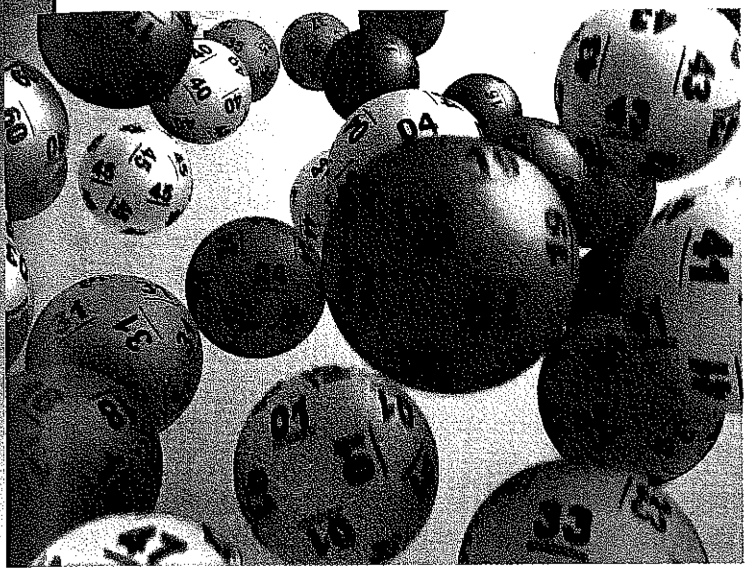
# 7.4 Down for the Count

## *Systematically Counting All Possible Outcomes*



Did you win?

> ***No priest or soothsayer that ever lived could hold his own against old probabilities.***
>
> OLIVER WENDELL HOLMES

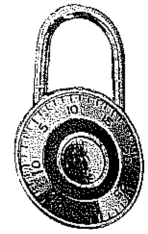
Let's count. To determine probabilities, we often have to do some serious counting. Since probability is often a fraction of the number of favorable outcomes divided by the total number of all possible outcomes, we must count how many outcomes are involved, and that's not always so easy. Perhaps you are thinking that you learned how to count in kindergarten, so you'll just skip this section. It's true that, when we were children, we learned to count one at a time, and that is a simple task for small collections of objects. But when we count big, complicated collections, such as how many lottery outcomes are possible, we find ourselves perplexed and prone to error.

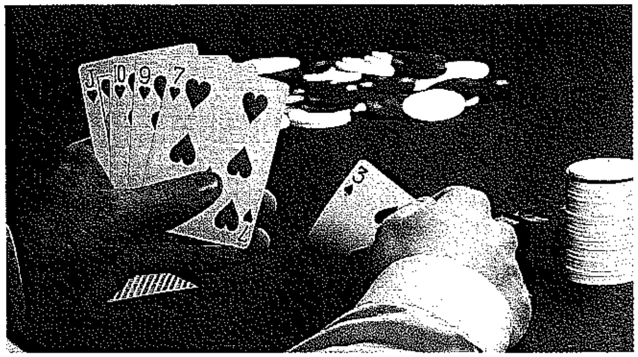
In principle, counting a collection of objects is easy. All we must do is

- count every object, and
- avoid counting the same object more than once.

These two rules sound so easy and obvious that we might think that counting is a piece of cake, but let's list a few things to count that might convince us that counting is not, in fact, a dessert item.

How many lock combinations are possible in a standard padlock? How many passwords are possible for your e-mail account? How many different poker hands are there? Experienced counters tend to group different counting scenarios into various categories, but, unless we are intending to become professional counters (known as *combinatorists*), sorting counting problems into a taxonomy of types can be a perilous enterprise. The perils center around the possibility of applying an incorrect counting method to a particular situation. Instead, let's explore some principles of analyzing questions on counting so that we can correctly analyze whatever case is at hand.

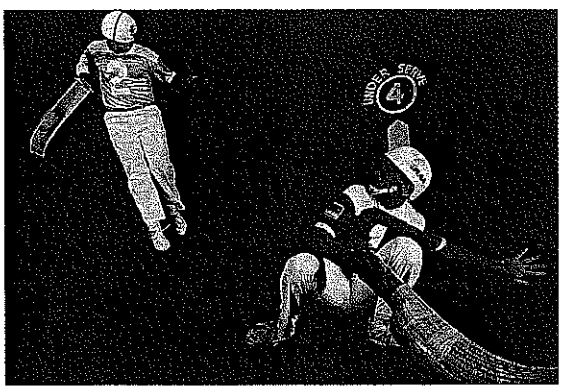




Darn!

## A Truly Merry Festival

In Florida, gambling is allowed on jai alai games. Jai alai (meaning *merry festival* in the Basque language) is the world's fastest ball game. The players use scoops (called *cestas*) to catch and throw the ball in one motion. During the day, six games are played in a type of round-robin play with eight players contesting each game. One can make a "super 6" bet for $2.00 by guessing who will win each of the six contests. Of course, the chance of anyone winning such a bet is rather small since one has to be correct on all six games. Thus, this type of wager is run like the lottery in that money that is bet is carried over into the jackpot day after day until someone finally wins.




I've got it!

On March 2, 1988, the pool of money at West Palm Beach Jai Alai reached an enormous amount, so one enterprising gambler decided not to gamble. He simply bet on every possible outcome, thus assuring himself of the prize money. Of course, there was the slight danger that some other person would also happen to win that day, in which case he would have had to share his bounty.

Our questions are as follows:

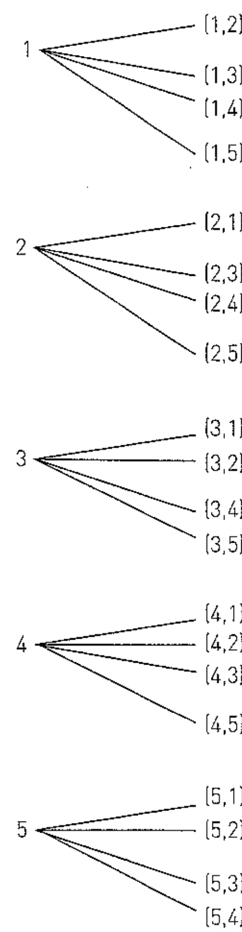
- If each bet costs $2.00, how big would the prize need to be to make it worthwhile to place every possible bet?
- How many bets would we have to make to be absolutely guaranteed of winning?

These questions present us with our first real counting issue. We have to pick the exact winner from each of the six contests. Naturally, we will have to guess all eight players as potential winners of the first game, because any one of them might win. So, for each of those we must choose all eight of the players of the second game. For each of those 64 patterns of potential winners in the first two games, we will have to choose all eight potential winners of the third game. So, there are $8 \times 8 \times 8$ potential sets of winners from the first three games. We see the pattern and conclude that there are $8^6$ different possible sets of winners for the six games, for a total of 262,144 possible bets costing a total of $524,288.

If the prize exceeds that amount, then we are sure to come out ahead. In the case of the West Palm Beach game in 1988, the payoff was $988,326 (minus $197,664 that the Internal Revenue Service deducted before issuing the check). Unfortunately, if someone else also wins, the prize must be shared, but fortunately for the Jai Alai bettors, they lucked out. This possibility of having to share the prize is what prevents wealthy people from actually buying every combination in lottery games whenever the prize exceeds the cost of buying every combination. Despite this nagging possibility, this method of making money by covering all the possibilities has been used throughout history. In 1729 the French writer François Voltaire used such a method to win the Parisian city lottery, and more recently, in 1992, a group of Australian investors essentially cornered the market on Virginia lottery tickets, winning about $27 million.

## Lottery

Let's dream about hitting the jackpot in the lottery and spending the remainder of our lives in the lap of luxury. In a typical lottery, we pay a dollar and choose six different numbers, each from 1 to 50; the order of the six numbers does not matter. If we guess all six, we win the big jackpot. How high would the lottery prize have to get before it would be worth our while to buy a ticket for each possible outcome, thus assuring ourselves the title of "winner"? Or, equivalently, how many collections of six different numbers from 1 to 50 are there?




Choosing six numbers from 50 is far too large a task to think about yet, so let's first examine a simpler task. Remember: When the going gets tough, the smart stop going and instead do something easy.

Rather than choosing six numbers from 50, let's figure out how many ways there are of choosing two numbers from 5. We could first list all pairs of numbers in all possible orders by writing down the pairs whose first number is 1, then 2, then 3, and so on.

It's easy to see how many of these ordered pairs we have, since for each of the five choices of first number there are four choices for second number; therefore there are five times four ordered pairs altogether.

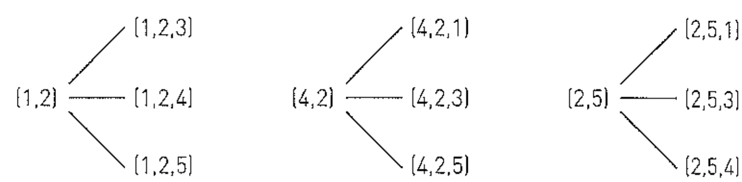
Notice that every unordered pair of numbers—for example, 2,4—occurs twice, once as 2,4 and once as 4,2. So, in our systematic list of ordered pairs, each pair of numbers appears twice instead of once. To get the actual number of unordered pairs, we need to take our 20 ordered pairs and divide by 2 (the number of times each pair that uses the same two numbers appears) to give the number of unordered pairs, namely, 10.

*When faced with a difficult challenge, it's wise to begin by considering related challenges that we can actually meet.*

Let's consider one more example before returning to our lottery question. Our goal here is to examine enough "easy" scenarios so that the hard lottery question at hand becomes "easy." Let's count how many unordered selections of three numbers there are taken from the collection of numbers 1, 2, 3, 4, 5. As before, our strategy is first to count how many ways we can select three numbers in which different orderings count as different selections and then see how many times each particular group of three numbers appears on that list.

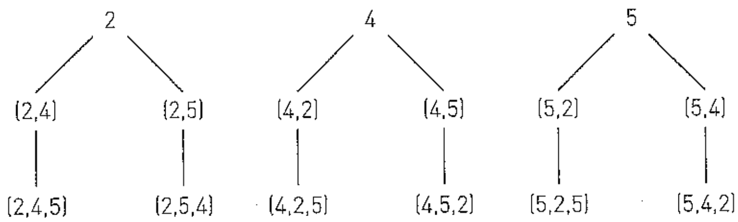
We know how to list all possible first two numbers since we already did that. But now, for each such pair of numbers, we have three choices for the third number. For example, the ordered pair 4,2 can be extended to three different ordered sets of three numbers, namely, 4,2,1 and 4,2,3 and 4,2,5. Likewise, the ordered pair 2,5 can be extended to 2,5,1 and 2,5,3 and 2,5,4. Although we have not written down every number on the chart,

Just three examples of orderings with a given starting pair.




we can compute how many such ordered triples of numbers there would be if we were to write out the entire chart. There are five choices for first number. For each of those first numbers, there are four choices for second number; thus, so far we have $5 \times 4$ choices for the first two ordered numbers. For each of those $5 \times 4$ choices of first two numbers, we have three remaining possibilities for the third number. Thus, we have $5 \times 4 \times 3 = 60$ ways of choosing three numbers in which we are viewing different orderings as different choices.

However, our real question was to count how many collections of three numbers there are in a set of five numbers if the order does *not* matter. So we ask ourselves how often a particular set of three numbers occurs in our list of ordered numbers. So let's just choose three numbers and see. Let's consider 2, 4, and 5. Where are all the places they occur in our list of 60 ordered groups of three? Well, any one of those three numbers could be first; and for each of the three possible first numbers, there are two choices for the second number. Once the first two numbers are chosen, the last number is determined. So our method of counting how many times 2,4,5 occurs (in any order) on our long list of ordered collections of three numbers is really the same method we employed before. Namely, we just look at a tree-like figure that has 2, 4, or 5 as the first position and branches out. Each of the three possibilities for first position has two choices for second position, and then the third position is determined. So the possible orderings of 2,4,5 are 2,4,5 and 2,5,4; then 4,2,5 and 4,5,2; then 5,2,4 and 5,4,2 for a total of $3 \times 2 = 6$.




So now we can answer the question of computing how many unordered collections of three numbers can be chosen from the numbers 1 to 5. The answer is to take the number of *ordered* sets of three numbers (which is $5 \times 4 \times 3$) and divide by the number of times the same set of three numbers occurs on that list (which is $3 \times 2 \times 1$). So the number of *unordered* ways to select three numbers from the numbers 1 to 5 is:

$$\frac{5 \times 4 \times 3}{3 \times 2 \times 1} = 10.$$

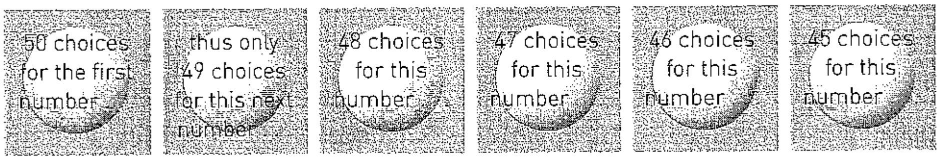
We can now face the lottery question. We will apply a similar type of analysis to figure out how many different ways we can select six different numbers, each from 1 to 50: We first count all *ordered* collections of six numbers and then divide by how many times each unordered collection was counted.

The number of different ways of selecting six different numbers from 1 to 50 is

$$\frac{50 \times 49 \times 48 \times 47 \times 46 \times 45}{6 \times 5 \times 4 \times 3 \times 2 \times 1},$$

which equals 15,890,700.

Why? We can choose any of the 50 numbers as the first number. For each first choice, we can choose any of the remaining 49 numbers second. So, we have $50 \times 49$ ways to choose the first two numbers. Continuing, we have $50 \times 49 \times 48 \times 47 \times 46 \times 45$ ways of choosing six numbers in specific orders.




So, the total number of possible ways of choosing six numbers from 50 if order matters =
$50 \times 49 \times 48 \times 47 \times 46 \times 45$.

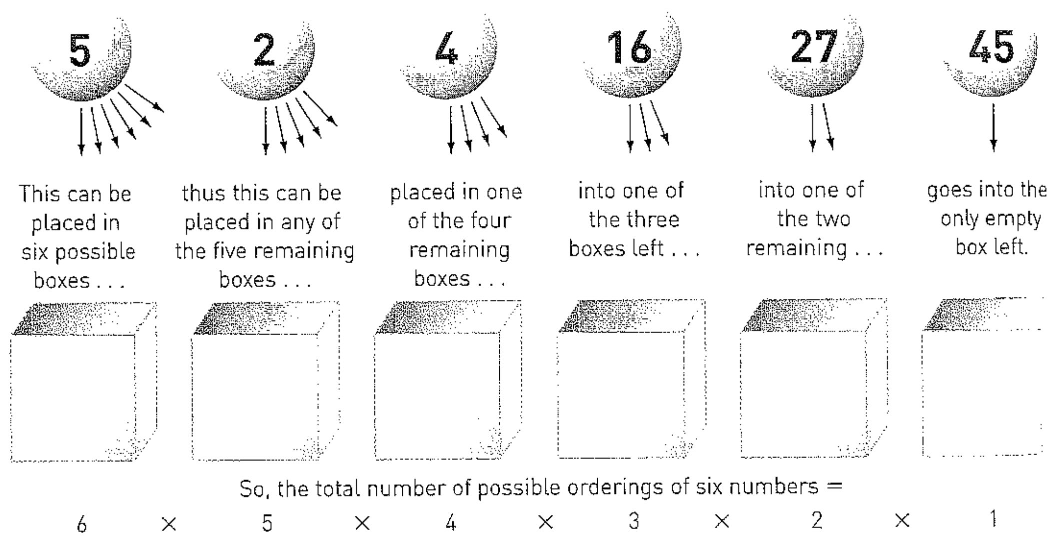
Since we counted the number of ways of selecting six numbers where order matters, we counted

2, 5, 4, 16, 27, 45;
5, 2, 4, 16, 27, 45;
2, 4, 5, 45, 27, 16; and so on,

separately since their orderings are different even though each ordering involves the same collection of six numbers; therefore, we did some major overcounting. In how many ways can we order these six numbers?

There are $6 \times 5 \times 4 \times 3 \times 2 \times 1$ ways to order those six numbers: Any of the six could be the first number, so there are six different possibilities for the first number. Once the first number is determined, any of the remaining five could be the second. Thus, for each choice of first number, there are five different possible choices for the second number. Similarly, there are four different choices for the third number, and so forth. Thus we have $6 \times 5 \times 4 \times 3 \times 2 \times 1$ many different orderings of each group of six numbers. So, since each group of six numbers is counted exactly $6 \times 5 \times 4 \times 3 \times 2 \times 1$ times in our count of $50 \times 49 \times 48 \times 47 \times 46 \times 45$, we simply divide

$50 \times 49 \times 48 \times 47 \times 46 \times 45$ by $6 \times 5 \times 4 \times 3 \times 2 \times 1$




to arrive at the number of different six-number groupings selected from 1 to 50.

## Generalized Counting

We can extend the calculations we just made to a more general situation. Suppose we have a total of $T$ objects and we wish to select $S$ of them. How many different ways can we do that? We see that the answer is that there are

$$\frac{T \times (T-1) \times (T-2) \times \cdots \times (T-(S-1))}{(S \times (S-1) \times (S-2) \times \cdots \times 2 \times 1)}$$

different ways of selecting $S$ objects from $T$ things where the order doesn't matter.

Calculators and computers have no difficulty counting these unwieldy collections. If from a set of 50 numbers we want to know how many different ways there are of choosing 6 where order does not matter, in real life we simply find the *combination* key on our calculator, enter the numbers 50 and 6, and we have the answer. The trick is to understand thoroughly when we are seeking the number of unordered sets of 6 from a set of 50 and when we really want something else. In this case, the calculator would report 15,890,700; and so if the Lotto jackpot exceeds \$15,890,700, we will make money if we buy every single possible combination of six numbers (for \$1 each)—assuming that we don't share the prize.

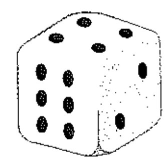
## Dealing with Cards

Some of the most challenging counting questions occur in dealing with playing cards. Let's use a standard 52-card deck of playing cards and consider, for example, five-card poker hands. First of all, how many different five-card hands could we be dealt? The answer is the number of ways of picking 5 things from that group of 52 things. We just figured out how to compute such a count (notice that order does not matter). The answer is $(52 \times 51 \times 50 \times 49 \times 48)/(5 \times 4 \times 3 \times 2 \times 1)$, which is 2,598,960. Among all those hands, we may now ask how many are of the various poker types, such as four of a kind, flush, straight, full house, and so forth. Many of these hands are not so easy to count—especially if we don't know how to play poker.

To count how many four-of-a-kind hands there are, let's first ask: How many hands have four aces? This question is not too difficult, because after we have been told that there are four aces, there can be only one additional (non-ace) card in the five-card hand. That card could be any one of the remaining 48 cards (52 minus the four aces). So, there are 48 poker hands that have four aces. There are, of course, the same number of poker hands that contain four kings, or four queens, and so on. So, the total number of hands that contain four of a kind is $48 \times 13$. If we are dealt a five-card hand, the probability of getting four of a kind is $(48 \times 13)/2{,}598{,}960$, or 0.00024—not likely.

Such a hand is rare indeed. Let's now see how it compares to a straight flush. A straight flush is a run of five consecutive cards all of the same suit. For example, 5, 6, 7, 8, 9 all of diamonds and 8, 9, 10, J, Q all of spades are both straight flushes. How many possible straight flushes are there? The answer requires some careful counting. Let's start with one suit, say spades. How many straight flushes are there in spades? Well, we could have the A, 2, 3, 4, 5, or the 2, 3, 4, 5, 6, and so forth until 10, J, Q, K, A. (The ace can be used either as the highest or lowest card.) That is a total of ten straight flushes in spades—the same for hearts, diamonds, and clubs. So the total number of straight flushes possible is $10 \times 4$. Therefore, a straight flush is rarer than four of a kind. The probability of being dealt a straight flush is $(10 \times 4)/2{,}598{,}960$, or 0.000014—which makes getting four of a kind look pretty easy. Counting the number of hands of each type is the method used to determine which hands beat which other hands. So, the counting we have done here shows why a straight flush beats four of a kind.

## "Or" and/or "And"



Suppose we have a die and a coin, and we are going to throw them both. Consider these three questions:

- What is the probability we will roll a 6 on the die **and** flip a heads on the coin?
- What is the probability we will roll a 6 on the die **or** flip a heads on the coin?
- Which of these two outcomes is more likely?

The first two questions are similar and bring up a significant feature of counting. As always, when we want to compute a probability, we must know how many total outcomes are possible, and then we need to know how many of those are the outcomes we desire. In this case, for every one of the six equally likely outcomes from rolling the die, there are two possibilities for the coin. Therefore, there are a total of 12 outcomes altogether.

| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| H | H, 1 | H, 2 | H, 3 | H, 4 | H, 5 | H, 6 |
| T | T, 1 | T, 2 | T, 3 | T, 4 | T, 5 | T, 6 |

Now let's count how many outcomes have a 6 on the die **and** a heads on the coin. Well, only one. So, the probability of rolling a 6 on the die and flipping a heads on the coin is 1/12. Notice that the probability of rolling a 6, 1/6, multiplied by the probability of flipping a heads, 1/2, yields the answer of 1/12. If we wish to compute the probability that one event happens **and** simultaneously another unrelated event also happens, we need only multiply the individual probabilities together. Why multiply? Because the number of outcomes of the two events is obtained by multiplying the number of outcomes for the first by the number of outcomes for the second.

| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| **H** | H, 1 | H, 2 | H, 3 | H, 4 | H, 5 | **H, 6** |
| **T** | T, 1 | T, 2 | T, 3 | T, 4 | T, 5 | T, 6 |

Heads **and** 6: 1 out of 12

**Probability of Two Events Both Occurring.**
*The probability of one thing happening* ***and*** *some unrelated thing happening is equal to the* ***product*** *of the individual probabilities.*

Now let's count how many of the outcomes have a 6 on the die **or** a heads on the coin. Let's be clear that when we say "6 on the die **or** a heads on the coin" we include the possibility that both a 6 appears and a heads appears. In mathematics, the word "or" always means that either of the options or both of the options happens. Well, we could have a 6 and either a heads or a tails on the coin. So, that's two ways. We could also have a heads on the coin and *any* number 1, 2, 3, 4, 5, or 6 on the die. That sounds like 6 more, but we must avoid the double-counting of a 6 and a heads. The total number of outcomes that have a 6 **or** a heads is 7. So, the probability of getting a 6 **or** a heads is 7/12.

| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| **H** | **H, 1** | **H, 2** | **H, 3** | **H, 4** | **H, 5** | **H, 6** |
| **T** | T, 1 | T, 2 | T, 3 | T, 4 | T, 5 | **T, 6** |

When computing the probability of getting something **or** something else, it is often more convenient to compute the probability of the opposite scenario. In the preceding case, for example, we were asked to consider the possibility of getting a 6 or a heads. The alternative is that we do not roll a 6 **and** we do not flip a heads. This question is easier, because we just saw how to find the probability that two such events happen simultaneously: We multiply the individual probabilities. What is the probability of not rolling a 6 on the die? There are five out of the six possibilities that lead to success, so the probability is 5/6. What is the probability of not flipping a heads? We know that it is 1/2. So, the probability of avoiding both a 6 **and** a heads is the product $(5/6) \times (1/2)$, which equals 5/12. We can now deduce that the probability of getting a 6 **or** a heads is $1 - 5/12$, which equals 7/12, as we saw before. So, the opposite of an event where *something happens* **or** *something else happens* is that *the first does not happen* ***and*** *the second does not happen.* With careful counting and multiplying probabilities, we can now find probabilities of several events happening together. Let's see these ideas in action.

### Dicey Issues

Recall that the seminal counting questions posed by our French nobleman concerned the frequency with which a 6 would appear in four rolls of a die. Let's count how many different ways a die can be thrown four times and then count how many of those outcomes contain a 6. These two numbers allow us to compute the probability of throwing a 6 in four rolls of a die. Let's count.

1. Any of the six numbers could come up first.
2. For each of the six possible first numbers, any of the six numbers could come up second.
3. For each of the 36 possible first and second rolls, any of the six numbers could come up third.
4. For each of the $36 \times 6 = 216$ possible first, second, and third rolls, any of the six numbers could come up fourth.

So, altogether there are

$$6^4 = 216 \times 6 = 1296$$

equally likely possible outcomes of rolling a die four times.

How many of those contain a 6? Well, the 6 could be in the first, **or** the second, **or** the third, **or** the fourth place. But beware: We must not double-count the outcomes of rolling two 6's in a row. We might also roll three 6's, and so on, so we have to be careful not to double, triple, and whatever count those outcomes involving 6's. Looking at the counting question in this way is sufficiently complicated and perplexing that it is best to try to think of an alternative approach.

*If the going gets tough, do something else.*

In this case, a little thought saves a great deal of toil. Instead of thinking about what's there, let's think about what's not. Often it is best to count what's missing rather than what's present. In this case, we want to know how many of the 1296 outcomes include a 6. Why don't we think about the alternative? Namely, how many outcomes do not contain *any* 6's? That is, let's count how many outcomes involve only the numbers 1 through 5. Well, that is pretty easy once we realize that we just answered that same kind of question. There are five possibilities for the first roll, five for the next, five for the third, and five for the fourth. So, there are $5 \times 5 \times 5 \times 5$ outcomes altogether that involve only the numbers 1 to 5. That is a grand total of 625.

Therefore, of the 1296 possible outcomes of rolling a die four times, 625 do not involve a 6, and so, the rest ($1296 - 625 = 671$) must involve a 6 somewhere. Therefore, the probability of rolling at least one 6 among the four rolls is 671/1296, which equals 0.5177. . . . Since this probability is greater than one-half, a person betting on rolling a 6 in four rolls will tend to win slightly more often than lose.

## Dicier Issues

When the French gambler tried his game of rolling a pair of dice 24 times and asked whether there would be a pair of 6's, his winning streak evaporated. He probably reasoned that in 24 rolls of the pair, the first die on average would be a 6 four times. Therefore, the other die would have those four chances to be a 6 as well. This unfortunately incorrect line of reasoning would lead him to think that his winning percentage for getting a pair of 6's in 24 rolls of a pair of dice should be the same as for getting one 6 in 4 rolls. Let's see why his winning streak came to a screeching halt.

Our whole goal is to carefully compute the number of possible outcomes of rolling two dice 24 times and seeing how many equally likely outcomes there are. With any one roll there are 36 possible outcomes. So, let's get rid of the red herring of two dice and instead concentrate on the fact that one of 36 outcomes is possible in each of the 24 repetitions. By the same reasoning as before, there are then a total of 36 to the 24th power outcomes, or about $2.245 \times 10^{37}$—which is one heck of a big number. We now need to know how often a particular outcome (a pair of 6's) from the 36 possible outcomes will occur at least once among the 24 trials. Since counting such a thing is difficult, we will instead count the number of outcomes that avoid a pair of 6's. That is to say, in how many ways can each of the 24 rolls result in one of the 35 outcomes other than a pair of 6's? Again, that number is just 35 to the 24th power, which is about $1.142 \times 10^{37}$. So, the probability of **not** getting a pair of 6's in 24 rolls is approximately

$$1.142 \times 10^{37}$$

divided by

$$2.245 \times 10^{37}.$$

That fraction equals

$$\frac{1142}{2245},$$

which is

$$0.508\ldots,$$

and so the probability of actually rolling a pair of 6's is approximately

$$1 - 0.508\ldots = 0.491\ldots.$$

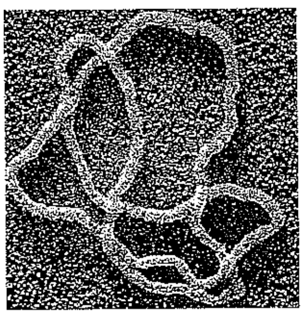
Thus, we are slightly more apt **not** to roll a pair of 6's during 24 rolls than we are to roll a pair of 6's, and our French *bon homme,* sadly, is a loser.

## From Dice to DNA

With cap pulled down and gloves pulled up, the murderer steals into the enclosed garden and confronts his victims. There is a blood-curdling scream as the murderer slashes the throat of his first victim. In a desperate and vain attempt to save his life, the second victim wrestles with the murderer, slightly injuring his assailant, but receiving a fatal wound himself. A few drops of the murderer's blood fall from his wound onto the carnage as he flees the gruesome scene.

Soon the police arrive and take blood samples from the scene. Most of the blood belongs to the victims, but a small portion belongs to the murderer. Those drops of blood might prove the culprit's guilt beyond any reasonable doubt, or they might not. Let's investigate.

DNA is the genetic material contained in each of our cells. All the cells of an individual have identical DNA; however, the DNA from one person differs from the DNA of another. DNA is composed of genes, each of which can be present in one of several forms, known as *alleles.* Human DNA has tens of thousands of genes, and each gene has several possible forms in which it might appear. By looking at the distinct alleles of the various genes, we will see how DNA can incriminate or exonerate a defendant.



A strand of DNA, but whose is it?

Suppose we examine 20 genes from a DNA sample, and each of these genes can occur as two different alleles. Let's further suppose that those two alleles are roughly equally present among the human population and that the choices of allele of these 20 genes are independent of one another. In other words, having one form of one gene does not make it more or less likely to have any particular form of any of the other genes. Let's try now to compute the probability that the 20 genes from a random DNA sample would have the same alleles as those in the murderer's DNA.

Each of the 20 genes can be present as one of two alleles. So, the first gene could be in the first or second form. For each of those possibilities, the second gene could be in its first or second form. So, there are four possible patterns for the first two genes. The third gene could also appear as either of two alleles. Thus, we observe that for 20 genes, there will be $2^{20}$—which equals 1,048,576—possible patterns for those 20 genes. So, the chance of another person having all 20 of those genes is 1 in 1,048,576.

DNA evidence is potent. How could a lawyer refute such incriminating evidence? The only possible way to refute such evidence is to recognize that the test is only as valid as the accuracy with which it is performed. If samples are switched, if the defendant's blood contaminates the blood from the murder scene, or if the samples are mislabeled, then the reported match may be completely unreliable. Thus, it could be argued that police and lab error, in a sense, wipe out the accuracy of DNA tests. In other words, the more convincing probabilistic analysis would take into account the probability that the laboratory or the police had made errors.

## A Look Back

COUNTING CAN BE hard. Careful counting can lead to a better understanding of probabilities and perhaps even to breaking the bank at Monte Carlo. In fact, if a casino catches an individual *successfully* counting cards at blackjack, that individual is banned from the casino. However, casinos generally encourage counting, because most people who try counting foul up and lose more money than ever. Counting helps us determine our chances for winning the lottery, being dealt a royal flush in poker, and having the same genetic profile as a murderer.

In all cases, careful counting consists of two challenges: counting every item and not counting anything twice. To deal with difficult counting challenges, we start with simpler versions of the same thing. Instead of choosing six numbers each from 1 to 50 for the state lottery, we start by assuming we are in a very small state where we choose two or three numbers from 1 to 5. From such small examples, we gain experience and see patterns that allow us to deal with the more difficult cases. Sometimes a useful strategy is to count the opposite of what we want. That is, instead of counting how many rolls of a pair of dice result in a 6 or a 3, we can count the number of rolls that result only in other numbers. Then we subtract from the total. Systematic organization is helpful for counting.

Complicated questions are often best approached by thinking hard about simple examples of similar situations. From the simpler situations we can more easily deduce patterns and methods that let us deal with the more complicated settings. Systematically organizing our thinking empowers us to deal with big issues using the same ideas that work for small ones.

*Look at simple cases.*

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*Find patterns and methods.*

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*Apply the methods systematically*
*to more complex cases.*

## *Mindscapes* *Invitations to Further Thought*

*In this section, Mindscapes marked* **(H)** *have hints for solutions at the back of the book. Mindscapes marked* **(ExH)** *have expanded hints for solutions at the back of the book. Mindscapes marked* **(S)** *have solutions.*

### I. Developing Ideas

1. **Have a heart.** If you draw a card from a regular deck of 52 cards, what are the chances that your card would be red? A heart? A queen? The queen of hearts?
2. **Have a head.** You flip a coin 10 times. What is the probability that you see at least one head?
3. **Sunny surprise.** Suppose the chances are 1 in 2 that tomorrow will be sunny. Suppose the chances are 1 in 10 that your math teacher will bring donuts to class tomorrow. What are the chances that it will be sunny **and** you get donuts in math class tomorrow?
4. **Elephant ears.** Suppose a quarter of all the elephants in an African wildlife refuge have ear tags. If there are exactly 60 tagged elephants, what is the total number of elephants in the refuge? If you pick an elephant at random, what is the probability it has a tag?
5. **Little deal.** In how many ways can you select three cards from a regular deck of 52 cards? (*Note:* The order of the cards doesn't matter.)

### II. Solidifying Ideas

6. **The gym lock.** A lock has a disk with 36 numbers written around its edge. The combination to the lock is made up of three numbers (such as 12-33-07 or 19-19-08). How many different possible combinations are there for this lock? What is the probability of randomly guessing the correct one?
7. **The dorm door.** A dormitory has an electronic lock. To unlock the door, students must enter their unique five-digit secret code into the keypad (made up of the digits from 0 to 9, each of which can be used more than once). How many different secret codes are there? Suppose there are 200 students living in the dorm. What is the probability of one of them randomly guessing a code and having the door unlock?
8. **28 cents.** How many different ways can you make 28 cents using current U.S. currency?
9. **82 cents.** How many different ways can you make 82 cents using current U.S. currency?
10. **Number please.** Someone you really wanted to go out on a date with gave you a beeper number. You didn't write it down, so you remember only the area code and the exchange (the first three digits

of the number). How many different numbers are there with that area code and exchange? What is the probability that you randomly pick the right number?

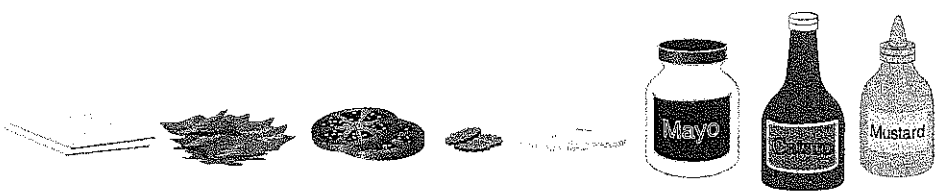
11. **Dealing with jack.** Suppose you deal three cards from a regular deck of 52 cards. What is the probability that they will all be jacks?

12. **MA Lotto (H).** To win the jackpot of the Massachusetts lottery game in bygone days, you had to correctly pick the six numbers selected from the numbers 1 through 36. What was the probability of winning the Massachusetts lottery?

13. **NY Lotto (H).** To win the jackpot of the New York lottery game in bygone days, you had to correctly pick the six numbers selected from the numbers 1 through 40. What was the probability of winning the New York lottery?

14. **OR Lotto.** To win the jackpot of the Oregon lottery game in bygone days, you had to correctly pick the six numbers selected from the numbers 1 through 42. What was the probability of winning the Oregon lottery?

15. **Burger King (S).** You take a summer job making hamburgers. The burgers can be made with any of the following: cheese, lettuce, tomato, pickles, onions, mayo, catsup, and mustard. How many different kinds of burgers can you make?




16. **More burgers.** Suppose you are working at a burger place where the burgers can be made with any of the following: cheese, lettuce, tomato, pickles, onions, mayo, catsup, and mustard. You have a picky clientele: They all want the works, but they wish to specify the order of the placement of the items! For example, one person may want burger, cheese, mayo, onion, catsup, pickle, mustard, tomato, lettuce. Someone else may want lettuce, tomato, burger, onion, cheese, mustard, mayo, catsup, pickle. How many different types of burgers-with-the-works are there?

17. **NetFlix.** You have 65 movies on your NetFlix list. In how many ways can you order them? (*Note:* Don't multiply it out.)

18. **Cineplex (ExH).** Your local Cineplex is showing eight movies. You want to see a different movie on each of Thursday, Friday, and Saturday nights. How many different three-night movie magic experiences can you choose from if the order in which you see the three movies does not matter to you? How many choices if the order does matter?

19. **One die.** You roll a fair die four times. What is the probability that you see at least one 1?

20. **Dressing for success.** You have 5 T-shirts, 10 shorts, and 3 pairs of underwear, and you wear 1 of each. How many different ensembles can you put together? *Bonus:* Assuming all these clothes are clean, how long could you go before doing the laundry?

21. **Band stand.** The Drew Aderburg Band is planning a concert tour of six cities. In how many different orders could the band cover the six cities?

22. **Monday's undies.** You are spending the weekend at a friend's parents' house. You need 3 pairs of underwear and have 10 clean pairs of underwear of different colors. How many different underwear triples can you pull out to impress your friend's folks?

23. **Counting classes.** Your institution will offer 200 courses next semester, each at a different time. You will take 4. How many different groups of 4 courses can you select from?

24. **Cranking tunes.** Your car stereo can be programmed to hold six radio stations of your choice. There are nine stations you really like. How many different ways can you program your six buttons with different collections of your favorite stations? (A different ordering of the same six stations counts as different programming.)

25. **The Great Books.** There are 20 Great Books, from which you know your English prof for next semester will select 10. You are an overachiever and want to figure out how many different combinations of 10 books the prof can pick. How many are there?

## III. Creating New Ideas

26. **Morning variety (S).** You wish to have a different breakfast every morning. Each day you choose exactly three of the following items: eggs, bagel, pancakes, coffee, orange juice. How many days can you go before you must eat a breakfast combination that you have already had?

27. **Indian poker.** You and a friend each pick a card from a regular deck of 52 cards but do not look at them. Both of you then hold your cards up by your foreheads so that you can each see the other's card but you can't see your own. Your friend is holding a 6. What is the probability that your card is higher?

28. **Crime story (ExH).** Suppose 20 witnesses saw someone commit a crime, and each supplied a piece of information. One witness said the perpetrator was wearing a certain type of shoe. Another said the perpetrator was taller than six feet. A third said the perpetrator had dark hair, and so on. Each piece of information distinguished the perpetrator only from half the people on Earth. The different pieces were independent in the sense that any combination of them

was possible. If there are roughly 6.6 billion people on Earth, approximately how many would fit all 20 of the pieces of information?

29. **There's a 4 (H).** Someone you really want to go out with gave you a beeper number. You didn't write it down and remember only the area code, the exchange (the first three digits of the number), and the fact that the last four digits contain at least one 4. How many different such numbers are there? What is the probability of your randomly picking the right number?

30. **Making up the test.** Your math prof says there will a 15-question test on probability. She also reports that the test will be made up of problems from the Mindscapes section of the chapter as follows: two questions each from Sections 7.2, 7.3, and 7.5, and three questions each from Sections 7.1 and 7.4 (no questions from the In Your Own Words category). How many different possible exams could she make up? What is the probability that this very question is on your exam?

31. **Moving up.** You have a part-time job in a department with 20 other people. Word comes out that 6 of the people from that department will be given a raise. If the 6 people are chosen at random, what is the probability that you will be one of the 6?

32. **Counterfeit bills.** You are given ten $100 bills and told that three of them are counterfeit. You randomly pick three of the bills and burn them. What is the probability that you burned the counterfeit bills?

33. **Car care.** A burglar wishes to break into a car that has a security system. There are five buttons (marked 1 through 5) to disarm the system, and he knows the code is a three-digit number like 253 or 422. How many possible security codes are there? Knowing this number of combinations, the burglar fears that he will enter the wrong number first—thus tripping the alarm (which everyone just ignores). So he writes down his first 20 guesses and then enters his next guess. Does this strategy improve his chances of success? Explain.

34. **Coins count.** On your bureau you have a half dollar, quarter, dime, nickel, and penny. How many different totals can be formed using exactly three coins? How about using four coins? How about using five coins?

35. **Math mania.** There are 90 students enrolled in Math 180. There are three different sections of 30 students, all meeting at the same time. How many different ways can the 90 students be placed into the three sections?

## IV. Further Challenges

36. **Party on.** You want to throw a party and can invite only 15 people. You want to invite 3 people from the soccer team (there are 10 people on the tiny team); 4 people from the orchestra (there are 20 people in the orchestra); and 8 people from your math class (there are 30 other

students in your class). Assuming no overlap of the groups, how many different invitation lists could you devise?

37. **No dice (H).** You roll a pair of dice 24 times. What is the probability of seeing at least one total that is 11?

38. **Three angles.** Draw 10 points on a piece of paper with no three points lying on the same straight line. How many different triangles can you make using these points as vertices?

39. **Four parties (ExH).** You want to have a party and you know 10 men and 10 women. Unfortunately, your common room can hold only 10 people. You wish to have enough parties so that each man and woman among your acquaintances would, at some point, meet at a party. How many parties are necessary to accomplish this task? Show that you need no more than four parties.

40. **Making the cut.** In 1988, the ignition keys for Ford Escorts were made out of a blank key with five cuts, each cut made of one of five different depths. How many different key types were there? In 1988, Ford sold roughly 380,000 Escorts. What is the probability that one Escort key will unlock a random Escort? (This story was reported in the April 1989 issue of the *Atlantic Monthly*.)

## V. In Your Own Words

41. **Personal perspectives.** Write a short essay describing the most interesting or surprising discovery you made in exploring the material in this section. If any material seemed puzzling or even unbelievable, address that as well. Explain why you chose the topics you did. Finally, comment on the aesthetics of the mathematics and ideas in this section.

42. **With a group of folks.** In a small group, discuss and work through the details involved in counting the five-card poker hands and the French dice games. After your discussion, write a brief narrative describing the methods in your own words.

43. **Creative writing.** Write an imaginative story (it can be humorous, dramatic, whatever you like) that involves or evokes the ideas of this section.

44. **Power beyond the mathematics.** Provide several real-life issues—ideally, from your own experience—that some of the strategies of thought presented in this section would effectively approach and resolve.