CS6122 - Probabilistic and smoothed analysis of Algorithms

Lecture 25: Analysis of 2-OPT Algorithm

March 10, 2023

Lecturer: Raghavendra Rao, B. V. *Scribe: Roy, Akash*

Topics: Continuation of analysis of the 2-OPT algorithm.

1 Analysis of theorem for ϕ perturbed graph

Theorem 1.1: Longest Path Theorem

Expected length of the longest path in any one of 2-OPT configuration graph where 2-OPT is run on ϕ perturbed graph is bounded by $O(m^{1+\epsilon}n\phi)$ for ϕ perturbed graph with n vertices and m edges.

The runtime of our 2-OPT algorithm depends upon the longest path in the 2-OPT configuration tree by intuition. Our theorem 3.1 captures the expected size of such longest paths. This will upperbound our running time for the 2-OPT algorithm. We'll prove a weaker version of theorem 3.1 for ϕ perturbed graph with n vertices and m edges.

Proof. Suppose G is a fixed graph on n vertices and m edges.
Weight function $w : E \to [0, 1]$ be ϕ perturbed weights such that $w(e) \sim f_e : [0, 1] \to [0, \phi]$.
Consider any 2-edges e_1 and e_2. Suppose after performing a 2-exchange the cost of the tour reduces and the new edges are e_3 and e_4. $(e_i)_{i\in 1,2,3,4}$ creates a four-cycle.
Say the change in weight is $\Delta(e_1, \ldots, e_4)$. We define the following

$$\Delta(e_1, \ldots, e_4) = \sum_{i\in 1,2} e_i - \sum_{i\in 3,4} e_i$$

Let $\Delta_{\min}$ be minimum across all such 4-cycles $\Delta_{\min} = \min_{(e_1,\ldots,e_4),\, \Delta(e_1,\ldots,e_4)\geq 0} \Delta(e_1, \ldots, e_4)$. Then the running time of the algorithm is $O(\frac{n}{\Delta_{\min}})$.

Lemma 1.1: $\Delta_{\min}$ do not have large values

Δ_{min} do not have large values is captured by the following probability measure.

$$\forall\, \epsilon > 0\, \exists \Pr[\Delta \leq \epsilon] \leq m^2\epsilon\phi$$

The up-above lemma shows that the probability of gaining during an 2-OPT step at most ϵ is at most $m^2\epsilon\phi$.

Proof of Lemma 1.1: Consider a fixed 2-exchange $\{e_1, \dots, e_4\}$. Then from definition

$$\Delta(e_1, \dots, e_4) = \sum_{i \in 1,2} e_i - \sum_{i \in 3,4} e_i$$

Now

$$\begin{aligned}&\Pr[\Delta(e_1, \dots, e_4) \leq \epsilon] \\ &= \Pr[(w(e_1) + w(e_2) - w(e_3) - w(e_4)) \in [0, \epsilon]]\end{aligned}$$

We set edge weight $w(e_1) = x_1$, $w(e_2) = x_2$, $w(e_3) = x_3$, $w(e_4) = x_4$ where $(x_i)_{i \in 1,2,3,4}$ are random variables. Now we fix values for $(x_i)_{i \in 2,3,4}$. For this fixed values $\Pr\left[\sum_{i \in 1,2} x_i - \sum_{i \in 3,4} x_i\right] \in [0, \epsilon]$ is equivalent to $x_1 \in [k, k+\epsilon]$ for some k.

$$\Pr\left[\sum_{i \in 1,2} (x_i) - \sum_{i \in 3,4} \in [0, \epsilon]\right] \leq \Pr\left[x_1 \in [k, k+\epsilon]\right]$$

The graph is a ϕ perturbed instance so $\Pr[\Delta(e_1, \dots, e_4) \leq \epsilon] \leq \epsilon\phi$. Now Using union bound we can say

$$\begin{aligned}\Pr[\Delta_{\min} \leq \epsilon] &= \Pr\left[\exists\, (e_1, e_2, e_3, e_4),\ \Delta(e_1, e_2, e_3, e_4) \leq \epsilon\right] \\ &\leq \sum_{e_1, e_2, e_3, e_4} \Pr[\Delta(e_1, e_2, e_3, e_4) \leq \epsilon] \\ &\leq m^2 \epsilon \phi\end{aligned}$$

Proof of Theorem 1.1: Suppose diameter of the 2-OPT graph is T. Then the following is true from lemma 1.1

$$\begin{aligned}\Pr[T \geq t] &\leq \Pr\left[\Delta_{\min} \leq \frac{n}{t}\right] \\ &\leq \frac{m^2 n \phi}{t}\end{aligned}$$

Expected Value of the 2-OPT diameter length then becomes

$$
\begin{aligned}
\mathbb{E}[T] &= \sum_{t=1}^{n!} \Pr[T \geq t] \\
&= \sum_{t=1}^{n!} \frac{m^2 n \phi}{t} \\
&= m^2 n \phi * O(\lg n!) \\
&\leq m^2 n^2 \phi \lg n
\end{aligned}
$$

This analysis shows that the bound on the 2-OPT configuration graph diameter's expected length is polynomial, which determines the runtime of the 2-OPT algorithm.
We'll see a much better analysis of the 2-OPT algorithm that'll improve the bound and the running time in the next lecture.