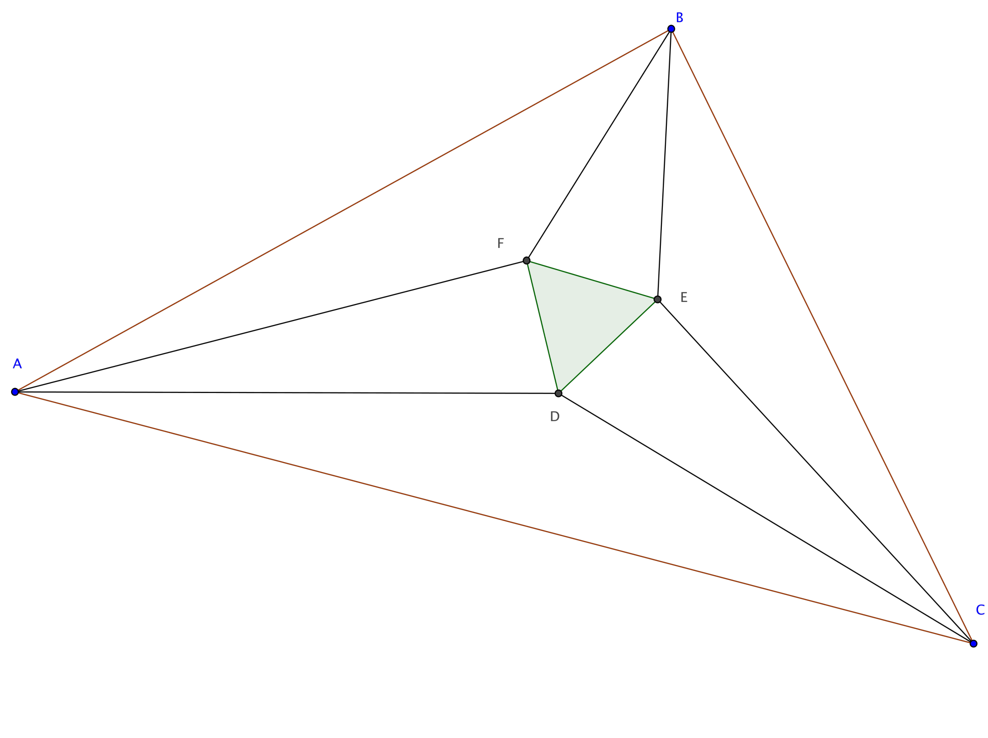
Transactions

in

Euclidean Geometry



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Regular rhombus

Mr. Hertzler

December 6, 2017

Communicated by: Ms. Koontz.

Theorem 6.2. Let ABCD be a rhombus. If angle A is congruent to B, then ABCD is regular

Proof. Let angles DAB, ABC, BCD, and CDA be called a, b, c, and d respectively. By Amos and Flesch I.1, the opposite angles in a rhombus are congruent, so a is congruent to c, and b is congruent to d. Since c is congruent to a, a is congruent to b, and b is congruent to d, then by Euclid common notion I.1, angles a, b, c, and d are mutually congruent. Since rhombus ABCD is equiangular and equilateral, by Euclid definition I.22, then rhombus ABCD is regular. □

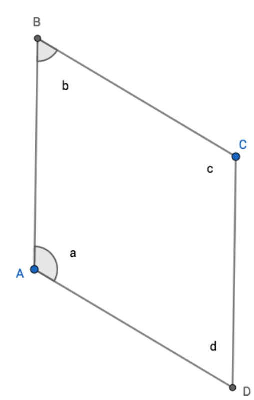

Figure 1: Above is rhombus ABCD where angle a is congruent to angle b.

Rhombus Regularity

Mr. Hertzler

December 6, 2017

Communicated by: Ms. Schafbuch.

The overall idea for this proof is using a rhombus that makes the theorem true, construct a new rhombus that proves to be a counterexample.

Theorem 6.3. Does theorem 6.2 hold if we replace "angle B" by "angle C"? State a result and prove it. Result: theorem 6.2 does not hold when "angle B" is replaced by "angle C".

Proof. Let ABCD be a rhombus. Let angle BAD be congruent to angle BCD. Join AC and BD by Euclid Postulate I.1. Name point E where AC and BD intersect. Pick a random point on line segment EC and name it F. Create circle EF. Name the intersection of circle EF and line segment AE, point G. Join BF, DF, BG, and DG by Euclid Postulate I.1. By Euclid Definition I.15, GE is congruent to EF. By Killburg Theorem 1.7, BE is congruent to DE and angles AEB, BEC, CED, and DEA are right angles. Triangle GEB is congruent to triangle FEB, triangle FEB is congruent to triangle FED, and triangle FED is congruent to triangle GED by Euclid I.4. By Euclid Common Notion I.1, triangles GEB, FEB, FED, and GED are mutually congruent. Since triangles GEB, FEB, FED, and GED are all congruent, then their corresponding components are congruent, thus BF, DF, DG, and BG are mutually congruent. Since vertices B, F, D, and G make up quadrilateral BFDG and all the sides are mutually congruent, then quadrilateral GBFD is a rhombus by Euclid Definition I.22. Since triangle BDF shares a base with triangle BDC, and the two sides of triangle BDF meet within triangle BDC, then by Euclid I.21, angle BFD is greater than BCD. Similarly, angle BGD is greater than angle BAD. Angle BGD is congruent to angle BGD by Amos Flesch Theorem I.1. Comparing the two rhombi, BGDF and ABCD, opposite angles BGD and BFD are congruent but greater than angles BAD and BCD. Angle GBF is congruent to angle GDF by Amos Flesch Theorem I.1. Angles GBF and GDF make up a portion of angles ABC and ADC respectively. By Euclid Common Notion I.5, Angles GBF and GDF are less than angles ABC and ADC respectively. Assuming that ABCD is a rhombus that makes theorem 6.3 true, then angles ABC, BCD, CDA, and DAB are mutually congruent. Rhombus BFDG, however, has angles DGB and DFB that are greater than angles DAB and DCB respectively. Angles GBF and GDF are less than angles ABC ADC respectively. Thus, rhombus GBFD is not regular, due to the nature of it's construction. Thus there exists a rhombus such that when one pair of opposite angles are congruent, the rhombus is not regular. □

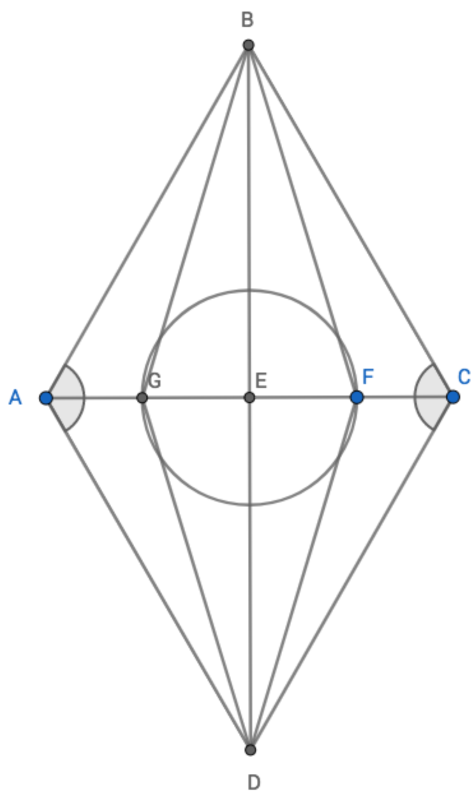

Figure 1: Above is rhombus ABCD with angles DAB congruent to DCB. Constructed within rhombus ABCD is rhombus GBFD.

Non-Congruent Triangles With Congruent Altitudes

Steven Flesch

December 6, 2017

Communicated by: Cameron Amos

I am going to make a construction of triangle ABC and another construction of triangle DEF. I am going to prove that the altitude of triangle ABC is congruent to the altitude of triangle DEF where triangle ABC is not congruent to triangle DEF and AC is congruent to DF.

Question R. Let ABC and DEF be two triangles. Suppose ABC is not congruent to DEF and AC is congruent to DF. Then the altitude of triangle ABC from point B is not congruent to the altitude of triangle DEF from point E.

Proof. I am going to show my construction of triangle of ABC first. Then, I am going to do the same thing for triangle DEF. Then, I will incorporate the two constructions to reach to my conclusion.

Construction of Triangle ABC First, draw circle AC. Then, draw circle CA, creating point B at the intersection of the two circles. Then, create AB, AC, and BC by Postulate 1.1. Triangle ABC is an equilateral triangle by Euclid I.1. By Mr. Avery's Theorem 6.1, triangle ABC is equiangular. Then, bisect angle BAC by Euclid I.9, creating point H at the intersection of the bisecting line of angle BAC and segment AC. Since triangle ABC is equiangular, angles ABC, BCA, and CAB are 2/3 of a right angle. Thus, angles ABH and CBH are 1/3 of a right angle. Therefore, angles AHB and CHB are a right angle since the interior angles of a triangle consist of two right angles.

Construction of Triangle DEF First, draw circle DF. Then, extend line DF, creating point G at the intersection of the line and the circle on the other end. Then, draw circle GD, creating point E at the intersection of the two circles. Also, create EG, ED, and EF by Postulate 1.1. Triangle EGD is equilateral by Euclid I.1. By Mr. Avery's Thm 6.1, triangle EGD is equiangular. Then, bisect angle GED by Euclid I.9, creating point I at the intersection of the bisecting line of angle GED and segment GD. Since triangle GED is equiangular, angles GED, EDG, and DGE are 2/3 of a right angle. Thus, angles GEI and DEI are 1/3 of a right angle. Therefore, angles GIE and DIE are a right angle since the interior angles of a triangle consist of two right angles.

Proof Using Both Constructions We are given AC is congruent to DF by Question R. Also, AC is congruent to DG because DG is the radius of circle DF along with segment DF. Also, AB is congruent to GE because AB is congruent to AC (ABC is equilateral triangle) is congruent to DF (given hypothesis) is congruent to GD (DF and DG are both radius of circle DF) is congruent to GE (DG and GE are both radius of circle GD). Also, angle ABH is congruent to angle GEI (both 1/3 of a right angle) and BAH is congruent to EGI (both 2/3 of a right angle). Triangle ABH is congruent to triangle GEI by Euclid I.26. Therefore, BH is congruent to GI. $\square$

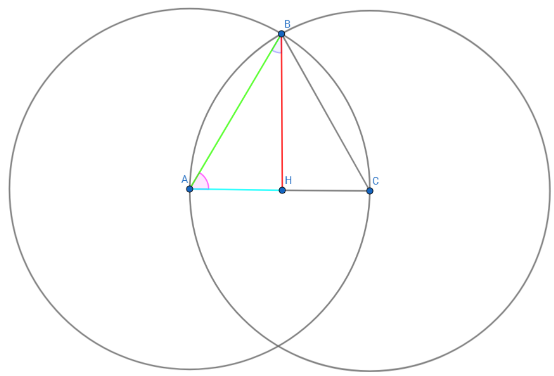

Figure 1: This is the construction of triangle **ABC** using Euclid I.1.

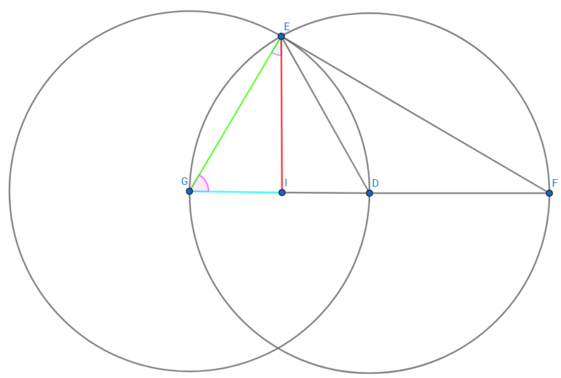

Figure 2: This is the construction of triangle **GED** using Euclid I.1. Triangle ABH is congruent to triangle GEI by Euclid I.26. Therefore, BG is congruent to EI.

Angles Formed by Intersecting Lines Outside a Circle

Grant Kilburg

December 6, 2017

Communicated by: Kayla Schafbuch

Theorem 10.2. Let Γ be a circle with center O. Let X be a point outside of the circle, and suppose that two lines l and m intersect at X so that l meets Γ at points A and A' and m meets Γ at B and B'. Then twice the angle AXB is congruent to the greater angle, AOB or $A'OB'$, less the lesser angle.

Proof. Let Γ be a circle with center O. Let X be a point outside of the circle, and suppose that two lines l and m intersect at X so that l meets Γ at points A and A' and m meets Γ at B and B'. By Postulate 1.1, construct line segments AO, BO, $A'O$, $B'O$, and AB'. By Euclid III.20, angle $AB'B$ equals one half angle AOB and angle $A'AB'$ equals one half angle $A'OB'$. By Euclid I.32, angle $AB'B$ is congruent to angle AXB taken together with angle $A'AB'$.

Since angle $AB'B$ is congruent to angle AXB and angle $A'AB'$ taken together, angle $AB'B$ equals one half angle AOB, and angle $A'AB'$ equals one half angle $A'OB'$, one half angle AOB is congruent to angle AXB taken together with one half angle $A'OB'$. Since one half angle AOB is congruent to angle AXB taken together with one half angle $A'OB'$, angle AXB is congruent to one half angle AOB less one half angle $A'OB'$. Thus, angle AXB is congruent to one half (angle AOB less angle $A'OB'$). Thus, twice angle AXB is congruent to angle AOB less angle $A'OB'$. Hence, twice the angle AXB is congruent to angle AOB, which is our greater angles, less angle $A'OB'$. $\square$

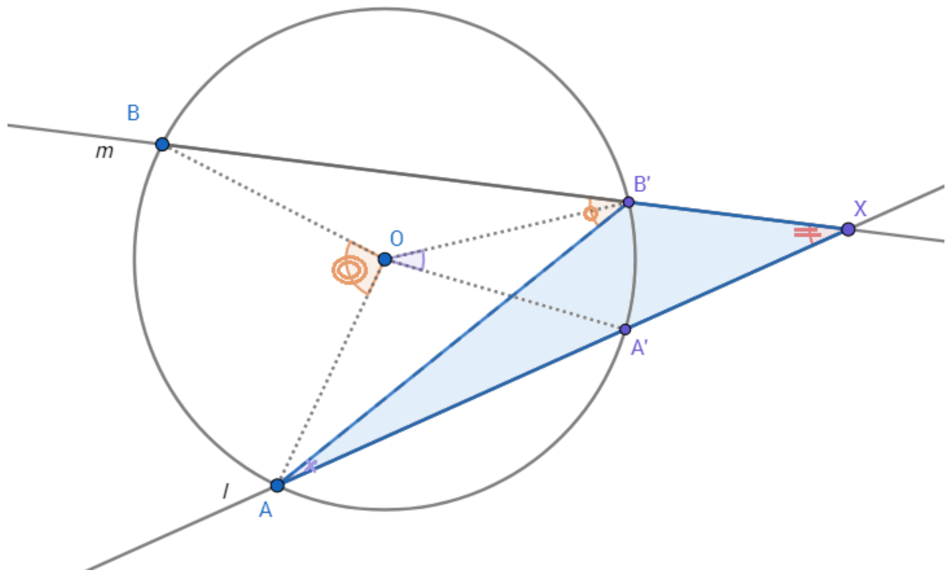

Figure 1: By Euclid I.32, angle AXB is congruent to one half angle AOB less one half angle $A'OB'$.

Squares on Diagonals and Sides of Parallelograms

Cameron Amos

December 6, 2017

Communicated by: Grant Kilburg.

Theorem 13.6. Let $ABCD$ be a parallelogram. Then the squares on the diagonals taken together have equal content with the squares on the four sides taken together.

Proof. By Postulate 1, draw BD and AC. AB is congruent to CD, BC is congruent to AD, angle BAD is congruent to angle BCD, angle ABC is congruent to angle CDA by Euclid I.34. Triangle ABD is congruent to triangle CDB and triangle ABC is congruent to triangle CDA by Euclid I.4. Since triangle ABD is congruent to triangle CDB, then the square of AD is congruent to the square of BC because AD is congruent to BC, the square of AB is congruent to the square of CD because AB is congruent to CD, and the square of BD is congruent to the square of BD because BD is congruent to BD. Since triangle ABC is congruent to triangle ADC, then the square of AB is congruent to the square of CD because AB is congruent to CD, the square of AD is congruent to the square of BC because AD is congruent to BC, and the square of AC is congruent to the square of AC because AC is congruent to AC.

Case 1. Suppose angles ABC, BCD, ADC, and BAD are right.

By Euclid I.47, the square of BD equals the square of AD plus the square of AB. Also, the square of BD equals the square of BC plus the square of CD by Euclid I.47. Therefore, the square of BD plus the square of BD equals the square of AD plus the square of AB plus the square of BC plus the square of CD. In other words, twice the square of BD equals the square of AD plus the square of AB plus the square of BC plus the square of CD.

By Euclid I.47, the square of AC equals the square of AB plus the square of BC. Also, the square of AC equals the square of CD plus the square of AD by Euclid I.47. Therefore, the square of AC plus the square of AC equals the square of AB plus the square of BC plus the square of CD plus the square of AD. In other words, twice the square of AC equals the square of AB plus the square of BC plus the square of CD plus the square of AD.

Twice the square of AC plus twice the square of BD equals twice (the square of AB plus the square of BC plus the square of CD plus the square of AD). Therefore, the square of BD plus the square of AC equals the square of AB plus the square of BC plus the square of CD plus the square of AD.

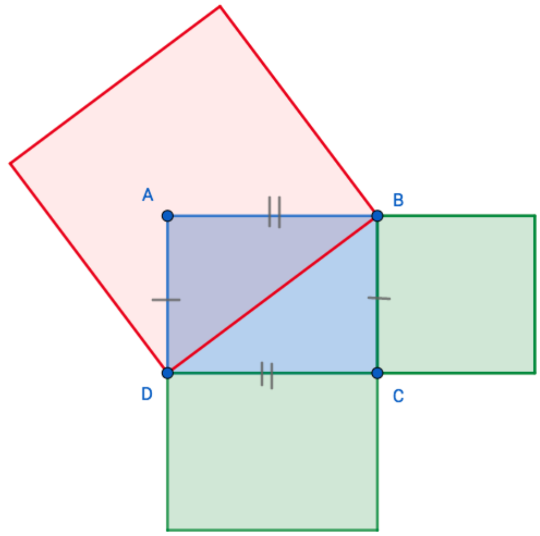

Figure 1: This figure represents Euclid I.47.

Case 2. Suppose a pair of opposite congruent angles are obtuse and the other pair of opposite congruent angles are acute in parallelogram $ABCD$. For this proof, we will say angles BAD and BCD are obtuse and angles ABC and ADC are acute.

By Euclid II.12, the square of BD equals the square of AB plus the square of AD plus twice rectangle (AD,AE). Also, the square of BD equals the square of BC plus the square of CD plus twice rectangle (CD,CF) by Euclid II.12. Therefore, twice the square of BD equals the square of AB plus the square of AD plus twice rectangle (AD,AE) plus the square of BC plus the square of CD plus twice rectangle (CD,CF).

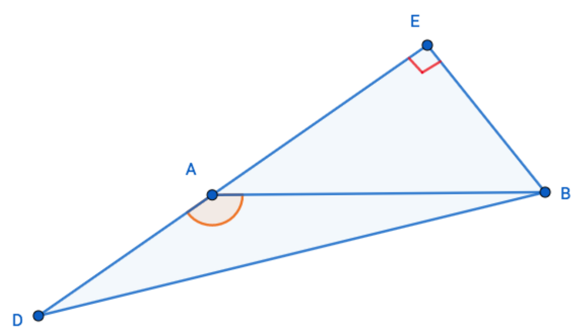

Figure 2: This figure represents the set up for Euclid II.12.

By Euclid II.13, the square of AC equals the square of CD plus the square of AD minus twice rectangle (CD,DG). Also, the square of AC equals the square of AB plus the square of BC minus twice rectangle (AB,BH) by Euclid II.13. Therefore, twice the square of AC equals the square of CD plus the square of AD minus twice rectangle (CD,DG) plus the square of AB plus the square of BC minus twice rectangle (AB,BH).

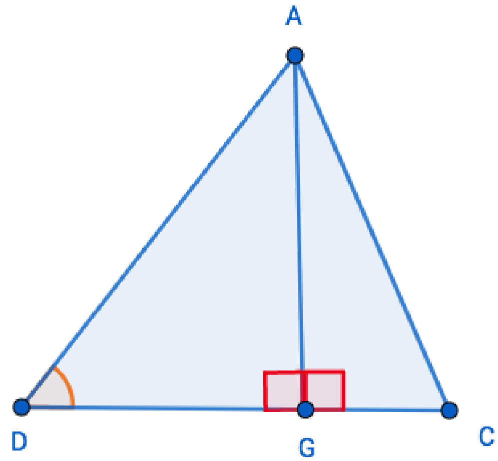

Figure 3: This figure represents the set up for Euclid II.13.

Taken together twice the square of BD and twice the square of AC gives twice (the square of AB plus the square of AD plus the square of BC plus the square of CD) plus twice rectangle (AD,AE) plus twice rectangle (CD,CF) minus twice rectangle (CD,DG) minus twice rectangle (AB,BH). After halving both sides of the previous statement, we are left with the square of BD plus the square of AC equals the square of AB plus the square of AD plus the square of BC plus the square of CD plus the rectangle (AD,AE) plus the rectangle (CD,CF) minus the rectangle (CD,DG) minus the rectangle (AB,BH)

Since the square of BD equals the square of AB plus the square of AD plus twice rectangle (AD,AE) and the square of BD equals the square of BC plus the square of CD plus twice rectangle (CD,CF), then we can set the statements equal to one another. Thus, the square of AB plus the square of AD plus twice rectangle (AD,AE) equals the square of BC plus the square of CD plus twice rectangle (CD,CF). Since the square of AD is congruent to the square of BC and the square of AB is congruent to the square of CD, then in the previous equation we can conclude twice rectangle (AD,AE) equals twice rectangle (CD,CF). After halving both sides of the statement, we have rectangle (AD,AE) equals rectangle (CD,CF). Thus, rectangle (AD,AE) and rectangle (CD,CF) have equal content.

Since the square of AC equals the square of CD plus the square of AD minus twice rectangle (CD,DG) and the square of AC equals the square of AB plus the square of BC minus twice rectangle (AB,BH), then we can set the statements equal to one another. Thus, the square of CD plus the square of AD minus twice rectangle (CD,DG) equals the square of AB plus the square of BC minus twice rectangle (AB,BH). Since the square of AD is congruent to the square of BC and the square of AB is congruent to the square of CD, then in the previous statement we can conclude negative twice rectangle (CD,DG) is equal to negative twice rectangle (AB,BH). After negative halving both sides of the statement, we have rectangle (CD,DG) equals rectangle (AB,BH). Thus, rectangle (CD,DG) and rectangle (AB,BH) have equal content.

Once again looking at parallelogram $ABCD$ with the additions from Euclid II.12 and Euclid II.13, similar to Figure 4, extend AB to point J and CB to point K by Postulate 2. Angle ABC is congruent to angle BCF by Euclid I.29. Since angle CHB is congruent to angle BFC, angle HBC is congruent to angle FCB, and BC is shared, then triangle HBC is congruent to triangle FCB by Euclid I.26. Since triangle HBC is congruent to triangle FCB, then BH is congruent to CF.

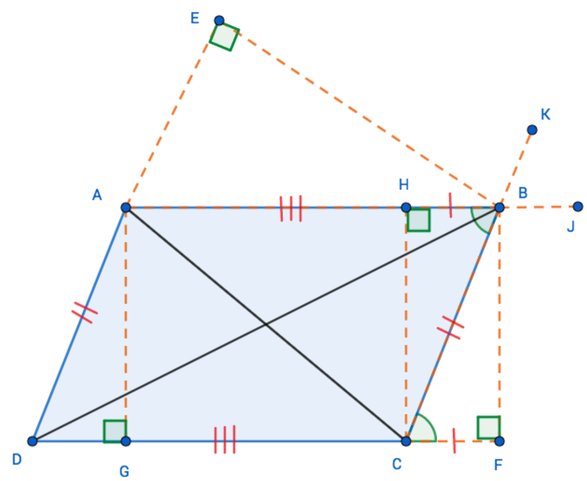

Figure 4: This figure represents the parallelogram $ABCD$ with the additions made throughout the proof from Euclid II.12, Euclid II.13, and Postulate 2. As well as representing BH is congruent to CF.

Since BH is congruent to CF and AB is congruent to CD, then rectangle (CD,CF) is congruent to rectangle (AB,BH). Since rectangle (CD,CF) is congruent to rectangle (AB,BH), then rectangle (CD,CF) and rectangle (AB,BH) have equal content. By Common Notion 1, rectangle (AD,AE), rectangle (CD,CF), rectangle (CD,DG), and rectangle (AB,BH) all have equal content.

Since the square of BD plus the square of AC equals the square of AB plus the square of AD plus the square of BC plus the square of CD plus the rectangle (AD,AE) plus the rectangle (CD,CF) minus the rectangle (CD,DG) minus the rectangle (AB,BH) and rectangle (AD,AE), rectangle (CD,CF), rectangle (CD,DG), and rectangle (AB,BH) all have equal content, we can conclude, the square of BD plus the square of AC equals the square of AB plus the square of AD plus the square of BC plus the square of CD.

Hence, the square of the diagonals, BD and AC, taken together have equal content with the squares of the four sides, AB, BC, CD, and AD, taken together. $\square$

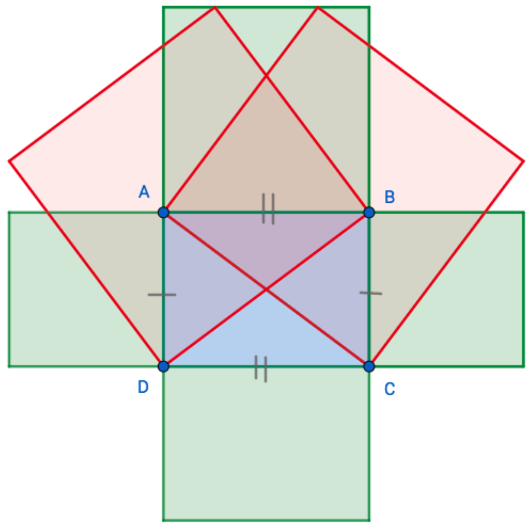

Figure 5: This figure represents the squares on the diagonal taken together have equal content with the squares on the four sides taken together in parallelogram $ABCD$.