Probability and Statistics: MA6.101

Tutorial 1

Topics Covered: Introductory Calculus, Maxima & Minima, Stationary Points, Sets, Basic Sequences and Series, Probability and Statistics, Linear Algebra

Linear Algebra

[Sriyansh]

Q1: Let

$$F : \mathbb{R}^2 \to \mathbb{R}^2, \quad F(x,y) = \big(u(x,y),\, v(x,y)\big) = \big(x^2 y,\ e^{xy} + x + y\big).$$

Compute the Jacobian matrix $J_F(x,y)$. What is the geometrical significance of the Jacobian Matrix and its determinant?

Q2: Let $P \in \mathbb{R}^{n\times n}$ be a matrix for which each row sums to 1, i.e.,

$$\forall i \in \{1,\dots,n\}, \quad \sum_{j=1}^{n} P_{ij} = 1.$$

Prove that the equation

$$\pi = \pi P$$

always has a non-zero solution $\pi^\top \in \mathbb{R}^n$

Let

$$P = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.2 & 0.3 & 0.5 \\ 0.1 & 0.1 & 0.8 \end{bmatrix}$$

Find the solution π for the given P.

Introductory Calculus

[Aryan R Chugh]

Q1: Evaluate the limit:

$$\lim_{x\to 0} \frac{\int_0^x \tan t\, dt}{x}$$

[Shreyas Mehta]

Q2: Evaluate the integral:

$$I = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (x^2 + y^2)\, dy\, dx$$

[Shreyas Mehta]

Q3: Let

$$I_n = \int_{-\infty}^{\infty} x^{2n} e^{-x^2}\, dx$$

for any non-negative integer n.

(a) Show that $I_0 = \sqrt{\pi}$ and

$$I_1 = \int_{-\infty}^{\infty} x^2 e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$$

(b) Using integration by parts, derive a recurrence relation of the form:

$$I_n = \frac{(2n-1)}{2} I_{n-1}$$

(c) [**Bonus**] Using mathematical induction and the recurrence above, prove that:

$$I_n = \frac{(2n-1)!! \cdot \sqrt{\pi}}{2^n}$$

where $(2n-1)!! = 1 \cdot 3 \cdot 5 \cdots (2n-1)$ denotes the **double factorial** of odd numbers.

Maxima / Minima and Stationary Points

[Aryan R Chugh]

Q1: Let

$$f(x) = \tan^{-1}(e^x - e^{-x}).$$

Find all stationary points and their nature.

Sets

[Shreyas Mehta]

Q1: **Bonferroni's inequality.**

(a) Prove that for any two events A and B, we have

$$\mathbb{P}(A \cap B) \geq \mathbb{P}(A) + \mathbb{P}(B) - 1.$$

(b) Generalize to the case of n events $A_1, A_2, \ldots, A_n$, by showing that

$$\mathbb{P}(A_1 \cap A_2 \cap \cdots \cap A_n) \geq \mathbb{P}(A_1) + \mathbb{P}(A_2) + \cdots + \mathbb{P}(A_n) - (n-1).$$

Permutation and Combinations

[Harshit Lalwani]

1. Eight students are running for the titles of "Felicity Coordinator" and "Corporate Head". In how many different ways can the two positions be assigned?
   A. 28
   B. 56
   C. 56
   D. 64
   E. 72
2. A trivia competition team must consist of 4 students chosen from 15. How many different teams are possible?
   A. 1345
   B. 1260
   C. 1365
   D. 1512
   E. 1780
3. Twelve distinct prizes are to be awarded to the top 3 winners (first, second, third). In how many ways can the prizes be distributed?
   A. 132
   B. 720
   C. 1320
   D. 1440
   E. 1560
4. At a food stall, a student selects one main course, one side, and one drink from 5, 3, and 4 options respectively. How many total meal combinations are possible?
   A. 45
   B. 60
   C. 80
   D. 72
   E. 120
5. A club with 20 members is electing a president, vice president, and secretary. Each must be distinct. In how many ways can the officers be elected?
   A. 684
   B. 68400
   C. 6840
   D. 6080
   E. 7200
6. Six cards are to be placed in six envelopes numbered 1 to 6 such that no card is placed in the envelope with the same number, and card 1 is placed in envelope 2. How many such arrangements exist?

A. 44
B. 52
C. 53
D. 64
E. 68

7. Evaluate the sum: $\sum_{r=1}^{15} \frac{r^2\binom{15}{r}}{\binom{15}{r-1}}$
   A. 540
   B. 600
   C. 640
   D. 680
   E. 720

8. Aanchik wants at least 10 matching pairs of socks in darkness. The drawer contains 100 red, 80 green, 60 blue, and 40 black socks. What is the minimum number he must pick to guarantee this?
   A. 21
   B. 23
   C. 24
   D. 50
   E. 30

9. A bag has 10 white and 3 black balls. Balls are drawn one-by-one without replacement until all 3 black balls are drawn. What is the probability that the last black ball appears on the 7th draw?
   A. $\frac{15}{286}$
   B. $\frac{1}{13}$
   C. $\frac{20}{286}$
   D. $\frac{13}{286}$
   E. $\frac{1}{26}$

10. Two friends agree to meet between 5–6 PM. Whoever arrives first waits 15 minutes and leaves. What is the probability they meet?
   A. $\frac{1}{4}$
   B. $\frac{1}{2}$
   C. $\frac{7}{16}$
   D. $\frac{3}{4}$
   E. $\frac{15}{32}$

Miscellaneous

[Aryan R Chugh]

Q1: In a logistics warehouse, a robotic system operates over a period of 30 days to stack boxes according to the following rules:

- On the k-th day ($1 \le k \le 30$):
  - The robot stacks exactly k boxes.
  - Each box weighs $\frac{1}{2^k}$ kilograms.
  - Placing each box takes k minutes.
  - The robot's overall efficiency decreases exponentially: the effective time to place each box is scaled by a factor of $\frac{1}{2^k}$.

Let:

$$W = \sum_{k=1}^{30} k \cdot \frac{1}{2^k} \quad \text{and} \quad T = \sum_{k=1}^{30} k^2 \cdot \frac{1}{2^k}$$

Tasks

(a) Evaluate W in closed form.

(b) Simplify or evaluate T.

(c) Compute the average time per kilogram over the 30-day period:

$$\frac{T}{W}$$

(d) Assume the robot continues this stacking process indefinitely (i.e., $k \to \infty$). Compute the exact values of:

$$W_\infty = \sum_{k=1}^{\infty} k \cdot \frac{1}{2^k}, \qquad T_\infty = \sum_{k=1}^{\infty} k^2 \cdot \frac{1}{2^k}$$

Hence, determine:

$$\lim_{n\to\infty} \frac{T_n}{W_n}$$