ADVANCED METHODS IN NONLINEAR CONTROL

Lecture Notes for Summer Term 2024

Dr.–Ing. Pascal Jerono
Karlsruhe Institute of Technology

Karlsruher Institut für Technologie

ADVANCED METHODS IN NONLINEAR CONTROL

Lecture Notes for Summer Term 2024

Dr.–Ing. Pascal Jerono

Karlsruhe Institute of Technology
Digital Process Engineering Group
Institute for Mechanical Process Engineering and Mechanics
Hertzstr. 16, Building 06.40
D–76187 Karlsruhe

pascal.jerono@kit.edu
http://www.mvm.kit.edu/dpe.php

Preface

The lecture notes are compiled using LaTeX. The presented numerical results are obtained using MATLAB and Maxima.

This lecture notes continue on the course „Nonlinear Process Control“ at KIT, which addresses the fundamental concepts of nonlinear control techniques.

Special thanks to Prof. Dr. habil. Alexander Schaum and Dr. Petro Feketa for providing the majority of the lecture notes from the eponymous course at Kiel University.

Pascal Jerono

Contents

Chapter 1

Lyapunov stability and Lyapunov's direct method

1.1 Stability in the sense of Lyapunov

Consider the autonomous nonlinear dynamical system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0, \tag{1.1}$$

with solutions

$$\boldsymbol{x}(t) = \boldsymbol{\phi}(t; \boldsymbol{x}_0).$$

A fundamental property of linear and nonlinear systems is the stability of **equilibrium points**, i.e. solutions $\boldsymbol{x}^*$ of the algebraic equation

$$\boldsymbol{0} = \boldsymbol{f}(\boldsymbol{x}^*, \boldsymbol{p}). \tag{1.2}$$

Throughout this text we will be concerned with stability and thus we need a formal definition. Here, we employ the common definitions associated to A. Lyapunov (Fuller, 1992; Khalil, 1996).

Definition 1.1

An equilibrium point $\boldsymbol{x}^*$ of (1.1) is said to be **stable** (*in the sense of Lyapunov*), if for any $\epsilon > 0$ there exists a constant $\delta > 0$ such that for any initial deviation from equilibrium within a δ-neighborhood, the solution trajectory is comprised within an ϵ-neighborhood, i.e.,

$$\forall\, \boldsymbol{x}_0 \,:\, \|\boldsymbol{x}_0 - \boldsymbol{x}^*\| \leq \delta \;\Rightarrow\; \|\boldsymbol{\phi}(t; \boldsymbol{x}_0) - \boldsymbol{x}^*\| \leq \epsilon \quad \forall\, t \geq 0. \tag{1.3}$$

If $\boldsymbol{x}^*$ is not stable, it is called **unstable**.

The concept of **instability** can be formalized by rigorously negating the statement of stability, i.e., *there exists an $\epsilon > 0$ so that for all $\delta > 0$ there exists a $T > 0$ and (at least one) $\boldsymbol{x}_0$ in the δ−neighboorhood (i.e., $\|\boldsymbol{x}_0 - \boldsymbol{x}^*\| \leq \delta$) for which $\|\boldsymbol{\phi}(T; \boldsymbol{x}_0) - \boldsymbol{x}^*\| > \epsilon$*, meaning that at least for one ϵ−neighborhood one will always find an initial condition, so that after some time $T > 0$ the trajectory will leave the ϵ−neighboorhood.

This concept is illustrated in Figure 1.1 (left), and is also referred to as **stability in the sense of Lyapunov**. Stability implies that the solutions stay arbitrarily close to the equilibrium whenever the initial condition is chosen sufficiently close to the equilibrium point. Note that stability implies boundedness of solutions, but not that these converge to an equilibrium point. Convergence in turn is ensured by the concept of attractivity.

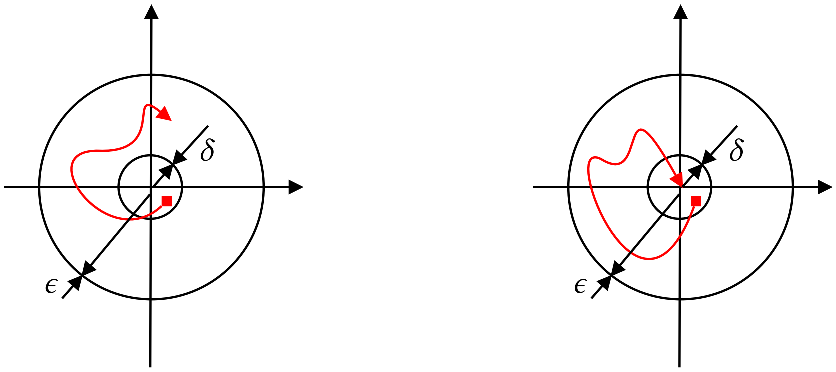

Figure 1.1: Qualitative illustration of the concepts of stability (left) and asymptotic stabilty (right).

Definition 1.2

The equilibrium point $\boldsymbol{x}^*$ is called **attractive** (or an **attractor**) for the set $\mathcal{D} \subseteq \mathbb{R}$, if

$$\forall\, \boldsymbol{x}_0 \in \mathcal{D} \,:\, \lim_{t\to\infty} \|\boldsymbol{\phi}(t;\boldsymbol{x}_0) - \boldsymbol{x}^*\| = 0. \tag{1.4}$$

Note that an equilibrium point which instead of attracting the trajectories does repell them is called a **repulsor**. Obviously, a repulsor is unstable, given that for small ϵ and for any δ the trajectories starting in the δ−neighborhood will eventually leave the ϵ−neighborhood.

The set $\mathcal{D}$ in Definition 1.2 is called the **domain of attraction**. Note that an equilibrium point may be attractive without being stable, i.e. that trajectories always have a large transient, so that for small ϵ no trajectory will stay for all times within the ϵ-neighborhood, but will return to it and converge to the equilibrium point.An example of such a behavior is given by Vinograd's system (Vinograd, 1957)

$$\dot{x}_1 = \frac{x_1^2(x_2 - x_1) + x_2^5}{(x_1^2 + x_2^2)\left(1 + (x_1^2 + x_2^2)^2\right)} \tag{1.5a}$$

$$\dot{x}_2 = \frac{x_2^2(x_2 - 2x_1)}{(x_1^2 + x_2^2)\left(1 + (x_1^2 + x_2^2)^2\right)} \tag{1.5b}$$

with phase portrait shown in Figure 1.2, illustrating a butterfly-shaped behavior where even for very small initial deviations from equilibrium there is a large transient returning asymptotically to the equilibrium point $\boldsymbol{x}^* = \boldsymbol{0}$.

It should be mentioned, that if one can demonstrate convergence within a domain $\mathcal{D}$, this does not necessarily imply that $\mathcal{D}$ is the maximal domain of attraction. The determination of the maximal domain of attraction for an attractive equilibrium of a nonlinear system is, in general, a non-trivial task, due to the fact that, in contrast to linear systems, nonlinear systems can have multiple attractors, each one with its own domain of attraction.

An example is given by the system

$$\dot{x} = x(1 - x^2), \quad x(0) = x_0$$

which has three equilibrium points, namely $x_1^* = -1, x_2^* = 0$ and $x_3^* = 1$, with $x_{1,3}^*$ being attractors and x_2^* being a repulsor. The domain of attraction of x_1^* is $\mathcal{D}_1 = (-\infty, 0)$ and of x_3^* is $\mathcal{D}_3 = (0, \infty)$.

The concept which combines the above two concepts is given by the asymptotic stability defined next.

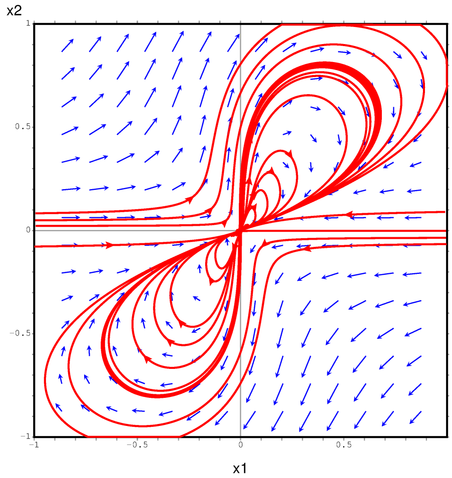

Figure 1.2: Phase portrait of the Vinograd system (1.5), with an unstable but attractive equilibrium point at $\boldsymbol{x}^* = \mathbf{0}$.

Definition 1.3

An equilibrium point $\boldsymbol{x}^*$ of (1.1) is said to be **asymptotically stable** if it is stable and attractive.

This concept is illustrated in Figure 1.1 (right). Note that in contrast to pure attractivity, the concept of asymptotic stability does not allow for large transients associated to small initial deviations, and is thus much stronger, and more of practical interest.

It must be noted that the asymptotic stability does not state anything about convergence speed. It only establishes that after an infinite time period the solution $\boldsymbol{x}(t)$, if starting in the domain of attraction, will approach $\boldsymbol{x}^*$ without ever reaching it actually. For most practical applications it is desirable to be able to state how long it will take to reach the equilibrium point at least sufficiently close. In this context, sufficiently close is often associated to a 95% to 99% convergence. A concept which allows to establish explicit statement about convergence speed is the exponential stability.

Definition 1.4

An equilibrium point $\boldsymbol{x}^*$ of (1.1) is said to be **exponentially stable** in a set $\mathcal{D}$, if it is stable, and there are constants $a, \lambda > 0$ so that

$$\forall\, \boldsymbol{x}_0 \in \mathcal{D} \,:\, \|\boldsymbol{\phi}(t;\boldsymbol{x}_0) - \boldsymbol{x}^*\| \leq a\|\boldsymbol{x}_0 - \boldsymbol{x}^*\|e^{-\lambda t}. \tag{1.6}$$

The constant a is known as the amplitude, and λ as the convergence rate.

Clearly, exponential stability implies asymptotic stability. Actually, the attractivity is ensured by the exponential term and the stability follows from the fact that by transitivity of the inequality relation it holds that for all $\boldsymbol{x}_0$ so that

$$\|\boldsymbol{x}_0 - \boldsymbol{x}^*\| \leq \delta \leq \frac{\epsilon}{a}$$

it follows that

$$\|\boldsymbol{\phi}(t,\boldsymbol{x}_0) - \boldsymbol{x}^*\| \leq a\|\boldsymbol{x}_0 - \boldsymbol{x}^*\|e^{-\lambda t} \leq a\|\boldsymbol{x}_0 - \boldsymbol{x}^*\| \leq a\delta \leq \epsilon$$

meaning that the trajectories stay for all $t \geq 0$ in the ϵ neighborhood.

The ratio

$$t_c = \frac{1}{\lambda}$$

is called the characteristic time constant for the exponential convergence assessment and it is straightforward to show that the bounding curve $\|\boldsymbol{x}_0 - \boldsymbol{x}\|e^{-\lambda t}$ converges to zero up to 98.5% within 4 characteristic times t_c.

Note that all the above stability and attractivity concepts can also be applied to sets. To examplify this, and for later reference, consider the definition of an attractive compact set.

Definition 1.5

A compact set $\mathcal{M}$ is called attractive for the domain $\mathcal{D}$, if

$$\forall\, \boldsymbol{x}_0 \in \mathcal{D} \,:\, \lim_{t\to\infty} \boldsymbol{\phi}(t, \boldsymbol{x}_0) \in \mathcal{M}.$$

Convergence to a set is of practical interest because in many situations it may be used to obtain a reduced model for analysis and design purposes, or may even be part of the design as in the backstepping control approach discussed in Section 2.1, or the *sliding-mode control* approach (see e.g. (Utkin, 1992)). In many situations it is an essential part of the stability assessment of a dynamical system (Seibert, 1969). Furthermore, in many applications it is more important to show the convergence to a set rather than to an equilibrium point.

Finally, it will be helpfull in the sequel to understand the concept of (positively) invariant sets $\mathcal{M}$ for a dynamical system.

Definition 1.6

A set $\mathcal{M} \subset \mathbb{R}^n$ is called **positively invariant**, if for all $\boldsymbol{x}_0 \in \mathcal{M}$ it holds that $\boldsymbol{\phi}(t; \boldsymbol{x}_0) \in \mathcal{M}$ for all $t \geq 0$.

The distinction **positive invariant** is used to distinguish the concept from **negative invariance**, referring to a reversion of time (i.e., letting time tending to minus infinity). The simplest example of a positively invariant set is an equilibrium point.

1.2 Lyapunov's direct method

A very usefull way to establish the stability of an equilibrium point for a nonlinear dynamical systems consists in Lyapunov's direct method. Motivated by studies on energy dissipation in physical processes, in particular in astronomy, Aleksandr Mikhailovich Lyapunov, generalized these considerations to functions which are positive for any non-zero argument (Fuller, 1992). In the sequel consider that the equilibrium point under consideration is the origin $\boldsymbol{x} = \boldsymbol{0}$. If other equilibria have to analyzed a linear coordinate shift $\tilde{\boldsymbol{x}} = \boldsymbol{x} - \boldsymbol{x}^*$ can be employed to move the equilibrim to the origin in the coordinate $\tilde{\boldsymbol{x}}$.

To summarize the results of Lyapunov and generalizations of it some definitions are in order.

Definition 1.7

A continuous functional $V \,:\, \mathcal{D} \subseteq \mathbb{R}^n \to \mathbb{R}$ is called

- positive semi-definite if $\forall\, \boldsymbol{x} \,:\, V(\boldsymbol{x}) \geq 0$.
- positive definite if $\forall\, \boldsymbol{x} \neq \boldsymbol{0} \,:\, V(\boldsymbol{x}) > 0$ and $V(\boldsymbol{x}) = 0$ only for $\boldsymbol{x} = \boldsymbol{0}$.
- negative semi-definite if $\forall\, \boldsymbol{x} \,:\, V(\boldsymbol{x}) \leq 0$.
- negative definite if $\forall\, \boldsymbol{x} \neq \boldsymbol{0} \,:\, V(\boldsymbol{x}) < 0$ and $V(\boldsymbol{x}) = 0$ only for $\boldsymbol{x} = \boldsymbol{0}$.

With these notions at hand, the following result can be stated.

Theorem 1.1

Let $V : \mathcal{D} \subset \mathbb{R}^n, V(\boldsymbol{x}) > 0$ be positive definite. If $\forall\, \boldsymbol{x} \in \mathcal{D} : \frac{\mathrm{d}V}{\mathrm{d}t}(\boldsymbol{x}) = \frac{\partial V(\boldsymbol{x})}{\partial \boldsymbol{x}} \dot{\boldsymbol{x}} \leq 0$, then $\boldsymbol{x} = \mathbf{0}$ is stable in the sense of Lyapunov.

Proof. Given that $V(\boldsymbol{x}) > 0$ and its continuity, there exists a function $W(\boldsymbol{x}) > 0$ such that

$$W(\boldsymbol{x}) \leq V(\boldsymbol{x}), \quad \forall\, \boldsymbol{x} \in \mathcal{D}. \tag{1.7}$$

Let $\epsilon > 0$ and set

$$m := \min_{\|\boldsymbol{x}\|=\epsilon} W(\boldsymbol{x}) > 0. \tag{1.8}$$

Choose $\delta > 0$ such that

$$\max_{0 \leq \|\boldsymbol{x}\| \leq \delta} V(\boldsymbol{x}) \leq m.$$

Given that $m > 0, V(\boldsymbol{x}) > 0$ and the continuity of V such a positive δ always exists. It follows from the fact that V is non-increasing over time ($\dot{V}(\boldsymbol{x}) < 0$) that

$$\forall\, \boldsymbol{x}_0 : \|\boldsymbol{x}_0\| \leq \delta \quad \Rightarrow \quad V(\boldsymbol{\phi}(t; \boldsymbol{x}_0)) \leq m.$$

By (1.7) this implies that

$$W(\boldsymbol{\phi}(t; \boldsymbol{x}_0)) \leq m.$$

By virtue of the definition of m in equation (1.8) it follows that

$$\forall\, \boldsymbol{x}_0 : \|\boldsymbol{x}_0\| \leq \delta \Rightarrow \|\boldsymbol{\phi}(t; \boldsymbol{x}_0)\| \leq \epsilon$$

showing that the origin is stable in the sense of Lyapunov. □

A function $V > 0$ that satisfies the conditions of Theorem 1.1 is called a *Lyapunov function*. Note that if $V > 0$ is continuously differentiable but it is not clear if $\frac{\mathrm{d}V}{\mathrm{d}t} \leq 0$ or the sign depends on some system parameters, then it is called a *Lyapunov function candidate*.

As can be seen from the proof, an essential part consists in that the sets

$$\Gamma_c = \{\boldsymbol{x} \in \mathbb{R}^n \,|\, V(\boldsymbol{x}) = c\} \tag{1.9}$$

defined by level curves of $V(\boldsymbol{x})$ are the boundaries of compact subsets $\mathcal{D}_c$ of the state space. In virtue of the non-increasing nature of V these sets are positively invariant. The geomtric idea of the proof of Theorem 1.1 is quite beautiful and will be shortly discussed. See Figure 1.3 for an illustration. The conditions of the theorem ensure that for a given ϵ there exists a value $c > 0$ such that the set $\mathcal{D}_c$ with the boundary Γ_c defined in (1.9) is completely contained in the ϵ-neigborhood $\mathcal{N}_\epsilon$ of the origin, i.e. it holds that

$$\mathcal{D}_c \subseteq \mathcal{N}_\epsilon.$$

Choosing $\delta > 0$ such that the δ-neighborhood $\mathcal{N}_\delta$ is completely contained in $\mathcal{D}_c$ one obtains that

$$\mathcal{N}_\delta \subseteq \mathcal{D}_c \subseteq \mathcal{N}_\epsilon$$

with $\mathcal{D}_c$ being positively invariant. Thus it holds that for all $\boldsymbol{x}_0$ with $\|\boldsymbol{x}_0\| \leq \delta$, i.e. $\boldsymbol{x}_0 \in \mathcal{N}_\delta$ the solution $\boldsymbol{\phi}(t; \boldsymbol{x}_0)$ is contained in $\mathcal{D}_c \subset \mathcal{N}_\epsilon$, implying that $\|\boldsymbol{\phi}(t, \boldsymbol{x}_0)\| \leq \epsilon$ for all $t \geq 0$.

The above only holds locally, unless $V(\boldsymbol{x})$ is strictly growing with $\|\boldsymbol{x}\|$. Thus the result is only local. The maximum compact set implied by the particular Lyapunov function can be explicitely determined. In

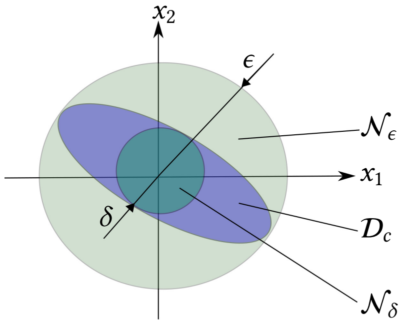

Figure 1.3: Geometrical idea behind the proof of Lyapunov's direct method in a two-dimensional state space.

the case that $\lim_{\|x\|\to\infty} V(\boldsymbol{x}) = \infty$ the function is called *radially unbounded*. For a radially unbounded Lyapunov function the above result becomes global, i.e. it holds with $\mathcal{D} = \mathbb{R}^n$.

By evaluating explicitely the inequality $\frac{\mathrm{d}V}{\mathrm{d}t}(\boldsymbol{x}) = 0$ which holds over the set

$$\mathcal{X}_0 = \left\{ \boldsymbol{x} \in \mathbb{R}^n \mid \frac{\mathrm{d}V(\boldsymbol{x})}{\mathrm{d}t} = 0 \right\} \tag{1.10}$$

one can apply the following result going back to Nikolay Nikolayevich Krasovsky and Joseph Pierre LaSalle and is known as the *invariance theorem*.

Theorem 1.2: Krasovsky-LaSalle

Let $\mathcal{D} \subseteq \mathbb{R}^n$ be a postively invariant compact set and $V \in C^1(\mathcal{D} \to \mathbb{R})$ positive defininte function with $\frac{\mathrm{d}V}{\mathrm{d}t}(\boldsymbol{x}) \leq 0$ for all $\boldsymbol{x} \in \mathcal{D}$. Then the trajectories $\boldsymbol{x}(t)$ converge to the largest positively invariant set $\mathcal{M} \subseteq \mathcal{X}_0$ with $\mathcal{X}_0$ defined in (1.10).

If the conditions of this theorem are satisfied, an additional condition implies the asymptotic stability of the origin as stated next.

Theorem 1.3

If the conditions of Theorem 1.2 are satisfied and it holds that $\mathcal{M} = \{\mathbf{0}\}$, then the origin $\boldsymbol{x} = \mathbf{0}$ is asymptotically stable.

A typical system where these results can be illustrated is given by the following Lienard oscillator

$$\ddot{x} + d\dot{x} + f(x) = 0 \tag{1.11}$$

with $d > 0$ and $f(x) > 0$ for $x > 0$, $f(x) = 0$ for $x = 0$ and $f(-x) = -f(x)$. The oscillator (1.11) can be written equivalently in state-space form with $x_1 = x$ and $x_2 = \dot{x}$ as

$$\dot{x}_1 = x_2 \tag{1.12a}$$
$$\dot{x}_2 = -f(x_1) - dx_2. \tag{1.12b}$$

Consider the following Lyapunov function candidate

$$V(\boldsymbol{x}) = \int_0^{x_1} f(\xi)d\xi + \frac{1}{2}x_2^2$$

motivated by the energy contained in the motion of $\boldsymbol{x}$ in form of potential and kinetic energy. The change in time of V is governed by

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t}(\boldsymbol{x}) &= f(x_1)\dot{x}_1 + x_2\dot{x}_2 \\ &= f(x_1)x_2 + x_2(-f(x_1) - dx_2) \\ &= -dx_2^2 \leq 0\end{aligned}$$

implying stability of the origin $\boldsymbol{x} = \boldsymbol{0}$ in virtue of Theorem 1.1. From Theorem 1.2 it is additionally known that $\boldsymbol{x}$ converges into the set

$$\mathcal{X}_0 = \{\boldsymbol{x} \in \mathbb{R}^2 \,|\, x_2 = 0\},$$

and more specifically into the largest positively invariant subset of $\mathcal{M} \subseteq \mathcal{X}_0$. This set in turn contains only trajectories for which $x_2(t) = 0$ for all times, given that it is positively invariant. This means that $\dot{x}_2(t) = 0$ for all times. Substituting $x_2 = 0, \dot{x}_2 = 0$ into (1.12b) this means that $f(x_1(t)) = 0$ for all times, showing that

$$\mathcal{M} = \{\boldsymbol{0}\}$$

given that $f(x_1) = 0$ only for $x_1 = 0$. Corollary 1.3 implies that the origin $\boldsymbol{x} = \boldsymbol{0}$ is asymptotically stable.

The asymptotic stability can also be concluded using Lyapunov's direct method if $\frac{\mathrm{d}V}{\mathrm{d}t}(\boldsymbol{x})$ is negative definite. This is stated in the next theorem.

Theorem 1.4

Let $V : \mathcal{D} \subseteq \mathbb{R}^n \to \mathbb{R}^n, V(\boldsymbol{x}) > 0$. If $\forall\, \boldsymbol{x} \in \mathcal{D} : \frac{\mathrm{d}V}{\mathrm{d}t}(\boldsymbol{x}) < 0$, then $\boldsymbol{x} = \boldsymbol{0}$ is locally asymptotically stable in $\mathcal{D}$.

Proof. In virtue of Theorem 1.1 we have that $\boldsymbol{x} = \boldsymbol{0}$ is stable in the sense of Lyapunov. It thus remains to show that $\lim_{t\to\infty} V(\boldsymbol{x}) = 0$ to conclude, by taking into account the positive definiteness and the continuity of $V(\boldsymbol{x})$, that $\lim_{t\to\infty} \|\boldsymbol{\phi}(t, \boldsymbol{x}_0)\| = 0$.

Assume that V does not converge to zero. Then there exists a positive constant $c > 0$ such that $\lim_{t\to\infty} V(\boldsymbol{x}(t)) = c > 0$. Let

$$S = \{\boldsymbol{x} \in \mathcal{D} \,|\, V(\boldsymbol{x}) \leq c\}$$

By assumption, for $\boldsymbol{x}_0 \notin S$, i.e. $V(\boldsymbol{x}_0) > c$ it holds that $\forall\, t \geq 0 : \boldsymbol{x}(t) \notin S$. Let Γ_S be the boundary of the set S, i.e.

$$\Gamma_S = \{\boldsymbol{x} \in \mathcal{D} \,|\, V(\boldsymbol{x}) = c\}.$$

We have that $\dot{V}(\boldsymbol{x})|_{\boldsymbol{x}\in\Gamma_S} < 0$. Introduce

$$-\gamma := \max_{\boldsymbol{x}\in\Gamma_S} \dot{V}(\boldsymbol{x}) < 0.$$

Now, let $\boldsymbol{x}_0 \notin S$, i.e. $V(\boldsymbol{x}_0) > c$ and let $t > t_* := \frac{V(\boldsymbol{x}_0) - c}{\gamma}$. Observe that

$$\begin{aligned}V(\boldsymbol{x}(t; \boldsymbol{x}_0)) &= V(\boldsymbol{x}_0) + \int_0^t \dot{V}(\boldsymbol{x}(\tau; \boldsymbol{x}_0) d\tau \\ &\leq V(\boldsymbol{x}_0) - \gamma t < V(\boldsymbol{x}_0) - \gamma t_* = c\end{aligned}$$

implying that $\forall\, t > t_*$ it holds that $\boldsymbol{x}(t; \boldsymbol{x}_0) \in \mathcal{S}$. This contradicts the initial assumption that $\forall\, t \geq 0 : \boldsymbol{x}(t) \notin S$, and thus c cannot be positive and it must hold that $c = 0$. This, in turn, implies that $\lim_{t\to\infty} V(\boldsymbol{x}(t)) = 0$, and thus $\forall\, \boldsymbol{x}_0 \in \mathcal{D} : \lim_{t\to\infty} \|\boldsymbol{x}(t)\| = 0$. □

At this place it is noteworthy that using Lyapunov functions one can establish a domain for which the equilibrium point is an attractor. This domain will always be included in the domain of attraction of the equilibrium point. Note that eventhough it is not possible to conclude if the domain of attraction established in this way is the complete domain of attraction or only a subset of it, unless the result is global.

As discussed above, in may cases it is not sufficient to conclude only the asymptotic stability and it becomes important to have a quantitative value for the convergence speed towards an equilibrium. This can be established if the equilibrium is exponentially stable (see Definition 1.4). Exponential stability can be concluded using Lyapunov functions if some additional properties are given. These are stated in the next theorem.

Theorem 1.5

Let $V : \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}, V(\boldsymbol{x}) > 0$ be a positive definite functional. If there exist constants $\alpha, \beta, \gamma > 0$ so that

$$(i) \quad \alpha\|\boldsymbol{x}\|^2 \leq V(\boldsymbol{x}) \leq \beta\|\boldsymbol{x}\|^2 \tag{1.13a}$$

$$(ii) \quad \frac{\mathrm{d}V}{\mathrm{d}t}(\boldsymbol{x}) \leq -\gamma V(\boldsymbol{x}) \tag{1.13b}$$

then $\boldsymbol{x} = \boldsymbol{0}$ is exponentially stable and (1.6) holds with $a = \sqrt{\beta/\alpha}$ and $\lambda = \gamma/2$.

Proof. In virtue of (1.13b) it holds that

$$V(\boldsymbol{x}(t)) \leq V(\boldsymbol{x}_0)e^{-\gamma t}.$$

From (1.13a) this implies that

$$\|\boldsymbol{x}(t)\|^2 \leq \frac{1}{\alpha}V(\boldsymbol{x}(t)) \leq \frac{1}{\alpha}V(\boldsymbol{x}_0)e^{-\gamma t} \leq \frac{\beta}{\alpha}e^{-\gamma t}\|\boldsymbol{x}_0\|^2$$

and finally

$$\|\boldsymbol{x}(t)\| \leq \sqrt{\frac{\beta}{\alpha}}\|\boldsymbol{x}_0\|e^{-\frac{\gamma}{2}t}.$$

Exponentially stability as defined in (1.6) follows with a and λ stated above.

□

The fact that the value of $V(\boldsymbol{x})$ monotonically decreases over time rules out the possibility of closed trajectories, given that they could only exist on level cures of V. Thus the use of Lyapunov functions is also an effective means for the preclusion of limit cycles.

Finally, it is possible to show that the existence of a Lyapunov function is intrinsically related to the stability properties of the equilibrium point as stated in the next theorem for the case that the origin is exponentially stable.

Theorem 1.6

Let $\boldsymbol{x} = \boldsymbol{0}$ be exponentially stable in $\mathcal{D} \subseteq \mathbb{R}^n$. Then there exists a Lyapunov function $V : \mathcal{D} \to \mathbb{R}$, $V(\boldsymbol{x}) > 0$ and a constant $\gamma > 0$ such that (1.13b) holds true.

Proof. By assumption there are constant a, λ such that $\|\boldsymbol{x}(t)\| \leq a\|\boldsymbol{x}_0\|e^{-\lambda t}$. Consider the functional

$$V(\boldsymbol{x}(t)) = \int_0^\infty \|\boldsymbol{x}(t+\tau)\|^2 d\tau.$$

It holds that

$$V(\boldsymbol{x}(t)) = 0, \quad \Leftrightarrow \|\boldsymbol{x}(t+\tau)\| = 0, \quad \forall \tau \geq 0,$$

showing that V is positive definite.

On the other hand, in virtue of the exponential stability of the origin it holds that

$$V(\boldsymbol{x}(t)) \leq \int_0^\infty a^2 \|\boldsymbol{x}(t)\|^2 e^{-2\lambda\tau} d\tau = \frac{a^2\|\boldsymbol{x}(t)\|^2}{2\lambda}$$

showing that for finite $\|\boldsymbol{x}_0\|$ the function $V(\boldsymbol{x})$ is quadratically bounded from above by the norm of $\boldsymbol{x}(t)$. Consider the rate of change of V at time t evaluated at the point $\boldsymbol{x}(t)$ given by

$$\begin{aligned}
\frac{\mathrm{d}V}{\mathrm{d}t}(\boldsymbol{x}(t)) &= \lim_{\tau \to 0^+} \frac{1}{\tau} \left(\int_\tau^\infty \|\boldsymbol{x}(t+s)\|^2 ds - \int_0^\infty \|\boldsymbol{x}(t+s)\|^2 ds \right) \\
&= - \lim_{\tau \to 0^+} \frac{1}{\tau} \int_0^\tau \|\boldsymbol{x}(t+s)\|^2 ds \\
&= -\|\boldsymbol{x}(t)\|^2 \\
&\leq -\frac{2\lambda}{a^2} V(\boldsymbol{x}(t))
\end{aligned}$$

showing that inequality (1.13b) holds with $\gamma = \frac{2\lambda}{a^2}$. □

This last results shows that the asymptotical and exponential stability of equilibria is intrinsically related to the existence of Lyapunov functions, in the sense that their existence implies stability properties and vice versa, the stability properties imply their existence. Thinking about the origins of the Lyapunov theory, stemming from potential and kinetic energies and interchanges between them allows to see that there are actually very fundamental relations in terms of simple scalar (energy) functions describing the complete behavior of potentially rather complex nonlinear systems.

Exercise 1.1.
Analyze the stability of the origin of the differential equations

$$\dot{x} = -x(1-x^2), \qquad x(0) = x_0$$

using Lyapunov's direct method. Is the origin unstable, stable in the sense of Lyapunov, asymptotically stable or exponentially stable?

Exercise 1.2.
Analyze the stability of the origin of the second order differential equation

$$\ddot{y} + d\dot{y} + g(y) = 0, \qquad g(y) \begin{cases} > 0, & \text{if} \quad y > 0 \\ = 0, & \text{if} \quad y = 0 \\ = -g(|y|), & \text{if} \quad y < 0 \end{cases}$$

using Lyapunov's direct method. Is the origin unstable, stable in the sense of Lyapunov, asymptotically stable or exponentially stable?

References

Fuller, A. T. (1992). „Lyapunov Centenary Issue“. In: *Int. J. of Control* 55 (3) (cit. on pp. 1, 4).
Khalil, H. (1996). *Nonlinear Systems.* 2nd. Prentice-Hall, Upper Saddle River, New Jersey (cit. on p. 1).
Seibert, P. (1969). „On stability relative to a set and to the whole space“. In: *5th Int. Conf. on Nonlin. Oscillations, 1969, V.2, Inst. Mat. Akad. Nauk USSR, Kiev*, pp. 448–457 (cit. on p. 4).
Utkin, V. I. (1992). *Sliding Modes in Control and Optimization.* Springer (cit. on p. 4).
Vinograd, R. E. (1957). „The inadequacy of the method of characteristics exponents for the study of nonlinear differential equations“. In: *Mat. Sbornik* 41 (83), pp. 431–438 (cit. on p. 2).

Chapter 2

Lyapunov–based design techniques

2.1 Integrator backstepping

Consider the following system

$$\dot{x}_1 = f_1(x_1) + g(x_1)x_2, \qquad x_1(0) = x_{10} \tag{2.1a}$$

$$\dot{x}_2 = u, \qquad x_2(0) = x_{20}, \tag{2.1b}$$

with smooth vector fields $f_1(x_1)$ and $g(x_1) \neq 0$, $\forall x_1 \in D_1 \subseteq \mathbb{R}$. Classical examples for such dynamics are motivated for systems where the actual actuator dynamics have to be taken into account and can be summarized in form of an integrator (e.g. a pump with supplied electric current). In the following a way to exploit the particular structure of this system dynamics for stabilization of the origin by state-feedback is discussed. Consider in a first, auxiliary step the state x_2 as *virtual* control input. By the condition that $g(x_1) \neq 0$ in D_1 it follows that a simple (linearizing) control can be assigned of the form

$$x_2 = \frac{v - f(x_1)}{g(x_1)} = \mu(x_1)$$

with closed-loop dynamics (for x_1) given by

$$\dot{x}_1 = v, \quad x_1(0) = x_{10}.$$

A desired behavior for x_1 can be introduced by adequately choosing v, e.g. as

$$v = -kx_1$$

to obtain exponential convergence with rate k. Another, more general approach could be to consider the Lyapunov function

$$V_1(x_1) = \frac{1}{2}x_1^2, \quad \frac{\mathrm{d}V_1(x_1)}{\mathrm{d}t} = x_1 v \overset{!}{=} -Q_1(x_1) \tag{2.2}$$

for some desired $Q_1(x_1) = x_1 q_1(x_1) > 0$ what is achieved using

$$\mu(x_1) = \frac{-q_1(x_1) - f_1(x_1)}{g(x_1)}. \tag{2.3}$$

Choosing $q_1(x_1) = kx_1$, the aforementioned control $v = -kx_1$ is recovered. So, consider for the moment that v is chosen such that (2.2) holds. Clearly, this control is not implementable, as it was assumed that

x_2 would be the control input. Thus, actually the difference betweeen x_2 and $\mu(x_1)$ has to be considered:

$$z = x_2 - \mu(x_1), \quad \dot{z} = u - \frac{\partial \mu(x_1)}{\partial x_1}\Big(f_1(x_1) + g(x_1) \underbrace{[z + \mu(x_1)]}_{=x_2} \Big). \tag{2.4}$$

Introducing the Lyapunov function candidate

$$V_z(z) = \frac{1}{2} z^2 \tag{2.5}$$

it follows that

$$\frac{\mathrm{d}V_z(z)}{\mathrm{d}t} = z \left(u - \frac{\partial \mu(x_1)}{\partial x_1}\Big(f_1(x_1) + g(x_1)[z + \mu(x_1)] \Big) \right).$$

Clearly, one can use this equation to find u in dependence of z and x_1 so that $\frac{\mathrm{d}V_z(z)}{\mathrm{d}t} < 0$. Nevertheless, this would neglect the dynamics of x_1 for the transient during which $x_2 \neq \mu(x_1)$. Thus, consider the system dynamics in (x_1, z) coordinates

$$\dot{x}_1 = f_1(x_1) + g(x_1)[z + \mu(x_1)], \qquad x_1(0) = x_{10} \tag{2.6a}$$

$$\dot{z} = u - \frac{\partial \mu(x_1)}{\partial x_1}\Big(f_1(x_1) + g(x_1)[z + \mu(x_1)] \Big), \qquad z(0) = z_0 \tag{2.6b}$$

and the joint Lyapunov function candidate

$$W(x_1, z) = V_1(x_1) + V_z(z). \tag{2.7}$$

The rate of change of $W(x_1, z)$ over time is given by

$$\begin{aligned} \frac{\mathrm{d}W(x_1, z)}{\mathrm{d}t} &= x_1 f_1(x_1) + x_1 g(x_1)[z + \mu(x_1)] + z \left(u - \frac{\partial \mu(x_1)}{\partial x_1}\Big(f_1(x_1) + g(x_1)[z + \mu(x_1)] \Big) \right) \\ &= -Q_1(x_1) + z \left(x_1 g(x_1) + u - \frac{\partial \mu(x_1)}{\partial x_1}\Big(f_1(x_1) + g(x_1)[z + \mu(x_1)] \Big) \right) \end{aligned}$$

Thus, to achieve the condition

$$\frac{\mathrm{d}W(x_1, z)}{\mathrm{d}t} \overset{!}{=} -Q_1(x_1) - Q_2(z) < 0$$

for some $Q_2(z) = z q_2(z)$ so that $Q_2(z) > 0$ it is sufficient to choose the control input u as

$$u = -q_2(z) - x_1 g(x_1) + \frac{\partial \mu(x_1)}{\partial x_1}\Big(f_1(x_1) + g(x_1)[z + \mu(x_1)] \Big) = \varpi(x_1, z) \tag{2.8}$$

with $\mu(x_1)$ given in (2.3). In terms of (x_1, x_2) this stabilizing controller can be written as

$$u = \alpha(x_1, x_2) := \varpi(x_1, z)\Big|_{z = x_2 - \mu(x_1)}.$$

This approach is known as **integrator backstepping** and can be summarized in the following two steps:

(i) Design x_2 as auxiliary (*virtual*) input variable to obtain the relationship $x_2 = \mu(x_1)$ for which asymptotic (or exponential) stability is ensured

(ii) Design the *actual* controller input $u = \varpi(x_1, x_2)$ so that the difference $z = x_2 - \mu(x_1)$ asymptotically (or exponentially) converges to zero, taking into account the (possibly open-loop unstable) of x_1 during the transient for which $x_2 \neq \mu(x_1)$.

This is put in the Lyapunov-framework in the way discussed above and can directly be generalized to the case where $\boldsymbol{x}_1$ is a vector in $\mathbb{R}^n$ and for the case that an integrator chain of n integrators separate the dynamics of $\boldsymbol{x}_1$ and the input u (so that the relative degree is at least n).

Theorem 2.1

Consider the system

$$\dot{\boldsymbol{x}}_1 = \boldsymbol{f}(\boldsymbol{x}_1) + \boldsymbol{g}(\boldsymbol{x}_1)x_2, \qquad x_1(0) = x_{10}$$
$$\dot{x}_2 = u, \qquad x_2(0) = x_{20}$$

with $\boldsymbol{x}_1 \in \mathbb{R}^n$, $x_2 \in \mathbb{R}$ and smooth vector fields $\boldsymbol{f}, \boldsymbol{g}$ with $\boldsymbol{f}(\boldsymbol{0}) = \boldsymbol{0}$. Let $\mu(x_1)$ be a continuously differentiable function with $\mu(0) = 0$ and $V_1(\boldsymbol{x}_1)$ a differentiable, positive definite (radially unbounded) function so that

$$\frac{\partial V_1(\boldsymbol{x}_1)}{\partial \boldsymbol{x}_1}\left(\boldsymbol{f}(\boldsymbol{x}_1) + \boldsymbol{g}(\boldsymbol{x}_1)\mu(\boldsymbol{x}_1)\right) \leq -Q_1(x_1) \leq 0. \tag{2.9}$$

Then the following holds true:

(i) If $Q(x_1) > 0$ then the feedback

$$u = \alpha(\boldsymbol{x}_1, x_2) = \frac{\partial \mu(\boldsymbol{x}_1)}{\partial \boldsymbol{x}_1}\left(\boldsymbol{f}(\boldsymbol{x}_1) + \boldsymbol{g}(\boldsymbol{x}_1)x_2\right) - \frac{\partial V_1(\boldsymbol{x}_1)}{\partial \boldsymbol{x}_1}\boldsymbol{g}(\boldsymbol{x}_1) - k(x_2 - \mu(x_1)) \tag{2.10}$$

with $k > 0$ asymptotically stabilizes the origin $[\boldsymbol{x}_1^T, x_2]^T = \boldsymbol{0}^T$ and

$$W(\boldsymbol{x}_1, x_2) = V_1(\boldsymbol{x}_1) + \frac{1}{2}(x_2 - \mu(x_1))^2$$

is an associated Lyapunov function.

(ii) If $Q_1(\boldsymbol{x}_1) \geq 0$ then for the closed-loop system with the feedback control (2.10) the trajectories converge into the largest positively invariant subset of

$$\mathcal{W} = \left\{ \begin{bmatrix} \boldsymbol{x}_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^{n+1} \,|\, Q_1(\boldsymbol{x}_1) = 0, x_2 = \mu(\boldsymbol{x}_1) \right\}. \tag{2.11}$$

Proof.

(i) According to the assumptions it holds that

$$\frac{\mathrm{d}W(\boldsymbol{x}_1, x_2)}{\mathrm{d}t} = \frac{\partial V_1(\boldsymbol{x}_1)}{\partial \boldsymbol{x}_1}\left(\boldsymbol{f}(\boldsymbol{x}_1) + \boldsymbol{g}(\boldsymbol{x}_1)[\mu(\boldsymbol{x}_1) + x_2 - \mu(x_1)]\right) + \\ + (x_2 - \mu(x_1))\left(u - \frac{\partial \mu(\boldsymbol{x}_1)}{\partial \boldsymbol{x}_1}\right)\left(\boldsymbol{f}(\boldsymbol{x}_1) + \boldsymbol{g}(\boldsymbol{x}_1)x_2\right)$$

and taking into account (2.9) and (2.10) it follows that

$$\frac{\mathrm{d}W(\boldsymbol{x}_1, x_2)}{\mathrm{d}t} = -Q_1(\boldsymbol{x}_1) - k(x_2 - \mu(\boldsymbol{x}_1))^2 < 0. \tag{2.12}$$

Given that $Q_1(\boldsymbol{x}_1) > 0$ by assumption, the asymptotic stability follows from Theorem 1.4.

(ii) For $Q_1(\boldsymbol{x}_1) \geq 0$ it follows from Theorem 1.2 that the trajectories converge into the largest positively invariant subset of

$$\mathcal{W}_0 = \left\{ \begin{bmatrix} \boldsymbol{x}_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^{n+1} \,\middle|\, \frac{\mathrm{d}W(\boldsymbol{x}_1, x_2)}{\mathrm{d}t} = 0 \right\}.$$

According to (2.12) this set is identical with (2.11).

□

Beyond this result, one can directly consider chains of integrators, like in the system

$$\begin{aligned} \dot{\boldsymbol{x}}_1 &= \boldsymbol{f}(\boldsymbol{x}_1) + \boldsymbol{g}(\boldsymbol{x}_1)x_2, & x_1(0) &= x_{10} \\ \dot{x}_2 &= x_3, & x_2(0) &= x_{20} \\ &\vdots \\ \dot{x}_n &= u, \quad x_n(0) = x_{n0} \end{aligned}$$

or the more general set-up where

$$\begin{aligned} \dot{\boldsymbol{x}}_1 &= \boldsymbol{f}_1(\boldsymbol{x}_1, \boldsymbol{x}_2), & \boldsymbol{x}_1(0) &= \boldsymbol{x}_{10} \\ \dot{\boldsymbol{x}}_2 &= \boldsymbol{f}_2(\boldsymbol{x}_1, \boldsymbol{x}_2) + \boldsymbol{u}, & \boldsymbol{x}_2(0) &= \boldsymbol{x}_{20} \end{aligned}$$

under the assumption that vector field $\boldsymbol{\alpha}(\boldsymbol{x}_1)$ and a Lyapunov function $V(\boldsymbol{x}_1)$ is known which proofs the asymptotic stability of the system

$$\dot{\boldsymbol{x}}_1 = \boldsymbol{f}_1(\boldsymbol{x}_1, \boldsymbol{\alpha}(\boldsymbol{x}_1)), \quad \boldsymbol{x}_1(0) = \boldsymbol{x}_{10}.$$

For more information, the interested reader is referred to the literature (Sepulchre et al., 1997; Kugi, 2017).

2.2 Dissipativity and passivity-based control

Consider the MIMO control system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + G(\boldsymbol{x})\boldsymbol{u}, \quad \boldsymbol{x}(0) = \boldsymbol{x}_0 \tag{2.13a}$$

$$\boldsymbol{y} = \boldsymbol{h}(\boldsymbol{x}) \tag{2.13b}$$

with state $\boldsymbol{x} \in \mathbb{R}^n$, input $\boldsymbol{u} \in \mathbb{R}^p$ and output $\boldsymbol{y} \in \mathbb{R}^m$ and smooth vector fields $\boldsymbol{f}(\boldsymbol{x}), \boldsymbol{g}_i(\boldsymbol{x})$, $i = 1, \ldots, p$, $G(\boldsymbol{x}) = [\boldsymbol{g}_1(\boldsymbol{x}), \ldots, \boldsymbol{g}_p(\boldsymbol{x})]$ and differentiable output map $\boldsymbol{h}(\boldsymbol{x})$.

The notion of passivity in systems theory is motivated by the notion of passivity in electrical engineering, where this concept refers to an electrical circuit in which the electric power consumption $P = UI$ is always positive, in the understanding that this implies that the circuit dos not produce energy by itself and the net flow of energy is always into the circuit. This has been extended to a general set-up for (open) dynamical (control) systems in a state space framework (Kalman, 1964; Kalman, 1963; P. J. Moylan, 1974; Hill and P. Moylan, 1976; Byrnes et al., 1991; Willems, 1972a; Willems, 1972b; Hill and P. Moylan, 1980; Brogliato et al., 2007). In order to state the notions of dissipativity and passivity the concept of internal (energy) storage is used, generalizing the concept of energy in terms of the system state $\boldsymbol{x}$. The second concept which is employed is a generalization of the power supply to a system represented in form of a general supply rate $\omega(\boldsymbol{u}, \boldsymbol{y})$.

With these concepts, the notions of dissipativity and passivity can be defined in the following way.

Definition 2.1

(i) The system (2.13) is called dissipative with respect to the supply rate $\omega(\boldsymbol{u}, \boldsymbol{y})$ if there exists a (positive semi-definite) storage function $S(\boldsymbol{x}) \geq 0$ so that

$$S(\boldsymbol{x}(t)) - S(\boldsymbol{x}_0) \leq \int_0^t \omega(\boldsymbol{u}(\tau), \boldsymbol{y}(\tau))\mathrm{d}\tau, \quad \forall\, \boldsymbol{x}_0, t \geq 0.$$

If S is differentiable, this relation can be written equivalently as

$$\frac{\mathrm{d}S(\boldsymbol{x})}{\mathrm{d}t} = \frac{\partial S(\boldsymbol{x})}{\partial \boldsymbol{x}} \big(\boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{G}(\boldsymbol{x})\boldsymbol{u}\big) \leq \omega(\boldsymbol{u}, \boldsymbol{y}).$$

(ii) System (2.13) with $m = p$ is called passive if it is dissipative with respect to the supply rate $\omega(\boldsymbol{u}, \boldsymbol{y}) = \boldsymbol{u}^T \boldsymbol{y}$.

(iii) A static map $\boldsymbol{y} = \boldsymbol{\varphi}(\boldsymbol{u})$ with $m = p$ is called passive if it holds that $\boldsymbol{u}^T \boldsymbol{y} = \boldsymbol{u}^T \boldsymbol{\varphi}(\boldsymbol{u}) \geq 0$.

Example 2.1. *A very simple example for a passive electrical element is a resistance R for which it holds by Ohm's law that $u_R = Ri_R$ and thus, taking i_R as the input and u_R as the output*

$$u_R = \varphi(i_R) = Ri_R, \quad \omega(i_R, u_R) = i_R u_R = i_R R i_R = R i_R^2 > 0.$$

Example 2.2. *A more interesting example with an internal memory is given by the RC circuit with a current source (see Fig. 2.1). For the voltage over the capacitance it holds that*

$$\frac{\mathrm{d}u_C}{\mathrm{d}t} = -\frac{1}{RC} u_C + \frac{1}{C} i_q, \quad u_C(0) = u_{C0}$$
$$y = u_C$$

so that taking the current source as input, i.e. $u = i_q$ and the electrical energy of the capacitance as storage function, i.e. $S = \frac{1}{2} C u_C^2 > 0$ it follows that

$$\frac{\mathrm{d}S(u_C)}{\mathrm{d}t} = -\frac{1}{R} u_C^2 + i_q u_C \leq i_q u_C = uy$$

implying the passivity of this simple electrical circuit.

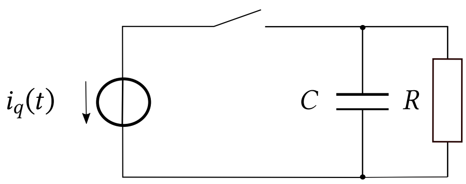

Figure 2.1: *RC*-circuit in example 2.1.

The main idea in passivity-based control is to use the storage function as Lyapunov-function candidate and exploit the structure of the supply rate for (output- or state-) feedback control design. Recalling the fundamental results from Lyapunov stability theory presented in Chapter 1, it is clear that as long as it is possible to find a control law in such a way that the rate of change in the storage is negative semi-definite, the stability follows if $S(\boldsymbol{x}) > 0$. In the case that even negative definiteness can be achieved the asymptotic (or exponential) stability of the origin can be ensured.

To analyze this feature further, note that in example 2.1 of the RC circuit an interesting additional property can be observed. To show the passivity of the system, in the final step the (negative definite) term $-\frac{1}{R}u_C^2 < 0$ has been neglected. Nevertheless, from the point of view of Lyapunov theory this term already ensures that for $u = 0$ one has

$$\frac{\mathrm{d}S(u_C)}{\mathrm{d}t} = -\frac{1}{R}u_C^2 < 0$$

implying the (exponential) asymptotic stability of the solution $u_C = 0$. There are a couple of different notions in the theory of dissipative and passive systems to handle such properties (see e.g. (Sepulchre et al., 1997; Khalil, 1996; Brogliato et al., 2007)). The one employed here is defined next.

Definition 2.2

System (2.13) is called strictly state dissipative with respect to the supply rate $\omega(\boldsymbol{u}, \boldsymbol{y})$ if there exists a non-negative (i.e., positive semi-definite) storage function $S(\boldsymbol{x}) \geq 0$ and a constant $\kappa > 0$ such that

$$\frac{\mathrm{d}S(\boldsymbol{x})}{\mathrm{d}t} \leq \omega(\boldsymbol{u}, \boldsymbol{y}) - \kappa\|\boldsymbol{x}\|^2.$$

It is called strictly (state) passive if it is strictly (state) dissipative with respect to the supply rate $\omega(\boldsymbol{u}, \boldsymbol{y}) = \boldsymbol{u}^T\boldsymbol{y}$.

In particular, when a system is strictly state passive with a positive definite storage function $S(\boldsymbol{x}) > 0$ it can be (exponentially) asymptotically stabilized using the simply linear feedback control law $\boldsymbol{u} = -K\boldsymbol{y}$ with $K \geq 0$, given that then

$$\frac{\mathrm{d}S(\boldsymbol{x})}{\mathrm{d}t} \leq -\kappa\|\boldsymbol{x}\|^2 + \boldsymbol{u}^T\boldsymbol{y} \overset{\boldsymbol{u}=-K\boldsymbol{y}}{=} -\kappa\|\boldsymbol{x}\|^2 - \boldsymbol{y}^T K\boldsymbol{y} \leq -\kappa\|\boldsymbol{x}\|^2 < 0.$$

This reasoning allready explains why passive systems have quite useful properties for control design, besides their importance in electrical engineering. In order to enable similar results for the more general class of dissipative systems typically some further structural constraints are introduced for the supply rate. For this purpose the studies are focussed on the case of quadratic supply rates of the form

$$\omega(\boldsymbol{u}, \boldsymbol{y}) = \begin{bmatrix}\boldsymbol{u}\\ \boldsymbol{y}\end{bmatrix}^T \begin{bmatrix}Q & S\\ S^T & R\end{bmatrix} \begin{bmatrix}\boldsymbol{u}\\ \boldsymbol{y}\end{bmatrix}$$

and the related dissipativity property is then called (Q, S, R)-dissipativity. This opens a wide field of oportunities which to cover goes beyond the scope of these notes and the interested reader is referred to the related literature (see e.g. (Willems, 1972b; Brogliato et al., 2007)).

Motivated by these initial considerations, in the following the focus is put on the class of passive systems. Consider again the case of a passive system with a positive definite storage function $S(\boldsymbol{x}) > 0$ and the feedback control

$$\boldsymbol{u} = -K\boldsymbol{y}, \quad K > 0.$$

Accordingly, in virtue of

$$\frac{\mathrm{d}S(\boldsymbol{x})}{\mathrm{d}t} \leq -\boldsymbol{y}^T K \boldsymbol{y} \leq 0$$

it follows from Krasovskiy-LaSalles invariance principle (see Theorem 1.2) that the state converges into the largest positively invariant subset of

$$\mathcal{Y}_0 = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{y} = \boldsymbol{h}(\boldsymbol{x}) \equiv \boldsymbol{0}\}. \tag{2.14}$$

The positive invariance goes at hand with the condition that $\boldsymbol{y} \equiv \boldsymbol{0}$, or equivalently, that $\boldsymbol{y}^{(k)} = 0$ for all $0 \leq k \in \mathbb{N}_0$. Using the notion of the Lie-derivative

$$L_f h_i(\boldsymbol{x}) = \frac{\partial h_i(\boldsymbol{x})}{\partial \boldsymbol{x}} \boldsymbol{f}(\boldsymbol{x}), \quad L_f^k h_i(\boldsymbol{x}) = \frac{\partial L_f^{k-1} h_i(\boldsymbol{x})}{\partial \boldsymbol{x}} \boldsymbol{f}(\boldsymbol{x}), \quad L_f^0 h_i(\boldsymbol{x}) = h_i(\boldsymbol{x})$$

for $i = 1, \ldots, m$, this restriction implies that

$$\begin{aligned} y_i &= h_i(\boldsymbol{x}) = 0 \\ \dot{y}_i &= L_f h_i(\boldsymbol{x}) = 0 \\ y_i^{(2)} &= L_f^2 h_i(\boldsymbol{x}) = 0 \\ &\vdots \\ y_i^{(n)} &= L_f^n h_i(\boldsymbol{x}) = 0 \end{aligned}$$

needs to hold true for all $i = 1, \ldots, m$. In vector form this implies that

$$\mathcal{O}(\boldsymbol{x}) = \begin{bmatrix} h_i(\boldsymbol{x}) \\ L_f h_i(\boldsymbol{x}) \\ \vdots \\ L_f^n h_m(\boldsymbol{x}) \end{bmatrix} = \boldsymbol{0}. \tag{2.15}$$

The map $\mathcal{O}(\boldsymbol{x})$ is the nonlinear observability map[1] (Isidori, 1995; Nijmeier and Schaft, 1990), so that in case that the system is completely observable –in the sense that this map is everywhere uniquely invertible – it turns out that only the zero vector $\boldsymbol{x} = \boldsymbol{0}$ is a solution of (2.15) and thus the asymptotic stability of $\boldsymbol{x} = \boldsymbol{0}$ follows.

In linear systems, if the map can be inverted along one trajectory, it can be inverted along any trajectory[2]. In nonlinear systems this is not true and it is worth introducing a new concept which corresponds to the invertibility along the solution (and thus uniqueness of this solution) for which $\boldsymbol{y} \equiv 0$.

Definition 2.3

The system (2.13) is called

(i) *zero–state observable* if $\boldsymbol{y} \equiv \boldsymbol{0}$ implies $\boldsymbol{x} = \boldsymbol{0}$, i.e. if the solution of $\mathcal{O}(\boldsymbol{x}) = \boldsymbol{0}$ is uniquely given by $\boldsymbol{x} = \boldsymbol{0}$.

(ii) *zero–state detectable* if $\boldsymbol{y} \equiv \boldsymbol{0}$ implies $\boldsymbol{x} \to \boldsymbol{0}$.

From the definition it should be clear, that zero-state observability implies zero-state detectability but

---

[1]It can be quickly shown, that for a linear system this map corresponds to the Kalman observability map $\mathcal{K}_o = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$.

[2]For linear systems the map (2.15) can actually be written as the Kalman observability matrix $\mathcal{K}_o$ times the state vector $\boldsymbol{x}$.

not *vice versa*. Note that if the system is not zero-state observable but zero-state detectable, then the map $\mathcal{O}(\boldsymbol{x})$ is not invertible, but all solutions $\boldsymbol{x}(t)$ which are mapped by $\mathcal{O}$ to the zero vector $\mathbf{0}$ converge asymptotically to zero. This implies that by Krasovskiy-LaSalles invariance principle all trajectories converge into a set where the origin is the unique attractor.

To simplify the notation and focus on the main concepts let us restrict the considered class of systems to the SISO case

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}) + g(\boldsymbol{x})u, \quad \boldsymbol{x}(0) = \boldsymbol{x}_0 \tag{2.16a}$$

$$y = h(\boldsymbol{x}). \tag{2.16b}$$

Looking on the constraint $y \equiv 0$ in the definition of *zero-state detectability* the notion of the *zero dynamics* comes into play. The zero dynamics is given by (2.16) with the constraint $y \equiv 0$ and u chosen so that this constraint holds true. To make this point clear, note that if the relative degree (Isidori, 1995; Sepulchre et al., 1997) of (2.16) is equal to one at $\boldsymbol{x} = 0$, i.e. there exists a neighborhood $\mathcal{N}_0$ of $\boldsymbol{x} = \mathbf{0}$ so that

$$L_g h(\boldsymbol{x}) \neq 0, \quad \forall\, \boldsymbol{x} \in \mathcal{N}_0,$$

and a diffeomorphic[3] state transformation $\boldsymbol{\Psi} : \mathcal{N}_0 \to \mathbb{R}^n$ of the form

$$\begin{bmatrix} z \\ \boldsymbol{\zeta} \end{bmatrix} = \boldsymbol{\Psi}(\boldsymbol{x}) = \begin{bmatrix} h(\boldsymbol{x}) \\ \boldsymbol{\Phi}(\boldsymbol{x}) \end{bmatrix}, \quad z \in \mathbb{R}, \quad \boldsymbol{\zeta} \in \mathbb{R}^{n-1}. \tag{2.17}$$

The dynamics in the new coordinates reads

$$\dot{z} = L_f h(\boldsymbol{x}) + L_g h(\boldsymbol{x})u, \qquad z(0) = z_0 \tag{2.18a}$$

$$\dot{\boldsymbol{\zeta}} = \boldsymbol{\varphi}(z, \boldsymbol{\zeta}, u), \qquad \boldsymbol{\zeta}(0) = \boldsymbol{\zeta}_0 \tag{2.18b}$$

$$y = z. \tag{2.18c}$$

Actually, it is possible to choose the map $\boldsymbol{\Phi}(\boldsymbol{x})$ so that the vector field $\boldsymbol{\varphi}$ does not depend on the input u, but this does not make a difference at this stage. The zero dynamics is given by

$$\dot{\boldsymbol{\zeta}} = \boldsymbol{\varphi}(0, \boldsymbol{\zeta}, u) =: \boldsymbol{\varphi}_0(\boldsymbol{\zeta}), \qquad \boldsymbol{\zeta}(0) = \boldsymbol{\zeta}_0 \tag{2.19a}$$

$$u = -\left.\frac{L_f h(\boldsymbol{x})}{L_g h(\boldsymbol{x})}\right|_{\boldsymbol{x} = \boldsymbol{\Psi}^{-1}\left(\begin{bmatrix} 0 \\ \boldsymbol{\zeta} \end{bmatrix}\right)} \tag{2.19b}$$

[3]A map is a diffeomorphism if it is continuously differentiable and invertible, with continuously differentiable inverse. Such maps conserve geometric and topological properties in state space and are frequently used in the theory of dynamical systems, in particular for control design purposes.

With these notions at hand the following result is a direct consequence of the passivity property and the application of Lyapunov's direct method.

Theorem 2.2

Let (2.16) be passive with positive definite storage function $S(\boldsymbol{x}) > 0$. Then the following holds true:

(i) The zero dynamics (2.19) is Lyapunov stable.

(ii) If (2.16) is zero-state observable then the output-feedback control $u = -ky$ asymptotically stabilizes the origin $\boldsymbol{x} = \mathbf{0}$.

Proof.

(i) As the storage function $S(\boldsymbol{x}) > 0$ is positive definite by assumption, it is a Lyapunov function candidate. From the passivity property it follows that for any state trajectory which is a solution of the zero dynamics (2.19) it holds that $\frac{\mathrm{d}S(x)}{\mathrm{d}t} \leq yu = 0$. For the zero dynamics it holds that $y(t) = 0$ for all $t \geq 0$, implying the Lyapunov stability in virtue of Theorem 1.1.

(ii) From Theorem 1.2 it follows that with $u = -ky$ the state trajectories $\boldsymbol{x}(t)$ converge into the largest positively invariant subset $\mathcal{M} \subseteq \mathcal{Y}_0$, with $\mathcal{Y}_0$ defined in (2.14). By the zero-state observability assumption this subset is given by $\mathcal{M} = \{\mathbf{0}\}$.

□

In the following the question is addressed if it is possible to passivate a system using feedback control. For this purpose, recall the state transformation (2.17) along with the dynamics in the new coordinates (2.18). According to the relative degree one property, it follows that using the control

$$u = \frac{v - L_f h(\boldsymbol{x})}{L_g h(\boldsymbol{x})} \tag{2.20}$$

and introducing the positive semi-definite storage function $S(\boldsymbol{x}) = \frac{1}{2}h(\boldsymbol{x})^2 = \frac{1}{2}z^2$, the system is passive with respect to the new input v and it holds that

$$\frac{\mathrm{d}S(\boldsymbol{x})}{\mathrm{d}t} = zv = yv.$$

From this relation alone, it is nevertheless not possible to conclude about the zero dynamics, given that here $S(\boldsymbol{x}) \geq 0$ is only positive semi-definite. The theory of Lyapunov has been extended to positive semi-definite Lyapunov functions, and is related to the concept of *conditional stability*, but treating this subject goes beyond the scope of the present notes. The reader can explore this interesting subject e.g. in the seminal work (Sepulchre et al., 1997) or the related literature. For the purpose at hand, we focus on positive definite storage functions $S(\boldsymbol{x}) > 0$.

The following concept further characterizes systems in dependence of the properties of the zero dynamics (2.19) (Byrnes et al., 1991; Sepulchre et al., 1997).

Definition 2.4

Let $L_g h(\mathbf{0}) \neq 0$. Then (2.16) is said to be

- *minimum phase* if $\boldsymbol{\zeta} = \mathbf{0}$ is a locally asymptotically stable equilibrium point of the zero dynamics (2.19)
- *weakly minimum phase* if there exists a positive definite Lyapunov function $V_0(\boldsymbol{\zeta}) > 0$ defined in a neighborhood $\mathcal{N}_{\boldsymbol{\zeta},\mathbf{0}}$ of $\boldsymbol{\zeta} = \mathbf{0}$ that is at least 2 times continuously differentiable and satisfies $L_{\boldsymbol{\varphi}_0} V_0(\boldsymbol{\zeta}) \leq 0$ for all $\boldsymbol{\zeta} \in \mathcal{N}_{\boldsymbol{\zeta},\mathbf{0}}$.

Note that the weakly minimum phase property implies the Lyapunov stability of $\boldsymbol{\zeta} = \mathbf{0}$ (see also the discussion in (Byrnes et al., 1991)).

Having these concepts at hand, the following result is stated without its proof which can be found in the related literature (Byrnes et al., 1991; Sepulchre et al., 1997) (and directly extends to the MIMO case).

Theorem 2.3

The system (2.16) is locally feedback equivalent to a passive system (i.e. there exists a feedback such that the closed-loop system is passive) with a C^2 positive definite storage function $S(\boldsymbol{x}) > 0$ if and only if it has relative degree $r = 1$ at $\boldsymbol{x} = \mathbf{0}$ and is weakly minimum phase.

Accordingly, a system which is feedback equivalent to a passive system can be stabilized by the feedback control

$$u = \frac{-ky - L_f h(\boldsymbol{x})}{L_g h(\boldsymbol{x})}.$$

A direct extension of this result holds for the case that the system is minimum phase. In this case the origin can be asymptotically stabilized using the above control law.

References

Brogliato, B., R. Lozano, B. Maschke, and O. Egeland (2007). *Dissipative Systems Analysis and Control: Theory and Applications.* 2nd. Springer-Verlag, London (cit. on pp. 14, 16).

Byrnes, C. I., A. Isidori, and J. C. Willems (1991). „Passivity, feedback equivalence and the global stabilization of minimum phase nonlinear systems“. In: *IEEE Trans. Auto. Contr.* 36 (11), pp. 1228–1240 (cit. on pp. 14, 19, 20).

Hill, D. J. and P. Moylan (1976). „The stability of nonlinear dissipative systems“. In: *IEEE Trans. Autom. Control.* 21 (5), pp. 708–711 (cit. on p. 14).

— (1980). „Dissipative dynamical systems: basic input-output and state properties“. In: *J. Franklin Inst.* 309 (5), pp. 327–357 (cit. on p. 14).

Isidori, A. (1995). *Nonlinear Control Systems.* Springer-Verlag, London (cit. on pp. 17, 18).

Kalman, R. E. (1963). „Lyapunov functions for the problem of Lur' e in automatic control“. In: *Proc. Nat. Acad. Sci. U.S.A.* 49 (2), pp. 201–205 (cit. on p. 14).

— (1964). „When is a linear control system optimal?“ In: *Trans. ASME J. of Basic Engng.* D 86, pp. 51–60 (cit. on p. 14).

Khalil, H. (1996). *Nonlinear Systems.* 2nd. Prentice-Hall, Upper Saddle River, New Jersey (cit. on p. 16).

Kugi, A. (2017). *Regelungssysteme 2 (Vorlesungsskript), TU Wien.* `https://www.acin.tuwien.ac.at/file/teaching/master/Regelungssysteme-2/Archiv/Regelungssysteme_2_SS2017.pdf` (cit. on p. 14).

Moylan, P. J. (1974). „Implications of passivity in a class of nonlinear systems“. In: *IEEE Trans. Auto. Contr.* Aug. Pp. 373–381 (cit. on p. 14).

Nijmeier, H. and A. van der Schaft (1990). *Nonlinear dynamical control systems.* Springer Verlag, New York (cit. on p. 17).

Sepulchre, R., M. Jankovic, and P. Kokotovic (1997). *Constructive Nonlinear Control.* Springer-Verlag, London (cit. on pp. 14, 16, 18–20).

Willems, J. C. (1972a). „Dissipative dynamical systems: Part I - general theory“. In: *Archive for Rational Mechanics and Analysis* 45 (5), pp. 321–351 (cit. on p. 14).

— (1972b). „Dissipative dynamical systems: Part II - Linear Systems with quadratic supply rates“. In: *Archive for Rational Mechanics and Analysis* 45 (5), pp. 352–393 (cit. on pp. 14, 16).

Chapter 3

Input-to-state stability

This chapter presents the concept of the input-to-state stability (ISS) for control systems with external input, Lyapunov-based approaches to verify the ISS property of a single system and interconnections of a large scale and, finally, the application of the ISS to feedback controller design. In particular, the ISS stabilization using backstepping will be considered. An overview of some ISS-related stability notions concludes the chapter. These include integral ISS, local ISS, and the ISS with respect to a set.

3.1 Introduction to ISS

Systems with inputs. Concept of solution

Consider a system of ordinary differential equations with external input

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0, \tag{3.1}$$

where $\boldsymbol{x}(t) \in \mathbb{R}^n$ and $\boldsymbol{u}(t) \in \mathbb{R}^m$ denote the state and the input at time $t \geq 0$, respectively, function $\boldsymbol{f} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is locally Lipschitz with $\boldsymbol{f}(\boldsymbol{0}, \boldsymbol{0}) = \boldsymbol{0}$, input $\boldsymbol{u} \in \mathcal{U} = L_\infty(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$ – the space of Lebesgue measurable essentially bounded functions equipped with the norm

$$\|\boldsymbol{u}\|_\infty := \operatorname*{ess\,sup}_{t \geq 0} |\boldsymbol{u}(t)| = \inf_{D \subset \mathbb{R},\, \mu(D)=0} \; \sup_{t \in \mathbb{R}_{\geq 0} \setminus D} |\boldsymbol{u}(t)|, \tag{3.2}$$

where $|\cdot|$ stands for the Euclidean norm.

Definition 3.1

A function $\phi : [a, b] \to \mathbb{R}$ is called **absolutely continuous** on $[a, b]$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any finite r and any pairwise disjoint sub-intervals (a_k, b_k) of $[a, b]$ with $\sum_{k=1}^{r} |b_k - a_k| < \delta$ it follows that $\sum_{k=1}^{r} |\phi(b_k) - \phi(a_k)| < \varepsilon$.

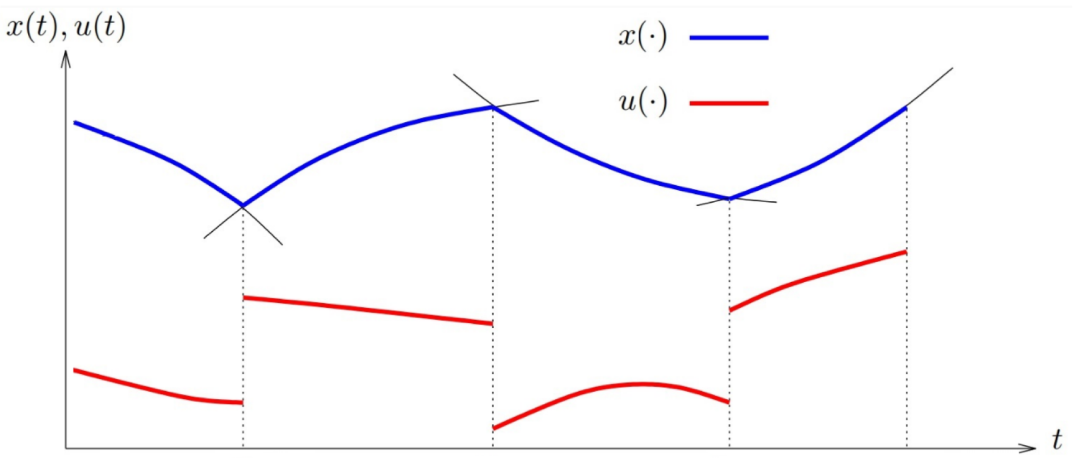

Figure 3.1: An illustration of an admissible input and the corresponding solution to (3.1).

Example 3.1. *(a) The function*

$$\phi(x) = \begin{cases} 0, & \text{if } \quad x = 0, \\ x \sin \frac{1}{x}, & \text{if } \quad x \neq 0 \end{cases}$$

is uniformly continuous but not absolutely continuous on a finite interval containing the origin.
(b) The function $\phi(x) = \sqrt{x}$, $x \in [0, c]$ *is absolutely continuous but not Lipschitz continuous.*

Proposition 3.1 ((Nielson, 1997)). *Absolutely continuous function is differentiable almost everywhere.*

Definition 3.2

An absolutely continuous function $t \mapsto \boldsymbol{\phi}(t, \boldsymbol{x}_0, \boldsymbol{u})$ is called a solution to the problem (3.1) for a given initial condition $\boldsymbol{x}_0 \in \mathbb{R}^n$ and a given input $\boldsymbol{u} \in \mathcal{U}$ if $\boldsymbol{\phi}(0, \boldsymbol{x}_0, \boldsymbol{u}(0)) = \boldsymbol{x}_0$ and $\dot{\boldsymbol{\phi}} = \boldsymbol{f}(\boldsymbol{\phi}, \boldsymbol{u})$ holds almost everywhere.

Theorem 3.1

Let $\boldsymbol{f}$ be Lipschitz continuous with respect to the first argument and uniformly continuous with respect to the second argument, i.e., there exists $L > 0$ such that for any $\boldsymbol{x}, \boldsymbol{y}\mathbb{R}^n$ and for any $\boldsymbol{v} \in \mathbb{R}^m$ it holds that $|\boldsymbol{f}(\boldsymbol{y}, \boldsymbol{v}) - \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{v})| \leq L|\boldsymbol{y} - \boldsymbol{x}|$. Then, for any input $\boldsymbol{u} \in \mathcal{U}$ and any initial condition $\boldsymbol{x}_0 \in \mathbb{R}^n$ there exists a unique solution $\boldsymbol{x} = \boldsymbol{\phi}(\cdot, \boldsymbol{x}_0, \boldsymbol{u})$ to (3.1).

Comparison functions and their properties

In order to define the global stability properties of solutions to (3.1) we recall the standard classes of comparison functions:

$$\begin{aligned}
\mathcal{P} \quad &:= \{\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \,|\, \gamma \text{ is continuous}, \gamma(0) = 0 \text{ and } \gamma(r) > 0 \text{ for } r > 0\}, \\
\mathcal{K} \quad &:= \{\gamma \in \mathcal{P} \,|\, \gamma \text{ is strictly increasing}\}, \\
\mathcal{K}_\infty \quad &:= \{\gamma \in \mathcal{K} \,|\, \gamma \text{ is unbounded}\}, \\
\mathcal{L} \quad &:= \{\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \,|\, \gamma \text{ is continuous and strictly decreasing with, } \lim_{t\to\infty} \gamma(t) = 0\}, \\
\mathcal{KL} \quad &:= \{\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \,|\, \beta \text{ is continuous, } \beta(s, \cdot) \in \mathcal{K} \text{ for any } s \geq 0,\ \beta(\cdot, r) \in \mathcal{L} \text{ for any } r > 0\}.
\end{aligned}$$

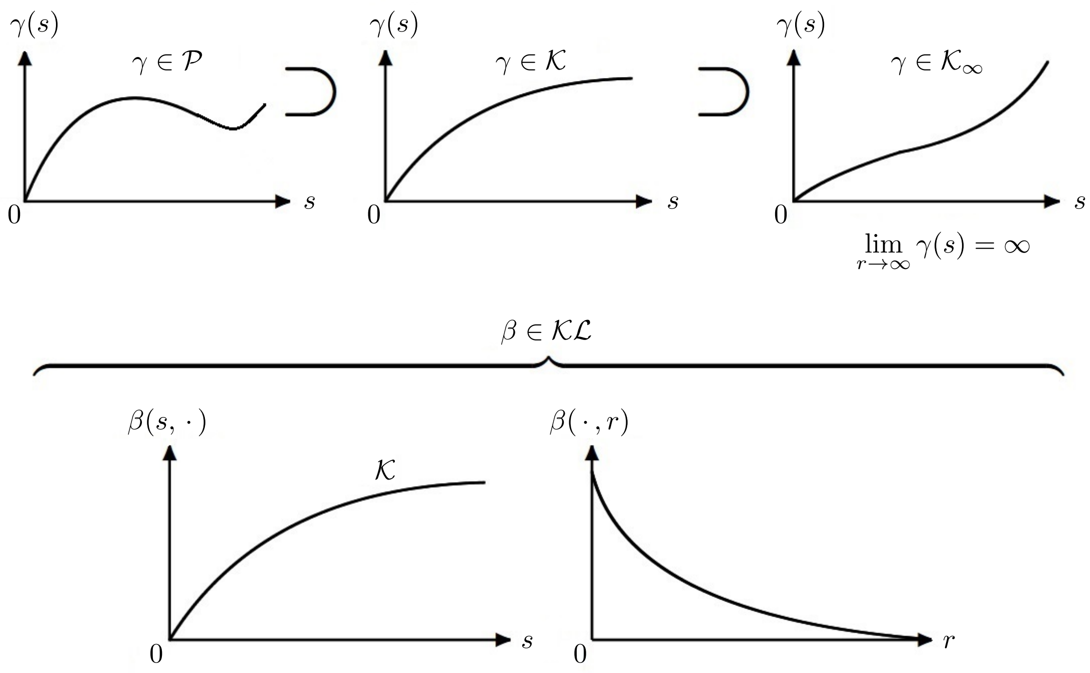

Figure 3.2: Examples of comparison functions.

> **Example 3.2.** *Examples of comparison functions:*
>
> *(a) Linear function $\gamma(r) = \alpha r$ for some $\alpha > 0$ belongs to the classes $\mathcal{P}, \mathcal{K}, \mathcal{K}_\infty$;*
>
> *(b) The function $\gamma(r) = \frac{r}{r+1}$ is of class $\mathcal{K}$ but not of class $\mathcal{K}_\infty$ since $\lim_{r\to\infty} \frac{r}{r+1} = 1 < \infty$;*
>
> *(c) The function $\beta(r,t) = Mre^{-\lambda t}$ for some $M, \lambda > 0$ is of class $\mathcal{KL}$.*

Global stability properties of solutions

Comparison functions can be used instead of "$\varepsilon - \delta$"-language to characterize the global stability properties of zero equilibrium to the ODE system (1.1) discussed in Chapter 1.

Definition 3.3
System (1.1) is called

- **globally stable** if there exists $\sigma \in \mathcal{K}_\infty$ such that for all $x_0 \in \mathbb{R}^n$ the corresponding solution satisfies

$$|\phi(t, x_0)| \le \sigma(|x_0|) \quad \text{for all} \quad t \ge 0; \tag{3.3}$$

- **globally asymptotically stable** (uniformly with respect to the states) if there exists $\beta \in \mathcal{KL}$ such that for all $x_0 \in \mathbb{R}^n$ the corresponding solution satisfies

$$|\phi(t, x_0)| \le \beta(|x_0|, t) \quad \text{for all} \quad t \ge 0. \tag{3.4}$$

Classical Lyapunov stability properties have been proposed as attributes of solutions to ODEs. In this chapter, by stability/asymptotic stability of the system we understand the corresponding stability property of its trivial solution, whose existence is explicitly assumed.

Remark 3.1

If (3.4) holds true with function $\beta(r,t) = Mre^{-\lambda t}$ for some $M, \lambda > 0$, then system (1.1) is called **globally exponentially stable**.

ISS control systems

In this section, we discuss the stability concept that unifies the internal Lyapunov-type stability of control system and its robustness with respect to the external disturbances.

Definition 3.4: (Sontag, 1989)

System (3.1) is called **input-to-state stable** (ISS) if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for any initial value $\boldsymbol{x}_0 \in \mathbb{R}^n$ and any input $\boldsymbol{u} \in \mathcal{U}$ the corresponding solution $\boldsymbol{x} = \boldsymbol{\phi}(\cdot, \boldsymbol{x}_0, \boldsymbol{u})$ exists on $[0, \infty)$ and satisfies

$$|\boldsymbol{\phi}(t, \boldsymbol{x}_0, \boldsymbol{u})| \leq \beta(|\boldsymbol{x}_0|, t) + \gamma(\|\boldsymbol{u}\|_\infty) \quad \text{for all} \quad t \geq 0. \tag{3.5}$$

Function $\gamma \in \mathcal{K}$ is called **ISS-gain** and describes the influence of the input $\boldsymbol{u} \in \mathcal{U}$ on the solutions of the system. Function $\beta \in \mathcal{KL}$ describes the transient behavior of the system.

Definition 3.5

System (3.1) is called **globally asymptotically stable at zero** (0-GAS) if the system (3.1) with $\boldsymbol{u} \equiv \mathbf{0}$ is GAS. System (3.1) possesses the **asymptotic gain property** (AG) if there exist $\gamma \in \mathcal{K}$ such that for any initial value $\boldsymbol{x}_0 \in \mathbb{R}^n$ and any input $\boldsymbol{u} \in \mathcal{U}$ it holds that

$$\limsup_{t\to\infty} |\boldsymbol{\phi}(t, \boldsymbol{x}_0, \boldsymbol{u})| \le \gamma(\|\boldsymbol{u}\|_\infty). \tag{3.6}$$

From (3.5), it immediately follows that the ISS implies 0-GAS and AG properties. Moreover, the converse statement is also true.

Theorem 3.2: (Sontag and Wang, 1996)

System (3.1) is ISS if and only if it is 0-GAS and AG.

The following example shows that 0-GAS is not sufficient to conclude ISS.

Example 3.3 ((Sontag, 2008)). *Consider a scalar nonlinear system*

$$\dot{x} = -x + (x^2 + 1)u. \tag{3.7}$$

System (3.7) *is 0-GAS, since it reduces to* $\dot{x} = -x$ *when* $u \equiv 0$. *On the other hand, solutions diverge even for some inputs that converge to zero. For example,*

$$\phi(t, \sqrt{2}, (2t+2)^{-\frac{1}{2}}) = \sqrt{2t+2} \xrightarrow{t\to\infty} \infty.$$

Hence, (3.7) *does not possess AG property. Even worse, the bounded input* $u \equiv 1$ *steers the state to infinity in a finite time.*

Although the ISS is much stronger property than the 0-GAS for nonlinear systems (see Example 3.3), these properties are equivalent for linear control systems.

Theorem 3.3: (Sontag, 2013)

A linear system

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u} \tag{3.8}$$

is ISS if and only if it is 0-GAS.

Proof. (ISS $\Rightarrow$ 0-GAS) Follows immediately from Definition 3.4 by substituting $\boldsymbol{u} \equiv \mathbf{0}$ into (3.5).

(0-GAS $\Rightarrow$ ISS) The explicit solution to (3.8) with initial condition $\boldsymbol{x}(0) = \boldsymbol{x}_0$ and input $\boldsymbol{u} \in \mathcal{U}$ is given by

$$\boldsymbol{\phi}(t, \boldsymbol{x}, \boldsymbol{u}) = e^{At}\boldsymbol{x}_0 + \int_0^t e^{A(t-s)} B\boldsymbol{u}(s) ds.$$

Since the (3.8) is 0-GAS, matrix A is Hurwitz and $\left\|e^{At}\right\| \leq Me^{-\lambda t}$ for some $M, \lambda > 0$ and for all $t \geq 0$. Then,

$$\begin{aligned}
|\boldsymbol{\phi}(t, \boldsymbol{x}, \boldsymbol{u})| &\leq |e^{At}\boldsymbol{x}_0| + \int_0^t \left\|e^{A(t-s)}\right\| |B\boldsymbol{u}(s)| ds \\
&\leq Me^{-\lambda t}|\boldsymbol{x}_0| + M \int_0^t e^{-\lambda(t-s)} ds \, \|B\| \, \|\boldsymbol{u}\|_\infty \\
&\leq Me^{-\lambda t}|\boldsymbol{x}_0| + M\frac{1}{\lambda}\left(1 - e^{-\lambda t}\right) \|B\| \, \|\boldsymbol{u}\|_\infty \\
&\leq Me^{-\lambda t}|\boldsymbol{x}_0| + \frac{M\,\|B\|}{\lambda} \|\boldsymbol{u}\|_\infty .
\end{aligned}$$

Hence, system (3.8) satisfies Definition 3.4 with $\beta(r, t) = Mre^{-\lambda t}$ and $\gamma(r) = \frac{M\|B\|}{\lambda} r$. □

3.2 Lyapunov characterization of the ISS

This section presents a Lyapunov-based technique for the ISS verification.

Definition 3.6: (Sontag and Wang, 1995a; Jiang et al., 1996)

A smooth function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is called an **ISS-Lyapunov function** for system (3.1) if

- there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that

$$\alpha_1(|\boldsymbol{x}|) \leq V(\boldsymbol{x}) \leq \alpha_2(|\boldsymbol{x}|) \quad \text{for all} \quad \boldsymbol{x} \in \mathbb{R}^n; \tag{3.9}$$

- there exist $\alpha \in \mathcal{P}$ and $\chi \in \mathcal{K}$ such that the following implication holds:

$$|\boldsymbol{x}| \geq \chi(|\boldsymbol{u}|) \quad \Rightarrow \quad \nabla V(\boldsymbol{x})\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \leq -\alpha(|\boldsymbol{x}|) \tag{3.10}$$

  for all $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{u} \in \mathbb{R}^m$.

If V is an ISS-Lyapunov function for (3.1), then V is a Lyapunov function (in the usual sense) for the autonomous system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \mathbf{0})$.

Remark 3.2

Replacing the inequality (3.10) in Definition 3.6 with

$$V(\boldsymbol{x}) \geq \chi(|\boldsymbol{u}|) \quad \Rightarrow \quad \nabla V(\boldsymbol{x})\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \leq -\alpha(V(\boldsymbol{x})) \tag{3.11}$$

leads to an equivalent definition of the ISS-Lyapunov function.

As an alternative to the implication form of the condition (3.10) in the Definition 3.6, one may use a dissipation type definition of the ISS-Lyapunov function:

Proposition 3.2 ((Sontag and Wang, 1995a)). *A smooth function V is an ISS-Lyapunov function for* (3.1) *if and only if there exist $\alpha_i \in \mathcal{K}_\infty$ $(1 \le i \le 4)$ such that* (3.9) *holds, and*

$$\nabla V(\boldsymbol{x}) \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \le -\alpha_3(|\boldsymbol{x}|) + \alpha_4(|\boldsymbol{u}|) \quad \textit{for all} \quad \boldsymbol{x} \in \mathbb{R}^n, \boldsymbol{u} \in \mathbb{R}^m.$$

Theorem 3.4: (Sontag and Wang, 1995a; Jiang et al., 1996)
The system (3.1) is ISS if and only if it has an ISS-Lyapunov function.

Example 3.4. *Let us study the ISS property of the system*

$$\dot{x} = -x^3 + u, \tag{3.12}$$

where $x(t), u(t) \in \mathbb{R}$. As a candidate ISS-Lypaunov function we pick positive-definite proper function $V(x) = \frac{x^2}{2}$. Then,

$$\dot{V} = -x^4 + xu = -(1-\varepsilon)x^4 - \varepsilon x^4 + xu \le -(1-\varepsilon)x^4 \quad \textit{if only} \quad |x| \ge \left(\frac{|u|}{\varepsilon}\right)^{\frac{1}{3}}$$

for any $\varepsilon \in (0,1)$. Then, V is an ISS-Lyapunov function for system (3.12) *in sense of Definition 3.6 with*

$$\alpha(r) = (1-\varepsilon)r^4 \quad \textit{and} \quad \chi(r) = \left(\frac{r}{\varepsilon}\right)^{\frac{1}{3}}.$$

From Theorem 3.4 we conclude the ISS of (3.12).

Exercise 3.1. *Let $x(t), u(t) \in \mathbb{R}$.*

(a) Check the ISS property of the system

$$\dot{x} \;=\; -x - 2x^3 + (1+x^2)u^2;$$

(b) Prove that the water tank system

$$\dot{x} \;=\; -\sqrt{2gx} + u$$

is ISS for some positive constant $g > 0$.

Exercise 3.2. *Let $x_1(t), x_2(t), u(t) \in \mathbb{R}$. Check the ISS property of the system*

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2^2, \\ \dot{x}_2 &= -x_2 + u. \end{aligned}$$

3.3 ISS feedback design

Consider a system

$$\dot{x} = f(x, u, d), \quad t \geq 0 \tag{3.13}$$

with $x(t) \in \mathbb{R}^n$, $u \in \mathcal{U}$, and $d \in \mathcal{D} := L_\infty(\mathbb{R}_{\geq 0}, \mathbb{R}^p)$ for some $n, p \in \mathbb{N}$. System (3.13) possesses two inputs:

- control input u, which we can choose (design) in order to archive some goal;
- external disturbance d, which we cannot influence.

Definition 3.7

System (3.13) is called **ISS stabilizable** if there exists a feedback $k : \mathbb{R}^n \to \mathbb{R}^m$ such that system

$$\dot{x} = f(x, k(x), d)$$

is ISS with respect to the disturbance d.

Definition 3.8

System (3.13) is called **GAS stabilizable** if there exists a feedback $k : \mathbb{R}^n \to \mathbb{R}^m$ such that system

$$\dot{x} = f(x, k(x), 0)$$

is GAS.

Example 3.5. *Consider a stabilization problem of the following system*

$$\dot{x} = x + (x^2 + 1)(u + d). \tag{3.14}$$

The term d may be interpreted as the actuator disturbance in the system. First, assume that $d \equiv 0$ and pick the feedback $k(x) = -\frac{2x}{1+x^2}$. Then, the closed-loop system takes the form

$$\dot{x} = -x. \tag{3.15}$$

System (3.15) is GAS. Hence, (3.14) is GAS stabilizable. However, in the presence of actuator disturbances this controller leads to the system $\dot{x} = -x + (x^2 + 1)d$. As seen before in Example 3.3, this system has solutions which diverge to infinity even for disturbances d that converge to zero. Moreover, the constant disturbance $d \equiv 1$ results in solutions that explode in finite time. The chosen feedback k does not stabilize the system in sense of ISS. A natural question arise, whether it is possible to modify the feedback controller k so that the resulting system is ISS with respect to the disturbances. The answer is given in Theorem 3.5.

Theorem 3.5: (Sontag, 1989)
Consider a control-affine system

$$\dot{\boldsymbol{x}} = \boldsymbol{g}_0(\boldsymbol{x}) + \boldsymbol{g}_1(\boldsymbol{x})(u + d) \tag{3.16}$$

with $\boldsymbol{x}(t) \in \mathbb{R}^n$, $u(t), d(t) \in \mathbb{R}$ and suppose that there is some differentiable feedback law $u = k(\boldsymbol{x})$ so that

$$\dot{\boldsymbol{x}} = \boldsymbol{g}_0(\boldsymbol{x}) + \boldsymbol{g}_1(\boldsymbol{x})k(\boldsymbol{x})$$

has $\boldsymbol{x} = \boldsymbol{0}$ as a GAS equilibrium. Then, there is a feedback law $u = \tilde{k}(\boldsymbol{x})$ such that

$$\dot{\boldsymbol{x}} = \boldsymbol{g}_0(\boldsymbol{x}) + \boldsymbol{g}_1(\boldsymbol{x})(\tilde{k}(\boldsymbol{x}) + d)$$

is ISS with input d.

Proof. Since $\dot{\boldsymbol{x}} = \boldsymbol{g}_0(\boldsymbol{x}) + \boldsymbol{g}_1(\boldsymbol{x})k(\boldsymbol{x})$ is GAS there exists a smooth Lyapunov function V such that

$$\nabla V(x)\left(\boldsymbol{g}_0(\boldsymbol{x}) + \boldsymbol{g}_1(\boldsymbol{x})k(\boldsymbol{x})\right) \leq -\alpha(|\boldsymbol{x}|) \tag{3.17}$$

for some $\alpha \in \mathcal{K}_\infty$ and any $\boldsymbol{x} \in \mathbb{R}^n$. Let us estimate the Lie derivative of V with respect to the perturbed system (3.16) under the feedback controller $u = \tilde{k}(\boldsymbol{x})$:

$$\begin{aligned}\nabla V(x)\left(\boldsymbol{g}_0(\boldsymbol{x}) + \boldsymbol{g}_1(\boldsymbol{x})(\tilde{k}(\boldsymbol{x}) + d)\right) =& \nabla V(x)\left(\boldsymbol{g}_0(\boldsymbol{x}) + \boldsymbol{g}_1(\boldsymbol{x})k(\boldsymbol{x})\right) \\ &+ \nabla V(x)\boldsymbol{g}_1(x)\left(\tilde{k}(\boldsymbol{x}) - k(\boldsymbol{x}) + d\right).\end{aligned} \tag{3.18}$$

Choosing $\tilde{k}(\boldsymbol{x}) := k(\boldsymbol{x}) - \nabla V(x)\boldsymbol{g}_1(x)$, from (3.18), we get

$$\begin{aligned}\nabla V(x)\left(\boldsymbol{g}_0(\boldsymbol{x}) + \boldsymbol{g}_1(\boldsymbol{x})(\tilde{k}(\boldsymbol{x}) + d)\right) &\leq -\alpha(|\boldsymbol{x}|) - \left(\nabla V(\boldsymbol{x})\boldsymbol{g}_1(\boldsymbol{x})\right)^2 + \nabla V(\boldsymbol{x})\boldsymbol{g}_1(\boldsymbol{x})d \\ &\leq -\alpha(|\boldsymbol{x}|) - \left(\nabla V(\boldsymbol{x})\boldsymbol{g}_1(\boldsymbol{x})\right)^2 + \frac{1}{2}\left(\nabla V(\boldsymbol{x})\boldsymbol{g}_1(\boldsymbol{x})\right)^2 + \frac{1}{2}|d|^2 \\ &= -\alpha(|\boldsymbol{x}|) - \frac{1}{2}\left(\nabla V(\boldsymbol{x})\boldsymbol{g}_1(\boldsymbol{x})\right)^2 + \frac{1}{2}|d|^2\end{aligned}$$

This means that V is an ISS-Lyapunov function of system $\dot{\boldsymbol{x}} = \boldsymbol{g}_0(\boldsymbol{x}) + \boldsymbol{g}_1(\boldsymbol{x})(\tilde{k}(\boldsymbol{x}) + d)$ and the feedback law $u = \tilde{k}(\boldsymbol{x})$ is an ISS stabilizing controller for (3.16). □

Remark 3.3
GAS stabilizability with differentiable feedback law implies ISS stabilizability also for a more general class of control-affine systems $\dot{\boldsymbol{x}} = \boldsymbol{g}_0(\boldsymbol{x}) + \sum_{i=1}^{m}(u_i + d_i)\boldsymbol{g}_i(x)$.

Example 3.6. *Find an ISS stabilizing feedback controller for the system*

$$\dot{x} = x + (x^2 + 1)u, \tag{3.19}$$

with $x(t), u(t) \in \mathbb{R}$ in the presence of actuating disturbance $d(t) \in \mathbb{R}$. System (3.19) *is GAS stabilizable. Indeed, let $u = -\frac{2x}{x^2+1}$. Then,* (3.19) *reduces to $\dot{x} = -x$ and it possesses a Lyapunov function $V(x) = \frac{1}{2}x^2$. According to Theorem 3.5 the following feedback controller*

$$u = -\frac{2x}{x^2 + 1} - x(x^2 + 1)$$

ISS stabilizes the system $\dot{x} = x + (x^2 + 1)(u + d)$.

ISS stabilization using backstepping

Consider a system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + G_1(\boldsymbol{x})\boldsymbol{z} + G_2(\boldsymbol{x})\boldsymbol{d}, \tag{3.20}$$
$$\dot{\boldsymbol{z}} = \boldsymbol{u} + F(\boldsymbol{x}, \boldsymbol{z})\boldsymbol{d}, \tag{3.21}$$

where $\boldsymbol{x}(t) \in \mathbb{R}^n$ and $\boldsymbol{z}(t) \in \mathbb{R}^m$ are the state vectors of system (3.20), (3.21), $\boldsymbol{u}(t) \in \mathbb{R}^m$ and $\boldsymbol{d}(t) \in \mathbb{R}^p$ are the vectors of control and perturbation, all functions in the right-hand sides of (3.20), (3.21) are assumed to be locally Lipschitz continuous and $\boldsymbol{f}(\boldsymbol{0}) = \boldsymbol{0}$. It is assumed that there exists a smooth control law $\boldsymbol{k} : \mathbb{R}^n \to \mathbb{R}^m$ such that by plugging $\boldsymbol{z} = \boldsymbol{k}(\boldsymbol{x})$ into (3.20) the system becomes ISS in state $\boldsymbol{x}$ and input $\boldsymbol{d}$. Our problem is as follows: to design a new smooth control $\boldsymbol{u} = \tilde{\boldsymbol{k}}(\boldsymbol{x}, \boldsymbol{z})$ so that the entire system (3.20), (3.21) becomes ISS w.r.t. the disturbance $\boldsymbol{d}$. This requires to "transfer" the control law $\boldsymbol{z} = \boldsymbol{k}(\boldsymbol{x})$ through the integrator.

Theorem 3.6: (Liberzon et al., 1999)

If system (3.20) is ISS stabilizable with a smooth control law $\boldsymbol{z} = \boldsymbol{k}(\boldsymbol{x})$ satisfying $\boldsymbol{k}(\boldsymbol{0}) = \boldsymbol{0}$, then the entire system (3.20), (3.21) is ISS stabilizable with a smooth control law $\boldsymbol{u} = \tilde{\boldsymbol{k}}(\boldsymbol{x}, \boldsymbol{z})$.

Proof. Since system (3.20) with the control law $\boldsymbol{z} = \boldsymbol{k}(\boldsymbol{x})$ is ISS, there exists a corresponding ISS-Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that for all $x \in \mathbb{R}^n$ and $d \in \mathbb{R}^p$

$$\alpha_1(|\boldsymbol{x}|) \leq V(\boldsymbol{x}) \leq \alpha_2(|\boldsymbol{x}|),$$
$$\nabla V(\boldsymbol{x}) \left(\boldsymbol{f}(\boldsymbol{x}) + G_1(x)\boldsymbol{k}(\boldsymbol{x}) + G_2(\boldsymbol{x})\boldsymbol{d}\right) \leq -\alpha(|\boldsymbol{x}|) + \sigma(|\boldsymbol{d}|)$$

for some functions $\alpha_1, \alpha_2, \alpha, \sigma \in \mathcal{K}_\infty$. Following (Liberzon et al., 1999; S. Dashkovskiy et al., 2011), we select

$$W(\boldsymbol{x}, \boldsymbol{z}) = V(\boldsymbol{x}) + \frac{1}{2}|\boldsymbol{z} - \boldsymbol{k}(\boldsymbol{x})|^2.$$

Its total time derivative is given by

$$\begin{aligned}
\dot{W}(\boldsymbol{x}, \boldsymbol{z}) &= \nabla V(\boldsymbol{x}) \left[\boldsymbol{f}(\boldsymbol{x}) + G_1(\boldsymbol{x})\boldsymbol{z} + G_2(\boldsymbol{x})\boldsymbol{d}\right] \\
&\quad + [\boldsymbol{z} - \boldsymbol{k}(\boldsymbol{x})] \left[\boldsymbol{u} + F(\boldsymbol{x}, \boldsymbol{z})\boldsymbol{d} - \nabla \boldsymbol{k}(\boldsymbol{x})[\boldsymbol{f}(\boldsymbol{x}) + G_1(\boldsymbol{x})\boldsymbol{z} + G_2(\boldsymbol{x})\boldsymbol{d}]\right] \\
&\leq -\alpha(|\boldsymbol{x}|) + \sigma(|\boldsymbol{d}|) \\
&\quad + [\boldsymbol{z} - \boldsymbol{k}(\boldsymbol{x})] \left[\boldsymbol{u} + \nabla V(\boldsymbol{x}) G_1(\boldsymbol{x}) + F(\boldsymbol{x}, \boldsymbol{z})\boldsymbol{d} - \nabla \boldsymbol{k}(\boldsymbol{x})[\boldsymbol{f}(\boldsymbol{x}) + G_1(\boldsymbol{x})\boldsymbol{z} + G_2(\boldsymbol{x})\boldsymbol{d}]\right].
\end{aligned}$$

Then, choosing feedback control

$$\boldsymbol{u} = \nabla k(\boldsymbol{x})[\boldsymbol{f}(\boldsymbol{x}) + G_1(\boldsymbol{x})\boldsymbol{z}] - \nabla V(\boldsymbol{x}) G_1(\boldsymbol{x}) - [\boldsymbol{z} - \boldsymbol{k}(\boldsymbol{x})] \left[1 + |F(\boldsymbol{x}, \boldsymbol{z})|^2 + |\nabla \boldsymbol{k}(\boldsymbol{x}) G_2(\boldsymbol{x})|^2\right]$$

we conclude that

$$
\begin{aligned}
\dot{W}(\boldsymbol{x}, z) &\leq -\alpha(|\boldsymbol{x}|) + \sigma(|\boldsymbol{d}|) \\
&\quad + [z - \boldsymbol{k}(\boldsymbol{x})][\boldsymbol{F}(\boldsymbol{x}, z)\boldsymbol{d} - \nabla \boldsymbol{k}(\boldsymbol{x}) G_2(\boldsymbol{x})\boldsymbol{d} - [z - \boldsymbol{k}(\boldsymbol{x})][1 + |\boldsymbol{F}(\boldsymbol{x}, z)|^2 + |\nabla \boldsymbol{k}(\boldsymbol{x}) G_2(\boldsymbol{x})|^2]] \\
&\leq -\alpha(|\boldsymbol{x}|) + \sigma(|\boldsymbol{d}|) - [z - \boldsymbol{k}(\boldsymbol{x})]^2 \\
&\quad + [z - \boldsymbol{k}(\boldsymbol{x})][[\boldsymbol{F}(\boldsymbol{x}, z) - \nabla \boldsymbol{k}(\boldsymbol{x}) G_2(\boldsymbol{x})]\boldsymbol{d} - [z - \boldsymbol{k}(\boldsymbol{x})][|\boldsymbol{F}(\boldsymbol{x}, z)|^2 + |\nabla \boldsymbol{k}(\boldsymbol{x}) G_2(\boldsymbol{x})|^2]] \\
&\leq -\alpha(|\boldsymbol{x}|) + \sigma(|\boldsymbol{d}|) - [z - \boldsymbol{k}(\boldsymbol{x})]^2 \\
&\quad + \frac{|z - \boldsymbol{k}(\boldsymbol{x})|^2}{2}[\boldsymbol{F}(\boldsymbol{x}, z) - \nabla \boldsymbol{k}(\boldsymbol{x}) G_2(\boldsymbol{x})]^2 + \frac{|\boldsymbol{d}|^2}{2} - |z - \boldsymbol{k}(\boldsymbol{x})|^2[|\boldsymbol{F}(\boldsymbol{x}, z)|^2 + |\nabla \boldsymbol{k}(\boldsymbol{x}) G_2(\boldsymbol{x})|^2] \\
&\leq -\alpha(|\boldsymbol{x}|) + \sigma(|\boldsymbol{d}|) - [z - \boldsymbol{k}(\boldsymbol{x})]^2 - \frac{|z - \boldsymbol{k}(\boldsymbol{x})|^2}{2}[|\boldsymbol{F}(\boldsymbol{x}, z)| + |\nabla \boldsymbol{k}(\boldsymbol{x}) G_2(\boldsymbol{x})|]^2 + \frac{|\boldsymbol{d}|^2}{2} \\
&\leq -\alpha(|\boldsymbol{x}|) + \sigma(|\boldsymbol{d}|) - [z - \boldsymbol{k}(\boldsymbol{x})]^2 + \frac{|\boldsymbol{d}|^2}{2} \\
&= -\alpha(|\boldsymbol{x}|) - [z - \boldsymbol{k}(\boldsymbol{x})]^2 + \sigma(|\boldsymbol{d}|) + \frac{|\boldsymbol{d}|^2}{2},
\end{aligned}
$$

which implies the ISS for (3.20), (3.21). □

3.4 Cascade and feedback interconnections

This section considers different types of interconnections of ISS systems. First, we show that a parallel (vacuous) connection of two ISS systems is ISS. Consider

$$
\begin{aligned}
\dot{z} &= f(z, \boldsymbol{v}), \\
\dot{\boldsymbol{x}} &= \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{u})
\end{aligned}
\tag{3.22}
$$

with states $\boldsymbol{x}(t) \in \mathbb{R}^n$, $z(t) \in \mathbb{R}^p$ and inputs $\boldsymbol{u}(t) \in \mathbb{R}^m$, $\boldsymbol{v}(t) \in \mathbb{R}^k$ for some $n, p, m, k \in \mathbb{N}$. Equations from (3.22) do not interchange any signals and, therefore, this type of connection is called vacuous. Assuming that both equations from (3.22) are ISS (from $\boldsymbol{v}$ to z and from $\boldsymbol{u}$ to $\boldsymbol{x}$, respectively) it follows that the corresponding solutions $z(t) = \boldsymbol{\phi}_1(t, z_0, \boldsymbol{v})$ and $\boldsymbol{x}(t) = \boldsymbol{\phi}_2(t, \boldsymbol{x}_0, \boldsymbol{u})$ satisfy

$$
\begin{aligned}
|z(t)| &\leq \beta_1(|z(0), t|) + \gamma_1(\|\boldsymbol{v}\|_\infty), \\
|\boldsymbol{x}(t)| &\leq \beta_2(|\boldsymbol{x}(0), t|) + \gamma_2(\|\boldsymbol{u}\|_\infty)
\end{aligned}
\tag{3.23}
$$

for some $\beta_1, \beta_2 \in \mathcal{KL}$, $\gamma_1, \gamma_2 \in \mathcal{K}$and all $t \geq 0$. Defining the new state $\tilde{\boldsymbol{x}} := (z^\top, \boldsymbol{x}^\top)^\top$ and the new input $\tilde{\boldsymbol{u}} := (\boldsymbol{v}^\top, \boldsymbol{u}^\top)^\top$ and introducing comparison functions $\beta(r, t) := \beta_1(r, t) + \beta_2(r, t)$ and $\gamma(r) := \gamma_1(r) + \gamma_2(r)$, we obtain

$$
|\tilde{\boldsymbol{x}}(t)| \leq \beta(|\tilde{\boldsymbol{x}}(0), t|) + \gamma(\|\tilde{\boldsymbol{u}}\|_\infty)
\tag{3.24}
$$

for all $t \geq 0$. Hence, the vacuous interconnection (3.22) is ISS from $\tilde{\boldsymbol{u}}$ to $\tilde{\boldsymbol{x}}$. Similarly to the case of two subsystems it is possible to conclude the ISS of $n \in \mathbb{N}$ vacuously connected ISS systems.

Next, consider a cascade-type connection of two ISS systems

$$
\begin{aligned}
\dot{z} &= f(z, \boldsymbol{x}), \\
\dot{\boldsymbol{x}} &= \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{u}),
\end{aligned}
\tag{3.25}
$$

where $\boldsymbol{x}(t) \in \mathbb{R}^n$, $z(t) \in \mathbb{R}^p$, $\boldsymbol{u} \in \mathcal{U}$ with $\boldsymbol{u}(t) \in \mathbb{R}^m$ for some $n, p, m \in \mathbb{N}$.

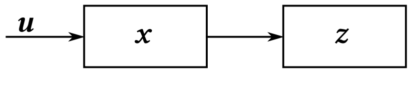

Figure 3.3: Cascade.

Theorem 3.7: (Sontag, 2008)

Let each subsystem in (3.25) be ISS. Then, the cascade (3.25) is ISS.

Corollary 3.1. *In the special case in which the* $\boldsymbol{x}$*-subsystem has no inputs, the cascade of a GAS and an ISS system is GAS.*

Combinations of parallel and sequential (cascade) connections lead to the connections of a tree structure. From the previous reasoning, it follows that the trees of ISS systems are ISS.

Next, we consider a more general type of feedback interconnection

$$\begin{aligned} \dot{\boldsymbol{x}}_1 &= \boldsymbol{f}_1(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{u}_1), \\ \dot{\boldsymbol{x}}_2 &= \boldsymbol{f}_2(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{u}_2), \end{aligned} \tag{3.26}$$

where, for $i = 1, 2$, $\boldsymbol{x}_i(t) \in \mathbb{R}^{n_i}$, $\boldsymbol{u}_i(t) \in \mathbb{R}^{m_i}$ for all $t \geq 0$, and $\boldsymbol{f}_i : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{m_i} \to \mathbb{R}^{n_i}$ is locally Lipschitz. We aim to relate stability properties of $\boldsymbol{x}_1$- and $\boldsymbol{x}_2$-subsystems with the stability of the whole interconnection (3.26). The following example shows that the ISS properties of both subsystems are not sufficient for the ISS of the feedback interconnection even in linear case.

Example 3.7. *Consider a linear system*

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{3.27}$$

with $x_1(t), x_2(t) \in \mathbb{R}$*. System* (3.27) *falls into the class of systems* (3.26) *with* $\boldsymbol{u}_1 = \boldsymbol{u}_2 \equiv 0$*. It is clear that the first equations, i.e.,* $\dot{x}_1 = -x_1 + 2x_2$ *is ISS from* x_2 *to* x_1 *and the second equation, i.e.,* $\dot{x}_2 = -x_2 + 2x_1$ *is ISS from* x_1 *to* x_2*. However system matrix* $A = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$ *has a positive eigenvalue* $\lambda = 1$ *and therefore system* (3.27) *is not GAS.*

Assume that, for $i = 1, 2$, there exists an ISS-Lyapunov function V_i for the $\boldsymbol{x}_i$-subsystem such that the following holds:

- there exist functions $\psi_{i1}, \psi_{i2} \in \mathcal{K}_\infty$ so that

$$\psi_{i1}(|\boldsymbol{x}_i|) \leq V_i(\boldsymbol{x}_i) \leq \psi_{i2}(|\boldsymbol{x}_i|) \tag{3.28}$$

  for all $\boldsymbol{x}_i \in \mathbb{R}^{n_i}$;

- there exist functions $\alpha_i \in \mathcal{K}_\infty$, $\chi_i, \gamma_i \in \mathcal{K}$ so that the implications

$$V_1(\boldsymbol{x}_1) \geq \max\{\chi_1(V_2(\boldsymbol{x}_2)), \gamma_1(|\boldsymbol{u}_1|)\} \quad \Rightarrow \quad \nabla V_1(\boldsymbol{x}_1)\boldsymbol{f}_1(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{u}_1) \leq -\alpha_1(V_1(\boldsymbol{x}_1)) \tag{3.29}$$

  and

$$V_2(\boldsymbol{x}_2) \geq \max\{\chi_2(V_1(\boldsymbol{x}_1)), \gamma_2(|\boldsymbol{u}_2|)\} \quad \Rightarrow \quad \nabla V_2(\boldsymbol{x}_2)\boldsymbol{f}_2(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{u}_2) \leq -\alpha_2(V_2(\boldsymbol{x}_2)) \tag{3.30}$$

  hold for all $\boldsymbol{x}_i(t) \in \mathbb{R}^{n_i}$, $\boldsymbol{u}_i(t) \in \mathbb{R}^{m_i}$.

The following result provides a nonlinear small-gain condition under which an ISS-Lyapunov functions for the interconnected system (3.26) may be expressed in terms of ISS-Lyapunov functions for two subsystems.

Theorem 3.8: (Jiang et al., 1996)

Assume that, for $i = 1, 2$, the $\boldsymbol{x}_i$-subsystem has an ISS-Lyapunov function V_i satisiying (3.28), (3.29) and (3.30). If

$$\chi_2 \circ \chi_1(r) < r \quad \text{for all} \quad r > 0 \tag{3.31}$$

then the interconnected system (3.26) is ISS. Moreover, there exist

- a $\mathcal{K}_\infty$-function σ continuously differentiable on $(0, \infty)$ with $\sigma'(r) > 0$ for all $r > 0$ such that

  $$\chi_2(r) < \sigma(r) < \chi_1^{-1}(r) \quad \text{for all} \quad r > 0;$$

- a locally Lipschitz on $\mathbb{R}^{n_1+n_2} \setminus \{0\}$ function $V : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}_{\geq 0}$ defined by

  $$V(\boldsymbol{x}_1, \boldsymbol{x}_2) = \max\{\sigma(V_1(\boldsymbol{x}_1)), V_2(\boldsymbol{x}_2)\}$$

  such that for almost all $\boldsymbol{x}_1 \in \mathbb{R}^{n_1}$, $\boldsymbol{x}_2 \in \mathbb{R}^{n_2}$ and for all $\boldsymbol{u}_1 \in \mathbb{R}^{m_1}$, $\boldsymbol{u}_2 \in \mathbb{R}^{m_2}$

  $$\nabla V(\boldsymbol{x}) \begin{pmatrix} \boldsymbol{f}_1(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{u}_1) \\ \boldsymbol{f}_2(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{u}_2) \end{pmatrix} \leq -\alpha(V(\boldsymbol{x}_1, \boldsymbol{x}_2)) \quad \text{whenever} \quad V(\boldsymbol{x}_1, \boldsymbol{x}_2) \geq \gamma(|(\boldsymbol{u}_1, \boldsymbol{u}_2)^\top|)$$

  for some $\alpha \in \mathcal{K}_\infty$, $\gamma \in \mathcal{K}$.

Remark 3.4

The existence of locally Lipschitz function V (not necessarily smooth) is sufficient for the ISS (Jiang et al., 1996).

Condition (3.31) is called a *small gain condition.* A generalization of this conditions to the case of $n \in \mathbb{N}$ interconnected ISS systems can be found in (S. N. Dashkovskiy et al., 2010).

3.5 ISS-related stability notions

In this section we provide an overview of some stability concepts that are weaker than the ISS.

Integral input-to-state stability

Definition 3.9: (Sontag, 1998)

System (3.1) is called **integral input-to-state stable** (iISS) if there exist $\beta \in \mathcal{KL}$ and $\alpha, \gamma \in \mathcal{K}_\infty$ such that for any initial value $\boldsymbol{x}_0 \in \mathbb{R}^n$ and any input $\boldsymbol{u} \in \mathcal{U}$ the corresponding solution $\boldsymbol{x} = \boldsymbol{\phi}(\cdot, \boldsymbol{x}_0, \boldsymbol{u})$ exists on $[0, \infty)$ and satisfies

$$\alpha(|\boldsymbol{\phi}(t, \boldsymbol{x}_0, \boldsymbol{u})|) \le \beta(|\boldsymbol{x}_0|, t) + \int_0^t \gamma(|\boldsymbol{u}(s)|)ds \quad \text{for all} \quad t \ge 0. \tag{3.32}$$

The iISS of system (3.1) implies a practically important asymptotic property for the corresponding solutions:

Theorem 3.9: (Sontag, 1998)

If system (3.1) is iISS, i.e., its solutions satisfy (3.32), then for any u such that

$$\int_0^\infty \gamma(|\boldsymbol{u}(s)|)ds < \infty,$$

and any $x_0 \in \mathbb{R}^n$ it follows that $\phi(t, \boldsymbol{x}_0, \boldsymbol{u}) \to 0$ as $t \to +\infty$.

The iISS property of system (3.1) is equivalent to the existence of a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_{\ge 0}$ satisfying

$$\begin{aligned} &\alpha_1(|\boldsymbol{x}|) \le V(\boldsymbol{x}) \le \alpha_2(|\boldsymbol{x}|) \quad \text{for all} \quad \boldsymbol{x} \in \mathbb{R}^n \\ &\nabla V(\boldsymbol{x})\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \le -\alpha_3(|\boldsymbol{x}|) + \alpha_4(|\boldsymbol{u}|) \quad \text{for all} \quad \boldsymbol{x} \in \mathbb{R}^n,\ \boldsymbol{u} \in \mathbb{R}^m \end{aligned}$$

for some $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\alpha_3 \in \mathcal{P}$, and $\alpha_4 \in \mathcal{K}_\infty$.

In contrast to the ISS case (see Proposition 3.2), where α_3 is required to be of class $\mathcal{K}_\infty$, it is sufficient to use positive definite α_3 for the iISS characterization. Also, there is no characterization for the iISS property via Lyapunov-like function in implication form similar to the ISS case (see Theorem 3.4).

Local input-to-state stability

Definition 3.10

System (3.1) is called **locally input-to-state stable** (LISS) if there exist $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}$, and constants $\rho_x, \rho_u > 0$ such that for any initial value $|\boldsymbol{x}_0| \le \rho_x$ and any input $\boldsymbol{u} \in \mathcal{U}$ with $\|\boldsymbol{u}\|_\infty \le \rho_u$ the corresponding solution $\boldsymbol{x} = \boldsymbol{\phi}(\cdot, \boldsymbol{x}_0, \boldsymbol{u})$ exists on $[0, \infty)$ and satisfies

$$|\boldsymbol{\phi}(t, \boldsymbol{x}_0, \boldsymbol{u})| \le \beta(|\boldsymbol{x}_0|, t) + \gamma(\|\boldsymbol{u}\|_\infty) \quad \text{for all} \quad t \ge 0. \tag{3.33}$$

The LISS property for system (3.1) is equivalent to the existence of a smooth function $V : D \to \mathbb{R}_{\ge 0}$, $0 \in \mathtt{int}(D) \subset \mathbb{R}^n$ satisfying conditions (3.9), (3.10) for all $\boldsymbol{x} \in D$, $u \in U := \{\boldsymbol{u} \in \mathbb{R}^m : |u| \le \rho\}$ for some constant $\rho > 0$.

From Definitions 3.4, 3.5, 3.9, and 3.10, it follows that

$$\text{ISS} \subset \text{iISS} \subset 0-\text{GAS} \quad \text{and} \quad \text{ISS} \subset \text{iISS} \subset \text{LISS}.$$

Input-to-state stability with respect to a set

Let $\mathcal{A} \subset \mathbb{R}^n$ be a nonempty compact set. The distance from any point $\boldsymbol{x} \in \mathbb{R}^n$ to $\mathcal{A}$ is defined by $|\boldsymbol{x}|_{\mathcal{A}} := \inf_{\boldsymbol{y}\in\mathcal{A}} |\boldsymbol{x} - \boldsymbol{y}|$.

Definition 3.11: (Sontag and Wang, 1995b)

System (3.1) is called **input-to-state stable with respect to $\mathcal{A}$** if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for any initial value $\boldsymbol{x}_0 \in \mathbb{R}^n$ and any input $\boldsymbol{u} \in \mathcal{U}$ the corresponding solution $\boldsymbol{x} = \boldsymbol{\phi}(\cdot, \boldsymbol{x}_0, \boldsymbol{u})$ exists on $[0, \infty)$ and satisfies

$$|\boldsymbol{\phi}(t, \boldsymbol{x}_0, \boldsymbol{u})|_{\mathcal{A}} \leq \beta(|\boldsymbol{x}_0|_{\mathcal{A}}, t) + \gamma(\|\boldsymbol{u}\|_\infty) \quad \text{for all} \quad t \geq 0. \tag{3.34}$$

The ISS w.r.t. $\mathcal{A}$ property for the system (3.1) is equivalent to the existence of a smooth function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ satisfying

$$\begin{aligned} &\alpha_1(|\boldsymbol{x}|_{\mathcal{A}}) \leq V(\boldsymbol{x}) \leq \alpha_2(|\boldsymbol{x}|_{\mathcal{A}}) \quad \text{for all} \quad \boldsymbol{x} \in \mathbb{R}^n \\ &|\boldsymbol{x}|_{\mathcal{A}} \geq \alpha_3(|\boldsymbol{u}|) \quad \Rightarrow \quad \nabla V(\boldsymbol{x})\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \leq -\alpha_4(V(\boldsymbol{x})) \quad \text{for all} \quad \boldsymbol{x} \in \mathbb{R}^n,\ \boldsymbol{u} \in \mathbb{R}^m \end{aligned}$$

for some $\alpha_i \in \mathcal{K}_\infty$ $(i = 1, \dots, 4)$.

References

Dashkovskiy, S. N., B. S. Rüffer, and F. R. Wirth (2010). „Small gain theorems for large scale systems and construction of ISS Lyapunov functions“. In: *SIAM Journal on Control and Optimization* 48.6, pp. 4089–4118 (cit. on p. 35).

Dashkovskiy, S., D. V. Efimov, and E. D. Sontag (2011). „Input to state stability and allied system properties“. In: *Automation and Remote Control* 72.8, p. 1579 (cit. on p. 32).

Jiang, Z.-P., I. M. Mareels, and Y. Wang (1996). „A Lyapunov formulation of the nonlinear small-gain theorem for interconnected ISS systems“. In: *Automatica* 32.8, pp. 1211–1215 (cit. on pp. 28, 29, 34, 35).

Liberzon, D., E. D. Sontag, and Y. Wang (1999). „On integral-input-to-state stabilization“. In: *Proceedings of the 1999 American Control Conference (Cat. No. 99CH36251)*. Vol. 3. IEEE, pp. 1598–1602 (cit. on p. 32).

Nielson, O. (1997). *An introduction to integration and measure theory*. John Wiley & Sons, Inc., New York. (cit. on p. 24).

Sontag, E. D. (1989). „Smooth stabilization implies coprime factorization“. In: *IEEE Transactions on Automatic Control* 34.4, pp. 435–443 (cit. on pp. 26, 31).

— (1998). „Comments on integral variants of ISS“. In: *Systems & Control Letters* 34.1-2, pp. 93–100 (cit. on p. 36).

— (2008). „Input to state stability: Basic concepts and results“. In: *Nonlinear and optimal control theory*. Springer, pp. 163–220 (cit. on pp. 27, 34).

— (2013). *Mathematical control theory: deterministic finite dimensional systems*. Vol. 6. Springer Science & Business Media (cit. on p. 27).

Sontag, E. D. and Y. Wang (1995a). „On characterizations of the input-to-state stability property“. In: *Systems & Control Letters* 24.5, pp. 351–359 (cit. on pp. 28, 29).

— (1996). „New characterizations of input-to-state stability“. In: *IEEE Transactions on Automatic Control* 41.9, pp. 1283–1294 (cit. on p. 27).

— (1995b). „On Characterizations of Input-to-State Stability with Respect to Compact Sets“. In: *IFAC Proceedings Volumes* 28.14. 3rd IFAC Symposium on Nonlinear Control Systems Design 1995, Tahoe City, CA, USA, 25-28 June 1995, pp. 203–208 (cit. on p. 37).