

What we know about $T1$

$T1$ is an unbounded linear map from the Fock space of QED onto itself. To be more specific, let \mathcal{H}_+ denote the Hilbert space of electron wavefunctions and \mathcal{H}_- denote the Hilbert space of positron wavefunctions. Then

$$T1 : \mathcal{F}(\mathcal{H}_+) \otimes \mathcal{F}(\mathcal{H}_-) \rightarrow \mathcal{F}(\mathcal{H}_+) \otimes \mathcal{F}(\mathcal{H}_-), \quad (1)$$

but by far not all matrix elements are nonzero, expressed in the fixed particle subspaces we already see more structure:

$$T1 : \mathcal{H}_+^{\otimes n} \otimes \mathcal{H}_-^{\otimes p} \rightarrow (\mathcal{H}_+^{\otimes n-1} \otimes \mathcal{H}_-^{\otimes p-1}) \oplus (\mathcal{H}_+^{\otimes n} \otimes \mathcal{H}_-^{\otimes p}) \oplus (\mathcal{H}_+^{\otimes n+1} \otimes \mathcal{H}_-^{\otimes p+1}). \quad (2)$$

So restricted to the vacuum sector this simplifies, the part from the vacuum sector is set to zero, because the vacuum polarization current should be zero.

$$T1 : \mathbb{C} \rightarrow (\mathcal{H}_+ \otimes \mathcal{H}_-) \quad (3)$$

We will construct the finite operator norm of $T1$ restricted to an arbitrary fixed particle sector, beginning with the vacuum sector. Let the vector potential of the electromagnetic field A be

$$A \in C_c^\infty(\mathbb{R}^4). \quad (4)$$

Then one obtains in physicists notation:

$$T1\Omega = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{d^3k}{\sqrt{2k^0}} \int_{\mathbb{R}^3} \frac{d^3p}{\sqrt{2p^0}} \int_{\mathbb{R}^4} d^4x \sum_{r,s=\pm 1} b_s^\dagger(k) d_r^\dagger(p) \bar{u}_s(k) \gamma^\mu v_r(p) \Omega e^{ix_\alpha(k^\alpha + p^\alpha)} A_\mu(x), \quad (5)$$

Where the γ^μ are the dirac matrices fulfilling the anticommutation relation

$$\{\gamma^\alpha \gamma^\beta\} = 2\eta^{\alpha\beta} \quad (6)$$

The u, v are given by:

$$u_s(p) = \begin{pmatrix} \sqrt{p \cdot \sigma} \xi_s \\ \sqrt{p \cdot \bar{\sigma}} \xi_s \end{pmatrix} \quad v_s(p) = \begin{pmatrix} \sqrt{p \cdot \sigma} \xi_{-s} \\ -\sqrt{p \cdot \bar{\sigma}} \xi_{-s} \end{pmatrix}, \quad (7)$$

with

$$\sigma = (\mathbb{1}, \vec{\sigma}), \quad \bar{\sigma} = (\mathbb{1}, -\vec{\sigma}) \quad (8)$$

$$\xi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \xi_{-1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (9)$$

Where $\vec{\sigma}$ is the vector containing the Pauli matrices. The creation and annihilation operator fulfill the following identity:

$$\{b_s(p), b_r^\dagger(p')\} = \delta^3(p - p') = \{d_s(p), d_r^\dagger(p')\} \quad (10)$$

Useful identities for sums of u and v are:

$$\sum_s u_s(p) \bar{u}_s(p) = \not{p} + m \quad \sum_s v_s(p) \bar{v}_s(p) = \not{p} - m \quad (11)$$

For the norm of the image of the vacuum we therefore arrive at:

$$\begin{aligned}
\|T1\Omega\|^2 &= \frac{1}{(2\pi)^6} \int_{\mathbb{R}^3} \frac{d^3k}{\sqrt{2k^0}} \int_{\mathbb{R}^3} \frac{d^3p}{\sqrt{2p^0}} \int_{\mathbb{R}^3} \frac{d^3k'}{\sqrt{2k'^0}} \int_{\mathbb{R}^3} \frac{d^3p'}{\sqrt{2p'^0}} \int_{\mathbb{R}^4} d^4x \int_{\mathbb{R}^4} d^4y \sum_{r,s,r',s'=\pm 1} \\
&\quad \Omega^\dagger d_{r'}(p') b_{s'}(k') \bar{v}_{r'}(p') \gamma^\epsilon u_{s'}(k') b_s^\dagger(k) d_r^\dagger(p) \bar{u}_s(k) \gamma^\mu v_r(p) \Omega \\
&\quad e^{ix_\alpha(k^\alpha+p^\alpha)} e^{-iy_\beta(k'^\beta+p'^\beta)} A_\mu(x) A_\epsilon(y) \\
&= \frac{1}{(2\pi)^6} \int_{\mathbb{R}^3} \frac{d^3k}{\sqrt{2k^0}} \int_{\mathbb{R}^3} \frac{d^3p}{\sqrt{2p^0}} \int_{\mathbb{R}^3} \frac{d^3k'}{\sqrt{2k'^0}} \int_{\mathbb{R}^3} \frac{d^3p'}{\sqrt{2p'^0}} \int_{\mathbb{R}^4} d^4x \int_{\mathbb{R}^4} d^4y \sum_{r,s,r',s'=\pm 1} \\
&\quad \Omega^\dagger \{d_{r'}(p') \{b_{s'}(k'), b_s^\dagger(k)\}, d_r^\dagger(p)\} \bar{v}_{r'}(p') \gamma^\epsilon u_{s'}(k') \bar{u}_s(k) \gamma^\mu v_r(p) \Omega \\
&\quad e^{ix_\alpha(k^\alpha+p^\alpha)} e^{-iy_\beta(k'^\beta+p'^\beta)} A_\mu(x) A_\epsilon(y) \\
&= \frac{1}{(2\pi)^6} \int_{\mathbb{R}^3} \frac{d^3k}{2k^0} \int_{\mathbb{R}^3} \frac{d^3p}{2p^0} \int_{\mathbb{R}^4} d^4x \int_{\mathbb{R}^4} d^4y \sum_{r,s=\pm 1} \bar{v}_r(p) \gamma^\epsilon u_s(k) \bar{u}_s(k) \gamma^\mu v_r(p) \\
&\quad e^{ix_\alpha(k^\alpha+p^\alpha)} e^{-iy_\beta(k^\beta+p^\beta)} A_\mu(x) A_\epsilon(y) \\
&= \frac{1}{(2\pi)^6} \int_{\mathbb{R}^3} \frac{d^3k}{2k^0} \int_{\mathbb{R}^3} \frac{d^3p}{2p^0} \int_{\mathbb{R}^4} d^4x \int_{\mathbb{R}^4} d^4y \sum_{r,s=\pm 1} \sum_{a=1}^4 \langle e_a | v_r(p) \bar{v}_r(p) \gamma^\epsilon u_s(k) \bar{u}_s(k) \gamma^\mu | e_a \rangle \\
&\quad e^{ix_\alpha(k^\alpha+p^\alpha)} e^{-iy_\beta(k^\beta+p^\beta)} A_\mu(x) A_\epsilon(y) \\
&= \frac{1}{(2\pi)^6} \int_{\mathbb{R}^3} \frac{d^3k}{2k^0} \int_{\mathbb{R}^3} \frac{d^3p}{2p^0} \int_{\mathbb{R}^4} d^4x \int_{\mathbb{R}^4} d^4y \text{tr} [(\not{p} - m) \gamma^\epsilon (\not{k} + m) \gamma^\mu] \\
&\quad e^{ix_\alpha(k^\alpha+p^\alpha)} e^{-iy_\beta(k^\beta+p^\beta)} A_\mu(x) A_\epsilon(y) \\
&= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^3} \frac{d^3k}{2k^0} \int_{\mathbb{R}^3} \frac{d^3p}{2p^0} (p_\zeta k_\omega \text{tr} [\gamma^\zeta \gamma^\epsilon \gamma^\omega \gamma^\mu] - m^2 \text{tr} [\gamma^\epsilon \gamma^\mu]) \\
&\quad \hat{A}_\mu(k+p) \hat{A}_\epsilon^*(k+p) \\
&= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^3} \frac{d^3k}{2k^0} \int_{\mathbb{R}^3} \frac{d^3p}{2p^0} 4 (p_\zeta k_\omega (\eta^{\zeta,\epsilon} \eta^{\omega,\mu} - \eta^{\zeta,\omega} \eta^{\epsilon,\mu} + \eta^{\zeta,\mu} \eta^{\epsilon,\omega}) - m^2 \eta^{\epsilon,\mu}) \\
&\quad \hat{A}_\mu(k+p) \hat{A}_\epsilon^*(k+p) \\
&= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^3} \frac{d^3k}{k^0} \int_{\mathbb{R}^3} \frac{d^3p}{p^0} \left(k^\mu \hat{A}_\mu(k+p) p^\epsilon \hat{A}_\epsilon^*(k+p) - p^\alpha k_\alpha \hat{A}^\mu(k+p) \hat{A}_\mu^*(k+p) \right. \\
&\quad \left. + k^\mu \hat{A}_\mu^*(k+p) p^\epsilon \hat{A}_\epsilon(k+p) - m^2 \hat{A}^\mu(k+p) \hat{A}_\mu^*(k+p) \right) \\
&\quad \stackrel{k^2=p^2=m^2}{=} \frac{1}{(2\pi)^2} \int_{\mathbb{R}^3} \frac{d^3k}{k^0} \int_{\mathbb{R}^3} \frac{d^3p}{p^0} \left(k^\mu \hat{A}_\mu(k+p) p^\epsilon \hat{A}_\epsilon^*(k+p) \right. \\
&\quad \left. + k^\mu \hat{A}_\mu^*(k+p) p^\epsilon \hat{A}_\epsilon(k+p) - \frac{1}{2} (k+p)^2 \hat{A}^\mu(k+p) \hat{A}_\mu^*(k+p) \right) \\
&= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^3} \frac{d^3k}{k^0} \int_{\mathbb{R}^3} \frac{d^3p}{p^0} \left(k^\mu \hat{A}_\mu(k+p) p^\epsilon \hat{A}_\epsilon^*(k+p) \right. \\
&\quad \left. + k^\mu \hat{A}_\mu^*(k+p) p^\epsilon \hat{A}_\epsilon(k+p) + \frac{1}{2} \hat{A}_\mu^*(k+p) \widehat{\square} \hat{A}^\mu(k+p) \right) \quad (12)
\end{aligned}$$

Up until this point the calculation is exact. If one wants to obtain a nice looking expression

for this bound one can add the terms $k^\alpha \hat{A}_\alpha k^\beta \hat{A}_\beta^* + p^\alpha \hat{A}_\alpha p^\beta \hat{A}_\beta^*$. These terms are clearly positive. One then ends up with:

$$\begin{aligned}
\|T1\Omega\|^2 &\leq \frac{1}{(2\pi)^2} \int_{\mathbb{R}^3} \frac{d^3k}{k^0} \int_{\mathbb{R}^3} \frac{d^3p}{p^0} \left((k^\mu + p^\mu) \hat{A}_\mu^*(k+p) (k^\epsilon + p^\epsilon) \hat{A}_\epsilon(k+p) \right. \\
&\quad \left. + \frac{1}{2} \hat{A}_\mu^*(k+p) \widehat{\square A}^\mu(k+p) \right) \\
&= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^3} \frac{d^3k}{k^0} \int_{\mathbb{R}^3} \frac{d^3p}{p^0} \left(\widehat{\partial^\alpha A_\alpha}^*(k+p) \widehat{\partial^\beta A_\beta}(k+p) \right. \\
&\quad \left. + \frac{1}{2} \hat{A}_\mu^*(k+p) \widehat{\square A}^\mu(k+p) \right) \\
&= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^3} \frac{d^3k}{k^0} \int_{\mathbb{R}^3} \frac{d^3p}{p^0} \left(\left\| \widehat{\partial^\beta A_\beta}(k+p) \right\|^2 + \frac{1}{2} \hat{A}_\mu^*(k+p) \widehat{\square A}^\mu(k+p) \right) \quad (13)
\end{aligned}$$

Taking $T1 : \mathbb{C} \rightarrow \mathcal{H}_+ \otimes \mathcal{H}_-$ for granted, we would like to define $T1$ on all of Fockspace. In order to do this, we introduce the “restriction of proper lifting” for the annihilation operator:

$$\forall \phi \in \mathcal{H} : \quad a(U\phi) \circ \tilde{U} = \tilde{U} \circ a(\phi) \quad (14)$$

Which is equivalent to the commutativity of the following diagram.

$$\begin{array}{ccc}
\mathcal{F} & \xrightarrow{\tilde{U}^A} & \mathcal{F} \\
\uparrow a & & \uparrow a \\
\mathcal{H} \otimes \mathcal{F} & \xrightarrow{U^A \otimes \tilde{U}^A} & \mathcal{H} \otimes \mathcal{F}
\end{array} \quad (15)$$

The respective condition for the creation operator, which can easily be derived from (14) is:

$$\forall \phi \in \mathcal{H} : \quad a^*(U^A \phi) \circ \tilde{U}^A = \tilde{U}^A \circ a^*(\phi) \quad (16)$$

Expanding U^A and \tilde{U}^A in a powerseries, one obtains the following commutation relations for the coefficients of said expansion:

$$U^A = \mathbb{1}_{\mathcal{H}} + \sum_{l=1}^{\infty} \quad (17)$$