# RECAP

- We looked at n-length random vectors $\mathbf{X}$ which are essentially multivariate random variables.
- Its CDF/pdf is simply joint CDF/pdf of components
- $E[\mathbf{X}] = [E[X_1], \dots E[X_n]]$. $C_{\mathbf{X}}$ is the covariance matrix.
- Let $\mathbf{Y} = G(\mathbf{X})$ where $G : \mathbb{R}^n \rightarrow \mathbb{R}^n$, continuous invertible with continuous partial derivatives. Let $H$ denote its inverse. Then $f_{\mathbf{Y}}(\mathbf{y}) = f_{\mathbf{X}}(H(\mathbf{y}))|J|$ where $J$ is the determinant of the Jacobian matrix.
- If $\mathbf{Y} = \mathbf{AX} + \mathbf{b}$, then $C_{\mathbf{Y}} = AC_{\mathbf{X}}A^T$ and

$$f_{\mathbf{Y}}(\mathbf{y}) = \frac{1}{|det(A)|} f_{\mathbf{X}}(A^{-1}(\mathbf{y} - \mathbf{b}))$$

# RECAP

- A random vector $\mathbf{Z}$ is called as a standard normal vector if its components $Z_i$ are independent and standard normal.

$$f_{\mathcal{Z}}(\mathbf{z}) = \frac{1}{(2\pi)^{n/2}} e^{\{-\frac{1}{2}\mathbf{z}^T\mathbf{z}\}}$$

- For $\mathbf{X} = A\mathbf{Z} + \boldsymbol{\mu}$. and $E[\mathbf{X}] = \boldsymbol{\mu}$ and $\Sigma := C_{\mathbf{X}} = AA^T$ and

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}\sqrt{det(C_{\mathbf{X}})}} e^{\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T C_X^{-1}(\mathbf{x}-\boldsymbol{\mu})\}}$$

- We say that $\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$

- A random vector $\mathbf{X}$ is Gaussian iff for some $A$ and $\boldsymbol{\mu}$, it can be written as $\mathbf{X} = A\mathbf{Z} + \boldsymbol{\mu}$

# Affine transformations preserve Gaussianity

- Suppose $\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$, then what can we say about $\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{b}$? Is it a Gaussian vector ?
- Easy to see that $E[\mathbf{Y}] = AE[\mathbf{X}] + b$ and $C_{\mathbf{Y}} = A\Sigma A^T$.
- Like in the univariate case, we can use MGF (for multivariate MGF see Bertsekas) to show the following

> Suppose $\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$. Now consider $\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{b}$. Then we have $\mathbf{Y} \sim \mathcal{N}(AE[\mathbf{X}] + b, A\Sigma A^T)$.

# Equivalent definitions of a Gaussian vector

The following are equivalent definitions (without proof)

$\mathbf{X} \sim \mathcal{N}(\mu, \Sigma)$ iff for some $A$ and $\boldsymbol{\mu}$, it can be written as $\mathbf{X} = A\mathbf{Z} + \boldsymbol{\mu}$

$\mathbf{X}$ is Gaussian with mean vector $\mu$ and covariance matrix $\Sigma$ (denoted by $\mathbf{X} \sim \mathcal{N}(\mu, \Sigma)$) iff it has the pdf

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}\sqrt{det(C_{\mathbf{X}})}} e^{\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T C_X^{-1}(\mathbf{x}-\boldsymbol{\mu})\}}$$

$\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$ iff for all vectors $\mathbf{a} \in \mathbb{R}^n$, it turns out that $\mathbf{a}^{\mathbf{T}}\mathbf{X}$ is univariate Gaussian $\mathcal{N}(a^T\boldsymbol{\mu}, a^T\Sigma a)$.

# Marginalization and Conditioning (without proof)

- Let $\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$ and partition $\mathbf{X}$ as $[\mathbf{X_1}, \mathbf{X_2}]^T$ where $\mathbf{X_1}$ is $m \times 1$ and $\mathbf{X_2}$ is $(n-m) \times 1$.
- We similarly have $\boldsymbol{\mu} = [\boldsymbol{\mu_1}, \boldsymbol{\mu_1}]^T$ and $\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$ where $\Sigma_{11}$ is $m \times m$ matrix and so on ..

**Marginalization property:** The $m-$dimensional marginal distribution of $\mathbf{X_1}$ is $\mathcal{N}(\boldsymbol{\mu_1}, \Sigma_{11})$ and $\mathbf{X_2}$ is $\mathcal{N}(\boldsymbol{\mu_2}, \Sigma_{22})$

**Conditioning property:** The $m-$dimensional conditional distribution of $\mathbf{X_1}$ given $\mathbf{X_2} = \mathbf{x_2}$ is

$$\mathcal{N}(\boldsymbol{\mu_1} + \Sigma_{12}\Sigma_{22}^{-1}(\mathbf{x_2} - \boldsymbol{\mu_2}), \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}).$$

Note the decrease in variance which does not depend on $\mathbf{x_2}$.

# Towards Bivariate Gaussians

- Suppose $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$ and $X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$. Also suppose $X_1$ and $X_2$ are independent. Then is $\mathbf{X} = [X_1, X_2]^T$ bivariate Gaussian ? What is the mean vector and Covariance matrix ?
- For $\mathbf{x} = [x_2, x_2]^T$, we have

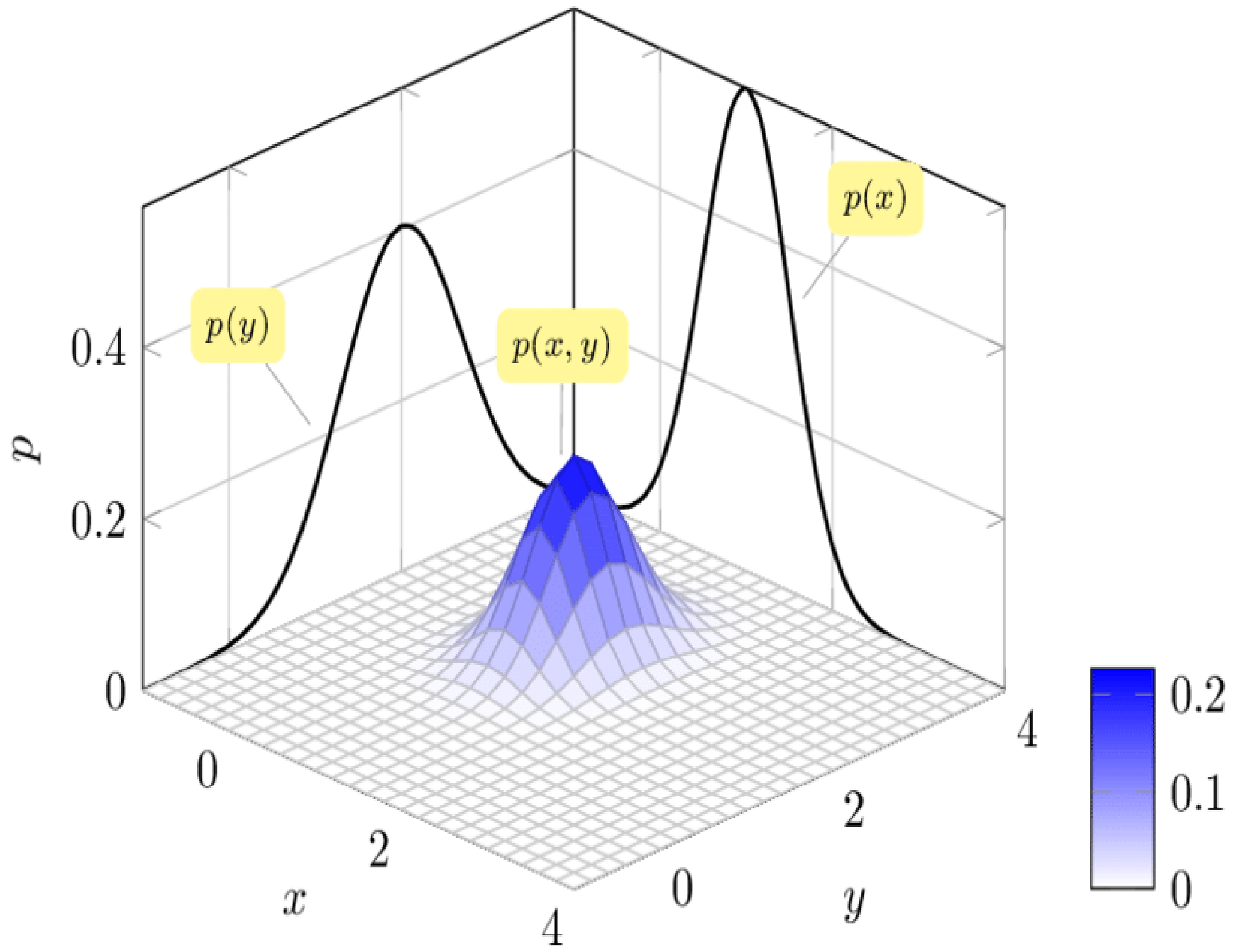
$$\begin{aligned} f_{\mathbf{X}}(\mathbf{x}) &= f_{X_1}(x_1) f_{X_2}(x_2) \\ &= \frac{1}{2\pi\sigma_1\sigma_2} e^{-\left(\frac{(x_1-\mu_1)^2}{2\sigma_1^2} + \frac{(x_2-\mu_2)^2}{2\sigma_2^2}\right)} \\ &= \frac{1}{(2\pi)\sqrt{det(\Sigma)}} e^{\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T \Sigma^{-1} (\mathbf{x}-\boldsymbol{\mu})\}} \end{aligned}$$

  where $\Sigma = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}$ and hence $\mathbf{X}$ is bivariate Gaussian.
- In general, a vector composed of independent Gaussians is a Gaussian vector. The converse is not true: a vector of dependent Gaussian components need not be Gaussian vector (EX 5.35 in probabilitycourse.com).

# Bivariate Gaussians

- In general, $X_1$ and $X_2$ need not be independent in which case we have a general bivariate Gaussian $\mathbf{X} = [X_1, X_2]^T \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$ where $\mu = [\mu_1, \mu_2]^T$ and $\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$
- Show that Bivariate Gaussian is closed under marginalization,i.e., $X_1 \sim \mathcal{N}(\mu_1, \Sigma_{11})$ and $X_2 \sim \mathcal{N}(\mu_2, \Sigma_{22})$.
- Show that Bivariate Gaussian is closed under conditioning. (For proof see Theorem 5.4, probabilitycourse.com).
- This means that Given $X_2 = x_2$, one can show that $f_{X_1|X_2}(x_1|x_2)$ is Gaussian.
- These two properties make multivariate Gaussians as efficient modelling tools and the handy in Gaussian processes and Bayesian optimization.

# Some Bivariate gaussian pdfs



# Some Bivariate gaussian pdfs