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ELEMENTS OF LINEAR ALGEBRA, VOLUME II CAHIER DES LIGNES

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PUBLISHED BY I MADE IT MYSELF.

SITES.UNI.EDU/THERON/LA17

First printing, September 2017

Contents

Introduction

This is a *workbook*. It is a collection of tasks that you should do to try to learn linear algebra for yourself.

It is not a *textbook*. We don't have a textbook, as such things are understood today. We have a primer, the first volume of this set: *Livre des Lignes*, which should serve as a basic source for reading about the main ideas, and we have this workbook. The books are separate so that you may have them both open at once. I hope this works.

It is important to focus on your own understanding when working on these tasks. At every stage possible, you should ask yourself, "Self, how do I know this?" and maybe also, "So, Self, do I know this for sure? Or do I still have doubts?"

If you are feeling at a loss for how to explain your thinking, try relying on simple geometric models and reasoning. It turns out that most of even the fanciest linear algebra is done that way. So, "Self, can you draw the relevant picture? Does that help?"

Note that some of these items are labeled as *tasks* and some are labeled as *challenges*. A task is something I think you should be able to do after having read the *Livre des Lignes*: the relevant procedure or problem is considered there somewhere. But a challenge is something that will require some independent thinking: not all of the work will correspond directly to the development in the *Livre*, and you might need to use some new idea, or new combination of ideas. Usually, the challenges are there to help set up ideas we will return to later.

Good luck. I'll see you in class, and I welcome your visits to office hours to talk about mathematics.

Vectors and Lines the Plane

Points and Vectors

Task 1. Write down three distinct points in the plane in proper notation. Plot those three points on a single diagram.

Task 2. Write down three distinct vectors in the plane in proper notation. Your three points from this task should NOT match any of the three points from the previous task. Plot those three vectors on a single diagram.

Definition. If $u_1, u_2, \ldots, u_n$ is a collection of vectors, and $\lambda_1, \lambda_2, \ldots$ is a collection of scalars, then the vector formed below is called a *linear combination* of the u_i's.

Defintion: Linear Combinations

$$\lambda_1 u_1 + \lambda_2 u_2 + \ldots \lambda_n u_n$$

Task 3. Find the sum of your three vectors from the last exercise. Then, choose some order of those three vectors so that they are u_1, u_2 and u_3, and compute the linear combination

$$3u_1 - 2u_2 + (1/2)u_3.$$

Task 4. Let's consider the vectors $u = \left(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right)$, $v = \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$, and $w = \left(\begin{smallmatrix} -3 \\ 1 \end{smallmatrix}\right)$. Compute all of the vectors in this list:

$$\frac{u+v}{2}, v-u, v-w, u+\left(\frac{v-u}{3}\right), u+\left(\frac{3(v-u)}{4}\right)$$

Then make a single diagram which contains u, v, w and all of those vectors from the list, plotted as accurately as you can.

What do you notice? Is anything interesting going on?

Task 5. Consider the vector $u = \left(\begin{smallmatrix} -4 \\ 2 \end{smallmatrix}\right)$. Find a vector v which has the property that $u + v$ is the zero vector, or explain why this is not possible.

Definition. An equation of the form $\lambda_1 u_1 + \ldots \lambda_n u_n = w$, where all of the vectors u_i and w are known, but the scalars λ_i are unknown, is called a *linear combination of vectors equation*. A *solution* to such an equation is a collection of scalars which make the equation true.

Definition: Linear Combination Equations and their Solutions

Task 6. For now, keep $u = \left(\begin{smallmatrix}-4\\2\end{smallmatrix}\right)$. Let $v = \left(\begin{smallmatrix}3\\5\end{smallmatrix}\right)$. How many solutions does the linear combination of vectors equation $\lambda u = v$ have?

How many solutions does the linear combination of vectors equation $\lambda u + \mu v = 0$ have? (Here, treat 0 as the zero vector.)

Task 7. We still use the notation u for the vector $u = \left(\begin{smallmatrix}-4\\2\end{smallmatrix}\right)$, but now use v for the vector $v = \left(\begin{smallmatrix}28\\-14\end{smallmatrix}\right)$. How many solutions does the linear combination of vectors equation $\lambda u = v$ have?

How many solutions does the linear combination of vectors equation $\lambda u + \mu v = 0$ have? (Again, treat 0 as the zero vector.)

Task 8. Find the midpoint between the points $P = (4, -2)$ and $Q = (3, 5)$. Then find the two points which divide the segment PQ into thirds.

How can vectors make this simpler than it first appears?

Challenge 9. Suppose you are given three points in the plane. Let's call them P, Q, and R. How can you use vectors to (quickly) determine if these three points are collinear?

Parametric Lines in the Plane

Task 10. Write down five different points which lie on the line described parametrically as:

$$t \mapsto \begin{pmatrix}-5\\3\end{pmatrix} + t\begin{pmatrix}1\\-1/2\end{pmatrix}.$$

Challenge 11. How many different solutions to the equation

Recall the definition above.

$$x\begin{pmatrix}3\\7\end{pmatrix} + y\begin{pmatrix}6\\14\end{pmatrix} = \begin{pmatrix}-3\\-7\end{pmatrix}$$

can you find? How many are they? Can you find a good way to describe all of the solutions? Is there a natural geometric way to describe all of the solutions? Is there an algebraic way to describe all of the solutions?

Task 12. Write down a parametric description for the line which passes through the origin $O = (0, 0)$ and the point $S = (-5, 5)$.

Is this the *only* way to write down such a parametric description for that line?

Task 13. Write down a parametric description for the line which passes through the points below.

$$T = (\pi, 0), \qquad J = (0, -\pi)$$

Now find a different parametric description for that line. (*Hint: Can you find a way to write a parametric description that doesn't use the number π?*)

Task 14. Compare the lines from the tasks 12 and 13. Do you note anything interesting? How do you know your observation is true?

Task 15. For each of the conditions below, either find an example of a 2-vector Y so that the equation

$$t\begin{pmatrix}3\\-1\end{pmatrix}=Y$$

has the given number of solutions, or explain why such an example is not possible.
a) exactly zero solutions;
b) exactly one solution;
c) exactly two solutions.

Task 16. For each of the conditions below, either find an example of a 2-vector Z so that the equation

$$\begin{pmatrix}1\\2\end{pmatrix}+t\begin{pmatrix}-1\\1\end{pmatrix}=Z$$

has the given number of solutions, or explain why such an example is not possible.
a) exactly zero solutions;
b) exactly one solution;
c) exactly two solutions.

Task 17. For each of the conditions below, either find an example of a 2-vector Y so that the equation

$$\begin{pmatrix}-2/5\\2\end{pmatrix}+tY=\begin{pmatrix}-1\\1\end{pmatrix}$$

has the given number of solutions, or explain why such an example is not possible.
a) exactly zero solutions;
b) exactly one solution;
c) exactly two solutions.

Task 18. For each of the conditions below, either find an example of a 2-vector Z so that the equation

$$x\begin{pmatrix}-5/3\\1\end{pmatrix}+y\begin{pmatrix}3\\7\end{pmatrix}=Z$$

has the given number of solutions, or explain why such an example is not possible.
a) exactly zero solutions;

b) exactly one solution;
c) exactly two solutions.

Task 19. For each of the conditions below, either find an example of a 2-vector Z so that the equation

$$x \begin{pmatrix} -5/3 \\ 1 \end{pmatrix} + y \begin{pmatrix} 1 \\ -3/5 \end{pmatrix} = Z$$

has the given number of solutions, or explain why such an example is not possible.
a) exactly zero solutions;
b) exactly one solution;
c) exactly two solutions.

Challenge 20. Find an example of four 2-vectors X, Y, Z, and W so that the equation

$$aX + bY + cZ = W$$

has at exactly two solutions, or explain why such an example is not possible.

The Dot Product: Norms and Angles

Task 21. Choose three different 2-vectors which have neither of their components equal to zero. Call these vectors u, v, and w.
a) Compute the norms of u, v, and w.
b) Compute the dot products $u \cdot v$, $v \cdot w$, and $u \cdot w$.
c) Find unit vectors u', v', and w' which point in the same directions as u, v, and w, respectively.
d) Find the angles between each of the pairs, u and v, u and w, v and w in radians.

Task 22. Fix some vector u. Draw a picture of u in the plane, and then shade the region of the plane which contains vectors v so that $u \cdot v > 0$.

Task 23. This task continues our quest for understanding the sign of a dot product geometrically.
a) Find an example of two 2-vectors v and w so that $\begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot v = 0$ and $\begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot w = 0$, or explain why such an example is not possible.
b) Let $v = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$. Find an example of a pair of 2-vectors u and w such that $v \cdot u < 0$ and $v \cdot w < 0$ and $w \cdot u = 0$, or explain why no such pair of vectors can exist.
c) Find an example of three 2-vectors u, v, and w so that $u \cdot v < 0$ and $u \cdot w < 0$ and $v \cdot w < 0$, or explain why no such example exists.

Task 24. What shape is the set of solutions $\binom{x}{y}$ to the equation

$$\begin{pmatrix} 3 \\ 7 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = 5?$$

That is, if we look at all possible vectors $\binom{x}{y}$ which make the equation true, what shape does this make in the plane? Draw this shape.

What happens if we change the vector $\binom{3}{7}$ to some other vector? What happens if we change the number 5 to some other number?

Task 25. a) Find an example of a number c so that the equation

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = c$$

has the vector $\binom{4}{7}$ as a solution, or explain why no such number exists.

b) Let $v = \binom{2}{1}$ and $w = \binom{-3}{4}$. Find an example of a number c so that

$$v \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = c \quad \text{and} \quad w \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = c,$$

or explain why this is not possible.

c) Let $P = \binom{-3}{4}$. Find an example of numbers c and d so that

$$\begin{pmatrix} 2 \\ -1 \end{pmatrix} \cdot P = c \quad \text{and} \quad \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot P = d,$$

or explain why no such example is possible.

Equations of Lines in the Plane

Task 26. Write down an equation for the set of vectors which are all orthogonal to $u = \binom{3}{-2}$.

Task 27. We begin with a line described parametrically by

$$t \mapsto \begin{pmatrix} 6 \\ -\pi \end{pmatrix} + t \begin{pmatrix} 34 \\ -19/3 \end{pmatrix}.$$

a) Find a normal vector for this line.
b) Plot the line and the normal vector you found.

Task 28. Find an equation for the line in the Task 27.

Task 29. We begin with a line described by the equation

$$-3x + y = 7.$$

a) Find a normal vector to the line
b) Plot this line and the normal vector you found.

Task 30. Find a parametric description for the line from Task 29.

Task 31. Consider the line described by the equation $4x + 7y = 3$. Find an equation for the line which is parallel to this one, but passes through the point indicated:
a) The origin O.
b) The point $P = (0, -10)$.

Challenge 32. Consider the line described by the equation $x - 2y = -2$. Find a line which is orthogonal to this one, and passes through the point $Q = (9, -1)$. Can you describe your line with an equation? Can you describe your line parametrically?

Vectors, Lines, and Planes in Space

Points, Vectors, and the Dot Product

Task 33. Write down four 3-vectors of your choosing, where none has any coordinate equal to 0. Call these vectors u_1, u_2, u_3, and u_4. Now choose any four non-zero scalars you like, call them λ_1, λ_2, λ_3 and λ_4. Compute the linear combination

$$\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 + \lambda_4 u_4.$$

Task 34. Consider the points $P = (1, 1, 1)$ and $Q = (-2, 7, 4)$ in $\mathbb{R}^3$. Find the coordinates of the point R which lies on the line through P and Q, one quarter of the way from P to Q.

Task 35. Write down two 3-vectors which have no coordinates equal to zero. Call them u and v. Find the following things:

- The dot product $u \cdot v$;
- The norms of u and v;
- unit vectors which point in the same directions as u and v, respectively; and
- the angle between u and v.

The next three tasks all involve linear combination equations. You may find it useful to review what that phrase means, and also what it means to find a solution to such an equation.

Task 36. This task concerns linear combination equations of the form $\lambda u = v$ for 3-vectors u and v and a scalar λ.

- Can you make an example of this equation which has no solutions? If so, make one. If not, explain why it is impossible.
- Can you make an example of this equation which has exactly one solution? If so, make one. If not, explain why it is impossible.
- Can you make an example of this equation which has exactly two solutions? If so, make one. If not, explain why it is impossible.
- Can you make an example of this equation which has infinitely many solutions? If so, make one. If not, explain why it is impossible.

Task 37. This task concerns linear combination equations of the form $\lambda u + \mu v = w$ for 3-vectors u, v, and w and scalars λ, and μ.

- Can you make an example of this equation which has no solutions? If so, make one. If not, explain why it is impossible.
- Can you make an example of this equation which has exactly one solution? If so, make one. If not, explain why it is impossible.
- Can you make an example of this equation which has exactly two solutions? If so, make one. If not, explain why it is impossible.
- Can you make an example of this equation which has infinitely many solutions? If so, make one. If not, explain why it is impossible.

Task 38. This task concerns linear combination equations of the form $xu_1 + yu_2 + zu_3 = b$ for 3-vectors u_1, u_2, u_3 and b and scalars x, y, and z.

- Can you make an example of this equation which has no solutions? If so, make one. If not, explain why it is impossible.
- Can you make an example of this equation which has exactly one solution? If so, make one. If not, explain why it is impossible.
- Can you make an example of this equation which has exactly two solutions? If so, make one. If not, explain why it is impossible.
- Can you make an example of this equation which has infinitely many solutions? If so, make one. If not, explain why it is impossible.

Challenge 39. Suppose that we are given a line in the plane described parametrically as $t \mapsto t\left(\begin{smallmatrix}-2\\5\end{smallmatrix}\right)$, and a point $P = (7, 4)$. Find the coordinates of the point Q which (1) lies on the line and (2) which is, of all the points on the line, *closest* to P.

Definition. Let n be a counting number. We define an n-vector to be a vertical stack of n real numbers, like so:

$$u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}.$$

The individual entries u_i of u are called its *components*. The collection of all n-vectors is called *n-space*, and denoted $\mathbb{R}^n$.

We may form *linear combinations* of n-vectors by doing scalar multiplication and addition in a component-by-component fashion.

Similarly, the notions of dot product, norm, and angle extend in the expected way to n-vectors.

Task 40. Create an example of five different 6-vectors, call them $v_1, v_2, \ldots v_5$. Make sure that at most one component of each vector is equal to zero. Then choose five different non-zero scalars $x_1, x_2, \dots x_5$. Compute the linear combination

$$x_1 v_1 + x_2 v_2 + x_3 v_3 + x_4 v_4 + x_5 v_5.$$

Task 41. Using the vectors v_1 and v_2 from your work in the last task, compute the following:
- The dot product of v_1 and v_2;
- The norms of v_1 and v_2;
- Unit vectors which point in the same directions as v_1 and v_2, respectively; and
- The angle between v_1 and v_2.

Task 42. This task has three parts:
a) Create an example of a linear combination equation for vectors u, v, w in $\mathbb{R}^3$ of the form $xu + yv = w$ which has no solution.
b) Unbundle your vectors to rewrite your linear combination equation as an equivalent system of three linear equations on the variables x and y.
c) How should we interpret this system of linear equations? What kind of thing does each equation describe?

Task 43. Repeat the last task, but instead, work with vectors u, v, w in $\mathbb{R}^4$.

Task 44. Consider the line in the plane given parametrically by

$$t \mapsto \begin{pmatrix} -2 \\ 4 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Find the coordinates of the point lying on this line which is closest to the point $P = (1, 5)$.

Task 45. Consider the line in the plane which is the set of solutions the equation $4x - 7y = -15$. Find the point on that line which is closest to the point $Q = (-5, 0)$.

Task 46. Make an example of a linear combination equation

$$xv_1 + yv_2 + zv_3 = b$$

for 4-vectors v_1, v_2, v_3, b which has no solution, or say why such an example is not possible.

Lines and Planes in $\mathbb{R}^3$

Task 47. Find a parametric description for the line in $\mathbb{R}^3$ which passes through the points $T = (1, 1, 1)$ and $J = (12, 3, -9)$.

Challenge 48. Find a parametric description for the line in $\mathbb{R}^3$ which passes through the point $S = (-5, -12, -17)$ and meets the y-axis in a right angle.

Task 49. Consider the plane which is the solution set to the equation $4x - 3y + 5z = 17$. Find three distinct points on this plane.

Task 50. Find a parametric description of the plane which passes through the origin and the points $T = (1, 1, 1)$ and $J = (12, 3, -9)$.

Task 51. Find an equation whose solution set is the plane which passes through the points $T = (1, 1, 1)$, $J = (12, 3, -9)$, and $H = (2, 3, 2)$.

Challenge 52. Consider the line in $\mathbb{R}^3$ given by the parametric description

$$t \mapsto \begin{pmatrix} -6 \\ -2 \\ 1 \end{pmatrix} + t \begin{pmatrix} 3 \\ -1/2 \\ 1 \end{pmatrix}.$$

Find the point on this line which is closest to $T = (1, 1, 1)$.

Challenge 53. Consider the plane in $\mathbb{R}^3$ which has the parametric description

$$(s, t) \mapsto s \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}.$$

Find the point on this plane which is closest to $J = (12, 3, -9)$.

Task 54. For each of the linear combination equations below, rewrite the equation as an equivalent system of linear equations.

a)

$$x \begin{pmatrix} -7 \\ 3 \end{pmatrix} + y \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$$

b)

$$x \begin{pmatrix} 2 \\ 0 \\ 7 \end{pmatrix} + y \begin{pmatrix} 12/5 \\ 1/3 \\ -5 \end{pmatrix} = \begin{pmatrix} -8 \\ -3 \\ 0 \end{pmatrix}$$

c)

$$x \begin{pmatrix} 2 \\ 0 \\ 7 \end{pmatrix} + y \begin{pmatrix} 12/5 \\ 1/3 \\ -5 \end{pmatrix} + z \begin{pmatrix} 0 \\ 9 \\ 0 \end{pmatrix} = \begin{pmatrix} -8 \\ -3 \\ 0 \end{pmatrix}$$

d)

$$x \begin{pmatrix} 0 \\ 7 \end{pmatrix} + y \begin{pmatrix} 12/5 \\ -5 \end{pmatrix} + z \begin{pmatrix} 9 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \end{pmatrix}$$

Task 55. For each of the four systems of linear equations in the last task, consider the types of things which should be solutions to that system. Fill in the blanks in the descriptive sentence below:

> This system describes the intersection of ________________ objects in $\mathbb{R}$—. Each object is a ________________.

Task 56. Consider the line in $\mathbb{R}^3$ given by the parametric description

$$t \mapsto \begin{pmatrix} -6 \\ -2 \\ 1 \end{pmatrix} + t \begin{pmatrix} 3 \\ -1/2 \\ 1 \end{pmatrix}.$$

Write a set of equations which have this line as their solution set.

Challenge: Then, write a different set of equations which have this line as their solution set.

Task 57. Consider the set of solutions in $\mathbb{R}^3$ of the equation below. Write a parametric description for the plane it describes.

$$4x - 3y + \frac{1}{5}z = 0$$

Task 58. Consider the set of solutions in $\mathbb{R}^3$ of the system of equations below. Write a parametric description of the line it describes.

$$\begin{cases} x - 3y + z = 1 \\ x + 6y + 2z = 3 \end{cases}$$

Task 59. Make an example of a system of two equations in three unknowns whose solution set is a plane rather than a line, or say why such an example is impossible.

Challenge 60. Consider the set of solutions in $\mathbb{R}^3$ of the system of equations below.

$$\begin{cases} x - 3y + z = 1 \\ x + 6y + 2z = 3 \end{cases}$$

Write a different system of equations which (1) has the same solution set, and (2) has the form

$$\begin{cases} x \quad\quad + bz = d \\ \quad\quad y + cz = e \end{cases}$$

for some numbers b, c, d, e.

Challenge 61. Consider the set of solutions in $\mathbb{R}^3$ of the system of equations below.

$$\begin{cases} x + 3y + z = 1 \\ 2x + 7y + 2z = 3 \end{cases}$$

Write a different system of equations which (1) has the same solution set, and (2) has the form

$$\begin{cases} x & & + & bz & = & d \\ & y & + & cz & = & e \end{cases}$$

for some numbers b, c, d, e.

Challenge 62. Consider the plane which is the solution set to the equation $5x - 6y - 2z = 0$. Find the point on this plane which is closest to the point $T = (1, 1, 1)$.

Challenge 63. Consider the line which is the solution set to the system of equations

$$\begin{cases} x & - & 3y & + & z & = & 1 \\ x & + & 6y & + & 2z & = & 3 \end{cases}$$

Find the point on this line which is closest to the point $T = (1, 1, 1)$.

Challenge 64. Find a parametric description for the line which is orthogonal to the plane $x - 3y + z = 0$ and passes through the point $J = (12, 3, -9)$.

Challenge 65. Find an equation for the plane which is orthogonal to the line

$$t \mapsto t \begin{pmatrix} -2 \\ -3 \\ -2 \end{pmatrix}$$

and passes through the point $T = (1, 1, 1)$.

Challenge 66. Find an equation whose solution set is the plane which is (1) parallel to the plane described by $x - 3y + z = 0$, and (2) passes through the point $J = (12, 3, -9)$.

Challenge 67. Find a set of equations whose solution set is the line which is parallel to the line

$$t \mapsto t \begin{pmatrix} -2 \\ -3 \\ -2 \end{pmatrix}$$

and passes through the point $T = (1, 1, 1)$.

Interlude: Thinking about Systems of Equations

Definition. Let m and n be counting numbers. A *system of m linear equations in n unknowns* has the form

$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & = & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = & b_m \end{cases},$$

where the unknowns are the variables $x_1, \dots, x_n$ we are meant to find, and all of the other letters a_{ij} and b_j are scalars that we are supposed to know already. The numbers a_{ij} are commonly called *coefficients*.

A *solution* of this system is a single set of values for the x_i's which makes all m of the equations true at the same time.

We have encountered many different systems of linear equations already, though they have not always been written in the standardized form above. For example, we have seen that to describe a line in $\mathbb{R}^3$, we need to use a system of $m = 2$ equations in $n = 3$ variables.

The goal of this set of tasks is to figure out what might count as a system of linear equations that "we shouldn't find difficult." For now, we will work with the special case where $m = n = 3$.

Task 68. By choosing particular values of the a_{ij}'s and the b_j's, write down an example system of 3 linear equations in 3 unknowns, x_1, x_2, and x_3, which you don't know how to solve, and you are pretty sure no one else in class can solve inside of five minutes.

Make it scary.

Task 69. By choosing particular values of the a_{ij}'s and the b_j's, write down an example system of 3 linear equations in 3 unknowns, x_1, x_2, and x_3, which you know how to solve, and you are pretty sure anyone else in class can solve almost instantly, just by looking at it.

Make it so clear that you can't fail to find the solution.

Task 70. By choosing particular values of the a_{ij}'s and the b_j's, write down an example system of 3 linear equations in 3 unknowns, x_1, x_2, and x_3, which you know a solution for, but is mildly disguised so that you do not think your classmates can find your solution just by looking. To make sure it is only mildly disguised, aim for something that is simple to check: If you give away the answer, your classmates should say, "Of Course!"

Just a little bit of a trick.

Task 71. By choosing particular values of the a_{ij}'s and the b_j's, write down an example system of 3 linear equations in 3 unknowns, x_1, x_2, and x_3, which you a solution for, but is just a bit more disguised. This one might take two or three steps of work to "see" the solution.

Maybe two tricks, or one applied twice.

Task 72. By choosing particular values of the a_{ij}'s and the b_j's, write down an example system of 3 linear equations in 3 unknowns, x_1, x_2, and x_3, which you know how to solve, but is pretty well disguised and you are pretty sure no one else will solve it inside of 2 minutes.

Design something to slow people down significantly.

Task 73. Think about the work you have done? What makes a system of linear equations straightforward? What things did you do to hide the answer?

Reflection Time!

Three Viewpoints & Five Questions

Systems of Equations vs. Linear Combination Equations

Task 74. Make an example of the following sort:
- A line in $\mathbb{R}^2$ defined parametrically; and
- a different line in $\mathbb{R}^2$ defined implicitly as the set of solutions to an equation.

a) Write down a system of linear equations whose solution helps you determine the points of intersection of these two lines. How many equations are there? How many unknowns?

b) Rewrite the system of equations from the previous task as a linear combination of vectors equation. What kind of vectors are there? How many terms does your equation have?

Task 75. Make an example of the following sort:
- A line in $\mathbb{R}^3$ defined parametrically; and
- a plane in $\mathbb{R}^3$ defined implicitly as the set of solutions to an equation.

a) Write down a system of linear equations whose solution helps you determine the points of intersection of these two lines. How many equations are there? How many unknowns?

b) Rewrite the system of equations from the previous task as a linear combination of vectors equation. What kind of vectors are there? How many terms does your equation have?

Task 76. Make an example of the following sort:
- A plane in $\mathbb{R}^3$ defined parametrically; and
- a different plane in $\mathbb{R}^3$ defined implicitly as the set of solutions to an equation.

a) Write down a system of linear equations whose solution helps you determine the points of intersection of these two planes. How many equations are there? How many unknowns?

b) Rewrite the system of equations from the previous task as a linear combination of vectors equation. What kind of vectors are there? How many terms does your equation have?

Task 77. Make an example of the following sort:
- three different planes in $\mathbb{R}^3$, each defined implicitly as the solution set of a linear equation.

a) Write down a system of linear equations whose solution helps you determine the points of intersection of these three planes. How many equations are there? How many unknowns?

b) Rewrite the system of equations from the previous task as a linear combination of vectors equation. What kind of vectors are there? How many terms does your equation have?

Task 78. Make an example of the following sort:
- three different planes in $\mathbb{R}^3$, each defined parametrically.

a) Write down a system of linear equations whose solution helps you determine the points of intersection of these three planes. How many equations are there? How many unknowns?

b) Rewrite the system of equations from the previous task as a linear combination of vectors equation. What kind of vectors are there? How many terms does your equation have?

Task 79. Consider the system of linear equations below.

$$\left\{\begin{array}{rcrcrcrcr}
2x_1 & + & 1x_2 & - & \frac{1}{2}x_3 & + & x_4 & = & 2 \\
-x_1 & + & 1x_2 & - & 5x_3 & + & 10x_4 & = & 12 \\
15x_1 & + & 2x_2 & + & 4x_3 & - & x_4 & = & 17 \\
-2x_1 & + & 3x_2 & - & \frac{1}{2}x_3 & + & 3x_4 & = & 27 \\
3x_1 & + & 5x_2 & + & x_3 & - & 5x_4 & = & -42 \\
\frac{3}{11}x_1 & + & 8x_2 & - & \frac{1}{2}x_3 & + & 3x_4 & = & 0 \\
9x_1 & + & 13x_2 & + & \frac{6}{5}x_3 & + & x_4 & = & 11
\end{array}\right.$$

a) Write a sentence which describes the row picture for this system of equations.

b) Translate this system of equations into a linear combination of vectors equation.

c) Write a sentence which describes the column picture of this linear combination of vectors equation.

d) Translate this system into a matrix-vector equation.

e) Write a sentence which describes the transformational picture of this matrix-vector equation.

Task 80. Consider the linear combination of vectors equations below.

$$x_1 \begin{pmatrix} 1 \\ 5 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 5 \end{pmatrix} + x_3 \begin{pmatrix} -3 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 4/5 \end{pmatrix} + x_5 \begin{pmatrix} 6 \\ 2 \end{pmatrix} + x_6 \begin{pmatrix} 9 \\ -9 \end{pmatrix} + x_7 \begin{pmatrix} \pi \\ 4 \end{pmatrix} + x_8 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + x_8 \begin{pmatrix} 0 \\ 5 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

a) Write a sentence which describes the column picture for this system of equations.

b) Translate this linear combination of vectors equation into a system of equations.

c) Write a sentence which describes the row picture of this system of equations.

d) Translate this system into a matrix-vector equation.

e) Write a sentence which describes the transformational picture of this matrix-vector equation.

Task 81. Consider the linear combination of vectors equations below.

$$\begin{pmatrix} 2 & 3 \\ 45 & -2 \\ 1 & 6 \\ -5 & 0 \\ 7 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 3 \\ -4 \\ \pi \end{pmatrix}$$

a) Write a sentence which describes the transformational picture for this matrix-vector equation.

b) Translate this matrix-vector equation into a system of linear equations.

c) Write a sentence which describes the row picture of this system of equations.

d) Translate this matrix-vector equation into a linear combination of vectors equation.

e) Write a sentence which describes the column picture of this linear combination of vectors equation.

Task 82. Consider the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

Choose four vectors of the appropriate shape to play the role of x, and compute Ax for each. What shape must those four vectors have? What shape must the resulting vectors Ax be?

Task 83. Consider the matrix

$$B = \begin{pmatrix} 0 & 1 & 4 \\ 1 & 1 & 3 \end{pmatrix}$$

Choose four vectors of the appropriate shape to play the role of x, and compute Bx for each. What shape must those four vectors have? What shape must the resulting vectors Bx be?

Task 84. Consider the matrix

$$C = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

Choose four vectors of the appropriate shape to play the role of x, and compute Cx for each. What shape must those four vectors have? What shape must the resulting vectors Cx be?

Task 85. Consider the matrix

$$D = \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \\ 1 & 1 \end{pmatrix}$$

Choose four vectors of the appropriate shape to play the role of x, and compute Dx for each. What shape must those four vectors have? What shape must the resulting vectors Dx be?

Task 86. Consider a 3×3 matrix A which is constructed by declaring the 3-vectors u_1, u_2, u_3 as its columns.

$$A = \begin{pmatrix} | & | & | \\ u_1 & u_2 & u_3 \\ | & | & | \end{pmatrix}$$

What are these vectors?

$$A \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad A \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad A \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Task 87. Suppose that A is a 3×3 matrix. Somehow, we know the following facts:

$$A \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad A \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ 7 \end{pmatrix}.$$

Find the vector

$$A \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Task 88. Consider the matrix B and the vector c given below:

$$B = \begin{pmatrix} 2 & 1 & 7 \\ 3 & 1 & 0 \end{pmatrix}, \quad c = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

We wish to solve the equation $Bx = c$. Make a row picture for this situation. Then make a column picture for this situation.

Definition of Transpose

Definition. Let A be an $m \times n$ matrix. The *transpose* of A is the $n \times m$ matrix obtained by switching the roles of rows and columns of A. So the matrix $A = (a_{ij})$ becomes the matrix $A^T = (a_{ji})$.

The usual notation is A^T, which is read as "A transpose," or "the transpose of A"

Task 89. Compute the transpose of the following matrices:

$$A_1 = \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 2 & 1 & 3 & 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 5 & 6 \\ 1 & -1 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 4 & 5 & 12 & 0 & 0 \\ -2 & -1 & 5 & 1 & 0 \\ 45 & -\pi & 1/2 & 1 & 5 \end{pmatrix}.$$

Definition. A square matrix which is equal to its transpose is called *symmetric*.

Task 90. Make three different examples of symmetric 2×2 matrices.

Task 91. It is often useful to think of an n-vector as a matrix with n rows and 1 column, that is, as an $n \times 1$ matrix. Using this perspective, compute $u^T v$ for the vectors u and v below.

$$u = \begin{pmatrix} -3 \\ 2 \\ 4 \end{pmatrix}, \quad v = \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$$

What familiar operation does this mimic?

Task 92. If we write an $m \times n$ matrix B as a bundle of columns

$$B = \begin{pmatrix} | & | & & | \\ b_1 & b_2 & \dots & b_n \\ | & | & & | \end{pmatrix},$$

how can we understand B^T in terms of those b_i's?

Task 93. Think about the work you have done in the last few tasks. How can we reinterpret the coordinates of the matrix-vector product Ax in a way that uses the transpose? How does this connect to a familiar operation?