Advanced Methods in Nonlinear Control (SS 2024) — Task 1

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Exercise 1.1.

Show via Lyapunov's direct method the asymptotic stability of the system

$$\dot{x} = -f(x)$$

with $f(x) > 0$ for $x > 0$, $f(0) = 0$ and $f(x) < 0$ for $x < 0$.

Hint: Use $V(x) = \int_0^x f(\xi)\mathrm{d}\xi$ as a Lyapunov function candidate.

Exercise 1.2.

Consider the system

$$\begin{aligned} \dot{x}_1 &= -x_1 - x_2 - x_1(x_1^2 + x_2^2), && t > 0, \quad x_1(0) = x_{10} \\ \dot{x}_2 &= x_1 - x_2 - x_2(x_1^2 + x_2^2), && t > 0, \quad x_2(0) = x_{20} \end{aligned}$$

and show the asymptotic stability of the origin using Lyapunov's direct method.

Exercise 2.1.

Consider the system

$$\begin{aligned} \dot{x}_1 &= x_1^3 + x_2, && t > 0, \quad x_1(0) = x_{10} \\ \dot{x}_2 &= u, && t > 0, \quad x_2(0) = x_{20}. \end{aligned}$$

Design a backstepping controller to ensure the asymptotic stability of the origin.

Bonus: Illustrate the behavior of the controller by showing a numerical simulation of the closed-loop system.

Exercise 3.1.

Consider a mass–spring–damper system with a nonlinear spring, that is modeled by

$$m\ddot{x} + d\dot{x} + k(x)x = F$$

with $m, d > 0$, $k : \mathbb{R} \to (0, \infty)$ and $F : [0, \infty) \to \mathbb{R}$. Show that the system with input–output pair $(F, \dot{x})$ is passive. Further show that with the proportional feedback control $F = -\kappa_c \dot{x}$ with $\kappa_c > -d$ the system can be asymptotically stabilized.

Hint: Bring the system into state–space form and use the total mechanical energy as storage function, i.e. $E_{\text{kin}} = \frac{1}{2}mv^2$ for a velocity v and $E_{\text{elastic}} = \int_0^s k(\xi)\xi\mathrm{d}\xi$ for the (nonlinear) spring *constant* $k(s)$ (which is not independent from the distance s). For the stability analysis recall the Krasovskii-LaSalle invariance theorem.

Exercise 4.1.

Consider the system

$$\dot{x} = x^3 + \left(x^2 + \sin(x) + \frac{3}{2}\right)u, \qquad t > 0, \quad x(0) = x_0$$

with an external input u which can be modified for control purpose.

Construct a GAS stabilizing feedback controller $u = k(x)$. Furtheremore consider the same systeme with a disturbance acting on the input, i.e.,

$$\dot{x} = x^3 + \left(x^2 + \sin(x) + \frac{3}{2}\right)(u + d), \quad t > 0, \quad x(0) = x_0$$

Construct (modify) a feedback controller $u = \tilde{k}(x)$ so that the states x are ISS with respect to the disturbance d.

Bonus: Illustrate the behavior of the controller $u = \tilde{k}(x)$ by showing a numerical simulation of the closed-loop system with some konstant or bounded function $d(t)$.