}$. What is $\lim_{n\to\infty} \hat{\mu}_n$?

▶ Cannot invoke SLLN as $\{X_i\}$ are not i.i.d.

▶ We will see later SLLN for Markov chains!

Finite dimensional distributions

▶ Consider a Markov dice with transition probability P.

▶ What is $P(X_0 = 4, X_1 = 5, X_2 = 6)$?

▶ $= P(X_2 = 6 | X_1 = 5, X_0 = 4)P(X_1 = 5 | X_0 = 4)P(X_0 = 4)$

▶ $= p_{65}p_{54}P(X_0 = 4)$.

▶ What is $P(X_0 = 4)$?

▶ This probability of starting in a particular state is called initial distribution of the markov chain.

Finite dimensional distributions

▶ Consider a DTMC $\{X_n, n \geq 0\}$ with transition matrix P.

▶ We assume M states and X_0 denotes the initial state.

▶ You can start in any starting state or may pick your starting state randomly.

▶ Let $\bar{\mu} = (\mu_1, \ldots, \mu_M)$ denote the initial distribution, i.e., $P(X_0 = x_0) = \mu_{x_0}$.

▶ How does one obtain the finite dimensional distribution $P(X_0 = x_0, X_1 = x_1, , X_2 = x_2)$?

▶ $P(X_0 = x_0, X_1 = x_1, , X_2 = x_2) = p_{x_1,x_2} p_{x_0,x_1} \mu_{x_0}$.

▶ In general,
$P(X_0 = x_0, X_1 = x_1, \ldots X_k = x_k) = p_{x_{k-1},x_k} \times \ldots \times p_{x_0,x_1} \mu_{x_0}$

Chapman Kolmogorov Equations for DTMC

▶ Consider a Markov coin and its transition probability matrix

$$P = \begin{bmatrix} p_{1,1} & p_{1,-1} \\ p_{-1,1} & p_{-1,-1} \end{bmatrix}.$$

▶ Given $X_0 = 1$, what is $P(X_2 = 1)$?

$$
\begin{aligned}
P(X_2 = 1 | X_0 = 1) &= P(X_2 = 1 | X_1 = 1, X_0 = 1)P(X_1 = 1 | X_0 = 1) \\
&+ P(X_2 = 1 | X_1 = -1, X_0 = 1)P(X_1 = -1 | X_0 = 1) \\
&= p_{1,1}^2 + p_{-1,1}p_{1,-1}
\end{aligned}
$$

▶ Here the first inequality follow from the fact that

$$P(C|A) = P(C|BA)P(B|A) + P(C|B^c A)P(B^c|A) \text{ HW: Verify}$$

▶ Similarly, $P(X_2 = -1 | X_0 = 1), P(X_2 = 1 | X_0 = -1), P(X_2 = -1 | X_0 = -1)$ can be obtained and these are elements of a two-step transition matrix $P^{(2)}$.

Chapman Kolmogorov Equations for DTMC

▶ The two step transition probability matrix $P^{(2)}$ is given by

$$P^{(2)} = \begin{bmatrix} p_{1,1}^2 + p_{1,-1}p_{-1,1} & p_{1,1}p_{1,-1} + p_{1,-1}p_{-1,-1} \\ p_{-1,1}p_{1,1} + p_{-1,-1}p_{-1,1} & p_{-1,1}p_{1,-1} + p_{-1,-1}^2 \end{bmatrix}.$$

▶ This implies that $P^{(2)} = P \times P = P^2$.

▶ In general, $P^{(n)} = P^n$.

▶ Chapman-Kolmogorov equations are a further generalization of this.

$$P^{(n+l)} = P^{(n)}P^{(l)}$$

▶ We wont see the proof of this.

Classification of states

▶ Consider a Markov process with state space $\mathcal{S}$

▶ We say that j is accessible from i if $p_{ij}^n > 0$ for some n.

▶ This is denoted by $i \rightarrow j$.

▶ if $i \rightarrow j$ and $j \rightarrow i$ then we say that i and j communicate. This is denoted by $i \leftrightarrow j$.

> A chain is said to be irreducible if $i \leftrightarrow j$ for all $i, j \in \mathcal{S}$.

▶ Are the examples of Markovian coin and dice we have considered till now irreducible? check!

Recurrent and Transient states

▶ We say that a state i is recurrent if
$F_{ii} = P(\text{ ever returning to } i \text{ having started in } i \text{ }) = 1.$

▶ F_{ii} is not easy to calculate. (Not part of this course)

▶ If a state is not recurrent, it is transient.

▶ For a transient state i, $F_{ii} < 1$.

▶ If $i \leftrightarrow j$ and i is recurrent, then j is recurrent.

Recurrent   Transient   Recurrent   Recurrent