Calculation for Generating Function

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Yesterday we arrived at the following expression for the generating function of the relevant part of c^+

$$G = \int_{\left(\mathbb{R}^+\right)^{2n}} dy \; i\left(\frac{\pi}{\sum_{j=1}^{2n} y_j}\right)^{d/2} \exp\left(\sum_{j=1}^{2n} y_j(q_j^2 - m^2) - \frac{\left(\sum_{j=1}^{2n} y_j q_j + i/2\xi\right)^2}{\sum_{j=1}^{2n} y_j}\right). \tag{1}$$

For this expression we can further treat $\sum_{j=1}^{2n} y_j = t$ as an independent variable and perform its integral. For this (and the next few steps) we introduce the following abbreviations

$$t := \sum_{j=1}^{2n} y_j \tag{2}$$

$$z_j := y_j/t \tag{3}$$

$$\overline{q} := \sum_{j=1}^{2n} z_j q_j \tag{4}$$

$$\overline{q^2} := \sum_{j=1}^{2n} z_j q_j^2 \tag{5}$$

$$\lambda := \sqrt{m^2 - \overline{q^2} + \overline{q}^2}. \tag{6}$$

For the following calculations we need $\mathrm{Re}(\lambda), -\mathrm{Re}(\xi^2) \geqslant 0$. We then arrive at

$$G = \frac{i}{\Gamma(2n)} E_z\left[\int_{\mathbb{R}^+} dt \; t^{2n-1}\left(\frac{\pi}{t}\right)^{d/2} \exp\left(-t(m^2 - \overline{q^2} + \overline{q}^2) - i\xi\overline{q} + \frac{\xi^2}{4t}\right)\right]. \tag{7}$$

This can be brought into a known integral expression for the modified Bessel function.

$$\begin{aligned} G &= \frac{i\pi^{d/2}}{\Gamma(2n)} E_z\left[e^{-i\xi\overline{q}}\lambda^{d-4n}\int_{\mathbb{R}^+}\frac{d\tau}{\tau^{d/2-2n+1}}e^{-\tau-\frac{-\xi^2\lambda^2}{4\tau}}\right] && (8)\\ &= \frac{i\pi^{d/2}}{\Gamma(2n)} E_z\left[e^{-i\xi\overline{q}}\lambda^{d-4n} 2(\lambda\sqrt{-\xi^2}/2)^{2n-d/2} K_{d/2-2n}(\lambda\sqrt{-\xi^2})\right] && (9)\\ &\overset{K_\nu(z)=K_{-\nu}(z)}{=} \frac{i\pi^{d/2}}{\Gamma(2n)} 2^{1-2n+d/2} E_z\left[e^{-i\xi\overline{q}}\left(\frac{\sqrt{-\xi^2}}{\lambda}\right)^{2n-d/2} K_{2n-d/2}(\lambda\sqrt{-\xi^2})\right]. && (10)\end{aligned}$$

In the limit $\xi^2 \to 0$ one should use the asymptotic behavior of K according to

$$K_\nu(z) \approx \frac{\Gamma(\nu)}{2}(z/2)^{-\nu} \quad \text{for } \mathrm{Re}(\nu) > 0, \text{ and } z \to 0 \tag{11}$$

to recover the case we discussed the last time.