Power Series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

this is an "infinite process"

we define

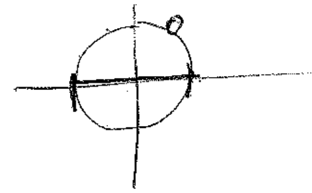
$$f(x) = \lim_{N \to \infty} S_N(x)$$

where $S_N(x) = \sum_{n=0}^{N} a_n x^n$

WHEN THE LIMIT MAKES SENSE!

Fun facts:

1. if $\sum a_n x^n$ converges anywhere then it converges on an interval of the form $(-R, R)$ or $[-R, R)$ or $(-R, R]$ or $[R, R]$



R = "radius of convergence"

(2) Inside the radius of convergence, you can do all of these operations term-by-term & they make sense & "behave properly"

* Add / subtract
* differentiate
* integrate
* multiply & divide.

How to divide P.S.

a_n, b_n known. ?

$$(*)\quad \frac{\sum a_n x^n}{\sum b_n x^n} = \sum c_n x^n$$

(☆) $\Rightarrow$

$$\sum a_n x^n = \left(\sum b_n x^n\right)\left(\sum c_n x^n\right)$$

$$\left(a_0 + a_1 x + a_2 x^2 + \cdots\right) = \left(b_0 + b_1 x + b_2 x^2 + \cdots\right) \times \left(c_0 + c_1 x + c_2 x^2 + \cdots\right)$$

$$= \left(b_0 c_0\right) + \left(\overset{b_0 c_1 + c_0 b_1}{\cancel{c_0 b_1 + c_1 b_0}}\right) x + \left(b_0 c_2 + b_1 c_1 + b_2 c_0\right) x^2 + \cdots$$

$$\Rightarrow \begin{cases} a_0 = b_0 c_0 \\ a_1 = b_0 c_1 + b_1 c_0 \\ a_2 = b_0 c_2 + b_1 c_1 + b_2 c_0 \\ \vdots \end{cases}$$

$$\begin{pmatrix} a & 0 & 0 \\ d & b & 0 \\ e & f & c \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} A \\ B \\ C \end{pmatrix}$$

How to Solve Diff Eqns.

__Example__ $y' = 3y \quad , y(0) = 7$

Use P.S. to find a solution.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

a_n's unknown.
$\longrightarrow$ go find them!

$$y'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1} \cancel{= \sum_{k=0}^{\infty} k a_{k+1} x^k}$$

let $k = n-1$

$$= \sum_{k=0}^{\infty} (k+1) a_{k+1} x^k$$

b/c
$k=-1$ term vanishes.

$$3y = 3\sum a_n x^n = \sum (3 a_n) x^n.$$

$$\text{ODE} \Rightarrow \sum_{k=0}^{\infty} (k+1) a_{k+1} x^k = \sum_{n=0}^{\infty} (3a_n) x^n$$

$$1 \cdot a_1 + 2 \cdot a_2 \cdot x + 3 \cdot a_3 \cdot x^2 + \cdots$$

$$= 3a_0 + 3a_1 x + 3a_2 x^2 + \cdots$$

$$\Rightarrow \begin{cases} a_1 = 3a_0 \\ 2a_2 = 3a_1 \\ 3a_3 = 3a_2 \\ 4a_4 = 3a_3 \\ \vdots \end{cases}$$

<u>Nice!</u> only need a_0!

$$a_1 = 3a_0, \quad a_2 = \frac{1}{2}(3a_1) = \frac{1}{2}(3(3a_0))$$

$$= \frac{1}{2} 3^2 a_0$$

$$a_3 = \frac{1}{3}(3a_2) = \frac{1}{3}\left(3\left(\frac{1}{2} 3^2 a_0\right)\right) = \frac{1}{3 \cdot 2} 3^3 a_0$$

$$a_4 = \frac{1}{4}(3a_3) = \frac{1}{4}\left(3\left(\frac{1}{3\cdot 2}3^3 a_0\right)\right)$$

$$= \frac{1}{4\cdot 3\cdot 2} 3^4 a_0 .$$

In general $\quad a_n = \dfrac{3^n}{n!} a_0$

So $\quad y(x) = \displaystyle\sum_{n=0}^{\infty} \left(\frac{3^n}{n!}\right) a_0 x^n$

$$= a_0 \sum_{n=0}^{\infty} \frac{1}{n!}(3x)^n$$

$$7 = y(0) = a_0 \sum_{n=0}^{\infty} \frac{1}{n!}(3\cdot 0)^n$$

$$= a_0(1 + 0 + 0 + \ldots)$$

$$= a_0 .$$

$$y(x) = 7 \cdot \sum_{n=0}^{\infty} \frac{1}{n!}(3x)^n = 7 \cdot e^{3x}$$

Examples:

$$y'' = -y$$

TRY ME At HOME!