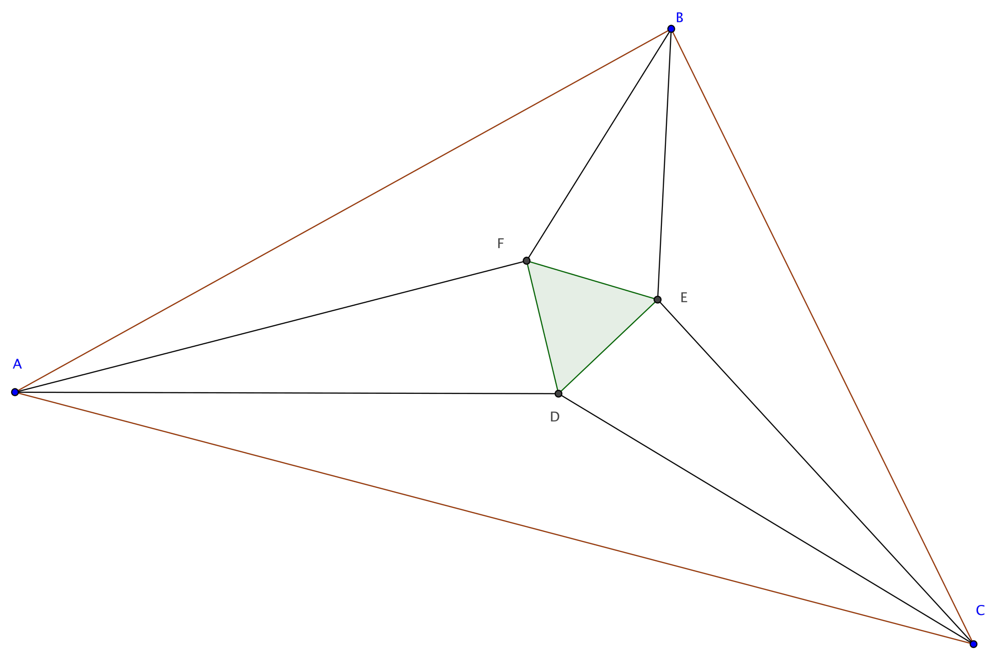
Transactions
in
Euclidean Geometry



Volume 2017F Issue # 8

Table of Contents

Regular Triangles

Mr. Hertzler

December 7, 2017

Communicated by: Mrs. Carstens

Theorem 6.1. An equilateral triangle is equiangular, hence regular.

Proof. Let ABC be an equilateral triangle. Since ABC is equilateral, then AB is congruent to BC is congruent to CA. Let angle ABC be called b, CAB be called a, and BCA be called c. Since CA is congruent to BC then by Euclid I.5, then angles a and b are considered the base angles of an isosceles triangle and thus congruent. Similarly AB and BC are congruent so by Euclid I.5, the angles a and c are considered the base angles of an isosceles triangle and are thus congruent. Since b is congruent to a, and a is congruent to c, then by Euclid Common Notion I.1, b is congruent to c. Since angles a, b, and c are mutually congruent, then triangle ABC is equiangular. Since triangle ABC is both equiangular and equilateral, then it is regular. □

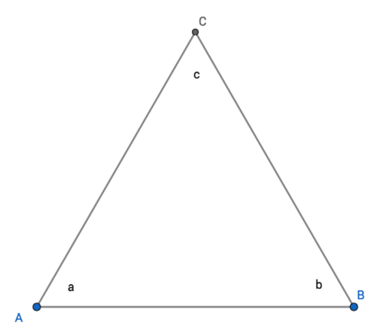


Figure 1: Above is the equilateral Triangle ABC with labeled angles a, b, and c.

Regular Pentagon

Mr. Hertzler

December 7, 2017

Communicated by: Ms. Thompson.

The overall method with this proof is to begin with a regular pentagon, then construct a new irregular pentagon that still holds the properties of the hypothesis in order to disprove the theorem by counterexample.

Theorem 6.4. Let ABCDE be an equilateral pentagon. If angle A is congruent to angle B, then ABCDE is regular.

Proof. Let ABCDE be a regular pentagon. Construct circles ED and CD. There are two intersections of circles ED and CD, one of the points is already D so let the other point be F. Join EF and CF by Euclid Postulate I.1. There is now pentagon ABCFE that has been constructed from the regular pentagon ABCDE. Angles AEF and BCF are a part of angles AED and BCD respectively. By Euclid Common Notion I.5, angles AEF and BCF are less than angles AED and BCD respectively. Since ABCDE is regular then it is equiangular. Since angles AEF and BCF are less than angles AED and BCD, then pentagon ABCFE is not regular since angle EAB is not congruent to AEF. Thus Theorem 6.4 is false by the nature of the construction of pentagon ABCFE. □

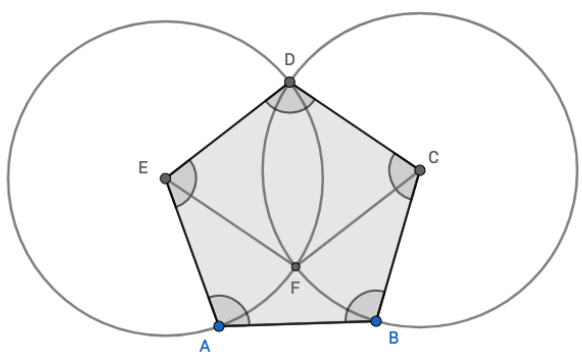


Figure 1: Above is regular pentagon ABCDE with which pentagon ABCFE is constructed from.

Hemispheres and Right Angles

Mr. Hertzler

December 7, 2017

Communicated by: Mr. Coyle

Theorem 7.4. If AB is the diameter of a circle and C lies on the circle, then angle ACB is a right angle.

Proof. Join AC and AB by Euclid postulate I.1. Bisect segment AB by Euclid I.10 and name the point F. Join FC by Euclid postulate I.1. Triangle ABC is now broken up into triangles ACF and BCF. The sum of the interior angles in triangle ACF add up to two right angles by Euclid I.32. Similarly, the sum of the interior angles in triangle BCF add up to two right angles by Euclid I.32. Triangle ACF is an isosceles triangle by Euclid definition I.20 since AF and CF are radii of circle F. Similarly triangle BCF is isosceles by Euclid definition I.20 since CF and BF are radii of circle C. By Euclid I.5 the base angles in isosceles triangles ACF and BCF are congruent. Let angles FCA and FAC be called L. Let angles FCB and FBC be called M. By Euclid I.32 M+M+L+L=2 right angles since angles M, M, L, and L are the interior angles of triangle ABC. M+M+L+L=2 right angles can be simplified to 2M+2L=2 right angles. Factor out a two so 2(M+L)=2 right angles. Divide both sides by two so M+L=1 right angle. Since M+L=1 right angle and makes up angle ACB, then ACB is a right angle when AB is the diameter of a circle and C lies on the circle. □

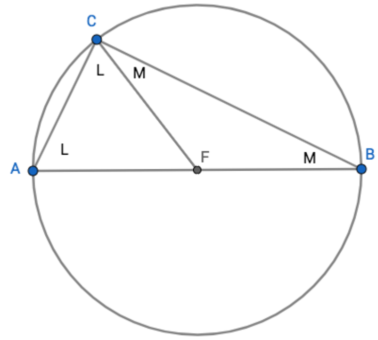


Figure 1: Here is a circle with diameter AB, where C is on the circle.

Right Angles and Hemispheres

Mr. Hertzler

December 7, 2017

Communicated by: Mrs. Bertacini

This proof will be broken up into two cases. Case 1 is the point is outside the circle. Case 2 is the point is inside the circle.

Theorem 7.5. If angle ACB is a right angle, then C lies on the circle with diameter AB.

Proof. Let AB be the diameter of a circle. Let the circle be named Z. Case 1: Point C is outside the circle. Let a random point be outside Z and name it C. Join AC and BC by Euclid Postulate 1. Let the intersection of AC and Z be named F and the intersection of BC and Z be named G. Pick a random point on the arc FG and name it E. Connect AE and BE by Euclid Postulate 1. By Hertzler 7.4, angle AEB is a right angle. By Euclid I.21, angle AEB is greater than angle ACB. Since angle AEB is a right angle, then ACB is not a right angle. Thus point C cannot be a right angle and outside the circle. Case 2: Point C is within the circle. Let a random point be inside Z and name it C. Join AC and BC by Euclid Postulate 1. Extend AC and BC by Euclid Postulate I.2. Let the intersection of the extension of AC and Z be named G and the intersection of BC and Z be named F. Pick a random point on the arc FG and name it E. Connect AE and BE by Euclid Postulate 1. By Hertzler 7.4, angle AEB is a right angle. By Euclid I.21, angle AEB is less than angle ACB. Since angle AEB is a right angle, then ACB is not a right angle. Thus point C cannot be a right angle and be inside the circle. Since C cannot be a right angle and exist either outside or inside the circle, then the only place C may lie is on the circle. □

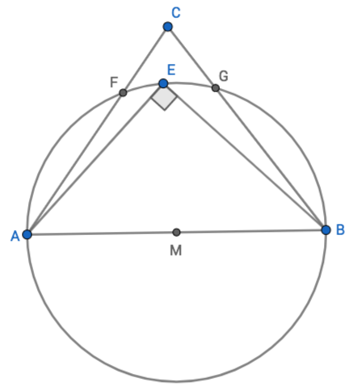


Figure 1: This is the construction of the argument in Case 1.

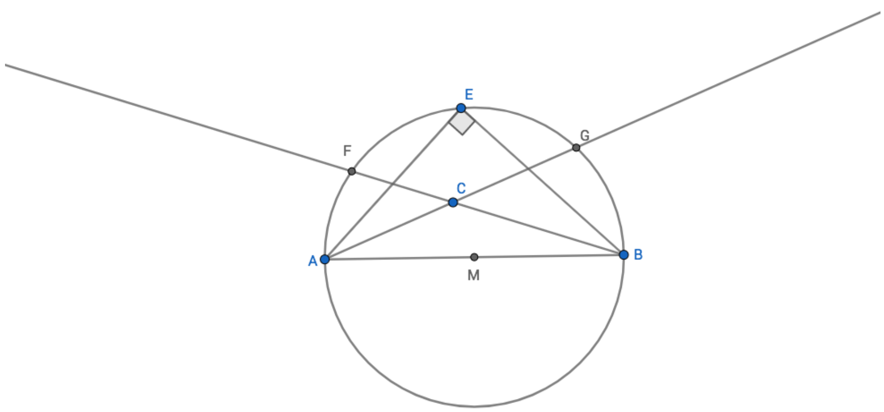


Figure 2: This is the construction of the argument in Case 2.

Angles Within a Circle

Cameron Amos

December 7, 2017

Communicated by: Katherine Bertacini.

The following proof is for a special case of Theorem 10.1. This special case is line l goes through the center of circle O.

Theorem 10.1. Let Γ be a circle with center O. Let X be a point in the interior of the circle, and suppose that two lines l and m intersect at X so that l meets Γ at points A and A' and m meets Γ at points B and B'. Then twice angle AXB is congruent to angle AOB and angle $A'OB'$ taken together.

Proof. By Postulate 1, Draw AB, BO, and $B'O$. Since BO and $B'O$ are the radii of circle O, then BO is congruent to $B'O$. Therefore, triangle BOB' is isosceles. Since triangle BOB' is isosceles, angle OBB' is congruent to angle $OB'B$ by Euclid I.5. Angle AXB plus angle $B'XA$ equals two right angles by Euclid I.13. Angle $B'XO$ plus angle XOB' plus angle $XB'O$ equals two right angles by Euclid I.32. The previous two equations are equal to one another, because they both equal two right angles. Therefore, angle AXB plus angle $B'XA$ equals angle $B'XO$ plus angle XOB' plus angle $XB'O$. Angle $B'XA$ is angle $B'XO$, so we can subtract that angle from both sides of the equation. Thus, angle AXB equals angle $B'OX$ plus angle $XB'O$. Since triangle BOB' is isosceles, angle $XB'O$ is congruent to angle XBO. Thus, angle XBO can be substituted for angle $XB'O$ in the equation, angle AXB equals angle $B'OX$ plus angle $XB'O$, giving the equation angle AXB equals angle $B'OX$ plus angle XBO. Thus angle XBO equals angle AXB minus angle $B'OX$.

By Euclid I.13, angle AOB plus angle BOX equals two right angles. By Euclid I.32, angle OXB plus angle BOX plus angle XBO equals two right angles. The previous two equations are equal to one another, because they both equal two right angles. Therefore, angle AOB plus angle BOX equals angle OXB plus angle BOX plus angle XBO. Angle BOX can be subtracted from both sides of the equation, giving angle AOB equals angle OXB plus angle XBO. Angle OXB is angle AXB, so angle AXB can be substituted for angle OXB into the equation, angle AOB equals angle OXB plus angle XBO. Thus giving the equation, angle AOB equals angle AXB plus angle XBO.

Since angle XBO equals angle AXB minus angle $B'OX$, angle AXB minus angle $B'OX$ can be substituted for angle XBO in the equation angle AOB equals angle AXB plus angle XBO. Therefore, we have angle AOB equals angle AXB plus angle AXB minus angle

$B'OX$. Through combining angle AXB and angle AXB and adding angle $B'OX$ to both sides of the equation, we have twice angle AXB equals angle AOB plus angle $B'OX$. Angle $B'OX$ is angle $A'OB'$, therefore angle $A'OB'$ can be substituted for angle $B'OX$ in the equation twice angle AXB equals angle AOB plus angle $B'OX$. Thus, twice angle AXB equals to angle AOB plus angle $A'OB'$. Hence, twice angle AXB is congruent to angle AOB and angle $A'OB'$ taken together. □

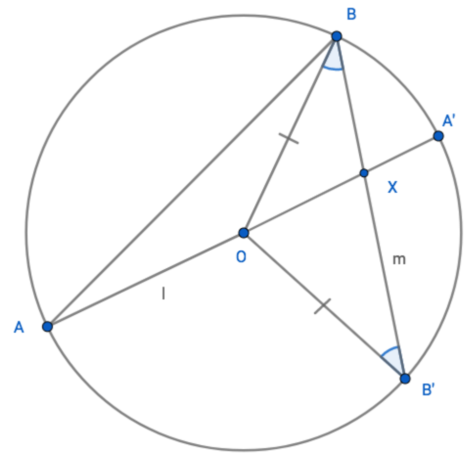


Figure 1: This figure represents isosceles triangle BOB' and congruent angles OBX and $OB'X$ in circle Γ.

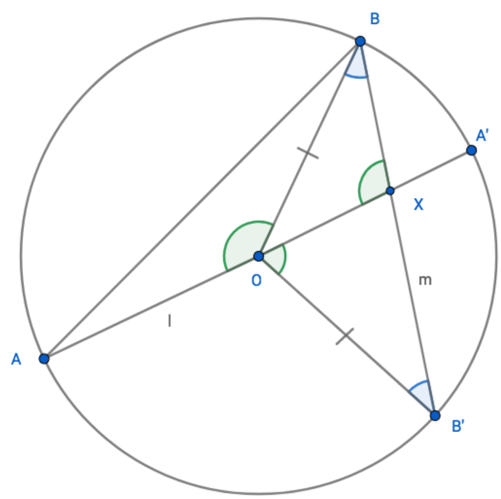


Figure 2: This figure shows angles AXB, AOB, and $A'OB'$ in circle Γ. From Theorem 10.1, twice angle AXB is congruent to angles AOB and $A'OB'$ taken together.

Theorem 10.5

Connor Coyle and Cameron Amos

December 7, 2017

Communicated by: Mr. Kilburg.

Theorem 10.5. A triangle has two angle bisectors which are congruent (measured from the vertex to the opposite side), then the triangle is isosceles.

Proof. Let l and m be lines such that line l bisects angle A and line m bisects angle B. Further, let l intersect BC at point E, line m intersect AB at point D, and let F be the point at which lines l and m intersect. Then, segment AF is congruent to CF and FD is congruent to FE by Theorem 8.1, which was proved by Mr. Otterbein. Using Euclid I.15, we get angle EFC congruent to DFA. By Euclid I.4, triangle AFD is congruent to triangle CFE because angle DFA is congruent to angle EFC, FD is congruent to FE, and AF is congruent to CF. By Euclid I.9 draw angle bisector of B. By using 8.2 theorem proved by Mr. Otterbein, we know that the angle bisector of B is concurrent with lines l and m, and that they meet at point F. Angle BDF is congruent to angle BEF by Euclid I.13. By Euclid I.26, triangle BEF is congruent to triangle BDF because angle BDF is congruent to angle BEF, angle DBF is congruent to EBF, and they share BF. Since BD is congruent to BE, and DA is congruent to EC, then BA is congruent to BC. Thus triangle ABC is isosceles. □

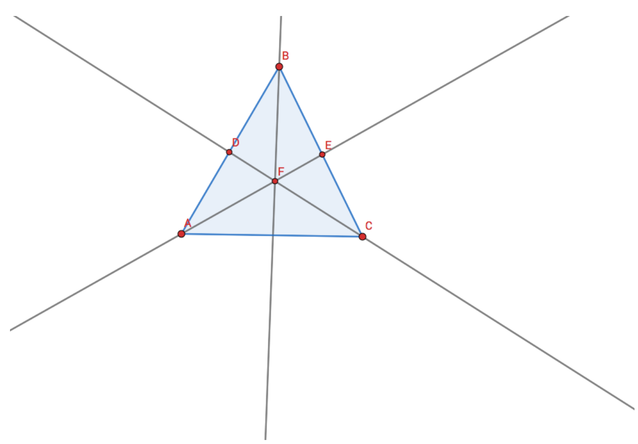


Figure 1: This is a picture of triangle ABC

Midpoint Construction

Kaelyn Koontz

December 7, 2017

Communicated by: Ms. Feldmann.

Theorem 11.2. Given a segment, find the midpoint.

Proof. First, you are given segment AB.

1. Draw circle AB.
2. Draw circle BA.
3. Name points C and D where the two circles intersect.
4. Draw segment CD.
5. Name point E where segment AB and segment CD intersect.

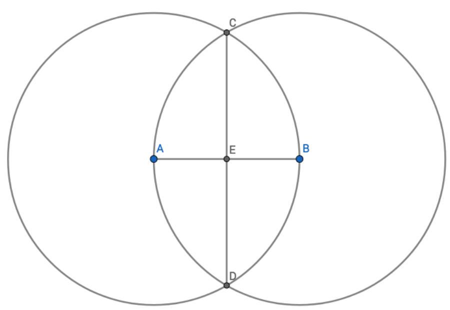


Figure 1: This is a picture of how to construct a midpoint.

When looking at triangle DAC and triangle DBC, we know that DA is congruent to DB and AC is congruent to BC because they have the same radius. Also since both triangles share side CD, then that side is congruent. Therefore, by Euclid I.8, triangle DAC is congruent to triangle DBC. Next we will look at triangle ACE and and BCE. Since triangle DAC is congruent to DBC, AC is congruent to BC and angle ACD is congruent to angle BCD. Also since triangle ACE and BCE share side CE, that side is congruent. By Euclid I.4, triangle ACE is congruent to triangle BCE. So, AE is congruent to BE. Therefore, E is the midpoint of segment AB. □

Midpoint of a Given Segment

Kayla Schafbuch

December 7, 2017

Communicated by: Emily Carstens.

Theorem 11.2. Given segment AB, we can construct its midpoint.

Proof. Construction
Step 1: Using a compass, draw circle AB.
Step 2: Using a compass, draw circle BA. Name points C and D where the circles intersect.
Step 3: Using a ruler, draw CD.

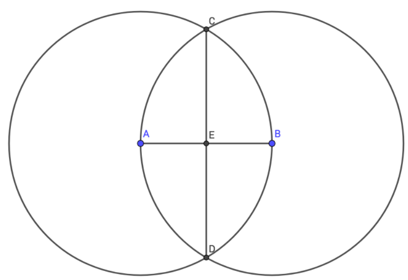


Figure 1: This is a picture of the construction above.

□

Proof. By Euclid's Postulate 1, construct AC, CB, BD, and AD. Segments AC, CB, BD, and DA are congruent to each other because circle AB has the same radius as circle BA. Since AD is congruent to AC, by Euclid I Definition 20, triangle DAC is isosceles. Then, angle ADC is congruent to angle ACD by Euclid I.5. By Euclid I.8, triangle DAC is congruent to triangle DBC. Since triangle DAC is congruent to triangle DBC, then triangle DBC is also isosceles. Therefore, angle ADC is congruent to angle ACD is congruent to BDC is congruent to BCD.
Similarly, by Euclid I Definition 20, triangle ABD is isosceles. Then, angle DAB is congruent to angle DBA by Euclid I.5. Triangle ABD is congruent to triangle ABC by Euclid I.8. Since triangle ABD is congruent to triangle ABC, then triangle ABC is also isosceles. Therefore, angle DAB is congruent to DBA is congruent to CAB is congruent to CBA.

By Euclid I.26, triangle AEC is congruent to triangle AED is congruent to BEC is congruent to BED. Then AE is congruent to EB, therefore E is the midpoint.

□

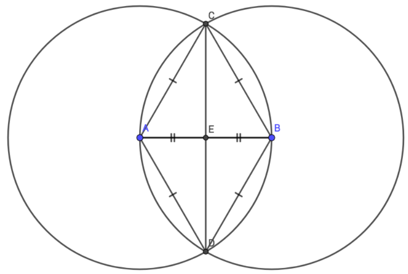


Figure 2: This picture shows segment AB and its midpoint E.

Congruent Angle Construction

Kaelyn Koontz

December 7, 2017

Communicated by: Rachelle Feldmann.

Theorem 11.5. Given an angle at a point A and given a ray emanating from a point B, construct an angle at B congruent to the angle at A having the given ray as a side.

Proof. First, we are given angle A and a ray emanating from point B.

1. Select a width on your compass and draw a circle with A as the center. Label the two points C and D where the circle intersects the angle.
2. With the same width, draw a circle with the center B. Label the point E where the circle intersects the ray.
3. Set your compass to the width CD and make the circle centered at point E with the radius of CD.
4. Where the two circles intersect, label that point F and draw segment BF.

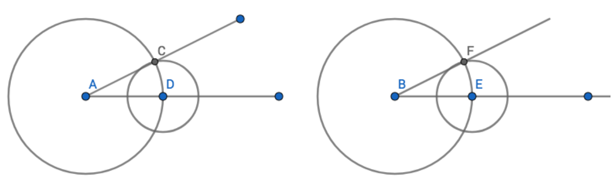


Figure 1: This is a picture of how to construct a congruent angle.

In order to prove that these are congruent angles, we will look at triangle ACD and triangle BFE. Segment AD is congruent to BE because they have the same radius. Segment AC is congruent to BF because they have the same radius. Since segment FE was created with the same radius as CD, then CD is congruent to FE. So, triangle ACD is congruent to triangle BFE by Euclid I.8. Therefore, the angle at A is congruent to the angle made at B. ☐

Inscribed Circle Construction

Kaelyn Koontz

December 7, 2017

Communicated by: Ms. Feldmann.

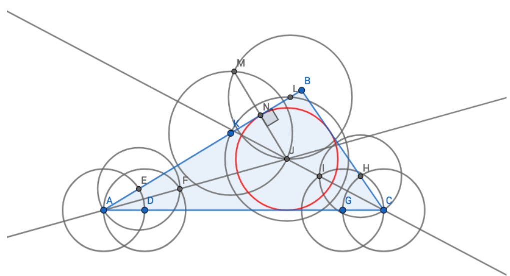


Figure 1: This is a picture of the construction of an inscribed circle.

Theorem 12.1. Construct a circle inscribed in a given triangle ABC.

Proof. We are given triangle ABC.

1. Select a width on your compass, and draw a circle with center A. Label the two points where the circle intersects the angle as point D and point E.

2. Draw circle DA and circle EA. Where the two circles intersect, label the intersection point F. Draw segment AF.

3. Select a width on your compass, and draw a circle with center C. Label the two points where the circle intersects the angle as point G and point H.

4. Draw circle GC and circle HC. Where the two circles intersect, label the intersection point I. Draw segment CI.

5. Where CI and AF intersect, label the intersection point J.

6. Select a width on your compass, and draw a circle with center J. Where the circle intersects AB, label the two points K and L.

7. Draw circle KJ and LJ. Where the two circles intersect, label the intersection point M.
8. Draw segment JM. This is a perpendicular line. Where JM intersects AB, label the intersection point N
9. Draw circle NJ.

A circle is inscribed in a circle if the circle lies in the interior of the figure and is tangent to each of the sides of the figure. By Avery 11.4, JN is perpendicular to AB. Circle NJ is tangent to AB, therefore Circle NJ is inscribed in triangle ABC. □