

# ACM 常用算法模板

therehello

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# 1 数据结构

## 1.1 并查集

```

1 struct dsu {
2     int n;
3     vector<int> fa, sz;
4     dsu(int _n) : n(_n), fa(n + 1), sz(n + 1, 1) { iota(fa.begin(), fa.end(), 0); }
5     int find(int x) { return x == fa[x] ? x : fa[x] = find(fa[x]); }
6     int merge(int x, int y) {
7         int fax = find(x), fay = find(y);
8         if (fax == fay) return 0; // 一个集合
9         sz[fay] += sz[fax];
10        return fa[fax] = fay; // 合并到哪个集合了
11    }
12    int size(int x) { return sz[find(x)]; }
13 };

```

## 1.2 树状数组

### 1.2.1 一维

```

1 template <class T>
2 struct fenwick {
3     int n;
4     vector<T> t;
5     fenwick(int _n) : n(_n), t(n + 1) {}
6     T query(int l, int r) {
7         auto query = [&](int pos) {
8             T res = 0;
9             while (pos) {
10                res += t[pos];
11                pos -= lowbit(pos);
12            }
13            return res;
14        };
15        return query(r) - query(l - 1);
16    }
17    void add(int pos, T num) {
18        while (pos <= n) {
19            t[pos] += num;
20            pos += lowbit(pos);
21        }
22    }
23 };

```

### 1.2.2 二维

```

1 template <class T>
2 struct Fenwick_tree_2 {
3     Fenwick_tree_2(int n, int m) : n(n), m(m), tree(n + 1, vector<T>(m + 1)) {}
4     T query(int l1, int r1, int l2, int r2) {
5         auto query = [&](int l, int r) {
6             T res = 0;
7             for (int i = l; i; i -= lowbit(i))

```

```

8         for (int j = r; j; j -= lowbit(j)) res += tree[i][j];
9         return res;
10    };
11    return query(l2, r2) - query(l2, r1 - 1) - query(l1 - 1, r2) + query(l1 - 1, r1 - 1);
12 }
13 void update(int x, int y, T num) {
14     for (int i = x; i <= n; i += lowbit(i))
15         for (int j = y; j <= m; j += lowbit(j)) tree[i][j] += num;
16 }
17 private:
18     int n, m;
19     vector<vector<T>> tree;
20 };

```

### 1.2.3 三维

```

1 template <class T>
2 struct Fenwick_tree_3 {
3     Fenwick_tree_3(int n, int m, int k)
4         : n(n), m(m), k(k), tree(n + 1, vector<vector<T>>(m + 1, vector<T>(k + 1))) {}
5     T query(int a, int b, int c, int d, int e, int f) {
6         auto query = [&](int x, int y, int z) {
7             T res = 0;
8             for (int i = x; i; i -= lowbit(i))
9                 for (int j = y; j; j -= lowbit(j))
10                     for (int p = z; p; p -= lowbit(p)) res += tree[i][j][p];
11             return res;
12         };
13         T res = query(d, e, f);
14         res -= query(a - 1, e, f) + query(d, b - 1, f) + query(d, e, c - 1);
15         res += query(a - 1, b - 1, f) + query(a - 1, e, c - 1) + query(d, b - 1, c - 1);
16         res -= query(a - 1, b - 1, c - 1);
17         return res;
18     }
19     void update(int x, int y, int z, T num) {
20         for (int i = x; i <= n; i += lowbit(i))
21             for (int j = y; j <= m; j += lowbit(j))
22                 for (int p = z; p <= k; p += lowbit(p)) tree[i][j][p] += num;
23     }
24 private:
25     int n, m, k;
26     vector<vector<vector<T>>> tree;
27 };

```

## 1.3 线段树

```

1 template <class Data, class Num>
2 struct Segment_Tree {
3     inline void update(int l, int r, Num x) { update(1, l, r, x); }
4     inline Data query(int l, int r) { return query(1, l, r); }
5     Segment_Tree(vector<Data>& a) {
6         n = a.size();
7         tree.assign(n * 4 + 1, {});
8         build(a, 1, 1, n);
9     }
10 private:

```

```

11     int n;
12     struct Tree {
13         int l, r;
14         Data data;
15     };
16     vector<Tree> tree;
17     inline void pushup(int pos) { tree[pos].data = tree[pos << 1].data + tree[pos << 1 | 1].data; }
18     inline void pushdown(int pos) {
19         tree[pos << 1].data = tree[pos << 1].data + tree[pos].data.lazytag;
20         tree[pos << 1 | 1].data = tree[pos << 1 | 1].data + tree[pos].data.lazytag;
21         tree[pos].data.lazytag = Num::zero();
22     }
23     void build(vector<Data>& a, int pos, int l, int r) {
24         tree[pos].l = l;
25         tree[pos].r = r;
26         if (l == r) {
27             tree[pos].data = a[l - 1];
28             return;
29         }
30         int mid = (tree[pos].l + tree[pos].r) >> 1;
31         build(a, pos << 1, l, mid);
32         build(a, pos << 1 | 1, mid + 1, r);
33         pushup(pos);
34     }
35     void update(int pos, int& l, int& r, Num& x) {
36         if (l > tree[pos].r || r < tree[pos].l) return;
37         if (l <= tree[pos].l && tree[pos].r <= r) {
38             tree[pos].data = tree[pos].data + x;
39             return;
40         }
41         pushdown(pos);
42         update(pos << 1, l, r, x);
43         update(pos << 1 | 1, l, r, x);
44         pushup(pos);
45     }
46     Data query(int pos, int& l, int& r) {
47         if (l > tree[pos].r || r < tree[pos].l) return Data::zero();
48         if (l <= tree[pos].l && tree[pos].r <= r) return tree[pos].data;
49         pushdown(pos);
50         return query(pos << 1, l, r) + query(pos << 1 | 1, l, r);
51     }
52 };
53 struct Num {
54     ll add;
55     inline static Num zero() { return {0}; }
56     inline Num operator+(Num b) { return {add + b.add}; }
57 };
58 struct Data {
59     ll sum, len;
60     Num lazytag;
61     inline static Data zero() { return {0, 0, Num::zero()}; }
62     inline Data operator+(Num b) { return {sum + len * b.add, len, lazytag + b}; }
63     inline Data operator+(Data b) { return {sum + b.sum, len + b.len, Num::zero()}; }
64 };

```

## 1.4 普通平衡树

### 1.4.1 树状数组实现

需要预先处理出来所有可能的数。

```

1 int lowbit(int x) { return x & -x; }
2
3 template <typename T>
4 struct treap {
5     int n, size;
6     vector<int> t;
7     vector<T> t2, S;
8     treap(const vector<T>& a) : S(a) {
9         sort(S.begin(), S.end());
10        S.erase(unique(S.begin(), S.end()), S.end());
11        n = S.size();
12        size = 0;
13        t = vector<int>(n + 1);
14        t2 = vector<T>(n + 1);
15    }
16    int pos(T x) { return lower_bound(S.begin(), S.end(), x) - S.begin() + 1; }
17    int sum(int pos) {
18        int res = 0;
19        while (pos) {
20            res += t[pos];
21            pos -= lowbit(pos);
22        }
23        return res;
24    }
25
26    // 插入cnt个x
27    void insert(T x, int cnt) {
28        size += cnt;
29        int i = pos(x);
30        assert(i <= n && S[i - 1] == x);
31        for (; i <= n; i += lowbit(i)) {
32            t[i] += cnt;
33            t2[i] += cnt * x;
34        }
35    }
36
37    // 删除cnt个x
38    void erase(T x, int cnt) {
39        assert(cnt <= count(x));
40        insert(x, -cnt);
41    }
42
43    // x的排名
44    int rank(T x) {
45        assert(count(x));
46        return sum(pos(x) - 1) + 1;
47    }
48
49    // 统计出现次数
50    int count(T x) { return sum(pos(x)) - sum(pos(x) - 1); }
51
52    // 第k小
53    T kth(int k) {

```



```

54     assert(0 < k && k <= size);
55     int cnt = 0, x = 0;
56     for (int i = __lg(n); i >= 0; i--) {
57         x += 1 << i;
58         if (x >= n || cnt + t[x] >= k) x -= 1 << i;
59         else cnt += t[x];
60     }
61     return S[x];
62 }
63
64 // 前k小的数之和
65 T pre_sum(int k) {
66     assert(0 < k && k <= size);
67     int cnt = 0, x = 0;
68     T res = 0;
69     for (int i = __lg(n); i >= 0; i--) {
70         x += 1 << i;
71         if (x >= n || cnt + t[x] >= k) x -= 1 << i;
72         else {
73             cnt += t[x];
74             res += t2[x];
75         }
76     }
77     return res + (k - cnt) * S[x];
78 }
79
80 // 小于x, 最大的数
81 T prev(T x) { return kth(sum(pos(x) - 1)); }
82
83 // 大于x, 最小的数
84 T next(T x) { return kth(sum(pos(x)) + 1); }
85 };

```

## 1.5 可持久化线段树

```

1  constexpr int MAXN = 200000;
2  vector<int> root(MAXN << 5);
3  struct Persistent_seg {
4      int n;
5      struct Data {
6          int ls, rs;
7          int val;
8      };
9      vector<Data> tree;
10     Persistent_seg(int n, vector<int>& a) : n(n) { root[0] = build(1, n, a); }
11     int build(int l, int r, vector<int>& a) {
12         if (l == r) {
13             tree.push_back({0, 0, a[l]});
14             return tree.size() - 1;
15         }
16         int mid = l + r >> 1;
17         int ls = build(l, mid, a), rs = build(mid + 1, r, a);
18         tree.push_back({ls, rs, tree[ls].val + tree[rs].val});
19         return tree.size() - 1;
20     }
21     int update(int rt, const int& idx, const int& val, int l, int r) {
22         if (l == r) {

```

```

23     tree.push_back({0, 0, tree[rt].val + val});
24     return tree.size() - 1;
25 }
26 int mid = l + r >> 1, ls = tree[rt].ls, rs = tree[rt].rs;
27 if (idx <= mid) ls = update(ls, idx, val, l, mid);
28 else rs = update(rs, idx, val, mid + 1, r);
29 tree.push_back({ls, rs, tree[ls].val + tree[rs].val});
30 return tree.size() - 1;
31 }
32 int query(int rt1, int rt2, int k, int l, int r) {
33     if (l == r) return l;
34     int mid = l + r >> 1;
35     int lcnt = tree[tree[rt2].ls].val - tree[tree[rt1].ls].val;
36     if (k <= lcnt) return query(tree[rt1].ls, tree[rt2].ls, k, l, mid);
37     else return query(tree[rt1].rs, tree[rt2].rs, k - lcnt, mid + 1, r);
38 }
39 };

```

## 1.6 st 表

```

1 auto lg = []() {
2     array<int, 10000001> lg;
3     lg[1] = 0;
4     for (int i = 2; i <= 10000000; i++) lg[i] = lg[i >> 1] + 1;
5     return lg;
6 }();
7 template <typename T>
8 struct st {
9     int n;
10    vector<vector<T>> a;
11    st(vector<T>& _a) : n(_a.size()) {
12        a.assign(lg[n] + 1, vector<int>(n));
13        for (int i = 0; i < n; i++) a[0][i] = _a[i];
14        for (int j = 1; j <= lg[n]; j++)
15            for (int i = 0; i + (1 << j) - 1 < n; i++)
16                a[j][i] = max(a[j - 1][i], a[j - 1][i + (1 << (j - 1))]);
17    }
18    T query(int l, int r) {
19        int k = lg[r - l + 1];
20        return max(a[k][l], a[k][r - (1 << k) + 1]);
21    }
22 };

```

## 2 图论

存图

```

1 struct Graph {
2     int n;
3     struct Edge {
4         int to, w;
5     };
6     vector<vector<Edge>> graph;
7     Graph(int _n) {
8         n = _n;
9         graph.assign(n + 1, vector<Edge>());
10    };
11    void add(int u, int v, int w) { graph[u].push_back({v, w}); }
12 };

```

### 2.1 最短路

#### 2.1.1 dijkstra

```

1 void dij(Graph& graph, vector<int>& dis, int t) {
2     vector<int> visit(graph.n + 1, 0);
3     priority_queue<pair<int, int>> que;
4     dis[t] = 0;
5     que.emplace(0, t);
6     while (!que.empty()) {
7         int u = que.top().second;
8         que.pop();
9         if (visit[u]) continue;
10        visit[u] = 1;
11        for (auto& [to, w] : graph.graph[u]) {
12            if (dis[to] > dis[u] + w) {
13                dis[to] = dis[u] + w;
14                que.emplace(-dis[to], to);
15            }
16        }
17    }
18 }

```

### 2.2 树上问题

#### 2.2.1 最近公公祖先

倍增法

```

1 vector<int> dep;
2 vector<array<int, 21>> fa;
3 dep.assign(n + 1, 0);
4 fa.assign(n + 1, array<int, 21>{{}});
5 void binary_jump(int root) {
6     function<void(int)> dfs = [&](int t) {
7         dep[t] = dep[fa[t][0]] + 1;
8         for (auto& [to] : graph[t]) {
9             if (to == fa[t][0]) continue;
10            fa[to][0] = t;
11            dfs(to);

```

```

12     }
13 };
14 dfs(root);
15 for (int j = 1; j <= 20; j++)
16     for (int i = 1; i <= n; i++) fa[i][j] = fa[fa[i][j - 1]][j - 1];
17 }
18 int lca(int x, int y) {
19     if (dep[x] < dep[y]) swap(x, y);
20     for (int i = 20; i >= 0; i--)
21         if (dep[fa[x][i]] >= dep[y]) x = fa[x][i];
22     if (x == y) return x;
23     for (int i = 20; i >= 0; i--) {
24         if (fa[x][i] != fa[y][i]) {
25             x = fa[x][i];
26             y = fa[y][i];
27         }
28     }
29     return fa[x][0];
30 }

```

### 树剖

```

1 int lca(int x, int y) {
2     while (top[x] != top[y]) {
3         if (dep[top[x]] < dep[top[y]]) swap(x, y);
4         x = fa[top[x]];
5     }
6     if (dep[x] < dep[y]) swap(x, y);
7     return y;
8 }

```

### 2.2.2 树链剖分

```

1 vector<int> fa, siz, dep, son, dfn, rnk, top;
2 fa.assign(n + 1, 0);
3 siz.assign(n + 1, 0);
4 dep.assign(n + 1, 0);
5 son.assign(n + 1, 0);
6 dfn.assign(n + 1, 0);
7 rnk.assign(n + 1, 0);
8 top.assign(n + 1, 0);
9 void hld(int root) {
10     function<void(int)> dfs1 = [&](int t) {
11         dep[t] = dep[fa[t]] + 1;
12         siz[t] = 1;
13         for (auto& [to, w] : graph[t]) {
14             if (to == fa[t]) continue;
15             fa[to] = t;
16             dfs1(to);
17             if (siz[son[t]] < siz[to]) son[t] = to;
18             siz[t] += siz[to];
19         }
20     };
21     dfs1(root);
22     int dfn_tail = 0;
23     for (int i = 1; i <= n; i++) top[i] = i;
24     function<void(int)> dfs2 = [&](int t) {
25         dfn[t] = ++dfn_tail;

```

```

26     rnk[dfn_tail] = t;
27     if (!son[t]) return;
28     top[son[t]] = top[t];
29     dfs2(son[t]);
30     for (auto& [to, w] : graph[t]) {
31         if (to == fa[t] || to == son[t]) continue;
32         dfs2(to);
33     }
34 };
35 dfs2(root);
36 }

```

## 2.3 强连通分量

```

1 void tarjan(Graph& g1, Graph& g2) {
2     int dfn_tail = 0, cnt = 0;
3     vector<int> dfn(g1.n + 1, 0), low(g1.n + 1, 0), exist(g1.n + 1, 0), belong(g1.n + 1, 0);
4     stack<int> sta;
5     function<void(int)> dfs = [&](int t) {
6         dfn[t] = low[t] = ++dfn_tail;
7         sta.push(t);
8         exist[t] = 1;
9         for (auto& [to] : g1.graph[t])
10             if (!dfn[to]) {
11                 dfs(to);
12                 low[t] = min(low[t], low[to]);
13             } else if (exist[to]) low[t] = min(low[t], dfn[to]);
14         if (dfn[t] == low[t]) {
15             cnt++;
16             while (int temp = sta.top()) {
17                 belong[temp] = cnt;
18                 exist[temp] = 0;
19                 sta.pop();
20                 if (temp == t) break;
21             }
22         }
23     };
24     for (int i = 1; i <= g1.n; i++)
25         if (!dfn[i]) dfs(i);
26     g2 = Graph(cnt);
27     for (int i = 1; i <= g1.n; i++) g2.w[belong[i]] += g1.w[i];
28     for (int i = 1; i <= g1.n; i++)
29         for (auto& [to] : g1.graph[i])
30             if (belong[i] != belong[to]) g2.add(belong[i], belong[to]);
31 }

```

## 2.4 拓扑排序

```

1 void toposort(Graph& g, vector<int>& dis) {
2     vector<int> in(g.n + 1, 0);
3     for (int i = 1; i <= g.n; i++)
4         for (auto& [to] : g.graph[i]) in[to]++;
5     queue<int> que;
6     for (int i = 1; i <= g.n; i++)
7         if (!in[i]) {
8             que.push(i);

```

```
9         dis[i] = g.w[i]; // dp
10     }
11     while (!que.empty()) {
12         int u = que.front();
13         que.pop();
14         for (auto& [to] : g.graph[u]) {
15             in[to]--;
16             dis[to] = max(dis[to], dis[u] + g.w[to]); // dp
17             if (!in[to]) que.push(to);
18         }
19     }
20 }
```

## 3 字符串

### 3.1 kmp

```

1 auto kmp(string& s) {
2     vector next(s.size(), -1);
3     for (int i = 1, j = -1; i < s.size(); i++) {
4         while (j >= 0 && s[i] != s[j + 1]) j = next[j];
5         if (s[i] == s[j + 1]) j++;
6         next[i] = j;
7     }
8     // next 意为长度
9     for (auto& i : next) i++;
10    return next;
11 }

```

### 3.2 哈希

```

1 constexpr int N = 1e6;
2 int pow_base[N + 1][2];
3 constexpr ll mod[2] = {(int)2e9 + 11, (int)2e9 + 33}, base[2] = {(int)2e5 + 11, (int)2e5 + 33};
4
5 struct Hash {
6     int size;
7     vector<array<int, 2>> a;
8     Hash() {}
9     Hash(const string& s) {
10         size = s.size();
11         a.resize(size);
12         a[0][0] = a[0][1] = s[0];
13         for (int i = 1; i < size; i++) {
14             a[i][0] = (a[i - 1][0] * base[0] + s[i]) % mod[0];
15             a[i][1] = (a[i - 1][1] * base[1] + s[i]) % mod[1];
16         }
17     }
18     array<int, 2> get(int l, int r) const {
19         if (l == 0) return a[r];
20         auto getone = [&](bool f) {
21             int x = (a[r][f] - 1ll * a[l - 1][f] * pow_base[r - l + 1][f]) % mod[f];
22             if (x < 0) x += mod[f];
23             return x;
24         };
25         return {getone(0), getone(1)};
26     }
27 };
28
29 auto _ = []() {
30     pow_base[0][0] = pow_base[0][1] = 1;
31     for (int i = 1; i <= N; i++) {
32         pow_base[i][0] = pow_base[i - 1][0] * base[0] % mod[0];
33         pow_base[i][1] = pow_base[i - 1][1] * base[1] % mod[1];
34     }
35     return true;
36 }();

```

### 3.3 manacher

```
1 auto manacher(const string& _s) {  
2     string s(_s.size() * 2 + 1, '$');  
3     for (int i = 0; i < _s.size(); i++) s[2 * i + 1] = _s[i];  
4     vector r(s.size(), 0);  
5     for (int i = 0, maxr = 0, mid = 0; i < s.size(); i++) {  
6         if (i < maxr) r[i] = min(r[mid * 2 - i], maxr - i);  
7         while (i - r[i] - 1 >= 0 && i + r[i] + 1 < s.size() && s[i - r[i] - 1] == s[i + r[i] + 1])  
8             ++r[i];  
9         if (i + r[i] > maxr) maxr = i + r[i], mid = i;  
10    }  
11    return r;  
12 }
```



## 4 数学

### 4.1 扩展欧几里得

需保证  $a, b \geq 0$

$$x = x + k * dx, y = y - k * dy$$

若要求  $x \geq p, k \geq \lceil \frac{p-x}{dx} \rceil$

若要求  $x \leq q, k \leq \lfloor \frac{q-x}{dx} \rfloor$

若要求  $y \geq p, k \leq \lfloor \frac{y-p}{dy} \rfloor$

若要求  $y \leq q, k \geq \lceil \frac{y-q}{dy} \rceil$

```

1 int __exgcd(int a, int b, int& x, int& y) {
2     if (!b) {
3         x = 1;
4         y = 0;
5         return a;
6     }
7     int g = __exgcd(b, a % b, y, x);
8     y -= a / b * x;
9     return g;
10 }
11
12 array<int, 2> exgcd(int a, int b, int c) {
13     int x, y;
14     int g = __exgcd(a, b, x, y);
15     if (c % g) return {INT_MAX, INT_MAX};
16     int dx = b / g;
17     int dy = a / g;
18     x = c / g % dx * x % dx;
19     if (x < 0) x += dx;
20     y = (c - a * x) / b;
21     return {x, y};
22 }

```

### 4.2 线性代数

#### 4.2.1 向量公约数

```

1 // 将这两个向量组转化为b.y=0的形式
2 array<vec, 2> gcd(vec a, vec b) {
3     while (b.y != 0) {
4         int t = a.y / b.y;
5         a = a - b * t;
6         swap(a, b);
7     }
8     return {a, b};
9 }
10
11 array<vec, 2> gcd(array<vec, 2> g, vec a) {
12     auto [b, c] = gcd(g[0], a);
13     g[0] = b;
14     g[1] = vec(gcd(g[1].x, c.x), 0);
15     if (g[1].x != 0) g[0].x %= g[1].x;
16     return g;
17 }

```

## 4.3 筛法

primes

```

1 constexpr int N = 1e7;
2 bitset<N + 1> ispr;
3 vector<int> primes;
4 bool _ = []() {
5     ispr.set();
6     ispr[0] = ispr[1] = 0;
7     for (int i = 2; i <= N; i++) {
8         if (!ispr[i]) continue;
9         primes.push_back(i);
10        for (int j = 2 * i; j <= N; j += i) ispr[j] = 0;
11    }
12    return 1;
13 }();

```

 $\varphi$ 

```

1 constexpr int N = 1e7;
2 array<int, N + 1> phi;
3 auto _ = []() {
4     iota(phi.begin() + 1, phi.end(), 1);
5     for (int i = 2; i <= N; i++) {
6         if (phi[i] == i)
7             for (int j = i; j <= N; j += i) phi[j] = phi[j] / i * (i - 1);
8     }
9     return true;
10 }();

```

 $\mu$ 

```

1 constexpr int N = 1e7;
2 bitset<N + 1> ispr;
3 array<int, N + 1> mu;
4 auto _ = []() {
5     mu.fill(1);
6     ispr.set();
7     mu[0] = ispr[0] = ispr[1] = 0;
8     for (int i = 2; i <= N; i++) {
9         if (!ispr[i]) continue;
10        mu[i] = -1;
11        for (int j = 2 * i; j <= N; j += i) {
12            ispr[j] = 0;
13            if (j / i % i == 0) mu[j] = 0;
14            else mu[j] *= -1;
15        }
16    }
17    return true;
18 }();

```

prime  $\varphi$ 

```

1 constexpr int N = 1e7;
2 bitset<N + 1> ispr;
3 array<int, N + 1> phi;
4 vector<int> primes;
5 bool _ = []() {
6     ispr.set();

```

```

7   ispr[0] = ispr[1] = 0;
8   iota(phi.begin() + 1, phi.end(), 1);
9   for (int i = 2; i <= N; i++) {
10      if (!ispr[i]) continue;
11      phi[i] = i - 1;
12      primes.push_back(i);
13      for (int j = 2 * i; j <= N; j += i) {
14         ispr[j] = 0;
15         phi[j] = phi[j] / i * (i - 1);
16      }
17   }
18   return 1;
19 }();

```

prime  $\mu$

```

1  constexpr int N = 1e7;
2  bitset<N + 1> ispr;
3  array<int, N + 1> mu;
4  vector<int> primes;
5  bool _ = []() {
6     mu.fill(1);
7     ispr.set();
8     mu[0] = ispr[0] = ispr[1] = 0;
9     for (int i = 2; i <= N; i++) {
10        if (!ispr[i]) continue;
11        mu[i] = -1;
12        primes.push_back(i);
13        for (int j = 2 * i; j <= N; j += i) {
14           ispr[j] = 0;
15           if (j / i % i == 0) mu[j] = 0;
16           else mu[j] *= -1;
17        }
18    }
19    return 1;
20 }();

```

prime  $\mu \varphi$

```

1  constexpr int N = 1e7;
2  bitset<N + 1> ispr;
3  array<int, N + 1> mu, phi;
4  vector<int> primes;
5  bool _ = []() {
6     mu.fill(1);
7     ispr.set();
8     mu[0] = ispr[0] = ispr[1] = 0;
9     iota(phi.begin() + 1, phi.end(), 1);
10    for (int i = 2; i <= N; i++) {
11       if (!ispr[i]) continue;
12       mu[i] = -1;
13       phi[i] = i - 1;
14       primes.push_back(i);
15       for (int j = 2 * i; j <= N; j += i) {
16          ispr[j] = 0;
17          if (j / i % i == 0) mu[j] = 0;
18          else mu[j] *= -1;
19          phi[j] = phi[j] / i * (i - 1);
20       }

```

```

21     }
22     return 1;
23 }();

1 constexpr int N = 1e7;
2 array<int, N + 1> minpr, mu, phi;
3 vector<int> primes;
4 bool _ = []() {
5     phi[1] = mu[1] = 1;
6     for (int i = 2; i <= N; i++) {
7         if (minpr[i] == 0) {
8             minpr[i] = i;
9             mu[i] = -1;
10            phi[i] = i - 1;
11            primes.push_back(i);
12        }
13        for (auto& j : primes) {
14            if (i * j > N) break;
15            minpr[i * j] = j;
16            if (j < minpr[i]) {
17                phi[i * j] = phi[i] * phi[j];
18                mu[i * j] = -mu[i];
19            } else {
20                mu[i * j] = 0;
21                phi[i * j] = phi[i] * j;
22                break;
23            }
24        }
25    }
26    return 1;
27 }();

```

#### 4.4 分解质因数

```

1 auto getprimes(int n) {
2     vector<array<int, 2>> res;
3     for (auto& i : primes) {
4         if (i > n / i) break;
5         if (n % i == 0) {
6             res.push_back({i, 0});
7             while (n % i == 0) {
8                 n /= i;
9                 res.back()[1]++;
10            }
11        }
12    }
13    if (n > 1) res.push_back({n, 1});
14    return res;
15 }

```

#### 4.5 pollard rho

```

1 using LL = __int128_t;
2
3 random_device rd;

```

```

4 mt19937 seed(rd());
5
6 ll power(ll a, ll b, ll mod) {
7     ll res = 1;
8     while (b) {
9         if (b & 1) res = (LL)res * a % mod;
10        a = (LL)a * a % mod;
11        b >>= 1;
12    }
13    return res;
14 }
15
16 bool isprime(ll n) {
17     static array primes{2, 3, 5, 7, 11, 13, 17, 19, 23};
18     static unordered_map<ll, bool> S;
19     if (n < 2) return 0;
20     if (S.count(n)) return S[n];
21     ll d = n - 1, r = 0;
22     while (!(d & 1)) {
23         r++;
24         d >>= 1;
25     }
26     for (auto& a : primes) {
27         if (a == n) return S[n] = 1;
28         ll x = power(a, d, n);
29         if (x == 1 || x == n - 1) continue;
30         for (int i = 0; i < r - 1; i++) {
31             x = (LL)x * x % n;
32             if (x == n - 1) break;
33         }
34         if (x != n - 1) return S[n] = 0;
35     }
36     return S[n] = 1;
37 }
38
39 ll pollard_rho(ll n) {
40     ll s = 0, t = 0;
41     ll c = seed() % (n - 1) + 1;
42     ll val = 1;
43     for (int goal = 1;; goal *= 2, s = t, val = 1) {
44         for (int step = 1; step <= goal; step++) {
45             t = ((LL)t * t + c) % n;
46             val = (LL)val * abs(t - s) % n;
47             if (step % 127 == 0) {
48                 ll g = gcd(val, n);
49                 if (g > 1) return g;
50             }
51         }
52         ll g = gcd(val, n);
53         if (g > 1) return g;
54     }
55 }
56 auto getprimes(ll n) {
57     unordered_set<ll> S;
58     auto get = [&](auto self, ll n) {
59         if (n < 2) return;
60         if (isprime(n)) {
61             S.insert(n);

```

```

62         return;
63     }
64     ll mx = pollard_rho(n);
65     self(self, n / mx);
66     self(self, mx);
67 };
68 get(get, n);
69 return S;
70 }

```

## 4.6 组合数

```

1 constexpr int N = 1e6;
2 array<modint, N + 1> fac, ifac;
3
4 modint C(int n, int m) {
5     if (m < 0 || m > n) return 0;
6     if (n <= mod) return fac[n] * ifac[m] * ifac[n - m];
7     // n >= mod 时需要这个
8     return C(n % mod, m % mod) * C(n / mod, m / mod);
9 }
10
11 auto _ = []() {
12     fac[0] = 1;
13     for (int i = 1; i <= N; i++) fac[i] = fac[i - 1] * i;
14     ifac[N] = fac[N].inv();
15     for (int i = N - 1; i >= 0; i--) ifac[i] = ifac[i + 1] * (i + 1);
16     return true;
17 }();

```

## 4.7 数论分块

求解形如  $\sum_{i=1}^n f(i)g(\lfloor \frac{n}{i} \rfloor)$  的合式

$$s(n) = \sum_{i=1}^n f(i)$$

```

1 modint sqrt_decomposition(int n) {
2     auto s = [&](int x) { return x; };
3     auto g = [&](int x) { return x; };
4     modint res = 0;
5     while (l <= R) {
6         int r = n / (n / l);
7         res = res + (s(r) - s(l - 1)) * g(n / l);
8         l = r + 1;
9     }
10    return res;
11 }

```

## 4.8 积性函数

### 4.8.1 定义

函数  $f(n)$  满足  $f(1) = 1$  且  $\forall x, y \in \mathbf{N}^*, \gcd(x, y) = 1$  都有  $f(xy) = f(x)f(y)$ , 则  $f(n)$  为积性函数。

函数  $f(n)$  满足  $f(1) = 1$  且  $\forall x, y \in \mathbf{N}^*$  都有  $f(xy) = f(x)f(y)$ , 则  $f(n)$  为完全积性函数。

### 4.8.2 例子

- 单位函数:  $\varepsilon(n) = [n = 1]$ 。(完全积性)
- 恒等函数:  $\text{id}_k(n) = n^k$ 。(完全积性)
- 常数函数:  $1(n) = 1$ 。(完全积性)
- 除数函数:  $\sigma_k(n) = \sum_{d|n} d^k$ 。  $\sigma_0(n)$  通常简记作  $d(n)$  或  $\tau(n)$ ,  $\sigma_1(n)$  通常简记作  $\sigma(n)$ 。
- 欧拉函数:  $\varphi(n) = \sum_{i=1}^n [\gcd(i, n) = 1]$ 。
- 莫比乌斯函数:  $\mu(n) = \begin{cases} 1 & n = 1 \\ 0 & \exists d > 1, d^2 | n, \text{ 其中 } \omega(n) \text{ 表示 } n \text{ 的本质不同质因子个数, 它是} \\ (-1)^{\omega(n)} & \text{otherwise} \end{cases}$   
一个加性函数。

## 4.9 狄利克雷卷积

对于两个数论函数  $f(x)$  和  $g(x)$ , 则它们的狄利克雷卷积得到的结果  $h(x)$  定义为:

$$h(x) = \sum_{d|x} f(d)g\left(\frac{x}{d}\right) = \sum_{ab=x} f(a)g(b)$$

可以简记为:  $h = f * g$ 。

### 4.9.1 性质

**交换律:**  $f * g = g * f$ 。

**结合律:**  $(f * g) * h = f * (g * h)$ 。

**分配律:**  $(f + g) * h = f * h + g * h$ 。

**等式的性质:**  $f = g$  的充要条件是  $f * h = g * h$ , 其中数论函数  $h(x)$  要满足  $h(1) \neq 0$ 。

### 4.9.2 例子

- $\varepsilon = \mu * 1 \iff \varepsilon(n) = \sum_{d|n} \mu(d)$
- $\text{id} = \varphi * 1 \iff \text{id}(n) = \sum_{d|n} \varphi(d)$
- $d = 1 * 1 \iff d(n) = \sum_{d|n} 1$
- $\sigma = \text{id} * 1 \iff \sigma(n) = \sum_{d|n} d$
- $\varphi = \mu * \text{id} \iff \varphi(n) = \sum_{d|n} d \cdot \mu\left(\frac{n}{d}\right)$

## 4.10 欧拉函数

```

1 constexpr int N = 1e6;
2 array<int, N + 1> phi;
3 auto _ = []() {
4     iota(phi.begin() + 1, phi.end(), 1);
5     for (int i = 2; i <= N; i++) {
6         if (phi[i] == i)
7             for (int j = i; j <= N; j += i) phi[j] = phi[j] / i * (i - 1);
8     }
9     return true;
10 }();

```

## 4.11 莫比乌斯反演

### 4.11.1 莫比乌斯函数性质

- $\sum_{d|n} \mu(d) = \begin{cases} 1 & n = 1 \\ 0 & n \neq 1 \end{cases}$ , 即  $\sum_{d|n} \mu(d) = \varepsilon(n)$ ,  $\mu * 1 = \varepsilon$
- $[\gcd(i, j) = 1] = \sum_{d|\gcd(i, j)} \mu(d)$

```

1 constexpr int N = 1e6;
2 array<int, N + 1> miu;
3 array<bool, N + 1> ispr;
4
5 auto _ = []() {
6     miu.fill(1);
7     ispr.fill(1);
8     for (int i = 2; i <= N; i++) {
9         if (!ispr[i]) continue;
10        miu[i] = -1;
11        for (int j = 2 * i; j <= N; j += i) {
12            ispr[j] = 0;
13            if ((j / i) % i == 0) miu[j] = 0;
14            else miu[j] *= -1;
15        }
16    }
17    return true;
18 }();

```

### 4.11.2 莫比乌斯变换/反演

$f(n) = \sum_{d|n} g(d)$ , 那么有  $g(n) = \sum_{d|n} \mu(d) f(\frac{n}{d}) = \sum_{n|d} \mu(\frac{d}{n}) f(d)$ 。

用狄利克雷卷积表示则为  $f = g * 1$ , 有  $g = f * \mu$ 。

$f \rightarrow g$  称为莫比乌斯反演,  $g \rightarrow f$  称为莫比乌斯反演。

## 4.12 杜教筛

杜教筛被用于处理一类数论函数的前缀和问题。对于数论函数  $f$ , 杜教筛可以在低于线性时间的复杂度内计算  $S(n) = \sum_{i=1}^n f(i)$ 。

$$S(n) = \frac{\sum_{i=1}^n (f * g)(i) - \sum_{i=2}^n g(i) S(\lfloor \frac{n}{i} \rfloor)}{g(1)}$$

可以构造恰当的数论函数  $g$  使得:

- 可以快速计算  $\sum_{i=1}^n (f * g)(i)$ 。
- 可以快速计算  $g$  的单点值, 用数论分块求解  $\sum_{i=2}^n g(i) S(\lfloor \frac{n}{i} \rfloor)$ 。

### 4.12.1 示例

```

1 ll sum_phi(ll n) {
2     if (n <= N) return sp[n];
3     if (sp2.count(n)) return sp2[n];
4     ll res = 0, l = 2;

```



```

5   while (1 <= n) {
6       ll r = n / (n / 1);
7       res = res + (r - 1 + 1) * sum_phi(n / 1);
8       l = r + 1;
9   }
10  return sp2[n] = (ll)n * (n + 1) / 2 - res;
11 }
12
13 ll sum_miu(ll n) {
14     if (n <= N) return sm[n];
15     if (sm2.count(n)) return sm2[n];
16     ll res = 0, l = 2;
17     while (1 <= n) {
18         ll r = n / (n / 1);
19         res = res + (r - 1 + 1) * sum_miu(n / 1);
20         l = r + 1;
21     }
22     return sm2[n] = 1 - res;
23 }

```

### 4.13 多项式

```

1  #define countr_zero(n) __builtin_ctz(n)
2  constexpr int N = 1e6;
3  array<int, N + 1> inv;
4
5  int power(int a, int b) {
6      int res = 1;
7      while (b) {
8          if (b & 1) res = 1ll * res * a % mod;
9          a = 1ll * a * a % mod;
10         b >>= 1;
11     }
12     return res;
13 }
14
15 namespace NFTS {
16     int g = 3;
17     vector<int> rev, roots{0, 1};
18     void dft(vector<int> &a) {
19         int n = a.size();
20         if (rev.size() != n) {
21             int k = countr_zero(n) - 1;
22             rev.resize(n);
23             for (int i = 0; i < n; ++i) rev[i] = rev[i >> 1] >> 1 | (i & 1) << k;
24         }
25         if (roots.size() < n) {
26             int k = countr_zero(roots.size());
27             roots.resize(n);
28             while ((1 << k) < n) {
29                 int e = power(g, (mod - 1) >> (k + 1));
30                 for (int i = 1 << (k - 1); i < (1 << k); ++i) {
31                     roots[2 * i] = roots[i];
32                     roots[2 * i + 1] = 1ll * roots[i] * e % mod;
33                 }
34                 ++k;
35             }

```

```

36     }
37     for (int i = 0; i < n; ++i)
38         if (rev[i] < i) swap(a[i], a[rev[i]]);
39     for (int k = 1; k < n; k *= 2) {
40         for (int i = 0; i < n; i += 2 * k) {
41             for (int j = 0; j < k; ++j) {
42                 int u = a[i + j];
43                 int v = 1ll * a[i + j + k] * roots[k + j] % mod;
44                 int x = u + v, y = u - v;
45                 if (x >= mod) x -= mod;
46                 if (y < 0) y += mod;
47                 a[i + j] = x;
48                 a[i + j + k] = y;
49             }
50         }
51     }
52 }
53 void idft(vector<int> &a) {
54     int n = a.size();
55     reverse(a.begin() + 1, a.end());
56     dft(a);
57     int inv_n = power(n, mod - 2);
58     for (int i = 0; i < n; ++i) a[i] = 1ll * a[i] * inv_n % mod;
59 }
60 } // namespace NFTS
61
62 struct poly {
63     poly &format() {
64         while (!a.empty() && a.back() == 0) a.pop_back();
65         return *this;
66     }
67     poly &reverse() {
68         ::reverse(a.begin(), a.end());
69         return *this;
70     }
71     vector<int> a;
72     poly() {}
73     poly(int x) {
74         if (x) a = {x};
75     }
76     poly(const vector<int> &a) : a(a) {}
77     int size() const { return a.size(); }
78     int &operator[](int id) { return a[id]; }
79     int at(int id) const {
80         if (id < 0 || id >= (int)a.size()) return 0;
81         return a[id];
82     }
83     poly operator-() const {
84         auto A = *this;
85         for (auto &x : A.a) x = (x == 0 ? 0 : mod - x);
86         return A;
87     }
88     poly mulXn(int n) const {
89         auto b = a;
90         b.insert(b.begin(), n, 0);
91         return poly(b);
92     }
93     poly modXn(int n) const {

```

```

94     if (n > size()) return *this;
95     return poly({a.begin(), a.begin() + n});
96 }
97 poly divXn(int n) const {
98     if (size() <= n) return poly();
99     return poly({a.begin() + n, a.end()});
100 }
101 poly &operator+=(const poly &rhs) {
102     if (size() < rhs.size()) a.resize(rhs.size());
103     for (int i = 0; i < rhs.size(); ++i)
104         if ((a[i] += rhs.a[i]) >= mod) a[i] -= mod;
105     return *this;
106 }
107 poly &operator-=(const poly &rhs) {
108     if (size() < rhs.size()) a.resize(rhs.size());
109     for (int i = 0; i < rhs.size(); ++i)
110         if ((a[i] -= rhs.a[i]) < 0) a[i] += mod;
111     return *this;
112 }
113 poly &operator*=(poly rhs) {
114     int n = size(), m = rhs.size(), tot = max(1, n + m - 1);
115     int sz = 1 << __lg(tot * 2 - 1);
116     a.resize(sz);
117     rhs.a.resize(sz);
118     NFTS::dft(a);
119     NFTS::dft(rhs.a);
120     for (int i = 0; i < sz; ++i) a[i] = 1ll * a[i] * rhs.a[i] % mod;
121     NFTS::idft(a);
122     return *this;
123 }
124 poly &operator/=(poly rhs) {
125     int n = size(), m = rhs.size();
126     if (n < m) return (*this) = poly();
127     reverse();
128     rhs.reverse();
129     (*this) *= rhs.inv(n - m + 1);
130     a.resize(n - m + 1);
131     reverse();
132     return *this;
133 }
134 poly &operator%=(poly rhs) { return (*this) -= (*this) / rhs * rhs; }
135 poly operator+(const poly &rhs) const { return poly(*this) += rhs; }
136 poly operator-(const poly &rhs) const { return poly(*this) -= rhs; }
137 poly operator*(poly rhs) const { return poly(*this) *= rhs; }
138 poly operator/(poly rhs) const { return poly(*this) /= rhs; }
139 poly operator%(poly rhs) const { return poly(*this) %= rhs; }
140 poly powModPoly(int n, poly p) {
141     poly r(1), x(*this);
142     while (n) {
143         if (n & 1) (r *= x) %= p;
144         (x *= x) %= p;
145         n >>= 1;
146     }
147     return r;
148 }
149 int inner(const poly &rhs) {
150     int r = 0, n = min(size(), rhs.size());
151     for (int i = 0; i < n; ++i) r = (r + 1ll * a[i] * rhs.a[i]) % mod;

```

```

152     return r;
153 }
154 poly derivation() const {
155     if (a.empty()) return poly();
156     int n = size();
157     vector<int> r(n - 1);
158     for (int i = 1; i < n; ++i) r[i - 1] = 11l * a[i] * i % mod;
159     return poly(r);
160 }
161 poly integral() const {
162     if (a.empty()) return poly();
163     int n = size();
164     vector<int> r(n + 1);
165     for (int i = 0; i < n; ++i) r[i + 1] = 11l * a[i] * ::inv[i + 1] % mod;
166     return poly(r);
167 }
168 poly inv(int n) const {
169     assert(a[0] != 0);
170     poly x(power(a[0], mod - 2));
171     int k = 1;
172     while (k < n) {
173         k *= 2;
174         x = (poly(2) - modXn(k) * x).modXn(k);
175     }
176     return x.modXn(n);
177 }
178 // 需要保证首项为 1
179 poly log(int n) const { return (derivation() * inv(n)).integral().modXn(n); }
180 // 需要保证首项为 0
181 poly exp(int n) const {
182     poly x(1);
183     int k = 1;
184     while (k < n) {
185         k *= 2;
186         x = (x * (poly(1) - x.log(k) + modXn(k))).modXn(k);
187     }
188     return x.modXn(n);
189 }
190 // 需要保证首项为 1, 开任意次方可以先 ln 再 exp 实现。
191 poly sqrt(int n) const {
192     poly x(1);
193     int k = 1;
194     while (k < n) {
195         k *= 2;
196         x += modXn(k) * x.inv(k);
197         x = x.modXn(k) * inv2;
198     }
199     return x.modXn(n);
200 }
201 // 减法卷积, 也称转置卷积  $\{\rm MULT\}(F(x), G(x)) = \sum_{i \geq 0} (\sum_{j \geq 0} f_{i+j} g_j) x^i$ 
202 //  $0 \leq i+j \leq n-1$ 
203 poly mulT(poly rhs) const {
204     if (rhs.size() == 0) return poly();
205     int n = rhs.size();
206     ::reverse(rhs.a.begin(), rhs.a.end());
207     return ((*this) * rhs).divXn(n - 1);
208 }
209 int eval(int x) {

```

```

210     int r = 0, t = 1;
211     for (int i = 0, n = size(); i < n; ++i) {
212         r = (r + 111 * a[i] * t) % mod;
213         t = 111 * t * x % mod;
214     }
215     return r;
216 }
217 // 多点求值新科技: https://jkloverdcoi.github.io/2020/08/04/转置原理及其应用/
218 // 模板例题: https://www.luogu.com.cn/problem/P5050
219 auto evals(vector<int> &x) const {
220     if (size() == 0) return vector(x.size(), 0);
221     int n = x.size();
222     vector ans(n, 0);
223     vector<poly> g(4 * n);
224     auto build = [&](auto self, int l, int r, int p) -> void {
225         if (r - l == 1) {
226             g[p] = poly({1, x[l] ? mod - x[l] : 0});
227         } else {
228             int m = (l + r) / 2;
229             self(self, l, m, 2 * p);
230             self(self, m, r, 2 * p + 1);
231             g[p] = g[2 * p] * g[2 * p + 1];
232         }
233     };
234     build(build, 0, n, 1);
235     auto solve = [&](auto self, int l, int r, int p, poly f) -> void {
236         if (r - l == 1) {
237             ans[l] = f[0];
238         } else {
239             int m = (l + r) / 2;
240             self(self, l, m, 2 * p, f.mulT(g[2 * p + 1]).modXn(m - 1));
241             self(self, m, r, 2 * p + 1, f.mulT(g[2 * p]).modXn(r - m));
242         }
243     };
244     solve(solve, 0, n, 1, mulT(g[1].inv(size())).modXn(n));
245     return ans;
246 }
247 }; // 全家桶测试: https://www.luogu.com.cn/training/3015#information
248
249 auto _ = []() {
250     inv[0] = inv[1] = 1;
251     for (int i = 2; i < inv.size(); i++) inv[i] = 111 * (mod - mod / i) * inv[mod % i] % mod;
252     return true;
253 }();

```

#### 4.14 盒子与球

$n$  个球,  $m$  个盒

球同	盒同	可空	公式
✓	✓	✓	$f_{n,m} = f_{n,m-1} + f_{n-m,m}$ 或 $[x^n]e^{\sum_{i=1}^m \sum_{j=1}^{\infty} \frac{x^{ij}}{j}}$
✓	✓	✗	$f_{n-m,m}$
✗	✓	✓	$\sum_{i=1}^m g_{n,i}$ 或 $\sum_{i=1}^m \sum_{j=0}^i \frac{j^n}{j!} \frac{(-1)^{i-j}}{(i-j)!}$
✗	✓	✗	$g_{n,m} = g_{n-1,m-1} + m * g_{n-1,m}$ 或 $\frac{1}{m!} \sum_{i=0}^m (-1)^i \binom{m}{i} (m-i)^n$
✓	✗	✓	$C_{n+m-1}^{m-1}$
✓	✗	✗	$C_{n-1}^{m-1}$
✗	✗	✓	$m^n$
✗	✗	✗	$m! * g_{n,m}$ 或 $\sum_{i=0}^m (-1)^i \binom{m}{i} (m-i)^n$

#### 4.14.1 球同，盒同，可空

```

1 int solve(int n, int m) {
2     vector a(n + 1, 0);
3     for (int i = 1; i <= m; i++)
4         for (int j = i, k = 1; j <= n; j += i, k++) a[j] = (a[j] + inv[k]) % mod;
5     auto p = poly(a).exp(n + 1);
6     return (p.a[n] + mod) % mod;
7 }

```

若要求不超过  $k$  个，答案为  $[x^n y^m] \prod_{i=0}^k \left( \sum_{j=0}^m x^{ij} y^j \right)$ 。

#### 4.14.2 球不同，盒同，可空

```

1 int solve(int n, int m) {
2     vector a(n + 1, 0);
3     vector b(n + 1, 0);
4     for (int i = 0; i <= n; i++) {
5         a[i] = ifac[i];
6         if (i & 1) a[i] = -a[i];
7         b[i] = 1ll * power(i, n) * ifac[i] % mod;
8     }
9     auto p = poly(a) * poly(b);
10    int ans = 0;
11    for (int i = 1; i <= min(n, m); i++) ans = (ans + p.a[i]) % mod;
12    return (ans + mod) % mod;
13 }

```

若要求不超过  $k$  个, 答案为  $m! \cdot [x^n y^m] \prod_{i=0}^k \left( \sum_{j=0}^n \frac{1}{i!^j} x^{ij} y^j \right)$ 。

#### 4.14.3 球同, 盒不同, 可空

若要求不超过  $k$  个, 答案为  $[x^n] \left( \sum_{i=0}^k x^i \right)^m = [x^n] \frac{(x^{k+1}-1)^m}{(x-1)^m}$ 。

也可以考虑容斥, 令  $f(i)$  表示至少有  $i$  个盒子装了  $> k$  个球方案数,  $f(i) = \binom{m}{i} \binom{n-(k+1)i+m-1}{m-1}$ 。  
总方案数则为  $\sum_{i=0}^m (-1)^i f(i)$ 。

#### 4.14.4 球同, 盒不同, 不可空

若要求不超过  $k$  个, 答案为  $[x^n] \left( \sum_{i=1}^k x^i \right)^m = [x^n] \frac{(x^{k+1}-x)^m}{(x-1)^m}$ 。

也可以考虑容斥, 令  $f(i)$  表示至少有  $i$  个盒子装了  $> k$  个球方案数,  $f(i) = \binom{m}{i} \binom{n-ki-1}{m-1}$ 。  
总方案数则为  $\sum_{i=0}^m (-1)^i f(i)$ 。

#### 4.14.5 球不同, 盒不同, 可空

若要求不超过  $k$  个, 答案为  $m! \cdot [x^n] \left( \sum_{i=0}^k \frac{1}{i!} x^i \right)^m$ 。

#### 4.14.6 球不同, 盒不同, 不可空

若要求不超过  $k$  个, 答案为  $m! \cdot [x^n] \left( \sum_{i=1}^k \frac{1}{i!} x^i \right)^m$ 。

### 4.15 线性基

```

1 // 线性基
2 struct basis {
3     int rnk = 0;
4     array<ull, 64> p{};
5
6     // 将x插入此线性基中
7     void insert(ull x) {
8         for (int i = 63; i >= 0; i--) {
9             if (!(x >> i & 1)) continue;
10            if (p[i] x ^= p[i];
11            else {
12                p[i] = x;
13                rnk++;
14                break;
15            }
16        }
17    }
18
19    // 将另一个线性基插入此线性基中
20    void insert(basis other) {
21        for (int i = 0; i <= 63; i++) {
22            if (!other.p[i]) continue;
23            insert(other.p[i]);
24        }
25    }
26
27    // 最大异或值

```

```

28     ull max_basis() {
29         ull res = 0;
30         for (int i = 63; i >= 0; i--)
31             if ((res ^ p[i]) > res) res ^= p[i];
32         return res;
33     }
34 };

```

## 4.16 矩阵快速幂

```

1 constexpr ll mod = 2147493647;
2 struct Mat {
3     int n, m;
4     vector<vector<ll>> mat;
5     Mat(int n, int m) : n(n), m(m), mat(n, vector<ll>(m, 0)) {}
6     Mat(vector<vector<ll>> mat) : n(mat.size()), m(mat[0].size()), mat(mat) {}
7     Mat operator*(const Mat& other) {
8         assert(m == other.n);
9         Mat res(n, other.m);
10        for (int i = 0; i < res.n; i++)
11            for (int j = 0; j < res.m; j++)
12                for (int k = 0; k < m; k++)
13                    res.mat[i][j] = (res.mat[i][j] + mat[i][k] * other.mat[k][j] % mod) % mod;
14        return res;
15    }
16 };
17 Mat ksm(Mat a, ll b) {
18     assert(a.n == a.m);
19     Mat res(a.n, a.m);
20     for (int i = 0; i < res.n; i++) res.mat[i][i] = 1;
21     while (b) {
22         if (b & 1) res = res * a;
23         b >>= 1;
24         a = a * a;
25     }
26     return res;
27 }

```



## 5 计算几何

### 5.1 整数

```

1 constexpr double inf = 1e100;
2
3 // 向量
4 struct vec {
5     static bool cmp(const vec &a, const vec &b) { return tie(a.x, a.y) < tie(b.x, b.y); }
6
7     ll x, y;
8     vec() : x(0), y(0) {}
9     vec(ll _x, ll _y) : x(_x), y(_y) {}
10
11     // 模
12     ll len2() const { return x * x + y * y; }
13     double len() const { return sqrt(x * x + y * y); }
14
15     // 是否在上半轴
16     bool up() const { return y > 0 || y == 0 && x >= 0; }
17
18     bool operator==(const vec &b) const { return tie(x, y) == tie(b.x, b.y); }
19     // 极角排序
20     bool operator<(const vec &b) const {
21         if (up() != b.up()) return up() > b.up();
22         ll tmp = (*this) ^ b;
23         return tmp ? tmp > 0 : cmp(*this, b);
24     }
25
26     vec operator+(const vec &b) const { return {x + b.x, y + b.y}; }
27     vec operator-() const { return {-x, -y}; }
28     vec operator-(const vec &b) const { return -b + (*this); }
29     vec operator*(ll b) const { return {x * b, y * b}; }
30     ll operator*(const vec &b) const { return x * b.x + y * b.y; }
31
32     // 叉积 结果大于0, a到b为逆时针, 小于0, a到b顺时针,
33     // 等于0共线, 可能同向或反向, 结果绝对值表示 a b 形成的平行四边形的面积
34     ll operator^(const vec &b) const { return x * b.y - y * b.x; }
35
36     friend istream &operator>>(istream &in, vec &data) {
37         in >> data.x >> data.y;
38         return in;
39     }
40     friend ostream &operator<<(ostream &out, const vec &data) {
41         out << fixed << setprecision(6);
42         out << data.x << " " << data.y;
43         return out;
44     }
45 };
46
47 ll cross(const vec &a, const vec &b, const vec &c) { return (a - c) ^ (b - c); }
48
49 // 多边形的面积a
50 double polygon_area(vector<vec> &p) {
51     ll area = 0;
52     for (int i = 1; i < p.size(); i++) area += p[i - 1] ^ p[i];
53     area += p.back() ^ p[0];
54     return abs(area / 2.0);

```

```

55 }
56
57 // 多边形的周长
58 double polygon_len(vector<vec> &p) {
59     double len = 0;
60     for (int i = 1; i < p.size(); i++) len += (p[i] - p[i-1]).len();
61     len += (p.back() - p[0]).len();
62     return len;
63 }
64
65 // 以整点为顶点的线段上的整点个数
66 ll count(const vec &a, const vec &b) {
67     vec c = a - b;
68     return gcd(abs(c.x), abs(c.y)) + 1;
69 }
70
71 // 以整点为顶点的多边形边上整点个数
72 ll count(vector<vec> &p) {
73     ll cnt = 0;
74     for (int i = 1; i < p.size(); i++) cnt += count(p[i-1], p[i]);
75     cnt += count(p.back(), p[0]);
76     return cnt - p.size();
77 }
78
79 // 判断点是否在凸包内，凸包必须为逆时针顺序
80 bool in_polygon(const vec &a, vector<vec> &p) {
81     int n = p.size();
82     if (n == 0) return 0;
83     if (n == 1) return a == p[0];
84     if (n == 2) return cross(a, p[1], p[0]) == 0 && (p[0] - a) * (p[1] - a) <= 0;
85     if (cross(a, p[1], p[0]) > 0 || cross(p.back(), a, p[0]) > 0) return 0;
86     auto cmp = [&](vec &x, const vec &y) { return ((x - p[0]) ^ y) >= 0; };
87     int i = lower_bound(p.begin() + 2, p.end() - 1, a - p[0], cmp) - p.begin() - 1;
88     return cross(p[(i + 1) % n], a, p[i]) >= 0;
89 }
90
91 // 凸包直径的两个端点
92 auto polygon_dia(vector<vec> &p) {
93     int n = p.size();
94     array<vec, 2> res{};
95     if (n == 1) return res;
96     if (n == 2) return res = {p[0], p[1]};
97     ll mx = 0;
98     for (int i = 0, j = 2; i < n; i++) {
99         while (abs(cross(p[i], p[(i + 1) % n], p[j])) <=
100             abs(cross(p[i], p[(i + 1) % n], p[(j + 1) % n])))
101             j = (j + 1) % n;
102         ll tmp = (p[i] - p[j]).len2();
103         if (tmp > mx) {
104             mx = tmp;
105             res = {p[i], p[j]};
106         }
107         tmp = (p[(i + 1) % n] - p[j]).len2();
108         if (tmp > mx) {
109             mx = tmp;
110             res = {p[(i + 1) % n], p[j]};
111         }
112     }

```

```

113     return res;
114 }
115
116 // 凸包
117 auto convex_hull(vector<vec> &p) {
118     sort(p.begin(), p.end(), vec::cmp);
119     int n = p.size();
120     vector sta(n + 1, 0);
121     vector v(n, false);
122     int tp = -1;
123     sta[++tp] = 0;
124     auto update = [&](int lim, int i) {
125         while (tp > lim && cross(p[i], p[sta[tp]], p[sta[tp] - 1]) >= 0) v[sta[tp--]] = 0;
126         sta[++tp] = i;
127         v[i] = 1;
128     };
129     for (int i = 1; i < n; i++) update(0, i);
130     int cnt = tp;
131     for (int i = n - 1; i >= 0; i--) {
132         if (v[i]) continue;
133         update(cnt, i);
134     }
135     vector<vec> res(tp);
136     for (int i = 0; i < tp; i++) res[i] = p[sta[i]];
137     return res;
138 }
139
140 // 闵可夫斯基和，两个点集的和构成一个凸包
141 auto minkowski(vector<vec> &a, vector<vec> &b) {
142     rotate(a.begin(), min_element(a.begin(), a.end(), vec::cmp), a.end());
143     rotate(b.begin(), min_element(b.begin(), b.end(), vec::cmp), b.end());
144     int n = a.size(), m = b.size();
145     vector<vec> c{a[0] + b[0]};
146     c.reserve(n + m);
147     int i = 0, j = 0;
148     while (i < n && j < m) {
149         vec x = a[(i + 1) % n] - a[i];
150         vec y = b[(j + 1) % m] - b[j];
151         c.push_back(c.back() + ((x ^ y) >= 0 ? (i++, x) : (j++, y)));
152     }
153     while (i + 1 < n) {
154         c.push_back(c.back() + a[(i + 1) % n] - a[i]);
155         i++;
156     }
157     while (j + 1 < m) {
158         c.push_back(c.back() + b[(j + 1) % m] - b[j]);
159         j++;
160     }
161     return c;
162 }
163
164 // 过凸多边形外一点求凸多边形的切线，返回切点下标
165 auto tangent(const vec &a, vector<vec> &p) {
166     int n = p.size();
167     int l = -1, r = -1;
168     for (int i = 0; i < n; i++) {
169         ll tmp1 = cross(p[i], p[(i - 1 + n) % n], a);
170         ll tmp2 = cross(p[i], p[(i + 1) % n], a);

```

```

171     if (l == -1 && tmp1 <= 0 && tmp2 <= 0) l = i;
172     else if (r == -1 && tmp1 >= 0 && tmp2 >= 0) r = i;
173 }
174 return array{1, r};
175 }
176
177 // 直线
178 struct line {
179     vec p, d;
180     line() : p(vec()), d(vec()) {}
181     line(const vec &p, const vec &d) : p(_p), d(_d) {}
182 };
183
184 // 点到直线距离
185 double dis(const vec &a, const line &b) { return abs((b.p - a) ^ (b.p + b.d - a)) / b.d.len(); }
186
187 // 点在直线哪边, 大于0在左边, 等于0在线上, 小于0在右边
188 ll side_line(const vec &a, const line &b) { return b.d ^ (a - b.p); }
189
190 // 两直线是否垂直
191 bool perpen(const line &a, const line &b) { return a.d * b.d == 0; }
192
193 // 两直线是否平行
194 bool parallel(const line &a, const line &b) { return (a.d ^ b.d) == 0; }
195
196 // 点的垂线是否与线段有交点
197 bool perpen(const vec &a, const line &b) {
198     vec p(-b.d.y, b.d.x);
199     bool cross1 = (p ^ (b.p - a)) > 0;
200     bool cross2 = (p ^ (b.p + b.d - a)) > 0;
201     return cross1 != cross2;
202 }
203
204 // 点到线段距离
205 double dis_seg(const vec &a, const line &b) {
206     if (perpen(a, b)) return dis(a, b);
207     return min((b.p - a).len(), (b.p + b.d - a).len());
208 }
209
210 // 点到凸包距离
211 double dis(const vec &a, vector<vec> &p) {
212     double res = inf;
213     for (int i = 1; i < p.size(); i++) res = min(dis_seg(a, line(p[i - 1], p[i] - p[i - 1])), res);
214     res = min(dis_seg(a, line(p.back(), p[0] - p.back())), res);
215     return res;
216 }
217
218 // 两直线交点
219 vec intersection(ll A, ll B, ll C, ll D, ll E, ll F) {
220     return {(B * F - C * E) / (A * E - B * D), (C * D - A * F) / (A * E - B * D)};
221 }
222
223 // 两直线交点
224 vec intersection(const line &a, const line &b) {
225     return intersection(a.d.y, -a.d.x, a.d.x * a.p.y - a.d.y * a.p.x, b.d.y, -b.d.x,
226         b.d.x * b.p.y - b.d.y * b.p.x);
227 }

```

## 三维

```

1 // 向量
2 struct vec3 {
3     static bool cmp(const vec3 &a, const vec3 &b) {
4         return tie(a.x, a.y, a.z) < tie(b.x, b.y, b.z);
5     }
6
7     ll x, y, z;
8     vec3() : x(0), y(0), z(0) {}
9     vec3(ll _x, ll _y, ll _z) : x(_x), y(_y), z(_z) {}
10
11 // 模
12 ll len2() const { return x * x + y * y + z * z; }
13 double len() const { return hypot(x, y, z); }
14
15 bool operator==(const vec3 &b) const { return tie(x, y, z) == tie(b.x, b.y, b.z); }
16 bool operator!=(const vec3 &b) const { return !(*this == b); }
17
18 vec3 operator+(const vec3 &b) const { return {x + b.x, y + b.y, z + b.z}; }
19 vec3 operator-(const vec3 &b) const { return {-x, -y, -z}; }
20 vec3 operator-(const vec3 &b) const { return -b + (*this); }
21 vec3 operator*(ll b) const { return {b * x, b * y, b * z}; }
22 ll operator*(const vec3 &b) const { return x * b.x + y * b.y + z * b.z; }
23
24 // 叉积 结果大于0, a到b为逆时针, 小于0, a到b顺时针,
25 // 等于0共线, 可能同向或反向, 结果绝对值表示 a b 形成的平行四边形的面积
26 vec3 operator^(const vec3 &b) const {
27     return {y * b.z - z * b.y, z * b.x - x * b.z, x * b.y - y * b.x};
28 }
29
30 friend istream &operator>>(istream &in, vec3 &data) {
31     in >> data.x >> data.y >> data.z;
32     return in;
33 }
34 friend ostream &operator<<(ostream &out, const vec3 &data) {
35     out << fixed << setprecision(6);
36     out << data.x << " " << data.y << " " << data.z;
37     return out;
38 }
39 };
40
41 struct line3 {
42     vec3 p, d;
43     line3() {}
44     line3(const vec3 &a, const vec3 &b) : p(a), d(b - a) {}
45 };
46
47 struct plane {
48     vec3 p, d;
49     plane() {}
50     plane(const vec3 &a, const vec3 &b, const vec3 &c) : p(a) {
51         d = (b - a) ^ (c - a);
52         assert(d != vec3());
53     }
54 };
55
56 // 线面是否垂直
57 bool perpen(const line3 &a, const plane &b) { return (a.d ^ b.d) == vec3(); }

```

```

58
59 // 线面是否平行
60 bool parallel(const line3 &a, const plane &b) { return a.d * b.d == 0; }

```

## 5.2 浮点数

```

1  using lf = double;
2
3  constexpr lf eps = 1e-8;
4  constexpr lf inf = 1e100;
5  const lf PI = acos(-1);
6
7  int sgn(lf a, lf b) {
8      lf c = a - b;
9      return c < -eps ? -1 : c < eps ? 0 : 1;
10 }
11
12 // 向量
13 struct vec {
14     static bool cmp(const vec &a, const vec &b) {
15         return sgn(a.x, b.x) ? a.x < b.x : sgn(a.y, b.y) < 0;
16     }
17
18     lf x, y;
19     vec() : x(0), y(0) {}
20     vec(lf _x, lf _y) : x(_x), y(_y) {}
21
22     // 模
23     lf len2() const { return x * x + y * y; }
24     lf len() const { return sqrt(x * x + y * y); }
25
26     // 与x轴正方向的夹角
27     lf angle() const {
28         lf angle = atan2(y, x);
29         if (angle < 0) angle += 2 * PI;
30         return angle;
31     }
32
33     // 逆时针旋转
34     vec rotate(const lf &theta) const {
35         return {x * cos(theta) - y * sin(theta), y * cos(theta) + x * sin(theta)};
36     }
37
38     vec e() const {
39         lf tmp = len();
40         return {x / tmp, y / tmp};
41     }
42
43     // 是否在上半轴
44     bool up() const { return sgn(y, 0) > 0 || sgn(y, 0) == 0 && sgn(x, 0) >= 0; }
45
46     bool operator==(const vec &other) const { return sgn(x, other.x) == 0 && sgn(y, other.y) == 0; }
47     // 极角排序
48     bool operator<(const vec &b) const {
49         if (up() != b.up()) return up() > b.up();
50         lf tmp = (*this) ^ b;
51         return sgn(tmp, 0) ? tmp > 0 : cmp(*this, b);

```

```

52     }
53
54     vec operator+(const vec &b) const { return {x + b.x, y + b.y}; }
55     vec operator-() const { return {-x, -y}; }
56     vec operator-(const vec &b) const { return -b + (*this); }
57     vec operator*(lf b) const { return {x * b, y * b}; }
58     vec operator/(lf b) const { return {x / b, y / b}; }
59     lf operator*(const vec &b) const { return x * b.x + y * b.y; }
60
61     // 叉积 结果大于0, a到b为逆时针, 小于0, a到b顺时针,
62     // 等于0共线, 可能同向或反向, 结果绝对值表示 a b 形成的平行四边形的面积
63     lf operator^(const vec &b) const { return x * b.y - y * b.x; }
64
65     friend istream &operator>>(istream &in, vec &data) {
66         in >> data.x >> data.y;
67         return in;
68     }
69     friend ostream &operator<<(ostream &out, const vec &data) {
70         out << fixed << setprecision(6);
71         out << data.x << " " << data.y;
72         return out;
73     }
74 };
75
76 lf cross(const vec &a, const vec &b, const vec &c) { return (a - c) ^ (b - c); }
77
78 lf angle(const vec &a, const vec &b) { return atan2(abs(a ^ b), a * b); }
79
80 // 多边形的面积
81 lf polygon_area(vector<vec> &p) {
82     lf area = 0;
83     for (int i = 1; i < p.size(); i++) area += p[i - 1] ^ p[i];
84     area += p.back() ^ p[0];
85     return abs(area / 2.0);
86 }
87
88 // 多边形的周长
89 lf polygon_len(vector<vec> &p) {
90     lf len = 0;
91     for (int i = 1; i < p.size(); i++) len += (p[i - 1] - p[i]).len();
92     len += (p.back() - p[0]).len();
93     return len;
94 }
95
96 // 判断点是否在凸包内, 凸包必须为逆时针顺序
97 bool in_polygon(const vec &a, vector<vec> &p) {
98     int n = p.size();
99     if (n == 0) return 0;
100    if (n == 1) return a == p[0];
101    if (n == 2) return sgn(cross(a, p[1], p[0]), 0) == 0 && sgn((p[0] - a) * (p[1] - a), 0) <= 0;
102    if (sgn(cross(a, p[1], p[0]), 0) > 0 || sgn(cross(p.back(), a, p[0]), 0) > 0) return 0;
103    auto cmp = [&](vec &x, const vec &y) { return sgn((x - p[0]) ^ y, 0) >= 0; };
104    int i = lower_bound(p.begin() + 2, p.end() - 1, a - p[0], cmp) - p.begin() - 1;
105    return sgn(cross(p[(i + 1) % n], a, p[i]), 0) >= 0;
106 }
107
108 // 凸包直径的两个端点
109 auto polygon_dia(vector<vec> &p) {

```

```

110     int n = p.size();
111     array<vec, 2> res{};
112     if (n == 1) return res;
113     if (n == 2) return res = {p[0], p[1]};
114     lf mx = 0;
115     for (int i = 0, j = 2; i < n; i++) {
116         while (sgn(abs(cross(p[i], p[(i + 1) % n], p[j])),
117                     abs(cross(p[i], p[(i + 1) % n], p[(j + 1) % n]))) <= 0)
118             j = (j + 1) % n;
119         lf tmp = (p[i] - p[j]).len();
120         if (tmp > mx) {
121             mx = tmp;
122             res = {p[i], p[j]};
123         }
124         tmp = (p[(i + 1) % n] - p[j]).len();
125         if (tmp > mx) {
126             mx = tmp;
127             res = {p[(i + 1) % n], p[j]};
128         }
129     }
130     return res;
131 }
132
133 // 凸包
134 auto convex_hull(vector<vec> &p) {
135     sort(p.begin(), p.end(), vec::cmp);
136     int n = p.size();
137     vector sta(n + 1, 0);
138     vector v(n, false);
139     int tp = -1;
140     sta[++tp] = 0;
141     auto update = [&](int lim, int i) {
142         while (tp > lim && sgn(cross(p[i], p[sta[tp]], p[sta[tp - 1]]), 0) >= 0) v[sta[tp--]] = 0;
143         sta[++tp] = i;
144         v[i] = 1;
145     };
146     for (int i = 1; i < n; i++) update(0, i);
147     int cnt = tp;
148     for (int i = n - 1; i >= 0; i--) {
149         if (v[i]) continue;
150         update(cnt, i);
151     }
152     vector<vec> res(tp);
153     for (int i = 0; i < tp; i++) res[i] = p[sta[i]];
154     return res;
155 }
156
157 // 闵可夫斯基和，两个点集的和构成一个凸包
158 auto minkowski(vector<vec> &a, vector<vec> &b) {
159     rotate(a.begin(), min_element(a.begin(), a.end(), vec::cmp), a.end());
160     rotate(b.begin(), min_element(b.begin(), b.end(), vec::cmp), b.end());
161     int n = a.size(), m = b.size();
162     vector<vec> c{a[0] + b[0]};
163     c.reserve(n + m);
164     int i = 0, j = 0;
165     while (i < n && j < m) {
166         vec x = a[(i + 1) % n] - a[i];
167         vec y = b[(j + 1) % m] - b[j];

```



```

168     c.push_back(c.back() + (sgn(x ^ y, 0) >= 0 ? (i++, x) : (j++, y)));
169 }
170 while (i + 1 < n) {
171     c.push_back(c.back() + a[(i + 1) % n] - a[i]);
172     i++;
173 }
174 while (j + 1 < m) {
175     c.push_back(c.back() + b[(j + 1) % m] - b[j]);
176     j++;
177 }
178 return c;
179 }
180
181 // 过凸多边形外一点求凸多边形的切线, 返回切点下标
182 auto tangent(const vec &a, vector<vec> &p) {
183     int n = p.size();
184     int l = -1, r = -1;
185     for (int i = 0; i < n; i++) {
186         if tmp1 = cross(p[i], p[(i - 1 + n) % n], a);
187         if tmp2 = cross(p[i], p[(i + 1) % n], a);
188         if (l == -1 && sgn(tmp1, 0) <= 0 && sgn(tmp2, 0) <= 0) l = i;
189         else if (r == -1 && sgn(tmp1, 0) >= 0 && sgn(tmp2, 0) >= 0) r = i;
190     }
191     return array{l, r};
192 }
193
194 // 直线
195 struct line {
196     vec p, d;
197     line() : p(vec()), d(vec()) {}
198     line(const vec &p, const vec &d) : p(_p), d(_d) {}
199 };
200
201 // 点到直线距离
202 if dis(const vec &a, const line &b) { return abs((b.p - a) ^ (b.p + b.d - a)) / b.d.len(); }
203
204 // 点在直线哪边, 大于0在左边, 等于0在线上, 小于0在右边
205 int side_line(const vec &a, const line &b) { return sgn(b.d ^ (a - b.p), 0); }
206
207 // 两直线是否垂直
208 bool perpen(const line &a, const line &b) { return sgn(a.d * b.d, 0) == 0; }
209
210 // 两直线是否平行
211 bool parallel(const line &a, const line &b) { return sgn(a.d ^ b.d, 0) == 0; }
212
213 // 点的垂线是否与线段有交点
214 bool perpen(const vec &a, const line &b) {
215     vec p(-b.d.y, b.d.x);
216     bool cross1 = sgn(p ^ (b.p - a), 0) > 0;
217     bool cross2 = sgn(p ^ (b.p + b.d - a), 0) > 0;
218     return cross1 != cross2;
219 }
220
221 // 点到线段距离
222 if dis_seg(const vec &a, const line &b) {
223     if (perpen(a, b)) return dis(a, b);
224     return min((b.p - a).len(), (b.p + b.d - a).len());
225 }

```

```

226
227 // 点到凸包距离
228 lf dis(const vec &a, vector<vec> &p) {
229     lf res = inf;
230     for (int i = 1; i < p.size(); i++) res = min(dis_seg(a, line(p[i - 1], p[i] - p[i - 1])), res);
231     res = min(dis_seg(a, line(p.back(), p[0] - p.back())), res);
232     return res;
233 }
234
235 // 两直线交点
236 vec intersection(lf A, lf B, lf C, lf D, lf E, lf F) {
237     return {(B * F - C * E) / (A * E - B * D), (C * D - A * F) / (A * E - B * D)};
238 }
239
240 // 两直线交点
241 vec intersection(const line &a, const line &b) {
242     return intersection(a.d.y, -a.d.x, a.d.x * a.p.y - a.d.y * a.p.x, b.d.y, -b.d.x,
243         b.d.x * b.p.y - b.d.y * b.p.x);
244 }
245
246 struct circle {
247     vec o;
248     lf r;
249     circle(const vec &o, lf _r) : o(o), r(_r){};
250
251     // 点与圆的关系 -1在圆内, 0在圆上, 1在圆外
252     int relation(const vec &a) const { return sgn((a - o).len(), r); }
253
254     // 圆与圆的关系 -3包含, -2内切, -1相交, 0外切, 1相离
255     int relation(const circle &a) const {
256         lf l = (a.o - o).len();
257         if (sgn(l, abs(r - a.r)) < 0) return -3;
258         if (sgn(l, abs(r - a.r)) == 0) return -2;
259         if (sgn(l, abs(r + a.r)) < 0) return -1;
260         if (sgn(l, abs(r + a.r)) == 0) return 0;
261         return 1;
262     }
263
264     lf area() { return PI * r * r; }
265 };
266
267 // 圆与直线交点
268 auto intersection(const circle &c, const line &l) {
269     lf d = dis(c.o, l);
270     vector<vec> res;
271     vec mid = l.p + l.d.e() * ((c.o - l.p) * l.d / l.d.len());
272     if (sgn(d, c.r) == 0) res.push_back(mid);
273     else if (sgn(d, c.r) < 0) {
274         d = sqrt(c.r * c.r - d * d);
275         res.push_back(mid + l.d.e() * d);
276         res.push_back(mid - l.d.e() * d);
277     }
278     return res;
279 }
280
281 // oab三角形与圆相交的面积
282 lf area(const circle &c, const vec &a, const vec &b) {
283     if (sgn(cross(a, b, c.o), 0) == 0) return 0;

```

```

284     vector<vec> p;
285     p.push_back(a);
286     line l(a, b - a);
287     auto tmp = intersection(c, l);
288     if (tmp.size() == 2) {
289         for (auto &i : tmp)
290             if (sgn((a - i) * (b - i), 0) < 0) p.push_back(i);
291     }
292     p.push_back(b);
293     if (p.size() == 4 && sgn((p[0] - p[1]) * (p[2] - p[1]), 0) > 0) swap(p[1], p[2]);
294     lf res = 0;
295     for (int i = 1; i < p.size(); i++)
296         if (c.relation(p[i - 1]) == 1 || c.relation(p[i]) == 1) {
297             lf ang = angle(p[i - 1] - c.o, p[i] - c.o);
298             res += c.r * c.r * ang / 2;
299         } else res += abs(cross(p[i - 1], p[i], c.o)) / 2.0;
300     return res;
301 }
302
303 // 多边形与圆相交的面积
304 lf area(vector<vec> &p, circle c) {
305     lf res = 0;
306     for (int i = 0; i < p.size(); i++) {
307         int j = i + 1 == p.size() ? 0 : i + 1;
308         if (sgn(cross(p[i], p[j], c.o), 0) <= 0) res += area(c, p[i], p[j]);
309         else res -= area(c, p[i], p[j]);
310     }
311     return abs(res);
312 }

```

### 三维

```

1  constexpr lf eps = 1e-8;
2
3  int sgn(lf a, lf b) {
4      lf c = a - b;
5      return c < -eps ? -1 : c < eps ? 0 : 1;
6  }
7
8  // 向量
9  struct vec3 {
10     lf x, y, z;
11     vec3() : x(0), y(0), z(0) {}
12     vec3(lf _x, lf _y, lf _z) : x(_x), y(_y), z(_z) {}
13
14     // 模
15     lf len2() const { return x * x + y * y + z * z; }
16     lf len() const { return hypot(x, y, z); }
17
18     bool operator==(const vec3 &b) const {
19         return sgn(x, b.x) == 0 && sgn(y, b.y) == 0 && sgn(z, b.z) == 0;
20     }
21     bool operator!=(const vec3 &b) const { return !(*this == b); }
22
23     vec3 operator+(const vec3 &b) const { return {x + b.x, y + b.y, z + b.z}; }
24     vec3 operator-(const vec3 &b) const { return {-x, -y, -z}; }
25     vec3 operator-(const vec3 &b) const { return -b + (*this); }
26     vec3 operator*(lf b) const { return {b * x, b * y, b * z}; }
27     lf operator*(const vec3 &b) const { return x * b.x + y * b.y + z * b.z; }

```

```

28
29 // 叉积 结果大于0, a到b为逆时针, 小于0, a到b顺时针,
30 // 等于0共线, 可能同向或反向, 结果绝对值表示 a b 形成的平行四边形的面积
31 vec3 operator^(const vec3 &b) const {
32     return {y * b.z - z * b.y, z * b.x - x * b.z, x * b.y - y * b.x};
33 }
34
35 friend istream &operator>>(istream &in, vec3 &a) {
36     in >> a.x >> a.y >> a.z;
37     return in;
38 }
39 friend ostream &operator<<(ostream &out, const vec3 &a) {
40     out << fixed << setprecision(6);
41     out << a.x << " " << a.y << " " << a.z;
42     return out;
43 }
44 };
45
46 struct line3 {
47     vec3 p, d;
48     line3() {}
49     line3(const vec3 &a, const vec3 &b) : p(a), d(b - a) {}
50 };
51
52 struct plane {
53     vec3 p, d;
54     plane() {}
55     plane(const vec3 &a, const vec3 &b, const vec3 &c) : p(a) {
56         d = (b - a) ^ (c - a);
57         assert(d != vec3());
58     }
59 };
60
61 // 线面是否垂直
62 bool perpen(const line3 &a, const plane &b) { return (a.d ^ b.d) == vec3(); }
63
64 // 线面是否平行
65 bool parallel(const line3 &a, const plane &b) { return sgn(a.d * b.d, 0) == 0; }
66
67 // 线面交点
68 vec3 intersection(const line3 &a, const plane &b) {
69     assert(!parallel(a, b));
70     double t = (b.p - a.p) * b.d / (a.d * b.d);
71     return a.p + a.d * t;
72 }

```

### 5.3 扫描线

```

1 #define ls (pos << 1)
2 #define rs (ls | 1)
3 #define mid ((tree[pos].l + tree[pos].r) >> 1)
4 struct Rectangle {
5     ll x_l, y_l, x_r, y_r;
6 };
7 ll area(vector<Rectangle>& rec) {
8     struct Line {
9         ll x, y_up, y_down;

```

```

10     int pd;
11 };
12 vector<Line> line(rec.size() * 2);
13 vector<ll> y_set(rec.size() * 2);
14 for (int i = 0; i < rec.size(); i++) {
15     y_set[i * 2] = rec[i].y_l;
16     y_set[i * 2 + 1] = rec[i].y_r;
17     line[i * 2] = {rec[i].x_l, rec[i].y_r, rec[i].y_l, 1};
18     line[i * 2 + 1] = {rec[i].x_r, rec[i].y_r, rec[i].y_l, -1};
19 }
20 sort(y_set.begin(), y_set.end());
21 y_set.erase(unique(y_set.begin(), y_set.end()), y_set.end());
22 sort(line.begin(), line.end(), [](Line a, Line b) { return a.x < b.x; });
23 struct Data {
24     int l, r;
25     ll len, cnt, raw_len;
26 };
27 vector<Data> tree(4 * y_set.size());
28 function<void(int, int, int)> build = [&](int pos, int l, int r) {
29     tree[pos].l = l;
30     tree[pos].r = r;
31     if (l == r) {
32         tree[pos].raw_len = y_set[r + 1] - y_set[l];
33         tree[pos].cnt = tree[pos].len = 0;
34         return;
35     }
36     build(ls, l, mid);
37     build(rs, mid + 1, r);
38     tree[pos].raw_len = tree[ls].raw_len + tree[rs].raw_len;
39 };
40 function<void(int, int, int, int)> update = [&](int pos, int l, int r, int num) {
41     if (l <= tree[pos].l && tree[pos].r <= r) {
42         tree[pos].cnt += num;
43         tree[pos].len = tree[pos].cnt ? tree[pos].raw_len
44             : tree[pos].l == tree[pos].r ? 0
45             : tree[ls].len + tree[rs].len;
46         return;
47     }
48     if (l <= mid) update(ls, l, r, num);
49     if (r > mid) update(rs, l, r, num);
50     tree[pos].len = tree[pos].cnt ? tree[pos].raw_len : tree[ls].len + tree[rs].len;
51 };
52 build(1, 0, y_set.size() - 2);
53 auto find_pos = [&](ll num) {
54     return lower_bound(y_set.begin(), y_set.end(), num) - y_set.begin();
55 };
56 ll res = 0;
57 for (int i = 0; i < line.size() - 1; i++) {
58     update(1, find_pos(line[i].y_down), find_pos(line[i].y_up) - 1, line[i].pd);
59     res += (line[i + 1].x - line[i].x) * tree[1].len;
60 }
61 return res;
62 }

```

## 6 杂项

### 6.1 快读

```

1 namespace IO {
2 constexpr int N = (1 << 20) + 1;
3 char Buffer[N];
4 int p = N;
5
6 char& get() {
7     if (p == N) {
8         fread(Buffer, 1, N, stdin);
9         p = 0;
10    }
11    return Buffer[p++];
12 }
13
14 template <typename T = int>
15 T read() {
16     T x = 0;
17     bool f = 1;
18     char c = get();
19     while (!isdigit(c)) {
20         f = c != '-';
21         c = get();
22     }
23     while (isdigit(c)) {
24         x = x * 10 + c - '0';
25         c = get();
26     }
27     return f ? x : -x;
28 }
29 } // namespace IO
30 using IO::read;

```

### 6.2 高精度

```

1 struct bignum {
2     string num;
3
4     bignum() : num("0") {}
5     bignum(const string& num) : num(num) { reverse(this->num.begin(), this->num.end()); }
6     bignum(ll num) : num(to_string(num)) { reverse(this->num.begin(), this->num.end()); }
7
8     bignum operator+(const bignum& other) {
9         bignum res;
10        res.num.pop_back();
11        res.num.reserve(max(num.size(), other.num.size()) + 1);
12        for (int i = 0, j = 0, x; i < num.size() || i < other.num.size() || j; i++) {
13            x = j;
14            j = 0;
15            if (i < num.size()) x += num[i] - '0';
16            if (i < other.num.size()) x += other.num[i] - '0';
17            if (x >= 10) j = 1, x -= 10;
18            res.num.push_back(x + '0');
19        }
20        res.num.capacity();

```

```

21     return res;
22 }
23
24 bignum operator*(const bignum& other) {
25     vector<int> res(num.size() + other.num.size() - 1, 0);
26     for (int i = 0; i < num.size(); i++)
27         for (int j = 0; j < other.num.size(); j++)
28             res[i + j] += (num[i] - '0') * (other.num[j] - '0');
29     int g = 0;
30     for (int i = 0; i < res.size(); i++) {
31         res[i] += g;
32         g = res[i] / 10;
33         res[i] %= 10;
34     }
35     while (g) {
36         res.push_back(g % 10);
37         g /= 10;
38     }
39     int lim = res.size();
40     while (lim > 1 && res[lim - 1] == 0) lim--;
41     bignum res2;
42     res2.num.resize(lim);
43     for (int i = 0; i < lim; i++) res2.num[i] = res[i] + '0';
44     return res2;
45 }
46
47 bool operator<(const bignum& other) {
48     if (num.size() == other.num.size())
49         for (int i = num.size() - 1; i >= 0; i--)
50             if (num[i] == other.num[i]) continue;
51             else return num[i] < other.num[i];
52     return num.size() < other.num.size();
53 }
54
55 friend istream& operator>>(istream& in, bignum& a) {
56     in >> a.num;
57     reverse(a.num.begin(), a.num.end());
58     return in;
59 }
60 friend ostream& operator<<(ostream& out, bignum a) {
61     reverse(a.num.begin(), a.num.end());
62     return out << a.num;
63 }
64 };

```

### 6.3 离散化

```

1 template <typename T>
2 struct Hash {
3     vector<int> S;
4     vector<T> a;
5     Hash(const vector<int>& b) : S(b) {
6         sort(S.begin(), S.end());
7         S.erase(unique(S.begin(), S.end()), S.end());
8         a = vector<T>(S.size());
9     }
10    T& operator[](int i) const {

```

```

11     auto pos = lower_bound(S.begin(), S.end(), i) - S.begin();
12     assert(pos != S.size() && S[pos] == i);
13     return a[pos];
14 }
15 };

```

## 6.4 模运算

```

1 constexpr int mod = 998244353;
2
3 template <typename T>
4 T power(T a, int b) {
5     T res = 1;
6     while (b) {
7         if (b & 1) res = res * a;
8         a = a * a;
9         b >>= 1;
10    }
11    return res;
12 }
13
14 struct modint {
15     int x;
16     modint(int _x = 0) : x(_x) {
17         if (x < 0) x += mod;
18         else if (x >= mod) x -= mod;
19     }
20     modint inv() const { return power(*this, mod - 2); }
21     modint operator+(const modint& b) { return x + b.x; }
22     modint operator-(const modint& b) { return -b.x; }
23     modint operator-(const modint& b) { return -b + *this; }
24     modint operator*(const modint& b) { return int((ll)x * b.x % mod); }
25     modint operator/(const modint& b) { return *this * b.inv(); }
26     friend istream& operator>>(istream& is, modint& other) {
27         ll _x;
28         is >> _x;
29         other = modint(_x);
30         return is;
31     }
32     friend ostream& operator<<(ostream& os, modint other) { return os << other.x; }
33 };

```

## 6.5 分数

```

1 struct frac {
2     ll a, b;
3     frac() : a(0), b(1) {}
4     frac(ll _a, ll _b) : a(_a), b(_b) {
5         assert(b);
6         if (a) {
7             int tmp = gcd(a, b);
8             a /= tmp;
9             b /= tmp;
10        } else *this = frac();
11    }
12     frac operator+(const frac& other) { return frac(a * other.b + other.a * b, b * other.b); }

```



```

13     frac operator-() const {
14         frac res = *this;
15         res.a = -res.a;
16         return res;
17     }
18     frac operator-(const frac& other) const { return -other + *this; }
19     frac operator*(const frac& other) const { return frac(a * other.a, b * other.b); }
20     frac operator/(const frac& other) const {
21         assert(other.a);
22         return *this * frac(other.b, other.a);
23     }
24     bool operator<(const frac& other) const { return (*this - other).a < 0; }
25     bool operator<=(const frac& other) const { return (*this - other).a <= 0; }
26     bool operator>=(const frac& other) const { return (*this - other).a >= 0; }
27     bool operator>(const frac& other) const { return (*this - other).a > 0; }
28     bool operator==(const frac& other) const { return a == other.a && b == other.b; }
29     bool operator!=(const frac& other) const { return !(*this == other); }
30 };

```

## 6.6 表达式求值

```

1 // 格式化表达式
2 string format(const string& s1) {
3     stringstream ss(s1);
4     string s2;
5     char ch;
6     while ((ch = ss.get()) != EOF) {
7         if (ch == ' ') continue;
8         if (isdigit(ch)) s2 += ch;
9         else {
10             if (s2.back() != ' ') s2 += ' ';
11             s2 += ch;
12             s2 += ' ';
13         }
14     }
15     return s2;
16 }
17
18 // 中缀表达式转后缀表达式
19 string convert(const string& s1) {
20     unordered_map<char, int> rank{{'+', 2}, {'-', 2}, {'*', 1}, {'/', 1}, {'^', 0}};
21     stringstream ss(s1);
22     string s2, temp;
23     stack<char> op;
24     while (ss >> temp) {
25         if (isdigit(temp[0])) s2 += temp + ' ';
26         else if (temp[0] == '(') op.push('(');
27         else if (temp[0] == ')') {
28             while (op.top() != '(') {
29                 s2 += op.top();
30                 s2 += ' ';
31                 op.pop();
32             }
33             op.pop();
34         } else {
35             while (!op.empty() && op.top() != '(' &&
36                 (temp[0] != '^' && rank[op.top()] <= rank[temp[0]])) {

```

```

37         rank[op.top()] < rank[temp[0]])) {
38             s2 += op.top();
39             s2 += ' ';
40             op.pop();
41         }
42         op.push(temp[0]);
43     }
44 }
45 while (!op.empty()) {
46     s2 += op.top();
47     s2 += ' ';
48     op.pop();
49 }
50 return s2;
51 }
52
53 // 计算后缀表达式
54 int calc(const string& s) {
55     stack<int> num;
56     stringstream ss(s);
57     string temp;
58     while (ss >> temp) {
59         if (isdigit(temp[0])) num.push(stoi(temp));
60         else {
61             int b = num.top();
62             num.pop();
63             int a = num.top();
64             num.pop();
65             if (temp[0] == '+') a += b;
66             else if (temp[0] == '-') a -= b;
67             else if (temp[0] == '*') a *= b;
68             else if (temp[0] == '/') a /= b;
69             else if (temp[0] == '^') a = ksm(a, b);
70             num.push(a);
71         }
72     }
73     return num.top();
74 }

```

## 6.7 日期

```

1  int month[] = {0, 31, 28, 31, 30, 31, 30, 31, 31, 30, 31, 30, 31};
2  int pre[13];
3  vector<int> leap;
4  struct Date {
5      int y, m, d;
6      bool operator<(const Date& other) const {
7          return array<int, 3>{y, m, d} < array<int, 3>{other.y, other.m, other.d};
8      }
9      Date(const string& s) {
10         stringstream ss(s);
11         char ch;
12         ss >> y >> ch >> m >> ch >> d;
13     }
14     int dis() const {
15         int yd = (y - 1) * 365 + (upper_bound(leap.begin(), leap.end(), y - 1) - leap.begin());
16         int md = pre[m - 1] + (m > 2 && (y % 4 == 0 && y % 100 != 0 || y % 400 == 0));

```

```

17     return yd + md + d;
18 }
19 int dis(const Date& other) const { return other.dis() - dis(); }
20 };
21 for (int i = 1; i <= 12; i++) pre[i] = pre[i - 1] + month[2];
22 for (int i = 1; i <= 1000000; i++)
23     if (i % 4 == 0 && i % 100 || i % 400 == 0) leap.push_back(i);

```

## 6.8 builtin 函数

如果是 long long 型，记得函数后多加个 ll。

- ctz, 从最低位连续的 0 的个数，如果传入 0 则行为未定义。
- clz, 从最高位连续的 0 的个数，如果传入 0 则行为未定义。
- popcount, 二进制 1 的个数。
- parity, 二进制 1 的个数奇偶性。

## 6.9 对拍

linux/Mac

```

1 #!/bin/bash
2
3 g++ $1 -o a -O2
4 g++ $2 -o b -O2
5 g++ random.cpp -o random -O2
6
7 cnt=0
8 while true; do
9     let cnt++
10    echo TEST:$cnt
11    ./random > in
12    ./a < in > out.a
13    ./b < in > out.b
14    if ! diff out.a out.b; then break; fi
15 done

```

windows

```

1 @echo off
2
3 g++ %1 -o a -O2
4 g++ %2 -o b -O2
5 g++ random.cpp -o random -O2
6
7 set cnt=0
8
9 :again
10    set /a cnt=cnt+1
11    echo TEST:%cnt%
12    .\random > in
13    .\a < in > out.a
14    .\b < in > out.b
15    fc out.a out.b > nul
16 if not errorlevel 1 goto again

```

## 6.10 编译常用选项

```
1 -Wall -Woverflow -Wextra -Wpedantic -Wfloat-equal -Wshadow -fsanitize=address,undefined
```

## 6.11 开栈

不同的系统/编译器可能命令不一样

```
1 ulimit -s
2 -Wl,--stack=0x10000000
3 -Wl,-stack_size -Wl,0x10000000
4 -Wl,-z,stack-size=0x10000000
```

## 6.12 clang-format

转储配置

```
1 clang-format -style=Google -dump-config > ./clang-format
```

.clang-format

```
1 BasedOnStyle: Google
2 IndentWidth: 4
3 AllowShortIfStatementsOnASingleLine: AllIfsAndElse
4 ColumnLimit: 100
```