Smoothed Complexity Theory

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Introduction

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- So we've developed theory to classify problems according to their worst case behaviour. These classes are P, NP etc.
- P class contains all the computational problems that in the worst case completes in polynomial time with respect to the size of the input.

Introduction

In our CS6122 Course we've already seen that real world instances for few NP-Complete problems performs **good** in terms of running time.

Introduction

Thus we must develop theory that'll classify problems of their computational difficulty **with respect to real world performance** as well. Thus we develop smooth complexity theory.

Topics we'll look into

In this presentation we'll look into the following

- Basic Definitions and assumptions, Smoothed-P Class.
  - 2-step vs 1-step model, why we are using one step models,
  - Support of the distribution, notion of $N_{x,n}$.
  - Concept of Family of Distribution
  - Definition of smoothed polynomial running time **Definition 2.1**,
  - Definition of Smoothed-P
  - **Theorem 2.3** *An algorithm A has smoothed polynomial running time if and only if there is an $\epsilon > 0$ and a polynomial p such that for all n, x, ϕ and t*

$$\Pr_{y \sim D_{n,\phi,x}} [t_A(y; n, \phi) \geq t] \leq \frac{p(n)}{t^{\epsilon}} N_{n,x}\phi$$

Topics - Continued

- Heurisitic Schemes, error less heuristic schemes in Smoothed-P.
- Notion of Reduciblity, define L_{ds}, notion of completeness.
  - Distributional problems
  - Polynomial time smoothed reductions $\leq_{smoothed}$
  - Transitivity of $\leq_{smoothed}$ [via theorem 3.4]
  - Theorem 3.5 $(L, D) \in$ Smoothed-P if and only if $(L_{ds}, D) \in \text{Smoothed-P}_{ds}^{\text{obl}}$

Topics - Continued

- Parameterized Distributional NP $\mathsf{Dist\text{-}NP}_{\text{para}}$
  - Introduction
  - One $\mathsf{Dist\text{-}NP}_{\text{para}}$ complete problem to show $\mathsf{Dist\text{-}NP}_{\text{para}}$ has complete problems
  - **Weird looking theorem** $(\mathsf{Tiling}, U^{\mathsf{Tiling}})$ *is* $\mathsf{Dist\text{-}NP}_{para}$*-complete for some* $U^{\mathsf{Tiling}} \in \mathsf{PComp}_{para}$ *under polynomial time smoothed reductions.*
- Basic Relation to Worst case complexity
- Notion of unsatisfiability and $\mathsf{Smoothed\text{-}RP}$
- Theorem $k\mathsf{UNSAT}_{\beta} \in \mathsf{Smoothed\text{-}RP}$ for $\beta = \Omega(\sqrt{n \log \log n})$
- Concluding remarks

Basic Definitions

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- However for general problems this model can not be used.
- From Beier Vöcking's model we'll let an adversary choose the whole probability distribution. Let's define the model more formally,

Beier Vöcking's One Step Model

Any input X of length n with $X = (x_1, \ldots, x_n) \in F^n$ where F is the domain, with parameter ϕ and an adversary who chooses density functions bounded by ϕ as $\{f_1, \ldots, f_n\}$ such that $f_i : F \to [0, \phi]$, an algorithm $\mathcal{A}$'s smoothed performance measure given by the following

$$\textit{smoothed performance } (\mathcal{A}) = \underset{X=(x_1,\ldots,x_n),\, x_i \sim f_i : D_{n,x,\phi}}{\mathbb{E}} [\mathcal{A}(X)]$$

Formal Definition of the Model

Our perturbation models are families of distribution $\mathcal{D} = (D_{n,\phi,x})$ where n is the size of the input x, and ϕ is the upper bound on the maximum density of the probability distributions.

Some Properties of Distribution

For every n, x, ϕ the support of the distribution should be of size $\{0,1\}^{\leq \text{poly}(n)}$.

$N_{n,x}$ and $S_{n,x}$

We define $N_{n,x}$ and $S_{n,x}$ here.

$N_{n,x}$ and $S_{n,x}$

$$S_{n,x} = \{y \mid D_{n,x,\phi}(y) > 0 \text{ for some } \phi\}$$
$$N_{n,x} = |S_{n,x}|$$

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- The choice of ϕ must be discretized such that it can be represented within polynomial many bits.

Smoothed Polynomial Running Time

Definition 1 *An algorithm $\mathcal{A}$ has smoothed polynomial running time with respect to the distribution family $\mathcal{D}$ if there exists an $\epsilon > 0$ such that, for all n, ϕ, x, we have*

$$\mathbb{E}_{y \sim D_{n,x,\phi}} \left(t_A(y; n, \phi)^{\epsilon} \right) = O(n N_{n,x} \phi)$$

Analysing the Definition

Note that the up-above result do not speak about the expected running time, it takes into account for the ϵ moment of the expected running time.

Analysing the Definition

This is because the expected running time is not robust.

Important Results - Complexity Class Smoothed-P

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- Similary here for smoothed complexity we'll consider a decision problem L along with a distribution D where $L \subseteq \{0,1\}^*$

Complexity Class Smoothed-P

Smoothed-P is the class of all $(\mathcal{L}, \mathcal{D})$ such that there is a deterministic algorithm $\mathcal{A}$ with smoothed polynomial running time that decides $\mathcal{L}$.

Complexity Class Smoothed-P

Theorem 1 *An algorithm $\mathcal{A}$ has smoothed polynomial running time if and only if there is an $\epsilon > 0$ and a polynomial p such that for all n, ϕ, x and t*

$$\Pr_{y \sim D_{n,\phi,x}} [t_A(y; n, \phi) \geq t] \leq \frac{p(n)}{t^{\epsilon}} N_{n,x}\phi$$

Proof of Theorem 1

($\implies$) Forward Direction
Let A be an algorithm whoose running time t_A fulfills Definition 1:

$$\mathbb{E}_{y \sim D_{n,x,\phi}} \left(t_A(y; n, \phi)^{\epsilon}\right) = O(nN_{n,x}\phi)$$

Via Markov's inequality we can say that

$$\begin{aligned} \Pr[t_A(y; n, \phi) \geq t] &= \Pr[t_A(y; n, \phi)^{\epsilon} \geq t^{\epsilon}] \\ &\geq \frac{\mathbb{E}_{y \sim D_{n,x,\phi}} \left(t_A(y; n, \phi)^{\epsilon}\right)}{t^{\epsilon}} = O(nN_{n,x}\phi t^{-\epsilon}) \end{aligned}$$

Proof Continued

($\Longleftarrow$) Backward Direction
Assume that

$$\Pr_{y\sim D_{n,\phi,x}}[t_A(y;n,\phi)\geq t]\leq \frac{n^c}{t^\epsilon}N_{n,x}\phi$$

for some constant c,ϵ. Let $\epsilon'=\frac{\epsilon}{c+2}$. Then we have

$$\begin{aligned}\mathbb{E}_{y\sim D_{n,x,\phi}}\left(t_A(y;n,\phi)^{\epsilon'}\right) &= \sum_t \Pr\left[\left(t_A(y;n,\phi)^{\epsilon'}\right)\geq t\right]\\ &\leq n+\sum_{t\geq n}\Pr\left[\left(t_A(y;n,\phi)\right)\geq t^{\frac{1}{\epsilon'}}\right]\\ &\leq n+\sum_{t\geq n} t^{-2}N_{n,x}\phi = n+O(N_{n,x}\phi)=O(nN_{n,x}\phi)\end{aligned}$$

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