Advanced Methods in Nonlinear Control (SS 2024) — Task 1

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Solution 1.1.
Consider the Lyapunov function candidate

$$V(x) = \int_0^x f(\xi)\mathrm{d}\xi.$$

With the properties of f given in the problem statement it follows that V is positive definite.

Furthermore it holds that

$$\frac{\mathrm{d}}{\mathrm{d}t}V(x) = f(x)\dot{x} = -f^2(x) < 0$$

is negative definite. Thus the asymptotic stability of the origin follows from Lyapunov's direct method.

Solution 1.2.
Consider the Lyapunov function candidate

$$V(\boldsymbol{x}) = \frac{1}{2}\|\boldsymbol{x}\|^2 = \frac{1}{2}(x_1^2 + x_2^2) > 0$$

it follows that

$$\begin{aligned}\frac{\mathrm{d}V(\boldsymbol{x}(t))}{\mathrm{d}t} &= -x_1^2 - x_1x_2 - x_1^2(x_1^2 + x_2^2) + x_1x_2 - x_2^2 - x_2^2(x_1^2 + x_2^2)\\ &= -(x_1^2 + x_2^2) - (x_1^2 + x_2^2)^2 < 0\end{aligned}$$

implying the local asymptotic stability of the origin. Given that $V(\boldsymbol{x})$ is radially unbounded it follows furhter that $\boldsymbol{x} = \boldsymbol{0}$ is globally asymptotically stable, i.e. $\forall\, \boldsymbol{x}_0 \in \mathbb{R}^2$ it holds that $\lim_{t\to\infty} \|\boldsymbol{x}(t)\| = 0$.

Actually one can further bound from above the right hand side of $\frac{\mathrm{d}V(\boldsymbol{x}(t))}{\mathrm{d}t}$ as

$$\frac{\mathrm{d}V(\boldsymbol{x}(t))}{\mathrm{d}t} \leq -2V(\boldsymbol{x}(t))$$

implying the exponential stability of $\boldsymbol{x} = \boldsymbol{0}$.

Solution 2.1.
For the system

$$\begin{aligned}\dot{x}_1 &= x_1^3 + x_2, & x_1(0) &= x_{10} \qquad (1a)\\ \dot{x}_2 &= u, & x_2(0) &= x_{20}. \qquad (1b)\end{aligned}$$

consider x_2 as virtual input together with the candidate Lyapunov function $V_1(x_1) = \frac{1}{2}x_1^2$. Next, require that

$$\frac{\mathrm{d}V_1}{\mathrm{d}t}(x_1) = x_1\left(x_1^3 + x_2\right) = -Q_1(x_1)$$

with $Q_1(x_1) > 0$. For the particular choice $Q(x_1) = k_1x_1^2$ with $k_1 > 0$ one obtains the condition

$$x_2 = -x_1^3 - k_1x_1.$$

Introduce the deviation variable $z = x_2 + k_1x_1 + x_1^3$ with

$$\dot{z} = u + (k_1 + 3x_1^2)(x_1^3 + x_2) = u + (k_1 + 3x_1^2)(x_1^3 + z - k_1x_1 - x_1^3).$$

With the candidate Lyapunov function $W(x_1, z) = V_1(x_1) + \frac{1}{2}z^2$ it follows that

$$\begin{aligned}\frac{\mathrm{d}W}{\mathrm{d}t}(x_1, z) &= x_1\left(x_1^3 + z - x_1^3 - k_1x_1\right) + z\left(u + (k_1 + 3x_1^2)(x_1^3 + z - k_1x_1 - x_1^3)\right)\\ &= -k_1x_1^2 + z\left(x_1 + u + (k_1 + 3x_1^2)(x_1^3 + z - k_1x_1 - x_1^3)\right)\end{aligned}$$

Requiring

$$z\left(x_1 + u + (k_1 + 3x_1^2)(x_1^3 + z - k_1 x_1 - x_1^3)\right) = -Q_2(z) < 0$$

it follows that $\frac{dW}{dt} < 0$ imlying the asymptotic stability of the origin.

For the particular choice $Q_2(z) = k_2 z^2 > 0$ with $k_2 > 0$ the associated control input is determined as

$$u = -k_2 z - x_1 - (k_1 + 3x_1^2)(x_1^3 + z - k_1 x_1 - x_1^3). \tag{2}$$

Solution 3.1.
In state space form with $x_1 = x$ and $x_2 = \dot{x}$, $u = F$ the given mass–spring–damper system is written as

$$\begin{aligned}
\dot{x}_1 &= x_2, && t > 0, \quad x_1(0) = x_{10}\\
\dot{x}_2 &= -\frac{d}{m}x_2 - \frac{1}{m}k(x_1)x_1 + \frac{1}{m}u, && t > 0, \quad x_2(0) = x_{20}\\
y &= x_2
\end{aligned}$$

Considering the total mechanical energy (i.e., the sum of potential and kinetic energies) as storage function

$$S(\boldsymbol{x}) = \frac{m}{2}x_2^2 + \int_0^{x_1} k(\xi)\xi \mathrm{d}\xi \succ 0$$

one obtains

$$\begin{aligned}
\frac{\mathrm{d}S}{\mathrm{d}t}(\boldsymbol{x}) &= m x_2 \left(-\frac{d}{m}x_2 - \frac{1}{m}k(x_1)x_1 + \frac{1}{m}u\right) + k(x_1)x_1x_2\\
&= -dx_2^2 + ux_2 \leq uy
\end{aligned}$$

showing the passivity of the system. Considering the output–feedback control $u = -\kappa_c y$ one has

$$\frac{\mathrm{d}S}{\mathrm{d}t}(\boldsymbol{x}) = -(d + \kappa_c)y^2.$$

This expression is negative semi–definite for $\kappa_c > -d$. By the Krasovskii–LaSalle invariance principle it follows that $\boldsymbol{x}(t)$ converges to the largest positively invariant subset of

$$\mathfrak{Y}_0 = \{\boldsymbol{x} \in \mathbb{R}^2 | x_2 = 0\}.$$

To characterize this set, consider the dynamics in $\mathfrak{Y}_0$, given by

$$\begin{aligned}
\dot{x}_1|_{\mathfrak{Y}_0} &= x_2|_{\mathfrak{Y}_0} = 0\\
\dot{x}_2|_{\mathfrak{Y}_0} &= -\frac{1}{m}k(x_1)x_1|_{\mathfrak{Y}_0}
\end{aligned}$$

so that the only positively invariant subset of $\mathfrak{Y}_0$ is the origin. This proofs the asymptotic convergence of $\boldsymbol{x}(t)$ to the origin.

Solution 4.1.
Consider the Lyapunov functional $V(x) = \frac{1}{2}x^2$ with the feedback controller $u = k(x)$. Then

$$\dot{V}(x) = x^4 + x\left(x^2 + \sin(x) + 1.5\right)k(x).$$

Choosing $k(x)$ according to

$$k(x) = -\frac{\alpha x + x^3}{x^2 + \sin(x) + 1.5}$$

implies that $\dot{V}(x) = -\alpha x^2$ is negative definite for some positive $\alpha > 0$. Therefore the closed–loop system is GAS.

Next, we choose $\tilde{k} := k(x) - \nabla V(x)\left(x^2 + \sin(x) + 1.5\right)$ which leads to

$$\nabla V(x)\left(x^3 + \left(x^2 + \sin(x) + 1.5\right)\left(\tilde{k}(x) + d\right)\right)$$

$$
\begin{aligned}
&= -\alpha x^2 - \left(\nabla V(x)\left(x^2 + \sin(x) + 1.5\right)\right)^2 + \nabla V(x)\left(x^2 + \sin(x) + 1.5\right) d \\
&\leq -\alpha x^2 - \left(\nabla V(x)\left(x^2 + \sin(x) + 1.5\right)\right)^2 + \frac{1}{2}\left(\nabla V(x)\left(x^2 + \sin(x) + 1.5\right)\right)^2 + \frac{1}{2}|d|^2 \\
&= -\alpha x^2 - \frac{1}{2}\left(\nabla V(x)\left(x^2 + \sin(x) + 1.5\right)\right)^2 + \frac{1}{2}|d|^2 \\
&= -\alpha x^2 - \frac{1}{2}\left(x^2 + x\sin(x) + 1.5x\right)^2 + \frac{1}{2}|d|^2
\end{aligned}
$$

and implies ISS. The resulting controller is therefore given by

$$
\tilde{k}(x) = -\frac{\alpha x + x^3}{x^2 + \sin(x) + 1.5} - x\left(x^2 + \sin(x) + 1.5\right)
$$

with $\alpha > 0$.