Electric field within a hollow tube

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Abstract

This project consists of solving the Laplace equation in two dimensions and finding the electric field E inside a square, hollow tube where the potential of each of the sides is known quantities.

1 Theory

The boundary conditions in this problem are given by the equations

$$\begin{aligned} (i)\ & V(x=0,y)=0 \\ (ii)\ & V(x=L,y)=0 \\ (iii)\ & V(x,y=0)=0 \\ (iv)\ & V(x,y=L)=V_0(x), \end{aligned}$$

where L is the length of the box. By using separation of variables one can solve the Laplace equation,

$$\partial_x^2 V + \partial_y^2 V = 0, \tag{1}$$

were V is the potential in the box. It is assumed that the potential will have the form $V(x,y)=X(x)Y(y)$. Putting this equation into equation (1), one finds the two ODE's:

$$\frac{1}{X}\frac{\mathrm{d}^2X}{\mathrm{d}x^2}+\frac{1}{Y}\frac{\mathrm{d}^2Y}{\mathrm{d}y^2}=0. \tag{2}$$

Equation (2) together with boundary conditions (i)-(iii) yield the following general solution for the potential $V(x,y)$,

$$V(\xi,\psi)=\sum_{N=0}^{\infty}C_N sinh(N\pi\psi)sin(N\pi\xi) \tag{3}$$

Where $\xi=x/L$ and $\psi=y/L$ for $N\in\mathbb{Z}^+$. $V(\xi,\psi)$ is expressed as a Fourier series. The constants C_n can be found using "Fouriers trick" and boundary condition (iv), this yield the following equation

$$C_N = \frac{2}{sinh(N\pi)} \int_0^1 sin(N\pi\xi) V_0(\xi L) d\xi. \tag{4}$$

$V(\xi, \psi)$ can therefore be determined by inserting equation (4) into equation (3). This is challenging to solve analytically as $V_0(\xi)$ is not a constant but dependent on the variable ξ. However, the problem is simple to solve numerically. To find the electric field $E(\xi, \psi)$ one simply uses the formula

$$E(\xi, \psi) = -\nabla V(\xi, \psi). \tag{5}$$

2 Results

Three different functions were used for the potential at the boundary condition $V_0(\xi)$, which where $V_0(\xi) = \sin(3\pi\xi)$, $V_0(\xi) = 1 - (\xi - \frac{1}{2})^{\frac{1}{4}}$ and $V_0(\xi) = \Theta(\xi L - \frac{L}{2})\Theta(\frac{3L}{4} - \xi L)$. Here Θ is the Heaviside step function.

2.1 Contour plot of $V(\xi, \psi)$

Figure 1 shows the different contour plots of the different potentials $V(\xi, \psi)$.

2.2 Plots at boundary $y = L$ and convergence

Figures 2, 3 and 4 shows $V(\xi, \psi = 1)$, i.e the potential plotted at the plotted at the boundary $y = L$, against the different $V_0(\xi)$. A convergence plot for $max|V(\xi) - V_0(\xi)|$ is also plotted.

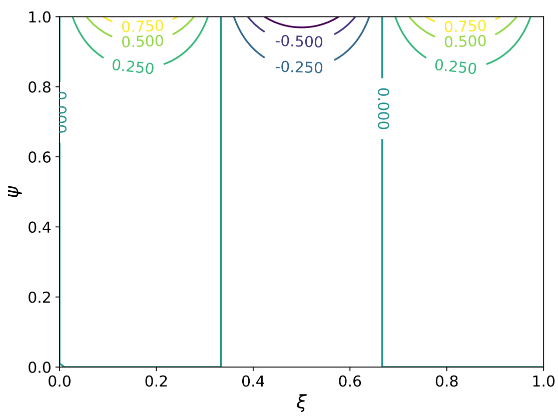
2.3 The electric field $E(\xi, \psi)$

The plots for the electric field $E(\xi, \psi)$ is shown in figure 5.

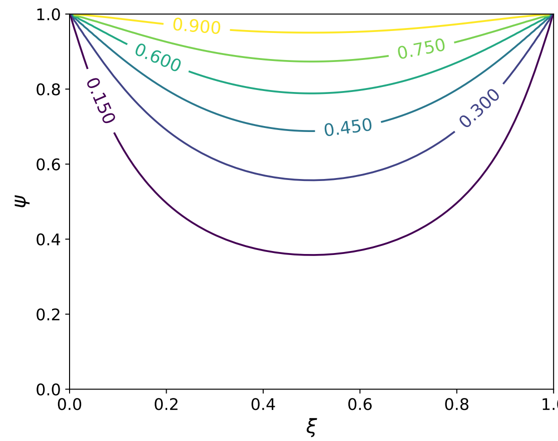
3 Discussion

One can see from figure 2 that there is close to no difference between the numerical and the analytical solution of $V(\xi, \psi)$, when $V_0(\xi) = \sin(3\pi\xi)$. From the convergence plot 3a one can see that the difference between the functions is in the range of $max|V(\xi) - V_0(\xi)| \in [10^{-16}, 10^{-14}]$ at $N = 2$ to $N = 100$. This is because the Fourier series of a sine function is the sine function itself. Therefore with $V_0 = \sin(3\pi\xi)$ one get an extremely precise result.

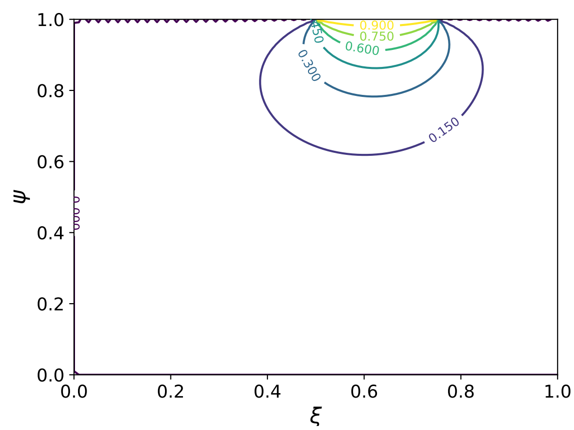
For $V_0(\xi) = 1 - (\xi - \frac{1}{2})^{\frac{1}{4}}$ and $V_0(\xi) = \Theta(\xi L - \frac{L}{2})\Theta(\frac{3L}{4} - \xi L)$ the lowest order of difference between the analytical and the numerical solution of $V(\xi, \psi)$ at the boundary $\psi = 1$, is in



(a) Contour plot of the potential $V(\xi, \psi)$, using $V_0(\xi) = \sin(3\pi\xi)$

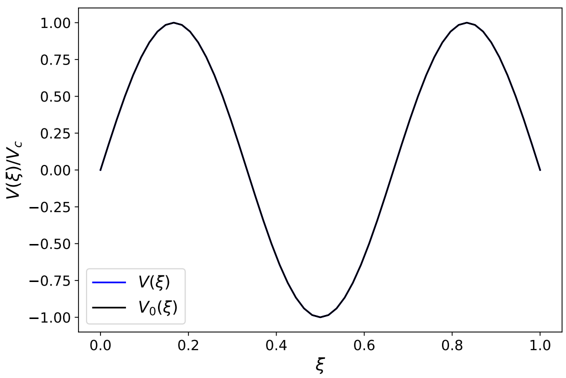


(b) Contour plot of the potential $V(\xi, \psi)$, using $V_0(\xi) = 1 - (\xi - \frac{1}{2})^{\frac{1}{4}}$

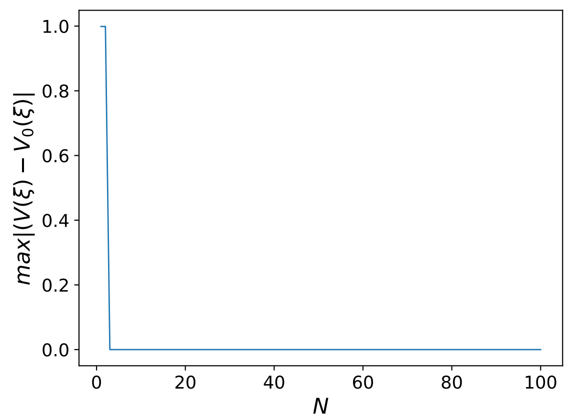


(c) Contour plot of the potential $V(\xi, \psi)$, using $V_0(\xi) = \Theta(\xi L - \frac{L}{2})\Theta(\frac{3L}{2} - \xi L)$

Figure 1: Contour plots of the potentials $V(\xi, \psi)$ using different $V_0(\xi)$. One can see the height of the curve by reading the number by every equipotential curve, which also is indicated in different colours.

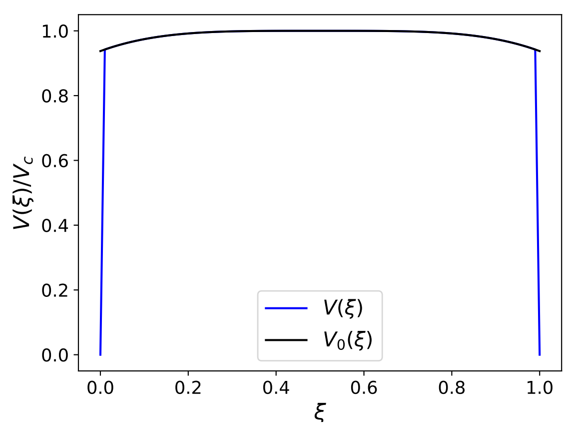


(a) Plot of V_0 and $V(\xi)$. V_C is a constant term and is here set equal to 1. The number of Fourier coefficients for $V(\xi)$ is $N = 100$.

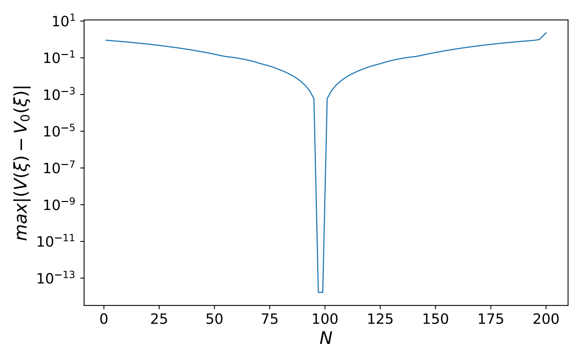


(b) Convergence plot where N is the number of Fourier coefficients.

Figure 2: Plot of $V(\xi)$ and $V_0(\xi)$ at the boundary of the tube, specifically for $\psi = 1$, in addition to a plot of the convergence between $V(\xi)$ and $V_0(\xi)$ at the boundary. Here $V_0(\xi) = \sin(3\pi\xi)$.

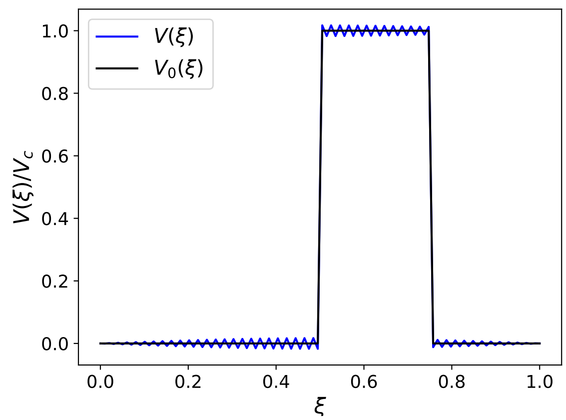


(a) Plot of V_0 and $V(\xi)$. V_C is a constant term and is here set equal to 1. The number of Fourier coefficients for $V(\xi)$ is $N = 99$.

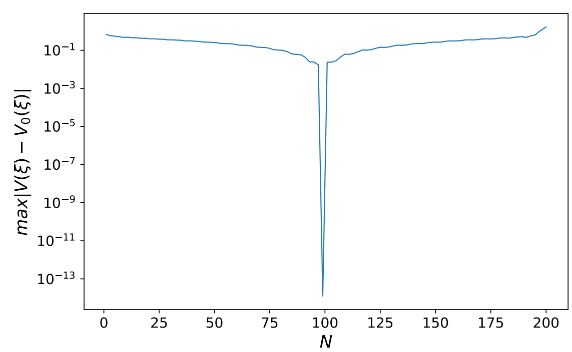


(b) Convergence plot where N is the number of-Fourier coefficients.

Figure 3: Plot of $V(\xi)$ and $V_0(\xi)$ at the boundary of the tube, specifically for $\psi = 1$. In addition there is a plot of the convergence between $V(\xi)$ and $V_0(\xi)$ at the boundary. Here $V_0(\xi) = 1 - (\xi - \frac{1}{2})^{\frac{1}{4}}$.

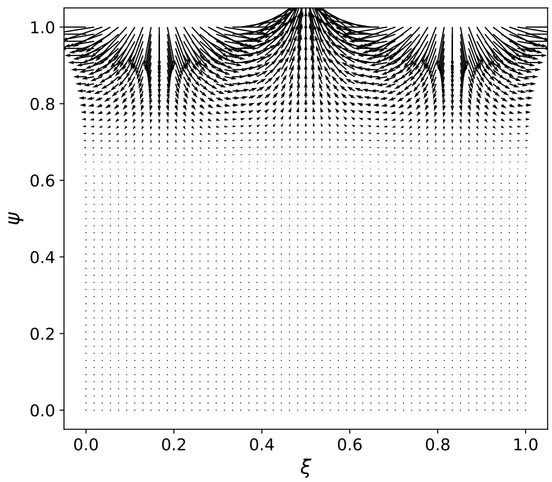


(a) Plot of V_0 and $V(\xi)$. V_C is a constant term and is here set equal to 1. The number of Fourier coefficients for $V(\xi)$ is $N = 99$.

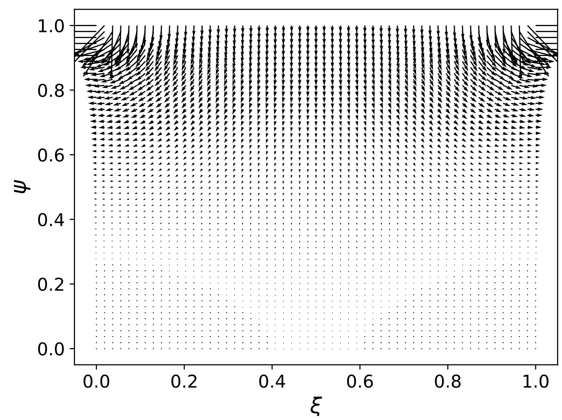


(b) Convergence plot where N is the number of Fourier coefficients.

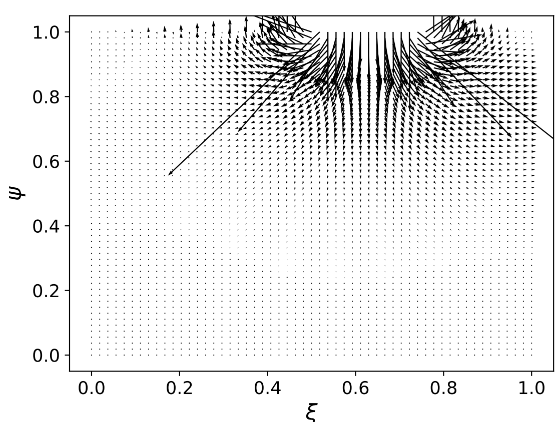
Figure 4: Plot of $V(\xi)$ and $V_0(\xi)$ at the boundary of the tube, specifically for $\psi = 1$, in addition to a plot of the convergence between $V(\xi)$ and $V_0(\xi)$ at the boundary. Here $V_0(\xi) = \Theta(\xi L - \frac{L}{2})\Theta(\frac{3L}{4} - \xi L)$.



(a) Quiver plot of the potential $E(\xi, \psi)$, using $V_0(\xi) = \sin(3\pi\xi)$.



(b) Quiver plot of the potential $E(\xi, \psi)$, using $V_0(\xi) = 1 - (\xi - \frac{1}{2})^{\frac{1}{4}}$.



(c) Quiver plot of the potential $V(\xi, \psi)$, using $V_0(\xi) = \Theta(\xi L - \frac{L}{2})\Theta(\frac{3L}{2} - \xi L)$.

Figure 5: Quiver plots of the potentials $E(\xi, \psi)$ using different $V_0(\xi)$. The strength of the electric field $E(\xi, \psi)$ is proportional to the lengths of the vectors plotted.

the power of 10^{-13}, as seen in figures 3b and 4b. The reason for the higher orders here, as compared to the sine wave, is because there are discontinuities in $V_0(\xi)$. As seen in 3b and 4b, more coefficients from the sum in equation 3, does not necessarily yield a more precise result. This phenomena is called *Gibbs phenomenon*, and states that *"In an interval which includes or reaches up to a point where $f(x)$ is discontinuous the Fourier's development of $f(x)$ cannot converge uniformly, since a uniformly convergent series of continuous functions necessarily represents a continuous function"*[2]. Therefore, the sum "overshoots" or "undershoots" depending in the number of coefficients used[3], which is at a minimum at approximately $N = 99$ for both the step function and exponential function.

The electric field $E(\xi, \psi)$ was found using equation (5) and the plots are shown in figure 5. These plots seem very reasonable as they have the same form as the contour plots in figure 1. Furthermore one can see from the plots in 5, that the Electric field $E(\xi, \psi)$ fulfils all the boundary conditions of $V(\xi, \psi)$.

4 Conclusion

The numerical method of solving the potential $V(\xi, \psi)$ is a good approximation, for the correct value of Fourier coefficients. Therefore, one can get accurate results for the electric field $E(\xi, \psi)$ in the hollow tube. Here the number of Fourier coefficients which yields the most accurate results are $N = 100$ for $V_0 = \sin(3\pi\xi)$, and N= 2 for $V_0(\xi) = 1 - (\xi - \frac{1}{2})^{\frac{1}{4}}$ and $V_0(\xi) = \Theta(\xi L - \frac{L}{2})\Theta(\frac{3L}{4} - \xi L)$.

References

[1] D.J.Griffiths *Introduction to Electrodynamics*, Cambridge University Press, 4^{th} edition, 2017

[2] Introduction to the Theory of Fourier's Series, 1906

[3] Gibbs Phenomenon, 2019