CS6122 - Probabilistic and smoothed analysis of Algorithms

Lecture 26: Improved Bound on 2-OPT Algorithm

March 13, 2023

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Topics: Improved bound on the 2-OPT algorithm for TSP tour.

1 Improved Analysis

Last class we looked at the upper bound on the running time of 2-OPT algorithm is $m^2n^2\phi\lg n$. In this lecture we'll try to improve this bound. In order to do this we need the concept of linked 2 exchanges. The idea is we start with some tour, be it Christofides or Double tree, then we start 2-OPT exchanges. Now let's suppose $(S,\ldots,S_t)$ be any sequence of exchanges. Suppose S_i and S_j be any two exchanges in the sequence where S_i happend before S_j. We say (S_i,S_j) are linked if there exists some edge e_i added in S_i and got removed in S_j.

Lemma 1.1: Lower bound on the number of linked disjoint pairs

In any sequence of t many exchanges by 2-OPT algorithm there are at least $\frac{2t-n}{7}$ disjoint pairs that are linked.

Proof. Suppose $(S,\ldots,S_t)$ be any sequence of exchanges. Iterativly construct a set $\mathcal{L}$ of linked 2 exchanges. Now construct $\mathcal{L}'$ from $\mathcal{L}$ by removing intersecting paris. Now $\mathcal{L}'$ has only disjoint pairs. Disjoint pair $(S_i,S_j),(S_i',S_j')$ is such that $S_i\neq S_j',S_i'$ and $S_j\neq S_j',S_i'$.
Observation. Every pair (S_i,S_j) is disjoint from $\mathcal{L}$ except 6 pairs. So

$$
\begin{aligned}
|\mathcal{L}'| &\geq \frac{|\mathcal{L}|}{7}\\
&\geq \frac{2t-n}{7}\\
&\geq \frac{t}{4}
\end{aligned}
$$

Lemma 1.2: Estimation Improvement in 2-OPT

In ϕ purturbed graphs of n vertices and m edges, the probability that there exists a pair of linked 2 exchanges with improvements by at most ϵ is $O(m^3\epsilon^2\phi^2)$

Proof. Let (S,S') be a linked pair. S has the following edges $\{e_1,e_2,e_3,e_4\}$ and S' has the following edges $\{e_3,e_5,e_6,e_7\}$. Suppose in the first exchange edeges e_1,e_2 are being exchanged with e_3,e_4, in the second exchange edeges e_4,e_5, are exhanged with edges e_6,e_7. We need to

bound the probability of $\Delta(e_1, e_2, e_3, e_4) \leq \epsilon$ and $\Delta(e_3, e_5, e_6, e_7) \leq \epsilon$. We'll use the principle of deffered decision to bound the probability. Probability of $\Delta\{e_1, e_2, e_3, e_4\} \in (0, \epsilon)$ is equivalent to the probability of $\mathsf{dist}(e_1) \in (\kappa, \kappa + \epsilon)$ and $\Delta\{e_4, e_5, e_6, e_7\} \in (0, \epsilon)$ is same as the probability of $\mathsf{dist}(e_6) \in (\kappa_2 - \epsilon, \kappa_2)$. Here e_1 is the edge going out and e_6 is edge coming in and $\kappa = \mathsf{dist}(e_3) + \mathsf{dist}(e_4) - \mathsf{dist}(e_2)$ and $\kappa_2 = \mathsf{dist}(e_4) + \mathsf{dist}(e_5) - \mathsf{dist}(e_7)$.
$\mathsf{dist}(e_1)$ and $\mathsf{dist}(e_6)$ are random variables with densities bound from above by $\phi\epsilon$. This makes $\Pr[\mathsf{dist}(e_1) \in (0, \epsilon) \wedge \Pr[\mathsf{dist}(e_6)] \in (0, \epsilon)] \leq \phi^2\epsilon^2$. There are at most $O(m^3)$ number of linked 2-exchanges, so via union bound we can conclde the proof of lemma 1.2 that $\Pr[\Delta^* \leq \epsilon] \leq m^3\epsilon^2\phi^2$.
Based on the Estimation Improvement in 2-OPT lemma and Lower bound on the number of linked disjoint pairs lemma we can state the following theorem.

Theorem 1.1: Estimation Improvement in 2-OPT

In ϕ purturbed graphs of n vertices and m edges, excepted length of the longest path in the 2-OPT configuration graph is $O(m^{\frac{3}{2}}n\phi)$

Proof: Suppose τ be the longest path in the configuration graph of the 2-OPT algorithm. Let Δ^* be the smallest improvement during any pair of linked 2 exchanges.
If $T \geq t$ then there must be a sequence of t 2-exchanges in the configuration graph of the 2-OPT algorithm. From lemma 1.1 we can say that there is at least $z = \frac{2t-n}{7}$ many pairs. From lemma 1.1

$$\Pr[T \geq t] \leq \Pr\left[\Delta^* \leq \frac{4n}{t}\right] \leq O(m^3t^{-2}\phi^2n^2)$$

$$\begin{aligned}
\mathbb{E}[T] &= \sum_{t=1}^{n!} \Pr[T \geq t] \\
&\leq \sum_{t=1}^{n!} \min(1, c_1m^3t^{-2}\phi^2n^2) \\
&\leq \sum_{t=1}^{n\phi m^{3/2}} 1 + \sum_{t=n\phi m^{3/2}+1}^{n!} c_1m^3t^{-2}\phi^2n^2 \\
&\leq m^{3/2}n\phi + \int_{n\phi m^{3/2}+1}^{\infty} c_1m^3t^{-2}\phi^2n^2dt \\
&= m^{3/2}n\phi + \left[-c_1m^3n^2\phi\right]_{m^{3/2}n^2\phi}^{\infty} \\
&= O(m^{\frac{3}{2}}n\phi)
\end{aligned}$$