# Applied Linear Algebra

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## 2.1 Real Vector Spaces

A vector space is the abstract reformulation of the quintessential properties of $n$-dimensional Euclidean Space $\mathbb{R}^n$, which is defined as the set of all real (column) vectors with $n$ entries. The basic laws of vector addition and scalar multiplication in $\mathbb{R}$.

**Definition 1.** AA vector space is a set of $V$ equipped with two operations:

- Addition: adding any pair of vectors v, w $\in V$ produces another vector v + w $\in V$;
- Scalar Multiplication: multiplying a vector v $\in V$ by a scalar $c \in \mathbb{R}$ produces a vector $c$v$\in V$

These are subject to the following axioms, valid for all u, v, w $\in V$ and all scalars $c, d \in \mathbb{R}$:

- Commutativity of Addition: v + w = w + v.
- Associativity of Addition: u + (v + w) = (u + v) + w.
- Additive Identity: There is a zero element $0 \in V$ satisfying v + 0 = v = 0 + v.
- Additive Inverse: For each v $\in V$ there is an element -v $\in V$ such that v+(-v)=0=(-v)+v.
- Distributivity: $(c+d)$v=($c$v) + ($d$v), and $c$(v+w)=($c$v)+($c$w).
- Associativity of Scalar Multiplication: $c$($d$v) = ($cd$)v.
- Unit for Scalar Multiplication: the scalar 1$\in \mathbb{R}$ satisfies 1v=v.

**Theorem 1.** Let $V$ be a vector space.

- $0 \times \vec{V} = \vec{0}$
- $-1\vec{V} = -\vec{V}$
- $c \times \vec{0} = \vec{0}$
- If $c \times \vec{V} = \vec{0}$, then $c = 0$ or $\vec{V} = \vec{0}$

Here are some examples of vector spaces:

- $\mathbb{R}^n = \left\{ \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix} \middle| r_1, r_2, r_n \in \mathbb{R} \right\}$
- $M_{m \times n}$ = The $m$ by $n$ matrices over $\mathbb{R}$.
- $\P^n$ = the polynomials of degree $\leq n$.

**Definition 2.** Let $V$ be a vector space over $F$. $W \leq V$ is a subspace of $V$ if $W$ is a vector space over $F$ under the same operation as $V$.

An example of definition (2.1.2). Let $V = \mathbb{R}^3 = \left\{ \left( \begin{smallmatrix} a \\ b \\ c \end{smallmatrix} \right) \middle| a, b, c \in \mathbb{R} \right\}$. $V$ is a vector space. If we let $W = \left\{ \left( \begin{smallmatrix} a \\ b \\ 0 \end{smallmatrix} \right) \middle| a, b \in \mathbb{R} \right\}$, then $W$ is a subspace of $V$.

**Theorem 2.** Let $V$ be a vector space. Let $W \leq V$. $W$ is a subspace of $V$ if

- $w \neq 0$.
- $\forall w_1 w_2 \in W; w_1 + w_2 \in W$.
- $\forall c \in F; \vec{W} \in W; c \cdot \vec{W} \in W$.

If we were to let $V = \mathbb{R}^3$ for $\left( \begin{smallmatrix} a \\ b \\ c \end{smallmatrix} \right)$. We can determine the following:

- $\{\vec{0}\}$ is a subspace of $V$.
- $\left\{ \left( \begin{smallmatrix} a \\ 0 \\ c \end{smallmatrix} \right) \middle| a, c \in \mathbb{R} \right\}$ is a subspace of $V$.
- Consider the equation $\left\{ \left( \begin{smallmatrix} x \\ x \\ 0 \end{smallmatrix} \right) \middle| x \in \mathbb{R} \right\} = W$ Show that $W$ is a subspace of $V$.
  - $\left( \begin{smallmatrix} 0 \\ 0 \\ 0 \end{smallmatrix} \right) \in W$ so $W \neq 0$.
  - $\left( \begin{smallmatrix} x \\ x \\ 0 \end{smallmatrix} \right), \left( \begin{smallmatrix} y \\ y \\ 0 \end{smallmatrix} \right) \in W$. Then $\left( \begin{smallmatrix} x \\ x \\ 0 \end{smallmatrix} \right) + \left( \begin{smallmatrix} y \\ y \\ 0 \end{smallmatrix} \right) = \left( \begin{smallmatrix} x+y \\ x+y \\ 0 \end{smallmatrix} \right)$
  - $\left( \begin{smallmatrix} x \\ x \\ 0 \end{smallmatrix} \right) \in W$, then $c \times \left( \begin{smallmatrix} x \\ x \\ 0 \end{smallmatrix} \right) = \left( \begin{smallmatrix} cx \\ cx \\ 0 \end{smallmatrix} \right) \in W$.
  - Therefore, we know that $W$ is a subspace of $V$ with respect to scalar multiplication and addition.
- $W = \left\{ \left( \begin{smallmatrix} x \\ y \\ 2x+3y \end{smallmatrix} \right) \middle| x, y \in \mathbb{R} \right\}$. $W$ is a subspace of $V$.

$\mathbb{R}^3$ only has 4 kinds of subspaces. $\mathbb{R}^3$, $\{\vec{0}\}$, planes passing through the origin and lines that are passing through the origin.

## 2.2 Subspaces

**Definition 3.** Let $I$ be an interval in $\mathbb{R}$. Let $\mathbb{F}(I)$ be the vector space of functions $\mathbb{F} = I \to \mathbb{R}$.

- $\mathbb{C}^0(I)$ = the continuous functions from $I \to \mathbb{R}$ is a subspace.
- $\P^n(I)$ = polynomials of degree $\leq n$ restricted to $\mathbb{F}(I)$. This is a subspace of $C^0(I)$.
- $\P^\infty(I)$ = all polynomials on $I$. This is a subspace of $\mathbb{F}(I)$.
- $\mathbb{C}^n(I)$ = the set of functions $f : I \to \mathbb{R}$ such that $f', f''...f^{(n)}$ all exist and are continuous.
- $\mathbb{C}^\infty(I)$ = functions from $I \to \mathbb{R}$ such that $f', f'', f'''$ all exist and are smooth functions.
- $A(I)$ = the functions in $\mathbb{C}^\infty(I)$ such that all $A \in I$, the power series $f(a) + \frac{f'(a)(x-a)}{1!} + \frac{f''(a)(x-a)^2}{2!} + \dots$ converges for all $x \in I$ sufficiently close to $a$.

**Problem.** Show that $v = \begin{pmatrix} x \\ y \\ 2x+y \end{pmatrix}$ is a subspace of $\mathbb{R}^3$.

- Because $\begin{pmatrix} 0 \\ 0 \\ 2(0)+0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ is in $v$, $v$ is not empty.
- Let $\vec{v_1} = \begin{pmatrix} x_1 \\ y_1 \\ 2x_1+y_1 \end{pmatrix}, \vec{v_2} = \begin{pmatrix} x_2 \\ y_2 \\ 2x_2+y_1 \end{pmatrix} \in V$

  $$v_1 + v_2 = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ 2x_1 + y_1 + 2x_2 + y_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ 2(x_1 + x_2) + (y_1 + y_2) \end{pmatrix} \in V$$

  so $v$ is closed with respect to addition.
- Let $r \in \mathbb{R}$ and $\begin{pmatrix} x \\ y \\ 2x+y \end{pmatrix} = \vec{v} \in v$.

## 2.3 Span and Linear Independence

If we let $V$ be a vector space over $\mathbb{R}$ and let $\vec{v_1}, \dots, \vec{v_n} \in V$, then we can determine that the

$$span(\{\vec{v_1}, \dots, \vec{v_n}\}) = \{c_1\vec{v_1} + \cdots + c_n\vec{v_n} | c_1, \dots, c_n \in \mathbb{R}\}$$

**Proposition 1.** The span of $\{\vec{v_1}, \vec{v_2}\}$ is a subspace of $V$.

*Proof.*

$$c_1\vec{v_1} + \cdots + c_n\vec{v_n}$$
$$k_1\vec{v_1} + \cdots + k_n\vec{v_n} \in span$$

If we add together both of the equations above we get

$$(c_1 + k_1)\vec{v_1} + \cdots + (c_n + k_n)\vec{v_n} \in span.$$
$$r(c_1\vec{v_1} + \cdots + c_n\vec{v_n})$$
$$= rc_1\vec{v_1} + \cdots + rc_n\vec{v_n} \in span$$

□

**Problem.** Let $V \in \mathbb{R}^3$. Also, we are going to let

$$\vec{v_1} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, span(\vec{v_1}) = \left\{ c \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \middle| c \in \mathbb{R} \right\}$$

$\left(\begin{smallmatrix} 1 \\ 2 \\ 0 \end{smallmatrix}\right)$ is a vector in 3-space. $c \times \left(\begin{smallmatrix} 1 \\ 2 \\ 0 \end{smallmatrix}\right)$ expands, contracts, changes direction. This is a line which goes through $\left(\begin{smallmatrix} 0 \\ 0 \\ 0 \end{smallmatrix}\right)$.

$$c \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

is in the $xy$-plane, let's solve for $y$ to find the equation of the line that is drawn by the vector:

$$\begin{pmatrix} c \\ 2c \\ 0 \end{pmatrix}$$
$$x = c$$
$$y = 2c$$
$$\frac{1}{2}y = c$$
$$\rightarrow x = \frac{1}{2}y$$
$$\rightarrow y = 2x$$

Now we are going to let $\vec{v_2} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. Consider the span of $(\{\vec{v_1}, \vec{v_2}\})$. The span of $(\{\vec{v_1}\vec{v_2}\})$ is a plane.

In $\mathbb{R}^3$, if $\vec{0} \neq \vec{v} \in \mathbb{R}$, then the span$\vec{v}$ is a line.

**Problem.** Let $v = ¶^2$. $v$ is the set of polynomials of degree $\leq 2 \in \mathbb{R}$.

- $span(1, x, x^2) = ¶^2$
- $span(4, 2x) = ¶^1$, which means all polynomials of degree $\leq 1$

**Definition 4.** Let $v$ be a vector space. $\vec{v_1}, \ldots, \vec{v_n}$ are linearly dependent if there exists $c_1, \ldots, c_n$ are not all zero, such that $c_1\vec{v_1} + \cdots + c_n\vec{v_n} = \vec{0}$, otherwise, $\vec{v_1}, \ldots, \vec{v_n}$ are linearly independent.

If we let $\vec{v_1} = \binom{1}{2}, \vec{v_2} = \binom{2}{4}$, we can do a simple test to see if they are linearly independent. We know that $2\vec{v_1} = \vec{v_2}$, which means that $2\vec{v_1} + -1\vec{v_2} = 0$. Because we can make $\vec{v_1} + \vec{v_2}$ by using a simple scalar value, these functions are linearly dependent.

**Problem.** Consider the following three matrices

$$\vec{v_1} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \vec{v_2} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \vec{v_3} = \begin{pmatrix} 8 \\ 1 \\ 11 \end{pmatrix}$$

Are these matrices linearly dependent or independent of each other? The following equation will let us set up a matrix to determine the results:

$$c_1\vec{v_1} + c_2\vec{v_2} + c_3\vec{v_3} = \vec{0}$$

We must try to see if there are any $c_n$ values that are not zero to make this true.

$$\begin{pmatrix} 1 & 2 & 8 \\ 2 & -1 & 1 \\ 1 & 3 & 11 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 8 & \vdots & 0 \\ 2 & -1 & 1 & \vdots & 0 \\ 1 & 3 & 0 & \vdots & 0 \end{pmatrix}$$

By doing some elementary row operations, we can find that in row echelon form, the matrix from equation (2.21) can be written as

$$\begin{pmatrix} 1 & 2 & 8 & \vdots & 0 \\ 9 & 1 & 3 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{pmatrix}$$

And from here we can solve for the different $c_n$ values.

$$\begin{aligned} 1c_1 + 2c_2 + 8c_3 &= 0 \\ c_1 + 3c_3 &= 0 \end{aligned}$$

$$\begin{aligned} c_3 &= -3c_3 \\ c_1 &= -2c_3 \\ c_3 &= c_3 \end{aligned}$$

Because we have this relationship where $c_1, c_2, c_3$ all depend on each other, we can tell that this is linearly independent.

### 2.3.1 Linear Independence and Dependence

## 2.4 Basis and Dimension

**Definition 5.** AA basis of a vector space $v$ is a collection of vectors $\vec{v_1}, \ldots, \vec{v_n}$ that (1) span $v$ and (2) are linearly dependent.

**Problem.** If we are looking at $\mathbb{R}^2$, with $e_1 = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$, and $e_2 = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ We can tell that this is a basis of $\mathbb{R}^2$. We can tell this because the span of $v$ is linearly independent. We can also see this because:

$$a \times e_1 + b \times e_2 = \begin{pmatrix} a \\ b \end{pmatrix}$$

**Problem.** Now we are going to look at an example in $\mathbb{R}^3$, with $e_1 = \left(\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right)$, $e_2 = \left(\begin{smallmatrix} 0 \\ 1 \\ 0 \end{smallmatrix}\right)$, and $e_3 = \left(\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right)$. We can figure out that this is a basis by doing the same technique as we did before:

$$\begin{aligned} c_1\vec{e_1} + c_2\vec{e_2} + c_3\vec{e_3} &= 0 \\ c_1 = c_2 = c_3 &= 0 \end{aligned}$$

Because $c_1$, $c_2$, and $c_3$ are all equal to zero, $\vec{e_1}$, $\vec{e_2}$, and $\vec{e_3}$ form a basis.

**Theorem 3.** If a vector space $v$ has a basis with $n$ elements, then every basis of $v$ has $n$ elements. We say $v$ has dimension $n$. We write down $v = n$.

**Theorem 4.** If the dimension of $v$ is $n$, then any collection of $n+1$ or more vectors must be linearly dependent.

**Theorem 5.** Suppose $v = n$

1. Every collection of more than $n$ vectors is linearly dependent.
2. No set of fewer than $n$ vectors spans $v$.
3. A set of $n$ vectors is a basis if and only if it spans $v$.
4. A set of $n$ vectors is a basis if and only if it is linearly dependent.

**Problem.** Assume $1, x, x^2$ is a basis for $\mathbb{P}^2$. We are going to multiply $1 \times 5$, $x \times 6$, and $x^2 \times 2$.

$$\begin{aligned} 5 + 6x + 2x^2 \\ c_1 \times 1 + c_2 \times x + c_3 \times x^2 = 0 \\ dim(\mathbb{P}^2) = 3 \end{aligned}$$

**Theorem 6.** h$\vec{v_1}, \ldots \vec{v_2}$ form a basis of $v$ if and only if for all $\vec{v} \in v$, there exist unique $c_1, \ldots, c_n$ such that $\vec{v} = c_1\vec{v_1} + \ldots c_n\vec{v_n}$

**Problem.** Let $v = \mathbb{R}^2$. Let $\vec{v} = \left(\begin{smallmatrix} 4 \\ 3 \end{smallmatrix}\right)$. We know from previous problems that $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ is a basis of $\mathbb{R}^2$. We can also figure out what our basis is by trying to figure out what our $c_1$ and $c_2$ values should be. Because the matrices we know are a basis consist of 1's and 0's, we can see that

$$4\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}.$$

The coordinates of $\vec{v}$ with respect to this basis, are $(4, 3)$. Let's consider a different basis. We are now going to look at the basis,

$$\left\{ \begin{pmatrix} 1 \\ -3 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right\}$$

Now we need to figure out the $a$ and $b$ values for this basis respectively.

$$a\begin{pmatrix}1\\-3\end{pmatrix}+b\begin{pmatrix}2\\-1\end{pmatrix}=\begin{pmatrix}4\\3\end{pmatrix}$$

The coordinates of $\vec{v}$ with respect to this basis are $(4,3)$. Let's consider the same problem but with the following basis:

$$\left\{\begin{pmatrix}1\\-3\end{pmatrix},\begin{pmatrix}2\\-1\end{pmatrix}\right\}$$

Now, we are going to do the same thing as before, where we solve for $a$ and $b$ in the following equation

$$a\begin{pmatrix}1\\-3\end{pmatrix}+b\begin{pmatrix}2\\-1\end{pmatrix}=\begin{pmatrix}4\\3\end{pmatrix}$$

We can set up this to be a system of equations that we can turn into a matrix

$$\begin{cases}1a+2b=4\\-3a+(-1)b=3\end{cases}\rightarrow\begin{bmatrix}1 & 2 & \vdots & 4\\-3 & -1 & \vdots & 3\end{bmatrix}$$

And now we can use the basic row operation $R_2 = R_2 + 3R_1$ in order to solve for $a$ and $b$:

$$\begin{bmatrix}1 & 2 & \vdots & 4\\0 & 5 & \vdots & 5\end{bmatrix}$$

$$\begin{aligned}5b &= 15 & a+2b &= 4\\ b &= 3 & 1+2*3 &= 4\\ & & a &= -2\end{aligned}$$

## 2.5 The fundamental Matrix Subspaces (Kernel and Image)

**Definition 6.** The image of an $m \times n$ matrix $A$ is the subspace spanned by the columns of A.

**Problem.** Let's consider the following equation.

$$\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\end{bmatrix}\begin{bmatrix}r_1\\r_2\\r_3\end{bmatrix}=\begin{bmatrix}x\\y\end{bmatrix}$$

When we multiply our the matrix, we see that the span of the columns give us all the possible $[\begin{smallmatrix} x \\ y \end{smallmatrix}]$ values. The span of the matrix

$$\begin{bmatrix} 1 \times r_1 & 2 \times r_2 & 3 \times r_3 \\ 4 \times r_1 & 5 \times r_2 & 6 \times r_3 \end{bmatrix}$$

would be the values $[\begin{smallmatrix} 1 \\ 4 \end{smallmatrix}]$, $[\begin{smallmatrix} 2 \\ 5 \end{smallmatrix}]$, and $[\begin{smallmatrix} 3 \\ 6 \end{smallmatrix}]$

**Definition 7.** AA space, $A$, is an $m \times m$ matrix, The kernel of $A$ is

$$\begin{aligned} A &= Ker(A) \\ &= \{\vec{x} | A\vec{x} = \vec{0}\} \end{aligned}$$

Using definition (2.5.2), if $A = [\begin{smallmatrix} 1 & 0 \\ 5 & 0 \end{smallmatrix}]$, then

$$\vec{x_1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \vec{x_2} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}.$$

Something to keep in mind: If $\vec{x_1}, \vec{x_2} \in Ker(A)$, then $r_1\vec{x_1} + r_2\vec{x_2} \in Ker(A)$. So the kernel of $A$ is a subspace of the domain of the function.

**Theorem 7.** Assume $\vec{x_1}$ solves $A\vec{x} = \vec{b}$. Then, $\vec{x_2}$ is another solution to $A\vec{x} = \vec{b}$ if and only if $\vec{x_2} = \vec{x_1} + \vec{z}$, where $z \in Ker(A)$

**Proposition 2.** Let $A$ be an $m \times n$ matrix. The following are true:

1. $Ker(A) = \{\vec{0}\}$
2. $rank(A) = n$
3. $A\vec{x} = \vec{b}$ has a unique solution for any $\vec{b}$ in the integer of $A$.
4. $A\vec{x} = \vec{b}$ has no free variables.
5. $A$ is non-singular.

**Definition 8.** Let $A$ be $m \times n$.

$$\begin{aligned} coimg(A) &= img(A^T) \\ coker(A) &= ker(A^T) \end{aligned}$$

The image of $A$ is the span of its columns. Thus, the coimage is the span of its radius. Also, the $\vec{r^T}$ in the cokernel of $A$ are those $\vec{r}$ such that $r \times A = \vec{0^T}$ since

$$\begin{aligned} (r \cdot A)^T &= (\vec{0^T})^T \\ A^T \cdot r^T &= \vec{0} \end{aligned}$$

**Theorem 8.** The Fundamental Theorem of Linear Algebra: Let $A$ be an $m \times n$ matrix and let r be its rank. Then

$$dim(coimg(A)) = dim(img(A)) = rank(A) = rank(A^T) = r$$
$$span(A_{rows}) = span(A_{columns})$$
$$dim(ker(A)) = n - r$$
$$dim(coker(A)) = m - r$$

Let's take a look at an example using the fundamental theorem of linear algebra. Let

$$A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 1 & -1 & 0 & 2 \\ -1 & 1 & 0 & -2 \\ 2 & 1 & 3 & 1 \end{bmatrix}$$

Let's do some row operations to get this matrix into row echelon form:

$$A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & -3 & -3 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

From this form, we can tell that $v_3$ and $v_4$ both depend on $v_1$ and $v_2$. Because there are only two pivot points within $A$ that are filled with values other than 0, $rank(A) = 0$. We also know that the dimension of the column space is equal to the dimension of the row space. Let's find the dimension of the kernel of $A$:

$$\begin{aligned} dim(ker(A)) + rank &= n \\ dim(key(A)) + 2 &= 4 \end{aligned}$$

From here, we know that both $y$ and $z$ are free variables.

$$\begin{aligned} w + 2x + 3y - z &= 0 \\ -3x - 3y + 3z &= 0 \\ x + y - z &= 0 \\ x &= -y + z \end{aligned}$$

$$\begin{aligned} w &= -2x - 3y + z \\ &= -2(-y + z) - 3y + z \\ &= -y - z \end{aligned}$$

Now we need to determine the basis for $ker(A)$.

$$\begin{pmatrix} -y-z \\ -y+z \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y \\ -y \\ -y \\ 0 \end{pmatrix} + \begin{pmatrix} -z \\ z \\ 0 \\ z \end{pmatrix} = y \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

Our basis for $ker(A)$ is $\left(\begin{smallmatrix} -1 \\ -1 \\ 1 \\ 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} -1 \\ 1 \\ 0 \\ 1 \end{smallmatrix}\right)$.

# 3 Inner Products and Norms

## 3.1 Inner Products

Dot products are a form of inner product.

Let's apply the dot product to the vectors $< v_1, v_2, \ldots, v_n > \cdot < w_1, w_2, \ldots, w_3 >$ to show that the dot product is an inner product.

$$= v_1 w_1 + v_2 w_2 + \ldots + v_n w_n \tag{1}$$

Therefore, $v \times v$ goes to $\mathbb{R}$. So we know that

$$\vec{v} \cdot \vec{v} = v_1^2 + v_2^2 + \ldots + v_n^2 \tag{2}$$

In general, we can assume that $||\vec{v}|| = \sqrt{\vec{v}\vec{v}}$. We should also keep in mind that $\vec{v} \cdot \vec{w} = ||\vec{v}||||\vec{w}|| \cos\theta$

**Definition 9.** AAn inner product of $V$ is a function $<,>: v \times v \to \mathbb{R}$ such that

-
$$< c\vec{u} + d\vec{v}, w > = c < \vec{u}, \vec{v} > +d < \vec{v}, \vec{u} > \tag{3}$$
$$< \vec{u}, c\vec{v} + d\vec{w} = c < \vec{u}, \vec{v} > +d < \vec{u}, \vec{w} > \tag{4}$$

- $< \vec{v}, \vec{w} > = < \vec{w}, \vec{v} >$
- $< \vec{v}, \vec{v} > \geq_0$ while $< 0, 0 > = \vec{0}$.

A vector space with an inner product is an inner product space.

**Definition 10.** IIf $V$ is an inner product space, then it's magnitude is

$$||\vec{v}|| = \sqrt{< \vec{v}, \vec{v} >} \tag{5}$$

Let's take a look at a weighted inner product on $\mathbb{R}^3$. We are going to let $r_1, r_2, r_3 > 0$. We can define $< \vec{v}, \vec{w} >$ as $r_1 v_1 w_1 + r_2 v_2 w_2 + r_3 v_3 w_3$

**Problem.** Let's define $[a,b] \leq \mathbb{R}$. Consider $\mathbb{C}^0[a,b]$. This is a vector space. Define

$$< f(x), g(x) >= \int_a^b f(x)g(x)dx \tag{6}$$

This is an inner product, so we also know that

$$||f|| = \sqrt{\int_a^b (f(x))^2 \, dx} \tag{7}$$

This equation is the $L^2$ norm.

## 3.2 Inequalities

Recall that $\vec{v} \cdot \vec{w} = ||\vec{v}|| ||\vec{w}|| \cos\theta$, where $\theta$ is the angle between $\vec{v}$ and $\vec{w}$. Now $-1 \leq \cos\theta \leq_1$, so we know that

$$||\vec{v} \cdot \vec{w}|| \leq ||\vec{v}|| ||\vec{w}|| \tag{8}$$

This is the Cauchy-Shuartz inequality.

**Theorem 9.** FFor any inner product space

$$|| < \vec{v}, \vec{w} > || \leq ||\vec{v}|| ||\vec{w}|| \tag{9}$$

**Definition 11.** IIf $\vec{v}, \vec{w} \in V$, we say $\vec{v}$ and $\vec{w}$ are orthogonal if $< \vec{v}, \vec{w} >= 0$

**Problem.** Let's look at an example of checking orthogonality of two equations $x, x^2 - y \in \mathbb{C}^0[0,1]$. In order to do this we need to find the $L^2$ norm of the equations.

$$\begin{aligned}
\left\langle x, x^2 - \frac{1}{2} \right\rangle &= \int_0^1 x \left( x^2 - \frac{1}{2} \right) dx \\
&= \int_0^1 \left( x^3 - \frac{1}{2} x \right) dx \\
&= \frac{1}{4}x^4 - \frac{1}{4}x^2 \Big|_0^1 = 0.
\end{aligned}$$

Because the result of the inner product was zero, we know that $x, x^2 - \frac{1}{2}$ are orthogonal in the $L^2$ norm.

**Theorem 10.** TThe triangle inequality states that if $V$ is an inner product space,

$$|| < \vec{v}, \vec{w} > || = ||\vec{v}|| + ||\vec{w}|| \tag{10}$$

Because we know that if we take the dot product of the same vector itself, $< a, b, c > \cdot < a, b, c >$, we get all of the items squared $< a^2, b^2, c^2 >$, and because we know that $||\vec{v}|| = \sqrt{< \vec{v}, \vec{v} >}$, we get an idea of size. So we can define the unit ball to be

$$\{\vec{v} \in V \Big| ||\vec{v}|| = 1\} \tag{11}$$

## 3.3 Norms

Equation (3.5) gives us the "size" of $\vec{V}$.

**Definition 12.** AA norm on $V$ is a function $|| \cdot || : V \to \mathbb{R}$ such that

- $||\vec{v}|| = 0$ if and only if $\vec{v} = 0$
- $||c\vec{v}|| = |c| \cdot ||\vec{v}||$
- $||\vec{v} + \vec{w}|| \leq ||\vec{v}|| + ||\vec{w}||$

If $|| < \vec{v}, \vec{w} > || \leq ||\vec{v}||||\vec{w}||$, then that is a norm. There are other norms to learn about.

**Problem.** Consider $V = \mathbb{R}^n$. We know that the magnitude of $\vec{V_p}$ is

$$\sqrt[p]{|\vec{v_1}|^p + |\vec{v_2}|^p + |\vec{v_3}|^p}. \tag{12}$$

So if $v = \mathbb{R}^2$, $p = 2$ we have

$$|| < x, y > ||_2 = \sqrt{x^2 + y^2}. \tag{13}$$

But if we were to have $p = 3$, we would have

$$|| < x, y > ||_3 = \sqrt[3]{x^3 + y^3} \tag{14}$$

In $|| \cdot ||_3$, the size is $\sqrt[3]{3^3 + 4^3} \approx_4 .5$

In the 4 term, $|| \cdot ||_4$, the unit circle is the $(x, y)$'s such that $\sqrt[4]{x^4 \cdot Y_4} = 1$

$$\boxed{x^4 + y^4 = 1} \tag{15}$$

Another norm on $R^n$ is the super-norm. This is where

$$|| < x_1, x_2, \ldots, x_n > ||_\infty = max\{|x_1|, |x_2|, \ldots, |x_n|\} \tag{16}$$

Here's a quick example: The super-norm for $< 3, 4 >$ is

$$|| < 3, 4 > ||_\infty = 4 \tag{17}$$

because the maximum value in the set is 4.

Something to keep in mind is $|| < x, y > || = |x| + |y|$.

**Theorem 11.** LLet $||\cdot||_A$ and $||\cdot||_B$ be two norms on $\mathbb{R}^n$. Then there exists positive numbers $0 < c < k$ such that

$$c \cdot ||\vec{v}||_A < ||\vec{v}||_B < k \cdot ||\vec{v}||_A \tag{18}$$

Let's consider $V \in \mathbb{R}^2$. Let's take a look at $||\cdot||_2$ and $||\cdot||_\infty$. Where $\vec{V} =< v_1, v_2 >$.

$$\frac{1}{\sqrt{2}} \cdot ||\vec{v}||_2 \leq ||\vec{v}||_\infty < 1 \cdot ||\vec{v}||_2 \tag{19}$$

We can also define norms on matrices.

**Theorem 12.** IIf $||\cdot||$ is a norm on $\mathbb{R}^2$ and $A$ is an $m \times n$ matrix, then

$$||A|| = max\{||A \cdot \vec{u}||\Big| ||\vec{u}|| = 1 \tag{20}$$

These matrix norms satisfy the following:

1. $||A \cdot \vec{v}|| \leq ||A|| \cdot ||\vec{v}||$
2. $||A \cdot B|| \leq ||A|| \cdot ||B||$
3. $||A^k|| \leq ||A||^k$

Let's take a quick look at $||A||_\infty$

**Definition 13.** TThe $i^{th}$ absolute row sum of $A$ is the sum of the absolute values of the entries in the $i^{th}$ row.

**Theorem 13.** y$||A||_\infty$ the maximum absolute row sum.

Here's an example of using the $||A||_\infty$ value. Let $A = \left(\begin{smallmatrix} -3 & 2 \\ 5 & 4 \end{smallmatrix}\right)$. We can determine that the maximum absolute row sum of $A$ is 8. This is because we can do

$$|-3| + |2| = 5 \tag{21}$$

$$|5| + |3| = \boxed{8} \tag{22}$$

## 3.4 Positive Definite Matrices

Consider the following two equations:

$$\vec{x} = x_1\vec{e_1} + x_2\vec{e_2} + \ldots + x_n\vec{e_n} \tag{23}$$

$$\vec{y} = y_1\vec{e_1} + y_2\vec{e_2} + \ldots + v_n\vec{e_n} \tag{24}$$

We can analyze this as

$$< \vec{x}, \vec{y} >= [x_1, x_2, \ldots, x_n] \cdot \begin{bmatrix} k_{11} & \ldots & k_{1n} \\ \vdots & \ddots & \vdots \\ k_{n_1} & \ldots & k_{nn} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots y_n \end{bmatrix} \tag{25}$$

$$k_{ij} =< e_i, e_j > \tag{26}$$
$$= \vec{x}^T k \vec{y} \tag{27}$$
$$k = k^T \tag{28}$$

This means that $k$ is symmetrical across the diagonal.

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{12} & k_{22} & k_{23} \\ k_{13} & k_{23} & k_{33} \end{bmatrix} \tag{29}$$

**Definition 14.** AA $n \times n$ matrix $A$ is a symmetrical positive definite matrix if $A = A^T$ and $x^t k < x > 0$.

**Theorem 14.** EEvery inner product on $\mathbb{R}^n$ is given by $< x, y >= \vec{x}^T k \vec{y}$ where $k$ is a symmetrical positive definite matrix. So $< \vec{x}, \vec{y} >$ is a dot product or a weighted product.

$$< \vec{x}, \vec{y} > = \vec{x}^T k \vec{y} \tag{30}$$
$$k^T = k \tag{31}$$
$$\vec{v}^T k \cdot \vec{v} > 0 \tag{32}$$
$$\vec{v} \neq_0 \tag{33}$$

Let's take a look at an example for this:

**Problem.** Let $k = \left[\begin{smallmatrix} 2 & 0 \\ 0 & 3 \end{smallmatrix}\right]$. First we need to check to see if $k^T = k$. By just looking at $k$, we can see that $k^T = k$. Next we need to do the following calculation to see if $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$ is the weighted inner product of the matrix $k$.

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 2x \\ 3x \end{bmatrix} \tag{34}$$
$$= 2x^2 + 2y^2 > 0 \tag{35}$$
$$\begin{bmatrix} x \\ y \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{36}$$

Therefore we know that

**Problem.** Let's consider the following problem

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 4 & -2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{37}$$

If we let $A$ be the numerical matrix, we can see that $A^T = A$. Let's simplify the equation from before

$$\begin{aligned} \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 4x - 2y \\ -2x + 3y \end{bmatrix} &= 4x^2 - 2xy - 2xy + 3x^2 & (38) \\ &= 4x^2 - 4xy + 3y^2 & (39) \\ (2x - y)^2 + 2y^2 > 0 \begin{bmatrix} x \\ y \end{bmatrix} &\neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} & (40) \end{aligned}$$

Now we can see that this is a positive definite matrix.

If we are given a symmetric matrix, $k$, the polynomial $x^T kx$ is a quadratic form of $k$.

**Problem.** Let's consider $k = \begin{bmatrix} 1 & -3 \\ -3 & 2 \end{bmatrix}$. Let's find the quadratic form of $k$. First we need to write $k$ like so:

$$\begin{aligned} \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & -3 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} x - 3y \\ -3x + 2y \end{bmatrix} & (41) \\ &= x^3 - 3xy - 3xy + 2y^2 & (42) \\ &= x^3 - 6xy + 2y^2 & (43) \\ & & (44) \end{aligned}$$

Therefore we know that the quadratic form of $k$ is $x^3 - 6xy - 2y^2$.

For a positive definite matrix, $k = k^T$ and $x^T kx > 0$ for all $\vec{x} \neq 0$

**Theorem 15.** EEvery inner product in $\mathbb{R}^n$ is given by

$$< x, y >= x^T ky \text{ for } xy \in \mathbb{R}^n \tag{45}$$

Let $v$ be an inner product space and $\vec{v_1}, \ldots, \vec{v_2}$. The gram matrix of $v$ is

$$K = \begin{bmatrix} < v_1, v_2 > & < v_1, v_2 > & \ldots & < v_1, v_n > \\ < v_2, v_1 > & < v_2, v_2 > & \ldots & < v_2, v_n > \\ \vdots & \vdots & \ddots & \vdots \\ < v_n, v_1 > & < v_n, v_2 > & \ldots & < v_n, v_n > \end{bmatrix} \tag{46}$$

**Definition 15.** AA is a matrix that is $n \times n$. A is a positive semidefinite matrix if $A^T = A$ and $\vec{x}^T A\vec{x} \geq 0$

**Theorem 16.** AAll gram matrices are positive semi-definite. They are positive definite if and only if $\vec{v_1}, \ldots, \vec{v_n}$ are linearly independent.

Suppose we are in $R^m$ and the inner produt is the dot product. Let $\vec{v_1}, \ldots \vec{v_n} \in \mathbb{R}^m$
Let $A = [v_1, v_2, v_3, \ldots, v_n]$. Then $K = A^T$. Let A be a gram matrix generated by $v_1, \ldots, v_n$ with the dot product.

$$K = A^T A = \begin{bmatrix} \vec{v_1} \\ \vec{v_2} \\ \vdots \\ \vec{v_n} \end{bmatrix} \begin{bmatrix} \vec{v_1} & \vec{v_2} & \ldots & \vec{v_n} \end{bmatrix} \tag{47}$$

$$= \begin{bmatrix} < v_1, v_1 > & < v_1, v_2 > & \ldots & < v_1, v_n > \\ < v_2, v_1 > & < v_2, v_2 > & \ldots & < v_2, v_n > \\ \vdots & \vdots & \ddots & \vdots \\ < v_n, v_1 > & < v_n, v_2 > & \ldots & < v_n, v_n > \end{bmatrix} \tag{48}$$

**Proposition 3.** GGiven an $m \times n$ matrix $A$. The following are true

1. The $m \times n$ matrix $k = A^T A$ is positive definite.
2. $A$ has linearly independent columns.
3. $rank(A) = n$
4. $Ker(A) = \{0\}$

**Theorem 17.** EEvery inner product on $\mathbb{R}^n$ is given by

$$< \vec{x}, \vec{y} >= \vec{x}^T \cdot c\vec{y} \tag{49}$$

where $C$ is a symetric, positive definite $n \times n$ matrix.

Let $\vec{v_1}, \ldots, \vec{v_n} \in \mathbb{R}^n$. Let $A = [\vec{v_1} \vdots \vec{v_2} \vdots \ldots \vdots \vec{v_n}]$. Then $K = A^T C A$ is the gram matrix with respect to the inner product $\vec{v}^T C \vec{w}$.

**Theorem 18.** SSuppose $A$ is an $m \times n$ matrix with linearly independent collumns. Suppose $C$ is any positive definite $m \times m$ matrix. Then $\vec{v}^T C \vec{w}$

**Definition 16.** TThe hilbert matrix $H = (h_{ij})$ where $h_{ij} = \frac{1}{i+j-1}$

The $3 \times 3$ hilbert matrix:

$$\begin{bmatrix} \frac{1}{1} & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix} \tag{50}$$

**Problem.** Make a gram matrix, not in $\mathbb{R}^m$. Let $V = \mathbb{C}^0[0,1]$. Use the $L^2$ inner product.

$$< f, g >= \int_0^1 f(x)g(x)dx. \tag{51}$$

$1, x, x^2$ are linearly independent.

$$\begin{aligned} < v_i, v_k > &= \int_0^1 x^{i-1}x^{j-1}dx && (52) \\ &= \int_0^1 x^{i+j-2}dx && (53) \\ &= \frac{1}{i+j-1}x^{i+j-1} && (54) \\ &= \frac{1}{i+j-1} && (55) \\ \to \begin{bmatrix} \frac{1}{1} & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix}. & && (56) \end{aligned}$$

# 4 Orthogonality

## 4.1

Recall that

$$\vec{v} \cdot \vec{w} = v_1 \cdot w_1 + \ldots + v_n \cdot w_n = ||\vec{v}||||\vec{w}|| \cos(\theta), \tag{57}$$

where $\theta$ is the angle between $\vec{v}$ and $\vec{w}$. Because $\cos\left(\frac{\pi}{2}\right) = 0$, we know that $\vec{v} \cdot \vec{w} = 0$ if and only if $\vec{v}$ is orthogonal to $\vec{w}$. In general, given $\vec{v}, \vec{w} \in \mathbb{R}^n$, $\vec{v}$ is orthogonal to $\vec{w}$ if and only if the angle between them is $\frac{\pi}{2}$.

**Definition 17.** Let $U$ be any inner product space. A basis $\vec{u}, \ldots, \vec{u_h} \in V$ is orthogonal if $\vec{u_j} \cdot \vec{u_i} = 0$ whenever $i \neq j$.
In addition, if $||u_i|| = 1$ for all $i$'s, the basis is orthonormal.

Here are a few examples of orthonormal and orthogonal basis's:

1. $\binom{1}{0}\binom{0}{1}$ is an orthonormal basis.
2. $\binom{2}{0}\binom{0}{1}$ is an orthogonal basis.
3. $\begin{pmatrix}1\\0\\0\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}$ is an orthonormal basis.

If $\vec{v_1}, \ldots, \vec{v_n}$ is an orthogonal basis, then

$$\frac{\vec{v_1}}{||\vec{v_1}||}, \frac{\vec{v_2}}{||\vec{v_2}||}, \ldots, \frac{\vec{v_n}}{||v_n||} \tag{58}$$

is an orthonormal basis.

**Problem.** Consider $\vec{u_1} = \binom{1}{2}$ and $\vec{u_2} = \binom{2}{-1}$. We know that these two matrices are not linearly dependent because $\vec{u_1}$ is not a multiple of $\vec{u_2}$. We can see that this is an orthogonal basis.

$$||\vec{u_1}|| = \sqrt{1^2 + 2^2} = \sqrt{5} \tag{59}$$

so we know

$$v_1 = \begin{pmatrix}\frac{1}{\sqrt{5}}\\\frac{1}{\sqrt{5}}\end{pmatrix}, v_2 = \begin{pmatrix}\frac{1}{\sqrt{5}}\\\frac{1}{\sqrt{5}}\end{pmatrix} \tag{60}$$

**Proposition 4.** Assume $\vec{v_1}, \ldots, \vec{v_n} \in V$ with $\vec{v_i} \neq \vec{0}$ for all $i$. Assume that $< v_i, v_j \geq 0$ whenever $i \neq j$, then $\{v_1, \ldots, v_n\}$ is linearly independent.

*Proof.* Assume that $c_1\vec{v_1} + \ldots + c_n\vec{n} = 0$. Let $i \in \{1, \ldots, n\}$

$$\begin{aligned} < c_1\vec{v_1} + \ldots + c_n\vec{v_n}, v_i > &= < 0, \vec{v_i} > \\ c_1 < v_1, v_i > + c_2 < v_2, v_i > + \ldots + c_i < v_i, v_i > + \ldots + c_n < v_i, v_n > \\ c_i < v_i, v_i > &= 0 \\ c_i &= 0. \end{aligned}$$

Now we know that $v_1, \ldots, v_n$ are linearly independent. □

**Corollary 1.** If $dim(v) = n$ and $\vec{v_1}, \ldots, \vec{v_n}$ are $n$ vectors such that $< v_i, v_j >= 0$ wherever $i \neq j$, then $\{\vec{v_1}, \ldots, \vec{v_n}\}$ is a basis.

**Theorem 19.** Let $\vec{u_1}, \ldots, \vec{u_n}$ be an orthonormal basis for $v$. Let $\vec{v} \in V$. Then we know $\vec{v} = c_1\vec{u_1} + \ldots + c_n\vec{u_n}$. In fact, $c_i =< \vec{v}, \vec{u_i} >$ and

$$||\vec{v}|| = \sqrt{< \vec{v}, \vec{u_1}, >^2 + < \vec{v}, \vec{u_2}^2 + < \vec{v}, \vec{u_n} >^2}. \tag{61}$$

**Problem.** $\mathbb{P}^2$ polynomials of degree $\leq 2$ on [0,1]. Use the $L^2$ norm.

$$< p, q >= \int_0^1 p(x)q(x)dx \tag{62}$$

Let $p_1 = 1, p_2 = x - \frac{1}{2}, p_3 = x^2 - x + \frac{1}{6}$.

$$< p_1, p_2 > = \int_0^1 x - \frac{1}{2}dx = 0 \tag{63}$$

$$< p_1, p_3 > = \int_0^1 x^2 - x + \frac{1}{6}dx = 0. \tag{64}$$

We have an orthogonal basis because of this. In order to check to see if it is orthonormal we must also do $< p_1, p_1 >, < p_2, p_2 >, < p_3, p_3 >$

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So if we have the basis $\{\binom{1}{2}, \binom{2}{1}\}, v_1 = \binom{1}{2}$. $v_2$ is equal to

$$\begin{pmatrix}2\\1\end{pmatrix} - \frac{\left\langle \begin{pmatrix}2\\1\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix} \right\rangle}{|| \begin{pmatrix}1\\2\end{pmatrix}} \cdot \begin{pmatrix}1\\2\end{pmatrix}$$

$$\begin{pmatrix}2\\1\end{pmatrix} - \frac{4}{\left(\sqrt{1^2+2^2}\right)^2}$$

$$\begin{pmatrix}2\\1\end{pmatrix} - \frac{4}{5}\begin{pmatrix}1\\2\end{pmatrix}$$

$$\begin{pmatrix}2\\1\end{pmatrix} - \begin{pmatrix}\frac{4}{5}\\\frac{8}{5}\end{pmatrix} = \begin{pmatrix}\frac{10}{5}\\\frac{5}{5}\end{pmatrix} - \begin{pmatrix}\frac{4}{5}\\\frac{8}{5}\end{pmatrix}$$

$$= \begin{pmatrix}\frac{6}{3}\\-\frac{3}{5}\end{pmatrix} \binom{2}{1} - \frac{< \binom{2}{1}, \binom{1}{2} >}{||\binom{1}{2}||^2} \cdot \binom{1}{2} \binom{2}{1} - \frac{4}{(\sqrt{1^2+2^2})^2}$$

$$\binom{2}{1} - \frac{4}{5}\binom{1}{2}$$

$$\binom{2}{1} - \begin{pmatrix}\frac{4}{5}\\\frac{8}{5}\end{pmatrix} = \begin{pmatrix}\frac{10}{5}\\\frac{5}{5}\end{pmatrix} - \begin{pmatrix}\frac{4}{5}\\\frac{8}{5}\end{pmatrix}$$

$$\begin{pmatrix}\frac{6}{3}\\-\frac{3}{5}\end{pmatrix}$$

.

We can conclude that our basis is $\{\binom{1}{2}, \begin{pmatrix}\frac{6}{5}\\-\frac{3}{5}\end{pmatrix}\}$. Let's now use our basis and rewrite it as

$$\begin{pmatrix}2\\3\end{pmatrix} = c_1 \begin{pmatrix}1\\2\end{pmatrix} + c_2 \begin{pmatrix}\frac{6}{5}\\-\frac{3}{5}\end{pmatrix}.$$

We can simplify this and solve for $c_1, c_2$

$$c_1 = < \begin{pmatrix}2\\3\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix} > = 2 + 6 = 8$$
$$c_2 = < \begin{pmatrix}2\\3\end{pmatrix}, \begin{pmatrix}\frac{6}{5}\\-\frac{3}{5}\end{pmatrix} > = \frac{12}{5} - \frac{9}{5} = \frac{3}{5}$$

.

Using Theorem 4.9 from the book, we do the following with an orthogonal basis to get its norm:

$$
\begin{aligned}
a_1 &= \frac{8}{||v_1||^2} = \frac{8}{(\sqrt{1^2+2^2})^2} = \frac{8}{5} \\
a_2 &= \frac{\frac{3}{5}}{||v_2||^2} = \frac{\frac{3}{5}}{\left(\frac{6}{5}\right)^2 + \left(-\frac{3}{5}\right)^2} \\
&= \frac{\frac{3}{5}}{\frac{36}{25} + \frac{9}{25}} = \frac{15}{25} \dots \\
\begin{pmatrix} 2 \\ 3 \end{pmatrix} &= a_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + a_2 \begin{pmatrix} \frac{6}{5} \\ -\frac{3}{5} \end{pmatrix} \\
&\frac{8}{5} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} \frac{6}{5} \\ -\frac{3}{5} \end{pmatrix} \\
&= \begin{pmatrix} \frac{8}{5} \\ \frac{16}{5} \end{pmatrix} + \begin{pmatrix} \frac{2}{5} \\ -\frac{1}{5} \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.
\end{aligned}
$$

**Problem.** Example of an orthogonal basis. Let $\mathbb{T}^n$ be the vector space of trigonometric polynomials.

$$\mathbb{T}^n = \sum_{0 \le j+k \le n} a_{jk} \sin^j(x) \cos^k(x).$$

Using the $L^2$ norm:

$$< f, g >= \int_{-\pi}^{\pi} f \cdot g.$$

An orthogonal basis is $\{1, \cos(x), \sin(x), \cos(2x), \sin(2x), \cos(3x), \sin(3x), \dots\}$. If we were going to do the $L^2$ norm for any of these equations we would need to do the following

$$\int_{-\pi}^{\pi} \sin(2x)\cos(4x)dx.$$

This equation is the Fourier series.

## 4.2

## 4.3 Orthogonal Matrices

**Definition 18.** AA square matrix $Q$ is orthogonal if $Q^t Q = Q \cdot Q^T = I$

If $Q$ is orthogonal, then

- $Q^{-1} = Q^T$
- $det(Q) = \pm 1$
- $Q \cdot Q^t = I$
- $det(Q)det(Q^T) = det(I)$
- $(det(Q))^2 = 1$
- $det(Q) = \pm 1$

Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. Suppose that $A$ is orthogonal, then

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \cdot \begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$a_{11}^2 + a_{12}^2 = 1$$
$$a_{11}a_{21} + a_{12}a_{22} = 0$$
$$a_{21}a_{11} + a_{22}a_{12} = 0$$
$$a_{21}^2 + a_{22}^2 = 1.$$

If we plot $(a_{11}, a_{12})$ on a graph, we can see that $\cos(\theta) = a_{12}$ and $\sin(\theta) = a_{11}$

**Proposition 5.** $Q$ is orthogonal if and only if its columns form an orthonormal basis.

*Proof.* Let $Q = [U_1 \vdots U_2 \vdots \dots \vdots U_n]$.

$$Q^T = \begin{bmatrix} U_1^T \\ U_2^T \\ \vdots \\ U_n^T \end{bmatrix}.$$

In $Q^TQ$, the $i, j^{th}$ entry is $U_1^T \cdot Uj$.

$$U_i^T \cdot U_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}.$$

So the $U_i$'so form an orthonormal basis. □

**Problem.** Let $A = \begin{bmatrix} 3 & 5 \\ 7 & \frac{1}{2} \end{bmatrix}$, and let $A$ be orthonormal. We know that $A \cdot A^T = A^T \cdot A = I$. Let's see if $A$ is an orthonormal basis.

$$\begin{bmatrix} 3 & 7 \\ 5 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 7 & \frac{1}{2} \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

This is not equal because $3 \times 7 + 3 \times 5 \neq 0$. Now let's try letting $A = \begin{bmatrix} 3 & -7 \\ 7 & 3 \end{bmatrix}$.

$$\begin{bmatrix} 3 & 7 \\ -7 & 3 \end{bmatrix} \begin{bmatrix} 3 & -7 \\ 7 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

This vector works with the zero values, but not the ones values, so we need to normalize this vector.

$$\left|\left|\begin{bmatrix} 3 \\ 7 \end{bmatrix}\right|\right| = \sqrt{4^2 + 7^2} = \sqrt{58} \qquad \left|\left|\begin{bmatrix} -7 \\ 3 \end{bmatrix}\right|\right| = \sqrt{(-7)^2 + 3^2} = \sqrt{58}$$

$$A = \begin{bmatrix} \frac{3}{58} & -\frac{7}{58} \\ \frac{7}{58} & \frac{3}{58} \end{bmatrix}.$$

Let's let $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and suppose $Q$ is orthogonal.

$$Q^T \cdot Q = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$\begin{aligned} a^2 + c^2 &= 1 \\ ab + cd &= 0 \\ ab + cd &= 0 \\ b^2 + d^2 &= 1. \end{aligned}$$

Given that the vectors $\begin{bmatrix} a \\ c \end{bmatrix}, \begin{bmatrix} b \\ d \end{bmatrix}$ lie on the unit circle, we can determine that

$$\begin{aligned} a &= \cos\theta & c &= \sin\theta \\ b &= \cos\phi & d &= \sin\phi, \end{aligned}$$

and we can determine that

$$0 = ab + cd = \cos\theta\cos\phi + \sin\theta\sin\phi = \cos(\theta - \phi).$$

If we use $\cos(\theta - \phi)$, we can determine that $\phi = \theta \pm \pi$, so $b = -\sin\theta, d = \cos\theta$ or $b = \sin\theta, d = -\cos\theta$. We either have $Q$ in one of two forms.

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \text{ or } \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}.$$

The determinant of the left matrix is 1, and the determinant of the right matrix is -1. They both give us a counter clockwise rotation by $\theta$, gives a reflection across the line with angle $\frac{\theta}{2}$

orthogonal matrices are square and $Q^t \cdot Q = QQ^t = I$. Every $2 \times 2$ orthogonal matrix has the form

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \text{ or } \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}..$$

In general, if $Q$ is orthogonal, then $det(Q) = \pm 1$

**Theorem 20.** The product of two orthogonal matrices is orthogonal. Recall, if $Q$ is orthogonal, then $Q^{-1} = Q^T$. The orthogonal $n \times n$ matrices satisfy

- Closed under the dot product.
- Multiplication is associative.
- They all have inverses.
- The identity matrix is orthogonal.

## 4.4 Vector Spaces

Up until now we've been trying to get a basis to be able to establish a location of vectors. Once we have determined an inner product, we can find an angle. Let $V$ be a subspace. Let $W \leq V$ be a finite dimensional subspace.

**Definition 19.** $\vec{z} \in V$ is orthogonal to $w$ if it is orthogonal to every vector in $w$.

**Note.** If $\vec{w_1}, \ldots, \vec{w_n}$ is a basis for $w$, then $\vec{z}$ is orthogonal to $w$ if and only if $\vec{z}$ is orthogonal to $w_1, \ldots, w_n$.

**Definition 20.** The orthogonal projection of $\vec{V}$ onto $w$ is the vector $\vec{w}$ such that $\vec{z} = \vec{v} - \vec{w}$, where $\vec{z}$ is orthogonal to $w$.

**Theorem 21.** Let $\vec{u_1}, \ldots, \vec{u_n}$ be an orthogonal basis for $w$. Let $\vec{v} \in V$. The orthogonal projection of $\vec{v}$ onto $w$ is

$$\vec{w} = c_1\vec{u_1} + \ldots + c_n\vec{u_n} \text{ where } c_i < v, v_i >.$$

**Note.** If $\vec{v_1}, \ldots, \vec{v_n}$ is an orthogonal basis for $w$ then

$$w = a_1\vec{v_1} + \ldots + a_n\vec{v_n}$$
$$a_i = \frac{\langle \vec{v}, \vec{v_i} \rangle}{\|v_i\|^2}.$$

**Definition 21.** Let $w, z \leq V$ be subspaces. $w$ is orthogonal to $z$ if every vector in $w$ is orthogonal to every vector in $z$. For example,

$$< \vec{w}, \vec{z} >= 0$$

for every $\vec{w} \in w, \vec{z} \in z$ .

**Note.** We only need to show this is true on the bases of $w$ and $z$.

**Definition 22.** Let $w \in v$ be a subspace. The orthogonal compliment of $w$, written $w^T$ is the set of vectors in $v$ orthogonal to $w$.

$$w^T = \{\vec{v} \in v | \langle \vec{v}, \vec{w} \rangle = 0, \forall w \in w\}.$$

**Theorem 22.** Let $w < v$ be a finite dimensional subspace. Every $\vec{v} \in v$ can be written uniquely as

$$\vec{v} = \vec{w} + \vec{z}$$

where $\vec{w} \in w$ and $\vec{z} \in w^T$.

# 5 Mimimization and Least Squares

**Theorem 23.** If $K$ is a positive definite (and hence symmetric) matrix, then the quadratic function has a unique minimizer, whcih is the solution to the linear system

$$Kx = f, \text{ namely } x^* = K^{-1}f.$$

The minimum value of $p(x)$ is equal to any of the following expressions:

$$p(X^*) = p(K^{-1}f) = c - f^T K^{-1} f = c - f^T x^* = c - (x^*)^T K x^*.$$

# 6 Equilibrium

# 7 Linearity

# 8 Eigenvalues and Singular Values

## 8.1

**Definition 23.** Let $A$ be an $nxn$ matrix. A scalar $\lambda$ is an eigenvalue of $A$ if

$$A\vec{v} = \lambda\vec{v}.$$

For some nonzero vector $\vec{v} \cdot \vec{v}$ is called the eigen vector corresponding to lambda.

**Example.** Consider $A = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$. Let's find the eigenvalues of this matrix.

$$\begin{aligned} 0 &= det\left(\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right) \\ &= det\begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix} \\ &= (2-\lambda)^2 - 1 \\ 0 &= (3-\lambda)(1-\lambda). \end{aligned}$$

Our eigenvalues are 3 and 1. Remember that $(A - \lambda I)\vec{v} = \vec{0}$. Consider $\lambda = 3$

$$\begin{aligned} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ -x + y &= 0 \\ x &= y \\ \begin{pmatrix} x \\ x \end{pmatrix} &= x\begin{pmatrix} 1 \\ 1 \end{pmatrix}. \end{aligned}$$

Now consider $\lambda = 1$.

$$\begin{aligned} A(\lambda I)\vec{v} &= \vec{0} \\ (A - \lambda I)\vec{v} \\ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ x + y &= 0 \\ y &= -x \\ \begin{pmatrix} x \\ -x \end{pmatrix} &= x \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \end{aligned}$$

**Example.** Consider $A = \left(\begin{smallmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{smallmatrix}\right) \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$. For eigenvalues, $0 = det(A - \lambda I)$.

$$\begin{aligned} 0 &= det \begin{pmatrix} 1-\lambda & 1 & 2 \\ 0 & 1-\lambda & 2 \\ 0 & 0 & 3-\lambda \end{pmatrix} \\ 0 &= (1-\lambda)(1-\lambda)(3-\lambda). \end{aligned}$$

This makes our eigenvalues 1 and 3. For $\lambda = 3$,

$$\begin{aligned} \begin{pmatrix} -2 & 1 & 2 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ -2y + 2z &= 0 \\ y &= 0 \\ -2x + z + 2z &= 0 \\ -2x + 3z &= 0 \\ 3z &= 2x \\ y = z &= \frac{2}{3}. \end{aligned}$$

This would make our eigenvector

$$\begin{pmatrix} x \\ \frac{2}{3}x \\ \frac{2}{3}x \end{pmatrix} = x \begin{pmatrix} 1 \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}.$$

**Definition 24.** Given an eigenvalue $\lambda$ of $A$, the corresponding eigenvectors form a subspace denoted $v_\lambda$. Note that $v_\lambda = ker(A - \lambda I)$

**Note.** $\lambda = 0$ is an eigenvalue of $A$ if and only if the $ker(A - \lambda I) = ker(A) = v_0 \neq \{0\}$. This is true if and only if $A$ is singular ($det(A) = 0$).

**Example.** Consider $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. The determinant of

$$\begin{pmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix}$$

is equal to zero. We can see that when $\lambda = 0$, the determinant is zero.

**Proposition 6.** If $A$ is a real matrix and $\lambda + i\mu$ is an eigenvalue of $A$ with eigenvector $\vec{v} = \vec{x} + i\vec{y}$, then $\lambda - i\mu$ is an eigenvalue of $A$ with eigenvector $\vec{x} - i\vec{y}$.

**Example.** Consider the matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

$$\begin{aligned} |A - \lambda I| &= 0 \\ \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} &= 0 \\ \lambda^2 + 1 &= 0 \\ \lambda^2 &= -1 \\ \lambda = \pm i &= 0 \pm i \\ (A - \lambda I)\vec{v} &= 0 \\ \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ -ix - y &= 0 \\ -ix &= y. \end{aligned}$$

We now know that if we let $x$ be anything, and $y = -ix$, then we have the eigenvector. We can rewrite it as

$$\begin{aligned} \begin{pmatrix} x \\ -ix \end{pmatrix} &= \square + i\square \\ &= \begin{pmatrix} x \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -ix \end{pmatrix} \\ &= x\begin{pmatrix} 1 \\ 0 \end{pmatrix} + x\begin{pmatrix} 0 \\ -i \end{pmatrix} \\ &= x\begin{pmatrix} 1 \\ 0 \end{pmatrix} + i\begin{pmatrix} 0 \\ -1 \end{pmatrix}. \end{aligned}$$

Now we also know that if $\lambda = -i$ then the eigenvector is

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} - i\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

## Lecture 1: 8.2, 8.3

Monday March 29, 2021

If $A$ is an $n \times n$ matrix with real entries, then

$$det(A - \lambda I) = p(\lambda) = \text{The characteristic polynomial.}$$

**Note.** If $A$ is $2 \times 2$, then $P_A(\lambda) = \lambda^2 - Tr(A) + det(A)$

Suppose that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The trace ($Tr(A)$) is $= a + d$, and the determinant if $A$ is $ad - bc$, so

$$\begin{aligned} p_a(\lambda) = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} &= (a-\lambda)(d-\lambda) - bc \\ &= ad - d\lambda - a\lambda + \lambda^2 - bc \\ &= \lambda^2 - (a+d)\lambda + (ad - bc). \end{aligned}$$

Remember, the trace is $a + d$, and the determinant is $ad - bc$. Recall that $A$ is real, so $P_A(\lambda)$ is real. If we set $P_A(\lambda) = 0$. By the fundamental theorem of algebra, $P_A(\lambda)$ factors into linear factors over $\mathbb{C}$. Consider the equation

$$x^{10} - 7x^9 + 8x^2 + \frac{1}{2} = 0.$$

This factors into 10 different roots. So if $A$ is $n \times n$, $P_A(\lambda)$ has at most $n$ roots in $\mathbb{C}$

**Theorem 24.** Let $A$ be $n \times n$ with real entries. Then $A$ has at most $n$ eigenvalues. If $a + bi$ is an eigenvalue, then so is $a - bi$

**Example.** The Jordan Block Matrix. Let's look at $J_{2,3} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$. Let's find the eigenvalues:

$$\begin{aligned} |J_{2,3} - \lambda I| &= 0 \\ \begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix} &= 0 \\ (2-\lambda)(2-\lambda)(1-\lambda) &= 0 \\ \lambda &= 2. \end{aligned}$$

Now let's find the eigenvector(s):

$$\begin{aligned} (a - lambdaI)\vec{v} &= \vec{0} \\ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ y &= 0 \\ z &= 0 \\ \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} &= x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \end{aligned}$$

**Theorem 25.** If $A$ is square, then $P_A(\lambda) = P_{A^T}(\lambda)$. So, $A$ and $A^T$ have the same eigenvalues. Probably not the same eigenvectors.

**Theorem 26.** Let $A$ be $n \times n$. The sum of the eigenvalues of $A$ is equal to the $Tr(A)$, and the product of the eigenvalues of $A$ is equal to $det(A)$.

**Example.** $J_{2,3} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$. The Trace of $J_{2,3} = 2+2+2$ (adding the diagonals). The determinant of $J_{2,3} = 2*2*2$ (multiplication of the diagonals).

$$P_{J_{2,3}}(\lambda) = (2-\lambda)(2-\lambda)(2-\lambda).$$

## 8.2

**Proposition 7.** If $\lambda_1, \ldots, \lambda_k$ are distinct eigenvalues of $A$, then the corresponding eigenvectors are linearly independent.

**Example.** Let's let $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix}$. Let's find the eigenvalues of $A$.

$$\begin{aligned} 0 = P_A(\lambda) &= \begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & 2-\lambda & -1 \\ 0 & 0 & 3-\lambda \end{vmatrix} \\ 0 &= (1-\lambda)(2-\lambda)(3-\lambda) \\ \lambda &= 1, 2, 3. \end{aligned}$$

For $\lambda = 1$

$$\begin{aligned} (A - \lambda I)\vec{v} &= \vec{0} \\ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ y &= 0 \\ y - z &= 0 \\ z &= 0 \\ \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} &= x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \end{aligned}$$

Now for $\lambda = 2$

$$\begin{aligned}
\begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
-x + y &= 0 \\
x &= y - z & &= 0 \\
\begin{pmatrix} x \\ x \\ 0 \end{pmatrix} &= x \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.
\end{aligned}$$

Now for $\lambda = 3$

$$\begin{aligned}
\begin{pmatrix} -2 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
-2x + y &= 0 \to y = 2x \\
-y - z &= 0 \to z = -y \\
z &= -2x \\
\begin{pmatrix} x \\ 2x \\ -2x \end{pmatrix} &= x \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}.
\end{aligned}$$

We now have the following eigenvectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}.$$

These are linearly independent so we have a basis for $\mathbb{R}^3$

**Theorem 27.** If $A$ is $n \times n$ and $a$ has $n$ distinct real (/complex) eigenvalues, then the corresponding eigenvectors form a basis for $\mathbb{R}^b(\mathbb{C}^n)$.

Now... vector spaces. Vectors have two main operations,

$$\begin{aligned}
&\vec{v} + \vec{w} \\
&c \cdot v.
\end{aligned}$$

Let $V, W$ be vector spaces over $\mathbb{R}$. We know that $\mathbb{R}^3 \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}$. In reality, $V, W$ are the same, but we must show that they are the same. We would need a function that preserves the operations from $V$ to $W$. A map (function) $f$ from $V$ to $W$ should have the following

$$\begin{aligned}
f(\vec{v_1} + \vec{v_2}) &= f(\vec{v_1}) + f(\vec{v_2}) \\
f(r \cdot \vec{v}) &= r f(\vec{v}).
\end{aligned}$$

## Lecture 2: 8.3

Wednesday, March 31st

Recall that $\lambda = 0$ is an eigenvalue of $A$, if and only if

$$(A - 0 \cdot I)\vec{v} = \vec{0}$$

has a $\vec{v} \neq \vec{0}$ solution if and only if

$$ker(A) \neq \{0\}$$

if and only if $A^{-1}$ does not exist.

**Theorem 28.** If $\lambda_1, \ldots, \lambda_k$ are distinct eigenvalues of $A$, then the corresponding eigenvectors are linearly independent.

**Example.** Consider $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. The eigenvalues for this matrix are $\lambda = 3, 1$. The eigenvector for $\lambda = 3$ is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and the eigenvector for $\lambda = 1$ is $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

**Example.** Consider $A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$. The characteristic polynomial is $P(\lambda) = (1-\lambda)^2(3-\lambda)$. For $\lambda = 1$, the eigenvector is $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, and for $\lambda = 3$, the eigenvector is $\begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}$

**Theorem 29.** If $A$ is $n \times n$ and $A$ has $n$ distinct real (or complex) eigenvalues, then the corresponding eigenvectors $\vec{v_1}, \ldots, \vec{v_n}$ form a basis for $\mathbb{R}^n$.

Consider $A = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}$. $A$ defines a map from $\mathbb{R}^2$ to $\mathbb{R}^2$. Our map is

$$L : \begin{pmatrix} x \\ y \end{pmatrix} \to A \begin{pmatrix} x \\ y \end{pmatrix}$$
$$L\left(r\vec{v}\right) = rL(\vec{v}).$$

$L$ is a linear transformation.

**Example.** Consider $A = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}$. What happens if we multiply $A$ by $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$?

$$\begin{aligned} A \begin{pmatrix} 2 \\ 1 \end{pmatrix} &= \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 8 \end{pmatrix}. \end{aligned}$$

Let's think of $\begin{pmatrix}2\\1\end{pmatrix}$ as $2e_1 + 1e_2$. This would make our equation

$$2\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}.$$

Let's also think about $\begin{pmatrix}1\\8\end{pmatrix}$ as

$$1\begin{pmatrix}1\\0\end{pmatrix} + 8\begin{pmatrix}0\\1\end{pmatrix}.$$

We can change the basis to

$$v_1 = \begin{pmatrix}1\\-1\end{pmatrix} \qquad\qquad v_2 = \begin{pmatrix}1\\-2\end{pmatrix}.$$

## Lecture 3: 8.2, 8.3

Monday March 29, 2021

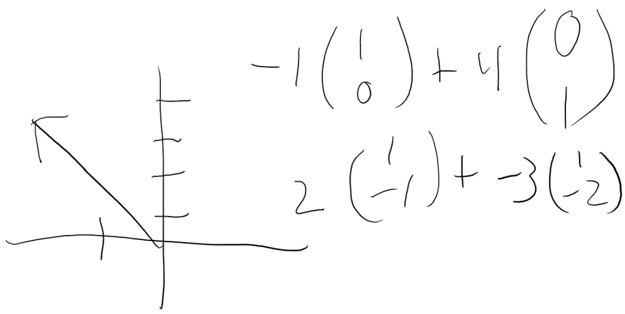


Figure 1: Example Line image

A linear transformation consists of the following:

$$L(v_1 + v_2) = L(v_1) + L(v_2)$$
$$L(r \cdot v) = rL(v).$$

Consider $A = \begin{pmatrix}1 & -1\\2 & 4\end{pmatrix}$. If everything is in the standard basis then

$$\begin{pmatrix}1 & -1\\2 & 4\end{pmatrix}\begin{pmatrix}-1\\4\end{pmatrix} = \begin{pmatrix}-5\\14\end{pmatrix}$$
$$-1\begin{pmatrix}1\\0\end{pmatrix} + 4\begin{pmatrix}0\\1\end{pmatrix} \to -5\begin{pmatrix}1\\0\end{pmatrix} + 14\begin{pmatrix}0\\1\end{pmatrix}.$$

It's harder to figure out what the vector is with the new basis from the picture, but the transformation has a nice description of

$$B = \begin{pmatrix}2 & 0\\0 & 3\end{pmatrix}.$$

If we take $B$ and hit it with the new coefficients:

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 4 \\ -9 \end{pmatrix}$$
$$2\begin{pmatrix} 1 \\ -1 \end{pmatrix} + -3\begin{pmatrix} 1 \\ -2 \end{pmatrix} \to 4\begin{pmatrix} 1 \\ -1 \end{pmatrix} + -9\begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

The left hand side is equal to $\begin{pmatrix} -1 \\ 4 \end{pmatrix}$ and the right hand side is equal to $\begin{pmatrix} -5 \\ 14 \end{pmatrix}$
If we take $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$, which looks like an eigenvector. The first basis that we had was $\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$, and the second basis was $\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix}\right\}$. So we took basis 1 and did a linear transofrmation to get basis 2 like the following

$$\begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix} \to \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}.$$

If $A$ is in the standard basis. Then $B = S^{-1}AS$, where $S = \left(\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}\right)$, where $\{\vec{v_1}, \ldots, \vec{v_n}\}$ is the new basis.

**Example.** Let $A = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}$.

$$\begin{aligned} S &= \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} \\ S^{-1} &= \frac{1}{-2 - -1} \begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} \\ B = S^{-1}AS &= \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ -2 & -6 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}. \end{aligned}$$

**Definition 25.** A square matrix $A$ is diagonalizable if there is a matrix $S$ and a diagonal matrix $\Lambda$ such that

$$\Lambda = S^{-1}AS$$

**Theorem 30.** $A$ is diagonalizable if and only if $A$ has $n$ linearly independent eigenvectors $\vec{v_1}, \ldots, \vec{v_n}$. In this case,

$$\Lambda = S^{-1}AS.$$

where $S = [\vec{v_1}, \ldots, \vec{v_n}]$, and $\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_n \end{bmatrix}$, where $\lambda_i$ is the eigenvalue for $\vec{v_i}$

If we are looking for the linearly independent solutions we need to do $A - \lambda I = 0$. From here we would get lambda values and create eigenvectors using $A - \lambda I \vec{v} = 0$

**Example.** Let $A = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$, which is our standard matrix. Once we find our

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Our basis is

$$\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

Let's take $A$ and hit it with coefficients in the standard basis. Let's try

$$A \cdot \begin{pmatrix} -3 \\ 2 \\ 2 \end{pmatrix}.$$

We need to think about the matrix on the right hand side as $-3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, so we can do the following:

$$\begin{pmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} -3 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \\ 3 \end{pmatrix}.$$

We know that our divided up equation from before (for the right matrix) is getting mapped to $4 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. We know that our $\Lambda$ must take $\begin{pmatrix} -3 \\ 2 \\ 2 \end{pmatrix}$, but we need to rewrite it in our new basis.

$$1 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + 1 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

If we multiply this by our $\lambda$, we can figure out out map.

$$\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 2\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}1\\1\\2\end{pmatrix}.$$

Our equation gets mapped to

$$1\begin{pmatrix}-1\\1\\0\end{pmatrix} + 1\begin{pmatrix}-1\\0\\1\end{pmatrix} + 2\begin{pmatrix}-1\\1\\1\end{pmatrix}.$$

This means that our vector is the same from before because we get $\begin{pmatrix}-4\\3\\3\end{pmatrix}$.

$Av_1 \to w_1$, while $\Lambda v_2 \to w_2$.

## Lecture 4

Monday April 12, 2021

## Lecture 5

Wednesday April 14, 2021

If we have an $n \times n$ matrix with $n$ distinct eigenvalues, we have $n$ linearly independent eigenvectors.

$$\Lambda = S^{-1}AS.$$

If the matrix is real symmetric, then we can get a real eigenvector basis so we can diagonalize like

$$S\Lambda S^{-1} = A.$$

If we do $A \cdot A$, then we get

$$\begin{aligned}&S\Lambda S^{-1}S\Lambda S^{-1}\\&\quad = S\Lambda S^{-1}\\A^k &= S\Lambda^k S^{-1}.\end{aligned}$$

Let's look at an example where this is not going to work.

**Example.** Let $M = \begin{bmatrix}1 & 1 & 1\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$. This matrix only has one eigenvalue of $\lambda = 1$. Let's find the eigenvectors.

$$\begin{bmatrix}0 & 1 & 1\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$y + z = 0 \to z = -y$$

$$\begin{bmatrix}x\\y\\-y\end{bmatrix} = x\begin{bmatrix}1\\0\\0\end{bmatrix} + y\begin{bmatrix}0\\1\\-1\end{bmatrix}.$$

Now supposed instead, if we let $M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. This matrix must have 3 linearly independent eigenvectors.

$$|M - \lambda I| = 0$$
$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$
$$(M - \lambda I)\vec{v} = 0$$
$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

**Theorem 31.** Let $A = A^T$ be an $n \times n$ real matrix. Let $\vec{v_1}, \ldots, \vec{v_n}$ be an eigenvector basis such that $\vec{v_1}, \ldots, \vec{v_r}$ correspond to nonzero eigenvalues and $\vec{v_{r+1}}, \ldots, \vec{v_n}$ correspond to the zero eigenvalue. Then $r = rank(A)$, $\vec{v_1}, \ldots, \vec{v_r}$ form an orthogonal basis for $IM(A) = coimg(A)$, and $\vec{v_{r+1}}, \ldots, \vec{v_n}$ form an orthogonal basis for the $ker(A) = coker(A)$.

**Theorem 32.** The spectral theorem. Let $A$ be a real symmetric matrix. Then there exists an orthogonal matrix $Q$ such that $A = Q\Lambda Q^{-1} = Q\Lambda Q^T$, where $\Lambda$ is a real diagonal matrix. The eigenvalues of $A$ appear on the diagonal of $\Lambda$, while the columns of $Q$ are orthonormal eigenvectors of $A$.

**Example.** Let $A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 01 & \end{bmatrix}$. Our eigenvalues are $1+\sqrt{2}, 1, 1-\sqrt{2}$, which means our eigenvectors are

$$\begin{bmatrix} \sqrt{2} \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -\sqrt{2} \\ -1 \\ 1 \end{bmatrix}.$$

The orthonormal basis of the eigenvectors is

$$\sqrt{(\sqrt{2})^2 + (-1)^2 + 1^2}$$
$$= \sqrt{6}$$
$$\begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}.$$

Now if we let $S = \begin{bmatrix} \sqrt{2} & 0 & -\sqrt{2} \\ -1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$ and we do

$$S^{-1}AS = \Lambda = \begin{bmatrix} 1+\sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1-\sqrt{2} \end{bmatrix}.$$

# 9 Iteration

# 10 Dynamics