7.5 Drizzling, Defending, and Doctoring

Probability in Our World and Our Lives

"I need those probabilities STAT."

> *I'm not a fan of facts. You see, the facts can change, but my opinion will never change, no matter what the facts are.*
>
> STEPHEN COLBERT

Probability has important applications in many real-life arenas from rolling dice to diagnosing diseases. Gambling involves random processes, so it is not surprising that probability is a key ingredient in Vegas, but probability also can play a central role in making decisions or making predictions in situations where randomness may not be involved at all. For example, we'll see that when a doctor makes a tricky diagnosis, the best method for guessing the right illness may involve probabilistic reasoning. Also, randomness and probability can help us train our dogs—if we want to teach our dogs to respond to a command, the best method may be to reward them at *random* intervals rather than with any regular pattern. In this way, the dog develops the hope (rather than expectation) that the next reward is just one more good deed away. Should random rewards be involved in child rearing, education, or governmental policies?

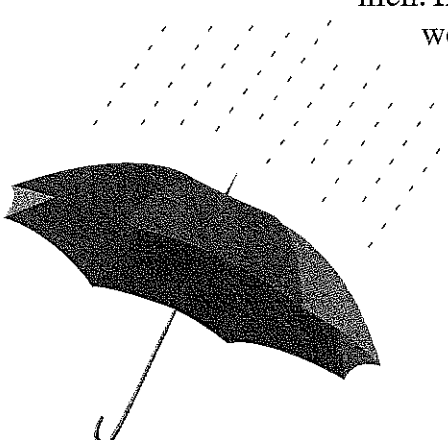
Randomness and probability definitely arise in game theory—the study of strategic decision-making. In game theory, often the optimal strategy is one that involves intentionally introducing randomness. Optimal business strategies or sports strategies frequently are probabilistic in nature. Should we run or pass on the football field; should we buy or sell in the trading pit? The best strategic decision-making process may be to flip a coin. But when optimal strategies involve probability, how can we judge whether decisions were made wisely? Even the very best possible strategy might result in some serious losses every once in a while.

Probability is a basic component of clear thinking as we cope with the randomness and uncertainty of our daily lives.

A Chance of Rain . . .

Predicting the future is always fraught with uncertainty. Those zany TV weather people do it every day when they declare a probability of rain tomorrow. We often need to know whether to take our umbrellas and rubbers when we leave home in the morning, so weather prediction has a noticeable impact on our daily lives. We wake up bleary-eyed from an all-too-short slumber, turn on the local news, and hear the prediction: *There is a 30% chance of rain today in the greater metropolitan area.* Now what are we going to do? What does a "30% chance of rain in the greater metropolitan area" even mean? Surprisingly, most people do not know its correct meaning. We'll now cast clear, sunny light on this cloudy issue of the probability of precipitation.

First, we need to know the "official" threshold of how much rain must fall before we can officially declare that "it rained." The answer is 0.01 inch. If we put out a rain gauge and more than 0.01 inch falls into it, then we report that it rained at that spot.



If a forecast of rain were to be given for one single spot, then the meaning of the "30% chance of rain today" prediction is pretty clear. It means that if 100 days had weather conditions nearly identical to those of today, we would expect to see rainfall of at least 0.01 inch at that spot on approximately 30 of those days. Historical weather records would help the weather forecasters make that "spot-on" prediction.

The issue is complicated when we hear that there is a 30% chance of rain in a large region, such as a greater metropolitan area. There are countless different points in the region and some points in the region may get more rain than other points, so we must face and understand these variations.

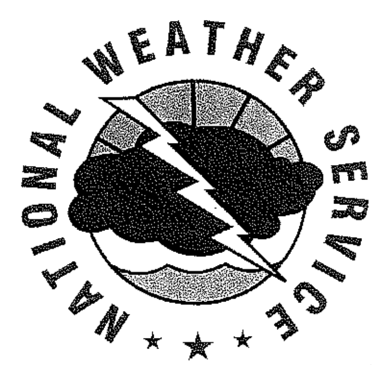
Let's consider a large region and assume that for every point in half this region there is a 40% probability of rainfall and that for every point in the other half there is a 20% chance of rain. Under these particular conditions, we would declare that the probability of rain for the entire region is the average of 20% and 40%, that is, the chance of rain for the entire region would equal to 30%. The 30% represents an average probability of precipitation weighted by the area. In this case we have [(½ the area) × (40%) + (½ the area) × (20%)] = 30% probability of precipitation for the whole area.

In an extreme case, we might have 30 square miles out of a 100-square-mile region in which the probability of rainfall is 100%, and in the remaining 70 square miles, the probability of rain is 0%. Again the probability of rain for the entire region is equal to the weighted average: (30/100 of the area) × (100%) + (70/100 of the area) × (0%) = 30% probability of precipitation for the whole area. Notice that there is no point in the region in which the probability of rain is actually equal to 30%. We remark that this weighted average foreshadows our discussion of *expected value* in Chapter 9. The 30% is summarizing two features of the situation: (1) It is reflecting the different probabilities of rain at different points in the region, and (2) it is weighing the various probabilities of rain together with the proportion of area that has those probabilities to describe a type of weighted average. In the case in which it will definitely rain on 30 square miles and definitely not rain on the remaining 70 square miles, the 30% probability of precipitation is as good a summary as we can provide for the whole region. But to know whether to pack our umbrella and rubbers, it would be much more useful to know whether our destination is in the area that gets no rain or in the area where it will definitely pour.

Unfortunately, many people believe that a 30% probability of precipitation in a region means that there is a 30% chance that it will rain at *some point* in the region. We now see that this interpretation is incorrect. In fact, a 30% probability of precipitation actually implies that *on average* 30% of the area will receive rain. If conditions were repeated many times where there is a 30% probability of precipitation for any point in the region, and for each repetition the percentage of the area of the region that got wet were recorded, then the average of those percentages would equal 30%.

If you're not sure what something means, find out the precise definition.

The definition of probability of precipitation is tricky. An unfortunately incorrect explanation has appeared on the National Weather Service Web site. Their official explanation is, at best, misleading: "Technically, the probability of precipitation (PoP) is defined as the likelihood of occurrence (expressed as a percent) of a measurable amount (0.01 inch or more) of liquid precipitation (or the water equivalent of frozen precipitation) during a specified period of time at any given point in the forecast area. Forecasts are normally issued for 12-hour time periods." The phrase "at any given point in the forecast area" cannot be correct since different points can have different likelihoods, particularly if some points in the forecast area lie in rainy terrains and others in drier areas. Whoops.



The definition *should* read: "Technically, the probability of precipitation (PoP) is defined as the likelihood of occurrence (expressed as a percent) of a measurable amount (0.01 inch or more) of liquid precipitation (or the water equivalent of frozen precipitation) during a specified period of time at a random point in the forecast area. Forecasts are normally issued for 12-hour time periods." Let's hope that someone from the National Weather Service reads this text and will correct the cloudy probability mistake.

Pass the Ball or Run? A Beautiful Mind

Whether we are or are not addicted to football, this sport does provide us with an excellent opportunity to illustrate a strategic decision-making process that employs probability and arises on the playing fields, in the Wall Street trading pits, and in many parts of our lives. *Game Theory* is the mathematical area that offers models for strategic decision making. It is heavily used in economics, business, games, sports, war, and other arenas in which strategic decisions must be made. We will consider a somewhat simple illustration of Game Theory that involves two competing teams, each of which must make one of two choices. The combination of those choices determines how each team fares. Our example presents itself at a moment of high drama on the football field. The fans are screaming because it is third down and long. The offense must decide whether to run or pass; and the defense must decide whether to align itself primarily against the run or the pass. What to do? The clock is ticking. . . .

What should you do if you know nothing about football and care even less about the game? You're not alone (in fact, one or more of the authors feels the same way), so there is no need to punt or even pout. Winning a football game is far more important to overpaid coaches than to most of us. But making decisions in the face of uncertainty is a challenge that each of us faces every day. So looking at the strategy employed on the gridiron is interesting and important, even if you don't care or know much about football.

Game Theory utilizes the concept of a *payoff matrix*, a grid of values that describes the payoffs for each combination of choices that the players could make. We introduce the payoff matrix and illustrate its import in the context of a strategic decision during a football game. In football, when the offense faces third down with many yards to go for a first down, the usual options for the offense are to either pass or run. The defending team can align itself to defend more effectively against the pass or defend more effectively against the run.

On page 647 is a possible payoff matrix for this scenario. Each number represents the payoff (in yards from the offensive team's point of view). The defense wants the offense to get as *few* yards as possible, so *lower* numbers are better for the defense.

If the offense chooses the strategy of always passing, the defense will quickly learn to always defend against the pass. The chart shows that this combination gives an expected value of 5 yards for the offense.

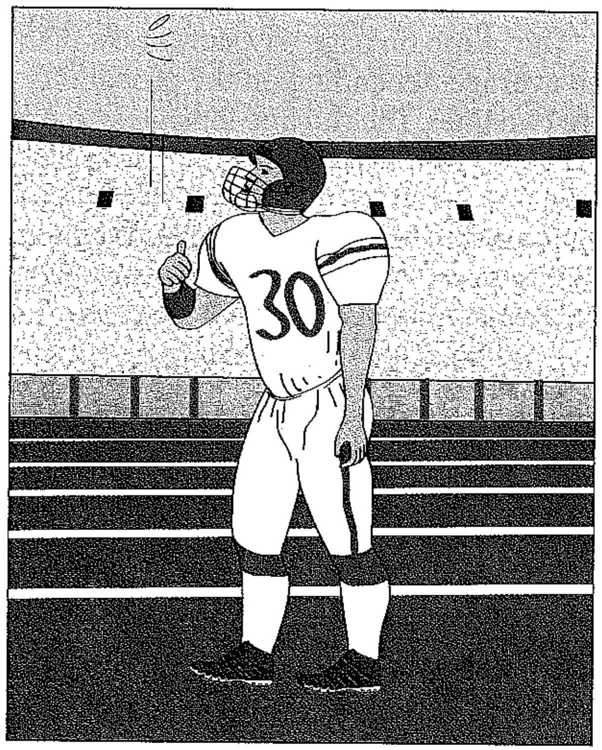
	Defend Against Pass	Defend Against Run
Pass	5 yards	7 yards
Run	6 yards	1 yards

Pass and Run are the options for the offense; Defend Aganist Pass and Defend Against Run are the options for the defense.

If the offense chooses the strategy of always running, the defense will learn to always defend against the run. That combination gives 1 yard for the offense. We might think that those are the only two strategies available for the offense; however, the trick is that the offense does not need to use a pure strategy of always passing or always running. Instead the offense can use the strategy of passing with a certain probability. For example, the offense could flip a coin and run if it comes up heads or pass if it comes up tails. Or instead of a coin, the offense could use some other random device to decide to pass with any chosen probability. Choosing a probabilistic strategy is a great idea, because as we will now see, with a probabilistic strategy, the offense can gain more yards on average than either pure strategy (of always passing or always running) yields.

Be open to new ideas.

The probabilistic Game Theory strategy confirms the intuitive idea that once in a while, at random, doing the nonobvious play is a good idea. Instead of deciding to always pass or to always run, we can decide to pass with some probability p and run with probability $(1 - p)$. So in the huddle, the quarterback could be flipping a coin or consulting a table of random numbers to determine what to call for that play.



Different choices for the probability of passing give the offense a different expectation for yards gained. For example, suppose the offense decides to flip a fair coin and pass when it comes up heads. By using the table, we can estimate how many yards the offense would expect to gain with that strategy. A 0.5 probability of passing means the offense will pass half the time and run half the time. Suppose the defense defends against the pass. Then when the offense passes, the offense will make 5 yards and when the offense runs, they will make 6 yards. So on average, when the defense defends against the pass and the offense passes with probability 0.5, the offense can expect to make

$$0.5 \times 5 + 0.5 \times 6 = 5.5 \text{ yards.}$$

Similarly, suppose the defense defends against the run. Then when the offense passes, the offense will make 7 yards and when the offense runs, they will make 1 yard. So on average when the defense defends against the run and the offense passes with probability 0.5, the offense can expect to make

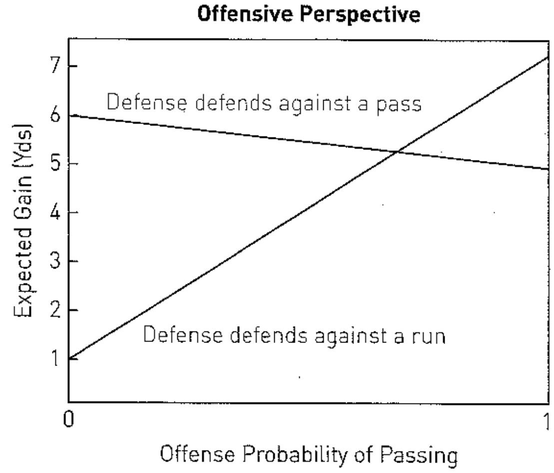
$$0.5 \times 7 + 0.5 \times 1 = 4 \text{ yards.}$$

Visualize information or data whenever possible.

We could do the exact same analysis for any choice of passing with probability p by simply putting in p for the first 0.5 and $(1 - p)$ for the second 0.5 in the two previous equations. We can visualize the information using a graph.

The graph on page 649 shows how many yards we would expect to gain on average depending on the probability p that we use for passing. There are two lines. The red line indicates how many yards we as the offense will expect to gain, on average, if the defense defends against the pass, and the blue line indicates how many yards we as the offense will expect to gain, on average, if the defense defends against the run. In each case we get a sliding scale for how many yards we would expect to gain depending on the probability p with which we pass.

For each probability p of passing, we can compute how many yards we as the offense would expect to gain on average if the defense defends against the pass (the red line). Specifically, the expected number of yards gained if the offense passes with probability p and defense defends against the pass is $p \times 5 + (1 - p) \times 6$. The graph is just a straight line descending from 6 yards down to 5 yards indicating our expected gain if the defense defends against the pass.



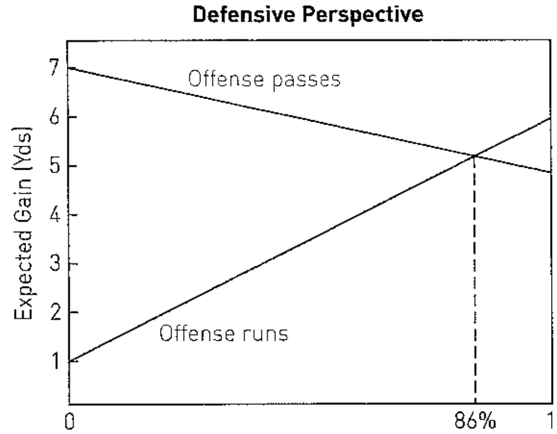
Similarly, if the defense defends against the run, we can display the number of yards we expect to make by the blue line. If we pass with probability 0 (that is, we always run), then we will get only 1 yard on average when the defense defends against the run. While if we pass with probability 1, then we will gain 7 yards when the defense defends against the run. The expected number of yards gained if the offense passes with probability p and the defense defends against the run is $p \times 7 + (1 - p) \times 1$.

The question is: How does the offense select with what probability p to pass in order to maximize the number of yards that it can expect to gain? The answer is that we look at where the red line and the blue line cross.

The red and blue lines cross at one point, specifically, when $p \times 5 + (1 - p) \times 6 = p \times 7 + (1 - p) \times 1$. Applying some basic algebra to solve for p, we can discover that in this example the crossing point occurs when our probability of passing p equals 0.71. At that point, we can expect on average to make 5.3 yards regardless of whether the defense defends against the pass or the run. For any other choice of probability of passing, the defense can always defend against the run or always defend against the pass and our expected number of yards will be less.

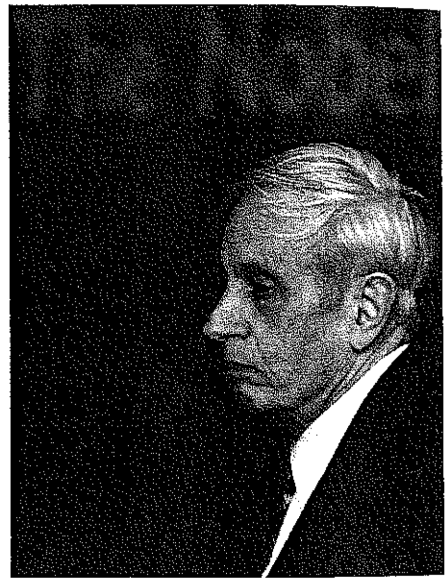
Thus our conclusion is that the offense should pass 71% of the time, but select the 71% at random. Passing with probability 71% gives an expected value of 5.3 yards gained for the offense, which is a higher value than either of the two pure strategies of always passing or always running.

Likewise, the defense would not want to always defend against the pass or always defend against the run, because then the offense could change its strategy to take advantage of that poor strategy. Using the payoff matrix values again and performing the same analysis for the defense as we did for the offense, we would find that the defense should defend against the pass 86% of the time (again at random).



So the next time you're watching a football game and the offense runs on third and long and gets tackled after a one-yard gain, you should not necessarily be angry or frustrated. Your team may, in fact, have been using the best *long-term* strategy, that is, your team may be choosing the play with an appropriate probability. Unfortunately, since the best strategy involves randomness, by definition, the results on each individual outcome will not always be great. But what you do know is that if your team always passes when it's third and long, you should hire a new coach.

Consider long-term solutions.

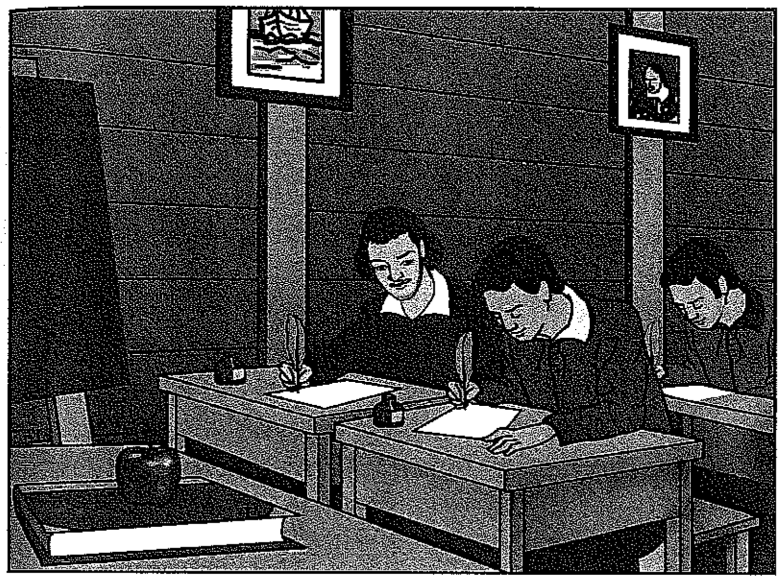


This combination of probabilistic strategies for the offense and defense is called a *Nash equilibrium*, that is, a strategy whereby no player can get an advantage by unilaterally changing his or her strategy. It was named for John Nash, who won the Nobel Prize in 1994 for his work on Game Theory and who became famous through the book and acclaimed movie *A Beautiful Mind.*

Believe It or Not—The Bayesian Model of Belief

Previously, we considered the experience of repeating trials that had random outcomes and we defined probability as an idea constructed to measure the likelihood of certain outcomes. For example, we said that if we rolled a fair die, then the probability of seeing a 4 was 1/6 because there were six equally likely outcomes from rolling a fair die and 4 was one of the possible six outcomes. This concept of probability is a wonderfully useful way to describe our uncertainty about what will happen when several random outcomes are possible. But another kind of uncertainty occurs in life when we talk about what we believe to be true, but we don't know for certain. Let's look at some examples.

Most scholars believe that Shakespeare wrote *Hamlet*, but a few scholars assert that the plays ascribed to Shakespeare were actually written by someone else. They point out that the Shakespearean plays contain many references to classical knowledge and display a nuanced understanding of sophisticated court behavior. Yet Shakespeare had only a few years of elementary school education and had no opportunity to experience court life. These scholars argue that the person who wrote the Shakespearean plays must have had a university education and must have lived among the nobility. So what do you think? Do you believe that Shakespeare wrote *Hamlet*? How would you express your level of uncertainty about this question?



Was it to be or not to be?

You might well say, "I am 98% sure that Shakespeare wrote *Hamlet*, although I admit that there is a small chance, say a 2% chance, that someone else wrote his plays."

This kind of assertion does not mean anything like what our dice-throwing probability meant. Our 98% certainty that Shakespeare wrote *Hamlet* does not mean that if 100 Shakespeares lived, then about 98 of them would have written *Hamlet*. Instead, the 98% figure is expressing a level of belief.

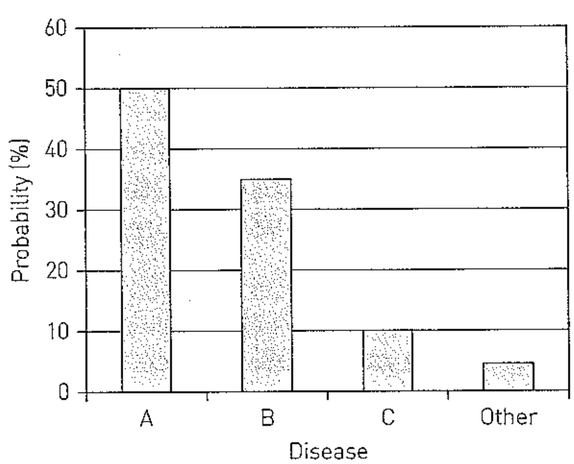
We often say things like, "I'm 95% sure I left the keys on the dresser." Or a doctor might tell us, "There is about an 80% chance that you have a cold, but there is a 20% chance that it's more serious."

One of the roles of mathematical thinking is to take everyday experiences and clarify them and, in particular, make them quantitative if possible. So here we'll take a mathematical look at how we can describe the strength of our beliefs about the world. One useful approach is named after Thomas Bayes, a mathematician and Presbyterian minister who lived from 1702 to 1761. The Bayesian analysis that we present below shows us one way to describe beliefs (in our example, what disease you have when you're sick) and how to update those beliefs when we get new evidence.

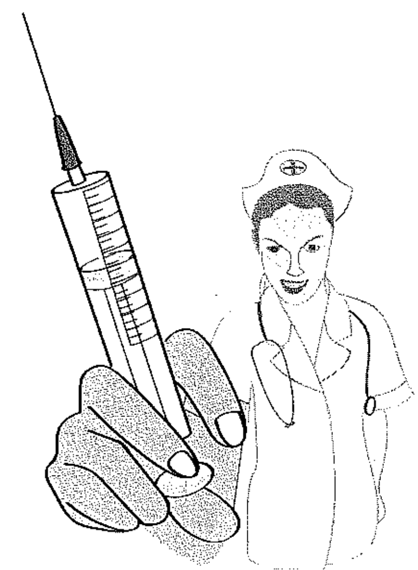
Calling All Doctors

One day you wake up feeling terrible. You have a fever of 101°, you are coughing, and you've lost your appetite. You feel so lousy that you decide not to go to class but instead to visit the doctor. The doctor looks in your ears and throat and listens to your chest and says, "Right, you're sick. I've seen thousands of patients with exactly those symptoms. Based on my extensive experience, I'm almost certain that you have one of three diseases A, B, or C and I know for certain that you can't have more than one of them. I'd say you have a 50% chance that it's disease A, a 35% chance that you have disease B, and a 10% chance that you have disease C. There is a 5% chance that it is something else."

This scenario offers us a perfect opportunity to develop a concept of probability to help us quantify our beliefs in an uncertain world. The fact is, in this situation, the truth is only one of four possible states: you have disease A, disease B, disease C, or another disease. So in some sense, there is no probability involved since only one of those states is true. On the other hand, we do not know which of them is true, so it is reasonable to describe our view of the world by ascribing a level of likelihood to each of the four possibilities. Since you definitely have one of those three specified diseases or some other disease, we know that the probabilities must add up to 1. So we could draw a little chart that shows our current state of belief.



The doctor could start treating you for disease A, the most likely disease, but might instead ask you to take a particular blood test that might give some additional information.



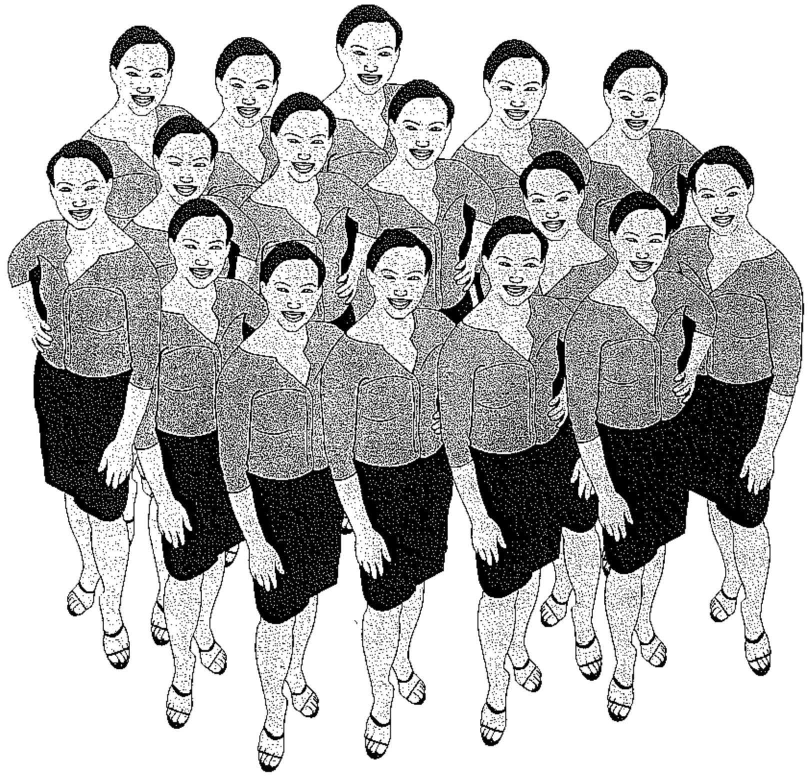
You bravely give a considerable amount of your lifeblood to the smiling, bloodsucking lab technician and go home to drink a few quarts of chicken soup. Two days later the doctor calls and says, "Your test indicates that you have *gorp* in your blood. Studies have shown that among people who have disease A, only 10% of them have *gorp*; while 30% of people with disease B have *gorp*; and 80% of people with disease C have *gorp*. Among people who do not have A, B, or C, 10% have *gorp*."

These additional facts give us information about the possible states of the world, so let's see how we can best apply this new information to develop a refined sense of the probabilities that we have disease A, disease B, disease C, or something else. Our goal is to ascribe a probability to each of the four possibilities by giving a numerical value to each possibility so that the sum of those probabilities adds up to 1. So the first question is whether there is a reasonable way to compute those probabilities or whether many different opinions are reasonable.

When we first think about assigning probabilities to the four possible states of the world, we might be inclined to simply ignore our previous sense of the likelihoods of A, B, C, or Other and simply use the evidence related to the *gorp* test. In other words, we might say, well, clearly you have disease C, since *gorp* appears much more frequently among people who have C than among people with either of the other diseases. However, that thinking would be incorrect, because that thinking would be ignoring all the previous evidence about the diseases. Instead, we need to devise a method to give appropriate weight to the previous probabilities of A, B, or C and then let the new evidence modify our opinion. A good way to think about this issue is to imagine a thousand yous.

A Thousand Yous

Suppose you lived in a large town and, fortunately for the town, one thousand residents were clones of you—that's right, there are one thousand yous. Now suppose you all got sick and had identical symptoms, but you did not have identical diseases. In fact, suppose that you had the various diseases that the doctor first suggested in proportion to the likelihoods he assigned. That is, 50% of the one thousand yous had disease A (in other words, 500 had disease A); 35% had disease B (that is, 350 had disease B); 10% had disease C (that is, 100 have disease C); and 5% or 50 had some other disease.



Now let's simply count how many of these thousand people would have *gorp* in their blood. Well, we were told that 10% of people with disease A have *gorp*, so of the 500 yous with disease A, 50 would have *gorp*. Similarly, since we learned that 30% of people with disease B have *gorp*, that would be 30% of 350, which equals 105 people with disease B also have *gorp*. The same reasoning shows that 80% of the 100 yous with disease C have *gorp*, which gives 80 people with disease C and *gorp*. And finally, 10% of the 50 yous with some other disease also have *gorp*, that is, 5 people have some other disease and *gorp*.

So among the 1000 people who initially came to see the doctor with the symptoms you had and then tested positive for *gorp*, we would see the following numbers having the various diseases:

- Disease A: 50
- Disease B: 105
- Disease C: 80
- Other disease: 5

We can record these data in a convenient table called a *two-way table*. The last column records how many yous are presumed to have each disease, A, B, C, or Other, and each row records how many of those have or do not have *gorp*. So reading the column labeled *Gorp* shows how many people whom we assumed to have each disease also have *gorp*.

	Gorp	No *Gorp*	Total
A	50	450	500
B	105	245	350
C	80	20	100
Other	5	45	50
Total	240	760	1000

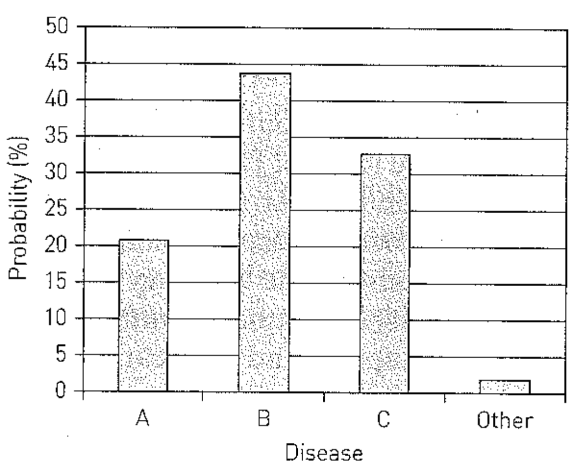
So a total of $50 + 105 + 80 + 5 = 240$ people have all the symptoms that the doctor can now use for evaluating the likelihood of the various diseases. We can simply look to see what fraction of these 240 have the various diseases:

- 50/240 have disease A
- 105/240 have disease B
- 80/240 have disease C
- 5/240 have another disease.

We can phrase this information in terms of probability of having the various diseases by saying you have a:

- 21% chance of having disease A
- 44% chance of having disease B
- 33% chance of having disease C
- 2% chance of having another disease

Using this analysis, the best course of treatment might be directed at disease B.



If you watch dramatic doctor programs on TV, the head doctor sometimes asks the assembled underling doctors, "What is the differential diagnosis?" That question is really asking for a *Bayesian analysis,* such as the one we just performed, of the consequences of some evidence. Given the symptoms, what is the likely disease? What further test will provide evidence that will discriminate among the possible diseases?

Definition of a Closed Mind

If you assign a 0% probability to some possible state of the world, then no amount of evidence will sway your opinion. So this analysis lets us give a mathematical description of a closed mind, namely, someone is completely closed-minded about a possibility if they believe there is literally no possibility that their own beliefs are wrong. Many people on many issues fail to view their world as having the possibility of being wrong. Many wars are fought because of that attitude.

Keep an open mind.

A Look Back

PROBABILITY IS applied in many areas of life, from weather prediction to sports strategies to the diagnosis of disease. We daily hear forecasts like, "There is a 30% chance of rain tomorrow in the greater metropolitan area." That prediction is really an average of the likelihoods of rain at all the various points in the region.

In sports or business, the best strategies may involve making decisions using randomness. Most of the time it may be better to do the expected thing, but once in a while, at random, it may be a good idea to do the unexpected.

Often we are not entirely certain about our world. We can use probability and, in particular, the ideas of Bayesian analysis to clarify the strength of our opinions and to learn how best to change our minds when we learn new evidence. In cases of medical conditions, the good use of evidence can improve the accuracy of diagnoses.

Using probability with clarity and understanding can be an important way to hone our perception and application of common sense.

Understand simple things deeply.

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Be open to new ideas.

•

Modify your thinking when the evidence points you in a new direction.

Mindscapes *Invitations to Further Thought*

In this section, Mindscapes marked **(H)** *have hints for solutions at the back of the book. Mindscapes marked* **(ExH)** *have expanded hints for solutions at the back of the book. Mindscapes marked* **(S)** *have solutions.*

I. Developing Ideas

1. **No pop quizzes.** Your instructor gives "mom quizzes" (she doesn't like pop quizzes). Suppose you expect a score of 100% when you study and a score of 60% when you don't. If you study only half the time, what would you expect your average score to be on all your quizzes?
2. **No easy quizzes (S).** Your instructor starts giving harder quizzes. When you study you expect a score of 80% and a score of 30% when you don't. If you study only half the time, what score would you expect on average?
3. **Silly sickness.** Based on your symptoms, your doctor says, "There's a 40% chance you have meemps, a 30% chance you have moosles, a 20% chance you have chicken pux, and a 10% chance you have some unknown silly disease." Create a bar graph similar to the one on page 655 to illustrate this silly diagnosis.
4. **Making an algebra-down.** Consider the equation $p \times 5 + (1 - p) \times 6 = p \times 7 + (1 - p) \times 1$ from the football scenario given in the section. Solve this equation for p and thus confirm the answer given in the text.
5. **A beautiful mind.** Who was John Nash? Give as many similarities and differences as you can between John Nash and Johnny Cash.

II. Solidifying Ideas

6. **Upon further study.** On your instructor's quizzes, you expect a score of 100% when you study and 60% when you don't. Suppose the probability that you study is 0.8. Given these facts, compute the expected average of all your quizzes.

7. **Power pass (ExH).** The matrix below gives the yards gained for the offense in each of four football scenarios. If the defense always defends against the pass and the offense passes with probability 0.6, find the number of yards the offense can expect to make on average.

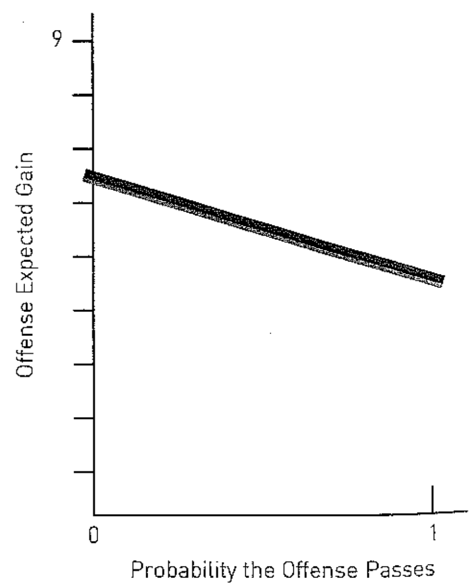
	Defend Against Pass	**Defend Against Run**
Pass	4	9
Run	6	1

8. **Pass with a *p*.** Suppose the defense referred to in the previous Mindscape always defends against the pass and the offense passes with probability p. Given the matrix in the previous Mindscape, write an expression (in terms of p) giving the number of yards the offense can expect to make on average.

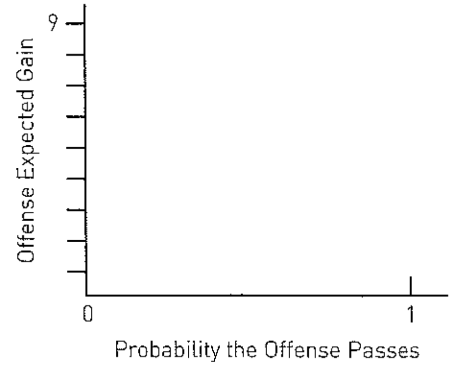
9. **Random run.** Suppose now in Mindscape II.7 the defense always defends against the run and the offense runs with probability 0.4. Find the number of yards the offense can expect to make on average.

10. **Another random run.** Suppose now the defense referred to in Mindscape II.7 always defends against the run and the offense passes with probability p. Write an expression (in terms of p) giving the number of yards the offense can expect to make on average.

11. **Lining up the defense (S).** Using the payoff matrix in Mindscape II.7, plot two points on the axes below as follows. First, let the probability the offense passes be 0 and suppose the defense defends against a pass. Plot the point showing the gain for the offense. Then, let the probability the offense passes be 1 and suppose the defense defends against a pass. Plot the point showing the gain for the offense. Draw the line through the two points to show the average offensive gain when the defense defends against a pass depending on the probability the offense passes. If you did Mindscape II.8, how does your answer there relate to this line?



12. De-lines of defense. Using the payoff matrix in Mindscape II.7, plot two points on the axes to the right as follows. First, let the probability the offense passes be 0 and suppose the defense defends against a run. Plot the point showing the gain for the offense. Then, let the probability the offense passes be 1 and suppose the defense defends against a run. Plot the point showing the gain for the offense. Draw the line through the two points to show the average offensive gain when the defense defends against a run depending on the probability the offense passes. If you did Mindscape II.10, how does your answer there relate to this line?



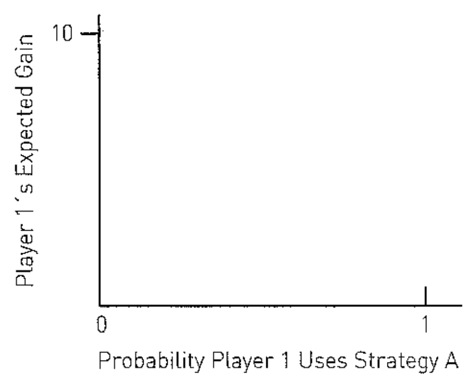
13. Approximate Nash (ExH). Take the two lines from the previous two Mindscapes and draw them on a single set of axes (as in the graph on page 649). Estimate the coordinates of the point of intersection. What do these coordinates tell you?

14. Precise Nash. Use your answers to Mindscapes II.8 and II.10 to help you find the exact coordinates at which the two lines in Mindscape II.12 intersect. (Or find equations of the two lines directly and solve them simultaneously.)

15. Positive payoff (H). Below is a payoff matrix for a game involving two players. Player 1 can use either Strategy A or Strategy B; Player 2 can defend against either Strategy A or Strategy B.

	Player 2 Defends Against Strategy A	**Player 2 Defends Against Strategy B**
Player 1 Strategy A	4	10
Player 1 Strategy B	10	7

Sketch a graph showing two lines on the axes to the right, one line giving the expected gain for Player 1 if Player 2 defends against Strategy A, the other giving the expected gain for Player 1 if Player 2 defends against strategy B. Label each line appropriately.



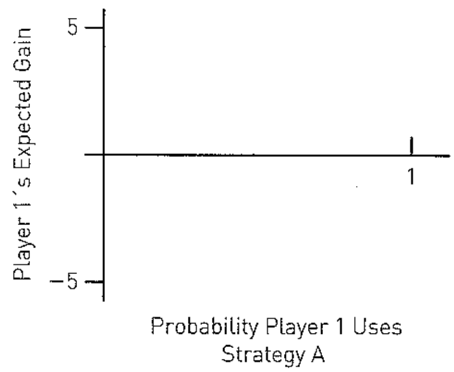
16. Estimating the equilibrium. Estimate the coordinates of the point of intersection of the two lines in the previous Mindscape.

17. Exacting the equilibrium. Find equations for the two lines in Mindscape II.15 and use them to solve for the exact coordinates of the point of intersection. (This point is called the *Nash equilibrium point.*)

18. **Payoffs need not be positive.** Here's a payoff matrix for another game. Notice that some of the payoff's are negative, indicating undesirable states.

	Player 2 Defends Against Strategy A	Player 2 Defends Against Strategy B
Player 1 Strategy A	−5	4
Player 1 Strategy B	2	−3

On the axes below, sketch a graph analogous to that requested in Mindscape II.15.



19. **Negative equilibrium.** Find equations for the two lines in the previous Mindscape and use them to solve for the Nash equilibrium point, in other words, the intersection of these two lines.

20. **Making negative sense.** In what kind of game or scenario might a negative payoff make sense?

III. Creating New Ideas

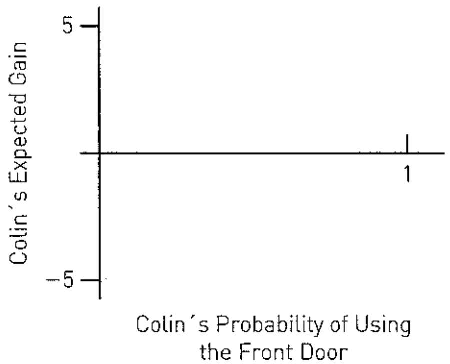
21. **Colin and Tubbes.** When first grader Colin comes home from school each day, his pet hippo Tubbes is waiting to greet him. Because Tubbes is so large and enthusiastic, Colin does not enjoy these blubbery greetings, so he tries to avoid them by switching between the back door and the front door to thwart Tubbes. When Colin succeeds in avoiding Tubbes, he feels a satisfaction "gain" of 5. If Colin enters the front door and finds Tubbes waiting, he feels a satisfaction gain of −5 (in effect, a slobbery loss). If Colin enters the back door and finds Tubbes waiting, his satisfaction loss is only −2 because his mother's presence in the kitchen helps keep Tubbes under control. Create a payoff matrix that illustrates this situation. (Let the top row give Colin's possible payoffs if he uses the front door. Let the first column give the payoffs if Tubbes is waiting at the front door.)

22. **Colin and Tubbes 50–50.** Suppose in Mindscape III.21 that Tubbes waits at the front door. If Colin chooses the front door with probability ½, what is his average satisfaction? Suppose Tubbes waits at the back

door and Colin chooses the back door with probability ½. What is Colin's average satisfaction?

23. **Averaging Colin and Tubbes.** Suppose p is the probability that Colin uses the front door in Mindscape III.21. Write an expression that gives Colin's satisfaction payoff if he uses the front door with probability p and Tubbes always chooses to wait at the front door.

24. **Colin and Tubbes line up (ExH).** Referring to Mindscape III.21, sketch a graph showing two lines on the axes below, one line giving the expected gain for Colin if Tubbes waits at the front door, the other giving the expected gain if Tubbes waits at the back door.



25. **Colin and Tubbes with a dash of Nash (S).** Find the Nash equilibrium point for the Colin and Tubbes scenario.

26. **Diagnosis.** Based on your symptoms, your doctor says, "There's a 40% chance you have disease A, a 30% chance you have disease B, a 20% chance you have disease C, and a 10% chance you have some unknown disease." Create a bar graph similar to the figure on page 655 to illustrate this diagnosis. Out of 1000 patients with identical symptoms, how many would you expect to have disease A? Disease B? Disease C? None of A, B, or C?

27. **Splоosh test (H).** Given the scenario in the previous Mindscape, one way to enhance the diagnosis is to test the blood for *sploosh*. Studies have shown that among people who have disease A, 20% have *sploosh* in their blood; among people with disease B, 50% have *sploosh*; among people with disease C, 10% have *sploosh*; and among people with neither disease A, B, nor C, 30% have *sploosh*. Suppose there are 1000 people with symptoms identical to yours. How many of those with disease A have *sploosh*? How many of those with disease B have *sploosh*? How many of those with disease C have *sploosh*? How many of those with neither disease A, B, nor C have *sploosh*?

28. **Diagnose the data.** Using the data from the previous Mindscape, create a table similar to the one on page 655. Compute the fraction of those with *sploosh* who also have disease A, disease B, disease C, or some unknown disease. Convert your fractions to percentages. Which disease is most likely for those with *sploosh*?

IV. Further Challenges

29. **My theory . . .** You are a paleontologist who finds part of a dinosaur bone. After careful study you determine there's a 50% chance the dinosaur was a Juliasaurus, a 15% chance it was a Noahsaurus, and a 35% chance it is a new dinosaur altogether. Later your graduate student finds a confirmed Maxxasaurus bone at the same site. You know from earlier research that of all confirmed discoveries of Juliasaurus bones, 5% have been found at sites that also contain Maxxasaurus bones. Of all confirmed discoveries of Noahsaurus bones, 30% have been found at sites containing Maxxasaurus bones. And among all discoveries that are neither Juliasaurus or Noahsaurus, 10% have been found at sites containing Maxxasaurus bones. Imagine you have 1000 identical bone specimens. Complete the table below to help you decide which dinosaur is the most likely source of your original discovery.

	Found with Maxxasaurus	**Not Found with Maxxasaurus**	**TOTAL**
Juliasaurus			
Noahsaurus			
Other			
TOTAL			1000

30. **The real (old) story.** What factors affect the reliability of the analysis in the previous Mindscape?

31. **Going stag.** Two Englishmen, Neville and Winston, go hunting. Each may choose to hunt a hare or a stag but must make a choice without knowing the other's decision. A solo hunter will catch a hare easily, but it takes both hunters in cooperation to catch a stag, the more valuable prey. The payoff matrix from Winston's point of view is given below.

	Neville Hunts a Stag	**Neville Hunts a Hare**
Winston Hunts a Stag	5	0
Winston Hunts a Hare	2	2

Find the Nash equilibrium point for this game. Is this point truly optimal for both hunters?

32. **Selling sweets.** Two snack cake companies, *Snacky Cake*s and *Little Eddie*, compete for the same consumers. If both spend the same amount on advertising, the effect is to cancel each other's message and there is no change in their sales. Yet the extra spending reduces profits (payoffs). Each would like to spend less on advertising, but if one reduces its budget and the other doesn't, then the latter company

would see increased sales over and above its additional spending on advertising. Fill in the table below to show a possible payoff matrix from the point of view of *Little Eddie* snack cakes.

	***Snacky Cakes* Spends More on Advertising**	***Snacky Cakes* Spends Less on Advertising**
Little Eddie Spends More on Advertising		
Little Eddie Spends Less on Advertising		

Create a graph illustrating your scenario. Is there a Nash equilibrium point?

33. Prisoner's dilemma. Two people are suspected of robbing a bank. They are being interrogated in separate rooms. If both stay silent, they can be convicted of a lesser crime and sentenced to only 6 months. If one confesses (or "defects") and the other does not, the confessor goes free as a reward for cooperating while the other suspect will be sent to prison for 10 years. If both defect, they each go to prison for 5 years. Create a payoff matrix from the point of view of Suspect #1. (Adopt the method from previous Mindscapes.)

V. In Your Own Words

34. Personal perspectives. Write a short essay describing the most interesting or surprising discovery you made in exploring the material in this section. If any material seemed puzzling or even unbelievable, address that as well. Explain why you chose the topics you did. Finally, comment on the aesthetics of the mathematics and ideas in this section.

35. With a group of folks. In a small group, discuss the ideas of strategy and game-playing introduced in this section. After your discussion, write a brief narrative describing these ideas in your own words.

36. Creative writing. Write an imaginative story (it can be humorous, dramatic, whatever you like) that involves or evokes the ideas of this section.

37. Power beyond the mathematics. Provide several real-life issues—ideally, from your own experience—that some of the strategies of thought presented in this section would effectively approach and resolve.