Applied Linear Algebra

MATH363

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§2.1 Real Vector Spaces

A vector space is the abstract reformulation of the quintessential properties of n-dimensional Euclidean Space $\mathbb{R}^n$, which is defined as the set of all real (column) vectors with n entries. The basic laws of vector addition and scalar multiplication in $\mathbb{R}$

Definition 2.1.1

A vector space is a set of V equipped with two operations:

- Addition: adding any pair of vectors v, w $\in V$ produces another vector v + w $\in V$;
- Scalar Multiplication: multiplying a vector v $\in V$ by a scalar $c \in \mathbb{R}$ produces a vector cv$\in V$

These are subject to the following axioms, valid for all u, v, w $\in V$ and all scalars $c, d \in \mathbb{R}$:

- Commutativity of Addition: v + w = w + v.
- Associativity of Addition: u + (v + w) = (u + v) + w.
- Additive Identity: There is a zero element $0 \in V$ satisyfing v + 0 = v = 0 + v.
- Additive Inverse: For each v $\in V$ there is an element -v $\in V$ such that v+(-v)=0=(-v)+v.
- Distributivity: $(c + d)$v=(cv) + (dv), and c(v+w)=(cv)+(cw).
- Assosiativity of Scalar Multiplication: $c(d$v$) = (cd)$v.

- Unit for Scalar Multiplication: the scalar 1$\in \mathbb{R}$ satisfies 1v=v.

Theorem 2.1.1

Let V be a vector space.

- $0 \times \vec{V} = \vec{0}$
- $-1\vec{V} = -\vec{V}$
- $c \times \vec{0} = \vec{0}$
- If $c \times \vec{V} = \vec{0}$, then $c = 0$ or $\vec{V} = \vec{0}$

Here are some examples of vector spaces:

- $\mathbb{R}^n = \left\{ \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix} \middle| r_1, r_2, r_n \in \mathbb{R} \right\}$
- $M_{m \times n}$ = The m by n matrices over $\mathbb{R}$.
- $\mathbb{P}^n$ = the polynomials of degree $\leq n$.

Definition 2.1.2

Let V be a vector space over F. $W \leq V$ is a subspace of V if W is a vector space over F under the same operation as V.

An example of definition (2.1.2). Let $V = \mathbb{R}^3 = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \middle| a, b, c \in \mathbb{R} \right\}$. V is a vector space. If we let $W = \left\{ \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} \middle| a, b \in \mathbb{R} \right\}$, then W is a subspace of V.

Theorem 2.1.2

et V be a vector space. Let $W \leq V$. W is a subspace of V if

- $w \neq 0$
- $\forall w_1 w_2 \in W; w_1 + w_2 \in W$
- $\forall c \in F; \vec{W} \in W; c \cdot \vec{W} \in W$

If we were to let $V = \mathbb{R}^3$ for $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$. We can determine the following:

- $\{\vec{0}\}$ is a subspace of V.
- $\left\{ \begin{pmatrix} a \\ 0 \\ c \end{pmatrix} \middle| a, c \in \mathbb{R} \right\}$ is a subspace of V.
- Consider the equation $\left\{ \begin{pmatrix} x \\ x \\ 0 \end{pmatrix} \middle| x \in \mathbb{R} \right\} = W$ Show that W is a subspace of V.
  - $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in W$ so $W \neq 0$.
  - $\begin{pmatrix} x \\ x \\ 0 \end{pmatrix}, \begin{pmatrix} y \\ y \\ 0 \end{pmatrix} \in W$. Then $\begin{pmatrix} x \\ x \\ 0 \end{pmatrix} + \begin{pmatrix} y \\ y \\ 0 \end{pmatrix} = \begin{pmatrix} x+y \\ x+y \\ 0 \end{pmatrix}$
  - $\begin{pmatrix} x \\ x \\ 0 \end{pmatrix} \in W$, then $c \times \begin{pmatrix} x \\ x \\ 0 \end{pmatrix} = \begin{pmatrix} cx \\ cx \\ 0 \end{pmatrix} \in W$.
  - Therefore we know that W is a subspace of V with respect to scalar multiplication and addition.
- $W = \left\{ \begin{pmatrix} x \\ y \\ 2x+3y \end{pmatrix} \middle| x, y \in \mathbb{R} \right\}$. W is a subspace of V.

$\mathbb{R}^3$ only has 4 kinds of subspaces. $\mathbb{R}^3$, $\{\vec{0}\}$, planes passing through the origin and lines that are passing through the origin.

§2.2 Subspaces

Definition 2.2.1

Let I be an interval in $\mathbb{R}$. Let $\mathbb{F}(I)$ be the vector space of functions $\mathbb{F} = I \to \mathbb{R}$.

- $\mathbb{C}^0(I) =$ the continuous functions from $I \to \mathbb{R}$ is a subspace.
- $\mathbb{P}^n(I) =$ polynomials of degree $\leq n$ restricted to $\mathbb{F}(I)$. This is a subspace of $C^0(I)$.
- $\mathbb{P}^\infty(I) =$ all polynomials on I. This is a subspace of $\mathbb{F}(I)$.
- $\mathbb{C}^n(I) =$ the set of functions $f : I \to \mathbb{R}$ such that $f', f''...f^{(n)}$ all exist and are continuous.

- $\mathbb{C}^\infty(I)$ = functions from $I \to \mathbb{R}$ such that f', f'', f''' all exist and are smooth functions.
- $A(I)$ = the functions in $\mathbb{C}^\infty(I)$ such that all $A \in I$, the power series $f(a) + \frac{f'(a)(x-a)}{1!} + \frac{f''(a)(x-a)^2}{2!} + \dots$ converges for all $x \in I$ sufficiently close to a.

Problem 1. *Show that* $v = \begin{pmatrix} x \\ y \\ 2x+y \end{pmatrix}$ *is a subspace of* $\mathbb{R}^3$.

- *Because* $\begin{pmatrix} 0 \\ 0 \\ 2(0)+0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ *is in* v, v *is not empty.*
- *Let* $\vec{v_1} = \begin{pmatrix} x_1 \\ y_1 \\ 2x_1+y_1 \end{pmatrix}, \vec{v_2} = \begin{pmatrix} x_2 \\ y_2 \\ 2x_2+y_1 \end{pmatrix} \in V$

$$v_1+v_2 = \begin{pmatrix} x_1+x_2 \\ y_1+y_2 \\ 2x_1+y_1+2x_2+y_2 \end{pmatrix} = \begin{pmatrix} x_1+x_2 \\ y_1+y_2 \\ 2(x_1+x_2)+(y_1+y_2) \end{pmatrix} \in V \quad (2.1)$$

so v *is closed with respect to addition.*

- *Let* $r \in \mathbb{R}$ *and* $\begin{pmatrix} x \\ y \\ 2x+y \end{pmatrix} = \vec{v} \in v$

§2.3 Span and Linear Independence

If we let V be a vector space over $\mathbb{R}$ and let $\vec{v_1}, \dots, \vec{v_n} \in V$, then we can determine that the

$$span(\{\vec{v_1}, \dots, \vec{v_n}\}) = \{c_1\vec{v_1} + \dots + c_n\vec{v_n} | c_1, \dots, c_n \in \mathbb{R}\} \quad (2.2)$$

Proposition 2.3.1

The span of $\{\vec{v_1}, \vec{v_2}\}$ is a subspace of V.

Proof.

$$c_1\vec{v_1} + \cdots + c_n\vec{v_n} \tag{2.3}$$
$$k_1\vec{v_1} + \cdots + k_n\vec{v_n} \in span \tag{2.4}$$
$$\tag{2.5}$$

If we add together both of the equations above we get

$$(c_1 + k_1)\vec{v_1} + \cdots + (c_n + k_n)\vec{v_n} \in span. \tag{2.6}$$
$$r(c_1\vec{v_1} + \cdots + c_n\vec{v_n}) \tag{2.7}$$
$$= rc_1\vec{v_1} + \cdots + rc_n\vec{v_n} \in span \tag{2.8}$$

■

Problem 2. *Let* $V \in \mathbb{R}^3$*. Also we are going to let*

$$\vec{v_1} = \begin{pmatrix}1\\2\\0\end{pmatrix}, span(\vec{v_1}) = \left\{ c\begin{pmatrix}1\\2\\0\end{pmatrix} \middle| c \in \mathbb{R} \right\} \tag{2.9}$$

$\left(\begin{smallmatrix}1\\2\\0\end{smallmatrix}\right)$ *is a vector in 3-space.* $c \times \left(\begin{smallmatrix}1\\2\\0\end{smallmatrix}\right)$ *expands, contracts, changes direction. This is a line which goes through* $\left(\begin{smallmatrix}0\\0\\0\end{smallmatrix}\right)$*.*

$$c\begin{pmatrix}1\\2\\0\end{pmatrix} \tag{2.10}$$

is in the xy-plane, let's solve for y to find the equation of the line that is drawn by the vector:

$$\begin{pmatrix} c \\ 2c \\ 0 \end{pmatrix} \tag{2.11}$$

$$x = c \tag{2.12}$$

$$y = 2c \tag{2.13}$$

$$\frac{1}{2}y = c \tag{2.14}$$

$$\to x = \frac{1}{2}y \tag{2.15}$$

$$\to y = 2x \tag{2.16}$$

$$\tag{2.17}$$

Now we are going to let $\vec{v_2} = \left(\begin{smallmatrix} 1 \\ 1 \\ 0 \end{smallmatrix}\right)$. *Consider the span of* $(\{\vec{v_1}, \vec{v_2}\})$. *The span of* $(\{\vec{v_1}\vec{v_2}\})$ *is a plane.*

In $\mathbb{R}^3$, if $\vec{0} \neq \vec{v} \in \mathbb{R}$, then the span$\vec{v}$ is a line.

Problem 3. *Let* $v = \mathbb{P}^2$. v *is the set of polynomials of degree* $\leq 2 \in \mathbb{R}$.

- $span(1, x, x^2) = \mathbb{P}^2$
- $span(4, 2x) = \mathbb{P}^1$, *which means all polynomials of degree* ≤ 1

Definition 2.3.1

Let v be a vector space. $\vec{v_1}, \dots, \vec{v_n}$ are linearly dependant if there exists $c_1, \dots, c_n$ are not all zero, such that $c_1\vec{v_1} + \cdots + c_n\vec{v_n} = \vec{0}$, otherwise, $\vec{v_1}, \dots, \vec{v_n}$ are linearly independent.

If we let $\vec{v_1} = \left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right)$, $\vec{v_2} = \left(\begin{smallmatrix} 2 \\ 4 \end{smallmatrix}\right)$, we can do a simple test to see if they are linearly independent. We know that $2\vec{v_1} = \vec{v_2}$, which means that $2\vec{v_1} + -1\vec{v_2} = 0$. Because we can make $\vec{v_1} + \vec{v_2}$ by using a simple scalar value, these functions are linearly dependent.

Problem 4. *Consider the following three matrices*

$$\vec{v_1} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \vec{v_2} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \vec{v_3} = \begin{pmatrix} 8 \\ 1 \\ 11 \end{pmatrix} \tag{2.18}$$

Are these matrices linearly dependent or independent from each other? The following equation will let us set up a matrix to determine the results:

$$c_1\vec{v_1} + c_2\vec{v_2} + c_3\vec{v_3} = \vec{0} \tag{2.19}$$

We must try to see if there are any c_n values that are not zero to make this true.

$$\begin{pmatrix} 1 & 2 & 8 \\ 2 & -1 & 1 \\ 1 & 3 & 11 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{2.20}$$

$$= \begin{pmatrix} 1 & 2 & 8 & \vdots & 0 \\ 2 & -1 & 1 & \vdots & 0 \\ 1 & 3 & 0 & \vdots & 0 \end{pmatrix} \tag{2.21}$$

By doing some elementary row operations, we can find that in row echelon form, the matrix from equation (2.21) can be written as

$$\begin{pmatrix} 1 & 2 & 8 & \vdots & 0 \\ 9 & 1 & 3 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{pmatrix} \tag{2.22}$$

And from here we can solve for the different c_n values.

$$1c_1 + 2c_2 + 8c_3 = 0 \tag{2.23}$$

$$c_1 + 3c_3 = 0 \tag{2.24}$$

$$\tag{2.25}$$

$$c_3 = -3c_3 \tag{2.26}$$

$$c_1 = -2c_3 \tag{2.27}$$

$$c_3 = c_3 \tag{2.28}$$

Because we have this relationship where c_1, c_2, c_3 all depend on each other, we can tell that this is linearly independent.

§2.3.1 Linear Independence and Dependence

§2.4 Basis and Dimension

Definition 2.4.1

A basis of a vector space v is a collection of vectors $\vec{v_1}, \ldots, \vec{v_n}$ that 1. span v and 2. are linearly dependent.

Problem 5. *If we are looking at $\mathbb{R}^2$, with $e_1 = \binom{1}{0}$, and $e_2 = \binom{0}{1}$ We can tell that this is a basis of $\mathbb{R}^2$. We can tell this because the span of v is linearly independent. We can also see this because:*

$$a \times e_1 + b \times e_2 = \begin{pmatrix} a \\ b \end{pmatrix} \tag{2.29}$$

Problem 6. *Now we are going to look at an example in $\mathbb{R}^3$, with $e_1 = \left(\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right)$, $e_2 = \left(\begin{smallmatrix} 0 \\ 1 \\ 0 \end{smallmatrix}\right)$, and $e_3 = \left(\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right)$. We can figure out that this is a basis by doing the same technique as we did before:*

$$c_1\vec{e_1} + c_2\vec{e_2} + c_3\vec{e_3} = 0 \tag{2.30}$$

$$c_1 = c_2 = c_3 = 0 \tag{2.31}$$

Because c_1, c_2, and c_3 are all equal to zero, $\vec{e_1}$, $\vec{e_2}$, and $\vec{e_3}$ form a basis.

Theorem 2.4.1

If a vector space v has a basis with n elements, then every basis of v has n elements. We say v has dimension n. We write down $v = n$

Theorem 2.4.2

If the dimension of v is n, then any collection of $n+1$ or more vectors must be linearly dependent.

Theorem 2.4.3

Suppose $v = n$

1. Every collection of more than n vectors is linearly dependent.
2. No set of fewer than n vectors spans v.
3. A set of n vectors is a basis if and only if it spans v.
4. A set of n vectors is a basis if and only if it is linearly dependent.

Problem 7. *Assume* $1, x, x^2$ *is a basis for* $\mathbb{P}^2$*. We are going to multiply* 1×5, $x \times 6$, *and* $x^2 \times 2$.

$$5 + 6x + 2x^2 \tag{2.32}$$
$$c_1 \times 1 + c_2 \times x + c_3 \times x^2 = 0 \tag{2.33}$$
$$dim(\mathbb{P}^2) = 3 \tag{2.34}$$
$$\tag{2.35}$$

Theorem 2.4.4

$\vec{v_1}, \dots \vec{v_2}$ form a basis of v if and only if for all $\vec{v} \in v$, there exist unique $c_1, \dots, c_n$ such that $\vec{v} = c_1\vec{v_1} + \dots c_n\vec{v_n}$

Problem 8. *Let* $v = \mathbb{R}^2$*. Let* $\vec{v} = \left(\begin{smallmatrix}4\\3\end{smallmatrix}\right)$*. We know from previous problems that* $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$ *is a basis of* $\mathbb{R}^2$*. We can also figure out what our basis is by trying to figure out what our* c_1 *and* c_2 *values should be. Because the matrices we know are a basis consist of 1's and 0's, we can see that*

$$4\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}4\\3\end{pmatrix}. \tag{2.36}$$

The coordinates of $\vec{v}$ *with respect to this basis, are* $(4,3)$*. Let's consider a different basis. We are now going to look at the basis,*

$$\left\{\begin{pmatrix}1\\-3\end{pmatrix}, \begin{pmatrix}2\\-1\end{pmatrix}\right\} \tag{2.37}$$

Now we need to figure out the a *and* b *values for this basis respectively.*

$$a\begin{pmatrix}1\\-3\end{pmatrix} + b\begin{pmatrix}2\\-1\end{pmatrix} = \begin{pmatrix}4\\3\end{pmatrix} \tag{2.38}$$

$$\tag{2.39}$$

The coordinates of $\vec{v}$ *with respect to this basis are* $(4,3)$*. Let's consider the same problem but with the following basis:*

$$\left\{\begin{pmatrix}1\\-3\end{pmatrix}, \begin{pmatrix}2\\-1\end{pmatrix}\right\} \tag{2.40}$$

Now, we are going to do the same thing as before, where we solve for a *and* b *in the following equation*

$$a\begin{pmatrix}1\\-3\end{pmatrix} + b\begin{pmatrix}2\\-1\end{pmatrix} = \begin{pmatrix}4\\3\end{pmatrix} \tag{2.41}$$

We can setup this to be a system of equations that we can turn into a matrix

$$\begin{cases}1a + 2b = 4\\ -3a + (-1)b = 3\end{cases} \rightarrow \left[\begin{array}{cc|c}1 & 2 & 4\\ -3 & -1 & 3\end{array}\right] \tag{2.42}$$

And now we can use the basic row operation $R_2 = R_2 + 3R_1$ in order to solve for a and b:

$$\left[\begin{array}{cc:c} 1 & 2 & 4 \\ 0 & 5 & 5 \end{array}\right] \tag{2.43}$$

$$5b = 15 \qquad a + 2b = 4 \tag{2.44}$$
$$b = 3 \qquad 1 + 2 * 3 = 4 \tag{2.45}$$
$$a = -2 \tag{2.46}$$

§2.5 The fundamental Matrix Subspaces (Kernel and Image)

Definition 2.5.1

The image of an $m \times n$ matrix A is the subspace spanned by the columns of A.

Problem 9. *Let's consider the following equation.*

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \tag{2.47}$$

When we multiply our the matrix, we see that the span of the columns give us all the possible $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$ values. The span of the matrix

$$\begin{bmatrix} 1 \times r_1 & 2 \times r_2 & 3 \times r_3 \\ 4 \times r_1 & 5 \times r_2 & 6 \times r_3 \end{bmatrix} \tag{2.48}$$

would be the values $\left[\begin{smallmatrix} 1 \\ 4 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 2 \\ 5 \end{smallmatrix}\right]$, and $\left[\begin{smallmatrix} 3 \\ 6 \end{smallmatrix}\right]$

Definition 2.5.2

A space, A, is an $m \times m$ matrix, The kernel of A is

$$A = Ker(A) \tag{2.49}$$
$$= \{\vec{x} | A\vec{x} = \vec{0}\} \tag{2.50}$$

Using definition (2.5.2), if $A = \left[\begin{smallmatrix} 1 & 0 \\ 5 & 0 \end{smallmatrix}\right]$, then

$$\vec{x_1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \vec{x_2} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}. \tag{2.51}$$

Something to keep in mind: If $\vec{x_1}, \vec{x_2} \in Ker(A)$, then $r_1\vec{x_1} + r_2\vec{x_2} \in Ker(A)$. So the kernel of A is a subspace of the domain of the function.

Theorem 2.5.1

Assume $\vec{x_1}$ solves $A\vec{x} = \vec{b}$. Then, $\vec{x_2}$ is another solution to $A\vec{x} = \vec{b}$ if and only if $\vec{x_2} = \vec{x_1} + \vec{z}$, where $z \in Ker(A)$

Proposition 2.5.1

Let A be an $m \times n$ matrix. The following are true:

1. $Ker(A) = \{\vec{0}\}$
2. $rank(A) = n$
3. $A\vec{x} = \vec{b}$ has a unique solution for any $\vec{b}$ in the integer of A.
4. $A\vec{x} = \vec{b}$ has no free variables.
5. A is non-singular.

Definition 2.5.3

Let A be $m \times n$.

$$coimg(A) = img(A^T) \tag{2.52}$$

$$coker(A) = ker(A^T) \tag{2.53}$$

(2.54)

The image of A is the span of its collumns. Thus the coimage is the span of its radius. Also the $\vec{r^T}$ in the cokernel of A are those $\vec{r}$ such that $r \times A = 0\vec{T}$ since

$$(r \cdot A)^T = (0\vec{T})^T \tag{2.55}$$

$$A^T \cdot r^T = \vec{0} \tag{2.56}$$

Theorem 2.5.2

The Fundamental Theorem of Linear Algebra: Let A be an mxn matrix and let r be its rank. Then

$$dim(coimg(A)) = dim(img(A)) = rank(A) = rank(A^T) = r \tag{2.57}$$

$$span(A_{rows}) = span(A_{columns}) \tag{2.58}$$

$$dim(ker(A)) = n - r \tag{2.59}$$

$$dim(coker(A)) = m - r \tag{2.60}$$

Let's take a look at an example using the fundamental theorem of linear algebra. Let

$$A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 1 & -1 & 0 & 2 \\ -1 & 1 & 0 & -2 \\ 2 & 1 & 3 & 1 \end{bmatrix} \tag{2.61}$$

Let's do some row operations to get this matrix into row echelon form:

$$A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & -3 & -3 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{2.62}$$

From this form, we can tell that v_3 and v_4 both depend on v_1 and v_2. Because there are only two pivot points within A that are filled with values other than 0, $rank(A) = 0$. We also know that the dimension of the column space is equal to the dimension of the row space. Let's find the dimension of the kernel of A:

$$\begin{aligned} dim(ker(A)) + rank &= n && (2.63) \\ dim(key(A)) + 2 &= 4 && (2.64) \\ & && (2.65) \end{aligned}$$

From here, we know that both y and z are free variables.

$$\begin{aligned} w + 2x + 3y - z &= 0 && (2.66) \\ -3x - 3y + 3z &= 0 && (2.67) \\ x + y - z &= 0 && (2.68) \\ x &= -y + z && (2.69) \end{aligned}$$

$$\begin{aligned} w &= -2x - 3y + z && (2.70) \\ &= -2(-y + z) - 3y + z && (2.71) \\ &= -y - z && (2.72) \end{aligned}$$

Now we need to determine the basis for $ker(A)$.

$$\begin{pmatrix} -y - z \\ -y + z \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y \\ -y \\ -y \\ 0 \end{pmatrix} + \begin{pmatrix} -z \\ z \\ 0 \\ z \end{pmatrix} = y \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \tag{2.73}$$

Our basis for $ker(A)$ is $\left(\begin{smallmatrix} -1 \\ -1 \\ 1 \\ 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} -1 \\ 1 \\ 0 \\ 1 \end{smallmatrix}\right)$.

Chapter 3

Inner Products and Norms

§3.1 Inner Products

Dot products are a form of inner product.

Let's apply the dot product to the vectors $< v_1, v_2, \ldots, v_n > \cdot < w_1, w_2, \ldots, w_3 >$ to show that the dot product is an inner product.

$$= v_1 w_1 + v_2 w_2 + \ldots + v_n w_n \tag{3.1}$$

Therefore, $v \times v$ goes to $\mathbb{R}$. So we know that

$$\vec{v} \cdot \vec{v} = v_1^2 + v_2^2 + \ldots + v_n^2 \tag{3.2}$$

In general, we can assume that $||\vec{v}|| = \sqrt{\vec{v}\vec{v}}$. We should also keep in mind that $\vec{v} \cdot \vec{w} = ||\vec{v}|| ||\vec{w}|| \cos\theta$

Definition 3.1.1

An inner product of V is a function $<,>: v \times v \to \mathbb{R}$ such that

- $$< c\vec{u} + d\vec{v}, w > = c < \vec{u}, \vec{v} > + d < \vec{v}, \vec{u} > \tag{3.3}$$
  $$< \vec{u}, c\vec{v} + d\vec{w} = c < \vec{u}, \vec{v} > + d < \vec{u}, \vec{w} > \tag{3.4}$$
- $< \vec{v}, \vec{w} > = < \vec{w}, \vec{v} >$
- $< \vec{v}, \vec{v} > \geq_0$ while $< 0, 0 > = \vec{0}$.

A vector space with an inner product is an inner product space.

Definition 3.1.2

If V is an inner product space, then it's magnitude is

$$||\vec{v}|| = \sqrt{<\vec{v},\vec{v}>} \tag{3.5}$$

Let's take a look at a weighted inner product on $\mathbb{R}^3$. We are going to let $r_1, r_2, r_3 > 0$. We can define $<\vec{v},\vec{w}>$ as $r_1v_1w_1 + r_2v_2w_2 + r_3v_3w_3$

Problem 10. *Let's define* $[a,b] \leq \mathbb{R}$. *Consider* $\mathbb{C}^0[a,b]$. *This is a vector space. Define*

$$< f(x), g(x) >= \int_a^b f(x)g(x)dx \tag{3.6}$$

This is an inner product, so we also know that

$$||f|| = \sqrt{\int_a^b (f(x))^2\, dx} \tag{3.7}$$

This equation is the L^2 *norm.*

§3.2 Inequalities

Recall that $\vec{v}\cdot\vec{w} = ||\vec{v}||||\vec{w}||\cos\theta$, where θ is the angle between $\vec{v}$ and $\vec{w}$. Now $-1 \leq \cos\theta \leq_1$, so we know that

$$||\vec{v}\cdot\vec{w}|| \leq ||\vec{v}||||\vec{w}|| \tag{3.8}$$

This is the Cauchy-Shuartz inequality.

Theorem 3.2.1

For any inner product space

$$|| < \vec{v},\vec{w} > || \leq ||\vec{v}||||\vec{w}|| \tag{3.9}$$

Definition 3.2.1

If $\vec{v},\vec{w} \in V$, we say $\vec{v}$ and $\vec{w}$ are orthogonal if $<\vec{v},\vec{w}>= 0$

Problem 11. *Let's look at an example of checking orthogonality of two equations* $x, x^2 - y \in \mathbb{C}^0[0,1]$*. In order to do this we need to find the* L^2 *norm of the equations.*

$$\begin{aligned}\left\langle x, x^2 - \frac{1}{2}\right\rangle &= \int_0^1 x\left(x^2 - \frac{1}{2}\right)dx \\ &= \int_0^1 \left(x^3 - \frac{1}{2}x\right)dx \\ &= \frac{1}{4}x^4 - \frac{1}{4}x^2\bigg|_0^1 = 0.\end{aligned}$$

Because the result of the inner product was zero, we know that $x, x^2 - \frac{1}{2}$ *are orthogonal in the* L^2 *norm.*

Theorem 3.2.2

The triangle inequality states that if V is an inner product space,

$$|| < \vec{v}, \vec{w} > || = ||\vec{v}|| + ||\vec{w}|| \tag{3.10}$$

Because we know that if we take the dot product of the same vector itself, $< a, b, c > \cdot < a, b, c >$, we get all of the items squared $< a^2, b^2, c^2 >$, and because we know that $||\vec{v}|| = \sqrt{< \vec{v}, \vec{v} >}$, we get an idea of size. So we can define the unit ball to be

$$\{\vec{v} \in V \bigg| ||\vec{v}|| = 1\} \tag{3.11}$$

§3.3 Norms

Equation (3.5) gives us the "size" of $\vec{V}$.

Definition 3.3.1

A norm on V is a function $|| \cdot || : V \to \mathbb{R}$ such that

- $||\vec{v}|| = 0$ if and only if $\vec{v} = 0$

- $||c\vec{v}|| = |c| \cdot ||\vec{v}||$
- $||\vec{v} + \vec{w}|| \leq ||\vec{v}|| + ||\vec{w}||$

If $|| < \vec{v}, \vec{w} > || \leq ||\vec{v}||||\vec{w}||$, then that is a norm. There are other norms to learn about.

Problem 12. *Consider $V = \mathbb{R}^n$. We know that the magnitude of $\vec{V_p}$ is*

$$\sqrt[p]{|\vec{v_1}|^p + |\vec{v_2}|^p + |\vec{v_3}|^p}. \tag{3.12}$$

So if $v = \mathbb{R}^2$, $p = 2$ we have

$$|| < x, y > ||_2 = \sqrt{x^2 + y^2}. \tag{3.13}$$

But if we were to have $p = 3$, we would have

$$|| < x, y > ||_3 = \sqrt[3]{x^3 + y^3} \tag{3.14}$$

In $|| \cdot ||_3$, the size is $\sqrt[3]{3^3 + 4^3} \approx_4 .5$

In the 4 term, $|| \cdot ||_4$, the unit circle is the (x, y)'s such that $\sqrt[4]{x^4 \cdot Y_4} = 1$

$$\boxed{x^4 + y^4 = 1} \tag{3.15}$$

Another norm on R^n is the super-norm. This is where

$$|| < x_1, x_2, \ldots, x_n > ||_\infty = max\{|x_1|, |x_2|, \ldots, |x_n|\} \tag{3.16}$$

Here's a quick example: The super-norm for $< 3, 4 >$ is

$$|| < 3, 4 > ||_\infty = 4 \tag{3.17}$$

because the maximum value in the set is 4.
Something to keep in mind is $|| < x, y > || = |x| + |y|$.

Theorem 3.3.1

Let $|| \cdot ||_A$ and $|| \cdot ||_B$ be two norms on $\mathbb{R}^n$. Then there exists positive numbers $0 < c < k$ such that

$$c \cdot ||\vec{v}||_A < ||\vec{v}||_B < k \cdot ||\vec{v}||_A \tag{3.18}$$

Let's consider $V \in \mathbb{R}^2$. Let's take a look at $|| \cdot ||_2$ and $|| \cdot ||_\infty$. Where $\vec{V} =< v_1, v_2 >$.

$$\frac{1}{\sqrt{2}} \cdot ||\vec{v}||_2 \leq ||\vec{v}||_\infty < 1 \cdot ||\vec{v}||_2 \tag{3.19}$$

We can also define norms on matrices.

Theorem 3.3.2

If $|| \cdot ||$ is a norm on $\mathbb{R}^2$ and A is an $m \times n$ matrix, then

$$||A|| = max\{||A \cdot \vec{u}|| \Big| ||\vec{u}|| = 1 \tag{3.20}$$

These matrix norms satisfy the following:

1. $||A \cdot \vec{v}|| \leq ||A|| \cdot ||\vec{v}||$
2. $||A \cdot B|| \leq ||A|| \cdot ||B||$
3. $||A^k|| \leq ||A||^k$

Let's take a quick look at $||A||_\infty$

Definition 3.3.2

The i^{th} absolute row sum of A is the sum of the absolute values of the entries in the i^{th} row.

Theorem 3.3.3

$||A||_\infty$ the maximum absolute row sum.

Here's an example of using the $||A||_\infty$ value. Let $A = \left(\begin{smallmatrix} -3 & 2 \\ 5 & 4 \end{smallmatrix}\right)$. We can determine that the maximum absolute row sum of A is 8. This is because we can do

$$|-3| + |2| = 5 \tag{3.21}$$
$$|5| + |3| = \boxed{8} \tag{3.22}$$

§3.4 Positive Definite Matrices

Consider the following two equations:

$$\vec{x} = x_1\vec{e_1} + x_2\vec{e_2} + \ldots + x_n\vec{e_n} \tag{3.23}$$
$$\vec{y} = y_1\vec{e_1} + y_2\vec{e_2} + \ldots + v_n\vec{e_n} \tag{3.24}$$

We can analyze this as

$$< \vec{x}, \vec{y} >= [x_1, x_2, \ldots, x_n] \cdot \begin{bmatrix} k_{11} & \ldots & k_{1n} \\ \vdots & \ddots & \vdots \\ k_{n_1} & \ldots & k_{nn} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots y_n \end{bmatrix} \tag{3.25}$$

$$k_{ij} =< e_i, e_j > \tag{3.26}$$
$$= \vec{x}^T k \vec{y} \tag{3.27}$$
$$k = k^T \tag{3.28}$$

This means that k is symmetrical across the diagonal.

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{12} & k_{22} & k_{23} \\ k_{13} & k_{23} & k_{33} \end{bmatrix} \tag{3.29}$$

Definition 3.4.1

A $n \times n$ matrix A is a symmetrical positive definite matrix if $A = A^T$ and $x^t k < x > 0$.

Theorem 3.4.1

Every inner product on $\mathbb{R}^n$ is given by $< x, y >= \vec{x}^T k\vec{y}$ where k is a symmetrical positive definite matrix. So $< \vec{x}, \vec{y} >$ is a dot product or a weighted product.

$$< \vec{x}, \vec{y} > = \vec{x}^T k\vec{y} \tag{3.30}$$

$$k^T = k \tag{3.31}$$

$$\vec{v}^T k \cdot \vec{v} > 0 \tag{3.32}$$

$$\vec{v} \neq_0 \tag{3.33}$$

Let's take a look at an example for this:

Problem 13. *Let $k = \left[\begin{smallmatrix} 2 & 0 \\ 0 & 3 \end{smallmatrix}\right]$. First we need to check to see if $k^T = k$. By just looking at k, we can see that $k^T = k$. Next we need to do the following calculation to see if $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$ is the weighted inner product of the matrix k.*

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 2x \\ 3x \end{bmatrix} \tag{3.34}$$

$$= 2x^2 + 2y^2 > 0 \tag{3.35}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3.36}$$

Therefore we know that

Problem 14. *Let's consider the following problem*

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 4 & -2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{3.37}$$

If we let A be the numerical matrix, we can see that $A^T = A$. Let's simplify the equation from before

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 4x - 2y \\ -2x + 3y \end{bmatrix} = 4x^2 - 2xy - 2xy + 3x^2 \tag{3.38}$$

$$= 4x^2 - 4xy + 3y^2 \tag{3.39}$$

$$(2x - y)^2 + 2y^2 > 0 \begin{bmatrix} x \\ y \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3.40}$$

Now we can see that this is a positive definite matrix.

If we are given a symmetric matrix, k, the polynomial $x^T kx$ is a quadratic form of k.

Problem 15. *Let's consider* $k = \begin{bmatrix} 1 & -3 \\ -3 & 2 \end{bmatrix}$. *Let's find the quadratic form of* k. *First we need to write* k *like so:*

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & -3 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} x - 3y \\ -3x + 2y \end{bmatrix} \tag{3.41}$$

$$= x^3 - 3xy - 3xy + 2y^2 \tag{3.42}$$

$$= x^3 - 6xy + 2y^2 \tag{3.43}$$

$$\tag{3.44}$$

Therefore we know that the quadratic form of k *is* $x^3 - 6xy - 2y^2$.

For a positive definite matrix, $k = k^T$ and $x^T kx > 0$ for all $\vec{x}0$

Theorem 3.4.2

Every inner product in $\mathbb{R}^n$ is given by

$$< x, y >= x^T ky \text{ for } xy \in \mathbb{R}^n \tag{3.45}$$

Let v be an inner product space and $\vec{v_1}, \ldots, \vec{v_2}$. The gram matrix of v

is

$$K = \begin{bmatrix} < v_1, v_2 > & < v_1, v_2 > & \dots & < v_1, v_n > \\ < v_2, v_1 > & < v_2, v_2 > & \dots & < v_2, v_n > \\ \vdots & \vdots & \ddots & \vdots \\ < v_n, v_1 > & < v_n, v_2 > & \dots & < v_n, v_n > \end{bmatrix} \tag{3.46}$$

Definition 3.4.2

A is a matrix that is $n \times n$. A is a positive semidefinite matrix if $A^T = A$ and $\vec{x}^T A \vec{x} \geq 0$

Theorem 3.4.3

All gram matrices are positive semi-definite. They are positive definite if and only if $\vec{v_1}, \dots, \vec{v_n}$ are linearly independent.

Suppose we are in R^m and the inner produt is the dot product. Let $\vec{v_1}, \dots \vec{v_n} \in \mathbb{R}^m$
Let $A = [v_1, v_2, v_3, \dots, v_n]$. Then $K = A^T$. Let A be a gram matrix generated by $v_1, \dots, v_n$ with the dot product.

$$K = A^T A = \begin{bmatrix} \vec{v_1} \\ \vec{v_2} \\ \vdots \\ \vec{v_n} \end{bmatrix} \begin{bmatrix} \vec{v_1} & \vec{v_2} & \dots & \vec{v_n} \end{bmatrix} \tag{3.47}$$

$$= \begin{bmatrix} < v_1, v_1 > & < v_1, v_2 > & \dots & < v_1, v_n > \\ < v_2, v_1 > & < v_2, v_2 > & \dots & < v_2, v_n > \\ \vdots & \vdots & \ddots & \vdots \\ < v_n, v_1 > & < v_n, v_2 > & \dots & < v_n, v_n > \end{bmatrix} \tag{3.48}$$

Proposition 3.4.1

Given an $m \times n$ matrix A. The following are true

1. The $m \times n$ matrix $k = A^T A$ is positive definite.
2. A has linearly independent columns.

3. $rank(A) = n$
4. $Ker(A) = \{0\}$

Theorem 3.4.4

Every inner product on $\mathbb{R}^n$ is given by

$$< \vec{x}, \vec{y} >= \vec{x}^T \cdot c\vec{y} \tag{3.49}$$

where C is a symetric, positive definite $n \times n$ matrix.

Let $\vec{v_1}, \ldots, \vec{v_n} \in \mathbb{R}^n$. Let $A = [\vec{v_1} \vdots \vec{v_2} \vdots \ldots \vdots \vec{v_n}]$. Then $K = A^T CA$ is the gram matrix with respect to the inner product $\vec{v}^T C\vec{w}$.

Theorem 3.4.5

Suppose A is an $m \times n$ matrix with linearly independent collumns. Suppose C is any positive definite $m \times m$ matrix. Then $\vec{v}^T C\vec{w}$

Definition 3.4.3

The hilbert matrix $H = (h_{ij})$ where $h_{ij} = \frac{1}{i+j-1}$

The 3×3 hilbert matrix:

$$\begin{bmatrix} \frac{1}{1} & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix} \tag{3.50}$$

Problem 16. *Make a gram matrix, not in $\mathbb{R}^m$. Let $V = \mathbb{C}^0[0,1]$. Use the L^2 inner product.*

$$< f, g >= \int_0^1 f(x)g(x)dx. \tag{3.51}$$

$1, x, x^2$ *are linearly independent.*

$$< v_i, v_k > = \int_0^1 x^{i-1} x^{j-1} dx \tag{3.52}$$

$$= \int_0^1 x^{i+j-2} dx \tag{3.53}$$

$$= \frac{1}{i+j-1} x^{i+j-1} \tag{3.54}$$

$$= \frac{1}{i+j-1} \tag{3.55}$$

$$\rightarrow \begin{bmatrix} \frac{1}{1} & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix}. \tag{3.56}$$

Chapter 4

Orthogonality

Chapter 5

Mimimization and Least Squares

Chapter 6

Equilibrium

Chapter 7

Linearity

Chapter 8

Eigenvalues and Singular Values

Chapter 9

Iteration

Chapter 10

Dynamics