But to us, probability is the very guide of life.

BISHOP JOSEPH BUTLER

Many, if not most, significant events in our lives arise from coincidence, randomness, and uncertainty. We meet friends and loved ones, we find intriguing opportunities, we fall into a profession or lifestyle. At a deeper level, the most basic interactions of molecules and subatomic particles are described in terms of probabilities and statistics. Nothing is more fundamental than chance. However, the uncertain and the unknown are not forbidding territories into which we dare not tread. There are ways to organize and understand them that can add meaning to our lives.

Probability enables us to better understand our uncertain world. It moves us from a vague sense of disordered randomness to a focused concept of measured proportion. Probability serves as the mathematical foundation of common sense, wisdom, and good judgment. Perhaps, too, it lets us view our world more truly as it is—a place where the totality follows rules of the aggregate while leaving individuals to their wild variation and unbridled possibilities.

Measure what is measurable and make measurable what is not so.

GALILEO GALILEI

We develop a measure of likelihood by looking at situations in which the future is uncertain but the possible outcomes are definite and easily described. Gambling games provide concrete and clear illustrations because dice and coins can teach us how to measure likelihood. We apply the principles we develop to measure the value of future possibilities, thus allowing us to weigh decisions involving the unknown future. Frequently, we must extrapolate the probable future from evidence from the past. This need presents us with the challenge of collecting and meaningfully interpreting data.

Surprises guide us in the study of the uncertain and the unknown. But we progress by considering concrete examples, by performing experiments to ground our theory in experience, and by looking at fallacies. Simple, clear cases let us develop principles that we can apply widely. So, even the uncertain and unknown are best understood by starting with the simple and the familiar.

7.1 Chance Surprises

Some Scenarios Involving Chance That Confound Our Intuition

The Dream (1921) by Max Beckmann. Analyzing what we see sometimes leads to surprising, counterintuitive results. (© 2009 Artists Rights Society (ARS), NY)

Chance, too, which seems to rush along with slack reins, is bridled and governed by law.

BOETHIUS

Many surprises lurk in the world of chance. We guess wrong because our intuition is untrained or mistrained. Each surprise is an enticement for us to find structure among the forces of the uncertain and unknown. In the following sections, we develop methods for analyzing chance scenarios. Let's begin with vintage TV.

Let's Make a Deal

Revisit the *Let's Make a Deal* scenario from Chapter 1 (Section 1.1, Story 7). The contestant selects one of three doors. The all-knowing host, Monty Hall, knowing the location of all prizes, opens another door to reveal one of the two mules rather than the lone Cadillac. The contestant now has the option to stick with the original guess or switch doors.

- *What we expect.* We probably expect that switching or sticking makes no difference. We might think that in either case the chance of getting the Cadillac is 1 out of 2, since there are two remaining closed doors.
- *Surprise.* Switching gives the contestant a 2 in 3 chance of winning the car. Why? The explanation is in Chapter 1 (Section 1.3, Story 7).

Try It On the *Heart of Mathematics* Web site, you will find a program that allows you to play the game or simulate playing this game many times to confirm the *Let's Make a Deal* probability. If you want to make it physical, here is an experiment that verifies the probability live. From a deck of regular playing cards, remove three cards—a king and two aces. The king represents the Cadillac, and the aces represent the mules. Have a friend act as the dealer. The dealer shuffles these three cards and places them facedown on a table, side by side, without looking at them. Once the cards are on the table, the dealer peeks under each card so that the location of the king is known to the dealer but not to you. Point to a card. The dealer then turns over one of the other two cards to reveal one of the aces (the mules). You now have the chance to switch cards. Stick with your original guess and see what happens. Have the dealer shuffle the cards again and repeat the exact same scenario—don't switch—at least 30 times and see what fraction of the time you end up with the king (the car). Now, do the same experiment, but this time try switching and again record how often you find the king. After repeating this experiment several times, you will discover that about one-third of the time you find the king if you stick to your original guess, and about two-thirds of the time you will find the king if you switch.

Reunion Scene—Take One

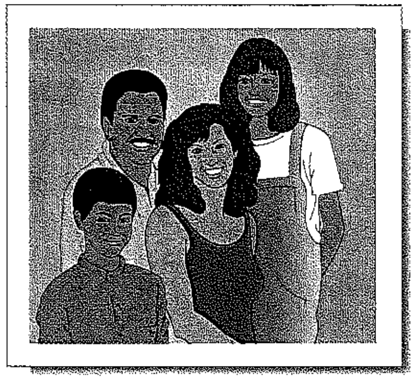
Suppose we return to our 25th college reunion and we see an old classmate.

CLASSMATE: I have two children.
WE: Wonderful! Is the *older* one a boy?
CLASSMATE: Yes! And . . .
(*At this point she chokes on an hors d'oeuvre and collapses.*)

Reunion Scene—Take Two

Suppose we return to our 25th college reunion and we see an old classmate.

CLASSMATE: I have two children.
WE: Wonderful! Is at least *one* of them a boy?
CLASSMATE: Yes! And . . .
(*At this point she chokes on an hors d'oeuvre and collapses.*)



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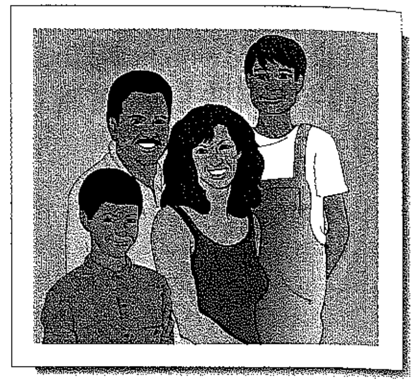

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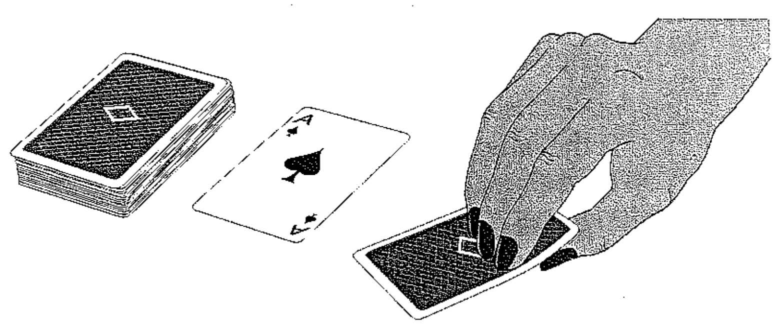
Naturally, we have some concern for the respiratory challenges of our former classmate; however, we probably would be more consumed with the following burning question: What is the chance that our classmate has two boys? Is the probability the same in both scenarios?

- *What we expect.* We probably expect that, in both cases, there is a 50–50 chance that she has two boys.
- *Surprise.* In Take One, the chance is exactly one-half (as expected); however, in Take Two the chance is only 1 in 3. Why? We'll see why in the next section.

Try It You can simulate this situation with a deck of cards. Think of the boys as the black cards and the girls as the red cards. Shuffle the deck and remove cards from the top of the deck in pairs. First look at all the pairs whose first card is black. What fraction of those have both cards black? Now start over. Shuffle the cards and take them off in pairs; however, this time look at all pairs that contain at least one black card (disregard the pairs of two red cards). What fraction of those have both cards black? Does this experiment tend to confirm our original intuition or the surprising result?



Is this card red?

The Birthday Question

How many people are needed in a room so that the probability of two people sharing the same birthday is roughly one-half?

- *What we expect.* We might expect that this experiment requires about 183 people. If 367 people were in the room, then we would be guaranteed at least two people who would share the same birthday, since 367 people can't all have different birthdays by the pigeonhole principle (see Chapter 2, Section 2.1). Therefore, if we want the chances of a matched pair of birthdays to be approximately 50–50, then it seems we would need about 183 people in the room (about half of the 367 people) for the chance of finding a shared birthday to be roughly 50%.
- *Surprise.* In a room containing only 23 people, the chance of two people sharing the same birthday is just *over* 50%. That is, in a random gathering of 23 people, we will more often than not find a pair of people with the same birthday. In a room with 183 people, the chance of finding a pair of people with the same birthday is over 99.999999%. Why? We'll see how to analyze this birthday surprise in the next section.

Try It The next time you are in a room with 40 people or so, ask them for their birthdays and see whether you find a common birthday. Feel free to wager, if you are so inclined.

A Look Back

MATTERS OF CHANCE can have satisfying explanations. However, we need to develop a sense of measuring uncertain events so that the experiences associated with the types of surprises discussed in this section come to appear natural and expected.

But to us, probability is the very guide of life.

BISHOP JOSEPH BUTLER

Thinking about situations that jar our intuition can lead to new and important insights.

Mindscapes Invitations to Further Thought

In this section, Mindscapes marked **(H)** *have hints for solutions at the back of the book. Mindscapes marked* **(ExH)** *have expanded hints at the back of the book. Mindscapes marked* **(S)** *have solutions.*

I. Developing Ideas

1. **Doors galore.** The 21st-century version of *Let's Make a Deal* has five doors instead of three. Two doors have cars behind them and the other three doors have mules. What percentage of doors have cars behind them?
2. **Birthday surprise.** How many people would you need to have in a room so that the chance that two (or more) of them share a birthday is over 50%?
3. **Opposite of heads.** Suppose you flip a coin 100 times, with 53 tosses landing heads up. What percentage of the tosses would be tails?
4. **Penny percent.** Suppose you flip a penny 50 times, with 28 tosses landing heads up. What percentage of the tosses would be heads? What percentage would be tails?
5. **Party time.** At a nephew's party, you decide to write down everyone's birthday. Here are your results:

Julia	Dec. 18	Isabel	Sept. 1
Max	Aug. 26	Colin	Dec. 31
Melinda	Oct. 1	Alexandra	Mar. 14
Zack	Jan. 17	Philip	Dec. 9
Drew	Apr. 18	Victoria	July 10
Margaret	June 16	Douglas	Oct. 31

What percentage of children have their birthdays in December? In February? What percentage have their birthdays in the same month as another child at the party?

II. Solidifying Ideas

6. **Flipping Lincoln.** Flip a penny 100 times and record how many pennies land heads up and tails up. What percentage of the pennies landed heads up?
7. **Flashing cards.** Shuffle a standard deck of 52 playing cards. Turn over the top 20 cards one by one and record how many are red. Of those cards you turned over, what percentage was black?

8. **King for a day.** Remove three cards from a deck of regular playing cards—a king and two aces. Shuffle the cards and choose one at random. Record whether it's a king or ace. Repeat this experiment 20 times. What percentage of the time did you select the king?

9. **A card deal stick.** Remove three cards from a deck of regular playing cards—a king and two aces. Have a friend act as the dealer. The dealer shuffles these three cards and places them facedown on a table, side by side, without looking at them. Once the cards are on the table, the dealer peeks under each card so that the location of the king is known to the dealer but not to you. Point to a card. The dealer then turns over one of the other two cards to reveal one of the aces. Stick with your original guess, turn over that card, and record whether you chose the king. Have the dealer scramble the cards again and repeat the scenario—again, don't switch—and record the result. Repeat this experiment 50 times (you can get very quick at it). What percentage of the time did you choose the king?

10. **A card deal switch.** Remove three cards from a deck of regular playing cards—a king and two aces. Have a friend act as the dealer. The dealer shuffles these three cards and places them facedown on a table, side by side, without looking at them. Once the cards are on the table, the dealer peeks under each card so that the location of the king is known to the dealer but not to you. Point to a card. The dealer then turns over one of the other two cards to reveal one of the aces. Now switch your guess to the other facedown card. Turn over that card and record whether you chose the king. Have the dealer shuffle the cards again and repeat the scenario—that is, switch your guess each time after the dealer turns over an ace—and record the result. Repeat this experiment 50 times (you can get very quick at it). What percentage of the time did you choose the king?

11. **A card reunion—black first (S).** Using a shuffled deck of cards, remove cards from the top of the deck in pairs. For each pair where the first card is black, record whether the second card is red or black. After you have gone through 10 pairs, reshuffle the deck and repeat until you have recorded 50 cases where the first card of the pair is black. What fraction of those pairs had both cards black?

12. **A card reunion (H).** Using a shuffled deck of cards, remove cards from the top of the deck in pairs. For each pair where at least one of the cards is black, record whether both cards are black or one is black and one is red. After you have gone through 10 pairs, reshuffle the deck and repeat until you have recorded 50 cases where at least one card of the pair is black. What fraction of those pairs had both cards black?

13. **Birthday bash.** The next time you are in a room with 40 people or so, ask them for their birthdays to see whether you find a common birthday.

14. **Presidential birthdays (ExH).** Have two presidents of the United States shared a birthday?
15. **Vice-presidential birthdays.** Have two vice-presidents of the United States shared a birthday?

III. In Your Own Words

16. **Personal perspectives.** Write a short essay describing the most interesting or surprising discovery you made in exploring the material in this section. If any material seemed puzzling or even unbelievable, address that as well. Explain why you chose the topics you did. Finally, comment on the aesthetics of the mathematics and ideas in this section.
17. **With a group of folks.** In a small group, discuss the surprises involving chance found in this section. After your discussion, write a brief narrative describing the surprising features in your own words.