Optimization of the Pursuit Equation

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Abstract:

The classic fox-rabbit pursuit equation first investigated by Pierre Bouguer is linearized and compared numerically to the full equation. Methods for minimizing the time-on-target and complications from these approaches are discussed.

The pursuit curve is the result of a pursuer $\vec{P}$ following a target $\vec{T}$ whose trajectory is arbitrary. For the purposes of this project, we will assume that $\vec{T}$ is continuous and differentiable at least once. Typically the pursuer's trajectory is defined as

$$(1) \qquad \frac{d\vec{P}}{dt} = k\frac{\vec{T}-\vec{P}}{\left|\vec{T}-\vec{P}\right|}$$

which ensures that the speed of the pursuer is proportional to that of the target. Though this is highly nonlinear, the special case of a target moving at constant velocity has an analytical solution when restricted to two dimensions, which will not be explored here.

Instead, this paper will focus on various linear formulations of the pursuer equation in two dimensions and compare these to the numerical results from (first equation). All vectors are expressed in terms of Cartesian coordinates. Equation (1) will be linearized for instances when the vector difference between the target and pursuer is small, and can be expressed as

$$(2) \qquad \begin{aligned} \vec{T}-\vec{P} &= d(\cos\theta\,\hat{x} + \sin\theta\,\hat{y}) \\ &= d(\alpha_1\hat{x} + \alpha_2\hat{y}) \end{aligned}$$

where $d \approx 1$. Performing the linearization and converting to index notation for compactness yields

$$(3) \qquad \frac{dP_i}{dt} \approx k\alpha_i + \frac{k}{d}(T_i - P_i)\sum_{j=1}^{2}(\varepsilon_{ij}\alpha_j)^2$$

with ε_{ij} representing the Levi-Civita symbol. This step makes the pursuer's motion equation amenable to analysis in the frequency domain. However, each component of the pursuer equations has a different transfer function. To increase the readability of equation (3), the sum over j will be replaced by ${\eta_i}^2$ Taking the Laplace transform of equation (3) results in

$$(4) \qquad \rho_i(s) = \frac{kd\alpha_i}{s(sd + k{\eta_i}^2)} + \tau_i\frac{k{\eta_i}^2}{(sd + k{\eta_i}^2)}$$

where $\rho_i \equiv \mathcal{L}\{P_i\}$ and $\tau_i \equiv \mathcal{L}\{T_i\}$. It's worth noting that equation (4) is simply the sum of two negative feedback loops. The inverse-Laplace transform of equation (4) is the sum of an exponential approach and a convolution between an exponential decay and the components of the target's trajectory.

In the case where the target is moving away from an initial position with a constant velocity, the Final Value Theorem can be used to determine the steady state error between the target's trajectory and the pursuer's motion in the limit of large values for time. In the time domain, the components of the target's trajectory are given by

(5) $$T_i = x_{oi} + v_{oi}t$$

and its frequency-space counterpart is

$$\tau_i = \frac{x_{oi}}{s} + \frac{v_{oi}}{s^2}.$$

Ideally the steady state error for the target and pursuer should be zero, indicating that the pursuer catches up with the target *eventually*.

(6) $$E_{ss} = \lim_{s\to 0} s(\tau_i - \rho_i) = \lim_{s\to 0} s\left[\tau_i \frac{sd}{(sd + k\eta_i{}^2)} - \frac{kd\alpha_i}{s(sd + k\eta_i{}^2)}\right]$$

Inserting equation (5.b) for τ_i in equation (6) leaves the following:

(7) $$E_{ss} = d\frac{v_{oi} - k\alpha_i}{k\eta_i{}^2}.$$

Making the prescription that the steady state error must go to zero leaves us with the following condition,

(8) $$k\alpha_i = v_{oi}.$$

Equation (8) tells us that in order for the pursuer to catch up with the target *eventually* (in the linearized case), the components of the pursuer's velocity must equal the components of the target's velocity.

There is cause, however, for trepidation. By linearizing equation (1) we have removed the pursuer's constant-speed restriction, giving it license to accelerate or decelerate. The fact that the pursuer's velocity changes is seen by using the Final Value Theorem on $s\rho_i(s)$, which is the frequency-space representation of the pursuer's velocity.

The pursuit equations can be integrated numerically, and the distance between the pursuer and target can be used to determine if the pursuer has "caught" the target. Several different trajectories were examined for the target, and the distance between the target and pursuer was plotted at various time points. These results are examined in the following figures.

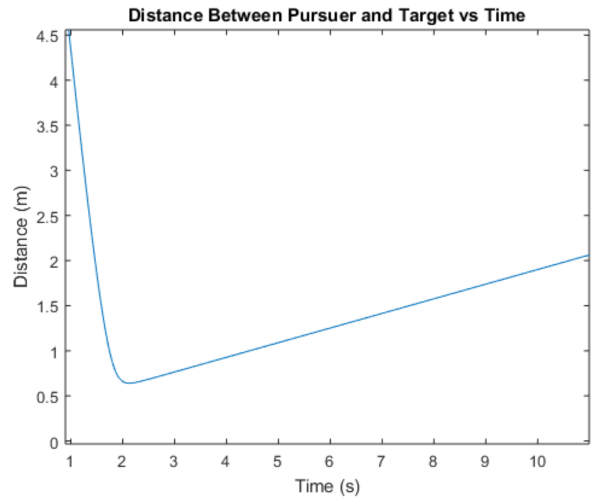

Figure 1
The pursuer's speed is less than the target's speed.
Target is moving at a constant velocity.

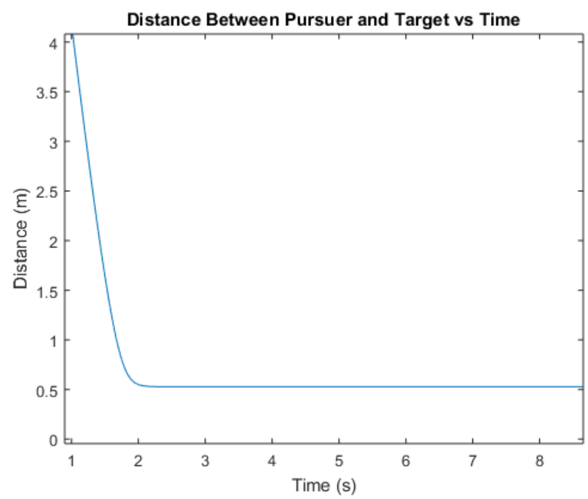

Figure 2
The pursuer's speed is equal to the target's speed.
Target is moving at a constant velocity.

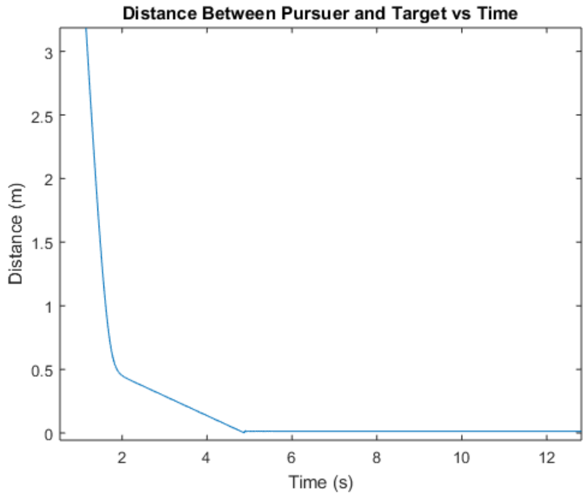

Figure 3
The pursuer's speed is greater than the target's speed. Target is moving at a constant velocity.

The condition listed in equation (8) states that the pursuer's velocity must be equal to the target's, corresponding to the setup in Figure (2). By examining the graph, we see that the pursuer comes within a threshold distance of the target, but never actually catches up with the target. It is clear that the results from the linear approximation do not match the numerical result, which is to be expected. However, it is worth noting that the result of the Final Value Theorem from equation (8) was only valid in the limit that time approaches infinity. In order to determine if the pursuer catches the target for some finite time, equation (4) can be brought back in to the time domain and set equal to the target's trajectory.

The three figures above suggest that the pursuer's speed must be greater than the target's speed in order for the pursuer to catch the target. This was suggestion was tested against more exotic target trajectories, including circular and trochoidal motion, with the same result.

With the pursuer's speed limited to a constant value, the only way to improve the time-on-target is to make modifications to the pursuit vector without altering its magnitude. One method is to force the pursuer to point ahead of the target by adding a constant multiple of the target's velocity to the pursuit vector. This makes equation (1) become

$$\text{(9)} \qquad \frac{d\vec{P}}{dt} = k\frac{\vec{T} + h\frac{d\vec{T}}{dt} - \vec{P}}{\left|\vec{T} + h\frac{d\vec{T}}{dt} - \vec{P}\right|}.$$

The linearization of which is

$$\text{(10)} \qquad \rho_i(s) = \frac{kd\alpha_i}{s(sd + k{\eta_i}^2)} + \tau_i\frac{k{\eta_i}^2(sh+1) - hT_{io}}{(sd + k{\eta_i}^2)},$$

And h has yet to be determined. By inspecting equation (10), we notice that by letting $h = \frac{d}{k{\eta_i}^2}$ the transfer function for τ_i simplifies to 1, leaving

$$\text{(11)} \quad \rho_i(s) = \frac{kd\alpha_i}{s(sd + k{\eta_i}^2)} + \tau_i - \frac{T_{io}d}{(sd + k{\eta_i}^2)},$$

which is an intriguing result because of the fact that $T_i - P_i$ can be calculated directly without prior knowledge of the target's trajectory. The result is shown below,

$$\text{(12)} \quad T_i - P_i = T_{io}e^{\frac{-k{\eta_i}^2}{d}t} - \frac{d\alpha_i}{{\eta_i}^2}\left(1 - e^{\frac{-k{\eta_i}^2}{d}t}\right).$$

As time increases, the second term becomes dominant, meaning that the pursuer ultimately ends up ahead of the target regardless of the target's trajectory. However, this doesn't necessarily imply that the pursuer never reaches the target.

The numerical simulations for Figures 1-3 were run-run using the addition to the pursuit vector, the results are shown below.

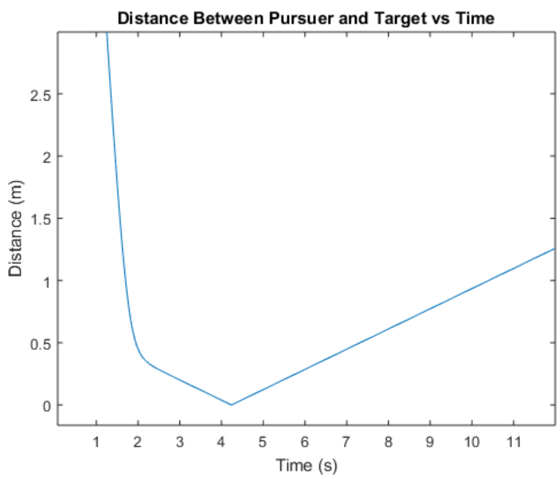

Figure 4
The pursuer's speed is less than the target's speed.
Target is moving at a constant velocity. Using equation (9).

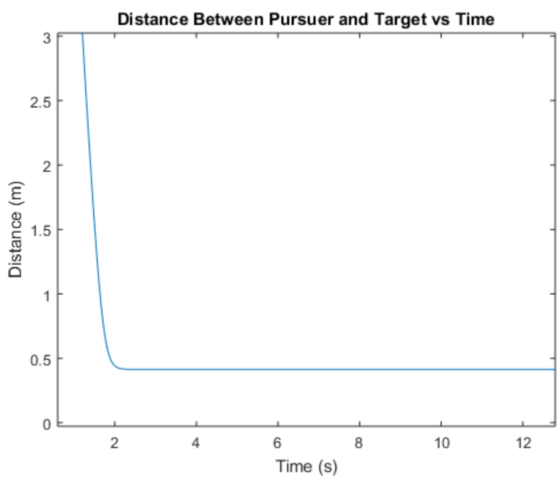

Figure 5
The pursuer's speed is equal to the target's speed.
Target is moving at a constant velocity. Using equation (9).

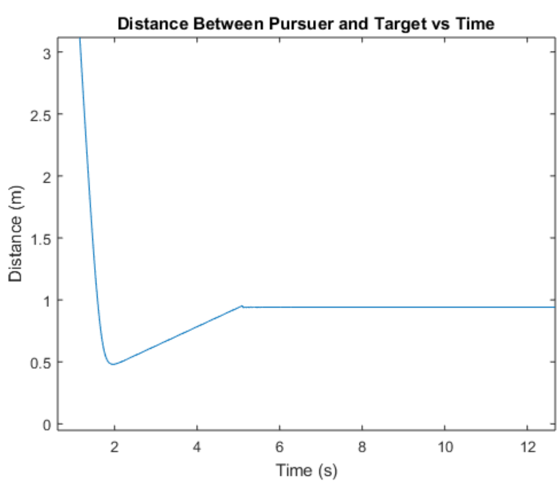

Figure 6
The pursuer's speed is greater than the target's speed.
Target is moving at a constant velocity. Using equation (9).

The results from Figures (4) and (6) are the most intriguing. In Figure (4) the pursuer is slower than the target but the pursuer still manages to come within a distance of $3.08 \times 10^{-4}\ m$ after $4.24\ s$. In Figure (6) the pursuer is moving more quickly than the target, but eventually ends up farther away from the target than in Figure (5). Thus, the result from equation (12) is validated for this specific target trajectory.

Note that requiring zero steady-state error for equation (11) yields the condition for the pursuer to catch the target.

$$E_{ss} = \lim_{s \to 0} s(\tau_i - \rho_i) = \frac{d\alpha_i}{{\eta_i}^2} = 0. \tag{13}$$

This result implies that the pursuer must be at the same position as the target.

By examining Figures (4) – (6), the general rule of thumb for chasing a target is to lead the target if the target is faster than the pursuer, and to run straight towards the target if the pursuer is faster than the target.

There are an infinite number of modifications that can be made to the pursuit vector, and one that has not been discussed is the possibility of introducing an annealing factor to the target's velocity. For ease of Laplace transformation, an annealing factor in the form of a decaying exponential can be attached to the target's velocity, giving a third possible pursuit equation:

$$\frac{d\vec{P}}{dt} = k \frac{\vec{T} + he^{-bt}\frac{d\vec{T}}{dt} - \vec{P}}{\left|\vec{T} + he^{-bt}\frac{d\vec{T}}{dt} - \vec{P}\right|} \tag{14}$$

This method is not examined thoroughly in this paper, but several arguments can be made for its validity. By adjusting the parameter *b* the pursuer whose speed matches the target can get closer to the target initially and remain closer to it for all other times. It also has the advantage of allowing the faster pursuer to avoid over-shooting the target, and affording the slower pursuer the same benefit of being able to catch the target, as in Figure (5).

The obvious disadvantage to equation (14) is the necessity of knowing the target’s trajectory and having to compute the convolution between the frequency domain counterparts of the decaying exponential and the target trajectory.