Analysis of Greedy Heuristic for Euclidean Travelling Salesman Problem

Analysis Report

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Abstract

In this report I discuss mathematical properties of naïve greedy heuristic for the euclidean travelling salesman problem. I define an euclidean functional for the greedy heuristic and explore subadditivity, superadditivity, monotonicity for the defined euclidean functional.

1 Introduction

1.1 Priliminary

Given collection of n points on the euclidean plane suppose G is a graph with those n points on the euclidean plane in $[0,1]^2$. A tour of G that visits all the vertices and have shortest total travelling cost is called a travelling salesman tour. Below I state a naïve greedy algorithm that computes a tour. This tour is not the minimum cost tour.

1.2 Naïve Greedy Algorithm For Euclidean TSP

Algorithm 1: Naive Greedy Algorithm

Input: Graph G

Edgeset $\mathbb{E} \leftarrow \phi$
while $card\mathbb{E} < n-1$ **do**
  Pick two points that has shortest euclidean distance between them, denote this e_1
  if *Subgraph* $E \cup \{e_1\}$ *is acyclic and* $\forall\ e = (u,v) \in E \cup \{e_1\}$ *degree(u) and degree(v)* $\leq$ *2* **then**
    $E \leftarrow E \cup \{e_1\}$
  else
    Ignore this edge.
  end
end
Output: $\mathbb{E} \cup$ *smallest among remaining edges* as the TSP tour.

1.3 Euclidean Functional

We define the cost of the TSP tour from the output of the algorithm 1 as the euclidean functional. We'll refer to that as NGA.

$$\mathsf{NGA} \stackrel{\mathsf{def}}{=} \text{cost of the TSP tour from the output of Algorithm 1}$$

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2 Mathematical Properties

2.1 Simple Superadditivity

Definition 2.1 (Simple Superadditivity). *A functional f is simple superadditive if $f(X \cup Y) \geq f(X) + f(Y)$*

Lemma 2.1. *Our euclidean functional NGA is not superadditive according to Definition 2.1.*

Proof. This can be shown using a simple graph. Suppose on euclidean plane $[0,1]^2$ there are two graphs G and F. $G.V = \{A, B\}$ and $F.V = \{C, D\}$. Length of the edges are following: $card(AB) = a$, $card(CD) = a$, $card(AC) = b$ and $card(BD) = b$. WLOG suppose $b \leq a$.

Thus greedy tour on the union of the two graphs will be $a + a + b + b = 2a + 2b$. This is clearly less than $2a + 2a = 4a$ which is the sum of tours for the individual graphs. ☐

2.2 Weak Superadditivity

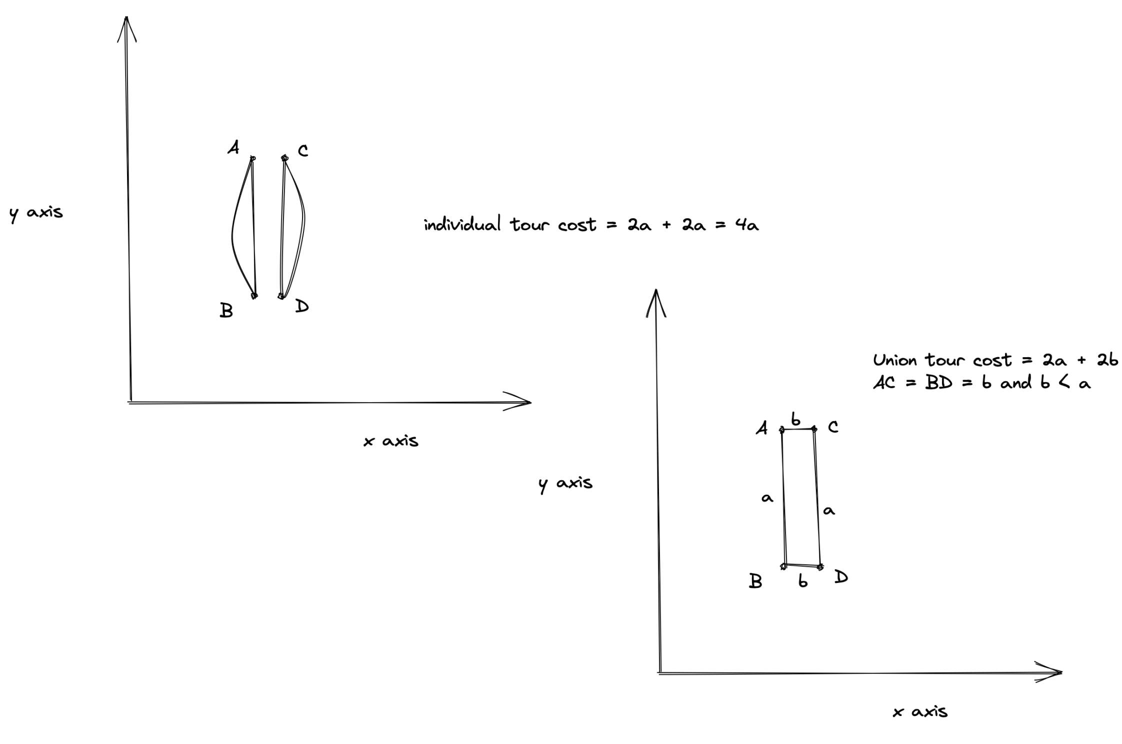
Simple superadditivity 2.1 is one of the most strongest forms of superadditivity. This is not often not required. Most of the time it is enough to define an approximate superadditivity over regions. This is called geometric superadditivity. There is also one more weaker version of superadditive property. First we define weak superadditivity.

Definition 2.2. *Weak Superadditivity A functional f is superadditive if $f(X \cup Y) \geq f(X) + f(Y)$. f is said to be weakly superadditive if we allow small error terms $f(X \cup Y) \geq f(X) + f(Y) - O(1)$.*

Lemma 2.2. *Our euclidean functional NGA does not show weak superadditivity as per the definition 2.2.*

Proof. If we could get an instance where the error term exceeds some constant term $O(1)$ then this would be a counter example to weak superadditivity.

Let's consider the following implementation of graph G and F as the following

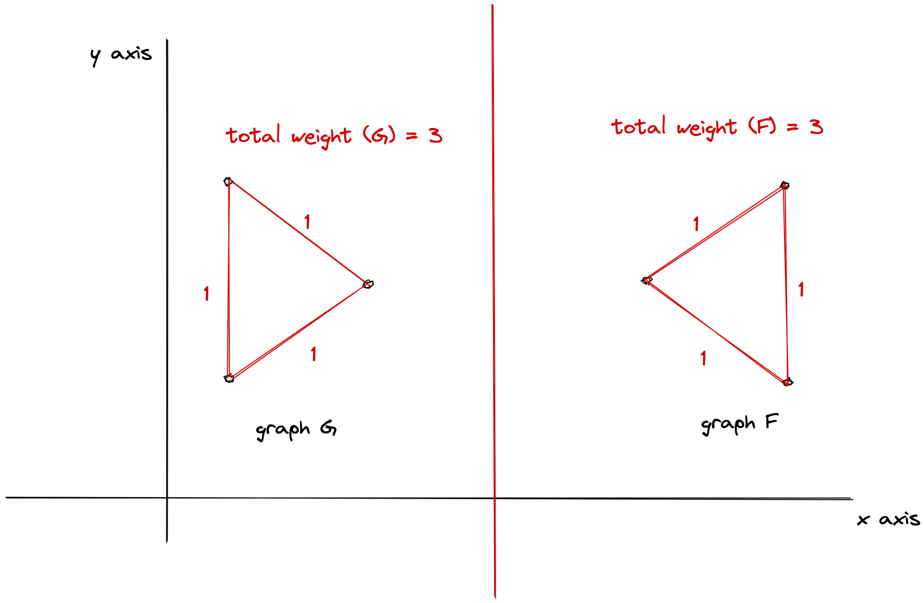


The graph G and F upabove clearly shows that union tour is strictly smaller than the sum of the individual tour. Hence it is now showing simple superadditive properties. Here the error term is not constant, it depends on the size of the boundary $O(diamR)$ because b depends on a and a is bounded by the size of the region. ☐

2.3 Simple Subadditivity

Definition 2.3 (Simple Subadditivity)**.** *A functional f is simple subadditive if $f(X \cup Y) \leq f(X) + f(Y)$*
Lemma 2.3. *Our euclidean functional* NGA *does not show simple subadditivity.*

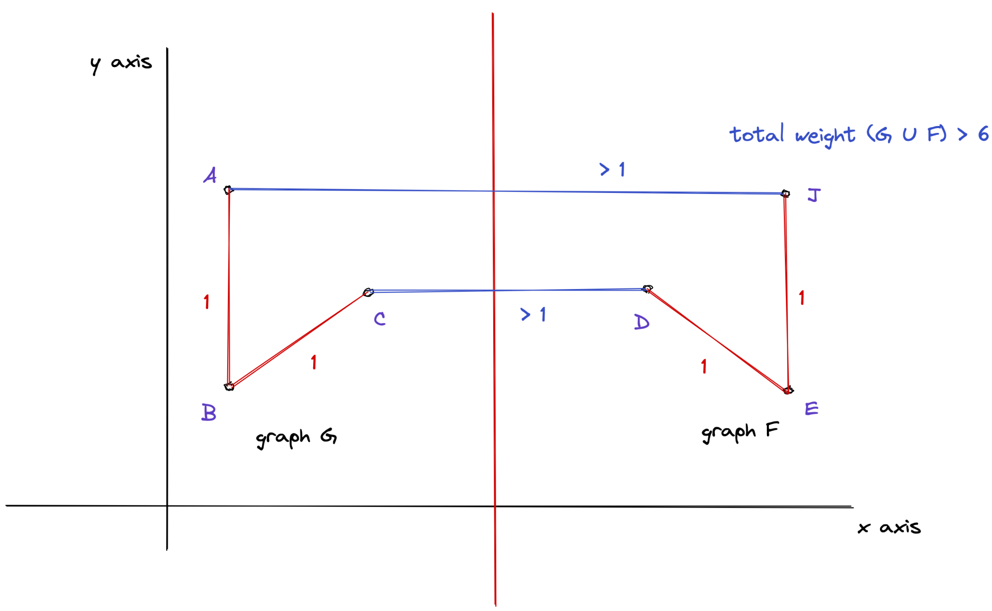
Proof. This can be shown using a counter example. The following is one simple 6 vertices graph with each G, F having 3 vertices each. The union of the graph $G' = G \cup F$ has 6 vertices. We show that the $\mathsf{NGA}(G)$ + $\mathsf{NGA}(F) \leq \mathsf{NGA}(G \cup F)$. Thus contradicting simple subadditivity definition.



When considered the individual graphs, the each of the individual NGA output is 3. So $\mathsf{NGA}(F) + \mathsf{NGA}(G) = 6$.

Let's consider the union of the two graphs $G \cup F$. The tour cost is > 6. Our NGA algorithm will choose AB, BC, DE, EJ edges first. Then it will not consider AC or the DJ edge because it'll violate the if condition in Algorithm 1's line number 4.

That's why it'll first consider the edge CD and at the end AJ to complete the tour.



For better results we should show this does not hold for arbitrary number of points n.

Suppose the following construction of graph G and F. Let us consider from A $\rightarrow$ B and $J \rightarrow E$ there is infinitely many points over the euclidean graph. Then this construction holds for arbitrary n over the euclidean region. $\square$

2.4 Geometric Subadditivity, Superadditivity

From the construction of the graphs if we divide the region into R_1, R_2 by going through the middle of the graph then all the proofs provided up above will also be a proof showing NGA do not show Geometric Subadditivity, Superadditivity with constant error term.

However I did not check the geometric subadditivity with slightly higher error term of $O(diamR)$.

2.5 Smoothness and Monotonicity Properties

Smoothness as defined in Yukitch's Book is given below.

Definition 2.4. *Smoothness An euclidean functional f is said to be smooth if $f(X \cup Y) - f(X) \leq c \mid Y \mid^{\frac{d-1}{d}}$ $\forall\ S, T \in [0,1]^d$.*

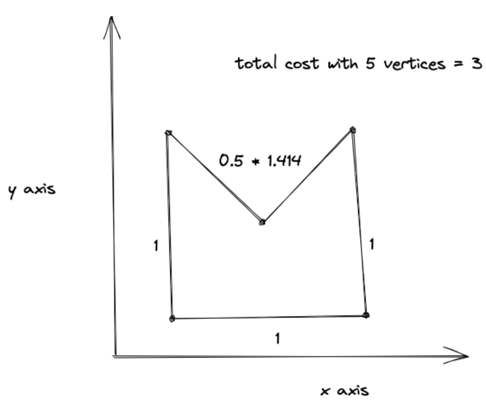
If a functional is monotone then we can easily prove that the functional along with subadditivity shows smoothness. So we explore the monotonicity of NGA algorithm.

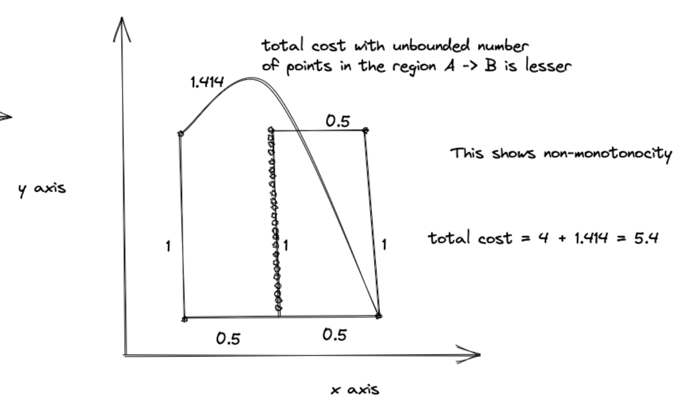
An euclidean functional of representing some graph algorithm is said to be showing monotonicity if reducing points from a graph reduces the functional value. If an euclidean functional is showing monotonicity then with subadditivity and growth bound euclidean functional shows smoothness.

Definition 2.5. *Monotonicity If $F \subseteq G$ in some arbitrary region $\mathcal{H}$ then an monotone euclidean function must hold the following inequality $f(F, \mathcal{H}) \leq f(G, \mathcal{H})$*

Lemma 2.4. *Our NGA functional do not show monotone property according to the definition above. This can be proved by the following family of counter example below.*

Proof. We prove this by constructing a counter example.





This shows a reduction on the tour cost according to NGA
on deployment of unboundedly many points into the euclidean plane

On Introduction of unbounded many points tour cost as computed by NGA algorithm reduces. Thus violating Monotonicity as defined by Definition 2.5. □