How the planar diagram code of a knot is changed by a "drag the underpass" move

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Let (a, b, c, d) denote the crossing being dragged and (e, f, g, h) denote the crossing directly before or after (a, b, c, d) which we will drag (a, b, c, d) underneath.

There are eight configurations for the crossings (a, b, c, d) and (e, f, g, h) depending on the orientation of the knot and which element the two tuples have in common. Each of these configurations takes one of two forms - the segments ac and eg can be drawn as either perpendicular (see Figure 1) or parallel (see Figure 2) to each other.

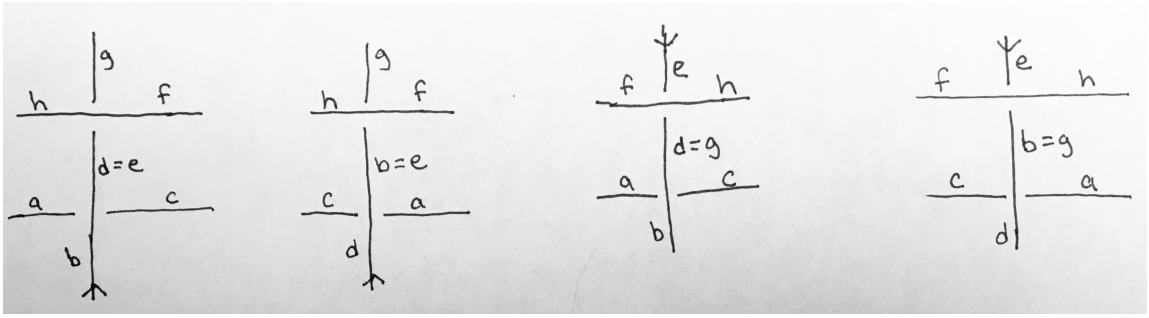

Figure 1: Knot segments with ac perpendicular to eg

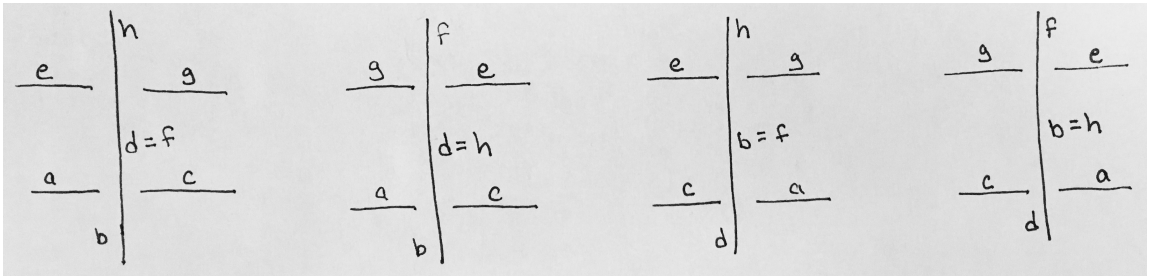

Figure 2: Knot segments with ac parallel to eg

In all cases, a "drag the underpass" move results in the knot being subdivided into four additional segments. This is illustrated in Figure 3, where the colored portions indicate the

sections of the original knot (left) that become new segments after dragging the underpass (a, b, c, d) (right).

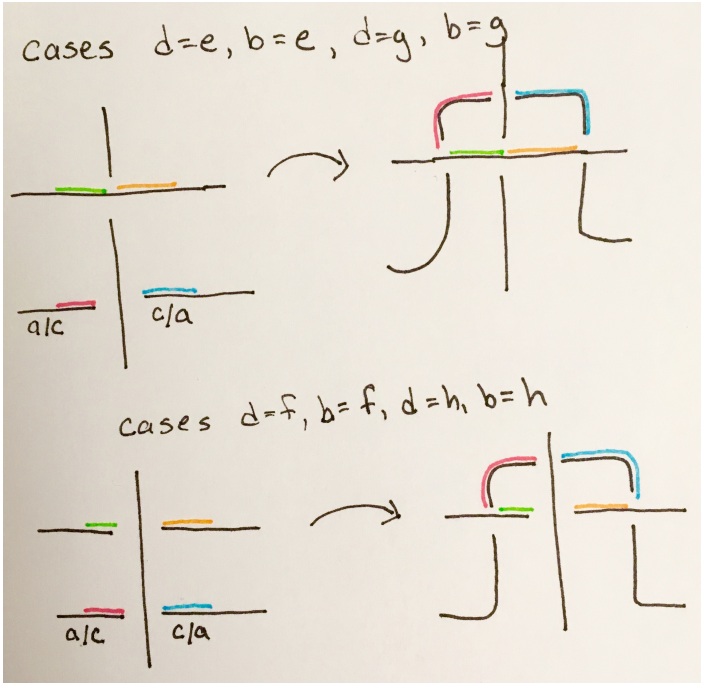

Figure 3: Knot segments before and after a "drag the underpass" move

The resulting change in PD code of every tuple other than (a, b, c, d) and (e, f, g, h) can be expressed by applying the function f to each element of the tuples.

If $d = e$, $b = e$, $d = g$ or $b = g$, then f is defined as:

$$f(x) = \begin{cases} x & x \leq min(a, c, f, h) \\ x + 2 & min(a, c, f, h) < x < max(a, c, f, h) \\ x + 4 & x \geq max(a, c, f, h) \end{cases}$$

Otherwise if $d = f$, $b = f$, $d = h$ or $b = h$, then f is defined as:

$$f(x) = \begin{cases} x & x \leq min(a, c, e, g) \\ x + 2 & min(a, c, e, g) < x < max(a, c, e, g) \\ x + 4 & x \geq max(a, c, e, g) \end{cases}$$

When we drag (a, b, c, d) under (e, f, g, h), the PD code tuple (a, b, c, d) is replaced by three tuples and the value of the elements in tuple (e, f, g, h) change. Exactly how the values of these tuples change depends on:

- which elements of the tuples (a, b, c, d) and (e, f, g, h) have the same value, and
- the order and orientation in which the segments of the knot are traversed.

Once we know which elements have the same value, there are $3! * 2 = 12$ sub-cases to consider - 3! for the number of ways to choose the order in which we traverse the segments then times 2 for the number of ways fh can be oriented.

Let y denote the element, f or h, we travel from toward the other. Most often, $|f-h| = 1$ and $y = min\,(f, h)$. However, there may be situations where $min\,(f, h) = 1$ and $max\,(f, h)$ equals the number of segments in the knot, in which case $y = max\,(f, h)$.

The order in which the segments are traversed can now be determined by ordering a, c and y from least to greatest, where the least element belongs to the first segment we traverse, the second greatest element to the second segment we traverse, and so on.

Case $d = e$

If $a < e < y, y = f$:
$(a, b, c, d) \to (a, f + 4, a + 1, f + 5)(a + 1, b + 3, a + 2, b + 4)(a + 2, f + 3, a + 3, f + 2)$
$(e, f, g, h) \to (b + 2, f + 3, b + 3, f + 4)$

If $a < e < y, y = h$:
$(a, b, c, d) \to (a, h + 3, a + 1, h + 2)(a + 1, b + 3, a + 2, b + 4)(a + 2, h + 4, a + 3, h + 5)$
$(e, f, g, h) \to (b + 2, h + 4, b + 3, h + 3)$

If $a < y < e, y = f$:
$(a, b, c, d) \to (a, f + 4, a + 1, f + 5)(a + 1, b + 5, a + 2, b + 6)(a + 2, f + 3, a + 3, f + 2)$
$(e, f, g, h) \to (b + 4, f + 3, b + 5, f + 4)$

If $a < y < e, y = h$:
$(a, b, c, d) \to (a, h + 3, a + 1, h + 2)(a + 1, b + 5, a + 2, b + 6)(a + 2, h + 4, a + 3, h + 5)$
$(e, f, g, h) \to (b + 4, h + 4, b + 5, h + 3)$

If $e < a < y, y = f$:
$(a, b, c, d) \to (a, f + 4, a + 1, f + 5)(a + 1, b + 1, a + 2, b + 2)(a + 2, f + 3, a + 3, f + 2)$
$(e, f, g, h) \to (b, f + 3, b + 1, f + 4)$

If $e < a < y, y = h$**:**
$(a, b, c, d) \to (a, h+3, a+1, h+2)(a+1, b+1, a+2, b+2)(a+2, h+4, a+3, h+5)$
$(e, f, g, h) \to (b, h+4, b+1, h+3)$

If $e < y < a, y = f$**:**
$(a, b, c, d) \to (a+2, f+2, a+3, f+3)(a+3, b+1, a+4, b+2)(a+4, f+1, a+5, f)$
$(e, f, g, h) \to (b, f+1, b+1, f+2)$

If $e < y < a, y = h$**:**
$(a, b, c, d) \to (a+2, h+1, a+3, h)(a+3, b+1, a+4, b+2)(a+4, h+2, a+5, h+3)$
$(e, f, g, h) \to (b, h+2, b+1, h+1)$

If $y < a < e, y = f$**:**
$(a, b, c, d) \to (a+2, f+2, a+3, f+3)(a+3, b+5, a+4, b+6)(a+4, f+1, a+5, f)$
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If $y < a < e, y = h$**:**
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$(a, b, c, d) \to (a+2, f+2, a+3, f+3)(a+3, b+3, a+4, b+4)(a+4, f+1, a+5, f)$
$(e, f, g, h) \to (b+2, f+1, b+3, f+2)$

If $y < e < a, y = h$**:**
$(a, b, c, d) \to (a+2, h+1, a+3, h)(a+3, b+3, a+4, b+4)(a+4, h+2, a+5, h+3)$
$(e, f, g, h) \to (b+2, h+2, b+3, h+1)$

Case $b = e$

If $a < e < y, y = f$**:**
$(a, b, c, d) \to (a, f+2, a+1, f+3)(a+1, d+4, a+2, d+3)(a+2, f+5, a+3, f+4)$
$(e, f, g, h) \to (d+2, f+3, d+3, f+4)$

If $a < e < y, y = h$**:**
$(a, b, c, d) \to (a, h+5, a+1, h+4)(a+1, d+4, a+2, d+3)(a+2, h+2, a+3, h+3)$
$(e, f, g, h) \to (d+2, h+4, d+3, h+3)$

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If $e < a < y, y = h$**:**
$(a, b, c, d) \to (a, h+5, a+1, h+4)(a+1, d+2, a+2, d+1)(a+2, h+2, a+3, h+3)$
$(e, f, g, h) \to (d, h+4, d+1, h+3)$

If $e < y < a, y = f$**:**
$(a, b, c, d) \to (a+2, f, a+3, f+1)(a+3, d+2, a+4, d+1)(a+4, f+3, a+5, f+2)$
$(e, f, g, h) \to (d, f+1, d+1, f+2)$

If $e < y < a, y = h$**:**
$(a, b, c, d) \to (a+2, h+3, a+3, h+2)(a+3, d+2, a+4, d+1)(a+4, h, a+5, h+1)$
$(e, f, g, h) \to (d, h+2, d+1, h+1)$

If $y < a < e, y = f$**:**
$(a, b, c, d) \to (a+2, f, a+3, f+1)(a+3, d+6, a+4, d+5)(a+4, f+3, a+5, f+2)$
$(e, f, g, h) \to (d+4, f+1, d+5, f+2)$

If $y < a < e, y = h$:
$(a, b, c, d) \to (a+2, h+3, a+3, h+2)(a+3, d+6, a+4, d+5)(a+4, h, a+5, h+1)$
$(e, f, g, h) \to (d+4, h+2, d+5, h+1)$

If $y < e < a, y = f$:
$(a, b, c, d) \to (a+2, f, a+3, f+1)(a+3, d+4, a+4, d+3)(a+4, f+3, a+5, f+2)$
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If $y < e < a, y = h$:
$(a, b, c, d) \to (a+2, h+3, a+3, h+2)(a+3, d+4, a+4, d+3)(a+4, h, a+5, h+1)$
$(e, f, g, h) \to (d+2, h+2, d+3, h+1)$

Case $d = g$

If $a < e < y, y = f$:
$(a, b, c, d) \to (a, f+3, a+1, f+2)(a+1, e+3, a+2, e+2)(a+2, f+4, a+3, f+5)$
$(e, f, g, h) \to (e+3, f+3, e+4, f+4)$

If $a < e < y, y = h$:
$(a, b, c, d) \to (a, h+4, a+1, h+5)(a+1, e+3, a+2, e+2)(a+2, h+3, a+3, h+2)$
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If $e < a < y, y = h$**:**
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If $e < y < a, y = f$**:**
$(a, b, c, d) \to (a+2, f+1, a+3, f)(a+3, e+1, a+4, e)(a+4, f+2, a+5, f+3)$
$(e, f, g, h) \to (e+1, f+1, e+2, f+2)$

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$(e, f, g, h) \to (e+1, h+2, e+2, h+1)$

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$(a, b, c, d) \to (a+2, h+2, a+3, h+3)(a+3, e+3, a+4, e+2)(a+4, h+1, a+5, h)$
$(e, f, g, h) \to (e+3, h+2, e+4, h+1)$

Case $b = g$

If $a < e < y, y = f$**:**
$(a, b, c, d) \to (a, f+5, a+1, f+4)(a+1, e+2, a+2, e+3)(a+2, f+2, a+3, f+3)$
$(e, f, g, h) \to (e+3, f+3, e+4, f+4)$

If $a < e < y, y = h$**:**
$(a, b, c, d) \to (a, h+2, a+1, h+3)(a+1, e+2, a+2, e+3)(a+2, h+5, a+3, h+4)$
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$(a, b, c, d) \to (a+2, f+3, a+3, f+2)(a+3, e, a+4, e+1)(a+4, f, a+5, f+1)$
$(e, f, g, h) \to (e+1, f+1, e+2, f+2)$

If $e < y < a, y = h$**:**
$(a, b, c, d) \to (a+2, h, a+3, h+1)(a+3, e, a+4, e+1)(a+4, h+3, a+5, h+2)$
$(e, f, g, h) \to (e+1, h+2, e+2, h+1)$

If $y < a < e, y = f$**:**
$(a, b, c, d) \to (a+2, f+3, a+3, f+2)(a+3, e+4, a+4, e+5)(a+4, f, a+5, f+1)$
$(e, f, g, h) \to (e+5, f+1, e+6, f+2)$

If $y < a < e, y = h$**:**
$(a, b, c, d) \to (a+2, h, a+3, h+1)(a+3, e+4, a+4, e+5)(a+4, h+3, a+5, h+2)$
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$(e, f, g, h) \to (e+3, h+2, e+4, h+1)$

Case $d = f$

If $a < e < min(b, h)$**:**
$(a, b, c, d) \to (a, e+3, a+1, e+2)(a+1, min(b, h)+5, a+2, h+4)(a+2, e+4, a+3, e+5)$
$(e, f, g, h) \to (e+3, b+4, e+4, min(b, h)+5)$

If $a < min(b, h) < e$**:**
$(a, b, c, d) \to (a, e+3, a+1, e+2)(a+1, min(b, h)+3, a+2, h+2)(a+2, e+4, a+3, e+5)$
$(e, f, g, h) \to (e+3, b+2, e+4, min(b, h)+3)$

If $e < a < min(b, h)$**:**
$(a, b, c, d) \to (a+2, e+1, a+3, e)(a+3, min(b, h)+5, a+4, h+4)(a+4, e+2, a+5, e+3)$
$(e, f, g, h) \to (e+1, b+4, e+2, min(b, h)+5)$

If $e < min(b, h) < a$**:**
$(a, b, c, d) \to (a + 2, e + 1, a + 3, e)(a + 3, min(b, h) + 3, a + 4, h + 2)(a + 4, e + 2, a + 5, e + 3)$
$(e, f, g, h) \to (e + 1, b + 2, e + 2, min(b, h) + 3)$

If $min(b, h) < a < e$**:**
$(a, b, c, d) \to (a, e + 3, a + 1, e + 2)(a + 1, min(b, h) + 1, a + 2, h)(a + 2, e + 4, a + 3, e + 5)$
$(e, f, g, h) \to (e + 3, b, e + 4, min(b, h) + 1)$

If $min(b, f) < e < a$**:**
$(a, b, c, d) \to (a + 2, e + 1, a + 3, e)(a + 3, min(b, h) + 1, a + 4, h)(a + 4, e + 2, a + 5, e + 3)$
$(e, f, g, h) \to (e + 1, b, e + 2, min(b, h) + 1)$

Case $d = h$

If $a < e < min(b, f)$**:**
$(a, b, c, d) \to (a, e + 4, a + 1, e + 5)(a + 1, min(b, f) + 5, a + 2, f + 4)(a + 2, e + 3, a + 3, e + 2)$
$(e, f, g, h) \to (e + 3, min(b, f) + 5, e + 4, b + 4)$

If $a < min(b, f) < e$**:**
$(a, b, c, d) \to (a, e + 4, a + 1, e + 5)(a + 1, min(b, f) + 3, a + 2, f + 2)(a + 2, e + 3, a + 3, e + 2)$
$(e, f, g, h) \to (e + 3, min(b, f) + 3, e + 4, b + 2)$

If $e < a < min(b, f)$**:**
$(a, b, c, d) \to (a + 2, e + 2, a + 3, e + 3)(a + 3, min(b, f) + 5, a + 4, f + 4)(a + 4, e + 1, a + 5, e)$
$(e, f, g, h) \to (e + 1, min(b, f) + 5, e + 2, b + 4)$

If $e < min(b, f) < a$**:**
$(a, b, c, d) \to (a + 2, e + 2, a + 3, e + 3)(a + 3, min(b, f) + 3, a + 4, f + 2)(a + 4, e + 1, a + 5, e)$
$(e, f, g, h) \to (e + 1, min(b, f) + 3, e + 2, b + 2)$

If $min(b, f) < a < e$**:**
$(a, b, c, d) \to (a, e + 4, a + 1, e + 5)(a + 1, min(b, f) + 1, a + 2, f)(a + 2, e + 3, a + 3, e + 2)$
$(e, f, g, h) \to (e + 3, min(b, f) + 1, e + 4, b)$

If $min(b, f) < e < a$**:**
$(a, b, c, d) \to (a + 2, e + 2, a + 3, e + 3)(a + 3, min(b, f) + 1, a + 4, f)(a + 4, e + 1, a + 5, e)$
$(e, f, g, h) \to (e + 1, min(b, f) + 1, e + 2, b)$

Case $b = f$

If $a < e < min(d, h)$**:**
$(a, b, c, d) \to (a, e + 5, a + 1, e + 4)(a + 1, h + 4, a + 2, min(d, h) + 5)(a + 2, e + 2, a + 3, e + 3)$
$(e, f, g, h) \to (e + 3, d + 4, e + 4, min(d, h) + 5)$

If $a < min(d, h) < e$**:**
$(a, b, c, d) \to (a, e + 5, a + 1, e + 4)(a + 1, h + 2, a + 2, min(d, h) + 3)(a + 2, e + 2, a + 3, e + 3)$
$(e, f, g, h) \to (e + 3, d + 2, e + 4, min(d, h) + 3)$

If $e < a < min(d, h)$**:**
$(a, b, c, d) \to (a + 2, e + 3, a + 3, e + 2)(a + 3, h + 4, a + 4, min(d, h) + 5)(a + 4, e, a + 5, e + 1)$
$(e, f, g, h) \to (e + 1, d + 4, e + 2, min(d, h) + 5)$

If $e < min(d, h) < a$**:**
$(a, b, c, d) \to (a + 2, e + 3, a + 3, e + 2)(a + 3, h + 2, a + 4, min(d, h) + 3)(a + 4, e, a + 5, e + 1)$
$(e, f, g, h) \to (e + 1, d + 2, e + 2, min(d, h) + 3)$

If $min(d, h) < a < e$**:**
$(a, b, c, d) \to (a, e + 5, a + 1, e + 4)(a + 1, h, a + 2, min(d, h) + 1)(a + 2, e + 2, a + 3, e + 3)$
$(e, f, g, h) \to (e + 3, d, e + 4, min(d, h) + 1)$

If $min(d,h) < e < a$**:**
$(a,b,c,d) \to (a+2, e+3, a+3, e+2)(a+3, h, a+4, min(d,h)+1)(a+4, e, a+5, e+1)$
$(e,f,g,h) \to (e+1, d, e+2, min(d,h)+1)$

Case $b = h$

If $a < e < min(d,f)$**:**
$(a,b,c,d) \to (a, e+2, a+1, e+3)(a+1, f+4, a+2, min(d,f)+5)(a+2, e+5, a+3, e+4)$
$(e,f,g,h) \to (e+3, min(d,f)+5, e+4, d+4$

If $a < min(d,f) < e$**:**
$(a,b,c,d) \to (a, e+2, a+1, e+3)(a+1, f+2, a+2, min(d,f)+3)(a+2, e+5, a+3, e+4)$
$(e,f,g,h) \to (e+3, min(d,f)+3, e+4, d+2)$

If $e < a < min(d,f)$**:**
$(a,b,c,d) \to (a+2, e, a+3, e+1)(a+3, f+4, a+4, min(d,f)+5)(a+4, e+3, a+5, e+2)$
$(e,f,g,h) \to (e+1, min(d,f)+5, e+2, d+4)$

If $e < min(d,f) < a$**:**
$(a,b,c,d) \to (a+2, e, a+3, e+1)(a+3, f+2, a+4, min(d,f)+3)(a+4, e+3, a+5, e+2)$
$(e,f,g,h) \to (e+1, min(d,f)+3, e+2, d+2)$

If $min(d,f) < a < e$**:**
$(a,b,c,d) \to (a, e+2, a+1, e+3)(a+1, f, a+2, min(d,f)+1)(a+2, e+5, a+3, e+4)$
$(e,f,g,h) \to (e+3, min(d,f)+1, e+4, d)$

If $min(d,f) < e < a$**:**
$(a,b,c,d) \to (a+2, e, a+3, e+1)(a+3, f, a+4, min(d,f)+1)(a+4, e+3, a+5, e+2)$
$(e,f,g,h) \to (e+1, min(d,f)+1, e+2, d)$

Example of a PD code change by dragging an underpass

Consider the knot diagram in Figure 4 which has the PD code
$(1,9,2,8)(3,7,4,6)(5,10,6,11)(7,3,8,2)(9,1,10,12)(11,4,12,5)$.

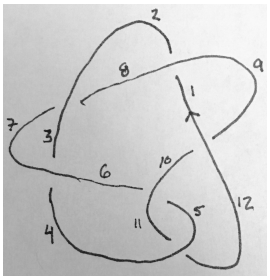

Figure 4: A labeled diagram of knot 6_1

Let $(a,b,c,d) = (3,7,4,6)$ and $(e,f,g,h) = (5,10,6,11)$. Inspecting the tuples we see that this knot fits the case $d = g$ with $a < e < y, y = f$, so

$(a,b,c,d) \to (a, f+3, a+1, f+2)(a+1, e+3, a+2, e+2)(a+2, f+4, a+3, f+5)$
$(e,f,g,h) \to (e+3, f+3, e+4, f+4)$

We apply the function f to each element of all the other tuples where f is defined as:

$$f(x) = \begin{cases} x & x \leq 3 \\ x+2 & 3 < x < 11 \\ x+4 & x \geq 11 \end{cases}$$

Hence the PD code after dragging the underpass is
$(1,11,2,10)(3,13,4,12)(4,8,5,7)(5,14,6,15)(8,13,9,14)(9,3,10,2)(11,1,12,16)(15,6,16,7)$,
which can be verified with the knot diagram in Figure 5.

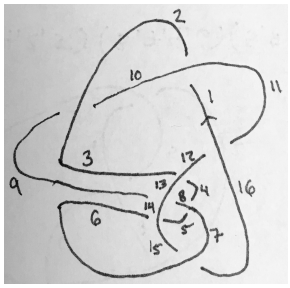

Figure 5: A labeled diagram of knot 6_1 after dragging the underpass