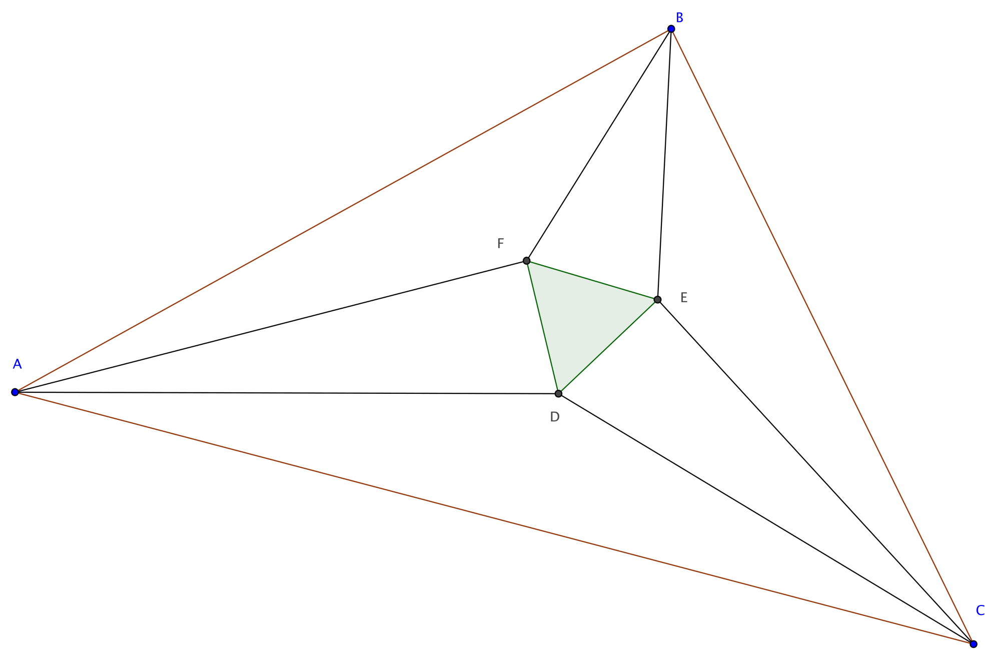
Transactions

in

# Euclidean Geometry




Volume 2017F Issue # 6

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# Not All Kites are Parallelograms

Steven Flesch

December 5, 2017

*Communicated by: Cameron Hertzler*

I am proving that the kite in my proof is not a parallelogram. I am using Ms. Carsten's construction of a kite from Thm 2.3.

**Counter example 2.4.** If $ABCD$ is a kite, then it is a parallelogram.

*Proof.* Let $ABCD$ be a kite. Since $ABCD$ is a parallelogram, then $AB$ is congruent to $CD$, and $AD$ is congruent to $BC$ by Euclid I.34. By Ms. Carsten's construction of a kite, $AB$ is congruent to $AD$, and $BC$ is congruent to $CD$. Then, draw line $BD$ by Postulate 1.1. Triangle $BAD$ is not congruent to triangle $BCD$ by Euclid I.4. Therefore, $AB$ is not congruent to $CD$, and $AD$ is not congruent to $BC$. Thus, we have a contradiction. Therefore, $ABCD$ is not a parallelogram. □

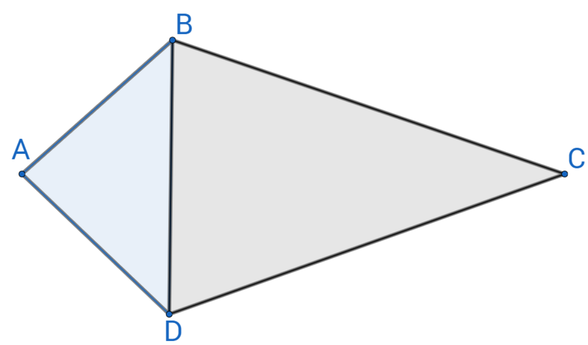


Figure 1: This is a representation of Euclid I.4 showing that triangle $BAD$ is not congruent to triangle $BCD$.

# Constructing a Kite.

Micah Otterbein

December 5, 2017

*Communicated by: Lakota Avery.*

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This is the proof of a construction of a kite given two line segments and an angle.

**Theorem I.** Given two line segments, XY and UV, and an angle, PQR. Construct a kite ABCD, using Ms. Carstens' 2.3 construction, such that AB is congruent to XY, BC is congruent to UV, and angle ABC is congruent to angle PQR.

*Proof.* Let A and E be points. Using Euclid's Postulate 1, construct the line AE. By Euclid I.23, construct the rectilinear angle FAE on the line AE, at point A, and congruent to angle PQR. By Euclid I.2, construct lines AG and AH at point A, which are congruent to XY and UV respectively. By Euclid's Postulate 3, construct circles AG and AH. Label point B at the intersection of circle AH and line segment AE. Label point D at the intersection of circle AG and line segment AF. By Euclid's Definition 15, all radii of a circle are congruent to one another. Thus, AD is congruent to AG and XY, and AB is congruent to AH and UV. By Euclid's Postulate 3, construct circles DA and BA. Label point C at the second intersection of circles DA and BA, not point A. By Euclid's Postulate 1, construct lines BC and DC. By Euclid's Definition 15, DC is congruent to AD, and BC is congruent to AB. Therefore, the quadrilateral ABCD is a kite constructed such that AB is congruent to XY, BC is congruent to UV, and angle ABC is congruent to angle PQR.

□

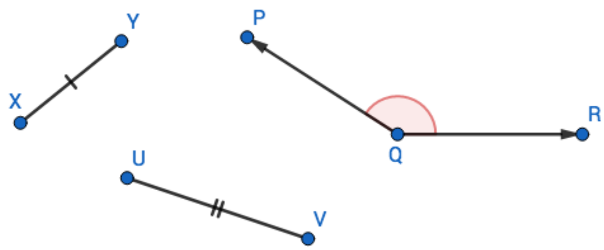


Figure 1: Given line segments XY and UV and given angle PQR.

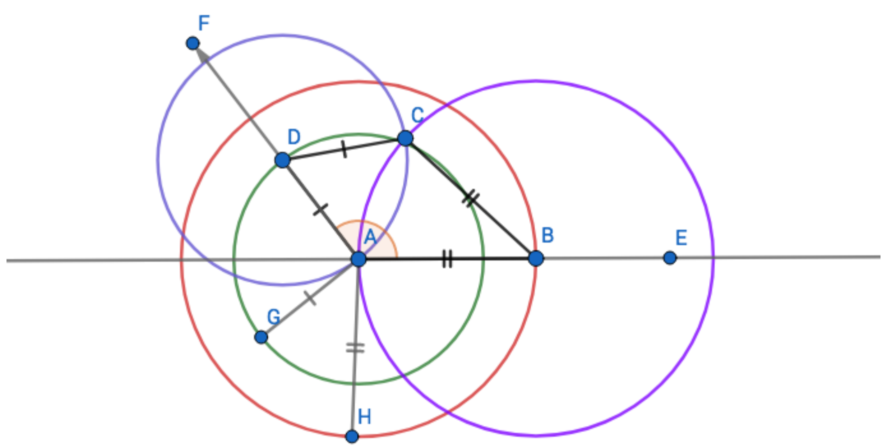


Figure 2: Constructed kite ABCD.

# Exterior Angles of Pentagons

Kaelyn Koontz

December 5, 2017

*Communicated by: Rachelle Feldmann.*

In this proof, we will discuss how the exterior angles of a pentagon add up to 4 right angles. In order for this method to work, the pentagon has to be simple. This means that we can split the figure using diagonals. As long as the figure is a simple pentagon, ABCDE can be convex or non-convex.

**Theorem 5.2.** The exterior angles of a pentagon, one choice made at each vertex, add up to four right angles.

*Proof.* Let ABCDE be a pentagon. Using Postulate 1.2, extend each line segment out from one point. Then using Euclid I.13, both the interior and the exterior angles on the line have to add up to the sum of two right angles. So, since there are 5 vertices in a pentagon, the sum of the interior and exterior angles adds up to 10 right angles. Next, using Postulate 1.1, draw AD and AC. So, the inside of ABCDE is now split into 3 triangles. Using Euclid I.32, the sum of the interior angles of a triangle equal two right angles. Since ABCDE is split into 3 triangles, the sum of the interior angles is equal to 6 right angles. Next, take the sum of the interior and exterior angles minus the sum of the interior angles, so ten minus six. This shows that the sum of the exterior angles is equal to four right angles.

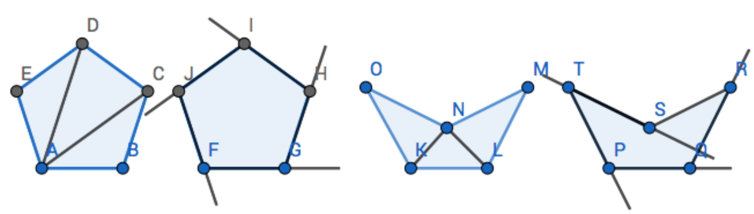


Figure 1: This demonstrates the two steps to finding the sum of the exterior angles of a convex and non-convex pentagon.

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# Tangent Lines from a Point to a Circle are Congruent

Lakota Avery

December 5, 2017

*Communicated by: Micah Otterbein*

**Theorem 9.1.** Let $AB$ and $AC$ be two tangent lines from a point $A$ outside a circle. Then $AB$ is congruent to $AC$.

*Proof.* By Euclid's Postulate I.1, draw line segments $AO, BO$, and $CO$. By Euclid's Proposition III.18, the radii $BO$ and $CO$ are perpendicular to the tangent lines $AB$ and $AC$ respectively. Then angles $ACO$ and $ABO$ are right angles. Line segments $BO$ and $CO$ are congruent to each other since they are both radii of the circle centered at $O$. Of course, line segment $AO$ is congruent to itself. Now we have the right triangles $ABO$ and $ACO$ with congruent hypotenuses $AO$ and congruent legs $BO$ and $CO$. Applying Hitchman's Theorem 7.2, triangles $ABO$ and $ACO$ are congruent to each other. Thus, line segments $AB$ and $AC$ are congruent.

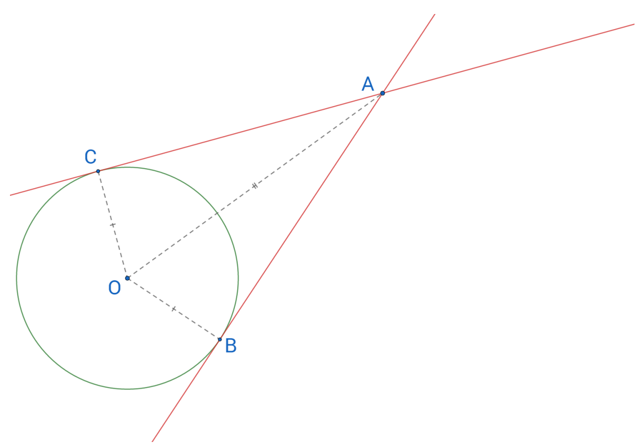


Figure 1: Two tangent lines $AB$ and $AC$ onto the circle with center $O$.

□

# Right Angles at the Intersections of Two Circles

Lakota Avery

December 5, 2017

*Communicated by: Micah Otterbein*

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**Theorem 9.2.** Let $Y$ and $Z$ be two circles with centers $G$ and $O$, respectively. Suppose that these circles meet at two points $A$ and $B$. If $GAO$ is a right angle, then $GBO$ is a right angle.

*Proof.* Let $GAO$ be a right angle. Line segments $GA, GB$ and $OA.OB$ are congruent to each other because they are radii of their respective circles. By Euclid's Postulate I.1, draw line segment $AB$. Now, there exist two triangles $GAB$ and $OAB$ that are isosceles since they both have a pair of congruent sides with a shared base $AB$. Then angles $GAB$ and $GBA$ are congruent, and angles $OAB$ and $OBA$ are congruent. We know that angles $GAB$ and $OAB$ must add up to a right angle since angle $GAO$ is a right angle. Therefore, angles $GBA$ and $OBA$ must add up to a right angle. Thus, angle $GBO$ is a right angle.

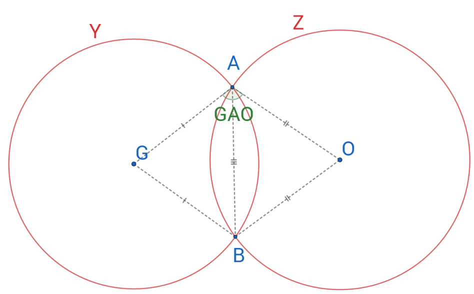


Figure 1: Quadrilateral $GAOB$ formed by centers $G$ and $O$, and intersections $A$ and $B$ of the two circles $Y$ and $Z$.

$\square$

# Rectangles: Permenantly Cyclic

Lakota Avery

December 5, 2017

*Communicated by: Micah Otterbein*

A quadrilateral $ABCD$ is said to be a *cyclic quadrilateral* if there is a circle $Z$ such that the four vertices $A, B, C$, and $D$ lie on $Z$.

**Theorem 9.3.** A rectangle is always a cyclic quadrilateral.

*Proof.* Let $ABCD$ be a rectangle. BY Euclid's Postulate I.1, draw the diagonals $AC$ and $BD$. By Kilburg's Theorem 3.3, the diagonals $AC$ and $BD$ are always congruent and bisect each other at a point $X$ in rectangles. Therefore, a circle can be drawn centered at $X$ and with radius $XA$. Since the circle will have radius $XA$ and $XA, XB, XC$, and $XD$ are all congruent with each other, then vertices $A, B, C$, and $D$ must lie on the circle. Thus, any rectangle $ABCD$ a cyclic quadrilateral.

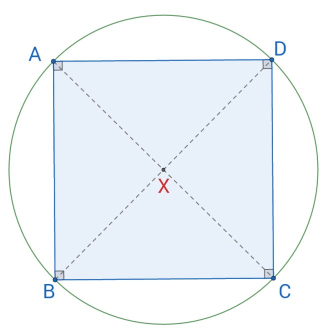


Figure 1: The inscribed rectangle $ABCD$.

$\square$

# Angles Formed by Intersecting Lines Within a Circle

Grant Kilburg

December 5, 2017

*Communicated by: Katherine Bertacini*

**Theorem 10.1.** Let $\Gamma$ be a circle with center $O$. Let $X$ be a point in the interior of the circle, and suppose that two lines $l$ and $m$ intersect at $X$ so that $l$ meets $\Gamma$ at points $A$ and $A'$ and $m$ meets $\Gamma$ at $B$ and $B'$. Then twice the angle $AXB$ is congruent to angle $AOB$ and angle $A'OB'$ taken together.

*Proof.* Let $\Gamma$ be a circle with center $O$. Let $X$ be a point in the interior of the circle, and suppose that two lines $l$ and $m$ intersect at $X$ so that $l$ meets $\Gamma$ at points $A$ and $A'$ and $m$ meets $\Gamma$ at $B$ and $B'$. By Postulate 1.1, construct line segments $AO$, $BO$, $A'O$, $B'O$, and $AB'$. By Euclid III.20, angle $AB'B$ is equal to one half angle $AOB$ and angle $A'AB'$ is equals one half angle $A'OB'$. By Euclid I.32, angle $AXB$ is congruent to angle $AB'B$ taken together with angle $A'AB'$. Since angle $AXB$ is congruent to angle $AB'B$ taken together with angle $A'AB'$, angle $AB'B$ is congruent to one half angle $AOB$, and angle $A'AB'$ is congruent to one half angle $A'OB'$, angle $AXB$ is congruent to one half angle $AOB$ taken together with one half angle $A'OB'$. Thus angle $AXB$ is congruent to one half of angle $AOB$ and angle $A'OB'$ taken together. Otherwise stated, twice the angle $AXB$ is congruent to angle $AOB$ and angle $A'OB'$ taken together. □

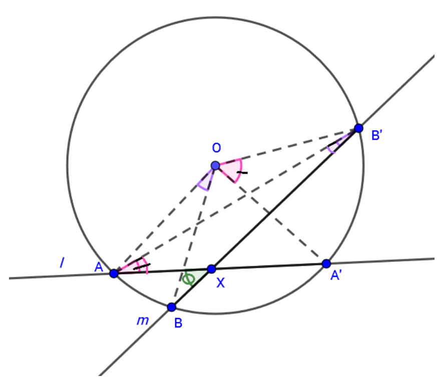


Figure 1: By Euclid I.32, angle $AXB$ is congruent to one half angle $AOB$ taken together with one half angle $A'OB'$.

# Angles Within a Circle Subtending Chords of Various Lengths

Grant Kilburg

December 5, 2017

*Communicated by: Steven Flesch*

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**Theorem 10.3.** If two chords of a circle subtend different acute angles at points of a circle, then the smaller angle belongs to the shorted chord.

*Proof.* Let Γ be a circle with center O of radius A. Let points B, C, and D be on Γ such that the chord AB is longer than the chord CD. Let A and B also meet at a point E on Γ such that angle AEB equals $\alpha$. Similarly, let C and D meet at a point F on Γ such that angle EFD equals $\beta$. By Euclid III.20, $\alpha$ equals one half angle AOB and $\beta$ equals one half angle COD.

Since chord AB is longer than chord CD, angle AOB is greater than angle COD by Euclid I.25. Similarly, one half angle AOB is greater than one half angle COD. Since $\alpha$ equals one half angle AOB and $\beta$ equals one half angle COD, $\alpha$ is greater than $\beta$. Hence, the smaller angle belongs to the shorter chord. □

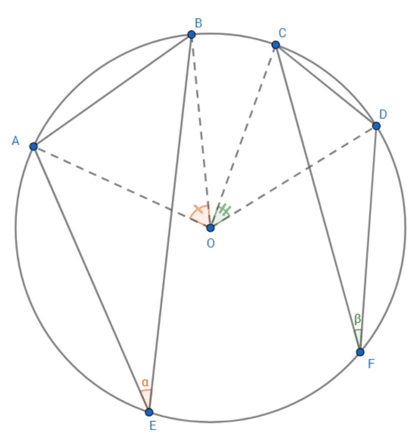


Figure 1: By Euclid I.25, angle AOB is greater than angle COD. Hence $\alpha$ is greater than $\beta$ by Euclid III.20.

# Angle Bisector Construction

Kaelyn Koontz

December 5, 2017

*Communicated by: Katherine Bertacini.*

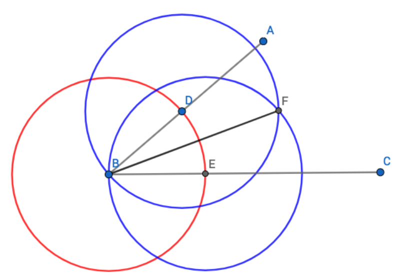


Figure 1: This is a picture of how to construct an angle bisector.

**Theorem 11.1.** Given an angle, construct the angle bisector.

*Proof.* First, you are given angle, in this case angle ABC.

1. Select a width on your compass, and draw a circle with center B. Label the two points where the circle intersects the angle as point D and point E.

2. Draw circle DB and circle EB.

3. Where the two circles intersect, label the intersection point F.

4. Draw segment BF.

When looking at triangle BDF and triangle BEF, segment BD and BE are congruent to each other because they have the same radius. Segment DF and EF are congruent to each other because circle DB and EB have the same radius. Since both triangles share side BF, then that side is also congruent. By Euclid I.8, triangle BDF and BEF are congruent. Since triangle BDF and BEF are congruent, angle DBF is congruent to angle EBF. Therefore, segment BF bisects angle ABC.

□

# Line Tangent to a Circle

Kayla Schafbuch

December 5, 2017

*Communicated by: Cameron Amos.*

This construction is proven by using Ms. Schafbuch's Theorem 11.2.

**Theorem 11.8.** Given a circle with center O and given a point A outside of the circle, we can construct line L tangent to circle O and through point A.

*Proof.* Construction. Step 1: Using a ruler, draw segment OA.
Step 2: Using a compass, draw circle OA.
Step 3: Using a compass, draw circle AO. Name points B and C where circle OA and AO intersect.
Step 4: Using a ruler, draw BC. Name point X where BC interects OA. Point X is known as the midpoint of segment OA.
Step 5: Using a compass, draw circle XO. Name point Y where circle XO intersects circle O.
Step 6: Using a ruler, draw AY.

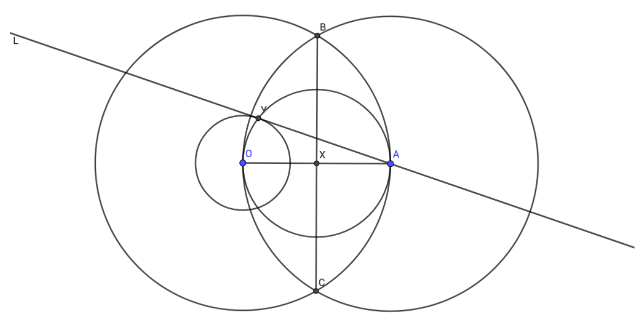


Figure 1: This figure describes the construction above.

□

*Proof.* By Euclid's Postulate 1, draw segments OY and XY. Segment OX is congruent to XA by Ms. Schafbuch's Theorem 11.2. Segment XY is congruent to segment XO since the length of XY and XO is the radius of circle XO. Therefore, segment OX is congruent to XA is congruent to XY. By Euclid's Definition 20, triangle OYX and triangle YXA are isosceles.

The sum of angles in a triangle equals the sum of two right angles by Euclid I.32. Therefore in triangle OYX, angle OXY = two right angles - angle XOY - angle OYX. In the triangle YXA, angle YXA = two right angles - angle XYA - angle YAX.

The sum of angle OXY and angle YXA is the same as two right angles since they form a straight line by Euclid I.13. Therefore, angle OXY + angle YXA = the sum of two right angles. When substituting 'two right angles - angle XOY - angle OYX' for angle OXY and 'two right angles - angle XYA - angle YAX' for angle YXA in the equation,

angle OXY + angle YXA = the sum of two right angles,

We get:

the sum of two right angles - angle XOY - OYX + the sum of two right angles - angle XYA - angle YAX = the sum of two right angles.

After subtracting four right angles from both sides of the equation, we get:
- XOY - OYX - XYA - YAX = the negative sum of two right angles.

Since triangle OYX is isosceles, its base angles are congruent by Euclid I.5. Therefore, angle XOY is congruent to OYX. Since triangle YXA is also isosceles, the same argument applies for its base angles. Therefore, angle XYA is congruent to angle YAX.

We can substitute angle OYX in for angle XOY, along with angle YAX in for XYA, in the equation,

- XOY - OYX - XYA - YAX = the negative sum of two right angles.

From substitution, we get:
- 2 of angle OYX - 2 of angle XYA = the negative sum of two right angles.

After dividing -2 from both sides of the equation, finally,

angle OYX + angle XYA = a right angle

Therefore, angle OYA is a right angle. By Euclid III.18, AY is tangent to circle O. □

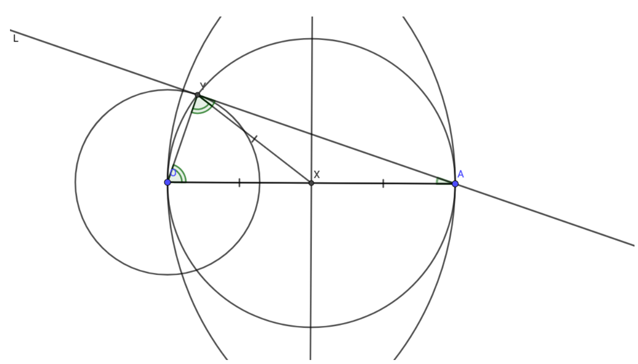

Figure 2: This figure shows line L tangent to circle O.