7.2 Predicting the Future in an Uncertain World

How to Measure Uncertainty Using the Idea of Probability

Le Tricheur à l'As de Carreau (1635) by Georges de la Tour. (The Cheater with the Ace of Diamonds.) *Watch those hands!* (The Louvre, Paris. Erich Lessing/Art Resource, NY)

> *To be, or not to be . . .*
>
> SHAKESPEARE

What will be? How can we cope with the unknown—the uncertain future or unpredictable present? Some seek insight from tea leaves, the stars, or the entrails of sheep. Some gaze deeply into the translucent beauty of a crystal ball. Let's not. Instead, let's gaze deeply into the powerful world of transcendent ideas and take our vague view of the future and give it some structure. That is to say, let's construct a means to measure the possibilities for a future we cannot know. Quantifying the likelihoods of various uncertain possibilities is an impressively grand idea. How can we sensibly measure what we admit is unknowable?

We adopt a strategy used in previous investigations. We have already confronted numerous mysteries, including infinity and the fourth dimension. We uncovered their secrets by first understanding basic ideas deeply. Clarifying fundamental ideas enabled us to effectively develop precise notions and led us to new discoveries. Now we wish to delve into the uncertain and the unknown, so we seek examples where we have an intuitive sense of how to measure the likelihood of a future event. We look

Understand simple things deeply.

at those examples with the goal of finding patterns and techniques that can be applied more broadly. A careful examination of our intuition often leads to new insights and discoveries.

Likelihood in Everyday Life

The notion of likelihood is a major component of our everyday lives. How likely is it that a certain scenario will actually happen? What are the chances? As we will continue to see, often the answers to such everyday questions are surprising and counterintuitive.

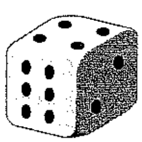
Tomorrow it will either snow or not snow. Does this fact imply that there is a 50–50 chance of snow? If we are reading this book in Hawaii in June, then we would not expect it to snow tomorrow. If, however, we are reading this book in Buffalo, New York, in June, then the answer is less clear. The point is, there is certainly a *chance* of snow, but is it *likely* to snow?

An amazing number of our actions and decisions are based on an intuitive sense of likelihood. In fact, "likelihood" often provides the foundation for what we think of as "common sense." Here is just a sample of some everyday issues and questions involving likelihoods, risks, and chances:

- Do you go to the dining hall for lunch the moment classes let out, or do you wait because you expect there to be long lines?
- While walking home at night, do you take the shortcut through the dark alley, or do you stick to the well-lit sidewalk? Why?
- When you are driving on a four-lane highway and are about to pass a car, you usually assume that the car will remain in its lane. Would you pass a car that was swerving in and out of its lane?
- Why do so many sexually active people practice safe sex?
- There are no nearby parking spaces. Do you park illegally in front of a store to run in for 5 minutes and take the chance of getting a ticket? What about 15 minutes? An hour?

Knowing how the future will unfold would be extremely valuable. We'd know which number will come up on the roulette wheel, which stock will skyrocket, and which numbered Ping-Pong balls will bubble out of the Lotto machine. We are constantly attempting to predict what will happen in our lives and act accordingly. To develop a measurement of likelihood, let's find some simple, concrete situations in which the future, though uncertain, presents clear, quantifiable alternatives.

On a Roll—the Measure of Likelihood


What are the chances?

Games of chance provide basic examples where the measurements of likelihood are reasonably clear. So let's measure uncertainties in the high-rolling domain of dice. Suppose we have an ordinary die with sides numbered from 1 to 6, and suppose it is a *fair die,* which means that no particular side is more likely to be rolled than any other. If we roll the die, what is the probability of rolling a 4? In other words, what number would you associate with the likelihood of a 4 coming up if you rolled a die once?

Probably you came up with 1/6. Why is 1/6 the probability of rolling a 4? Well, there are six possible outcomes of rolling a die. We could roll a 1, a 2, a 3, a 4, a 5, or a 6. All of these outcomes are equally likely with a fair die. Exactly one of these outcomes (rolling a 4) is the outcome whose likelihood we are assessing. Thus, there is exactly one way out of the six equally likely possible outcomes to roll a 4, and so there is a 1 in 6 chance of rolling a 4. The number 1/6 captures the idea that rolling a 4 is *one* of the *six* equally likely outcomes that are possible when the die is rolled.

Let's put our intuition to the test and experiment by rolling a die a bunch of times and recording the outcomes. We did some experimenting with 100 rolls of a die. Here are our results.

Number Appearing on Die	1	2	3	4	5	6
Times Rolled (out of 100)	18	16	20	17	15	14

We see that 17 out of our 100 rolls were a 4. Thus, 17/100 (or 0.17) of the time we saw a 4. This experiment seems to support our thinking that the probability of rolling a 4 is 1/6 or 0.1666. . . .

Let's now try to determine the probability of an even number appearing on the rolled die. As before, there are a total of six possible outcomes, each equally likely. However, now more than one outcome would lead to success (an even number): We could roll a 2, a 4, or a 6. Thus, there are a total of three different ways of rolling an even number. If we divide the total number of ways of rolling an even number by the total number of possible outcomes, we have 3/6, which equals 1/2, or 0.5. So the probability of rolling an even number is 1/2. This answer makes sense because half the numbers on the die are even; therefore, half the time we would expect to roll an even number.

A Measure of Likelihood—Probability

The concept of dividing the number of successful outcomes by the total number of possible outcomes provides us with a measure of likelihood. Notice that this fraction will always be a number between 0 and 1, where the closer this fraction is to 1, the greater our confidence that the successful outcome will actually occur, and the closer the fraction is to 0, the lower our confidence that the successful outcome will occur. Let's extend this concept of measuring likelihood into a precise definition.

Suppose we perform a certain activity in which there are only finitely many possible outcomes, any one of which is just as likely as any other to occur. Now we focus our attention on a specific collection of those outcomes (in the case of rolling a die, for example, we could think of the outcomes 2, 4, 6). A collection of outcomes is called an *event*. We then define the *probability* *of a particular event* to be the number of different outcomes in that particular event divided by the total number of possible outcomes. Let's say that again. Suppose that a certain activity (say, rolling a die) will result in a total of T possible outcomes, all of which are equally likely to occur (for rolling an ordinary, 6-sided fair die, T would be 6). We now consider a specific event, which we'll call E. (For instance, E might be the even numbers.) If we know that there are N different outcomes in the event E (in this case, N would equal 3, since there are three different outcomes that are even numbers), then we define the probability of the event E to be the number N/T. (In our example, this probability would be 3/6, or just 1/2.) So,

Probability.

The probability of the event E occurring =

$$\frac{N}{T} = \frac{(\textit{number of different outcomes in } E)}{(\textit{total number of equally likely outcomes})}.$$

Notice that N is some number from 0 to T. Therefore, the smallest the probability could be is $0/T = 0$, and the largest the probability could be is $T/T = 1$. Observe that the larger the probability of an event, the more likely it is that an outcome in the event will occur.

Relative Frequency

As we repeat an experiment again and again, we can keep track of the number of times a particular outcome occurs. We can calculate the *relative frequency* of that particular outcome by dividing the number of times a particular outcome occurred by the number of times we repeated the experiment. In other words:

Relative Frequency.

Relative frequency of an outcome =

$$\frac{(\textit{the number of times that outcome occurred})}{(\textit{the total number of times the experiment was repeated})}.$$

For example, in our first die-rolling experiment, we saw that a 4 appeared in 17 out of 100 rolls. So, the relative frequency of rolling a 4 in this repeated die-rolling experiment is 17/100, which equals 0.17. The probability of rolling a 4 is equal to $1/6 = 0.1666\ldots$. Notice how close 0.17 is to $0.1666\ldots$. It seems reasonable that the more times we repeat an experiment and compute the relative frequency of an outcome, the closer that frequency should be to the actual probability of that outcome. This insight is known as the *Law of Large Numbers.*

Law of Large Numbers.

If an experiment is repeated a large number of times, then the relative frequency of a particular outcome will tend to be close to the probability of that particular outcome.

The Birth of Probability

Probability started with dice. The French nobleman Antoine Gombauld, the Chevalier de Méré, was a famous 17th-century French gambler. He loved dice games. One of his favorites was betting that a 6 would appear at least once in 4 consecutive rolls of a die. After some time, Gombauld became bored with this game of chance and devised a new game by scaling up from one die to two dice. In the new game, he bet there would be at least one pair of 6's in 24 consecutive rolls of a pair of dice. Soon he noticed that he tended to lose with his new game. Bothered by this discovery, in 1654, Gombauld wrote a letter to the French mathematician Blaise Pascal, who, in turn, mentioned this problem to Pierre de Fermat. The two mathematicians solved the mystery. They computed that the probability that a 6 would appear at least once in 4 consecutive rolls of a die is equal to 0.52. Since this probability is slightly greater than 0.5, over the long run, Gombauld would win slightly more often than he would lose. However, the probability of seeing at least one pair of double 6's in 24 consecutive rolls of a pair of dice is equal to 0.49. (We will verify both of these probabilities ourselves in Section 7.4.) Since this probability is slightly less than 0.5, Gombauld would lose more, on average, than he would win.

This observation by the gambler Gombauld and the answer given by Fermat and Pascal led to the birth of the study of probability. You may be amused by Pascal's view of humanity. In a letter to Fermat referring to Gombauld, Pascal wrote:

> *He is very intelligent but he is not a mathematician; this as you know is a great defect.*

Analyzing Our Chance Surprises

Armed with the ideas of probability and relative frequency, let's take another look at the chance surprises from the previous section, as well as some additional dicey issues.

All Boys?

The two reunion scenarios from the last section asked what we can deduce from the slightly different dialogues: (1) her older child is a boy versus (2) at least one of her children is a boy. In each case, what is the probability that the speaker has two boys? Our analysis must begin with a careful listing of the possibilities.

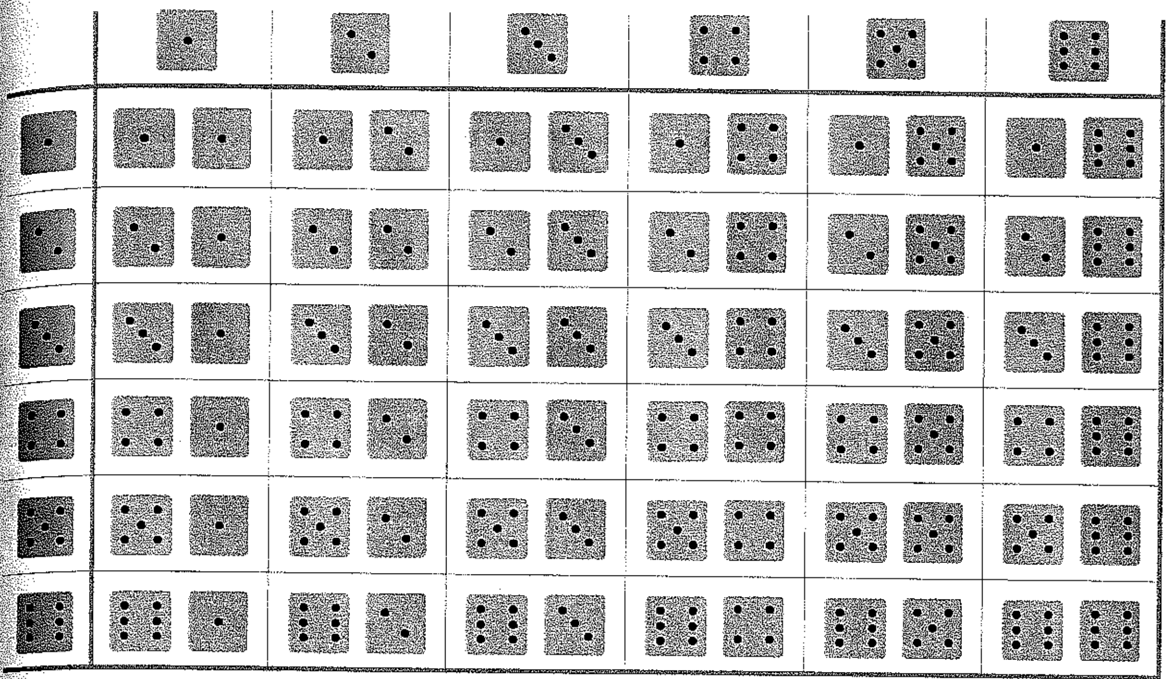
A person with two children may have had first a boy then a girl, first a girl then a boy, first a boy then a boy, or first a girl then a girl. These are the four equally likely ways to have two children. Enumerating these possibilities helps us analyze the scenarios; whereas if we rely on vague intuition, we could easily be led astray.

In the first scenario, we know that her older child is a boy. So, either she had first a boy then a girl or first a boy then another boy. Thus, the probability of her having two sons is 1/2. However, in the second scenario, we know that the three possibilities are first boy then girl, first girl then boy, or first boy then boy. In only one of these three equally likely possibilities would she have two sons. Therefore, the probability of her having two boys is only one in three, or 1/3. The number of equally likely possibilities—3—divided into the number of those we are interested in—1—gives the probability.

More Dicey Issues

Given the genesis of probability, it seems only fitting that we roll some more dice. Suppose we now roll *two* fair dice. What is the probability of rolling "snake eyes" (rolling a sum of 2)?

Well, there's only one way of getting the numbers on the two dice to add up to 2: Each die must be showing a 1. So, there is only one outcome to yield "snake eyes." We now need to figure out the total number of possible outcomes. A reasonable guess is 11, because when we roll two dice, we see a total of 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, or 12. The trouble with this guess is that not all these outcomes are equally likely. For example, we have already seen that there is only one way to roll a 2: Each die must be showing a 1. However, there are two different ways of rolling a 3: The first die could be a 1 and the second die could be a 2; *or* the first die could be a 2 and the second die could be a 1. These two possibilities are different outcomes. To get a clear picture of all the possible outcomes, it is better to color the dice different colors to distinguish one from the other. The table on page 589 illustrates all the possible outcomes of rolling two dice, a red one and a green one. Notice that there are a lot more than 11 different outcomes.



Notice that there are 36 possible, equally likely outcomes. Since only one of them produces "snake eyes," we conclude that the probability of rolling "snake eyes" is 1/36, which equals 0.0277. . . . This probability is pretty small, so we would not expect to see "snake eyes" too often. What is the probability of rolling a total of 4? What is the probability of rolling a total of 13? What is the probability of rolling a total of 7? Figure out these probabilities using the table.

The Probability of Success versus Failure

Let's think about the probability of an event *not* happening. What is the probability of rolling two dice and getting anything other than a total of 7? There are 30 outcomes that do not give us 7. Thus, the probability of not rolling 7 is 30/36, which equals 5/6 = 0.8333. . . . Therefore, it is likely that we will not roll a total of 7. How does this answer relate to the probability of rolling 7? Do you notice an interesting connection between these two probabilities? The probability of rolling a 7 is 6/36 = 1/6 = 0.1666. . . . When we add the probability of rolling a total of 7 to the probability of not rolling a total of 7, we get exactly 1. Take a few moments to extend this observation into a general principle. Once you have formulated a specific idea of the relationship between the probability that an event will happen and the probability that an event will not happen, continue reading.

When we roll a pair of dice, there are 36 equally likely outcomes. Six of these outcomes add up to 7, and the other 30 outcomes add up to something other than 7. Thus, the 36 total outcomes can be divided into the successes (6 outcomes) and the nonsuccesses (30 outcomes). So, the probability of getting 7 (6/36) plus the probability of not getting 7 (30/36) must add up to 1 (36/36). This insight lets us find the probability of an event if we know the probability that the event *won't* happen. The probability that an event E will happen is equal to 1 minus the probability that the event E will not happen. This relationship may be stated as:

It Either Happens or It Doesn't.
The probability that the event E will *happen = 1 − (the probability that the event* E will not *happen).*

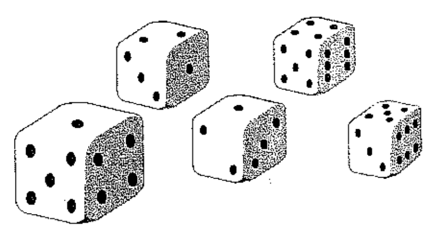
So, the probability of something happening can be found easily if we know the probability that the thing will not happen. Often, as we will discover, computing the probability that something will not happen is easier than finding the probability that it will happen.

What is the probability of rolling a pair of dice and seeing numbers whose sum is greater than 3? We could count all the entries in the chart where the sum exceeds 3, or we could count the opposite outcomes: those rolls whose sum is less than or equal to 3. There are only three such outcomes: 1 and 1; 1 and 2; 2 and 1. Therefore, it is easy to see that the probability of *not* rolling a sum that *is* greater than 3 is 3/36. So, by our previous observation, the probability of rolling a sum that *is* greater than 3 equals $1 - 3/36$, which equals $33/36 = 11/12 = 0.91666\ldots$. This high probability indicates that it is very likely that we will roll a sum greater than 3. This example also illustrates the power of looking at an issue in a different way. Looking at a situation from another perspective may lead to an easy and elegant solution.

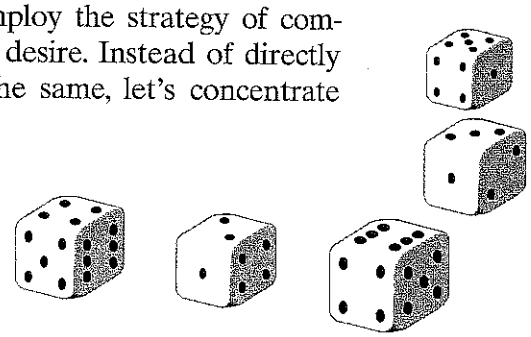
Yahtzee

Maybe you have played the game Yahtzee, which is a dice-lover's dream since it involves rolling five dice at once. Players score points when several dice reveal the same number, so let's answer the following question:

The Yahtzee Pair Question.
What is the probability of rolling five dice and getting at least one matched pair; that is, having at least two of the dice showing the same number?



This question gives us a great excuse to employ the strategy of computing the opposite probability from what we desire. Instead of directly computing the probability of getting a pair the same, let's concentrate on computing the probability of *not* getting a matched pair, that is, the probability of rolling five dice and seeing five different numbers. We know that 1 − (the probability of not getting any matched pair) = (the probability of getting at least one matched pair). Since five dice are a lot to shake at once in our minds, let's instead start not as a high roller but as a low roller.



Starting with Two Dice

Let's first consider the warm-up challenge of finding the probability that rolling two dice will give a pair of the same number. The first die will come up with some number, so the question could be rephrased: What is the probability that the second die shows the same number as the first? Well, there are six possible numbers that can come up on the first die, and for the second die to match the first, it must land on that one and only number that agrees with the first die. So, the probability of a match is 1/6. To check our reasoning, let's again compute the probability, but this time let's use the chart that we used for the previous dice-rolling experiment.

Which entries on that chart correspond to the two dice being the same? The entries along the diagonal starting at the upper-left corner have both numbers the same. There are six entries on that diagonal, so the probability of the two dice being the same is $6/(6 \times 6)$, which again, luckily, equals 1/6. We're definitely on a roll!

Not Different Numbers

It will be useful to consider a different point of view for computing the probability of rolling two dice and seeing the same number. This time, let's first find the probability of the opposite outcome—the probability that the two dice do *not* show the same number. It is this strategy that will enable us to solve the original Yahtzee Pair Question.

Looking at a situation in a new way may lead to an easy solution.

As before, there are 6×6 different possible, equally likely outcomes from rolling two dice. How many of these 36 possible outcomes produce different numbers? There are six possible numbers for the first die. However, once that first number is known, the second die must avoid that particular number like the plague, thus leaving only five possible numbers to ensure that the two dice are different. So, for every one of the six possible numbers for the first die, we have five possible numbers for the second die (all the numbers except the first die's number). This gives a total number of $6 \times 5 = 30$ outcomes in which the two numbers are different. Therefore, the probability of rolling two dice showing different numbers is

$$\frac{6\times 5}{6\times 6} = \frac{5}{6} = 0.83333\ldots.$$

Using our previous relation between an event happening and it not happening, we conclude that the probability that two dice come up with the same number is equal to

$$1-\frac{5}{6} = \frac{1}{6} = 0.16666\ldots,$$

yet again confirming our previous computations. We're on our way to a winning streak.

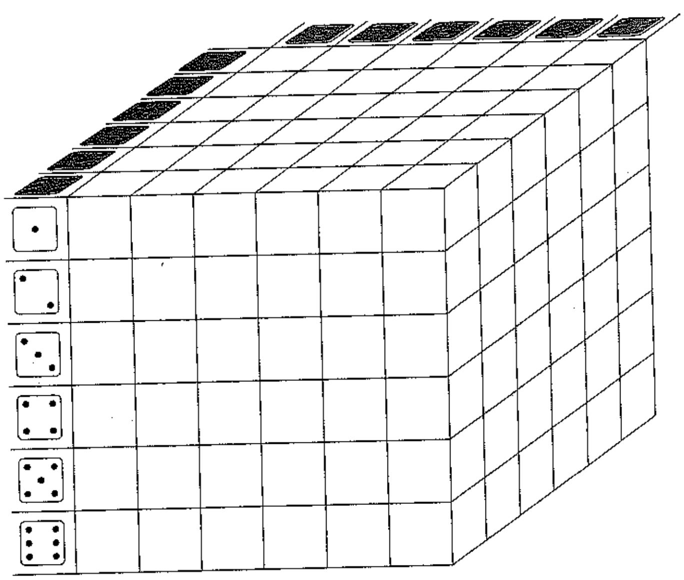
Another Die

Look for patterns.

So, the probability that two rolled dice will each land on the same number is fairly low: 1/6. What if we roll three dice? What is the probability that two or more of the three come up with the same number? We examine this new situation with an eye toward finding a pattern that will allow us to answer the question for larger numbers of dice.

Let's again consider the opposite outcome; namely, that all three dice show different numbers. Let's first ask: How many possible triples of numbers are possible outcomes for rolling three dice? There are six possibilities

A 3-dimensional table of all possible outcomes of rolling three dice (not filled in!)



for the first die, six possibilities for the second die, and six possibilities for the third. We can imagine creating a cubical graph to record each of those triple numbers. Instead of just a red die and a green die, we can add a yellow die. Therefore, there must be

$$6 \times 6 \times 6 = 216$$

possible triples of numbers that could arise when rolling three dice.

How many of these triples have the property that all three numbers are different? Theoretically, we could just look at our cubical chart and count; however, that would be difficult to see in practice. So instead let's think. The first die can show any number from 1 to 6, so there are six possibilities. To be different from the first, the second die can show any number *except* for the number of the first die, so the second die has five allowable numbers. The third die has to avoid the numbers of both the first and the second die, which leaves four allowable numbers. How many in all? Well, for each of the six possible numbers for the first die, we have any one of five possible numbers for the second die, and for each of those combinations, we have four possible numbers for the third die. Thus, there would be

$$6 \times 5 \times 4 = 120$$

possible triples of numbers in which all three numbers are different.

So, when rolling three dice, the probability of getting three different numbers is equal to

$$\frac{6 \times 5 \times 4}{6 \times 6 \times 6} = 0.55555\ldots.$$

Therefore, the probability of the opposite (having at least two dice out of the three share the same number) is

$$1 - \frac{6 \times 5 \times 4}{6 \times 6 \times 6} = 0.44444\ldots.$$

Can you now determine the probability of a match if four dice are rolled?

Try It The previous reasoning can be used to show that the probability of getting a pair of matched numbers if four dice are rolled is equal to

$$1 - \frac{6 \times 5 \times 4 \times 3}{6 \times 6 \times 6 \times 6} = 0.72222\ldots.$$

We've Got the Pattern

We now see the pattern. We can now answer the Yahtzee Pair Question completely. The probability of seeing at least two equal numbers when rolling five dice is 1 minus the probability of seeing five different numbers when rolling five dice. So the probability of seeing at least two equal numbers when rolling five dice is:

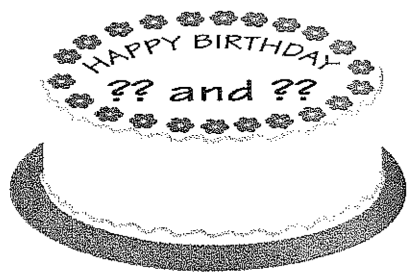
$$1 - \frac{6 \times 5 \times 4 \times 3 \times 2}{6 \times 6 \times 6 \times 6 \times 6} = 0.907407407407\ldots$$

This analysis shows us that when we roll five dice, more than 90% of the time, we will get at least one pair (pretty likely!). Not only do we now see the pattern that solved the Yahtzee Pair Question, but we have also developed all the probability ideas required to answer the Birthday Question.

When Must a Pair Share a Birthday Cake?

We now return to the seemingly straightforward, harmless question posed in the previous section:

> **The Birthday Question.**
> *How many people are needed in a room so that the probability that there are at least two people whose birthdays are the same day is roughly one-half?*



Let's make the assumption that it is equally likely to be born on one day as on any other—no day is more or less popular for celebrating birthdays. That is, the probability that someone is born on any given day, say December 9, is 1/365, since there are 365 days in the year (let's pretend we never leap) that are all equally likely candidates for one's birthday, and exactly one of them is December 9. Our strategy for answering this Birthday Question is to follow the path of ideas laid out in answering the Yahtzee Question.

Starting with Small Crowds

Let's first find the probability that two people share the same birthday. The first person has some birthday, so the question could be rephrased: What is the probability that the second person has the same birthday as the first person? Well, there are 365 possible days for a birthday, so the probability of that happening is 1/365. To check our reasoning, let's again compute the probability, but this time let's use a chart, as we did for the dice-rolling experiment.

How many different possible pairs of dates are there for the birthdays of two people? Well, there are 365 possibilities for the first person and 365 possibilities for the second person. We could make a huge chart similar to the two-dice chart, only this one would have 365 rows and 365 columns. Making the chart would be difficult, but figuring out how many outcomes are represented in the chart is easy: $365 \times 365 = 133{,}225$. Which entries in that chart correspond to the two people having the same birthday? The entries along the diagonal starting at the upper-left corner have both dates the same (starting with January 1 and ending with December 31). There are 365 entries on that diagonal, so the probability of the two people having the same birthday is $365/(365 \times 365)$, which again, much to our delight, equals $1/365$.

Different Birthdays

Now let's first find the probability of the opposite outcome—the probability that they do *not* share the same birthday. It is this strategy that will enable us to solve the original Birthday Question.

Looking at a situation in a new way may lead to an easy solution.

As before, there are 365×365 different pairs of birthdays. How many of these 133,225 possible outcomes produce a pair of different birthdays? There are 365 possible and allowable birthdays for the first person. However, once that person's birthday is known, the second person must avoid that particular date as one would avoid stale birthday cake. So, for every one of the 365 possible dates for the first person, we have 364 possible dates for the second person (all the dates except the first person's date). This gives a total number of $365 \times 364 = 132{,}860$ pairs of dates in which the two dates differ. Therefore, the probability of two people having different birthdays is

$$\frac{365 \times 364}{365 \times 365} = \frac{364}{365} = 0.9972\ldots$$

Using our previous relation between an event happening and it not happening, we conclude that the probability that two people have the same birthday is equal to

$$1 - \frac{364}{365} = \frac{1}{365} = 0.00273\ldots,$$

yet again confirming our previous computations.

A Few More People

Look for patterns.

In a room with only two people, the probability that they share a common birthday is extremely low. What if there were three people in the room? What is the probability that two or more of the three share the same birthday?

Let's again consider the opposite outcome, namely: All three people have different birthdays. Let's first ask: How many possible triples of dates are there for the birthdays of any three people? There are 365 possibilities for the first person, 365 possibilities for the second person, and 365 possibilities for the third person. Therefore, there must be

$$365 \times 365 \times 365 = 48{,}627{,}125$$

possible triples of dates.

How many of these triples have the property that all three dates are different? The first person's birthday can be any date, so there are 365 possibilities for that person. The second person's birthday can be any date, except for the date of the first person, so the second person has 364 possible dates. The third person has to avoid the dates of both the first and the second person, which leaves 363 possible dates. How many in all? As with the dice we multiply these numbers together to discover that there are

$$365 \times 364 \times 363 = 48{,}228{,}180$$

possible triple dates in which the three dates are different.

So, when we have three people, the probability that they have three different birthdays is equal to

$$\frac{(365 \times 364 \times 363)}{(365 \times 365 \times 365)} = 0.9917\ldots.$$

Therefore, the probability of the opposite (having at least two people out of three share the same birthday) is

$$1 - \frac{(365 \times 364 \times 363)}{(365 \times 365 \times 365)} = 1 - 0.9917\ldots = 0.0082\ldots.$$

Although this probability is still extremely small and nowhere near the 0.5 probability that we seek, we do notice that having a birthday match with three people is about *three times* as likely as with two people. Can you now determine the probability of a match if four people are in the room?

Try It The previous reasoning can be used to show that the probability of having a pair of matched birthdays among four people is equal to

$$1 - \frac{(365 \times 364 \times 363 \times 362)}{(365 \times 365 \times 365 \times 365)} = 0.01635\ldots,$$

which, although still nowhere near 1/2, is almost twice as large as the probability of finding a match among three people. We can continue to compute the probabilities in this manner. For example, a match among five people would have a probability of

$$1 - \frac{(365 \times 364 \times 363 \times 362 \times 361)}{(365 \times 365 \times 365 \times 365 \times 365)} = 0.0271.$$

We've Got the Pattern

We now see the pattern. If we continue for various numbers of people, we could produce the following chart.

Number of People in the Room	Probability of at Least Two Sharing the Same Birthday
5	0.027 . . .
10	0.116 . . .
15	0.252 . . .
20	0.411 . . .
25	0.568 . . .
30	0.706 . . .
40	0.891 . . .
50	0.970 . . .
60	0.994 . . .
70	0.9991 . . .
80	0.99991 . . .
90	0.999993 . . .

It is truly surprising how quickly the probability heads toward 1. With only 50 people, it is almost a sure thing that there will be a match. With 90 people, we are essentially 100% confident of a match; yet 90 is a far cry from 366 people, which guarantees a match for sure. We also have an answer to our Birthday Question: The probability of a birthday match with 23 people is 0.5072. . . .

Retraining Our Intuition

If our intuition leads us astray, we need to look at the situation in different ways until not only our reason but also our intuition is convinced.

Why is the actual answer of 23 people so much lower than we first guessed? When dealing with many small probabilities simultaneously, our intuition often does not accurately correspond to reality. Before thinking about the Birthday Question, our intuition was probably influenced by some simple yet wrong reasoning. We might have reasoned that because 366 people are required to guarantee that two people will share the same birthday, then 183 people will be required for a 0.5 probability. Now we see that such reasoning is far from correct. Somehow, to make this birthday principle real to us, we must retrain our intuition.

A helpful technique in retraining the intuition is to try the birthday experiment in several actual gatherings to see that, in fact, pairs of people will share the same birthday. Another approach is to examine situations similar to the Birthday Question and discover the answer in the new setting. We could try analogous experiments in other settings, such as with cards, to experience the underlying principles at work. In the Mindscapes we invite you to try several. The surprising answer to the Birthday Question illustrates the power of analyzing simple cases carefully and then seeing how the principles apply to a harder case.

The Birthday Question is just one of several instances where probability and everyday intuition diverge. The "Cool dice" that you saw in Chapter 1 and that appear in Mindscape IV.36 present another cool counterintuitive fact that you can share with your friends (and use to get rich). The four funky dice required for this exercise are included in your kit that accompanies this text.

Our entire discussion of probability is implicitly based on a concept known as *randomness*. What is randomness? Is it synonymous with unpredictability? We will either visit the subtle notion of randomness in the next section . . . or not.

A Look Back

PROBABILITY PROVIDES us with a quantitative method to analyze the uncertain and the unknown. It is a measure of the likelihood of an event, such as rolling a sum of 7 with two dice. For an activity (like rolling two dice) with only finitely many equally likely outcomes, the probability of a particular event is the number of different outcomes in that particular event divided by the total number of possible outcomes. So the probability of rolling a sum of 7 is $6/36 = 1/6$. Using this basic definition and careful analysis, we can understand many probabilistic situations, some leading to surprising results. Perhaps the most famous and surprising example of unexpected probability is the Birthday Question, whose answer is that, in a group of 23 people, it is slightly more likely than not that two of them have the same birthday.

We develop ideas about probability by starting with familiar situations in which the probabilities are intuitively clear—for example, the simple cases of rolling dice. From those examples, we extrapolate the basic idea of probability. We then formulate a specific definition. Finally, we explore consequences of that definition and discover some surprising results.

Carefully analyzing simple and familiar events opens the door for us to understand more complex and puzzling situations. We can more easily see patterns and develop insights from simple and clear examples than we can from complex or muddy examples. So, focusing on the simple and familiar allows us to concentrate on uncovering the essential principles.

Analyze simple things deeply.

•

Deduce general principles.

•

Apply them to more complex settings.

Mindscapes *Invitations to Further Thought*

In this section, Mindscapes marked **(H)** *have hints for solutions at the back of the book. Mindscapes marked* **(ExH)** *have expanded hints at the back of the book. Mindscapes marked* **(S)** *have solutions.*

I. Developing Ideas

1. **Black or white?** Your friend chooses his sartorial color scheme by putting all of his black and white T-shirts in a drawer, then closing his eyes and reaching into the drawer, and selecting a shirt. The probability that he wears a white T-shirt is 3/5. What is the probability that he wears a black T-shirt?
2. **Eleven cents.** You have a dime and a penny. Flip them both, noting whether each coin lands heads up or tails up. List all possible outcomes. Let E be the event that you get at least one head. List all the outcomes that give E. What is the probability that E occurs?
3. **Yummm.** You have a small bag of candy-coated chocolates that melt in your mouth; three are red, four are yellow, two are green, and five are blue. If you take a piece out of the bag at random, what is the probability it is green? What is the probability it is blue? What is the probability that you will eat it?
4. **Rubber duckies.** A game at a carnival has 75 rubber ducks floating in water. The ducks are numbered 1 to 75, with the numbers written on their undersides so they can't be seen. To play, you select a duck and see what number it has. If the number is less than 60, you win a consolation prize. If the number is at least 60 but less than or equal to 70, you win a stuffed duck. If the number is greater than 70 you win a giant, stuffed banana. What is the probability you win a stuffed duck? What is the probability you don't win a duck or a banana?
5. **Legally large.** What does the Law of Large Numbers assert?

II. Solidifying Ideas

6. **Lincoln takes a hit.** On your wall is a poster containing equal-sized pictures of each of the presidents of the United States. You take a dart, close your eyes, and throw it randomly at the poster. What is the probability that you will hit Lincoln?

7. **Giving orders.** Order the following events in terms of likelihood. Start with the least likely event and end with the most likely.

   - You randomly select an ace from a regular deck of 52 playing cards.
   - There is a full moon at night.
   - You roll a die and a 6 appears.
   - A politician fulfills all his or her campaign promises.
   - You randomly select the queen of hearts from a regular deck of 52 playing cards.
   - Someone flies safely from Chicago to New York City, but his or her luggage may or may not have been so lucky.
   - You randomly select a black card from a regular deck of 52 playing cards.

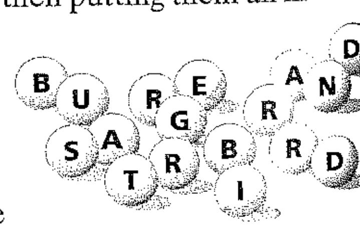
8. **Two heads are better.** Simultaneously flip a dime and a quarter. If you see two tails, ignore that flip. If you see at least one head, record whether you see one or two heads. Repeat this experiment 30 times. Calculate the number of double heads divided by 30. How close is this answer to the computed probability of having two boys in the second reunion scenario?

9. **Tacky probabilities.** Before doing the following experiment, think a bit, and then guess the probability and record it. Cup five identical, standard thumbtacks in your hands. Shake them, and then toss them slightly upward and let them fall onto a smooth, tiled floor. Count how many of the tacks land completely on their flat side and how many land resting against their points. Repeat this experiment 10 times, then use your data to estimate the probability of a tossed thumb tack landing point-side down.

Tack landing flat

Tack landing against the point

10. **BURGER AND STARBIRD.** Suppose you randomly select a letter from BURGER AND STARBIRD. Imagine writing these letters on Ping-Pong balls—one letter per ball—then putting them all in a barrel and removing one. What is the probability of pulling out an R? What is the probability of pulling out a B? What is the probability of pulling out a letter that appears in the first half of the alphabet? What is the probability of pulling out a vowel?



11. **Monty Hall.** Read and rework the *Let's Make a Deal* scenario from Chapter 1, "Fun and Games." Work through the solution. Next, find a friend and simulate the *Let's Make a Deal* situation, keeping track of the outcomes under the two possible strategies—the switch strategy and the stick strategy. Perform the experiment approximately 40 times and record the results. Do the experimental data accord with the analysis of the probabilities? You can simulate *Let's Make a Deal* by visiting the *Heart of Mathematics* Web site.

12. **7 or 11 (S).** What is the probability of rolling a sum of 7 or 11 with two fair dice?

13. **D and D.** You simultaneously flip a dime and roll a die. Make a table of all the possible outcomes. What is the probability of seeing Roosevelt and a 4? Suppose now that someone else flipped and rolled, did not show you the result, but reported that the die shows a 2. What is the probability that the dime is showing tails? Justify your answer.

14. **The top 10 (ExH).** Suppose you have 10 marbles. They are marked with the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, or 10. They are placed in a jar, and you reach in and select one. What is the probability that the number you select has a factor of 3? What is the probability that the number you select is a prime number? What is the probability that the number you select is even? What is the probability that the number you select is evenly divisible by 13?

15. **One five and dime (H).** Someone simultaneously flips a penny, a nickel, and a dime. Make a list of all the possible outcomes. What is the probability of seeing three presidents? What is the probability of seeing exactly two presidents? Suppose now that you do not see the outcome, but you are told that a president is showing. Now, what is the probability of seeing three presidents? Suppose, instead, that you are told that Lincoln is showing. What now is the probability of seeing three presidents? Why do the answers differ?

16. **Five flip.** Someone flips five coins, but you don't see the outcome. The person reports that no tails are showing. What is the probability that the person flipped five heads?

17. **Flipped out.** We take a coin and flip it 10,000,000 times (okay, we have a lot of time on our hands). We notice that 6,010,375 times it landed on heads. What do you suspect about the coin?

18. **Spinning wheel.** A roulette wheel has 36 spaces marked from 1 to 36, half of which are marked red and half black. In addition, there are two green spaces marked 0 and 00. What is the probability of the little ball landing on 13? What is the probability of it landing on a red spot?

19. **December 9.** Choose two people at random. What is the probability that they were both born on December 9?

20. **High roller (H).** Using two fair dice, what is the probability of rolling a sum that exceeds 4?

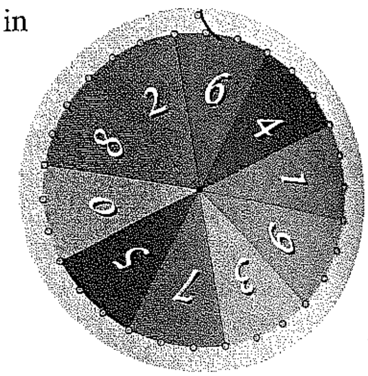
21. **Double dice.** You roll two fair dice. What is the probability you will roll a double (two 1's, two 2's, two 3's, and so on)?

22. **Silly puzzle.** After a professor explains the Birthday Question to her class of 20, she points out that the probability of having a birthday match in the class is around 0.4. A student raises her hand and states that she is certain that there will be a birthday match. She knows no one's birthday except her own. Explain why she was able to make this statement with such certainty.

23. **Just do it.** Find groups of roughly 35 people together (in a class, dorm, or dining hall) and have each person in turn shout out his or her birthday. Is there more than one pair of matches? Record your results.

24. **No matches (S).** Suppose 40 people are in a room. What is the probability that no two people share the same birthday?

25. **Spinner winner.** If you were to spin the wheel illustrated to the right, it is equally likely to stop at any point. You win if it stops on a space that is 6 or higher. Otherwise you lose. What is the probability of winning?



III. Creating New Ideas

26. **Flip side (S).** Someone flips three coins behind a screen and says, "I flipped at least two heads." What is the probability that the flipper flipped three heads?

27. **Other flip side.** Someone flips three coins behind a screen and says, "I didn't flip all tails." What is the probability that the flipper flipped all three heads?

28. **Blackjack.** From a regular deck of 52 playing cards, you turn over a 5 and then a 6. What is the probability that the next card you turn over will be a face card?

29. **Be rational (ExH).** Suppose someone has randomly selected two numbers from the set of the first one million natural numbers and used them to make a fraction. Reduce the fraction to its lowest terms. Is there a 0.5 probability that both the numerator and the denominator are odd numbers? Why or why not?

30. **Well red (H).** Someone shows you three cards. One is red on both sides, another is blue on both sides, and the last is red on one side and blue on the other. The cards are shuffled, and you are then shown one side of one card. You see red. What is the probability that the other side is blue? Is it 0.5? Explain.

31. **Regular dice.** Dungeons and Dragons players use dice in the shape of each of the regular solids (see Section 4.5). The faces are always numbered 1 through the number of total faces there are. You shake all five dice. What is the probability of your throwing a total of 6?

32. **Take your seat.** You decide to fly to California on EconoJet Airlines. You are randomly assigned a seat. Seats are numbered by row from 1 to 40 and in each row by A, B, C, or D, and amazingly, there is only one window seat in each row. The plane is boarded from the rear in groups of 10 rows at a time. What is the probability that you will be in the first group to board the plane? What is the probability that you get a window seat?

33. **Eight flips.** What is the probability of flipping a half dollar eight times and a head appearing at least once?

34. **Lottery (S).** The lottery in an extremely small state consists of picking two different numbers from 1 to 10. Ten Ping-Pong balls numbered 1 through 10 are dropped in a fish bowl, and two are selected. Suppose you bet on 2 and 9. What is the probability that you match at least one number? What is the probability you match both numbers?

35. **Making the grade.** What is wrong with the following statement? "The way I figure, the probability I get a 4.0 average this term is 0.2. The probability I get below a 4.0 average this term is 0.9." Explain. Given the statement, guess the person's actual GPA.

IV. Further Challenges

36. **Cool dice (ExH).** Find the four dice from your kit (see the kit instructions). Show that, no matter which die someone picks, there is always another die such that the probability of rolling a higher number on the second die is greater than 0.5. That is, there is no best die. Order the dice A, B, C, D such that B beats A, C beats B, D beats C, and A beats D. This phenomenon is like a conference of sports teams in which B generally beats A, C generally beats B, and D generally beats C, but, because D has some weakness that one of A's strengths can take advantage of, A generally beats D. Play this dice game with several friends. What is the probability that the die with 6's and 2's will beat the die with 4's and blanks?

37. **Don't squeeze.** Five shoppers buy Charmin toilet paper. One Charmin out of 10 in this batch is defective—it's unsqueezable. You want to save everyone from this catastrophe, so you stop them at the door and ask to squeeze their Charmin. After squeezing 5 rolls, what is the probability that you have located 1 or more defective Charmins?

38. **Birthday cards.** Using a regular deck of 52 playing cards, select a card at random, record it, and then put it back in the deck. Shuffle the

cards and select another card at random and record it, put it back, and so on. How many cards do you draw before you select the same card for the second time? Do this experiment several times. Calculate the probability of choosing 10 times and seeing 10 different cards.

39. Too many boys. Long, long ago and far, far away, an emperor believed that there were too, too many males and not enough females. To correct this wrong, the emperor decreed that, as soon as a woman gave birth to a male child, she would not be permitted to have any more children. If the woman gave birth to a female, she would be allowed to continue bearing children. What was the result of this decree? After the decree, what fraction of the babies will be male? Carefully explain your answer.

40. Three paradox (H). The correct probability of tossing three coins and having all three landing the same is 1/4. What's wrong with the following dubious reasoning?

> When we toss three coins, we know for a fact that two of the coins will land the same; therefore, we only have to get the third one to match. Thus, the probability is 1/2.

V. In Your Own Words

41. Personal perspectives. Write a short essay describing the most interesting or surprising discovery you made in exploring the material in this section. If any material seemed puzzling or even unbelievable, address that as well. Explain why you chose the topics you did. Finally, comment on the aesthetics of the mathematics and ideas in this section.

42. With a group of folks. In a small group, discuss and work through the details involved in the answer to the Birthday Question. After your discussion, write a brief narrative describing the reasoning in your own words.

43. Creative writing. Write an imaginative story (it can be humorous, dramatic, whatever you like) that involves or evokes the ideas of this section.

44. Power beyond the mathematics. Provide several real-life issues—ideally, from your own experience—that some of the strategies of thought presented in this section would effectively approach and resolve.

7.

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