Planar diagram codes and Reidemeister moves of type 2

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Identifying that a knot can be simplified by a Reidemeister move of type 2 based on its planar diagram code

A knot is said to be "simplifiable" by a Reidemeister move of type 2 if two crossings of the knot can be eliminated by moving one loop completely over or under another without creating any new crossings.

Claim. A knot can be simplified by a Reidemeister move of type type if and only if two consecutive tuples in a planar diagram (PD) code of the knot are of one of the following forms:

1. $(a, \gamma, \beta, b)(\beta, \gamma, c, d)$ with $|a - \beta| = 1$
2. $(a, b, \beta, \gamma)(\beta, c, d, \gamma)$ with $|a - \beta| = 1$
3. $(m, b, 1, \beta)(1, a, 2, \beta)$ where m is the maximum value in the PD code
4. $(m, \beta, 1, b)(1, \beta, 2, a)$ where m is the maximum value in the PD code

Note, we consider the last and first tuples in the PD code to be consecutive since the crossings they represent are directly connected in the knot.

Proof. We begin by showing that if a knot can be simplified by a Reidemeister move of type 2, then the PD code tuples representing the crossings to be eliminated are of one of the four forms stated above.

First, consider the PD codes of a knot that can be simplified by a Reidemeister move of type 2 which are obtained by choosing a starting point that is not in the loop. Depending on the orientation of the knot, there are eight possible diagrams. These are depicted in Figure 1, supposing that we traverse the knot from a_1 toward a_2 and then from b_1 toward b_2.

Note that the PD codes of these segments have one of two forms, $(a, \gamma, \beta, b)(\beta, \gamma, c, d)$ or $(a, b, \beta, \gamma)(\beta, c, d, \gamma)$, depending on whether we first turn into the loop or way from it as we move counter-clockwise to determine the PD code. Also note that since a and β are consecutive segments of the knot, by the construction of the PD code $\beta = (a + 1)\text{mod}(\text{the}$ maximum possible value in the PD code). Also by the construction of the PD code, $a_1 < a_2$

and $b_1 < b_2$, so a is not the maximum possible value in the PD code. Hence $\beta = a + 1$ and $|a - \beta| = 1$.

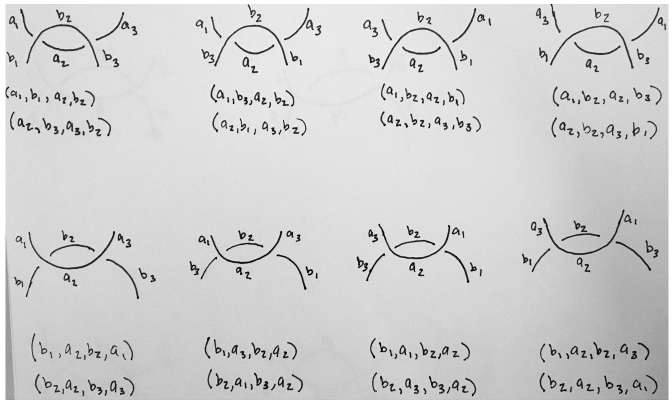

Figure 1: Knot segments which can be simplified by a Reidemeister move of type 2 and the corresponding PD code tuples

Second, consider the PD codes of a knot that can be simplified by a Reidemeister move of type 2 which are obtained by choosing a starting point that is in one of the arcs. There are eight possible diagrams and these are depicted in figure 2 and figure 3, supposing that we traverse the knot from 1 toward 2 and then from b_1 toward b_2.

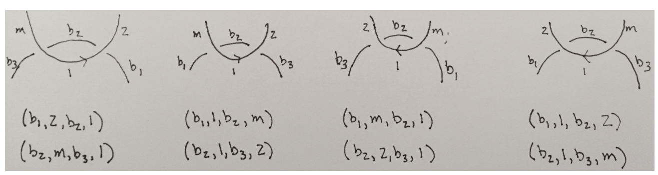

Figure 2: Diagrams of knot segments with a starting point in the arc forming an overpass

Note that in the cases that segment which has a value of 1 forms an overpass (figure 2), the PD codes of these diagram segments have either the form $(a, \gamma, \beta, b)(\beta, \gamma, c, d)$ or $(a, b, \beta, \gamma)(\beta, c, d, \gamma)$ with $|a - \beta| = 1$.

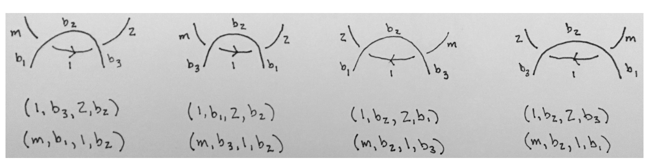

Figure 3: Diagrams of knot segments with a starting point in the arc forming an underscross

In the cases that the segment which has a value of 1 dead-ends into a segment creating an underpass (figure 3), the PD codes of the segments have either the form $(m, b, 1, \beta)(1, a, 2, \beta)$ or $(m, \beta, 1, b)(1, \beta, 2, a)$ where m is the maximum value in the PD code.

Hence if a knot can be simplified by a Reidemeister move of type 2, then the PD code tuples representing the crossings to be eliminated are of one of the following forms:

1. $(a, \gamma, \beta, b)(\beta, \gamma, c, d)$ with $|a - \beta| = 1$

2. $(a, b, \beta, \gamma)(\beta, c, d, \gamma)$ with $|a - \beta| = 1$

3. $(m, b, 1, \beta)(1, a, 2, \beta)$ where m is the maximum value in the PD code

4. $(m, \beta, 1, b)(1, \beta, 2, a)$ where m is the maximum value in the PD code

We finish by showing that every pair of PD code tuples with one of the forms enumerated below corresponds to a knot segment which can be simplified by a Reidemeister move of type 2:

1. $(a, \gamma, \beta, b)(\beta, \gamma, c, d)$ with $|a - \beta| = 1$

2. $(a, b, \beta, \gamma)(\beta, c, d, \gamma)$ with $|a - \beta| = 1$

3. $(m, b, 1, \beta)(1, a, 2, \beta)$ where m is the maximum value in the PD code

4. $(m, \beta, 1, b)(1, \beta, 2, a)$ where m is the maximum value in the PD code

Let us consider the diagrams that correspond to PD codes of the form $(a, \gamma, \beta, b)(\beta, \gamma, c, d)$ or $(a, b, \gamma, \beta)(\gamma, c, d, \beta)$ with $|a - \beta| = 1$.

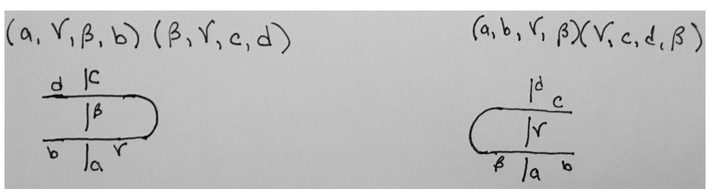

Figure 4: Knot segment diagrams with consecutive PD code tuples of the form $(a, \gamma, \beta, b)(\beta, \gamma, c, d)$ or $(a, b, \gamma, \beta)(\gamma, c, d, \beta)$

From inspection of 4 we see that these knot segments can be simplified by a Reidemeister move of type 2.

Now let us consider the diagrams the correspond to PD codes of the form $(m, b, 1, \beta)(1, a, 2, \beta)$ or $(m, \beta, 1, b)(1, \beta, 2, a)$ where m is the maximum value in the PD code.

We see from inspection of the diagrams in figure 5 that these knot segments can also be simplified by a Reidemeister move of type 2.

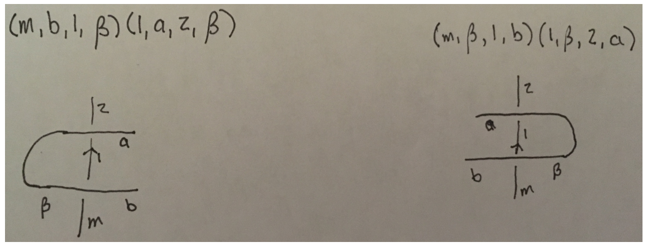

Figure 5: Knot segment diagrams with consecutive PD code tuples of the form $(m, b, 1, \beta)(1, a, 2, \beta)$ or $(m, \beta, 1, b)(1, \beta, 2, a)$

Therefore, two consecutive tuples in a planar diagram code indicate that the knot can be simplified by a Reidemeister move of type 2 if and only if the tuples are of one of the forms:

1. $(a, \gamma, \beta, b)(\beta, \gamma, c, d)$ with $|a - \beta| = 1$
2. $(a, b, \beta, \gamma)(\beta, c, d, \gamma)$ with $|a - \beta| = 1$
3. $(m, b, 1, \beta)(1, a, 2, \beta)$ where m is the maximum value in the PD code
4. $(m, \beta, 1, b)(1, \beta, 2, a)$ where m is the maximum value in the PD code

□

How the planar diagram code of a knot changes when the knot is simplified by a Reidemeister move of type 2

To adjust the PD code of the knot by performing a Reidemeister move of type 2:

1. Remove from the PD code the pair of consecutive tuples that correspond to the crossings being eliminated (the tuples are of one of the four forms identified in the previous section).
2. Let n_1, n_2 be the values of the knot segments that were eliminated from the knot (values β, γ if the tuples were of form 1 or 2 and values $1, \beta$ if the tuples were of form 3 or 4). Without loss of generality, assume $n_1 < n_2$. Let

$$k = \begin{cases} n_2 - 1 & n_1 = 1 \\ n_2 - 2 & n_1 > 1 \end{cases}$$

3. Apply $g \circ f$ to each element of the remaining PD code tuples, where f is defined by

$$f(x) = \begin{cases} x & x < n_1 \\ x-1 & x > n_1, n_1 = 1 \\ x-2 & x > n_1, n_1 > 1 \end{cases}$$

and g is defined by

$$g(x) = \begin{cases} x \mod (m-2) & x < k \\ (x-1) \mod (m-2) & x > k, k = 1 \\ (x-2) \mod (m-2) & x > k, k > 1 \end{cases}$$

with m being the maximum value in the original PD code.