Applied Linear Algebra

MATH363

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Vector Spaces and Bases

§2.1 Real Vector Spaces

A vector space is the abstract reformulation of the quintessential properties of n-dimensional Euclidean Space $\mathbb{R}^n$, which is defined as the set of all real (column) vectors with n entries. The basic laws of vector addition and scalar multiplication in $\mathbb{R}$

Definition 2.1.1

A vector space is a set of V equipped with two operations:

- Addition: adding any pair of vectors v, w $\in V$ produces another vector v + w $\in V$;
- Scalar Multiplication: multiplying a vector v $\in V$ by a scalar $c \in \mathbb{R}$ produces a vector cv$\in V$

These are subject to the following axioms, valid for all u, v, w $\in V$ and all scalars $c, d \in \mathbb{R}$:

- Commutativity of Addition: v + w = w + v.
- Associativity of Addition: u + (v + w) = (u + v) + w.
- Additive Identity: There is a zero element $0 \in V$ satisfyng v + 0 = v = 0 + v.
- Additive Inverse: For each v $\in V$ there is an element -v $\in V$ such that v+(-v)=0=(-v)+v.
- Distributivity: $(c+d)$v=(cv) + (dv), and c(v+w)=(cv)+(cw).
- Assosiativity of Scalar Multiplication: $c(d$v$) = (cd)$v.
- Unit for Scalar Multiplication: the scalar 1$\in \mathbb{R}$ satisfies 1v=v.

Theorem 2.1.1

Let V be a vector space.

- $0 \times \vec{V} = \vec{0}$
- $-1\vec{V} = -\vec{V}$
- $c \times \vec{0} = \vec{0}$
- If $c \times \vec{V} = \vec{0}$, then $c = 0$ or $\vec{V} = \vec{0}$

Here are some examples of vector spaces:

- $\mathbb{R}^n = \left\{ \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix} \middle| r_1, r_2, r_n \in \mathbb{R} \right\}$
- $M_{m\times n}$ = The m by n matrices over $\mathbb{R}$.
- $\mathbb{P}^n$ = the polynomials of degree $\leq n$.

Definition 2.1.2

Let V be a vector space over F. $W \leq V$ is a subspace of V if W is a vector space over F under the same operation as V.

An example of definition (2.1.2). Let $V = \mathbb{R}^3 = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \middle| a, b, c \in \mathbb{R} \right\}$. V is a vector space. If we let $W = \left\{ \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} \middle| a, b \in \mathbb{R} \right\}$, then W is a subspace of V.

Theorem 2.1.2

et V be a vector space. Let $W \leq V$. W is a subspace of V if

- $w \neq 0$
- $\forall w_1 w_2 \in W; w_1 + w_2 \in W$
- $\forall c \in F; \vec{W} \in W; c \cdot \vec{W} \in W$

If we were to let $V = \mathbb{R}^3$ for $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$. We can determine the following:

- $\{\vec{0}\}$ is a subspace of V.
- $\left\{ \begin{pmatrix} a \\ 0 \\ c \end{pmatrix} \middle| a, c \in \mathbb{R} \right\}$ is a subspace of V.
- Consider the equation $\left\{ \begin{pmatrix} x \\ x \\ 0 \end{pmatrix} \middle| x \in \mathbb{R} \right\} = W$ Show that W is a subspace of V.
  - $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in W$ so $W \neq 0$.
  - $\begin{pmatrix} x \\ x \\ 0 \end{pmatrix}, \begin{pmatrix} y \\ y \\ 0 \end{pmatrix} \in W$. Then $\begin{pmatrix} x \\ x \\ 0 \end{pmatrix} + \begin{pmatrix} y \\ y \\ 0 \end{pmatrix} = \begin{pmatrix} x+y \\ x+y \\ 0 \end{pmatrix}$
  - $\begin{pmatrix} x \\ x \\ 0 \end{pmatrix} \in W$, then $c \times \begin{pmatrix} x \\ x \\ 0 \end{pmatrix} = \begin{pmatrix} cx \\ cx \\ 0 \end{pmatrix} \in W$.
  - Therefore we know that W is a subspace of V with respect to scalar multiplication and addition.
- $W = \left\{ \begin{pmatrix} x \\ y \\ 2x+3y \end{pmatrix} \middle| x, y \in \mathbb{R} \right\}$. W is a subspace of V.

$\mathbb{R}^3$ only has 4 kinds of subspaces. $\mathbb{R}^3$, $\{\vec{0}\}$, planes passing through the origin and lines that are passing through the origin.

§2.2 Subspaces

Definition 2.2.1

Let I be an interval in $\mathbb{R}$. Let $\mathbb{F}(I)$ be the vector space of functions $\mathbb{F} = I \to \mathbb{R}$.

- $\mathbb{C}^0(I) =$ the continuous functions from $I \to \mathbb{R}$ is a subspace.
- $\mathbb{P}^n(I) =$ polynomials of degree $\leq n$ restricted to $\mathbb{F}(I)$. This is a subspace of $C^0(I)$.
- $\mathbb{P}^\infty(I) =$ all polynomials on I. This is a subspace of $\mathbb{F}(I)$.
- $\mathbb{C}^n(I) =$ the set of functions $f : I \to \mathbb{R}$ such that $f', f''...f^{(n)}$ all exist and are continuous.
- $\mathbb{C}^\infty(I) =$ functions from $I \to \mathbb{R}$ such that f', f'', f''' all exist and are smooth functions.
- $A(I) =$ the functions in $\mathbb{C}^\infty(I)$ such that all $A \in I$, the power series $f(a) + \frac{f'(a)(x-a)}{1!} + \frac{f''(a)(x-a)^2}{2!} + \dots$ converges for all $x \in I$ sufficiently close to a.

Problem 1. *Show that* $v = \begin{pmatrix} x \\ y \\ 2x+y \end{pmatrix}$ *is a subspace of* $\mathbb{R}^3$.

- *Because* $\begin{pmatrix} 0 \\ 0 \\ 2(0)+0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ *is in* v, v *is not empty.*
- *Let* $\vec{v_1} = \begin{pmatrix} x_1 \\ y_1 \\ 2x_1+y_1 \end{pmatrix}, \vec{v_2} = \begin{pmatrix} x_2 \\ y_2 \\ 2x_2+y_1 \end{pmatrix} \in V$

$$v_1 + v_2 = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ 2x_1 + y_1 + 2x_2 + y_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ 2(x_1 + x_2) + (y_1 + y_2) \end{pmatrix} \in V \tag{2.1}$$

  so v *is closed with respect to addition.*

- *Let* $r \in \mathbb{R}$ *and* $\begin{pmatrix} x \\ y \\ 2x+y \end{pmatrix} = \vec{v} \in v$

§2.3 Span and Linear Independence

If we let V be a vector space over $\mathbb{R}$ and let $\vec{v_1}, \dots, \vec{v_n} \in V$, then we can determine that the

$$span(\{\vec{v_1}, \dots, \vec{v_n}\}) = \{c_1\vec{v_1} + \cdots + c_n\vec{v_n} | c_1, \dots, c_n \in \mathbb{R}\} \tag{2.2}$$

Proposition 2.3.1

The span of $\{\vec{v_1}, \vec{v_2}\}$ is a subspace of V.

Proof.

$$c_1\vec{v_1} + \cdots + c_n\vec{v_n} \tag{2.3}$$
$$k_1\vec{v_1} + \cdots + k_n\vec{v_n} \in span \tag{2.4}$$
$$\tag{2.5}$$

If we add together both of the equations above we get

$$(c_1 + k_1)\vec{v_1} + \cdots + (c_n + k_n)\vec{v_n} \in span. \tag{2.6}$$
$$r(c_1\vec{v_1} + \cdots + c_n\vec{v_n}) \tag{2.7}$$
$$= rc_1\vec{v_1} + \cdots + rc_n\vec{v_n} \in span \tag{2.8}$$

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Problem 2. *Let $V \in \mathbb{R}^3$. Also we are going to let*

$$\vec{v_1} = \begin{pmatrix}1\\2\\0\end{pmatrix}, span(\vec{v_1}) = \left\{ c\begin{pmatrix}1\\2\\0\end{pmatrix} \middle| c \in \mathbb{R} \right\} \tag{2.9}$$

$\left(\begin{smallmatrix}1\\2\\0\end{smallmatrix}\right)$ *is a vector in 3-space.* $c \times \left(\begin{smallmatrix}1\\2\\0\end{smallmatrix}\right)$ *expands, contracts, changes direction. This is a line which goes through* $\left(\begin{smallmatrix}0\\0\\0\end{smallmatrix}\right)$.

$$c\begin{pmatrix}1\\2\\0\end{pmatrix} \tag{2.10}$$

is in the xy-plane, let's solve for y to find the equation of the line that is drawn by the vector:

$$\begin{pmatrix}c\\2c\\0\end{pmatrix} \tag{2.11}$$
$$x = c \tag{2.12}$$
$$y = 2c \tag{2.13}$$
$$\frac{1}{2}y = c \tag{2.14}$$
$$\rightarrow x = \frac{1}{2}y \tag{2.15}$$
$$\rightarrow y = 2x \tag{2.16}$$
$$\tag{2.17}$$

Now we are going to let $\vec{v_2} = \left(\begin{smallmatrix}1\\1\\0\end{smallmatrix}\right)$. Consider the span of $(\{\vec{v_1}, \vec{v_2}\})$. The span of $(\{\vec{v_1}\vec{v_2}\})$ is a plane.

In $\mathbb{R}^3$, if $\vec{0} \neq \vec{v} \in \mathbb{R}$, then the span$\vec{v}$ is a line.

Problem 3. *Let* $v = \mathbb{P}^2$. v *is the set of polynomials of degree* $\leq 2 \in \mathbb{R}$.

- $span(1, x, x^2) = \mathbb{P}^2$
- $span(4, 2x) = \mathbb{P}^1$, *which means all polynomials of degree* ≤ 1

Definition 2.3.1

Let v be a vector space. $\vec{v_1}, \ldots, \vec{v_n}$ are linearly dependant if there exists $c_1, \ldots, c_n$ are not all zero, such that $c_1\vec{v_1} + \cdots + c_n\vec{v_n} = \vec{0}$, otherwise, $\vec{v_1}, \ldots, \vec{v_n}$ are linearly independent.

If we let $\vec{v_1} = \binom{1}{2}, \vec{v_2} = \binom{2}{4}$, we can do a simple test to see if they are linearly independent. We know that $2\vec{v_1} = \vec{v_2}$, which means that $2\vec{v_1} + -1\vec{v_2} = 0$. Because we can make $\vec{v_1} + \vec{v_2}$ by using a simple scalar value, these functions are linearly dependent.

Problem 4. *Consider the following three matrices*

$$\vec{v_1} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \vec{v_2} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \vec{v_3} = \begin{pmatrix} 8 \\ 1 \\ 11 \end{pmatrix} \tag{2.18}$$

Are these matrices linearly dependent or independent from each other? The following equation will let us set up a matrix to determine the results:

$$c_1\vec{v_1} + c_2\vec{v_2} + c_3\vec{v_3} = \vec{0} \tag{2.19}$$

We must try to see if there are any c_n *values that are not zero to make this true.*

$$\begin{pmatrix} 1 & 2 & 8 \\ 2 & -1 & 1 \\ 1 & 3 & 11 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{2.20}$$

$$= \begin{pmatrix} 1 & 2 & 8 & \vdots & 0 \\ 2 & -1 & 1 & \vdots & 0 \\ 1 & 3 & 0 & \vdots & 0 \end{pmatrix} \tag{2.21}$$

By doing some elementary row operations, we can find that in row echelon form, the matrix from equation (2.21) can be written as

$$\begin{pmatrix} 1 & 2 & 8 & \vdots & 0 \\ 9 & 1 & 3 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{pmatrix} \tag{2.22}$$

And from here we can solve for the different c_n *values.*

$$1c_1 + 2c_2 + 8c_3 = 0 \tag{2.23}$$
$$c_1 + 3c_3 = 0 \tag{2.24}$$
$$\tag{2.25}$$
$$c_3 = -3c_3 \tag{2.26}$$
$$c_1 = -2c_3 \tag{2.27}$$
$$c_3 = c_3 \tag{2.28}$$

Because we have this relationship where c_1, c_2, c_3 all depend on each other, we can tell that this is linearly independent.

§2.3.1 Linear Independence and Dependence

§2.4 Basis and Dimension

Definition 2.4.1

A basis of a vector space v is a collection of vectors $\vec{v_1}, \ldots, \vec{v_n}$ that 1. span v and 2. are linearly dependent.

Problem 5. *If we are looking at $\mathbb{R}^2$, with $e_1 = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$, and $e_2 = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$ We can tell that this is a basis of $\mathbb{R}^2$. We can tell this because the span of v is linearly independent. We can also see this because:*

$$a \times e_1 + b \times e_2 = \begin{pmatrix} a \\ b \end{pmatrix} \tag{2.29}$$

Problem 6. *Now we are going to look at an example in $\mathbb{R}^3$, with $e_1 = \left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right)$, $e_2 = \left(\begin{smallmatrix}0\\1\\0\end{smallmatrix}\right)$, and $e_3 = \left(\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right)$. We can figure out that this is a basis by doing the same technique as we did before:*

$$c_1\vec{e_1} + c_2\vec{e_2} + c_3\vec{e_3} = 0 \tag{2.30}$$
$$c_1 = c_2 = c_3 = 0 \tag{2.31}$$

Because c_1, c_2, and c_3 are all equal to zero, $\vec{e_1}$, $\vec{e_2}$, and $\vec{e_3}$ form a basis.

Theorem 2.4.1

If a vector space v has a basis with n elements, then every basis of v has n elements. We say v has dimension n. We write down $v = n$

Theorem 2.4.2

If the dimension of v is n, then any collection of $n+1$ or more vectors must be linearly dependent.

Theorem 2.4.3

Suppose $v = n$

1. Every collection of more than n vectors is linearly dependent.
2. No set of fewer than n vectors spans v.
3. A set of n vectors is a basis if and only if it spans v.
4. A set of n vectors is a basis if and only if it is linearly dependent.

Problem 7. *Assume* $1, x, x^2$ *is a basis for* $\mathbb{P}^2$*. We are going to multiply* 1×5*,* $x \times 6$*, and* $x^2 \times 2$*.*

$$5 + 6x + 2x^2 \tag{2.32}$$

$$c_1 \times 1 + c_2 \times x + c_3 \times x^2 = 0 \tag{2.33}$$

$$dim(\mathbb{P}^2) = 3 \tag{2.34}$$

$$\tag{2.35}$$

Theorem 2.4.4

$\vec{v_1}, \ldots \vec{v_2}$ form a basis of v if and only if for all $\vec{v} \in v$, there exist unique $c_1, \ldots, c_n$ such that $\vec{v} = c_1\vec{v_1} + \ldots c_n\vec{v_n}$

Problem 8. *Let* $v = \mathbb{R}^2$*. Let* $\vec{v} = \left(\begin{smallmatrix}4\\3\end{smallmatrix}\right)$*. We know from previous problems that* $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$ *is a basis of* $\mathbb{R}^2$*. We can also figure out what our basis is by trying to figure out what our* c_1 *and* c_2 *values should be. Because the matrices we know are a basis consist of 1's and 0's, we can see that*

$$4\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}4\\3\end{pmatrix}. \tag{2.36}$$

The coordinates of $\vec{v}$ with respect to this basis, are $(4, 3)$. Let's consider a different basis. We are now going to look at the basis,

$$\left\{ \begin{pmatrix} 1 \\ -3 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right\} \tag{2.37}$$

Now we need to figure out the a and b values for this basis respectively.

$$a \begin{pmatrix} 1 \\ -3 \end{pmatrix} + b \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \tag{2.38}$$

$$\tag{2.39}$$

The coordinates of $\vec{v}$ with respect to this basis are $(4, 3)$. Let's consider the same problem but with the following basis:

$$\left\{ \begin{pmatrix} 1 \\ -3 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right\} \tag{2.40}$$

Now, we are going to do the same thing as before, where we solve for a and b in the following equation

$$a \begin{pmatrix} 1 \\ -3 \end{pmatrix} + b \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \tag{2.41}$$

We can setup this to be a system of equations that we can turn into a matrix

$$\begin{cases} 1a + 2b = 4 \\ -3a + (-1)b = 3 \end{cases} \rightarrow \begin{bmatrix} 1 & 2 & \vdots & 4 \\ -3 & -1 & \vdots & 3 \end{bmatrix} \tag{2.42}$$

And now we can use the basic row operation $R_2 = R_2 + 3R_1$ in order to solve for a and b:

$$\begin{bmatrix} 1 & 2 & \vdots & 4 \\ 0 & 5 & \vdots & 5 \end{bmatrix} \tag{2.43}$$

$$5b = 15 \qquad a + 2b = 4 \tag{2.44}$$

$$b = 3 \qquad 1 + 2 * 3 = 4 \tag{2.45}$$

$$a = -2 \tag{2.46}$$

§2.5 The fundamental Matrix Subspaces (Kernel and Image)

Definition 2.5.1

The image of an $m \times n$ matrix A is the subspace spanned by the columns of A.

Problem 9. *Let's consider the following equation.*

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \tag{2.47}$$

When we multiply our the matrix, we see that the span of the columns give us all the possible $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$ *values. The span of the matrix*

$$\begin{bmatrix} 1 \times r_1 & 2 \times r_2 & 3 \times r_3 \\ 4 \times r_1 & 5 \times r_2 & 6 \times r_3 \end{bmatrix} \tag{2.48}$$

would be the values $\left[\begin{smallmatrix} 1 \\ 4 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 2 \\ 5 \end{smallmatrix}\right]$, *and* $\left[\begin{smallmatrix} 3 \\ 6 \end{smallmatrix}\right]$

Definition 2.5.2

A space, A, is an $m \times m$ matrix, The kernel of A is

$$\begin{aligned} A &= Ker(A) && (2.49) \\ &= \{\vec{x} | A\vec{x} = \vec{0}\} && (2.50) \end{aligned}$$

Using definition (2.5.2), if $A = \left[\begin{smallmatrix} 1 & 0 \\ 5 & 0 \end{smallmatrix}\right]$, then

$$\vec{x_1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \vec{x_2} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}. \tag{2.51}$$

Something to keep in mind: If $\vec{x_1}, \vec{x_2} \in Ker(A)$, then $r_1\vec{x_1} + r_2\vec{x_2} \in Ker(A)$. So the kernel of A is a subspace of the domain of the function.

Theorem 2.5.1

Assume $\vec{x_1}$ solves $A\vec{x} = \vec{b}$. Then, $\vec{x_2}$ is another solution to $A\vec{x} = \vec{b}$ if and only if $\vec{x_2} = \vec{x_1} + \vec{z}$, where $z \in Ker(A)$

Proposition 2.5.1

Let A be an $m \times n$ matrix. The following are true:

1. $Ker(A) = \{\vec{0}\}$
2. $rank(A) = n$
3. $A\vec{x} = \vec{b}$ has a unique solution for any $\vec{b}$ in the integer of A.
4. $A\vec{x} = \vec{b}$ has no free variables.
5. A is non-singular.

Definition 2.5.3

Let A be $m \times n$.

$$coimg(A) = img(A^T) \tag{2.52}$$
$$coker(A) = ker(A^T) \tag{2.53}$$
$$\tag{2.54}$$

The image of A is the span of its collumns. Thus the coimage is the span of its radius. Also the $\vec{r^T}$ in the cokernel of A are those $\vec{r}$ such that $r \times A = \vec{0^T}$ since

$$(r \cdot A)^T = (\vec{0^T})^T \tag{2.55}$$
$$A^T \cdot r^T = \vec{0} \tag{2.56}$$

Theorem 2.5.2

The Fundamental Theorem of Linear Algebra: Let A be an mxn matrix and let r be its rank. Then

$$dim(coimg(A)) = dim(img(A)) = rank(A) = rank(A^T) = r \tag{2.57}$$

$$span(A_{rows}) = span(A_{columns}) \tag{2.58}$$

$$dim(ker(A)) = n - r \tag{2.59}$$

$$dim(coker(A)) = m - r \tag{2.60}$$

Let's take a look at an example using the fundamental theorem of linear algebra. Let

$$A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 1 & -1 & 0 & 2 \\ -1 & 1 & 0 & -2 \\ 2 & 1 & 3 & 1 \end{bmatrix} \tag{2.61}$$

Let's do some row operations to get this matrix into row echelon form:

$$A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & -3 & -3 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{2.62}$$

From this form, we can tell that v_3 and v_4 both depend on v_1 and v_2. Because there are only two pivot points within A that are filled with values other than 0, $rank(A) = 0$. We also know that the dimension of the column space is equal to the dimension of the row space. Let's find the dimension of the kernel of A:

$$\begin{aligned} dim(ker(A)) + rank &= n && (2.63)\\ dim(key(A)) + 2 &= 4 && (2.64)\\ & && (2.65) \end{aligned}$$

From here, we know that both y and z are free variables.

$$\begin{aligned} w + 2x + 3y - z &= 0 && (2.66)\\ -3x - 3y + 3z &= 0 && (2.67)\\ x + y - z &= 0 && (2.68)\\ x &= -y + z && (2.69) \end{aligned}$$

$$\begin{aligned} w &= -2x - 3y + z && (2.70)\\ &= -2(-y + z) - 3y + z && (2.71)\\ &= -y - z && (2.72) \end{aligned}$$

Now we need to determine the basis for $ker(A)$.

$$\begin{pmatrix} -y-z \\ -y+z \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y \\ -y \\ -y \\ 0 \end{pmatrix} + \begin{pmatrix} -z \\ z \\ 0 \\ z \end{pmatrix} = y\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \tag{2.73}$$

Our basis for $ker(A)$ is $\left(\begin{smallmatrix} -1 \\ -1 \\ 1 \\ 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} -1 \\ 1 \\ 0 \\ 1 \end{smallmatrix}\right)$.

Chapter 3

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