Numerical exercise TFY4305 Nonlinear dynamics

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1 Introduction

This project is threefold, question 1 considers Cobweb analysis of Poincare maps, question 2 considers the Lorentz system and in question 3 one looks at the Rössler system.

2 Problem 1 - Poincaré maps, cobweb constructions and stable limit cycles

2.1 Warm-up

In the first part of the warmup we are asked to find the Poincaré map of the vector field $\dot{r} = r(1-r^2),\ \dot{\theta} = 4\pi$. Furthermore, the question is asking to show that the system has a stable limit cycle at $r = 1$, which easily can be done by using cobweb analysis. With a start value of r_0 and a time of flight $t = \frac{1}{2}$, we can find the Poincaré maps $r_1 = P(r_0)$ by first solving for r_1,

$$\int_{r_0}^{r_1} \frac{dr}{r(1-r^2)} = \int_0^{\frac{1}{2}} dt = \frac{\pi}{2}. \tag{1}$$

Solving this integral yields $r_1 = [1 + e^{-1}(r_0^2 - 1)]^{-\frac{1}{2}}$, which in turn gives the solution of the Poincaré map to be

$$P(r) = [1 + e^{-1}(r^{-2} - 1)]^{-\frac{1}{2}}. \tag{2}$$

The resulting plot for the cobweb analysis is show in figure 1 below. As one can see, the cobweb converges towards $r \backsim 1$, which implies that there is a stable limit cycle at that point.

Furthermore, the first 30 values of the iterative map starting form $r_0 = 0.1$ is plotted in figure (2). The figure illustrates that a particle with this start value will end up rotating in a circle with $r \backsim 1$ if we wait a sufficient amount of time.

2.2 Challenge

For the first challenge question, we are asked to find the Poincare map of the vector field given by $\dot{r} = A(r-\pi)e^{-\omega t},\ \dot{\theta} = \pi$. This can be found by solving the following equation;

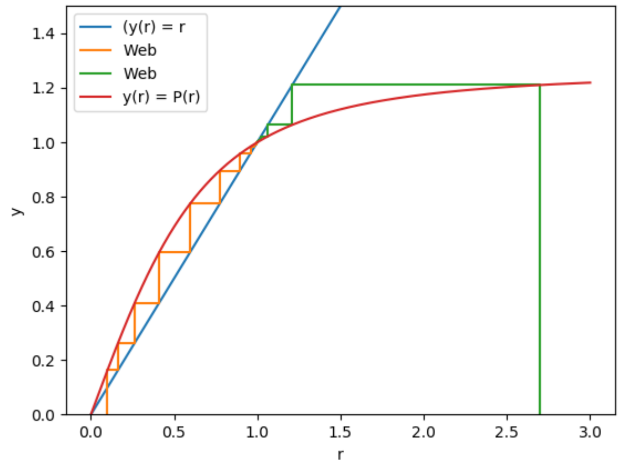


Figure 1: Cobweb analysis of the Poincaré function given by $P(r) = [1 + e^{-1}(r^{-2} - 1)]^{-\frac{1}{2}}$.

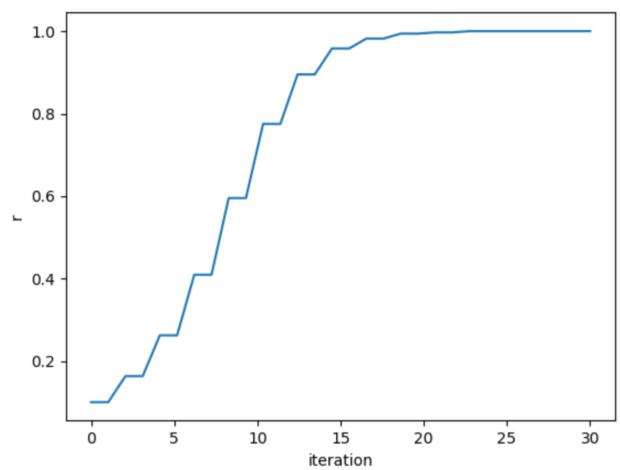


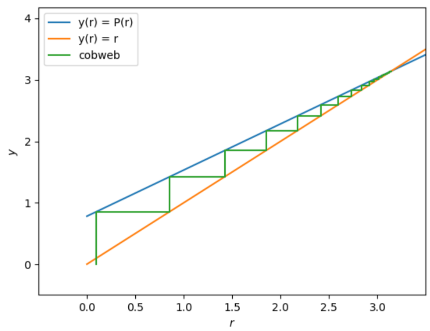
Figure 2: The iterative map as a function of iterations. The start value was $r_0 = 0.1$.

$$\int_{r_0}^{r_1} \frac{dr'}{r' - \pi} = \int_0^t Ae^{-\omega t'} dt. \tag{3}$$

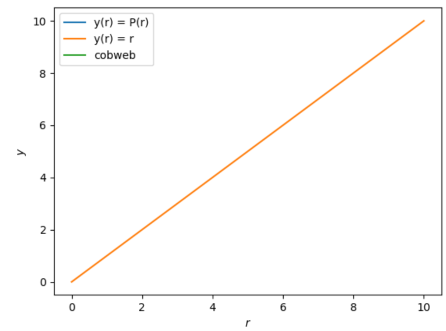
Where the solution of this integral equation yields $r = \pi + (r_0 - \pi)e^{-\frac{A}{\omega}(e^{-2\omega}-1)}$. This in turn implies that we have the Poincaré map

$$P(r) = \pi + (r - \pi)e^{-\frac{A}{\omega}(e^{-2\omega}-1)}. \tag{4}$$

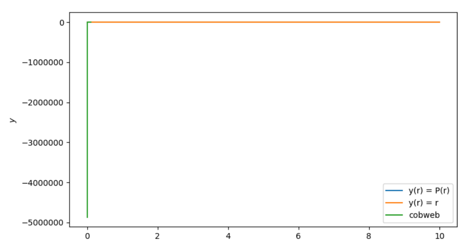
This Poincaré map is shown for different values for A and ω in figure (4). One can see from figure (3a) that for negative values of A and positive values for ω one will get the right slope to be able to construct a cobweb. You will also get a cobweb for positive values for A and negative values for ω, only a slightly different plot which is not shown here. Using different signs for A and ω one gets an exponentially growing curve, which diverges for large values of r. This is seen in figure (3b); the Poincaré map is so much larger than $y = r$ in magnitude so the cobweb diverges. From equation (2) one can see that for small ω, $P(r)$ blows up, which is consistent with what we get in figure (3c), i.e one can see that the cobweb diverges. In fact, we can get a stable limit cycle in $r = \pi$, for $A < 0$, as long as ω is not too large in magnitude.



(a) Cobweb construction of the Poincaré map with values $A = -2$ and $\omega = 7$.



(b) Cobweb construction of the Poincaré map with values $A = 2$ and $\omega = -7$.



(c) Cobweb construction of the Poincaré map with values $A = 2$ and $\omega = 7$

Figure 3: Graphical representation of the Poincaré map of the function $r = \pi + (r_0 - \pi)e^{-\frac{A}{\omega}(e^{-2\omega}-1)}$ for differing values of A and ω.

3 Problem 2- The Lorentz equations

The Lorentz system is a set of differential equations that was first studied by Edward Lorenz and Ellen Fetter. The set of coupled ordinary differential equations are given by,

$$\dot{x} = \sigma(y - x) \tag{5}$$
$$\dot{y} = x(\rho - z) - y \tag{6}$$
$$\dot{z} = xy - \beta z. \tag{7}$$

Where β, σ and ρ are some constants for the system. The Lorentz system have a chaotic solution for certain values of these constants which is called the Lorentz attractor. In the following section we will look at the Lorentz attractor.

3.1 Warm-up

In the warm-up we are asked to solve the Lorentz equation numerically up to the dimensionless time T = 50. The values for β, σ and ρ are given as 8/3, 10 and 28 respectively. These constants are found to give chaotic solutions. The initial conditions used were $x_0 = y_0 = z_0 = 0.001$. Furthermore, a time step of $dt = 0.001$ was used. The resulting plot is shown in figure (4).

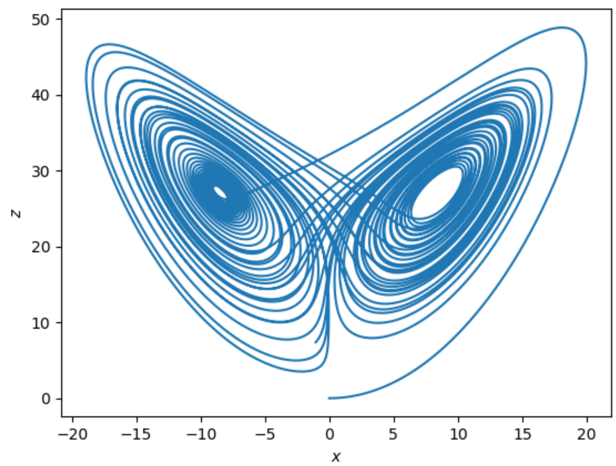


Figure 4: Plot of the Lorentz system.

3.2 Challenge

In this challange we are asked to subtract some information from the Lorentz attractor. First, we can see how the plot looks for z as a function of time t, as is shown in figure (5).

At first glance the plot may seem chaotic, but there could be some structure to it. Lorentz's idea was that z_n could predict z_{n+1}. To check this, we can numerically integrate the equations (5), (6), (7) over a long time ($T = 10^5$ was used in this calculation), then measure the z valuate at each peak and then plot z_{n+1} against z_n. This was done in figure (6).

By using this method Lorentz's was able to show that there was an order in the chaos. The function $f(z_n) = z_{n+1}$, shown in figure (6), is what is known as the Lorentz's map. Note that there are single data points plotted in figure (6), but since there are so many data points it does indeed look like a graph. Technically it isn't a continuous function, but we can gain some insight if we treat it as such. For example, by looking at $f(z_n)$ one can deduce that there are no stable limit cycles for the Lorentz's attractor. We can see this in the fact that $|f'(z)| > 1 \; \forall z$. This means if there are any limit cycles in the system they would be unstable.

The Lorentz's attractor is also what is known as a strange attractor, which is to say that the system is very sensitive to initial conditions. More precisely, the separation distance

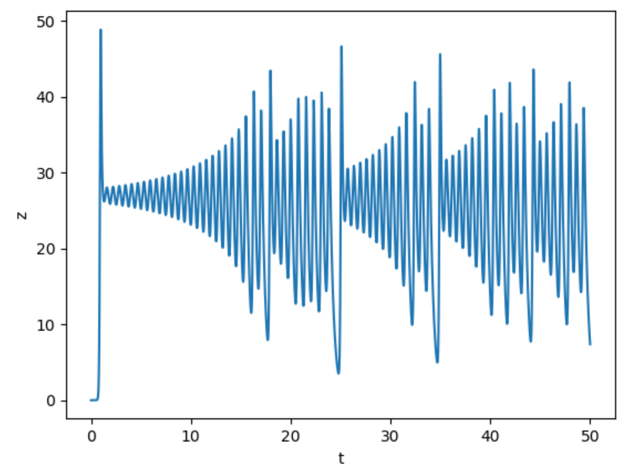


Figure 5: Plot of the $z(t)$ for the Lorentz's system.

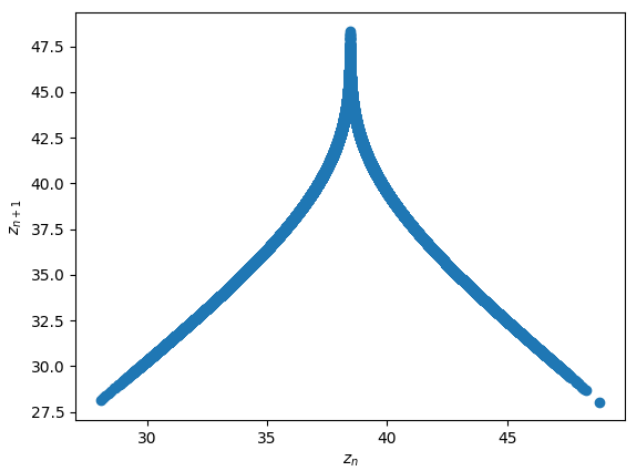


Figure 6: Plot of the peak z_{n+1} against peak z_n.

$||\delta(t)||$ between two endpoints $\vec{x_0}$ and $\vec{x_1}$ on two different trajectories increases as

$$||\delta(t)|| = ||\vec{x_1} - \vec{x_0}|| \backsim \delta_0 e^{\lambda t}. \tag{8}$$

Where λ is the maximum Lyapunov exponent (but it will be referred to as just the Lyapunov exponent from now on). The Lyapunov exponent is therefore a measure on how fast two different initial conditions leads to differences in the two trajectories given by the initial conditions. In order to find this exponent we could start by finding a random point near the origin, $x_0 \backsim 10^{-6}$ was used in this calculation. Secondly, we integrate the Lorentz's equations for this initial condition out to about $T = 50$; this will be $\vec{x_0}$. For the other random point x_1, we set the constraint that $||\vec{x_1} - \vec{x_0}|| = 10^{-6}$. Note that one practical way to construct a random point with a certain norm from another fixed point is to construct a ball of radius $r = 10^{-6}$ around x_0, where the angles θ and ϕ are randomly generated, i.e the norm stays constant, but where you are on the sphere is random. Furthermore integrating the Lorentz equation further over a given time interval, which we set to be 30, using both initial conditions $\vec{x_1}$ and $\vec{x_0}$. Then plotting the logarithm of a function that is the separation between the endpoints at each iteration, as a function of dimensionless time. This has been done in figure (7).

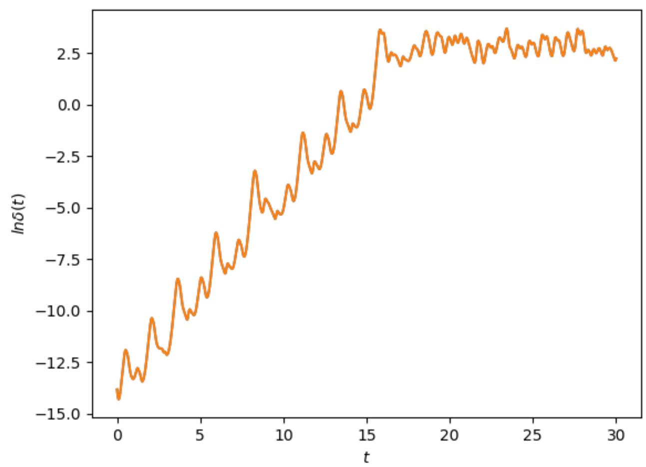


Figure 7: Plot of the logarithm of the separation distance $ln|\delta(t)|$ from two trajectories with different initial conditions, as a function of dimensionless time t.

As one can see in this figure, the separation distance grows exponentially until some cutoff time T_c, which here has a value $\backsim 15$. If we run the program several times, one gets different plots and slightly different T_c, because of the randomly generated start conditions. However if we take an average after running the program several times one sees that $T_c = 15$ is a good approximation. So the graph flattens out after the cutoff time T_c, this is because the exponentially divergence must stop when the separation is comparable to the diameter of the attractor, as certainly the paths can't diverge further than that. As we want to make a linear fitting to the curve in figure (7), it is useful to only integrate up to $T_c = 15$. The Lyapunov exponent will be the slope of this linear fitting. The linear fitting is shown in figure (8). Here the Lyapunov exponent is found to be $\lambda = 0.95$ for this single iteration.

Note that the slope for the the curve in figure (8) is different for each initial condition. In order to find a more accurate λ one can take run the system a sufficiently high number of trajectories to find the average Lyapunov exponent. If we take the average at $N = 1000$ iterations we find that the average Lyapunov exponent is given as $\lambda = 0.903$. Ten of these different separation distances, with the line given by $\delta_0 e^{0.9t}$, are plotted in figure (9).

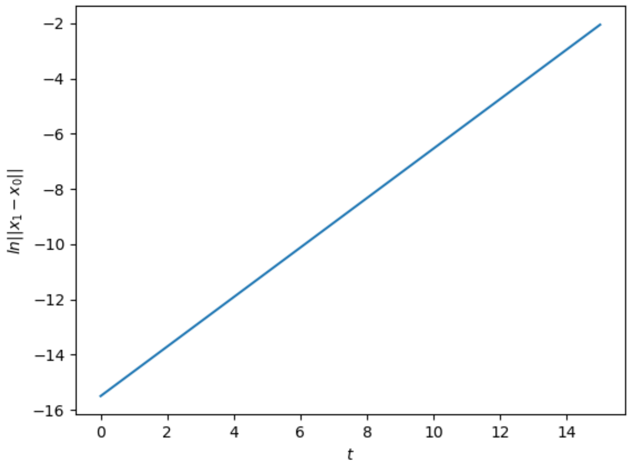


Figure 8: The graph $||\delta(t)|| = ||\vec{x_1} - \vec{x_0}|| = 0.95t - 14.17$. The Lyapunov exponent is $\lambda = 0.95$.

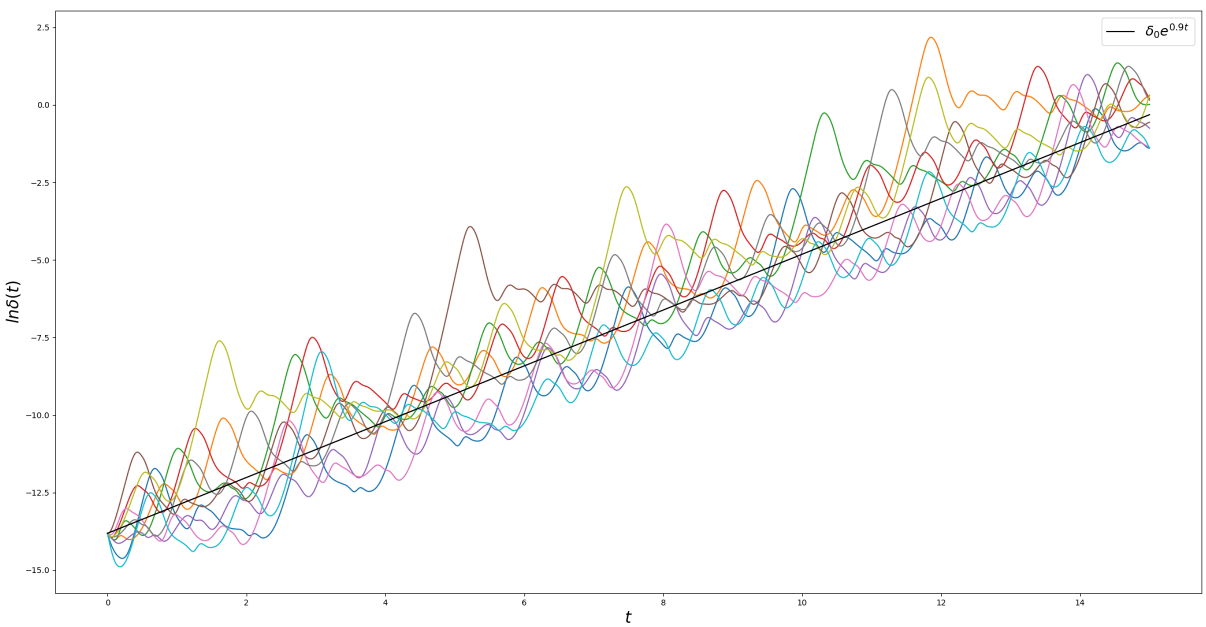


Figure 9: Plot of ten different iterations of $ln|\delta(t)|$, with the linear fitted curve $\delta_0 e^{0.9t}$.

4 Problem 3 - The Rössler system

The Rössler system is a set of three non-linear differential equations developed by Otto Rössler after he gained some inspiration from the actions made by a taffy pulling machine. The equations are given given by;

$$\dot{x} = -y - z \tag{9}$$

$$\dot{y} = x + ay \tag{10}$$

$$\dot{z} = b + z(x - c), \tag{11}$$

for some constants a, b and c. The Rössler system is comparable to the Lorentz's attractor in the way that it also is a strange attractor, only simpler, because it only has one quadratic non linearity, zx. This attractor also has chaotic solutions for certain values for the constants a, b and c, which will be studied in the following sections.

4.1 Warm-up

In the warm-up we are asked to make a plot of the Rössler system in the xz-plane, using the values $a = b = 0.2,\ c = 5.7$. The initial conditions where $x_0 = y_0 = z_0 = 0.001$. In addition a time step of $dt = 0.001$ was used. The Rössler system is shown in figure (10). Similar to the Lorentz's attractor, this attractor is also almost completely flat, except for some movement in the z-directions. In fact, for small values of z, the system can be approximated as two-dimensional.

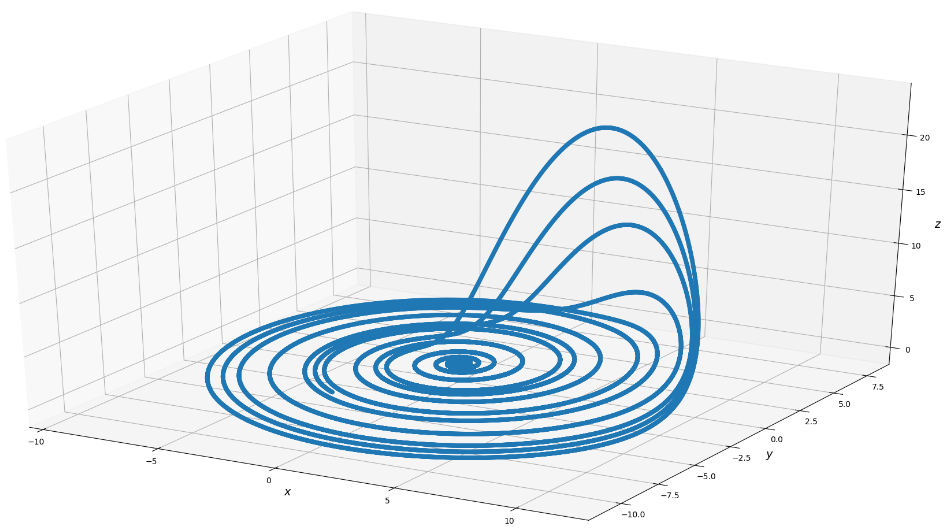


Figure 10: Chaotic solution for the Rössler system.

4.2 Challenge

We want to do the same analysis for the Rössler system, as what was done in the Lorentz's system and compare the results. Therefore, we can start with plotting the z value against t, which is shown in figure (11).

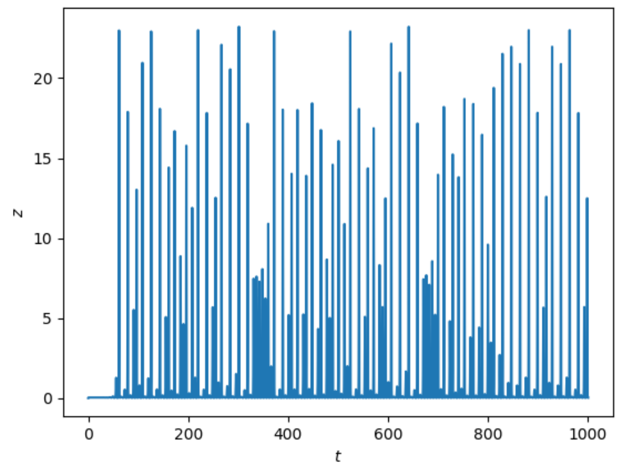


Figure 11: Plot of $z(t)$ in the chaotic solution for Rössler system.

This plot is similar to the same plot as for $z(t)$ in the Lorentz's system in the way that it seems chaotic, but again, there could be some structure to the system. This structure can be seen in figure (12), where peak z_{n+1} is plotted as a function of z_n.

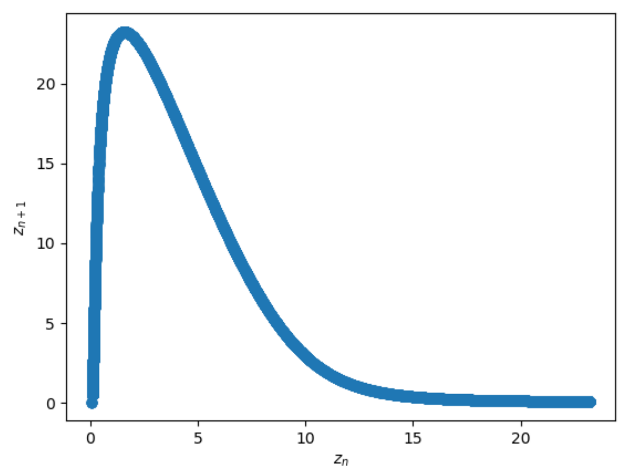


Figure 12: Plot of the peak z_{n+1} against z_n.

Here a time period of $T = 10^5$ were used. Same as before, the data from the chaotic time series fall neatly on a curve. As seen from the Lorentz's system as well, there is clearly an order to the chaotic system. Again we can do an approximation with regarding this curve as a function given by $f(z_n) = z_{n+1}$.

As for the separation distance between two endpoints on the attractor, we can see in figure (13) that there is a saturation after some cutoff time T_c. Here, we have a cutoff time because the endpoints on the trajectories can't diverge further than the diameter of the attractor, which is again finite. After running the system several times, it seems as $T_c = 175$

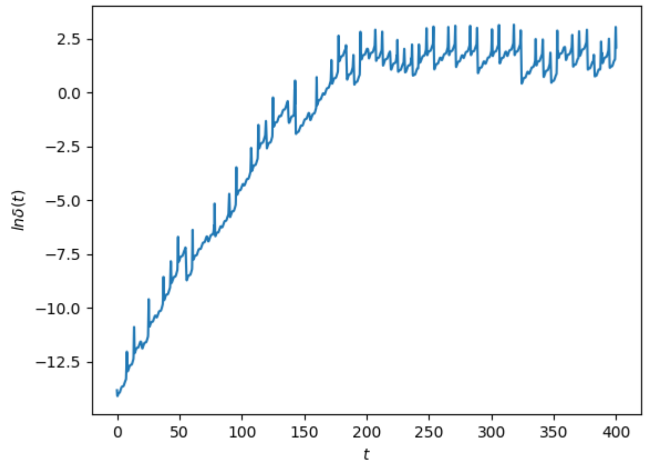


Figure 13: Plot of the logarithm of the separation distance $ln|\delta(t)|$ from two trajectories with different initial conditions, as a function of dimensionless time t.

is a good approximation. Note that the system was integrated out to T = 200 in order to find $\vec{x_0}$ and then to another $T = 400$ to find T_c.

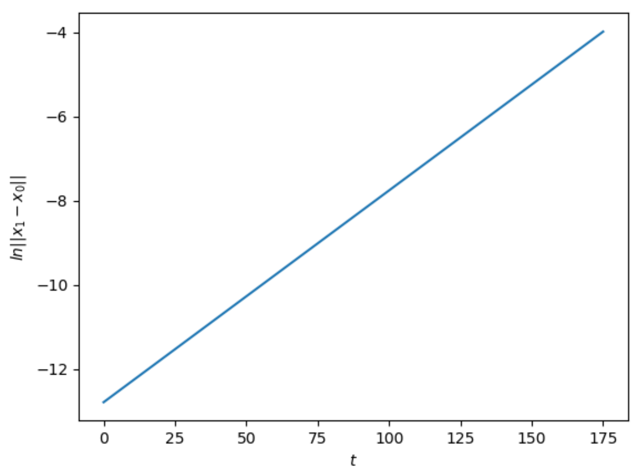


Figure 14: The graph $||\delta(t)|| = ||\vec{x_1} - \vec{x_0}|| = 0.05t - 12.78$. The Lyapunov exponent is $\lambda = 0.05$.

The linear fitting was done in figure (15), where the graph $||\delta(t)|| = ||\vec{x_1} - \vec{x_0}|| = 0.95t - 12.78$ was found, which means that the Lyapunov exponent for this initial condition was found to be $\lambda = 0.05$. Again the Lyapunov exponent will vary slightly for each iteration, because random initial condition is being used. In order to find a more accurate representation of the Lyapunov exponent we take the average over a large number of iterations. After using $N = 1000$ iteration, the average Lyapunov exponent was found to be; $\lambda = 0.07$. Ten of these different separation distances, with the line given by $\delta_0 e^{0.07t}$, are plotted in figure (14). As the maximum Lyapunov exponent can be used as a measure of how rapidly the endpoints on two trajectories diverge, one can conclude that the trajectories on the Lorentz's attractor diverge faster then for the Rössler system. Furthermore, one can say that the slower the endpoints diverge the easier the chaotic system would be to predict, therefore predictions made in the Rössler system will be more accurate then for the Lorentz's system.

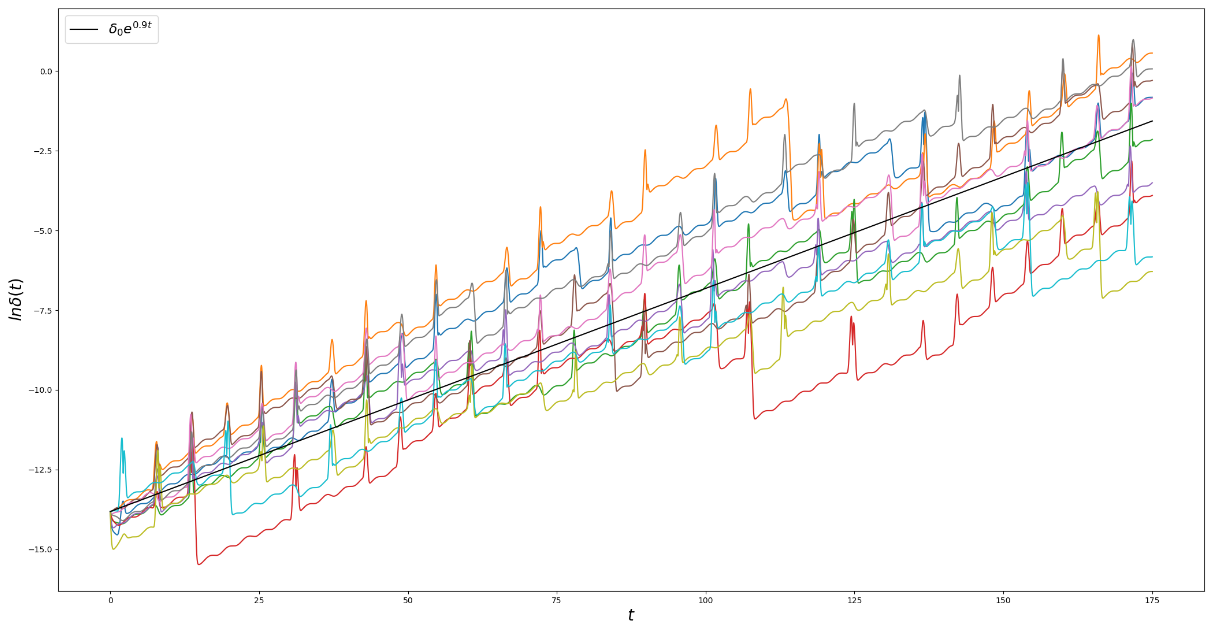


Figure 15: The graph $||\delta(t)|| = ||\vec{x_1} - \vec{x_0}|| = 0.07t - 12.78$. The Liapunov exponent is $\lambda = 0.07$.

References

[1] Strogatz's Nonlinear Dynamics And Chaos, 2^{nd} edition Westview press 2015