The Phase of the Second Quantised Time Evolution Operator

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Abstract

abstract to be written

1 Introduction

We follow the necessary definitions in [**?**].

is das ok hier ivp2 so ausgiebig zu zitieren? Mir scheint aber, die ganzen Sachen müssen eben genau nochmal eingeführt werden.

Definition 1. *For a Cauchy surface Σ, we define $\mathcal{H}_\Sigma$ to be the Hilbert space of $\mathcal{C}^4$ valued, square integrable functions on Σ. Furthermore, let $\mathrm{Pol}(\mathcal{H}_\Sigma)$ denote the set of all closed, linear subspaces $V \subset \mathcal{H}_\Sigma$ such that both V and $V^\perp$ are infinite dimensional. Any $V \in \mathrm{Pol}(\mathcal{H}_\Sigma)$ is called a polarisation of $\mathcal{H}$. For $V \in \mathrm{Pol}$, let $P^V_\Sigma : \mathcal{H}_\Sigma \to V$ denote the orthogonal projection of $\mathcal{H}_\Sigma$ onto V.*
The Fock space corresponding to polarisation V on Cauchy surface Σ is then defined by

$$\mathcal{F}(V,\mathcal{H}_\Sigma) := \bigoplus_{c\in\mathbb{Z}} \mathcal{F}_c(V,\mathcal{H}_\Sigma), \quad \mathcal{F}_c(V,\mathcal{H}_\Sigma) := \bigoplus_{\substack{n,m\in\mathbb{N}_0 \\ c=m-n}} \left(V^\perp\right)^{\wedge n} \otimes \overline{V}^{\wedge m}, \tag{1}$$

where $\bigoplus$ denotes the Hilbert space direct sum, $\wedge$ the antisymmetric tensor product of Hilbert spaces, and $\overline{V}$ the conjugate complex vector space of V, which coincides with V as a set and has the same vector space operations as V with the exception of the scalar multiplication, which is replaced by $(z,\psi) \mapsto z^*\psi$ for $z \in \mathbb{C}), \psi \in V$.

Each polarisation V splits the Hilbert space $\mathcal{H}_\Sigma$ into a direct sum, i.e., $\mathcal{H}_\Sigma = V^\perp \oplus V$. The "standard“ polarisation $\mathcal{H}^+_\Sigma$ and $\mathcal{H}^-_\Sigma$ are determined by the orthogonal projectors P^+_Σ and P^-_Σ onto the free positive and negative energy Dirac solutions, respectively, restricted to Σ:

$$\mathcal{H}^+_\Sigma := P^+_\Sigma \mathcal{H} = (1 - P^-_\Sigma)\mathcal{H}_\Sigma, \quad \mathcal{H}^-_\Sigma := P^-_\Sigma \mathcal{H}_\Sigma. \tag{2}$$

Given two Cauchy surfaces Σ, Σ' and two polarisations $V \in \mathrm{Pol}(\mathcal{H}_\Sigma)$ and $W \in \mathrm{Pol}(\Sigma_{\Sigma'})$ a sensible lift of the one particle Dirac evolution $U^A_{\Sigma'\Sigma} : \mathcal{H} \to \mathcal{H}_\Sigma$ should be given by a unitary operator $\tilde{U}^A_{\Sigma',\Sigma} : \mathcal{F}(V,\mathcal{H}_\Sigma) \to \mathcal{F}(W,\mathcal{H}_{\Sigma'})$ that fulfils

$$\tilde{U}^A_{\Sigma',\Sigma} \psi_{V,\Sigma}(f) (\tilde{U}^A_{\Sigma',\Sigma})^{-1} = \psi_{W,\Sigma'}(U^A_{\Sigma',\Sigma} f), \quad \forall f \in \mathcal{H}_\Sigma. \tag{3}$$

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Here, $\psi_{V,\Sigma}$ denotes the Dirac field operator corresponding to Fock space $\mathcal{F}(V,\Sigma)$, i.e.,

$$\psi_{V,\Sigma}(f) := b_\Sigma\big(P_\Sigma^{V^\perp} f\big) + d_\Sigma^*\big(P_\Sigma^{V} f\big), \quad \forall f \in \mathcal{H}_\Sigma, \tag{4}$$

where b_Σ, d_Σ^* denote the annihilation and creation operators on the $V^\perp$ and $\overline{V}$ sectors of $\mathcal{F}_c(V,\mathcal{H}_\Sigma)$, respectively. Note that $P_\Sigma^{V^\perp} : \mathcal{H}_\Sigma \to \overline{V}$ is anti-linear; thus, $\psi_{V,\Sigma}(f)$ is anti-linear in its argument f. The condition under which such a lift $\tilde{U}^A_{\Sigma',\Sigma}$ exists can be inferred from a straight-forward application of Shale and Stinespring's well-known theorem [?]

Theorem 1 (Shale-Stinespring)**.** *The following statements are equivalent:*

- *There is a unitary operator $\tilde{U}^A_{\Sigma\Sigma'} : \mathcal{F}(V,\mathcal{H}_\Sigma) \to \mathcal{F}(W,\mathcal{H}_{\Sigma'})$ which fulfils (3).*
- *The off-diagonals $P_{\Sigma'}^{W^\perp} U^A_{\Sigma'\Sigma} P_\Sigma^V$ and $P_{\Sigma'}^{W} U^A_{\Sigma'\Sigma}$ are Hilbert-Schmidt operators.*

Please note that condition (3) is for fixed polarisations V, W and general external field A not always satisfied; see e.g. [?]. However, when carefully adapting the choices of polarisation V to $A|_\Sigma$ and W to $A|_{\Sigma'}$ one can always fulfil condition (3) and therefore construct a lift $\tilde{U}^A_{\Sigma'\Sigma}$, see [?, ?, ?].
Furthermore condition (3) does not fix the phase of the lift $\tilde{U}^A_{\Sigma'\Sigma}$. Considering Bogolyubov's formula

$$j^\mu(x) = i\tilde{U}^A_{\Sigma_{\text{in}},\Sigma_{\text{out}}} \frac{\delta \tilde{U}^A_{\Sigma_{\text{out}},\Sigma_{\text{in}}}}{\delta A_\mu(x)}, \tag{5}$$

we notice that the current operator depends in a rather sensitive way on the phase of $\tilde{U}^A$. Since the current is experimentally accessible we would like to fix the phase by additional physical constraints. This paper is a step this direction.
Throughout this paper $A, A', B, C, F, G, H \in C_c^\infty(\mathbb{R}^4, \mathbb{R}^4)$ denote external electromagnetic four-potentials. Furthermore, $\Sigma, \Sigma', \Sigma_{\text{in}}, \Sigma_{\text{ou}}$ denote Cauchy surfaces, Σin is in the remote past and Σout in the far future. We introduce the standard polarisation in the remote past

$$P^- := P^-_{\Sigma_{\text{in}}}, \quad P^+ = 1 - P^-, \tag{6}$$

given by the negative respectively positive energy subspaces of $\mathcal{H}_{\Sigma_{\text{in}}}$.
... some glue text

Definition 2. *We define for all four potentials A, B*

$$S_{A,B} := U^A_{\Sigma_{\text{in}},\Sigma_{\text{out}}} U^B_{\Sigma_{\text{out}},\Sigma_{\text{in}}}. \tag{7}$$

Using the notation of [?, ?] we choose for all $S_{A,B} \in U_{\text{res}}(\mathcal{H}_{\Sigma_{\text{in}}}, \mathcal{H}^-)$ such that $P^- S_{A,B} P^-$ is invertible the lift

$$\overline{S}_{A,B} = \mathcal{R}_{P^- S_{B,A} P^- |P^- S_{B,A} P^-|^{-1}} \mathcal{L}_{S_{A,B}}. \tag{8}$$

Furthermore, we define for any complex number $z \in \mathbb{C}\backslash\{0\}$

$$\overset{\circ}{z} := \frac{z}{|z|} \tag{9}$$

and for four-potentials A, B, C such that $P^{\pm} S_{A,B} P^{\mp}, P^{\pm} S_{B,C} P^{\mp}, P^{\pm} S_{C,A} P^{\mp} \in I_2(\mathcal{H}_{\Sigma_{\text{in}}})$, the complex number of unit magnitude

should I remove all outermost factors fo P^-, or will that decrease ease of reading?

$$\gamma_{A,B,C} := \det_{\mathcal{H}^-}(P^- - P^- S_{A,C} P^+ S_{C,A} P^- - P^- S_{A,B} P^+ S_{B,C} P^- S_{C,A} P^-), \tag{10}$$

$$\Gamma_{A,B,C} := \mathring{\gamma}_{A,B,C}, \tag{11}$$

where Ω is a vacuum vector corresponding to the standard polarisation on $\mathcal{H}_{\Sigma_{\text{in}}}$. Here, Γ is defined whenever $\gamma \neq 0$. Lastly we introduce the partial derivative in the direction of any four-potential F by

$$\partial_F T(F) := \partial_\varepsilon T(\varepsilon F)|_{\varepsilon=0} \tag{12}$$

and for four-potentials A, B, C the function

$$c_A(F, G) := -i \partial_F \partial_G \Im \operatorname{tr}[P^- S_{A,A+F} P^+ S_{A,A+G} P^-]. \tag{13}$$

2 Main Result

Definition 3. *We define a causal splitting as a function*

$$c^+ : (C_c^\infty(\mathbb{R}^4, \mathbb{R}^4))^3 \to \mathbb{C}, \tag{14}$$

$$(A, F, G) \mapsto c_A^+(F, G), \tag{15}$$

smooth in the first and linear in the second and third argument, satisfying

$$c_A(F, G) = c_A^+(F, G) - c_A^+(G, F), \tag{16}$$

$$\partial_H c_{A+H}^+(F, G) = \partial_G c_{A+G}^+(F, H), \tag{17}$$

$$\forall F \prec G : c_A^+(F, G) = 0. \tag{18}$$

Definition 4. *Given a lift $\hat{S}_{A,B}$ of the one-particle scattering operator $S_{A,B}$ we define the associated current by Bogolyubov's formula:*

$$j_A^{\hat{S}}(F) := i \partial_F \left\langle \Omega, \hat{S}_{A,A+F} \Omega \right\rangle. \tag{19}$$

Theorem 2. *Given a causal splitting c^+, there is a second quantised scattering operator $\tilde{S}$, lift of the one-particle scattering operator S with the following properties*

$$\forall A, B, C \in C_c^\infty : \tilde{S}_{A,B} \tilde{S}_{B,C} = \tilde{S}_{A,C} \tag{20}$$

$$\forall F \prec G : \tilde{S}_{A,A+F} = \tilde{S}_{A+G,A+F+G} \tag{21}$$

and the associated current satisfies

$$\partial_G j_{A+G}^{\tilde{S}}(F) = \begin{cases} -2ic_A(F, G) & \text{for } G \prec F \\ 0 & \text{otherwise.} \end{cases} \tag{22}$$

3 Proofs

Throughout this section we will assume we have a function c^+ fulfilling (16),(17) and (18). Since the phase of a lift relative to any other lift is fixed by a single matrix element, we may use the vacuum expectation values to characterise the phase of a lift. The function c captures the dependence of this object on variation of the external fields, the connection between vacuum expectation values and c becomes clearer with the next lemma.

Lemma 1. *The function Γ has the following properties for all four-potentials A, B, C such that the expressions occurring in each equation are well defined, as well as $\alpha, \beta \in \mathbb{R}$:*

$$\Gamma_{A,B,C} = \det_{\mathcal{H}^-}(P^- S_{C,A} \overset{\circ}{P^-} S_{A,B} P^- S_{B,C}) \tag{1}$$

$$\Gamma_{A,B,C} = \langle \Omega, \overline{S}_{A,B} \overset{\circ}{\overline{S}}_{B,C} \overline{S}_{C,A} \Omega \rangle \tag{2}$$

$$\Gamma_{A,B,C} = \Gamma_{B,C,A} = \frac{1}{\Gamma_{B,A,C}} \tag{23}$$

$$\Gamma_{A,A,B} = 1 \tag{24}$$

$$\Gamma_{A,B,C}\Gamma_{B,A,D}\Gamma_{A,C,D}\Gamma_{C,B,D} = 1 \quad \Leftarrow \text{is that necessary?} \tag{3}$$

$$\overline{S}_{A,C} = \Gamma_{A,B,C} \overline{S}_{A,B} \overline{S}_{B,C} \tag{25}$$

$$c_A(B, C) = \partial_B \partial_C \ln \Gamma_{A,A+B,A+C}. \tag{26}$$

In order to construct the desired lift, we first construct a reference lift $\hat{S}$, that is well defined for any four-potential A such that $\operatorname{supp} \vec{A} \cap \Sigma_{\text{in}} = \varnothing$. Afterwards we will study the dependence of the relative phase between this global lift $\hat{S}_{0,A}$ and a local lift given by $\hat{S}_{0,B}\overline{S}_{B,A}$ for $B - A$ small as a multiplication operator on one-particle wave functions. By exploiting properties of this phase we will be able to construct a global lift that has the desired properties. Since $C_c^\infty(\mathbb{R}^4, \mathbb{R}^4)$ is star shaped, we may reach any four-potential A from 0 through the straight line $\{tA \mid t \in [0, 1]\}$.

Definition 5. *For any four-potentials A, B and any two lifts $S'_{A,B}, S''_{A,B}$ of the one particle scattering operator $S_{A,B}$ we define*

$$\frac{S'_{A,B}}{S''_{A,B}} \tag{27}$$

to be the unique complex number $z \in S^1$ such that

$$\frac{S'_{A,B}}{S''_{A,B}} S''_{A,B} = S'_{A,B} \tag{28}$$

holds. Furthermore, for any four-potential A we define the lift $\hat{S}_{0,A}$ as the unique solution of the differential equation

$$A, B \text{ linearly dependent} \Rightarrow \partial_B \frac{\hat{S}_{0,A+B}}{\hat{S}_{0,A} \overline{S}_{A,A+B}} = 0, \tag{29}$$

subject to the boundary condition $\hat{S}_{0,0} = \mathbb{1}$

Remark 1. *The lift $\hat{S}_{0,A}$ can also be constructed differently: pick $N \in \mathbb{N}$ such that $\|\mathbb{1} - S_{0,N^{-1}A}\| < 1$ holds true. Then $P^- S_{nN^{-1}A,(n+1)N^{-1}A} P^-$ is invertible for $N > n \in \mathbb{N}_0$. Now,*

$$\hat{S}_{0,A} = \prod_{n=0}^{N-1} \overline{S}_{nN^{-1}A,(n+1)N^{-1}A}. \tag{30}$$

This can be seen as follows: by (25) we notice that if $\Gamma_{\alpha A,\beta A,\gamma A}$ were equal to 1 for all $\alpha, \beta, \gamma \in \mathbb{R}^+$ small enough, the claim would follow by taking the continuum limit $1/N \to 0$

in (31)*. However, we do have this equality, as the following calculation shows:*

$$\ln\Gamma_{A,\beta A,\gamma A} = \int_1^\beta d\beta' \partial_{\beta'} \ln\Gamma_{A,\beta' A,\gamma A} + \overbrace{\ln\Gamma_{A,A,\gamma A}}^{=0} \tag{31}$$

$$= \int_1^\beta d\beta' \left(\int_1^\gamma d\gamma' \partial_{\gamma'} \partial_{\beta'} \ln\Gamma_{A,\beta' A,\gamma' A} + \partial_{\beta'} \overbrace{\ln\Gamma_{A,\beta' A,A}}^{=0} \right) \tag{32}$$

$$= \int_1^\beta d\beta' \int_1^\gamma d\gamma' c_A(\beta' A,\gamma' A) = \int_1^\beta d\beta' \int_1^\gamma d\gamma' \beta'\gamma' \overbrace{c_A(A,A)}^{=0} = 0, \tag{33}$$

where we have without loss of generality restricted to $\alpha = 1$ and used various properties of lemma 1.

Definition 6. *Let $A, B \in \mathcal{A}$ such that $\|1 - S_{A,B}\| < 1$ holds. We define $\theta_{A,B} \in [-\pi, \pi[$ by*

$$e^{i\theta_{A,B}} := \frac{\hat{S}_{0,B}}{\hat{S}_{0,A}\overline{S}_{A,B}}. \tag{34}$$

Lemma 2. *For all $A, F, G \in \mathcal{A}$ such that $\|1 - S_{A,F}\| < 1$, $\|1 - S_{F,G}\| < 1$, $\|1 - S_{A,G}\| < 1$ hold, as well as for all $H, K \in \mathcal{A}$, we have*

$$\theta_{A,F} = -\theta_{F,A} \tag{35}$$

$$e^{i(\theta_{F,A}+\theta_{A,G}+\theta_{G,F})} = \Gamma_{F,A,G} \tag{36}$$

$$i\partial_{\varepsilon_1}\partial_{\varepsilon_2}\theta_{A+\varepsilon_1 H, A+\varepsilon_2 K} = c_A(H,K). \tag{37}$$

Proof. Pick $A, F, G \in \mathcal{A}$ as in the lemma. We start off by analysing

$$\hat{S}_{0,F}\overline{S}_{F,G} \overset{(35)}{=} e^{i\theta_{A,F}}\hat{S}_{0,A}\overline{S}_{A,F}\overline{S}_{F,G} \tag{38}$$

$$\overset{(25)}{=} e^{i\theta_{A,F}}\Gamma^{-1}_{A,F,G}\hat{S}_{0,A}\overline{S}_{A,G}. \tag{39}$$

Exchanging A and F in this equation and bringing the phases to the other side leads to

$$\hat{S}_{0,F}\overline{S}_{F,G} = e^{-i\theta_{F,A}}\Gamma_{F,A,G}\hat{S}_{0,A}\overline{S}_{A,G}\ , \tag{40}$$

taking (23) into account this means that

$$\theta_{A,F} = -\theta_{F,A} \tag{41}$$

holds true. Equation (40) solved for $\hat{S}_{0,A}\overline{S}_{A,G}$ also gives us

$$\hat{S}_{0,G} \overset{(35)}{=} e^{i\theta_{A,G}}\hat{S}_{0,A}\overline{S}_{A,G} \overset{(40)}{=} e^{i\theta_{A,G}}e^{-i\theta_{A,F}}\Gamma_{A,F,G}\hat{S}_{0,F}\overline{S}_{F,G}. \tag{42}$$

The latter equation compared with

$$\hat{S}_{0,G} \overset{(35)}{=} e^{i\theta_{F,G}}\hat{S}_{0,F}\overline{S}_{F,G}, \tag{43}$$

yields a direct connection between Γ and θ:

$$e^{i\theta_{A,G}-i\theta_{A,F}}\Gamma_{A,F,G} = e^{i\theta_{F,G}}, \tag{44}$$

or by (36)

$$\Gamma_{A,F,G} = e^{i\theta_{F,G}+i\theta_{A,F}+i\theta_{G,A}}. \tag{45}$$

Finally, in this equation we replace $F = A + \varepsilon_1 H$ as well as $G = A + \varepsilon_2 K$, where $\varepsilon_1, \varepsilon_2$ is small enough so that θ and Γ are still well defined. Then we take the logarithm and derivatives to find

$$i\partial_{\varepsilon_1}\partial_{\varepsilon_2}\theta_{A+\varepsilon_1 H, A+\varepsilon_2 K} = \partial_{\varepsilon_1}\partial_{\varepsilon_2}\ln\Gamma_{A,A+\varepsilon_1 H,A+\varepsilon_2 K} \overset{(26)}{=} c_A(H,K). \tag{46}$$

□

So we find that θ is an anti derivative of c. In the following we will characterise θ more thoroughly by c and c^+.

Definition 7. *We define the one form $\chi \in \Omega^1(\mathcal{A})$ by*

$$\chi_A(B) := \partial_B \theta_{A,A+B} \tag{47}$$

for all $A, B \in \mathcal{A}$. Furthermore for a differential form $\omega \in \Omega^p(\mathcal{A})$ for some $p \in \mathbb{N}$ we define the exterior derivative of ω, $d\omega \in \Omega^{p+1}(\mathcal{A})$ by

$$(d\omega)_A(B_1,\dots,B_{p+1}) := \sum_{k=1}^{p+1}(-1)^{k+1}\partial_{B_k}\omega_{A+B_k}(B_1,\dots,\cancel{B_k},\dots,B_{p+1}), \tag{48}$$

for general $A, B_1, \dots, B_{p+1} \in \mathcal{A}$, where the notation $\cancel{B_k}$ denotes that B_k is not to be inserted as an argument.

Lemma 3. *The differential form χ fulfils*

$$(d\chi)_A(F,G) = c_A(F,G) \tag{49}$$

for all $A, F, G \in \mathcal{A}$.

Proof. Pick $A, F, G \in \mathcal{A}$, we calculate

$$\begin{aligned}(d\omega)_A(F,G) &= \partial_F\partial_G\theta_{A+F,A+F+G} - \partial_F\partial_G\theta_{A+G,A+F+G} &(50)\\ &= \partial_F\partial_G(\theta_{A,A+F+G} + \theta_{A+F,A+G}) - \partial_F\partial_G(\theta_{A,A+F+G} + \theta_{A+G,A+F}) &(51)\\ &\overset{(36)}{=} 2\partial_F\partial_G\theta_{A+F,A+G} \overset{(38)}{=} -2ic_A(F,G). &(52)\end{aligned}$$

□

Now since $dc = 0$, we might have by Poincaré's lemma a way independent of θ to construct a differential form ω such that $d\omega = c$. In order to execute this plan, we first need to prove Poincaré's lemma for our setting:

Lemma 4 (Poincaré). *Let $\omega \in \Omega^p(\mathcal{A})$ for $p \in \mathbb{R}$ be closed, i.e. $d\omega = 0$. Then ω is also exact, moreover we have*

$$\omega = d\int_0^1 \iota_t^* i_X f^* \omega dt, \tag{53}$$

where Y, ι_t and f are given by $X : \mathbb{R}\times\mathcal{A} \to \mathbb{R}\times\mathcal{A}, (t,B) \mapsto (1,0)$, $\iota : \mathcal{A} \to \mathbb{R}\times\mathcal{A}, B \mapsto (t,B)$ and $f : \mathbb{R}\times\mathcal{A} \mapsto \mathcal{A}, (t,B) \mapsto tB$. The function f_t is then given by $f_t = f(t,\cdot)$.

Proof. Pick some $\omega \in \Omega^p(\mathcal{A})$. We will first show the more general formula

$$f_b^*\omega - f_a^*\omega = d\int_a^b \iota_t^* i_X f^*\omega \; dt + \int_a^b \iota_t^* i_X f^* d\omega dt. \tag{54}$$

The lemma follows then by $b = 1, a = 0$, $f_1^*\omega = \omega, f_0^*\omega = 0$ and $d\omega = 0$ for a closed ω. First, to prove (55). We begin by summarising the right hand side of (55):

$$d\int_a^b \iota_t^* i_X f^*\omega \; dt + \int_a^b \iota_t^* i_X f^* d\omega dt = \int_a^b (d\iota_t^* i_X f^*\omega + \iota_t^* i_X f^* d\omega) dt. \tag{55}$$

Next we look at both of these terms separately. Let therefore $p \in \mathbb{N}$, $t, s_k \in \mathbb{R}$ and $A, B_k \in \mathcal{A}$ for each $p + 1 \geqslant k \in \mathbb{N}$. First, we calculate $d\iota_t^* i_X f^*\omega$

$$(f^*\omega)_{(t,A)}((s_1, B_1), \ldots, (s_p, B_p)) = \omega_{tA}(s_1 A + tB_1, \ldots, s_p A + tB_p) \tag{56}$$

$$(i_X f^*\omega)_{(t,A)}((s_1, B_1), \ldots, (s_{p-1}, B_{p-1})) = \omega_{tA}(A, s_1 A + tB_1, \ldots, s_{p-1} A + tB_{p-1} \tag{57}$$

$$(\iota_t^* i_X f^*\omega)_A(B_1, \ldots, B_{p-1}) = t^{p-1}\omega_{tA}(A, B_1, \ldots, B_{p-1}) \tag{58}$$

$$(d\iota_t^* i_X f^*\omega)_A(B_1, \ldots, B_p) = \partial_\varepsilon|_{\varepsilon=0} \sum_{k=1}^p (-1)^{k+1} t^{p-1} \omega_{tA+\varepsilon t B_k}(A, B_1, \ldots, \cancel{B_k}, \ldots, B_p) \tag{59}$$

$$+ \partial_\varepsilon|_{\varepsilon=0} \sum_{k=1}^p (-1)^{k+1} \omega_{tA}(A + \varepsilon B_k, B_1, \ldots, \cancel{B_k}, \ldots, B_p) \tag{60}$$

$$= \partial_\varepsilon|_{\varepsilon=0} \sum_{k=1}^p t^p (-1)^{k+1} \omega_{tA+\varepsilon B_k}(A, B_1, \ldots, \cancel{B_k}, \ldots, B_p) + pt^{p-1}\omega_{tA}(B_1, \ldots, B_p). \tag{61}$$

Now, we calculate $\iota_t^* i_X f^* d\omega$:

$$(d\omega)_A(B_1,\cdots,B_{p+1}) = \partial_\varepsilon|_{\varepsilon=0}\sum_{k=1}^{p+1}(-1)^{k+1}\omega_{A+\varepsilon B_k}(B_1,\ldots,\cancel{B_k},\ldots,B_{p+1}) \tag{62}$$

$$(f^*d\omega)(t,A)((s_1,B_1),\ldots,(s_{p+1},B_{p+1})) = (d\omega)_{tA}(s_1A+tB_1,\ldots,s_{p+1}A+tB_{p+1}) \tag{63}$$

$$= \partial_\varepsilon|_{\varepsilon=0}\sum_{k=1}^{p+1}(-1)^{k+1}\omega_{tA+\varepsilon(s_kA+tB_k)}(s_1A+tB_1,\ldots,\cancel{s_kA+tB_k},\ldots,s_pA+tB_p) \tag{64}$$

$$+ \partial_\varepsilon|_{\varepsilon=0}\omega_{(t+\varepsilon)A}(s_1A+tB_1,\ldots,s_pA+tB_p) \tag{65}$$

$$(i_Xf^*d\omega)_{(t,A)}((s_1,B_1),\ldots,(s_p,B_p)) = \partial_\varepsilon|_{\varepsilon=0}\omega_{(t+\varepsilon)A}(s_1A+tB_1,\ldots,s_pA+tB_p) \tag{66}$$

$$+ \partial_\varepsilon|_{\varepsilon=0}\sum_{k=1}^{p}(-1)^{k}\omega_{tA+\varepsilon(s_kA+tB_k)}(A,s_1A+tB_1,\ldots,\cancel{s_kA+tB_k},\ldots,s_pA+tB_p) \tag{67}$$

$$= \partial_\varepsilon|_{\varepsilon=0}\sum_{k=1}^{p}(s_kt^{p-1}(-1)^{k+1}\omega_{(t+\varepsilon)A}(A,B_1,\ldots,B_p) + t^p\omega_{tA+\varepsilon tB_k}(A,B_1,\ldots,,\cancel{B_k},\ldots,B_p)) \tag{68}$$

$$+ \partial_\varepsilon|_{\varepsilon=0}\sum_{k=1}^{p}(-1)^{k}t^{p-1}(\omega_{(t+s_k\varepsilon)A}(A,B_1,\ldots,\cancel{B_k},\ldots,B_p) + \omega_{tA+\varepsilon tB_k}(A,B_1,\ldots,\cancel{B_k},\ldots,B_p)) \tag{69}$$

$$= t^p\partial_\varepsilon|_{\varepsilon=0}\left(\omega_{(t+\varepsilon)A}(B_1,\ldots,B_p) + \sum_{k=1}^{p}(-1)^k\omega_{tA+\varepsilon B_k}(A,B_1,\ldots,\cancel{B_k},\ldots,B_p)\right) \tag{70}$$

$$(\iota_t^*i_Xf^*d\omega)_A(B_1,\ldots,B_p) = t^p\partial_\varepsilon|_{\varepsilon=0}\Big(\omega_{(t+\varepsilon)A}(B_1,\ldots,B_p) \tag{71}$$

$$+ \sum_{k=1}^{p}(-1)^k\omega_{tA+\varepsilon B_k}(A,B_1,\ldots,\cancel{B_k},\ldots,B_p)\Big) \tag{72}$$

Adding (62) and (73) we find for (56):

$$\int_a^b(d\iota_t^*i_Xf^*\omega + \iota_t^*i_Xf^*d\omega)dt = \tag{73}$$

$$\int_a^b\Big(t^p\partial_\varepsilon|_{\varepsilon=0}\omega_{(t+\varepsilon)A}(B_1,\ldots,B_p) + pt^{p-1}\omega_{tA}(B_1,\ldots,B_p)\Big)dt \tag{74}$$

$$= \int_a^b\frac{d}{dt}(t^p\omega_{tA}(B_1,\ldots,B_p))dt = \int_a^b\frac{d}{dt}(f_t^*\omega)_A(B_1,\ldots,B_p))dt \tag{75}$$

$$= (f_b^*\omega)_A(B_1,\ldots,B_p) - (f_a^*\omega)_A(B_1,\ldots,B_p). \tag{76}$$

□

Definition 8. *For a closed exterior form $\omega \in \Omega^p(\mathcal{A})$ we define the form $\prod[\omega]$*

$$\prod[\omega] := \int_0^1 \iota_t^*i_Xf^*\omega dt. \tag{77}$$

For $A, B_1, \ldots, B_{p-1} \in \mathcal{A}$ it takes the form

$$\prod[\omega]_A(B_1,\ldots,B_p) = \int_0^1 t^{p-1}\omega_{tA}(A,B_1,\ldots,B_{p-1})dt. \tag{78}$$

By lemma 4 we know $d\prod[\omega] = \omega$.

Now we found two one forms each produces c when the exterior derivative is taken. The next lemma informs us about their relationship.

Lemma 5. *The following equality holds*

$$\chi = -2i\prod[c]. \tag{79}$$

Proof. We have $d(c + 2i\prod[c]) = 0$ so by lemma 4 we know that there is $v : \mathcal{A} \to \mathbb{R}$ such that

$$dv = \chi + 2i\prod[c] \tag{80}$$

holds. Now (30) translates into the following ODE for θ:

$$\partial_B \theta_{0,B} = 0, \quad \partial_B \theta_{A,A+B}|_{A=B} = 0 \tag{81}$$

for all $A, B \in \mathcal{A}$. This means that

$$\chi_0(B) = 0 = \prod[c]_0(B), \quad \chi_{A,A} = 0 = \prod[c]_A(A) \tag{82}$$

hold. This implies

$$\partial_\varepsilon v_{A+\varepsilon A} = 0, \quad \partial_\varepsilon v_{\varepsilon A} = 0, \tag{83}$$

which means that v is constant. $\square$

Recall equation (17):

$$\forall A, F, G, H : \partial_H c^+_{A+H}(F, G) = \partial_G c^+_{A+G}(F, H). \tag{84}$$

For a fixed $F \in \mathcal{A}$, this condition can be read as $d(c^+_\cdot(F, \cdot)) = 0$. As a consequence we can apply lemma 4 to define a one form.

Definition 9. *For any $F \in \mathcal{A}$, we define*

$$\beta_A(F) := 2i\prod[c^+_\cdot(F, \cdot)]_A. \tag{85}$$

Lemma 6. *The following two equations hold:*

$$d\beta = -2ic \tag{86}$$
$$d(\beta - \chi) = 0. \tag{87}$$

Proof. We start with the exterior derivative of β. Pick $A, F, G \in \mathcal{A}$:

$$d\beta_A(F, G) = \partial_F \beta_{A+F}(G) - \partial_G \beta_{A+G}(F) \tag{88}$$
$$= d\Big(\prod[c^+_\cdot(G, \cdot)]\Big)_A(F) - d\Big(\prod[c^+_\cdot(F, \cdot)]\Big)_A(G) \tag{89}$$
$$= 2ic^+_A(G, F) - 2ic^+_A(F, G) \overset{(16)}{=} -2ic_A(F, G). \tag{90}$$

This proves the first equality. The second equality follows directly by $d\chi = -2ic$. $\square$

Definition 10. *Since $\beta - \chi$ is closed, we may use lem 4 again to define the phase*

$$\alpha := \prod[\beta - \chi]. \tag{91}$$

Furthermore, for all $A, B \in \mathcal{A}$ we define the corrected second quantised scattering operator

$$\tilde{S}_{0,A} := e^{i\alpha_A}\hat{S}_{0,A} \tag{92}$$
$$\tilde{S}_{A,B} := \tilde{S}^{-1}_{0,A}\tilde{S}_{0,B}. \tag{93}$$

Corollary 1. *We have* $\tilde{S}_{A,B}\tilde{S}_{B,C} = \tilde{S}_{A,C}$ *for all* $A, B, C \in \mathcal{A}$.

Theorem 3. *The corrected second quantised scattering operator fulfils the following causality condition for all* $A, F, G \in \mathcal{A}$ *such that* $F \prec G$*:*

$$\tilde{S}_{A,A+F} = \tilde{S}_{A+G,A+G+F}. \tag{94}$$

Proof. Let $A, F, G \in \mathcal{A}$ such that $F \prec G$ We note that for the first quantised scattering operator we have

$$S_{A+G,A+G+F} = S_{A,A+F}, \tag{95}$$

so by definition of $\overline{S}$ we also have

$$\overline{S}_{A+G,A+G+F} = \overline{S}_{A,A+F}. \tag{96}$$

So for any lift this equality is true up to a phase, meaning that

$$f(A, F, G) := \frac{\tilde{S}_{A+G,A+G+F}}{\tilde{S}_{A,A+F}} \tag{97}$$

is well defined. We see immediately

$$f(A, 0, G) = 1 = F(A, F, 0). \tag{98}$$

Pick $F_1, F_2 \prec G_1, G_1$. We abbreviate $F = F_1 + F_2, G = G_1 + G_2$, we calculate

$$\begin{aligned}
f(A, F, G) &= \frac{\tilde{S}_{A+G,A+F+G}}{\tilde{S}_{A,A+F}} && (99)\\
&= \frac{\tilde{S}_{A+G,A+F+G}}{\tilde{S}_{A+G_1,A+G_1+F}} \frac{\tilde{S}_{A+G_1,A+G_1+F}}{\tilde{S}_{A,A+F}} && (100)\\
&= \frac{\tilde{S}_{A+G,A+G+F_1}\tilde{S}_{A+G+F_1,A+F+G}}{\tilde{S}_{A+G_1,A+F_1+G_1}\tilde{S}_{A+G_1+F_1,A+G_1+F}} \frac{\tilde{S}_{A+G_1,A+G_1+F}}{\tilde{S}_{A,A+F}} && (101)\\
&= \frac{\tilde{S}_{A+G,A+G+F_1}}{\tilde{S}_{A+G_1,A+F_1+G_1}} \frac{\tilde{S}_{A+G+F_1,A+F+G}}{\tilde{S}_{A+G_1+F_1,A+G_1+F}} f(A, G_1, F_1 + F_2) && (102)\\
&= f(A + G_1, F_1, G_2)f(A + G_1 + F_1, G_2, F_2)f(A, G_1, F_1 + F_2). && (103)
\end{aligned}$$

Taking the logarithm and differentiating we find:

$$\partial_{F_2}\partial_{G_2} \ln f(A, F_1 + F_2, G_1 + G_2) = \partial_{F_2}\partial_{G_2} \ln f(A + F_1 + G_1, F_2, G_2). \tag{104}$$

Next we pick $F_2 = \alpha_1 F_1$ and $G_2 = \alpha_2 G_2$ for $\alpha_1, \alpha_2 \in \mathbb{R}^+$ small enough so that

$$\begin{aligned}
\|1 - S_{A+F+G,A+F_1+G_1}\| &< 1 && (105)\\
\|1 - S_{A+F+G,A+F_1+G}\| &< 1 && (106)\\
\|1 - S_{A+F+G,A+F+G_1}\| &< 1 && (107)
\end{aligned}$$

hold. We abbreviate $A' = A + G_1 + F_1$ and compute

$$f(A', F_2, G_2) = \frac{e^{i\alpha_{A'+F_2+G_2}+i\theta_{A',A'+F_2+G_2}-i\alpha_{A'+G_2}-i\theta_{A',A'+G_2}}}{e^{i\alpha_{A'+F_2}+i\theta_{A',A'+F_2}-i\alpha_{A'}-i\theta_{A',A'}}} \frac{\overline{S}_{A'+G_2,A'}\overline{S}_{A',A'+F_2+G_2}}{\overline{S}_{A',A'}\overline{S}_{A',A'+F_2}}. \tag{108}$$

The second factor in this product can be simplified significantly:

$$\frac{\overline{S}_{A'+G_2,A'}\overline{S}_{A',A'+F_2+G_2}}{\overline{S}_{A',A'}\overline{S}_{A',A'+F_2}} = \frac{\overline{S}_{A'+G_2,A'}\overline{S}_{A',A'+F_2+G_2}}{\overline{S}_{A',A'+F_2}} \tag{109}$$

$$\overset{(25)}{=} \Gamma^{-1}_{A'+G_2,A',A'+F_2+G_2}\frac{\overline{S}_{A'+G_2,A'+F_2+G_2}}{\overline{S}_{A',A'+F_2}} \tag{110}$$

$$\overset{(97)}{=} \Gamma_{A',A'+G_2,A'+F_2+G_2} \overset{(37)}{=} e^{i\theta_{A',A'+G_2}+i\theta_{A'+G_2,A'+G_2+F_2}+i\theta_{A'+F_2+G_2,A'}}. \tag{111}$$

So in total we find

$$f(A',F_2,G_2) = \frac{e^{i\alpha_{A'+F_2+G_2}+i\theta_{A',A'+F_2+G_2}-i\alpha_{A'+G_2}-i\theta_{A',A'+G_2}}}{e^{i\alpha_{A'+F_2}+i\theta_{A',A'+F_2}-i\alpha_{A'}-i\theta_{A',A'}}}\times \tag{112}$$

$$e^{i\theta_{A',A'+G_2}+i\theta_{A'+G_2,A'+G_2+F_2}+i\theta_{A'+F_2+G_2,A'}} \tag{113}$$

$$= \exp(i\alpha_{A'+F_2+G_2} - i\alpha_{A'+G_2} - i\alpha_{A'+F_2} + i\alpha_{A'} + i\theta_{A'+G_2,A'+G_2+F_2} - i\theta_{A',A'+F_2}). \tag{114}$$

Most of the terms in the exponent do not depend on F_2 and G_2, so taking the mixed logarithmic derivative things simplify:

$$\partial_{G_2}\partial_{F_2}\ln f(A',F_2,G_2) = i\partial_{G_2}\partial_{F_2}(\alpha_{A'+F_2+G_2} + \theta_{A'+G_2,A'+G_2+F_2}) \tag{115}$$

$$\overset{(92),(48)}{=} i\partial_{G_2}(\beta_{A'+G_2}(F_2) - \chi_{A'+G_2}(F_2) + \chi_{A'+G_2}(F_2)) \tag{116}$$

$$\overset{(87)}{=} -2c^{+}_{A'}(F_2,G_2) \overset{F_2 \precsim G_2}{=} 0. \tag{117}$$

So by (105) we also have

$$\partial_{F_2}\partial_{G_2}\ln f(A,F_1+F_2,G_1+G_2) = 0 = \partial_{\alpha_1}\partial_{\alpha_2}\ln f(A,F_1(1+\alpha_1),G_1(1+\alpha_2)) \tag{118}$$

But then we can integrate and obtain

$$0 = \int_{-1}^{0} d\alpha_1 \int_{-1}^{0} d\alpha_2 \partial_{\alpha_1}\partial_{\alpha_2}\ln f(A,F_1(1+\alpha_1),G_1(1+\alpha_2)) \tag{119}$$

$$= \ln f(A,F_1,G_1) - \ln f(A,0,G_1) - \ln f(A,F_1,0) + \ln f(A,0,0) \tag{120}$$

$$\overset{(99)}{=} \ln f(A,F_1,G_1). \tag{121}$$

remembering equation (98), the definition of f, this ends our proof. □

Using $\tilde{S}$ we introduce the current associated to it.

Definition 11. *Let $A, F \in \mathcal{A}$, define*

$$j_A(F) := i\partial_F\left\langle \Omega, \tilde{S}_{A,A+F}\Omega\right\rangle = i\partial_F \ln\left\langle \Omega, \tilde{S}_{A,A+F}\Omega\right\rangle. \tag{122}$$

Theorem 4. *For general $A, F \in \mathcal{A}$ we have*

$$j_A(F) = -\beta_A(F). \tag{123}$$

So in particular for $G \in \mathcal{A}$

$$\partial_G j_{A+G}(F) = -2ic_A(F,G). \tag{124}$$

holds.

Proof. Pick $A, F \in \mathcal{A}$ as in the theorem. We calculate

$$i\partial_F \ln \left\langle \Omega, \tilde{S}_{A,A+F}\Omega \right\rangle \tag{125}$$

$$= i\partial_F \left(i\alpha_{A+F} - i\alpha_A + \ln \left\langle \Omega, \hat{S}_{0,A}^{-1}\hat{S}_{0,A+F}\Omega \right\rangle \right) \tag{126}$$

$$= i\partial_F \left(i\alpha_{A+F} + i\theta_{A,A+F} + \ln \left\langle \Omega, \overline{S}_{A,A+F}\Omega \right\rangle \right) \tag{127}$$

The last summand vanishes, as can be seen by the following calculation

$$\partial_F \ln \left\langle \Omega, \overline{S}_{A,A+F}\Omega \right\rangle = \partial_F \ln \det P^-\overline{S}_{A,A+F}P^-\overline{S}_{A,A+F}^{-1}P^- \tag{128}$$

$$= \partial_F \ln \det(P^- - P^-\overline{S}_{A,A+F}P^+\overline{S}_{A+F,A}P^-) \stackrel{*}{=} -\partial_F \operatorname{tr}(P^-\overline{S}_{A,A+F}P^+\overline{S}_{A+F,A}P^-) \tag{129}$$

$$= -\partial_F \operatorname{tr}(P^-\overline{S}_{A,A+F}P^+\overline{S}_{A,A}P^-) - \partial_F \operatorname{tr}(P^-\overline{S}_{A,A}P^+\overline{S}_{A+F,A}P^-) = 0, \tag{130}$$

where for the marked identity we used that for any $f : \mathbb{R} \to I_1$ we have

$$\partial_\varepsilon \det(1 + f(\varepsilon)) = \operatorname{tr}(\partial_\varepsilon f(\varepsilon)). \tag{131}$$

So we are left with

$$j_A(F) = -\partial_F(\alpha_{A+F} + \theta_{A,A+F}) = -(\beta_A(F) - \chi_A(F) + \chi_A(F)) = -\beta_A(F). \tag{132}$$

Finally by taking the derivative with respect to $G \in \mathcal{A}$ and using the definition of β we find

$$\partial_G j_{A+G}(F) = -2ic_A^+(F, G). \tag{133}$$

$\square$