Planar diagram codes and Reidemeister moves of type 2

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Identifying that a knot can be simplified by a Reidemeister move of type 2 based on its planar diagram code

A knot is said to be "simplifiable" by a Reidemeister move of type 2 if two crossings of the knot can be eliminated by moving one loop completely over another without creating any new crossings.

Claim. Two consecutive tuples in a planar diagram (PD) code indicate that the knot can be simplified by a Reidemeister move of type 2 if and only if the tuples are of the form $(a, \gamma, \beta, b)(\beta, \gamma, c, d)$ or $(a, b, \gamma, \beta)(\gamma, c, d, \beta)$.

Proof. We begin by showing that if a knot can be simplified by a Reidemeister move of type 2, then the PD code tuples representing the crossings to be eliminated are of the form $(a, \gamma, \beta, b)(\beta, \gamma, c, d)$ or $(a, b, \gamma, \beta)(\gamma, c, d, \beta)$.

There are eight possible configurations of a knot segment which can be simplified by a Reidemeister move of type 2, depending on the orientation of the knot. These are depicted in Figure 1, supposing that the knot is oriented such that we traverse the knot from a_1 toward a_3 and then from b_1 toward b_3.

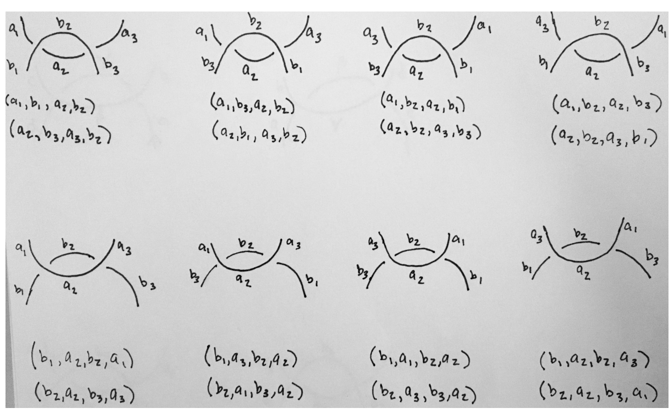

Figure 1: Knot segments which can be simplified by a Reidemeister move of type 2 and the corresponding PD code tuples

Note that the PD codes of these segments have one of two forms, $(a, \gamma, \beta, b)(\beta, \gamma, c, d)$ or $(a, b, \gamma, \beta)(\gamma, c, d, \beta)$, depending on whether we first turn into the loop or way from it as we move counter-clockwise to determine the PD code.

We will finish by showing that every PD code tuple of the form $(a, \gamma, \beta, b)(\beta, \gamma, c, d)$ or $(a, b, \gamma, \beta)(\gamma, c, d, \beta)$ corresponds to a knot segment which can be simplified by a Reidemeister move of type 2.

Let us consider the diagrams that correspond to PD codes of the form $(a, \gamma, \beta, b)(\beta, \gamma, c, d)$ or $(a, b, \gamma, \beta)(\gamma, c, d, \beta)$.

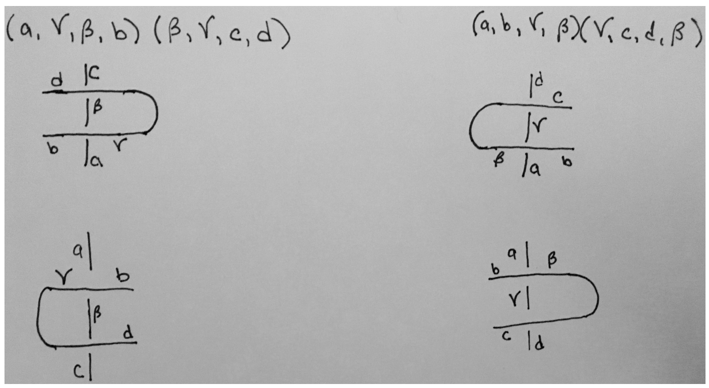

Figure 2: Knot segment diagrams with consecutive PD code tuples of the form $(a, \gamma, \beta, b)(\beta, \gamma, c, d)$ or $(a, b, \gamma, \beta)(\gamma, c, d, \beta)$

From inspection we see that these knot segments can be simplified by a Reidemeister move of type 2.

Hence two consecutive tuples in a planar diagram (PD) code indicate that the knot can be simplified by a Reidemeister move of type 2 if and only if the tuples are of the form $(a, \gamma, \beta, b)(\beta, \gamma, c, d)$ or $(a, b, \gamma, \beta)(\gamma, c, d, \beta)$.

$\square$

How the planar diagram code of a knot changes when the knot is simplified by a Reidemeister move of type 2

Suppose that $(a, \gamma, \beta, b)(\beta, \gamma, c, d)$ or $(a, b, \gamma, \beta)(\gamma, c, d, \beta)$ is a pair of PD code tuples which indicates that a knot can be simplified by a Reidemeister move of type 2. To adjust the PD code of the knot by performing a Reidemeister move of type 2:

1. Remove the tuples $(a, \gamma, \beta, b)(\beta, \gamma, c, d)$ or $(a, b, \gamma, \beta)(\gamma, c, d, \beta)$ from the PD code.

2. Apply the function f to each element of the remaining tuples, where f is defined by

$$f(x) = \begin{cases} x & x < min(\gamma, \beta) \\ x - 2 & min(\gamma, \beta) < x < max(\gamma, \beta) \\ x - 4 & x > max(\gamma, \beta) \end{cases}$$