

Applied Linear Algebra

Cameron Williamson

Spring 2020

Contents

1	Linear Algebraic Systems	2
2	Vector Spaces and Bases	2
2.1	Real Vector Spaces	2
2.2	Subspaces	3
2.3	Span and Linear Independence	4
2.3.1	Linear Independence and Dependence	7
2.4	Basis and Dimension	7
2.5	The fundamental Matrix Subspaces (Kernel and Image)	9
3	Inner Products and Norms	12
3.1	Inner Products	12
3.2	Inequalities	13
3.3	Norms	14
3.4	Positive Definite Matrices	15
4	Orthogonality	20
4.1	20
4.2	23
4.3	Orthogonal Matrices	23
4.4	Vector Spaces	26
5	Minimization and Least Squares	27
6	Equilibrium	27
7	Linearity	27
8	Eigenvalues and Singular Values	27
9	Iteration	29
10	Dynamics	29

1 Linear Algebraic Systems

2 Vector Spaces and Bases

2.1 Real Vector Spaces

A vector space is the abstract reformulation of the quintessential properties of n -dimensional Euclidean Space \mathbb{R}^n , which is defined as the set of all real (column) vectors with n entries. The basic laws of vector addition and scalar multiplication in \mathbb{R} .

Definition 1. A vector space is a set of V equipped with two operations:

- Addition: adding any pair of vectors $v, w \in V$ produces another vector $v + w \in V$;
- Scalar Multiplication: multiplying a vector $v \in V$ by a scalar $c \in \mathbb{R}$ produces a vector $cv \in V$

These are subject to the following axioms, valid for all $u, v, w \in V$ and all scalars $c, d \in \mathbb{R}$:

- Commutativity of Addition: $v + w = w + v$.
- Associativity of Addition: $u + (v + w) = (u + v) + w$.
- Additive Identity: There is a zero element $0 \in V$ satisfying $v + 0 = v = 0 + v$.
- Additive Inverse: For each $v \in V$ there is an element $-v \in V$ such that $v + (-v) = 0 = (-v) + v$.
- Distributivity: $(c + d)v = (cv) + (dv)$, and $c(v + w) = (cv) + (cw)$.
- Associativity of Scalar Multiplication: $c(dv) = (cd)v$.
- Unit for Scalar Multiplication: the scalar $1 \in \mathbb{R}$ satisfies $1v = v$.

Theorem 1. Let V be a vector space.

- $0 \times \vec{V} = \vec{0}$
- $-1\vec{V} = -\vec{V}$
- $c \times \vec{0} = \vec{0}$
- If $c \times \vec{V} = \vec{0}$, then $c = 0$ or $\vec{V} = \vec{0}$

Here are some examples of vector spaces:

- $\mathbb{R}^n = \left\{ \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix} \mid r_1, r_2, r_n \in \mathbb{R} \right\}$
- $M_{m \times n}$ = The m by n matrices over \mathbb{R} .
- \mathbb{P}^n = the polynomials of degree $\leq n$.

Definition 2. Let V be a vector space over F . $W \leq V$ is a subspace of V if W is a vector space over F under the same operation as V .

An example of definition (2.1.2). Let $V = \mathbb{R}^3 = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\}$. V is a vector space. If we let $W = \left\{ \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} \mid a, b \in \mathbb{R} \right\}$, then W is a subspace of V .

Theorem 2. Let V be a vector space. Let $W \leq V$. W is a subspace of V if

- $w \neq 0$.
- $\forall w_1, w_2 \in W; w_1 + w_2 \in W$.
- $\forall c \in F; \vec{w} \in W; c \cdot \vec{w} \in W$.

If we were to let $V = \mathbb{R}^3$ for $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$. We can determine the following:

- $\{\vec{0}\}$ is a subspace of V .
- $\left\{ \begin{pmatrix} a \\ 0 \\ c \end{pmatrix} \mid a, c \in \mathbb{R} \right\}$ is a subspace of V .
- Consider the equation $\left\{ \begin{pmatrix} x \\ x \\ 0 \end{pmatrix} \mid x \in \mathbb{R} \right\} = W$ Show that W is a subspace of V .
 - $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in W$ so $W \neq 0$.
 - $\begin{pmatrix} x \\ x \\ 0 \end{pmatrix}, \begin{pmatrix} y \\ y \\ 0 \end{pmatrix} \in W$. Then $\begin{pmatrix} x \\ x \\ 0 \end{pmatrix} + \begin{pmatrix} y \\ y \\ 0 \end{pmatrix} = \begin{pmatrix} x+y \\ x+y \\ 0 \end{pmatrix}$
 - $\begin{pmatrix} x \\ x \\ 0 \end{pmatrix} \in W$, then $c \times \begin{pmatrix} x \\ x \\ 0 \end{pmatrix} = \begin{pmatrix} cx \\ cx \\ 0 \end{pmatrix} \in W$.
 - Therefore, we know that W is a subspace of V with respect to scalar multiplication and addition.
- $W = \left\{ \begin{pmatrix} x \\ y \\ 2x+3y \end{pmatrix} \mid x, y \in \mathbb{R} \right\}$. W is a subspace of V .

\mathbb{R}^3 only has 4 kinds of subspaces. \mathbb{R}^3 , $\{\vec{0}\}$, planes passing through the origin and lines that are passing through the origin.

2.2 Subspaces

Definition 3. Let I be an interval in \mathbb{R} . Let $\mathbb{F}(I)$ be the vector space of functions $\mathbb{F} = I \rightarrow \mathbb{R}$.

- $\mathbb{C}^0(I)$ = the continuous functions from $I \rightarrow \mathbb{R}$ is a subspace.
- $\mathbb{P}^n(I)$ = polynomials of degree $\leq n$ restricted to $\mathbb{F}(I)$. This is a subspace of $\mathbb{C}^0(I)$.
- $\mathbb{P}^\infty(I)$ = all polynomials on I . This is a subspace of $\mathbb{F}(I)$.
- $\mathbb{C}^n(I)$ = the set of functions $f : I \rightarrow \mathbb{R}$ such that $f', f'' \dots f^{(n)}$ all exist and are continuous.
- $\mathbb{C}^\infty(I)$ = functions from $I \rightarrow \mathbb{R}$ such that $f', f'', f''' \dots$ all exist and are smooth functions.
- $A(I)$ = the functions in $\mathbb{C}^\infty(I)$ such that all $A \in I$, the power series $f(a) + \frac{f'(a)(x-a)}{1!} + \frac{f''(a)(x-a)^2}{2!} + \dots$ converges for all $x \in I$ sufficiently close to a .

Problem. Show that $v = \begin{pmatrix} x \\ y \\ 2x+y \end{pmatrix}$ is a subspace of \mathbb{R}^3 .

- Because $\begin{pmatrix} 0 \\ 0 \\ 2(0)+0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ is in v , v is not empty.
- Let $\vec{v}_1 = \begin{pmatrix} x_1 \\ y_1 \\ 2x_1+y_1 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} x_2 \\ y_2 \\ 2x_2+y_2 \end{pmatrix} \in V$

$$v_1 + v_2 = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ 2x_1 + y_1 + 2x_2 + y_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ 2(x_1 + x_2) + (y_1 + y_2) \end{pmatrix} \in V$$

so v is closed with respect to addition.

- Let $r \in \mathbb{R}$ and $\begin{pmatrix} x \\ y \\ 2x+y \end{pmatrix} = \vec{v} \in v$.

2.3 Span and Linear Independence

If we let V be a vector space over \mathbb{R} and let $\vec{v}_1, \dots, \vec{v}_n \in V$, then we can determine that the

$$\text{span}(\{\vec{v}_1, \dots, \vec{v}_n\}) = \{c_1\vec{v}_1 + \dots + c_n\vec{v}_n | c_1, \dots, c_n \in \mathbb{R}\}$$

Proposition 1. The span of $\{\vec{v}_1, \vec{v}_2\}$ is a subspace of V .

Proof.

$$\begin{aligned} c_1 \vec{v}_1 + \cdots + c_n \vec{v}_n \\ k_1 \vec{v}_1 + \cdots + k_n \vec{v}_n \in \text{span} \end{aligned}$$

If we add together both of the equations above we get

$$\begin{aligned} (c_1 + k_1) \vec{v}_1 + \cdots + (c_n + k_n) \vec{v}_n &\in \text{span}. \\ r(c_1 \vec{v}_1 + \cdots + c_n \vec{v}_n) \\ &= rc_1 \vec{v}_1 + \cdots + rc_n \vec{v}_n \in \text{span} \end{aligned}$$

□

Problem. Let $V \in \mathbb{R}^3$. Also, we are going to let

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \text{span}(\vec{v}_1) = \left\{ c \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \mid c \in \mathbb{R} \right\}$$

$\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ is a vector in 3-space. $c \times \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ expands, contracts, changes direction. This is a line which goes through $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

$$c \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

is in the xy -plane, let's solve for y to find the equation of the line that is drawn by the vector:

$$\begin{aligned} \begin{pmatrix} c \\ 2c \\ 0 \end{pmatrix} \\ x = c \\ y = 2c \\ \frac{1}{2}y = c \\ \rightarrow x = \frac{1}{2}y \\ \rightarrow y = 2x \end{aligned}$$

Now we are going to let $\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Consider the span of $(\{\vec{v}_1, \vec{v}_2\})$. The span of $(\{\vec{v}_1, \vec{v}_2\})$ is a plane.

In \mathbb{R}^3 , if $\vec{0} \neq \vec{v} \in \mathbb{R}$, then the $\text{span}\vec{v}$ is a line.

Problem. Let $v = \mathbb{P}^2$. v is the set of polynomials of degree $\leq 2 \in \mathbb{R}$.

- $\text{span}(1, x, x^2) = \mathbb{P}^2$
- $\text{span}(4, 2x) = \mathbb{P}^1$, which means all polynomials of degree ≤ 1

Definition 4. Let v be a vector space. $\vec{v}_1, \dots, \vec{v}_n$ are linearly dependent if there exists c_1, \dots, c_n are not all zero, such that $c_1\vec{v}_1 + \dots + c_n\vec{v}_n = \vec{0}$, otherwise, $\vec{v}_1, \dots, \vec{v}_n$ are linearly independent.

If we let $\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, we can do a simple test to see if they are linearly independent. We know that $2\vec{v}_1 = \vec{v}_2$, which means that $2\vec{v}_1 + -1\vec{v}_2 = \vec{0}$. Because we can make $\vec{v}_1 + \vec{v}_2$ by using a simple scalar value, these functions are linearly dependent.

Problem. Consider the following three matrices

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \vec{v}_3 = \begin{pmatrix} 8 \\ 1 \\ 11 \end{pmatrix}$$

Are these matrices linearly dependent or independent of each other? The following equation will let us set up a matrix to determine the results:

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$$

We must try to see if there are any c_n values that are not zero to make this true.

$$\begin{pmatrix} 1 & 2 & 8 \\ 2 & -1 & 1 \\ 1 & 3 & 11 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 8 & \vdots & 0 \\ 2 & -1 & 1 & \vdots & 0 \\ 1 & 3 & 0 & \vdots & 0 \end{pmatrix}$$

By doing some elementary row operations, we can find that in row echelon form, the matrix from equation (2.21) can be written as

$$\begin{pmatrix} 1 & 2 & 8 & \vdots & 0 \\ 9 & 1 & 3 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{pmatrix}$$

And from here we can solve for the different c_n values.

$$1c_1 + 2c_2 + 8c_3 = 0$$

$$c_1 + 3c_3 = 0$$

$$c_3 = -3c_3$$

$$c_1 = -2c_3$$

$$c_3 = c_3$$

Because we have this relationship where c_1, c_2, c_3 all depend on each other, we can tell that this is linearly independent.

2.3.1 Linear Independence and Dependence

2.4 Basis and Dimension

Definition 5. A basis of a vector space v is a collection of vectors $\vec{v}_1, \dots, \vec{v}_n$ that (1) span v and (2) are linearly independent.

Problem. If we are looking at \mathbb{R}^2 , with $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ We can tell that this is a basis of \mathbb{R}^2 . We can tell this because the span of v is linearly independent. We can also see this because:

$$a \times e_1 + b \times e_2 = \begin{pmatrix} a \\ b \end{pmatrix}$$

Problem. Now we are going to look at an example in \mathbb{R}^3 , with $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, and $e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. We can figure out that this is a basis by doing the same technique as we did before:

$$c_1 \vec{e}_1 + c_2 \vec{e}_2 + c_3 \vec{e}_3 = 0$$

$$c_1 = c_2 = c_3 = 0$$

Because c_1, c_2 , and c_3 are all equal to zero, \vec{e}_1, \vec{e}_2 , and \vec{e}_3 form a basis.

Theorem 3. If a vector space v has a basis with n elements, then every basis of v has n elements. We say v has dimension n . We write down $\dim v = n$.

Theorem 4. If the dimension of v is n , then any collection of $n + 1$ or more vectors must be linearly dependent.

Theorem 5. Suppose $\dim v = n$

1. Every collection of more than n vectors is linearly dependent.
2. No set of fewer than n vectors spans v .
3. A set of n vectors is a basis if and only if it spans v .
4. A set of n vectors is a basis if and only if it is linearly independent.

Problem. Assume $1, x, x^2$ is a basis for \mathbb{P}^2 . We are going to multiply 1×5 , $x \times 6$, and $x^2 \times 2$.

$$\begin{aligned} 5 + 6x + 2x^2 \\ c_1 \times 1 + c_2 \times x + c_3 \times x^2 = 0 \\ \dim(\mathbb{P}^2) = 3 \end{aligned}$$

Theorem 6. $\vec{v}_1, \dots, \vec{v}_n$ form a basis of v if and only if for all $\vec{v} \in v$, there exist unique c_1, \dots, c_n such that $\vec{v} = c_1\vec{v}_1 + \dots + c_n\vec{v}_n$

Problem. Let $v = \mathbb{R}^2$. Let $\vec{v} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$. We know from previous problems that $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is a basis of \mathbb{R}^2 . We can also figure out what our basis is by trying to figure out what our c_1 and c_2 values should be. Because the matrices we know are a basis consist of 1's and 0's, we can see that

$$4 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}.$$

The coordinates of \vec{v} with respect to this basis, are $(4, 3)$. Let's consider a different basis. We are now going to look at the basis,

$$\left\{ \begin{pmatrix} 1 \\ -3 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right\}$$

Now we need to figure out the a and b values for this basis respectively.

$$a \begin{pmatrix} 1 \\ -3 \end{pmatrix} + b \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

The coordinates of \vec{v} with respect to this basis are $(4, 3)$. Let's consider the same problem but with the following basis:

$$\left\{ \begin{pmatrix} 1 \\ -3 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right\}$$

Now, we are going to do the same thing as before, where we solve for a and b in the following equation

$$a \begin{pmatrix} 1 \\ -3 \end{pmatrix} + b \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

We can set up this to be a system of equations that we can turn into a matrix

$$\begin{cases} 1a + 2b = 4 \\ -3a + (-1)b = 3 \end{cases} \rightarrow \begin{bmatrix} 1 & 2 & \vdots & 4 \\ -3 & -1 & \vdots & 3 \end{bmatrix}$$

And now we can use the basic row operation $R_2 = R_2 + 3R_1$ in order to solve for a and b :

$$\begin{bmatrix} 1 & 2 & \vdots & 4 \\ 0 & 5 & \vdots & 5 \end{bmatrix}$$

$$\begin{array}{ll} 5b = 15 & a + 2b = 4 \\ b = 3 & 1 + 2 * 3 = 4 \\ & a = -2 \end{array}$$

2.5 The fundamental Matrix Subspaces (Kernel and Image)

Definition 6. The image of an $m \times n$ matrix A is the subspace spanned by the columns of A .

Problem. Let's consider the following equation.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

When we multiply our the matrix, we see that the span of the columns give us all the possible $\begin{bmatrix} x \\ y \end{bmatrix}$ values. The span of the matrix

$$\begin{bmatrix} 1 \times r_1 & 2 \times r_2 & 3 \times r_3 \\ 4 \times r_1 & 5 \times r_2 & 6 \times r_3 \end{bmatrix}$$

would be the values $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$, and $\begin{bmatrix} 3 \\ 6 \end{bmatrix}$

Definition 7. A space, A , is an $m \times m$ matrix, The kernel of A is

$$\begin{aligned} A &= Ker(A) \\ &= \{\vec{x} | A\vec{x} = \vec{0}\} \end{aligned}$$

Using definition (2.5.2), if $A = \begin{bmatrix} 1 & 0 \\ 5 & 0 \end{bmatrix}$, then

$$\vec{x}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} 0 \\ 5 \end{bmatrix}.$$

Something to keep in mind: If $\vec{x}_1, \vec{x}_2 \in Ker(A)$, then $r_1\vec{x}_1 + r_2\vec{x}_2 \in Ker(A)$. So the kernel of A is a subspace of the domain of the function.

Theorem 7. Assume \vec{x}_1 solves $A\vec{x} = \vec{b}$. Then, \vec{x}_2 is another solution to $A\vec{x} = \vec{b}$ if and only if $\vec{x}_2 = \vec{x}_1 + \vec{z}$, where $z \in Ker(A)$

Proposition 2. Let A be an $m \times n$ matrix. The following are true:

1. $Ker(A) = \{\vec{0}\}$
2. $rank(A) = n$
3. $A\vec{x} = \vec{b}$ has a unique solution for any \vec{b} in the image of A .
4. $A\vec{x} = \vec{b}$ has no free variables.
5. A is non-singular.

Definition 8. Let A be $m \times n$.

$$\begin{aligned} coimg(A) &= img(A^T) \\ coker(A) &= ker(A^T) \end{aligned}$$

The image of A is the span of its columns. Thus, the coimage is the span of its rows. Also, the r^T in the cokernel of A are those \vec{r} such that $r \cdot A = \vec{0}^T$ since

$$\begin{aligned} (r \cdot A)^T &= (\vec{0}^T)^T \\ A^T \cdot r^T &= \vec{0} \end{aligned}$$

Theorem 8. The Fundamental Theorem of Linear Algebra: Let A be an $m \times n$ matrix and let r be its rank. Then

$$\begin{aligned} \dim(\text{coimg}(A)) &= \dim(\text{img}(A)) = \text{rank}(A) = \text{rank}(A^T) = r \\ \text{span}(A_{\text{rows}}) &= \text{span}(A_{\text{columns}}) \\ \dim(\ker(A)) &= n - r \\ \dim(\text{coker}(A)) &= m - r \end{aligned}$$

Let's take a look at an example using the fundamental theorem of linear algebra. Let

$$A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 1 & -1 & 0 & 2 \\ -1 & 1 & 0 & -2 \\ 2 & 1 & 3 & 1 \end{bmatrix}$$

Let's do some row operations to get this matrix into row echelon form:

$$A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & -3 & -3 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

From this form, we can tell that v_3 and v_4 both depend on v_1 and v_2 . Because there are only two pivot points within A that are filled with values other than 0, $\text{rank}(A) = 2$. We also know that the dimension of the column space is equal to the dimension of the row space. Let's find the dimension of the kernel of A :

$$\begin{aligned} \dim(\ker(A)) + \text{rank} &= n \\ \dim(\ker(A)) + 2 &= 4 \end{aligned}$$

From here, we know that both y and z are free variables.

$$\begin{aligned} w + 2x + 3y - z &= 0 \\ -3x - 3y + 3z &= 0 \\ x + y - z &= 0 \\ x &= -y + z \end{aligned}$$

$$\begin{aligned} w &= -2x - 3y + z \\ &= -2(-y + z) - 3y + z \\ &= -y - z \end{aligned}$$

Now we need to determine the basis for $\ker(A)$.

$$\begin{pmatrix} -y - z \\ -y + z \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y \\ -y \\ -y \\ 0 \end{pmatrix} + \begin{pmatrix} -z \\ z \\ 0 \\ z \end{pmatrix} = y \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

Our basis for $\ker(A)$ is $\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$.

3 Inner Products and Norms

3.1 Inner Products

Dot products are a form of inner product.

Let's apply the dot product to the vectors $\langle v_1, v_2, \dots, v_n \rangle \cdot \langle w_1, w_2, \dots, w_n \rangle$ to show that the dot product is an inner product.

$$= v_1 w_1 + v_2 w_2 + \dots + v_n w_n \quad (1)$$

Therefore, $v \times v$ goes to \mathbb{R} . So we know that

$$\vec{v} \cdot \vec{v} = v_1^2 + v_2^2 + \dots + v_n^2 \quad (2)$$

In general, we can assume that $\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}$. We should also keep in mind that $\vec{v} \cdot \vec{w} = \|\vec{v}\| \|\vec{w}\| \cos \theta$

Definition 9. An inner product of V is a function $\langle, \rangle: v \times v \rightarrow \mathbb{R}$ such that

•

$$\langle c\vec{u} + d\vec{v}, \vec{w} \rangle = c \langle \vec{u}, \vec{v} \rangle + d \langle \vec{v}, \vec{u} \rangle \quad (3)$$

$$\langle \vec{u}, c\vec{v} + d\vec{w} \rangle = c \langle \vec{u}, \vec{v} \rangle + d \langle \vec{u}, \vec{w} \rangle \quad (4)$$

- $\langle \vec{v}, \vec{w} \rangle = \langle \vec{w}, \vec{v} \rangle$
- $\langle \vec{v}, \vec{v} \rangle \geq 0$ while $\langle 0, 0 \rangle = 0$.

A vector space with an inner product is an inner product space.

Definition 10. If V is an inner product space, then its magnitude is

$$\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle} \quad (5)$$

Let's take a look at a weighted inner product on \mathbb{R}^3 . We are going to let $r_1, r_2, r_3 > 0$. We can define $\langle \vec{v}, \vec{w} \rangle$ as $r_1 v_1 w_1 + r_2 v_2 w_2 + r_3 v_3 w_3$

Problem. Let's define $[a, b] \subseteq \mathbb{R}$. Consider $\mathbb{C}^0[a, b]$. This is a vector space. Define

$$\langle f(x), g(x) \rangle = \int_a^b f(x)g(x)dx \quad (6)$$

This is an inner product, so we also know that

$$\|f\| = \sqrt{\int_a^b (f(x))^2 dx} \quad (7)$$

This equation is the L^2 norm.

3.2 Inequalities

Recall that $\vec{v} \cdot \vec{w} = \|\vec{v}\| \|\vec{w}\| \cos \theta$, where θ is the angle between \vec{v} and \vec{w} . Now $-1 \leq \cos \theta \leq 1$, so we know that

$$\|\vec{v} \cdot \vec{w}\| \leq \|\vec{v}\| \|\vec{w}\| \quad (8)$$

This is the Cauchy-Schwarz inequality.

Theorem 9. For any inner product space

$$\|\langle \vec{v}, \vec{w} \rangle\| \leq \|\vec{v}\| \|\vec{w}\| \quad (9)$$

Definition 11. If $\vec{v}, \vec{w} \in V$, we say \vec{v} and \vec{w} are orthogonal if $\langle \vec{v}, \vec{w} \rangle = 0$

Problem. Let's look at an example of checking orthogonality of two equations $x, x^2 - \frac{1}{2} \in \mathbb{C}^0[0, 1]$. In order to do this we need to find the L^2 norm of the equations.

$$\begin{aligned} \left\langle x, x^2 - \frac{1}{2} \right\rangle &= \int_0^1 x \left(x^2 - \frac{1}{2} \right) dx \\ &= \int_0^1 \left(x^3 - \frac{1}{2}x \right) dx \\ &= \left. \frac{1}{4}x^4 - \frac{1}{4}x^2 \right|_0^1 = 0. \end{aligned}$$

Because the result of the inner product was zero, we know that $x, x^2 - \frac{1}{2}$ are orthogonal in the L^2 norm.

Theorem 10. The triangle inequality states that if V is an inner product space,

$$\|\langle \vec{v}, \vec{w} \rangle\| = \|\vec{v}\| + \|\vec{w}\| \quad (10)$$

Because we know that if we take the dot product of the same vector itself, $\langle a, b, c \rangle \cdot \langle a, b, c \rangle$, we get all of the items squared $\langle a^2, b^2, c^2 \rangle$, and because we know that $\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$, we get an idea of size. So we can define the unit ball to be

$$\{\vec{v} \in V \mid \|\vec{v}\| = 1\} \quad (11)$$

3.3 Norms

Equation (3.5) gives us the "size" of \vec{V} .

Definition 12. A norm on V is a function $\|\cdot\| : V \rightarrow \mathbb{R}$ such that

- $\|\vec{v}\| = 0$ if and only if $\vec{v} = 0$
- $\|c\vec{v}\| = |c| \cdot \|\vec{v}\|$
- $\|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|$

If $\|\langle \vec{v}, \vec{w} \rangle\| \leq \|\vec{v}\| \|\vec{w}\|$, then that is a norm. There are other norms to learn about.

Problem. Consider $V = \mathbb{R}^n$. We know that the magnitude of \vec{V}_p is

$$\sqrt[p]{|\vec{v}_1|^p + |\vec{v}_2|^p + |\vec{v}_3|^p}. \quad (12)$$

So if $v = \mathbb{R}^2$, $p = 2$ we have

$$\|\langle x, y \rangle\|_2 = \sqrt{x^2 + y^2}. \quad (13)$$

But if we were to have $p = 3$, we would have

$$\|\langle x, y \rangle\|_3 = \sqrt[3]{x^3 + y^3} \quad (14)$$

In $\|\cdot\|_3$, the size is $\sqrt[3]{3^3 + 4^3} \approx 4.5$

In the 4 term, $\|\cdot\|_4$, the unit circle is the (x, y) 's such that $\sqrt[4]{x^4 + y^4} = 1$

$$\boxed{x^4 + y^4 = 1} \quad (15)$$

Another norm on \mathbb{R}^n is the super-norm. This is where

$$\|\langle x_1, x_2, \dots, x_n \rangle\|_\infty = \max\{|x_1|, |x_2|, \dots, |x_n|\} \quad (16)$$

Here's a quick example: The super-norm for $\langle 3, 4 \rangle$ is

$$\|\langle 3, 4 \rangle\|_\infty = 4 \quad (17)$$

because the maximum value in the set is 4.

Something to keep in mind is $\|\langle x, y \rangle\| = |x| + |y|$.

Theorem 11. Let $\|\cdot\|_A$ and $\|\cdot\|_B$ be two norms on \mathbb{R}^n . Then there exists positive numbers $0 < c < k$ such that

$$c \cdot \|\vec{v}\|_A < \|\vec{v}\|_B < k \cdot \|\vec{v}\|_A \quad (18)$$

Let's consider $V \in \mathbb{R}^2$. Let's take a look at $\|\cdot\|_2$ and $\|\cdot\|_\infty$. Where $\vec{V} = \langle v_1, v_2 \rangle$.

$$\frac{1}{\sqrt{2}} \cdot \|\vec{v}\|_2 \leq \|\vec{v}\|_\infty < 1 \cdot \|\vec{v}\|_2 \quad (19)$$

We can also define norms on matrices.

Theorem 12. If $\|\cdot\|$ is a norm on \mathbb{R}^2 and A is an $m \times n$ matrix, then

$$\|A\| = \max\{\|A \cdot \vec{u}\| \mid \|\vec{u}\| = 1\} \quad (20)$$

These matrix norms satisfy the following:

1. $\|A \cdot \vec{v}\| \leq \|A\| \cdot \|\vec{v}\|$
2. $\|A \cdot B\| \leq \|A\| \cdot \|B\|$
3. $\|A^k\| \leq \|A\|^k$

Let's take a quick look at $\|A\|_\infty$

Definition 13. The i^{th} absolute row sum of A is the sum of the absolute values of the entries in the i^{th} row.

Theorem 13. $\|A\|_\infty$ the maximum absolute row sum.

Here's an example of using the $\|A\|_\infty$ value. Let $A = \begin{pmatrix} -3 & 2 \\ 5 & 4 \end{pmatrix}$. We can determine that the maximum absolute row sum of A is 8. This is because we can do

$$|-3| + |2| = 5 \quad (21)$$

$$|5| + |3| = \boxed{8} \quad (22)$$

3.4 Positive Definite Matrices

Consider the following two equations:

$$\vec{x} = x_1 \vec{e}_1 + x_2 \vec{e}_2 + \dots + x_n \vec{e}_n \quad (23)$$

$$\vec{y} = y_1 \vec{e}_1 + y_2 \vec{e}_2 + \dots + y_n \vec{e}_n \quad (24)$$

We can analyze this as

$$\langle \vec{x}, \vec{y} \rangle = [x_1, x_2, \dots, x_n] \cdot \begin{bmatrix} k_{11} & \dots & k_{1n} \\ \vdots & \ddots & \vdots \\ k_{n1} & \dots & k_{nn} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \quad (25)$$

$$k_{ij} = \langle e_i, e_j \rangle \quad (26)$$

$$= \vec{x}^T k \vec{y} \quad (27)$$

$$k = k^T \quad (28)$$

This means that k is symmetrical across the diagonal.

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{12} & k_{22} & k_{23} \\ k_{13} & k_{23} & k_{33} \end{bmatrix} \quad (29)$$

Definition 14. A $n \times n$ matrix A is a symmetrical positive definite matrix if $A = A^T$ and $x^t k < x > 0$.

Theorem 14. Every inner product on \mathbb{R}^n is given by $\langle x, y \rangle = \vec{x}^T k \vec{y}$ where k is a symmetrical positive definite matrix. So $\langle \vec{x}, \vec{y} \rangle$ is a dot product or a weighted product.

$$\langle \vec{x}, \vec{y} \rangle = \vec{x}^T k \vec{y} \quad (30)$$

$$k^T = k \quad (31)$$

$$\vec{v}^T k \cdot \vec{v} > 0 \quad (32)$$

$$\vec{v} \neq 0 \quad (33)$$

Let's take a look at an example for this:

Problem. Let $k = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. First we need to check to see if $k^T = k$. By just looking at k , we can see that $k^T = k$. Next we need to do the following calculation to see if $\begin{bmatrix} x \\ y \end{bmatrix}$ is the weighted inner product of the matrix k .

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 2x \\ 3x \end{bmatrix} \quad (34)$$

$$= 2x^2 + 2y^2 > 0 \quad (35)$$

$$\begin{bmatrix} x \\ y \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (36)$$

Therefore we know that

Problem. Let's consider the following problem

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 4 & -2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad (37)$$

If we let A be the numerical matrix, we can see that $A^T = A$. Let's simplify the equation from before

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 4x - 2y \\ -2x + 3y \end{bmatrix} = 4x^2 - 2xy - 2xy + 3y^2 \quad (38)$$

$$= 4x^2 - 4xy + 3y^2 \quad (39)$$

$$(2x - y)^2 + 2y^2 > 0 \begin{bmatrix} x \\ y \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (40)$$

Now we can see that this is a positive definite matrix.

If we are given a symmetric matrix, k , the polynomial $x^T k x$ is a quadratic form of k .

Problem. Let's consider $k = \begin{bmatrix} 1 & -3 \\ -3 & 2 \end{bmatrix}$. Let's find the quadratic form of k . First we need to write k like so:

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & -3 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} x - 3y \\ -3x + 2y \end{bmatrix} \quad (41)$$

$$= x^3 - 3xy - 3xy + 2y^2 \quad (42)$$

$$= x^3 - 6xy + 2y^2 \quad (43)$$

$$(44)$$

Therefore we know that the quadratic form of k is $x^3 - 6xy + 2y^2$.

For a positive definite matrix, $k = k^T$ and $x^T k x > 0$ for all $\vec{x} \neq 0$

Theorem 15. Every inner product in \mathbb{R}^n is given by

$$\langle x, y \rangle = x^T k y \text{ for } x, y \in \mathbb{R}^n \quad (45)$$

Let v be an inner product space and $\vec{v}_1, \dots, \vec{v}_n$. The gram matrix of v is

$$K = \begin{bmatrix} \langle v_1, v_1 \rangle & \langle v_1, v_2 \rangle & \dots & \langle v_1, v_n \rangle \\ \langle v_2, v_1 \rangle & \langle v_2, v_2 \rangle & \dots & \langle v_2, v_n \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle v_n, v_1 \rangle & \langle v_n, v_2 \rangle & \dots & \langle v_n, v_n \rangle \end{bmatrix} \quad (46)$$

Definition 15. A is a matrix that is $n \times n$. A is a positive semidefinite matrix if $A^T = A$ and $\vec{x}^T A \vec{x} \geq 0$

Theorem 16. All gram matrices are positive semi-definite. They are positive definite if and only if $\vec{v}_1, \dots, \vec{v}_n$ are linearly independent.

Suppose we are in \mathbb{R}^m and the inner product is the dot product. Let $\vec{v}_1, \dots, \vec{v}_n \in \mathbb{R}^m$

Let $A = [v_1, v_2, v_3, \dots, v_n]$. Then $K = A^T A$. Let A be a gram matrix generated by v_1, \dots, v_n with the dot product.

$$K = A^T A = \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vdots \\ \vec{v}_n \end{bmatrix} \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \end{bmatrix} \quad (47)$$

$$= \begin{bmatrix} \langle v_1, v_1 \rangle & \langle v_1, v_2 \rangle & \dots & \langle v_1, v_n \rangle \\ \langle v_2, v_1 \rangle & \langle v_2, v_2 \rangle & \dots & \langle v_2, v_n \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle v_n, v_1 \rangle & \langle v_n, v_2 \rangle & \dots & \langle v_n, v_n \rangle \end{bmatrix} \quad (48)$$

Proposition 3. Given an $m \times n$ matrix A . The following are true

1. The $m \times n$ matrix $k = A^T A$ is positive definite.
2. A has linearly independent columns.
3. $\text{rank}(A) = n$
4. $\text{Ker}(A) = \{0\}$

Theorem 17. Every inner product on \mathbb{R}^n is given by

$$\langle \vec{x}, \vec{y} \rangle = \vec{x}^T \cdot C \vec{y} \quad (49)$$

where C is a symmetric, positive definite $n \times n$ matrix.

Let $\vec{v}_1, \dots, \vec{v}_n \in \mathbb{R}^n$. Let $A = [\vec{v}_1 : \vec{v}_2 : \dots : \vec{v}_n]$. Then $K = A^T C A$ is the gram matrix with respect to the inner product $\vec{v}^T C \vec{w}$.

Theorem 18. Suppose A is an $m \times n$ matrix with linearly independent columns. Suppose C is any positive definite $m \times m$ matrix. Then $\vec{v}^T C \vec{w}$

Definition 16. The hilbert matrix $H = (h_{ij})$ where $h_{ij} = \frac{1}{i+j-1}$

The 3×3 hilbert matrix:

$$\begin{bmatrix} \frac{1}{3} & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix} \quad (50)$$

Problem. Make a gram matrix, not in \mathbb{R}^m . Let $V = \mathbb{C}^0[0, 1]$. Use the L^2 inner product.

$$\langle f, g \rangle = \int_0^1 f(x)g(x)dx. \quad (51)$$

$1, x, x^2$ are linearly independent.

$$\langle v_i, v_k \rangle = \int_0^1 x^{i-1}x^{j-1}dx \quad (52)$$

$$= \int_0^1 x^{i+j-2}dx \quad (53)$$

$$= \frac{1}{i+j-1}x^{i+j-1} \quad (54)$$

$$= \frac{1}{i+j-1} \quad (55)$$

$$\rightarrow \begin{bmatrix} \frac{1}{3} & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix}. \quad (56)$$

4 Orthogonality

4.1

Recall that

$$\vec{v} \cdot \vec{w} = v_1 \cdot w_1 + \dots + v_n \cdot w_n = \|\vec{v}\| \|\vec{w}\| \cos(\theta), \quad (57)$$

where θ is the angle between \vec{v} and \vec{w} . Because $\cos(\frac{\pi}{2}) = 0$, we know that $\vec{v} \cdot \vec{w} = 0$ if and only if \vec{v} is orthogonal to \vec{w} . In general, given $\vec{v}, \vec{w} \in \mathbb{R}^n$, \vec{v} is orthogonal to \vec{w} if and only if the angle between them is $\frac{\pi}{2}$.

Definition 17. Let U be any inner product space. A basis $\vec{u}_1, \dots, \vec{u}_h \in V$ is orthogonal if $\vec{u}_j \cdot \vec{u}_i = 0$ whenever $i \neq j$. In addition, if $\|\vec{u}_i\| = 1$ for all i 's, the basis is orthonormal.

Here are a few examples of orthonormal and orthogonal basis's:

1. $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is an orthonormal basis.
2. $\begin{pmatrix} 2 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is an orthogonal basis.
3. $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is an orthonormal basis.

If $\vec{v}_1, \dots, \vec{v}_n$ is an orthogonal basis, then

$$\frac{\vec{v}_1}{\|\vec{v}_1\|}, \frac{\vec{v}_2}{\|\vec{v}_2\|}, \dots, \frac{\vec{v}_n}{\|\vec{v}_n\|} \quad (58)$$

is an orthonormal basis.

Problem. Consider $\vec{u}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\vec{u}_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$. We know that these two matrices are not linearly dependent because \vec{u}_1 is not a multiple of \vec{u}_2 . We can see that this is an orthogonal basis.

$$\|\vec{u}_1\| = \sqrt{1^2 + 2^2} = \sqrt{5} \quad (59)$$

so we know

$$v_1 = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}, v_2 = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{pmatrix} \quad (60)$$

Proposition 4. Assume $\vec{v}_1, \dots, \vec{v}_n \in V$ with $\vec{v}_i \neq \vec{0}$ for all i . Assume that $\langle v_i, v_j \rangle \geq 0$ whenever $i \neq j$, then $\{v_1, \dots, v_n\}$ is linearly independent.

Proof. Assume that $c_1 \vec{v}_1 + \dots + c_n \vec{v}_n = \vec{0}$. Let $i \in \{1, \dots, n\}$

$$\begin{aligned} \langle c_1 \vec{v}_1 + \dots + c_n \vec{v}_n, v_i \rangle &= \langle \vec{0}, \vec{v}_i \rangle \\ c_1 \langle v_1, v_i \rangle + c_2 \langle v_2, v_i \rangle + \dots + c_i \langle v_i, v_i \rangle + \dots + c_n \langle v_n, v_i \rangle &= 0 \\ c_i \langle v_i, v_i \rangle &= 0 \\ c_i &= 0. \end{aligned}$$

Now we know that v_1, \dots, v_n are linearly independent. \square

Corollary 1. If $\dim(V) = n$ and $\vec{v}_1, \dots, \vec{v}_n$ are n vectors such that $\langle v_i, v_j \rangle = 0$ whenever $i \neq j$, then $\{\vec{v}_1, \dots, \vec{v}_n\}$ is a basis.

Theorem 19. Let $\vec{u}_1, \dots, \vec{u}_n$ be an orthonormal basis for V . Let $\vec{v} \in V$. Then we know $\vec{v} = c_1 \vec{u}_1 + \dots + c_n \vec{u}_n$. In fact, $c_i = \langle \vec{v}, \vec{u}_i \rangle$ and

$$\|\vec{v}\|^2 = \langle \vec{v}, \vec{u}_1 \rangle^2 + \langle \vec{v}, \vec{u}_2 \rangle^2 + \dots + \langle \vec{v}, \vec{u}_n \rangle^2. \quad (61)$$

Problem. \mathbb{P}^2 polynomials of degree ≤ 2 on $[0,1]$. Use the L^2 norm.

$$\langle p, q \rangle = \int_0^1 p(x)q(x)dx \quad (62)$$

Let $p_1 = 1, p_2 = x - \frac{1}{2}, p_3 = x^2 - x + \frac{1}{6}$.

$$\langle p_1, p_2 \rangle = \int_0^1 x - \frac{1}{2} dx = 0 \quad (63)$$

$$\langle p_1, p_3 \rangle = \int_0^1 x^2 - x + \frac{1}{6} dx = 0. \quad (64)$$

We have an orthogonal basis because of this. In order to check to see if it is orthonormal we must also do $\langle p_1, p_1 \rangle, \langle p_2, p_2 \rangle, \langle p_3, p_3 \rangle$

INSERT NOTES FROM 03.08 HERE

So if we have the basis $\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$, $v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. v_2 is equal to

$$\begin{aligned}
& \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \frac{\langle \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \rangle}{\| \begin{pmatrix} 1 \\ 2 \end{pmatrix} \|} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\
& \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \frac{4}{(\sqrt{1^2 + 2^2})^2} \\
& \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \frac{4}{5} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\
& \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \begin{pmatrix} \frac{4}{5} \\ \frac{8}{5} \end{pmatrix} = \begin{pmatrix} \frac{10}{5} \\ \frac{5}{5} \end{pmatrix} - \begin{pmatrix} \frac{4}{5} \\ \frac{8}{5} \end{pmatrix} \\
& = \begin{pmatrix} \frac{6}{5} \\ -\frac{3}{5} \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \frac{\langle \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \rangle}{\| \begin{pmatrix} 1 \\ 2 \end{pmatrix} \|^2} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \frac{4}{(\sqrt{1^2 + 2^2})^2} \\
& \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \frac{4}{5} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\
& \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \begin{pmatrix} \frac{4}{5} \\ \frac{8}{5} \end{pmatrix} = \begin{pmatrix} \frac{10}{5} \\ \frac{5}{5} \end{pmatrix} - \begin{pmatrix} \frac{4}{5} \\ \frac{8}{5} \end{pmatrix} \\
& \begin{pmatrix} \frac{6}{5} \\ -\frac{3}{5} \end{pmatrix} .
\end{aligned}$$

We can conclude that our basis is $\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} \frac{6}{5} \\ -\frac{3}{5} \end{pmatrix} \}$. Let's now use our basis and rewrite it as

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} \frac{6}{5} \\ -\frac{3}{5} \end{pmatrix} .$$

We can simplify this and solve for c_1, c_2

$$\begin{aligned}
c_1 &= \langle \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \rangle = 2 + 6 = 8 \\
c_2 &= \langle \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} \frac{6}{5} \\ -\frac{3}{5} \end{pmatrix} \rangle = \frac{12}{5} - \frac{9}{5} = \frac{3}{5}
\end{aligned}$$

Using Theorem 4.9 from the book, we do the following with an orthogonal basis to get its norm:

$$\begin{aligned}
a_1 &= \frac{8}{\|v_1\|^2} = \frac{8}{(\sqrt{1^2 + 2^2})^2} = \frac{8}{5} \\
a_2 &= \frac{\frac{3}{5}}{\|v_2\|^2} = \frac{\frac{3}{5}}{\left(\frac{6}{5}\right)^2 + \left(-\frac{3}{5}\right)^2} \\
&= \frac{\frac{3}{5}}{\frac{36}{25} + \frac{9}{25}} = \frac{15}{25} \dots \\
\begin{pmatrix} 2 \\ 3 \end{pmatrix} &= a_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + a_2 \begin{pmatrix} \frac{6}{5} \\ -\frac{3}{5} \end{pmatrix} \\
&= \frac{8}{5} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} \frac{6}{5} \\ -\frac{3}{5} \end{pmatrix} \\
&= \begin{pmatrix} \frac{8}{5} \\ \frac{16}{5} \end{pmatrix} + \begin{pmatrix} \frac{2}{5} \\ -\frac{1}{5} \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.
\end{aligned}$$

Problem. Example of an orthogonal basis. Let \mathbb{T}^n be the vector space of trigonometric polynomials.

$$\mathbb{T}^n = \sum_{0 \leq j+k \leq n} a_{jk} \sin^j(x) \cos^k(x).$$

Using the L^2 norm:

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f \cdot g.$$

An orthogonal basis is $\{1, \cos(x), \sin(x), \cos(2x), \sin(2x), \cos(3x), \sin(3x), \dots\}$. If we were going to do the L^2 norm for any of these equations we would need to do the following

$$\int_{-\pi}^{\pi} \sin(2x) \cos(4x) dx.$$

This equation is the Fourier series.

4.2

4.3 Orthogonal Matrices

Definition 18. A square matrix Q is orthogonal if $Q^t Q = Q \cdot Q^T = I$

If Q is orthogonal, then

- $Q^{-1} = Q^T$
- $\det(Q) = \pm 1$
- $Q \cdot Q^t = I$
- $\det(Q) \det(Q^T) = \det(I)$
- $(\det(Q))^2 = 1$
- $\det(Q) = \pm 1$

Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. Suppose that A is orthogonal, then

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \cdot \begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{aligned} a_{11}^2 + a_{12}^2 &= 1 \\ a_{11}a_{21} + a_{12}a_{22} &= 0 \\ a_{21}a_{11} + a_{22}a_{12} &= 0 \\ a_{21}^2 + a_{22}^2 &= 1. \end{aligned}$$

If we plot (a_{11}, a_{12}) on a graph, we can see that $\cos(\theta) = a_{12}$ and $\sin(\theta) = a_{11}$

Proposition 5. Q is orthogonal if and only if its columns form an orthonormal basis.

Proof. Let $Q = [U_1 : U_2 : \dots : U_n]$.

$$Q^T = \begin{bmatrix} U_1^T \\ U_2^T \\ \vdots \\ U_n^T \end{bmatrix}.$$

In $Q^T Q$, the i, j^{th} entry is $U_i^T \cdot U_j$.

$$U_i^T \cdot U_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}.$$

So the U_i 's form an orthonormal basis. □

Problem. Let $A = \begin{bmatrix} 3 & 5 \\ 7 & \frac{1}{2} \end{bmatrix}$, and let A be orthonormal. We know that $A \cdot A^T = A^T \cdot A = I$. Let's see if A is an orthonormal basis.

$$\begin{bmatrix} 3 & 7 \\ 5 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 7 & \frac{1}{2} \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

This is not equal because $3 \times 7 + 3 \times 5 \neq 0$. Now let's try letting $A = \begin{bmatrix} 3 & -7 \\ 7 & 3 \end{bmatrix}$.

$$\begin{bmatrix} 3 & 7 \\ -7 & 3 \end{bmatrix} \begin{bmatrix} 3 & -7 \\ 7 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

This vector works with the zero values, but not the ones values, so we need to normalize this vector.

$$\begin{aligned} \left\| \begin{bmatrix} 3 \\ 7 \end{bmatrix} \right\| &= \sqrt{4^2 + 7^2} = \sqrt{58} & \left\| \begin{bmatrix} -7 \\ 3 \end{bmatrix} \right\| &= \sqrt{(-7)^2 + 3^2} = \sqrt{58} \\ A &= \begin{bmatrix} \frac{3}{\sqrt{58}} & -\frac{7}{\sqrt{58}} \\ \frac{7}{\sqrt{58}} & \frac{3}{\sqrt{58}} \end{bmatrix}. \end{aligned}$$

Let's let $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and suppose Q is orthogonal.

$$Q^T \cdot Q = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$a^2 + c^2 = 1$$

$$ab + cd = 0$$

$$ab + cd = 0$$

$$b^2 + d^2 = 1.$$

Given that the vectors $\begin{bmatrix} a \\ c \end{bmatrix}, \begin{bmatrix} b \\ d \end{bmatrix}$ lie on the unit circle, we can determine that

$$a = \cos \theta$$

$$c = \sin \theta$$

$$b = \cos \phi$$

$$d = \sin \phi,$$

and we can determine that

$$0 = ab + cd = \cos \theta \cos \phi + \sin \theta \sin \phi = \cos(\theta - \phi).$$

If we use $\cos(\theta - \phi)$, we can determine that $\phi = \theta \pm \pi$, so $b = -\sin \theta, d = \cos \theta$ or $b = \sin \theta, d = -\cos \theta$. We either have Q in one of two forms.

$$\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \text{ or } \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix}.$$

The determinant of the left matrix is 1, and the determinant of the right matrix is -1. They both give us a counter clockwise rotation by θ , gives a reflection across the line with angle $\frac{\theta}{2}$

orthogonal matrices are square and $Q^t \cdot Q = QQ^t = I$. Every 2×2 orthogonal matrix has the form

$$\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \text{ or } \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix}.$$

In general, if Q is orthogonal, then $\det(Q) = \pm 1$

Theorem 20. The product of two orthogonal matrices is orthogonal. Recall, if Q is orthogonal, then $Q^{-1} = Q^T$. The orthogonal $n \times n$ matrices satisfy

- Closed under the dot product.
- Multiplication is associative.
- They all have inverses.
- The identity matrix is orthogonal.

4.4 Vector Spaces

Up until now we've been trying to get a basis to be able to establish a location of vectors. Once we have determined an inner product, we can find an angle. Let V be a subspace. Let $W \leq V$ be a finite dimensional subspace.

Definition 19. $\vec{z} \in V$ is orthogonal to w if it is orthogonal to every vector in w .

Note. If $\vec{w}_1, \dots, \vec{w}_n$ is a basis for w , then \vec{z} is orthogonal to w if and only if \vec{z} is orthogonal to w_1, \dots, w_n .

Definition 20. The orthogonal projection of \vec{V} onto w is the vector \vec{w} such that $\vec{z} = \vec{v} - \vec{w}$, where \vec{z} is orthogonal to w .

Theorem 21. Let $\vec{u}_1, \dots, \vec{u}_n$ be an orthogonal basis for w . Let $\vec{v} \in V$. The orthogonal projection of \vec{v} onto w is

$$\vec{w} = c_1 \vec{u}_1 + \dots + c_n \vec{u}_n \text{ where } c_i = \langle \vec{v}, \vec{u}_i \rangle / \|\vec{u}_i\|^2.$$

Note. If $\vec{v}_1, \dots, \vec{v}_n$ is an orthogonal basis for w then

$$w = a_1 \vec{v}_1 + \dots + a_n \vec{v}_n$$

$$a_i = \frac{\langle \vec{v}, \vec{v}_i \rangle}{\|\vec{v}_i\|^2}.$$

Definition 21. Let $w, z \leq V$ be subspaces. w is orthogonal to z if every vector in w is orthogonal to every vector in z . For example,

$$\langle \vec{w}, \vec{z} \rangle = 0$$

for every $\vec{w} \in w, \vec{z} \in z$.

Note. We only need to show this is true on the bases of w and z .

Definition 22. Let $w \in v$ be a subspace. The orthogonal complement of w , written w^T is the set of vectors in v orthogonal to w .

$$w^T = \{ \vec{v} \in v \mid \langle \vec{v}, \vec{w} \rangle = 0, \forall w \in w \}.$$

5 Mimimization and Least Squares

6 Equilibrium

7 Linearity

8 Eigenvalues and Singular Values

Definition 23. Let A be an $n \times n$ matrix. A scalar λ is an eigenvalue of A if

$$A\vec{v} = \lambda\vec{v}.$$

For some nonzero vector \vec{v} . \vec{v} is called the eigen vector corresponding to lambda.

Example. Consider $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. Let's find the eigenvalues of this matrix.

$$\begin{aligned} 0 &= \det \left(\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right) \\ &= \det \begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix} \\ &= (2-\lambda)^2 - 1 \\ 0 &= (3-\lambda)(1-\lambda). \end{aligned}$$

Our eigenvalues are 3 and 1. Remember that $(A - \lambda I)\vec{v} = \vec{0}$. Consider $\lambda = 3$

$$\begin{aligned} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ -x + y &= 0 \\ x &= y \\ \begin{pmatrix} x \\ x \end{pmatrix} &= x \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \end{aligned}$$

Now consider $\lambda = 1$.

$$\begin{aligned} A(\lambda I)\vec{v} &= \vec{0} \\ (A - \lambda I)\vec{v} &= \vec{0} \\ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ x + y &= 0 \\ y &= -x \\ \begin{pmatrix} x \\ -x \end{pmatrix} &= x \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \end{aligned}$$

Example. Consider $A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$. For eigenvalues, $0 = \det(A - \lambda I)$.

$$0 = \det \begin{pmatrix} 1-\lambda & 1 & 2 \\ 0 & 1-\lambda & 2 \\ 0 & 0 & 3-\lambda \end{pmatrix}$$

$$0 = (1-\lambda)(1-\lambda)(3-\lambda).$$

This makes our eigenvalues 1 and 3. For $\lambda = 3$,

$$\begin{pmatrix} -2 & 1 & 2 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} -2y + 2z &= 0 \\ y &= 0 \\ -2x + z + 2z &= 0 \\ -2x + 3z &= 0 \\ 3z &= 2x \\ y = z &= \frac{2}{3}. \end{aligned}$$

This would make our eigenvector

$$\begin{pmatrix} x \\ \frac{2}{3}x \\ \frac{2}{3}x \end{pmatrix} = x \begin{pmatrix} 1 \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}.$$

Definition 24. Given an eigenvalue λ of A , the corresponding eigenvectors form a subspace denoted v_λ . Note that $v_\lambda = \ker(A - \lambda I)$

Note. $\lambda = 0$ is an eigenvalue of A if and only if the $\ker(A - \lambda I) = \ker(A) = v_0 \neq \{0\}$. This is true if and only if A is singular ($\det(A) = 0$).

Example. Consider $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. The determinant of

$$\begin{pmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix}$$

is equal to zero. We can see that when $\lambda = 0$, the determinant is zero.

Proposition 6. If A is a real matrix and $\lambda + i\mu$ is an eigenvalue of A with eigenvector $\vec{v} = \vec{x} + i\vec{y}$, then $\lambda - i\mu$ is an eigenvalue of A with eigenvector $\vec{x} - i\vec{y}$.

9 Iteration

10 Dynamics