To show HAMPATH is NP Complete.

We already know 3SAT is an NP Complete problem.
Enough to show $3SAT \leq_m^P HAMPATH$

$$HAMPATH = \{ G \mid G \text{ has hamiltonian path} \}$$

Hamiltonian path is a path such that a walk along the path visits every vertex exactly once.

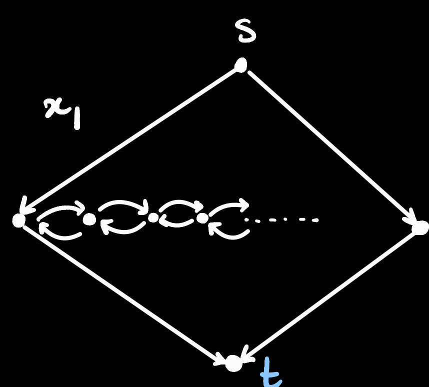
Say a reduction f.

$$\Phi = (x_1 \vee \bar{x}_2 \vee x_3) \wedge (\bar{x}_1 \vee x_2 \vee x_4) \wedge \dots \wedge (\dots)$$

Given an instance of Φ, our reduction f should reduce Φ to a graph G in such a way that

$\Phi = 1$ if and only if G has a hamiltonian path

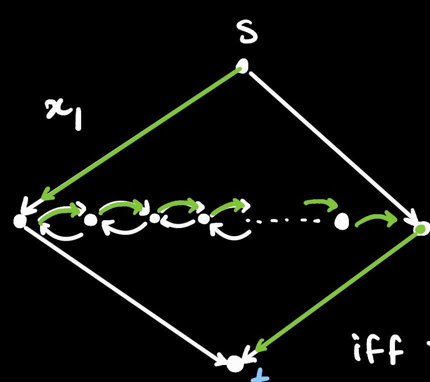
Let's define a substructure in the graph



This is a variable gadget.

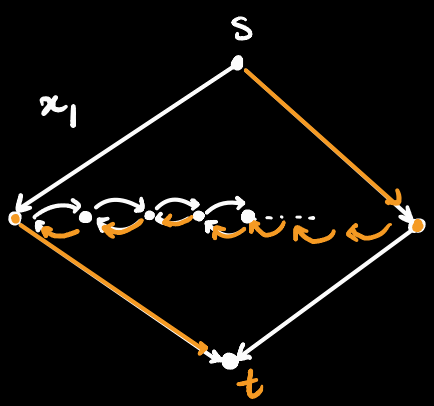
I define two walks Walk A & walk B are two different type of walk on the variable gadget.

They are the following:



This is one type of walk to reach to t.

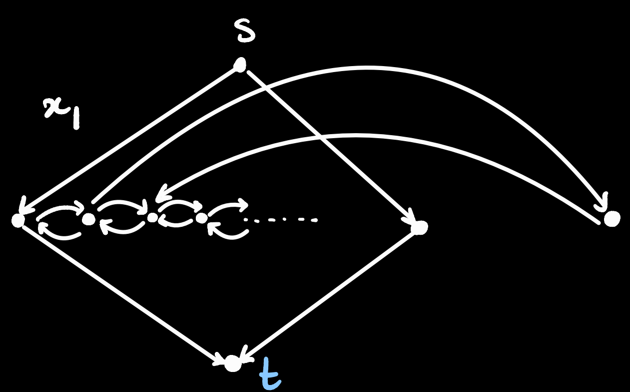
This walk is followed iff the variable is set to TRUE.



This is another type of walk to reach t.

This walk is followed for a variable gadget iff the variable is set to false.

One clause gadget is a single vertex connected with the variable gadget.



Observation

A key observation is that for type II walk it's impossible to cover the clause gadget. Hence hamiltonian path is not possible if x_1 is set to false.

Now with the variable gadget & the clause gadget defined let's construct a G such that

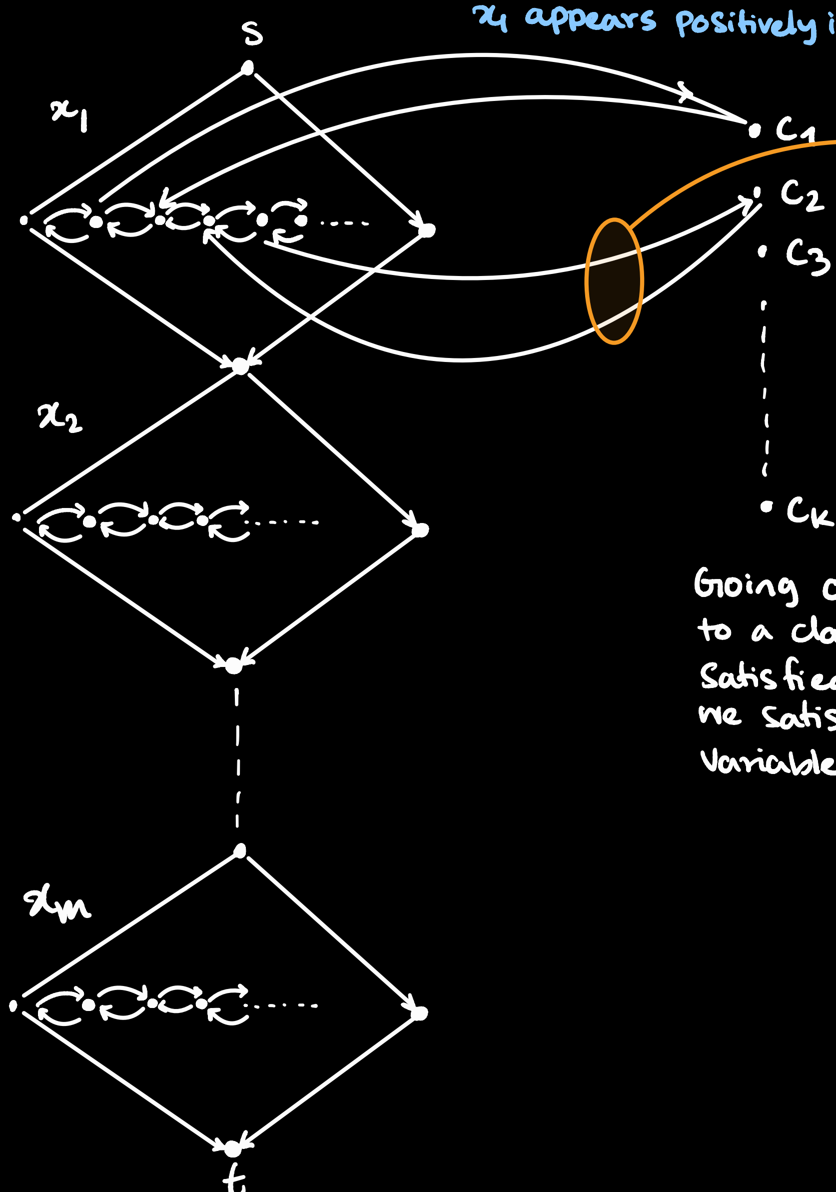
$$\Phi \in 3SAT \Leftrightarrow G \text{ has a hamiltonian path}$$

Say Φ is a boolean formula having m variable and k clauses

$$\Phi = C_1 \wedge C_2 \wedge C_3 \wedge \dots \wedge C_m$$
$$= (x_1 \vee \dots) \wedge (\bar{x}_1 \vee \dots) \wedge \dots$$

Graph G:

x_1 appears positively in clause C_1



x_1 appears in $\bar{x}_1$ form thats why the order is reversed to support type II walk.

Going out from variable gadget to a clause C_i shows that we've satisfied clause C_i what else can we satisfy so we go back to the variable gadget.

This is the total construction of graph G from Φ.

Now to show

Φ is satisfiable iff there is a Hamiltonian path $\mathcal{H}$ in G.

Theorem: Φ is Satisfiable iff G has a Hampath.

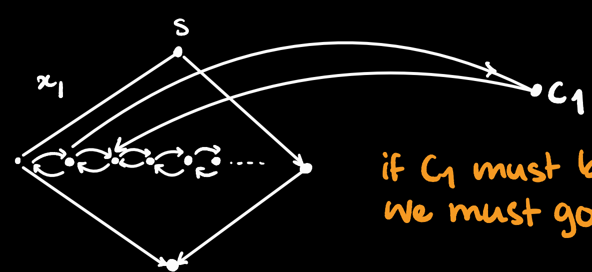
Proof: ($\Rightarrow$) forward direction is straight forward.

φ is satisfiable means we can now walk on the graph G, type 1 or type 2 depending upon the literal type. Then there is a clear hamiltonian path from s to t. Because for all clause to be satisfied some of the literals must [variable gadget] set to true/false & based on that we'll be able to cover the clause gadgets without duplication.

($\Leftarrow$) There is a hamiltonian path P in graph G from s to t.

Following this path p starting from s whatever way is the path assign every literal that value either true or false.

Because of the construction of graph



if C_1 must be satisfied by x_1 then we must go left $\boxed{\text{from Observation}}$

means we should set the variable x_1 to true.

This is how every clause will be satisfied.

$\therefore$ above theorem is true.

This graph G from Φ can also be done using polynomial time.
Hence

$$3SAT \leq_m HAMPATH$$

$\therefore$ HAMPATH IS NP-COMPLETE.