Lucas Paul Unterweger

Sophia Oberbrinkmann

Assignment 1
Adv. Macro II
2023-04-25

0 Preliminary

We hereby declare that the answers to the given assignment are entirely our own, resulting from our own work effort only. Our team members contributed to the answers of the assignment in the following proportions:

⇒ Unterweger Lucas: 50%

⇒ Oberbrinkmann Sophia: 50%

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Signed by:	Lucas Paul Unterweger
Date:	25.04.2023 16:39:05

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1 Business cycle stylized facts (4 points)

For this question, we chose to use data from **Ireland** which we gathered from *EUROSTAT*. The R code which we used to generate these results is attached to the submission of the assignment or can be found on GitHub.

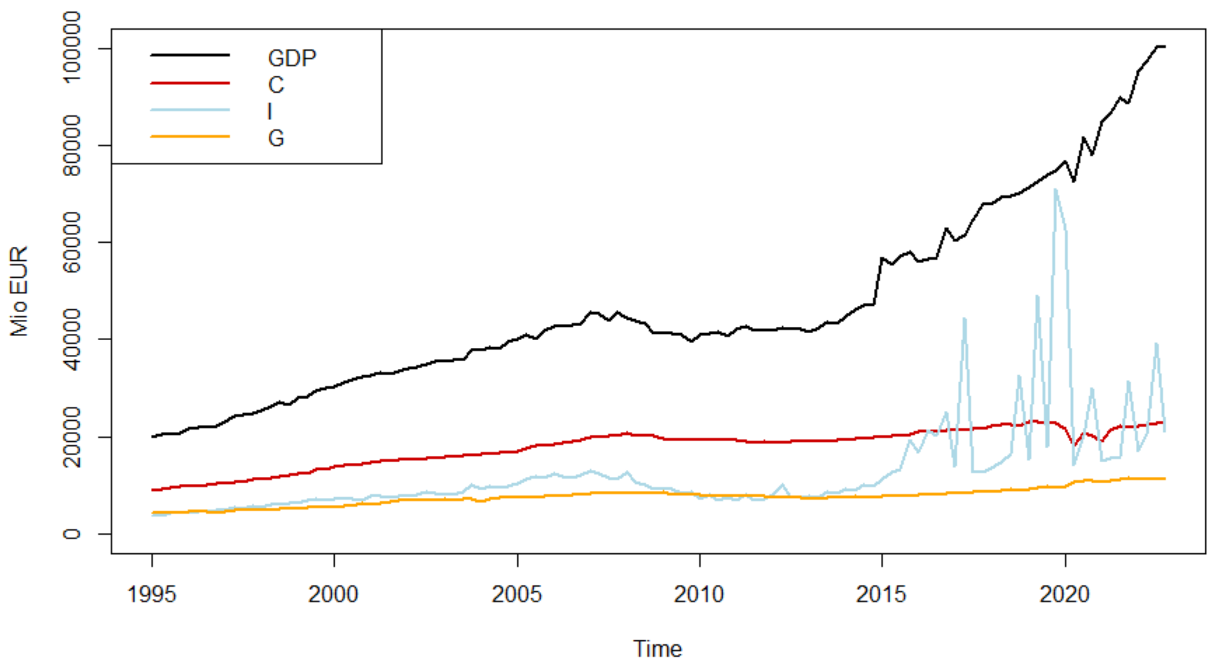

Figure 1: GDP, Investment, Consumption and State Expenditures of Ireland in Mio EUR

The sample means of C/Y, I/Y and G/Y are the following:

Sample Mean of	Value
C/GDP	0.4063485
I/GDP	0.2530171
G/GDP	0.1719054

To obtain the stationary (cyclical) component of these macroeconomic time series we are going to apply HP-filtering to the logged time series. To compute these using R, we will use the *mFilter* package. The resulting plots are given below:

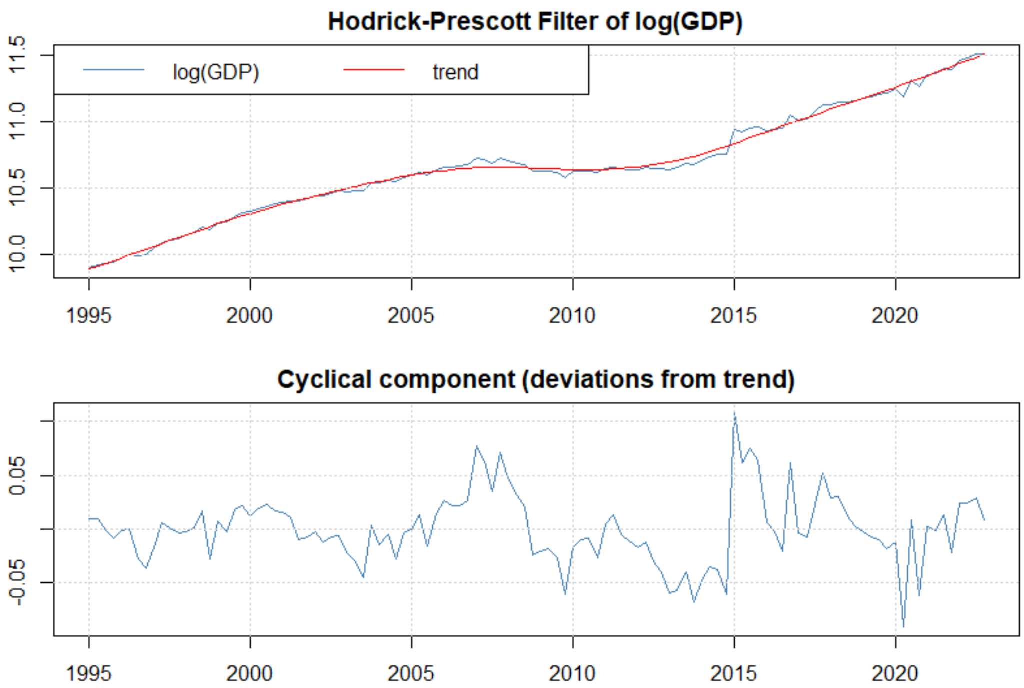

Figure 2: HP Filter applied to logged GDP

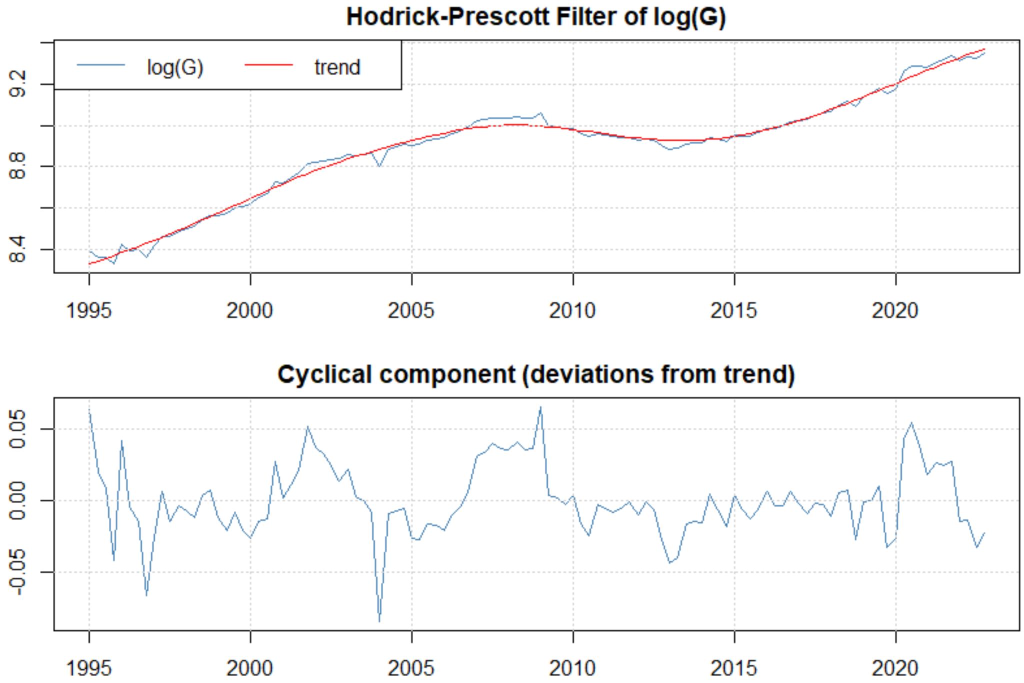

Figure 3: HP Filter applied to logged G

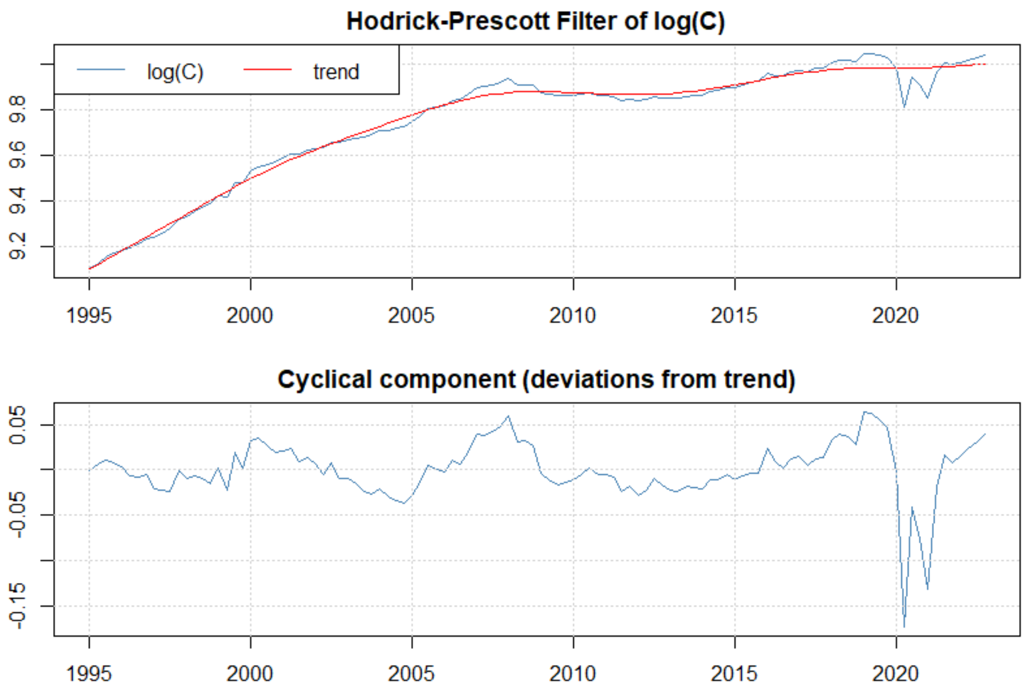

Figure 4: HP Filter applied to logged C

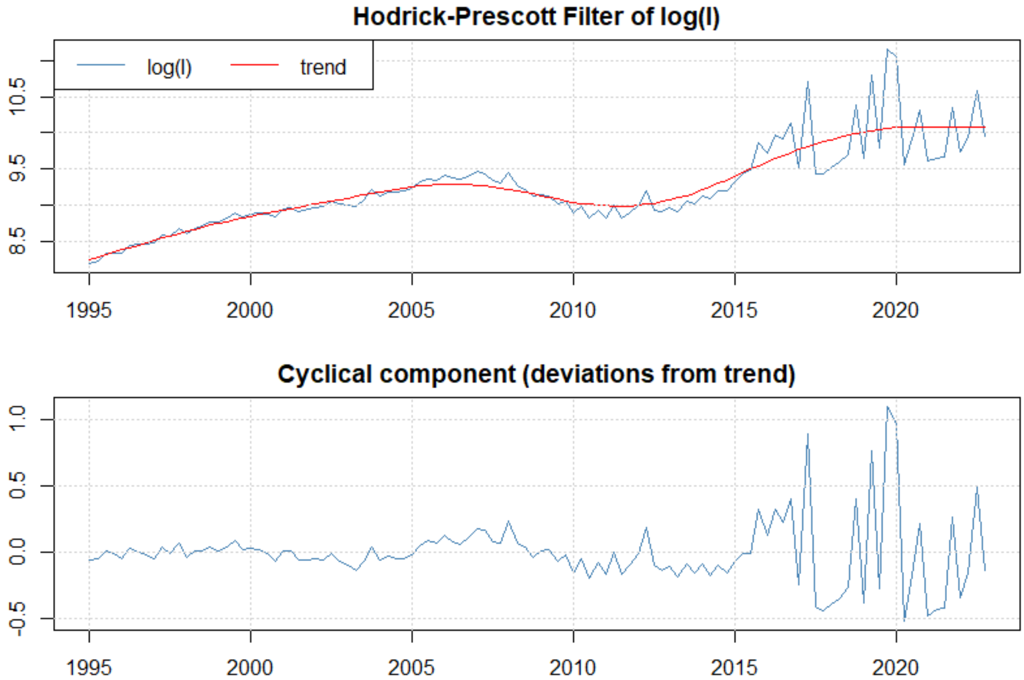

Figure 5: HP Filter applied to logged I

We continue by displaying some business cycle stylized facts of the cyclical component of these time series. A table with the main key indicators can be found below:

Variable	Standard Deviation	Rel. Standard Deviation	Cont. Output Correlation
GDP	0.0323092	1.0000000	1.0000000
I	0.2546824	7.8826591	0.1112197
C	0.0319381	0.9885142	0.4768935
G	0.0247057	0.7646655	0.1549818

Table 2: Stylized facts of the cyclical component for the entire period

Splitting up the time series into a pre 2008 and a post 2008 time series results in the following two tables:

Variable	Standard Deviation	Rel. Standard Deviation	Cont. Output Correlation
GDP	0.0158288	1.000000	1.0000000
I	0.0505051	3.190716	0.4005525
C	0.0159168	1.005561	0.4630622
G	0.0255812	1.616120	0.1368501

Table 3: Stylized business cycle facts for the pre 2008 preiod

Variable	Standard Deviation	Rel. Standard Deviation	Cont. Output Correlation
GDP	0.0380350	1.0000000	1.0000000
I	0.3435733	9.0330832	0.0549968
C	0.0378345	0.9947279	0.3921984
G	0.0191117	0.5024774	0.0065691

Table 4: Stylized business cycle facts for the post 2008 period

Interpretation:

GDP, Consumption and Investment fluctuate more in the post 2008 period compared to the pre 2008 period and therefore we conclude that the fluctuation in the whole period is rather determined by the second period. In contrast to this result, the governmental expenditures fluctuate more in the first (post 2008) than in the second (pre 2008) period, which can also be seen by examining the cyclical component plots from above. As expected, Investment fluctuates much more than Consumption. The Literature suggests that Investment is about three times more volatile than output. Especially in the post 2008 period our results accelerate this rate.

The *Relative Standard Deviation* is calculated by the standard deviation of one economic variable divided by the standard deviation of output and hence the relative Standard Deviation of Output is 1. Investment, again, has a higher relative standard deviation compared to Output. Comparing Consumption and Governmental Expenditure, the relative volatility of Consumption is higher in the second period whereas the relative volatility of governmental expenditure is higher in the fist period. Overall, Consumption has a higher relative volatility.

The overall quite low correlation of Output and Investment is rather surprising and was expected to be higher, however this might be a special property of Ireland and its high volatility in investment in the recent decade. Thus especially in the second period the correlation is very low. Compared to that, the correlation of Consumption and Output is higher, but still not as high as expected. Governmental Expenditure also has a relatively low correlation with Output.

2 A real business cycle model (5 points)

Expected discounted utility of the representative household:

$$U = E_0 \left[\sum_{t=0}^{\infty} \beta_t \frac{C_t^{1-\sigma}}{1-\sigma} + \theta \log(1 - N_t) \right]$$

subject to

$$C_t + K_{t+1} + P_t EN_t \leq A_t K_t^{\alpha} N_t^{\gamma} EN_t^{1-\alpha-\gamma} + (1-\delta) K_t$$

2.1 Social Planner's Intertemporal Optimization Problem

We start by setting up our Lagrangian system where we maximize U w.r.t. the constraint given above. The Lagrangian is then given by

$$L_t = E_t \left[\sum_{t=0}^{\infty} \beta^t \left\{ \frac{C_t^{1-\sigma} - 1}{1-\sigma} + \theta \log(1 - N_t) + \lambda_t \left(A_t K_t^{\alpha} N_t^{\gamma} EN_t^{1-\alpha-\gamma} + (1-\delta) K_t - C_t - K_{t+1} - P_t EN_t \right) \right\} \right]$$

We can now use *first order optimality conditions* and optimize w.r.t the *current consumption* C_t, *tomorrows capital stock* K_{t+1}, *hours worked* N_t, *energy consumption* EN_t and the *Lagrange multiplier* λ_t. This results in the following five equations:

1st FOC w.r.t. C_t:

$$\begin{aligned} \frac{\partial L}{\partial C_t} &= E_t \beta^t \left\{ (1-\sigma) \frac{C_t^{-\sigma}}{1-\sigma} - \lambda_t \right\} \stackrel{!}{=} 0 \\ \iff C_t^{-\sigma} &= \lambda_t \end{aligned}$$

2nd FOC w.r.t. N_t:

$$\begin{aligned} \frac{\partial L}{\partial N_t} &= E_t \beta^t \left\{ \frac{(-1) \cdot \theta}{1 - N_t} + \lambda_t A_t K_t^{\alpha} \gamma N_t^{\gamma-1} EN_t^{1-\alpha-\gamma} \right\} \stackrel{!}{=} 0 \\ \iff \frac{\theta}{1 - N_t} &= \lambda_t A_t K_t^{\alpha} \gamma N_t^{\gamma-1} EN_t^{1-\alpha-\gamma} \\ \iff \frac{\theta}{1 - N_t} \cdot \frac{1}{A_t K_t^{\alpha} \gamma N_t^{\gamma-1} EN_t^{1-\alpha-\gamma}} &= \lambda_t \end{aligned}$$

3rd FOC w.r.t. K_{t+1}:

$$\begin{aligned} \frac{\partial L}{\partial K_{t+1}} &= E_t \beta^t \{-\lambda_t\} + E_t \beta^{t+1} \left\{ \lambda_{t+1} (A_{t+1} \alpha K_{t+1}^{\alpha-1} N_{t+1}^{\gamma} EN_{t+1}^{1-\alpha-\gamma} + (1-\delta)) \right\} \stackrel{!}{=} 0 \\ \iff \beta^t \lambda_t &= E_t \beta^{t+1} \lambda_{t+1} A_{t+1} \alpha K_{t+1}^{\alpha-1} N_{t+1}^{\gamma} EN_{t+1}^{1-\alpha-\gamma} + \lambda_{t+1}(1-\delta) \\ \iff \lambda_t &= \beta E_t \lambda_{t+1} A_{t+1} \alpha K_{t+1}^{\alpha-1} N_{t+1}^{\gamma} EN_{t+1}^{1-\alpha-\gamma} + \lambda_{t+1}(1-\delta) \end{aligned}$$

4th FOC w.r.t. EN:

$$\frac{\partial L}{\partial EN_t} = E_t\beta^t \left\{\lambda_t A_t K_t^\alpha N_t^\gamma (1-\alpha-\gamma) EN_t^{-\alpha-\gamma} - \lambda_t P_t\right\} \overset{!}{=} 0$$
$$\iff \lambda_t P_t = \lambda_t A_t K_t^\alpha N_t^\gamma (1-\alpha-\gamma) EN_t^{-\alpha-\gamma}$$
$$\iff P_t = A_t K_t^\alpha N_t^\gamma (1-\alpha-\gamma) EN_t^{-\alpha-\gamma}$$

5th FOC w.r.t. λ_t:

$$\frac{\partial L}{\partial \lambda_t} = A_t K_t^\alpha N_t^\gamma EN_t^{1-\alpha-\gamma} + (1-\delta)K_t - C_t - K_{t+1} - P_t EN_t \overset{!}{=} 0$$
$$\iff A_t K_t^\alpha N_t^\gamma EN_t^{1-\alpha-\gamma} + (1-\delta)K_t = C_t + K_{t+1} + P_t EN_t$$

By assumption, we know that investment is governed by the *capital law of motion*, which is given by the following equation:

$$I_t = K_{t+1} - (1-\delta)K_t$$

We will continue by equating the 1st and 2nd as well as the 1st and 3rd FOC, which results in

$$C_t^{-\sigma} = \frac{\theta}{1-N_t} \cdot \frac{1}{A_t K_t^\alpha \gamma N_t^{\gamma-1} EN_t^{1-\alpha-\gamma}}$$

$$C_t^{-\sigma} = \beta E_t \left[\lambda_{t+1} A_{t+1} \alpha K_{t+1}^{\alpha-1} N_{t+1}^\gamma EN_{t+1}^{1-\alpha-\gamma} + \lambda_{t+1}(1-\delta)\right]$$

We also know that

$$Y_t = A_t K_t^\alpha N_t^\gamma EN_t^{1-\alpha-\gamma}$$

Using that and the exogenous processes given, we can finally set up our system of 8 equations:

$$C_t^{-\sigma} A_t K_t^\alpha \gamma N_t^{\gamma-1} EN_t^{1-\alpha-\gamma}(1-N_t) = \theta \tag{1}$$
$$C_t^{-\sigma} = \beta E_t \left[C_{t+1}^{-\sigma}\left(A_{t+1}\alpha K_{t+1}^{\alpha-1} N_{t+1}^\gamma EN_{t+1}^{1-\alpha-\gamma} + (1-\delta)\right)\right] \tag{2}$$
$$P_t = A_t K_t^\alpha N_t^\gamma (1-\alpha-\gamma) EN_t^{-\alpha-\gamma} \tag{3}$$
$$A_t K_t^\alpha N_t^\gamma EN_t^{1-\alpha-\gamma} + (1-\delta)K_t = C_t + K_{t+1} + P_t EN_t \tag{4}$$
$$I_t = K_{t+1} - (1-\delta)K_t \tag{5}$$
$$Y_t = A_t K_t^\alpha N_t^\gamma EN_t^{1-\alpha-\gamma} \tag{6}$$
$$\log(A_{t+1}) = \rho_A \log(A_t) + \epsilon_{A,t+1} \tag{7}$$
$$\log(P_{t+1}) = \rho_P \log(P_t) + \epsilon_{P,t+1} \tag{8}$$

2.2 Economic Interpretation of Conditions

Interpretation of the FOC:

- 1st FOC w.r.t. C_t: The Lagrange multiplier has the the interpretation of the value of relaxing the constraint by one unit (Slide 51). Here it gives us the additional utility value of consuming one unit more.

- 2nd FOC w.r.t. N_t: Here the Lagrange multiplier tells us the additional utility value of working one more hour.
- 3rd FOC w.r.t. K_{t+1}: If the resource constraint is relaxed by one unit, the additional unit could either be eaten up, which yields additional utility (Captures by the LHS), or it could be saved and used in the future by increasing the capital stock. This is captured by the RHS.
- 4th FOC w.r.t. EN: The total Price of Energy must equal the amount of Output that is contributed to Energy.
- 5th FOC w.r.t. λ_t: This is the resource constraint and it holds with equaltity.

Interpretation of the equilibrium conditions:

- (1) This equation can be thought of as the equilibrium in the labour market. In other words, the wage has to be equal to the marginal product of labour.
- (2) This equation combines the FOC w.r.t C_t and the FOC w.r.t. K_{t+1}. This is the inter-temporal Euler equation. It describes the evolution of economic variables along the optimal path.
- (3) If we rewrite it as $P_t EN_t = Y_t(1-\alpha-\gamma)$, which can be interpreted as in equilibrium, the total cost of energy $(P_t \cdot EN_t)$ is equal to the total output share of energy.
- (4) This is an equilibrium condition which states, that Production plus the not depreciated capital should equal consumption plus the capital in the next period plus the total price of Energy. It simply states everything produced is consumed or used as capital and nothing is "left over".
- (5) This equations represents the capital law of motion.Ccapital in period (k+1) is created by the not depreciated capital in period (k) plus investment.
- (6) This equation is the production function.
- (7) This equation captures the shocks to total factor productivity in form of a (log) autoregressive process.
- (8) This equation captures the shocks to the energy price in form of a (log) autoregressive process.

3 Understanding impulse response and model simulation (13.5 points)

3.1 Steady State

For this question, the following parameter values are given:

Parameter	Value	Parameter	Value
β	0.99	σ	1
θ	3.48	α	0.3
δ	0.025	$\bar{A}$	1
$\bar{P}$	1	ρ_A	0.95
σ_A	0.007	ρ_P	0.5

σ_P	0.00001	γ	0.65

We follow the recommended approach from the assignment and start with step (1) by using the steady-state capital Euler equation, which is given by:

$$C_t^{-\sigma} = \beta E_t \left[C_{t+1}^{-\sigma} A_{t+1} \alpha K_{t+1}^{\alpha-1} N_{t+1}^{\gamma} EN_{t+1}^{1-\alpha-\gamma} + \lambda_{t+1}(1-\delta) \right]$$

We now use the steady-state notation:

$$\begin{aligned}
\bar{C}^{-\sigma} &= \beta E_t \left[\bar{C}^{-\sigma} (\bar{A}\alpha \bar{K}^{\alpha-1} \bar{N}^{\gamma} \bar{EN}^{1-\alpha-\gamma} + (1-\delta)) \right] \\
\iff 1 &= \beta E_t \left[(\bar{A}\alpha \bar{K}^{\alpha-1} \bar{N}^{\gamma} \bar{EN}^{1-\alpha-\gamma} \frac{\bar{K}}{\bar{K}} + (1-\delta)) \right] \\
\iff 1 &= \beta E_t \left[(\alpha \bar{Y} \frac{1}{\bar{K}} + (1-\delta)) \right] \\
\iff 1 &= \beta (\alpha \bar{Y} \frac{1}{\bar{K}} + (1-\delta)) \\
\iff 1 - \beta(1-\delta) &= \beta\alpha \frac{\bar{Y}}{\bar{K}} \\
\iff \frac{1-(1-\delta)\beta}{\beta\alpha} &= \frac{\bar{Y}}{\bar{K}} \\
\iff \frac{\beta\alpha}{1-(1-\delta)\beta} &= \frac{\bar{K}}{\bar{Y}}
\end{aligned}$$

by plugging in the given values we gain:

$$\iff \frac{\bar{K}}{\bar{Y}} = 8.547$$

In step (2) we use the equation resulting from the FOC w.r.t EN_t to gain:

$$\begin{aligned}
\bar{P} &= \bar{A}\bar{K}^{\alpha}\bar{N}^{\gamma}(1-\alpha-\gamma)\bar{EN}^{-\alpha-\gamma} \\
\iff \bar{P} &= \bar{A}\bar{K}^{\alpha}\bar{N}^{\gamma}(1-\alpha-\gamma)\bar{EN}^{-\alpha-\gamma}\frac{\bar{EN}}{\bar{EN}} \\
\iff \bar{P} &= (1-\alpha-\gamma)\frac{\bar{Y}}{\bar{EN}} \\
\iff \frac{\bar{P}}{(1-\alpha-\gamma)} &= \frac{\bar{Y}}{\bar{EN}} \\
\iff 0.05 &= \frac{\bar{EN}}{\bar{Y}}
\end{aligned}$$

For step (3), we want to gain a value for $\frac{\bar{C}}{\bar{Y}}$. We can do this by examining the steady-state version of the resource constraint given by:

$$
\begin{aligned}
&\bar{A}\bar{K}^{\alpha}\bar{N}^{\gamma}\bar{EN}^{1-\alpha-\gamma} + (1-\delta)\bar{K} = \bar{C} + \bar{K} + \bar{P}\bar{EN} \\
&\iff \bar{Y} + (1-\delta)\bar{K} = \bar{C} + \bar{K} + \bar{P}\bar{EN} \\
&\iff \bar{Y} + (1-\delta)\bar{K} - \bar{K} - \bar{EN} = \bar{C} \\
&\iff 1 - \delta\frac{\bar{K}}{\bar{Y}} - \frac{\bar{EN}}{\bar{Y}} = \frac{\bar{C}}{\bar{Y}} \\
&\frac{\bar{C}}{\bar{Y}} = 1 - \delta \cdot 8.547 - 0.05 = 0.736
\end{aligned}
$$

In step (4) we are using the consumption-labour equation to gain a value for $\bar{N}$. Hence, we rewrite equation (1) from our system to gain:

$$
\begin{aligned}
&\bar{C}^{-\sigma} = \frac{\theta}{1-\bar{N}} \cdot \frac{1}{\bar{A}\bar{K}^{\alpha}\gamma\bar{N}^{\gamma-1}\bar{EN}^{1-\alpha-\gamma}} \\
&\iff \bar{C}^{-\sigma} = \frac{\theta}{1-\bar{N}} \cdot \frac{1}{\bar{A}\bar{K}^{\alpha}\gamma\bar{N}^{\gamma-1}\bar{EN}^{1-\alpha-\gamma}\frac{\bar{N}}{\bar{N}}} \\
&\iff \bar{C}^{-1} = \frac{\theta}{1-\bar{N}} \cdot \frac{1}{\gamma\frac{\bar{Y}}{\bar{N}}} \\
&\iff \frac{(1-\bar{N})\gamma}{\bar{C}\theta} = \frac{\bar{N}}{\bar{Y}} \\
&\iff \frac{\gamma \cdot (1-\bar{N})}{\theta\bar{C}} = \frac{\bar{N}}{\bar{Y}} \\
&\iff \frac{\gamma\bar{Y}}{\theta\bar{C}} = \frac{\bar{N}}{(1-\bar{N})} \\
&\iff \frac{\theta\bar{C}}{\gamma\bar{Y}} = \frac{(1-\bar{N})}{\bar{N}} \\
&\iff \frac{\theta}{\gamma} \cdot \frac{\bar{C}}{\bar{Y}} = \frac{1}{\bar{N}} - 1 \\
&\iff \frac{3.48}{0.65} \cdot 0.736 + 1 = \frac{1}{\bar{N}} \\
&\iff \bar{N} = 0.202339
\end{aligned}
$$

In step (5), we can now use the steady state production function to gain a value for $\bar{Y}$. Thus, we use equation (6) from our system and rewrite:

$$
\begin{aligned}
&\bar{Y} = \bar{A}\bar{K}^{\alpha}\bar{N}^{\gamma}\bar{EN}^{1-\alpha-\gamma} \\
&\iff \frac{\bar{Y}}{\bar{Y}^{1-\alpha-\gamma}\bar{Y}^{\alpha}} = \left(\frac{\bar{K}}{Y}\right)^{\alpha} \bar{N}^{\gamma} \left(\frac{\bar{EN}}{Y}\right)^{1-\alpha-\gamma} \\
&\iff \bar{Y}^{\gamma} = 8.547^{\alpha} \cdot 0.202^{\gamma} \cdot 0.05^{1-\alpha-\gamma} \\
&\iff \bar{Y} = 8.547^{\frac{\alpha}{\gamma}} \cdot 0.202^{\frac{\gamma}{\gamma}} \cdot 0.05^{\frac{1-\alpha-\gamma}{\gamma}} \\
&\iff \bar{Y} = 0.432576
\end{aligned}
$$

Using these values, we can now compute values for all steady states, which we summarize in the table below:

Variable	Formula	Steady State Value
$\bar{Y}$		0.432576
$\bar{K}$	$\bar{K} = \frac{\bar{K}}{\bar{Y}}\bar{Y} = 8.547 \cdot 0.432576$	3.69713
$\bar{N}$		0.202339
$\bar{C}$	$\bar{C} = \frac{\bar{C}}{\bar{Y}}\bar{Y} = 0.736 \cdot 0.432576$	0.318519
$\bar{I}$	$\bar{I} = \bar{K} - (1-\delta)\bar{K} = 3.69713 - 0.975 \cdot 3.69713$	0.0924281
$\bar{EN}$	$\bar{EN} = \frac{\bar{EN}}{\bar{Y}}\bar{Y} = 0.05 \cdot 0.432576$	0.0216288
$\bar{A}$		1
$\bar{P}$		1

3.2 Dynare Code

Our code file which solves the RBC model can be found on GitHub or in the appendix. The policy functions are given in the table below:

	y	c	k	inve	n	en	a	p
Constant	-0.837997	-1.144073	1.307557	-2.381324	-1.597811	-3.833730	0	0
k(-1)	0.086822	0.506355	0.941026	-1.358943	-0.334646	0.086821	0	0
p(-1)	-0.052990	-0.004116	-0.005535	-0.221416	-0.038985	-0.552989	0	0.5
a(-1)	1.650131	0.458910	0.143881	5.755233	0.950189	1.650130	0.95	0
ea	1.736980	0.483063	0.151453	6.058140	1.000199	1.736978	1	0
ep	-0.105979	-0.008231	-0.011071	-0.442832	-0.077970	-1.105978	0	1

3.3 Negative Shock To Productivity

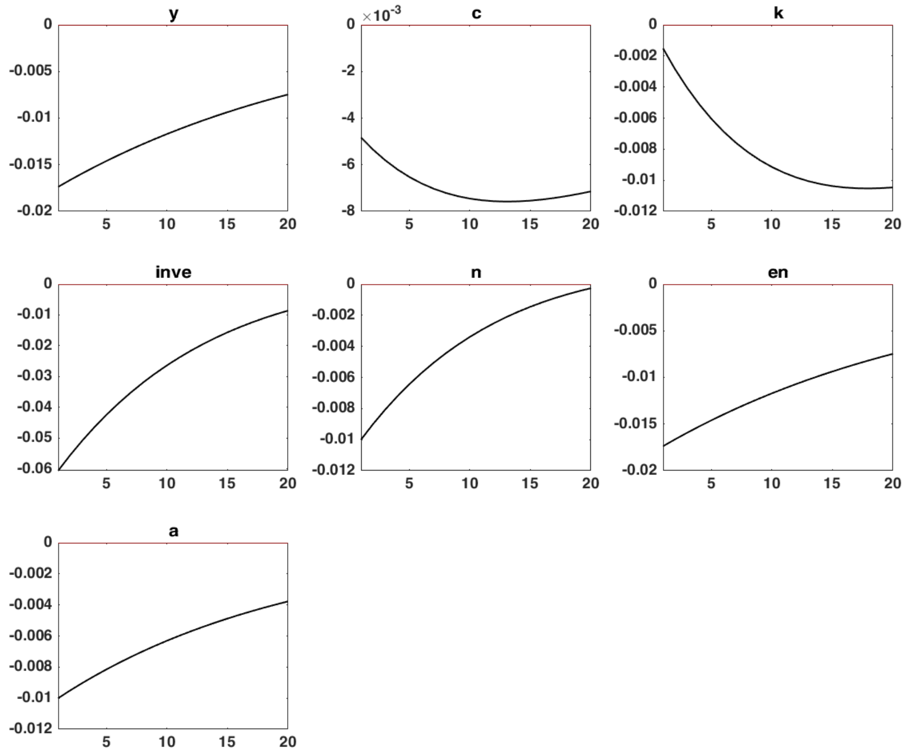

Figure 6: Impulse Response Functions (IRF) to a negative 1% shock to Total Factor Productivity

Intepretation:
The impulse response functions of a negative 1% TFP shock show the adjustment of the economy to a reduction in its productive capacity. TFP measures the efficiency with which labor, energy and capital inputs are used to produce output in the economy. Therefore, a negative TFP shock leads to a decline in the economy's productivity (as capital is already pre-set for this period), leading to a decline in output, income, consumption, and investment. As the shock in TFP is persistent ($\rho_A = 0.95$), the economy keeps decreasing capital. The persistence of the decline in output is due to the slow adjustment of capital, energy and labor inputs to the long-lasting revival of pre-shock TFP levels.

The decline in output leads to a fall in consumption, investment and labor supply and Energy required. Consumption falls less than output because households can smooth their consumption over time by using their savings. Investment falls because the economy needs time time decrease the capital stock, hence they need to pull back some investments. The fall in labor supply reflects the decrease in the economy's production capacity, as fewer workers are needed to produce the lower level of output. As we did not assume any energy prices shocks in this scenario, p doesn't change at all.

In summary, the impulse response functions of a negative TFP shock show that the shock has negative effects on the economy's output, consumption, investment, labor supply, and Energy. The persistence of the decline in output shows how long-lasting the negative effect is on the economy's productivity. The economy gradually returns to its pre-shock level.

3.4 Positive Shock To Energy Prices

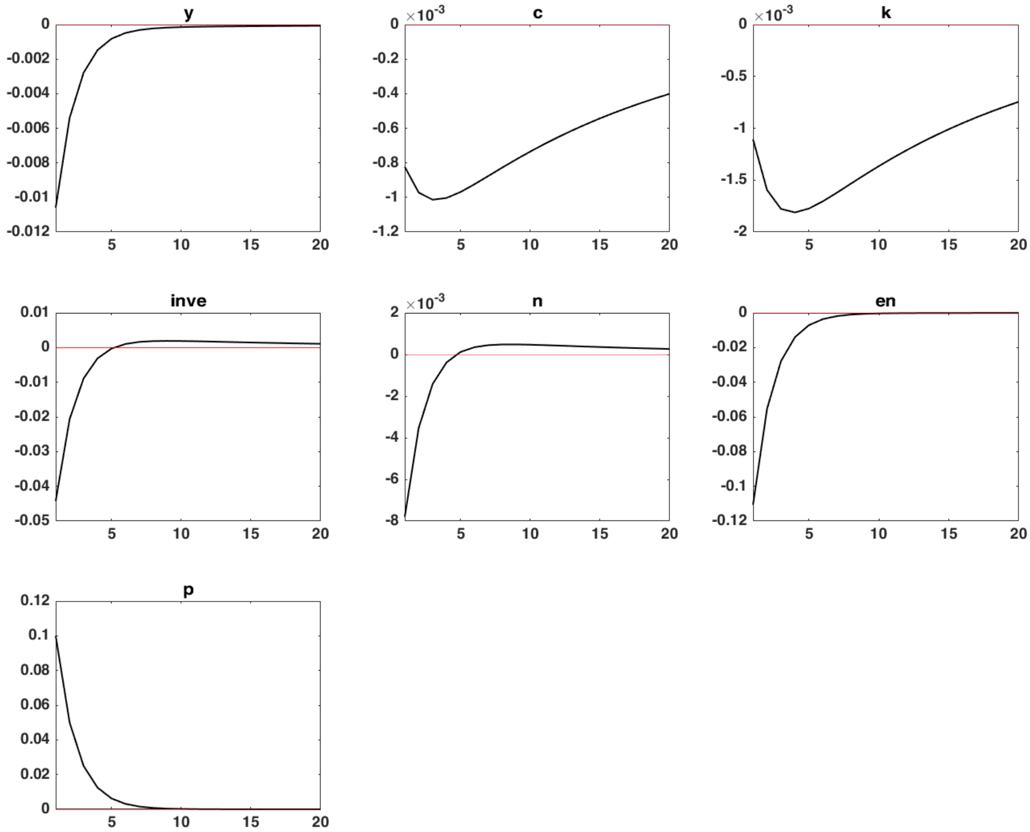

Figure 7: Impulse Response Functions (IRF) to a positive 10% energy price shock

Interpretation:

The impulse response functions of a positive 10% Energy price shock show the adjustment of the economy to an increase in the energy price, which increases the price of energy inputs used in production. This shock affects the production costs and leads to a change in the relative prices of goods and services in the economy. As we know that $\rho_P = 0.5$, the shock decreases quite fast and is almost gone five periods later.

The immediate effect of the energy price shock is a decline in output. The reduction in output is because the increase in energy prices leads to a rise in production costs, and firms are unable to produce the same level of output as before. The decline of output is not very persistent, as mentioned above, and the original level of Output is reached again after a few periods. Energy declines and recovers in a similar way as output.

The positive energy price shock leads to an immediate decline in consumption due to the rise in energy prices. Investment falls immediately following the energy price shock as the capital stock needs to decrease, which can be achieved by the pull back of investments. As the shock is gone after roughly five periods, the capital starts to return to its steady state after five periods which can also be seen by the fact that investment already recovered after roughly five periods. The same can be said about the deviation in consumption. The energy price shock also leads to a decline in labor supply immediately. This is because firms reduce their production, and hence, they require fewer workers to produce the same level of output.

The impulse response functions of a positive energy price shock show that the shock has a negative effect on output, consumption, investment and labor supply in the short run. However, the economy gradually adjusts to the new higher energy prices, and the negative effects on output and other variables are not permanent.

3.5 Log-Linearization

To gather the log-linearized version of our model we will proceed by transforming every equation in answer 2.1. The numbers refer to to the equation numbering used before.

Equation 1:
We rewrite in terms of the transformed variables where for some variable X_t it holds that $X_t = \bar{X}e^{\hat{X}_t}$. Thus we rewrite (1) as:

$$
\begin{aligned}
&C_t^{-\sigma} A_t K_t^{\alpha} \gamma N_t^{\gamma-1} E N_t^{1-\alpha-\gamma}(1-N_t) = \theta \\
&\iff C_t^{-\sigma} A_t K_t^{\alpha} \gamma N_t^{\gamma-1} E N_t^{1-\alpha-\gamma} - C_t^{-\sigma} A_t K_t^{\alpha} \gamma N_t^{\gamma} E N_t^{1-\alpha-\gamma} = \theta \\
&\iff C_t^{-\sigma} A_t K_t^{\alpha} \gamma N_t^{\gamma-1} E N_t^{1-\alpha-\gamma} = \theta + C_t^{-\sigma} A_t K_t^{\alpha} \gamma N_t^{\gamma} E N_t^{1-\alpha-\gamma} \\
&\iff C_t^{-\sigma} A_t K_t^{\alpha} \gamma N_t^{\gamma-1} E N_t^{1-\alpha-\gamma} = \theta + C_t^{-\sigma} \gamma Y_t \\
&\iff \frac{\gamma C_t^{-\sigma} Y_t}{N_t} = \theta + C_t^{-\sigma} \gamma Y_t \\
&\iff \gamma C_t^{-\sigma} Y_t = \theta N_t + N_t C_t^{-\sigma} \gamma Y_t \\
&\iff C_t^{-\sigma} Y_t = \frac{\theta}{\gamma} N_t + N_t C_t^{-\sigma} Y_t \\
&\iff \frac{Y_t}{C_t} = \frac{\theta}{\gamma} N_t + \frac{N_t Y_t}{C_t} \\
&\iff Y_t = \frac{\theta}{\gamma} N_t C_t + N_t Y_t \\
&\iff \bar{Y} e^{\hat{Y}_t} = \frac{\theta}{\gamma} \bar{N} e^{\hat{N}_t} \bar{C} e^{\hat{C}_t} + \bar{N} e^{\hat{N}_t} \bar{Y} e^{\hat{Y}_t}
\end{aligned}
$$

Through log-linearization we thus gain

$$
\bar{Y}\hat{Y}_t = \frac{\theta}{\gamma}\bar{C}\bar{N}\hat{C}_t + \left(\frac{\theta}{\gamma}\bar{C}\bar{N} + \bar{N}\bar{Y}\right)\hat{N}_t + \bar{N}\bar{Y}\hat{Y}_t
$$

Equation 2:
Using (2) from above, we use the same procedure to rewrite the right and left hand side of the equation.

1. LHS:

$$
\hat{C_t}^{-\sigma} = \bar{C}^{-\sigma} e^{-\sigma \hat{C}_t} \simeq \bar{C}^{-\sigma} e^{0} - \sigma \bar{C}^{-\sigma}(\hat{C}_t - 0)
$$

2. RHS:

$$
\begin{aligned}
&\beta E_t\left[C_{t+1}^{-\sigma}\left(A_{t+1}\alpha K_{t+1}^{\alpha-1}N_{t+1}^{\gamma}EN_{t+1}^{1-\alpha-\gamma}+(1-\delta)\right)\right]\\
&=\beta E_t\left[(\bar{C}e^{\hat{C_{t+1}}})^{-\sigma}\left(\alpha(\bar{A}e^{\hat{A_{t+1}}})(\bar{K}e^{\hat{K_{t+1}}})^{\alpha-1}(\bar{N}e^{\hat{N_{t+1}}})^{\gamma}(\bar{EN}e^{\hat{EN}_{t+1}})^{1-\alpha-\gamma}+(1-\delta)\right)\right]\\
&\simeq\beta\left[\bar{C}^{-\sigma}\left(\alpha\bar{A}\bar{K}^{\alpha-1}\bar{N}^{\gamma}\bar{EN}^{1-\alpha-\gamma}+(1-\delta)\right)\right]\\
&\quad+\beta(-\sigma)\left[\bar{C}^{-\sigma}\left(\alpha\bar{A}\bar{K}^{\alpha-1}\bar{N}^{\gamma}\bar{EN}^{1-\alpha-\gamma}+(1-\delta)\right)\right]\cdot E_t[\hat{C_{t+1}}]\\
&\quad+\beta\left[\bar{C}^{-\sigma}\left(\alpha\bar{A}\bar{K}^{\alpha-1}\bar{N}^{\gamma}\bar{EN}^{1-\alpha-\gamma}\right)\right]\cdot E_t[\hat{A_{t+1}}]\\
&\quad+\beta(\alpha-1)\left[\bar{C}^{-\sigma}\left(\alpha\bar{A}\bar{K}^{\alpha-1}\bar{N}^{\gamma}\bar{EN}^{1-\alpha-\gamma}\right)\right]\cdot E_t[\hat{K_{t+1}}]\\
&\quad+\beta\gamma\left[\bar{C}^{-\sigma}\left(\alpha\bar{A}\bar{K}^{\alpha-1}\bar{N}^{\gamma}\bar{EN}^{1-\alpha-\gamma}\right)\right]\cdot E_t[\hat{N_{t+1}}]\\
&\quad+\beta(1-\gamma-\alpha)\left[\bar{C}^{-\sigma}\left(\alpha\bar{A}\bar{K}^{\alpha-1}\bar{N}^{\gamma}\bar{EN}^{1-\alpha-\gamma}\right)\right]\cdot E_t[\hat{EN}_{t+1}]
\end{aligned}
$$

Now, we can combine the LHS with the RHS. Also, we can further simplify the equation by substracting the steady state from both sides twice and by using the fact that $\alpha\bar{A}\bar{K}^{\alpha-1}\bar{N}^{\gamma}\bar{EN}^{1-\alpha-\gamma}=[1-\beta(1-\delta)]$. This leaves us with the log-linearized equation:

$$
-\sigma\hat{C}_t=E_t[-\sigma\hat{C_{t+1}}+[1-\beta(1-\delta)][\hat{A_{t+1}}+(\alpha-1)\hat{K_{t+1}}+\gamma\hat{N_{t+1}}+(1-\alpha-\gamma)\hat{EN}_{t+1}]
$$

Equation 3:
Equation (3) is given by

$$
P_t=A_tK_t^{\alpha}N_t^{\gamma}(1-\alpha-\gamma)EN_t^{-\alpha-\gamma}
$$

Using (6) this can be rewritten as

$$
P_t=\frac{Y_t}{EN_t}(1-\alpha-\gamma)\iff EN_tP_t=Y_t(1-\alpha-\gamma)
$$

We now rewrite this as deviations and gain

$$
\bar{EN}e^{\hat{EN}_t}\bar{P}e^{\hat{P}_t}=\bar{Y}e^{\hat{Y}_t}(1-\alpha-\gamma)
$$

By using log-linearization on both sides (LHS = RHS) we get our log-linearized version as

$$
\begin{aligned}
&\bar{P}\bar{EN}+\bar{P}\bar{EN}(\hat{P}_t+\hat{EN}_t)=\bar{Y}(1-\alpha-\gamma)+\bar{Y}(1-\alpha-\gamma)\hat{Y}_t\\
\iff&\bar{P}\bar{EN}\hat{P}_t+\bar{P}\bar{EN}\hat{EN}_t=\bar{Y}(1-\alpha-\gamma)\hat{Y}_t
\end{aligned}
$$

Equation 4:
Due the linear structure of the resource constraint, we can see that

$$
A_tK_t^{\alpha}N_t^{\gamma}EN_t^{1-\alpha-\gamma}+(1-\delta)K_t=C_t+K_{t+1}+P_tEN_t
$$

rewritten with the transformed variables yields

$$
(\bar{A}e^{\hat{a}_t})(\bar{K}e^{\hat{K}_t})^{\alpha}(\bar{N}e^{\hat{N}_t})^{\gamma}(\bar{EN}e^{\hat{EN}_t})^{(1-\alpha-\gamma)}+(1-\delta)\bar{K}e^{\hat{K}_t}=\bar{C}e^{\hat{C}_t}+\bar{K}e^{\hat{K_{t+1}}}+\bar{P}e^{\hat{P}_t}\bar{EN}e^{\hat{EN}_t}
$$

- RHS $\simeq \bar{C} + \bar{K} + \bar{P}\bar{EN} + \bar{C}\hat{C}_t + \bar{K}\hat{K_{t+1}} + \bar{P}\bar{EN}(\hat{P}_t + \hat{EN}_t)$
- LHS $\simeq$

$$\begin{aligned}
&\bar{A}\bar{K}^{\alpha}\bar{N}^{\gamma}\bar{EN}^{1-\alpha-\gamma} + (1-\delta)\bar{K} \\
&\quad + \alpha\bar{A}\bar{K}^{\alpha-1}\bar{N}^{\gamma}\bar{EN}^{1-\alpha-\gamma}\hat{A}_t \\
&\quad + [\alpha\bar{A}\bar{K}^{\alpha-1}\bar{N}^{\gamma}\bar{EN}^{1-\alpha-\gamma} + (1-\delta)\bar{K}]\hat{K}_t \\
&\quad + (1-\alpha-\gamma)\bar{A}\bar{K}^{\alpha-1}\bar{N}^{\gamma}\bar{EN}^{1-\alpha-\gamma}\hat{EN}_t \\
&\quad + \gamma\bar{A}\bar{K}^{\alpha-1}\bar{N}^{\gamma}\bar{EN}^{1-\alpha-\gamma}\hat{N}_t
\end{aligned}$$

Setting LHS = RHS, substracting the steady state from both sides and dividing by $\bar{Y}$ yields the final equation

$$\frac{\bar{C}}{\bar{Y}}\hat{C}_t + \frac{\bar{K}}{\bar{Y}}\hat{K_{t+1}} + \frac{\bar{P}\bar{EN}}{\bar{Y}}(\hat{P}_t + \hat{EN}_t) = \hat{A}_t + \alpha\hat{K}_t + (1-\alpha-\gamma)\hat{EN}_t + \gamma\hat{N}_t + (1-\delta)\frac{\bar{K}}{\bar{Y}}\hat{K}_t$$

Equation 5: We continue by examining equation (5), which is given by and can be rewritten as

$$I_t = K_{t+1} - (1-\delta)K_t \implies \bar{I}e^{\hat{I_{t+1}}} = \bar{K}e^{\hat{K_{t+1}}} - (1-\delta)\bar{K}e^{\hat{K}_t}$$

Using log-linearization we can gain

$$\bar{I} + \bar{I}\hat{I}_t = \bar{K} - (1-\delta)\bar{K} + \bar{K}\hat{K_{t+1}} - (1-\delta)\bar{K}\hat{K}_t$$

$$\iff \bar{I}\hat{I}_t = \bar{K}\hat{K_{t+1}} - (1-\delta)\bar{K}\hat{K}_t$$

Equation 6: The next equation (6) is given by

$$Y_t = A_t K_t^{\alpha} N_t^{\gamma} EN_t^{1-\alpha-\gamma}$$

Rewriting it as usual yields

$$\bar{Y}e^{\hat{Y}_t} = \bar{A}e^{\hat{A}_t}(\bar{K}e^{\hat{K}_t})^{\alpha}(\bar{N}e^{\hat{N}_t})^{\gamma}(\bar{EN}e^{\hat{EN}_t})^{1-\alpha-\gamma}$$

Expanding around the steady state results in

$$LHS \simeq \bar{Y} + \bar{Y}\hat{Y}_t$$

as well as

$$\begin{aligned}
RHS &\simeq \bar{A}\bar{K}^{\alpha}\bar{N}^{\gamma}\bar{EN}^{1-\alpha-\gamma} \\
&\quad + \bar{A}\bar{K}^{\alpha}\bar{N}^{\gamma}\bar{EN}^{1-\alpha-\gamma}\hat{A}_t \\
&\quad + \alpha\bar{A}\bar{K}^{\alpha}\bar{N}^{\gamma}\bar{EN}^{1-\alpha-\gamma}\hat{K}_t \\
&\quad + \gamma\bar{A}\bar{K}^{\alpha}\bar{N}^{\gamma}\bar{EN}^{1-\alpha-\gamma}\hat{N}_t \\
&\quad + (1-\alpha-\gamma)\bar{A}\bar{K}^{\alpha}\bar{N}^{\gamma}\bar{EN}^{1-\alpha-\gamma}\hat{EN}_t
\end{aligned}$$

Setting LHS equal to RHS and substracting the steady state yields

$$\bar{Y}\hat{Y}_t = \bar{A}\bar{K}^{\alpha}\bar{N}^{\gamma}\bar{EN}^{1-\alpha-\gamma}\hat{A}_t + \cdots + (1-\alpha-\gamma)\bar{A}\bar{K}^{\alpha}\bar{N}^{\gamma}\bar{EN}^{1-\alpha-\gamma}\hat{EN}_t$$

$$\iff \hat{Y}_t = \hat{A}_t + \alpha\hat{K}_t + \gamma\hat{N}_t + (1-\alpha-\gamma)\hat{EN}_t$$

Equations (7) and (8) are already in log-linearized form, thus we can write:
Equation 7:

$$\log(A_{t+1}) = \rho_A \log(A_t) + \epsilon_{A,t+1}$$

Equation 8:

$$\log(P_{t+1}) = \rho_P \log(P_t) + \epsilon_{P,t+1}$$

which completes our set of log-linearized equations.

3.6 Matrix Modelling

Using the previously derived equations, we can setup matrices A and B which satisfy the relationship

$$A \cdot E_t z_{t+1} = B z_t$$

where $z_t = [\hat{K}_t \quad \hat{A}_t \quad \hat{P}_t \quad \hat{C}_t \quad \hat{I}_t \quad \hat{Y}_t \quad \hat{N}_t \quad \hat{EN}_t]^T$. The matrices which will be used to solve the system using the Klein algorithm look like

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ [1-\beta(1-\delta)](\alpha-1) & [1-\beta(1-\delta)] & 0 & -\sigma & 0 & 0 & [1-\beta(1-\delta)]\gamma & [1-\beta(1-\delta)](1-\gamma-\alpha) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \bar{K}/\bar{Y} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \bar{K} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

as well as

$$B = \begin{bmatrix} 0 & 0 & 0 & \frac{\theta}{\gamma}\bar{C}\bar{N} & 0 & -\bar{Y}+\bar{N}\bar{Y} & \bar{N}(\frac{\theta}{\gamma}\bar{C}+\bar{Y}) & 0 \\ 0 & 0 & 0 & -\sigma & 0 & 0 & 0 & 0 \\ 0 & 0 & \bar{P}\bar{EN} & 0 & 0 & -\bar{Y}(1-\alpha-\gamma) & 0 & \bar{P}\bar{EN} \\ \alpha+(1-\delta)\frac{\bar{K}}{\bar{Y}} & 1 & -\frac{\bar{P}\bar{EN}}{\bar{Y}} & -\frac{\bar{C}}{\bar{Y}} & 0 & 0 & \gamma & -\frac{\bar{P}\bar{EN}}{\bar{Y}}+1-\alpha-\gamma \\ (1-\delta)\bar{K} & 0 & 0 & 0 & \bar{I} & 0 & 0 & 0 \\ -\alpha & -1 & 0 & 0 & 0 & 1 & -\gamma & -(1-\alpha-\gamma) \\ 0 & \rho_A & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho_P & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Again, our code file can be found on GitHub or in the appendix. The computed matrices **G** and **H** can be found below:

$$H = \begin{bmatrix} 0.941 & 0.1515 & -0.0111 \\ 0 & 0.9500 & 0 \\ 0 & 0 & 0.5 \end{bmatrix} \quad G = \begin{bmatrix} -1.3589 & 6.0581 & -0.4428 \\ 0.0868 & 1.7370 & -0.1060 \\ 0.5064 & 0.4831 & -0.0082 \\ -0.3346 & 1.0002 & -0.0780 \\ 0.0868 & 1.7370 & -1.1060 \end{bmatrix}$$

These matrices represent the following matrices:

$$H = \begin{bmatrix} \phi_{kk} & \phi_{ka} & \phi_{kp} \\ 0 & \rho_p & 0 \\ 0 & 0 & \rho_a \end{bmatrix} \quad G = \begin{bmatrix} \phi_{ik} & \phi_{ia} & \phi_{ip} \\ \phi_{yk} & \phi_{ya} & \phi_{yp} \\ \phi_{ck} & \phi_{ca} & \phi_{cp} \\ \phi_{nk} & \phi_{na} & \phi_{np} \\ \phi_{enk} & \phi_{ena} & \phi_{enp} \end{bmatrix}$$

Thus, all in all the system looks like

$$\begin{bmatrix} \hat{i}_t \\ \hat{y}_t \\ \hat{c}_t \\ \hat{n}_t \\ \hat{en}_t \end{bmatrix} = \begin{bmatrix} \phi_{ik} & \phi_{ia} & \phi_{ip} \\ \phi_{yk} & \phi_{ya} & \phi_{yp} \\ \phi_{ck} & \phi_{ca} & \phi_{cp} \\ \phi_{nk} & \phi_{na} & \phi_{np} \\ \phi_{enk} & \phi_{ena} & \phi_{enp} \end{bmatrix} \cdot \begin{bmatrix} \hat{k}_t \\ \hat{a}_t \\ \hat{p}_t \end{bmatrix}$$

which corresponds with our previously reported policy values.

4 Appendix

4.1 Dynare Code for question 3(b)

```
// Assignment 2 - Dynare Code
//
// Oberbrinkmann - Unterweger


var c k n y inve en a p;
varexo ea ep;
parameters alfa betta gama delta sigma theta rhoa rhop sigmaa sigmap;

// Parameter values
alfa = 0.3;
betta = 0.99;
sigma = 1;
theta = 3.48;
gama = 0.65;
delta = 0.025;
rhoa = 0.95;
rhop = 0.5;
sigmap = 0.1;
sigmaa = 0.01;


// Steady States
ASS = 1;
PSS = 1;
YSS = 0.432576;
KSS = 3.69713;
NSS = 0.202339;
CSS = 0.318519;
ENSS = 0.0216288;
ISS = 0.0924281;



// Model Block
model;
    exp(c)^(-sigma)*exp(y) = (theta/gama)*exp(n) + exp(n)*exp(c)^(-sigma)*exp(y); //exp(c
        )^(-sigma) = (theta/(1-exp(n)))*(1/(exp(a)*(exp(k(-1)))^alfa * gama * exp(n)^(
        gama-1)*exp(en)^(1-alfa-gama)));
    exp(c)^(-sigma) = betta * exp(c(+1))^(-sigma)*(exp(a(+1))*alfa*exp(k)^(alfa-1)*exp(n
        (+1))^gama*exp(en(+1))^(1-alfa-gama)+(1-delta));
    exp(p) = exp(a)*exp(k(-1))^(alfa)*exp(n)^(gama)*(1-alfa-gama)*exp(en)^(-alfa-gama);
    exp(a)*exp(k(-1))^alfa*exp(n)^gama*exp(en)^(1-alfa-gama)+(1-delta)*exp(k(-1)) = exp(c
        ) + exp(k) + exp(p)*exp(en);
    exp(inve) = exp(k) - (1-delta)*exp(k(-1));
    exp(y) = exp(a)*exp(k(-1))^alfa*exp(n)^gama*exp(en)^(1-alfa-gama);
    (a)    = rhoa * (a(-1)) + ea;
    (p)    = rhop * (p(-1)) + ep;
end;

initval;
  k    = log(KSS);
  c    = log(CSS);
  a    = log(ASS);
  y    = log(YSS);
  inve = log(ISS);
  en = log(ENSS);
  n = log(NSS);
  p = log(PSS);
```

```
end;

shocks;
var ep = 0;       // var stands for variance, else (var e; stderr sigma;)
var ea = 0;
end;

steady;

stoch_simul(periods=2100,irf=20,order=1) y c k inve n en a p;
```

4.2 Matlab Code for question 3(f)

```
% Assignment 2 - Matrix Solution
%
% Oberbrinkmann - Unterweger

% We start by clearing the environment
format compact;
clc; clear all; close all;

% The given parameters values are:
% Parameter values
alfa = 0.3;
betta = 0.99;
sigma = 1;
theta = 3.48;
gama = 0.65;
delta = 0.025;
rhoa = 0.95;
rhop = 0.5;
sigmap = 0.00001;
sigmaa = 0.0007;
ASS = 1;
PSS = 1;


% Steady States
YSS = 0.432576;
KSS = 3.69713;
NSS = 0.202339;
CSS = 0.318519;
ENSS = 0.0216288;
ISS = 0.0924281;

% We need to bring our log-linearized model into matrix form
% First we construct the empty matrices A and B with the
% dimensions 8x8 as we are dealing with 8 variables:
A  = zeros(8,8);
B  = zeros(8,8);

% We also define the column indices for the variables
ik = 1; % column index for capital
ia = 2; % column index for productivity
ip = 3; % column index for energy prices
iinve = 4; % column index for investment
iy = 5; % column index for output
ic = 6; % column index for consumption
in = 7; % column index for hours worked
ien = 8; % column index for energy consumption


% equation 1: CHECK
B(1,iy) = -YSS+NSS*YSS;
B(1,ic) = (theta/gama)*CSS*NSS;
B(1,in) = (theta/gama)*CSS*NSS + NSS*YSS;

% equation 2: CHECK
A(2,ik) = (1-betta*(1-delta))*(alfa-1);
A(2,ia) = (1-betta*(1-delta));
A(2,ic) = -sigma;
A(2,in) = (1-betta*(1-delta))*gama;
A(2,ien) = (1-betta*(1-delta))*(1-alfa-gama);
B(2,ic) = -sigma;

```

```
% equation 3: CHECK
B(3,iy) = -YSS*(1-alfa-gama);
B(3,ip) = PSS*ENSS;
B(3,ien) = PSS*ENSS;

% equation 4: CHECK
A(4,ik) = KSS/YSS;
B(4,ik) = alfa+(1-delta)*(KSS/YSS);
B(4,ia) = 1;
B(4,ic) = -CSS/YSS;
B(4,ip) = -(PSS*ENSS)/YSS;
B(4,in) = gama;
B(4,ien)= -((PSS*ENSS)/YSS) + 1 - alfa - gama;

% equation 5: CHECK

A(5,ik) = KSS;
B(5,ik) = (1-delta)*KSS;
B(5,iinve) = ISS;

% equation 6: CHECK
B(6,ik) = -alfa;
B(6,ia) = -1;
B(6,iy) = 1;
B(6,in) = -gama;
B(6,ien) = -(1 - alfa - gama);

% equation 7:CHECK
A(7,ia) = 1;
B(7,ia) = rhoa;

% equation 8: CHECK
A(8,ip) = 1;
B(8,ip) = rhop;


% Use solab.m
[G,H]=solab(A,B,3)
```