



Applied Linear Algebra

MATH363

DR. MCKENZIE
GONZAGA UNIVERSITY



EDITED BY
CAMERON WILLIAMSON

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Chapter 1

Linear Algebraic Systems

Chapter 2

Vector Spaces and Bases

§2.1 Real Vector Spaces

A vector space is the abstract reformulation of the quintessential properties of n -dimensional Euclidean Space \mathbb{R}^n , which is defined as the set of all real (column) vectors with n entries. The basic laws of vector addition and scalar multiplication in \mathbb{R}

Definition 2.1.1

A vector space is a set of V equipped with two operations:

- Addition: adding any pair of vectors $v, w \in V$ produces another vector $v + w \in V$;
- Scalar Multiplication: multiplying a vector $v \in V$ by a scalar $c \in \mathbb{R}$ produces a vector $cv \in V$

These are subject to the following axioms, valid for all $u, v, w \in V$ and all scalars $c, d \in \mathbb{R}$:

- Commutativity of Addition: $v + w = w + v$.
- Associativity of Addition: $u + (v + w) = (u + v) + w$.
- Additive Identity: There is a zero element $0 \in V$ satisfying $v + 0 = v = 0 + v$.
- Additive Inverse: For each $v \in V$ there is an element $-v \in V$ such that $v + (-v) = 0 = (-v) + v$.
- Distributivity: $(c + d)v = (cv) + (dv)$, and $c(v + w) = (cv) + (cw)$.
- Associativity of Scalar Multiplication: $c(dv) = (cd)v$.

- Unit for Scalar Multiplication: the scalar $1 \in \mathbb{R}$ satisfies $1\mathbf{v} = \mathbf{v}$.

Theorem 2.1.1

Let V be a vector space.

- $0 \times \vec{V} = \vec{0}$
- $-1\vec{V} = -\vec{V}$
- $c \times \vec{0} = \vec{0}$
- If $c \times \vec{V} = \vec{0}$, then $c = 0$ or $\vec{V} = \vec{0}$

Here are some examples of vector spaces:

- $\mathbb{R}^n = \left\{ \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix} \mid r_1, r_2, r_n \in \mathbb{R} \right\}$
- $M_{m \times n} =$ The m by n matrices over \mathbb{R} .
- $\mathbb{P}^n =$ the polynomials of degree $\leq n$.

Definition 2.1.2

Let V be a vector space over F . $W \leq V$ is a subspace of V if W is a vector space over F under the same operation as V .

An example of definition (2.1.2). Let $V = \mathbb{R}^3 = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\}$. V is a vector space. If we let $W = \left\{ \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} \mid a, b \in \mathbb{R} \right\}$, then W is a subspace of V .

Theorem 2.1.2

Let V be a vector space. Let $W \leq V$. W is a subspace of V if

- $w \neq 0$
- $\forall w_1, w_2 \in W; w_1 + w_2 \in W$
- $\forall c \in F; \vec{w} \in W; c \cdot \vec{w} \in W$

If we were to let $V = \mathbb{R}^3$ for $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$. We can determine the following:

- $\{\vec{0}\}$ is a subspace of V .
- $\left\{ \begin{pmatrix} a \\ 0 \\ c \end{pmatrix} \middle| a, c \in \mathbb{R} \right\}$ is a subspace of V .
- Consider the equation $\left\{ \begin{pmatrix} x \\ x \\ 0 \end{pmatrix} \middle| x \in \mathbb{R} \right\} = W$ Show that W is a subspace of V .
 - $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in W$ so $W \neq 0$.
 - $\begin{pmatrix} x \\ x \\ 0 \end{pmatrix}, \begin{pmatrix} y \\ y \\ 0 \end{pmatrix} \in W$. Then $\begin{pmatrix} x \\ x \\ 0 \end{pmatrix} + \begin{pmatrix} y \\ y \\ 0 \end{pmatrix} = \begin{pmatrix} x+y \\ x+y \\ 0 \end{pmatrix}$
 - $\begin{pmatrix} x \\ x \\ 0 \end{pmatrix} \in W$, then $c \times \begin{pmatrix} x \\ x \\ 0 \end{pmatrix} = \begin{pmatrix} cx \\ cx \\ 0 \end{pmatrix} \in W$.
 - Therefore we know that W is a subspace of V with respect to scalar multiplication and addition.
- $W = \left\{ \begin{pmatrix} x \\ y \\ 2x+3y \end{pmatrix} \middle| x, y \in \mathbb{R} \right\}$. W is a subspace of V .

\mathbb{R}^3 only has 4 kinds of subspaces. \mathbb{R}^3 , $\{\vec{0}\}$, planes passing through the origin and lines that are passing through the origin.

§2.2 Subspaces

Definition 2.2.1

Let I be an interval in \mathbb{R} . Let $\mathbb{F}(I)$ be the vector space of functions $\mathbb{F} = I \rightarrow \mathbb{R}$.

- $\mathbb{C}^0(I)$ = the continuous functions from $I \rightarrow \mathbb{R}$ is a subspace.
- $\mathbb{P}^n(I)$ = polynomials of degree $\leq n$ restricted to $\mathbb{F}(I)$. This is a subspace of $\mathbb{C}^0(I)$.
- $\mathbb{P}^\infty(I)$ = all polynomials on I . This is a subspace of $\mathbb{F}(I)$.
- $\mathbb{C}^n(I)$ = the set of functions $f : I \rightarrow \mathbb{R}$ such that $f', f'' \dots f^{(n)}$ all exist and are continuous.

- $\mathbb{C}^\infty(I)$ = functions from $I \rightarrow \mathbb{R}$ such that f', f'', f''' all exist and are smooth functions.
- $A(I)$ = the functions in $\mathbb{C}^\infty(I)$ such that all $A \in I$, the power series $f(a) + \frac{f'(a)(x-a)}{1!} + \frac{f''(a)(x-a)^2}{2!} + \dots$ converges for all $x \in I$ sufficiently close to a .

Problem 1. Show that $v = \begin{pmatrix} x \\ y \\ 2x+y \end{pmatrix}$ is a subspace of \mathbb{R}^3 .

- Because $\begin{pmatrix} 0 \\ 0 \\ 2(0)+0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ is in v , v is not empty.
- Let $\vec{v}_1 = \begin{pmatrix} x_1 \\ y_1 \\ 2x_1+y_1 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} x_2 \\ y_2 \\ 2x_2+y_2 \end{pmatrix} \in V$

$$v_1 + v_2 = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ 2x_1 + y_1 + 2x_2 + y_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ 2(x_1 + x_2) + (y_1 + y_2) \end{pmatrix} \in V \quad (2.1)$$

so v is closed with respect to addition.

- Let $r \in \mathbb{R}$ and $\begin{pmatrix} x \\ y \\ 2x+y \end{pmatrix} = \vec{v} \in v$

§2.3 Span and Linear Independence

If we let V be a vector space over \mathbb{R} and let $\vec{v}_1, \dots, \vec{v}_n \in V$, then we can determine that the

$$\text{span}(\{\vec{v}_1, \dots, \vec{v}_n\}) = \{c_1\vec{v}_1 + \dots + c_n\vec{v}_n \mid c_1, \dots, c_n \in \mathbb{R}\} \quad (2.2)$$

Proposition 2.3.1

The span of $\{\vec{v}_1, \vec{v}_2\}$ is a subspace of V .

Proof.

$$c_1\vec{v}_1 + \cdots + c_n\vec{v}_n \quad (2.3)$$

$$k_1\vec{v}_1 + \cdots + k_n\vec{v}_n \in \text{span} \quad (2.4)$$

$$(2.5)$$

If we add together both of the equations above we get

$$(c_1 + k_1)\vec{v}_1 + \cdots + (c_n + k_n)\vec{v}_n \in \text{span}. \quad (2.6)$$

$$r(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) \quad (2.7)$$

$$= rc_1\vec{v}_1 + \cdots + rc_n\vec{v}_n \in \text{span} \quad (2.8)$$

■

Problem 2. Let $V \in \mathbb{R}^3$. Also we are going to let

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \text{span}(\vec{v}_1) = \left\{ c \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \middle| c \in \mathbb{R} \right\} \quad (2.9)$$

$\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ is a vector in 3-space. $c \times \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ expands, contracts, changes direction. This is a line which goes through $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

$$c \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \quad (2.10)$$

is in the xy -plane, let's solve for y to find the equation of the line that is drawn by the vector:

$$\begin{pmatrix} c \\ 2c \\ 0 \end{pmatrix} \quad (2.11)$$

$$x = c \quad (2.12)$$

$$y = 2c \quad (2.13)$$

$$\frac{1}{2}y = c \quad (2.14)$$

$$\rightarrow x = \frac{1}{2}y \quad (2.15)$$

$$\rightarrow y = 2x \quad (2.16)$$

$$(2.17)$$

Now we are going to let $\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Consider the span of $(\{\vec{v}_1, \vec{v}_2\})$. The span of $(\{\vec{v}_1, \vec{v}_2\})$ is a plane.

In \mathbb{R}^3 , if $\vec{0} \neq \vec{v} \in \mathbb{R}$, then the $\text{span}\vec{v}$ is a line.

Problem 3. Let $v = \mathbb{P}^2$. v is the set of polynomials of degree $\leq 2 \in \mathbb{R}$.

- $\text{span}(1, x, x^2) = \mathbb{P}^2$
- $\text{span}(4, 2x) = \mathbb{P}^1$, which means all polynomials of degree ≤ 1

Definition 2.3.1

Let v be a vector space. $\vec{v}_1, \dots, \vec{v}_n$ are linearly dependant if there exists c_1, \dots, c_n are not all zero, such that $c_1\vec{v}_1 + \dots + c_n\vec{v}_n = \vec{0}$, otherwise, $\vec{v}_1, \dots, \vec{v}_n$ are linearly independent.

If we let $\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$, we can do a simple test to see if they are linearly independent. We know that $2\vec{v}_1 = \vec{v}_2$, which means that $2\vec{v}_1 + -1\vec{v}_2 = \vec{0}$. Because we can make $\vec{v}_1 + \vec{v}_2$ by using a simple scalar value, these functions are linearly dependent.

Problem 4. Consider the following three matrices

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \vec{v}_3 = \begin{pmatrix} 8 \\ 1 \\ 11 \end{pmatrix} \quad (2.18)$$

Are these matrices linearly dependent or independent from each other? The following equation will let us set up a matrix to determine the results:

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0} \quad (2.19)$$

We must try to see if there are any c_n values that are not zero to make this true.

$$\begin{pmatrix} 1 & 2 & 8 \\ 2 & -1 & 1 \\ 1 & 3 & 11 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (2.20)$$

$$= \begin{pmatrix} 1 & 2 & 8 & : & 0 \\ 2 & -1 & 1 & : & 0 \\ 1 & 3 & 0 & : & 0 \end{pmatrix} \quad (2.21)$$

By doing some elementary row operations, we can find that in row echelon form, the matrix from equation (2.21) can be written as

$$\begin{pmatrix} 1 & 2 & 8 & : & 0 \\ 9 & 1 & 3 & : & 0 \\ 0 & 0 & 0 & : & 0 \end{pmatrix} \quad (2.22)$$

And from here we can solve for the different c_n values.

$$1c_1 + 2c_2 + 8c_3 = 0 \quad (2.23)$$

$$c_1 + 3c_3 = 0 \quad (2.24)$$

$$(2.25)$$

$$c_3 = -3c_3 \quad (2.26)$$

$$c_1 = -2c_3 \quad (2.27)$$

$$c_3 = c_3 \quad (2.28)$$

Because we have this relationship where c_1, c_2, c_3 all depend on each other, we can tell that this is linearly independent.

§2.3.1 Linear Independence and Dependence

§2.4 Basis and Dimension

Definition 2.4.1

A basis of a vector space v is a collection of vectors $\vec{v}_1, \dots, \vec{v}_n$ that 1. span v and 2. are linearly dependent.

Problem 5. If we are looking at \mathbb{R}^2 , with $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ We can tell that this is a basis of \mathbb{R}^2 . We can tell this because the span of v is linearly independent. We can also see this because:

$$a \times e_1 + b \times e_2 = \begin{pmatrix} a \\ b \end{pmatrix} \quad (2.29)$$

Problem 6. Now we are going to look at an example in \mathbb{R}^3 , with $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, and $e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. We can figure out that this is a basis by doing the same technique as we did before:

$$c_1 \vec{e}_1 + c_2 \vec{e}_2 + c_3 \vec{e}_3 = 0 \quad (2.30)$$

$$c_1 = c_2 = c_3 = 0 \quad (2.31)$$

Because c_1 , c_2 , and c_3 are all equal to zero, \vec{e}_1 , \vec{e}_2 , and \vec{e}_3 form a basis.

Theorem 2.4.1

If a vector space v has a basis with n elements, then every basis of v has n elements. We say v has dimension n . We write down $\dim v = n$

Theorem 2.4.2

If the dimension of v is n , then any collection of $n + 1$ or more vectors must be linearly dependent.

Theorem 2.4.3

Suppose $\dim v = n$

1. Every collection of more than n vectors is linearly dependent.
2. No set of fewer than n vectors spans v .
3. A set of n vectors is a basis if and only if it spans v .
4. A set of n vectors is a basis if and only if it is linearly independent.

Problem 7. Assume $1, x, x^2$ is a basis for \mathbb{P}^2 . We are going to multiply 1×5 , $x \times 6$, and $x^2 \times 2$.

$$5 + 6x + 2x^2 \tag{2.32}$$

$$c_1 \times 1 + c_2 \times x + c_3 \times x^2 = 0 \tag{2.33}$$

$$\dim(\mathbb{P}^2) = 3 \tag{2.34}$$

$$\tag{2.35}$$

Theorem 2.4.4

$\vec{v}_1, \dots, \vec{v}_n$ form a basis of v if and only if for all $\vec{v} \in v$, there exist unique c_1, \dots, c_n such that $\vec{v} = c_1\vec{v}_1 + \dots c_n\vec{v}_n$

Problem 8. Let $v = \mathbb{R}^2$. Let $\vec{v} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$. We know from previous problems that $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is a basis of \mathbb{R}^2 . We can also figure out what our basis is by trying to figure out what our c_1 and c_2 values should be. Because the matrices we know are a basis consist of 1's and 0's, we can see that

$$4 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}. \quad (2.36)$$

The coordinates of \vec{v} with respect to this basis, are $(4, 3)$. Let's consider a different basis. We are now going to look at the basis,

$$\left\{ \begin{pmatrix} 1 \\ -3 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right\} \quad (2.37)$$

Now we need to figure out the a and b values for this basis respectively.

$$a \begin{pmatrix} 1 \\ -3 \end{pmatrix} + b \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \quad (2.38)$$

$$(2.39)$$

The coordinates of \vec{v} with respect to this basis are $(4, 3)$. Let's consider the same problem but with the following basis:

$$\left\{ \begin{pmatrix} 1 \\ -3 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right\} \quad (2.40)$$

Now, we are going to do the same thing as before, where we solve for a and b in the following equation

$$a \begin{pmatrix} 1 \\ -3 \end{pmatrix} + b \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \quad (2.41)$$

We can setup this to be a system of equations that we can turn into a matrix

$$\begin{cases} 1a + 2b = 4 \\ -3a + (-1)b = 3 \end{cases} \rightarrow \begin{bmatrix} 1 & 2 & \vdots & 4 \\ -3 & -1 & \vdots & 3 \end{bmatrix} \quad (2.42)$$

And now we can use the basic row operation $R_2 = R_2 + 3R_1$ in order to solve for a and b :

$$\begin{bmatrix} 1 & 2 & \vdots & 4 \\ 0 & 5 & \vdots & 5 \end{bmatrix} \quad (2.43)$$

$$5b = 15 \quad a + 2b = 4 \quad (2.44)$$

$$b = 3 \quad 1 + 2 * 3 = 4 \quad (2.45)$$

$$a = -2 \quad (2.46)$$

§2.5 The fundamental Matrix Subspaces (Kernel and Image)

Definition 2.5.1

The image of an $m \times n$ matrix A is the subspace spanned by the columns of A .

Problem 9. Let's consider the following equation.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \quad (2.47)$$

When we multiply our the matrix, we see that the span of the columns give us all the possible $\begin{bmatrix} x \\ y \end{bmatrix}$ values. The span of the matrix

$$\begin{bmatrix} 1 \times r_1 & 2 \times r_2 & 3 \times r_3 \\ 4 \times r_1 & 5 \times r_2 & 6 \times r_3 \end{bmatrix} \quad (2.48)$$

would be the values $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$, and $\begin{bmatrix} 3 \\ 6 \end{bmatrix}$

Definition 2.5.2

A space, A , is an $m \times m$ matrix, The kernel of A is

$$A = \text{Ker}(A) \quad (2.49)$$

$$= \{\vec{x} | A\vec{x} = \vec{0}\} \quad (2.50)$$

Using definition (2.5.2), if $A = \begin{bmatrix} 1 & 0 \\ 5 & 0 \end{bmatrix}$, then

$$\vec{x}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} 0 \\ 5 \end{bmatrix}. \quad (2.51)$$

Something to keep in mind: If $\vec{x}_1, \vec{x}_2 \in \text{Ker}(A)$, then $r_1\vec{x}_1 + r_2\vec{x}_2 \in \text{Ker}(A)$. So the kernel of A is a subspace of the domain of the function.

Theorem 2.5.1

Assume \vec{x}_1 solves $A\vec{x} = \vec{b}$. Then, \vec{x}_2 is another solution to $A\vec{x} = \vec{b}$ if and only if $\vec{x}_2 = \vec{x}_1 + \vec{z}$, where $z \in \text{Ker}(A)$

Proposition 2.5.1

Let A be an $m \times n$ matrix. The following are true:

1. $\text{Ker}(A) = \{\vec{0}\}$
2. $\text{rank}(A) = n$
3. $A\vec{x} = \vec{b}$ has a unique solution for any \vec{b} in the integer of A .
4. $A\vec{x} = \vec{b}$ has no free variables.
5. A is non-singular.

Definition 2.5.3

Let A be $m \times n$.

$$\text{coimg}(A) = \text{img}(A^T) \quad (2.52)$$

$$\text{coker}(A) = \ker(A^T) \quad (2.53)$$

$$(2.54)$$

The image of A is the span of its columns. Thus the coimage is the span of its rows. Also the r^T in the cokernel of A are those r^T such that $r \times A = 0^T$ since

$$(r \cdot A)^T = (0^T)^T \quad (2.55)$$

$$A^T \cdot r^T = 0 \quad (2.56)$$

Theorem 2.5.2

The Fundamental Theorem of Linear Algebra: Let A be an $m \times n$ matrix and let r be its rank. Then

$$\dim(\text{coimg}(A)) = \dim(\text{img}(A)) = \text{rank}(A) = \text{rank}(A^T) = r \quad (2.57)$$

$$\text{span}(A_{\text{rows}}) = \text{span}(A_{\text{columns}}) \quad (2.58)$$

$$\dim(\ker(A)) = n - r \quad (2.59)$$

$$\dim(\text{coker}(A)) = m - r \quad (2.60)$$

Let's take a look at an example using the fundamental theorem of linear algebra. Let

$$A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 1 & -1 & 0 & 2 \\ -1 & 1 & 0 & -2 \\ 2 & 1 & 3 & 1 \end{bmatrix} \quad (2.61)$$

Let's do some row operations to get this matrix into row echelon form:

$$A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & -3 & -3 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (2.62)$$

From this form, we can tell that v_3 and v_4 both depend on v_1 and v_2 . Because there are only two pivot points within A that are filled with values other than 0, $\text{rank}(A) = 2$. We also know that the dimension of the column space is equal to the dimension of the row space. Let's find the dimension of the kernel of A :

$$\dim(\ker(A)) + \text{rank} = n \quad (2.63)$$

$$\dim(\ker(A)) + 2 = 4 \quad (2.64)$$

$$(2.65)$$

From here, we know that both y and z are free variables.

$$w + 2x + 3y - z = 0 \quad (2.66)$$

$$-3x - 3y + 3z = 0 \quad (2.67)$$

$$x + y - z = 0 \quad (2.68)$$

$$x = -y + z \quad (2.69)$$

$$w = -2x - 3y + z \quad (2.70)$$

$$= -2(-y + z) - 3y + z \quad (2.71)$$

$$= -y - z \quad (2.72)$$

Now we need to determine the basis for $\ker(A)$.

$$\begin{pmatrix} -y - z \\ -y + z \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y \\ -y \\ y \\ 0 \end{pmatrix} + \begin{pmatrix} -z \\ z \\ 0 \\ z \end{pmatrix} = y \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \quad (2.73)$$

Our basis for $\ker(A)$ is $\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$.

Chapter 3

Inner Products and Norms

§3.1 Inner Products

Dot products are a form of inner product.

Let's apply the dot product to the vectors $\langle v_1, v_2, \dots, v_n \rangle \cdot \langle w_1, w_2, \dots, w_n \rangle$ to show that the dot product is an inner product.

$$= v_1 w_1 + v_2 w_2 + \dots + v_n w_n \quad (3.1)$$

Therefore, $v \cdot v$ goes to \mathbb{R} . So we know that

$$\vec{v} \cdot \vec{v} = v_1^2 + v_2^2 + \dots + v_n^2 \quad (3.2)$$

In general, we can assume that $\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}$. We should also keep in mind that $\vec{v} \cdot \vec{w} = \|\vec{v}\| \|\vec{w}\| \cos \theta$

Definition 3.1.1

An inner product of V is a function $\langle, \rangle: V \times V \rightarrow \mathbb{R}$ such that

•

$$\langle c\vec{u} + d\vec{v}, \vec{w} \rangle = c \langle \vec{u}, \vec{v} \rangle + d \langle \vec{v}, \vec{u} \rangle \quad (3.3)$$

$$\langle \vec{u}, c\vec{v} + d\vec{w} \rangle = c \langle \vec{u}, \vec{v} \rangle + d \langle \vec{u}, \vec{w} \rangle \quad (3.4)$$

• $\langle \vec{v}, \vec{w} \rangle = \langle \vec{w}, \vec{v} \rangle$

• $\langle \vec{v}, \vec{v} \rangle \geq 0$ while $\langle 0, 0 \rangle = 0$.

A vector space with an inner product is an inner product space.

Definition 3.1.2

If V is an inner product space, then it's magnitude is

$$||\vec{v}|| = \sqrt{\langle \vec{v}, \vec{v} \rangle} \quad (3.5)$$

Let's take a look at a weighted inner product on \mathbb{R}^3 . We are going to let $r_1, r_2, r_3 > 0$. We can define $\langle \vec{v}, \vec{w} \rangle$ as $r_1 v_1 w_1 + r_2 v_2 w_2 + r_3 v_3 w_3$

Problem 10. Let's define $[a, b] \subseteq \mathbb{R}$. Consider $\mathbb{C}^0[a, b]$. This is a vector space. Define

$$\langle f(x), g(x) \rangle = \int_a^b f(x)g(x)dx \quad (3.6)$$

This is an inner product, so we also know that

$$||f|| = \sqrt{\int_a^b (f(x))^2 dx} \quad (3.7)$$

This equation is the L^2 norm.

§3.2 Inequalities

Recall that $\vec{v} \cdot \vec{w} = ||\vec{v}|| ||\vec{w}|| \cos \theta$, where θ is the angle between \vec{v} and \vec{w} . Now $-1 \leq \cos \theta \leq 1$, so we know that

$$||\vec{v} \cdot \vec{w}|| \leq ||\vec{v}|| ||\vec{w}|| \quad (3.8)$$

This is the Cauchy-Shuartz inequality.

Theorem 3.2.1

For any inner product space

$$||\langle \vec{v}, \vec{w} \rangle|| \leq ||\vec{v}|| ||\vec{w}|| \quad (3.9)$$

Definition 3.2.1

If $\vec{v}, \vec{w} \in V$, we say \vec{v} and \vec{w} are orthogonal if $\langle \vec{v}, \vec{w} \rangle = 0$

Problem 11. *Let's look at an example of checking orthogonality of two equations $x, x^2 - \frac{1}{2} \in \mathbb{C}^0[0, 1]$. In order to do this we need to find the L^2 norm of the equations.*

$$\begin{aligned}\left\langle x, x^2 - \frac{1}{2} \right\rangle &= \int_0^1 x \left(x^2 - \frac{1}{2} \right) dx \\ &= \int_0^1 \left(x^3 - \frac{1}{2}x \right) dx \\ &= \left. \frac{1}{4}x^4 - \frac{1}{4}x^2 \right|_0^1 = 0.\end{aligned}$$

Because the result of the inner product was zero, we know that $x, x^2 - \frac{1}{2}$ are orthogonal in the L^2 norm.

Theorem 3.2.2

The triangle inequality states that if V is an inner product space,

$$\| \langle \vec{v}, \vec{w} \rangle \| = \|\vec{v}\| + \|\vec{w}\| \quad (3.10)$$

Because we know that if we take the dot product of the same vector itself, $\langle a, b, c \rangle \cdot \langle a, b, c \rangle$, we get all of the items squared $\langle a^2, b^2, c^2 \rangle$, and because we know that $\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$, we get an idea of size. So we can define the unit ball to be

$$\{\vec{v} \in V \mid \|\vec{v}\| = 1\} \quad (3.11)$$

§3.3 Norms

Equation (3.5) gives us the "size" of \vec{V} .

Definition 3.3.1

A norm on V is a function $\|\cdot\| : V \rightarrow \mathbb{R}$ such that

- $\|\vec{v}\| = 0$ if and only if $\vec{v} = 0$

- $||c\vec{v}|| = |c| \cdot ||\vec{v}||$
- $||\vec{v} + \vec{w}|| \leq ||\vec{v}|| + ||\vec{w}||$

If $||\langle \vec{v}, \vec{w} \rangle|| \leq ||\vec{v}|| ||\vec{w}||$, then that is a norm. There are other norms to learn about.

Problem 12. Consider $V = \mathbb{R}^n$. We know that the magnitude of \vec{V}_p is

$$\sqrt[p]{|\vec{v}_1|^p + |\vec{v}_2|^p + |\vec{v}_3|^p}. \quad (3.12)$$

So if $v = \mathbb{R}^2$, $p = 2$ we have

$$||\langle x, y \rangle||_2 = \sqrt{x^2 + y^2}. \quad (3.13)$$

But if we were to have $p = 3$, we would have

$$||\langle x, y \rangle||_3 = \sqrt[3]{x^3 + y^3} \quad (3.14)$$

In $||\cdot||_3$, the size is $\sqrt[3]{3^3 + 4^3} \approx 4.5$

In the 4 term, $||\cdot||_4$, the unit circle is the (x, y) 's such that $\sqrt[4]{x^4 + y^4} = 1$

$$\boxed{x^4 + y^4 = 1} \quad (3.15)$$

Another norm on \mathbb{R}^n is the super-norm. This is where

$$||\langle x_1, x_2, \dots, x_n \rangle||_\infty = \max\{|x_1|, |x_2|, \dots, |x_n|\} \quad (3.16)$$

Here's a quick example: The super-norm for $\langle 3, 4 \rangle$ is

$$||\langle 3, 4 \rangle||_\infty = 4 \quad (3.17)$$

because the maximum value in the set is 4.

Something to keep in mind is $||\langle x, y \rangle|| = |x| + |y|$.

Theorem 3.3.1

Let $||\cdot||_A$ and $||\cdot||_B$ be two norms on \mathbb{R}^n . Then there exists positive

numbers $0 < c < k$ such that

$$c \cdot \|\vec{v}\|_A < \|\vec{v}\|_B < k \cdot \|\vec{v}\|_A \quad (3.18)$$

Let's consider $V \in \mathbb{R}^2$. Let's take a look at $\|\cdot\|_2$ and $\|\cdot\|_\infty$. Where $\vec{V} = \langle v_1, v_2 \rangle$.

$$\frac{1}{\sqrt{2}} \cdot \|\vec{v}\|_2 \leq \|\vec{v}\|_\infty < 1 \cdot \|\vec{v}\|_2 \quad (3.19)$$

We can also define norms on matrices.

Theorem 3.3.2

If $\|\cdot\|$ is a norm on \mathbb{R}^2 and A is an $m \times n$ matrix, then

$$\|A\| = \max\{\|A \cdot \vec{u}\| \mid \|\vec{u}\| = 1\} \quad (3.20)$$

These matrix norms satisfy the following:

1. $\|A \cdot \vec{v}\| \leq \|A\| \cdot \|\vec{v}\|$
2. $\|A \cdot B\| \leq \|A\| \cdot \|B\|$
3. $\|A^k\| \leq \|A\|^k$

Let's take a quick look at $\|A\|_\infty$

Definition 3.3.2

The i^{th} absolute row sum of A is the sum of the absolute values of the entries in the i^{th} row.

Theorem 3.3.3

$\|A\|_\infty$ the maximum absolute row sum.

Here's an example of using the $\|A\|_\infty$ value. Let $A = \begin{pmatrix} -3 & 2 \\ 5 & 4 \end{pmatrix}$. We can determine that the maximum absolute row sum of A is 8. This is because we can do

$$|-3| + |2| = 5 \quad (3.21)$$

$$|5| + |4| = \boxed{8} \quad (3.22)$$

§3.4 Positive Definite Matrices

Consider the following two equations:

$$\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + \dots + x_n\vec{e}_n \quad (3.23)$$

$$\vec{y} = y_1\vec{e}_1 + y_2\vec{e}_2 + \dots + y_n\vec{e}_n \quad (3.24)$$

We can analyze this as

$$\langle \vec{x}, \vec{y} \rangle = [x_1, x_2, \dots, x_n] \cdot \begin{bmatrix} k_{11} & \dots & k_{1n} \\ \vdots & \ddots & \vdots \\ k_{n1} & \dots & k_{nn} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots y_n \end{bmatrix} \quad (3.25)$$

$$k_{ij} = \langle e_i, e_j \rangle \quad (3.26)$$

$$= \vec{x}^T k \vec{y} \quad (3.27)$$

$$k = k^T \quad (3.28)$$

This means that k is symmetrical across the diagonal.

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{12} & k_{22} & k_{23} \\ k_{13} & k_{23} & k_{33} \end{bmatrix} \quad (3.29)$$

Definition 3.4.1

A $n \times n$ matrix A is a symmetrical positive definite matrix if $A = A^T$ and $x^T k x > 0$.

Theorem 3.4.1

Every inner product on \mathbb{R}^n is given by $\langle x, y \rangle = \vec{x}^T k \vec{y}$ where k is a symmetrical positive definite matrix. So $\langle \vec{x}, \vec{y} \rangle$ is a dot product or a weighted product.

$$\langle \vec{x}, \vec{y} \rangle = \vec{x}^T k \vec{y} \quad (3.30)$$

$$k^T = k \quad (3.31)$$

$$\vec{v}^T k \cdot \vec{v} > 0 \quad (3.32)$$

$$\vec{v} \neq \vec{0} \quad (3.33)$$

Let's take a look at an example for this:

Problem 13. Let $k = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. First we need to check to see if $k^T = k$. By just looking at k , we can see that $k^T = k$. Next we need to do the following calculation to see if $\begin{bmatrix} x \\ y \end{bmatrix}$ is the weighted inner product of the matrix k .

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 2x \\ 3x \end{bmatrix} \quad (3.34)$$

$$= 2x^2 + 2y^2 > 0 \quad (3.35)$$

$$\begin{bmatrix} x \\ y \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (3.36)$$

Therefore we know that

Problem 14. Let's consider the following problem

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 4 & -2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad (3.37)$$

If we let A be the numerical matrix, we can see that $A^T = A$. Let's simplify the equation from before

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 4x - 2y \\ -2x + 3y \end{bmatrix} = 4x^2 - 2xy - 2xy + 3x^2 \quad (3.38)$$

$$= 4x^2 - 4xy + 3y^2 \quad (3.39)$$

$$(2x - y)^2 + 2y^2 > 0 \begin{bmatrix} x \\ y \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (3.40)$$

Now we can see that this is a positive definite matrix.

If we are given a symmetric matrix, k , the polynomial $x^T k x$ is a quadratic form of k .

Problem 15. Let's consider $k = \begin{bmatrix} 1 & -3 \\ -3 & 2 \end{bmatrix}$. Let's find the quadratic form of k . First we need to write k like so:

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & -3 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} x - 3y \\ -3x + 2y \end{bmatrix} \quad (3.41)$$

$$= x^3 - 3xy - 3xy + 2y^2 \quad (3.42)$$

$$= x^3 - 6xy + 2y^2 \quad (3.43)$$

$$(3.44)$$

Therefore we know that the quadratic form of k is $x^3 - 6xy - 2y^2$.

For a positive definite matrix, $k = k^T$ and $x^T k x > 0$ for all $\vec{x} \neq 0$

Theorem 3.4.2

Every inner product in \mathbb{R}^n is given by

$$\langle x, y \rangle = x^T k y \text{ for } x, y \in \mathbb{R}^n \quad (3.45)$$

Let v be an inner product space and $\vec{v}_1, \dots, \vec{v}_2$. The gram matrix of v

is

$$K = \begin{bmatrix} \langle v_1, v_1 \rangle & \langle v_1, v_2 \rangle & \dots & \langle v_1, v_n \rangle \\ \langle v_2, v_1 \rangle & \langle v_2, v_2 \rangle & \dots & \langle v_2, v_n \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle v_n, v_1 \rangle & \langle v_n, v_2 \rangle & \dots & \langle v_n, v_n \rangle \end{bmatrix} \quad (3.46)$$

Definition 3.4.2

A is a matrix that is $n \times n$. A is a positive semidefinite matrix if $A^T = A$ and $\vec{x}^T A \vec{x} \geq 0$

Theorem 3.4.3

All gram matrices are positive semi-definite. They are positive definite if and only if $\vec{v}_1, \dots, \vec{v}_n$ are linearly independent.

Suppose we are in \mathbb{R}^m and the inner product is the dot product. Let $\vec{v}_1, \dots, \vec{v}_n \in \mathbb{R}^m$. Let $A = [\vec{v}_1, \vec{v}_2, \vec{v}_3, \dots, \vec{v}_n]$. Then $K = A^T A$. Let A be a gram matrix generated by v_1, \dots, v_n with the dot product.

$$K = A^T A = \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vdots \\ \vec{v}_n \end{bmatrix} \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \end{bmatrix} \quad (3.47)$$

$$= \begin{bmatrix} \langle v_1, v_1 \rangle & \langle v_1, v_2 \rangle & \dots & \langle v_1, v_n \rangle \\ \langle v_2, v_1 \rangle & \langle v_2, v_2 \rangle & \dots & \langle v_2, v_n \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle v_n, v_1 \rangle & \langle v_n, v_2 \rangle & \dots & \langle v_n, v_n \rangle \end{bmatrix} \quad (3.48)$$

Proposition 3.4.1

Given an $m \times n$ matrix A . The following are true

1. The $m \times n$ matrix $k = A^T A$ is positive definite.
2. A has linearly independent columns.

3. $\text{rank}(A) = n$
4. $\text{Ker}(A) = \{0\}$

Theorem 3.4.4

Every inner product on \mathbb{R}^n is given by

$$\langle \vec{x}, \vec{y} \rangle = \vec{x}^T \cdot C \vec{y} \quad (3.49)$$

where C is a symmetric, positive definite $n \times n$ matrix.

Let $\vec{v}_1, \dots, \vec{v}_n \in \mathbb{R}^n$. Let $A = [\vec{v}_1 : \vec{v}_2 : \dots : \vec{v}_n]$. Then $K = A^T C A$ is the gram matrix with respect to the inner product $\vec{v}^T C \vec{w}$.

Theorem 3.4.5

Suppose A is an $m \times n$ matrix with linearly independent columns. Suppose C is any positive definite $m \times m$ matrix. Then $\vec{v}^T C \vec{w}$

Definition 3.4.3

The hilbert matrix $H = (h_{ij})$ where $h_{ij} = \frac{1}{i+j-1}$

The 3×3 hilbert matrix:

$$\begin{bmatrix} \frac{1}{1} & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix} \quad (3.50)$$

Problem 16. Make a gram matrix, not in \mathbb{R}^m . Let $V = \mathbb{C}^0[0, 1]$. Use the L^2 inner product.

$$\langle f, g \rangle = \int_0^1 f(x)g(x)dx. \quad (3.51)$$

$1, x, x^2$ are linearly independent.

$$\langle v_i, v_k \rangle = \int_0^1 x^{i-1} x^{j-1} dx \quad (3.52)$$

$$= \int_0^1 x^{i+j-2} dx \quad (3.53)$$

$$= \frac{1}{i+j-1} x^{i+j-1} \quad (3.54)$$

$$= \frac{1}{i+j-1} \quad (3.55)$$

$$\rightarrow \begin{bmatrix} \frac{1}{1} & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix}. \quad (3.56)$$

Chapter 4

Orthogonality

Chapter 5

Mimization and Least Squares

Chapter 6

Equilibrium

Chapter 7

Linearity

Chapter 8

Eigenvalues and Singular Values

Chapter 9

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Chapter 10

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