Electron-Positron Pair Creation in External Fields

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Abstract

In this project we investigate the phenomenon of creation of matter-antimatter pairs of particles, more precisely electron-positron pairs, out of the vacuum subject to strong external electromagnetic fields. Although this phenomenon was predicted already in 1929 and was observed in many experiments, its rigorous mathematical description still lies at the frontier of human understanding of nature. Dirac introduced the heuristic description of the vacuum of quantum electrodynamics (QED) as a homogeneous sea of particles. Although the picture of pair creation as lifting a particle out of the Dirac sea, leaving a hole in the sea, is very explanatory the mathematical formulation of the time evolution of the Dirac sea faces many mathematical challenges. A straightforward interaction of sea particles and the radiation field is ill-defined and physically important quantities such as the total charge current density are badly divergent due to the infinitely many occupied states in the sea. Nevertheless, in the last century physicists and mathematicians

have developed strong methods called "perturbative renormalisation theory" that allow at least to treat the scattering regime perturbatively. Non-perturbative methods have to be developed in order to give an adequate theoretical description of upcoming next-generation experiments allowing a study of the time evolution. Our endeavour focuses on the so-called *external field model of QED* in which one neglects the interaction between the sea particles and only allows an interaction with a prescribed electromagnetic field. It is the goal of this project to develop the necessary non-perturbative methods in this model to give a rigorous construction of the scattering and the time evolution operator.
Keywords: Second Quantised Dirac Equation, non-perturbative QED, external-field QED, Scattering Operator

Contents

Chapter 1

Introduction

Todo: Historische Einleitung durch Anfänge relativistischer Quantenphysik, Strahlungskatastrophe (unbounded below). No potentials resultat Lukas

Chapter 2

Direct Interaction in Relativistic Quantum Mechanics

As we have seen in the last chapter, having interaction mediated by potentials in a Dirac equation does not seem to be a viable option. One alternative approach to this problem is to reformulate Diracs equation as an integral equation and to introduce interaction afterwards. For the benefit of the unfamiliar reader, we will first follow the heuristic derivation of this type of equation in [6], then briefly review the mathematical results that had been established in the past and finally discuss new results for which the author is at least partially to blame.

2.1 Overview

2.1.1 Derivation

Todo: Check whether copying from [6] like this is fine.

We now follow the heuristic derivation of [6] for an equation for a multi-time wave function for two particles that expresses direct interaction along light-like configurations. This type of equation will then keep us occupied for the rest of this chapter. The derivatin is organised as follows: We start out reformulating Diracs equation for a single particle as an integral equation. The reformulated version is then extended to two particles in an Poincaré invariant manner. Extending the equation is conveniently done in the framework of multi-time wavefunctions.
Diracs equation for one particle subject to a potential V takes the form

$$i\partial_t \phi(t,\vec{x}) = \big(H^0 + V(t,\vec{x})\big)\phi(t,\vec{x}), \tag{2.1}$$

here ϕ denotes the wavefunction in question, $\vec{x} \in \mathbb{R}^3, t \in \mathbb{R}$ and H^0 is the Hamiltonian associated with a free Dirac particle. We denote by S^{ret} the retarded Green's function of the non interacting Dirac equation, that is S^{ret} satisfies

$$\big(i\partial_t - H^0\big)S^{\text{ret}} = \delta^4, \tag{2.2}$$

$$S^{\text{ret}}(t,\vec{x}) = 0 \quad \text{for } \ t < t_0. \tag{2.3}$$

Then inverting the differential operator $i\partial_t - H^0$ in (2.1) results in

$$\phi(t,\vec{x}) = \phi^0(t,\vec{x}) + \int_{t_0}^{\infty} dt' \int d^3\vec{x}'\, S^{\text{ret}}(t-t',\vec{x}-\vec{x}')V(t',\vec{x}')\phi(t',\vec{x}'), \tag{2.4}$$

where ϕ^0 is the solution of the non interacting equation subject to the initial condition $\phi^0(t_0) = \phi_0$. Equations (2.4) and (2.1) subject to $\phi(t_0) = \phi_0$ yield equivalent descriptions, as can be verified directly:

An action of $i\partial_t - H^0$ on (2.4) shows that a solution thereof also solves (2.1). Also the initial condition is fulfilled, as the integral term vanishes for $t = t_0$. Conversely equation (2.1) can be considered a free Dirac equation involving an inhomogeneous term of the form $V\phi$, whose solutions are known to be of the form (2.4). Executing the analogous procedure for the two particle Dirac equation

$$i\partial_t \phi(t, \vec{x}_1, \vec{x}_2) = \big(H_1^0 + H_2^0 + V(t, \vec{x}_1, \vec{x}_2)\big)\phi(t, \vec{x}_1, \vec{x}_2), \tag{2.5}$$

subject to the initial condition $\phi(t_0) = \phi_0$, results in the integral equation

$$\begin{aligned}\phi(t, \vec{x}_1, \vec{x}_2) = \phi^0(t, \vec{x}_1, \vec{x}_2) + \int_{t_0}^{\infty} dt' \int d^3\vec{x}_1'\, d^3\vec{x}_2'\, S_1^{\mathrm{ret}}(t - t', \vec{x}_1 - \vec{x}_1') \\ \times S_2^{\mathrm{ret}}(t - t', \vec{x}_2 - \vec{x}_2') V(t', \vec{x}_1', \vec{x}_2') \phi(t', \vec{x}_1', \vec{x}_2'),\end{aligned} \tag{2.6}$$

where $\phi^0(t)$ is a solution to the free Dirac equation for two particles subject to $\phi^0(t_0) = \phi_0$ and S_k^{ret} is the retarded Green's function of the free Dirac equation of particle number k. Here it is crucial to notice that The Green's function of the free two particle Dirac equation factorises into a product of two Green's functions of the Dirac equation for one particle.

Since equation (2.6) contains only one temporal variable, but six spatial ones, it is not obvious how it might be considered a relativistic equation at all. So we will the step from one to two particles, but before we do so let us first rewrite equation (2.4) in a more suggestive way:

$$\psi(x) = \psi^0(x) + \int d^4x'\, S^{\mathrm{ret}}(x - x') V(x') \psi(x'), \tag{2.7}$$

where non bold letters denote elements of Minkowski spacetime and we replaced ϕ by ψ in order to make a visible switch to relativistic notation. Furthermore we replaced the lower bound in the temporal

integral domain by $-\infty$ in order to render the total domain of integral Poincaré invariant.
Equation (2.7) immediately inspres us to write down the following generalisation

$$\begin{aligned}\psi(x_1, x_2) =& \psi^0(x_1, x_2) \\ &+ \int d^4x_1'\, d^4x_2'\, S_1^{\mathrm{ret}}(x_1 - x_1')S_2^{\mathrm{ret}}(x_2 - x_2')K(x_1', x_2')\psi(x_1', x_2').\end{aligned} \tag{2.8}$$

we integrate over all of $\mathbb{R}^8$ and ψ^0 a solution of the free Dirac equation both in x_1 and x_2 and their respective spinor indices:

$$D_1\psi^0(x_1, x_2) = 0, \tag{2.9}$$
$$D_2\psi^0(x_1, x_2) = 0, \tag{2.10}$$

For the object K, called the "interaction kernel", the optimal choice is not yet known. However, for (2.8) to be Poincaré invariant, it should be invariant itself. A simple way to ensure this is to let it only depend directly the squared Minkowski distance $(x_1 - x_2)^2$. A choice that shows some resemblence of Wheeler-Feynman electrodynamics is

$$K(x_1, x_2) = i\, \frac{e_1 e_2}{4\pi} \gamma_1^\mu \gamma_{2,\mu}\, \delta((x_1 - x_2)^2). \tag{2.11}$$

In an equation incorporating (2.11) the interaction between the particles happens along light-like distances. The constant in front of (2.11) is fixed by the non-relativistic limit, recovering an equation very much like the Breit equation, see [8, section 3.6].
Summarising this results in the equation

$$\begin{aligned}\psi(x_1, x_2) = \psi^0(x_1, x_2) + i\, \frac{e_1 e_2}{4\pi} \int d^4x_1'\, d^4x_2' \\ \times S_1^{\mathrm{ret}}(x_1 - x_1')S_2^{\mathrm{ret}}(x_2 - x_2')\, \gamma_1^\mu \gamma_{2,\mu} \delta((x_1' - x_2')^2)\psi(x_1', x_2').\end{aligned} \tag{2.12}$$

Despite the fact that the motivation for (2.12) holds for Dirac particles, it is also conceivable to replace ψ, S^{ret} and all the constant factor by quantities related to the Klein Gordon equation and arrive at

$$\begin{aligned}\psi(x_1, x_2) = \psi^0(x_1, x_2) + \lambda \int d^4x_1'\, d^4x_2' \\ \times\, G_1^{\text{ret}}(x_1 - x_1')G_2^{\text{ret}}(x_2 - x_2')\, \delta((x_1' - x_2')^2)\psi(x_1', x_2'),\end{aligned} \tag{2.13}$$

where ψ and $\psi 0$ are no longer spinor valued, ψ^0 is a solution to the free Klein Gordone equation in both x_1 and x_2,

$$(\Box_{x_1} + m_1^2)\psi = 0 \tag{2.14}$$
$$(\Box_{x_2} + m_2^2)\psi = 0 \tag{2.15}$$

and G^{ret} is the retarded Green's function of the Klein Gordon equation. In fact, most of the rigorous results about equations of a similar type as the ones motivated in this chapter are about the Klein Gordon version (2.13).

2.1.2 Previous Results on Directly Interacting Particles

In this chapter we summarise the most important existence results on equations of the type of (2.8). Because this line of work is still fairely young, it can still readily be summarised. The results are taken from [8] and [9]. The theorems are about slightly different versions of equations of the type of (2.13). I tried to contain the necessary notation to within each of the theorems. I only cited the theorems that are about a four dimensional spacetime, there are also results about three and two dimensional spacetime, the interested reader is refered to [6, 9]. The versions of equation (2.13) in the theorems are considerably modified:

- The spacetime of equation (2.13) is $\mathbb{R}^4$, Minkowski spacetime, whilst the spacetime of all the theorems in flat spacetime is $\mathbb{R}^+ \times \mathbb{R}^3 =: \frac{1}{2}\mathbb{M}$. That is, there is a beginning in time. This modification has technical reasons. However, as current cosmological models of our universe do have a beginning in time this modification does not necessarily mean that the equation can no longer describe certain aspects of physics. As these cosmological models have nonzero curature the authors of [9] have shown existence of solutions of versions of equation (2.13) on Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime. In section 2.2 and 2.3 we will also employ this simplification and show existence on FLRW spacetime. This is not an attempt to treat general curved spacetimes, it is done as an act of consistency. We introduce a beginning in time and try to justify this by cosmological arguments and hence we treat a spacetime commonly used in cosmology.

Todo: really also in the second paper?

- The interaction kernel K which we motivated to be proportional to $\delta((x_1 - x_2)^2)$ is replaced by various less singular objects. This modification is purely technical and we do not justify it. The previous results approach the singular K introduced in the last section to different degrees. In section 2.2 where we treat Dirac particles we will use a rather soft interaction kernel. The new result about Klein Gordon particles in section 2.3 employs the fully singular $\delta((x_1 - x_2)^2)$ kernel.

Theorem 1 (Thm 3.4 ($d = 3$) of [8])**.** *Let $T > 0, \lambda \in \mathbb{C}$, for every bounded $K : \mathbb{R}^8 \to \mathbb{C}$ and every $\psi^0 \in \mathcal{B}_3 = L^\infty([0,T]^2, L^2(\mathbb{R}^6))$ the equation*

$$\psi(t_1, \vec{x}_1, t_2, \vec{x}_2) = \psi^0(t_1, \vec{x}_1, t_2, \vec{x}_2) + \frac{\lambda}{(4\pi)^2} \int d\vec{x}_1' d\vec{x}_2'$$
$$\times \frac{H(t_1 - |\vec{x}_1 - \vec{x}_1'|)}{|\vec{x}_1 - \vec{x}_1'|} \frac{H(t_2 - |\vec{x}_2 - \vec{x}_2'|)}{|\vec{x}_2 - \vec{x}_2'|} K(t_1 - |\vec{x}_1 - \vec{x}_1'|, \vec{x}_1', t_2 - |\vec{x}_2 - \vec{x}_2'|, \vec{x}_2')$$
$$\times \psi(t_1 - |\vec{x}_1 - \vec{x}_1'|, \vec{x}_1', t_2 - |\vec{x}_2 - \vec{x}_2'|, \vec{x}_2')$$

has a unique solution $\psi \in \mathcal{B}_3$.

Theorem 2 (Thm 3.5 of [8]). *Let* $T > 0, \lambda \in \mathbb{C}$, *for every bounded* $f : \mathbb{R}^8 \to \mathbb{C}$ *and every* $\psi^0 \in \mathcal{B}_3$ *the equation*

$$\psi(t_1, \vec{x}_1, t_2, \vec{x}_2) = \psi^0(t_1, \vec{x}_1, t_2, \vec{x}_2) + \frac{\lambda}{(4\pi)^2} \int d^3\vec{x}_1' d^3\vec{x}_2'$$
$$\times \frac{H(t_1 - |\vec{x}_1 - \vec{x}_1'|)}{|\vec{x}_1 - \vec{x}_1'|} \frac{H(t_2 - |\vec{x}_2 - \vec{x}_2'|)}{|\vec{x}_2 - \vec{x}_2'|} \frac{f(t_1 - |\vec{x}_1 - \vec{x}_1'|, \vec{x}_1', t_2 - |\vec{x}_2 - \vec{x}_2'|, \vec{x}_2')}{|\vec{x}_1' - \vec{x}_1|}$$
$$\times \psi(t_1 - |\vec{x}_1 - \vec{x}_1'|, \vec{x}_1', t_2 - |\vec{x}_2 - \vec{x}_2'|, \vec{x}_2'),$$

has a unique solution $\psi \in \mathcal{B}_3$.

The next results will be about FLRW spacetime. We have to introduce some notation before we can present them. Here we cite [7, sec 3.2].

Todo: entscheide ob man hier vielleicht Matt+mein KG paper zitieren kann, dann muss man nicht umformulieren.

2.2 Directly Interacting Dirac Particles

2.3 KG-Particles Interacting Directly Along Lightcones

Chapter 3

Quantum Field Theoretic Approach to Interactions

Todo: quick introduction into Hadamard stuff and Dirk & Franz stuff, erwähne Ruijsnaars stuff

rewrite!

In order to be able to state our main conjecture (**??**) precisely I will need to introduce the one-particle dynamics for electrons and their scattering operator, see section 3.4.1 below, and then move on to the second quantised dynamics and its corresponding scattering operator in section 3.4.2. Second quantization is the canonical method of turning a one-particle theory into one of an arbitrary and possibly changing number of particles. The informal series expansion of the one-particle scattering operator U is derived from Dirac's equation of motion for the electron. In section **??** the convergence of this expansion is shown. The informal expansion of the second quantized scattering operator S is then derived from U by second quantisation in section 3.4.2. At this point I have gathered enough tools to present the main conjecture **??** in section **??**. After the main conjecture is known, I present several of my own results in sections **??**, **??** and **??** about the first order, the second order and all other odd orders. These results give an intuition on how to control the convergence of the informal expansion of the scattering operator S.

3.1 The Relationship Between Hadamard States, the Fermionic Projector, and Admissible Polarisation Classes

insert QFT communications paper, once ready

necessary: more thorough introduction to Franz and Dirk stuff for handling the geometry part

3.2 Analyticity of the One Particle Scattering Operator

In this section we analyse the construction of the one particle scattering operator S_A carried out in [1] and answer the question whether operators like

$$P^+ \partial_B S_A^* S_{A+B} P^- \tag{3.1}$$

are Hilbert-Schmidt operators. This will turn out to be important for the geometric construction carried out in chapter 3.3.
Since this section is heavily inspired from [1], we need to introduce some notation from this paper.

Definition 3. *We define the set $\mathcal{V}$ of four potentials*

$$\mathcal{V} := C_c^\infty(\mathbb{R}^4, \mathbb{R}^4). \tag{3.2}$$

Let $A \in \mathcal{V}$, we define the the integral operator $Q^A : \mathcal{H} \circlearrowleft$ by giving its integral kernel, which is also denoted by Q^A:

$$\mathbb{R}^3 \times \mathbb{R}^3 \ni (p,q) \mapsto Q^A(p,q) := \frac{Z^A_{+-}(p,q) - Z^A_{-+}(p,q)}{i(E(p)+E(q))} \tag{3.3}$$

$$\text{with } Z^A_{\pm\mp}(p,q) := P_\pm(p) Z^A(p-q) P_\mp(q), \tag{3.4}$$

$$Z^A = -ie\gamma^0\gamma^\alpha \hat{A}_\alpha, \tag{3.5}$$

$$\hat{A}_\mu(t) : \mathcal{H} \circlearrowleft, \hat{A}_\mu(t)\psi := \hat{A}_\mu(t) * \psi, \tag{3.6}$$

$$\hat{A}_\mu := \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} A_\mu(x) e^{-ipx} d^3x, \tag{3.7}$$

$$\text{and } E(p) := \sqrt{m^2 + |p|^2}. \tag{3.8}$$

We introduce the standard polarisation for the free Dirac equation

think of some more intuitive notation for the projector.

$$P^- := 1_{\mathrm{spec}(H^0)<0}, \quad P^+ = 1 - P^-. \tag{3.9}$$

Lemma 4. *For general $A, F \in \mathcal{V}$ we have the well known equations for the one-particle time evolution operators*

$$U^A(t_1,t_0) = U^0(t_1,t_0) + \int_{t_0}^{t_1} dt U^0(t_1,t) Z^A(t) U^A(t,t_0) \tag{3.10}$$

$$U^{A+F}(t_1,t_0) = U^A(t_1,t_0) + \int_{t_0}^{t_1} dt U^A(t_1,t) Z^F(t) U^{A+F}(t,t_0). \tag{3.11}$$

Proof. … □

Theorem 5 (Analyticity of S)**.** *Let $n \in \mathbb{N}$, $A, H_k \in \mathcal{V}$ for $k \leqslant n$, pick t_1 after $\operatorname{supp} A \cup \bigcup_{k\leqslant n} \operatorname{supp} H_k$ and t_0 before $\operatorname{supp} A \cup \bigcup_{k\leqslant n} \operatorname{supp} H_k$ then*

think about generalisation to higher derivatives

$$\partial_{H_1} U^{A+H_1}(t_1,t_0) = \int_{t_0}^{t_1} dt\ U^A(t_1,t) Z^{H_1}(t) U^A(t,t_0) \tag{3.12}$$

holds with respect to the topology induced by the norm

$$T : \mathcal{H} \circlearrowleft \|T\|_{\mathrm{op}+I_2} = \|T\| + \|P^+TP^-\|_{I_2} + \|P^-TP^+\|_{I_2}, \tag{3.13}$$

where $\|\cdot\|$ *is the operator norm and* $\|\cdot\|_{I_2}$ *is the Hilbert-Schmidt norm. Additionally the derivatives*

$$\partial_{H_1} \dots \partial_{H_k} U^{A+\sum_{b=1}^{k} F_b}(t_1, t_0) \tag{3.14}$$

all exist with respect to the topology just mentioned. Furthermore let

how well does cyclicity of the trace hold for operators?

$\in I_2(\mathcal{F})$ *then the following formulas hold:*

$$\begin{aligned} &\partial_{H_1} \operatorname{tr}(T_1 P^+ U^A(t_0, t_1) U^{A+H_1}(t_1, t_0) P^-) \\ &= \operatorname{tr}(T_1 P^+ U^A(t_0, t_1) \partial_{H_1} U^{A+H_1}(t_1, t_0) P^-) \end{aligned} \tag{3.15}$$

$$\begin{aligned} &\partial_{H_1} \operatorname{tr}(T_1 P^+ U^A(t_0, t_1) U^{A+H_1}(t_1, t_0) P^+) \\ &= \operatorname{tr}(T_1 P^- U^A(t_0, t_1) \partial_{H_1} U^{A+H_1}(t_1, t_0) P^+). \end{aligned} \tag{3.16}$$

Proof. Following [1] we will throughout this proof make use of the following shorthand notation. For operator valued maps $T_1, T_2 : \mathbb{R}^2 \to (\mathcal{H} \circlearrowleft)$ we define for $t_1, t_0 \in \mathbb{R}$

$$T_1 T_2 = \int_{t_0}^{t_1} dt \; T_1(t_1, t) T_2(t, t_0), \tag{3.17}$$

as a map of the same type as T_1 and T_2 whenever this is well defined. Furthermore for operator valued functions $C_1, C_2 : \mathbb{R} \to (\mathcal{H} \circlearrowleft)$ we define

$$T_1 C_1 = T_1(t_1, t_0) C_1(t_0) \tag{3.18}$$

$$C_1 T_1 = C(t_1) T_1(t_1, t_0) \tag{3.19}$$

$$C_1 C_2 = C_1(t_1), \tag{3.20}$$

as maps of the same type as T_1, T_1 and C_1 respectively. Pick$n \in \mathbb{N}$, $A, H_b \in \mathcal{V}$ for $b \leqslant k$ and t_1, t_0 as in the theorem. Furhtermore we introduce the integral operator $Q'^A : \mathcal{H} \circlearrowleft$ by it's kernel

$$\mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \ni (t, p, q) \mapsto Q'^A(t, p, q) = \partial_t Q^A(t, p, q), \tag{3.21}$$

where the time dependence is due to the time dependence of the four-potential A. The following notion of even and odd part of an arbitrary bounded linear operator $T : \mathcal{F} \circlearrowleft$ on Fock space will come in handy:

$$T_{\text{odd}} := P^+TP^- + P^-TP^+T_{\text{ev}} := P^+TP^+ + P^-TP^-. \tag{3.22}$$

We abbreviate

$$H := \sum_{b=1}^{k} H_b, \quad B := A + H \tag{3.23}$$

Because of the choice of t_1, t_0 we have

$$R^{A+H}(t_1, t_0) := [(1 - Q^{A+H})U^{A+H}(1 + Q^{A+H})](t_1, t_0) = U^{A+H}(t_1, t_0), \tag{3.24}$$

because $A + H = 0$ both at t_1 and t_0.
So it suffices to study the family of operators R. From the analysis carried out in [1] we know that R^B for $B \in \mathcal{V}$ is the limit in the sense of the operator norm of the sequence

$$R_0^B := 0, \quad R_{n+1}^B := U^0 F^B R_n^B + U^0 + G^B, \tag{3.25}$$

where F and G are given By

$$F^B := (-Q'^B + Z_{\text{ev}}^B - Q^B Z^B)(1 + Q^B), \tag{3.26}$$

$$G^B := -U^0(1 - Q^B Q^B) + U^0(-Q'^B \tag{3.27}$$

$$+Z_{\text{ev}} - Q^B Z^B) Q^B Q^B U^B (1 + Q^B). \tag{3.28}$$

Lemma 6. *We are going to prove that for $n \in \mathbb{N}$ and*

$$R_n^A \xrightarrow[n\to\infty]{\|\cdot\|_{\mathrm{op}+I_2}} R^A \tag{3.29}$$

$$\partial_F R_n^{A+F} \xrightarrow[n\to\infty]{\|\cdot\|_{\mathrm{op}+I_2}} \partial_F R^{A+F} \tag{3.30}$$

holds.

Proof. First we introduce the auxiliary norms for operators T and W depending on one and two scalar variables respectively.

$$\|T\|_{\mathrm{op}+I_2,\gamma} := \sup_{t\in[t_1,t_0]} e^{-\gamma(t-t_0)} \left(\|T(t)\| + \|T_{\mathrm{odd}}(t)\|_{I_2}\right) \tag{3.31}$$

$$\|T\|_{\mathrm{op},\gamma} := \sup_{t\in[t_1,t_0]} e^{-\gamma(t-t_0)} \|T(t)\| \tag{3.32}$$

$$\|W\|_{\mathrm{op},\infty} := \sup_{t,t'\in[t_1,t_0]} \|W(t,t')\| \tag{3.33}$$

$$\|T\|_{I_2,\gamma} := \sup_{t\in[t_1,t_0]} e^{-\gamma(t-t_0)} \|T(t)\|_{I_2}, \tag{3.34}$$

$$\|W\|_{I_2,\infty} := \sup_{t,t'\in[t_1,t_0]} \|W(t,t')\|_{I_2}, \tag{3.35}$$

for some $\gamma \geqslant 0$.
From the analysis of [1] follows that the recursive equation [1, (3.42)]

$$R_n^B = U^0 F_{\mathrm{ev}}^B R_{n-1}^B + U^0 F_{\mathrm{odd}}^B U^0 F^B R_{n-2}^B \tag{3.36}$$

$$+U^0 F_{\mathrm{odd}}^B + U^0 F_{\mathrm{odd}}^B U^0 + U^0 + G^B \tag{3.37}$$

is fulfilled by the same sequence of operators for $n \geqslant 2$. Hence we have for $n \geqslant 2$:

$$\begin{aligned} R^B - R_n^B &= U^0 F_{\mathrm{ev}}^B (R^B - R_{n-1}^B) \\ &\quad + U^0 F_{\mathrm{odd}}^B U^0 F^B (R^B - R_{n-2}^B). \end{aligned} \tag{3.38}$$

Abbreviating $U^0 F^B_{\mathrm{ev}} =: a, U^0 F^B_{\mathrm{odd}} U^0 F^B := b$, we obtain

$$\begin{aligned} R^B - R^B_n &= a(R^B - R^B_{n-1}) \\ &\quad + b(R^B - R^B_{n-2}). \end{aligned} \tag{3.39}$$

Furthermore, we introduce the notation for $k \in \mathbb{N}$

$$\forall k \in \mathbb{N}[k] := \{l \in \mathbb{R} \mid l \leqslant k\} \tag{3.40}$$

$$\forall u \subseteq [k] : \partial_u := \prod_{k \in u} \partial_{H_k}, \tag{3.41}$$

$$R^B - R^B_n := \Delta^n, \tag{3.42}$$

where the product of derivatives is to be understood as the mixed derivative with respect to all the factors. Now we estimate for some set $u \subset [k]$

$$\|\partial_u \Delta^n_{\mathrm{odd}}\|_{I_2} \leqslant \sup_{t \in [t_0, t_1]} e^{-\gamma(t-t_0)} \left\| \sum_{w \subseteq u} (\partial_{u \setminus w} a \ \partial_w \Delta^{n-1}_{\mathrm{odd}})(t, t_0) \right\|_{I_2} \tag{3.43}$$

$$+ \sup_{t \in [t_0, t_1]} e^{-\gamma(t-t_0)} \left\| P^+ \sum_{w \subseteq u} (\partial_{u \setminus w} b \ \partial_w \Delta^{n-2})(t, t_0) P^- \right\|_{I_2} \tag{3.44}$$

$$+ \sup_{t \in [t_0, t_1]} e^{-\gamma(t-t_0)} \left\| P^- \sum_{w \subseteq u} (\partial_{u \setminus w} b \ \partial_w \Delta^{n-2})(t, t_0) P^+ \right\|_{I_2} \tag{3.45}$$

$$\leqslant \sup_{t \in [t_0, t_1]} e^{-\gamma(t-t_0)} \sum_{w \subseteq u} \|\partial_{u \setminus w} (a \ \partial_w \Delta^{n-1}_{\mathrm{odd}})(t, t_0)\|_{I_2} \tag{3.46}$$

$$+2 \sup_{t \in [t_0, t_1]} e^{-\gamma(t-t_0)} \sum_{w \subseteq u} \|(\partial_{u \setminus w} b \ \partial_w \Delta^{n-2})(t, t_0)\|_{I_2} \tag{3.47}$$

$$\leqslant \sup_{t \in [t_0, t_1]} e^{-\gamma(t-t_0)} \sum_{w \subseteq u} \int_{t_0}^{t} dt' \|\partial_{u \setminus w} a(t, t') \ \partial_w \Delta^{n-1}_{\mathrm{odd}}(t', t_0)\|_{I_2} \tag{3.48}$$

$$+2 \sup_{t\in[t_0,t_1]} e^{-\gamma(t-t_0)} \sum_{w\subseteq u} t_{t_0}^{t} dt' \|\partial_{u\backslash w} b(t,t')\ \partial_w \Delta^{n-2}(t',t_0)\|_{I_2} \tag{3.49}$$

$$\leqslant \sup_{t\in[t_0,t_1]} e^{-\gamma(t-t_0)} \sum_{w\subseteq u} \int_{t_0}^{t} dt' \|\partial_{u\backslash w} a\|_{\mathrm{op},\infty} \|\partial_w \Delta^{n-1}_{\mathrm{odd}}(t',t_0)\|_{I_2} \tag{3.50}$$

$$+2 \sup_{t\in[t_0,t_1]} e^{-\gamma(t-t_0)} \sum_{w\subseteq u} \int_{t_0}^{t} dt' \|\partial_{u\backslash w} b(t,t')\|_{I_2} \|\partial_w \Delta^{n-2}(t',t_0)\|_{\mathrm{op}} \tag{3.51}$$

$$\leqslant \sup_{t\in[t_0,t_1]} e^{-\gamma t} \sum_{w\subseteq u} \int_{t_0}^{t} dt' e^{\gamma t'} \|\partial_{u\backslash w} a\|_{\mathrm{op},\infty} \|\partial_w \Delta^{n-1}_{\mathrm{odd}}(\cdot,t_0)\|_{I_2,\gamma} \tag{3.52}$$

$$+2 \sup_{t\in[t_0,t_1]} e^{-\gamma t} \sum_{w\subseteq u} \int_{t_0}^{t} dt' e^{\gamma t'} \|\partial_{u\backslash w} b\|_{I_2,\infty} \|\partial_w \Delta^{n-2}(t',t_0)\|_{\mathrm{op},\gamma} \tag{3.53}$$

$$\leqslant \frac{1}{\gamma} \sum_{w\subseteq u} \|\partial_{u\backslash w} a\|_{\mathrm{op},\infty} \|\partial_w \Delta^{n-1}_{\mathrm{odd}}(\cdot,t_0)\|_{I_2,\gamma} \tag{3.54}$$

$$+\frac{2}{\gamma} \sup_{t\in[t_0,t_1]} \sum_{w\subseteq u} \|\partial_{u\backslash w} b\|_{I_2,\infty} \|\partial_w \Delta^{n-2}(t',t_0)\|_{\mathrm{op},\gamma} \tag{3.55}$$

$$\tag{3.56}$$

, this can be brought into the vectorised form:

$$\begin{pmatrix} R^B - R^B_n \\ R^B - R^B_{n-1} \end{pmatrix} = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix} \begin{pmatrix} R^B - R^B_{n-1} \\ R^B - R^B_{n-2} \end{pmatrix}, \tag{3.57}$$

which by induction yields

$$\begin{pmatrix} R^B - R^B_n \\ R^B - R^B_{n-1} \end{pmatrix} = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}^{n-1} \begin{pmatrix} R^B - R^B_1 \\ R^B \end{pmatrix}. \tag{3.58}$$

Taking the derivative of (3.39) we find

$$\begin{aligned} \partial_H(R^{A+F} - R^{A+H}_n) = U^0 F^A_{\mathrm{ev}} \partial_H (R^{A+H} - R^{A+H}_{n-1}) \\ +U^0 F^A_{\mathrm{odd}} U^0 F^A \partial_H (R^{A+H} - R^{A+H}_{n-2}) + \partial_H U^0 F^{A+H}_{\mathrm{ev}} (R^A - R^A_{n-1}) \\ +\partial_H U^0 F^{A+H}_{\mathrm{odd}} U^0 F^{A+H} (R^A - R^A_{n-2}). \end{aligned} \tag{3.59}$$

Using the furher abbreviations

$$c := \partial_H U^0 F_{\mathrm{ev}}^{A+H} \tag{3.60}$$
$$d := \partial_H U^0 F_{\mathrm{odd}}^{A+H} U^0 F^{A+H} \tag{3.61}$$
$$T_n := R^A - R_n^A \tag{3.62}$$
$$\dot{T}_n := \partial_H (R^{A+H} - R_n^{A+H}), \tag{3.63}$$

we vectorise the recursive equation and arrive at

$$\begin{pmatrix} \dot{T}_n \\ \dot{T}_{n-1} \end{pmatrix} = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \dot{T}_{n-1} \\ \dot{T}_{n-2} \end{pmatrix} + \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} T_{n-1} \\ T_{n-2} \end{pmatrix}. \tag{3.64}$$

This yields by induction

$$\begin{pmatrix} \dot{T}_n \\ \dot{T}_{n-1} \end{pmatrix} = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}^{n-1} \begin{pmatrix} \dot{T}_1 \\ \dot{T}_0 \end{pmatrix} \tag{3.65}$$
$$+ \sum_{k=0}^{n-2} \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}^{k} \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} T_{n-1-k} \\ T_{n-2-k} \end{pmatrix}. \tag{3.66}$$

Inserting (3.57) then gives

$$\begin{pmatrix} \dot{T}_n \\ \dot{T}_{n-1} \end{pmatrix} = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}^{n-1} \begin{pmatrix} \dot{T}_1 \\ \dot{T}_0 \end{pmatrix} \tag{3.67}$$
$$+ \sum_{k=0}^{n-2} \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}^{k} \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}^{n-2-k} \begin{pmatrix} T_1 \\ T_0 \end{pmatrix}. \tag{3.68}$$

□

□

3.3 Geometric Construction of the Phase

Next we introduce the set of four potentials we work with, as well as the argument of a complex number and an invertible bounded operator. For complex numbers the convention we chose here differs slightly from the standard in the literature, which is why we also use a slightly non standard name for this function.

Definition 7 (polar decomposition and spectral projections)**.** *We denote by $\mathcal{H} = L^2(\mathbb{R}^3, \mathbb{C})$. For $X : \mathcal{H} \to \mathcal{H}$ bounded*

$$\mathrm{AG}(X) := X|X|^{-1}. \tag{3.69}$$

Furthermore, we define for any complex number $z \in \mathbb{C}\backslash\{0\}$

$$\mathrm{ag}(z) := \frac{z}{|z|}. \tag{3.70}$$

In abuse of notation we will define the expression

$$\partial_t \ln f(t) := \frac{\partial_t f(t)}{f(t)}, \tag{3.71}$$

for any differentiable $f : \mathbb{R} \to \mathbb{C}\backslash\{0\}$, even if the expression $\ln f(t)$ cannot be interpreted as the principle branch of the logarithm.
We also introduce $S^1 := \{z \in \mathbb{C} \mid |z| = 1\}$.

Definition 8 (scattering operator and phases)**.** *We define for all $A, B \in \mathcal{V}$*

$$S_{A,B} := U^A_{\Sigma_{\mathrm{in}},\Sigma_{\mathrm{out}}} U^B_{\Sigma_{\mathrm{out}},\Sigma_{\mathrm{in}}}, \tag{3.72}$$

vllt direkt mit S_A definieren, warte ab bis abschätzung der ableitungen geschehen ist

where Σ_{out} and Σ_{int} are Cauchy-surfaces of Minkowski spacetime such that

$$\forall (x, y) \in \operatorname{supp} A \cup \operatorname{supp} B \times \Sigma_{\mathrm{in}} : (x-y)^2 \geqslant 0 \Rightarrow x^0 > y^0, \tag{3.73}$$

$$\forall (x, y) \in \operatorname{supp} A \cup \operatorname{supp} B \times \Sigma_{\mathrm{out}} : (x-y)^2 \geqslant 0 \Rightarrow x^0 < y^0 \tag{3.74}$$

holds. Let

$$\mathrm{dm} := \{(A,B) \in \mathcal{V}^2 \mid P^- S_{A,B} P^- \textit{ and} \tag{3.75}$$
$$P^- S_{B,A} P^- : \mathcal{H}^- \circlearrowleft \textit{ are invertible}\}, \tag{3.76}$$

we define

$$\operatorname{dom} \overline{S} := \{(A,B) \in \mathrm{dm} \mid \overline{A\ B} \times \overline{A\ B} \subseteq \mathrm{dm}\}, \tag{3.77}$$

where $\overline{A\ B}$ *is the line segment connecting* A *and* B *in* $\mathcal{V}$*. Furthermore, we choose for all* $A, B \in \operatorname{dom} \overline{S}$ *the lift*

$$\overline{S}_{A,B} = \mathcal{R}_{\mathrm{AG}((P^- S_{A,B} P^-)^{-1})} \mathcal{L}_{S_{A,B}}. \tag{3.78}$$

For $(A,B),(B,C),(C,A) \in \operatorname{dom} \overline{S}$*, we define the complex numbers*

$$\gamma_{A,B,C} := \mathrm{ag}(\det_{\mathcal{H}^-}(P^- S_{A,B} P^- S_{B,C} P^- S_{C,A} P^-)), \tag{3.79}$$
$$\Gamma_{A,B,C} := \mathrm{ag}(\gamma_{A,B,C}). \tag{3.80}$$

We will see in lemma 13 that $\gamma_{A,B,C} \neq 0$ *and* $P^- S_{A,B} P^- S_{B,C} P^- S_{C,A} - 1$ *is traceclass, so that* $\Gamma_{A,B,C}$ *is well-defined. Lastly we introduce the partial derivative in the direction of any four-potential* F *by*

$$\partial_F T(F) := \partial_\varepsilon T(\varepsilon F)|_{\varepsilon=0} \tag{3.81}$$

and for $A, B, C \in \mathcal{V}$ *the function*

$$c_A(F,G) := -i \partial_F \partial_G \Im \operatorname{tr}[P^- S_{A,A+F} P^+ S_{A,A+G} P^-]. \tag{3.82}$$

Lemma 9 (properties of $\operatorname{dom} \overline{S}$)**.** *The set* $\operatorname{dom} \overline{S}$ *has the following properties:*

1. *contains the diagonal:* $\{(A,A) \mid A \in \mathcal{V}\} \subseteq \operatorname{dom} \overline{S}$*.*

2. *openness:* $\forall n \in \mathbb{N} : \left\{ s \in \mathbb{R}^{2n} \mid \left(\sum_{k=1}^{n} s_k A_k, \sum_{k=n+1}^{2n} s_k A_k \right) \in \operatorname{dom} \overline{S} \right\}$ *is an open subset of* $\mathbb{R}^{2n}$ *for all* $A_1, \ldots A_{2n} \in \mathcal{V}$.

3. *symmetry:* $(A, A') \in \operatorname{dom} \overline{S} \iff (A', A) \in \operatorname{dom} \overline{S}$

4. *star-shaped:* $(A, tA) \in \operatorname{dom} \overline{S} \Rightarrow \forall s \in \overline{1 \quad t} : (A, sA) \in \operatorname{dom} \overline{S}$

5. *well defined-ness of* $\overline{S}$*:* $\operatorname{dom} \overline{S} \subseteq \{A, B \in \mathcal{V} \mid P^- S_{A,B} P^- : \mathcal{H}^- \circlearrowleft$ *is invertible*$\}$.

We will only prove openness, as the other properties follow directly from the definition (3.77)*. So pick* $n \in \mathbb{N}$*,* $A_i \in \mathcal{V}$ *for* $i \in \mathbb{N}, i \leqslant 2n$ *and* $s \in \mathbb{R}^{2n}$ *such that* $\left(\sum_{k=1}^{n} s_k A_k, \sum_{k=n+1}^{2n} s_k A_k \right) \in \operatorname{dom} \overline{S}$*. We have to find a neighbourhood* $U \subseteq \mathbb{R}^{2n}$ *of* s *such that* $\{ \left(\sum_{k=1}^{n} s'_k A_k, \sum_{k=n+1}^{2n} s'_k A_k \right) \mid s' \in U \} \subseteq \operatorname{dom} \overline{S}$ *holds. In doing so we have to ensure that the square*

$$\overline{\sum_{k=1}^{n} s_k A_k \qquad \sum_{k=n+1}^{2n} s_k A_k}^{\,2} \tag{3.83}$$

stays a subsets of dm *for all* $s' \in U$*. Now pick a metric* d *on* $\mathbb{R}^n$ *and define*

$$r := \inf \left\{ d(s, s') \mid \overline{\sum_{k=1}^{n} s'_k A_k \qquad \sum_{k=n+1}^{2n} s'_k A_k}^{\,2} \cap \operatorname{dm}^c \neq \varnothing \right\}.$$

It cannot be the case that $r = 0$*, because the metric is continuous, the square compact in* $\mathbb{R}^{2n}$ *and the set of invertible bounded operators (defining* dm*) is open in the topology generated by the operator norm. If* $r = \infty$ *then* $U = \mathbb{R}^{2n}$ *will suffice. If* $r \in \mathbb{R}^+$ *then* $U = B_r(s, t)$ *the open ball of radius* r *around* s *works.*

reicht das, oder soll ich das ausführlicher schreiben?

3.3.1 Main Result of Construction

Definition 10 (causal splitting). *We define a causal splitting as a function*

$$c^+ : \mathcal{V}^3 \to \mathbb{C}, \tag{3.84}$$
$$(A, F, G) \mapsto c^+_A(F, G), \tag{3.85}$$

such that c^+ restricted to any finite dimensional subspace is smooth in the first argument and linear in the second and third argument. Furthermore c^+ should satisfy

$$c_A(F, G) = c^+_A(F, G) - c^+_A(G, F), \tag{3.86}$$
$$\partial_H c^+_{A+H}(F, G) = \partial_G c^+_{A+G}(F, H), \tag{3.87}$$
$$\forall F \prec G : c^+_A(F, G) = 0. \tag{3.88}$$

Definition 11 (current). *Given a lift $\hat{S}_{A,B}$ of the one-particle scattering operator $S_{A,B}$ for which the derivative in the following expression exists, we define the associated current by Bogolyubov's formula:*

$$j^{\hat{S}}_A(F) := i\partial_F \left\langle \Omega, \hat{S}_{A,A+F}\Omega \right\rangle. \tag{3.89}$$

Theorem 12 (existence of causal lift). *Given a causal splitting c^+, there is a second quantised scattering operator $\tilde{S}$, lift of the one-particle scattering operator S with the following properties*

$$\forall A, B, C \in \mathcal{V} : \tilde{S}_{A,B}\tilde{S}_{B,C} = \tilde{S}_{A,C} \tag{3.90}$$
$$\forall F \prec G : \tilde{S}_{A,A+F} = \tilde{S}_{A+G,A+F+G} \tag{3.91}$$

and the associated current satisfies

$$\partial_G j^{\tilde{S}}_{A+G}(F) = \begin{cases} -2ic_A(F, G) & \textit{for } G \prec F \\ 0 & \textit{otherwise.} \end{cases} \tag{3.92}$$

3.3.2 Proofs

Since the phase of a lift relative to any other lift is fixed by a single matrix element, we may use the vacuum expectation values to characterise the phase of a lift. The function c captures the dependence of this object on variation of the external fields, the connection between vacuum expectation values and c becomes clearer with the next lemma.

Lemma 13 (properties of Γ)**.** *The function Γ has the following properties for all $A, B, C, D \in \mathcal{V}$ such that the expressions occurring in each equation are well defined:*

$$\gamma_{A,B,C} \neq 0 \tag{3.93}$$
$$\Gamma_{A,B,C} = \det_{\mathcal{H}^-}(P^- - P^- S_{A,C} P^+ S_{C,A} P^- \\ - P^- S_{A,B} P^+ S_{B,C} P^- S_{C,A} P^-) \tag{3.94}$$
$$\Gamma^{-1}_{A,B,C} = \mathrm{ag}(\langle \Omega, \overline{S}_{A,B} \overline{S}_{B,C} \overline{S}_{C,A} \Omega \rangle) \tag{3.95}$$
$$\Gamma_{A,B,C} = \Gamma_{B,C,A} = \frac{1}{\Gamma_{B,A,C}} \tag{3.96}$$
$$\Gamma_{A,A,B} = 1 \tag{3.97}$$
$$\Gamma_{A,B,C} \Gamma_{B,A,D} \Gamma_{A,C,D} \Gamma_{C,B,D} = 1 \tag{3.98}$$
$$\Gamma_{A,B,C} = \Gamma_{D,B,C} \Gamma_{A,D,C} \Gamma_{A,B,D} \tag{3.99}$$
$$\overline{S}_{A,C} = \Gamma_{A,B,C} \overline{S}_{A,B} \overline{S}_{B,C} \tag{3.100}$$
$$c_A(B, C) = \partial_B \partial_C \ln \Gamma_{A,A+B,A+C}. \tag{3.101}$$

Proof. Pick $A, B, C \in \mathcal{V}$ such that $\|1 - S_{X,Y}\| < 1$ for $X, Y \in \{A, B, C\}$. By definition γ is

$$\gamma_{A,B,C} = \det_{\mathcal{H}^-}(P^- S_{A,B} P^- S_{B,C} P^- S_{C,A} P^-). \tag{3.102}$$

The operator whose determinant we take in the last line is a product

$$P^- S_{A,B} P^- S_{B,C} P^- S_{C,A} P^- = P^- S_{A,B} P^- \quad P^- S_{B,C} P^- \quad P^- S_{C,A} P^- . \tag{3.103}$$

The three factors appearing in this product are all invertible, hence the product is also invertible as operators of type $\mathcal{H}^- \to \mathcal{H}^-$ because of the conditions of $\{A, B, C\}$ imply that $\|P^- - P^- S_{X,Y} P^-\| < 1$ which means that the Von Neumann series of the inverse converges, therefore if the determinant exists we have $\gamma_{A,B,C} \neq 0$. To see that it does exist, we reformulate

$$\gamma_{A,B,C} = \det_{\mathcal{H}^-}(P^- S_{A,B} P^- S_{B,C} P^- S_{C,A} P^-) \tag{3.104}$$

$$= \det_{\mathcal{H}^-}(P^- S_{A,C} P^- S_{C,A} P^- - P^- S_{A,B} P^+ S_{B,C} P^- S_{C,A} P^-) \tag{3.105}$$

$$= \det_{\mathcal{H}^-}(P^- - P^- S_{A,C} P^+ S_{C,A} P^- - P^- S_{A,B} P^+ S_{B,C} P^- S_{C,A} P^-), \tag{3.106}$$

now we know by a classic result of Ruisnaars [?] that $P^+ S_{X,Y} P^-$ is a Hilbert-Schmidt operator for our setting, hence γ and also Γ are well defined.

Equation (3.106) also proves (3.94). Next we show (3.95). Borrowing notation from [1, section 2] to identify $\Omega = \bigwedge \Phi$ with the injection $\Phi : \mathcal{H}^- \hookrightarrow \mathcal{H}$ and $\bigwedge$ is used to construct the infinite wedge spaces that are the perspective of Fock space introduced in [1]. We begin by reformulating the right hand side of (3.95)

$$\langle \Omega, \overline{S}_{A,B} \overline{S}_{B,C} \overline{S}_{C,A} \Omega \rangle \tag{3.107}$$

$$= \left\langle \bigwedge \Phi, \bigwedge \left(S_{A,B} S_{B,C} S_{C,A} \Phi \, \mathrm{AG}(P^- S_{C,A} P^-)^{-1} \right.\right. \\ \left.\left. \mathrm{AG}(P^- S_{B,C} P^-)^{-1} \, \mathrm{AG}(P^- S_{A,B} P^-)^{-1} \right) \right\rangle$$

$$= \left\langle \bigwedge \Phi, \bigwedge \left(\Phi \, \mathrm{AG}(P^- S_{C,A} P^-)^{-1} \right.\right. \\ \left.\left. \times \, \mathrm{AG}(P^- S_{B,C} P^-)^{-1} \, \mathrm{AG}(P^- S_{A,B} P^-)^{-1} \right) \right\rangle \tag{3.108}$$

$$= \det_{\mathcal{H}^-} \big((\Phi)^* \big[\Phi \,\mathrm{AG}(P^- S_{C,A} P^-)^{-1} \,\mathrm{AG}(P^- S_{B,C} P^-)^{-1} \times \,\mathrm{AG}(P^- S_{A,B} P^-)^{-1}\big]\big) \tag{3.109}$$

$$= \det_{\mathcal{H}^-} \big() \,\mathrm{AG}(P^- S_{C,A} P^-)^{-1} \,\mathrm{AG}(P^- S_{B,C} P^-)^{-1} \times \mathrm{AG}(P^- S_{A,B} P^-)^{-1}\big) \tag{3.110}$$

$$= \frac{1}{\det_{\mathcal{H}^-} \mathrm{AG}(P^- S_{A,B} P^-) \,\mathrm{AG}(P^- S_{B,C} P^-) \,\mathrm{AG}(P^- S_{C,A} P^-)}. \tag{3.111}$$

We first note that $\det_{\mathcal{H}^-} |P^- S_{X,Y} P^-| \in \mathbb{R}^+$ for $X, Y \in \{A, B, C\}$. This is well defined because

$$\langle \Omega, \overline{S}_{X,Y} \Omega \rangle = \big\langle \bigwedge \Phi, \bigwedge (S_{X,Y} \Phi \,\mathrm{AG}(P^- S_{X,Y} P^-)^{-1}) \big\rangle \tag{3.112}$$

$$= \det_{\mathcal{H}^-} \big(\Phi^* S_{X,Y} \Phi \,\mathrm{AG}(P^- S_{X,Y} P^-)^{-1}\big) \tag{3.113}$$

$$= \det_{\mathcal{H}^-} \big(P^- S_{X,Y} P^- \,\mathrm{AG}(P^- S_{X,Y} P^-)^{-1}\big) \tag{3.114}$$

$$= \det_{\mathcal{H}^-} \big(\mathrm{AG}(P^- S_{X,Y} P^-)^{-1} P^- S_{X,Y} P^-\big) \tag{3.115}$$

$$= \det_{\mathcal{H}^-} \big(\mathrm{AG}(P^- S_{X,Y} P^-)^{-1} \,\mathrm{AG}(P^- S_{X,Y} P^-) |P^- S_{X,Y} P^-|\big) \tag{3.116}$$

$$= \det_{\mathcal{H}^-} |P^- S_{X,Y} P^-| \tag{3.117}$$

holds. Moreover this determinant does not vanish, since the $P^- S_{X,Y} P^-$ is invertible. Also clearly the eigenvalues are positive since $|P^- S_{X,Y} P^-|$ is an absolute value. We continue with the result of (3.111). Thus, we find

$$\langle \Omega, \overline{S}_{A,B} \overline{S}_{B,C} \overline{S}_{C,A} \Omega \rangle^{-1} \tag{3.118}$$

$$= \det_{\mathcal{H}^-} \big(\mathrm{AG}(P^- S_{A,B} P^-) \,\mathrm{AG}(P^- S_{B,C} P^-) \,\mathrm{AG}(P^- S_{C,A} P^-)\big) \tag{3.119}$$

$$= \det_{\mathcal{H}^-} \big(\mathrm{AG}(P^- S_{A,B} P^-) \,\mathrm{AG}(P^- S_{B,C} P^-) P^- S_{C,A} P^- \times |P^- S_{C,A} P^-|^{-1}\big) \tag{3.120}$$

$$= \det_{\mathcal{H}^-} \left(\mathrm{AG}(P^- S_{A,B} P^-)\, \mathrm{AG}(P^- S_{B,C} P^-) P^- S_{C,A} P^-\right) \times \det_{\mathcal{H}^-} |P^- S_{C,A} P^-|^{-1} \tag{3.121}$$

$$= \det_{\mathcal{H}^-} \left(P^- S_{C,A} P^- \,\mathrm{AG}(P^- S_{A,B} P^-)\, \mathrm{AG}(P^- S_{B,C} P^-)\right) \times \det_{\mathcal{H}^-} |P^- S_{C,A} P^-|^{-1} \tag{3.122}$$

$$= \frac{\det_{\mathcal{H}^-} \left(P^- S_{A,B} P^- P^- S_{B,C} P^- P^- S_{C,A} P^-\right)}{\det_{\mathcal{H}^-} |P^- S_{A,B} P^-| \cdot \det_{\mathcal{H}^-} |P^- S_{B,C} P^-|} \times \frac{1}{\det_{\mathcal{H}^-} |P^- S_{C,A} P^-|}. \tag{3.123}$$

Now since the denominator of this fraction is real we can use (3.94) to identity

$$\mathrm{ag}(\langle \Omega, \overline{S}_{A,B} \overline{S}_{B,C} \overline{S}_{C,A} \Omega\rangle) = \Gamma^{-1}_{A,B,C}, \tag{3.124}$$

which proves (3.95).
For the first equality in (3.96) we use $\det X(1+Y)X^{-1} = \det(1+Y)$ for any Y trace-class and X bounded and invertible. So we can cyclicly permute the factors $P^- S_{X,Y} P^-$ in the determinant and find

$$\begin{aligned}\Gamma_{A,B,C} &= \mathrm{ag}(\det_{\mathcal{H}^-} P^- S_{A,B} P^- S_{B,C} P^- S_{C,A} P^-) \\ &= \mathrm{ag}(\det_{\mathcal{H}^-} P^- S_{C,A} P^- S_{A,B} P^- S_{B,C} P^-) = \Gamma_{C,A,B}.\end{aligned}$$

For the second equality of (3.96) we use (3.94) to represent both $\Gamma_{A,B,C}$ and $\Gamma_{B,A,C}$. Using this and the manipulations of the determinant we already employed, we arrive at

$$\Gamma_{A,B,C} \Gamma_{B,A,C} \tag{3.125}$$

$$= \mathrm{ag}(\det_{\mathcal{H}^-}(P^- S_{A,B} P^- S_{B,C} P^- S_{C,A} P^-)) \tag{3.126}$$

$$\times\, \mathrm{ag}(\det_{\mathcal{H}^-}(P^- S_{B,A} P^- S_{A,C} P^- S_{C,B} P^-)) \tag{3.127}$$

$$
\begin{aligned}
&= \operatorname{ag}(\det_{\mathcal{H}^-}(P^- S_{A,B} P^- S_{B,C} P^- S_{C,A} P^-)) && (3.128)\\
&\quad \times (\operatorname{ag}(\det_{\mathcal{H}^-}(P^- S_{B,C} P^- S_{C,A} P^- S_{A,B} P^-)))^* && (3.129)\\
&= \operatorname{ag}(\det_{\mathcal{H}^-}(P^- S_{A,B} P^- S_{B,C} P^- S_{C,A} P^-)) && (3.130)\\
&\quad \times (\operatorname{ag}(\det_{\mathcal{H}^-}(P^- S_{A,B} P^- S_{B,C} P^- S_{C,A} P^-)))^* && (3.131)\\
&= |\operatorname{ag}(\det_{\mathcal{H}^-}(P^- S_{A,B} P^- S_{B,C} P^- S_{C,A} P^-))|^2 = 1, && (3.132)
\end{aligned}
$$

which proves (3.96).
Next, using (3.79) inserting twice the same argument yields

$$\gamma_{A,A,C} = \det_{\mathcal{H}^-} P^- S_{A,C} P^- S_{C,A} P^- = \det_{\mathcal{H}^-} (P^- S_{C,A} P^-)^* P^- S_{C,A} P^- \in \mathbb{R}^+, \tag{3.133}$$

hence (3.97) follows.
For proving (3.98) we will use the definition of Γ directly and repeatedly use that we can cyclicly permute operator groups of the form $P^- S_{X,Y} P^-$ for $X, Y \in \{A, B, C, D\}$ in the determinant, i.e.

$$\det P^- S_{X,Y} P^- O = \det O P^- S_{X,Y} P^-. \tag{$\circlearrowleft$}$$

This is possible, because $P^- S_{X,Y} P^-$ is bounded and invertible. Furthermore we will use that

$$\det O_1 O_2 = \det O_1 \det O_2 \tag{$\leftrightarrow$}$$

holds whenever both O_1 and O_2 have a determinant. Moreover for any $(P^- S_{X,Y} P^-)^* P^- S_{X,Y} P^-$ is the modulus squared of an invertible operator and hence its determinant is positive which means that

$$\operatorname{ag} \det (P^- S_{X,Y} P^-)^* P^- S_{X,Y} P^- = 1. \tag{ag$|\,|$}$$

These three rules will be repeatedly used. We calculate

$$\Gamma_{A,B,C}\Gamma_{B,A,D}\Gamma_{A,C,D}\Gamma_{C,B,D} \tag{3.134}$$
$$= \operatorname{ag}\det_{\mathcal{H}^-} P^-S_{A,B}P^-S_{B,C}P^-S_{C,A}P^-$$
$$\times \operatorname{ag}\det_{\mathcal{H}^-} P^-S_{B,A}P^-S_{A,D}P^-S_{D,B}P^- \;\; \Gamma_{A,C,D}\Gamma_{C,B,D} \tag{3.135}$$
$$\overset{(\circlearrowleft)}{=} \operatorname{ag}\det_{\mathcal{H}^-} P^-S_{A,D}P^-S_{D,B}P^-S_{B,A}P^-$$
$$\times \operatorname{ag}\det_{\mathcal{H}^-} P^-S_{A,B}P^-S_{B,C}P^-S_{C,A}P^- \;\; \Gamma_{A,C,D}\Gamma_{C,B,D} \tag{3.136}$$
$$\overset{(\leftrightarrow)}{=} \operatorname{ag}\det_{\mathcal{H}^-} \big(P^-S_{A,D}P^-S_{D,B}\big[P^-S_{B,A}P^-S_{A,B}P^-\big]$$
$$\times S_{B,C}P^-S_{C,A}P^-\big) \; \Gamma_{A,C,D}\Gamma_{C,B,D} \tag{3.137}$$
$$\overset{(\circlearrowleft)}{=} \operatorname{ag}\det_{\mathcal{H}^-} P^-S_{B,C}P^-S_{C,A}P^-S_{A,D}P^-S_{D,B}\big[P^-S_{B,A}P^-S_{A,B}P^-\big]$$
$$\times \Gamma_{A,C,D}\Gamma_{C,B,D} \tag{3.138}$$
$$\overset{(\leftrightarrow)}{=} \operatorname{ag}\det_{\mathcal{H}^-} P^-S_{B,C}P^-S_{C,A}P^-S_{A,D}P^-S_{D,B}P^-$$
$$\times \operatorname{ag}\det_{\mathcal{H}^-} P^-S_{B,A}P^-S_{A,B}P^- \;\; \Gamma_{A,C,D}\Gamma_{C,B,D} \tag{3.139}$$
$$\overset{(\operatorname{ag}|\,|)}{=} \operatorname{ag}\det_{\mathcal{H}^-} P^-S_{B,C}P^-S_{C,A}P^-S_{A,D}P^-S_{D,B}P^- \;\; \Gamma_{A,C,D}\Gamma_{C,B,D} \tag{3.140}$$
$$\overset{(\circlearrowleft)}{=} \operatorname{ag}\det_{\mathcal{H}^-} P^-S_{A,D}P^-S_{D,B}P^-S_{B,C}P^-S_{C,A}P^- \;\; \Gamma_{A,C,D}\Gamma_{C,B,D} \tag{3.141}$$
$$= \operatorname{ag}\det_{\mathcal{H}^-} P^-S_{A,C}P^-S_{C,D}P^-S_{D,A}P^-$$
$$\times \operatorname{ag}\det_{\mathcal{H}^-} P^-S_{A,D}P^-S_{D,B}P^-S_{B,C}P^-S_{C,A}P^- \;\; \Gamma_{C,B,D} \tag{3.142}$$
$$\overset{(\leftrightarrow)}{=} \operatorname{ag}\det_{\mathcal{H}^-} \big(P^-S_{A,C}P^-S_{C,D}P^-\big[P^-S_{D,A}P^-P^-S_{A,D}P^-\big]$$

$$\times S_{D,B}P^- S_{B,C}P^- S_{C,A}P^-\big) \;\; \Gamma_{C,B,D} \tag{3.143}$$

$$\overset{(\circlearrowleft)}{=} \operatorname{ag}\det_{\mathcal{H}^-}\big(P^- S_{D,B}P^- S_{B,C}P^- S_{C,A}P^- S_{A,C}P^- S_{C,D}P^- \\ \times \big[P^- S_{D,A}P^- P^- S_{A,D}P^-\big]\big) \;\; \Gamma_{C,B,D} \tag{3.144}$$

$$\overset{(\leftrightarrow)}{=} \operatorname{ag}\det_{\mathcal{H}^-} P^- S_{D,B}P^- S_{B,C}P^- S_{C,A}P^- S_{A,C}P^- S_{C,D}P^- \\ \times \operatorname{ag}\det_{\mathcal{H}^-} P^- S_{D,A}P^- P^- S_{A,D}P^- \;\; \Gamma_{C,B,D} \tag{3.145}$$

$$\overset{(\operatorname{ag}|\,|)}{=} \operatorname{ag}\det_{\mathcal{H}^-}\big(P^- S_{D,B}P^- S_{B,C}P^-\big[P^- S_{C,A}P^- S_{A,C}P^-\big] \\ \times P^- S_{C,D}P^-\big)\Gamma_{C,B,D} \tag{3.146}$$

$$\overset{(\circlearrowleft)}{=} \operatorname{ag}\det_{\mathcal{H}^-} P^- S_{C,D}P^- S_{D,B}P^- S_{B,C}P^-\big[P^- S_{C,A}P^- S_{A,C}P^-\big] \\ \times \Gamma_{C,B,D} \tag{3.147}$$

$$\overset{(\leftrightarrow)}{=} \operatorname{ag}\det_{\mathcal{H}^-} P^- S_{C,D}P^- S_{D,B}P^- S_{B,C}P^- \\ \times \operatorname{ag}\det P^- S_{C,A}P^- S_{A,C}P^- \;\; \Gamma_{C,B,D} \tag{3.148}$$

$$\overset{(\operatorname{ag}|\,|)}{=} \operatorname{ag}\det_{\mathcal{H}^-} P^- S_{C,D}P^- S_{D,B}P^- S_{B,C}P^- \;\; \Gamma_{C,B,D} \tag{3.149}$$

$$= \operatorname{ag}\det_{\mathcal{H}^-} P^- S_{C,D}P^- S_{D,B}P^- S_{B,C}P^- \\ \times \operatorname{ag}\det_{\mathcal{H}^-} P^- S_{C,B}P^- S_{B,D}P^- S_{D,C}P^- \tag{3.150}$$

$$= |\operatorname{ag}\det_{\mathcal{H}^-} P^- S_{C,D}P^- S_{D,B}P^- S_{B,C}P^-|^2 = 1. \tag{3.151}$$

Equation (3.99) is a direct consequence of (3.98) and (3.96).
For (3.100) we realise that according to [1] that two lifts can only differ by a phase, that is

$$\overline{S}_{A,C} = \alpha \overline{S}_{A,B}\overline{S}_{B,C} \tag{3.152}$$

for some $\alpha \in \mathbb{C}$, $|\alpha| = 1$.

In order to identify α we recognise that $\overline{S}_{X,Y} = \overline{S}_{Y,X}^{-1}$ for four potentials X, Y and find

$$1\alpha^{-1} = \overline{S}_{A,B}\overline{S}_{B,C}\overline{S}_{C,A}. \tag{3.153}$$

Now we take the vacuum expectation value on both sides of this equation and use (3.95) to find

$$\alpha^{-1} = \langle \Omega, \overline{S}_{A,B}\overline{S}_{B,C}\overline{S}_{C,A}\Omega\rangle = \Gamma_{A,B,C}^{-1}. \tag{3.154}$$

Finally we prove (3.101). We start from the right hand side of this equation and work our way towards the left hand side of it. In the following calculation we will repeatedly make use of the fact that $(P^- S_{A,A+B} P^- S_{A+B,A} P^-)$ is the absolute value squared of an invertible operator and has a determinant, which is therefore positive. For the marked equality we will use that for a differentiable function $z : \mathbb{R} \to \mathbb{C}$ at points t where $z(t) \in \mathbb{R}^+$ holds, we have

$$\begin{aligned}(z/|z|)'(t) = \frac{z'}{|z|}(t) + \frac{-z}{|z|^2}\frac{z'z^* + z^{*\prime}z}{2|z|}(t) = \frac{z'}{2|z|}(t) - \frac{z^2 z^{*\prime}}{2|z|^3}(t) \\ = i(\Im(z'))/z(t).\end{aligned} \tag{3.155}$$

Furthermore, we will use the following expressions for the derivative of the determinant which holds for all functions $M : \mathbb{R} \to (\mathcal{H} \to \mathcal{H})$ such that $M(t) - 1$ is traceclass and M is invertible for all $t \in \mathbb{R}$

$$\partial_\varepsilon \det M(\varepsilon)|_{\varepsilon=0} = \det M(0)\,\mathrm{tr}(M^{-1}(0)\partial_\varepsilon M(\varepsilon)|_\varepsilon), \tag{3.156}$$

likewise we need the following expression for the derivative of M^{-1} for $M : \mathbb{R} \to (\mathcal{H} \to \mathcal{H})$ such that $M(t)$ is invertible and bounded for every $t \in \mathbb{R}$

$$\partial_\varepsilon M^{-1}(\varepsilon)|_{\varepsilon=0} = -M^{-1}(0)\partial_\varepsilon M(\varepsilon)|_{\varepsilon=0}M^{-1}(0). \tag{3.157}$$

We compute

$$
\begin{aligned}
& \partial_B \partial_C \ln \Gamma_{A,A+B,A+C} && (3.158)\\
&\overset{(3.94)}{=} \partial_B \partial_C \ln \operatorname{ag}(\det_{\mathcal{H}^-}(P^- S_{A,A+B} P^- S_{A+B,A+C} P^- S_{A+C,A} P^-)) && (3.159)\\
&= \partial_B \frac{\partial_C \operatorname{ag}(\det_{\mathcal{H}^-}(P^- S_{A,A+B} P^- S_{A+B,A+C} P^- S_{A+C,A} P^-))}{\operatorname{ag}(\det_{\mathcal{H}^-}(P^- S_{A,A+B} P^- S_{A+B,A} P^-))} && (3.160)\\
&= \partial_B \partial_C \operatorname{ag}(\det_{\mathcal{H}^-}(P^- S_{A,A+B} P^- S_{A+B,A+C} P^- S_{A+C,A} P^-)) && (3.161)\\
&\overset{*}{=} i\partial_B \frac{\Im \partial_C \det_{\mathcal{H}^-}(P^- S_{A,A+B} P^- S_{A+B,A+C} P^- S_{A+C,A} P^-)}{\det_{\mathcal{H}^-}(P^- S_{A,A+B} P^- S_{A+B,A} P^-)} && (3.162)\\
&= i\partial_B \Big[\frac{\det_{\mathcal{H}^-}(P^- S_{A,A+B} P^- S_{A+B,A} P^-)}{\det_{\mathcal{H}^-}(P^- S_{A,A+B} P^- S_{A+B,A} P^-)} \\
&\quad \times \Im \operatorname{tr}((P^- S_{A,A+B} P^- S_{A+B,A} P^-)^{-1} \\
&\quad \times \partial_C P^- S_{A,A+B} P^- S_{A+B,A+C} P^- S_{A+C,A} P^-) \Big] && (3.163)
\end{aligned}
$$

The fraction in front of the trace equals 1. As a next step we replace the second but last projector $P^- = 1 - P^+$, the resulting first summand vanishes, because the dependence on C cancels. This results in

$$
\begin{aligned}
(3.163) = &-i\partial_B \Im \operatorname{tr}((P^- S_{A,A+B} P^- S_{A+B,A} P^-)^{-1} \\
&\times \partial_C P^- S_{A,A+B} P^- S_{A+B,A+C} P^+ S_{A+C,A} P^-). \quad (3.164)
\end{aligned}
$$

Now, because $P^+ P^- = 0$ only one summand of the product rule survives:

$$
\begin{aligned}
(3.164) = &-i\partial_B \Im \operatorname{tr}((P^- S_{A,A+B} P^- S_{A+B,A} P^-)^{-1} \\
&\times \partial_C P^- S_{A,A+B} P^- S_{A+B,A} P^+ S_{A+C,A} P^-). \quad (3.165)
\end{aligned}
$$

Next we use $(MN)^{-1} = N^{-1} M^{-1}$ for invertible operators M and N for the first factor in the trace and cancel as much as possible of the second factor:

$$
\begin{aligned}
(3.165) &= -i\partial_B \Im \operatorname{tr}((P^- S_{A+B,A} P^-)^{-1} P^- S_{A+B,A} \\
&\qquad \times P^+ \partial_C S_{A+C,A} P^-) && (3.166) \\
&= -i \Im \operatorname{tr}(\partial_B [(P^- S_{A+B,A} P^-)^{-1} P^- S_{A+B,A} \\
&\qquad \times P^+ \partial_C S_{A+C,A} P^-]) && (3.167) \\
&= -i \Im \operatorname{tr}(\partial_B P^- S_{A+B,A} P^+ \partial_C S_{A+C,A} P^-) && (3.168) \\
&= -i \Im \operatorname{tr}(\partial_B P^- S_{A,A+B} P^+ \partial_C S_{A,A+C} P^-) && (3.169) \\
&= -i\partial_B \partial_C \Im \operatorname{tr}(P^- S_{A,A+B} P^+ S_{A,A+C} P^-) && (3.170)
\end{aligned}
$$

which proves the claim.

□

In order to construct the lift announced in theorem 12, we first construct a reference lift $\hat{S}$, that is well defined on all of $\mathcal{V}$. Afterwards we will study the dependence of the relative phase between this global lift $\hat{S}_{0,A}$ and a local lift given by $\hat{S}_{0,B}\overline{S}_{B,A}$ for $B - A$ small. By exploiting properties of this phase and the causal splitting c^+ we will construct a global lift that has the desired properties.
Since $\mathcal{V}$ is star shaped, we may reach any four-potential A from 0 through the straight line $\{tA \mid t \in [0,1]\}$.

Definition 14 (ratio of lifts)**.** *For any $A, B \in \mathcal{V}$ and any two lifts $S'_{A,B}, S''_{A,B}$ of the one particle scattering operator $S_{A,B}$ we define the ratio*

$$\frac{S'_{A,B}}{S''_{A,B}} \in S^1 \tag{3.171}$$

to be the unique complex number $z \in S^1$ such that

$$z\ S''_{A,B} = S'_{A,B} \tag{3.172}$$

holds.

Theorem 15 (existence of global lift)**.** *There is a unique map $\hat{S}_{0,\cdot} : \mathcal{V} \to U(\mathcal{F})$ which maps $A \in \mathcal{V}$ to a lift of $S_{0,A}$ and solves the differential equation*

$$A, B \in \mathcal{V} \text{ linearly dependent} \Rightarrow \partial_B \frac{\hat{S}_{0,A+B}}{\hat{S}_{0,A}\overline{S}_{A,A+B}} = 0, \tag{3.173}$$

subject to the initial condition $\hat{S}_{0,0} = 1$.

The proof of theorem 15 is divided into two lemmas due to its length. We will introduce the integral flow ϕ_A associated with the differential equation (3.173) for some $A \in \mathcal{V}$. We will then study the properties of ϕ_A in the two lemmas and finally construct $\hat{S}_{0,A} = 1\phi_A(0,1)$. In the first lemma we will establish the existence of a local solution. The solution will be constructed along the line $\overline{0\ \ A}$. In the second lemma we patch local solutions together to a global one.

Lemma 16 (ϕ local existence and uniqueness)**.** *There is a unique $\phi_A : \{(t,s) \in \mathbb{R}^2 \mid (tA, sA) \in \operatorname{dom}\overline{S}\} \to U(\mathcal{F})$ for every $A \in \mathcal{V}$ satisfying*

$$\forall (t,s) \in \operatorname{dom}\phi_A : \phi_A(t,s) \text{ is a lift of } S_{tA,sA} \tag{3.174}$$

$$\forall (t,s),(s,l),(l,t) \in \operatorname{dom}\phi_A : \phi_A(t,s)\phi_A(s,l) = \phi_A(t,l) \tag{3.175}$$

$$\forall t \in \mathbb{R} : \phi_A(t,t) = 1 \tag{3.176}$$

$$\forall s \in \mathbb{R} : \partial_t \left.\frac{\phi_A(s,t)}{\overline{S}_{sA,tA}}\right|_{t=s} = 0. \tag{3.177}$$

Proof. We first define the phase

$$z : \{(A,B) \in \operatorname{dom}\overline{S} \mid A, B \text{ linearly dependent}\} \to S^1 \tag{3.178}$$

by the differential equation

$$\frac{d}{dx} \ln z(tA, xA) = -\left.\left(\frac{d}{dy} \ln \Gamma_{tA,xA,yA}\right)\right|_{y=x} \tag{3.179}$$

and the initial condition

$$z(A, A) = 1 \tag{3.180}$$

for any $A \in \mathcal{V}$. The phase z takes the form

$$z(tA, xA) = \exp\left(-\int_t^x dx' \left(\frac{d}{dx'} \ln \Gamma_{tA,yA,x'A} \right)\bigg|_{y=x'} \right). \tag{3.181}$$

Please note that both differential equation and initial condition are invariant under rescaling of the potential A, so z is well defined. We will now construct a local solution to (3.173) and define ϕ_A using this solution. Pick $A \in \mathcal{V}$ the expression

$$\hat{S}_{0,sA} = \hat{S}_{0,A}\overline{S}_{A,sA}z(A, sA) \tag{3.182}$$

solves (3.173) locally. Local here means that s is close enough to 1 such that $(A, sA) \in \operatorname{dom} \overline{S}$. Calculating the argument of the derivative of (3.173) we find:

$$0 = \frac{\hat{S}_{0,(s+\varepsilon)A}}{\hat{S}_{0,sA}\overline{S}_{sA,(s+\varepsilon)A}} = \frac{\hat{S}_{0,A}\overline{S}_{A,(s+\varepsilon)A}z(A, (s+\varepsilon)A)}{\hat{S}_{0,A}\overline{S}_{A,sA}\overline{S}_{sA,(s+\varepsilon)}z(A, sA)} \tag{3.183}$$

$$\overset{(3.99)}{=} \frac{\hat{S}_{0,A}\overline{S}_{A,sA}\overline{S}_{sA,(s+\varepsilon)}\Gamma_{A,sA,(s+\varepsilon)A}z(A, (s+\varepsilon)A)}{\hat{S}_{0,A}\overline{S}_{tA,sA}\overline{S}_{sA,(s+\varepsilon)}z(A, sA)} \tag{3.184}$$

$$= \frac{\Gamma_{tA,sA,(s+\varepsilon)A}z(tA, (s+\varepsilon)A)}{z(A, sA)} \tag{3.185}$$

Now we take the derivative with respect to ε at $\varepsilon = 0$, cancel the factor that does not depend on ε and relabel $s = x$ to obtain

$$0 = \left(\frac{d}{dy} (\Gamma_{A,xA,yA}\ z(A, yA)) \right)\bigg|_{y=x} \tag{3.186}$$

$$\Longleftrightarrow \frac{d}{dx} \ln z(tA, xA) = \left(-\frac{d}{dy} \ln \Gamma_{tA,xA,yA} \right)\bigg|_{y=x}, \tag{3.187}$$

which is exactly the defining differential equation of z. The initial condition of z equation (3.180) is necessary to match the initial condition in (3.182) for $s = 1$. The connection to ϕ from the statement of the lemma can now be made. We define

$$\phi_A(t,s) := z(tA, sA)\overline{S}_{tA,sA}, \tag{3.188}$$

for $(tA, sA) \in \operatorname{dom}\overline{S}$. Since $\overline{S}$ is a lift of S, we see that (3.174) holds. Equation (3.176) follows from (3.180) and $\overline{S}_{tA,tA} = 1$ for general $t \in \mathbb{R}$. Equation (3.177) follows by plugging in (3.188) and using the differential equation for z (3.180):

$$\begin{aligned} \partial_s \left.\frac{\phi_A(t,s)}{\overline{S}_{tA,sA}}\right|_{s=t} &= \partial_s \left.\frac{z(tA,sA)\overline{S}_{tA,sA}}{\overline{S}_{tA,sA}}\right|_{s=t} && (3.189)\\ &= \partial_t \left. z(tA,sA)\right|_{t=s} = 0. && (3.190)\end{aligned}$$

It remains to see that (3.175), i.e. that

$$\phi_A(t,s)\phi_A(s,l) = \phi_A(t,l) \tag{3.191}$$

holds for $(tA, sA), (sA, lA), (tA, lA) \in \operatorname{dom}\overline{S}$. In order to do so we plug in the definition (3.188) of ϕ_A and obtain

$$\begin{aligned} \phi_A(t,s)\phi_A(s,l) &= \phi_A(t,l) && (3.192)\\ \Longleftrightarrow\quad z(tA,sA)z(sA,lA)\overline{S}_{tA,sA}\overline{S}_{sA,lA} &= z(tA,lA)\overline{S}_{tA,lA} && (3.193)\\ \Longleftrightarrow\quad z(tA,sA)z(sA,lA)\overline{S}_{tA,sA}\overline{S}_{sA,lA} & && (3.194)\\ = z(tA,lA)\overline{S}_{tA,sA}\overline{S}_{sA,lA}&\Gamma_{tA,sA,lA} && (3.195)\\ \Longleftrightarrow\quad z(tA,sA)z(sA,lA)z(tA,lA)^{-1} &= \Gamma_{tA,sA,lA}. && (3.196)\end{aligned}$$

In order to check the validity of the last equality we plug in the ntegral formula (3.181) for z, we also abbreviate $\frac{d}{dx} = \partial_x$

$$
\begin{aligned}
& z(tA, sA)z(sA, lA)z(tA, lA)^{-1} && (3.197)\\
&= e^{-\int_t^s dx'\,(\partial_{x'} \ln \Gamma_{tA,yA,x'A})\big|_{y=x'} - \int_s^l dx'\,(\partial_{x'} \ln \Gamma_{sA,yA,x'A})\big|_{y=x'}} && (3.198)\\
&\quad \times e^{+\int_t^l dx'\,(\partial_{x'} \ln \Gamma_{tA,yA,x'A})\big|_{y=x'}} && (3.199)\\
&= e^{-\int_l^s dx'\,(\partial_{x'} \ln \Gamma_{tA,yA,x'A})\big|_{y=x'} - \int_s^l dx'\,(\partial_{x'} \ln \Gamma_{sA,yA,x'A})\big|_{y=x'}} && (3.200)\\
&\overset{(3.99)}{=} e^{-\int_l^s dx'\,(\partial_{x'} \ln \Gamma_{sA,yA,x'A})\big|_{y=x'} - \int_l^s dx'\,(\partial_{x'} \ln \Gamma_{tA,sA,x'A})\big|_{y=x'}} && (3.201)\\
&\quad \times e^{-\int_l^s dx'\,(\partial_{x'} \ln \Gamma_{tA,yA,sA})\big|_{y=x'} - \int_s^l dx'\,(\partial_{x'} \ln \Gamma_{sA,yA,x'A})\big|_{y=x'}} && (3.202)\\
&= e^{-\int_l^s dx'\,(\partial_{x'} \ln \Gamma_{tA,sA,x'A})\big|_{y=x'}} && (3.203)\\
&= e^{-\int_l^s dx' \partial_{x'} \ln \Gamma_{tA,sA,x'A}} && (3.204)\\
&\overset{(3.97)}{=} \Gamma_{tA,sA,lA}, && (3.205)
\end{aligned}
$$

which proves the validity of the consistency relation (3.191).
In order to prove uniqueness we pick $A \in \mathcal{V}$ and assume there is ϕ' also defined on $\operatorname{dom}\phi_A$ and satisfies (3.174)-(3.177). Then we may use (3.174) to conclude that for any $(t, s) \in \operatorname{dom}\phi_A$ there is $\gamma(t, s) \in S^1$ such that

$$\phi_A(t, s) = \phi'(t, s)\gamma(t, s) \tag{3.206}$$

holds true. Picking l such that $(t, s), (s, l), (t, l) \in \operatorname{dom}\phi_A$ and using (3.175) we find

$$
\begin{aligned}
& \phi'(t, s)\gamma(t, s) = \phi_A(t, s) = \phi_A(t, l)\phi_A(l, s) && (3.207)\\
&= \gamma(t, l)\phi'(t, l)\gamma(l, s)\phi'(l, s) = \gamma(t, l)\gamma(l, s)\phi'(t, s), && (3.208)
\end{aligned}
$$

hence we have

$$\gamma(t, s) = \gamma(t, l)\gamma(l, s). \tag{3.209}$$

From property (3.176) we find

$$\gamma(t,t) = 1, \tag{3.210}$$

for any t. Using equation (3.177) we conclude that

$$0 = \partial_t \left. \frac{\phi'(s,t)}{\overline{S}_{sA,tA}} \right|_{t=s} = \partial_t \left. \frac{\phi_A(s,t)\gamma(s,t)}{\overline{S}_{sA,tA}} \right|_{t=s} \tag{3.211}$$

$$= \partial_t \, \gamma(s,t) \left. \frac{\phi_A(s,t)}{\overline{S}_{sA,tA}} \right|_{t=s} = \partial_t \, \gamma(s,t)\big|_{t=s} + \partial_t \left. \frac{\phi_A(s,t)}{\overline{S}_{sA,tA}} \right|_{t=s} \tag{3.212}$$

$$= \partial_t \, \gamma(s,t)\big|_{t=s} . \tag{3.213}$$

Finally we find for general $(s,t) \in \operatorname{dom} \phi_A$:

$$\partial_x \gamma(s,x)|_{x=t} = \partial_x(\gamma(s,t)\gamma(t,x))|_{x=t} = \gamma(s,t)\partial_x\gamma(t,x)|_{x=t} = 0. \tag{3.214}$$

So $\gamma(t,s) = 1$ everywhere. We conclude $\phi_A = \phi'$.

□

Lemma 17 (ϕ global existence and uniqueness)**.** *For any $A \in \mathcal{V}$ the map ϕ_A constructed in lemma 16 can be uniquely extended to all of $\mathbb{R}^2$ keeping its defining properties*

$$\forall (t,s) \in \mathbb{R}^2 : \phi_A(t,s) \text{ is a lift of } S_{tA,sA} \tag{3.215}$$

$$\forall (t,s),(s,l),(l,t) \in \mathbb{R}^2 : \phi_A(t,s)\phi_A(s,l) = \phi_A(t,l) \tag{3.216}$$

$$\forall t \in \mathbb{R} : \phi_A(t,t) = 1 \tag{3.217}$$

$$\forall s \in \mathbb{R} : \partial_t \left. \frac{\phi_A(s,t)}{\overline{S}_{sA,tA}} \right|_{t=s} = 0. \tag{3.218}$$

Proof. Pick $A \in \mathcal{V}$. For $x \in \mathbb{R}$ we define the set

$$U_x := \{y \in \mathbb{R} \mid (xA, yA) \in \operatorname{dom} \overline{S}\}, \tag{3.219}$$

which according to properties 2 and 4 of lemma 9 is an open interval and fulfills that $\bigcup_{x\in\mathbb{R}} U_x \times U_x$ is an open neighbourhood of the diagonal $\{(x,x) \mid x \in \mathbb{R}\}$. Therefore ϕ_A is defined for arguments that are close enough to each other. Since properties (3.218) and (3.217) only concern the behavior of ϕ_A at the diagonal any extension fulfils them. We pick a sequence $(x_k)_{k\in\mathbb{N}} \subset \mathbb{R}$ such that

$$\bigcup_{k\in\mathbb{N}_0} U_{x_k} = \mathbb{R} \tag{3.220}$$

holds and

$$\forall n \in \mathbb{N}_0 : \bigcup_{k=0}^{n} U_{x_k} =: \mathrm{dom}_n \tag{3.221}$$

is an open interval. Please note that such a sequence always exists. We are going to prove that for any $n \in \mathbb{R}_0$ There is a function ψ_n : $\mathrm{dom}_n \times \mathrm{dom}_n \to U(\mathcal{F})$, which satisfies the conditions

$$\forall (t,s) \in \mathrm{dom}_n \times \mathrm{dom}_n : \psi_n(t,s) \text{ is a lift of } S_{tA,sA} \tag{3.222}$$
$$\forall s,k,l \in \mathrm{dom}_n : \psi_n(k,s)\psi_n(s,l) = \psi_n(k,l) \tag{3.223}$$
$$\forall x,y \in \mathrm{dom}_n : (xA,yA) \in \mathrm{dom}\,\overline{S} \Rightarrow \psi_n(x,y) = \phi_A(x,y) \tag{3.224}$$

and is the unique function to do so, i.e. any other function fulfilling properties (3.222)-(3.224) possibly being defined on a larger domain coincides with ψ_n on $\mathrm{dom}_n \times \mathrm{dom}_n$.

We start with $\psi_0 = \phi_A$ restricted to $U_{x_0} \times U_{x_0}$. This function is a restriction of ϕ_A and because of lemma 16 it fulfils all of the required properties directly.

For the induction step we define ψ_{n+1} on the domain $\mathrm{dom}_{n+1} \times \mathrm{dom}_{n+1}$ by

$$\psi_{n+1}(x,y) := \begin{cases} \psi_n(x,y) & \text{for } x,y \in \mathrm{dom}_n \\ \phi_A(x,y) & \text{for } x,y \in U_{x_{n+1}} \\ \psi_n(x,t)\phi_A(t,y) & \text{for } x \in \mathrm{dom}_n, y \in U_{x_{n+1}} \\ \phi_A(x,t)\psi_n(t,y) & \text{for } y \in \mathrm{dom}_n, x \in U_{x_{n+1}}. \end{cases} \tag{3.225}$$

In order to complete the induction step we have to show that ψ_{n+1} is well defined and fulfils properties (3.222)-(3.224) with n replaced by $n+1$ and is the unique function to do so.
To see that ψ_{n+1} is well defined we have to check that the cases in the definition agree when they overlap.

1. If we have $x, y \in \mathrm{dom}_n \cap U_{x_{n+1}}$ all four cases overlap; however, the alternative definitions all equal $\phi_A(x,y)$:

$$\psi_n(x,y) \overset{(3.224)}{=} \phi_A(x,y) \overset{(3.191)}{=} \phi_A(x,t)\phi_A(t,y) \overset{(3.224)}{=} \begin{cases} \psi_A(x,t)\phi_n(t,y) \\ \phi_A(x,t)\psi_n(t,y). \end{cases} \tag{3.226}$$

2. Furthermore, if we have $x \in \mathrm{dom}_n$, $y \in \mathrm{dom}_n \cap U_{x_{n+1}}$ case one and three overlap. Here both calternatives are equal to $\psi_n(x,y)$, since $x, y \in \mathrm{dom}_n$ and we obtain:

$$\psi_n(x,y) \overset{(3.223)}{=} \psi_n(x,t)\psi_n(t,y) \overset{(3.224)}{=} \psi_n(x,t)\phi_A(t,y). \tag{3.227}$$

3. Additionally, if $y \in \mathrm{dom}_n$, $x \in \mathrm{dom}_n \cap U_{x_{n+1}}$ case one and four overlap. Here they are equal to $\psi_n(x,y)$, since $x, y \in \mathrm{dom}_n$ a quick calculation yields:

$$\psi_n(x,y) \overset{(3.223)}{=} \psi_n(x,t)\psi_n(t,y) \overset{(3.224)}{=} \psi_A(x,t)\psi_n(t,y). \tag{3.228}$$

4. Moreover, if we have $y \in U_{x_{n+1}}$, $x \in \mathrm{dom}_n \cap U_z$ case two and three overlap. Here both candidate definitions are equal to $\phi_A(x,y)$, since $x, t \in U_z$ we arrive at:

$$\phi_A(x,y) \overset{(3.191)}{=} \phi_A(x,t)\phi_A(t,y) \overset{(3.224)}{=} \psi_n(x,t)\phi_A(t,y). \tag{3.229}$$

5. Also, if we have $x \in U_{x_{n+1}}$, $y \in \mathrm{dom}_n \cap U_{x_{n+1}}$ case two and four overlap. In this case both alternatives are equal to $\phi_A(x,y)$, since $y, t \in U_{x_{n+1}}$ we get:

$$\phi_A(x,y) \overset{(3.191)}{=} \phi_A(x,t)\phi_A(t,y) \overset{(3.224)}{=} \phi_A(x,t)\psi_n(t,y). \tag{3.230}$$

We proceed to show the induction claim, starting with $(3.222)_{n+1}$. By the induction hypothesis we know that $\psi_n(x,y)$ as well as $\phi_A(x,y)$ are lifts of $S_{xA,yA}$ for any (x,y) in their domain of definition. Therefore we have for $x, y \in \mathrm{dom}_n \cup U_{x_{n+1}}$

$$\psi_{n+1}(x,y) = \begin{cases} \psi_n(x,y) & \text{for } x,y \in \mathrm{dom}_n, \\ \phi_A(x,y) & \text{for } x,y \in U_{x_{n+1}}, \\ \psi_n(x,t)\phi_A(t,y) & \text{for } x \in \mathrm{dom}_n, y \in U_{x_{n+1}}, \\ \phi_A(x,t)\psi_n(t,y) & \text{for } y \in \mathrm{dom}_n, x \in U_{x_{n+1}}, \end{cases} \tag{3.225}$$

where each of the lines is a lift of $S_{xA,yA}$ whenever the expression is defined.

Equation $(3.223)_{n+1}$ we will again show in a case by case manner depending on the s, k and l:

1. $s, k, l \in \mathrm{dom}_n$: $(3.223)_{n+1}$ follows directly from the induction hypothesis;

2. $s, k \in \mathrm{dom}_n$ and $l \in U_{x_{n+1}}$:

$$\begin{aligned}\psi_{n+1}(s,k)\psi_{n+1}(k,l) &= \psi_n(s,k)\psi_n(k,t)\phi_A(t,l) \\ &\overset{(3.223)}{=} \psi_n(s,t)\phi_A(t,l) = \psi_{n+1}(s,l),\end{aligned} \tag{3.231}$$

3. $s, l \in \mathrm{dom}_n$ and $k \in U_{x_{n+1}}$:

$$\begin{aligned}\psi_{n+1}(s,k)\psi_{n+1}(k,l) &= \psi_n(s,t)\phi_A(t,k)\phi_A(t,k)\psi_n(t,l) \\ &\overset{(3.176),(3.175)}{=} \psi_n(s,t)\psi_n(t,l) \overset{(3.223)}{=} \psi_n(s,l) = \psi_{n+1}(s,l),\end{aligned}$$

4. $s \in \mathrm{dom}_n$ and $k, l \in U_{x_{n+1}}$:

$$\begin{aligned}\psi_{n+1}(s,k)\psi_{n+1}(k,l) &= \psi_n(s,t)\phi_A(t,k)\phi_A(k,l)\\ &\overset{(3.175)}{=} \psi_n(s,t)\phi_A(t,l) = \psi_{n+1}(s,l),\end{aligned}$$

5. $k, l \in \mathrm{dom}_n$ and $s \in U_{x_{n+1}}$:

$$\begin{aligned}\psi_{n+1}(s,k)\psi_{n+1}(k,l) &= \phi_A(s,t)\psi_n(t,k)\psi_n(k,l)\\ &\overset{(3.223)}{=} \phi_A(s,t)\psi_n(t,l) = \psi_{n+1}(s,l),\end{aligned}$$

6. $k \in \mathrm{dom}_n$ and $s, l \in U_{x_{n+1}}$:

$$\begin{aligned}\psi_{n+1}(s,k)\psi_{n+1}(k,l) &= \phi_A(s,t)\psi_n(t,k)\psi_n(k,t)\phi_A(t,l)\\ \overset{(3.223)}{=} \phi_A(s,t)\psi(t,t)\phi_A(t,l) &\overset{(3.224),(3.176)}{=} \phi_A(s,t)\phi_A(t,l)\\ &\overset{(3.175)}{=} \phi_A(s,l) = \psi_{n+1}(s,l),\end{aligned}$$

7. $l \in \mathrm{dom}_n$ and $s, k \in U_{x_{n+1}}$:

$$\begin{aligned}\psi_{n+1}(s,k)\psi_{n+1}(k,l) &= \phi_A(s,k)\phi_A(k,t)\psi_n(t,l)\\ &\overset{(3.175)}{=} \phi_A(s,t)\psi_n(t,l) = \psi_{n+1}(s,l),\end{aligned}$$

8. and if $s, k, l \in U_z$:

$$\begin{aligned}\psi_{n+1}(s,k)\psi_{n+1}(k,l) &= \phi_A(s,k)\phi_A(k,l)\\ &\overset{(3.175)}{=} \phi_A(s,l) = \psi_{n+1}(s,l).\end{aligned}$$

To see $(3.224)_{n+1}$, i.e. that ψ_{n+1} is coincides with ϕ_A where both functions are defined pick $x, y \in \mathrm{dom}_{n+1}$ such that $(xA, yA) \in \mathrm{dom}\,\overline{S})$. Recall the definition of ψ_{n+1}

$$\psi_{n+1}(x,y) = \begin{cases} \psi_n(x,y) & \text{for } x,y \in \mathrm{dom}_n, \\ \phi_A(x,y) & \text{for } x,y \in U_{x_{n+1}}, \\ \psi_n(x,t)\phi_A(t,y) & \text{for } x \in \mathrm{dom}_n, y \in U_{x_{n+1}}, \\ \phi_A(x,t)\psi_n(t,y) & \text{for } y \in \mathrm{dom}_n, x \in U_{x_{n+1}}, \end{cases} \tag{3.225}$$

Therefore if $x, y \in \mathrm{dom}_{n+1}$ we may use the induction hypothesis directly and if $x, y \in U_{x_{n+1}}$ we also arrived a the claim we want to prove. Excluding these cases, we are left with rows number three and four of this definition with the restriction

3. $x \in \mathrm{dom}_n \setminus U_{x_{n+1}}, y \in U_{x_{n+1}} \setminus \mathrm{dom}_n$ and
4. $y \in \mathrm{dom}_n \setminus U_{x_{n+1}}, x \in U_{x_{n+1}} \setminus \mathrm{dom}_n$,

respectively. Because t appearing in the definition of ψ_{n+1} satisfies $t \in \mathrm{dom}_n \cap U_{x_{n+1}}$, we have in both cases $t \in \overline{x\, y}$. By using property 4 of lemma 9 we infer from $(xA, yA) \in \mathrm{dom}\,\overline{S}$ that in both cases $(xA, tA), (tA, yA) \in \mathrm{dom}\,\overline{S}$ also holds. Hence we may apply the induction hypothesis $(3.224)_n$.
Now only uniqueness is left to show. So let $\tilde{\psi}$ defined at least on$\mathrm{dom}_{n+1} \times \mathrm{dom}_{n+1}$ fulfil

$$\forall (t,s) \in \mathrm{dom}_n \times \mathrm{dom}_n : \tilde{\psi}(t,s) \text{ is a lift of } S_{tA,sA} \tag{3.232}$$

$$\forall s,k,l \in \mathbb{R} : \tilde{\psi}(k,s)\tilde{\psi}(s,l) = \tilde{\psi}(k,l), \tag{3.233}$$

$$\forall (x,y) \in \mathrm{dom}_n : (xA, yA) \in \mathrm{dom}\,\overline{S} \Rightarrow \tilde{\psi}(x,y) = \phi_A(x,y). \tag{3.234}$$

Now pick $x, y \in (\mathrm{dom}_n \cup U_z)$. Let without loss of generality $y > x$. Pick for each $z \in [x,y]$ an open set $U_z \subseteq \mathrm{dom}\,\overline{S}$ then

$$\bigcup_{z \in [x,y]} U_z \supseteq [x,y] \tag{3.235}$$

is an open cover of $[x, y]$. Since $[x, y]$ is compact there is a finite selection of $z_i \in [x, y], 1 \leqslant i \leqslant n \in \mathbb{N}$ such that also

$$\bigcup_{i=1}^{n} U_{z_i} \supseteq [x, y] \tag{3.236}$$

is an open cover of $[x, y]$. Without loss of generality let $z_1 = x, z_n = y$. We next reduce $\tilde{\psi}(x, y)$ to the following product

$$\tilde{\psi}(x, y) = \prod_{i=1}^{n=1} \tilde{\psi}(z_i, z_{i+1}) = \prod_{i=1}^{n=1} \phi_A(z_i, z_{i+1}), \tag{3.237}$$

by repeated application of equation (3.233) and application of (3.234). By the very same procedure we can reduce $\psi_{n+1}(x, y)$ to the same product, proving $\psi_{n+1}(x, y) = \tilde{\psi}(x, y)$. This ends the induction.
Now we have established an extension ψ_n of ϕ_A fulfilling properties (3.222)-(3.224) and coinciding with any other function on its domain dom_n that also satisfies these properties. The domain is of the form

$$\bigcup_{k=0}^{n} U_{x_k} \tag{3.238}$$

with x_k such that $\bigcup_{k=0}^{l} U_{x_k}$ is connected for all l. This also yields a global unique extension of ϕ_A because we can pick points $(x_n)_{n \in \mathbb{N}}$ such that $\bigcup_{k=0}^{n} U_{x_k}$ is connected for all n and

$$\mathbb{R} = \bigcup_{n \in \mathbb{N}} U_{x_n} \tag{3.239}$$

holds.

□

Lemma 17 enables us to define a global lift.

Definition 18 (global lift)**.** *For any $A \in \mathcal{V}$ we define*

$$\hat{S}_{0,A} := \phi_A(1,0). \tag{3.240}$$

using lemma 17 we are now in a position to prove theorem 15.

proof of theorem 15. The operator $\hat{S}$ fulfils the claimed differential equation (3.173) due to the global multiplication property (3.216) and the differential equation (3.218). Its uniqueness is inherited from the uniqueness of ϕ_A for $A \in \mathcal{V}$ from lemma 17. □

Definition 19 (relative phase)**.** *Let $(A, B) \in \operatorname{dom} \overline{S}$, we define $z(A, B) \in S^1$ by*

$$z(A, B) := \frac{\hat{S}_{0,B}}{\hat{S}_{0,A}\overline{S}_{A,B}}. \tag{3.241}$$

Please note that for such A, B the lift $\bar{S}_{A,B}$ is well defined. This means that the product in the denominator is a lift of $S_{0,B}$ and according to definition 18 the ratio is well defined.

Remark 20. *The global function z defined here is an extension of the funtion z appearing locally in the the proof of lemma 16, cf. formula (3.178).*
Please note that z is smooth when restricted to $\mathcal{W}^2 \cap \operatorname{dom} \overline{S}$ for any finite dimensional subspace $\mathcal{W} \subseteq \mathcal{V}$.

Lemma 21 (properties of the relative phase)**.** *For all $(A, F), (F, G), (G, A) \in \operatorname{dom} \overline{S}$, as well as or all $H, K \in \mathcal{V}$, we have*

$$z(A, F) = z(F, A)^{-1} \tag{3.242}$$
$$z(F, A)z(A, G)z(G, F) = \Gamma_{F,A,G} \tag{3.243}$$
$$\partial_H \partial_K \ln z(A + H, A + K) = c_A(H, K). \tag{3.244}$$

Proof. Pick $A, F, G \in \mathcal{V}$ as in the lemma. We start off by analysing

$$\hat{S}_{0,F}\overline{S}_{F,G} \stackrel{(3.241)}{=} z(A,F)\hat{S}_{0,A}\overline{S}_{A,F}\overline{S}_{F,G} \tag{3.245}$$

$$\stackrel{(3.99)}{=} z(A,F)\Gamma^{-1}_{A,F,G}\hat{S}_{0,A}\overline{S}_{A,G}. \tag{3.246}$$

Exchanging A and F in this equation yields

$$\hat{S}_{0,A}\overline{S}_{A,G} = z(F,A)\Gamma^{-1}_{F,A,G}\hat{S}_{0,F}\overline{S}_{F,G}. \tag{3.247}$$

This is equivalent to

$$\hat{S}_{0,F}\overline{S}_{F,G} = z(F,A)^{-1}\Gamma_{F,A,G}\hat{S}_{0,A}\overline{S}_{A,G}\ . \tag{3.248}$$

Comparing the last equation with formula (3.246) and taking the permutation properties (3.96) of Γ into account this implies that

$$z(A,F) = z(F,A)^{-1} \tag{3.249}$$

holds true. Equation (3.246) solved for $\hat{S}_{0,A}\overline{S}_{A,G}$ also gives us

$$\hat{S}_{0,G} \stackrel{(3.241)}{=} z(A,G)\hat{S}_{0,A}\overline{S}_{A,G} \tag{3.250}$$

$$\stackrel{(3.246)}{=} z(A,G)z(A,F)^{-1}\Gamma_{A,F,G}\hat{S}_{0,F}\overline{S}_{F,G}. \tag{3.251}$$

The latter equation compared with

$$\hat{S}_{0,G} \stackrel{(3.241)}{=} z(F,G)\hat{S}_{0,F}\overline{S}_{F,G}, \tag{3.252}$$

yields a direct connection between Γ and z:

$$\frac{z(A,G)}{z(A,F)}\Gamma_{A,F,G} = z(F,G), \tag{3.253}$$

which we rewrite using the antisymmetry (3.242) of z as

$$\Gamma_{A,F,G} = z(F,G)z(A,F)z(G,A). \tag{3.254}$$

Finally, in this equation, we substitute $F = A + \varepsilon_1 H$ as well as $G = A + \varepsilon_2 K$, where $\varepsilon_1, \varepsilon_2$ is small enough so that z and Γ are still well defined. Then we take the second logarithmic derivative to find

$$\begin{aligned}\partial_{\varepsilon_1}\partial_{\varepsilon_2} \ln z(A+\varepsilon_1 H, A+\varepsilon_2 K) &= \partial_{\varepsilon_1}\partial_{\varepsilon_2} \ln \Gamma_{A,A+\varepsilon_1 H, A+\varepsilon_2 K} \\ &\overset{(3.101)}{=} c_A(H,K).\end{aligned} \tag{3.255}$$

□

So we find that c_A is the second mixed logarithmic derivative of z. In the following we will characterise z more thoroughly by c and c^+.

Definition 22 (p-forms of four potentials, phase integral)**.** *For $p \in \mathbb{N}$, we introduce the set Ω^p of p-forms to consist of all maps $\omega : \mathcal{V}\times\mathcal{V}^p \to \mathbb{C}$ such that ω is linear and antisymmetric in its p last arguments and smooth in its first argument when restricted to any finite dimensional subspace of $\mathcal{V}$.*
Additionally, we define the 1-form $\chi \in \Omega^1(\mathcal{V})$ by

$$\chi_A(B) := \partial_B \ln z(A, A+B) \tag{3.256}$$

for all $A, B \in \mathcal{V}$. Furthermore, for $p \in \mathbb{N}$ and any differential form $\omega \in \Omega^p(\mathcal{V})$, we define its exterior derivative, $d\omega \in \Omega^{p+1}(\mathcal{V})$ by

$$(d\omega)_A(B_1,\ldots,B_{p+1}) := \sum_{k=1}^{p+1} (-1)^{k+1} \partial_{B_k} \omega_{A+B_k}(B_1,\ldots,\cancel{B_k},\ldots,B_{p+1}), \tag{3.257}$$

for $A, B_1, \ldots, B_{p+1} \in \mathcal{V}$, where the notation $\cancel{B_k}$ denotes that B_k is dropped as an argument.

Lemma 23 (connection between c and the relative phase)**.** *The differential form χ fulfils*

$$(d\chi)_A(F,G) = 2c_A(F,G) \tag{3.258}$$

for all $A, F, G \in \mathcal{V}$.

Proof. Pick $A, F, G \in \mathcal{V}$, we calculate

$$\begin{aligned}(d\chi)_A(F,G) &= \partial_F\partial_G \ln z(A+F, A+F+G) \\ &\quad - \partial_F\partial_G \ln z(A+G, A+F+G) && (3.259)\\ &= \partial_F\partial_G(\ln z(A, A+F+G) + \ln z(A+F, A+G)) && (3.260)\\ &\quad - \partial_F\partial_G(\ln z(A, A+F+G) + \ln z(A+G, A+F)) && (3.261)\\ &\overset{(3.242)}{=} 2\partial_F\partial_G \ln z(A+F, A+G) \overset{(3.244)}{=} 2c_A(F,G). && (3.262)\end{aligned}$$

□

Now since $dc = 0$, we might use Poincaré's lemma as a method independent of z to construct a differential form ω such that $d\omega = c$. In order to execute this plan, we first need to prove Poincaré's lemma for our setting:

Lemma 24 (Poincaré)**.** *Let $\omega \in \Omega^p(\mathcal{V})$ for $p \in \mathbb{R}$ be closed, i.e. $d\omega = 0$. Then ω is also exact, more precisely we have*

$$\omega = d\int_0^1 \iota_t^* i_X f^* \omega dt, \tag{3.263}$$

where X, ι_t for $t \in \mathbb{R}$ and f are given by

$$X : \mathbb{R} \times \mathcal{V} \to \mathbb{R} \times \mathcal{V}, \tag{3.264}$$
$$(t, B) \mapsto (1, 0) \tag{3.265}$$
$$\forall t \in \mathbb{R} : \iota_t : \mathcal{V} \to \mathbb{R} \times \mathcal{V}, \tag{3.266}$$
$$B \mapsto (t, B) \tag{3.267}$$
$$f : \mathbb{R} \times \mathcal{V} \mapsto \mathcal{V}, \tag{3.268}$$
$$(t, B) \mapsto tB \tag{3.269}$$
$$\forall t \in \mathbb{R} : f_t := f(t, \cdot). \tag{3.270}$$

Proof. Pick some $\omega \in \Omega^p(\mathcal{V})$. We will first show the more general formula

$$f_b^* \omega - f_a^* \omega = d \int_a^b \iota_t^* i_X f^* \omega \; dt + \int_a^b \iota_t^* i_X f^* d\omega dt. \tag{3.271}$$

The lemma follows then by $b = 1, a = 0$, $f_1^* \omega = \omega$, $f_0^* \omega = 0$ and $d\omega = 0$ for a closed ω. We begin by rewriting the right hand side of (3.271):

$$\begin{aligned} d \int_a^b \iota_t^* i_X f^* \omega \; dt + \int_a^b \iota_t^* i_X f^* d\omega dt \\ = \int_a^b (d\iota_t^* i_X f^* \omega + \iota_t^* i_X f^* d\omega) dt. \end{aligned} \tag{3.272}$$

Next we look at both of these terms separately. Let therefore $p \in \mathbb{N}$, $t, s_k \in \mathbb{R}$ and $A, B_k \in \mathcal{V}$ for each $p + 1 \geqslant k \in \mathbb{N}$. First, we calculate $d\iota_t^* i_X f^* \omega$:

$$\begin{aligned} &(f^* \omega)_{(t,A)}((s_1, B_1), \ldots, (s_p, B_p)) \\ &\qquad = \omega_{tA}(s_1 A + tB_1, \ldots, s_p A + tB_p) \end{aligned} \tag{3.273}$$

$$\Rightarrow (i_X f^* \omega)_{(t,A)}((s_1, B_1), \ldots, (s_{p-1}, B_{p-1})) \tag{3.274}$$

$$= \omega_{tA}(A, s_1 A + tB_1, \ldots, s_{p-1}A + tB_{p-1}$$

$$\Rightarrow (\iota_t^* i_X f^* \omega)_A(B_1, \ldots, B_{p-1}) = t^{p-1}\omega_{tA}(A, B_1, \ldots, B_{p-1}) \qquad (3.275)$$

$$\Rightarrow (d\iota_t^* i_X f^* \omega)_A(B_1, \ldots, B_p)$$

$$= \partial_\varepsilon|_{\varepsilon=0} \sum_{k=1}^{p} (-1)^{k+1} t^{p-1} \omega_{tA+\varepsilon t B_k}(A, B_1, \ldots, \cancel{B_k}, \ldots, B_p) \qquad (3.276)$$

$$+ \partial_\varepsilon|_{\varepsilon=0} \sum_{k=1}^{p} (-1)^{k+1} t^{p-1} \omega_{tA}(A + \varepsilon B_k, B_1, \ldots, \cancel{B_k}, \ldots, B_p) \qquad (3.277)$$

$$= \partial_\varepsilon|_{\varepsilon=0} \sum_{k=1}^{p} t^p (-1)^{k+1} \omega_{tA+\varepsilon B_k}(A, B_1, \ldots, \cancel{B_k}, \ldots, B_p)$$

$$+ p t^{p-1} \omega_{tA}(B_1, \ldots, B_p). \qquad (3.278)$$

Now, we calculate $\iota_t^* i_X f^* d\omega$:

$$(d\omega)_A(B_1, \cdots, B_{p+1})$$

$$= \partial_\varepsilon|_{\varepsilon=0} \sum_{k=1}^{p+1} (-1)^{k+1} \omega_{A+\varepsilon B_k}(B_1, \ldots, \cancel{B_k}, \ldots, B_{p+1}) \qquad (3.279)$$

$$(f^* d\omega)(t, A)((s_1, B_1), \ldots, (s_{p+1}, B_{p+1})) \qquad (3.280)$$

$$= (d\omega)_{tA}(s_1 A + tB_1, \ldots, s_{p+1}A + tB_{p+1})$$

$$= \partial_\varepsilon|_{\varepsilon=0} \sum_{k=1}^{p+1} (-1)^{k+1} \qquad (3.281)$$

$$\times\, \omega_{tA+\varepsilon(s_k A + tB_k)}(s_1 A + tB_1, \ldots, \cancel{s_k A + tB_k}, \ldots, s_p A + tB_p)$$

$$(i_X f^* d\omega)_{(t,A)}((s_1, B_1), \ldots, (s_p, B_p)) \qquad (3.282)$$

$$= \partial_\varepsilon|_{\varepsilon=0} \omega_{(t+\varepsilon)A}(s_1 A + tB_1, \ldots, s_p A + tB_p)$$

$$
\begin{aligned}
&+ \partial_\varepsilon|_{\varepsilon=0} \sum_{k=1}^{p} (-1)^k \\
&\times \omega_{tA+\varepsilon(s_kA+tB_k)}(A, s_1A + tB_1, \ldots, \cancel{s_kA + tB_k}, \ldots, s_pA + tB_p) \\
&= t^p \partial_\varepsilon|_{\varepsilon=0} \omega_{(t+\varepsilon)A}(B_1, \ldots, \cancel{B_k}, \ldots, B_p)) && (3.283)\\
&\qquad + \sum_{k=1}^{p} s_k t^{p-1} (-1)^{k+1} \partial_\varepsilon|_{\varepsilon=0} \omega_{(t+\varepsilon)A}(A, B_1, \ldots, B_p) && (3.284)\\
&+ \partial_\varepsilon|_{\varepsilon=0} \sum_{k=1}^{p} (-1)^k t^{p-1} (\omega_{(t+s_k\varepsilon)A}(A, B_1, \ldots, \cancel{B_k}, \ldots, B_p) && (3.285)\\
&\qquad + \omega_{tA+\varepsilon tB_k}(A, B_1, \ldots, \cancel{B_k}, \ldots, B_p)) && (3.286)\\
&= t^p \partial_\varepsilon|_{\varepsilon=0} \Big(\omega_{(t+\varepsilon)A}(B_1, \ldots, B_p) && (3.287)\\
&\qquad + \sum_{k=1}^{p} (-1)^k \omega_{tA+\varepsilon B_k}(A, B_1, \ldots, \cancel{B_k}, \ldots, B_p) \Big)
\end{aligned}
$$

$$
\begin{aligned}
(\iota_t^* i_X f^* d\omega)_A(B_1, \ldots, B_p) &= t^p \partial_\varepsilon|_{\varepsilon=0} \Big(\omega_{(t+\varepsilon)A}(B_1, \ldots, B_p) && (3.288)\\
&\qquad + \sum_{k=1}^{p} (-1)^k \omega_{tA+\varepsilon B_k}(A, B_1, \ldots, \cancel{B_k}, \ldots, B_p) \Big)
\end{aligned}
$$

Adding (3.278) and (3.288) we find for (3.272):

$$
\begin{aligned}
&\int_a^b (d\iota_t^* i_X f^* \omega + \iota_t^* i_X f^* d\omega) dt = && (3.289)\\
&\int_a^b \Big(t^p \partial_\varepsilon|_{\varepsilon=0} \omega_{(t+\varepsilon)A}(B_1, \ldots, B_p) + p t^{p-1} \omega_{tA}(B_1, \ldots, B_p) \Big) dt && (3.290)\\
&= \int_a^b \frac{d}{dt} (t^p \omega_{tA}(B_1, \ldots, B_p)) dt = \int_a^b \frac{d}{dt} (f_t^* \omega)_A(B_1, \ldots, B_p) dt && (3.291)\\
&= (f_b^* \omega)_A(B_1, \ldots, B_p) - (f_a^* \omega)_A(B_1, \ldots, B_p). && (3.292)
\end{aligned}
$$

□

Definition 25 (integral of a closed p form)**.** *For a closed exterior form $\omega \in \Omega^p(\mathcal{V})$ we define the form $\prod[\omega]$*

$$\prod[\omega] := \int_0^1 \iota_t^* i_X f^* \omega dt. \tag{3.293}$$

For $A, B_1, \dots, B_{p-1} \in \mathcal{V}$ it takes the form

$$\prod[\omega]_A(B_1, \dots, B_p) = \int_0^1 t^{p-1} \omega_{tA}(A, B_1, \dots, B_{p-1}) dt. \tag{3.294}$$

By lemma 24 we know $d\prod[\omega] = \omega$ if $d\omega = 0$.

Now we found two one forms each produces c when the exterior derivative is taken. The next lemma informs us about their relationship.

Lemma 26 (inversion of lemma 23)**.** *The following equality holds*

$$\chi = 2\prod[c]. \tag{3.295}$$

Proof. We have $d(\chi - 2\prod[c]) = 0$ so by lemma 24 we know that there is $v : \mathcal{V} \to \mathbb{R}$ such that

$$dv = \chi - 2\prod[c] \tag{3.296}$$

holds. Now (3.173) translates into the following ODE for z:

$$\partial_B \ln z(0, B) = 0, \quad \partial_\varepsilon \ln z(A, (1+\varepsilon)A)|_{\varepsilon=0} = 0 \tag{3.297}$$

for all $A, B \in \mathcal{V}$. This means that

$$\chi_0(B) = 0 = \prod[c]_0(B), \quad \chi_{A,A} = 0 = \prod[c]_A(A) \tag{3.298}$$

hold. This implies

$$\partial_\varepsilon v_{\varepsilon A} = 0, \quad \partial_\varepsilon v_{A+\varepsilon A} = 0, \tag{3.299}$$

which means that v is constant. □

From this point on we will assume the existence of a function c^+ fulfilling (3.86),(3.87) and (3.88). Recall equation (3.87):

$$\forall A, F, G, H : \partial_H c^+_{A+H}(F, G) = \partial_G c^+_{A+G}(F, H). \tag{3.300}$$

For a fixed $F \in \mathcal{V}$, this condition can be read as $d(c^+_{\cdot}(F, \cdot)) = 0$. As a consequence we can apply lemma 24 to define a one form.

Definition 27 (integral of the causal splitting)**.** *For any $F \in \mathcal{V}$, we define*

$$\beta_A(F) := 2 \prod [c^+_{\cdot}(F, \cdot)]_A. \tag{3.301}$$

Lemma 28 (relation between the integral of the causal splitting and the phase integral)**.** *The following two equations hold:*

$$d\beta = -2c \tag{3.302}$$
$$d(\beta + \chi) = 0. \tag{3.303}$$

Proof. We start with the exterior derivative of β. Pick $A, F, G \in \mathcal{V}$:

$$d\beta_A(F, G) = \partial_F \beta_{A+F}(G) - \partial_G \beta_{A+G}(F) \tag{3.304}$$
$$= d\Big(\prod [c^+_{\cdot}(G, \cdot)]\Big)_A(F) - d\Big(\prod [c^+_{\cdot}(F, \cdot)]\Big)_A(G) \tag{3.305}$$
$$= 2c^+_A(G, F) - 2c^+_A(F, G) \overset{(3.86)}{=} -2c_A(F, G). \tag{3.306}$$

This proves the first equality. The second equality follows directly by $d\chi = 2c$. □

Definition 29 (corrected lift)**.** *Since $\beta + \chi$ is closed, we may use lem 24 again to define the phase*

$$\alpha := \prod [\beta + \chi]. \tag{3.307}$$

Furthermore, for all $A, B \in \mathcal{V}$ we define the corrected second quantised scattering operator

$$\tilde{S}_{0,A} := e^{-\alpha_A} \hat{S}_{0,A} \tag{3.308}$$
$$\tilde{S}_{A,B} := \tilde{S}_{0,A}^{-1} \tilde{S}_{0,B}. \tag{3.309}$$

Corollary 30 (group structure of the corrected lift)**.** *We have $\tilde{S}_{A,B}\tilde{S}_{B,C} = \tilde{S}_{A,C}$ for all $A, B, C \in \mathcal{V}$.*

Theorem 31 (causality of the corrected lift)**.** *The corrected second quantised scattering operator fulfils the following causality condition for all $A, F, G \in \mathcal{V}$ such that $F \prec G$:*

$$\tilde{S}_{A,A+F} = \tilde{S}_{A+G,A+G+F}. \tag{3.310}$$

Proof. Let $A, F, G \in \mathcal{V}$ such that $F \prec G$ We note that for the first quantised scattering operator we have

$$S_{A+G,A+G+F} = S_{A,A+F}, \tag{3.311}$$

so by definition of $\overline{S}$ we also have

$$\overline{S}_{A+G,A+G+F} = \overline{S}_{A,A+F}. \tag{3.312}$$

So for any lift this equality is true up to a phase, meaning that

$$f(A, F, G) := \frac{\tilde{S}_{A+G,A+G+F}}{\tilde{S}_{A,A+F}} \tag{3.313}$$

is well defined. We see immediately

$$f(A, 0, G) = 1 = F(A, F, 0). \tag{3.314}$$

Pick $F_1, F_2 \prec G_1, G_2$. We abbreviate $F = F_1 + F_2, G = G_1 + G_2$ and we calculate

$$f(A, F, G) = \frac{\tilde{S}_{A+G,A+F+G}}{\tilde{S}_{A,A+F}} \tag{3.315}$$

$$= \frac{\tilde{S}_{A+G,A+F+G}}{\tilde{S}_{A+G_1,A+G_1+F}} \frac{\tilde{S}_{A+G_1,A+G_1+F}}{\tilde{S}_{A,A+F}} \tag{3.316}$$

$$= \frac{\tilde{S}_{A+G,A+G+F_1}\tilde{S}_{A+G+F_1,A+F+G}}{\tilde{S}_{A+G_1,A+F_1+G_1}\tilde{S}_{A+G_1+F_1,A+G_1+F}} \frac{\tilde{S}_{A+G_1,A+G_1+F}}{\tilde{S}_{A,A+F}} \tag{3.317}$$

$$= \frac{\tilde{S}_{A+G,A+G+F_1}}{\tilde{S}_{A+G_1,A+F_1+G_1}} \frac{\tilde{S}_{A+G+F_1,A+F+G}}{\tilde{S}_{A+G_1+F_1,A+G_1+F}} f(A, G_1, F_1 + F_2) \tag{3.318}$$

$$= f(A + G_1, F_1, G_2) f(A + G_1 + F_1, G_2, F_2) \times f(A, G_1, F_1 + F_2). \tag{3.319}$$

Taking the logarithm and differentiating we find:

$$\partial_{F_2}\partial_{G_2} \ln f(A, F_1 + F_2, G_1 + G_2) = \partial_{F_2}\partial_{G_2} \ln f(A + F_1 + G_1, F_2, G_2). \tag{3.320}$$

Next we pick $F_2 = \alpha_1 F_1$ and $G_2 = \alpha_2 G_1$ for $\alpha_1, \alpha_2 \in \mathbb{R}^+$ small enough so that

$$\|1 - S_{A+F+G,A+F_1+G_1}\| < 1 \tag{3.321}$$

$$\|1 - S_{A+F+G,A+F_1+G}\| < 1 \tag{3.322}$$

$$\|1 - S_{A+F+G,A+F+G_1}\| < 1 \tag{3.323}$$

hold. We abbreviate $A' = A + G_1 + F_1$, use (3.241) and compute

$$\begin{aligned}
&f(A', F_2, G_2) = \\
&\frac{e^{-\alpha_{A'+F_2+G_2}+\alpha_{A'+G_2}} z(A', A'+F_2+G_2) z^{-1}(A', A'+G_2)}{e^{-\alpha_{A'+F_2}+\alpha_{A'}} z(A', A'+F_2) z^{-1}(A', A')} \\
&\qquad \times \frac{\overline{S}_{A'+G_2,A'} \overline{S}_{A',A'+F_2+G_2}}{\overline{S}_{A',A'} \overline{S}_{A',A'+F_2}}. \qquad (3.324)
\end{aligned}$$

The second factor in this product can be simplified significantly:

$$\frac{\overline{S}_{A'+G_2,A'} \overline{S}_{A',A'+F_2+G_2}}{\overline{S}_{A',A'} \overline{S}_{A',A'+F_2}} = \frac{\overline{S}_{A'+G_2,A'} \overline{S}_{A',A'+F_2+G_2}}{\overline{S}_{A',A'+F_2}} \qquad (3.325)$$

$$\overset{(3.99)}{=} \Gamma^{-1}_{A'+G_2,A',A'+F_2+G_2} \frac{\overline{S}_{A'+G_2,A'+F_2+G_2}}{\overline{S}_{A',A'+F_2}} \qquad (3.326)$$

$$\overset{(3.312)}{=} \Gamma_{A',A'+G_2,A'+F_2+G_2} \qquad (3.327)$$

$$\begin{aligned}
&\overset{(3.243)}{=} z(A', A'+G_2) z(A'+G_2, A'+G_2+F_2) \\
&\qquad \times z(A'+F_2+G_2, A'). \qquad (3.328)
\end{aligned}$$

So in total we find

$$\begin{aligned}
&f(A', F_2, G_2) = \\
&\frac{e^{-\alpha_{A'+F_2+G_2}+\alpha_{A'+G_2}} z(A', A'+F_2+G_2) z^{-1}(A', A'+G_2)}{e^{-\alpha_{A'+F_2}+\alpha_{A'}} z(A', A'+F_2) z^{-1}(A', A')} \qquad (3.329) \\
&\qquad \times z(A', A'+G_2) z(A'+G_2, A'+G_2+F_2) \\
&\qquad \times z(A'+F_2+G_2, A') \\
&= \exp(-\alpha_{A'+F_2+G_2} + \alpha_{A'+G_2} + \alpha_{A'+F_2} - \alpha_{A'}) \qquad (3.330) \\
&\qquad \times z(A'+G_2, A'+G_2+F_2) z^{-1}(A', A'+F_2). \qquad (3.331)
\end{aligned}$$

Most of the factors do not depend on F_2 and G_2, so taking the mixed logarithmic derivative things simplify:

$$\partial_{G_2}\partial_{F_2} \ln f(A', F_2, G_2) = \partial_{G_2}\partial_{F_2}(-\alpha_{A'+F_2+G_2} + \ln z(A' + G_2, A' + G_2 + F_2)) \tag{3.332}$$

$$\overset{(3.307),(3.256)}{=} \partial_{G_2}(-\beta_{A'+G_2}(F_2) - \chi_{A'+G_2}(F_2) + \chi_{A'+G_2}(F_2)) \tag{3.333}$$

$$\overset{(3.302)}{=} 2c^+_{A'}(F_2, G_2) \overset{F_2<G_2,(3.88)}{=} 0. \tag{3.334}$$

So by (3.320) we also have

$$\partial_{F_2}\partial_{G_2} \ln f(A, F_1 + F_2, G_1 + G_2) = 0 \tag{3.335}$$

$$= \partial_{\alpha_1}\partial_{\alpha_2} \ln f(A, F_1(1 + \alpha_1), G_1(1 + \alpha_2)) \tag{3.336}$$

But then we can integrate and obtain

$$0 = \int_{-1}^{0} d\alpha_1 \int_{-1}^{0} d\alpha_2 \partial_{\alpha_1}\partial_{\alpha_2} \ln f(A, F_1(1 + \alpha_1), G_1(1 + \alpha_2)) \tag{3.337}$$

$$= \ln f(A, F_1, G_1) - \ln f(A, 0, G_1) - \ln f(A, F_1, 0) + \ln f(A, 0, 0) \tag{3.338}$$

$$\overset{(3.314)}{=} \ln f(A, F_1, G_1). \tag{3.339}$$

remembering equation (3.313), the definition of f, this ends our proof. □

Using $\tilde{S}$ we introduce the current associated to it.

Theorem 32 (evaluation of the current of the corrected lift)**.** *For general $A, F \in \mathcal{V}$ we have*

$$j_A(F) = -i\beta_A(F). \tag{3.340}$$

So in particular for $G \in \mathcal{V}$

$$\partial_G j_{A+G}(F) = -2ic^+_A(F, G). \tag{3.341}$$

holds.

Proof. Pick $A, F \in \mathcal{V}$ as in the theorem. We calculate

$$i\partial_F \ln \left\langle \Omega, \tilde{S}_{A,A+F}\Omega \right\rangle \tag{3.342}$$

$$= i\partial_F \left(-\alpha_{A+F} - \alpha_A + \ln \left\langle \Omega, \hat{S}_{0,A}^{-1}\hat{S}_{0,A+F}\Omega \right\rangle \right) \tag{3.343}$$

$$= i\partial_F \left(-\alpha_{A+F} + \ln z(A, A+F) + \ln \left\langle \Omega, \overline{S}_{A,A+F}\Omega \right\rangle \right) \tag{3.344}$$

The last summand vanishes, as can be seen by the following calculation

$$\partial_F \ln \left\langle \Omega, \overline{S}_{A,A+F}\Omega \right\rangle \tag{3.345}$$

$$= i\partial_F \ln \det_{\mathcal{H}^-}(P^- S_{A,A+F} P^- \operatorname{AG}(P^- S_{A,A+F} P^-)^{-1}) \tag{3.346}$$

$$= i\partial_F \ln \det_{\mathcal{H}^-} |P^- S_{A,A+F} P^-| \tag{3.347}$$

$$= \frac{i}{2}\partial_F \ln \det_{\mathcal{H}^-}((P^- S_{A,A+F} P^-)^* P^- S_{A,A+F} P^-) \tag{3.348}$$

$$= \frac{i}{2}\partial_F \det(P^- S_{A+F,A} P^- S_{A,A+F} P^-) \tag{3.349}$$

$$= \frac{i}{2} \operatorname{tr}(\partial_F P^- S_{A+F,A} P^- S_{A,A+F} P^-) \tag{3.350}$$

$$= \frac{i}{2} \operatorname{tr}(\partial_F P^- S_{A,A+F} P^- + \partial_F P^- S_{A+F,A} P^-) = 0 \tag{3.351}$$

where we made use of (3.156). So we are left with

$$j_A(F) = i\partial_F(-\alpha_{A+F} + \ln z(A, A+F)) \tag{3.352}$$

$$= i(-\beta_A(F) - \chi_A(F) + \chi_A(F)) = -i\beta_A(F). \tag{3.353}$$

Finally by taking the derivative with respect to $G \in \mathcal{V}$ and using the definition of β we find

$$\partial_G j_{A+G}(F) = -2ic_A^+(F, G). \tag{3.354}$$

□

3.4 Simple Formula for the Scattering Operator

3.4.1 Defining One-Particle Scattering-Matrix

In order to introduce the one-particle dynamics I introduce Diracs equation (3.355) and reformulate it in integral form in equation (**??**). By iterating this equation we will naturally be led to the informal series expansion of the scattering operator equation (**??**), whose convergence is discussed in the next section.
Throughout this thesis I will consider four-potentials A, F or G to be smooth functions in $C_c^\infty(\mathbb{R}^4) \otimes \mathbb{C}^4$, where the index c denotes that the elements have compact support. The Dirac equation for a wave function $\phi \in L^2(\mathbb{R}^3) \otimes \mathbb{C}^4$ is

$$0 = (i\not{\partial} - e\not{A} - m\mathbb{1})\phi, \tag{3.355}$$

where m is the mass of the electron, $\mathbb{1} : \mathbb{C}^4 \to \mathbb{C}^4$ is the identity and crossed out letters mean that their four-index is contracted with Dirac matrices

$$\not{A} := A_\alpha \gamma^\alpha, \tag{3.356}$$

where Einstein's summation convention is used. These matrices fulfil the anti-commutation relation

$$\forall \alpha, \beta \in \{0, 1, 2, 3\} : \{\gamma^\alpha, \gamma^\beta\} := \gamma^\alpha\gamma^\beta + \gamma^\beta\gamma^\alpha = g^{\alpha\beta}, \tag{3.357}$$

where g is the Minkowski metric. I work with the $+ - --$ metric signature and the Dirac representation of this algebra. Squared four dimensional objects always refer to the Minkowski square, meaning for all $a \in \mathbb{C}^4$, $a^2 := a^\alpha a_\alpha$.
In order to define Lorentz invariant measures for four dimensional integrals I employ the same notation as in [3]. The standard volume

form over $\mathbb{R}^4$ is denoted by $\mathrm{d}^4x = \mathrm{d}x^0\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3$, the product of forms is understood as the wedge product. The symbol d^3x means the 3-form $\mathrm{d}^3x = \mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3$ on $\mathbb{R}^4$. Contraction of a form ω with a vector v is denoted by $\mathfrak{i}_v(\omega)$. The notation $\mathfrak{i}_v(\omega)$ is also used for the spinor matrix valued vector $\gamma = (\gamma^0, \gamma^1, \gamma^2, \gamma^3) = \gamma^\alpha e_\alpha$:

$$\mathfrak{i}_\gamma(\mathrm{d}^4x) := \gamma^\alpha \mathfrak{i}_{e_\alpha}(\mathrm{d}^4x), \tag{3.358}$$

with $(e_\alpha)_\alpha$ being the canonical basis of $\mathbb{C}^4$. Let $\mathcal{C}_A$ be the space of solutions to (3.355) which have compact support on any spacelike hyperplane Σ. Let ϕ, ψ be in $\mathcal{C}_\mathcal{A}$, the scalar product $\langle \cdot, \cdot \rangle$ of elements of $\mathcal{C}_A$ is defined as

Todo: delete derivation; simply state series representation and convergence result citing something

3.[illegible].2 Construction of the Second Quantised Scattering-Matrix

In the following I outline how the construction of the second quantised scattering operator is to be carried out, we will naturally be led to an informal power series representation for the scattering operator S. First I fix some more notation in agreement with [2]. Using a general Hilbertspace $\mathcal{H}$ as a one-particle Hilbertspace. A closed subspace $\mathcal{H}^+$ of $\mathcal{H}$ is called polarisation if both $\mathcal{H}^+$ and $\mathcal{H}^- := (\mathcal{H}^+)^\perp$ are infinite dimensional, where by $\perp$ I denote the orthogonal complement. With a polarisation $\mathcal{H}^+$ comes also the orthogonal projection operator P^+ onto the subspace $\mathcal{H}^+$ and its complement $P^- = 1 - P^+$. For one particle operators C we introduce the notation $C_{\#\ddagger} := P^\# C P^\ddagger$, where $\#, \ddagger \in \{+, -\}$. One constructs the Fock space associated with $\mathcal{H}$ and a polarisation $\mathcal{H}^+$ of $\mathcal{H}$ in the following way. We define $\overline{\mathcal{H}^-}$ identical with $\mathcal{H}^-$ as a set, but scalar multiplication as $\mathbb{C} \times \overline{\mathcal{H}} \ni (a, \psi) \mapsto \overline{a}\psi$ where the bar denotes complex conjugation of complex numbers. A wedge $\wedge$ in the exponent denotes that only elements which are antisymmetric with respect to permutations are allowed. This antisymmetric product as well as the tensor product are to be understood in the Hilbert space sense. The Factor $(\mathcal{H}^{\pm})^0$ is understood as $\mathbb{C}$. We now define Fock space as

$$\mathcal{F} := \bigoplus_{m,p=0}^{\infty} \left(\mathcal{H}^{+}\right)^{\wedge m} \otimes \left(\overline{\mathcal{H}^{-}}\right)^{\wedge p}. \tag{3.359}$$

I will denote the sectors of Fock space of fixed particle numbers by $\mathcal{F}_{m,p}$. The element of $\mathcal{F}_{0,0}$ of norm 1 will be denoted by Ω. The simplest and yet interesting example of this construction is the Fock space constructed on a hyperplane prior to the support of an external field, in this case $\mathcal{H} = L^2(\mathbb{R}^3, \mathbb{C}^4)$ and $\mathcal{H}^+$ consists of the wavefunctions that can be constructed from the generalised eigenfunctions of positive energy with respect to the free Dirac Hamiltonian.

The annihilation operator a acts on an arbitrary sector of Fock space $\mathcal{F}_{m,p}$, for any $m, p \in \mathbb{N}_0$ with either of the operator types

$$a : \overline{\mathcal{H}} \otimes \mathcal{F}_{m,p} \to \mathcal{F}_{m-1,p} \oplus \mathcal{F}_{m,p+1} \tag{3.360}$$

$$a : \overline{\mathcal{H}} \times \mathcal{F}_{m,p} \to \mathcal{F}_{m-1,p} \oplus \mathcal{F}_{m,p+1} \tag{3.361}$$

regardless of the exact type of the annihilation operator I will denote it by a. Also here the tensor product is understood in the algebraic sense. I start out by defining a on elements of $\{\bigwedge_{l=1}^{m} \varphi_l \otimes \bigwedge_{c=1}^{p} \phi_c \mid \forall c : \varphi_c \in \mathcal{H}^+, \phi_c \in \mathcal{H}^-\}$ which spans a dense subset of $\mathcal{F}_{m,p}$, then one continues this operator uniquely by linearity and finally by the bounded linear extension theorem to all of $\mathcal{F}_{m,p}$ and then again by linearity to all of $\overline{\mathcal{H}} \otimes \mathcal{F}_{m,p}$.

$$a\left(\phi \otimes \bigwedge_{l=1}^{m} \varphi_l \otimes \bigwedge_{c=1}^{p} \phi_c\right) = a\left(\phi, \bigwedge_{l=1}^{m} \varphi_l \otimes \bigwedge_{c=1}^{p} \phi_c\right) \tag{3.362}$$

$$= \sum_{k=1}^{m} (-1)^{1+k} \langle P^+\phi, \varphi_k\rangle \bigwedge_{\substack{l=1\\ l\neq k}}^{m} \varphi_l \otimes \bigwedge_{c=1}^{p} \phi_c + \bigwedge_{l=1}^{m} \varphi_l \otimes P^-\phi \wedge \bigwedge_{c=1}^{p} \phi_c \tag{3.363}$$

where $\langle,\rangle$ denotes that the scalar product of $\mathcal{H}$. The first summand on the right hand side is taken to vanish for $m = 0$. For $\varphi \in \mathcal{H}$ I will also use the abbreviation $a(\varphi) := a(\varphi, \cdot)$.
Now we turn to the construction of the S-matrix, the second quantised analogue of U^A. This construction is carried out axiomatically. The first axiom makes sure that the following diagram, and the analogue for the adjoint of the annihilation operator commute.

$$\begin{tikzcd} \mathcal{F} \arrow[r, "S^A"] & \mathcal{F} \\ \overline{\mathcal{H}} \otimes \mathcal{F} \arrow[u, "a"] \arrow[r, "U^A \otimes S^A"] & \overline{\mathcal{H}} \otimes \mathcal{F} \arrow[u, "a"] \end{tikzcd} \tag{3.364}$$

Axiom 33. *The S operator fulfils the "lift condition".*

$$\forall \phi \in \mathcal{H}: \quad S^A \circ a(\phi) = a\left(U^A\phi\right) \circ S^A, \qquad \text{(lift condition)}$$
$$\forall \phi \in \mathcal{H}: \quad S^A \circ a^*(\phi) = a^*\left(U^A\phi\right) \circ S^A, \qquad \text{(adjoint lift condition)}$$

where a^* is the adjoint of the annihilation operator, the creation operator.

There is a convergent power series of the one-particle scattering operator U^A:

$$U^A = \sum_{k=0}^{\infty} \frac{1}{k!} Z_k(A), \tag{3.365}$$

where $Z_k(A)$ are bounded operators on $\mathcal{H}$, which are homogeneous of degree k in A. We try an analogous formal power series ansatz for the second quantised scattering operator S^A

$$S^A = \sum_{k=0}^{\infty} \frac{1}{k!} T_k(A). \tag{3.366}$$

Here T_k are assumed to be homogeneous of degree k in A; however, they will only turn out to be bounded on fixed particle number subspaces $\mathcal{F}_{m,p}$ of Fock space. It is the goal of the following sections to show that this ansatz indeed works. That is, we can identify operators T_k such that (3.366) holds up to a global phase and furthermore the question of convergence can be settled if one assumes that the phase is analytic in the external field A. In order to fully characterise S^A it is enough to characterise all of the T_k operators. Using the (lift condition) one can derive commutation relations for the operators T_k by plugging in (3.365) and (3.366) into (lift condition) and (adjoint lift condition) and collecting all terms with the same degree of homogeneity. They are given by

$$\left[T_m(A), a^{\#}(\phi)\right] = \sum_{j=1}^{m} \binom{m}{j} a^{\#}\left(Z_j(A)\phi\right) T_{m-j}(A), \tag{3.367}$$

where $a^{\#}$ is either a or a^*. Together T_k and $\langle T_k \rangle$ characterise the operator T_k on the whole algebraic direct sum, it can then be further extended to all of Fock space.

Before we go on to construct a concrete form of the scattering operator, we will first define a certain kind of unitary operator on Fock space.

3.4.3 Differential second quantisation

Let $B : \mathcal{H} \to \mathcal{H}$ be a bounded operator on $\mathcal{H}$, such that iB is self adjoint and B_{+-} is a Hilbert-Schmidt operator. We would like to construct a version $d\Gamma(B)$ of B that acts on Fock space and also is skew adjoint. The strategy of this section is to construct an operator in two steps that is essentially self adjoint of the Fock space of finitely many particles, a dense subset of Fock space. It is denoted by

Definition 34.

$$\mathcal{F}' := \bigoplus_{m,p=0}^{\infty} \mathcal{F}_{m,p}, \tag{3.368}$$

where $\bigoplus$ *refers to the algebraic direct sum.*

Because $B_{-+} : \mathcal{H}^+ \to \mathcal{H}^-$ is compact, there is an ONB $(\varphi_n)_{n\in\mathbb{N}}$ of $\mathcal{H}^+$ and likewise an ONB $(\varphi_{-n})_{n\in\mathbb{N}}$ of $\mathcal{H}^-$ such that it takes the canonical form of compact operators

$$B_{-+} = \sum_{n\in\mathbb{N}} \lambda_n |\varphi_{-n}\rangle\langle\varphi_n|, \quad \lambda_n \geqslant 0. \tag{3.369}$$

Here the numbers λ_n fulfil $\sum_{k=1}^{\infty} \lambda_k^2 = \|B_{-+}\|_{\mathrm{HS}} < \infty$. As a consequence we have

$$B_{+-} = -\sum_{n\in\mathbb{N}} \lambda_n |\varphi_n\rangle\langle\varphi_{-n}|. \tag{3.370}$$

With respect to this basis we define the set of finite linear combinations of product states of finitely many particles

Definition 35. *We define*

$$\mathcal{F}^0 := \mathrm{span}\left\{ \prod_{k=1}^{m} a^*(\varphi_{L_k}) \prod_{c=1}^{p} a(\varphi_{-C_c})\Omega \mid m, p \in \mathbb{N}, (L_k)_k, (C_c)_c \subset \mathbb{N} \right\}, \tag{3.371}$$

we will refer to a subset of this set for fixed values of m *and* p *by* $\mathcal{F}^0_{m,p}$.

In order to do so, the following splitting turns out to be advantageous.

Definition 36. *We define the following operators of type $\mathcal{F}^0 \to \mathcal{F}$*

$$\mathrm{d}\Gamma(B_{++}) := \sum_{n\in\mathbb{N}} a^*(B_{++}\varphi_n)a(\varphi_n) \tag{3.372}$$

$$\mathrm{d}\Gamma(B_{--}) := -\sum_{n\in\mathbb{N}} a(\varphi_{-n})a^*(B_{--}\varphi_{-n}) \tag{3.373}$$

$$\mathrm{d}\Gamma(B_{-+}) := \sum_{n\in\mathbb{N}} a^*(B_{-+}\varphi_n)a(\varphi_n) \tag{3.374}$$

where the sum converges in the strong operator topology and $(\varphi_n)_n, (\varphi_{-n})_n$ are arbitrary ONBs of $\mathcal{H}^+$ and $\mathcal{H}^-$.

Lemma 37. *The operators $\mathrm{d}\Gamma(B_{++}), \mathrm{d}\Gamma(B_{--})$ and $\mathrm{d}\Gamma(B_{-+})$ restricted to $|_{\mathcal{F}^0_{m,p}}$ they have the following type*

$$\mathrm{d}\Gamma(B_{++})|_{\mathcal{F}^0_{m,p}} : \quad \mathcal{F}^0_{m,p} \to \mathcal{F}_{m,p} \tag{3.375}$$

$$\mathrm{d}\Gamma(B_{--})|_{\mathcal{F}^0_{m,p}} : \quad \mathcal{F}^0_{m,p} \to \mathcal{F}_{m,p} \tag{3.376}$$

$$\mathrm{d}\Gamma(B_{-+})|_{\mathcal{F}^0_{m,p}} : \quad \mathcal{F}^0_{m,p} \to \mathcal{F}_{m-1,p-1} \tag{3.377}$$

and fulfil the following bounds for all m, p

$$\|\mathrm{d}\Gamma(B_{++})|_{\mathcal{F}^0_{m,p}}\| \leqslant (m+1)\|B_{++}\| \tag{3.378}$$

$$\|\mathrm{d}\Gamma(B_{--})|_{\mathcal{F}^0_{m,p}}\| \leqslant (p+1)\|B_{--}\| \tag{3.379}$$

$$\|\mathrm{d}\Gamma(B_{-+})|_{\mathcal{F}^0_{m,p}}\| \leqslant \|B_{-+}\|_{HS}. \tag{3.380}$$

Proof. Pick $\alpha \in \mathcal{F}^0_{m,p}$ for $m, p \in \mathbb{N}_0$, α can be expressed in terms of a general ONB $(\tilde{\varphi}_k)_{k\in\mathbb{N}}$ of $\mathcal{H}^+$ and $(\tilde{\varphi}_{-k})_{k\in\mathbb{N}}$ of $\mathcal{H}^-$

$$\alpha = \sum_{\substack{L,C\subset\mathbb{N}\\|L|=m,|C|=p}} \alpha_{L,C} \prod_{l=1}^{m} a^*(\tilde{\varphi}_{L_l}) \prod_{c=1}^{p} a(\tilde{\varphi}_{-C_c})\Omega. \tag{3.381}$$

In this expansion only finitely many coefficients $\alpha_{\cdot,\cdot}$ are nonzero. Our operators all map the vacuum onto the zero vector, so commuting them through the products of of creation and annihilation operators in the expansion of α we can make the action of them more explicit:

$$\mathrm{d}\Gamma(B_{++})\alpha = \sum_{\substack{L,C\subset\mathbb{N}\\|L|=m,|C|=p}} \alpha_{L,C} \sum_{b=1}^{m}\prod_{l=1}^{b-1} a^*(\tilde\varphi_{L_l}) \sum_{n\in\mathbb{N}} a^*(B_{++}\varphi_n)\langle\varphi_n,\tilde\varphi_{L_b}\rangle \prod_{l=b+1}^{m} a^*(\tilde\varphi_l)\prod_{c=1}^{p} a(\tilde\varphi_{-C_c})\Omega \tag{3.382}$$

$$= \sum_{\substack{L,C\subset\mathbb{N}\\|L|=m,|C|=p}} \alpha_{L,C} \sum_{b=1}^{m}\prod_{l=1}^{b-1} a^*(\tilde\varphi_{L_l})\ \ a^*(B_{++}\tilde\varphi_{L_b})\prod_{l=b+1}^{m} a^*(\tilde\varphi_l)\prod_{c=1}^{p} a(\tilde\varphi_{-C_c})\Omega. \tag{3.383}$$

We notice, that $\mathrm{d}\Gamma(B_{++})\alpha \in \mathcal{F}_{m,p}$ holds. What is left to show for the first operator is therefore its norm. For estimating this we see that B_{++} in the last line can be replaced by

$$B^L_{L_b} := \left(1 - \sum_{\substack{l=1\\l\neq b}}^{m} |\tilde\varphi_{L_l}\rangle\langle\tilde\varphi_{L_l}|\right) B_{++}, \tag{3.384}$$

due to the antisymmetry of fermions. Expanding

$$\|\mathrm{d}\Gamma(B_{++})\alpha\|^2 = \langle \mathrm{d}\Gamma(B_{++})\alpha, \mathrm{d}\Gamma(B_{++})\alpha\rangle$$
$$= \sum_{\substack{L,C,L',C'\subset\mathbb{N}\\|L'|=|L|=m,|C'|=|C|=p}} \overline{\alpha_{L,C}}\alpha_{L',C'} \sum_{b,b'=1}^{m} \left\langle \prod_{l=1}^{b-1} a^*(\tilde\varphi_{L_l})\ \ a^*(B^L_{L_b}\tilde\varphi_{L_b})\right.$$

$$\prod_{l=b+1}^{m} a^*(\tilde{\varphi}_{L_l}) \prod_{c=1}^{p} a(\tilde{\varphi}_{-C_c})\Omega, \prod_{l=1}^{b'-1} a^*(\tilde{\varphi}_{L'_l}) \;\; a^*(B^{L'}_{L'_{b'}} \tilde{\varphi}_{L'_b}) \prod_{l=b'+1}^{m} a^*(\tilde{\varphi}_{L'_l}) \prod_{c=1}^{p} a(\tilde{\varphi}_{-C'_c})\Omega \Big\rangle \tag{3.385}$$

we see that in fact C and C' need to agree, because we can just commute the corresponding annihilation operators from one end of the scalar product to the other. Furthermore only a single wavefunction on each side of the scalar product is modified, this implies that in order for the scalar product not to vanish $|L \cap L'| \geqslant m-2$ has to hold. If $L \neq L'$ the double sum over n, n' has only the contribution where $b = L_l \notin L'$ and $b' = L'_{l'} \notin L$ are selected. Otherwise the full sum contributes, yielding

$$\begin{aligned}
&\|\mathrm{d}\Gamma(B_{++})\alpha\|^2 = \\
&= \sum_{\substack{L,C\subset\mathbb{N}\\|C|=p\\|L|=m-1}} \sum_{n\neq n'\in\mathbb{N}\setminus L} \overline{\alpha_{L\cup\{n\},C}}\alpha_{L\cup\{n'\},C} \langle B_n^{L\cup\{n\}}\tilde{\varphi}_n, B_{n'}^{L\cup\{n'\}}\tilde{\varphi}_{n'}\rangle (-1)^{g(L,n)+g(L,n')} \\
&+ \sum_{\substack{L,C\subset\mathbb{N}\\|L|=m,|C|=p}} |\alpha_{L,C}|^2 \sum_{b,b'=1}^{m} \Big\langle \prod_{l=1}^{b-1} a^*(\tilde{\varphi}_{L_l}) \;\; a^*(B^L_{L_b}\tilde{\varphi}_{L_b}) \\
&\prod_{l=b+1}^{m} a^*(\tilde{\varphi}_{L_l})\Omega, \prod_{l=1}^{b'-1} a^*(\tilde{\varphi}_{L_l}) \;\; a^*(B^L_{L_{b'}}\tilde{\varphi}_{L_b}) \prod_{l=b'+1}^{m} a^*(\tilde{\varphi}_{L_l}))\Omega \Big\rangle,
\end{aligned} \tag{3.386}$$

where $g(L,n) := |\{l \in L \mid l < n\}|$ keeps track of the number of anti commutations. In the first sum we add and subtract the terms where $n = n'$. The enlarged sum can then be reformulated

$$\sum_{\substack{L,C\subset\mathbb{N}\\|L|=m-1\\|C|=p}}\sum_{n,n'\in\mathbb{N}\setminus L}\overline{\alpha_{L\cup\{n\},C}}\alpha_{L\cup\{n'\},C}\big\langle B_n^{L\cup\{n\}}\tilde{\varphi}_n, B_{n'}^{L\cup\{n'\}}\tilde{\varphi}_{n'}\big\rangle(-1)^{g(L,n)+g(L,n')}$$

$$= \sum_{\substack{L,C\subset\mathbb{N}\\|L|=m-1,|C|=p}}\left\|\sum_{n\in\mathbb{N}\setminus L}\alpha_{L\cup\{n\},C}B_n^{L\cup\{n\}}\tilde{\varphi}_n(-1)^{g(L,n)}\right\|^2$$

$$= \sum_{\substack{L,C\subset\mathbb{N}\\|L|=m-1\\|C|=p}}\left\|\left(1-\sum_{l\in L}|\tilde{\varphi}_l\rangle\langle\tilde{\varphi}_l|\right)B_{++}\sum_{n\in\mathbb{N}\setminus L}\alpha_{L\cup\{n\},C}\tilde{\varphi}_n(-1)^{g(L,n)}\right\|^2 \tag{3.387}$$

Now the operator product inside the norm has operator norm $\|B_{++}\|$ and so we can estimate the whole object by

$$(3.387) \leqslant \|\alpha\|^2\|B_{++}\|^2. \tag{3.388}$$

Now for the first term in (3.386) we need to estimate the term we added to complete the norm square, this is done as follows

$$\sum_{\substack{L,C\subset\mathbb{N}\\|L|=m-1,|C|=p}}\sum_{n\in\mathbb{N}\setminus L}|\alpha_{L\cup\{n\},C}|^2\|B_n^{L\cup\{n\}}\tilde{\varphi}_n\|^2$$

$$\leqslant \sum_{\substack{L,C\subset\mathbb{N}\\|L|=m,|C|=p}}\|B_{++}\|^2|\alpha_{L,C}|^2 = \|\alpha\|^2\|B_{++}\|^2. \tag{3.389}$$

What remains is the second sum in (3.386), for this term there are two cases. If $b = b'$ then the scalar product is equal to $\langle B^L_{L_b}\tilde{\varphi}_b, B^L_{L_b}\tilde{\varphi}_b\rangle$. If $b \neq b'$ the scalar product is, up to a sign, equal to $\langle B^L_{L_b}\tilde{\varphi}_b, \tilde{\varphi}_b\rangle\langle\tilde{\varphi}_{b'}, B^L_{L_{b'}}\tilde{\varphi}_{b'}\rangle$. However both of these terms can be estimated by $\|B_{++}\|^2$. So all m^2

summands of this sum contribute $\|B_{++}\|^2$. Overall this estimate yields

$$\begin{aligned}&\|\mathrm{d}\Gamma(B_{++})\alpha\|^2 \leqslant (3.388) + (3.389) + \|\alpha\|^2 m^2 \|B_{++}\|^2\\ &= \|\alpha\|^2(2+m^2)\|B_{++}\|^2.\end{aligned}$$

For convenience of notation the estimate can be weakened to

$$\|\mathrm{d}\Gamma(B_{++})\alpha\| \leqslant (m+1)\|B_{++}\|, \tag{3.390}$$

because for all $m \neq 0$ this estimate is and upper bound on what we found, but for $m = 0$ the operator $\mathrm{d}\Gamma(B_{++})$ is actually the zero operator. A completely analogous argument works for $\mathrm{d}\Gamma(B_{--})$.
So lets move on to $\mathrm{d}\Gamma(B_{-+})$. Applying it to the same $\alpha \in \mathcal{F}^0_{m,p}$ again we permute all the operators to the right, where they annihilate the vacuum. The remaining terms are

$$\begin{aligned}&\sum_{n\in\mathbb{N}} a^*(B_{-+}\varphi_n)a(\varphi_n) \sum_{\substack{L,C\subset\mathbb{N}\\ |L|=m,|C|=p}} \alpha_{L,C} \prod_{l=1}^{m} a^*(\tilde\varphi_{L_l}) \prod_{c=1}^{p} a(\tilde\varphi_{-C_c})\Omega\\ &= \sum_{\substack{L,C\subset\mathbb{N}\\ |L|=m,|C|=p}} \alpha_{L,C} \sum_{b=1}^{m}\sum_{d=1}^{p} (-1)^{m-1+b+d}\langle B_{+-}\tilde\varphi_{-C_d}, \tilde\varphi_{L_b}\rangle\\ &\qquad \prod_{\substack{l=1\\ l\neq b}}^{m} a^*(\tilde\varphi_{L_l}) \prod_{\substack{c=1\\ c\neq d}}^{p} a(\tilde\varphi_{-C_c})\Omega.\end{aligned} \tag{3.391}$$

By counting the remaining creation and annihilation operators we immediately see that $\mathrm{d}\Gamma(B_{-+})\alpha \in \mathcal{F}_{m-1,p-1}$. For estimating the norm of this vector, we switch basis from $(\tilde\varphi_{\pm n})_n$ to $(\varphi'_{\pm n})_n$, the basis where B_{-+} takes its canonical form. Then the scalar product involving B_{-+} reduces to $\lambda_{L_b}\delta_{L_b,C_d}$. We estimate

$$
\begin{aligned}
\|\mathrm{d}\Gamma(B_{-+})\alpha\|^2 = &\sum_{\substack{L,L',C,C'\subset\mathbb{N}\\|L|=|L'|=m\\|C|=|C'|=p}} \sum_{a,a'=1}^{m}\sum_{b,b'=1}^{p} \overline{\alpha}_{L,C}\alpha_{L',C'}(-1)^{b+d+b'+d'}\lambda_{L_b}\lambda_{L'_{b'}} \\
&\delta_{L_b,C_d}\delta_{L'_{b'},C'_{d'}}\left\langle \prod_{\substack{l=1\\l\neq b}}^{m} a^*(\varphi'_{L_l})\prod_{\substack{c=1\\c\neq d}}^{p} a(\varphi'_{-C_c})\Omega, \prod_{\substack{l=1\\l\neq b'}}^{m} a^*(\varphi'_{L'_l})\prod_{\substack{c=1\\c\neq d'}}^{p} a(\varphi'_{-C'_c})\Omega\right\rangle .
\end{aligned}
\tag{3.392}
$$

The scalar product in the second line tells us that $L\backslash\{L_b\} = L'\backslash\{L'_{b'}\}$ and $C\backslash\{C_d\} = C'\backslash\{C'_{d'}\}$ have to hold in order for the term not to vanish. So this means that L and L' as well as C and C' can respectively differ at most by one element which then has to be in the intersection $L \cap C$. Because this sum is really just a finite sum, we can reorder it in the following way

$$
\begin{aligned}
\|\mathrm{d}\Gamma(B_{-+})\alpha\|^2 = &\sum_{\substack{L,C\subset\mathbb{N}\\|L|=m-1\\|C|=p-1}} \sum_{b,b'\in\mathbb{N}\backslash(L\cup C)} \lambda_b\lambda_{b'}\overline{\alpha}_{L\cup\{b\},C\cup\{b\}}\alpha_{L\cup\{b'\},C\cup\{b'\}} \\
&(-1)^{g(L,b)+g(C,b)+g(L,b')+g(C,b')},
\end{aligned}
\tag{3.393}
$$

where $g(L,b) = |\{l \in L \mid l < b\}|$ as before. This expression can be rewritten in terms of a scalar product in $\ell^2(\mathbb{N})$

$$
\begin{aligned}
\|\mathrm{d}\Gamma(B_{-+})\alpha\|^2 = &\sum_{\substack{L,C\subset\mathbb{N}\\|L|=m-1\\|C|=p-1}} \left|\left\langle 1_{(L\cup C)^c}\alpha_{L\cup\{\cdot\},C\cup\{\cdot\}}(-1)^{g(L,\cdot)+g(C,\cdot)}, \lambda_\cdot\right\rangle_{\ell^2}\right|^2 \\
\leqslant &\sum_{\substack{L,C\subset\mathbb{N}\\|L|=m-1\\|C|=p-1}} \sum_{b\in\mathbb{N}} 1_{(L\cup C)^c}(b)|\alpha_{L\cup\{b\},C\cup\{b\}}|^2\sum_{d\in\mathbb{N}}\lambda_d^2
\end{aligned}
\tag{3.394}
$$

$$\leqslant \|\alpha\|^2 \|B_{-+}\|_{\mathrm{HS}}^2. \tag{3.395}$$

□

Corollary 38. *The operators* $\mathrm{d}\Gamma(B_{--})$ *and* $\mathrm{d}\Gamma(B_{++})$ *can be extended by continuity on* $\mathcal{F}^0_{m,p}$ *to unbounded operators on all of* $\mathcal{F}'$*. The operator* $\mathrm{d}\Gamma(B_{-+})$ *can be continuously extended to all of* $\mathcal{F}$*.*

Lemma 39. *The operator* $(\mathrm{d}\Gamma(B_{-+}))^*$ *acts on elements of* $\mathcal{F}^0$ *as*

$$-\sum_{n\in\mathbb{N}} a^*(B_{+-}\varphi_{-n})a(\varphi_{-n}) =: -\mathrm{d}\Gamma(B_{+-}). \tag{3.396}$$

So also $\mathrm{d}\Gamma(B_{+-}) : \mathcal{F}^0 \to \mathcal{F}$ *can be extended continuously to all of* $\mathcal{F}$*. Moreover* $\mathrm{d}\Gamma(B_{-+}) + \mathrm{d}\Gamma(B_{+-})$ *is skew-adjoint.*

Proof. Pick $\beta, \alpha \in \mathcal{F}^0$. We expand those states with respect to the basis $(\varphi'_k)_{k\in\mathbb{Z}\setminus\{0\}}$. Consider

$$\begin{aligned}
\langle \beta, \mathrm{d}\Gamma(B_{-+})\alpha\rangle &= \left\langle \beta, \sum_{n\in\mathbb{N}} a^*(B_{-+}\varphi_n)a(\varphi_n)\alpha \right\rangle \\
&= \sum_{n\in\mathbb{N}} \langle \beta, a^*(B_{-+}\varphi_n)a(\varphi_n)\alpha\rangle = \sum_{n\in\mathbb{N}} \langle a^*(\varphi_n)a(B_{-+}\varphi_n)\beta, \alpha\rangle \\
&= \left\langle \sum_{n\in\mathbb{N}} a^*(\varphi_n)a(B_{-+}\varphi_n)\beta, \alpha \right\rangle = \left\langle \sum_{n\in\mathbb{N}} \lambda_n a^*(\varphi_n)a(\varphi_{-n})\beta, \alpha \right\rangle \\
&= \left\langle \sum_{n\in\mathbb{N}} a^*(-B_{+-}\varphi_{-n})a(\varphi_{-n})\beta, \alpha \right\rangle = -\langle \mathrm{d}\Gamma(B_{+-})\beta, \alpha\rangle,
\end{aligned} \tag{3.397}$$

So we see that $\mathrm{d}\Gamma(B_{+-})$ and $\mathrm{d}\Gamma(B_{-+})^*$ agree on $\mathcal{F}^0$ which is dense. So they are the same bounded and continuous operator on all of Fock space. □

Lemma 40. *The operator* $\mathrm{d}\Gamma(B) : \mathcal{F}' \to \mathcal{F}$,

$$\mathrm{d}\Gamma(B) := \mathrm{d}\Gamma(B_{++}) + \mathrm{d}\Gamma(B_{+-}) + \mathrm{d}\Gamma(B_{-+}) + \mathrm{d}\Gamma(B_{--}) \tag{3.398}$$

is skew symmetric.

Proof. Since the sum of skew symmetric operators is skew symmetric, it suffices to show skew symmetry of $\mathrm{d}\Gamma(B_{++})$ and $\mathrm{d}\Gamma(B_{--})$. Moreover since both of these operators are extended versions of operators of the same name of type $\mathcal{F}^0 \to \mathcal{F}$ it suffices to show skew symmetry on this domain. We will only do the calculation for $\mathrm{d}\Gamma(B_{++})$, the other calculation is analogous. First we notice that

$$\mathrm{d}\Gamma(B_{++}) = \sum_{n\in\mathbb{N}} a^*(B_{++}\varphi_n)a(\varphi_n) = \sum_{n\in\mathbb{N}}\sum_{m\in\mathbb{N}} \langle\varphi_m, B_{++}\varphi_n\rangle a^*(\varphi_m)a(\varphi_n) \tag{3.399}$$

holds. Pick $\alpha, \beta \in \mathcal{F}^0$. Consider

$$\begin{aligned}
\langle\beta, \mathrm{d}\Gamma(B_{++})\alpha\rangle &= \sum_{L,L',C,C'\subset\mathbb{N}} \overline{\beta}_{L',C'}\alpha_{L,C} \Bigg\langle \prod_{l=1}^{|L'|} a^*(\varphi_{L'_l}) \prod_{c=1}^{|C'|} a(\varphi_{-C'_c})\Omega, \\
&\qquad \sum_{n\in\mathbb{N}} a^*(B_{++}\varphi_n)a(\varphi_n) \prod_{l=1}^{|L|} a^*(\varphi_{L_l}) \prod_{c=1}^{|C|} a(\varphi_{-C_c})\Omega \Bigg\rangle \\
&= \sum_{L,L',C,C'\subset\mathbb{N}} \overline{\beta}_{L',C'}\alpha_{L,C} \sum_{n\in\mathbb{N}} \Bigg\langle \prod_{l=1}^{|L'|} a^*(\varphi_{L'_l}) \prod_{c=1}^{|C'|} a(\varphi_{-C'_c})\Omega, \\
&\qquad a^*(B_{++}\varphi_n)a(\varphi_n) \prod_{l=1}^{|L|} a^*(\varphi_{L_l}) \prod_{c=1}^{|C|} a(\varphi_{-C_c})\Omega \Bigg\rangle \\
&= \sum_{L,L',C,C'\subset\mathbb{N}} \overline{\beta}_{L',C'}\alpha_{L,C} \sum_{n\in\mathbb{N}} \Bigg\langle a^*(\varphi_n)a(B_{++}\varphi_n) \prod_{l=1}^{|L'|} a^*(\varphi_{L'_l}) \prod_{c=1}^{|C'|} a(\varphi_{-C'_c})\Omega, \\
&\qquad \prod_{l=1}^{|L|} a^*(\varphi_{L_l}) \prod_{c=1}^{|C|} a(\varphi_{-C_c})\Omega \Bigg\rangle
\end{aligned}$$

$$= \sum_{L,L',C,C' \subset \mathbb{N}} \overline{\beta}_{L',C'} \alpha_{L,C} \left\langle \sum_{n \in \mathbb{N}} \sum_{m \in \mathbb{N}} \langle B_{++} \varphi_n, \varphi_m \rangle a^*(\varphi_n) a(\varphi_m) \right.$$
$$\left. \prod_{l=1}^{|L'|} a^*(\varphi_{L'_l}) \prod_{c=1}^{|C'|} a(\varphi_{-C'_c}) \Omega, \prod_{l=1}^{|L|} a^*(\varphi_{L_l}) \prod_{c=1}^{|C|} a(\varphi_{-C_c}) \Omega \right\rangle.$$

Now because $B_{++}^* = -B_{++}$ we see that

$$\langle \beta, \mathrm{d}\Gamma(B_{++}) \alpha \rangle = -\langle \mathrm{d}\Gamma(B_{++}) \beta, \alpha \rangle \tag{3.400}$$

holds. □

Now we would like to define $e^{\mathrm{d}\Gamma(B)}$, in order to do so, we will show that $\mathrm{d}\Gamma(B)$ is essentially skew-adjoint. One way of doing so is by Nelson's analytic vector theorem.

Theorem 41 (Nelson's analytic vector theorem)**.** *Let C be a symmetric operator on a Hilbert space $\mathcal{H}$. If $D(C)$ contains a total set $S \subset \bigcap_{n=1}^{\infty} D(C^n)$ of analytic vectors, then C is essentially self adjoint. A vector $\phi \in \bigcap_{n=1}^{\infty} D(C^n)$ is called analytic if there is $t > 0$ such that $\sum_{k=0}^{\infty} \frac{\|C^n \phi\|}{n!} t^n < \infty$ holds. A set S is said to be total if $\overline{span(S)} = \mathcal{H}$*

For a proof see e.g. [10].

Lemma 42. *For any $\alpha \in \mathcal{F}', t > 0$ the operator $\mathrm{d}\Gamma(B) : \mathcal{F}' \to \mathcal{F}$ satisfies*

$$\sum_{k=0}^{\infty} \frac{\|\mathrm{d}\Gamma(B)^k \alpha\|}{k!} t^k < \infty. \tag{3.401}$$

Proof. By definition of $\mathcal{F}'$ there are $m, p \in \mathbb{N}$ such that $\alpha \in \bigoplus_{l=0}^{m} \bigoplus_{c=0}^{p} \mathcal{F}_{l,p}$. Fix $t > 0$. We dissect α into its parts of fixed particle numbers:

$$\sum_{k=0}^{\infty} \frac{\|\mathrm{d}\Gamma(B)^k \alpha\|}{k!} t^k \leqslant \sum_{l=0}^{m} \sum_{c=0}^{p} \sum_{k=0}^{\infty} \frac{\|\mathrm{d}\Gamma(B)^k \alpha_{l,c}\|}{k!} t^k. \tag{3.402}$$

Using the following abbreviations

$$\Gamma_{-1} := \mathrm{d}\Gamma(B)_{-+} \tag{3.403}$$
$$\Gamma_0 := \mathrm{d}\Gamma(B)_{++} + \mathrm{d}\Gamma(B)_{--} \tag{3.404}$$
$$\Gamma_{+1} := \mathrm{d}\Gamma(B)_{+-} \tag{3.405}$$
$$\beta := \max\{\|B_{++}\| + \|B_{--}\|, \|B_{-+},\|_{\mathrm{HS}}\} \tag{3.406}$$

we estimate

$$\begin{aligned}\|\mathrm{d}\Gamma(B)^k \alpha_{l,c}\| &\leqslant \sum_{x\in\{-1,0,+1\}^k} \left\| \prod_{b=1}^k \Gamma_{x_b} \alpha_{l,c} \right\| \\ \leqslant \sum_{x\in\{-1,0,+1\}^k} \prod_{b=1}^k &\left\| \Gamma_{x_b} |_{\mathcal{F}_{l+\sum_{d=1}^{b-1} x_d, c+\sum_{d=1}^{b-1} x_d}} \right\| \|\alpha_{l,c}\| \qquad (3.407) \\ \leqslant 3^k \|\alpha\| \max_{x\in\{-1,0,+1\}^k} &\prod_{b=1}^k \left\| \Gamma_{x_b} |_{\mathcal{F}_{l+\sum_{d=1}^{b-1} x_d, c+\sum_{d=1}^{b-1} x_d}} \right\| . \qquad (3.408)\end{aligned}$$

At this point the factors only depend on the number of particles the Fock space vector attains as we act on it with the operators $\Gamma_.$. As these bounds increase with the particle number we can restrict the set $\{-1, 0, +1\}$ in the last line to $\{0, +1\}$. We notice that the bound in (3.407) will only increase if we exchange each pair $x_i = 1, x_h = 0$ by the pair $x_{\max\{i,h\}} = 1, x_{\min\{i,h\}} = 0$ so that the norm of the operator that acts like a particle number operator is taken after the particle number is increased. The maximum therefore has the form $(c + l + 2 + 2d)^{k-d}$, which we bound by $2^k(c/2 + d/2 + 1 + d)^{k-d}$. For maximising this object we treat d as a continuous variable take the derivative and set it to zero. From the form of the function to be maximised it is clear that it is equal to 1 for $d = k$ and at $d = -c/2 - l/2$, but for k large it will be bigger in between. We abbreviate $y = c/2 + l/2 + 1$.

$$0 = (y+d)^{k-d}(-\ln(y+d) + \frac{k-d}{y+d}) \tag{3.409}$$

$$\iff \frac{k-d}{y+d} = \ln(y+d) \tag{3.410}$$

$$\iff \frac{k+y}{y+d} - 1 = -1 + \ln(e(y+d)) \tag{3.411}$$

$$\iff e(k+y) = e(y+d)\ln(e(y+d)) \tag{3.412}$$

$$\iff e(k+y) = \ln(e(y+d))e^{\ln(e(y+d))} \tag{3.413}$$

$$\iff W_0(e(k+y)) = \ln(e(y+d)) \tag{3.414}$$

$$\iff e^{W_0(e(k+y))-1} - y = d, \tag{3.415}$$

where we made use of the Lambert W function, which is the inverse function of $x \mapsto xe^x$ and has multiple branches; however as $e(y+d) > 0$ W_0 is the only real branch which is applicable here, it corresponds to the inverse of $x \mapsto xe^x$ for $x > -1$. From the form of the maximising value we see, that it is always bigger than $-y$. Plugging this back onto our function we find its maximum

$$\begin{aligned}
&\max_{d \in]-y,\infty[} (y+d)^{k-d} = e^{(W_0(e(k+y))-1)(k+y)-(W_0(e(k+y))-1)e^{W_0(e(k+y))-1}} \\
&= e^{-(k+y)+(k+y)W_0(e(k+y))+e^{W_0(e(k+y))-1}-((k+y)e)/e} \\
&= e^{-2(k+y)+(k+y)W_0((k+y)e)+\frac{e(k+y)}{eW_0((k+y)e)}} \\
&= e^{(k+y)(-2+W_0((k+y)e)+W_0((k+y)e)^{-1})},
\end{aligned} \tag{3.416}$$

where we repeatedly used $W_0(x)e^{W_0(x)} = x$. Putting things together we find

$$\|\Gamma(B)^k \alpha_{l,c}\| \leqslant (6\beta)^k \|\alpha\| e^{(k+y)(-2+W_0((k+y)e)+W_0((k+y)e)^{-1})}. \tag{3.417}$$

Dividing this by $k!$ and using the lower bound given by Sterling's formula we would like to prove that

$$\sum_{k=1}^{\infty}(6\beta t)^k e^{k(1-\ln(k))-\frac{1}{2}\ln(k)+(k+y)(-2+W_0((k+y)e)+W_0((k+y)e)^{-1})} < \infty \tag{3.418}$$

holds, where we neglected constant factors and the summand $k = 0$ which do not matter for the task at hand. Next we are going to use an inequality about the growth of W_0 proven in [5]. For any $x \geqslant e$

$$W_0(x) \leqslant \ln(x) - \ln(\ln(x)) + \frac{e}{e-1}\frac{\ln(\ln(x))}{\ln(x)} \tag{3.419}$$

holds true. Plugging this into our sum the exponent is bounded from above by

$$\begin{aligned}
&k(1-\ln(k)) - \frac{1}{2}\ln(k) + (k+y)\Big[-1+\ln(k+y)-\ln(1+\ln(k+y)) \\
&+\frac{e}{e-1}\frac{\ln(1+\ln(k+y))}{1+\ln(k+y)} + W_0((k+y)e)^{-1}\Big] \\
&= -y + k\ln\left(1+\frac{y}{k}\right) + y\ln(k+y) - \frac{1}{2}\ln(k) + \\
&(k+y)\left[\ln(1+\ln(k+y))\frac{1-(e-1)\ln(k+y)}{(e-1)(1+\ln(k+y))} + W_0((k+y)e)^{-1}\right] \\
&\leqslant y\ln(k+y) - \frac{1}{2}\ln(k) + (k+y)W_0((k+y)e)^{-1} + \\
&(k+y)\ln(1+\ln(k+y))\frac{1-(e-1)\ln(k+y)}{(e-1)(1+\ln(k+y))}.
\end{aligned} \tag{3.420}$$

Now it is important to notice that the only remaining term that grows faster than linearly in magnitude is the last summand. This term;

however, is negative for large k, as the fraction converges to $-(e-1)$ for large k, while the double logarithm in front grows without bounds. So there is a k^* big enough such that for all $k > k^*$ (3.420) is smaller than $-k(\ln(6\beta t)+1)$, proving that (3.418) in fact holds.

$\square$

Theorem 43. *The operator* $\mathrm{d}\Gamma(B) : \mathcal{F}' \to \mathcal{F}$ *is essentially skew adjoint and hence by Stones theorem generates a strongly continuous unitary group* $\left(e^{t\ \widehat{\mathrm{d}\Gamma(B)}}\right)_t$*, where* $\widehat{\mathrm{d}\Gamma(B)}$ *is the closure of* $\mathrm{d}\Gamma(B)$*.*

Proof. In order to apply Nelson's analytic vector theorem we pick $C = \mathcal{F}'$. Pick $\alpha \in \mathcal{F}'$. We need to show that there is $t > 0$ such that

$$\sum_{k=0}^{\infty} \frac{\|\mathrm{d}\Gamma(B)^k \alpha\|}{k!} t^k < \infty \tag{3.421}$$

holds. This is guaranteed by the last lemma. $\square$

Lastly in this chapter, we will investigate the commutation properties of $\mathrm{d}\Gamma(B)$ with general creation and annihilation operators. These properties are the reason we are interested in this operator, they will prove to be very useful in the next chapter.

Theorem 44. *For* $\psi \in \mathcal{H}$ *we have*

$$[\mathrm{d}\Gamma(B), a^{\#}(\psi)] = a^{\#}(B\psi), \tag{3.422}$$

where $a^{\#}$ *can be either* a *or* a^**.*

Proof. Because $\mathrm{d}\Gamma(B)$ is defined as the extension of an operator on $\mathcal{F}^0$ it suffices to show the desired identity on this space. In order to do so we first restrict $\psi \in \mathrm{span}\{\varphi_n | n \in \mathbb{Z}\backslash\{0\}\}$. We will first cover the case $a(\psi)$. As a first step we decompose $\mathrm{d}\Gamma(B)$ into its four parts

$$[\mathrm{d}\Gamma(B), a(\psi)] = [\mathrm{d}\Gamma(B_{++}) + \mathrm{d}\Gamma(B_{-+}) + \mathrm{d}\Gamma(B_{-+}) + \mathrm{d}\Gamma(B_{--}), a(\psi)]\,, \tag{3.423}$$

each of those parts is evaluated directly. We begin with the B_{++} part, this can be expressed as

$$[\mathrm{d}\Gamma(B_{++}), a(\psi)] \tag{3.424}$$
$$= \sum_{n\in\mathbb{N}} a^*(B_{++}\varphi_n)a(\varphi_n)a(\psi) - \sum_{n\in\mathbb{N}} a(\psi)a^*(B_{++}\varphi_n)a(\varphi_n) \tag{3.425}$$
$$= \sum_{n\in\mathbb{N}} [-\langle\psi, B_{++}\varphi_n\rangle a(\varphi_n) + a(\psi)a^*(B_{++}\varphi_n)a(\varphi_n)] \tag{3.426}$$
$$- \sum_{n\in\mathbb{N}} a(\psi)a^*(B_{++}\varphi_n)a(\varphi_n). \tag{3.427}$$

Let $\alpha \in \mathcal{F}^0$. Now applying the expression in the last two lines to α online finitely many elements of the sum in fact contribute. So we may split the firs sum into two and observe the cancellation between the last two terms. Continuing we find

$$[\mathrm{d}\Gamma(B_{++}), a(\psi)]\,\alpha = -\sum_{n\in\mathbb{N}}\langle\psi, B_{++}\varphi_n\rangle a(\varphi_n)\alpha \tag{3.428}$$
$$= -a\left(\sum_{n\in\mathbb{N}}\langle B_{++}\varphi_n, \psi\rangle\varphi_n\right)\alpha = a\left(\sum_{n\in\mathbb{N}}\langle\varphi_n, B_{++}\psi\rangle\varphi_n\right)\alpha \tag{3.429}$$
$$= a(B_{++}\psi)\alpha, \tag{3.430}$$

where we used $B^* = -B^*$. Now for a general $\psi \in \mathcal{H}$ we pick a sequence $(\psi_k)_{k\in\mathbb{N}} \subset \mathrm{span}\{\varphi_n | n \in \mathbb{Z}\backslash\{0\}\}$ such that $\lim_{k\to\infty}\psi_k = \psi$. Now because of the calculation we have the equality

$$[\mathrm{d}\Gamma(B_{++}), a(\psi_k)]\,\alpha = a(B_{++}\psi_k)\alpha \tag{3.431}$$

for each $k \in \mathbb{N}$ and hence if a limit exists it also holds in the limit. Now on the right hand side, because a is a bounded operator on all of $\mathcal{F}$ clearly the limit exists and is equal to $a(B_{++}\psi)\alpha$. On the left hand side we know $\mathrm{d}\Gamma(B)$ to be bounded and hence continuous on every $\mathcal{F}_{m,p}$ for every $m, p \in \mathbb{N}$. Furhtermore since $\alpha \in \mathcal{F}^0$ there is are $m', p' \in \mathbb{N}$ such that $\alpha \in \bigoplus_{m=0}^{m'} \bigoplus_{p=0}^{p'} \mathcal{F}_{m,p}$ holds and hence we can exchange the limit also with $\mathrm{d}\Gamma(B)$ and find

$$[\mathrm{d}\Gamma(B_{++}), a(\psi)]\,\alpha = a(B_{++}\psi)\alpha \tag{3.432}$$

for general $\psi \in \mathcal{H}$. The final extension of this equation to all $\alpha \in \mathcal{F}'$ happens via the continuous linear extension theorem on $\mathcal{F}_{m,p}$ for each $m, p \in \mathbb{N}$. The proof in all seven other cases are completely analogous. Putting thins together again we obtain

$$\begin{aligned} [\mathrm{d}\Gamma(B_{++}), a(\psi)] + [\mathrm{d}\Gamma(B_{-+}), a(\psi)] \qquad (3.433)\\ + [\mathrm{d}\Gamma(B_{+-}), a(\psi)] + [\mathrm{d}\Gamma(B_{--}), a(\psi)] = \qquad (3.434)\\ a(B_{++}\psi) + a(B_{+-}\psi) + a(B_{-+}\psi) + a(B_{--}\psi) \iff \qquad (3.435)\\ [\mathrm{d}\Gamma(B), a(\psi)] = a(B\psi) \qquad (3.436) \end{aligned}$$

on all of $\mathcal{F}'$.

$\square$

3.4.4 Presentation and Proof of the Formula

In this chapter we verify the formula for the S-matrix directly. For a heuristic derivation in the appendix in section3.4.5

Theorem 45. *For A such that*

$$\|\mathbb{1} - U^A\| < 1. \tag{3.437}$$

The second quantized scattering operator fulfils

$$S^A = e^{i\varphi^A} e^{\mathrm{d}\Gamma(\ln(U^A))} \tag{3.438}$$

for some phase $\varphi^A \in \mathbb{R}$, that may depend on the external field A.

Proof. In order to establish this theorem we need to verify that the expression given in (45) for the scattering operator is a well defined object and fulfils (lift condition) and (adjoint lift condition). Because these conditions uniquely fix the Operator S^A up to a phase this suffices as a proof.
Well definedness is established, by theorem 43, because for unitary U^A with $\|1 - U^A\| < 1$ the power series of the logarithm converges and fulfils

$$\|\ln(U^A)\| = \|\ln(1 - (1 - U^A))\| = \left\| -\sum_{k=1}^{\infty} \frac{(1 - U^A)^k}{k} \right\| \tag{3.439}$$

$$\leqslant \sum_{k=1}^{\infty} \frac{\|1 - U^A\|^k}{k} = -\ln(1 - \|1 - U^A\|) \tag{3.440}$$

implying that the power series of the logarithm around the identity is a well defined map from the one particle operators of norm less than one to the bounded one particle operators. Moreover this operator fulfils $[\ln(U^A)]^* = \ln(U^A)^* = \ln(U^A)^{-1} = -\ln(U^A)$, so $\mathrm{d}\Gamma(\ln U^A)$ is a well defined unbounded operator that is essentially self adjoint on the finite particle sector of Fockspace $\mathcal{F}'$.

Let $\varphi \in \mathcal{H}$, for any $k \in \mathbb{N}_0$ we see applying the commutation relation of $\mathrm{d}\Gamma$:

$$\begin{aligned}
\mathrm{d}\Gamma(\ln U) \sum_{l=0}^{k} \binom{k}{l} a^{\#}\left((\ln U)^{l}\varphi\right)(\mathrm{d}\Gamma(\ln U))^{k-l} = \\
\sum_{l=0}^{k} \binom{k}{l} a^{\#}\left((\ln U)^{l+1}\varphi\right)(\mathrm{d}\Gamma(\ln U))^{k-l} \\
+ \sum_{l=0}^{k} \binom{k}{l} a^{\#}\left((\ln U)^{l}\varphi\right)(\mathrm{d}\Gamma(\ln U))^{k-l+1} \\
= \sum_{b=0}^{k+1} \left(\binom{k}{b-1} + \binom{k}{b}\right) a^{\#}\left((\ln U)^{b}\varphi\right)(\mathrm{d}\Gamma(\ln U))^{k+1-b} \\
= \sum_{b=0}^{k+1} \binom{k+1}{b} a^{\#}\left((\ln U)^{b}\varphi\right)(\mathrm{d}\Gamma(\ln U))^{k+1-b},
\end{aligned}$$

so we see that for $k \in \mathbb{N}_0$

$$(\mathrm{d}\Gamma(\ln U))^{k} a^{\#}(\varphi) = \sum_{b=0}^{k} \binom{k}{b} a^{\#}\left((\ln U)^{b}\varphi\right)(\mathrm{d}\Gamma(\ln U))^{k-b} \tag{3.441}$$

holds. Let $\alpha \in \mathcal{F}'$. Using what we just obtained, we conclude

$$\begin{aligned}
e^{\mathrm{d}\Gamma(\ln U)} a^{\#}(\varphi) &= \sum_{k=0}^{\infty} \frac{1}{k!} (\mathrm{d}\Gamma(\ln U))^{k} a^{\#}(\varphi)\alpha \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{b=0}^{k} \binom{k}{b} a^{\#}\left((\ln U)^{b}\varphi\right)(\mathrm{d}\Gamma(\ln U))^{k-b} \alpha \\
&\overset{*}{=} \sum_{c=0}^{\infty} \sum_{l=0}^{\infty} \frac{1}{c!l!} a^{\#}\left((\ln U)^{c}\varphi\right)(\mathrm{d}\Gamma(\ln U))^{l} \alpha \\
&= a^{\#}\left(e^{\ln U}\varphi\right) e^{\mathrm{d}\Gamma(\ln U)}\alpha = a^{\#}(U\varphi)\, e^{\mathrm{d}\Gamma(\ln U)}\alpha.
\end{aligned}$$

For the marked equality changing oder of summation is justified, because by the bounds $\|a^{\#}((\ln U)^c\varphi)\| \leqslant \|\ln U\|^c$ and lemma 42 the sum obtained by changing the order of summands converges absolutely. Clearly multiplying the second quantised operator by an additional phase as in (45) does not influence this calculation. $\square$

Todo: decide whether to use or delete this part.

As a preparation for calculating the vacuum polarisation current we proof the following

Lemma 46. *Let $P_k, P_l \in Q$ then the following holds*

$$[G(P_k), G(P_l)] = \operatorname{tr}\left(P_{\substack{k\\-+}} P_{\substack{l\\+-}}\right) - \operatorname{tr}\left(P_{\substack{l\\-+}} P_{\substack{k\\+-}}\right) + G\left([P_k, P_l]\right). \tag{3.442}$$

For a proof of this lemma let $P_k, P_l \in Q$, we compute

$$
\begin{aligned}
&[G(P_k), G(P_l)] \overset{(??)}{=} \\
&= \sum_{n,b\in\mathbb{N}} \left[a^*\left(P_k\varphi_n\right) a(\varphi_n), a^*\left(P_l\varphi_b\right) a(\varphi_n)\right] \\
&- \sum_{-b,n\in\mathbb{N}} \left[a^*\left(P_k\varphi_n\right) a(\varphi_n), a(\varphi_b) a^*\left(P_l\varphi_b\right)\right] \\
&- \sum_{-n,b\in\mathbb{N}} \left[a(\varphi_n) a^*\left(P_k\varphi_n\right), a^*\left(P_l\varphi_b\right) a(\varphi_b)\right] \\
&+ \sum_{n,b\in\mathbb{N}} \left[a(\varphi_n) a^*\left(P_k\varphi_n\right), a(\varphi_b) a^*\left(P_l\varphi_b\right)\right] \\
&= \sum_{n,b\in\mathbb{N}} \left(a^*\left(P_k\varphi_n\right) \langle\varphi_n, P_l\varphi_b\rangle a(\varphi_b) - a^*\left(P_l\varphi_b\right) \langle\varphi_b, P_k\varphi_n\rangle a(\varphi_n)\right) \\
&- \sum_{-b,n\in\mathbb{N}} \left(-\langle\varphi_b, P_k\varphi_n\rangle a(\varphi_n) a^*\left(P_l\varphi_b\right) + a(\varphi_b) a^*\left(P_k\varphi_n\right) \langle\varphi_n, P_l\varphi_b\rangle\right) \\
&- \sum_{-n,b\in\mathbb{N}} \left(-\langle\varphi_n, P_l\varphi_b\rangle a^*\left(P_k\varphi_n\right) a(\varphi_b) + a^*\left(P_l\varphi_b\right) a(\varphi_n) \langle\varphi_b, P_k\varphi_n\rangle\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{n,b\in-\mathbb{N}} \left(a(\varphi_n)\left\langle \varphi_b, P_k\varphi_n\right\rangle a^*\left(P_l\varphi_b\right) - a(\varphi_b)\left\langle \varphi_n, P_l\varphi_b\right\rangle a^*\left(P_k\varphi_n\right)\right) \\
&= \sum_{b\in\mathbb{N}} a^*\left(P_k P_{\substack{l\\++}}\varphi_b\right) a(\varphi_b) - \sum_{n\in\mathbb{N}} a^*\left(P_l P_{\substack{k\\++}}\varphi_n\right) a(\varphi_n) \\
&+ \sum_{n\in\mathbb{N}} a(\varphi_n) a^*\left(P_l P_{\substack{k\\-+}}\varphi_n\right) - \sum_{-b\in\mathbb{N}} a(\varphi_b) a^*\left(P_k P_{\substack{l\\+-}}\varphi_b\right) \\
&+ \sum_{b\in\mathbb{N}} a^*\left(P_k P_{\substack{l\\-+}}\varphi_b\right) a(\varphi_b) - \sum_{-n\in\mathbb{N}} a^*\left(P_l P_{\substack{k\\+-}}\varphi_n\right) a(\varphi_n) \\
&+ \sum_{-n\in\mathbb{N}} a(\varphi_n) a^*\left(P_l P_{\substack{k\\--}}\varphi_n\right) - \sum_{-b\in\mathbb{N}} a(\varphi_b) a^*\left(P_k P_{\substack{l\\--}}\varphi_b\right) \\
= &\sum_{n\in\mathbb{N}} a^*\left(P_k P_l\varphi_n\right) a(\varphi_n) - \sum_{n\in\mathbb{N}} a^*\left(P_l P_{\substack{k\\++}}\varphi_n\right) a(\varphi_n) \\
&+ \operatorname{tr}\left(P_{\substack{l\\+-}} P_{\substack{k\\-+}}\right) - \sum_{n\in\mathbb{N}} a^*\left(P_l P_{\substack{k\\-+}}\varphi_n\right) a(\varphi_n) \\
&- \operatorname{tr}\left(P_{\substack{l\\-+}} P_{\substack{k\\+-}}\right) + \sum_{-b\in\mathbb{N}} a(\varphi_b) a^*\left(P_l P_{\substack{k\\+-}}\varphi_b\right) \\
&+ \sum_{-b\in\mathbb{N}} a(\varphi_b) a^*\left(P_l P_{\substack{k\\--}}\varphi_b\right) - \sum_{-b\in\mathbb{N}} a(\varphi_b) a^*\left(P_k P_l\varphi_b\right) \\
&= \operatorname{tr}\left(P_{\substack{l\\+-}} P_{\substack{k\\-+}}\right) - \operatorname{tr}\left(P_{\substack{l\\-+}} P_{\substack{k\\+-}}\right) \\
&+ \sum_{n\in\mathbb{N}} a^*\left(\left[P_k, P_l\right]\varphi_n\right) a(\varphi_n) + \sum_{-b\in\mathbb{N}} a(\varphi_b) a^*\left(\left[P_l, P_k\right]\varphi_b\right) \\
&= \operatorname{tr}\left(P_{\substack{l\\+-}} P_{\substack{k\\-+}}\right) - \operatorname{tr}\left(P_{\substack{l\\-+}} P_{\substack{k\\+-}}\right) + G\left(\left[P_k, P_l\right]\right)
\end{aligned}
$$

$\square$

Definition 47. *For $k \in \mathbb{N}_0$, $X, Y \in \mathcal{B}(\mathcal{H})$ the nested commutator $[X,Y]_k$ is defined inductively as*

$$[X,Y]_0 := Y$$

$$[X,Y]_{k+1} := [X,[X,Y]_k] \quad \forall k \in \mathbb{N}_0.$$

Lemma 48. *For $m \in \mathbb{N}$ and $B, C \in Q$ the following holds*

$$[G(B),G(C)]_m = \operatorname{tr}\left(P_- B P_+ [B,C]_{m-1}\right) - \operatorname{tr}\left(P_+ B P_- [B,C]_{m-1}\right) + G\left([B,C]_m\right). \quad (3.443)$$

Proof: Proof by Induction is the first thing that comes to mind, looking at the claim. Indeed, $m = 1$ is the consequence of the lemma 46. For m general we have

$$\begin{aligned}
[G(B),G(C)]_{m+1} &= [G(B),[G(B),G(C)]_m] \\
&\overset{\text{ind.hyp.}}{=} [G(B), \operatorname{tr}\left(P_- B P_+ [B,C]_{m-1}\right) - \operatorname{tr}\left(P_+ B P_- [B,C]_{m-1}\right) + G\left([B,C]_m\right)] \\
&= [G(B), G\left([B,C]_m\right)] \\
&\overset{\text{lemma 46}}{=} \operatorname{tr}\left(P_- B P_+ [B,C]_m\right) - \operatorname{tr}\left(P_+ B P_- [B,C]_m\right) + G\left([B,[B,C]_m]\right) \\
&= \operatorname{tr}\left(P_- B P_+ [B,C]_m\right) - \operatorname{tr}\left(P_+ B P_- [B,C]_m\right) + G\left([B,C]_{m+1}\right)
\end{aligned} \quad (3.444)$$

□

Lemma 49. *For external potentials A, X small enough the derivatives of the scattering operator can be computed to fulfil*

$$\partial_\varepsilon|_{\varepsilon=0} e^{G \ln U^{A+\varepsilon X}} = e^{G \ln U^A} \dot{j}^0_A(X) + e^{G \ln U^A} G((U^A)^{-1} \partial_\varepsilon U^{A+\varepsilon X}) \quad (3.445)$$

$$\partial_\varepsilon|_{\varepsilon=0} e^{-G \ln U^{A+\varepsilon X}} = -e^{-G \ln U^A} \dot{j}^0_A(X) + G(\partial_\varepsilon \left(U^{A+\varepsilon X}\right)^{-1} U^A) e^{-G \ln U^A}, \quad (3.446)$$

with

$$\dot{j}^0_A(X) := \sum_{l \in \mathbb{N}_0} \frac{(-1)^{l+1}}{(l+2)!} \left(\operatorname{tr} P_- \ln U^A P_+ \left[\ln U^A, \partial_\varepsilon \ln U^{A+\varepsilon X}\right]_l - \operatorname{tr} P_+ \ln U^A P_- \left[\ln U^A, \partial_\varepsilon \ln U^{A+\varepsilon X}\right]_l \right). \quad (3.447)$$

Proof: We start out by employing Duhamel's and Hadamard's formulas. These are

ref!! + restrictions, something better than this

$$\partial_\alpha \, e^{Y+\alpha X}\big|_{\alpha=0} = \int_0^1 dt e^{(1-t)Y} X e^{tY} \qquad \text{(Duhamel's formula)}$$

and

$$e^X Y e^{-X} = \sum_{k=0}^{\infty} \frac{1}{k!}[X,Y]_k. \qquad \text{(Hadamard's formula)}$$

So one gets

$$\begin{aligned} \partial_\varepsilon|_{\varepsilon=0} e^{G\ln U^{A+\varepsilon X}} &= \int_0^1 dz e^{(1-z)G\ln U^A} \partial_\varepsilon|_{\varepsilon=0} G\ln U^{A+\varepsilon X} e^{zG\ln U^A} \quad (3.448) \\ &= e^{G\ln U^A} \int_0^1 dz \sum_{l\in\mathbb{N}_0} \frac{1}{l!} \left[-zG\ln U^A, \partial_\varepsilon|_{\varepsilon=0} G\ln U^{A+\varepsilon X}\right]_l \\ &= e^{G\ln U^A} \int_0^1 dz \sum_{l\in\mathbb{N}_0} \frac{(-z)^l}{l!} \partial_\varepsilon|_{\varepsilon=0} \left[G\ln U^A, G\ln U^{A+\varepsilon X}\right]_l. \end{aligned}$$

At this point we see that for $l = 0$ the summand vanishes. For all other values of l we use lemma 48, yielding

$$\begin{aligned} \partial_\varepsilon|_{\varepsilon=0} e^{G\ln U^{A+\varepsilon X}} &= e^{G\ln U^A} \int_0^1 dz \sum_{l\in\mathbb{N}} \frac{(-z)^l}{l!} \partial_\varepsilon|_{\varepsilon=0} \left(G\left(\left[\ln U^A, \ln U^{A+\varepsilon X}\right]\right)\right. \\ &\quad + \operatorname{tr} P_- \ln U^A P_+ \left[\ln U^A, \partial_\varepsilon|_{\varepsilon=0} \ln U^{A+\varepsilon X}\right]_{l-1} \\ &\quad \left. - \operatorname{tr} P_+ \ln U^A P_- \left[\ln U^A, \partial_\varepsilon|_{\varepsilon=0} \ln U^{A+\varepsilon X}\right]_{l-1}\right). \quad (3.449) \end{aligned}$$

The last two terms together result in the first term of (3.445) after performing the integration and shifting the summation index. For the first term we will use linearity and continuity of G and use the same identities backwards to give

contiuity of G!!

$$
\begin{aligned}
&e^{G \ln U^A} \int_0^1 dz \sum_{l \in \mathbb{N}} \frac{(-z)^l}{l!} \partial_\varepsilon|_{\varepsilon=0} G\left(\left[\ln U^A, \ln U^{A+\varepsilon X}\right]\right) \\
&= e^{G \ln U^A} G\left(\int_0^1 dz \sum_{l \in \mathbb{N}} \frac{1}{l!}\left[-z \ln U^A, \partial_\varepsilon|_{\varepsilon=0} \ln U^{A+\varepsilon X}\right]\right) \\
&= e^{G \ln U^A} G\left(e^{-\ln U^A} \int_0^1 dz e^{\ln U^A} e^{-z \ln U^A} \partial_\varepsilon|_{\varepsilon=0} \ln U^{A+\varepsilon X} e^{z \ln U^A}\right) \\
&= e^{G \ln U^A} G\left(e^{\ln\left(U^A\right)^{-1}} \partial_\varepsilon|_{\varepsilon=0} e^{\ln U^{A+\varepsilon X}}\right) \\
&= e^{G \ln U^A} G\left(\left(U^A\right)^{-1} \partial_\varepsilon|_{\varepsilon=0} U^{A+\varepsilon X}\right).
\end{aligned} \tag{3.450}
$$

Putting things together results in the first equality we wanted to prove. For the second one the computation is completely analogous, except for after applying Duhamel's formula as in (3.448) we substitute $u = 1-z$. The minus sign in front of the first term then arises by the chain rule, where as the second term does not share the sign change with the first, since we have to revert the use of the chain rule in the second half of the calculation when we apply (Duhamel's formula) backwards. □

Definition 50. *We use Bogoliubov's formula to define the vacuum* ref!! *ectation value of the current*

$$j_A(F) = i\partial_\varepsilon \left\langle \Omega, S^{A^*} S^{A+\varepsilon F} \Omega \right\rangle\Big|_{\varepsilon=0}. \tag{3.451}$$

Theorem 51. *The vacuum expectation value of the current of the* *scattering operator takes the form*

$$j_A(F) = -\partial_\varepsilon\, \varphi(A+\varepsilon F)|_{\varepsilon=0}$$
$$-2\int_0^1 dz(1-z)\Im\operatorname{tr}\left(P_+ \ln U^A P_- e^{-z\ln U^A}\partial_\varepsilon \ln U^{A+\varepsilon F}\Big|_{\varepsilon=0} e^{z\ln U^A}\right)$$
$$= -\partial_\varepsilon\, \varphi(A+\varepsilon F)|_{\varepsilon=0}$$
$$-2\Im\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+2)!}\operatorname{tr}\left(P_+\ln U^A P_-\left[\ln U^A, \partial_\varepsilon \ln U^{A+\varepsilon F}\Big|_{\varepsilon=0}\right]_k\right)$$

Proof: By theorem 45 and abbreviating $\varphi(A) = \sum_{n\in\mathbb{N}}\frac{C_n(A)}{n!}$we see that the current can be written in the form

$$\begin{aligned} j_A(F) &= i\partial_\varepsilon\left\langle\Omega, S^{A*}S^{A+\varepsilon F}\Omega\right\rangle\Big|_{\varepsilon=0}\\ &= i\partial_\varepsilon\left\langle\Omega, e^{-i\varphi(A)}e^{-G(\ln(U^A))}e^{i\varphi(A+\varepsilon F)}e^{G(\ln(U^{A+\varepsilon F}))}\Omega\right\rangle\Big|_{\varepsilon=0}\\ &= -\partial_\varepsilon\,\varphi(A+\varepsilon F)|_{\varepsilon=0} + i\langle\Omega, \partial_\varepsilon\, e^{-G(\ln(U^A))}e^{G(\ln(U^{A+\varepsilon F}))}\Omega\rangle\Big|_{\varepsilon=0}, \end{aligned} \tag{3.452}$$

so the first summand works out just as claimed. For the second summand we employ lemma 49 and note that the vacuum expectation value of G vanishes no matter its argument.

$$\begin{aligned} &i\langle\Omega, \partial_\varepsilon\, e^{-G(\ln(U^A))}e^{G(\ln(U^{A+\varepsilon F}))}\Omega\rangle\Big|_{\varepsilon=0}\\ &\quad= -i\partial_\varepsilon\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+2)!}\operatorname{tr}\left(P_-\ln U^A P_+[\ln U^A, \ln U^{A+\varepsilon F}]_k\right)\Big|_{\varepsilon=0}\\ &\quad\quad+ i\partial_\varepsilon\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+2)!}\operatorname{tr}\left(P_+\ln U^A P_-[\ln U^A, \ln U^{A+\varepsilon F}]_k\right)\Big|_{\varepsilon=0} \end{aligned} \tag{3.453}$$

In order to apply Hadamard's formula once again in the opposite direction, we introduce two auxiliary integrals. The second term then becomes

$$
\begin{aligned}
& i\partial_\varepsilon \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+2)!} \operatorname{tr}\left(P_+ \ln U^A P_- [\ln U^A, \ln U^{A+\varepsilon F}]_k\right)\Big|_{\varepsilon=0} \\
&= i\partial_\varepsilon \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_0^1 dt \int_0^1 s^k t^{k+1} \operatorname{tr}\left(P_+ \ln U^A P_- [\ln U^A, \ln U^{A+\varepsilon F}]_k\right)\Big|_{\varepsilon=0} \\
&= i\partial_\varepsilon \sum_{k=0}^{\infty} \frac{1}{k!} \int_0^1 dt \int_0^1 ds\, t \operatorname{tr}\left(P_+ \ln U^A P_- [-ts \ln U^A, \ln U^{A+\varepsilon F}]_k\right)\Big|_{\varepsilon=0} \\
&= i\partial_\varepsilon \sum_{k=0}^{\infty} \frac{1}{k!} \int_0^1 dz \int_z^1 ds\; \operatorname{tr}\left(P_+ \ln U^A P_- [-z \ln U^A, \ln U^{A+\varepsilon F}]_k\right)\Big|_{\varepsilon=0} \\
&= i\partial_\varepsilon \sum_{k=0}^{\infty} \frac{1}{k!} \int_0^1 dz (1-z) \operatorname{tr}\left(P_+ \ln U^A P_- [-z \ln U^A, \ln U^{A+\varepsilon F}]_k\right)\Big|_{\varepsilon=0} \\
&= i\partial_\varepsilon \int_0^1 dz (1-z) \operatorname{tr}\left(P_+ \ln U^A P_- \sum_{k=0}^{\infty} \frac{1}{k!} [-z \ln U^A, \ln U^{A+\varepsilon F}]_k\right)\Big|_{\varepsilon=0} \\
&\overset{\text{(Hadamard's formula)}}{=} i\partial_\varepsilon \int_0^1 dz (1-z) \operatorname{tr}\left(P_+ \ln U^A P_- e^{-z \ln U^A} \ln U^{A+\varepsilon F} e^{z \ln U^A}\right)\Big|_{\varepsilon=0} \\
&= i \int_0^1 dz (1-z) \operatorname{tr}\left(P_+ \ln U^A P_- e^{-z \ln U^A} \partial_\varepsilon \ln U^{A+\varepsilon F}\big|_{\varepsilon=0} e^{z \ln U^A}\right).
\end{aligned}
\tag{3.454}
$$

The calculation for the first term of (3.453) is identical. At this point we notice that (3.454) and the term where the projectors are exchanged are complex conjugates of one another. So summarising we find

$$
\begin{aligned}
&j_A(F) = -\partial_\varepsilon \, \varphi(A+\varepsilon F)\big|_{\varepsilon=0} \\
&-2\int_0^1 dz(1-z)\Im \operatorname{tr}\left(P_+ \ln U^A P_- e^{-z\ln U^A} \partial_\varepsilon \ln U^{A+\varepsilon F}\big|_{\varepsilon=0} e^{z\ln U^A}\right).
\end{aligned}
$$

□

Theorem 52. *Independent of the phase that is used to correct the scattering operator the following formula holds true for any four potentials A, F, H, with A small enough so that the relevant series converge.*

$$
\begin{aligned}
&\partial_\varepsilon|_{\varepsilon=0}(j_{A+\varepsilon H}(F) - j_{A+\varepsilon F}(H)) = \\
&\qquad 2\Im \operatorname{tr}\left(P_+ \left(U^A\right)^{-1} \partial_\varepsilon|_{\varepsilon=0} U^{A+\varepsilon F} P_- \left(U^A\right)^{-1} \partial_\delta|_{\delta=0} U^{A+\delta H}\right) \qquad (3.455)
\end{aligned}
$$

Proof: We compute $\partial_\varepsilon|_{\varepsilon=0} j_{A+\varepsilon F}(H)$.

$$
\begin{aligned}
&-i\partial_\varepsilon|_{\varepsilon=0} j_{A+\varepsilon H}(F) = \\
&\partial_\varepsilon|_{\varepsilon=0}\partial_\delta|_{\delta=0}\langle \Omega, e^{i\varphi(A+\varepsilon H+\delta F)-i\varphi(A+\varepsilon H)} e^{-G\ln U^{A+\varepsilon H}} e^{G\ln U^{A+\varepsilon H+\delta F}} \Omega\rangle
\end{aligned}
$$

We first act with the derivative with respect to H, fixing F.

$$
\begin{aligned}
&-i\partial_\varepsilon|_{\varepsilon=0} j_{A+\varepsilon H}(F) = \\
&\partial_\delta|_{\delta=0} i(\partial_\varepsilon|_{\varepsilon=0}\varphi(A+\varepsilon H+\delta F) - \partial_\varepsilon|_{\varepsilon=0}\varphi(A+\varepsilon H)) e^{i\varphi(A+\delta F)-i\varphi(A)} \\
&\langle \Omega, e^{-G\ln U^A} e^{G\ln U^{A+\delta F}} \Omega\rangle \\
&+\partial_\delta|_{\delta=0} e^{i\varphi(A+\delta F)-i\varphi(A)} \langle \Omega \partial_\varepsilon|_{\varepsilon=0} e^{-G\ln U^{A+\varepsilon H}} e^{G\ln U^{A+\delta F}} \Omega\rangle \\
&+\partial_\delta|_{\delta=0} e^{i\varphi(A+\delta F)-i\varphi(A)} \langle \Omega e^{-G\ln U^A} \partial_\varepsilon|_{\varepsilon=0} e^{G\ln U^{A+\varepsilon H+\delta F}} \Omega\rangle
\end{aligned}
$$

In computing further one can notice a few cancellations. For the first summand the first factor vanishes if δ is set to zero, so the only the first summand in the product rule will not vanish. For the second and third summand we will use lemma 49, giving

$$
\begin{aligned}
&-i\partial_\varepsilon|_{\varepsilon=0} j_{A+\varepsilon H}(F) = \\
&\partial_\delta|_{\delta=0} i\partial_\varepsilon|_{\varepsilon=0}\varphi(A+\varepsilon H+\delta F) \\
&-\partial_\delta|_{\delta=0} e^{i\varphi(A+\delta F)-i\varphi(A)} j^0_A(H)\langle\Omega, e^{-G\ln U^A} e^{G\ln U^{A+\delta F}}\Omega\rangle \\
&+\partial_\delta|_{\delta=0} e^{i\varphi(A+\delta F)-i\varphi(A)}\langle\Omega, G\left(\partial_\varepsilon|_{\varepsilon=0}\left(U^{A+\varepsilon H}\right)^{-1} U^A\right) e^{-G\ln U^A} e^{G\ln U^{A+\delta F}}\Omega\rangle \\
&+\partial_\delta|_{\delta=0} e^{i\varphi(A+\delta F)-i\varphi(A)} j^0_{A+\delta F}(H)\langle\Omega, e^{-G\ln U^A} e^{G\ln U^{A+\delta F}}\Omega\rangle \\
&+\partial_\delta|_{\delta=0} e^{i\varphi(A+\delta F)-i\varphi(A)}\langle\Omega, e^{-G\ln U^A} e^{G\ln U^{A+\delta F}} G\left(\left(U^{A+\delta F}\right)^{-1}\partial_\varepsilon|_{\varepsilon=0} U^{A+\varepsilon H+\delta F}\right)\Omega\rangle.
\end{aligned}
$$

Now there are a few further simplifications to appreciate: since $\langle\Omega, G\Omega\rangle = 0$, in the third and last summand only the derivatives with respect to δ which produce by lemma 49 another factor of G will contribute to the sum. For the other summands except for the first we can spot the appearance of j^0. Respecting all this results in

$$
\begin{aligned}
&-i\partial_\varepsilon|_{\varepsilon=0} j_{A+\varepsilon H}(F) = i\partial_\delta|_{\delta=0}\partial_\varepsilon|_{\varepsilon=0}\varphi(A+\varepsilon H+\delta F) \\
&-i\partial_\delta|_{\delta=0}\varphi(A+\delta F) j^0_A(H) - j^0_A(H) j^0_A(F) \\
&+\langle\Omega, G\left(\partial_\varepsilon|_{\varepsilon=0}\left(U^{A+\varepsilon H}\right)^{-1} U^A\right) G\left(\left(U^A\right)^{-1}\partial_\delta|_{\delta=0} U^{A+\delta F}\right)\Omega\rangle \\
&+i\partial_\delta|_{\delta=0}\varphi(A+\delta F) j^0_A(H) + \partial_\delta|_{\delta=0} j^0_{A+\delta F}(H) + j^0_A(H) j^0_A(F) \\
&+\langle\Omega, G\left(\left(U^A\right)^{-1}\partial_\delta|_{\delta=0} U^{A+\delta F}\right) G\left(\left(U^A\right)^{-1}\partial_\varepsilon|_{\varepsilon=0} U^{A+\varepsilon H}\right)\Omega\rangle.
\end{aligned}
$$

A few more terms cancel in the second and fourth line, also since $\partial_\varepsilon|_{\varepsilon=0}\left(U^{A+\varepsilon H}\right)^{-1} U^{A+\varepsilon H} = 0$ we can combine the two products of G into a commutator:

$$
\begin{aligned}
&- i\partial_\varepsilon|_{\varepsilon=0} j_{A+\varepsilon H}(F) = i\partial_\delta|_{\delta=0}\partial_\varepsilon|_{\varepsilon=0}\varphi(A+\varepsilon H+\delta F) \\
&+ \partial_\delta|_{\delta=0} j^0_{A+\delta F}(H) \\
&+ \langle \Omega, \left[G\left(\left(U^A\right)^{-1} \partial_\delta|_{\delta=0} U^{A+\delta F} \right), G\left(\left(U^A\right)^{-1} \partial_\varepsilon|_{\varepsilon=0} U^{A+\varepsilon H} \right) \right] \Omega \rangle.
\end{aligned}
$$

So we can once again apply lemma 46, which results in exactly right hand side of the equation we claimed to produce in the statement of this theorem. So all that is left is to recognise that one can combine the first two summands into $-i\partial_\varepsilon j_{A+\varepsilon H}(F)$, which is a direct consequence of theorem 51. $\square$

3.4.5 Quantitative Estimates

Since we do not only want to give an expression for the time evolution operator, but also give bounds on the numerical errors which are due to truncate the occurring series we need to look at these series a little closer. The series involve powers of the second quantisation operator G, so we start by examining these in greater depth. In order to do so we define an object closely related to G.

Todo: probably this part cannot be made rigorous. Decide whether to keep it as heuristics

Definition 53.

$$
\begin{aligned}
L : \{M \subset B(\mathcal{H}) \mid |M| < \infty\} \times \{M \subset B(\mathcal{H}) \mid |M| < \infty\} \to B(\mathcal{F}) \\
L(\{A_1, \ldots, A_c\}, \{B_1, \ldots, B_m\}) := \prod_{l=1}^{m} a(\varphi_{-k_l}) \\
\prod_{l=1}^{c} a^*(A_l \varphi_{n_l}) \prod_{l=1}^{m} a^*(B_l \varphi_{-k_l}) \prod_{l=1}^{c} a(\varphi_{n_l}), \quad (3.456)
\end{aligned}
$$

where for notational reasons we chose to list the occurring one-particle operators in a specific order; however, the order does not matter, since *commutation of the relevant creation and annihilation operators always results an overall factor of one.*

Since this operator L occurs when computing powers of G we compute is product with some G with the following

Lemma 54. *For any $a, b, \in \mathbb{N}_0$ and apropriate one particle operators A_k, B_l, C for $1 \leqslant k \leqslant a$, $1 \leqslant l \leqslant b$ we have the following equality*

$$L\Big(\bigcup_{l=1}^{a}\{A_l\};\bigcup_{l=1}^{b}\{B_l\}\Big)G(C) = \tag{3.457}$$

$$(-1)^{a+b}L\Big(\bigcup_{l=1}^{a}\{A_l\}\cup\{C\};\bigcup_{l=1}^{b}\{B_l\}\Big) \tag{3.458}$$

$$+(-1)^{a+b+1}L\Big(\bigcup_{l=1}^{a}\{A_l\};\bigcup_{l=1}^{b}\{B_l\}\cup\{C\}\Big) \tag{3.459}$$

$$+\sum_{f=1}^{a}L\Big(\bigcup_{\substack{l=1\\ l\neq f}}^{a}\{A_l\}\cup\{A_f P_+ C\};\bigcup_{l=1}^{b}\{B_l\}\Big) \tag{3.460}$$

$$+\sum_{f=1}^{a}L\Big(\bigcup_{\substack{l=1\\ f\neq l}}^{a}\{A_l\}\cup\{-CP_- A_f\};\bigcup_{l=1}^{b}\{B_l\}\Big) \tag{3.461}$$

$$-\sum_{f=1}^{a}L\Big(\bigcup_{\substack{l=1\\ f\neq l}}^{a}\{A_l\};\bigcup_{l=1}^{b}\{B_l\}\cup\{A_f P_+ C\}\Big) \tag{3.462}$$

$$+\sum_{f=1}^{b}L\Big(\bigcup_{l=1}^{a}\{A_l\};\bigcup_{\substack{l=1\\ l\neq f}}^{b}\{B_l\}\cup\{-CP_- B_f\}\Big) \tag{3.463}$$

$$+(-1)^{a+b+1}\sum_{f=1}^{a}\operatorname{tr}\Big(P_+ C P_- A_f\Big)L\Big(\bigcup_{\substack{l=1\\ l\neq f}}^{a}\{A_l\};\bigcup_{l=1}^{b}\{B_l\}\Big) \tag{3.464}$$

$$+ (-1)^{a+b+1} \sum_{\substack{f_1,f_2=1\\ f_1\neq f_2}}^{a} L\Big(\bigcup_{\substack{l=1\\ l\neq f_1,f_2}}^{a} \{A_l\} \cup \{-A_{f_2}P_+CP_-A_{f_1}\}; \bigcup_{l=1}^{b}\{B_l\}\Big) \tag{3.465}$$

$$+ (-1)^{a+b+1} \sum_{f=1}^{b}\sum_{g=1}^{a} L\Big(\bigcup_{\substack{l=1\\ l\neq g}}^{a}\{A_l\}; \bigcup_{\substack{l=1\\ l\neq f}}^{b}\{B_l\} \cup \{-A_gP_+CP_-B_f\}\Big). \tag{3.466}$$

Proof: The proof of this equality is a rather long calculation, where (3.456) is used repeatedly. We break up the calculation into several parts. Let us start with

$$L\left(\bigcup_{l=1}^{a}\{A_l\}; \bigcup_{l=1}^{b}\{B_l\}\right) L(C;) =$$
$$\prod_{l=1}^{b} a(\varphi_{-k_l}) \prod_{l=1}^{a} a^*(A_l\varphi_{n_l}) \prod_{l=1}^{b} a^*(B_l\varphi_{-k_l}) \prod_{l=1}^{a} a(\varphi_{n_l}) a^*(C\varphi_m) a(\varphi_m). \tag{3.467}$$

We (anti)commute the creation operator involving C to its place at the end of the second product, after that the term will be normally ordered and can be rephrased in terms of Ls. During the commutation the creation operator in question can be picked up by any of the annihilation operators in the rightmost product. For each term where that happens we can perform the sum over the basis of $\mathcal{H}^-$ related to the annihilation operator whose anticommutator triggered. After this sum the corresponding term is also normally ordered and can be rephrased in terms of an L after some reshuffling which may only produce signs. So performing these steps we get

$$\begin{aligned}
&L\left(\bigcup_{l=1}^{a}\{A_l\};\bigcup_{l=1}^{b}\{B_l\}\right)L(C;)= \\
&\quad\sum_{f=a}^{1}(-1)^{a-f}\prod_{l=1}^{b}a(\varphi_{-k_l})\prod_{l=1}^{f-1}a^*(A_l\varphi_{n_l})a^*(A_fP_+C\varphi_m) \\
&\quad\prod_{l=f+1}^{a}a^*(A_l\varphi_{n_l})\prod_{l=1}^{b}a^*(B_l\varphi_{-k_l})\prod_{\substack{l=1\\l\neq f}}^{a}a(\varphi_{n_l})a(\varphi_m) \\
&\quad+L\left(\bigcup_{l=1}^{a}\{A_l\}\cup\{C\};\bigcup_{l=1}^{b}\{B_l\}\right) \\
&=\sum_{f=1}^{a}L\left(\bigcup_{\substack{l=1\\l\neq f}}^{a}\{A_l\}\cup\{A_fP_+C\};\bigcup_{l=1}^{b}\{B_l\}\right) \\
&\quad+L\left(\bigcup_{l=1}^{a}\{A_l\}\cup\{C\};\bigcup_{l=1}^{b}\{B_l\}\right). \qquad (3.468)
\end{aligned}$$

Now the remaining case is more laborious, that is why we will split off and treat some of the appearing terms separately. We start off analogous to before

$$\begin{aligned}
&L\left(\bigcup_{l=1}^{a}\{A_l\};\bigcup_{l=1}^{b}\{B_l\}\right)L(;C)= \\
&\prod_{l=1}^{b}a(\varphi_{-k_l})\prod_{l=1}^{a}a^*(A_l\varphi_{n_l})\prod_{l=1}^{b}a^*(B_l\varphi_{-k_l})\prod_{l=1}^{a}a(\varphi_{n_l})a(\varphi_{-m})a^*(C\varphi_{-m}).
\end{aligned}\qquad (3.469)$$

This time we need to (anti)commute the rightmost annihilation operator all the way to the end of the first product and the creation operator to the end of the second but last product. So there will be several qualitatively different terms. From the first step alone we get

$$
\begin{aligned}
& L\left(\bigcup_{l=1}^{a}\{A_l\};\bigcup_{l=1}^{b}\{B_l\}\right)L(;C) = \\
& (-1)^a \sum_{f=b}^{1}(-1)^{b-f}\prod_{l=1}^{b}a(\varphi_{-k_l})\prod_{l=1}^{a}a^*(A_l\varphi_{n_l})\prod_{\substack{l=1\\ l\neq f}}^{b}a^*(B_l\varphi_{-k_l}) \\
& \qquad\qquad \prod_{l=1}^{a}a(\varphi_{n_l})a^*(CP_-B_f\varphi_{-k_f}) \qquad (3.470) \\
& +(-1)^{a+b} \sum_{f=a}^{1}(-1)^{b-f}\prod_{l=1}^{b}a(\varphi_{-k_l})\prod_{\substack{l=1\\ l\neq f}}^{a}a^*(A_l\varphi_{n_l}) \\
& \qquad\qquad \prod_{l=1}^{b}a^*(B_l\varphi_{-k_l})\prod_{l=1}^{a}a(\varphi_{n_l})a^*(CP_-\varphi_{n_f}) \qquad (3.471) \\
& +(-1)^b\prod_{l=1}^{b}a(\varphi_{-k_l})a(\varphi_{-m})\prod_{l=1}^{a}a^*(A_l\varphi_{n_l}) \\
& \qquad\qquad \prod_{l=1}^{b}a^*(B_l\varphi_{-k_l})\prod_{l=1}^{a}a(\varphi_{n_l})a^*(C\varphi_{-m}). \qquad (3.472)
\end{aligned}
$$

We will discuss terms (3.470), (3.471) and (3.472) separately. In Term (3.470) we need to commute the last creation operator into its place in the third product, it can be picked up by one of the annihilation operators of the last product, but after performing the sum over the corresponding basis the resulting term can be rephrased in terms of an L operator by commuting only creation operators of the second and third product. Performing these steps yields the identity

$$
\begin{aligned}
(3.470) = &\sum_{f=1}^{b} L\left(\bigcup_{l=1}^{a}\{A_l\};\bigcup_{\substack{l=1\\ l\neq f}}^{b}\{B_l\}\cup\{CP_-B_f\}\right)\\
&+(-1)^{a+b+1}\sum_{f=1}^{b}\sum_{g=1}^{a} L\left(\bigcup_{\substack{l=1\\ l\neq g}}^{a}\{A_l\};\bigcup_{\substack{l=1\\ l\neq f}}^{b}\{B_l\}\{A_gP_+CP_-B_f\}\right).
\end{aligned}
\tag{3.473}
$$

For (3.471) the last creation operator needs to be commuted to the end of the second product. It can be picked up by one of the annihilation operators of the last product, but here we have to distinguish between two cases. If the index of this annihilation operator equals f the resulting commutator will be $\operatorname{tr} P_+CP_-A_f$ otherwise one can again perform the sum over the corresponding index and express the whole Product in terms of an L operator. All this results in

$$
\begin{aligned}
(3.471) = &\sum_{f=1}^{a} L\left(\bigcup_{\substack{l=1\\ l\neq f}}^{a}\{A_l\}\cup\{CP_-A_f\};\bigcup_{l=1}^{b}\{B_l\}\right)\\
&+(-1)^{a+b}\sum_{f=1}^{a} L\left(\bigcup_{\substack{l=1\\ l\neq f}}^{a}\{A_l\};\bigcup_{l=1}^{b}\{B_l\}\right)\operatorname{tr}(P_+CP_-A_f)
\end{aligned}
$$

$$+(-1)^{a+b+1}\sum_{\substack{f_1,f_2=1\\ f_1\neq f_2}}^{a} L\left(\bigcup_{\substack{l=1\\ l\neq f_1,f_2}}^{a}\{A_l\}\cup\{A_{f_2}P_+CP_-A_{f_1}\};\bigcup_{l=1}^{b}\{B_l\}\right). \tag{3.474}$$

For (3.472) the procedure is basically the same as for (3.470), it results in

$$\begin{aligned}(3.472) = (-1)^{a+b}L\left(\bigcup_{l=1}^{a}\{A_l\};\bigcup_{l=1}^{b}\{B_l\}\right)\\ +\sum_{f=1}^{a} L\left(\bigcup_{\substack{l=1\\ l\neq f}}^{a}\{A_l\}\cup\{CP_-A_f\};\bigcup_{l=1}^{b}\{B_l\}\cup\{A_fP_+C\}\right).\end{aligned} \tag{3.475}$$

Putting the results of the calculation together results in the claimed equation, after pulling in some factors of -1 into L. □

We carry on with defining the important quantities for powers of G. First we introduce for each $k\in\mathbb{N}$ a linear bounded operator on $\mathcal{H}$, X_k which fulfils $\operatorname{tr} P_+X_kP_-X_k<\infty \wedge \operatorname{tr} P_-X_kP_+X_k<\infty$.

Definition 55. *Let*

$$Y := \{X_k \mid k\in\mathbb{N}\}.$$

Let for $n\in\mathbb{N}$

$$\langle n\rangle := \{X_l \mid l\in\mathbb{N}, l\leqslant n\}.$$

Definition 56. *Let for $b\subset Y$, such that $|b|<\infty$*

$$\begin{aligned}f_b : \{l\in\mathbb{N}\mid l\leqslant |b|\}\to b\\ \forall k<|b| : f_b(k)=X_l\wedge f_b(k+1)=X_m\to l<m\end{aligned} \tag{3.476}$$

Definition 57. *For any set b, we denote by $S(b)$ the symmetric group (group of permutations) over b.*

Definition 58. *Let for $b \subset Y$, such that $|b| < \infty$ and $\sigma_b \in S(b)$*

$$VZ^b_{\sigma_b} : \{k \in \mathbb{N} \mid k < |b|\} \to \{-1, 1\}$$
$$VZ^b_{\sigma_b}(k) := sgn\big[f_b^{-1}(\sigma_b(f_b(k+1))) - f_b^{-1}(\sigma_b(f_b(k)))\big]$$

In what is to follow the order of one particle operators will be changed in all possible ways, to keep track of this by use of a compact notation we introduce

Definition 59.

$$W : \{(b, \sigma_b) \mid b \subseteq Y \wedge |b| < \infty \wedge \sigma_b \in S(b)\} \to B(\mathcal{H})$$
$$W(b, \sigma_b) := \left(\prod_{k=1}^{|b|-1} \sigma_b(f_b(k)) P_{VZ^b_{\sigma_b}(k)} VZ^b_{\sigma_b}(k)\right) \sigma_b(f_b(|b|))$$

Definition 60. *Let l be any finite subset of Y. Denote by X^l_{max} the operator $X_k \in l$ such that for any $X_c \in l$ the relation $k \geqslant c$ is fulfilled. Furthermore define*

$$PT : \{T \subset \mathcal{P}(Y) \mid |T| < \infty, \forall b \in T : |b| < \infty\} \to \mathbb{C}$$
$$\textit{for: } T = \varnothing : PT(T) = 1, \ \textit{otherwise:}$$
$$PT(T) = \sum_{\substack{\forall l \in T: \\ \sigma_l \in S(l \setminus \{X^l_{max}\})}} \prod_{l \in T} \mathrm{tr}[P_+ X^l_{max} P_- W(l, \sigma_l)]$$

There is one more function left to define

Definition 61.

$$Op : \{R \in \mathcal{P}(Y) \mid |R| < \infty\} \times \{D \subset \mathcal{P}(Y) \mid |T| < \infty\} \to \mathcal{B}(\mathcal{F})$$
$$Op(R, D) = \sum_{\substack{\forall l \in D: \\ \sigma_l \in S(l)}} \sum_{a \subseteq R \cup \bigcup_{l \in D} \{W(l, \sigma_l)\}} L(a, a^c)(-1)^{|a| + \frac{(|R|+|D|)(|R|+|D|+1)}{2}}$$

Now we are able to state the main theorem which will help us do quantitative estimates.

Theorem 62. *Let $n \in \mathbb{N}$, $X_1, \ldots, X_n \in Y$ then the following equation holds*

$$\prod_{k=1}^{n} G(X_k) = \sum_{\substack{\langle n\rangle = \dot{\bigcup}_{l\in T} l \,\dot\cup\, \dot{\bigcup}_{l\in D} l \,\dot\cup\, R \\ \forall l\in T\cup D:|l|\geqslant 2}} PT(T)\,Op(R,D), \tag{3.477}$$

where the abbreviation $\langle n\rangle := \{X_k \mid k \leqslant n\}$ was used.

Proof: The proof will be by induction on n. Since the formula in the claim reduces to 1 for $n = 0$ we will not spend any more time on the start of the induction. The general strategy of the proof is to break up the right hand side of (3.477) for $n+1$ into small pieces and show for each piece that it corresponds to one of the contributions of lemma 54, while also each term in this lemma is represented by one of the terms obtained by breaking up (3.477).

As a first step we break the right hand side of (3.477) into three pieces separated by in which set X_{n+1} ends up in :

$$\sum_{\substack{\langle n+1\rangle = \dot{\bigcup}_{l\in T} l \,\dot\cup\, \dot{\bigcup}_{l\in D} l \,\dot\cup\, R \\ \forall l\in T\cup D:|l|\geqslant 2}} \mathrm{PT}(T)\mathrm{Op}(R,D) =$$

$$\sum_{\substack{\langle n+1\rangle = \dot{\bigcup}_{l\in T} l \,\dot\cup\, \dot{\bigcup}_{l\in D} l \,\dot\cup\, R \\ \exists l\in T: X_{n+1}\in l \\ \forall l\in T\cup D:|l|\geqslant 2}} \mathrm{PT}(T)\mathrm{Op}(R,D) \tag{3.478}$$

$$+ \sum_{\substack{\langle n+1\rangle = \dot{\bigcup}_{l\in T} l \,\dot\cup\, \dot{\bigcup}_{l\in D} l \,\dot\cup\, R \\ \exists l\in D: X_{n+1}\in l \\ \forall l\in T\cup D:|l|\geqslant 2}} \mathrm{PT}(T)\mathrm{Op}(R,D) \tag{3.479}$$

$$+\sum_{\substack{\langle n+1\rangle=\biguplus_{l\in T} l \uplus \biguplus_{l\in D} l \uplus R\\ X_{n+1}\in R\\ \forall l\in T\cup D:|l|\geqslant 2}} \mathrm{PT}(T)\mathrm{Op}(R,D), \tag{3.480}$$

We will discuss each term separately. For term (3.478) the term containing X_{n+1} is in one of the elements l' of T, but each such element has to have more than one element. So if we were to sum over the partitions of $\langle n\rangle$ instead, the rest of $l'\backslash\{X_{n+1}\}$ is either an element of D or, if it contains only one element, of R. Picking D instead of T is at this stage an arbitrary choice, but this choice leads to the terms of lemma 54. Alls this means that one correct rewriting of term (3.478) is

$$(3.478)=\sum_{\substack{\langle n\rangle=\biguplus_{l\in T} l \uplus \biguplus_{l\in D} l \uplus R\\ \forall l\in D\cup T:|l|>2}} \sum_{b\in D\cup\{\{r\}|r\in R\}} \mathrm{PT}(T\cup\{\{X_{n+1}\cup f\}\})\mathrm{Op}(R\backslash b, D\backslash\{b\}). \tag{3.481}$$

Next we pull one factor and the corresponding sum out of PT and write out Op. Then we see that the sums over permutations can be merged into one. There we take the convention that for any set f such that $|f|=1$ holds, we define σ_f to be the identity on that set.
This results in

$$(3.481)=\sum_{\substack{\langle n\rangle=\biguplus_{l\in T} l \uplus \biguplus_{l\in D} l \uplus R\\ \forall l\in D\cup T:|l|>2}} \sum_{b\in D\cup\{\{r\}|r\in R\}} \sum_{\sigma_b\in S(b)} \mathrm{tr}[P_+X_{n+1}P_-W(b,\sigma_b)]\mathrm{PT}(T) \sum_{\substack{\forall l\in D\backslash\{b\}\\ \sigma_l\in S(l)}} \sum_{a\subseteq R\backslash b\cup\bigcup_{l\in D\backslash\{b\}}\{W(l,\sigma_l)\}} L(a,a^c)(-1)^{|a|+\frac{(|R|+|D|-1)(|R|+|D|)}{2}}$$

$$
\begin{aligned}
&= \sum_{\substack{\langle n\rangle = \biguplus_{l\in T} l \uplus \biguplus_{l\in D} l \uplus R \\ \forall l\in D\cup T: |l|>2}} \sum_{\substack{\forall l\in D \\ \sigma_l\in S(l)}} \sum_{b\in D\cup\{\{r\}|r\in R\}} \mathrm{PT}(T) \\
&\sum_{a\subseteq R\backslash b\cup\bigcup_{l\in D\backslash\{b\}}\{W(l,\sigma_l)\}} L(a,a^c)(-1)^{|a|+\frac{(|R|+|D|-1)(|R|+|D|)}{2}} \\
&\qquad \mathrm{tr}[P_+X_{n+1}P_-W(b,\sigma_b)] \\
&= \sum_{\substack{\langle n\rangle = \biguplus_{l\in T} l \uplus \biguplus_{l\in D} l \uplus R \\ \forall l\in D\cup T: |l|>2}} \sum_{\substack{\forall l\in D \\ \sigma_l\in S(l)}} \mathrm{PT}(T) \sum_{a\subseteq R\cup\bigcup_{l\in D}\{W(l,\sigma_l)\}} \\
&\sum_{b\in D\cup\{\{r\}|r\in R\}} \mathbb{1}_{W(b,\sigma_b)\in a} L(a\backslash\{b\},a^c)(-1)^{|a|+1} \\
&(-1)^{\frac{(|R|+|D|-1)(|R|+|D|)}{2}} \mathrm{tr}[P_+X_{n+1}P_-W(b,\sigma_b)] \\
&= \sum_{\substack{\langle n\rangle = \biguplus_{l\in T} l \uplus \biguplus_{l\in D} l \uplus R \\ \forall l\in D\cup T: |l|>2}} \mathrm{PT}(T) \sum_{\substack{\forall l\in D \\ \sigma_l\in S(l)}} \sum_{a\subseteq R\cup\bigcup_{l\in D}\{W(l,\sigma_l)\}} \\
&\sum_{b\in a} L(a\backslash\{b\},a^c)(-1)^{|a|+\frac{(|R|+|D|+1)(|R|+|D|)}{2}} \\
&(-1)^{1+|R|+|D|} \mathrm{tr}[P_+X_{n+1}P_-W(b,\sigma_b)] \\
&= \sum_{\substack{\langle n\rangle = \biguplus_{l\in T} l \uplus \biguplus_{l\in D} l \uplus R \\ \forall l\in D\cup T: |l|>2}} \mathrm{PT}(T) \sum_{\substack{\forall l\in D \\ \sigma_l\in S(l)}} \sum_{a\subseteq R\cup\bigcup_{l\in D}\{W(l,\sigma_l)\}} \\
&(-1)^{|a|+\frac{(|R|+|D|+1)(|R|+|D|)}{2}} (3.464)_{L(a,a^c)G(X_{n+1})},
\end{aligned}
\tag{3.482}
$$

where the notation in the last line is to be taken as “apply Lemma 54 apply it to $L(a,a^c)G(X_{n+1})$ and pick only term (3.464)”. We will use this abbreviating notation also for the next terms.

The next term is (3.479). Here we need a few more notational conventions. For any set $b\subseteq\langle n\rangle$ and corresponding permutation $\sigma_b\in S(b)$, we denote by the same symbol σ_b the continuation of σ_b to $b\cup\{X_{n+1}\}$, where for this continuation X_{n+1} is a fixed point. Furthermore we define for any set $b \subseteq \langle n \rangle$, σ_c^b by

$$\begin{aligned} &\sigma_c^b \in S(b \cup \{X_{n+1}\}), \\ \forall k \leqslant |b| :&\sigma_c^b(f_{b\cup\{X_{n+1}\}}(k)) = f_{b\cup\{X_{n+1}\}}(k+1) \\ &\sigma_c^b(X_{n+1}) = f_b(1). \end{aligned} \tag{3.483}$$

Finally we define for sets $b_1, b_2 \subseteq \langle n \rangle, b_1 \cap b_2 = \varnothing$ and corresponding permutations $\sigma_{b_1} \in S(b_1), \sigma_{b_2} \in S(b_2)$ the permutation σ_{b_1,b_2}^{n+1} by

$$\begin{aligned} M_{b_1,b_2}^{n+1} &:= b_1 \cup b_2 \cup \{X_{n+1}\} \\ \sigma_{b_1,b_2}^{n+1} &\in S(M_{b_1,b_2}^{n+1}) \\ \forall 1 \leqslant k \leqslant |b_1| : \sigma_{b_1,b_2}^{n+1}(f_{M_{b_1,b_2}^{n+1}}(k)) &= \sigma_{b_1}(f_{b_1}(k)) \\ \sigma_{b_1,b_2}^{n+1}(f_{M_{b_1,b_2}^{n+1}}(|b_1|+1)) &= X_{n+1} \\ \forall |b_1| + 2 \leqslant k \leqslant |b_1| + |b_2| + 1 &: \\ \sigma_{b_1,b_2}^{n+1}(f_{M_{b_1,b_2}^{n+1}}(k)) &= \sigma_{b_2}(f_{b_2}(k - |b_1| - 1)) \end{aligned} \tag{3.484}$$

The beginning of the treatment of term (3.479) is analogous to (3.478), we rewrite the partition of $\langle n+1 \rangle$ into one of $\langle n \rangle$ with an additional sum over where the other operators packed to together with X_{n+1} come from. This splits into three parts, either X_{n+1} is put at the beginning of the compound operator, or its put at the end of the compound object, or to the left as well as to the right are operators with smaller index. Since the overall sign is decided by how often the operator index rises or falls, this separation into cases is helpful. The last case we then rewrite as picking two sets of operators, one of which will be in front of X_{n+1} and the other one behind this operator.
The calculation is as follows

$$
\begin{aligned}
(3.479) = &\sum_{\substack{\langle n+1\rangle = \biguplus_{l\in T} l \uplus \biguplus_{l\in D} l \uplus R \\ \exists l\in D: X_{n+1}\in l \\ \forall l\in T\cup D: |l|\geqslant 2}} \mathrm{PT}(T)\mathrm{Op}(R,D) \\
= &\sum_{\substack{\langle n\rangle = \biguplus_{l\in T} l \uplus \biguplus_{l\in D} l \uplus R \\ \forall l\in D\cup T: |l|\geqslant 2}} \mathrm{PT}(T) \sum_{b\in D\cup\{\{r\}|r\in R\}} \mathrm{Op}(R\backslash b, D\cup\{b\cup\{X_{n+1}\}\backslash\{b\}\}) \\
= &\sum_{\substack{\langle n\rangle = \biguplus_{l\in T} l \uplus \biguplus_{l\in D} l \uplus R \\ \forall l\in D\cup T: |l|\geqslant 2}} \mathrm{PT}(T) \sum_{b\in D\cup\{\{r\}|r\in R\}} \sum_{\substack{\forall l\in D\cup\{b\cup\{X_{n+1}\}\} \\ \sigma_l\in S(l)}} \\
&\sum_{a\subseteq R\backslash b\cup\bigcup_{l\in D\cup\{b\cup\{X_{n+1}\}\}\backslash\{b\}}\{W(l,\sigma_l)\}} L(a,a^c)(-1)^{|a|+\frac{(|R|+|D|)(|R|+|D|+1)}{2}} \\
= &\sum_{\substack{\langle n\rangle = \biguplus_{l\in T} l \uplus \biguplus_{l\in D} l \uplus R \\ \forall l\in D\cup T: |l|\geqslant 2}} \mathrm{PT}(T) \sum_{b\in D\cup\{\{r\}|r\in R\}} \sum_{\substack{\forall l\in D \\ \sigma_l\in S(l)}} \Bigg[\\
&\sum_{a\subseteq R\backslash b\cup\bigcup_{l\in D\backslash\{b\}}\{W(l,\sigma_l)\}\cup\{W(b\cup\{X_{n+1}\},\sigma_b)\}} L(a,a^c)(-1)^{|a|+\frac{(|R|+|D|)(|R|+|D|+1)}{2}}
\end{aligned}
\tag{3.485}
$$

$$
+ \sum_{a\subseteq R\backslash b\cup\bigcup_{l\in D\backslash\{b\}}\{W(l,\sigma_l)\}\cup\{W(b\cup\{X_{n+1}\},\sigma_c^b\circ\sigma_b)\}} L(a,a^c)(-1)^{|a|+\frac{(|R|+|D|)(|R|+|D|+1)}{2}} \Bigg]
\tag{3.486}
$$

$$
\begin{aligned}
+ &\sum_{\substack{\langle n\rangle = \biguplus_{l\in T} l \uplus \biguplus_{l\in \overline{D}} l \uplus \overline{R} \\ \forall l\in \overline{D}\cup T: |l|\geqslant 2}} \mathrm{PT}(T) \sum_{\substack{b_1,b_2\in\overline{D}\cup\{\{r\}|r\in\overline{R}\} \\ b_1\neq b_2}} \sum_{\substack{\forall l\in\overline{D} \\ \sigma_l\in S(l)}} \\
&\sum_{a\subseteq\tilde{R}\cup\bigcup_{l\in\tilde{D}}\{W(l,\sigma_l)\}\cup\{W(b_1\cup\{X_{n+1}\}\cup b_2,\sigma_{b_1,b_2}^{n+1})\}} L(a,a^c)(-1)^{|a|+\frac{(|\overline{R}|+|\overline{D}|-1)(|\overline{R}|+|\overline{D}|)}{2}}
\end{aligned}
\tag{3.487}
$$

where $\tilde{R} = \overline{R}\backslash(b_1\cup b_2)$ and $\tilde{D} := \overline{D}\cup\{b_1\cup\{X_{n+1}\}\cup b_2\}\backslash\{b_1,b_2\}$. For the term (3.487) we had to reshuffle the outermost sum a bit. For each term in the original sum where X_{n+1} is neither the first nor the last factor in its product (we will call the set of factors in front of X_{n+1} α and the factors behind it β) there is a different splitting of $\langle n\rangle$ into $\overline{R}$ and $\overline{D}$ such that α and β are separate elements of $\overline{D} \cup \{\{r\} \mid r \in \overline{R}\}$. So we replace the original sum over D and R into one of $\overline{D}$ and $\overline{R}$. Since this is a one to one correspondence and the sum is finite this is always possible. The exponent of the sign also changes for this reason, since $|R|+|D| = |\overline{R}|+|\overline{D}|-1$ holds. Continuing with (3.485) the next steps are similar to the last steps in treating (3.478). They are

$$
\begin{aligned}
(3.485) = & \sum_{\substack{\langle n\rangle=\overset{\vartheta}{\bigcup}_{l\in T} l \,\vartheta\, \overset{\vartheta}{\bigcup}_{l\in D} l \,\vartheta\, R \\ \forall l\in D\cup T:|l|\geqslant 2}} \mathrm{PT}(T) \sum_{b\in D\cup\{\{r\}|r\in R\}} \sum_{\substack{\forall l\in D \\ \sigma_l\in S(l)}} \\
& \sum_{a\subseteq R\backslash b\cup\bigcup_{l\in D\backslash\{b\}}\{W(l,\sigma_l)\}\cup\{W(b\cup\{X_{n+1}\},\sigma_b)\}} L(a,a^c)(-1)^{|a|+\frac{(|R|+|D|)(|R|+|D|+1)}{2}} \\
= & \sum_{\substack{\langle n\rangle=\overset{\vartheta}{\bigcup}_{l\in T} l \,\vartheta\, \overset{\vartheta}{\bigcup}_{l\in D} l \,\vartheta\, R \\ \forall l\in D\cup T:|l|\geqslant 2}} \mathrm{PT}(T) \sum_{b\in D\cup\{\{r\}|r\in R\}} \sum_{\substack{\forall l\in D \\ \sigma_l\in S(l)}} \\
& \sum_{a\subseteq R\cup\bigcup_{l\in D}\{W(l,\sigma_l)\}} (-1)^{|a|+\frac{(|R|+|D|)(|R|+|D|+1)}{2}} \mathbb{1}_{W(b,\sigma_b)\in a} \\
& \big[L(a\backslash\{W(b,\sigma_b)\}\cup\{W(b\cup\{X_{n+1}\},\sigma_b)\},a^c) \\
& - L(a\backslash\{W(b,\sigma_b)\},a^c\cup\{W(b\cup\{X_{n+1}\},\sigma_b)\})\big] \\
= & \sum_{\substack{\langle n\rangle=\overset{\vartheta}{\bigcup}_{l\in T} l \,\vartheta\, \overset{\vartheta}{\bigcup}_{l\in D} l \\ \forall l\in D\cup T:|l|\geqslant 2}} \mathrm{PT}(T) \sum_{\substack{\forall l\in D \\ \sigma_l\in S(l)}} \\
& \sum_{a\subseteq R\cup\bigcup_{l\in D}\{W(l,\sigma_l)\}} (-1)^{|a|+\frac{(|R|+|D|)(|R|+|D|+1)}{2}} \sum_{W(b,\sigma_b)\in a} \\
& \big[L(a\backslash\{W(b,\sigma_b)\}\cup\{W(b\cup\{X_{n+1}\},\sigma_b)\},a^c)
\end{aligned}
$$

$$
\begin{aligned}
&- L(a\backslash\{W(b,\sigma_b)\}, a^c \cup \{W(b\cup\{X_{n+1}\},\sigma_b)\})\big] \\
&= \sum_{\substack{\langle n\rangle = \biguplus_{l\in T} l \uplus \biguplus_{l\in D} l \uplus R \\ \forall l\in D\cup T: |l|\geqslant 2}} \mathrm{PT}(T) \sum_{\substack{\forall l\in D \\ \sigma_l\in S(l)}} \sum_{a\subseteq R\cup\bigcup_{l\in D}\{W(l,\sigma_l)\}} \\
&(-1)^{|a|+\frac{(|R|+|D|)(|R|+|D|+1)}{2}} ((3.460)+(3.462))_{L(a,a^c)G(X_{n+1})}.
\end{aligned}
\tag{3.488}
$$

Almost the same procedure applies to (3.486). It yields

$$
\begin{aligned}
(3.486) &= \sum_{\substack{\langle n\rangle = \biguplus_{l\in T} l \uplus \biguplus_{l\in D} l \uplus R \\ \forall l\in D\cup T: |l|\geqslant 2}} \mathrm{PT}(T) \sum_{b\in D\cup\{\{r\}|r\in R\}} \sum_{\substack{\forall l\in D \\ \sigma_l\in S(l)}} \\
&\sum_{a\subseteq R\backslash b\cup\bigcup_{l\in D\backslash\{b\}}\{W(l,\sigma_l)\}\cup\{W(b\cup\{X_{n+1}\},\sigma_c^b\circ\sigma_b)\}} L(a,a^c)(-1)^{|a|+\frac{(|R|+|D|)(|R|+|D|+1)}{2}} \\
&= \sum_{\substack{\langle n\rangle = \biguplus_{l\in T} l \uplus \biguplus_{l\in D} l \uplus R \\ \forall l\in D\cup T: |l|\geqslant 2}} \mathrm{PT}(T) \sum_{b\in D\cup\{\{r\}|r\in R\}} \sum_{\substack{\forall l\in D \\ \sigma_l\in S(l)}} \\
&\sum_{a\subseteq R\cup\bigcup_{l\in D}\{W(l,\sigma_l)\}} (-1)^{|a|+\frac{(|R|+|D|)(|R|+|D|+1)}{2}} \\
&\big[\mathbb{1}_{W(b,\sigma_c^b\circ\sigma_b)\in a} L(a\backslash\{W(b,\sigma_b)\}\cup\{W(b\cup\{X_{n+1}\},\sigma_c^b\circ\sigma_b)\}, a^c) \\
&+ \mathbb{1}_{W(b,\sigma_c^b\circ\sigma_b)\in a^c} L(a\backslash\{W(b,\sigma_b)\}, a^c\cup\{W(b\cup\{X_{n+1}\},\sigma_c^b\circ\sigma_b)\})\big] \\
&= \sum_{\substack{\langle n\rangle = \biguplus_{l\in T} l \uplus \biguplus_{l\in D} l \uplus R \\ \forall l\in D\cup T: |l|\geqslant 2}} \mathrm{PT}(T) \sum_{\substack{\forall l\in D \\ \sigma_l\in S(l)}} \\
&\sum_{a\subseteq R\cup\bigcup_{l\in D}\{W(l,\sigma_l)\}} (-1)^{|a|+\frac{(|R|+|D|)(|R|+|D|+1)}{2}} \\
&\Big[\sum_{W(b,\sigma_b)\in a} L(a\backslash\{W(b,\sigma_b)\}\cup\{W(b\cup\{X_{n+1}\},\sigma_c^b\circ\sigma_b)\}, a^c) \\
&+ \sum_{W(b,\sigma_b)\in a^c} L(a, a^c\backslash\{W(b,\sigma_b)\}\cup\{W(b\cup\{X_{n+1}\},\sigma_c^b\circ\sigma_b)\})\Big]
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{\substack{\langle n\rangle = \overset{\vartheta}{\bigcup}_{l\in T} l \,\vartheta\, \overset{\vartheta}{\bigcup}_{l\in D} l \,\vartheta\, R \\ \forall l\in D\cup T: |l|\geqslant 2}} \mathrm{PT}(T) \sum_{\substack{\forall l\in D \\ \sigma_l\in S(l)}} \sum_{a\subseteq R\cup\bigcup_{l\in D}\{W(l,\sigma_l)\}} \\
&(-1)^{|a|+\frac{(|R|+|D|)(|R|+|D|+1)}{2}} ((3.461)+(3.463))_{L(a,a^c)G(X_{n+1})}.
\end{aligned} \tag{3.489}
$$

Also for (3.487) the procedure is almost the same. We bring the sums into a form such that one can read off the terms generated by the induction. We begin be renaming the sets which we had to change by resumming back to the names of the original sets.

$$
\begin{aligned}
(3.487) &= \sum_{\substack{\langle n\rangle = \overset{\vartheta}{\bigcup}_{l\in T} l \,\vartheta\, \overset{\vartheta}{\bigcup}_{l\in \overline{D}} l \,\vartheta\, \overline{R} \\ \forall l\in \overline{D}\cup T: |l|\geqslant 2}} \mathrm{PT}(T) \sum_{\substack{b_1,b_2\in\overline{D}\cup\{\{r\}|r\in\overline{R}\} \\ b_1\neq b_2}} \sum_{\substack{\forall l\in \overline{D} \\ \sigma_l\in S(l)}} \\
&\sum_{a\subseteq\tilde{R}\cup\bigcup_{l\in\tilde{D}}\{W(l,\sigma_l)\}\cup\{W(b_1\cup\{X_{n+1}\}\cup b_2,\sigma^{n+1}_{b_1,b_2})\}} L(a,a^c)(-1)^{|a|+\frac{(|\overline{R}|+|\overline{D}|-1)(|\overline{R}|+|\overline{D}|)}{2}} \\
&= \sum_{\substack{\langle n\rangle = \overset{\vartheta}{\bigcup}_{l\in T} l \,\vartheta\, \overset{\vartheta}{\bigcup}_{l\in D} l \,\vartheta\, R \\ \forall l\in D\cup T: |l|\geqslant 2}} \mathrm{PT}(T) \sum_{\substack{b_1,b_2\in D\cup\{\{r\}|r\in R\} \\ b_1\neq b_2}} \sum_{\substack{\forall l\in D \\ \sigma_l\in S(l)}} \\
&\sum_{a\subseteq R\cup\bigcup_{l\in D}\{W(l,\sigma_l)\}\cup\{W(b_1\cup\{X_{n+1}\}\cup b_2,\sigma^{n+1}_{b_1,b_2})\}} L(a,a^c)(-1)^{|a|+\frac{(|R|+|D|-1)(|R|+|D|)}{2}} \\
&= \sum_{\substack{\langle n\rangle = \overset{\vartheta}{\bigcup}_{l\in T} l \,\vartheta\, \overset{\vartheta}{\bigcup}_{l\in D} l \,\vartheta\, R \\ \forall l\in D\cup T: |l|\geqslant 2}} \mathrm{PT}(T) \sum_{\substack{b_1,b_2\in D\cup\{\{r\}|r\in R\} \\ b_1\neq b_2}} \sum_{\substack{\forall l\in D \\ \sigma_l\in S(l)}} \sum_{a\subseteq R\cup\bigcup_{l\in D}\{W(l,\sigma_l)\}} \\
&(-1)^{|R|+|D|+\frac{(|R|+|D|)(|R|+|D|+1)}{2}} \mathbb{1}_{W(b_1,\sigma_1)\in a} \\
&\Big[+(-1)^{|a|+1}\mathbb{1}_{W(b_2,\sigma_2)\in a} L\Big(a\backslash\{W(b_1,\sigma_1),W(b_2,\sigma_2)\}\cup \\
&\cup\{W(b_1\cup\{X_{n+1}\}\cup b_2,\sigma^{n+1}_{b_1,b_2})\},a^c\Big) \\
&+(-1)^{|a|+1}\mathbb{1}_{W(b_2,\sigma_2)\in a^c} L\Big(a\backslash\{W(b_1,\sigma_1)\},a^c\backslash\{W(b_2,\sigma_2)\}\cup
\end{aligned}
$$

$$
\begin{aligned}
&\cup\{W(b_1 \cup \{X_{n+1}\} \cup f_2, \sigma^{n+1}_{b_1,b_2})\}\Big)\Big] \\
&= \sum_{\substack{\langle n\rangle = \biguplus_{l\in T} l \uplus \biguplus_{l\in D} l \uplus R \\ \forall l\in D\cup T: |l|\geqslant 2}} \mathrm{PT}(T) \sum_{\substack{\forall l\in D \\ \sigma_l\in S(l)}} \sum_{a\subseteq R\cup\bigcup_{l\in D}\{W(l,\sigma_l)\}} \\
&(-1)^{|R|+|D|+\frac{(|R|+|D|)(|R|+|D|+1)}{2}} \\
&\Big[(-1)^{|a|+1} \sum_{\substack{b_1,b_2\in a \\ b_1\neq b_2}} L\Big(a\backslash\{W(b_1,\sigma_1), W(b_2,\sigma_2)\}\cup \\
&\cup \{W(b_1 \cup \{X_{n+1}\} \cup b_2, \sigma^{n+1}_{b_1,b_2})\}, a^c\Big) \\
&+ (-1)^{|a|+1} \sum_{b_1\in a, b_2\in a^c} L\Big(a\backslash\{W(b_1,\sigma_1)\}, a^c\backslash\{W(b_2,\sigma_2)\}\cup \\
&\cup\{W(b_1 \cup \{X_{n+1}\} \cup f_2, \sigma^{n+1}_{b_1,b_2})\}\Big)\Big] \\
&= \sum_{\substack{\langle n\rangle = \biguplus_{l\in T} l \uplus \biguplus_{l\in D} l \uplus R \\ \forall l\in D\cup T: |l|\geqslant 2}} \mathrm{PT}(T) \sum_{\substack{\forall l\in D \\ \sigma_l\in S(l)}} \sum_{a\subseteq R\cup\bigcup_{l\in D}\{W(l,\sigma_l)\}} \\
&(-1)^{|a|+\frac{(|R|+|D|)(|R|+|D|+1)}{2}} ((3.465) + (3.466))_{L(a,a^c)G(X_{n+1})}
\end{aligned}
$$

Lastly we will discuss term (3.480); luckily, this term is less involved than the other two. The general procedure; however, stays the same. First we reformulate the partition of $\langle n+1\rangle$ into one of $\langle n\rangle$, where the terms acquire modifications. Secondly we massage these terms until the involved sums look exactly like the one in our induction hypothesis (3.477) and realise that the terms are produced by lemma 54. For term (3.480) this results in

$$
(3.480) = \sum_{\substack{\langle n+1\rangle = \biguplus_{l\in T} l \uplus \biguplus_{l\in D} l \uplus R \\ X_{n+1}\in R \\ \forall l\in T\cup D: |l|\geqslant 2}} \mathrm{PT}(T)\mathrm{Op}(R,D)
$$

$$= \sum_{\substack{\langle n\rangle=\biguplus_{l\in T} l \uplus \biguplus_{l\in D} l \uplus R \\ \forall l\in T\cup D:|l|\geqslant 2}} \mathrm{PT}(T)\mathrm{Op}(R\cup\{X_{n+1}\},D)$$

$$= \sum_{\substack{\langle n\rangle=\biguplus_{l\in T} l \uplus \biguplus_{l\in D} l \uplus R \\ \forall l\in T\cup D:|l|\geqslant 2}} \mathrm{PT}(T) \sum_{\substack{\forall l\in D: \\ \sigma_l\in S(l)}} \sum_{a\subseteq R\cup\{X_{n+1}\}\cup\bigcup_{l\in D}\{W(l,\sigma_l)\}}$$

$$L(a,a^c)(-1)^{|a|+\frac{(|R|+1+|D|)(|R|+|D|+2)}{2}}$$

$$= \sum_{\substack{\langle n\rangle=\biguplus_{l\in T} l \uplus \biguplus_{l\in D} l \uplus R \\ \forall l\in T\cup D:|l|\geqslant 2}} \mathrm{PT}(T) \sum_{\substack{\forall l\in D: \\ \sigma_l\in S(l)}} \sum_{a\subseteq R\cup\bigcup_{l\in D}\{W(l,\sigma_l)\}} (-1)^{|a|+\frac{(|R|+1+|D|)(|R|+|D|)}{2}}$$

$$\Big(-L(a\cup\{X_{n+1}\},a^c)+L(a,a^c\cup\{X_{n+1}\})\Big)(-1)^{|R|+|D|+1}$$

$$= \sum_{\substack{\langle n\rangle=\biguplus_{l\in T} l \uplus \biguplus_{l\in D} l \uplus R \\ \forall l\in T\cup D:|l|\geqslant 2}} \mathrm{PT}(T) \sum_{\substack{\forall l\in D: \\ \sigma_l\in S(l)}} \sum_{a\subseteq R\cup\bigcup_{l\in D}\{W(l,\sigma_l)\}} (-1)^{|a|+\frac{(|R|+1+|D|)(|R|+|D|)}{2}}$$

$$\Big(L(a\cup\{X_{n+1}\},a^c)(-1)^{|R|+|D|}+L(a,a^c\cup\{X_{n+1}\})(-1)^{|R|+|D|+1}\Big)$$

$$= \sum_{\substack{\langle n\rangle=\biguplus_{l\in T} l \uplus \biguplus_{l\in D} l \uplus R \\ \forall l\in T\cup D:|l|\geqslant 2}} \mathrm{PT}(T) \sum_{\substack{\forall l\in D: \\ \sigma_l\in S(l)}} \sum_{a\subseteq R\cup\bigcup_{l\in D}\{W(l,\sigma_l)\}} (-1)^{|a|+\frac{(|R|+1+|D|)(|R|+|D|)}{2}}$$

$$\Big((3.458)+(3.459)\Big)_{L(a,a^c)G(X_{n+1})}. \qquad \blacksquare$$

Summarising we showed

$$\sum_{\substack{\langle n+1\rangle=\biguplus_{l\in T} l \uplus \biguplus_{l\in D} l \uplus R \\ \forall l\in T\cup D:|l|\geqslant 2}} \mathrm{PT}(T)\mathrm{Op}(R,D)$$

$$= \sum_{\substack{\langle n\rangle=\biguplus_{l\in T} l \uplus \biguplus_{l\in D} l \uplus R \\ \forall l\in D\cup T:|l|>2}} \mathrm{PT}(T) \sum_{\substack{\forall l\in D \\ \sigma_l\in S(l)}} \sum_{a\subseteq R\cup\bigcup_{l\in D}\{W(l,\sigma_l)\}}$$

$$(-1)^{|a|+\frac{(|R|+|D|+1)(|R|+|D|)}{2}}$$

$$\Big((3.464)+(3.460)+(3.462)+(3.461)+(3.463)$$

$$+(3.465)+(3.466)+(3.458)+(3.459)\Big)_{L(a,a^c)G(X_{n+1})}$$

$$
\begin{aligned}
&= \sum_{\substack{\langle n\rangle = \biguplus_{l\in T} l \uplus \biguplus_{l\in D} l \uplus R \\ \forall l\in D\cup T : |l|>2}} \mathrm{PT}(T) \sum_{\substack{\forall l\in D \\ \sigma_l \in S(l)}} \sum_{a\subseteq R\cup \bigcup_{l\in D}\{W(l,\sigma_l)\}} \\
&(-1)^{|a|+\frac{(|R|+|D|+1)(|R|+|D|)}{2}} L(a,a^c) G(X_{n+1}) \\
&= \sum_{\substack{\langle n\rangle = \biguplus_{l\in T} l \uplus \biguplus_{l\in D} l \uplus R \\ \forall l\in D\cup T : |l|>2}} \mathrm{PT}(T)\mathrm{Op}(R,D) G(X_{n+1}) \\
&= \prod_{l=1}^{n} G(X_l) \quad G(X_{n+1}),
\end{aligned}
$$

which ends our proof by induction. □

Appendix

Heuristic Construction of S-Matrix expression

In the following I derive a recursive equation for the coefficients of the expansion of the second quantized scattering operator. The starting point of this derivation is the commutator of T_m, equation (3.367).

Guessing Equations

Why at this point one might suspect that such a representation exists is, because looking at equation (3.367) for a while, one comes to the conclusion that if one replaces T_m by

$$T_m - \frac{1}{2} \sum_{k=1}^{m-1} \binom{m}{k} T_k T_{m-k}, \tag{490}$$

no T_k with $k > m-2$ will occur on the right hand side of the resulting equation. So if one subtracts the right polynomial in T_k for suitable k one might achieve a commutator which contains only the creation

Todo: place proper reference to definition of G operator

respectively annihilation operator concatenated with some one particle operator. From our treatment of T_1 we know which operators have such commutation relations.
So having this in Mind we start with the ansatz

$$\Gamma_m := \sum_{g=2}^{m} \sum_{\substack{b \in \mathbb{N}^g \\ |b|=m}} c_b \prod_{k=1}^{g} T_{b_k}. \tag{491}$$

Now in order to show that T_m and Γ_m agree up to operators which have a commutation relation of the form (**??**), we calculate $\left[T_m - \Gamma_m, a^{\#}(\varphi_n)\right]$ for arbitrary $n \in \mathbb{Z}$ and try to choose the coefficients c_b of (491) such that all contributions vanish which do not have the form $a^{\#}\left(\prod_k Z_{\alpha_k}\right)$ for any suitable $(\alpha_k)_k \subset \mathbb{N}$. If one does so, one is led to a system of equations of which I wrote down a few to give an overview of its structure. The objects α_k, β_l in the system of equations can be any natural Number for any $k, l \in \mathbb{N}$.

$$\begin{aligned}
0 &= c_{\alpha_1,\beta_1} + c_{\beta_1,\alpha_1} + \binom{\alpha_1+\beta_1}{\alpha_1} \\
0 &= c_{\alpha_1,\alpha_2,\beta_1} + c_{\beta_1,\alpha_1,\alpha_2} + c_{\alpha_2,\alpha_1,\beta_1} + \binom{\alpha_2+\beta_1}{\alpha_2} c_{\alpha_1,\alpha_2+\beta_1} \\
&\qquad + \binom{\alpha_1+\beta_1}{\alpha_1} c_{\alpha_1+\beta_1,\alpha_2} \\
0 &= c_{\alpha_1,\alpha_2,\alpha_3,\beta_1} + c_{\alpha_1,\alpha_2,\beta_1,\alpha_3} + c_{\alpha_1,\beta_1,\alpha_2,\alpha_3} + c_{\beta_1,\alpha_1,\alpha_2,\alpha_3} \\
&+ \binom{\alpha_1+\beta_1}{\beta_1} c_{\alpha_1+\beta_1,\alpha_2,\alpha_3} + \binom{\alpha_2+\beta_1}{\beta_1} c_{\alpha_1,\alpha_2+\beta_1,\alpha_3} \\
&\qquad + \binom{\alpha_3+\beta_1}{\beta_1} c_{\alpha_1,\alpha_2,\alpha_3+\beta_1} \\
0 &= c_{\alpha_1,\alpha_2,\beta_1,\beta_2} + c_{\alpha_1,\beta_1,\alpha_2,\beta_2} + c_{\beta_1,\alpha_1,\alpha_2,\beta_2} + c_{\alpha_1,\beta_1,\beta_2,\alpha_2}
\end{aligned}$$

$$+ c_{\beta_1,\alpha_1,\beta_2,\alpha_2} + c_{\beta_1,\beta_2,\alpha_1,\alpha_2} + \binom{\alpha_1+\beta_1}{\alpha_1}\big(c_{\alpha_1+\beta_1,\alpha_2,\beta_2}$$
$$+ c_{\alpha_1+\beta_1,\beta_2,\alpha_2}\big) + \binom{\alpha_1+\beta_2}{c}_{\beta_1,\alpha_1+\beta_2,\alpha_1}$$
$$+ \binom{\alpha_2+\beta_1}{\alpha_2} c_{\alpha_1,\alpha_2+\beta_1,\beta_2} + \binom{\alpha_2+\beta_2}{\alpha_2}\big(c_{\alpha_1,\beta_1,\alpha_2+\beta_2}$$
$$+ c_{\beta_1,\alpha_1,\alpha_2+\beta_2}\big) + \binom{\alpha_1+\beta_1}{\alpha_1}\binom{\alpha_2+\beta_2}{\alpha_2} c_{\alpha_1+\beta_1,\alpha_2+\beta_2}$$

$$0 = c_{\alpha_1,\beta_1,\beta_2,\beta_3,\beta_4} + c_{\beta_1,\alpha_1,\beta_2,\beta_3,\beta_4} + c_{\beta_1,\beta_2,\alpha_1,\beta_3,\beta_4}$$
$$+ c_{\beta_1,\beta_2,\beta_3,\alpha_1,\beta_4} + c_{\beta_1,\beta_2,\beta_3,\beta_4,\alpha_1}$$
$$+ \binom{\alpha_1+\beta_1}{\alpha_1} c_{\alpha_1+\beta_1,\beta_2,\beta_3,\beta_4} + \binom{\alpha_1+\beta_2}{\alpha_1} c_{\beta_1,\alpha_1+\beta_2,\beta_3,\beta_4}$$
$$+ \binom{\alpha_1+\beta_3}{\alpha_1} c_{\beta_1,\beta_2,\alpha_1+\beta_3,\beta_4} + \binom{\alpha_1+\beta_4}{\alpha_1} c_{\beta_1,\beta_2,\beta_3,\alpha_1+\beta_4}$$

$$0 = c_{\alpha_1,\alpha_2,\beta_1,\beta_2,\beta_3} + c_{\alpha_1,\beta_1,\alpha_2,\beta_2,\beta_3} + c_{\beta_1,\alpha_1,\alpha_2,\beta_2,\beta_3}$$
$$+ c_{\alpha_1,\beta_1,\beta_2,\alpha_2,\beta_3} + c_{\beta_1,\alpha_1,\beta_2,\alpha_2,\beta_3} + c_{\beta_1,\beta_2,\alpha_1,\alpha_2,\beta_3}$$
$$+ c_{\alpha_1,\beta_1,\beta_2,\beta_3,\alpha_2} + c_{\beta_1,\alpha_1,\beta_2,\beta_3,\alpha_2} + c_{\beta_1,\beta_2,\alpha_1,\beta_3,\alpha_2}$$
$$+ c_{\beta_1,\beta_2,\beta_3,\alpha_1,\alpha_2} + \binom{\alpha_1+\beta_1}{\beta_1}\big(c_{\alpha_1+\beta_1,\alpha_2,\beta_2,\beta_3}$$
$$+ c_{\alpha_1+\beta_1,\beta_2,\alpha_2,\beta_3} + c_{\alpha_1+\beta_1,\beta_2,\beta_3,\alpha_2}\big)$$
$$+ \binom{\alpha_2+\beta_1}{\beta_1} c_{\alpha_1,\alpha_2+\beta_1,\beta_2,\beta_3}$$
$$+ \binom{\alpha_2+\beta_2}{\beta_2}\big(c_{\beta_1,\alpha_1,\alpha_2+\beta_2,\beta_3} + c_{\alpha_1,\beta_1,\alpha_2+\beta_2,\beta_3}\big)$$
$$+ \binom{\alpha_1+\beta_2}{\beta_2}\big(c_{\beta_1,\alpha_1+\beta_2,\alpha_2,\beta_3} + c_{\beta_1,\alpha_1+\beta_2,\beta_3,\alpha_2}\big)$$
$$+ \binom{\alpha_2+\beta_3}{\beta_3}\big(c_{\alpha_1,\beta_1,\beta_2,\alpha_2+\beta_3} + c_{\beta_1,\alpha_1,\beta_2,\alpha_2+\beta_3}$$

$$
\begin{aligned}
&+ c_{\beta_1,\beta_2,\alpha_1,\alpha_2+\beta_3}\big) + \binom{\alpha_1+\beta_3}{\beta_3} c_{\beta_1,\beta_2,\alpha_1+\beta_3,\alpha_2} \\
&+ \binom{\alpha_1+\beta_1}{\alpha_1}\binom{\alpha_2+\beta_2}{\alpha_2} c_{\alpha_1+\beta_1,\alpha_2+\beta_2,\beta_3} \\
&+ \binom{\alpha_1+\beta_2}{\alpha_1}\binom{\alpha_2+\beta_3}{\alpha_2} c_{\beta_1,\alpha_1+\beta_2,\alpha_2+\beta_3} \\
&+ \binom{\alpha_1+\beta_1}{\alpha_1}\binom{\alpha_2+\beta_3}{\alpha_2} c_{\alpha_1+\beta_1,\beta_2,\alpha_2+\beta_3} \\
&\qquad\qquad \vdots
\end{aligned}
$$

Solving the first few equations and plugging the solution into the consecutive equations one can see that at least the first equations are solved by

$$c_{\alpha_1,\dots,\alpha_k} = \frac{(-1)^k}{k}\begin{pmatrix} \sum_{l=1}^k \alpha_l \\ \alpha_1\ \alpha_2\ \cdots\ \alpha_k \end{pmatrix}, \tag{492}$$

where the last factor is the multinomial coefficient of the indices $\alpha_1, \dots, \alpha_k \in \mathbb{N}$.

Recursive equation for Coefficients of the second quantised scattering operator

For the rest of this chapter, we are going to derive a concrete form of the second quantised scattering matrix. In order to turn the "conjectures" into "theorems" not only would one have to turn the rough sketches of the combinatorics into proofs, one also would have to show linearity (over real numbers) and continuity of $\mathrm{d}\Gamma(B)$ in B. However, since the final result can be verified to be well defined and to fulfil the lift conditions, this will not be necessary. We will nonetheless come across various combinatorial assertions that we are going to prove rigorously. These will be clearly marked: "lemma" and "proof".

We are going to use the following definition of binomial coefficients:

Definition 63. *For $a \in \mathbb{C}, b \in \mathbb{Z}$ we define*

$$\binom{a}{b} := \begin{cases} \prod_{l=0}^{b-1} \frac{a-l}{l+1} & \text{for } b \geqslant 0 \\ 0 & \text{otherwise.} \end{cases} \tag{493}$$

Defining the binomial coefficient for negative lower index to be zero has the merit, that one can extend the range of validity of many rules and sums involving binomial coefficients, also one does not have to worry about the range of summation in many cases.
The coefficients which we have already guessed more generally to be

Conjecture 64. *For any $n \in \mathbb{N}$ the n-th expansion coefficient of the second quantised scattering operator T_n is given by*

$$\begin{aligned} T_n = &\sum_{g=2}^{n} \sum_{\substack{\vec{b} \in \mathbb{N}^g \\ |\vec{b}|=n}} \frac{(-1)^g}{g} \binom{n}{\vec{b}} \prod_{l=1}^{g} T_{b_l} + C_n \mathbb{1}_{\mathcal{F}} \\ &+ d\Gamma\left(\sum_{g=1}^{n} \sum_{\substack{\vec{b} \in \mathbb{N}^g \\ |\vec{b}|=n}} \frac{(-1)^{g+1}}{g} \binom{n}{\vec{b}} \prod_{l=1}^{g} Z_{b_l} \right), \end{aligned} \tag{494}$$

for some $C_n \in \mathbb{C}$ which depends on the external field A. The last summand will henceforth be abbreviated by Γ_n.

Motivation: The way we will prove this is to compute the commutator of the difference between T_n and the first summand of (494) with the creation and annihilation operator of an element of the basis of $\mathcal{H}$. This will turn out to be exactly equal to the corresponding commutator of the second summand of (494), since two operators on Fock space only have the same commutator with general creation and annihilation operators if they agree up to multiples of the identity this will conclude the motivation of this conjecture.

In order to simplify the notation as much as possible, I will denote by $a^{\#}z$ either $a(z(\varphi_p))$ or $a^*(z(\varphi_p))$ for any one particle operator z and any element φ_p of the orthonormal basis $(\varphi_p)_{p\in\mathbb{Z}\setminus\{0\}}$ of $\mathcal{H}$. (We need not decide between creation and annihilation operator, since the expressions all agree)

In order to organize the bookkeeping of all the summands which arise from iteratively making use of the commutation rule (3.367) we organize them by the looking at a spanning set of the possible terms that arise my choice is

$$\left\{ a^{\#} \prod_{k=1}^{m_1} Z_{\alpha_k} \prod_{k=1}^{m_2} T_{\beta_k} \middle| m_1 \in \mathbb{N}, m_2 \in \mathbb{N}_0, \alpha \in \mathbb{N}^{m_1}, \beta \in \mathbb{N}^{m_2}, |\alpha| + |\beta| = n \right\}. \tag{495}$$

As a first step of computing the commutator in question we look at the summand corresponding to a fixed value of the summation index g of

$$-\sum_{g=1}^{n} \sum_{\substack{\vec{b}\in\mathbb{N}^g \\ |\vec{b}|=n}} \frac{(-1)^g}{g} \binom{n}{\vec{b}} \prod_{l=1}^{g} T_{b_l}. \tag{496}$$

We need to bring this object into the form of a sum of terms which are multiples of elements of the set (495). This we will commit ourselves to for the next few pages. First we apply the product rule for the commutator:

$$\left[\sum_{\substack{\vec{l}\in\mathbb{N}^g\\|\vec{l}|=n}}\frac{(-1)^g}{g}\binom{n}{\vec{l}}\prod_{k=1}^{g}T_{l_k},a^{\#}\right]$$
$$=\sum_{\substack{\vec{l}\in\mathbb{N}^g\\|\vec{l}|=n}}\frac{(-1)^g}{g}\binom{n}{\vec{l}}\sum_{\tilde{k}=1}^{g}\prod_{j=1}^{\tilde{k}-1}T_{l_j}\left[T_{l_{\tilde{k}}},a^{\#}\right]\prod_{j=\tilde{k}+1}^{g}T_{l_j}$$
$$=\sum_{\substack{\vec{l}\in\mathbb{N}^g\\|\vec{l}|=n}}\frac{(-1)^g}{g}\binom{n}{\vec{l}}\sum_{\tilde{k}=1}^{g}\prod_{j=1}^{\tilde{k}-1}T_{l_j}\sum_{\sigma_{\tilde{k}}=1}^{l_{\tilde{k}}}\binom{l_{\tilde{k}}}{\sigma_{\tilde{k}}}a^{\#}Z_fT_{l_{\tilde{k}}-\sigma_{\tilde{k}}}\prod_{j=\tilde{k}+1}^{g}T_{l_j},$$

in the second step we used (3.367). Now we commute all the T_ls to the left of $a^{\#}$ to its right:

$$=\sum_{\substack{\vec{l}\in\mathbb{N}^g\\|l|=n}}\frac{(-1)^g}{g}\binom{n}{\vec{l}}\sum_{\tilde{k}=1}^{g}\sum_{\substack{\forall 1\leqslant j<\tilde{k}\\0\leqslant\sigma_j\leqslant l_j}}\sum_{\sigma_{\tilde{k}}=1}^{l_{\tilde{k}}}\prod_{j=1}^{\tilde{k}}\binom{l_j}{\sigma_j}a^{\#}\prod_{j=1}^{\tilde{k}}Z_{\sigma_j}\prod_{j=1}^{\tilde{k}}T_{l_j-\sigma_j}\prod_{j=\tilde{k}+1}^{g}T_{l_j}. \tag{497}$$

At this point we notice that the multinomial coefficient can be combined with all the binomial coefficients to form a single multinomial coefficient of degree $g+\tilde{k}$. Incidentally this is also the amount of Z operators plus the amount of T operators in each product. Moreover the indices of the Multinomial index agree with the indices of the Z and T operators in the product. Because of this, we see that if we fix an element of the spanning set (495) $a^{\#}\prod_{k=1}^{m_1}Z_{\alpha_k}\prod_{k=1}^{m_2}T_{\beta_k}$, each summand of (497) which contributes to this element, has the prefactor

$$\frac{(-1)^g}{g}\binom{n}{\alpha_1\ \cdots\alpha_{m_1}\ \beta_1\cdots\beta_{m_2}} \tag{498}$$

no matter which summation index $l \in \mathbb{N}^g$ it corresponds to. In order to do the matching one may ignore the indices σ_j and $l_j - \sigma_j$ which vanish, because the corresponding operators Z_0 and T_0 are equal to the identity operator on $\mathcal{H}$ respectively Fock space.
Since we know that

$$\left[d\Gamma\left(\sum_{g=1}^{n} \sum_{\substack{\vec{b}\in\mathbb{N}^g \\ |\vec{b}|=n}} \frac{(-1)^g}{g} \binom{n}{\vec{b}} \prod_{l=1}^{g} Z_{b_l} \right), a^{\#} \right] = a^{\#} \sum_{g=1}^{n} \sum_{\substack{\vec{b}\in\mathbb{N}^g \\ |\vec{b}|=n}} \frac{(-1)^g}{g} \binom{n}{\vec{b}} \prod_{l=1}^{g} Z_{b_l}$$

holds, all that is left to show is that

$$\left[-\sum_{g=1}^{n} \sum_{\substack{\vec{b}\in\mathbb{N}^g \\ |\vec{b}|=n}} \frac{(-1)^g}{g} \binom{n}{\vec{b}} \prod_{l=1}^{g} T_{b_l}, a^{\#} \right] = a^{\#} \sum_{g=1}^{n} \sum_{\substack{\vec{b}\in\mathbb{N}^g \\ |\vec{b}|=n}} \frac{(-1)^{g+1}}{g} \binom{n}{\vec{b}} \prod_{l=1}^{g} Z_{b_l} \tag{499}$$

also holds. For which we need to count the summands which are multiples of each element of (495) corresponding to each g in (496). So let us fix some element $K(m_1, m_2)$ of (495) corresponding to some $m_1 \in \mathbb{N}, m_2 \in \mathbb{N}_0, \alpha \in \mathbb{N}^{m_1}$ and $\beta \in \mathbb{N}^{m_2}$. Rephrasing this problem, we can ask which products

$$\prod_{l=1}^{g} T_{\gamma_l} \tag{500}$$

for suitable g and $(\gamma_l)_l$ produce, when commuted with a creation or annihilation operator, multiples of $K(m_1, m_2)$? We will call this number of total contributions weighted with the factor $-\frac{(-1)^g}{g}$ borrowed from (496) $\#K(m_1, m_2)$. Looking at the commutation relations (3.367) we split the set of indices $\{\gamma_1 \dots \gamma_g\}$ into three sets A, B and C, where the commutation relation has to be used in such a way, that

$$\begin{aligned} \forall k : \gamma_k \in A &\iff \exists j \leqslant m_1 : \gamma_k = \alpha_j, \\ \wedge \forall k : \gamma_k \in B &\iff \exists j \leqslant m_2 : \gamma_k = \beta_j \\ \wedge \forall k : \gamma_k \in C &\iff \exists j \leqslant m_1, l \leqslant m_2 : \gamma_k = \alpha_j + \beta_l \end{aligned}$$

holds. Unfortunately not every splitting corresponds to a contribution and not every order of multiplication of a legal splitting corresponds to a contribution either. However $\prod_j T_{\alpha_j} \prod_j T_{\beta_j}$ gives a contribution and it is in fact the longes product that does. We may apply the commutation relations backwards to obtain any shorter valid combination and hence all combinations. Transforming the commutation rule for T_k read from right to left into a game results in the following rules. Starting from the string

$$A_1 A_2 \dots A_{m_1} B_1 B_2 \dots B_{m_2}, \tag{501}$$

representing the longes product, where here and in the following A's represent operators T_k which will turn into Z_k by the commutation rule, B's represent operators T_k which will stay T_k after commutation and C's represent operators T_k which will produce both a Z_l in the creation/annihilation operator and a T_{k-l} behind that operator. The indices are merely there to keep track of which operator moved where. So the game consists in the answering how many strings can we produce by applying the following rules to the initial string?

1. You may replace any occurrence of $A_k B_j$ by $B_j A_k$ for any j and k.

2. You may replace any occurrence of A_kB_j by $C_{k,j}$ for any j and k.

Where we have to count the number of times we applied the second rule, or equivalently the number $\#C$ of C's in the resulting string, because the summation index g in (496) corresponds to $m_1+m_2-\#C$. Fix $\#C \in \{0,\dots,\min(m_1,m_2)\}$. A valid string has $m_1+m_2-\#C$ characters, because the number of its Cs is $\#C$, its number of As is $m_1-\#C$ and its number of Bs is $m_2-\#C$. Ignoring the labelling of the As, Bs and Cs there are $\binom{m_1+m_2-\#C}{\#C\ (m_1-\#C)\ (m_2-\#C)}$ such strings. Now if we consider one such string without labelling, e.g.

$$CAABACCBBACBBABBBB, \tag{502}$$

there is only one correct labelling to be restored, namely the one where each A and the first index of any C receive increasing labels from left to right and analogously for B and the second index of any C, resulting for our example in

$$C_{1,1}A_2A_3B_2A_4C_{5,3}C_{6,4}B_5B_6A_7C_{8,7}B_8B_9A_9B_{10}B_{11}B_{12}B_{13}. \tag{503}$$

So any unlabelled string corresponds to exactly one labelled string which in turn corresponds to exactly one choice of operator product $\prod T$. So returning to our Operators, we found the number $\#K(m_1,m_2)$ it is

$$\#K(m_1,m_2) = -\sum_{g=\max(m_1,m_2)}^{m_1+m_2} \frac{(-1)^g}{g}\binom{g}{(m_1+m_2-g)\ (g-m_1)\ (g-m_2)}, \tag{504}$$

where the total minus sign comes from the total minus sign in front of (499) with respect to (494).

Now since we introduced the slightly non-standard definition of binomial coefficients used in [4] we can make use of the rules for summing binomial coefficients derived there. As a first step to evaluate (504) we split the trinomial coefficient into binomial ones and make use of the absorption identity

$$\forall a \in \mathbb{C}\ \forall b \in \mathbb{Z} : b\binom{a}{b} = a\binom{a-1}{b-1} \tag{absorption}$$

for $m_2, m_1 \neq 0$ as follows

$$\begin{aligned}
&\#K(m_1, m_2)\\
&= -\sum_{g=\max(m_1,m_2)}^{m_1+m_2} \frac{(-1)^g}{g}\binom{g}{(m_1+m_2-g)\ (g-m_1)\ (g-m_2)}\\
&= -\sum_{g=\max(m_1,m_2)}^{m_1+m_2} \frac{(-1)^g}{g}\binom{g}{m_2}\binom{m_2}{g-m_1}\\
&\overset{(\text{absorption})}{=} -\sum_{g=\max(m_1,m_2)}^{m_1+m_2} \frac{(-1)^g}{m_2}\binom{g-1}{m_2-1}\binom{m_2}{g-m_1}\\
&= \frac{-1}{m_2}\sum_{g=\max(m_1,m_2)}^{m_1+m_2} (-1)^g\binom{g-1}{m_2-1}\binom{m_2}{g-m_1}\\
&\overset{m_1>0}{=} \frac{-1}{m_2}\sum_{g\in\mathbb{Z}}(-1)^g\binom{m_2}{g-m_1}\binom{g-1}{m_2-1}\\
&\overset{*}{=} \frac{-1}{m_2}(-1)^{m_2-m_1}\binom{m_1-1}{-1} = 0,
\end{aligned}$$

where for the second but last equality $m_1 > 0$ is needed for the $g = 0$ summand not to contribute and for the marked equality we used summation rule (5.24) of [4]. So all the coefficients vanish that fulfil

$m_1, m_2 \neq 0$. The sum for the remaining cases is readily computed, since there is just one summand. Summarising we find

$$\#K(m_1, m_2) = \delta_{m_2,0} \frac{(-1)^{1+m_1}}{m_1} + \delta_{m_1,0} \frac{(-1)^{1+m_2}}{m_2},$$

where the second summand can be ignored, since terms with $m_1 = 0$ are irrelevant for our considerations.
So the left hand side of (499) can be evaluated

$$\left[-\sum_{g=1}^{n} \sum_{\substack{\vec{b} \in \mathbb{N}^g \\ |\vec{b}|=n}} \frac{(-1)^g}{g} \binom{n}{\vec{b}} \prod_{l=1}^{g} T_{b_l}, a^{\#} \right]$$
$$= \sum_{g=1}^{n} \sum_{\substack{\vec{b} \in \mathbb{N}^g \\ |b|=n}} \frac{(-1)^{g+1}}{g} \binom{n}{\vec{b}} a^{\#} \prod_{l=1}^{g} Z_{b_l},$$

which is exactly equal to the right hand side of (499). This ends the motivation of the conjecture.

Solution to Recursive Equation

So we found a recursive equation for the T_ns, now we need to solve it. In order to do so we need the following lemma about combinatorial distributions

Lemma 65. *For any $g \in \mathbb{N}, k \in \mathbb{N}$*

$$\sum_{\substack{\vec{g} \in \mathbb{N}^g \\ |\vec{g}|=k}} \binom{k}{\vec{g}} = \sum_{l=0}^{g} (-1)^l (g-l)^k \binom{g}{l} \tag{505}$$

holds. The reader interested in terminology may be eager to know, that the right hand side is equal to $g!$ *times the Stirling number of the second kind* $\left\{ {k \atop g} \right\}$.

Proof: We would like to apply the multinomial theorem but there are all the summands missing where at least one of the entries of $\vec{g}$ is zero, so we add an appropriate expression of zero. We also give the expression in question a name, since we will later on arrive at a recursive expression.

$$\begin{aligned} F(g,k) &:= \sum_{\substack{\vec{g}\in\mathbb{N}^g \\ |\vec{g}|=k}} \binom{k}{\vec{g}} = \sum_{\substack{\vec{g}\in\mathbb{N}_0^g \\ |\vec{g}|=k}} \binom{k}{\vec{g}} - \sum_{\substack{\vec{g}\in\mathbb{N}_0^g \\ |\vec{g}|=k \\ \exists l: g_l=0}} \binom{k}{\vec{g}} \\ &= g^k - \sum_{\substack{\vec{g}\in\mathbb{N}_0^g \\ |\vec{g}|=k \\ \exists l: g_l=0}} \binom{k}{\vec{g}} = g^k - \sum_{n=1}^{g-1} \sum_{\substack{\vec{g}\in\mathbb{N}_0^g \\ |\vec{g}|=k}} \binom{k}{\vec{g}} 1_{\exists! i_1 \dots i_n : \forall i_l \neq i_k \wedge \forall l: g_{i_l}=0} \end{aligned} \tag{506}$$

where in the last line the indicator function is to enforce there being exactly n different indices i_l for which $g_{i_l} = 0$ holds. Now since it does not matter which entries of the vector vanish because the multinomial coefficient is symmetric and its value is identical to the corresponding multinomial coefficient where the vanishing entries are omitted, we can further simplify the sum:

$$F(g,k) = g^k - \sum_{n=1}^{g-1} \binom{g}{n} \sum_{\substack{\vec{g}\in\mathbb{N}^n \\ |\vec{g}|=k}} \binom{k}{\vec{g}}$$

The inner sum turns out to be $F(g-n,k)$, so we found the recursive relation for F:

$$F(g,k) = g^k - \sum_{n=1}^{g-1} \binom{g}{n} F(n,k) = g^k - \sum_{n=1}^{g-1} \binom{g}{n} F(g-n,k), \quad (507)$$

where for the last equality we used the symmetry of binomial coefficients. By iteratively applying this equation, we find the following formula, which we will now prove by induction

$$\forall d \in \mathbb{N}_0 : F(g,k) = \sum_{l=0}^{d} (-1)^l (g-l)^k \binom{g}{l} \\ + (-1)^{d+1} \sum_{n=1}^{g-d-1} \binom{n+d-1}{d} \binom{g}{n+d} F(g-d-n,k). \quad (508)$$

We already showed the start of the induction, so what's left is the induction step. Before we do so the following remark is in order: We are only interested in the case $d = g$ and the formula seems meaningless for $d > g$; however, the additional summands in the left sum vanish, where as the right sum is empty for these values of d since the upper bound of the summation index is lower than its lower bound.
For the induction step, pick $d \in \mathbb{N}_0$, use (508) and pull the first summand out of the second sum, on this summand we apply the recursive relation (507) resulting in

$$F(g,k) = \sum_{l=0}^{d} (-1)^l (g-l)^k \binom{g}{l} \\ + (-1)^{d+1} \sum_{n=2}^{g-d-1} \binom{n+d-1}{d} \binom{g}{n+d} F(g-d-n,k)$$

$$
\begin{aligned}
&+ (-1)^{d+1}\binom{d}{d}\binom{g}{d+1}F(g-d-1,k) \\
&\overset{(507)}{=} \sum_{l=0}^{d+1}(-1)^l (g-l)^k \binom{g}{l} \\
&+ (-1)^{d+1}\sum_{n=2}^{g-d-1}\binom{n+d-1}{d}\binom{g}{n+d}F(g-d-n,k) \\
&- (-1)^{d+1}\binom{g}{d+1}\sum_{n=1}^{g-d-2}\binom{g-d-1}{n}F(g-d-1-n,k) \\
&= \sum_{l=0}^{d+1}(-1)^l (g-l)^k \binom{g}{l} \\
&+ (-1)^{d+1}\sum_{n=1}^{g-d-2}\binom{n+d}{d}\binom{g}{n+d+1}F(g-d-1-n,k) \\
&- (-1)^{d+1}\binom{g}{d+1}\sum_{n=1}^{g-d-2}\binom{g-d-1}{n}F(g-d-1-n,k). \qquad (509)
\end{aligned}
$$

After the index shift we can combine the last two sums.

$$
\begin{aligned}
F(g,k) &= \sum_{l=0}^{d+1}(-1)^l (g-l)^k \binom{g}{l} \\
&+ \sum_{n=1}^{g-d-2}\left[\binom{g}{d+1}\binom{g-d-1}{n} - \binom{n+d}{d}\binom{g}{n+d+1}\right] \\
&\qquad (-1)^{d+2}F(g-d-1-n,k). \qquad (510)
\end{aligned}
$$

In order to combine the two binomials we reassemble $\binom{g}{d+1}\binom{g-d-1}{n}$ into $\binom{g}{n+d+1}\binom{n+d+1}{d+1}$, which can be seen to be possible by representing everything in terms of factorials. This results in

$$
\begin{aligned}
F(g,k) &= \sum_{l=0}^{d+1}(-1)^l(g-l)^k\binom{g}{l} \\
&+(-1)^{d+2}\sum_{n=1}^{g-d-2}\left[\binom{n+d+1}{d+1}-\binom{n+d}{d}\right]\binom{g}{n+d+1}F(g-d-1-n,k) \\
&= \sum_{l=0}^{d+1}(-1)^l(g-l)^k\binom{g}{l} \\
&+ (-1)^{d+2}\sum_{n=1}^{g-d-2}\binom{n+d}{d+1}\binom{g}{n+d+1}F(g-d-1-n,k), \quad (511)
\end{aligned}
$$

where we used the addition formula for binomials:

$$\forall n \in \mathbb{C}\forall k \in \mathbb{Z} : \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}. \tag{512}$$

This concludes the proof by induction. By setting $d = g$ in equation (508) we arrive at the desired result. □

Using the previous lemma, we are able to show the next

Lemma 66. *For any $k \in \mathbb{N}\backslash\{1\}$ the following equation holds*

$$\sum_{g=1}^{k}\frac{(-1)^g}{g}\sum_{\substack{\vec{g}\in\mathbb{N}^g \\ |\vec{g}|=k}}\binom{k}{\vec{g}} = 0. \tag{513}$$

Proof: Let $k \in \mathbb{N}\backslash\{1\}$, as a first step we apply lemma 65. We change the order of summation, use (absorption), extend the range of summation and shift summation index to arrive at

$$
\begin{aligned}
\sum_{g=1}^{k} \frac{(-1)^g}{g} \sum_{l=0}^{g} (-1)^l (g-l)^k \binom{g}{l} &= \sum_{g=1}^{k} \frac{1}{g} \sum_{l=0}^{g} (-1)^{g-l} (g-l)^k \binom{g}{g-l} \\
= \sum_{g=1}^{k} \sum_{p=0}^{g} (-1)^p p^k \frac{1}{g} \binom{g}{p} &= \sum_{g=1}^{k} \sum_{p=0}^{g} (-1)^p p^k \frac{1}{p} \binom{g-1}{p-1} \\
= \sum_{g=1}^{k} \sum_{p\in\mathbb{Z}} (-1)^p p^{k-1} \binom{g-1}{p-1} &= \sum_{p\in\mathbb{Z}} (-1)^p p^{k-1} \sum_{g=1}^{k} \binom{g-1}{p-1} \\
&= \sum_{p\in\mathbb{Z}} (-1)^p p^{k-1} \sum_{g=0}^{k-1} \binom{g}{p-1}. \qquad (514)
\end{aligned}
$$

Now we use equation (5.10) of [4]:

$$
\forall m, n \in \mathbb{N}_0 : \sum_{k=0}^{n} \binom{k}{m} = \binom{n+1}{m+1}, \qquad \text{(upper summation)}
$$

which can for example be proven by induction on n.
We furthermore rewrite the power of the summation index p in terms of the derivative of an exponential and change order summation and differentiation. This results in

$$
\begin{aligned}
&\sum_{g=1}^{k} \frac{(-1)^g}{g} \sum_{l=0}^{g} (-1)^l (g-l)^k \binom{g}{l} = \sum_{p\in\mathbb{Z}} (-1)^p p^{k-1} \binom{k}{p} \\
&= \sum_{p=0}^{k} (-1)^p \left. \frac{\partial^{k-1}}{\partial \alpha^{k-1}} e^{\alpha p} \right|_{\alpha=0} \binom{k}{p} = \left. \frac{\partial^{k-1}}{\partial \alpha^{k-1}} \sum_{p=0}^{k} (-1)^p e^{\alpha p} \binom{k}{p} \right|_{\alpha=0} \\
&= \left. \frac{\partial^{k-1}}{\partial \alpha^{k-1}} \left(1 - e^{\alpha p}\right)^k \right|_{\alpha=0} = \left. (-1)^k \frac{\partial^{k-1}}{\partial \alpha^{k-1}} \left(\sum_{l=1}^{\infty} \frac{(\alpha p)^l}{l!} \right)^k \right|_{\alpha=0}
\end{aligned}
$$

$$= (-1)^k \frac{\partial^{k-1}}{\partial \alpha^{k-1}} ((\alpha p)^k + \mathcal{O}((\alpha p)^{k+1}))\bigg|_{\alpha=0} = 0.$$

□

We are now in a position to state the solution to the recursive equation (494) and motivate that it is in fact a solution.

Conjecture 67. *For $n \in \mathbb{N}$ the solution of the recursive equation (494) solely in terms of G_a and C_a is given by*

$$T_n = \sum_{g=1}^{n} \sum_{\substack{\vec{b} \in \mathbb{N}^g \\ |\vec{b}|=n}} \sum_{\vec{d} \in \{0,1\}^g} \frac{1}{g!} \binom{n}{\vec{b}} \prod_{l=1}^{g} F_{b_l, d_l}, \tag{515}$$

where F is given by

$$F_{a,b} = \begin{cases} \Gamma_a & \text{for } b = 0 \\ C_a & \text{for } b = 1 \end{cases}. \tag{516}$$

For the readers convenience we remind her, that Γ_a and the constants C_n are defined in theorem 64.

Motivation: The structure of this proof will be induction over n. For $n = 1$ the whole expression on the right hand side collapses to $C_1 + \Gamma_1$, which we already know to be equal to T_1. For arbitrary $n + 1 \in \mathbb{N} \backslash \{1\}$ we apply the recursive equation (494) once and use the induction hypothesis for all $k \leqslant n$ and thereby arrive at the rather convoluted expression

$$
\begin{aligned}
T_{n+1} &\overset{(494)}{=} \Gamma_{n+1} + C_{n+1} + \sum_{g=2}^{n+1} \sum_{\substack{\vec{b}\in\mathbb{N}^g \\ |\vec{b}|=n+1}} \frac{(-1)^g}{g} \binom{n+1}{\vec{b}} \prod_{l=1}^{g} T_{b_l} \\
&\overset{\text{induction hyp}}{=} \Gamma_{n+1} + C_{n+1} + \sum_{g=2}^{n+1} \sum_{\substack{\vec{b}\in\mathbb{N}^g \\ |\vec{b}|=n+1}} \frac{(-1)^g}{g} \binom{n+1}{\vec{b}} \prod_{l=1}^{g} \\
&\qquad \sum_{g_l=1}^{b_l} \sum_{\substack{\vec{c}_l\in\mathbb{N}^{g_l} \\ |\vec{c}_l|=b_l}} \sum_{\vec{e}_l\in\{0,1\}^{g_l}} \frac{1}{g_l!} \binom{b_l}{\vec{c}_l} \prod_{k=1}^{g_l} F_{c_{l,k},e_{l,k}}.
\end{aligned}
\tag{517}
$$

If we were to count the contributions of this sum to a specific product $\prod F_{c_j,e_j}$ for some choice of $(c_j)_j, (e_j)_j$ we would first recognize that all the multinomial factors in (517) combine to a single one whose indices are given by the first indices of all the F factors involved. Other than this factor each contribution adds $\frac{(-1)^g}{g} \prod_{l=1}^{g} \frac{1}{g_l!}$ to the sum. So we need to keep track of how many contributions there are and which distributions of g_l they belong to.

Fix some product $\prod F := \prod_{j=1}^{\tilde{g}} F_{\tilde{b}_j,\tilde{d}_j}$. In the sum (517) we pick some initial short product of length g and split each factor into g_l pieces to arrive at one of length $\tilde{g}$ if the product is to contribute to $\prod F$. So clearly $\sum_{l=1}^{g} g_l = \tilde{g}$ holds for any contribution to $\prod F$. The reverse is also true, for any g and $g_1, \ldots, g_g \in \mathbb{N}$ such that $\sum_{l=1}^{g} g_l = \tilde{g}$ holds the corresponding expression in (517) contributes to $\prod F$. Furthermore $\prod F$ and $g, g_1, \ldots g_g$ is enough to uniquely determine the summand of (517) the contribution belongs to. For an illustration of this splitting see

$$\underbrace{\underbrace{F^1_{3,1}F^2_{2,0}F^3_{7,1}}_{g_1=3}\ \underbrace{F^4_{5,0}}_{g_2=1}\ \underbrace{F^5_{4,1}F^6_{2,1}}_{g_3=2}\ \underbrace{F^7_{1,1}F^8_{3,0}F^9_{4,1}}_{g_4=3}\ \underbrace{F^{10}_{4,1}F^{11}_{1,0}}_{g_5=2}}_{g=5}$$

$$g_1 + g_2 + g_3 + g_4 + g_5 = 11 = \tilde{g},$$

where I labelled the factors in the upper right index for the readers convenience. We recognize that the sum we are about to perform is by no means unique for each order of n but only depends on the number of appearing factors and the number of splittings performed on them. By the preceding argument we need

$$\sum_{g=2}^{\tilde{g}} \frac{(-1)^g}{g} \sum_{\substack{\vec{g}\in\mathbb{N}^g \\ |\vec{g}|=\tilde{g}}} \prod_{l=1}^{g} \frac{1}{g_l!} = \frac{1}{\tilde{g}!} \tag{518}$$

to hold for $\tilde{g} > 1$, in order to find agreement with the proposed solution (516). Now proving (518) is done by realizing, that one can include the right hand side into the sum as the $g = 1$ summand, dividing the equation by $\tilde{g}!$ and using lemma 66 with $k = \tilde{g}$. The remaining case, $\tilde{g} = 1$, can directly be read off of (517). This ends the motivation of this conjecture.

Conjecture 68. *For $n \in \mathbb{N}$, T_n can be written as*

$$\frac{1}{n!}T_n = \sum_{\substack{1\leqslant c+g\leqslant n \\ c,g\in\mathbb{N}_0}} \sum_{\substack{\vec{g}\in\mathbb{N}^g \\ \vec{c}\in\mathbb{N}^c \\ |\vec{c}|+|\vec{g}|=n}} \frac{1}{c!g!} \prod_{l=1}^{c} \frac{1}{c_l!} C_{c_l} \prod_{l=1}^{g} \frac{1}{g_l!} \Gamma_{g_l}. \tag{519}$$

Please note that for ease of notation we defined $\mathbb{N}^0 := \{1\}$.

Motivation: By an argument completely analogous to the combinatorial argument in the motivation of conjecture (64) we see that we can disentangle the Fs in (515) into Γs and Cs if we multiply by a factor of $\binom{c+g}{c}$ where c is the number of Cs and g is the number of Γs giving

$$T_n = \sum_{\substack{1\leqslant c+g\leqslant n\\ c,g\in\mathbb{N}_0}} \sum_{\substack{\vec{g}\in\mathbb{N}^g\\ \vec{c}\in\mathbb{N}^c\\ |\vec{c}|+|\vec{g}|=n}} \binom{c+g}{c} \frac{1}{(c+g)!} \binom{n}{\vec{g}\oplus\vec{c}} \prod_{l=1}^{c} C_{c_l} \prod_{l=1}^{g} \Gamma_{g_l}, \tag{520}$$

which directly reduces to the equation we wanted to prove, by plugging in the multinomials in terms of factorials.

Conjecture 69. *As a formal power series, the second quantized scattering operator can be written in the form*

$$S = e^{\sum_{l\in\mathbb{N}} \frac{C_l}{l!}} e^{\sum_{l\in\mathbb{N}} \frac{\Gamma_l}{l!}}. \tag{521}$$

Proof: We plug conjecture 68 into the defining Series for the T_ns giving

$$S = \sum_{n\in\mathbb{N}_0} \frac{1}{n!} T_n \tag{522}$$

$$= \mathbb{1}_{\mathcal{F}} + \sum_{n\in\mathbb{N}} \sum_{\substack{1\leqslant c+g\leqslant n\\ c,g\in\mathbb{N}_0}} \sum_{\substack{\vec{g}\in\mathbb{N}^g\\ \vec{c}\in\mathbb{N}^c\\ |\vec{c}|+|\vec{g}|=n}} \frac{1}{c!g!} \prod_{l=1}^{c} \frac{1}{c_l!} C_{c_l} \prod_{l=1}^{g} \frac{1}{g_l!} \Gamma_{g_l} \tag{523}$$

$$= \mathbb{1}_{\mathcal{F}} + \sum_{\substack{1\leqslant c+g\\ c,g\in\mathbb{N}_0}} \sum_{\substack{\vec{g}\in\mathbb{N}^g\\ \vec{c}\in\mathbb{N}^c}} \frac{1}{c!g!} \prod_{l=1}^{c} \frac{1}{c_l!} C_{c_l} \prod_{l=1}^{g} \frac{1}{g_l!} \Gamma_{g_l} \tag{524}$$

$$= \sum_{c,g\in\mathbb{N}_0} \sum_{\substack{\vec{g}\in\mathbb{N}^g\\ \vec{c}\in\mathbb{N}^c}} \frac{1}{c!g!} \prod_{l=1}^{c} \frac{1}{c_l!} C_{c_l} \prod_{l=1}^{g} \frac{1}{g_l!} \Gamma_{g_l} \tag{525}$$

$$= \sum_{c\in\mathbb{N}_0} \frac{1}{c!} \sum_{\vec{c}\in\mathbb{N}^c} \prod_{l=1}^{c} \frac{1}{c_l!} C_{c_l} \sum_{g\in\mathbb{N}_0} \frac{1}{g!} \sum_{\vec{g}\in\mathbb{N}^g} \prod_{l=1}^{g} \frac{1}{g_l!} \Gamma_{g_l} \tag{526}$$

$$= \sum_{c\in\mathbb{N}_0} \frac{1}{c!} \prod_{l=1}^{c} \sum_{k\in\mathbb{N}} \frac{1}{k!} C_k \sum_{g\in\mathbb{N}_0} \frac{1}{g!} \prod_{l=1}^{g} \sum_{b\in\mathbb{N}} \frac{1}{b!} \Gamma_b \tag{527}$$

$$= \sum_{c\in\mathbb{N}_0} \frac{1}{c!} \left(\sum_{k\in\mathbb{N}} \frac{1}{k!} C_k \right)^c \sum_{g\in\mathbb{N}_0} \frac{1}{g!} \left(\sum_{b\in\mathbb{N}} \frac{1}{b!} \Gamma_b \right)^g \tag{528}$$

$$= e^{\sum_{l\in\mathbb{N}} \frac{1}{l!} C_l} e^{\sum_{l\in\mathbb{N}} \frac{1}{l!} \Gamma_l}. \tag{529}$$

Conjecture 70. *For A such that*

$$\|\mathbb{1} - U^A\| < 1. \tag{530}$$

The second quantized scattering operator fulfils

$$S = e^{\sum_{n\in\mathbb{N}} \frac{C_n}{n!}} e^{\mathrm{d}\Gamma(\ln(U))} \tag{531}$$

where C_n must be imaginary for any $n \in \mathbb{N}$ in order to satisfy unitarity.

Motivation: First the remark about $C_n \in i\mathbb{R}$ for any n is a direct consequence of the second factor of (531) begin unitary. This in turn follows directly from $\mathrm{d}\Gamma^*(K) = -\mathrm{d}\Gamma(K)$ for any K in the domain of $\mathrm{d}\Gamma$. That $\ln U$ is in the domain of $\mathrm{d}\Gamma$ follows from $(\ln U)^* = \ln U^* = \ln U^{-1} = -\ln U$ and $\|U - \mathbb{1}\| < 1$.

We are going to change the sum in the second exponential of (521), so let's take a closer look at that: by exchanging summation we can step by step simplify

$$
\begin{aligned}
\sum_{l\in\mathbb{N}} \frac{\Gamma_l}{l!} &= \sum_{n\in\mathbb{N}} \frac{1}{n!} \mathrm{d}\Gamma \left(\sum_{g=1}^{n} \sum_{\substack{\vec{b}\in\mathbb{N}^g \\ |\vec{b}|=n}} \frac{(-1)^{g+1}}{g} \binom{n}{\vec{b}} \prod_{l=1}^{g} Z_{b_l} \right) \\
&= \mathrm{d}\Gamma \left(\sum_{n\in\mathbb{N}} \frac{1}{n!} \sum_{g=1}^{n} \sum_{\substack{\vec{b}\in\mathbb{N}^g \\ |\vec{b}|=n}} \frac{(-1)^{g+1}}{g} \binom{n}{\vec{b}} \prod_{l=1}^{g} Z_{b_l} \right) \\
&= \mathrm{d}\Gamma \left(\sum_{n\in\mathbb{N}} \sum_{g=1}^{n} \sum_{\substack{\vec{b}\in\mathbb{N}^g \\ |\vec{b}|=n}} \frac{(-1)^{g+1}}{g} \prod_{l=1}^{g} \frac{Z_{b_l}}{b_l!} \right) \\
&= \mathrm{d}\Gamma \left(\sum_{g\in\mathbb{N}} \sum_{\vec{b}\in\mathbb{N}^g} \frac{(-1)^{g+1}}{g} \prod_{l=1}^{g} \frac{Z_{b_l}}{b_l!} \right) \\
&= \mathrm{d}\Gamma \left(\sum_{g\in\mathbb{N}} \frac{(-1)^{g+1}}{g} \prod_{l=1}^{g} \left(\sum_{b_l\in\mathbb{N}} \frac{Z_{b_l}}{b_l!} \right) \right) \\
&= \mathrm{d}\Gamma \left(\sum_{g\in\mathbb{N}} \frac{(-1)^{g+1}}{g} \left(\sum_{b\in\mathbb{N}} \frac{Z_b}{b!} \right)^{g} \right) \\
&= \mathrm{d}\Gamma \left(\sum_{g\in\mathbb{N}} \frac{(-1)^{g+1}}{g} \left(U - \mathbb{1} \right)^{g} \right) = \mathrm{d}\Gamma \left(- \sum_{g\in\mathbb{N}} \frac{1}{g} \left(\mathbb{1} - U \right)^{g} \right) \\
&= \mathrm{d}\Gamma \left(\ln \left(\mathbb{1} - \left(\mathbb{1} - U \right) \right) \right) = \mathrm{d}\Gamma \left(\ln \left(U \right) \right). \qquad (532)
\end{aligned}
$$

The last conjecture is proven directly in section 3.4.4

Bibliography

[1] D-A Deckert, D Dürr, F Merkl, and M Schottenloher, *Time-evolution of the external field problem in quantum electrodynamics*, Journal of Mathematical Physics **51** (2010), no. 12, 122301.

[2] D-A Deckert and F Merkl, *External field qed on cauchy surfaces for varying electromagnetic fields*, Communications in Mathematical Physics **345** (2016), no. 3, 973–1017.

[3] D-A Deckert and Franz Merkl, *Dirac equation with external potential and initial data on cauchy surfaces*, Journal of Mathematical Physics **55** (2014), no. 12, 122305.

[4] Ronald L Graham, Donald E Knuth, and Oren Patashnik, *Concrete mathematics: a foundation for computer science*, Addison-Wesley, Reading, 1994.

[5] Abdolhossein Hoorfar and Mehdi Hassani, *Inequalities on the lambert w function and hyperpower function*, J. Inequal. Pure and Appl. Math **9** (2008), no. 2, 5–9.

[6] Matthias Lienert, *Direct interaction along light cones at the quantum level*, Journal of Physics A: Mathematical and Theoretical **51** (2018), no. 43, 435302.

[7] Matthias Lienert and Markus Nöth, *Existence of relativistic dynamics for two directly interacting dirac particles in 1+ 3 dimensions*, arXiv preprint arXiv:1903.06020 (2019).

[8] Matthias Lienert and Roderich Tumulka, *A new class of volterra-type integral equations from relativistic quantum physics*, arXiv preprint arXiv:1803.08792 (2018).

[9] ———, *Interacting relativistic quantum dynamics of two particles on spacetimes with a big bang singularity*, Journal of Mathematical Physics **60** (2019), no. 4, 042302.

[10] Michael Reed and Barry Simon, *Methods of modern mathematical physics, vol. ii*, 1975.

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