Analysis of Greedy Heuristic for Euclidean Travelling Salesman Problem

Analysis Report

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Abstract

In this report I discuss mathematical properties of naïve greedy heuristic for the euclidean travelling salesman problem. I define an euclidean functional for the greedy heuristic and explore subadditivity, superadditivity, and smoothness for the defined euclidean functional.

1 Introduction

1.1 Priliminary

Given collection of n points on the euclidean plane suppose G is a graph with those n points on the euclidean plane in $[0,1]^2$. A tour of G that visits all the vertices and have shortest total travelling cost is called a travelling salesman tour. Below I state a naïve greedy algorithm that computes a tour. This tour is not the minimum cost tour.

1.2 Naïve Greedy Algorithm For Euclidean TSP

Algorithm 1: Naive Greedy Algorithm

Input: Graph G

Edgeset $\mathbb{E} \leftarrow \phi$
while $card\mathbb{E} < n - 1$ **do**
  Pick two points that has shortest euclidean distance between them, denote this e_1
  if *Subgraph* $E \cup \{e_1\}$ *is acyclic and* $\forall\ e = (u, v) \in E \cup \{e_1\}$ *degree(u) and degree(v)* ≤ 2 **then**
    $E \leftarrow E \cup \{e_1\}$
  else
    Ignore this edge.
  end
end
Output: $\mathbb{E} \cup$ *smallest among remaining edges* as the TSP tour.

1.3 Euclidean Functional

We define the output of the algorithm 1 as the euclidean functional. We'll refer to that as NGA.

$$\mathsf{NGA} \stackrel{\text{def}}{=} \text{Output of Algorithm 1}$$

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2 Mathematical Properties

2.1 Simple Superadditivity

Definition 2.1: Simple Superadditivity

A functional f is simple superadditive if $f(X \cup Y) \geq f(X) + f(Y)$

Our euclidean functional NGA is not superadditive according to definition 2.1. This can be shown using a simple graph. Suppose on euclidean plane $[0,1]^2$ there are two graphs G and F. $G.V = \{A, B\}$ and $F.V = \{C, D\}$. Length of the edges are following: $card(AB) = a$, $card(CD) = a$, $card(AC) = b$ and $card(BD) = b$. WLOG suppose $b \leq a$.

Thus greedy tour on the union of the two graphs will be $a + a + b + b = 2a + 2b$. This is clearly less than $2a + 2a = 4a$ which is the sum of tours for the individual graphs.

Weak Superadditivity

Simple superadditivity 2.1 is one of the most strongest forms of superadditivity. This is not often not required. Most of the time it is enough to define an approximate superadditivity over regions. This is called geometric superadditivity. There is also one more weaker version of superadditive property. First we define weak superadditivity.

Definition 2.2: Weak Superadditivity

A functional f is superadditive if $f(X \cup Y) \geq f(X) + f(Y)$. f is said to be weakly superadditive if we allow small error terms $f(X \cup Y) \geq f(X) + f(Y) - O(1)$.

Our euclidean functional NGA does not show weak superadditivity.

2.2 Geometric Superadditivity

Definition 2.3: Geometric Superadditivity

A functional f is geometric superadditive if $f(X \cup Y) \geq f(X) + f(Y)$

2.3 Simple Subadditivity

Definition 2.4: Simple Subadditivity

A functional f is simple subadditive if $f(X \cup Y) \leq f(X) + f(Y)$

Our euclidean functional NGA does not show simple subadditivity. This can be shown using a counter example.

The following is one simple 6 vertices graph with each G_1, G_2 having 3 vertices each. The union of the graph $G = G_1 \cup G_2$ has 6 vertices. We show that the $\mathsf{NGA}(G_1) + \mathsf{NGA}(G_2) \leq \mathsf{NGA}(G)$. Thus contradicting simple subadditivity definition.

2.4 Smoothness and Monotonicity Properties

Smoothness as defined in Yukitch's Book is given below.

Definition 2.5: Smoothness

An euclidean functional f is said to be smooth if $f(X \cup Y) - f(X) \leq c \mid Y \mid^{\frac{d-1}{d}} \ \forall \ S, T \in [0,1]^d$.

If a functional is monotone then we can easily prove that the functional along with subadditivity shows smoothness. So we explore the monotonicity of NGA algorithm.

Lemma 2.1: Monotonicity and smoothness

An euclidean functional of representing some graph algorithm is said to be showing monotonicity if reducing points from a graph reduces the functional value. If an euclidean functional is showing monotonicity then with subadditivity and growth bound euclidean functional shows smoothness.