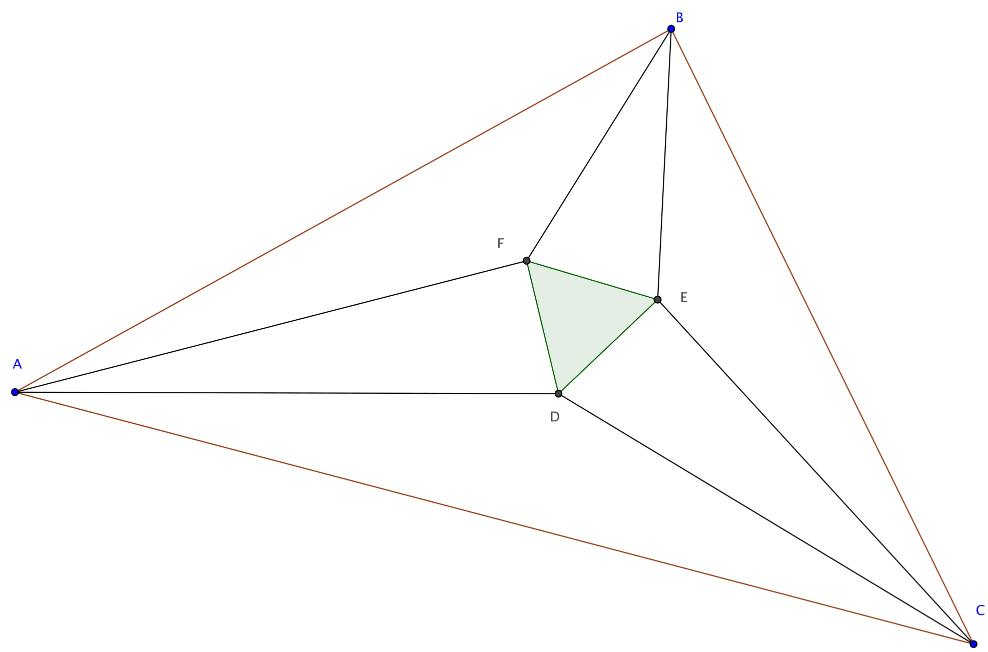
Transactions

in

Euclidean Geometry



Volume 2017F Issue # 4

Table of Contents

Square Construction

Katherine Bertacini, Rachelle Feldmann, and Kaelyn Koontz

November 17, 2017

Communicated by: Emily Carstens.

In this construction, we will make square ADCE. We are given segment AB and are able to use different circles to show the congruence of the sides in order to create square ADCE.

Theorem F. The construction of ADCE is a square.

Proof. First, we are given line segment AB. Draw circle AB and circle BA. Using Postulate 1.2, extend line AB to a point C which falls upon circle AB. Then draw circle BC. Using Euclid I.11, starting from point B draw a perpendicular line up to a point on circle AB and name it point D. Starting from point B, draw a perpendicular line down to a point on circle AB and name it point E. Using Postulate 1.2, draw line AD, DC, CE, and EA.

Using Euclid I.4, triangle ABE is congruent to triangle EBC which is congruent to triangle CBD which is congruent to triangle DBA.This shows that AD is congruent to DC which is congruent to CE which is congruent to EA. Using Euclid I.32, we know that in a triangle the angles have to add up to the sum of 2 right angles. Angle ABE is a right angle from Euclid I.11. So, in triangle ABE, we know that the sum of angles BAE and AEB have to add up to one right angle. Since we have four congruent triangles, we know that the congruent adjacent angles have to add up to the sum of a right angle.

Thus, ADCE is a square.

□

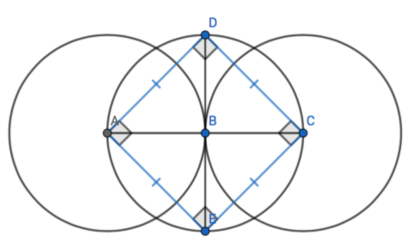


Figure 1: This is a picture of the construction of a square.

Squares and Rectangles

Rachelle Feldmann and Kaelyn Koontz

November 17, 2017

Communicated by: Ms. Bertacini.

This is a special case of a square being a rectangle. By using square ADCE from Theorem F, we are able to apply the definition of a rectangle and say that square ADCE is a rectangle. Then we can prove that each pair of opposites sides of ADCE are congruent to each other.

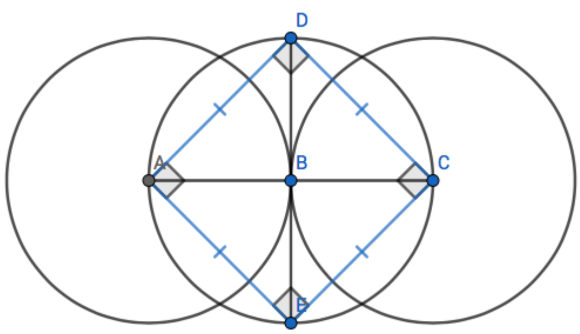


Figure 1: This is square ADCE.

Definition 3.1. A rectangle is a quadrilateral which has all four interior angles that are right angles.

Theorem 3.2. Let ACDE be a rectangle. Then each pair of opposite sides of ACDE is a pair of congruent sides.

Proof. Let ADCE be the square from Theorem F. Using Definition 3.1, we can say square ADCE is a rectangle because it has four interior right angles. Hence, square ADCE is a rectangle. Notice that sides AD, DC, CE, and EA are all congruent. This means that opposite sides AD and CE are congruent to one another, while sides DC and DA are congruent to one another. Therefore in rectangle ADCE, each pair of opposite sides is a pair of congruent sides. □

Kites and Parallelograms

Kayla Schafbuch

November 17, 2017

Communicated by: Cameron Hertzler.

In class, we were trying to prove the opposite angles in a kite were congruent. This can be proven true when the kite is a parallelogram. So, the question that arose was if all kites are parallelograms.

Question H. Are all kites parallelograms?

Definition 1. A kite is a quadrilateral with two pairs of adjacent and congruent sides.

Definition 2. Parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.

Theorem 2.4. [Flesch] Assume ABCD is a kite, but is neither a rhombus nor a square, with AB congruent to AD, BC congruent to DC, AB not congruent to DC, and AD not congruent BC. By Euclid I.34, kite ABCD is not a parallelogram.

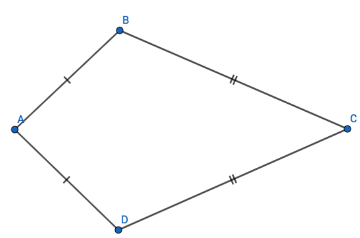


Figure 1: Kite ABCD.

Mr. Flesch proved kite ABCD was not a parallelogram. Therefore, not all kites are parallelograms.

Polygons

Rachelle Feldmann

November 17, 2017

Communicated by: Ms. Koontz .

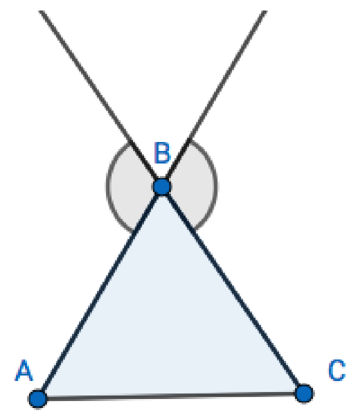


Figure 1: Triangle ABC.

Theorem 5.1. Suppose that A, B, C are three consecutive vertices's of a polygon. If at the vertex B we extend one of the two sides through B to a ray, then we create a new angle, called an exterior angle to the polygon at B. This construction has a choice in it. In principle, this could be a problem. Describe the problem, then state and prove a theorem that resolves the issue.

Proof. The choice presented in the problem is which angle at B to use when a ray is extended from either sides, A or C. Using Euclid 1.15, vertical angles are congruent. The angle produced at B created by the extended ray at A or C are the same angle. Therefore the choice does not make a difference because they are the same angle. □

Simple and Non-Simple Polygons

Kayla Schafbuch

November 17, 2017

Communicated by: Ashlyn Thompson.

As a class, we came up with many different polygons. Our definition of polygons allows weird shapes to be classified as polygons, such as the shapes in Figure 2.

Question W. How should we define the terms "simple polygon" or "non-simple polygon"?

By looking at the different polygons constructed, there are some we see as simple and some that we see as non-simple by intuition.

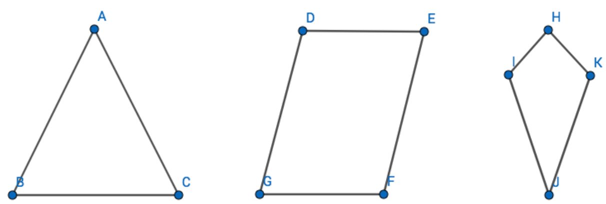


Figure 1: Polygons we see as simple.

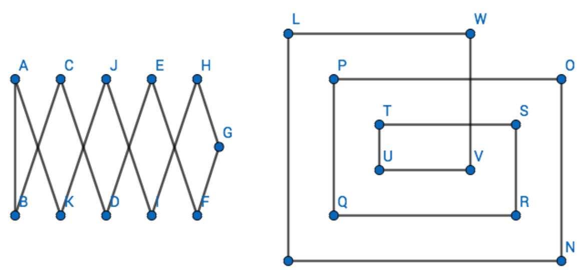


Figure 2: Polygons we see as non-simple.

The polygons we see as simple, each of their sides only meet two other sides of the figure, not allowing the sides of the polygon to cross at any point besides the given points. The polygons we see as non-simple, there is a side that meets more than two sides of the figure, allowing the sides of the polygon to meet at a point not given. Thus, we have new definitions.

Definition 2. A polygon is called simple when each side meets only two other sides of the figure.

Definition 3. A polygon is called non-simple when there exists a side that meets more than two other sides of the figure.

Interior Angles of Quadrilaterals

Katherine Bertacini and Kaelyn Koontz

November 17, 2017

Communicated by: Micah Otterbein.

In this proof, we will discuss that by using the diagonals, one can split a quadrilateral into two triangles to find the sum of the interior angles of the quadrilateral. First, it is important to point out that in order for this to work, the quadrilateral must have at least one diagonal that is inside the figure. Quadrilaterals that are convex have 2 diagonals that are inside the figure. Quadrilaterals that are non-convex may have one diagonal in the figure. Non-simple quadrilaterals will not have any diagonals that are fully in the figure.

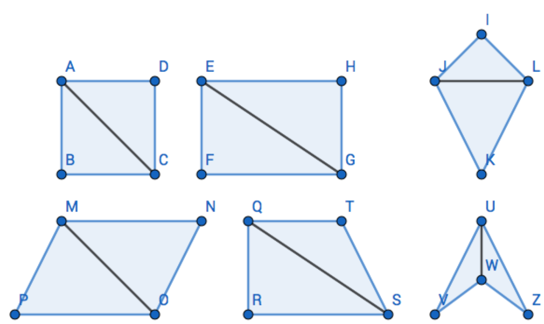


Figure 1: This demonstrates how we can draw the diagonal to split the quadrilateral into two triangles.

Theorem N. Let ABCD be a simple quadrilateral. Then the angles ABC, BCD, CDA and DAB taken together make 4 right angles.

Proof. In quadrilateral ABCD, by Euclid's Postulate 1.1, draw the diagonal AC or BD. Then by Euclid I.32, we know that the sum of the interior angles of a triangle equal 2 right angles. So, by adding the sums of the interior angles of the 2 triangles, the sum of the interior angles of quadrilateral ABCD is equal to 4 right angles.

□

A Congruence Theorem for Right Triangles

Theron Hitchman

November 17, 2017

Communicated by: The Editor.

There are many well-known triangle congruence theorems, each of which gives a set of sufficient conditions to conclude that a pair of triangles is congruent given that some collection of corresponding sides and angles are congruent. Here, we give a new triangle congruence theorem, which is applicable to pairs of right triangles. That is, we assume not just that the triangles have a congruent corresponding angle, but that this angle is also known to be a right angle.

A clear statement requires just a little bit of new terminology for the different types of sides in a right triangle.

Definition 1. In a right triangle, the side opposite the right angle is called the *hypotenuse* of T. The other two sides are called *legs* of T.

Now we are prepared

Theorem 7.2. Suppose that S and T are right triangles. If the hypotenuse of S is congruent to the hypotenuse of T, and if there is a leg of S which is congruent to a leg of T, then S and T are congruent triangles.

Proof. The idea of the argument is to construct a new triangle U, so that the following things are both true at once: U is congruent to T, and U and S form two halves of an isosceles triangle split by its altitude. Then we will have that S is congruent to U and U is congruent to T, from which we can deduce the desired result.

Let S be the triangle ABC with a right angle at A, and let T be the triangle DEF with a right angle at D. The hypotenuse of S is BC and the hypotenuse of T is EF. By assumption these segments are congruent.

We also know that one leg of each triangle is congruent to a leg of the other. Without loss of generality, we assume that AC is congruent to DF.

First, we shall show how to construct the new triangle U. Extend the segment AB to a ray from B through A. By Euclid Proposition I.2, construct a circle about A with radius DE. This circle and the ray meet at a point X which is different from B. (They may also meet at B, but this coincidence is not important to us.) We want to show that the triangle AXC is congruent to the triangle DEF.

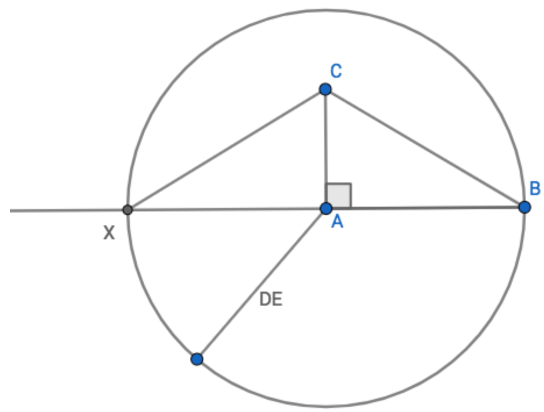


Figure 1: The construction of U to share a side of S.

By construction, the segment AX is congruent to DE. By hypothesis, AC is congruent to DF. Since the line CA cuts the line BX at C, the angles BAC and CAX make a pair of right angles by Euclid Proposition I.13. Thus, by Euclid Proposition I.4, we have that triangle AXC is congruent to triangle DEF.

Our next goal is to show that the triangles ABC and AXC are congruent.

Note that since the angles BAC and CAX are right angles, and the points B, A, and X are collinear, the figure BCX is a triangle. Since triangle AXC is congruent to DEF, we know that CX is congruent to EF. By hypothesis, EF is congruent to BC. Thus, CX is congruent to CB and the triangle BCX is an isosceles triangle. By Euclid Proposition I.5, we deduce that the angles CBA is congruent to the angle CXA. Shifting our attention back to the pairs of triangles AXC and ABC, we see that they have two pairs of congruent corresponding angles and two pairs of congruent corresponding sides. Hence, they are congruent by Euclid Proposition I.26.

Finally, since triangle ABC is congruent to triangle AXC and triangle AXC is congruent to triangle DEF, we learn that ABC is congruent to DEF. Since this was the desired result, the proof is complete.

$\square$

Finding the Intersection of the Perpendicular Bisectors of a Triangle

Grant Kilburg

November 17, 2017

Communicated by: Cameron Amos

Theorem 8.4. The three perpendicular bisectors of any triangle are concurrent.

Proof. Let ABC be a triangle. By Euclid I.10, bisect line segment AC at a point D. Construct the perpendicular bisector r of line segment AC through D by Euclid I.11. Similarly, bisect line segment BC at a point E by Euclid I.10. By Euclid I.11, construct the perpendicular bisector s of line segment BC through E. By Mr. Otterbein's Theorem, r and s must cross. Let X be the point of this intersection. By Postulate 1.1, construct line segments AX, BX, and CX. Since D is the midpoint of AC, AD is congruent to CD. Further, since r is the perpendicular bisector of AC, angle ADX is congruent to angle CDX. Therefore, by Euclid I.4 (SAS), triangle ADX is congruent to triangle CDX as AD is congruent to CD, angle ADX is congruent to angle CDX, and line segment DX is shared. Hence, AX is congruent to CX. Similarly, since E is the midpoint of BC, BE is congruent to CE. Further, because s is the perpendicular bisector of BC, angle BEX is congruent to angle CEX. Therefore, by Euclid I.4 (SAS), triangle BEX is congruent to triangle CEX as BE is congruent to CE, angle BEX is congruent to angle CEX, and line segment EX is shared. Hence, BX is congruent to CX.

Since AX is congruent to BX and BX is congruent to CX, AX is also congruent to CX by Common Notion 1. By Euclid I.10, bisect line segment AB at a point F. Draw line segment FX by Postulate 1.1. Since F is the midpoint of AB, AF is congruent to BF. Further, we know that BX is congruent to AX and XF is shared. Thus, triangle AXF is congruent to triangle BXF by Euclid I.8 (SSS). Since triangle AXF is congruent to triangle BXF, angle AFX is congruent to angle BXF. By Definition 10, angle AFX and angle BFX must by right angles, making line segment XF perpendicular to line segment AB. Since X lies on XF which is perpendicular to AB, and X is the intersection of lines r and s which are perpendicular to line segments AC and BC respectively, X is the intersection of the perpendicular bisectors of triangle ABC. Hence, the three perpendicular bisectors of any triangle are concurrent. $\square$

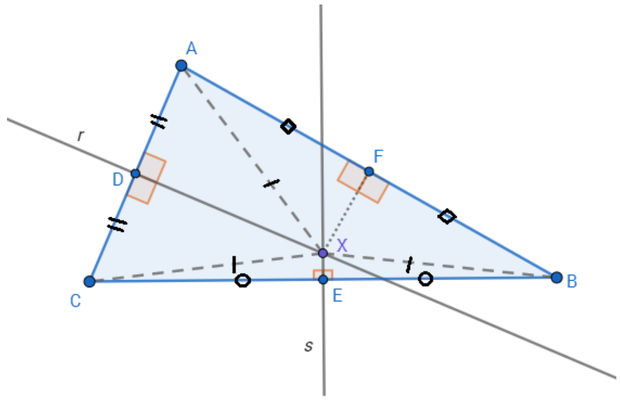


Figure 1: Since triangle ABC is an acute triangle, the perpendicular bisectors of triangle ABC are concurrent (i.e. they meet at a point) *inside* the triangle.

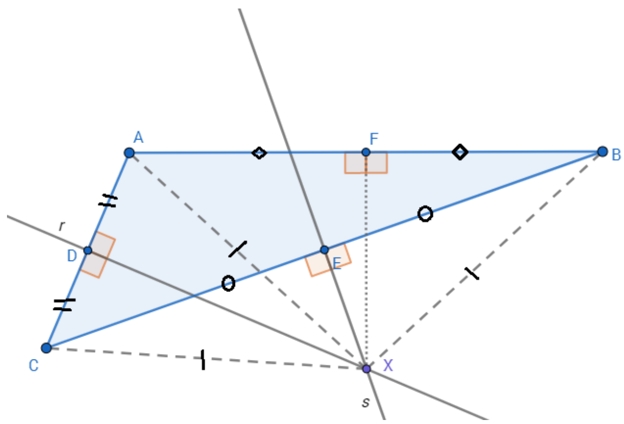


Figure 2: Since triangle ABC is an obtuse triangle, the perpendicular bisectors of triangle ABC meet at a point *outside* the triangle.

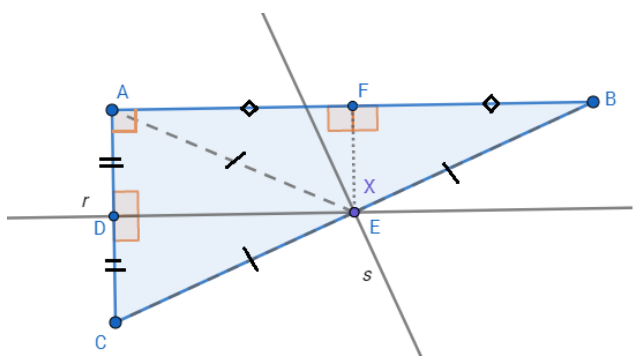


Figure 3: Since triangle ABC is a right triangle, the perpendicular bisectors of triangle ABC meet at the *midpoint* of the hypotenuse the triangle.

Inside and Outside a Quadrilateral

Emily Carstens

November 17, 2017

Communicated by: Mr. Kilburg.

Definition of Inside. Let ABCD be a convex quadrilateral. We say that a point P lies *inside* ABCD when P lies on a segment XY where X and Y lie on different sides of our quadrilateral.

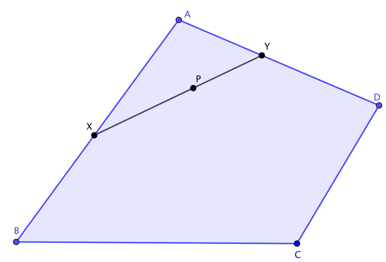


Figure 1: This figure shows how the following definition works in a convex quadrilateral.

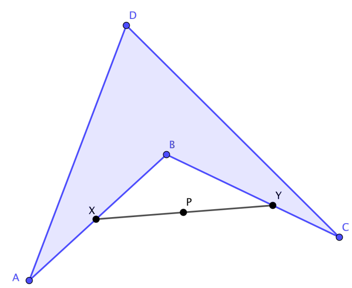


Figure 2: This is a figure of a nonconvex quadrilateral. As you can see, the definition does not hold for this figure because our point P lies outside ABCD.

Right Angles Within Two Circles

Cameron Amos

November 17, 2017

Communicated by: Ashlyn Thompson.

Theorem 9.2. Let Γ and Ω be two circles with centers G and O respectively. Suppose that these circles meet at two points, A and B. If angle GAO is a right angle, then angle GBO is a right angle.

Proof. By Postulate.1, draw GO. Since G is the center of circle Γ, then GA is congruent to GB. Since O is the center of circle Ω, then OA is congruent to OB. Since GA is congruent to GB, OA is congruent to OB, and GO is congruent to GO, then triangle GAO is congruent to triangle GBO by Euclid I.8. Since triangle GAO is congruent to triangle GBO, then angle GAO is congruent to angle GBO. Since angle GAO is a right angle and angle GAO is congruent to angle GBO, then angle GBO is a right angle.

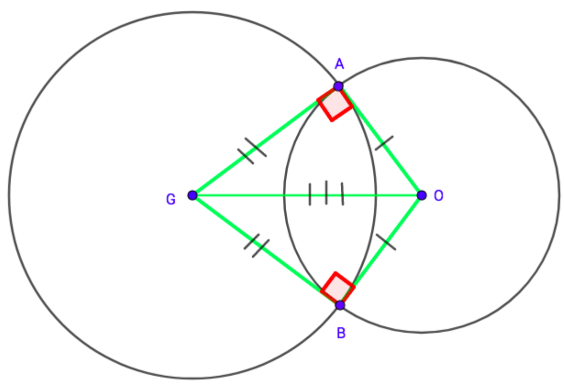


Figure 1: This is a picture representing two right angles, GAO and GBO, within circle Γ and circle Ω.

$\square$