The method of Frobenius for regular singular points of 2nd order homogeneous linear ODEs

<u>Basic Idea</u>: x_0 is a regular singular pt of

(1) $y'' + P(x)y' + Q(x)y = 0.$

Look for a solution of the form

(2) $y = x^m(a_0 + a_1 x + a_2 x^2 + \cdots) = \sum_{n=0}^{\infty} a_n x^{m+n}$

where the a_i's and the number m are unknown. Also, we might as well assume $a_0 \neq 0$.

The tricky business is that this method will <u>not</u> always give you two independent solutions. (which is what we expect)

General nonsense approach. let $x_0 = 0$ for simplicity

Write y as in (2)

(3) $$\left\{ \begin{aligned} xP(x) &= \sum_{n=0}^{\infty} p_n x^n \\ x^2 Q(x) &= \sum_{n=0} q_n x^n \end{aligned} \right\}$$

Then let's collect up the bits of (1):

$$y' = \sum_{n=0}^{\infty} a_n(m+n)\, x^{m+n-1}$$

$$y'' = \left(\sum_{n=0}^{\infty} a_n (m+n)(m+n-1)\, x^n\right) x^{m-2}$$

$$P(x)\,y' = \ldots = x^{m-2} \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n-1} p_{n-k} a_k (m+k) + p_0 a_n (m+n) \right] x^n$$

$$Q(x)\,y = \ldots = x^{m-2} \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n-1} q_{n-k} a_k + q_0 a_n \right) x^n$$

Put everything together & cancel the common x^{m-2} factor to get...

(4) $$0 = \sum_{n=0}^{\infty} \left\{ a_n\left[(m+n)(m+n-1) + (m+n)p_0 + q_0\right] + \sum_{k=0}^{n-1} a_k\left[(m+k)p_{n-k} + q_{n-k}\right] \right\} x^n$$

If everything is analytic, then all of those complicated coefficients are zeros!

Things are getting hairy, so let's write

$$f(t) = t(t-1) + t p_0 + q_0 .$$

Then the vanishing of the coefficients of (4) leads us to this table:

$$(5)\begin{cases} a_0 f(m) = 0 \\ a_1 f(m+1) + a_0(mp_1+q_1) = 0 \\ a_2 f(m+2) + a_0(mp_2+q_2) + a_1((m+1)p_1+q_1) = 0 \\ \quad\vdots \\ a_n f(m+n) + a_0(mp_n+q_n) + \dots + a_{n-1}[(m+n-1)p_1+q_1] = 0 \end{cases}$$

And the goal is to use these to solve for the a_i's.

Since $a_0 \neq 0$ we have to have $f(m) = 0$
$\longrightarrow$ tells us what m should be!

$$(6) \quad 0 = f(m) = m(m-1) + mp_0 + q_0$$

this is called the "indicial equation" of (1). Its roots m_1 and m_2 are called the "exponents" of the ODE (1) at $x=0$.

Note that if we know a_0, we can use the equations (5) to solve for all the other a_n's ... unless at some point $f(m+n) = 0$. Then everything breaks.

<u>Theorem</u>: If $m_2 \le m_1$ are the exponents of (1) then the method of Fröbenius finds at least one solution for us in the form

$$y_1 = x^{m_1} \sum_{n=0}^{\infty} a_n x^n$$

Furthermore, if $m_1 - m_2 \notin \{0, 1, 2, \ldots\}$ then we can also find a second solution

$$y_2 = x^{m_2} \sum_{n=0}^{\infty} b_n x^n$$

What happens if $m_1 - m_2 \in \{0, 1, 2, \ldots\}$?

[A] If $m_1 = m_2$ there is not another Frobenius type solution.

If $m_1 - m_2 = n$ is a positive integer we will find an equation in (5) where $f(m_2 + n) = 0$ "$f(m_1)$"
two things can happen
Consider the eqn from (5)

(7) $\quad a_n f(m_2 + n) = -a_0 (m_2 p_n + q_n) + \cdots$ stuff

[B] If RHS (7) $\neq 0$, the calculation must stop & there is no solution.

[C] If RHS (7) $= 0$, then choose any value for a_n and continue on to find a second ~~value~~ soln.

Note: If there is not a second Frobenius soln, then some more p.s. manipulation will show

$$y_2 = A\,y_1 \log x + x^{m_2}\sum_{n=0}^{\infty} c_n x^n$$

will work.

Example: $x^2 y'' - 3xy' + (4x+4)y = 0$

In std form

$$y'' - \frac{3}{x}y' + \frac{4x+4}{x^2}y = 0 \quad , x=0 \text{ reg sing pt}$$

$$P(x) = \frac{-3}{x} \qquad xP(x) = -3$$

$$Q(x) = \frac{4x+4}{x^2} \qquad x^2Q(x) = 4x+4$$

So if $y = x^m \sum a_n x^n$, $y' = \sum a_n (m+n) x^{m+n-1}$

$$y'' = \sum a_n (m+n)(m+n-1) x^{m+n-2}$$

$$P(x)y' = \sum_{n=0}^{\infty} -3a_n (m+n) x^{m+n-2}$$

$$Q(x)y = \left(\sum_{n=0}^{\infty} a_n x^{m+n}\right)\left(\frac{4x+4}{x^2}\right) = \left(\sum a_n x^{m+n-2}\right)(4x+4)$$

$$= \sum_{n=0}^{\infty} 4a_n x^{m+n-2} + \sum_{n=0}^{\infty} 4a_n x^{m+n-1}$$

$$= \sum_{n=0}^{\infty} 4(a_n + a_{n-1}) x^{m+n-2}$$

careful here $a_{-1} = 0$.

finally, combine & factor out x^{m-2}

$$0 = \sum \left\{ a_n (m+n)(m+n-1) - 3a_n (m+n) + 4(a_n + a_{n-1}) \right\} x^n$$

$$0 = \sum \left[a_n \left\{ (m+n)(m+n-1) - 3(m+n) + 4 \right\} + 4a_{n-1} \right] x^n$$

So we get

$n=0 \qquad a_0\{m(m-1) - 3m + 4\} = 0$

$n=1 \qquad a_1\{(m+1)(m) - 3(m+1) + 4\} + 4a_0 = 0$

$n=2 \qquad a_2\{(m+2)(m+1) - 3(m+2) + 4\} + 4a_1 = 0$

etc...

Indicial Equation is

$$0 = m(m-1) - 3m + 4 = m^2 - m - 3m + 4$$

$$= m^2 - 4m + 4 = (m-2)^2$$

$$\Rightarrow m_1 = m_2 = 2$$

$n=0 \qquad a_0\{2\cdot 1 - 3\cdot\cancel{2} + 4\} = 0$ ✓

$n=1 \qquad a_1\{3\cdot 2 - 3\cdot 3 + 4\} + 4a_0 = 0$

$\qquad \Rightarrow a_1 + 4a_0 = 0 \quad \Rightarrow a_1 = -4a_0$

$n=2 \qquad a_2\{4\cdot 3 - 3\cdot 4 + 4\} + 4a_1 = 0$

$\qquad \Rightarrow 4a_2 + 4a_1 = 0 \quad \Rightarrow a_2 = -a_1$

$n=3 \qquad a_3\{5\cdot 4 - 3\cdot 5 + 4\} + 4a_2 = 0$

$\qquad \Rightarrow 9a_3 + 4a_2 = 0 \quad \Rightarrow a_3 = -\frac{4}{9}a_2$

$n=4 \qquad a_4\{6\cdot 5 - 3\cdot 6 + 4\} + 4a_3$

$\qquad \Rightarrow 16a_4 + 4a_3 = 0 \quad \Rightarrow a_4 = -\frac{1}{4}a_3$

So $y = a_0 x^2\left(1 - 4x + 4x^2 - \frac{16}{9}x^3 + \frac{4}{9}x^4 - \dots\right)$

Homework

(1) The equation below has only one Frobenius soln

$$4x^2y'' - 8x^2y' + (4x^2+1)y = 0$$

find it.

(2) Find two independent Frobenius solutions to

$$xy'' + 2y' + xy = 0$$

Note: for both $x=0$ is a regular singular pt. work at that spot.