Advanced Methods in Nonlinear control summary

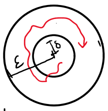
Sonntag, 12. Januar 2025 18:47

Lyapunov Stability:

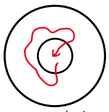
$\dot{x} = f(x)$, $x(0) = x_0$ $x(t) = \Phi(t, x_0)$

Equilibrium for $\dot{x} = 0 \Rightarrow f(x^*, p) = 0$

Stability: $\forall x_0 : \|x_0 - x^*\| \leq \delta \longrightarrow \|\Phi(t; x_0) - x^*\| < \varepsilon$



Lyapunov

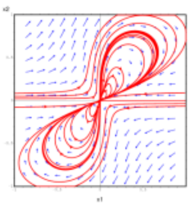


asymptotic

Attractive equilibrium: $\forall x_0 \in D : \lim_{t\to\infty} \|\Phi(t, x_0) - x^*\| = 0$

domain of attraction

attraction ≠ stability (e.g. large transients in Vinograd Systems)
Nonlinear systems can have multiple attractors
attraction + stability = asymptotically stable



— attractive but unstable

exponential stability: $\forall x_0 \in D : \|\Phi(t, x_0) - x^*\| \leq a\|x_0 - x^*\| e^{-\lambda t}$

Amplitude convergence rate

$t_c = \lambda^{-1}$ = characteristic time constant

Definition for attraction can also be used for sets:

$\forall x_0 \in D : \lim_{t\to\infty} \Phi(t, x_0) \in M \rightarrow M$ is attractive for the domain D

positive invariance for sets: $M \subseteq \mathbb{R}^n$ is positive invariant, if

$\forall x_0 \in M \Rightarrow \Phi(t, x_0) \in M \;\forall t \geq 0$. e.g. x^*

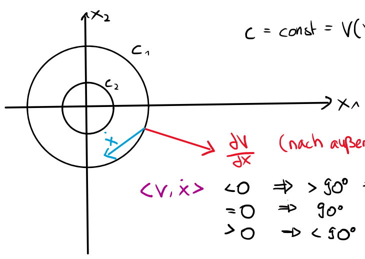
Lyapunov's direct method

semi
· positive definite : $\forall x : V(x) \geq 0$, and $V(x) = 0$ only for $x = 0$
· neg. definite : $\forall x : V(x) \leq 0$, and $V(x) = 0$ only for $x = 0$

$V : D \subset \mathbb{R}^n$, $V(x) > 0$.

$$\boxed{\text{If } \frac{dV}{dt} = \frac{dV}{dx}\dot{x} \leq 0 \Rightarrow \text{stable in the sense of Lyapunov} \quad < 0 \Rightarrow \text{asymptotically stable!}}$$

Geometrische Interpretation:



$c = \text{const} = V(x_i)$ (level curves)

$\frac{dV}{dx}$ (nach außen, da $V(x) > 0$ und steigend)

$\langle V, \dot{x} \rangle$ $< 0 \Rightarrow > 90° \Rightarrow$ konverg.
$= 0 \Rightarrow 90° \Rightarrow$ bleibt auf lvl. curve
$> 0 \Rightarrow < 90° \Rightarrow$ divergiert

only holds locally, unless $\frac{dV}{dx} > 0 \;\forall \|x\| \Rightarrow$

radially unbounded: $\lim_{\|x\|\to\infty} V(x) \longrightarrow \infty$

Krasovsky Lasalle:

$D \subseteq \mathbb{R}^n$ is a positively invariant compact set. $V \in C^1(D \to \mathbb{R})$ is a pos. def. function

$\frac{dV}{dt} \leq 0 \;\forall x \in D$. Then $x(t)$ converges to the largest positively invariant set

$M \subseteq \mathcal{X}_0$, with $\mathcal{X}_0 = \{ x \in \mathbb{R}^n \mid \frac{dV(x)}{dt} = 0 \}$

Assume it holds: $M = \{0\}$, then the origin $x=0$ is asymptotically stable

if
$$\alpha\|x\|^2 \le V(x) \le \beta\|x\|^2$$
$$\frac{dV}{dt}(x) \le -\gamma V(x)$$
then $x=0$ is exponentially stable and $a = \sqrt{\beta/\alpha}$, $\lambda = \gamma/2$

and vice versa: if $x=0$ is exp. stable, then $\gamma>0$ exists, as well as a Lyapunov function $V(x)>0$

Integrator Backstepping:

System:
$$\dot{x}_1 = f(x_1) + g(x_1)x_2$$
$$\dot{x}_2 = u$$

$\Rightarrow$
1. x_2 als fiktiven Input festlegen $\mu(x_1)$
2. Lyapunov Funktion für x_1, $V(x_1) = \frac{1}{2}x_1^2$
3. Lyapunov-Funktion für $z = x_2 - \mu(x_1)$; $V(z) = \frac{1}{2}z^2$
4. $V(x_1) + V(z) = W$ ← Lyapunov-Funktion zur Stabilis. des Gesamtsystems

$$u = \frac{\partial\mu(x_1)}{\partial x_1}(f(x_1) + g(x_1)\cdot x_2) - \frac{\partial V_1(x_1)}{\partial x_1} g(x_1) - k\cdot(x_2 - \mu(x_1))$$

Disspativity and passivity based control: using the storage function as Lyapunov candidates

MIMO: $\dot{x} = f(x) + G(x)\cdot u$, $x_0 = x(0)$, $G(x) = [g_1(x) \dots g_p(x)]$, $u \in \mathbb{R}^p$
$y = h(x)$, $y \in \mathbb{R}^m$

ω = supply-rate

i Dissipative: $S(x(t)) - S(x_0) \le \int_0^t \omega(u(\tau), y(\tau))\,d\tau \quad \forall\, x_0, t \ge 0$

$$\Leftrightarrow \frac{dS(x)}{dt} = \frac{\partial S}{\partial x}(f(x) + G(x)u) \le \omega(u,y)$$

ii Passive: if $m=p$ and system is dissipative with $\omega(u,y) = u^T y$

iii A static map $y = \varphi(u)$ with $m=p$ is passive if $u^T y = u^T\varphi(u) \ge 0$

Example: $u_R = R\cdot I$ let $u = I$ and $y = u_R = \varphi(I) = R\cdot I$ ← static map

$\Rightarrow$ ii) $u^T\cdot y = I\cdot u_R = RI^2 > 0$ → passive

Example: $\frac{du_C}{dt} = -\frac{1}{RC}u_C + \frac{1}{C}i_q$ $\quad u = i_q$

$y = u_C$

take $S = \frac{1}{2}Cu_C^2 > 0$

$$\Rightarrow \frac{dS}{dt} = -\frac{1}{R}u_C^2 + i_q u_C \le i_q u_C = u\cdot y$$

strictly state dissipative: $\frac{dS(x)}{dt} \le \omega(u,y) - K\|x\|^2$

strictly state passive if $\omega = u^T y$

if $S(x)>0$ and strictly state passive: $u = -K\cdot y$ asymptotically stabilizes the system $(K \ge 0)$ since
$$\frac{dS(x)}{dt} \le -K\|x\|^2 + u^T y = -K\|x\|^2 - y^T\cdot K\cdot y \le -K\|x\|^2 < 0$$

↳ Krasovsky Lasalle: $Y_0 = \{x \in \mathbb{R}^n \mid y = h(x) = 0\}$

since pos.invariant, $y \equiv 0 \Rightarrow y^k = 0$, $k \in \mathbb{N}_0$

with $L_f h_i(x) = \frac{\partial h_i}{\partial x} f(x)$ $\quad i \in (1 \dots m)$

$y_i = h_i(x) = 0$

$\dot{y}_i = L_f h_i(x) = 0$

$\vdots$

$y^n = L_f^n h_i(x) = 0$

$$\Theta = \begin{bmatrix} h_i(x) \\ \vdots \\ L_f^n h_i(x) \end{bmatrix} = 0$$ only for $x = 0$ $\Rightarrow$ completely observable + asymptotically stable

zero state observable: $y \equiv 0 \Rightarrow x = 0$ $\quad (\theta(x) = 0$ only if $x = 0)$

zero state detectable: $y \equiv 0 \Rightarrow x \to 0$

↳ trajectories converge into set, where origin is the unique attractor

$$\dot{x} = f(x) + g(x)u \qquad , r = 1 \qquad (2.16)$$
$$y = h(x)$$

Detectability: zero dynamics

$L_g h \neq 0$

$$\begin{bmatrix} z \\ \xi \end{bmatrix} = \Psi(x) = \begin{bmatrix} h(x) \\ \Phi(x) \end{bmatrix}, \quad z \in \mathbb{R}, \ \xi \in \mathbb{R}^{n-1}$$

$$\dot{z} = L_f h(x) + L_g h(x) u$$

$$\dot{\xi} = \varphi(z, \xi, u) \xrightarrow{\text{zero dynamics}} \varphi(0, \xi, u) = \varphi_0(\xi) \qquad u = -\frac{L_f h(x)}{L_g h(x)} \Big|_{x = \Psi^{-1}\begin{bmatrix} 0 \\ \xi \end{bmatrix}}$$

$$y = z$$

passive System and $S(x) > 0$:

i) zero-dynamics are Lyapunov-stable

ii) if system is even zero state observable, $u = -ky$ asymptotically stabilizes the origin $x = 0$

$\frac{dS}{dt} \leq y \cdot u = 0$ — 0 for zero dynamics

→ k. LaSalle and zero state observ. says this subset is given by $M = \{0\}$

It is possible to passivate Systems with respect to v

Problem if S is only semi pos. def.

if $L_g h \neq 0$:

i) minimum phase if $\xi = 0$ is asympt. stable equilibr.

ii) weakly min. Phase if $\exists$ pos. def $V_0(\xi) > 0$ in neighbourhood of $\xi = 0$ that is C^2 and $L_{f_0} V_0(\xi) \leq 0 \ \forall \xi \in \mathcal{N}_{\xi_0}$

→ weakly min phase implies Lyapunov stability at $\xi = 0$

⇒ (2.16) is feedback equivalent to a passive System if $\exists S(x) > 0$, $S(x) \in C^2$, $r = 1$ at $x = 0$ and weakly min. phase

($\hat{=}$ closed loop sys. is passive) and can be stabilized by

$$u = \frac{-ky - L_f h(x)}{L_g h(x)}$$

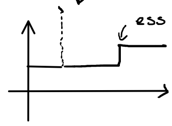
INPUT-TO-STATE Stabiltiy

$$\dot{x} = f(x, u) \qquad u \in \mathcal{U}$$

$$\|u\|_\infty = \operatorname*{ess\,sup}_{t \geq 0} |u(t)| = \inf_{D \subset \mathbb{R},\ \mu(D) = 0} \ \sup_{t \in \mathbb{R}_{\geq 0} \setminus D} |u(t)| \quad \to$$

point of 0 measure

ess sup $|(u(t)|$



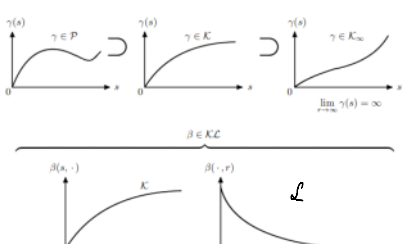
uniformly continuous $\supseteq$ absolutely continuous $\supseteq$ Lipschitz continuous

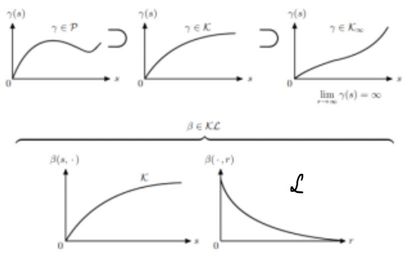
$[a, b] < \varepsilon$, $\varphi(a) - \varphi(b) < \delta$, $\sum_{k=1}^{r} |\Phi(b_k) - \Phi(a_k)| < \delta$, $|f(b, u) - f(a, u)| \leq L|y - x|$

(finites r)

comparison functions

$\mathcal{P} := \{\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \mid \gamma$ is continuous, $\gamma(0) = 0$ and $\gamma(r) > 0$ for $r > 0\}$,
$\mathcal{K} := \{\gamma \in \mathcal{P} \mid \gamma$ is strictly increasing$\}$,
$\mathcal{K}_\infty := \{\gamma \in \mathcal{K} \mid \gamma$ is unbounded$\}$,
$\mathcal{L} := \{\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \mid \gamma$ is continuous and strictly decreasing with, $\lim_{t \to \infty} \gamma(t) = 0\}$,
$\mathcal{KL} := \{\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \mid \beta$ is continuous, $\beta(s, \cdot) \in \mathcal{K}$ for any $s \geq 0$, $\beta(\cdot, r) \in \mathcal{L}$ for any $r > 0\}$.





Stability in autonomous systems

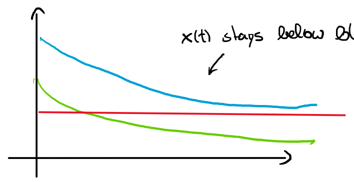
$\dot{x} = f(x)$ is globally stable if $\exists\ \sigma \in K_\infty : |\Phi(t,x_0)| \leq \sigma(|x_0|) \quad \forall x_0 \in \mathbb{R}^n, t \geq 0$ sol < Bound

is globally asympt. stable if $\exists \beta \in K\mathcal{L} : |\Phi(t,x_0)| \leq \beta(|x_0|,t) \quad \forall x_0 \in \mathbb{R}, t \geq 0$ → sol. decreases with time

↳ is exp. stable, if $\beta(r,t) = M r \cdot e^{-\lambda t}$, $M, \lambda > 0$

Stability in Systems with inputs

$\dot{x} = f(x,u)$ is ISS if $\exists \beta \in K\mathcal{L}, \gamma \in K : |\Phi(t,x_0,u)| \leq \underbrace{\underbrace{\beta(|x_0|,t)}_{\text{transient behaviour}} + \underbrace{\gamma(\|u\|_\infty)}_{\text{ISS-Gain}}}_{\text{upper bound}} \quad \forall t \geq 0$



$\dot{x} = f(x,u)$ is 0-GAS (0-globally asympt. stable) if it is GAS with $u \equiv 0$

$\dot{x} = f(x,u)$ has AG (Asymptotic gain property) if $\exists \gamma \in K_\infty : \lim_{t\to\infty} \sup |\Phi(t,x_0,u)| \leq \gamma(\|u\|_\infty) \quad \forall u \in \mathcal{U}$ and $\forall x \in \mathbb{R}^n$

AG: solution does not „explode" for any input u

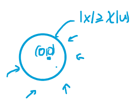
The system is ISS only if it is 0-GAS && AG

A linear system $\dot{x} = Ax + Bu$ is ISS if and only if it is 0-GAS

Lyapunov Characterization of the ISS

$\dot{x} = f(x,u)$

1. $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and $\alpha_{1,2} \in K_\infty : \alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \quad \forall x \in \mathbb{R}^n$
2. $\exists \alpha \in P$ and $\chi \in K : |x| \geq \chi(|u|) \Rightarrow \nabla V(x) f(x,u) \leq -\alpha(|x|)$
   or alternatively $\nabla V(x) f(x,u) \leq -\alpha_3(|x|) + \alpha_4(|u|)$



if 1. and 2. hold ⇒ Lyapunov-function

ISS-Feedback-Design

$\dot{x} = f(x,u,d)$ — disturbance $\in \mathcal{D} := L_\infty(\mathbb{R}_{\geq 0}, \mathbb{R}^P)$

if $\exists$ → Feedback $k(x)$, so that System is ISS: ISS stabilizable
→ " " " $\dot{x} = f(x,k(x),0)$ is GAS: Gas-stabilizable

consider input affine System:

$\dot{x} = g_0(x) + g_1(x)(u+d)$.

if $\exists\ u = k(x)$: $x = 0$ is GAS-equilibrium ⇒ $\exists\ u = \tilde{k}(x) = k(x) - \nabla V(x) g_1(x)$ so System is ISS

ISS-Backstepping

$\dot{x} = f(x) + G_1(x) z + G_2(x) \cdot d$, $x \in \mathbb{R}^n$

$\dot{z} = u + F(x,z) \cdot d$, $z \in \mathbb{R}^m$

⇒ z as virtual input

if System is ISS with $z = k(x)$ and $k(0) = 0$, then the entire System is stabilizable with

$u = \tilde{k}(x,z) = \nabla k(x)(f(x) + G_1(x) z) - \nabla V(x) G_1(x) - (z - k(x))(1 + |F(x,z)|^2 + |\nabla k(x) G_2(x)|^2)$

Analog to „normal" Backstepping

Cascade and Feedback interconnections

$$\dot z = f(z,v)$$
$$\dot x = g(x,u)$$

Parallel systems are both ISS $\Rightarrow$ whole system is ISS

$$\dot z = f(z,x) \qquad u \to \boxed{x} \to \boxed{z}$$
$$\dot x = g(x,u)$$

in a cascade: if each subsystem is ISS, then the cascade is ISS

in the general case, this is not enough

$$\dot x_1 = f(x_1,x_2,u_1)$$
$$\dot x_2 = g(x_1,x_2,u_2)$$

$\Rightarrow$ Small-gain condition

- $\psi_{i,1} \le V_i(x_i) \le \psi_{i,2}$, $\psi \in K_\infty$ $\forall x \in \mathbb{R}^n$, $\chi \in K$, $\alpha_i \in K_\infty$, $\gamma \in K$

- $V_1(x_1) \ge \max(\chi_1(V_2(x_2)), \gamma(|u_1|)) \Rightarrow \nabla V_1(x_1) f(x_1,x_2,u_1) \le -\alpha_1(V(x_1))$
  and
  $V_2(x_2) \ge \max(\chi_2(V_1(x_1)), \gamma(|u_2|)) \Rightarrow \nabla V_2(x_2) g(x_1,x_2,u_2) \le -\alpha_2(V(x_2))$

$$\chi_1 \circ \chi_2(r) < r \quad \forall r$$

also it holds that:

- a $\mathcal{K}_\infty$-function σ continuously differentiable on $(0,\infty)$ with $\sigma'(r) > 0$ for all $r > 0$ such that
  $$\chi_2(r) < \sigma(r) < \chi_1^{-1}(r) \quad \text{for all} \quad r > 0;$$
- a locally Lipschitz on $\mathbb{R}^{n_1+n_2} \setminus \{0\}$ function $V : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}_{\ge 0}$ defined by
  $$V(x_1,x_2) = \max\{\sigma(V_1(x_1)), V_2(x_2)\}$$
  such that for almost all $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$ and for all $u_1 \in \mathbb{R}^{m_1}$, $u_2 \in \mathbb{R}^{m_2}$
  $$\nabla V(x) \begin{pmatrix} f_1(x_1,x_2,u_1) \\ f_2(x_1,x_2,u_2) \end{pmatrix} \le -\alpha(V(x_1,x_2)) \quad \text{whenever} \quad V(x_1,x_2) \ge \gamma(|(u_1,u_2)^T|)$$
  for some $\alpha \in \mathcal{K}_\infty$, $\gamma \in \mathcal{K}$.

and the existence of a locally Lipschitz function is sufficient to show ISS

ISS-related stability notions

Integral-Input-State stability (iISS)

$\beta \in KL \qquad \alpha, \gamma \in K_\infty$

$$\alpha(\phi(t,x_0,u)) \le \beta(|x_0|,t) + \int_0^t \gamma(|u(s)|)\,ds \quad \forall t \ge 0$$

$\Rightarrow$ if u is chosen, so that $\int_0^\infty \gamma(|u(s)|)\,ds < \infty \Rightarrow \phi(t,x_0,u) \xrightarrow{t\to\infty} 0$

alternative definition

$$\alpha_1(|x|) \le V(x) \le \alpha_2(|x|) \quad \text{and} \quad \nabla V(x) f(x,u) \le -\alpha_3(|x|) + \alpha_4(|u|)$$

mit $\alpha_{1,2,4} \in K_\infty$ and $\alpha_3 \in K$ (for ISS $\alpha_3 = K_\infty$!)

Local ISS (LISS)

$\phi(t,x_0,u) \le \beta(|x_0|,t) + \gamma(\|u\|_\infty)$ $\forall t \ge 0$, $|x| < \rho_x$, $\|u\|_\infty < \rho_u$, ρ_x and $\rho_u > 0$

ISS $\subset$ iISS $\subset$ LISS and ISS $\subset$ iISS $\subset$ 0-GAS

ISS with a set $\mathcal{A}$

$|\phi(t,x_0,u)|_{\mathcal{A}} \le \beta(|x_0|_{\mathcal{A}},t) + \gamma(\|u\|_\infty)$, $\beta \in KL$, $\gamma \in K$