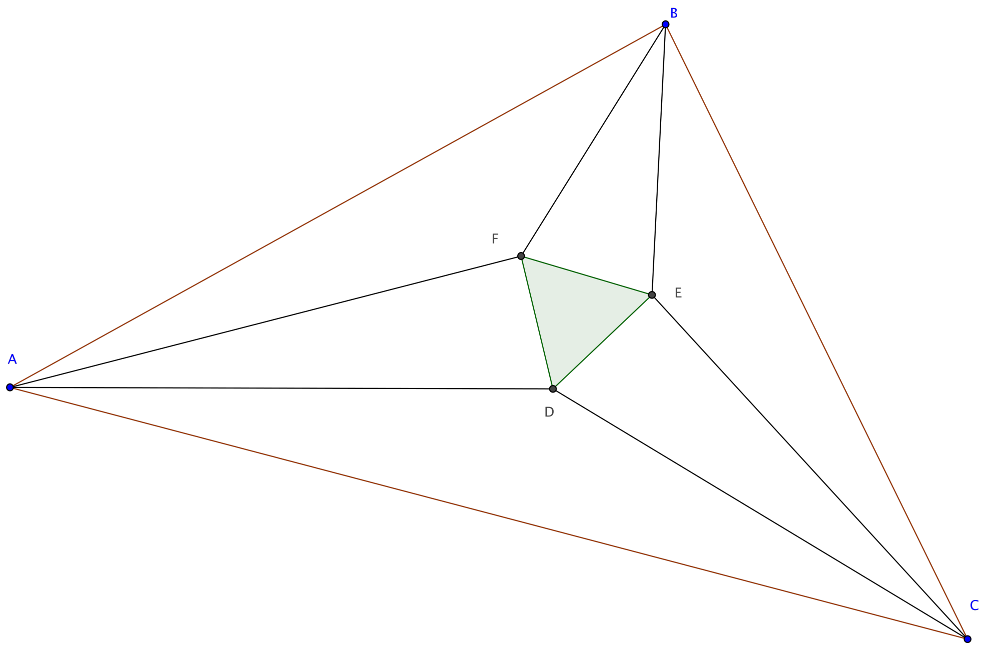
Transactions

in

Euclidean Geometry



Volume 2017F Issue # 2

Table of Contents

Finding a Square from a Rhombus with a Pair of Congruent Interior Angles

Lakota Avery

November 15, 2017

Communicated by: Mr. Micah Otterbein

Theorem B. Let $ABCD$ be a rhombus. If $\angle BAC$ is congruent to $\angle BDC$, then ABCD is a square.

Proof. Let $ABCD$ be a rhombus. Draw $\overline{AC}$ and $\overline{BD}$ by Euclid Postulate I.1. Suppose $\angle BAC$ and $\angle BDC$ are congruent. We know that $ABCD$ is a parallelogram by Mr. Flesch's Theorem 1.6. Since $ABCD$ is a parellelogram, we can say that $\angle BAC$ is congruent to $\angle ACD$ and similarly $\angle BDC$ is congruent to $\angle ABD$ by Euclid I.29. The diagonals $\overline{AC}$ and $\overline{BD}$ bisect all angles at points A, B, C, D by Mr. Hertzler's Theorem E, making every angle congruent to each other. We know the diagonals meet at a point E by Ms. Thompson's Theorem 1.2. All angles at point E must be right angles by Kilburg's Theorem 1.7.

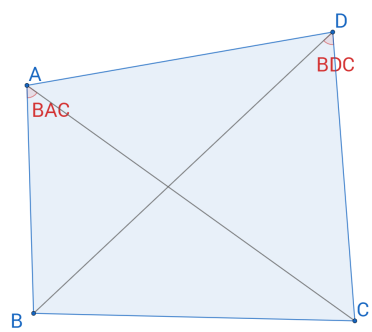


Figure 1: Rhombus $ABCD$

For triangles AEB, BEC, CED, and DEA in rhombus $ABCD$, the angles must add up to two right angles by Euclid I.32. Since there is already a right angle in each triangle at point E, the remaining angles must add up to a right angle. Since all angles are congruent, they must add up to a right angle at the points A, B, C, D. Since we have all sides congruent and all angles being right angles, we conclude that $ABCD$ is a square. □

Square in a Rhombus

Katherine Bertacini

November 15, 2017

Communicated by: Ms. Koontz.

Theorem C. Let ECFD be a rhombus that is not a square. Then one pair of opposite angles of ECFD is greater than a right angle and the other pair is less. This is a special case with a square inside a rhombus.

Proof. First, let ACBD be a square from Theorem F.

1. Make line AB using Postulate 1.1.
2. Extend line AB using Postulate 1.2.
3. Make circle AB, and circle BA.
4. Make points E and F, on the extended line AB from circles AB and BA.
5. Then make line segments EC, CF, FD, and DE from Postulate 1.1.
6. Hence, ECFD be a rhombus.

Therefore, in this construction of a rhombus angles ECF and EDF are greater than a right angle, also angle DEC and DFC are less than a right angle. □

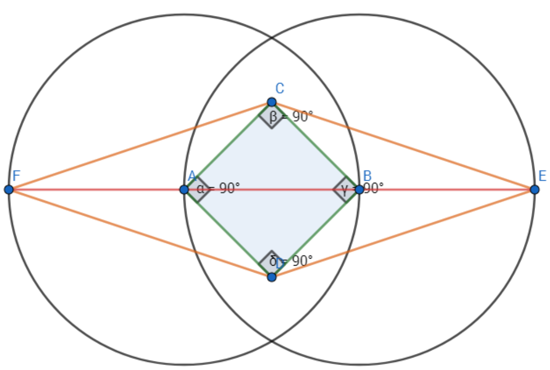


Figure 1: This is a picture of the square inside the rhombus.

Diagonals of a Kite

Grant Kilburg

October 23, 2017

Communicated by: Ashlyn Thompson

Theorem K. If ABCD is a kite with sides AB congruent to AD and DC congruent to BC, and the diagonals of ABCD intersect, then line segment AC will bisect line segment BD.

Proof. Let ABCD be a kite with sides AB, AD and DC, BC congruent. By Postulate 1.1, construct line segments AC and BD so that the two intersect. Let X be the point at which they cross. By Theorem I, line segment AC bisects angle DAB and angle DCB. Since AC bisects angles DAB and DCB, then angle DAX is congruent to angle BAX and angle DCX is congruent to angle BCX. Since AD is congruent to AB, angle DAX is congruent to angle BAX, and line segment AX is reflexive, triangle DAX is congruent to triangle BAX by Euclid I.4 (SAS). Since triangle DAX is congruent to triangle BAX, line segment BX is congruent to line segment DX. Thus, X is the midpoint of segment BD. Hence, line segment AC bisects line segment BD. □

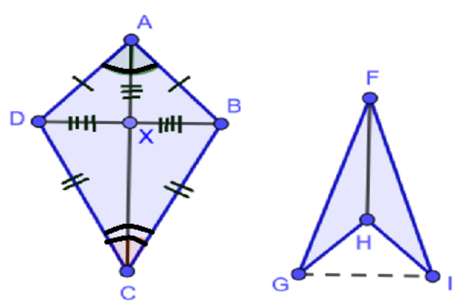


Figure 1: Since X is the midpoint of BD, AC bisects BD. The diagonals of kite FGHI do not cross. Thus they do not bisect each other.

Note: The diagonals of kite FGHI do not intersect. Since the diagonals of a kite must intersect for this proof, only kites with the form of ABCD will work.

The Diagonals of a Kite must Cross

Emily Carstens

November 15, 2017

Communicated by: Mr. Hertzler.

Theorem 2.2. If ABCD is a kite with AB congruent to BC and AD congruent to DC, then the diagonals, AC and BD, of the kite ABCD must cross.

Proof. Segment AB is congruent to segment BC by the definition of a kite. Construct segment AC using Euclid's Postulate I.1. Triangle ABC is isosceles because it has a pair of congruent sides AB and BC. By Euclid's Proposition 5, angle BAC is congruent to angle BCA.

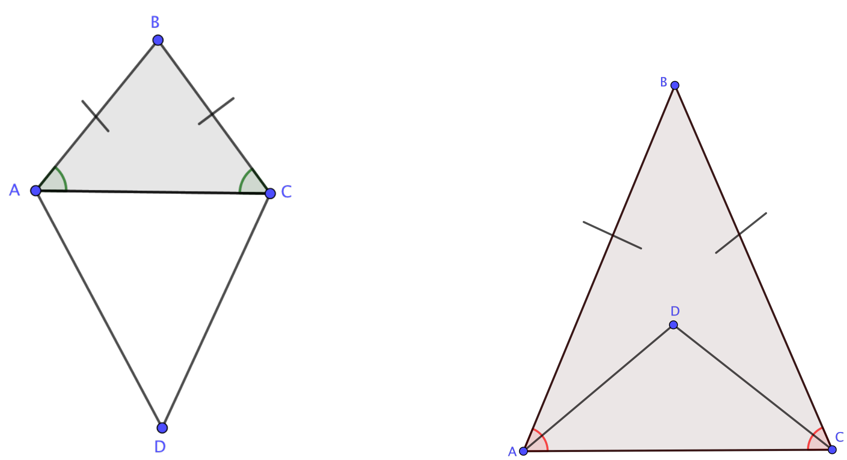
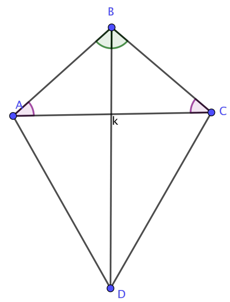


Figure 1: These figures show the isosceles triangles ABC in both a convex and nonconvex kites. They are the shaded portions of the kites.

By Euclid's Proposition 32, the angles of triangle ABC, which are angle BAC, angle BCA and angle ABC, add up to two right angles. Because angle BAC is congruent to angle BCA, then those angles must be less than a right angle for Euclid's Proposition 32 to hold.

Construct segment BD using Euclid's Postulate 1. By Theorem I, segment BD bisects angle ABC, thus angle ABD is congruent to angle DBC. Also, angle ABD plus angle DBC equals angle ABC. Using Euclid's Common Notion 1, we can say angle BAC plus angle BCA plus angle ABD plus angle DBC = 2 right angles. Since angle ABD is congruent to angle DBC, then those angles must be less than a right angle for Euclid's Proposition 32 to hold.

Since angle BAC and angle ABD are less than a right angle, then by Euclid's Postulate 5 segments AC and BD must cross.



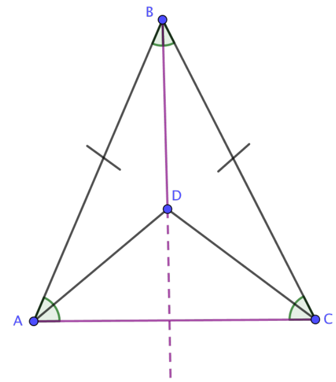


Figure 2: These figures have the angles that are less than right angles marked and show the diagonals that cross.

□

The Rectangle

Cameron Amos

October 23, 2017

Communicated by: Micah Otterbein.

Theorem 3.5. Let $ABCD$ be a quadrilateral such that angles ABC and ADC are right angles. If segments AB and CD are parallel, then $ABCD$ is a rectangle.

Proof. Let $ABCD$ be a quadrilateral such that ABC and ADC are right angles. Let segments AB and CD be parallel. By Postulate 2, extend BA to a point E, extend AB to a point F, extend DC to a point G, and extend CD to a point H. By Euclid.I13, angle CBF is a right angle. By Euclid.I29, angle BCD is a right angle. By Euclid.I13, angle ADH is a right angle. By Euclid.I29, angle BAD is a right angle. Therefore, angles ABC, BCD, ADC, and BAD are all right angles. Thus, $ABCD$ is a rectangle. □

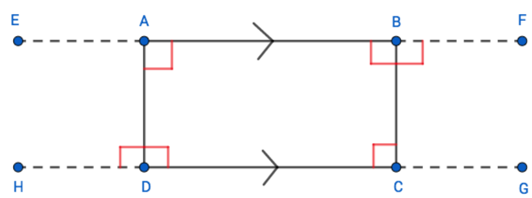


Figure 1: This is a picture representing rectangle ABCD.

Parallelism of the Midline

Grant Kilburg

November 15, 2017

Communicated by: Ashlyn Thompson

From our previous work with parallelograms, we noticed that the diagonals of a parallelogram always seemed to bisect one another. We therefore made the following conjecture which we named Conjecture S. Conjecture S says that *"If the diagonals of a quadrilateral bisect one another, then the quadrilateral is a parallelogram."* This theorem is proven based on the assumption that Conjecture S is true.

Theorem 3.6. Suppose conjecture S is true. Let ABC be a triangle, D the midpoint of AB and E the midpoint of AC. Then the line through E and D, called the *midline*, is parallel to the line through B and C.

Proof. Since D is the midpoint of AB, AD is congruent to BD. Since E is the midpoint of AC, AE is congruent to CE. It remains to show that DE is parallel to BC. By Postulate 1.3, construct circle ED. By Postulate 1.2, extend line segment DE. Let F be the point at which DE intersects circle ED. Since E is the center of circle ED, and both D and F fall on the boundary, segment DE is congruent to segment EF by Postulate 1.3. By Postulate 1.1, construct line segment FC. Since DF and AC are straight lines that cut each other, angle DEA is congruent to angle FEC by Euclid I.15. Since angle DEA is congruent to angle FEC, AE is congruent to CE, and DE is congruent to FE, triangle DEA is congruent to angle FEC by Euclid I.4 (SAS).

By Postulate 1.1, construct line segments DC and AF. Since DE is congruent to EF and AE is congruent to CE, and AC and DF are the diagonals of AFCD, then AFCD is a parallelogram by Conjecture S. Since AFCD is a parallelogram, AF is parallel to DC and AD is parallel to FC, by definition of a parallelogram. Since AD is parallel to FC, and DB and AD fall on the same line, BD is also parallel to CF. Furthermore, since D is the midpoint of AB, DA is congruent to BD. Since DA is congruent to BD, and DA is congruent to CF, BD is congruent to CF by Common Notion 1. Hence, BD is parallel to CF and BD is congruent to CF. Since BD is parallel to CF and BD is congruent to CF, and DF and BC join BD and CF, DF is parallel to BC by Euclid I.33. Since DE is the midpoint of DB, it follows that DE is parallel to BC. Hence, assuming that Conjecture S is true, the midline through DE is parallel to the line through BC. □

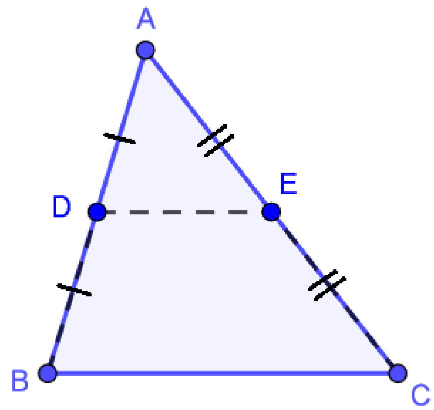


Figure 1: Triangle ABC with D the midpoint of AB and E the midpoint of AC.

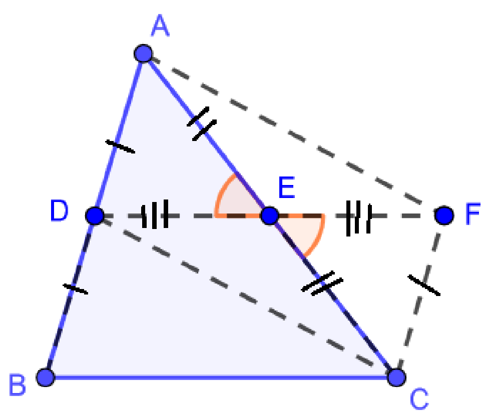


Figure 2: By Euclid I.33, DE is parallel to BC.

A Characterization of Parallelograms

Theron J Hitchman

October 11, 2017

Communicated by: The Editor

Theorem S. Let ABCD be a quadrilateral. Suppose that the diagonals AC and BD bisect one another. Then ABCD is a parallelogram.

Proof. Let the point of intersection of the diagonals be X. By hypothesis, X is the midpoint of both AC and BD. So we conclude that AX is congruent to XC and BX is congruent to XD. Since the straight lines AC and BD meet at X, by Euclid Proposition I.15, angle AXC is congruent to angle BXD.
Since AX is congruent to CX, XB is congruent to XD, and angle AXB is congruent to angle CXD, by Euclid Proposition I.4 triangle AXB is congruent to angle CXD. Since these triangles are congruent, we deduce that angle ABX is congruent to angle CDX.

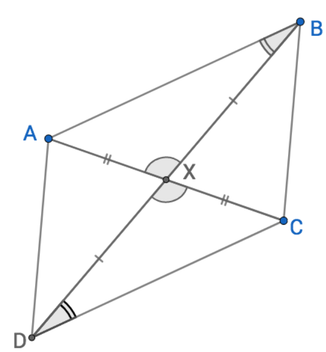


Figure 1: This is a picture of math.

Since AC meets BD at X, the points A and C lie on opposite sides of line BD. Therefore, the line BD falls upon the lines AB and DC, making the angles ABD and angles CDB a pair of alternate interior angles. Note that angle ABD coincides with angle ABX, and angle CDB coincides with CDX. We deduce that angles ABD and CDB are congruent. So by Euclid Proposition I.27, AB is parallel to CD.
In a similar way, the triangles BXC and DXA are congruent, and following the same line of argument we deduce that AD is parallel to BC.
Since both pairs of opposite sides are parallel, ABCD is a parallelogram.

$\square$

Parallelograms within Quadrilaterals

Lakota Avery

November 15, 2017

Communicated by: Mr. Micah Otterbein

Theorem 3.7. Let $ABCD$ be a quadrilateral. The midpoints of the four sides are the vertices of a parallelogram.

Proof. Let $ABCD$ be a quadrilateral. Let the midpoints of the four sides be labeled E, F, G, and H, and let the quadrilateral $EFGH$ be formed from those points. Draw AC and BD by Euclid's Postulate I.1. With triangles ABC and ADC, we can say that lines EF and GH are parallel to line BD by Kilburg's Theorem 3.6. Similarly, lines FG and EH are parallel to line AC. Since EF and GH are parallel to the same line AC, they are parallel to each other by Euclid's Proposition I.30. Similarly, FG and EH are parallel to each other, thus creating the parallelogram $EFGH$. □

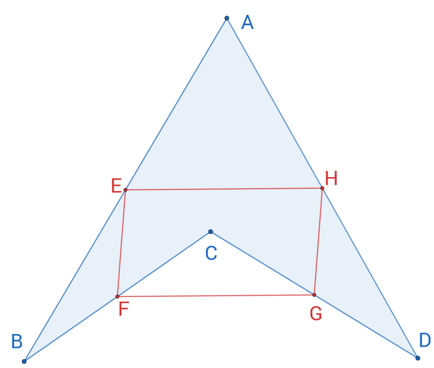


Figure 1: Quadrilateral $ABCD$ with midpoints E, F, G, H.

An Equilateral Triangle is a Regular Triangle

Lakota Avery

November 15, 2017

Communicated by: Micah Otterbein

Theorem 6.1. An equilateral triangle is equiangular, hence regular.

Proof. Let ABC be an equilateral triangle. We know that AB and AC are congruent by the definition of an equilateral triangle. By Euclid's Proposition I.5, we can say that angles BCA and ABC are congruent. Similary, we know that AC and BC are congruent by the definition of an equilateral triangle. Again, by Euclid's Proposition I.5, we can say that angles ABC and BAC are congruent. By Euclid's Common Notions I.1, since angle BAC is congruent to angle ABC, and angle ABC is congruent to angle BCA, then angle BAC is congruent to angle BCA. Therefore, the equilateral triangle ABC is equiangular. Thus, triangle ABC is a regular triangle.

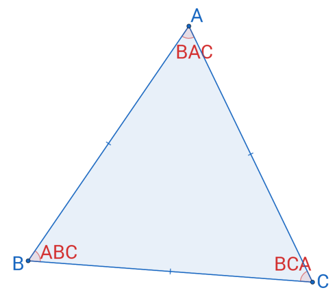


Figure 1: An equilateral triangle ABC.

□

Angles in an Equilateral Triangle

Ashlyn Thompson

November 15, 2017

Comunicated by: Grant Kilburg

Theorem 6.1. An equilateral triangle is equiangular.

Proof. By way of contradiction, let ABC be an equilateral triangle that is not equiangular. Since triangle ABC is not equiangular then two angles are congruent with the remaining angle being obtuse or two angles are congruent with the remaining angle being acute or one angle is a right angle or all angles have different measures.

Case 1. Suppose two angles in triangle ABC are congruent, with the remaining angle being obtuse. By Euclid I.32, the sum of the interior angles of a triangle are equal to two right angles. Since one angle is obtuse, the remaining angles taken together must be acute. Therefore, the remaining two angles must be acute. Hence, the obtuse angle is the greatest angle in triangle ABC. By Euclid I.19, the side subtending the obtuse angle is the greater side. Therefore, triangle ABC is an equilateral triangle, and is not an equilateral triangle.

Case 2. Suppose two angles in triangle ABC are congruent, with the remaining angle being acute. By Euclid I.32, the three interior angles of a triangle taken together are equal to two right angles. Since the remaining angle is acute, the congruent angles taken together must be obtuse. Additionally, since the three interior angles of a triangle taken together are equal to two right angles, and one angle is acute, the two congruent angles cannot both be obtuse. Hence, the two congruent angles must both be acute. Since triangle ABC is not equiangular, one of the angles must have a greater measure. Therefore, by Euclid I.19, the side subtending that greater angle is the greater side. Thus, triangle ABC is an equilateral triangle, and is not an equilateral triangle.

Case 3. Suppose that triangle ABC is a right triangle. By Euclid I.32, the sum the three interior angles of triangle ABC taken together are equal to two right angles. Hence, the remaining two angles taken together must be acute. Therefore, the remaining angles are each acute, and the right angle is the greatest angle in triangle ABC. By Euclid I.19, the side subtending the greater angle is the greater side. Hence, triangle ABC is equilateral and not equilateral.

Case 4. Suppose that all angles are not congruent in triangle ABC. Since all angles are not congruent, there must be a greater angle. By Euclid I.19, the side subtending the greater angle is the greater side. Hence, triangle ABC is equilateral and not equilateral.

Thus, the theorem is true.

$\square$

A Regular Rhombus

Emily Carstens

November 15, 2017

Communicated by: Ms. Thompson.

Theorem 6.2. Let ABCD be a rhombus. If angle A is congruent to angle B, then ABCD is regular.

Proof. Let ABCD be a rhombus. By definition of a rhombus, all sides of the rhombus are congruent.

Draw segment AC using Euclid's Postulate I.1. Segment AC is shared side of triangle ABC and triangle ADC. Triangle ABC is congruent to triangle ADC by Euclid's Proposition I.8. Thus angle B is congruent to angle D.

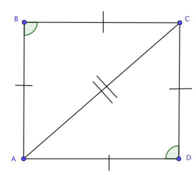


Figure 1: This figure shows the congruent triangles ABC and ADC that make angle B and angle D congruent.

Draw segment BD using Euclid's Postulate I.1. Segment BD is a shared side of triangle BAD and triangle BCD. Triangle BAD is congruent to triangle BCD by Euclid's Proposition I.8. Thus angle A is congruent to angle C.

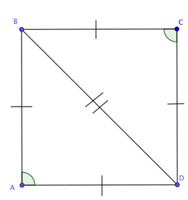


Figure 2: This figure shows the congruent triangles BAD and BCD that make angle A and angle C congruent.

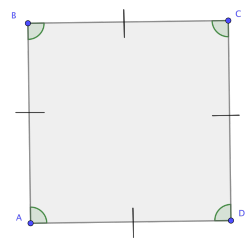


Figure 3: This figure shows the regular rhombus ABCD.

Since angle A is congruent to angle B, angle B is congruent to angle D and angle A is congruent to angle C, then all four angles are congruent. Thus ABCD is regular.

□

Isosceles Triangle Within Regular Pentagon

Cameron Amos

November 15, 2017

Communicated by: Connor Coyle.

I will prove that triangle ACD is an isosceles triangle in regular pentagon $ABCDE$.

Theorem 6.5. Let ABCDE be a regular pentagon. The triangle ACD is isosceles.

Proof. Since $ABCDE$ is a regular pentagon, then AB, BC, CD, DE, and EA are all congruent and angle A, angle B, angle C, angle D, and angle E are all congruent. By Postulate 1, draw AC and AD. By EuclidI.4, triangle ABC is congruent to triangle AED. Since triangle ABC is congruent to triangle AED, then AC is congruent to AD. Therefore, triangle ACD is an isosceles triangle. □

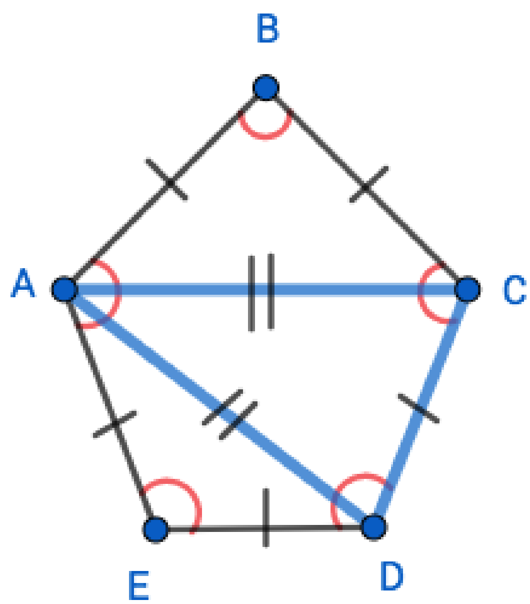


Figure 1: This is a picture representing isosceles triangle ACD in regular pentagon $ABCDE$.