Sec 7.1: Eigenvalues and Eigenvectors

HW:

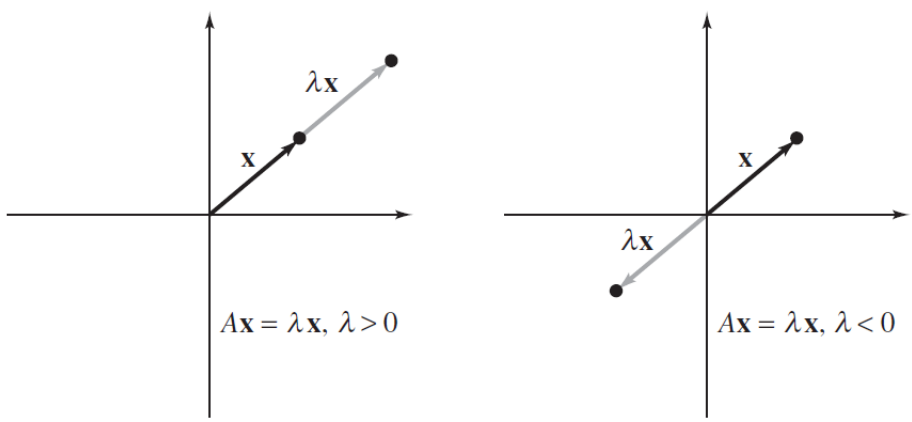
Question: If A is an n x n matrix, do there exist nonzero vectors $\mathbf{x}$ in R^n such that A$\mathbf{x}$ is a scalar multiple of $\mathbf{x}$?

$$A\mathbf{x} = \lambda\mathbf{x}$$

Vocab:

- The scalar (λ) is known as an **eigenvalue** of the matrix A.
- The (nonzero) vector $\mathbf{x}$ is called an **eigenvector** of A corresponding to λ
- The terms eigenvalue and eigenvector are derived from the German word *Eigenwert* meaning "proper value."

Geometrically:



Example 1. *For the matrix* $A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$ *verify that* $\boldsymbol{x} = (1, 0)$ *is an eigenvector of* A *corresponding to the eigenvalue* $\lambda = 2$.

Example 2. *Write $A\boldsymbol{x} = \lambda\boldsymbol{x}$ as a homogeneous system.*

Note: The homogeneous system of equations has **nonzero** solutions if and only if the coefficient matrix $(\lambda I - A)$ is **not** invertible. That is, if and only if the determinant of $(\lambda I - A)$ is zero.

Theorem 1. *Eigenvalues and Eigenvectors of a Square Matrix*
Let A be an n x n matrix.

1. *Eigenvalues: An eigenvalue of A is a scalar λ such that*

$$det(\lambda I - A) = 0$$

2. *Eigenvectors: The eigenvectors of A corresponding to λ are the* ***nonzero*** *solutions of*

$$(\lambda I - A)\boldsymbol{x} = \boldsymbol{0}$$

Definition 1. *Characteristic Equation*
The equation $det(\lambda I - A) = 0$ is called the ***characteristic equation*** *of A. The eigenvalues of an n x n matrix correspond to the roots of the characteristic polynomial. The characteristic equation is a polynomial of degree n in the variable λ. (We will only consider the real roots of the equation and therefore only the real eigenvalues.)*

$$|(\lambda I - A)| = \lambda^n + c_{n-1}\lambda^{n-1} + \ldots c_1\lambda + c_0$$

Theorem 2. *Eigenvectors of λ form a Subspace*

If A is an n x n matrix with an eigenvalues λ, then the ***set*** *of all eigenvectors of λ together with the zero vector is a subspace of R^n. This subspace is called the* ***eigenspace*** *of λ.*

$$\{\boldsymbol{0}\ \} \cup \{\boldsymbol{x} \mid \boldsymbol{x} \text{ is an eigenvector of } \lambda\}$$

Example 3. *Find the eigenvalues and corresponding eigenvectors of A.*

$$A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}$$

a) <u>Characteristic Polynomial</u>:

b) <u>Eigenvalues</u>:

c) <u>Eigenvectors</u>:

d) <u>Eigenspace</u>:

Theorem 3. *Eigenvalues of a Triangular Matrix*
If A is an n x n matrix, then its eigenvalues are the entries on its main diagonal.

Note: The proof follows from the fact that the determinant of a triangular matrix is the product of its diagonal elements. (If necessary, recall the rules of determinants from Chapter 3.)

Example 4. *Find the eigenvalues for the following matrix.*

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 5 & 3 & -3 \end{bmatrix}$$

Definition 2. *Diagonal Matrix*
A matrix that is both upper and lower triangular is called a ***diagonal*** *matrix.*

Eigenvalues and Eigenvectors of Linear Transformations

Eigenvectors and eigenvalues can also be defined in terms of linear transformations.

Definition 3. *Let V be a* ***finite*** *dimensional vector space and let $T : V \to V$ be a linear transformation. Then λ is an eigenvalue for T if there exists a non-zero vector $\boldsymbol{x}$ in V such that*

$$T(\boldsymbol{x}) = \lambda \boldsymbol{x}$$

Let $T : R^3 \to R^3$ whose matrix relative to the standard basis is:

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Then the matrix of T relative to the nonstandard basis B' is a diagonal matrix. (From Sec 6.4)

$$A' = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \qquad B' = \{(1,1,0),(1,-1,0),(0,0,1)\}$$

Goal: For a given transformation T, find a basis B' whose corresponding matrix is diagonal.

Example 5. *Find eigenvalues and corresponding eigenspace of*

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

a) Characteristic Equation:

b) Eigenvalues:

c) Basis for Eigenspaces:

Surprising Results from Previous Example

- Let $T : R^3 \to R^3$ be the standard matrix whose matrix is A.
- Let B' be the basis of R^3 made up of the three linearly independent eigenvectors.
- Then the transition matrix of T relative to B' is diagonal.
- The eigenvalues of A are the main diagonal entries of the matrix A'.

$$A' = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Eigenvalues of A

Nonstandard basis:

$$B' = \{(1, 1, 0), (1, -1, 0), (0, 0, 1)\}$$

Eigenvectors of A