Assignment 2

AKASH ROY | CS22M007

Colab: Neha, Abhijit

1. (12 points) (a) (5 points) Obtain profits $p_1, \ldots, p_n$ and weights $w_1, \ldots, w_n$ for the knapsack problem so that $|\mathcal{P}|$ is exponential in n (e.g $2^{\Omega(n)}$). Prove your answer.

   (b) (7 points) Let $I = (p, W, w)$ be the instance in your answer to the above question (1 (a)). Now, consider the following distribution on the instances for knapsack obtained from I. For $1 \le i \le n$, let β_i be sampled uniformly from $[-\epsilon, \epsilon]$. Set $w'_i = \max\{w_i + \beta_i, 0.001\}$. What is the expected size of $\mathcal{P}$? Prove your answers.

(a) I is an instance of knapsack problem.

Wt. = [1, 2, 3 ... n] weight of i for each item i = 1, 2, 3 ... n

Pt. : [1, 2, 3 ... n] profit of i for each item i = 1, 2, 3 ... n

For this all the solutions are pareto optimal solution.
hence size $|P| = 2^n$

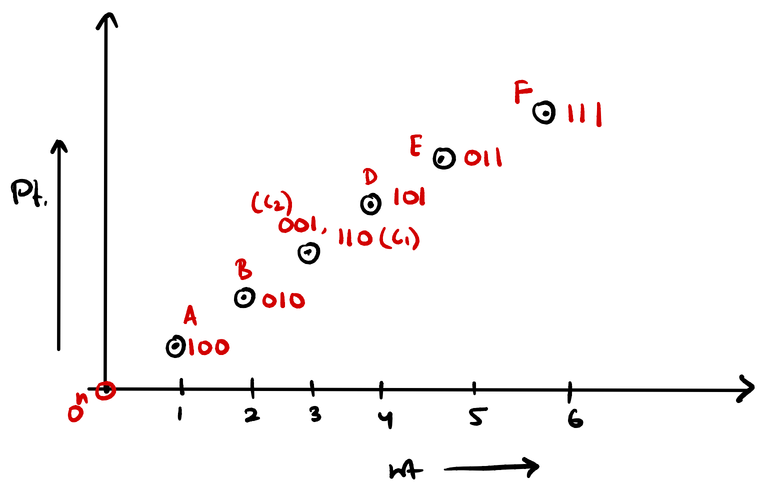
Proof. Base Case:

Suppose I is an instance of knapsack problem.

Wt. = [1, 2, 3] weight of i for each item i = 1, 2, 3

Pt. : [1, 2, 3] profit of i for each item i = 1, 2, 3

Pt. & Wt. graph for this instance:



$P_0 = \{0^n\}$

$P_1 = \{A, 0^n\}$

$P_2 = \{C_1, A, B, 0^n\}$

$P_3 = \{0^n, A, B, C_1, C_2, D, E, F\}$

hypothesis: for pf. = [1 ... n] & weights = [1 ... n] all solutions are pareto-optimal.

proof: To show for profit = [1 ... n, n+1] & wt. = [1 ... n, n+1] all solutions = 2^{n+1} are pareto optimal.

Solution Space Size $= 2^{n+1} = 2^n + 2^n$.

take 2^n items from Solution Set of $[n]$ items.

set $n+1^{th}$ item $= 0$. $= 2^n$ Solutions for $[n+1]$
& $n+1^{th}$ item $= 1$. $= 2^n$ more " " $[n+1]$

Need To Show all of these items are pareto optimal.

The Last item has distinct weight $n+1$. Thus adding Last item to every Solutions in 2^n will create new 2^n solutions all of them will be pareto optimal.

$\forall i \in$ Solutions of 1 to n, new points will be $i+(n+1)$ with profits $i+(n+1)$.

But there was solutions previously with wt. $(i+1)+n$ when added the n^{th} item.

Similarly if $n+1^{th}$ item is 0 then it is also pareto optimal.

Thus all the solutions are pareto optimal.

(b) Suppose β_i be in $[-\epsilon, \epsilon]$. $\forall \epsilon$ we can obtain δ s.t. $\delta >> 2\epsilon$ and $\delta = \text{argmin}\{\forall i,j \in S \; |w^T i - w^T j|\}$

hence

$\forall \epsilon$ we can obtain I such that

$\forall (i,j) \in$ solution set $|w^T i - w^T j| > 2\epsilon$. Hence

the pertubation will not affect the # of pareto optimal solutions. Expected # of pareto optimal solution will be still exponential.

Similarly for every instance I we can get a 'ϵ' large enough s.t. it'll eat up all the pareto optimal instances to it's left.

2. (10 points) In the class we had proved smoothed number of pareto optimal solutions for a Knapsack instance when the weights are ϕ-perturbed instances and the profits being chosen by the adversary. It is known that choosing the weights arbitrarily and profits as per ϕ-perturbations would also lead to similar bounds. Give a detailed discussion on what kind of changes that would be needed to the arguments done in the class to obtain a bound on the expected value of $|\mathcal{P}|$ when the profits are ϕ-perturbed and the weights are chosen by an adversary.

Modifications to incorporate a different pertubation model.

Pertubation Model

1. Choose $w = (w_1, \ldots, w_n)$ arbitrarily
2. draw profits $P = (P_1, \ldots, P_n)$ s.t. p_i is drawn according to $f_i : [0,1] \to [0, \phi]$

- Changes: Suppose $K \in \mathbb{N}$ be an integer. Profits are in $[0,1]$

$0 \le P_i \le 1$

hence $\forall x \quad p^T x \le n$.

divide the profit axis into equal $\frac{n}{K}$ chunks.

we'll look into # of pareto optimal sol$\underline{n}$. in these intervals.

$I_0 \ldots I_{K-1}$ will be those intervals.

Let $\underbrace{x^K}_{\text{non-empty profit intervals}} = 1 + \left|\{ j \mid \exists x \in P,\ p^T x \in I_j \}\right|$

for Large K, $|P| = x^K$ then $E[|P|] = E[x^K]$

Suppose F_K be an event: $\exists x \ne y \in \{0,1\}^n$ with $|p^T x - p^T y| \le \frac{n}{K}$

We need to find probability of event that 2 non-trivial Solutions lie into the Same interval.

lemma: $\boxed{\Pr[F_K] \le 2^{2n+1} \cdot \frac{n\phi}{K} \quad \forall K \in \mathbb{N}}$

Proof. Let $x \ne y \in \{0,1\}^n$ be fixed. Suppose $x_i \ne y_i$

Wlog: $x_i = 0$ & $y_i = 1$ all profits except P_i is fixed.

$$(p^T x - p^T y) = \alpha - p_i \quad \text{for some } \alpha$$

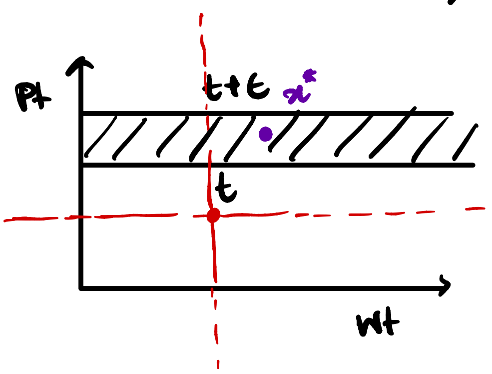
$$\Pr\left[p^T x - p^T y \le \frac{n}{k}\right] = \Pr\left[|\alpha - p_i| \le \frac{n}{k}\right]$$
$$= \Pr\left[p_i \in \left[\alpha - \frac{n}{k}, \alpha + \frac{n}{k}\right]\right]$$
$$\le \frac{2n\phi}{k}$$

By taking union bound over all pair $x \neq y$

$$\Pr[F_k] \le 2^{2n} \cdot 2 \cdot \frac{n\phi}{k}$$
$$\le 2^{n+1} \frac{n\phi}{k} \qquad k \simeq O(2^{n+1} n\phi)$$

lemma: $\forall k \in \mathbb{N} \quad E[x^k] \le n^2\phi + 1$ – to prove this we need to modify the following lemma:

lemma: $\forall t \geq 0 \ \& \ \forall \epsilon > 0$



$$\Pr\left[\exists x \in P \mid p^T x \in (t, t+\epsilon]\right] \le n\phi\epsilon$$

$x^* \leftarrow$ highest weight solution & $p^T x < t$

r.v. $\Lambda(t) \le \epsilon \Leftrightarrow \exists x \in P;\ p^T x \in (t, t+\epsilon]$

here x^* is the winner.

A solution x is a looser if $w^T x > w^T x^*$ & $p^T x > t$
hence $p^T x > p^T x^*$

Suppose $\hat{x}$ be the looser with the least profit.

if x^* is the most optimal solution then x^* might be not defined.

After this rest of the argument holds.

3. (12 points) Let $I = (p, W, w)$ be an instance of the Knapsack problem. Let $\mathcal{P}_i$ denote the set of all Pareto optimal solutions for the instance that includes only the first i items.

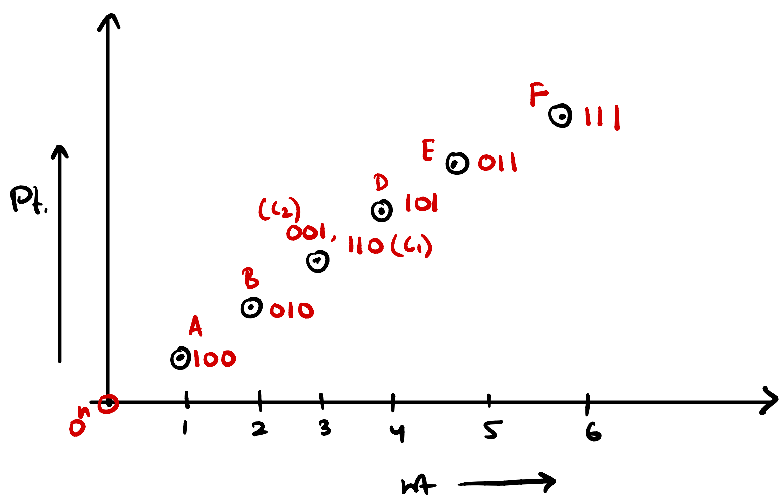
(a) (6 points) We had shown in the class that $\mathcal{P}_i \subseteq \mathcal{P}_{i-1} \cup \mathcal{P}_{i-1}^{+i}$. Are there instances for which this inclusion is an equality? Give an example. Also discuss about instances for which this inclusion is strict. Give an example.

(b) (6 points) Is it possible that $|\mathcal{P}_i| < |\mathcal{P}_{i-1}|$ for some instance I? Prove your answer.

(a) Suppose I is an instance of knapsack problem.

Wt. = [1, 2, 3] weight of i for each item i = 1,2,3
Pt. : [1, 2, 3] profit of i for each item i = 1,2,3

Pt. & Wt. graph for this instance:



$P_0 = \{0^n\}$
$P_1 = \{A, 0^n\}$
$P_2 = \{C_1, A, B, 0^n\}$
$P_3 = \{0^n, A, B, C_1, C_2, D, E, F\}$

to show that $P_{i-1}^{+i} \cup P_{i-1} \subseteq P_i$ so the equality holds
we need to show $\forall i \in [2]$ $P_{i-1}^{+i} \cup P_{i-1} \subseteq P_i$

for this instance: $P_0^{+1} = \{A\}$
$P_1^{+2} = \{C_1, B\}$
$P_2^{+3} = \{F, D, E, C_2\}$

Now we need to check for every $i \in [3]$

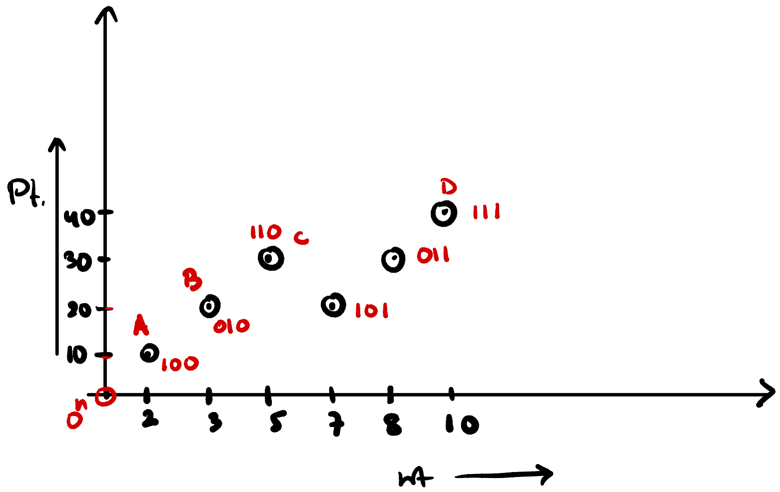
$i = 1$ | $P_0 \cup P_0^{+1} = \{0^n\} \cup \{A\} \subseteq P_1$
$i = 2$ | $P_1 \cup P_1^{+2} = \{A, 0^n, B, C_1\} \subseteq P_2$
$i = 3$ | $P_2 \cup P_2^{+3} = \{C_1, A, B, 0^n\} \cup \{F, D, E, C_2\}$
$\subseteq \{C_1, A, B, 0, C_2, F, D, E\} \subseteq P_3$

This shows equality for $\boxed{P_{i-1}^{+i} \cup P_{i-1} = P_i}$

- Consider I be an instance that makes $P_i \subseteq P_{i-1} \cup P_{i-1}^{+i}$ inequality strict.

Pt.= [10, 20, 10] weight of i for each item i = 1,2,3
Wt. = [2,3,5] profit of i for each item i = 1,2,3

Pt. & Wt. graph for this instance:



To show strict inequality we need to show

$$\exists i, \exists x \quad x \in P_{i-1}^{+i} \cup P_{i-1} \ \& \ x \notin P_i$$

$P_0 = \{0^n\}$
$P_1 = \{0^n, A\}$
$P_2 = \{0^n, A, B, C\}$
$P_3 = \{A, B, C, D\}$

$P_0^{+1} = \{A\}$
$P_1^{+2} = \{B, C\}$
$P_2^{+3} = \{001, 101, 011, D\}$

Suppose $i=3$ then $P_2 \cup P_2^{+3} = \{0, A, B, C\} \cup \{001, 101, 011, D\} \not\subseteq P_3$

This shows a strict inclusion.

(b) To become $P_i < P_{i-1}$ $\exists x \in P_{i-1}$ & $x \notin P_i$

then $\nexists y \in P_{i-1}$ y dominates x.

$\exists z \in P_i$ z dominates x.

z comes from $P_{i-1} \cup P_{i-1}^{+i}$

$z \notin P_{i-1}$ otherwise $x \notin P_{i-1}$

$\therefore z \in P_{i-1}^{+i}$

$\therefore \forall x \in P_{i-1}$ & $x \notin P_i$ $\exists z \in P_{i-1}^{+i}$ & $z \in P_i$ Such that z dominates x.

if one z dominates more than one x then $|P_{i-1}| > |P_i|$

Observation:

for following instance of weight & profit.

Weights = [2, 1, 9, 6, 3]

Profits = [6, 4, 9, 7, 4]

$|P_5| < |P_4|$

Cardinality of $P_5 = 10$

Cardinality of $P_4 = 11$

details of this instance to be shared later.

4. (16 points) We consider the following variant of TSP. In this variant, we are given a graph $G = (V, E)$ with edge lengths $w : E \to [0, 1]$ and delay $c : E \to [0, 1]$. A maximum permissible delay D is given. The objective is to find a minimum weight Hamiltonian cycle with total delay at most D. Let us denote this modified problem by dTSP.

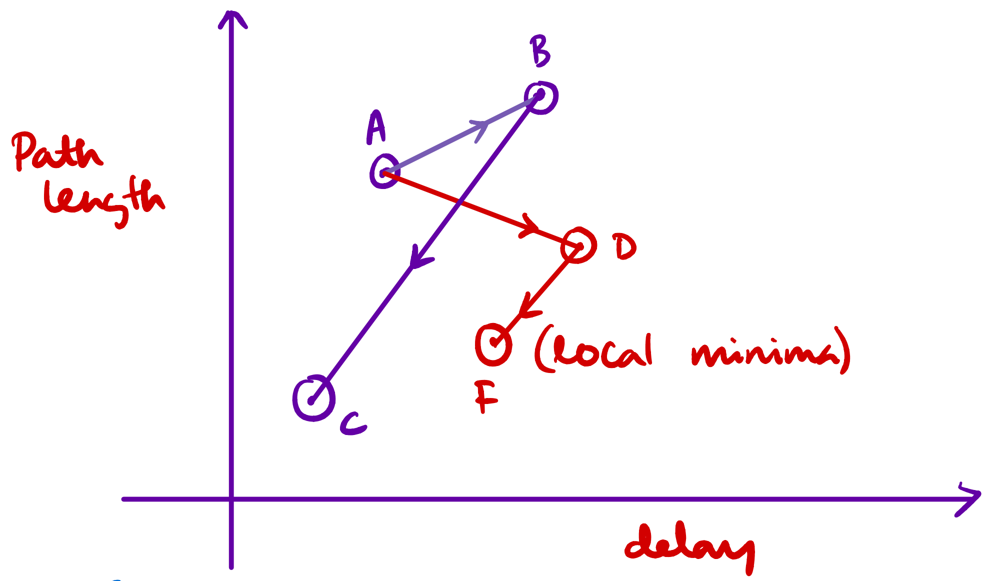
(c) (4 points) Give a modified version of the 2OPT algorithm for the dTSP problem.

Answer to (d) depends upon how we defined 2-OPT Algorithm here.
Suppose we define 2OPT in following 3 ways:

exchange 2-edges e_1, e_2 till no more exchanges possible

- (A) If overall path length & delay reduces.
- (B) If overall path length reduces & delay [$\sum d_i < D$]
- (C) If either of the overall path length or delay reduces.

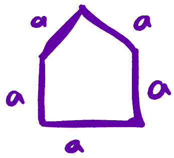
(d) (4 points) Does the modified 2OPT for dTSP always output a Pareto optimal solution? Give a detailed justification.



Any algorithm from @ A,B,C will choose the path ADF in the 2-OPT algorithm.

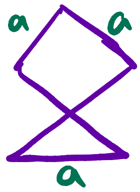
Suppose ∃ a path in the 2-OPT configuration graph [ABC] in first 2-OPT exchange both w.r.t. of path & delay increases but next-2OPT will find a global minima [most optimal hence pareto optimal]. But Any A,B,C will never choose path AB.

One example Showing upabove phenomena



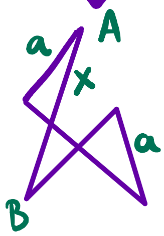
total tour cost = 5a = 3a + 2a.

doing one 2-exchange



→ this step will not be taken by 2-OPTs

$3a + 2\sqrt{2}a$ = total cost



global minima

$2a + 2\sqrt{2}a + x$ → length of AB

Say $x << a$

then 2OPT algorithm fails to find the global optimal.

(a) (4 points) Can the pareto optimal solutions of dTSP be computed in time polynomial in the number of pareto optimal solutions? Give detailed justifications of your answers.

Defn. pareto optimal Solutions: Suppose x & y be some hamiltonian cycle with delay $d(x)$ & $d(y)$ & path length $p(x)$ & $p(y)$.

y is said to be dominated by x if $p(x) \leq p(y)$ or $d(x) \leq d(y)$ either one of these is strict.

$P_i = \{$ pareto optimal set considering till i^{th} edge $\}$

S = Solution Space $= \{$ ham cycle with $\Sigma d_i < D \}$

(e_i) is a hamiltonian cycle $\in S$. I define $S_i = \{ e_0 \cdots e_i \in \{0,1\}^i, e_{i+1} \cdots e_n = 0 \}$

each $x \in S_i$ is a subgraph of Solutions (e_i) in S.

trivially $P_i \subseteq P_{i-1} \cup P_{i-1}^{+i}$ — ① take all the $x \in P_{i-1}$ & then add the i^{th} edge to it.

lemma: $\exists$ a optimal solution which is also pareto optimal. — ②

Proof: Suppose x is an optimal solution. x is not a pareto optimal solution.

$\therefore \exists y$ such that y dominates x.

Then $\text{delay}(y) < \text{delay}(x) < D$

Since x is optimal so $\text{path}(y) = \text{path}(x)$.

hence y is also optimal.

from obs. ① & lemma ② Simple Ullman like algorithm can be designed.

Ⓐ ALGORITHM:

① $P_0 = \{0^n\}$

② for $1 \ldots n$ do

$Q_i = P_{i-1} \cup P_{i-1}^{+i}$

$P_i = \{ x \mid \nexists y \in Q_i ; y \text{ dominates } x \}$ — Step Ⓜ

③ return $\arg\min \{ \text{Path}(x) \in P_n \mid \Sigma d_i(x) < D \}$

with definition P_i Algorithm A runs in polynomial time in size of P_n.

at Step (M) we need to check for every $x \in P_i$; x must be also $\in S_i$

Checking $x \in S_i$ takes $O(n \times n!)$ time.

dTSP as binary optimization problem:

$$\min \sum_{ij} w_{ij} x_{ij} \quad - (1)$$

$$x_{ij} \in \{0,1\} \quad \forall i,j$$

$$x_{ij} = x_{ji} \rightarrow \text{Symmetry TSP.}$$

$$\sum_{i \in S, j \in \bar{S}} x_{ij} \geq 2 \quad \forall S \subseteq V,\ S \neq \phi, V$$

$$\sum_{i,j} d_{ij} x_{ij} < D \quad - \text{delay bound}$$

Now this is an linear program with $O(2^n) + O(n^2)$ constraints.
Not possible to solve in polynomial (n) time.

(b) (4 points) Can the expected number of Pareto optimal solutions on ϕ perturbed instances be polynomial in N and ϕ? (Here, N is the size of the instance.)

Setting $d=1$ for all edges in dTSP & $D > 2^n$ makes the problem TSP.

TSP is Strongly NP-Complete. Thus unlikely to have sol. in Poly(N, ϕ)

Specific definitions of P_i up-above Seems unlikely that $\exists$ a Polytime solvable algorithm even smoothed.