Bayesian Optimization of a Wearable Assistive Device Using an Estimator Stopping Process

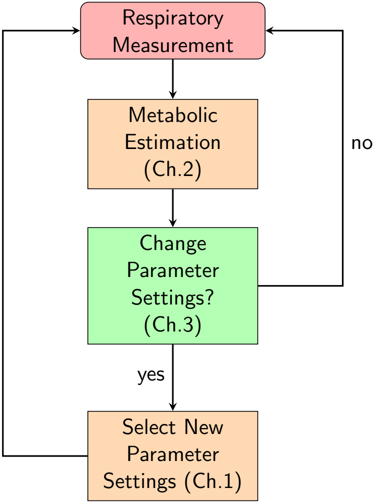
Charles Liu
cliu02@g.harvard.edu

IACS, Harvard University

May 4, 2018

Soft Exosuit

Process Overview



Standard Gaussian Process Model

Observations represent underlying process with some Gaussian noise

$$y \sim f + \mathcal{N}(0, \sigma_n^2)$$
$$f \sim \mathbb{G}(0, \kappa)$$
$$\kappa(x_i, x_j | \sigma_\theta^2, l) = \sigma_\theta^2 \exp(-\frac{1}{2} d^2(\frac{x_i}{l}, \frac{x_j}{l}))$$

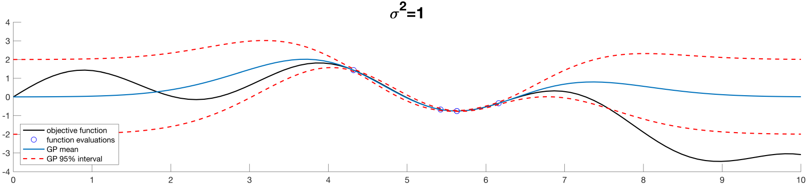
Gaussian Process Regression

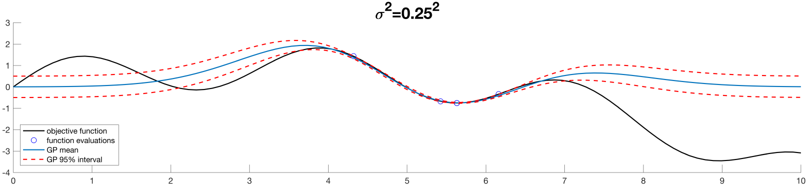
Given some training data, closed form posterior distribution at any point x

$$
\begin{aligned}
\bar{\mu}(x) &= K(X,x)^T[K(X,X)+\sigma_n^2 I]^{-1}Y \\
\bar{\sigma}^2(x) &= \kappa(x,x|\theta) - K(X,x)^T[K(X,X)+\sigma_n^2 I]^{-1}K(X,x) \\
K(X,x)_i &= \kappa(x_i,x|\theta) \\
K(X,X)_{ij} &= \kappa(x_i,x_j|\theta),
\end{aligned}
$$

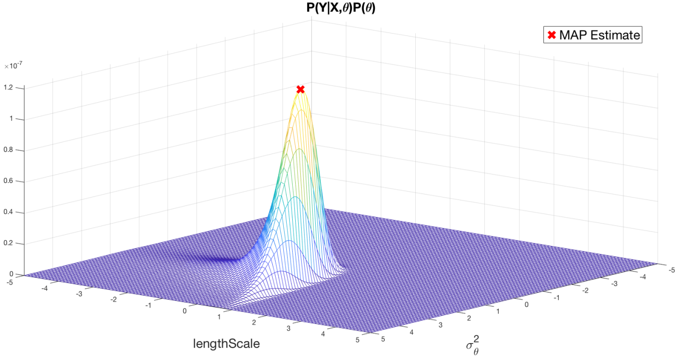
where $X = \{x_i\}_{i=1}^n$ and $Y = \{y_i\}_{i=1}^n$.

σ_θ^2 Effect





Hyperparameter Estimation



Expected Improvement

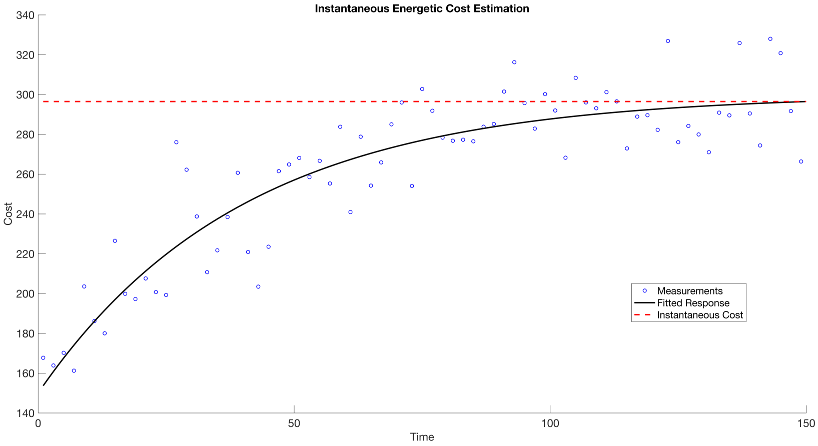
$$
\begin{aligned}
EI(x|\mathbb{S}) &= \int_{\infty}^{\infty} max(0, y^* - y)p(y|x)dy \\
&= z\bar{\sigma}(x)\Phi(z) + \bar{\sigma}(x)\phi(z) \\
z &= \frac{y^* - \bar{\mu}(x) + \xi}{\bar{\sigma}(x)},
\end{aligned}
$$

where y^* is the best value observed so far, $\Phi(z)$ and $\phi(z)$ are the standard normal CDF and PDF functions, and ξ is a scaling parameter to adjust the tradeoff between exploration-exploitation

Bayesian Optimization

Objective Function $F(x)$
Acquisition Function $g(\mu, \sigma^2)$
Specify Exploration Points $\mathbb{E} = \{e_1, e_2, ..., e_n\}$
Training Samples $\mathbb{S}$
for $i = 1$ **to** n
  $\mathbb{S} = \mathbb{S} \cup \{e_i, F(e_i)\}$
end
while $t < T$
  Update GP Hyperparameters θ
  Given $f(x|\theta, \mathbb{S}) \sim \mathcal{N}(\mu_x, \sigma_x^2)$,
  $x^* = \arg\max_{x \in \mathbb{X}} g(\mu_x, \sigma_x^2)$
  $\mathbb{S} = \mathbb{S} \cup \{x^*, F(x^*)\}$
end

Instantaneous Energetic Cost



Kalman Filter

System Dynamics

$$
\begin{aligned}
x(t+1) &= F(x(t), v(t), t) \\
z(t) &= H(x(t), w(t), t)
\end{aligned}
$$

Update Equations

$$
\begin{aligned}
\hat{x}(t) &= \hat{x}(t|t-1) + Ky \\
P_x(t) &= P_x(t|t-1) - KP_y(t|t-1)K^T \\
K &= P_{xy}(t|t-1)P_y^{-1}(t|t-1) \\
y &= z(t) - H(\hat{x}(t|t-1), w(t), t)
\end{aligned}
$$

Unscented Transform

$$
\begin{aligned}
\chi_0 &= \bar{x} \\
\chi_i &= \bar{x} + (\sqrt{(N+\lambda)P_x})_i & i &= 1, ..., N \\
\chi_i &= \bar{x} - (\sqrt{(N+\lambda)P_x})_{i-N} & i &= N+1, ..., 2N \\
W_0^{(m)} &= \lambda/(N+\lambda) \\
W_0^{(c)} &= \lambda/(N+\lambda) + (1-\alpha^2+\beta) \\
W_i^m &= W_i^c = 1/\{(2(N+\lambda))\} & i &= 1, ..., 2N,
\end{aligned}
$$

where $\lambda = \alpha^2(N+\kappa) - N$, and α, κ, and β are scaling parameters. Given $y = g(x)$,

$$
\bar{y} \approx \sum_{i=0}^{2N} W_i^{(m)} g(\chi_i)
$$

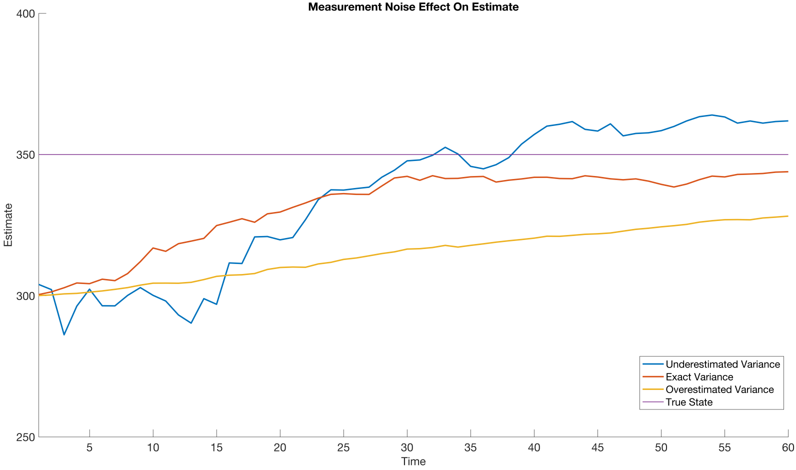
$$
P_y \approx \sum_{i=0}^{2N} W_i^{(c)} (g(\chi_i) - \bar{y})(g(\chi_i) - \bar{y})^T
$$

UKF Model

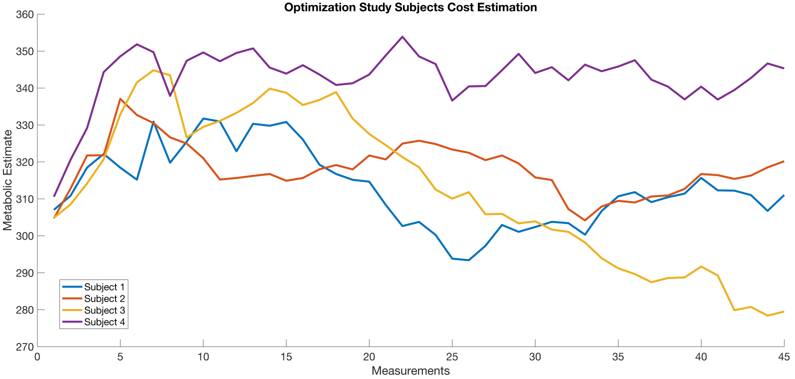
Time constants τ_0, τ characterizing rate of change of cost constants c_0, c

$$
\begin{aligned}
x &= [c_0 \ c \ \tau_0 \ \tau] \\
F(x(t), v(t), t) &= x(t) + v(t) \\
H(x(t), w(t), t) &= c(1 - e^{\frac{-t}{\tau}}) + c_0 e^{\frac{-t}{\tau_0}} + w(t)
\end{aligned}
$$

Measurement Noise



Estimator in Subject Trials



Stopping Problem Formulation

N	Finite horizon
X_t	State at time t
$P(X_t \mid X_{t-1})$	State transitions, typically Markovian
λ	Discount Factor $\in (0, 1]$
$r(X)$	Bounded reward function for continuing at state X
$g(X)$	Bounded reward function for stopping at state X

Optimal Stopping Point

Via backward induction

$$J_N(x) = g(x)$$
$$J_n(x) = max\{g(x), r(x) + \lambda \mathbb{E}_{P(y|x)}[J_{n+1}(y)]\}$$

Optimal stopping point

$$\tau = \min_t J_t(X_t) = g(X_t)$$

Distribution of Interest

Given

$$\text{Current Estimate: } \hat{x}_t \sim \mathcal{N}(\mu_{x_t}, \sigma^2_{x_t})$$
$$\text{Best Estimate: } \hat{x}^* \sim \mathcal{N}(\mu_{x^*}, \sigma^2_{x^*})$$

We are concerned with

$$\hat{x}_t - \hat{x}^* \sim \mathcal{N}(\mu, \sigma^2)$$
$$\mu = \mu_{x_t} - \mu_{x^*}$$
$$\sigma^2 = \sigma^2_{x_t} + \sigma^2_{x^*}$$

σ-Offset Model

$$\begin{aligned} X_t &= \mu \\ P(X_t|X_{t-1}) &= P(X_t) \sim \mathcal{N}(\mu, \sigma^2) \\ r(x) &= K\sigma - x \\ g(x) &= 0 \end{aligned}$$

Stopping condition $X_t > K\sigma$

Success/Failure Model

Define success as

$$\Phi(\frac{\mu}{\sigma}) > K,$$

for risk tolerance K. Define model as

$$X_t = (\alpha_t + \alpha_0, \beta_t + \beta_0)$$

$$P(X_{t+1}) = \begin{cases} (\alpha_t + \alpha_0 + 1, \beta_t + \beta_0) \\ \qquad \text{w.p. } \frac{\alpha_t + \alpha_0}{\alpha_t + \alpha_0 + \beta_t + \beta_0} \\ (\alpha_t + \alpha_0, \beta_t + \beta_0 + 1) \\ \qquad \text{w.p. } \frac{\beta_t + \beta_0}{\alpha_t + \alpha_0 + \beta_t + \beta_0} \end{cases}$$

$$r(X_t) = \frac{\alpha_t + \alpha_0}{\alpha_t + \alpha_0 + \beta_t + \beta_0},$$

with smoothing priors α_0, β_0 and success/failures α_t, β_t

Multi-Armed Bandit Problem

Which is the better option?

- Option 1: 20 successes, 20 failures
- Option 2: 3 successes, 4 failures

Define Gittins Index as

$$\nu(x) = (1 - \lambda) \min_{K} \{J_0(x) = g(x) = K\}$$
$$\nu((30, 30)) = 0.5133$$
$$\nu((4, 6)) = 0.5289$$

In the case of single option, must set another threshold $\nu(x) < V$

Adaptive Threshold

Use the exploration phase of Bayesian optimization to tune thresholds in either model, with final distribution for each point as "best" value. Consider a window of values at time t

e_{00}	e_{01}	e_{02}	e_{03}	e_{04}	e_{05}	e_{06}	e_{07}	e_{08}	e_{09}	
e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}	e_{16}	e_{17}	e_{18}	e_{19}	$\cdots$
e_{20}	e_{21}	e_{22}	e_{23}	e_{24}	e_{25}	e_{26}	e_{27}	e_{28}	e_{29}	

$\vdots$

Use most risk averse threshold in window: max σ-offset, min for Gittins

Hip-Only Trials

Multi-Joint Trials

Future Work

Acknowledgements