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ELEMENTS OF LINEAR ALGEBRA, VOLUME II CAHIER DES LIGNES

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PUBLISHED BY I MADE IT MYSELF.

SITES.UNI.EDU/THERON/LA17

First printing, September 2017

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Introduction

This is a *workbook*. It is a collection of tasks that you should do to try to learn linear algebra for yourself.

It is not a *textbook*. We don't have a textbook, as such things are understood today. We have a primer, the first volume of this set: *Livre des Lignes*, which should serve as a basic source for reading about the main ideas, and we have this workbook. The books are separate so that you may have them both open at once. I hope this works.

It is important to focus on your own understanding when working on these tasks. At every stage possible, you should ask yourself, "Self, how do I know this?" and maybe also, "So, Self, do I know this for sure? Or do I still have doubts?"

If you are feeling at a loss for how to explain your thinking, try relying on simple geometric models and reasoning. It turns out that most of even the fanciest linear algebra is done that way. So, "Self, can you draw the relevant picture? Does that help?"

Good luck. I'll see you in class, and I welcome your visits to office hours to talk about mathematics.

Vectors and Lines the Plane

Points and Vectors

Task 1. Write down three distinct points in the plane in proper notation. Plot those three points on a single diagram.

Task 2. Write down three distinct vectors in the plane in proper notation. Your three points from this task should NOT match any of the three points from the previous task. Plot those three vectors on a single diagram.

Definition. If $u_1, u_2, \ldots, u_n$ is a collection of vectors, and $\lambda_1, \lambda_2, \ldots$ is a collection of scalars, then the vector formed below is called a *linear combination* of the u_i's.

Defintion: Linear Combinations

$$\lambda_1 u_1 + \lambda_2 u_2 + \ldots \lambda_n u_n$$

Task 3. Find the sum of your three vectors from the last exercise. Then, choose some order of those three vectors so that they are u_1, u_2 and u_3, and compute the linear combination

$$3u_1 - 2u_2 + (1/2)u_3.$$

Task 4. Let's consider the vectors $u = \left(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right)$, $v = \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$, and $w = \left(\begin{smallmatrix} -3 \\ 1 \end{smallmatrix}\right)$. Compute all of the vectors in this list:

$$\frac{u+v}{2}, v-u, v-w, u+\left(\frac{v-u}{3}\right), u+\left(\frac{3(v-u)}{4}\right)$$

Then make a single diagram which contains u, v, w and all of those vectors from the list, plotted as accurately as you can.

What do you notice? Is anything interesting going on?

Task 5. Consider the vector $u = \left(\begin{smallmatrix} -4 \\ 2 \end{smallmatrix}\right)$. Find a vector v which has the property that $u + v$ is the zero vector, or explain why this is not possible.

Definition. An equation of the form $\lambda_1 u_1 + \ldots \lambda_n u_n = w$, where all of the vectors u_i and w are known, but the scalars λ_i are unknown, is called a *linear combination of vectors equation*. A *solution* to such an equation is a collection of scalars which make the equation true.

Definition: Linear Combination Equations and their Solutions

Task 6. For now, keep $u = \left(\begin{smallmatrix}-4\\2\end{smallmatrix}\right)$. Let $v = \left(\begin{smallmatrix}3\\5\end{smallmatrix}\right)$. How many solutions does the linear combination of vectors equation $\lambda u = v$ have?

How many solutions does the linear combination of vectors equation $\lambda u + \mu v = 0$ have? (Here, treat 0 as the zero vector.)

Task 7. We still use the notation u for the vector $u = \left(\begin{smallmatrix}-4\\2\end{smallmatrix}\right)$, but now use v for the vector $v = \left(\begin{smallmatrix}28\\-14\end{smallmatrix}\right)$. How many solutions does the linear combination of vectors equation $\lambda u = v$ have?

How many solutions does the linear combination of vectors equation $\lambda u + \mu v = 0$ have? (Again, treat 0 as the zero vector.)

Task 8. Find the midpoint between the points $P = (4, -2)$ and $Q = (3, 5)$. Then find the two points which divide the segment PQ into thirds.

How can vectors make this simpler than it first appears?

Challenge 9. Suppose you are given three points in the plane. Let's call them P, Q, and R. How can you use vectors to (quickly) determine if these three points are collinear?

Parametric Lines in the Plane

Task 10. Write down five different points which lie on the line described parametrically as:

$$t \mapsto \begin{pmatrix}-5\\3\end{pmatrix} + t\begin{pmatrix}1\\-1/2\end{pmatrix}.$$

Challenge 11. How many different solutions to the equation

Recall the definition above.

$$x\begin{pmatrix}3\\7\end{pmatrix} + y\begin{pmatrix}6\\14\end{pmatrix} = \begin{pmatrix}-3\\-7\end{pmatrix}$$

can you find? How many are they? Can you find a good way to describe all of the solutions? Is there a natural geometric way to describe all of the solutions? Is there an algebraic way to describe all of the solutions?

Task 12. Write down a parametric description for the line which passes through the origin $O = (0, 0)$ and the point $S = (-5, 5)$.

Is this the *only* way to write down such a parametric description for that line?

Task 13. Write down a parametric description for the line which passes through the points below.

$$T = (\pi, 0), \qquad J = (0, -\pi)$$

Now find a different parametric description for that line. (*Hint: Can you find a way to write a parametric description that doesn't use the number π?*)

Task 14. Compare the lines from the tasks 12 and 13. Do you note anything interesting? How do you know your observation is true?

Task 15. For each of the conditions below, either find an example of a 2-vector Y so that the equation

$$t\begin{pmatrix}3\\-1\end{pmatrix} = Y$$

has the given number of solutions, or explain why such an example is not possible.

a) exactly zero solutions;
b) exactly one solution;
c) exactly two solutions.

Task 16. For each of the conditions below, either find an example of a 2-vector Z so that the equation

$$\begin{pmatrix}1\\2\end{pmatrix} + t\begin{pmatrix}-1\\1\end{pmatrix} = Z$$

has the given number of solutions, or explain why such an example is not possible.

a) exactly zero solutions;
b) exactly one solution;
c) exactly two solutions.

Task 17. For each of the conditions below, either find an example of a 2-vector Y so that the equation

$$\begin{pmatrix}-2/5\\2\end{pmatrix} + tY = \begin{pmatrix}-1\\1\end{pmatrix}$$

has the given number of solutions, or explain why such an example is not possible.

a) exactly zero solutions;
b) exactly one solution;
c) exactly two solutions.

Task 18. For each of the conditions below, either find an example of a 2-vector Z so that the equation

$$x\begin{pmatrix}-5/3\\1\end{pmatrix} + y\begin{pmatrix}3\\7\end{pmatrix} = Z$$

has the given number of solutions, or explain why such an example is not possible.

a) exactly zero solutions;

b) exactly one solution;
c) exactly two solutions.

Task 19. For each of the conditions below, either find an example of a 2-vector Z so that the equation

$$x\begin{pmatrix}-5/3\\1\end{pmatrix}+y\begin{pmatrix}1\\-3/5\end{pmatrix}=Z$$

has the given number of solutions, or explain why such an example is not possible.
a) exactly zero solutions;
b) exactly one solution;
c) exactly two solutions.

Challenge 20. Find an example of four 2-vectors X, Y, Z, and W so that the equation

$$aX+bY+cZ=W$$

has at exactly two solutions, or explain why such an example is not possible.

The Dot Product: Norms and Angles

Task 21. Choose three different 2-vectors which have neither of their components equal to zero. Call these vectors u, v, and w.
a) Compute the norms of u, v, and w.
b) Compute the dot products $u\cdot v$, $v\cdot w$, and $u\cdot w$.
c) Find unit vectors u', v', and w' which point in the same directions as u, v, and w, respectively.
d) Find the angles between each of the pairs, u and v, u and w, v and w in radians.

Task 22. Fix some vector u. Draw a picture of u in the plane, and then shade the region of the plane which contains vectors v so that $u\cdot v>0$.

Task 23. This task continues our quest for understanding the sign of a dot product geometrically.
a) Find an example of two 2-vectors v and w so that $\left(\begin{smallmatrix}1\\2\end{smallmatrix}\right)\cdot v=0$ and $\left(\begin{smallmatrix}1\\2\end{smallmatrix}\right)\cdot w=0$, or explain why such an example is not possible.
b) Let $v=\left(\begin{smallmatrix}3\\-1\end{smallmatrix}\right)$. Find an example of a pair of 2-vectors u and w such that $v\cdot u<0$ and $v\cdot w<0$ and $w\cdot u=0$, or explain why no such pair of vectors can exist.
c) Find an example of three 2-vectors u, v, and w so that $u\cdot v<0$ and $u\cdot w<0$ and $v\cdot w<0$, or explain why no such example exists.

Task 24. What shape is the set of solutions $\binom{x}{y}$ to the equation

$$\begin{pmatrix} 3 \\ 7 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = 5?$$

That is, if we look at all possible vectors $\binom{x}{y}$ which make the equation true, what shape does this make in the plane? Draw this shape.

What happens if we change the vector $\binom{3}{7}$ to some other vector? What happens if we change the number 5 to some other number?

Task 25. a) Find an example of a number c so that the equation

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = c$$

has the vector $\binom{4}{7}$ as a solution, or explain why no such number exists.

b) Let $v = \binom{2}{1}$ and $w = \binom{-3}{4}$. Find an example of a number c so that

$$v \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = c \quad \text{and} \quad w \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = c,$$

or explain why this is not possible.

c) Let $P = \binom{-3}{4}$. Find an example of numbers c and d so that

$$\begin{pmatrix} 2 \\ -1 \end{pmatrix} \cdot P = c \quad \text{and} \quad \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot P = d,$$

or explain why no such example is possible.

Equations of Lines in the Plane

Task 26. Write down an equation for the set of vectors which are all orthogonal to $u = \binom{3}{-2}$.

Task 27. We begin with a line described parametrically by

$$t \mapsto \begin{pmatrix} 6 \\ -\pi \end{pmatrix} + t \begin{pmatrix} 34 \\ -19/3 \end{pmatrix}.$$

a) Find a normal vector for this line.
b) Plot the line and the normal vector you found.

Task 28. Find an equation for the line in the Task 27.

Task 29. We begin with a line described by the equation

$$-3x + y = 7.$$

a) Find a normal vector to the line
b) Plot this line and the normal vector you found.

Task 30. Find a parametric description for the line from Task 29.

Task 31. Consider the line described by the equation $4x + 7y = 3$. Find an equation for the line which is parallel to this one, but passes through the point indicated:
a) The origin O.
b) The point $P = (0, -10)$.

Challenge 32. Consider the line described by the equation $x - 2y = -2$. Find a line which is orthogonal to this one, and passes through the point $Q = (9, -1)$. Can you describe your line with an equation? Can you describe your line parametrically?