

# Applied Linear Algebra

Cameron Williamson

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# 1 Linear Algebraic Systems

## 2 Vector Spaces and Bases

### 2.1 Real Vector Spaces

A vector space is the abstract reformulation of the quintessential properties of  $n$ -dimensional Euclidean Space  $\mathbb{R}^n$ , which is defined as the set of all real (column) vectors with  $n$  entries. The basic laws of vector addition and scalar multiplication in  $\mathbb{R}$ .

**Definition 1.** A vector space is a set of  $V$  equipped with two operations:

- Addition: adding any pair of vectors  $v, w \in V$  produces another vector  $v + w \in V$ ;
- Scalar Multiplication: multiplying a vector  $v \in V$  by a scalar  $c \in \mathbb{R}$  produces a vector  $cv \in V$

These are subject to the following axioms, valid for all  $u, v, w \in V$  and all scalars  $c, d \in \mathbb{R}$ :

- Commutativity of Addition:  $v + w = w + v$ .
- Associativity of Addition:  $u + (v + w) = (u + v) + w$ .
- Additive Identity: There is a zero element  $0 \in V$  satisfying  $v + 0 = v = 0 + v$ .
- Additive Inverse: For each  $v \in V$  there is an element  $-v \in V$  such that  $v + (-v) = 0 = (-v) + v$ .
- Distributivity:  $(c + d)v = (cv) + (dv)$ , and  $c(v + w) = (cv) + (cw)$ .
- Associativity of Scalar Multiplication:  $c(dv) = (cd)v$ .
- Unit for Scalar Multiplication: the scalar  $1 \in \mathbb{R}$  satisfies  $1v = v$ .

**Theorem 1.** Let  $V$  be a vector space.

- $0 \times \vec{V} = \vec{0}$
- $-1\vec{V} = -\vec{V}$
- $c \times \vec{0} = \vec{0}$
- If  $c \times \vec{V} = \vec{0}$ , then  $c = 0$  or  $\vec{V} = \vec{0}$

Here are some examples of vector spaces:

- $\mathbb{R}^n = \left\{ \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix} \mid r_1, r_2, r_n \in \mathbb{R} \right\}$
- $M_{m \times n}$  = The  $m$  by  $n$  matrices over  $\mathbb{R}$ .
- $\mathbb{P}^n$  = the polynomials of degree  $\leq n$ .

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**Definition 2.** Let  $V$  be a vector space over  $F$ .  $W \leq V$  is a subspace of  $V$  if  $W$  is a vector space over  $F$  under the same operation as  $V$ .

An example of definition (2.1.2). Let  $V = \mathbb{R}^3 = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\}$ .  $V$  is a vector space. If we let  $W = \left\{ \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} \mid a, b \in \mathbb{R} \right\}$ , then  $W$  is a subspace of  $V$ .

**Theorem 2.** Let  $V$  be a vector space. Let  $W \leq V$ .  $W$  is a subspace of  $V$  if

- $w \neq 0$ .
- $\forall w_1, w_2 \in W; w_1 + w_2 \in W$ .
- $\forall c \in F; \vec{w} \in W; c \cdot \vec{w} \in W$ .

If we were to let  $V = \mathbb{R}^3$  for  $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ . We can determine the following:

- $\{\vec{0}\}$  is a subspace of  $V$ .
- $\left\{ \begin{pmatrix} a \\ 0 \\ c \end{pmatrix} \mid a, c \in \mathbb{R} \right\}$  is a subspace of  $V$ .
- Consider the equation  $\left\{ \begin{pmatrix} x \\ x \\ 0 \end{pmatrix} \mid x \in \mathbb{R} \right\} = W$  Show that  $W$  is a subspace of  $V$ .
  - $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in W$  so  $W \neq 0$ .
  - $\begin{pmatrix} x \\ x \\ 0 \end{pmatrix}, \begin{pmatrix} y \\ y \\ 0 \end{pmatrix} \in W$ . Then  $\begin{pmatrix} x \\ x \\ 0 \end{pmatrix} + \begin{pmatrix} y \\ y \\ 0 \end{pmatrix} = \begin{pmatrix} x+y \\ x+y \\ 0 \end{pmatrix}$
  - $\begin{pmatrix} x \\ x \\ 0 \end{pmatrix} \in W$ , then  $c \times \begin{pmatrix} x \\ x \\ 0 \end{pmatrix} = \begin{pmatrix} cx \\ cx \\ 0 \end{pmatrix} \in W$ .
  - Therefore, we know that  $W$  is a subspace of  $V$  with respect to scalar multiplication and addition.
- $W = \left\{ \begin{pmatrix} x \\ y \\ 2x+3y \end{pmatrix} \mid x, y \in \mathbb{R} \right\}$ .  $W$  is a subspace of  $V$ .

$\mathbb{R}^3$  only has 4 kinds of subspaces.  $\mathbb{R}^3$ ,  $\{\vec{0}\}$ , planes passing through the origin and lines that are passing through the origin.

## 2.2 Subspaces

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**Definition 3.** Let  $I$  be an interval in  $\mathbb{R}$ . Let  $\mathbb{F}(I)$  be the vector space of functions  $\mathbb{F} = I \rightarrow \mathbb{R}$ .

- $\mathbb{C}^0(I)$  = the continuous functions from  $I \rightarrow \mathbb{R}$  is a subspace.
- $\mathbb{P}^n(I)$  = polynomials of degree  $\leq n$  restricted to  $\mathbb{F}(I)$ . This is a subspace of  $\mathbb{C}^0(I)$ .
- $\mathbb{P}^\infty(I)$  = all polynomials on  $I$ . This is a subspace of  $\mathbb{F}(I)$ .
- $\mathbb{C}^n(I)$  = the set of functions  $f : I \rightarrow \mathbb{R}$  such that  $f', f'' \dots f^{(n)}$  all exist and are continuous.
- $\mathbb{C}^\infty(I)$  = functions from  $I \rightarrow \mathbb{R}$  such that  $f', f'', f''' \dots$  all exist and are smooth functions.
- $A(I)$  = the functions in  $\mathbb{C}^\infty(I)$  such that all  $A \in I$ , the power series  $f(a) + \frac{f'(a)(x-a)}{1!} + \frac{f''(a)(x-a)^2}{2!} + \dots$  converges for all  $x \in I$  sufficiently close to  $a$ .

**Problem.** Show that  $v = \begin{pmatrix} x \\ y \\ 2x+y \end{pmatrix}$  is a subspace of  $\mathbb{R}^3$ .

- Because  $\begin{pmatrix} 0 \\ 0 \\ 2(0)+0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$  is in  $v$ ,  $v$  is not empty.
- Let  $\vec{v}_1 = \begin{pmatrix} x_1 \\ y_1 \\ 2x_1+y_1 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} x_2 \\ y_2 \\ 2x_2+y_2 \end{pmatrix} \in V$

$$v_1 + v_2 = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ 2x_1 + y_1 + 2x_2 + y_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ 2(x_1 + x_2) + (y_1 + y_2) \end{pmatrix} \in V$$

so  $v$  is closed with respect to addition.

- Let  $r \in \mathbb{R}$  and  $\begin{pmatrix} x \\ y \\ 2x+y \end{pmatrix} = \vec{v} \in v$ .

### 2.3 Span and Linear Independence

If we let  $V$  be a vector space over  $\mathbb{R}$  and let  $\vec{v}_1, \dots, \vec{v}_n \in V$ , then we can determine that the

$$\text{span}(\{\vec{v}_1, \dots, \vec{v}_n\}) = \{c_1\vec{v}_1 + \dots + c_n\vec{v}_n | c_1, \dots, c_n \in \mathbb{R}\}$$

---

**Proposition 1.** The span of  $\{\vec{v}_1, \vec{v}_2\}$  is a subspace of  $V$ .

*Proof.*

$$\begin{aligned} c_1\vec{v}_1 + \cdots + c_n\vec{v}_n \\ k_1\vec{v}_1 + \cdots + k_n\vec{v}_n \in \text{span} \end{aligned}$$

If we add together both of the equations above we get

$$\begin{aligned} (c_1 + k_1)\vec{v}_1 + \cdots + (c_n + k_n)\vec{v}_n &\in \text{span}. \\ r(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) \\ &= rc_1\vec{v}_1 + \cdots + rc_n\vec{v}_n \in \text{span} \end{aligned}$$

□

**Problem.** Let  $V \in \mathbb{R}^3$ . Also, we are going to let

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \text{span}(\vec{v}_1) = \left\{ c \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \mid c \in \mathbb{R} \right\}$$

$\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$  is a vector in 3-space.  $c \times \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$  expands, contracts, changes direction. This is a line which goes through  $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ .

$$c \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

is in the  $xy$ -plane, let's solve for  $y$  to find the equation of the line that is drawn by the vector:

$$\begin{aligned} \begin{pmatrix} c \\ 2c \\ 0 \end{pmatrix} \\ x = c \\ y = 2c \\ \frac{1}{2}y = c \\ \rightarrow x = \frac{1}{2}y \\ \rightarrow y = 2x \end{aligned}$$

Now we are going to let  $\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ . Consider the span of  $(\{\vec{v}_1, \vec{v}_2\})$ . The span of  $(\{\vec{v}_1, \vec{v}_2\})$  is a plane.

In  $\mathbb{R}^3$ , if  $\vec{0} \neq \vec{v} \in \mathbb{R}$ , then the  $\text{span}\vec{v}$  is a line.

**Problem.** Let  $v = \mathbb{P}^2$ .  $v$  is the set of polynomials of degree  $\leq 2 \in \mathbb{R}$ .

- $\text{span}(1, x, x^2) = \mathbb{P}^2$
- $\text{span}(4, 2x) = \mathbb{P}^1$ , which means all polynomials of degree  $\leq 1$

**Definition 4.** Let  $v$  be a vector space.  $\vec{v}_1, \dots, \vec{v}_n$  are linearly dependent if there exists  $c_1, \dots, c_n$  are not all zero, such that  $c_1\vec{v}_1 + \dots + c_n\vec{v}_n = \vec{0}$ , otherwise,  $\vec{v}_1, \dots, \vec{v}_n$  are linearly independent.

If we let  $\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ , we can do a simple test to see if they are linearly independent. We know that  $2\vec{v}_1 = \vec{v}_2$ , which means that  $2\vec{v}_1 + -1\vec{v}_2 = \vec{0}$ . Because we can make  $\vec{v}_1 + \vec{v}_2$  by using a simple scalar value, these functions are linearly dependent.

**Problem.** Consider the following three matrices

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \vec{v}_3 = \begin{pmatrix} 8 \\ 1 \\ 11 \end{pmatrix}$$

Are these matrices linearly dependent or independent of each other? The following equation will let us set up a matrix to determine the results:

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$$

We must try to see if there are any  $c_n$  values that are not zero to make this true.

$$\begin{pmatrix} 1 & 2 & 8 \\ 2 & -1 & 1 \\ 1 & 3 & 11 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 8 & \vdots & 0 \\ 2 & -1 & 1 & \vdots & 0 \\ 1 & 3 & 0 & \vdots & 0 \end{pmatrix}$$

By doing some elementary row operations, we can find that in row echelon form, the matrix from equation (2.21) can be written as

$$\begin{pmatrix} 1 & 2 & 8 & \vdots & 0 \\ 9 & 1 & 3 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{pmatrix}$$

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And from here we can solve for the different  $c_n$  values.

$$\begin{aligned} 1c_1 + 2c_2 + 8c_3 &= 0 \\ c_1 + 3c_3 &= 0 \end{aligned}$$

$$\begin{aligned} c_3 &= -3c_3 \\ c_1 &= -2c_3 \\ c_3 &= c_3 \end{aligned}$$

Because we have this relationship where  $c_1, c_2, c_3$  all depend on each other, we can tell that this is linearly independent.

### 2.3.1 Linear Independence and Dependence

## 2.4 Basis and Dimension

**Definition 5.** A basis of a vector space  $v$  is a collection of vectors  $\vec{v}_1, \dots, \vec{v}_n$  that (1) span  $v$  and (2) are linearly independent.

**Problem.** If we are looking at  $\mathbb{R}^2$ , with  $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ , and  $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$  We can tell that this is a basis of  $\mathbb{R}^2$ . We can tell this because the span of  $v$  is linearly independent. We can also see this because:

$$a \times e_1 + b \times e_2 = \begin{pmatrix} a \\ b \end{pmatrix}$$

**Problem.** Now we are going to look at an example in  $\mathbb{R}^3$ , with  $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ ,  $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ , and  $e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ . We can figure out that this is a basis by doing the same technique as we did before:

$$\begin{aligned} c_1 \vec{e}_1 + c_2 \vec{e}_2 + c_3 \vec{e}_3 &= 0 \\ c_1 = c_2 = c_3 &= 0 \end{aligned}$$

Because  $c_1, c_2$ , and  $c_3$  are all equal to zero,  $\vec{e}_1, \vec{e}_2$ , and  $\vec{e}_3$  form a basis.

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**Theorem 3.** If a vector space  $v$  has a basis with  $n$  elements, then every basis of  $v$  has  $n$  elements. We say  $v$  has dimension  $n$ . We write down  $\dim v = n$ .

**Theorem 4.** If the dimension of  $v$  is  $n$ , then any collection of  $n + 1$  or more vectors must be linearly dependent.

**Theorem 5.** Suppose  $\dim v = n$

1. Every collection of more than  $n$  vectors is linearly dependent.
2. No set of fewer than  $n$  vectors spans  $v$ .
3. A set of  $n$  vectors is a basis if and only if it spans  $v$ .
4. A set of  $n$  vectors is a basis if and only if it is linearly independent.

**Problem.** Assume  $1, x, x^2$  is a basis for  $\mathbb{P}^2$ . We are going to multiply  $1 \times 5$ ,  $x \times 6$ , and  $x^2 \times 2$ .

$$\begin{aligned} 5 + 6x + 2x^2 \\ c_1 \times 1 + c_2 \times x + c_3 \times x^2 = 0 \\ \dim(\mathbb{P}^2) = 3 \end{aligned}$$

**Theorem 6.**  $\vec{v}_1, \dots, \vec{v}_n$  form a basis of  $v$  if and only if for all  $\vec{v} \in v$ , there exist unique  $c_1, \dots, c_n$  such that  $\vec{v} = c_1\vec{v}_1 + \dots + c_n\vec{v}_n$

**Problem.** Let  $v = \mathbb{R}^2$ . Let  $\vec{v} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ . We know from previous problems that  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$  is a basis of  $\mathbb{R}^2$ . We can also figure out what our basis is by trying to figure out what our  $c_1$  and  $c_2$  values should be. Because the matrices we know are a basis consist of 1's and 0's, we can see that

$$4 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}.$$

The coordinates of  $\vec{v}$  with respect to this basis, are  $(4, 3)$ . Let's consider a different basis. We are now going to look at the basis,

$$\left\{ \begin{pmatrix} 1 \\ -3 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right\}$$

Now we need to figure out the  $a$  and  $b$  values for this basis respectively.



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$$a \begin{pmatrix} 1 \\ -3 \end{pmatrix} + b \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

The coordinates of  $\vec{v}$  with respect to this basis are  $(4, 3)$ . Let's consider the same problem but with the following basis:

$$\left\{ \begin{pmatrix} 1 \\ -3 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right\}$$

Now, we are going to do the same thing as before, where we solve for  $a$  and  $b$  in the following equation

$$a \begin{pmatrix} 1 \\ -3 \end{pmatrix} + b \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

We can set up this to be a system of equations that we can turn into a matrix

$$\begin{cases} 1a + 2b = 4 \\ -3a + (-1)b = 3 \end{cases} \rightarrow \begin{bmatrix} 1 & 2 & \vdots & 4 \\ -3 & -1 & \vdots & 3 \end{bmatrix}$$

And now we can use the basic row operation  $R_2 = R_2 + 3R_1$  in order to solve for  $a$  and  $b$ :

$$\begin{bmatrix} 1 & 2 & \vdots & 4 \\ 0 & 5 & \vdots & 5 \end{bmatrix}$$

$$\begin{array}{ll} 5b = 15 & a + 2b = 4 \\ b = 3 & 1 + 2 * 3 = 4 \\ & a = -2 \end{array}$$

## 2.5 The fundamental Matrix Subspaces (Kernel and Image)

**Definition 6.** The image of an  $m \times n$  matrix  $A$  is the subspace spanned by the columns of  $A$ .

**Problem.** Let's consider the following equation.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

When we multiply our the matrix, we see that the span of the columns give us all the possible  $\begin{bmatrix} x \\ y \end{bmatrix}$  values. The span of the matrix

$$\begin{bmatrix} 1 \times r_1 & 2 \times r_2 & 3 \times r_3 \\ 4 \times r_1 & 5 \times r_2 & 6 \times r_3 \end{bmatrix}$$

would be the values  $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ ,  $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$ , and  $\begin{bmatrix} 3 \\ 6 \end{bmatrix}$

**Definition 7.** A space,  $A$ , is an  $m \times m$  matrix, The kernel of  $A$  is

$$\begin{aligned} A &= Ker(A) \\ &= \{\vec{x} | A\vec{x} = \vec{0}\} \end{aligned}$$

Using definition (2.5.2), if  $A = \begin{bmatrix} 1 & 0 \\ 5 & 0 \end{bmatrix}$ , then

$$\vec{x}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} 0 \\ 5 \end{bmatrix}.$$

Something to keep in mind: If  $\vec{x}_1, \vec{x}_2 \in Ker(A)$ , then  $r_1\vec{x}_1 + r_2\vec{x}_2 \in Ker(A)$ . So the kernel of  $A$  is a subspace of the domain of the function.

**Theorem 7.** Assume  $\vec{x}_1$  solves  $A\vec{x} = \vec{b}$ . Then,  $\vec{x}_2$  is another solution to  $A\vec{x} = \vec{b}$  if and only if  $\vec{x}_2 = \vec{x}_1 + \vec{z}$ , where  $z \in Ker(A)$

**Proposition 2.** Let  $A$  be an  $m \times n$  matrix. The following are true:

1.  $Ker(A) = \{\vec{0}\}$
2.  $rank(A) = n$
3.  $A\vec{x} = \vec{b}$  has a unique solution for any  $\vec{b}$  in the image of  $A$ .
4.  $A\vec{x} = \vec{b}$  has no free variables.
5.  $A$  is non-singular.

**Definition 8.** Let  $A$  be  $m \times n$ .

$$\begin{aligned} coimg(A) &= img(A^T) \\ coker(A) &= ker(A^T) \end{aligned}$$

The image of  $A$  is the span of its columns. Thus, the coimage is the span of its rows. Also, the  $r^T$  in the cokernel of  $A$  are those  $\vec{r}$  such that  $r \cdot A = \vec{0}^T$  since

$$\begin{aligned} (r \cdot A)^T &= (\vec{0}^T)^T \\ A^T \cdot r^T &= \vec{0} \end{aligned}$$

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**Theorem 8.** The Fundamental Theorem of Linear Algebra: Let  $A$  be an  $m \times n$  matrix and let  $r$  be its rank. Then

$$\begin{aligned} \dim(\text{coimg}(A)) &= \dim(\text{img}(A)) = \text{rank}(A) = \text{rank}(A^T) = r \\ \text{span}(A_{\text{rows}}) &= \text{span}(A_{\text{columns}}) \\ \dim(\ker(A)) &= n - r \\ \dim(\text{coker}(A)) &= m - r \end{aligned}$$

Let's take a look at an example using the fundamental theorem of linear algebra. Let

$$A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 1 & -1 & 0 & 2 \\ -1 & 1 & 0 & -2 \\ 2 & 1 & 3 & 1 \end{bmatrix}$$

Let's do some row operations to get this matrix into row echelon form:

$$A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & -3 & -3 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

From this form, we can tell that  $v_3$  and  $v_4$  both depend on  $v_1$  and  $v_2$ . Because there are only two pivot points within  $A$  that are filled with values other than 0,  $\text{rank}(A) = 2$ . We also know that the dimension of the column space is equal to the dimension of the row space. Let's find the dimension of the kernel of  $A$ :

$$\begin{aligned} \dim(\ker(A)) + \text{rank} &= n \\ \dim(\ker(A)) + 2 &= 4 \end{aligned}$$

From here, we know that both  $y$  and  $z$  are free variables.

$$\begin{aligned} w + 2x + 3y - z &= 0 \\ -3x - 3y + 3z &= 0 \\ x + y - z &= 0 \\ x &= -y + z \end{aligned}$$

$$\begin{aligned} w &= -2x - 3y + z \\ &= -2(-y + z) - 3y + z \\ &= -y - z \end{aligned}$$

---

Now we need to determine the basis for  $\ker(A)$ .

$$\begin{pmatrix} -y - z \\ -y + z \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y \\ -y \\ -y \\ 0 \end{pmatrix} + \begin{pmatrix} -z \\ z \\ 0 \\ z \end{pmatrix} = y \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

Our basis for  $\ker(A)$  is  $\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}$  and  $\begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$ .

### 3 Inner Products and Norms

#### 3.1 Inner Products

Dot products are a form of inner product.

Let's apply the dot product to the vectors  $\langle v_1, v_2, \dots, v_n \rangle \cdot \langle w_1, w_2, \dots, w_n \rangle$  to show that the dot product is an inner product.

$$= v_1 w_1 + v_2 w_2 + \dots + v_n w_n \quad (1)$$

Therefore,  $v \times v$  goes to  $\mathbb{R}$ . So we know that

$$\vec{v} \cdot \vec{v} = v_1^2 + v_2^2 + \dots + v_n^2 \quad (2)$$

In general, we can assume that  $\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}$ . We should also keep in mind that  $\vec{v} \cdot \vec{w} = \|\vec{v}\| \|\vec{w}\| \cos \theta$

**Definition 9.** An inner product of  $V$  is a function  $\langle, \rangle: v \times v \rightarrow \mathbb{R}$  such that

•

$$\langle c\vec{u} + d\vec{v}, \vec{w} \rangle = c \langle \vec{u}, \vec{v} \rangle + d \langle \vec{v}, \vec{u} \rangle \quad (3)$$

$$\langle \vec{u}, c\vec{v} + d\vec{w} \rangle = c \langle \vec{u}, \vec{v} \rangle + d \langle \vec{u}, \vec{w} \rangle \quad (4)$$

- $\langle \vec{v}, \vec{w} \rangle = \langle \vec{w}, \vec{v} \rangle$
- $\langle \vec{v}, \vec{v} \rangle \geq 0$  while  $\langle 0, 0 \rangle = 0$ .

A vector space with an inner product is an inner product space.

**Definition 10.** If  $V$  is an inner product space, then its magnitude is

$$\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle} \quad (5)$$

Let's take a look at a weighted inner product on  $\mathbb{R}^3$ . We are going to let  $r_1, r_2, r_3 > 0$ . We can define  $\langle \vec{v}, \vec{w} \rangle$  as  $r_1 v_1 w_1 + r_2 v_2 w_2 + r_3 v_3 w_3$

---

**Problem.** Let's define  $[a, b] \subseteq \mathbb{R}$ . Consider  $\mathbb{C}^0[a, b]$ . This is a vector space. Define

$$\langle f(x), g(x) \rangle = \int_a^b f(x)g(x)dx \quad (6)$$

This is an inner product, so we also know that

$$\|f\| = \sqrt{\int_a^b (f(x))^2 dx} \quad (7)$$

This equation is the  $L^2$  norm.

### 3.2 Inequalities

Recall that  $\vec{v} \cdot \vec{w} = \|\vec{v}\| \|\vec{w}\| \cos \theta$ , where  $\theta$  is the angle between  $\vec{v}$  and  $\vec{w}$ . Now  $-1 \leq \cos \theta \leq 1$ , so we know that

$$\|\vec{v} \cdot \vec{w}\| \leq \|\vec{v}\| \|\vec{w}\| \quad (8)$$

This is the Cauchy-Schwarz inequality.

**Theorem 9.** For any inner product space

$$\|\langle \vec{v}, \vec{w} \rangle\| \leq \|\vec{v}\| \|\vec{w}\| \quad (9)$$

**Definition 11.** If  $\vec{v}, \vec{w} \in V$ , we say  $\vec{v}$  and  $\vec{w}$  are orthogonal if  $\langle \vec{v}, \vec{w} \rangle = 0$

**Problem.** Let's look at an example of checking orthogonality of two equations  $x, x^2 - \frac{1}{2} \in \mathbb{C}^0[0, 1]$ . In order to do this we need to find the  $L^2$  norm of the equations.

$$\begin{aligned} \left\langle x, x^2 - \frac{1}{2} \right\rangle &= \int_0^1 x \left( x^2 - \frac{1}{2} \right) dx \\ &= \int_0^1 \left( x^3 - \frac{1}{2}x \right) dx \\ &= \left. \frac{1}{4}x^4 - \frac{1}{4}x^2 \right|_0^1 = 0. \end{aligned}$$

Because the result of the inner product was zero, we know that  $x, x^2 - \frac{1}{2}$  are orthogonal in the  $L^2$  norm.

**Theorem 10.** The triangle inequality states that if  $V$  is an inner product space,

$$\|\langle \vec{v}, \vec{w} \rangle\| = \|\vec{v}\| + \|\vec{w}\| \quad (10)$$

---

Because we know that if we take the dot product of the same vector itself,  $\langle a, b, c \rangle \cdot \langle a, b, c \rangle$ , we get all of the items squared  $\langle a^2, b^2, c^2 \rangle$ , and because we know that  $\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$ , we get an idea of size. So we can define the unit ball to be

$$\{\vec{v} \in V \mid \|\vec{v}\| = 1\} \quad (11)$$

### 3.3 Norms

Equation (3.5) gives us the "size" of  $\vec{V}$ .

**Definition 12.** A norm on  $V$  is a function  $\|\cdot\| : V \rightarrow \mathbb{R}$  such that

- $\|\vec{v}\| = 0$  if and only if  $\vec{v} = 0$
- $\|c\vec{v}\| = |c| \cdot \|\vec{v}\|$
- $\|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|$

If  $\|\langle \vec{v}, \vec{w} \rangle\| \leq \|\vec{v}\| \|\vec{w}\|$ , then that is a norm. There are other norms to learn about.

**Problem.** Consider  $V = \mathbb{R}^n$ . We know that the magnitude of  $\vec{V}_p$  is

$$\sqrt[p]{|\vec{v}_1|^p + |\vec{v}_2|^p + |\vec{v}_3|^p}. \quad (12)$$

So if  $v = \mathbb{R}^2$ ,  $p = 2$  we have

$$\|\langle x, y \rangle\|_2 = \sqrt{x^2 + y^2}. \quad (13)$$

But if we were to have  $p = 3$ , we would have

$$\|\langle x, y \rangle\|_3 = \sqrt[3]{x^3 + y^3} \quad (14)$$

In  $\|\cdot\|_3$ , the size is  $\sqrt[3]{3^3 + 4^3} \approx 4.5$

In the 4 term,  $\|\cdot\|_4$ , the unit circle is the  $(x, y)$ 's such that  $\sqrt[4]{x^4 + y^4} = 1$

$$\boxed{x^4 + y^4 = 1} \quad (15)$$

Another norm on  $\mathbb{R}^n$  is the super-norm. This is where

$$\|\langle x_1, x_2, \dots, x_n \rangle\|_\infty = \max\{|x_1|, |x_2|, \dots, |x_n|\} \quad (16)$$

Here's a quick example: The super-norm for  $\langle 3, 4 \rangle$  is

$$\|\langle 3, 4 \rangle\|_\infty = 4 \quad (17)$$

because the maximum value in the set is 4.

Something to keep in mind is  $\|\langle x, y \rangle\| = |x| + |y|$ .

---

**Theorem 11.** Let  $\|\cdot\|_A$  and  $\|\cdot\|_B$  be two norms on  $\mathbb{R}^n$ . Then there exists positive numbers  $0 < c < k$  such that

$$c \cdot \|\vec{v}\|_A < \|\vec{v}\|_B < k \cdot \|\vec{v}\|_A \quad (18)$$

Let's consider  $V \in \mathbb{R}^2$ . Let's take a look at  $\|\cdot\|_2$  and  $\|\cdot\|_\infty$ . Where  $\vec{V} = \langle v_1, v_2 \rangle$ .

$$\frac{1}{\sqrt{2}} \cdot \|\vec{v}\|_2 \leq \|\vec{v}\|_\infty < 1 \cdot \|\vec{v}\|_2 \quad (19)$$

We can also define norms on matrices.

**Theorem 12.** If  $\|\cdot\|$  is a norm on  $\mathbb{R}^2$  and  $A$  is an  $m \times n$  matrix, then

$$\|A\| = \max\{\|A \cdot \vec{u}\| \mid \|\vec{u}\| = 1\} \quad (20)$$

These matrix norms satisfy the following:

1.  $\|A \cdot \vec{v}\| \leq \|A\| \cdot \|\vec{v}\|$
2.  $\|A \cdot B\| \leq \|A\| \cdot \|B\|$
3.  $\|A^k\| \leq \|A\|^k$

Let's take a quick look at  $\|A\|_\infty$

**Definition 13.** The  $i^{th}$  absolute row sum of  $A$  is the sum of the absolute values of the entries in the  $i^{th}$  row.

**Theorem 13.**  $\|A\|_\infty$  the maximum absolute row sum.

Here's an example of using the  $\|A\|_\infty$  value. Let  $A = \begin{pmatrix} -3 & 2 \\ 5 & 4 \end{pmatrix}$ . We can determine that the maximum absolute row sum of  $A$  is 8. This is because we can do

$$|-3| + |2| = 5 \quad (21)$$

$$|5| + |3| = \boxed{8} \quad (22)$$

### 3.4 Positive Definite Matrices

Consider the following two equations:

$$\vec{x} = x_1 \vec{e}_1 + x_2 \vec{e}_2 + \dots + x_n \vec{e}_n \quad (23)$$

$$\vec{y} = y_1 \vec{e}_1 + y_2 \vec{e}_2 + \dots + y_n \vec{e}_n \quad (24)$$

---

We can analyze this as

$$\langle \vec{x}, \vec{y} \rangle = [x_1, x_2, \dots, x_n] \cdot \begin{bmatrix} k_{11} & \dots & k_{1n} \\ \vdots & \ddots & \vdots \\ k_{n1} & \dots & k_{nn} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \quad (25)$$

$$k_{ij} = \langle e_i, e_j \rangle \quad (26)$$

$$= \vec{x}^T k \vec{y} \quad (27)$$

$$k = k^T \quad (28)$$

This means that  $k$  is symmetrical across the diagonal.

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{12} & k_{22} & k_{23} \\ k_{13} & k_{23} & k_{33} \end{bmatrix} \quad (29)$$

**Definition 14.** A  $n \times n$  matrix  $A$  is a symmetrical positive definite matrix if  $A = A^T$  and  $x^t k x > 0$ .

**Theorem 14.** Every inner product on  $\mathbb{R}^n$  is given by  $\langle x, y \rangle = \vec{x}^T k \vec{y}$  where  $k$  is a symmetrical positive definite matrix. So  $\langle \vec{x}, \vec{y} \rangle$  is a dot product or a weighted product.

$$\langle \vec{x}, \vec{y} \rangle = \vec{x}^T k \vec{y} \quad (30)$$

$$k^T = k \quad (31)$$

$$\vec{v}^T k \cdot \vec{v} > 0 \quad (32)$$

$$\vec{v} \neq 0 \quad (33)$$

Let's take a look at an example for this:

**Problem.** Let  $k = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ . First we need to check to see if  $k^T = k$ . By just looking at  $k$ , we can see that  $k^T = k$ . Next we need to do the following calculation to see if  $\begin{bmatrix} x \\ y \end{bmatrix}$  is the weighted inner product of the matrix  $k$ .

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 2x \\ 3x \end{bmatrix} \quad (34)$$

$$= 2x^2 + 2y^2 > 0 \quad (35)$$

$$\begin{bmatrix} x \\ y \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (36)$$

Therefore we know that



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**Problem.** Let's consider the following problem

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 4 & -2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad (37)$$

If we let  $A$  be the numerical matrix, we can see that  $A^T = A$ . Let's simplify the equation from before

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 4x - 2y \\ -2x + 3y \end{bmatrix} = 4x^2 - 2xy - 2xy + 3y^2 \quad (38)$$

$$= 4x^2 - 4xy + 3y^2 \quad (39)$$

$$(2x - y)^2 + 2y^2 > 0 \begin{bmatrix} x \\ y \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (40)$$

Now we can see that this is a positive definite matrix.

If we are given a symmetric matrix,  $k$ , the polynomial  $x^T k x$  is a quadratic form of  $k$ .

**Problem.** Let's consider  $k = \begin{bmatrix} 1 & -3 \\ -3 & 2 \end{bmatrix}$ . Let's find the quadratic form of  $k$ . First we need to write  $k$  like so:

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & -3 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} x - 3y \\ -3x + 2y \end{bmatrix} \quad (41)$$

$$= x^3 - 3xy - 3xy + 2y^2 \quad (42)$$

$$= x^3 - 6xy + 2y^2 \quad (43)$$

$$(44)$$

Therefore we know that the quadratic form of  $k$  is  $x^3 - 6xy + 2y^2$ .

For a positive definite matrix,  $k = k^T$  and  $x^T k x > 0$  for all  $\vec{x} \neq 0$

**Theorem 15.** Every inner product in  $\mathbb{R}^n$  is given by

$$\langle x, y \rangle = x^T k y \text{ for } x, y \in \mathbb{R}^n \quad (45)$$

Let  $v$  be an inner product space and  $\vec{v}_1, \dots, \vec{v}_n$ . The gram matrix of  $v$  is

$$K = \begin{bmatrix} \langle v_1, v_1 \rangle & \langle v_1, v_2 \rangle & \dots & \langle v_1, v_n \rangle \\ \langle v_2, v_1 \rangle & \langle v_2, v_2 \rangle & \dots & \langle v_2, v_n \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle v_n, v_1 \rangle & \langle v_n, v_2 \rangle & \dots & \langle v_n, v_n \rangle \end{bmatrix} \quad (46)$$

**Definition 15.**  $A$  is a matrix that is  $n \times n$ .  $A$  is a positive semidefinite matrix if  $A^T = A$  and  $\vec{x}^T A \vec{x} \geq 0$

---

**Theorem 16.** All gram matrices are positive semi-definite. They are positive definite if and only if  $\vec{v}_1, \dots, \vec{v}_n$  are linearly independent.

Suppose we are in  $\mathbb{R}^m$  and the inner product is the dot product. Let  $\vec{v}_1, \dots, \vec{v}_n \in \mathbb{R}^m$ . Let  $A = [v_1, v_2, v_3, \dots, v_n]$ . Then  $K = A^T A$ . Let  $A$  be a gram matrix generated by  $v_1, \dots, v_n$  with the dot product.

$$K = A^T A = \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vdots \\ \vec{v}_n \end{bmatrix} \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \end{bmatrix} \quad (47)$$

$$= \begin{bmatrix} \langle v_1, v_1 \rangle & \langle v_1, v_2 \rangle & \dots & \langle v_1, v_n \rangle \\ \langle v_2, v_1 \rangle & \langle v_2, v_2 \rangle & \dots & \langle v_2, v_n \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle v_n, v_1 \rangle & \langle v_n, v_2 \rangle & \dots & \langle v_n, v_n \rangle \end{bmatrix} \quad (48)$$

**Proposition 3.** Given an  $m \times n$  matrix  $A$ . The following are true

1. The  $m \times n$  matrix  $k = A^T A$  is positive definite.
2.  $A$  has linearly independent columns.
3.  $\text{rank}(A) = n$
4.  $\text{Ker}(A) = \{0\}$

**Theorem 17.** Every inner product on  $\mathbb{R}^n$  is given by

$$\langle \vec{x}, \vec{y} \rangle = \vec{x}^T \cdot C \vec{y} \quad (49)$$

where  $C$  is a symmetric, positive definite  $n \times n$  matrix.

Let  $\vec{v}_1, \dots, \vec{v}_n \in \mathbb{R}^n$ . Let  $A = [\vec{v}_1 : \vec{v}_2 : \dots : \vec{v}_n]$ . Then  $K = A^T C A$  is the gram matrix with respect to the inner product  $\vec{v}^T C \vec{w}$ .

**Theorem 18.** Suppose  $A$  is an  $m \times n$  matrix with linearly independent columns. Suppose  $C$  is any positive definite  $m \times m$  matrix. Then  $\vec{v}^T C \vec{w}$

**Definition 16.** The hilbert matrix  $H = (h_{ij})$  where  $h_{ij} = \frac{1}{i+j-1}$

The  $3 \times 3$  hilbert matrix:

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$$\begin{bmatrix} \frac{1}{3} & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix} \quad (50)$$

**Problem.** Make a gram matrix, not in  $\mathbb{R}^m$ . Let  $V = \mathbb{C}^0[0, 1]$ . Use the  $L^2$  inner product.

$$\langle f, g \rangle = \int_0^1 f(x)g(x)dx. \quad (51)$$

$1, x, x^2$  are linearly independent.

$$\langle v_i, v_k \rangle = \int_0^1 x^{i-1}x^{j-1}dx \quad (52)$$

$$= \int_0^1 x^{i+j-2}dx \quad (53)$$

$$= \frac{1}{i+j-1}x^{i+j-1} \quad (54)$$

$$= \frac{1}{i+j-1} \quad (55)$$

$$\rightarrow \begin{bmatrix} \frac{1}{3} & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix}. \quad (56)$$

## 4 Orthogonality

### 4.1

Recall that

$$\vec{v} \cdot \vec{w} = v_1 \cdot w_1 + \dots + v_n \cdot w_n = \|\vec{v}\| \|\vec{w}\| \cos(\theta), \quad (57)$$

where  $\theta$  is the angle between  $\vec{v}$  and  $\vec{w}$ . Because  $\cos(\frac{\pi}{2}) = 0$ , we know that  $\vec{v} \cdot \vec{w} = 0$  if and only if  $\vec{v}$  is orthogonal to  $\vec{w}$ . In general, given  $\vec{v}, \vec{w} \in \mathbb{R}^n$ ,  $\vec{v}$  is orthogonal to  $\vec{w}$  if and only if the angle between them is  $\frac{\pi}{2}$ .

**Definition 17.** Let  $U$  be any inner product space. A basis  $\vec{u}_1, \dots, \vec{u}_h \in V$  is orthogonal if  $\vec{u}_j \cdot \vec{u}_i = 0$  whenever  $i \neq j$ . In addition, if  $\|\vec{u}_i\| = 1$  for all  $i$ 's, the basis is orthonormal.

Here are a few examples of orthonormal and orthogonal basis's:

1.  $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$  is an orthonormal basis.
2.  $\begin{pmatrix} 2 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$  is an orthogonal basis.
3.  $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$  is an orthonormal basis.

If  $\vec{v}_1, \dots, \vec{v}_n$  is an orthogonal basis, then

$$\frac{\vec{v}_1}{\|\vec{v}_1\|}, \frac{\vec{v}_2}{\|\vec{v}_2\|}, \dots, \frac{\vec{v}_n}{\|\vec{v}_n\|} \quad (58)$$

is an orthonormal basis.

**Problem.** Consider  $\vec{u}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$  and  $\vec{u}_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ . We know that these two matrices are not linearly dependent because  $\vec{u}_1$  is not a multiple of  $\vec{u}_2$ . We can see that this is an orthogonal basis.

$$\|\vec{u}_1\| = \sqrt{1^2 + 2^2} = \sqrt{5} \quad (59)$$

so we know

$$v_1 = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}, v_2 = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{pmatrix} \quad (60)$$

---

**Proposition 4.** Assume  $\vec{v}_1, \dots, \vec{v}_n \in V$  with  $\vec{v}_i \neq \vec{0}$  for all  $i$ . Assume that  $\langle v_i, v_j \rangle \geq 0$  whenever  $i \neq j$ , then  $\{v_1, \dots, v_n\}$  is linearly independent.

*Proof.* Assume that  $c_1 \vec{v}_1 + \dots + c_n \vec{v}_n = \vec{0}$ . Let  $i \in \{1, \dots, n\}$

$$\begin{aligned} \langle c_1 \vec{v}_1 + \dots + c_n \vec{v}_n, v_i \rangle &= \langle \vec{0}, \vec{v}_i \rangle \\ c_1 \langle v_1, v_i \rangle + c_2 \langle v_2, v_i \rangle + \dots + c_i \langle v_i, v_i \rangle + \dots + c_n \langle v_n, v_i \rangle &= 0 \\ c_i \langle v_i, v_i \rangle &= 0 \\ c_i &= 0. \end{aligned}$$

Now we know that  $v_1, \dots, v_n$  are linearly independent.  $\square$

**Corollary 1.** If  $\dim(V) = n$  and  $\vec{v}_1, \dots, \vec{v}_n$  are  $n$  vectors such that  $\langle v_i, v_j \rangle = 0$  whenever  $i \neq j$ , then  $\{\vec{v}_1, \dots, \vec{v}_n\}$  is a basis.

**Theorem 19.** Let  $\vec{u}_1, \dots, \vec{u}_n$  be an orthonormal basis for  $V$ . Let  $\vec{v} \in V$ . Then we know  $\vec{v} = c_1 \vec{u}_1 + \dots + c_n \vec{u}_n$ . In fact,  $c_i = \langle \vec{v}, \vec{u}_i \rangle$  and

$$\|\vec{v}\|^2 = \langle \vec{v}, \vec{u}_1 \rangle^2 + \langle \vec{v}, \vec{u}_2 \rangle^2 + \dots + \langle \vec{v}, \vec{u}_n \rangle^2. \quad (61)$$

**Problem.**  $\mathbb{P}^2$  polynomials of degree  $\leq 2$  on  $[0,1]$ . Use the  $L^2$  norm.

$$\langle p, q \rangle = \int_0^1 p(x)q(x)dx \quad (62)$$

Let  $p_1 = 1, p_2 = x - \frac{1}{2}, p_3 = x^2 - x + \frac{1}{6}$ .

$$\langle p_1, p_2 \rangle = \int_0^1 x - \frac{1}{2} dx = 0 \quad (63)$$

$$\langle p_1, p_3 \rangle = \int_0^1 x^2 - x + \frac{1}{6} dx = 0. \quad (64)$$

We have an orthogonal basis because of this. In order to check to see if it is orthonormal we must also do  $\langle p_1, p_1 \rangle, \langle p_2, p_2 \rangle, \langle p_3, p_3 \rangle$

INSERT NOTES FROM 03.08 HERE

So if we have the basis  $\left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right\}$ ,  $v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ .  $v_2$  is equal to

---


$$\begin{aligned}
& \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \frac{\langle \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \rangle}{\| \begin{pmatrix} 1 \\ 2 \end{pmatrix} \|} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\
& \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \frac{4}{(\sqrt{1^2 + 2^2})^2} \\
& \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \frac{4}{5} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\
& \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \begin{pmatrix} \frac{4}{5} \\ \frac{4}{5} \end{pmatrix} = \begin{pmatrix} \frac{10}{5} \\ \frac{5}{5} \end{pmatrix} - \begin{pmatrix} \frac{4}{5} \\ \frac{4}{5} \end{pmatrix} \\
& = \begin{pmatrix} \frac{6}{5} \\ \frac{3}{5} \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \frac{\langle \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \rangle}{\| \begin{pmatrix} 1 \\ 2 \end{pmatrix} \|^2} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \frac{4}{(\sqrt{1^2 + 2^2})^2} \\
& \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \frac{4}{5} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\
& \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \begin{pmatrix} \frac{4}{5} \\ \frac{4}{5} \end{pmatrix} = \begin{pmatrix} \frac{10}{5} \\ \frac{5}{5} \end{pmatrix} - \begin{pmatrix} \frac{4}{5} \\ \frac{4}{5} \end{pmatrix} \\
& \begin{pmatrix} \frac{6}{5} \\ \frac{3}{5} \end{pmatrix} .
\end{aligned}$$

We can conclude that our basis is  $\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} \frac{6}{5} \\ -\frac{3}{5} \end{pmatrix} \}$ . Let's now use our basis and rewrite it as

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} \frac{6}{5} \\ -\frac{3}{5} \end{pmatrix} .$$

We can simplify this and solve for  $c_1, c_2$

$$\begin{aligned}
c_1 &= \langle \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \rangle = 2 + 6 = 8 \\
c_2 &= \langle \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} \frac{6}{5} \\ -\frac{3}{5} \end{pmatrix} \rangle = \frac{12}{5} - \frac{9}{5} = \frac{3}{5}
\end{aligned}$$

Using Theorem 4.9 from the book, we do the following with an orthogonal basis to get its norm:

---


$$\begin{aligned}
a_1 &= \frac{8}{\|v_1\|^2} = \frac{8}{(\sqrt{1^2 + 2^2})^2} = \frac{8}{5} \\
a_2 &= \frac{\frac{3}{5}}{\|v_2\|^2} = \frac{\frac{3}{5}}{\left(\frac{6}{5}\right)^2 + \left(-\frac{3}{5}\right)^2} \\
&= \frac{\frac{3}{5}}{\frac{36}{25} + \frac{9}{25}} = \frac{15}{25} \dots \\
\begin{pmatrix} 2 \\ 3 \end{pmatrix} &= a_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + a_2 \begin{pmatrix} \frac{6}{5} \\ -\frac{3}{5} \end{pmatrix} \\
&= \frac{8}{5} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} \frac{6}{5} \\ -\frac{3}{5} \end{pmatrix} \\
&= \begin{pmatrix} \frac{8}{5} \\ \frac{16}{5} \end{pmatrix} + \begin{pmatrix} \frac{2}{5} \\ -\frac{1}{5} \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.
\end{aligned}$$

**Problem.** Example of an orthogonal basis. Let  $\mathbb{T}^n$  be the vector space of trigonometric polynomials.

$$\mathbb{T}^n = \sum_{0 \leq j+k \leq n} a_{jk} \sin^j(x) \cos^k(x).$$

Using the  $L^2$  norm:

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f \cdot g.$$

An orthogonal basis is  $\{1, \cos(x), \sin(x), \cos(2x), \sin(2x), \cos(3x), \sin(3x), \dots\}$ . If we were going to do the  $L^2$  norm for any of these equations we would need to do the following

$$\int_{-\pi}^{\pi} \sin(2x) \cos(4x) dx.$$

This equation is the Fourier series.

## 4.2

### 4.3 Orthogonal Matrices

**Definition 18.** A square matrix  $Q$  is orthogonal if  $Q^t Q = Q \cdot Q^T = I$

If  $Q$  is orthogonal, then

- $Q^{-1} = Q^T$
- $\det(Q) = \pm 1$
- $Q \cdot Q^t = I$
- $\det(Q) \det(Q^T) = \det(I)$
- $(\det(Q))^2 = 1$
- $\det(Q) = \pm 1$

---

Let  $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ . Suppose that  $A$  is orthogonal, then

$$\begin{aligned} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \cdot \begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ a_{11}^2 + a_{12}^2 &= 1 \\ a_{11}a_{21} + a_{12}a_{22} &= 0 \\ a_{21}a_{11} + a_{22}a_{12} &= 0 \\ a_{21}^2 + a_{22}^2 &= 1. \end{aligned}$$

If we plot  $(a_{11}, a_{12})$  on a graph, we can see that  $\cos(\theta) = a_{12}$  and  $\sin(\theta) = a_{11}$

**Proposition 5.**  $Q$  is orthogonal if and only if its columns form an orthonormal basis.

*Proof.* Let  $Q = [U_1 : U_2 : \dots : U_n]$ .

$$Q^T = \begin{bmatrix} U_1^T \\ U_2^T \\ \vdots \\ U_n^T \end{bmatrix}.$$

In  $Q^T Q$ , the  $i, j^{th}$  entry is  $U_i^T \cdot U_j$ .

$$U_i^T \cdot U_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}.$$

So the  $U_i$ 's form an orthonormal basis. □

**Problem.** Let  $A = \begin{bmatrix} 3 & 5 \\ 7 & \frac{1}{2} \end{bmatrix}$ , and let  $A$  be orthonormal. We know that  $A \cdot A^T = A^T \cdot A = I$ . Let's see if  $A$  is an orthonormal basis.

$$\begin{bmatrix} 3 & 7 \\ 5 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 7 & \frac{1}{2} \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

This is not equal because  $3 \times 7 + 3 \times 5 \neq 0$ . Now let's try letting  $A = \begin{bmatrix} 3 & -7 \\ 7 & 3 \end{bmatrix}$ .

$$\begin{bmatrix} 3 & 7 \\ -7 & 3 \end{bmatrix} \begin{bmatrix} 3 & -7 \\ 7 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

This vector works with the zero values, but not the ones values, so we need to normalize this vector.

$$\begin{aligned} \left\| \begin{bmatrix} 3 \\ 7 \end{bmatrix} \right\| &= \sqrt{4^2 + 7^2} = \sqrt{58} & \left\| \begin{bmatrix} -7 \\ 3 \end{bmatrix} \right\| &= \sqrt{(-7)^2 + 3^2} = \sqrt{58} \\ A &= \begin{bmatrix} \frac{3}{\sqrt{58}} & -\frac{7}{\sqrt{58}} \\ \frac{7}{\sqrt{58}} & \frac{3}{\sqrt{58}} \end{bmatrix}. \end{aligned}$$



---

Let's let  $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ , and suppose  $Q$  is orthogonal.

$$Q^T \cdot Q = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$a^2 + c^2 = 1$$

$$ab + cd = 0$$

$$ab + cd = 0$$

$$b^2 + d^2 = 1.$$

Given that the vectors  $\begin{bmatrix} a \\ c \end{bmatrix}, \begin{bmatrix} b \\ d \end{bmatrix}$  lie on the unit circle, we can determine that

$$a = \cos \theta$$

$$c = \sin \theta$$

$$b = \cos \phi$$

$$d = \sin \phi,$$

and we can determine that

$$0 = ab + cd = \cos \theta \cos \phi + \sin \theta \sin \phi = \cos(\theta - \phi).$$

If we use  $\cos(\theta - \phi)$ , we can determine that  $\phi = \theta \pm \pi$ , so  $b = -\sin \theta, d = \cos \theta$  or  $b = \sin \theta, d = -\cos \theta$ . We either have  $Q$  in one of two forms.

$$\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \text{ or } \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix}.$$

The determinant of the left matrix is 1, and the determinant of the right matrix is -1. They both give us a counter clockwise rotation by  $\theta$ , gives a reflection across the line with angle  $\frac{\theta}{2}$

orthogonal matrices are square and  $Q^t \cdot Q = QQ^t = I$ . Every  $2 \times 2$  orthogonal matrix has the form

$$\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \text{ or } \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix}.$$

In general, if  $Q$  is orthogonal, then  $\det(Q) = \pm 1$

**Theorem 20.** The product of two orthogonal matrices is orthogonal. Recall, if  $Q$  is orthogonal, then  $Q^{-1} = Q^T$ . The orthogonal  $n \times n$  matrices satisfy

- Closed under the dot product.
- Multiplication is associative.
- They all have inverses.
- The identity matrix is orthogonal.

## 4.4 Vector Spaces

Up until now we've been trying to get a basis to be able to establish a location of vectors. Once we have determined an inner product, we can find an angle. Let  $V$  be a subspace. Let  $W \leq V$  be a finite dimensional subspace.

**Definition 19.**  $\vec{z} \in V$  is orthogonal to  $w$  if it is orthogonal to every vector in  $w$ .

**Note.** If  $\vec{w}_1, \dots, \vec{w}_n$  is a basis for  $w$ , then  $\vec{z}$  is orthogonal to  $w$  if and only if  $\vec{z}$  is orthogonal to  $w_1, \dots, w_n$ .

**Definition 20.** The orthogonal projection of  $\vec{V}$  onto  $w$  is the vector  $\vec{w}$  such that  $\vec{z} = \vec{v} - \vec{w}$ , where  $\vec{z}$  is orthogonal to  $w$ .

**Theorem 21.** Let  $\vec{u}_1, \dots, \vec{u}_n$  be an orthogonal basis for  $w$ . Let  $\vec{v} \in V$ . The orthogonal projection of  $\vec{v}$  onto  $w$  is

$$\vec{w} = c_1 \vec{u}_1 + \dots + c_n \vec{u}_n \text{ where } c_i = \langle \vec{v}, \vec{u}_i \rangle / \|\vec{u}_i\|^2.$$

**Note.** If  $\vec{v}_1, \dots, \vec{v}_n$  is an orthogonal basis for  $w$  then

$$w = a_1 \vec{v}_1 + \dots + a_n \vec{v}_n$$

$$a_i = \frac{\langle \vec{v}, \vec{v}_i \rangle}{\|\vec{v}_i\|^2}.$$

**Definition 21.** Let  $w, z \leq V$  be subspaces.  $w$  is orthogonal to  $z$  if every vector in  $w$  is orthogonal to every vector in  $z$ . For example,

$$\langle \vec{w}, \vec{z} \rangle = 0$$

for every  $\vec{w} \in w, \vec{z} \in z$ .

**Note.** We only need to show this is true on the bases of  $w$  and  $z$ .

**Definition 22.** Let  $w \in v$  be a subspace. The orthogonal complement of  $w$ , written  $w^T$  is the set of vectors in  $v$  orthogonal to  $w$ .

$$w^T = \{ \vec{v} \in v \mid \langle \vec{v}, \vec{w} \rangle = 0, \forall \vec{w} \in w \}.$$

**Theorem 22.** Let  $w < v$  be a finite dimensional subspace. Every  $\vec{v} \in v$  can be written uniquely as

$$\vec{v} = \vec{w} + \vec{z}$$

where  $\vec{w} \in w$  and  $\vec{z} \in w^T$ .

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## 5 Mimimization and Least Squares

## 6 Equilibrium

## 7 Linearity

## 8 Eigenvalues and Singular Values

### 8.1

**Definition 23.** Let  $A$  be an  $n \times n$  matrix. A scalar  $\lambda$  is an eigenvalue of  $A$  if

$$A\vec{v} = \lambda\vec{v}.$$

For some nonzero vector  $\vec{v}$ .  $\vec{v}$  is called the eigen vector corresponding to lambda.

**Example.** Consider  $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ . Let's find the eigenvalues of this matrix.

$$\begin{aligned} 0 &= \det \left( \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right) \\ &= \det \begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix} \\ &= (2-\lambda)^2 - 1 \\ 0 &= (3-\lambda)(1-\lambda). \end{aligned}$$

Our eigenvalues are 3 and 1. Remember that  $(A - \lambda I)\vec{v} = \vec{0}$ . Consider  $\lambda = 3$

$$\begin{aligned} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ -x + y &= 0 \\ x &= y \\ \begin{pmatrix} x \\ x \end{pmatrix} &= x \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \end{aligned}$$

Now consider  $\lambda = 1$ .

$$\begin{aligned} A(\lambda I)\vec{v} &= \vec{0} \\ (A - \lambda I)\vec{v} &= \vec{0} \\ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ x + y &= 0 \\ y &= -x \\ \begin{pmatrix} x \\ -x \end{pmatrix} &= x \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \end{aligned}$$

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**Example.** Consider  $A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$ . For eigenvalues,  $0 = \det(A - \lambda I)$ .

$$0 = \det \begin{pmatrix} 1-\lambda & 1 & 2 \\ 0 & 1-\lambda & 2 \\ 0 & 0 & 3-\lambda \end{pmatrix}$$

$$0 = (1-\lambda)(1-\lambda)(3-\lambda).$$

This makes our eigenvalues 1 and 3. For  $\lambda = 3$ ,

$$\begin{pmatrix} -2 & 1 & 2 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} -2y + 2z &= 0 \\ y &= 0 \\ -2x + z + 2z &= 0 \\ -2x + 3z &= 0 \\ 3z &= 2x \\ y = z &= \frac{2}{3}. \end{aligned}$$

This would make our eigenvector

$$\begin{pmatrix} x \\ \frac{2}{3}x \\ \frac{2}{3}x \end{pmatrix} = x \begin{pmatrix} 1 \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}.$$

**Definition 24.** Given an eigenvalue  $\lambda$  of  $A$ , the corresponding eigenvectors form a subspace denoted  $v_\lambda$ . Note that  $v_\lambda = \ker(A - \lambda I)$

**Note.**  $\lambda = 0$  is an eigenvalue of  $A$  if and only if the  $\ker(A - \lambda I) = \ker(A) = v_0 \neq \{0\}$ . This is true if and only if  $A$  is singular ( $\det(A) = 0$ ).

**Example.** Consider  $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ . The determinant of

$$\begin{pmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix}$$

is equal to zero. We can see that when  $\lambda = 0$ , the determinant is zero.

**Proposition 6.** If  $A$  is a real matrix and  $\lambda + i\mu$  is an eigenvalue of  $A$  with eigenvector  $\vec{v} = \vec{x} + i\vec{y}$ , then  $\lambda - i\mu$  is an eigenvalue of  $A$  with eigenvector  $\vec{x} - i\vec{y}$ .

**Example.** Consider the matrix  $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ .

$$\begin{aligned}
 |A - \lambda I| &= 0 \\
 \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} &= 0 \\
 \lambda^2 + 1 &= 0 \\
 \lambda^2 &= -1 \\
 \lambda &= \pm i = 0 \pm i \\
 (A - \lambda I)\vec{v} &= 0 \\
 \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
 -ix - y &= 0 \\
 -ix &= y.
 \end{aligned}$$

We now know that if we let  $x$  be anything, and  $y = -ix$ , then we have the eigenvector. We can rewrite it as

$$\begin{aligned}
 \begin{pmatrix} x \\ -ix \end{pmatrix} &= \square + i\square \\
 &= \begin{pmatrix} x \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -ix \end{pmatrix} \\
 &= x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x \begin{pmatrix} 0 \\ -i \end{pmatrix} \\
 &= x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \end{pmatrix}.
 \end{aligned}$$

Now we also know that if  $\lambda = -i$  then the eigenvector is

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} - i \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

## Lecture 1: 8.2, 8.3

Monday March 29, 2021

If  $A$  is an  $n \times n$  matrix with real entries, then

$$\det(A - \lambda I) = p(\lambda) = \text{The characteristic polynomial.}$$

**Note.** If  $A$  is  $2 \times 2$ , then  $P_A(\lambda) = \lambda^2 - \text{Tr}(A)\lambda + \det(A)$

Suppose that  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ . The trace ( $\text{Tr}(A)$ ) is  $a + d$ , and the determinant of  $A$  is  $ad - bc$ , so

$$\begin{aligned}
 p_a(\lambda) &= \begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} = (a - \lambda)(d - \lambda) - bc \\
 &= ad - d\lambda - a\lambda + \lambda^2 - bc \\
 &= \lambda^2 - (a + d)\lambda + (ad - bc).
 \end{aligned}$$

Remember, the trace is  $a + d$ , and the determinant is  $ad - bc$ . Recall that  $A$  is real, so  $P_A(\lambda)$  is real. If we set  $P_A(\lambda) = 0$ . By the fundamental theorem of algebra,  $P_A(\lambda)$  factors into linear factors over  $\mathbb{C}$ . Consider the equation

$$x^{10} - 7x^9 + 8x^2 + \frac{1}{2} = 0.$$

This factors into 10 different roots. So if  $A$  is  $n \times n$ ,  $P_A(\lambda)$  has at most  $n$  roots in  $\mathbb{C}$

**Theorem 23.** Let  $A$  be  $n \times n$  with real entries. Then  $A$  has at most  $n$  eigenvalues. If  $a + bi$  is an eigenvalue, then so is  $a - bi$

**Example.** The Jordan Block Matrix. Let's look at  $J_{2,3} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ . Let's find the eigenvalues:

$$\begin{aligned} |J_{2,3} - \lambda I| &= 0 \\ \begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix} &= 0 \\ (2-\lambda)(2-\lambda)(2-\lambda) &= 0 \\ \lambda &= 2. \end{aligned}$$

Now let's find the eigenvector(s):

$$\begin{aligned} (A - \lambda I)\vec{v} &= \vec{0} \\ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ y &= 0 \\ z &= 0 \\ \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} &= x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \end{aligned}$$

**Theorem 24.** If  $A$  is square, then  $P_A(\lambda) = P_{A^T}(\lambda)$ . So,  $A$  and  $A^T$  have the same eigenvalues. Probably not the same eigenvectors.

**Theorem 25.** Let  $A$  be  $n \times n$ . The sum of the eigenvalues of  $A$  is equal to the  $\text{Tr}(A)$ , and the product of the eigenvalues of  $A$  is equal to  $\det(A)$ .

**Example.**  $J_{2,3} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$ . The Trace of  $J_{2,3} = 2 + 2 + 2$  (adding the

diagonals). The determinant of  $J_{2,3} = 2 * 2 * 2$  (multiplication of the diagonals).

$$P_{J_{2,3}}(\lambda) = (2 - \lambda)(2 - \lambda)(2 - \lambda).$$

## 8.2

**Proposition 7.** If  $\lambda_1, \dots, \lambda_k$  are distinct eigenvalues of  $A$ , then the corresponding eigenvectors are linearly independent.

**Example.** Let's let  $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix}$ . Let's find the eigenvalues of  $A$ .

$$\begin{aligned} 0 = P_A(\lambda) &= \begin{vmatrix} 1 - \lambda & 1 & 0 \\ 0 & 2 - \lambda & -1 \\ 0 & 0 & 3 - \lambda \end{vmatrix} \\ 0 &= (1 - \lambda)(2 - \lambda)(3 - \lambda) \\ \lambda &= 1, 2, 3. \end{aligned}$$

For  $\lambda = 1$

$$\begin{aligned} (A - \lambda I)\vec{v} &= \vec{0} \\ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ y &= 0 \\ y - z &= 0 \\ z &= 0 \\ \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} &= x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \end{aligned}$$

Now for  $\lambda = 2$

$$\begin{aligned} \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ -x + y &= 0 \\ x &= y - z \\ \begin{pmatrix} x \\ x \\ 0 \end{pmatrix} &= x \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}. \end{aligned} \quad = 0$$

Now for  $\lambda = 3$

$$\begin{pmatrix} -2 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} -2x + y &= 0 \rightarrow y = 2x \\ -y - z &= 0 \rightarrow z = -y \\ z &= -2x \end{aligned}$$

$$\begin{pmatrix} x \\ 2x \\ -2x \end{pmatrix} = x \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}.$$

We now have the following eigenvectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}.$$

These are linearly independent so we have a basis for  $\mathbb{R}^3$

**Theorem 26.** If  $A$  is  $n \times n$  and  $a$  has  $n$  distinct real (/complex) eigenvalues, then the corresponding eigenvectors form a basis for  $\mathbb{R}^n(\mathbb{C}^n)$ .

Now... vector spaces. Vectors have two main operations,

$$\begin{aligned} \vec{v} + \vec{w} \\ c \cdot v. \end{aligned}$$

Let  $V, W$  be vector spaces over  $\mathbb{R}$ . We know that  $\mathbb{R}^3 \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}$ . In reality,  $V, W$  are the same, but we must show that they are the same. We would need a function that preserves the operations from  $V$  to  $W$ . A map (function)  $f$  from  $V$  to  $W$  should have the following

$$\begin{aligned} f(\vec{v}_1 + \vec{v}_2) &= f(\vec{v}_1) + f(\vec{v}_2) \\ f(r \cdot \vec{v}) &= r f(\vec{v}). \end{aligned}$$

## Lecture 2: 8.3

Wednesday, March 31st

Recall that  $\lambda = 0$  is an eigenvalue of  $A$ , if and only if

$$(A - 0 \cdot I)\vec{v} = \vec{0}$$

has a  $\vec{v} \neq \vec{0}$  solution if and only if

$$\ker(A) \neq \{0\}$$

if and only if  $A^{-1}$  does not exist.



**Theorem 27.** If  $\lambda_1, \dots, \lambda_k$  are distinct eigenvalues of  $A$ , then the corresponding eigenvectors are linearly independent.

**Example.** Consider  $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ . The eigenvalues for this matrix are  $\lambda = 3, 1$ . The eigenvector for  $\lambda = 3$  is  $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ , and the eigenvector for  $\lambda = 1$  is  $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ .

**Example.** Consider  $A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$ . The characteristic polynomial is  $P(\lambda) = (1-\lambda)^2(3-\lambda)$ . For  $\lambda = 1$ , the eigenvector is  $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ , and for  $\lambda = 3$ , the eigenvector is  $\begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}$ .

**Theorem 28.** If  $A$  is  $n \times n$  and  $A$  has  $n$  distinct real (or complex) eigenvalues, then the corresponding eigenvectors  $\vec{v}_1, \dots, \vec{v}_n$  form a basis for  $\mathbb{R}^n$ .

Consider  $A = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}$ .  $A$  defines a map from  $\mathbb{R}^2$  to  $\mathbb{R}^2$ . Our map is

$$L : \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow A \begin{pmatrix} x \\ y \end{pmatrix}$$

$$L(r\vec{v}) = rL(\vec{v}).$$

$L$  is a linear transformation.

**Example.** Consider  $A = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}$ . What happens if we multiply  $A$  by  $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ ?

$$A \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 8 \end{pmatrix}.$$

Let's think of  $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$  as  $2e_1 + 1e_2$ . This would make our equation

$$2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Let's also think about  $\begin{pmatrix} 1 \\ 8 \end{pmatrix}$  as

$$1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 8 \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We can change the basis to

$$v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

### Lecture 3: 8.2, 8.3

Monday March 29, 2021

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A linear transformation consists of the following:

$$\begin{aligned} L(v_1 + v_2) &= L(v_1) + L(v_2) \\ L(r \cdot v) &= rL(v). \end{aligned}$$

Consider  $A = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}$ . If everything is in the standard basis then

$$\begin{aligned} \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} -1 \\ 4 \end{pmatrix} &= \begin{pmatrix} -5 \\ 14 \end{pmatrix} \\ -1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 4 \begin{pmatrix} 0 \\ 1 \end{pmatrix} &\rightarrow -5 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 14 \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \end{aligned}$$

It's harder to figure out what the vector is with the new basis from the picture, but the transformation has a nice description of

$$B = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}.$$

If we take  $B$  and hit it with the new coefficients:

$$\begin{aligned} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ -3 \end{pmatrix} &= \begin{pmatrix} 4 \\ -9 \end{pmatrix} \\ 2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + -3 \begin{pmatrix} 1 \\ -2 \end{pmatrix} &\rightarrow 4 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + -9 \begin{pmatrix} 1 \\ -2 \end{pmatrix}. \end{aligned}$$

The left hand side is equal to  $\begin{pmatrix} -1 \\ 4 \end{pmatrix}$  and the right hand side is equal to  $\begin{pmatrix} -5 \\ 14 \end{pmatrix}$

If we take  $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ , which looks like an eigenvector. The first basis that we had was  $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ , and the second basis was  $\left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\}$ . So we took basis 1 and did a linear transformation to get basis 2 like the following

$$\begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}.$$

If  $A$  is in the standard basis. Then  $B = S^{-1}AS$ , where  $S = (\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n)$ , where  $\{\vec{v}_1, \dots, \vec{v}_n\}$  is the new basis.

**Example.** Let  $A = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}$ .

$$\begin{aligned} S &= \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} \\ S^{-1} &= \frac{1}{-2 - (-1)} \begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} \\ B = S^{-1}AS &= \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ -2 & -6 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}. \end{aligned}$$

**Definition 25.** A square matrix  $A$  is diagonalizable if there is a matrix  $S$  and a diagonal matrix  $\Lambda$  such that

$$\Lambda = S^{-1}AS$$

**Theorem 29.**  $A$  is diagonalizable if and only if  $A$  has  $n$  linearly independent eigenvectors  $\vec{v}_1, \dots, \vec{v}_n$ . In this case,

$$\Lambda = S^{-1}AS.$$

where  $S = [\vec{v}_1, \dots, \vec{v}_n]$ , and  $\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_n \end{bmatrix}$ , where  $\lambda_i$  is the eigenvalue for  $\vec{v}_i$

If we are looking for the linearly independent solutions we need to do  $A - \lambda I = 0$ . From here we would get lambda values and create eigenvectors using  $A - \lambda I \vec{v} = 0$

**Example.** Let  $A = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ , which is our standard matrix. Once we find our

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Our basis is

$$\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

Let's take  $A$  and hit it with coefficients in the standard basis. Let's try

$$A \cdot \begin{pmatrix} -3 \\ 2 \\ 2 \end{pmatrix}.$$

We need to think about the matrix on the right hand side as  $-3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ , so we can do the following:

$$\begin{pmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} -3 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \\ 3 \end{pmatrix}.$$

We know that our divided up equation from before (for the right matrix) is getting mapped to  $4 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ . We know that our  $\Lambda$  must take  $\begin{pmatrix} -3 \\ 2 \\ 2 \end{pmatrix}$ , but we need to rewrite it in our new basis.

$$1 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + 1 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

If we multiply this by our  $\lambda$ , we can figure out our map.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.$$

Our equation gets mapped to

$$1 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

This means that our vector is the same from before because we get  $\begin{pmatrix} -4 \\ 3 \\ 3 \end{pmatrix}$ .

$Av_1 \rightarrow w_1$ , while  $\Lambda v_2 \rightarrow w_2$ .

## 9 Iteration

## 10 Dynamics