Ordinary Differential Equations

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1 Introduction to Ordinary Differential Equations

Differential equations come from real-world problems and problems in applied mathematics. When mathematics is applied to real-world problems, it is often the case that finding a relation between a function and its rate of change is easier than finding a formula for the function itself; it is this relation between an unknown function and its derivatives that produces a differential equation. To give a very simple example, a biologist studying the growth of a population with size at time t given by the function $P(t)$, might make the very simple, but logical, assumption that a population grows at a rate directly proportional to its size. In mathematical notation, the equation for $P(t)$ could then be written as:

$$\frac{dp}{dt} = rP(t)$$

Where the constant of proportionality, r would probably be determined experimentally by biologists working in the field. Equations used for modeling population growth can be much more complicated than this, sometimes involving scores of interacting populations with different properties.

1.1 Basic Terminology

Definition 1. A differential equation is any equation involving an unknown function and one or more if its derivatives.

The following are examples of differential equations:

1. $P'(t) = rP(t)(1 - P(t)/N) - H$ harvested population growth
2. $\frac{d^2x}{d\tau^2} + 0.9\frac{dx}{d\tau} + 2x = 0$ spring mass equation
3. $I''(t) + 4I(t) = sin(\omega t)$ RCL circuit showing beats
4. $y''(t) + \mu(y^2(t) - 1)y'(t) + y(t)$ ban der Pol equation
5. $\frac{\partial^2}{\partial x^2}u(x, y) + \frac{\partial^2}{\partial y^2}u(x, y) = 0$ Laplace's equation

1.1.1 Ordinary vs. Partial Differential Equations

Differential equations fall into two very broad categories, called ordinary differential equations and partial differential equations. If the unknown function in the equation is a function of only one variable, the equation is called an ordinary differential equation. If the unknown function in the equation depends on more than one independent variable, the equation is called a partial differential equation, and in this case, the derivatives appearing in the equation will be partial derivatives.

1.1.2 Independent Variables, Dependent Variables, and Parameters

Three different types of quantities can appear in a differential equation. The unknown function, for which the equation is to be solved, is called the dependent variable, and when considering ordinary differential equations, the dependent variable is a function of a single independent variable. In addition to the independent and dependent variables, a third type of variable, called a parameter, may appear in the equation. A parameter is a quantity that remains fixed in any specification of the problem, but can very from problem to problem.

1.1.3 Order of a Differential Equation

Another important way in which differential equations are classified is in terms of their order.

Definition 2. The order of a differential equation is the order of the highest derivative of the unknown function that appears in the equation.

The differential equation 1 is a first-order equation and the others are all second-order. Even though equation 5 is a partial differential equation, it is still said to be of second order since no derivatives of order higher than two appear in the equation.

1.1.4 What is a solution

Given a differential equation, what is a solution? We must realize that we are looking for a function, and therefore it needs to be defined on some interval of its independent variable.

Definition 3. An analytic solution of a differential equation is a sufficiently differentiable function that, if substituted into the equation, together with the necessary derivatives, makes the equation an identity (a true statement for all values of the independent variable) over some interval of the independent variables.

Problem. Show that the function $p(t) = e^{-2t}$ is a solution to the differential equation:

$$x'' + 3x' + 3x = 0$$

Solution. To show that it is a solution, compute the first and second derivatives of $p(t)$:

$$\begin{aligned} p'(t) &= -2e^{-2t} \\ p''(t) &= 4e^{-2t} \end{aligned}$$

When the three functions $p(t)$, $p'(t)$, and $p''(t)$ are substituted into the differential equation in place of x, x', and x'', it becomes:

$$
\begin{aligned}
(4e^{-2t}) + 3(-2e^{-2t}) + 2(e^{-2t}) &\equiv 0 \\
(4 - 6 + 2)(e^{-2t}) &\equiv 0 \\
(0)(e^{-2t}) &\equiv 0
\end{aligned}
$$

which is an identity (in the independent variable t for all real values of t). When showing that both sides of an equation are identical for all values of the variables, we will use the equivalence sign $\equiv$.

Problem. Show that the function $\phi(t) = (1 - t^2)^{1/2} \equiv \sqrt{1 - t^2}$ is a solution of the differential equation $x' = -t/x$.

Solution. First, notice that $\phi(t)$ is not even defined outside the interval $-1 \leq t \leq 1$. In the interval $-1 < t < 1$, $\phi(t)$ can be differentiated by the chain rule (for powers of functions):

$$
\phi'(t) = \frac{1}{2}(1 - t^2)^{-\frac{1}{2}}(-2t) = -\frac{t}{(1 - t^2)^{\frac{1}{2}}}
$$

The right-hand side of the equation $x' = -t/x$, with $phi(x)$ substituted for x, is

$$
-\frac{t}{\phi(t)} = -\frac{t}{(1 - t^2)^{\frac{1}{2}}}
$$

which is identically equal to $\phi(t)$ wherever ϕ and ϕ' are both defined. Therefore, $\phi(t)$ is a solution to the differential equation $x' = -t/x$ on the interval (-1, 1).

1.2 Systems of Differential Equations

Problem. Show that the functions $x(t) = e^{-t}, y(t) = -4e^{-t}$ form a solution of the system of differential equations

$$
x'(t) = 3x + yy'(t) = -4x - 2y
$$

Solution. The derivatives that we need are $x'(t) = -e^{-}t$ and $y'(t) = -(-4e^{-t}) = 4e^{-t}$. Then substitution into the second equation gives:

$$
\begin{aligned}
3x + y = (2e^{-t}) + (-4e^{-t}) = (3 - 4)e^{-t} = -e^{-t} &\equiv x'(t), \\
-4x - 2y = -4(e^{-t}) - 2(-4e^{-t}) = (-4 + 8)e^{-t} = 4e^{-t} &\equiv y'(t);
\end{aligned}
$$

therefore, the given functions of x and y form a solution for the system.

1.3 Families of Solutions, Initial-Value Problems

In this section the solutions of some very simple differential equations will be examined in order to give us an understanding of the terms n-parameter family of solutions and general solution of a differential equation. We will also be shown how to use certain types of information to pick one particular solution out of a set of solutions.

While we do not yet have any formal methods for solving differential equations, there are some very simple equations that can be solved by inspection. One of these is:

$$x' = x$$

This first-order differential equation asks you to find a function $x(t)$ which is equal to its own derivative at every value of t.

Definition 4. A first-order differential equation with one initial condition specified is called an initial-value problem, usually abbreviated as an IVP. The solution of an IVP will be called a particular solution of the differential equation.

Problem. Solve the IVP $x' = x, x(0) = \frac{1}{2}$.

Solution. Since we just found that the general solution of $x' = x$ is $x(t) = Ce^t$, we only need to use the initial condition to determine the value of C. This will pick out one particular curve in the family. Substituting $t = 0$ and $x(0) = \frac{1}{2}$ into the general solution,

$$x(0) = Ce^0 = C = \frac{1}{2}.$$

With $C = \frac{1}{2}$, the solution of the IVP is $x(t) = \frac{1}{2}e^t$. This particular solution is the dotted curve shown in Figure 1.1, with the initial point $(0, \frac{1}{2})$ circled.

2 First-Order Differential Equations

In this chapter, methods will be given for solving first- order differential equations. First-order means that the first derivative of the unknown function is the highest derivative appearing in the equation. This implied that the most general first-order differential equation has the form $F(t, x, t') = 0$ for some function F, in this chapter, we will assume that the equation can be solved explicitly for x'. This means that our first-order differential equations can always be put in the form:

$$x' = f(t, x)$$

where f denotes an arbitrary function of two variables. To see why such an assumption makes sense, supposed the differential equation is:

$$(x'(x))^2 + 4x'(t) + 3x(t) = t.$$

It would be messy, but not impossible to use the quadratic formula to extract two differential equations of the form $x' = f(t, x)$ from this quadratic equation. However, one could also imagine equations where solving for $x'(t)$ is not even possible, and in such a case, some of our methods might not be applicable.

The material in this chapter will cover several analytic methods for solving first-order differential equations, each requiring the function f to have a special form. Two different graphical methods are also described; one for the general equation depending only on x. Numerical methods for first-order equations are introduced and theoretical issues of existence and uniqueness of solutions are discussed.

2.1 Separable First-Order Equations

The first analytic method we will consider applies to first-order equations that can be written in the form

$$\frac{dx}{dt} = g(t)h(t);$$

that is, when the function $f(t, x)$ can be factored into a product of a function of t times a function of x. Such a differential equation is called separable.

Problem. Determine which of the following first-order differential equations are separable. Hint: try to factor the right- hand side if the equation does not initially appear to be separable.

$$x' = xt + 2x \to x' = x(t + 2) \to g(t) = t + 2, h(x) = x$$

$$x' = x + \cos(t)$$

$$x' = xt^2 + t^2 - tx \to x' = (t^2 - t)(x+1) \to g(t) = (t^2 - t), h(x) = (x+1)$$

$$x' = x^2 + x + 3 \to x' = (1)(x^2 + x + 3) \to g(t) = 1, h(x) = x^2 + x + 3$$

1. If $h(x) = 1$, the separable equation $x' = g(t)$ is just an integrating problem and the solution is
$$x = \int g(t)dt;$$
that is, x is just the **indefinite integral** of the function $g(t)$. Remember that this means that x can be *any* function $G(t)$ such that $G'(t) = g(t)$, and this introduces an arbitrary constant into the solution. As an example, the solution of $x' = t + 1$ is
$$x(t) = \int (t+1)dt = \frac{t^2}{2} + t + c$$
Even in this simple case the solution is an infinite one-parameter family of functions.

2. If $g(t) = 1$, the separable equation $x' = h(x)$ is called an **autonomous** first-order differential equation. Unless $h(x)$ is a constant, it is no longer possible to solve the equation by simple integration, and the method given below must be used. Autonomous first-order differential equations are important and will be investigated more thoroughly in section 2.7. In the above examples, only the last equation is autonomous. The other three contain functions of t (other than the unknown function $x(t)$) on the right-and side.

Problem. Solve the differential Equation $\frac{dx}{dt} = -tx^2$.

Solution. Split dx/dt into two pieces, dx and dt, and do a bit of algebra to write:

$$-\frac{dx}{x^2} = tdt$$

Integrate each side with respect to its own variable to obtain:

$$\int \left(-\frac{1}{x^2}\right) = \int tdt \to \frac{1}{x} = \frac{t^2}{x} + C.$$

where the arbitrary constants on each side have been collected on the right. Solve this equation for x to obtain the one parameter family of solutions

$$x = \frac{1}{(t^2/2) + C}.$$

We should check that the function $x(t)$ does satisfy the differential equation for any value of the constant C. It appears that this method works, but splitting dx/dt into two pieces is not a mathematically condoned operation; therefore, a justification of the method needs to be given.

If an equation is separable, and $x'(t)$ is written as dx/dt, both sides of the equation $dx/dt = g(t)h(x)$ can be divided by $h(x)$, and the equation becomes

$$\frac{1}{h(x(t))}\left(\frac{dx}{dt}\right)dt = \int g(t)dt + C.$$

The method of simple substitution can be applied to the integral on the left. If we substitute $u = x(t)$, then $du = (dx/dt)dt$, and the equation becomes

$$\int \frac{1}{h(u)}du = \int g(t)dt + C.$$

Now let $H(u)$ be any function such that $H'(u) = 1/h(u)$ and $G(t)$ any function with $G'9t) = g(t)$. Then the equation above implies that

$$H(u) + C_1 = G(t) + C_2 \rightarrow H(u) = G(t) + C,$$

Where C is the constant $C_2 - C_1$

Replacing u again by $x(t)$:

$$H(x(t)) = G(t) + C$$

Check carefully that the expression $H(x) = G(t) + C$ is exactly the same as the solution obtained above. It is an **implicit solution** of $-dx/x^2 = tdt$; that is, it defines a relationship between the unknown function x and its independent variable t. If it can be solved explicitly for x as a function of t, the result is called an **explicit solution of the differential equation. As expected, the integration produces an infinite on-parameter family of solutions

Theorem 1. TTo solve a separable first-order differential equation, $x'(t) = g(t)h(x)$:

- Write the equation in the form $dx/dt = g(t)h(x)$.
- Multiply both sides by dt, divide by $h(x)$, and integrate, to put the equation in the form
$$\int \frac{1}{h(x)} dx = \int g(t) dt.$$
- Find any function $H(x)$ such that $H'(x) = 1/h(x)$ and any function $G(t)$ such that
- Write the solution as $H(x) = G(t) + C$
- If possible, solve the equation from the previous step explicitly for x, as a function of t.

2.2 Solving Linear ODEs

Definition 5. An ODE is linear for the dependent variable y if it is homogeneous when $g(x) = 0$ and otherwise it is non-homogeneous.

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

The standard form of a linear ODE is

$$\frac{dy}{dx} + P(x)y = f(x).$$

If the homogeneous ODE is separable: $y = e^{-\int P(x)dx}$, and $y_c = cy_1(x)$ where $y_1 = e^{-\int P(x)dx}$. For a non-homogeneous ODE, we need to use the process of "variation of parameters". First we need to find the function u so that $y_p = u(x)y_1(x) = u(x)e^{-\int P(x)dx}$, and then we follow the following steps

1. Put in standard form
2. Determine $P(x)$ and integrating factor $e^{\int P(x)dx}$.
3. Multiply std form by integrating factor.
4. Write
$$\frac{d}{dx}\left[e^{\int P(x)dx} y\right] = e^{\int P(x)dx} f(x).$$
5. Integrate both sides.

2.3

2.4 Existence and Uniqueness of Initial Value Problems

Given an initial value problem, how can I know that a solution exists, and if so, the uniqueness of that solution.

Problem. $\frac{dx}{dt} = \sqrt{x}$, $x(0) = 0$ has two solutions. These solutions are $x(t) = \frac{t^2}{4}$ and 0. We can change the initial condition to be $x(0) = 5$. We can separate our values to get:

$$x(t) = \left(\frac{t}{2} + \sqrt{5}\right)^2$$

If we change the initial condition to $x(5) = -5$ we get no solutions. We can also change the initial condition to $x(5) = 0$ which gives us two solutions:

$$x(t) = \left(\frac{t-5}{2}\right)^2$$
$$x(t) = 0$$

2.5 Exact Ordinary Differential Equations

This highlights an analytic technique to find the solutions to an ordinary differential equation. This technique depends on if we are able to write an ODE in the form $\frac{d}{dx}(F(x, y(x))) = 0$. How can we tell if this is possible?

We can determine this by using the multi-variable chain rule from calculus 3:

$$\frac{d}{dx}(F(x, y(x))) = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

Now in order to actually solve the ordinary differential equation, we will turn the equation above into something that we can work with:

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0$$

Now we ask, does $f(x, y)$ exist such that $\frac{\partial F}{\partial x} = M(x, y)$ and $\frac{\partial F}{\partial y} = N(x, y)$? We can test this by checking to see if we can do the following:

$$\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x} \rightarrow \boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

If it passed the test, we use the equations to find $F(x, y)$, and then we can use $F(x, y(x)) = C$ to find the solution to the problem.

Problem. Calculus 3 example. Consider a field $\vec{v} =< 2y + 4x, 2x - 5y >$. Is this a gradient field? If so, what is its potential function. In order to find this, there must be a function $f(x, y)$ such that $\nabla f = \vec{v}$.

$$\partial f \partial y\rangle =< 6xy, 2x - 5y >$$

$$\frac{\partial f}{\partial x} = 6xy, \qquad \frac{\partial f}{\partial y} = 2x - 5y$$

Therefore, we know that $f_{xy} = f_{yx}$ Now we just need to set the two equations equal to each other and solve:

$$\frac{\partial}{\partial y}(2y + 4x) \stackrel{?}{=} \frac{\partial}{\partial x}(2x - 5y)$$

And we can solve that equation algebraically and get $2 = 2$, therefore this is the solution to the ordinary differential equation

Problem. Solve $(4x+2y)+(2x-5y)\frac{dy}{dx} = 0$. We should set $M(x,y) = 4x+2y$ and set $N(x,y) = (2x - 5y)$.

1. Testing to see if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$$\frac{\partial}{\partial y}(4x + 2y) \stackrel{?}{=} \frac{\partial}{\partial x}(2x - 5y)$$

2. Now we need to find $F(x,y)$:

$$\frac{\partial F}{\partial x} = 4x + 2y \qquad \frac{\partial F}{\partial y} = 2x - 5y$$

$$F(x,y) = \int (4x + 2y)dx = 2x^2 + 2xy + c_y, \quad F(x,y) = \int (2x - 5y)dy = 2xy - \frac{5}{2}y^2 + c_x$$

We can determine that $c_y = \frac{5}{2}y^2$ and $c_x = 2x^2$ by looking at the two equations next to each other. Therefore, we know that $F(x,y) = 2x^2 + 2xy - \frac{5}{2}y^2$.

Our resulting solution is:

$$\boxed{2xy - \frac{5}{2}y^2 + 2x^2 = C}$$

This was an implicitly defined solution, so $F(x,y) = 2xy - \frac{5}{2}y^2 + 2x$, and our ode is:

$$\frac{d}{dx}\left[2xy(y) - \frac{5}{2}(y(x))^2 + 2x^2\right] = \frac{d}{dx}[C]$$
$$(2y + 2xy') - \frac{5}{2}2(y) + 4x = 0$$

Example 2 Solve the differential equation,

$$2x^2y + e^y + (x^3 + xe^y - 2y)\frac{dy}{dx} = 0.$$

We can see that $3x^2y + e^y = M(x, y)$ and $(x^3 + xe^y - 2y)\frac{dy}{dx} = 0$.

We can test this by checking to see if:

$$\frac{\partial M}{\partial y} \stackrel{?}{=} \frac{\partial N}{\partial x}$$

and by taking the partial derivatives of each of the equations, we can see that $3x^2 + e^y = 3x^2 + e^y$. Now that we know that this is an exact ordinary differential equation, we need to solve for $F(x, y)$. In order to find $F(x, y)$, we must do

$$\frac{\partial F}{\partial x} = M \qquad\qquad \frac{\partial F}{\partial y} = N.$$

This means that

$$\begin{aligned} F(x, y) &= \int (3x^2y + e^y)dx \\ &= x^3y + xe^y + C_y \end{aligned} \qquad\qquad \begin{aligned} F(x, y) &= \int (x^3 + xe^y - 2y)dy \\ &= x^3y + xe^y - y^2 + C_x \end{aligned}$$

So, $F(x, y) = x^3yxe^y - y^2$, and our solution is:

$$\boxed{x^3y + xe^y - y^2 = C}$$

Now we need to find a solution that satisfies the initial value problem, $y(-2) = 1$.

$$\begin{aligned} (-2)^3(1) + (-2)e^1 - (1)^2 &= C \\ -8 - 2e - 1 &= C \\ -9 - 2e &= C \end{aligned}$$

Which results in our particular solution being:

$$\boxed{x^3y + xe^y - y^2 = -9 - 2e}$$

2.5.1 Substitution

Sometimes we can make a substitution that turns one ordinary differential equation into one that we can solve.

Problem. Let's take the equation $\frac{dy}{dx} = \frac{1-x-y}{x+y}$. In order to do this you need to trade out $y(x)$ for $u(x)$:

$$\frac{dy}{dx} = \frac{du}{dx} - 1$$

$$\frac{du}{dx} - 1 = \frac{1-u}{u} = \frac{1}{u} - 1$$

After we do those steps, we can see that this is a separable ordinary differential equation, and we can use the normal method to solve a separable equation:

$$\begin{aligned}
\int udu &= \int 1dx \\
\frac{u^2}{2} &= x + C \\
u^2 &= 2x + D \leftarrow D = 2C \\
(x+y)^2 &= 2x + D \rightarrow x + y = \pm\sqrt{2x+D} \\
y &= -x \pm \sqrt{2x+D}
\end{aligned}$$

Where the initial condition determines the last equation above. Now we need to solve for the initial condition of $y(6) = 4$, and we can do this by just plugging in the values, and determining the sign on the square root:

$$\begin{aligned}
4 - 6 &\pm \sqrt{12+D} \\
10 &= \pm\sqrt{12+D} \\
D &= 88 \rightarrow y(x) = -x + \sqrt{2x+88}
\end{aligned}$$

2.5.2 Bernoulli Ordinary Differential Equations

We use Bernoulli's solution to an ordinary differential equation if the equation can be written in the form $\frac{dx}{dt} + p(t)x = q(t)x^n$, where if $n = 0$, the equation is linear and where if $n = 1$, the equation is both linear and separable. We can do a substitution for Bernoulli ordinary differential equations where we let $v = x^{1-n}$, and we trade out $x(t) \rightarrow v(t)$ to turn the differential into a linear differential. Another formula we can use for plugging in to find the separable ordinary differential equation is $\frac{du}{dx} + (1-n)p(x)u = (1-n)q(x)$, but we do not have an example to show for that method.

Problem. Consider the Ordinary differential equation $t\frac{dy}{dt} + y = \frac{1}{y^2}$, and solve. Our first step for this problem is to divide both sides by t to rearrange it into the Bernoulli form:

$$\frac{dy}{dt} + \frac{1}{t}y = \frac{1}{t}y^-2$$

Because y is our dependent variable, we will be looking for our n value, and we will find it on the y on the right-hand side. This gives us $n = -2$. Now we need to let $v(t) = y^{1-(-2)} = y^3$ and this means that $y(t) = (v(t))^{1/3}$. Now wee need to take the differential of this equation in order to find a substitution for $\frac{dy}{dt}$:

$$\frac{dy}{dt} = \frac{1}{3y^2}\frac{dv}{dt}$$
$$\frac{dv}{dt} = 3y^{-2}\frac{dy}{dt}$$

Once we have found a substitution for $\frac{dy}{dt}$, we can simply plug in our equation and solve for $v(t)$:

$$\frac{1}{3y^2}\frac{dv}{dt} + y = \frac{1}{y^2}$$

$$\frac{t}{3}\frac{dv}{dt} + y^3 = 1$$

We solve for $v(t)$, and then we can use the integrating factors method to solve the ordinary differential equation:

$$\begin{aligned}
\frac{dv}{dt} + \frac{3}{t}v &= \frac{3}{t} \\
\mu(t) &= e^{\int \frac{3}{t}dt} = e^{3ln(t)} = t^3 \\
t^3\frac{dv}{dt} + 3t^2v &= 3t^2 \\
\frac{d}{dt}(t^3v) &= 3t^2 \\
\int t^3v &= \int 3t^2dt = t^3 + C \\
v(t) &= 1 + \frac{C}{t^3} \rightarrow y(t) = (v(t))^{\frac{1}{3}}
\end{aligned}$$

Now we just need to let $v(t) = y^3$ and plug this back into the equation to get $y(t) = \left(1 + \frac{C}{t^3}\right)^{\frac{1}{3}}$ as our final solution.

Homogeneous ordinary differential equations can be written in the form $\frac{dx}{dt} = f\left(\frac{x}{t}\right)$. This means that if we just substitute v for $\frac{x}{t}$, we will turn the Homogeneous equation into a separable ordinary differential equation.

Problem. Consider the homogeneous ODE, $(5y-2x)\frac{dy}{dx} = 4x+2y$. We can divide the $5y-2x$, so we will end up with the equation

$$\frac{dy}{dx} = \frac{4x+2y}{5y-2x} = \frac{4+\frac{2y}{x}}{5\frac{y}{x}-2}$$

The reason we can make this change is that we are dividing everything on the left side by x to get it in the form $\frac{y}{x}$. Our next step is to trade out $y(x)$ for $v(x)$:

$$\frac{4+2v}{5v+2} = v + x\frac{dv}{dx}$$

Which we can further simplify into $\frac{dv}{dx} = \frac{1}{x}\left(\frac{4+2v}{5v-2} - v\right)$. Now, we are able to solve this as a separable ordinary differential equation:

$$\begin{aligned} \frac{dv}{dx} &= \frac{1}{x}\left(\frac{4+4v-5v^2}{5v-2}\right) \\ \frac{5v-2}{4+4v-5v^2}dv &= \frac{1}{x}dx \end{aligned}$$

Now we need to let $u - 4 + 4v - 5v^2$, and let $du = (4-10v)dv = -2(5v-2)dv$. And then we integrate with respect to u to end up with

$$\begin{aligned} -\frac{1}{2}\int \frac{1}{u}du &= \int \frac{1}{x}dx \\ -\frac{1}{2}ln|u| &= ln|x| + C \\ ln|4+4v-5v^2| &= -2ln|x| + D \\ 4+4v+5v^2 &= e^{ln|x|^-2+D} \\ &= e^{ln|x|^{-2}}e^D \\ 4+4\left(\frac{y}{x}\right) - 5\left(\frac{y}{x}^2\right) &= e^{ln|x|^{-2}}A \end{aligned}$$

$$\boxed{4x^2+4xy-5y^2 = A}$$

2.6

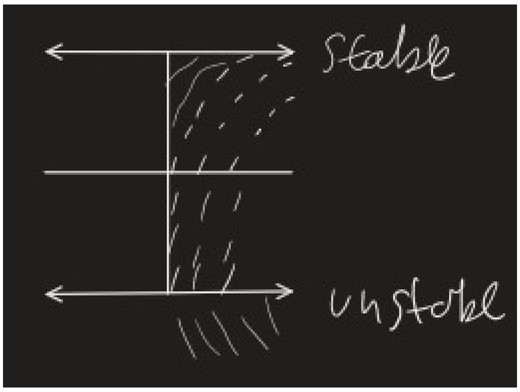
2.7 Phase Lines and Equilibrium Solutions

This is a qualitative technique for autonomous ordinary differential equations. This is used for finding the long term behavior of solutions for various initial conditions. An equilibrium solution is a constant function satisfying the ODE:

$$y(t) = k \rightarrow \frac{dy}{dt} = 0$$

and we need to look for the roots of $f(y)$.

Problem. Consider the ODE: $\frac{dy}{dx} = 4 - y^2$, we know that the equilibrium solutions (stationary points / fixed points / critical points) are $0 = 4 - y^2$ and $y = \pm 2$. If we were to plot this as a slope field, we would see an image like the following:



We need to classify each of the lines here as stable (a sink), unstable (a source), or semi-stable (a node). From this slope diagram we can make a phase diagram.

The arrows in this diagram indicate whether $\frac{dy}{dx}$ is above or below 0. We can see that the long term behavior of the solution:

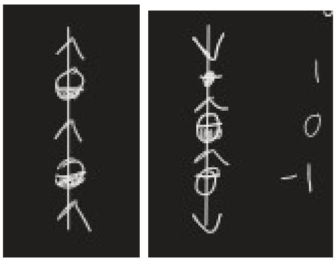
$$\begin{aligned} y_0 = y(t_0) > 2, &\to y(t) \to 2 \\ -2 < 2_0, &\to y(t) \to 2 \\ y_0 < -2, &\, y(t) \to -\infty \end{aligned}$$

Let's take a look at another example problem:

Problem. Compare the pivots of the two ordinary differential equations, $\frac{dy}{dt} = y^2(1-y)^2$, and $\frac{dy}{dx} = y^2(1-y^2)$.

$$\begin{aligned} \frac{dy}{dt} &= y^2(1-y)^2 \\ 0 &= y^2(1-y)^2 \end{aligned} \qquad \begin{aligned} \frac{dy}{dt} &= y^2(1-y^2) \\ 0 &= y^2(1-y^2) \\ y &= 0, \pm 1 \end{aligned}$$

From finding the equilibrium points of the differential equations, we can now craft a pivot diagram:



From this we can see that in the long term, the equations are going to look like this:

$$\begin{aligned} y_0 < 0, y(t) &\to 0 \\ 0 < y_0 < 1, y(t) &\to 1 \\ 1 < y_0, y(t) &\to \infty \end{aligned}$$

2.7.1 Bifurcations

For ordinary differential equations with a parameter, there are times that a small change in the value of the parameter results in a huge change in the behavior of the solutions. It causes a change in the number or stability of the equilibrium solutions.

Problem. Find the bifurcation value for

$$\frac{dp}{dt} = 0.5p\left(1 - \frac{p}{100}\right) - H$$

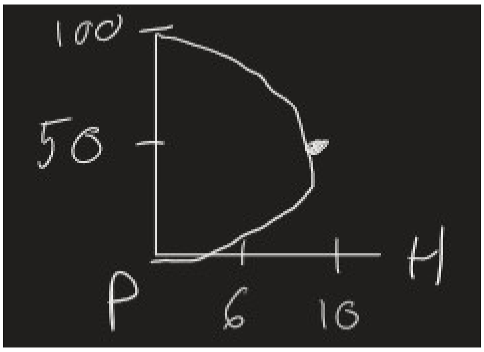
The first step for finding a bifurcation for an equation is finding the equilibrium solutions. In order to do that, we set equation (2.69) equal to 0 and then rearrange it, so it is in the form

$$p^2 - 100p + 250H = 0$$

In this case, we are going to need to use the quadratic formula to find the equilibrium points:

$$\begin{aligned} p = & \frac{100 \pm \sqrt{10000 - 1000H}}{2} \\ & = 50 \pm 5\sqrt{100 - 10H} \end{aligned}$$

We can determine that our bifurcation value is $H = 10$, because if we take what's under the square root, and set it equal to zero we can find those values. Now we can plot our bifurcations:



2.8 Numerical Methods

[1]

[1] We got here from 4.1

3 Second-order Differential Equations

3.1 General Theory of Homogeneous Linear Equations

The standard form for a general second order ordinary differential equation is

$$x''(t) + p(t)x'(t) + q(t)x(t) = f(t).$$

For us, $p(t)$ and $q(t)$ are typically going to be constants. Also, we know that if $f(t) = 0$, the equation is homogeneous.

Lemma 1. GGiven an equation in the form

$$x''(t) + p(t)x'(t) + q(t)x(t) = 0,$$

if $x_1(t)$ and $x_2(t)$ are solutions to equation (3.2), then

$$x(t) = C_1x_1(t) + C_2x_2(t)$$

is also a solution to equation (3.1) for all C_1, C_2's. Equation (3.3) is the general solution to a homogeneous ODE

Let's take a look at an application of lemma (3.1.1).

Example. Consider the equation

$$t^2x''' + 2tx'' - 6x' = 0.$$

This equation is a third order, linear and homogeneous, but not autonomous ordinary differential equation. Let's put this differential equation into standard form:

$$x'' + \frac{2}{t}x'' + \frac{6}{t^2}x' = 0$$

As long as $t_0 \neq 0$, we have a unique solution if given 3 initial conditions. The domain of this ordinary differential equation is $t > 0, t < 0$. Earlier, in section 3.1, we showed that $1, t^3, \frac{1}{t^2}$ were all solutions. Because this ordinary differential equation is linear and homogeneous, we know that

$$x(t) = C_1 1 + C_2t^3 + C_3\frac{1}{t^2},$$

is also a solution for all C_1, C_2, C_3. Just note, equation (3.6) is a linear combination of $1, t^3, \frac{1}{t^2}$. So, this is a general solution as long as the functions, $1, t^3, \frac{1}{t^2}$ are all "different" from each other.

What is "different"? Different means no one function in a set can be written as a linear combination of other functions in the set. Different means that all the functions are linearly independent. More generally, if

$$C_1 f_1(t) + C_2 f_2(t) + \cdots + C_n f_n(t) = 0$$

Theorem 2. We can instead test for linear independence by using the wronskian,

$$w(f_1, f_2)(t) = det \begin{vmatrix} f_1(t) & f_2(t) \\ f_1'(t) & f_2'(t) \end{vmatrix}$$

- If $w(t) = 0$ for all t's, then $\{f_1, f_2\}$ are linearly dependent.
- If $w(t) = 0$ for all t's, then $\{f_1, f_2\}$ are linearly independent.

Now, if we try to use the wronskian on problem 16, we do it like so:

$$\begin{aligned} w(t) &= \begin{vmatrix} 1 & t^3 & \frac{1}{t^2} \\ 0 & 3t^2 & -\frac{2}{t^2} \\ 0 & 6t & \frac{6}{t^4} \end{vmatrix} \\ &\to 1 \times \begin{vmatrix} 3t^2 & -\frac{2}{t^2} \\ 6t & \frac{6}{t^4} \end{vmatrix} \\ &= \frac{18}{t^2} + \frac{12}{t^2} = \frac{30}{t^2} \neq 0 \end{aligned}$$

So we know that $\{1, t^3, \frac{1}{t^3}\}$ are linearly independent.

Let's consider another problem where we need to show that the solutions are linearly dependent.

Example. Let's consider the equation $y''(x) = -9y$, which has the solutions $\{\cos(3x), \sin(3x)\}$ and let's show that the two solutions are linearly independent:

$$w(x) = \begin{bmatrix} \cos(3t) & \sin(3t) \\ -3\sin(3t) & 3\cos(3t) \end{bmatrix} = 3\cos^2(3x) + 3\sin^2(3x) = 3 \neq 0$$

Which means that both solutions are valid for this ordinary differential equation. We can turn our original equation into $y''(x) + 9y = 0$ which is a homogeneous ordinary differential equation. Our $p(x) = 0$, and because we know that $w(x) = Ce^{-\int p(x)dx}$, we know that the solution to our equation is 0.

Theorem 3. What to we do when solving linear non-homogeneous ordinary differential equations?

1. Solve the associated homogeneous ODE $\rightarrow X_h(t)$, which will have many unknown constants
2. Find one solution to non-homogeneous ODE $\rightarrow X_p(t)$

The general solution to the original non-homogeneous ODE is

$$x(t) = x_h(t) + x_p(t)$$

Let's try out this new theorem on an example problem.

Example. Consider the differential equation $y'' - 5y' - 6y = 4x$. We claim that

$$y(x) = c_1 e^{-x} + c_2 e^{6x} - \frac{2}{3} + \frac{5}{9}$$

Are $\{e^{-x}, e^{6x}\}$ linearly independent? Verify both solutions to equation (3.14).

$$\begin{aligned} w(x) &= \begin{bmatrix} e^{-x} & e^{6x} \\ -e^{-x} & 6e^{6x} \end{bmatrix} \\ &= 6e^{5x} + e^{5x} = 7e^{5x} \end{aligned}$$

Now we need to verify that this is a solution:

$$\begin{aligned} y'(x) &= -C_1 e^{-x} + 6C_2 e^{6x} - \frac{2}{3} \\ y''(x) &= C_1 e^{-x} + 36C_2 e^{6x} \end{aligned}$$

Let's see if the left-hand side is equal to $4x$ like it was above.

$$\begin{aligned} y'' - 5y' - 6y = \left[C_1 e^{-x} + 36C_2 e^{6x}\right] &- 5\left[-C_1 e^{-x} + 6C_2 e^{6x} - \frac{2}{3}\right] \\ &- 6\left[C_1 e^{-x} + C_2 e^{6x} - \frac{2}{3} + \frac{5}{9}\right] \end{aligned}$$

$$\begin{aligned} &= C_1[e^{-x} + te^{-x} - 6e^{-x}] + C_2[36e^{6x} - 30e^{6x} - 6e^{6x}] + \left[\frac{10}{3} + 4x - \frac{10}{3}\right] \\ &= 0 + 0 + 4x. \end{aligned}$$

3.2 Homogeneous Linear Equations with Constant Coefficients

Homogeneous ordinary differential equations are ones that come in the form

$$ay'' + by' + cy = 0$$

The first method of solving homogeneous linear equations with constant coefficients is the method of lucky guess. In this method, we let $y(x) = e^{rx}$. If we take the multiple derivatives of e^{rx}, we can find what $y'(x)$ and $y''(x)$ equal. Now we can plug in the e^{rx} into our equation to get

$$\begin{aligned} ar^2e^{rx} + bre^{rx} + ce^{rx} &= 0 \\ e^{rx}[ar^2 + br + c] &= 0. \end{aligned}$$

$$\boxed{ar^2 + br + c = 0}$$

There are 3 distinct cases for the roots of this equation.

1. 2 real distinct roots.
2. 1 repeated root.
3. 2 complex conjugate roots $(A + Bi)$.

Let's take a look at an example of this method in action.

Example. Consider the equation

$$y'' - 5y' - 6y = 0.$$

We are going to let $y(x) = e^{rx}$. We can skip adding e^{rx} to the beginning of each term because we can just divide that over in our minds. After going through that we get

$$\begin{aligned} r^2 - 5r - 6 &= 0 \\ (r - 6)(r + 1) &= 0 \\ r &= 6, -1. \end{aligned}$$

From this, we know that our two solutions are $y_1(x) = e^{6x}, y_2(x) = e^{-x}$. Our general solution is

$$\boxed{y(x) = C_1e^{6x} + C_2e^{-x}}$$

Now all we need to do is check the wronskian $W(x)$ to make sure that $\{e^{r_1x}, e^{r_2x}\}$ are linearly independent.

Let's explore a case of a real repeated root. Let's look at the equation

$$y'' - 8y' + 16 = 0.$$

We can let $y(t) = e^{rt}$, to get

$$r^2 - 8r + 16 = 0.$$

Our roots for r are 4 and 4. One of our solutions should be $y_1(t) = e^{4t}$, but because we have a repeated root, I claim that $y_2(t) = te^{4t}$. We can verify that this is a repeated root by putting it in a wronskian:

$$\begin{aligned} w(t) &= \begin{vmatrix} e^{4t} & te^{4t} \\ 4e^{4t} & e^{4t} + 4te^{4t} \end{vmatrix} \\ &= e^{8t} + 4te^{8t} - 4te^{8t} \\ &= e^{8t} \neq 0. \end{aligned}$$

Because the wronskian result was not equal to zero, we know that these are two linearly independent solutions. Our general solution in this case would be

$$y(t) = C_1 e^{4t} + C_2 te^{4t}.$$

If we were to solve a 4th order ordinary differential equation, our general solution would look something like this:

$$y(t) = C_1 e^{2t} + C_2 e^{-3t} + C_3 te^{-3t} + C_4 t^2 e^{-3t}.$$

As you can see, the main difference between the second and third order ordinary differential equations is the amount of roots you need to take. Let's take a look at the third case, having 2 complex conjugate values for $r = \alpha \pm i\beta$.

Consider the equation

$$y'' - 2y' + 5y = 0.$$

Let $y(t) = e^{rt}$. Our previous equation will turn into $r^2 - 2r + 5 = 0$, and we will need to use the quadratic formula for this case.

$$\begin{aligned} r &= \frac{2 \pm \sqrt{4 - 20}}{2} \\ &= \frac{2 \pm \sqrt{-16}}{2} \\ &= 1 \pm 2i. \end{aligned}$$

Because of Euler's identity,

$$e^{i\theta} = \cos(\theta) + i\sin(\theta),$$

we can determine that the general solution of this ordinary differential equation is

$$y(t) = C_1 e^{1t}\cos(2t) + C_2 e^{1t}\sin(2t).$$

Example. Solve the homogeneous linear ordinary differential equation,

$$x''' + 3x'' - 4x' - 12x = 0.$$

Let $x(t) = e^{rt}$, this means that

$$\begin{aligned} x' &= re^{rt} \\ x'' &= r^2 e^{rt} \\ x''' &= r^3 e^{rt}. \end{aligned}$$

Our expanded out equation is

$$e^{rt}[r^3 + 3r^2 - 4r - 12] = 0.$$

From this, we can see that our roots are $r = 2, -2, -3$, which means that our general solution is

$$x(t) = C_1 e^{2t} + C_2 e^{-2t} + C_3 e^{-3t}.$$

3.3 The Spring-Mass Equation

The spring-mass equation is a variation of Hooke's law, which states

$$F_{spring} = -kx(k > 0).$$

This tries to bring mass back to equilibrium where $x(t)$ is the distance of the mass from its equilibrium position at time t. The differential version of the equation is

$$mx'' + bx' + kx = F(t).$$

Where mx'' is the acceleration, bx' is the resistance friction, kx is the spring force and $F(t)$ is any extra force acting on the system. Where is gravity within this equation? It drops out because the stretching of the string due to gravity puts the block attached to the string into equilibrium.

When you first attach the mass and the mass stops moving,

$$m(0) = -k(s) - b(0) - mg \to mg = ks \to k.$$

Example. A mass weighing 10 pounds stretches a string $\frac{1}{4}ft$. Assuming no dampening, find the amplitude period of oscillations of the mass if it is released from a point $\frac{1}{10}ft$below it's equilibrium position with an initial upward velocity of $\frac{1}{20}\frac{ft}{s}$.
Because the mass weighs 10 pounds, $mg = 10$, which means that $m = \frac{10}{32} = \frac{5}{16}\frac{lbs^2}{ft}$. The first thing we need to do is find K. This can be done by setting $mg = ks$ and solving for k.

$$\begin{aligned} 10 &= \frac{1}{4}k \\ k &= \frac{10}{\frac{1}{4}} = 40 \\ [k] &= \frac{10lb}{\frac{1}{ft}} = \left[\frac{lb}{ft}\right]. \end{aligned}$$

Our ordinary differential equation is going to be

$$\frac{5}{16}x'' + 40x = 0.$$

We are given the initial values of $x'(0) = -\frac{1}{20}$ and $x(0) = -\frac{1}{10}$. We get the ordinary differential equation

$$\rightarrow x'' + 128x = 0$$
.

Furthermore, we are going to let $x(t) = e^{rt}$. This gives us

$$\begin{aligned} r^2 + 128 &= 0 \\ r^2 &= -128 \\ r &= \pm\sqrt{-128} = \pm_8\sqrt{2}i. \end{aligned}$$

The general solution to our ordinary differential equation in this situation is

$$x(t) = C_1 \cos\left(8\sqrt{2}t\right) + C_2 \sin\left(8\sqrt{2}t\right).$$

This gives us that the angular frequency of our system, $\beta = 8\sqrt{2}\left[\frac{1}{s}\right]$, and the period of our system, $T = \frac{\pi}{4\sqrt{2}}s$.
NOTE: In general, the natural frequency of a spring mass system is $\omega_0 = 8\sqrt{2}$.
Now we need to apply our initial conditions to find C_1, C_2.

$$\begin{aligned} x'(t) &= -8\sqrt{2}C_1 \sin\left(8\sqrt{2}t\right) + 8\sqrt{2}C_2 \cos\left(8\sqrt{2}t\right) \\ x(0) &= C_1 = -\frac{1}{10} \\ x'(0) &= 8\sqrt{2}C_2 = \frac{1}{10}. \end{aligned}$$

This gives us that our equation of motion is

$$\begin{aligned} x(t) &= -\frac{1}{10}\cos\left(8\sqrt{2}t\right) + \frac{1}{160\sqrt{2}}\sin\left(8\sqrt{t}\right) \\ &= R\sin\left(\beta t + \phi\right) \\ C_1 &= R\sin\phi \\ C_2 &= R\cos\phi \\ C_1^2 + C_2^2 &= R^2 \\ \sqrt{\left(-\frac{1}{10}\right)^2 + \left(\frac{1}{160\sqrt{2}}\right)^2} &= R \\ \sqrt{\frac{513}{51200}} &= R \end{aligned}$$

$$\boxed{R \approx 0.100ft}.$$

$$
\begin{aligned}
\phi &= tan^{-1}\left(\frac{C_1}{C_2}\right) + \frac{1}{60}\sqrt{2}\sin(8\sqrt{2}t) \\
&= tan^{-1}\left(-16\sqrt{2}\right) \\
\phi &= -1.4226rad \\
x(t) &= -\frac{1}{10}\cos\left(8\sqrt{2}t\right) + \frac{1}{160\sqrt{2}}\sin\left(8\sqrt{2}t\right) \\
&\approx 0.1001\sin\left(8\sqrt{2}t - 1.5266\right).
\end{aligned}
$$

$x(t)$ is the displacement of mass from equilibrium at time t. The following is the general formula for a mass-spring system.

$$mx'' + bx' + kx = 0 \text{ where } mbk > 0.$$

Let's consider a dampening situation. If we let $x(t) = e^{rt}$, then

$$
\begin{aligned}
mr^2 + br + k &= 0 \\
\rightarrow r &= \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}.
\end{aligned}
$$

- Repeated r values. For this case, we are going to get $b^2 - 4mk = 0$, which is a critically dampened system. This would make $r = -\frac{b}{2m}, -\frac{b}{2m}$, which would make our general solution

  $$x(t) = C_1 e^{-\frac{b}{2m}t} + C_2 t e^{-\frac{b}{2m}t}.$$

  If we take the $\lim_{t\to\infty} x(t)$, we can see that this equation goes to zero.

- 2 real distinct roots.

  $$r_1 = \frac{-b + \sqrt{b^2 - 4mk}}{2m} < b, r_2 = \frac{-b - \sqrt{b^2 - 4mk}}{2m} < 0.$$

  If $b^2 - 4mk > 0$, we have an over dampened system. $r < r_1 < 0$. The solution to this equation would be

  $$x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}.$$

  The limit as this ordinary differential equation goes to infinity is zero.

- 2 real complex conjugate roots. If

  $$r = -\frac{b}{2m} \pm i\frac{\sqrt{4mk - b^2}}{2m}.$$

  Our $\alpha = -\frac{b}{2m}$, and our $\beta = i\frac{\sqrt{4mk-b^2}}{2m}$. Our general solution in this scenario would be

  $$x(t) = C_1 e^{-\frac{b}{2m}t}\cos\left(i\frac{\sqrt{4mk - b^2}}{2m}t\right) + C_2 e^{-\frac{b}{2m}t}\sin\left(i\frac{\sqrt{4mk - b^2}}{2m}t\right).$$

  $$\lim_{t\to\infty} x(t) = 0.$$

3.4 Non-homogeneous Linear Equations

3.4.1 Method of Undetermined Coefficients

When can we use the method of undetermined coefficients? This method can be used with

- Most constant coefficient ordinary differential equations
- Non-homogeneous term must be of the form:
  - Polynomials.
  - e^{cx}
  - $\sin(bx), \cos(bx)$

Theorem 4. After finding the homogeneous function $y_h(x)$:

1. Make your first guess at the form of $y_p(x)$ with unknown constants based on $f(x)$.
2. Plug $y_p(x)$ into ordinary differential equation. (Left-hand side equals right-hand side for all x's).
3. Solve linear system.
4. Plug back into the guess $y_p(x)$.
5. General solution is $y(x) = y_h(x) + y_p(x)$.

Example. Consider the following ordinary differential equation.

$$x'' - 3x' - 4x = -7t.$$

In order to solve for $x_h(t)$, we set the right-hand side equal to zero and let $x(t) = re^{rt}$.

$$\begin{aligned} r^2 - 3r - 4 &= 0 \\ (r-4)(r+1) &= 0 \\ r &= 4, 1. \end{aligned}$$

This gives us that our general homogeneous solution is

$$x_h(t) = C_1 e^{4t} + C_2 e^{-t}.$$

1. Now we make our initial guess of $x_p(t) = At + B$. Then we need to make sure that it does not match one of the homogeneous solutions from above.
2. $x_p'(t) = A$, $x_p''(t) = 0$

   Now we must plug our $x_p(t)$ and $x_p''(t)$ into our original ordinary differential equation. This gives us

   $$0 - 3A - 4(At + B) = -7t.$$

Now We need to separate all the terms that have a t attached to them, and all the solutions that are constant as

$$\begin{aligned} t &: -4A = -7 \\ 1 &: -3A - 4B = 0. \end{aligned}$$

3. Now we must simply solve for both A and B. After solving, we get that

$$x_p(t) = \frac{7}{4}t - \frac{21}{16}.$$

This gives us the general solution as

$$C_1e^{4t} + C_2e^{-t} + \frac{7}{4}t - \frac{21}{16}.$$

Just a caution, we need to make sure that we apply our initial and boundary conditions to our general solution only.

Example. Consider the ordinary differential equation

$$x'' - 3x' - 4x = 6e^{-4t}.$$

We can see that $x_h(t) = C_1e^{4t} + C_2e^{-t}$. Our initial guess is going to be $x_p(t) = Ae^{-4t}$. Using our initial guess we can do the following

$$\begin{aligned} x_p'' - 3x_p - 4x_p &= 6e^{-4t} \\ 16Ae^{-4t} + 12Ae^{-4t} - 4Ae^{-4t} &= 6e^{-4t} \\ 24A = 6 \therefore A = \frac{1}{4}x_p(t) &= \frac{1}{4}e^{-4t}. \end{aligned}$$

Because our initial guess matched on of the solutions in the homogeneous equation we need to modify our equation or else we will be incorrect, but how do we know how to modify our guess? We need to add a t and check again to make sure that our solution is not matching. Let's try again with Ate^{-4t}

$$\begin{aligned} x_p' &= 4e^{4t} + 4Ate^{4t} \\ x_p'' &= 8Ae^{4t} + 16Ae^{4t}. \end{aligned}$$

After plugging x_p and x_p'' into our ordinary differential equation and solving the system at the end, we get the general result as

$$x(t) = C_1e^{4t} + C_2e^{-t} + \frac{6}{5}te^{4t}.$$

3.4.2 Variation of Parameters

This method is more tedious and difficult than the method described in section (3.4.1), but it is more universal and will find a particular solution to any non-homogeneous linear ordinary differential equation.

Theorem 5. To solve something using variation of parameters,

1. Find $y_h(x) = C_1 y_1(x) + C_2 y_2(x)$.
2. Let $y(x) = v_1(x)y_1(x) + v_2(x)y_2(x)$, where $y_1(x), y_2(x)$ are from the homogeneous solution. Our goal is to find v_1, v_2.
3. Solve system for v_1', v_2'. In order to do this, we can use the following system

$$\begin{aligned} y_1'(x)v_1'(x) + y_2'(x)v_2'(x) &= f(x) \\ y_1(x)v_1'(x) + y_2(x)v_2'(x) &= 0. \end{aligned}$$

   In order to do this, we need to use Cramer's rule. Cramer's rule tells us to find

$$\begin{aligned} w(x) &= \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} \\ v_1'(x) &= \frac{\begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}}{w(x)} \\ v_2'(x) &= \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}}{w(x)}. \end{aligned}$$

4. Integrate to find $v_1(x)$ and $v_2(x)$. $v_1(x) = \square + c, v_2(x) = \square + c$.
5. Simplify.

Example. Solve the ordinary differential equation $x'' - 2x' + x = \frac{e^t}{t}$. First we need to find the homogeneous ordinary differential equation by letting $x_h(t) = e^{rt}$ and solving for r. When all is done we end up with $x_1(t) = e^t, x_2(t) = te^t$.

1. Let $x(t) = v_1(t)e^t + v_2(t)te^t$
2. Set up a system of equations.

$$\begin{aligned} e^t v_1' + te^t v_2' &= 0 \\ e^t v_1' + \left(e^t + te^t\right) v_2' &= \frac{e^t}{t}. \end{aligned}$$

   Now we can solve this to get $v_1' = -1, v_2' = \frac{1}{t}$.
3. Now we need to integrate both $v_1(t)$ and $v_2(t)$ like so

$$\begin{aligned} v_1(t) &= \int -1 dt = -t + C_1 \\ v_2(t) &= \int \frac{1}{t} dt = \ln|t| + C_2. \end{aligned}$$

4. So,

$$\begin{aligned} x(t) &= (-t + C_1)e^t + (\ln|t| + C_2)\,te^t \\ &= -te^t + te^t \ln|t| + C_1 e^t + C_2 te^t. \end{aligned}$$

Another way we can solve this is using the wronskian,

$$\begin{aligned} w(t) &= \begin{vmatrix} e^t & te^t \\ e^t & te^t + e^t \end{vmatrix} = \ldots = e^{2t} \\ v_1'(t) &= \frac{\begin{vmatrix} 0 & te^t \\ \frac{e^t}{t} & te^t + e^t \end{vmatrix}}{w(t)} = \ldots = -1 \\ v_2'(t) &= \frac{\begin{vmatrix} e^t & 0 \\ e^t & \frac{e^t}{t} \end{vmatrix}}{w(t)} = \ldots = \frac{1}{t}. \end{aligned}$$

3.5 The Forced Spring-Mass System

The equations from a forced spring-mass system are in the form

$$mx'' + bx' + kx = f(x).$$

In this section we are going to examine the effects of $f(t)$ on the system. There are three cases for the dampening of a system, under damped, over damped, and critically dampened. We are going to be looking at the under damped case first.

Example. Consider the equation

$$x'' + w_0^2 x = A\cos(\omega t) \text{ where } \omega_0 = \sqrt{\frac{k}{m}}.$$

If we were to make our guess based off of this information, we should guess that $x_p(t) = B\cos(\omega t) + C\sin(\omega t)$. $x_h(t) = C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t)$, which is the natural frequency of this equation. If $\omega \neq \omega_0$, then we don't have to modify. Otherwise, we will need to modify our equation to

$$x_p(t) = Bt\cos(\omega_0 t) + Ct\sin(\omega_0 t).$$

Beats happen when ω is close, but not equal to ω_0. Through a trigonometric identity,

$$x(t) = \frac{-2A}{\omega_0^2 - \omega^2}\sin\left(\frac{\omega + \omega_0}{2} \cdot t\right)\sin\left(\frac{\omega - \omega_0}{2}t\right).$$

Example. A mass of $5kg$ stretches a spring $10cm$ and is acted on by an external force of $10\sin\left(\frac{t}{2}\right) N$ and moves in a viscous medium that imparts a force of 2 N when the speed is $4\frac{cm}{s}$. Find the initial value problem that models setting the mass in motion from its equilibrium position with an upward velocity of $3\frac{cm}{s}$.

Let t be time since the mass in set in motion in seconds. Let x be displacement (where up is positive) of the mass from equilibrium.

$$\begin{aligned} mg &= ks \\ 5.98 &= k(0.1m) \\ \rightarrow k &= 490\frac{kg}{s^2}. \end{aligned}$$

$$\begin{aligned} 5x'' + bx' + 490x = 10\sin\left(\frac{t}{2}\right) \\ x(0) = 0 \\ x'(0) = 0.03. \end{aligned}$$

They told us that $f_{damping} = -bx'$. We know that the force damping when at a velocity of $0.4\frac{m}{s^2}$, so we can solve for b and see that $b = 50$. This makes our ordinary differential equation

$$x'' + 10x' + 98x = 2\sin\left(\frac{t}{2}\right).$$

We should expect to see trigonometric functions because of the value of $f(x)$. Let's try the equation

$$x_p(t) = A\cos\left(\frac{t}{2}\right) + B\sin\left(\frac{t}{2}\right).$$

We can use $b - 4mk$ to determine the numbers required for our homogeneous solution.

Theorem 6. Here is the outline for solving these problems.

1. Find $x_h(t) = C_1e^{\alpha t}\cos(\beta t) + C_2e^{\alpha}\sin(\beta t)$
2. Find $x_p(t) = A\cos(\beta t) + B\sin(\beta t)$
3. State general solutions.
4. Apply initial conditions to find C_1, C_2.
5. Find the particular solution

Note. The maximum amplitude occurs when

$$\begin{aligned} \omega &= \sqrt{\omega_0^2 - 2\lambda} \\ \omega_0 &= \sqrt{\frac{k}{m}}. \end{aligned}$$

3.6 Solving Cauchy Euler Ordinary Differential Equations

3.6.1 Method of Lucky Guess

If an equation is not in the form

$$ax^2y'' + bxy' + cy = 0,$$

then we cannot use the method of lucky guess. This method is common in polar, cylindrical, and spherical coordinates. The first thing we need to do when using this method is put it in standard form,

$$y'' + \frac{b}{a}\frac{1}{x}y' + \frac{c}{ax^2}y = 0.$$

Now, we just need to let $y(x) = x^r$ to find our r-values.

Example. Solve the following ordinary differential equation.

$$x^2y''' + 2xy'' - 6y' = 0.$$

Let $y(x) = x^r$. Using this we can find that $y' = rx^r, y'' = r(r-1)x^{r-2}, y''' = r(r-1)(r-2)x^{r-3}$. Now let's plug our y's into our ordinary differential equation and solve for r like the following.

$$\begin{aligned} x^2r(r-1)(r-2)x^{r-3} + 2xr(r-1)x^{r-2} - 6rx^{r-1} &= 0 \\ x^{r-1}\left[\left(r^3 - 3r^2 + 2r\right) + \left(2r^2 - 2r\right) - 6r\right] &= 0 \\ r\left[r^2 - 3r + 2 + 2r - 2 - 6\right] &= 0 \\ r\left[r^2 - r - 6\right] &= 0 \\ r(r-3)(r+2) &= 0 \\ &\to r = 0, 3, -2.. \end{aligned}$$

From here we can determine the following.

$$\begin{aligned} y_1(x) &= x^0 = 1 \\ y_2(x) &= x^3 \\ y_3(x) &= x^2. \end{aligned}$$

So the general solution of our ordinary differential equation is

$$y(x) = C_1 + C_2x^3 + \frac{C_3}{x^2}.$$

If r is repeated, then $y_1(x) = x^r, y_2 = x^r \ln x, y_n(\ln x)^n$. If $r = \alpha \pm i\beta$, then we need to set

$$\begin{aligned} y_1(x) &= x^\alpha \cos(\beta \ln x) \\ y_2(x) &= x^\alpha \sin(\beta \ln x). \end{aligned}$$

This gives us two equations,

$$ay'' + by' + cy = 0 \qquad\qquad ax^2y'' + bxy + cy = 0.$$

Let's take a look at an example for this.

Example. Consider the equation $4x^2y'' + y = 0$

1. Let $y(x) = e^r$, after taking the derivatives we can see that
$$\begin{aligned} x^r[4r(r-1)+1] &= 0 \text{ for all } x\text{'s} \\ 4r^2 - 4r + 1 &= 0 \\ (2r-1)^2 = 0 \to r &= \frac{1}{2}, \frac{1}{2}. \end{aligned}$$
This gives us that our general solution is
$$C_1 x^{\frac{1}{2}} + C_2 x^{\frac{1}{2}} \ln x, x > 0.$$

2. Let's consider a different approach.
$$\begin{aligned} x^r[4r(r-1)+17] &= 0 \\ 4r^2 - 4r + 17 &= 0 \\ (2r-1)^2 &= 1 - 17 \\ |2r-1| &= 4i \\ 2r - 1 &= \pm_4 i \\ r &= \frac{1}{2} \pm 2i. \end{aligned}$$
This tells us that the general solution is
$$y(x) = C_1 x^{\frac{1}{2}} \cos(2\ln x) + C_2 x^{\frac{1}{2}}(2\ln x), x > 0.$$

The equation $ax^2 \frac{d^2y}{dt^2} + bx\frac{dy}{dt} + cy = 0$ can be transformed via $x = e^t, t = \ln x$ to
$$a\frac{d^2y}{dt^2} + (b-a)\frac{dy}{dt} + cy = 0.$$
Let's take a look at another example.

Example. Consider the following ordinary differential equation,
$$\rho\frac{d^2u}{d\rho^2} + 2\frac{du}{d\rho} = 0.$$
Let $u(a) = A, u(b) = B$. We can see that $u(\rho) = \rho^r, u' = r\rho^{r-1}, u'' = r(r-1)\rho^{r-2}$. Now in order to solve this ordinary differential equation, we need to determine the roots of r, and we need to solve for both C_1, C_2.
$$\begin{aligned} \rho^{r-1}[r(r-1)+2r] &= 0 \\ r^2 - r + 2r &= 0 \\ r^2 + r &= 0 \\ r(r+1) &= 0 \\ r &= 0, 1. \end{aligned}$$
Now, we can determine that $u(\rho) = C_1 + C_2\rho^{-1}$ as long as $\rho > 0$. We can now use the following system to solve for C_1, C_2,
$$\begin{cases} \rho(a) = C_1 + \frac{C_2}{a} = A & C_2 = \frac{A-B}{\frac{1}{a}-\frac{1}{b}} = \frac{ab(B-A)}{a-b} \\ \rho(b) = C_1 + \frac{C_2}{b} = B & C_1 = \frac{aA-bB}{a-b} \end{cases}$$

Let's take a look at another example.

Example. Consider the ordinary differential equation

$$x^2y'' - 2xy' + 2y = -x, \text{ where } x > 0.$$

From here, we need to put our equation into standard form, so it looks like

$$y'' - \frac{2}{x}y' + \frac{2}{x^2}y = -\frac{1}{x}, \text{ where } -\frac{1}{x} = f(x).$$

The first thing we need to do now is solve for $y_h(x)$, which means we have to do the following

$$\begin{aligned} x^2y'' - 2xy' + 2y &= 0 \\ r &= 2, 1 \\ y_1(x) &= x^2 \\ y_2(x) &= x \\ y_h(x) &= C_1x^2 + C_2x. \end{aligned}$$

Now we are going to use variation of parameters to solve for C_1, C_2. In order to do this, let $y(x) = v_1(x)x^2 + v_2(x)x$. We can either use a linear system to solve this, or we can use the wronskian. Using the linear system, we end up with

$$\begin{cases} x^2v_1' + xv_2' = 0 \\ 2xv_1' + 1v_2' = -\frac{1}{x} \end{cases}$$

If we were to use the wronskian we would do the following

$$\begin{aligned} w(x) &= \begin{vmatrix} x^2 & x \\ 2x & 1 \end{vmatrix} = x^2 - 2x^2 = -x^2 \\ v_1' &= \frac{\begin{vmatrix} 0 & x \\ -\frac{1}{x} & 1 \end{vmatrix}}{-x^2} = -\frac{1}{x^2} \\ v_2' &= \frac{\begin{vmatrix} x^2 & 0 \\ 2x & -\frac{1}{x} \end{vmatrix}}{-x^2} = -\frac{x}{x^2} = \frac{1}{x} \\ v_1 &= \int -\frac{1}{x^2}dx = \frac{1}{x} + C_1 \\ v_2 &= \int \frac{1}{x}dx = \ln|x| + C_2. \end{aligned}$$

So $y(x) = v_1x^2 + v_2x$. After simplification, we end up with

$$y(x) = C_1x^2 + C_3x + x\ln|x|.$$

3.6.2 Taylor and Frobenius Series Solutions to Linear Ordinary Differential Equations

This is a backup strategy if the method of lucky guess does not work. In this strategy, the main point is to let

$$y(x) = \sum_{n=0}^{\infty} C_n(x - x_0)^n.$$

Our goal is to use our initial conditions to solve the ordinary differential equation and find C_n's. Sometimes we need a more generalized series, and that is when we are going to use the frobenius series. Here is the general pathway through one of these problems.

$$\begin{aligned} y(x) &= (x - x_0)^r \sum_{n=0}^{\infty} C_n (x - x_0)^n \\ x_0 &\to 0 \\ y(x) &= \sum_{n=0}^{\infty} C_n x^{n+r}. \end{aligned}$$

Given an ordinary differential equation with initial conditions, $y(0) = y_0, y'(0) = v_0$, let

$$\begin{aligned} y(x) &= C_0 + C_1 x + C_2 x^2 + C_3 x^3 \\ y'(x) &= C_1 + 2C_2 x + 3C_3 x^2. \end{aligned}$$

Now we apply our initial conditions, which tell us the first two terms in the solution. There are three main ordinary differential equations that we need to know that force us to use the Taylor or Frobenius series solutions,

- Legendre's ODE

$$(1 - t^2)x'' - 2tx' + \lambda(\lambda + 1)x = 0.$$

- Bessel's ODE

$$t^2 x'' + tx' + (t^2 - v^2)x = 0.$$

- Airy's ODE

$$x'' - tx = 0.$$

Theorem 7. Consider the equation

$$x'' + p(t)x' + q(t)x = 0,$$

where $f(x) = 0$. A singular point of the ordinary differential equation is a value of t where p, q, or f is not analytic (cannot be written as Taylor series). If we attempt to solve a singular point ODE, we will end up dividing by 0. An ordinary point are non-singular points.

Consider Bessel's ordinary differential equation, $x'' + \frac{1}{t}x' + \left(1 - \frac{2^2}{t^2}\right) = 0$ This equation has a singular point of $t = 0$. A point is regular if tp (in this case $t - t_0$) and $t^2 q$ $\left((t - t_0)^2\right)$ are analytic. Otherwise, they are irregular points. In the case of Bessel's ordinary differential equation, there is a regular singular point.

Theorem 8. If you expect a Taylor series at $t = t_0$ (ordinary point) and the radius of convergence is at least as big as the distance to the nearest singular point (can be complex), then there exist 2 linearly independent series solutions. If $t = t_0$ is a singular point, you need to use a frobenius series.

Example. Consider Airy's ordinary differential equation, $x'' - t = 0$. The two linearized solutions are $A_i(t)$ and $B_i(t)$. We are given the initial conditions $x(0) = 1, x'(0) = -2$. From here, let $x(t) = 1 - 2t + C_2t^3 + C_4t^4$, this gives us that

$$\begin{aligned} x'(t) &= -2 + 2C_2t + 3C_3t^2 + 4C_4t^3 \\ x'' &= C_2 + 6C_3t + 12C_4t^2. \end{aligned}$$

If we plug the equations above into Airy's equation, we get

$$\left[2C_2 + 6C_3t + 12C_4t^2 + \ldots\right] - t\left[1 - 2t + C_2t^2 + C_3t^3 + C_4t^4 + \ldots\right] = 0.$$

We can pull out specific values in order to solve for C_2, C_3, C_4, like the following

$$\begin{aligned} 1 &: 2C_2 = 0 && \to C_2 = 0 \\ t &: 6C_3 - 1 = 0 && \to C_3 = \frac{1}{6} \\ t^2 &: 12C_4 + 2 = 0 && \to C_3 = -\frac{1}{6}. \end{aligned}$$

From here, we can determine that our general solution is

$$\boxed{x(t) = 1 - 2t + \frac{1}{6}t^3 - \frac{1}{6}t^4 + \ldots}$$

For general initial conditions $(x(0) = C_0, x'(0) = C_1)$.

$$\begin{aligned} & x'' - tx \\ & [2C_2 + 6C_3t + 12C_4t^2 + \ldots] - t[C_0 + C_1t + C_2t^2 + \ldots] \\ & . \end{aligned}$$

$$\begin{aligned} 1 &: 2C_2 = 0 && \to C_2 = 0 \\ t &: 6C_3 - C_0 = 0 && \to C_3 = \frac{1}{6}C_0 \\ t^2 &: 12C_4 - C_1 = 0 && \to C_4 = \frac{1}{12}C_1 \\ . \end{aligned}$$

This would make our particular solution

$$x(t) = C_0 + C_1t + \left(\frac{1}{6}C_0\right)t^3 + \left(\frac{1}{12}C_1\right)t^4$$
.

Another important skill that plays in with Taylor series ordinary differential equations is re-indexing. This is basically just an u-substitution for a series. Our goal is to rewrite

$$\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2}.$$

In order to have t^k, we need to let $k = n-2 \to n = k+2$, which gives us a series that looks like

$$\sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2}t^k.$$

Let's take a look at re indexing when it comes to Airy's equation.

$$x'' - tx = 0.$$

Example.

$$x(t) = \sum_{n=0}^{\infty} C_n t^n$$
$$x'(t) = \sum_{n=1}^{\infty} C_n n t^{n-1}$$
$$x''(t) = \sum_{n=2}^{\infty} C_n n(n-1)t^{n-2} = .$$

We can rewrite Airy's ODE as

$$\sum_{n=2}^{\infty} C_n n(n-1)t^{n-2} - t\sum_{n=0}^{\infty} C_n t^n = 0.$$

Because we have a difference of 3 in the exponents, let $n-2 = k+1, n = k+3$ or $k = n-3$. This makes our Taylor series

$$\sum_{k=-1}^{\infty} C_{k+3}(k+3)(k+2)t^{k+1} - \sum_{n=0}^{\infty} C_n t^{n+1} = 0.$$

Now if we choose another variable m and take both k, n and put them in terms of m, we get

$$C_2(2)(1)t^0 + \sum_{m=0}^{\infty} \left[C_{m+3}(m+3)(m+2) - C_m\right] t^{m+1} = 0.$$

From here, we can solve for C_2 using $2C_2 = 0$ for the 1's values. This leaves us with

$$m \geq 0 : C_{m+3}(m+3)(m+2) - C_m = 0.$$

Our general recursion relation is

$$C_{m+3} = \frac{C_m}{(m+3)(m+2)}.$$

If we follow our recursion relationship

$$
\begin{aligned}
m = 0: &\qquad C_3 = \frac{C_0}{3 \times 2} = \frac{1}{6} C_0 \\
m = 1: &\qquad C_4 = \frac{C_1}{4 \times 3} = \frac{1}{12} C_1 \\
m = 2 &\qquad C_5 = \frac{C_2}{5 \times 4} = 0 \\
m = 3: &\qquad C_6 = \frac{C_3}{6 \times 3} = \frac{1}{18} C_3 = \frac{1}{18} \times \frac{1}{6} C_0 = \frac{1}{108} C_0.
\end{aligned}
$$

Every third coefficient is 0, all other coefficients can be expressed in terms of C_1, C_2.

$$
\begin{aligned}
x(t) &= C_0 + C_1 t + C_2 t^2 + \dots \\
&= C_0 + C_1 t + \frac{1}{6} C_0 t^3 + \frac{1}{16} c_1 t^4 + \dots \\
&= C_0 \left[1 + \frac{1}{6} t^3 + \frac{1}{108} t^6 + \dots \right] + C_1 \left[t + \frac{1}{12} t^4 + \dots \right].
\end{aligned}
$$

Let's take a look at another example to solidify understanding.

Example. Consider the equation $(x + 2)y'' + 3y' + 4y = 0$. Let $y(x) = \sum_{n=0}^{\infty} C_n x^n$. This makes our differential

$$
(x + 2) \sum_{n=2}^{\infty} C_n n(n-1) x^{n-2} + 3 \sum_{n=1}^{\infty} C_n n x^{n-1} + 4 \sum_{n=0}^{\infty} C_n x^n = 0.
$$

Let $n - 2 = k - 1$, which would make $n = k + 1, k = n - 1$. Also let $n = j - 1, j = n + 1$ After changing our values, our ordinary differential equation, our equation is

$$
\sum_{n=2}^{\infty} C_n n(n-1) x^{n-1} + \sum_{k=1}^{\infty} 2C_{k+1}(k+1) k x^{k-1} + \sum_{n=1}^{\infty} 3c_n n x^{n-1} + \sum_{j=1}^{\infty} 4C_{j-1} x^{j-1} = 0.
$$

Now rewrite $n, k, j \to m$. Let $k, n, j = 1$. This makes our differential

$$
\begin{aligned}
0 =& 2C_2 \times 2 \times 1 \times x^0 + 3C_1 \times 1 \times x^0 + 4C_0 \times x^0 \\
&+ \sum_{m=2}^{\infty} \left[C_m \left[m(m-1) + 3m \right] + 2C_{m+1}(m+1)m + 4C_m - 1 \right] x^m.
\end{aligned}
$$

After simplification, we can pull out like terms to solve for C_2 and the recursion relation.

$$
\begin{aligned}
1: 4C_2 + 3C_1 + 4C_0 = 0 &\qquad \to C_2 = \frac{-3C_1 - 4C_0}{4} \\
x^{m-1}: C_m(m^2 + 2m) + 2C_{m+1}(m+1)m + 4C_m - 1 = 0.
\end{aligned}
$$

Our general recursion relation as long as $m \geq 2$ is

$$
C_{m+1} = \frac{-m(m+2)C_m - 4C_m - 1}{m(m+1)}.
$$

If we set $m = 2$, we can solve and get results for C_1, C_2. One approach to finding two linearly independent solutions is to set $y(x) = C_0 + C_1 x + C_2 x^2$, and then use the values calculated from C_{m+1} to get

$$
\begin{aligned}
&= C_0 + C_1 x + \left(-\frac{3}{4}C_1 - C_0\right) x^2 + (\text{from above}) x^3 \\
&= C_0 \left[1 - x^2 + \square x^3 + \ldots\right] + C_1 \left[x - \frac{3}{4}x^2 + 1\right] x^3.
\end{aligned}
$$

An alternative for finding two linearly independent solutions is solving with $C_0 = 1, C_1 = 0$. This makes our initial conditions $x(0) = 1, x'(0) = 0$. This makes our $C_2 = -1, C_3 = \frac{2}{3}, C_4 = -\frac{1}{4}$. We also need to solve with the initial conditions $x(0) = 0, x'(0) = 1$. This makes our solutions

$$
\begin{aligned}
y_1(x) &= 1 + 0x - 1x^2 + \frac{2}{3}x^3 - \frac{1}{4}x^4 \\
y_2(x) &= 0 + 1x - \frac{3}{4}x^2 + \frac{1}{6}x^3 + \frac{1}{48}x^4.
\end{aligned}
$$

The general solution is

$$
y(x) = C_1 y_1(x) + C_2 y_2(x),
$$

where $C_1 = x(0), C_2 = x'(0)$. A Frobenius series is when you expand around a singular point (a point that is hard for the equation). Consider Bessel's ordinary differential equation.

Example.

$$
x^2 y'' + x y' + \left(x^2 - \frac{1}{4}\right) y = 0 \text{ with } \nu = \frac{1}{2}.
$$

Here we are going to let $y(x) = \sum_{n=0}^{\infty} C_n x^{n+r}$, and our goal is to find n and c_n. We can easily find that

$$
\begin{aligned}
y' &= \sum_{n=0}^{\infty} C_n (n+r) x^{n+r-1} \\
y'' &= \sum_{n=0}^{\infty} C_n (n+r)(n+r-1) x^{n+r-2}.
\end{aligned}
$$

Our ordinary differential equation written out is

$$
\sum_{n=0}^{\infty} C_n (n+r)(n+r-1) x^{n+r} + \sum_{n=0}^{\infty} C_n (n+r) x^{n+r} + \sum_{n=0}^{\infty} C_n x^{n+r+2} - \sum_{n=0}^{\infty} \frac{1}{4} C_n x^n = 0.
$$

Let $k + r = n + r + 2$. This gives us that

$$
\sum_{n=0}^{\infty} C_n \left[(n+r)(n+r-1) + (n+r) - \frac{1}{4}\right] x^{n+r} + \sum_{k=2}^{\infty} C_{k-2} x^{k+r} = 0.
$$

If we transform the k into an n , we get that

$$\sum_{n=2}^{\infty}\left(C_n\left[(n+r)(n+r-1+1)-\frac{1}{4}\right]+C_n-2\right)x^{n+r}+$$
$$C_0\left(r(r-1)+r-\frac{1}{4}\right)x^r+C_1\left[(r+1)(r)+(r+1)-\frac{1}{4}\right]x^{r+1}.$$

Now we can pull out like terms to create a system of equations to solve for C values:

$$\begin{aligned} x^r: &\qquad r^2-\frac{1}{4}=0 \text{ or } C_0=0\\ x^{r+1}: &\qquad (r+1)^2-\frac{1}{4}=0 \text{ or } C_1=0\\ x^{r+h}: &\qquad C_n=\frac{-C_n-2}{(n+r)^2-\frac{1}{4}}. \end{aligned}$$

If $C_0 \neq 0, r=\pm\frac{1}{2}$ or if $C_1 \neq 0, r+1=\pm\frac{1}{2}$. There are two linearly dependent solutions

- $r=\frac{1}{2}$
- $r=-\frac{1}{2}$

If we take $r=\frac{1}{2}$ and simplify C_n, we can see that

$$C_n=\frac{-C_n-2}{n(n+1)}, n \geq 2.$$

Using this, we can solve for a certain number of terms to get that

$$y_1(x)=x^{\frac{1}{2}}\left[C_0-\frac{1}{3!}C_0x^2+\frac{1}{5!}x^4+\ldots\right].$$

This equation simplifies down to $\frac{C_0}{\sqrt{x}}\sin(x)$.

4 Systems of Ordinary Differential Equations

4.1

Systems of ordinary differential equations have 1 dependent variable and 1 independent variable. Let's look at an example.

Example. Here is a system of 3 first order ordinary differential equations of dimension 3

$$\begin{aligned} x' &= 4x - 7y + z^2 \\ y' &= 2t - e^t y - z \\ z' &= z^2 + y^2 - yx. \end{aligned}$$

We can also turn higher order ODE's into systems of first-order ODEs.

Example. Consider the equation $x'' + 7x'' + 7tx' + 6x = t^2$. If we let $y = x', z = x'' = y', x''' = z'$, then using the ODE,

$$z' = t^2 - 6x + 7ty - 7z,$$

we can make a system of nonlinear, nonautonomous, nonhomogeneous ordinary differential equations.[2]

$$\begin{aligned} x' &= y \\ y' &= z \\ z' &= t^2 - 6x + ty - 7z. \end{aligned}$$

4.2 Matrix Algebra

A matrix is a rectangular array of numbers.

Example. Consider $A = \begin{pmatrix} 2 & 3 & 1 \\ 4 & 5 & 0 \end{pmatrix}$. A has 2 rows and 3 columns, so its size is 2×3. The a_{ij} is an entry in A's i^{th} row and j^{th} column.

Example. Consider $B = \begin{pmatrix} 5 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$, which is a 3×3 matrix. This is also a square matrix, due to the size. B is also a diagonal matrix, because it's only nonzero entries are on the main diagonals.

An upper triangular matrix looks like

$$\begin{pmatrix} 5 & 2 & 0 \\ 0 & -1 & 3 \\ 0 & 0 & 3 \end{pmatrix}.$$

[2] After this system we went to chapter 2.6

A lower triangular matrix looks like

$$\begin{pmatrix} 5 & 0 & 0 \\ \pi & -1 & 0 \\ -1 & -\frac{1}{2} & 3 \end{pmatrix}.$$

We can add and subtract matrices element-wise, but only if the matrices have the same size. We can also multiply a matrix by a scalar

$$3\begin{pmatrix} 4 & 5 \\ 6 & 7 \end{pmatrix} = \begin{pmatrix} 12 & 15 \\ 18 & 21 \end{pmatrix}.$$

A row vector is $\begin{pmatrix} 2 & 3 & 1 \end{pmatrix}$, in this case the size would be 1×3. A column vector looks like $\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$ with the size 3×1. The zero vector is of size $n \times n$, and has 0 in every index in the matrix. The transpose of a matrix reverses all of the columns and rows. Consider

$$\begin{pmatrix} 2 & 3 & 1 \\ 4 & 5 & 6 \end{pmatrix}^T = \begin{pmatrix} 2 & 4 \\ 3 & 5 \\ 1 & 6 \end{pmatrix}.$$

Here are the properties of a transpose:

- $\left(A^T\right)^T = A$
- $(A + B)^T = A^T + B^T$
- $(kA)^T = kA^T$
- $(AB)^T = B^T A^T$

Now we are going to start talking about matrix multiplication. Matrix multiplication is like the dot product. The $(i, j)^{th}$ entry of AB is the dot product of the i^{th} row of A w/ j^{th} column of B. Consider the following

$$\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = (1 \times 4) + (2 \times 5) + (3 \times 6) = 32.$$

In order to do matrix multiplication, rows of A must match the columns of B and vice versa. Matrix multiplication is not commutative.

Example. If we let $A = \begin{pmatrix} 1 & 4 \\ 5 & 10 \\ 8 & 12 \end{pmatrix}$ and $B = \begin{pmatrix} -4 & 7 & -3 \\ 1 & -3 & 2 \end{pmatrix}$. Does $AB = BA$? This is not true, because matrices are equal if and only if all entries are equal. If we just multiply BA, we get

$$\begin{pmatrix} -4 & 7 & -3 \\ 1 & -3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 4 \\ 5 & 10 \\ 7 & 12 \end{pmatrix} = \begin{pmatrix} 2 & 8 \\ 2 & -2 \end{pmatrix}.$$

Here are some properties of matrices.

Note. We must maintain the order from left to right.

- $A(B+C) = AB + BC$
- $(A+B)C = AC + BC$
- $k(AB) = (kA)B = A(kB)$
- $ABC = (AB)C = A(BC)$

The identity matrix is a square diagonal matrix with ones on the diagonal. For example, the identity matrix of dimension 3 and 4 are the following

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

With the identity matrix, if we multiply any matrix by the identity matrix, then we get the same answer. Back in lesson 2, we were given the general solution to a Cauchy-Euler ordinary differential equation as

$$x(t) = C_1(1) + C_2t^3 + C_3\frac{1}{t^2}.$$

Let's find a particular solution satisfying $x(1) = 2, x'(1) = 4, x'(1) = 0$. We can do this by taking the derivative and second derivative of our functions to get

$$x'(t) = 3C_2t^2 - 2C_3t^{-3}$$
$$x''(t) = 6C_2t + 6C_3t^{-4}.$$

Now using our initial conditions

$$x(1) = C_1 + C_2 + C_3 = 2$$
$$x'(1) = 3C_2 - 2C_3 = 4$$
$$x''(1) = 6C_2 + 6C_3 = 0.$$

We can rewrite this in matrix form as

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & -2 \\ 0 & 6 & 6 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}.$$

We will call the first matrix, A, the second matrix, $\vec{x}$, and the third matrix, $\vec{b}$, so we must solve $A\vec{x} = \vec{b}$ for $\vec{x}$. We can solve this by doing the following techniques

1. Cramer's Rule (See book appendix)
2. Gaussian elimination
3. (Left)-multiply by A^{-1}
4. MATLAB: rref or A/B

Theorem 9. This is the invertible matrix theorem. If A is square, either all of case 1 or all of case 2 statements are true. Either A is non-singular [good] or A is singular [bad].
Considering case 1

- $det(A) \neq 0$
- $A\vec{x} = \vec{b}$ has exactly one solution $\vec{x}$ for each $\vec{b}$.
- $A\vec{x} = \vec{0}$ has only the trivial solution, $\vec{x} = \vec{0}$.
- The rows (and columns) are linearly independent.
- A^{-1} exists.

Considering Case 2

- $det(A) = 0$
- $A\vec{x} = \vec{b}$ has zero or infinite solutions
- $A\vec{x} = \vec{0}$ has infinite solutions.
- The rows (and columns) are linearly dependent.
- A^{-1} does not exist.

A^{-1} is a square matrix such that

$$A^{-1}A = AA^{-1} = I.$$

We find A^{-1} through the following methods

1. $rref(A|I) \to (I|A^{-1})$
2. $A^{-1} = \frac{1}{det(A)} \cdot adj(A)$ where $adj(A)$ is the transpose of the cofactor matrix.

How do we use A^{-1}? Given $A\vec{x} = \vec{b}$, we must solve for x. In order to do this, we left multiply by A^{-1} to get

$$\begin{aligned} A^{-1}A\vec{x} &= A^{-1}\vec{b} \\ I\vec{x} &= A^{-1}\vec{b} \\ \vec{x} = A^{-1}\vec{b}. \end{aligned}$$

If given a general 2×2 matrix, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we can find the inverse by doing

$$\begin{aligned} A^{-1} &= \frac{1}{ad - bc}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}^T \\ &= \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. \end{aligned}$$

The trace of A ($trace(A)$) is the sum along the diagonal.
Let's write a linear system of first order differential equations in matrix form. If the equation is not first order, use Sect (4.1) technique to turn into 1st order system.

Example. Consider the following system

$$\begin{aligned} x' &= x - y + z \\ y' &= 4x + y - 5z + 7 \\ z' &= 7x - 8y - 9z. \end{aligned}$$

The $+7$ in the second equation makes this a nonhomogeneous ordinary differential equation with constant coefficients. We can rewrite this as $\vec{x}'(t) = A(t)\vec{x} + \vec{b}(t)$, where $\vec{x}(t) =$ a vector of dependent variables. In order to do this, let $\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$. Using the equation from before, we can set

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 4 & 1 & -5 \\ 7 & -8 & -9 \end{bmatrix} + \begin{bmatrix} 0 \\ 7 \\ 0 \end{bmatrix}.$$

We can notice that all of our nonhomogeneous terms get moved into the $\vec{b}(t)$ matrix.

Note. We might have $\begin{bmatrix} c \end{bmatrix} \vec{x}' = A\vec{x} + \vec{b}$. From here we would need to do the following

$$\begin{aligned} c^{-1}c\vec{x}' &= c^{-1}(A\vec{x} + \vec{b}) \\ \vec{x}' &= c^{-1}A\vec{x} + c^{-1}\vec{b}. \end{aligned}$$

Example. Consider the following ordinary differential equation and put it into matrix form.

$$\begin{aligned} x'' + 4x' - 4x + 4y &= 0 \\ y'' - 4y' + 5y - 2x' + 3x &= \sin(t). \end{aligned}$$

In order to solve this we need to let $u = x'$, which makes $u' = x''$, and we need to let $v = y'$, which lets $u' = -4x' + 4x - 4y$ and $v' = y'' = 4y' - 5y + 2u - 3x + \sin(t)$. This makes our system

$$\begin{aligned} u' &= -4u + 4x - 4y \\ v' &= 4v - 5y + 2u - 3x + \sin(t) \\ x' &= u \\ y' &= v. \end{aligned}$$

Now we can turn this into our $\vec{x'} = A\vec{x} + \vec{b}$. This makes our equation

$$\begin{bmatrix} u' \\ v' \\ x' \\ y' \end{bmatrix} = \begin{bmatrix} -4 & 0 & 4 & -4 \\ 2 & 4 & -3 & -5 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \sin(t) \\ 0 \\ 0 \end{bmatrix}.$$

4.3 Eigenvalues and Eigenvectors of Square Matrices

We are going to be using λ for the eigenvalues and $\vec{v}$ for the eigenvectors.

Definition 6. λ and $\vec{v}$ are an eigenvalue/eigenvector of A if

$$A\vec{v} = \lambda\vec{v}, v \neq 0.$$

Our steps to find the eigenvector are the following:

1. Find λ's with $det(A - \lambda I) = 0$ (polynomial in λ).
2. For each λ find its eigenvector $\vec{v}$ using $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = 0$

We need to make sure that $\vec{v}$ is a non-zero solution to the equation. This means that we will have infinite solutions because every multiple of an eigenvector is also an eigenvector.

Example. For $A = \begin{bmatrix} 4 & 2 \\ 5 & 1 \end{bmatrix}$, find eigenvalues and eigenvectors. In order to find the eigen vector we take the deteminant of $A - \lambda I$ and set it equal to zero.

$$\begin{aligned} det(\begin{bmatrix} 4 & 2 \\ 5 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) &= 0 \\ \begin{vmatrix} 4-\lambda & 2 \\ 5 & 1-\lambda \end{vmatrix} &= 0 \\ (4-\lambda)(1-\lambda) - 10 &= 0 \\ 4 - 5\lambda + \lambda^2 - 10 &= 0 \\ \lambda^2 - 5\lambda - 6 &= 0 \\ \lambda &= 6, -1. \end{aligned}$$

Now we need to solve for the eigenvectors. First, let $\lambda = 6$. Let $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$.

$$\begin{aligned} \begin{bmatrix} 4-6 & 2 \\ 5 & 1-6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} -2 & 2 \\ 5 & -5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \end{aligned}$$

If we put this back into equation form we get

$$\begin{aligned} &-2v_1 + 2v_2 = 0 \\ &5v_1 - 5v_2 = 0 \\ &\rightarrow v_1 = v_2. \end{aligned}$$

So this means that $v_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvalue for $\lambda = 6$ for all $v_2 \neq 0$. Let's now solve for $\lambda = 1$:

$$\begin{aligned} \begin{bmatrix} 5 & 2 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ 5v_1 - 1v_2 = 0 \rightarrow v_2 &= \frac{-5}{2}v_1. \end{aligned}$$

This means that any multiple of the vector $\begin{bmatrix} 1 \\ -\frac{5}{2} \end{bmatrix}$ is an eigenvector for $\lambda = 1$.

Let's show that if $\vec{v}$ is an eigenvector for A with λ, then $c\vec{v}$ is an eigenvector for A with λ if $c \neq 0$.. We know that $A\vec{v} = \lambda\vec{v}$. Is $A(c\vec{v}) = \lambda(c\vec{v})$? Yes. Let $X(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$. If we solve for $\vec{X}' = A\vec{x}$,

$$\begin{aligned} \begin{bmatrix} x \\ y \end{bmatrix}' &= \begin{bmatrix} 4 & 2 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ x' &= 4x + 2y \\ y' &= 5x + y. \end{aligned}$$

We can now write down our solutions as

$$\begin{aligned} \vec{X}_1(t) &= e^{6t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} e^{6t} \\ e^{6t} \end{bmatrix} \\ \vec{X}_2(t) &= e^{-t} \begin{bmatrix} -2 \\ 5 \end{bmatrix} = \begin{bmatrix} -2e^{-t} \\ 5e^{-t} \end{bmatrix}. \end{aligned}$$

This means that we can determine that

$$\vec{X}(t) = c_1\vec{x}_1 + c_2\vec{x}_2$$

is also a solution for all c_1, c_2. This means that our general solution is going to be

$$\vec{X} = c_1 e^{6t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -2 \\ 5 \end{bmatrix}.$$

We can also take our c_1, c_2 and distribute them to get

$$\vec{X}(t) = \begin{bmatrix} c_1 e^{6t} - 2c_2 e^{-2} \\ c_2 e^{6t} + 5c_2 e^{-t} \end{bmatrix},$$

with $x(t)$ being the top equation and $y(t)$ being the bottom equation. We can verify the solutions in order to convince ourselves that this works. We can verify by using our $\begin{bmatrix} -2e^{-t} \\ 5e^{-t} \end{bmatrix}$. Our left hand side of the equation is going to be

$$\vec{X}_2' = -e^{-t} \begin{bmatrix} -2 \\ 5 \end{bmatrix} = e^{t} \begin{bmatrix} 2 \\ -5 \end{bmatrix}.$$

Our right hand side of the original equation is going to be

$$A\vec{X}_2 = \begin{bmatrix} 4 & 2 \\ 5 & 1 \end{bmatrix} e^{t} \begin{bmatrix} -2 \\ 5 \end{bmatrix}.$$

After simplification of this equation, we get that the result is $e^{t} \begin{bmatrix} 2 \\ -5 \end{bmatrix}$.

Example. Find the particular solution satisfying $x(0) = 2, y(0) = -2$ for the general solution above. Now we just go through the following steps.

$$\begin{aligned} \vec{X}(0) = \begin{bmatrix} 2 \\ -2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 5 \end{bmatrix} \\ 2 = c_1 - 2c_2 \\ -2 = c_1 + 5c_2. \end{aligned}$$

Now we just solve for c_1, c_2 to get $c_1 = \frac{6}{7}, c_2 = -\frac{4}{7}$, so our general solution is

$$\vec{X}(t) = \frac{6}{7}e^{6t}\begin{bmatrix}1\\1\end{bmatrix} - \frac{4}{7}e^{-t}\begin{bmatrix}-2\\5\end{bmatrix}.$$

How can we verify that two solutions are linearly independent[3]? We can solve this by using the wronskian. For the last equation we can check this by doing

$$w(t) = \begin{vmatrix}e^{6t} & -2e^{-t}\\ e^{6t} & 5e^{-t}\end{vmatrix} = 5e^{5t} + 2e^{5t} \neq 0.$$

Because the wronskian does not equal zero, the solutions are linearly independent.

Example.

1. Is $\vec{v} = \begin{bmatrix}3\\2\end{bmatrix}$ an eigenvector for $A = \begin{bmatrix}2 & 3\\2 & 1\end{bmatrix}$. This can be tested by checking to see if $A\vec{v} = \lambda\vec{v}$ for some λ?

   $$A\vec{v} = \begin{bmatrix}2 & 3\\2 & 1\end{bmatrix}\begin{bmatrix}3\\2\end{bmatrix} = \begin{bmatrix}12\\8\end{bmatrix} = \lambda\begin{bmatrix}3\\2\end{bmatrix}.$$

   This is true for all $\lambda = 4$

2. Is $\lambda = -1$ and eigenvalue for $A = \begin{bmatrix}2 & 3\\2 & 1\end{bmatrix}$? We can check this by asking if $det(A - (-1)I) = 0$?

   $$\begin{vmatrix}2+1 & 3\\2 & 1+1\end{vmatrix} = \begin{vmatrix}3 & 3\\2 & 2\end{vmatrix} = 6 - 6 = 0.$$

   Because this determinant equals zero, $\lambda = -1$ is an eigenvalue of A

For repeated (real) eigenvalues, we can have many different options that can come out. For a $2 \times 2, \lambda_1 = \lambda_2$. Here are our options

1. Only one eigenvector (and its multiples).
2. 2 linearly independent eigenvectors.

For option 2, an example would be

$$\vec{X}_1 = e^{\lambda t}\begin{bmatrix}2\\3\end{bmatrix}$$
$$\vec{X}_2 = e^{\lambda t}\begin{bmatrix}4\\0\end{bmatrix}.$$

Our general solution is going to be

$$\vec{X}(t) = C_1\vec{X}_1 + C_2\vec{X}_2 = e^{2t}\begin{bmatrix}2C_1 + 4C_2\\3C_1\end{bmatrix}.$$

This means that every vector is an eigenvector. This only occurs if our eigenvector is a multiple of the identity matrix.

[3]This is a 4.4 idea but we are not quite there yet.

Example. Find all eigenpairs of the matrix $A = \begin{bmatrix} 4 & 5 \\ -2 & 6 \end{bmatrix}$. First we need to find the eigenvalues by doing

$$\begin{aligned} \begin{vmatrix} 4-\lambda & 5 \\ -2 & 6-\lambda \end{vmatrix} &= (4-\lambda)(6-\lambda)+10 \\ &= 24 - 10\lambda + \lambda^2 + 10 \\ &= \lambda^2 - 10\lambda + 34 = 0. \end{aligned}$$

We can solve this by completing the square by doing

$$\begin{aligned} (\lambda^2 - 10\lambda + 25) &= -34 + 25 \\ (\lambda-5)^2 &= -9 \\ |\lambda - 5| &= 3i \\ \lambda - 5 &= \pm 3i \\ \lambda &= 5 \pm 3i. \end{aligned}$$

Now we need to find the eigenvectors by solving $A\vec{v} = \lambda\vec{v}$ or doing $(A - \lambda I)\vec{v} = \vec{0}$. Now solving for $\lambda_1 = 5 + 3i$.

$$\begin{aligned} \begin{bmatrix} 4-(5+3i) & 5 \\ -2 & 6-(5+3i) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} -1-3i & 5 \\ -2 & 1-3i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ (-1-3i)v_1 + 5v_2 &= 0 \\ -2v_1 + (1-3i)v_2 &= 0. \end{aligned}$$

The two last equations above are equivalent so we can work with either of them, we are going to be working with $5v_2 = (1+3i)v_1$. If we choose $v_1 = 5, v_2 = 1+\lambda i$ we can solve $v_2 = \frac{1+3i}{5}v_1$ for all v_1.

Example. Show that $\vec{v} = \begin{bmatrix} 2+2i \\ -1 \end{bmatrix}$ is an eigenvector of $A = \begin{bmatrix} 2 & 8 \\ -1 & -2 \end{bmatrix}$ We can set $A\vec{v} = \lambda\vec{v}$ for some λ to get

$$\begin{bmatrix} 2 & 8 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 2+2i \\ -1 \end{bmatrix} = \begin{bmatrix} 4+4i-8 \\ -2-2i+3 \end{bmatrix} = \begin{bmatrix} 4i-4 \\ -2i \end{bmatrix} \stackrel{?}{=} \lambda \begin{bmatrix} 2+2i \\ -1 \end{bmatrix}.$$

We can check to see if $\lambda = 2i$ by doing

$$\begin{aligned} 2i = (2+2i) &\stackrel{?}{=} 4i - 4 \\ 4i - 4 &= 4i - 4. \end{aligned}$$

This means that $\vec{v}$ is an eigenvector for A with $\lambda = 2i$.

How can we turn a complex eigenpair of A into a solution of $\vec{x}' = A\vec{x}$? Last time we found that $\lambda_1 = 5 + 3i$ had $\vec{v}_1 = \begin{bmatrix} 5 \\ 1+3i \end{bmatrix}$ or $\begin{bmatrix} 1-3i \\ 2 \end{bmatrix}$

Note. We know that $\vec{x}_1(t) = e^{(5+3i)t}\begin{bmatrix} 5 \\ 1+3i \end{bmatrix}$ is a solution, and so is $\vec{x}_1(t) = e^{5-3i)t}\begin{bmatrix} 5 \\ 1-3i \end{bmatrix}$ is a solution, so

$$\vec{x}(t) = C_1\vec{x}_1(t) = C_2\vec{x}_2(t).$$

How are we able to turn these into real-valued solutions? The real and imaginary parts of a solution must also be solutions. We can do the following calculations

$$\begin{aligned}
\vec{x}_1(t) &= e^{5t}e^{3it}\begin{bmatrix} 5 \\ 1+3i \end{bmatrix} \\
&= e^{5t}\left(\cos(3t) + i\sin(3t)\right)\begin{bmatrix} 5 \\ 1+3i \end{bmatrix} \\
&= e^{5t}\begin{bmatrix} 5\cos(3t) + 5i\sin(3t) \\ \cos(3t) + i\sin(3t) + 3i\cos(3t) - 3\sin(3t) \end{bmatrix} \\
&= e^{5t}\begin{bmatrix} 5\cos(3t) \\ \cos(3t) - 3\sin(3t) \end{bmatrix} + ie^{5t}\begin{bmatrix} 5\sin(3t) \\ 3\cos(3t) + \sin(3t) \end{bmatrix}.
\end{aligned}$$

The two sides of the addition are considered $\vec{x}_3(t), \vec{x}_4(t)$ respectively. (x_4 does not include the i). This makes our general solution

$$\vec{x}(t) = C_3\vec{x}_3(t) + C_4\vec{x}_4(t).$$

Example. Find the eigenvalues of $A = \begin{bmatrix} 2 & -7 & 0 \\ 5 & 10 & 4 \\ 0 & 5 & 2 \end{bmatrix}$. First we need to find the eigenvalues by doing $det(A - \lambda I) = 0$.

$$\begin{aligned}
\begin{vmatrix} 2-\lambda & -7 & 0 \\ 5 & 10-\lambda & 4 \\ 0 & 5 & 2-\lambda \end{vmatrix} &= (2-\lambda)\begin{vmatrix} 10-\lambda & 4 \\ 5 & 2-\lambda \end{vmatrix} + (7)\begin{vmatrix} 5 & 4 \\ 0 & 2-\lambda \end{vmatrix} \\
&= (2-\lambda)[(10-\lambda)(2-\lambda) - 20] + 7.5(2-\lambda) \\
&= (2-\lambda)[20 - 12\lambda + \lambda^2 - 20 + 35] \\
&= (2-\lambda)(\lambda^2 - 12\lambda + 35) \\
&= (2-\lambda)(\lambda-5)(\lambda-7) = 0 \\
\lambda &= 2, 5, 7.
\end{aligned}$$

How can we find a second linearly independent solution to $\vec{x}' = A\vec{x}$ if A only had 1 eigenpair?

Example. Consider the equation $B = \begin{bmatrix} 3*-18 & \\ 2 & -9 \end{bmatrix}$, which has the eigenvalues $\lambda = -3, -3$ with an eigenvector of $\vec{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. We know that $\vec{x}_1 = (t)e^{-3t}\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is one solution. We need to find $\vec{x}_2$. We can say that $\vec{x}_2(t) = te^{\lambda t}\vec{v} + e^{\lambda t}\vec{w}$, where $(B-\lambda I)\vec{w}=\vec{v}$ Let's find our $\vec{w}$.

$$\begin{aligned}\begin{bmatrix}3-(-3) & -18\\ 2 & -9-(-3)\end{bmatrix}\begin{bmatrix}w_1\\ w_2\end{bmatrix} &= \begin{bmatrix}3\\ 1\end{bmatrix}\\ \begin{bmatrix}6 & -18\\ 2 & -6\end{bmatrix}\begin{bmatrix}w_1\\ w_2\end{bmatrix} &= \begin{bmatrix}3\\ 1\end{bmatrix}\\ 6w_1-18w_2 &= 3\\ 2w_1-6w_2 &= 1.\end{aligned}$$

We can choose any of the w_1, w_2 combinations that work. If we choose that $w_2=0, w_1=\frac{1}{2}$. This makes our general solution

$$\vec{x}(t)=Ce^{-3t}\begin{bmatrix}3\\ 1\end{bmatrix}+C_2e^{-3t}\left(t\begin{bmatrix}3\\ 1\end{bmatrix}+\begin{bmatrix}0\\ \frac{1}{2}\end{bmatrix}\right).$$