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Chapter 2

Vector Spaces and Bases

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§2.3.1 Linear Independence and Dependence

§2.4 Basis and Dimension

Definition 2.4.1

A basis of a vector space v is a collection of vectors $\vec{v_1}, \ldots, \vec{v_n}$ that 1. span v and 2. are linearly dependent.

Problem 1. *If we are looking at* $\mathbb{R}^2$, *with* $e_1 = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$, *and* $e_2 = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$ *We can tell that this is a basis of* $\mathbb{R}^2$. *We can tell this because the span of* v *is linearly independent. We can also see this because:*

$$a \times e_1 + b \times e_2 = \begin{pmatrix} a \\ b \end{pmatrix} \tag{2.1}$$

Problem 2. *Now we are going to look at an example in* $\mathbb{R}^3$, *with* $e_1 = \left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right)$, $e_2 = \left(\begin{smallmatrix}0\\1\\0\end{smallmatrix}\right)$, *and* $e_3 = \left(\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right)$. *We can figure out that this is a basis by doing the same technique as we did before:*

$$c_1\vec{e_1} + c_2\vec{e_2} + c_3\vec{e_3} = 0 \tag{2.2}$$
$$c_1 = c_2 = c_3 = 0 \tag{2.3}$$

Because c_1, c_2, *and* c_3 *are all equal to zero,* $\vec{e_1}$, $\vec{e_2}$, *and* $\vec{e_3}$ *form a basis.*

Theorem 2.4.1

If a vector space v has a basis with n elements, then every basis of v has n elements. We say v has dimension n. We write down $v = n$

Theorem 2.4.2

If the dimension of v is n, then any collection of $n + 1$ or more vectors must be linearly dependent.

Theorem 2.4.3

Suppose $v = n$

1. Every collection of more than n vectors is linearly dependent.
2. No set of fewer than n vectors spans v.
3. A set of n vectors is a basis if and only if it spans v.
4. A set of n vectors is a basis if and only if it is linearly dependent.

Problem 3. *Assume* $1, x, x^2$ *is a basis for* $\mathbb{P}^2$*. We are going to multiply* 1×5*,* $x \times 6$*, and* $x^2 \times 2$*.*

$$5 + 6x + 2x^2 \tag{2.4}$$
$$c_1 \times 1 + c_2 \times x + c_3 \times x^2 = 0 \tag{2.5}$$
$$dim(\mathbb{P}^2) = 3 \tag{2.6}$$
$$\tag{2.7}$$

Theorem 2.4.4

$\vec{v_1}, \ldots \vec{v_2}$ form a basis of v if and only if for all $\vec{v} \in v$, there exist unique $c_1, \ldots, c_n$ such that $\vec{v} = c_1\vec{v_1} + \ldots c_n\vec{v_n}$

Problem 4. *Let* $v = \mathbb{R}^2$*. Let* $\vec{v} = \left(\begin{smallmatrix}4\\3\end{smallmatrix}\right)$*. We know from previous problems that* $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$ *is a basis of* $\mathbb{R}^2$*. We can also figure out what our basis is by trying to figure out what our* c_1 *and* c_2 *values should be. Because the matrices we know are a basis consist of 1's and 0's, we can see that*

$$4\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}4\\3\end{pmatrix}. \tag{2.8}$$

The coordinates of $\vec{v}$ with respect to this basis, are $(4, 3)$. Let's consider a different basis. We are now going to look at the basis,

$$\left\{ \begin{pmatrix} 1 \\ -3 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right\} \tag{2.9}$$

Now we need to figure out the a and b values for this basis respectively.

$$a \begin{pmatrix} 1 \\ -3 \end{pmatrix} + b \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \tag{2.10}$$

$$\tag{2.11}$$

The coordinates of $\vec{v}$ with respect to this basis are $(4, 3)$. Let's consider the same problem but with the following basis:

$$\left\{ \begin{pmatrix} 1 \\ -3 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right\} \tag{2.12}$$

Now, we are going to do the same thing as before, where we solve for a and b in the following equation

$$a \begin{pmatrix} 1 \\ -3 \end{pmatrix} + b \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \tag{2.13}$$

We can setup this to be a system of equations that we can turn into a matrix

$$\begin{cases} 1a + 2b = 4 \\ -3a + (-1)b = 3 \end{cases} \rightarrow \begin{bmatrix} 1 & 2 & \vdots & 4 \\ -3 & -1 & \vdots & 3 \end{bmatrix} \tag{2.14}$$

And now we can use the basic row operation $R_2 = R_2 + 3R_1$ in order to solve for a and b:

$$\begin{bmatrix} 1 & 2 & \vdots & 4 \\ 0 & 5 & \vdots & 5 \end{bmatrix} \tag{2.15}$$

$$5b = 15 \qquad a + 2b = 4 \tag{2.16}$$

$$b = 3 \qquad 1 + 2 * 3 = 4 \tag{2.17}$$

$$a = -2 \tag{2.18}$$

§2.5 The fundamental Matrix Subspaces (Kernel and Image)

Definition 2.5.1

he image of an $m \times n$ matrix A is the subspace spanned by the columns of A.