1. Niemandsland

As a preparation for calculating the vacuum polarisation current we proof the following

Lemma 1.1. *Let $P_k, P_l \in Q$ then the following holds*

$$[G(P_k), G(P_l)] = \operatorname{tr}\left(P_{k\atop -+} P_{l\atop +-}\right) - \operatorname{tr}\left(P_{l\atop -+} P_{k\atop +-}\right) + G([P_k, P_l]). \qquad (1)$$

For a proof of this lemma let $P_k, P_l \in Q$, we compute

$$[G(P_k), G(P_l)] \overset{(??)}{=}$$

$$= \sum_{n,b \in \mathbb{N}} [a^*(P_k \varphi_n) a(\varphi_n), a^*(P_l \varphi_b) a(\varphi_n)]$$

$$- \sum_{-b,n \in \mathbb{N}} [a^*(P_k \varphi_n) a(\varphi_n), a(\varphi_b) a^*(P_l \varphi_b)]$$

$$- \sum_{-n,b \in \mathbb{N}} [a(\varphi_n) a^*(P_k \varphi_n), a^*(P_l \varphi_b) a(\varphi_b)]$$

$$+ \sum_{n,b \in \mathbb{N}} [a(\varphi_n) a^*(P_k \varphi_n), a(\varphi_b) a^*(P_l \varphi_b)]$$

$$= \sum_{n,b \in \mathbb{N}} \left(a^*(P_k \varphi_n) \langle \varphi_n, P_l \varphi_b \rangle a(\varphi_b) - a^*(P_l \varphi_b) \langle \varphi_b, P_k \varphi_n \rangle a(\varphi_n)\right)$$

$$- \sum_{-b,n \in \mathbb{N}} \left(- \langle \varphi_b, P_k \varphi_n \rangle a(\varphi_n) a^*(P_l \varphi_b) + a(\varphi_b) a^*(P_k \varphi_n) \langle \varphi_n, P_l \varphi_b \rangle\right)$$

$$- \sum_{-n,b \in \mathbb{N}} \left(- \langle \varphi_n, P_l \varphi_b \rangle a^*(P_k \varphi_n) a(\varphi_b) + a^*(P_l \varphi_b) a(\varphi_n) \langle \varphi_b, P_k \varphi_n \rangle\right)$$

$$+ \sum_{n,b \in -\mathbb{N}} \left(a(\varphi_n) \langle \varphi_b, P_k \varphi_n \rangle a^*(P_l \varphi_b) - a(\varphi_b) \langle \varphi_n, P_l \varphi_b \rangle a^*(P_k \varphi_n)\right)$$

$$= \sum_{b \in \mathbb{N}} a^*\left(P_k P_{l\atop ++} \varphi_b\right) a(\varphi_b) - \sum_{n \in \mathbb{N}} a^*\left(P_l P_{k\atop ++} \varphi_n\right) a(\varphi_n)$$

$$+ \sum_{n \in \mathbb{N}} a(\varphi_n) a^*\left(P_l P_{k\atop -+} \varphi_n\right) - \sum_{-b \in \mathbb{N}} a(\varphi_b) a^*\left(P_k P_{l\atop +-} \varphi_b\right)$$

$$+ \sum_{b \in \mathbb{N}} a^*\left(P_k P_{l\atop -+} \varphi_b\right) a(\varphi_b) - \sum_{-n \in \mathbb{N}} a^*\left(P_l P_{k\atop +-} \varphi_n\right) a(\varphi_n)$$

$$+ \sum_{-n \in \mathbb{N}} a(\varphi_n) a^*\left(P_l P_{k\atop --} \varphi_n\right) - \sum_{-b \in \mathbb{N}} a(\varphi_b) a^*\left(P_k P_{l\atop --} \varphi_b\right)$$

$$= \sum_{n \in \mathbb{N}} a^*(P_k P_l \varphi_n) a(\varphi_n) - \sum_{n \in \mathbb{N}} a^*\left(P_l P_{k\atop ++} \varphi_n\right) a(\varphi_n)$$

$$+ \operatorname{tr}\left(P_{l\atop +-} P_{k\atop -+}\right) - \sum_{n \in \mathbb{N}} a^*\left(P_l P_{k\atop -+} \varphi_n\right) a(\varphi_n)$$

$$- \operatorname{tr}\left(P_{l\atop -+} P_{k\atop +-}\right) + \sum_{-b \in \mathbb{N}} a(\varphi_b) a^*\left(P_l P_{k\atop +-} \varphi_b\right)$$

$$+ \sum_{-b \in \mathbb{N}} a(\varphi_b) a^*\left(P_l P_{k\atop --} \varphi_b\right) - \sum_{-b \in \mathbb{N}} a(\varphi_b) a^*(P_k P_l \varphi_b)$$

1

$$= \mathrm{tr}\left(P_{\underset{+-}{l}}P_{\underset{-+}{k}}\right) - \mathrm{tr}\left(P_{\underset{-+}{l}}P_{\underset{+-}{k}}\right)$$

$$+ \sum_{n\in\mathbb{N}} a^*\left([P_k,P_l]\,\varphi_n\right)a(\varphi_n) + \sum_{-b\in\mathbb{N}} a(\varphi_b)a^*\left([P_l,P_k]\,\varphi_b\right)$$

$$= \mathrm{tr}\left(P_{\underset{+-}{l}}P_{\underset{-+}{k}}\right) - \mathrm{tr}\left(P_{\underset{-+}{l}}P_{\underset{+-}{k}}\right) + G\left([P_k,P_l]\right)$$

$\square$

Definition 1.2. *For $k \in \mathbb{N}_0$, $X,Y \in \mathcal{B}(\mathcal{H})$ the nested commutator $[X,Y]_k$ is defined inductively as*

$$[X,Y]_0 := Y$$
$$[X,Y]_{k+1} := [X,[X,Y]_k] \quad \forall k \in \mathbb{N}_0.$$

Lemma 1.3. *For $m \in \mathbb{N}$ and $B,C \in Q$ the following holds*

$$[G(B),G(C)]_m = \mathrm{tr}\left(P_-BP_+[B,C]_{m-1}\right) - \mathrm{tr}\left(P_+BP_-[B,C]_{m-1}\right)$$
$$+ G\left([B,C]_m\right). \quad (2)$$

Proof: Proof by Induction is the first thing that comes to mind, looking at the claim. Indeed, $m = 1$ is the consequence of the lemma 1.1. For m general we have

$$[G(B),G(C)]_{m+1} = [G(B),[G(B),G(C)]_m]$$
$$\overset{\text{ind.hyp.}}{=} [G(B),\mathrm{tr}\left(P_-BP_+[B,C]_{m-1}\right) - \mathrm{tr}\left(P_+BP_-[B,C]_{m-1}\right) + G\left([B,C]_m\right)]$$
$$= [G(B),G\left([B,C]_m\right)]$$
$$\overset{\text{lemma } 1.1}{=} \mathrm{tr}\left(P_-BP_+[B,C]_m\right) - \mathrm{tr}\left(P_+BP_-[B,C]_m\right) + G\left([B,[B,C]_m]\right)$$
$$= \mathrm{tr}\left(P_-BP_+[B,C]_m\right) - \mathrm{tr}\left(P_+BP_-[B,C]_m\right) + G\left([B,C]_{m+1}\right) \quad (3)$$

$\square$

Lemma 1.4. *For external potentials A,X small enough the derivatives of the scattering operator can be computed to fulfil*

$$\partial_\varepsilon|_{\varepsilon=0}e^{G\ln U^{A+\varepsilon X}} = e^{G\ln U^A}j_A^0(X) + e^{G\ln U^A}G((U^A)^{-1}\partial_\varepsilon U^{A+\varepsilon X}) \quad (4)$$
$$\partial_\varepsilon|_{\varepsilon=0}e^{-G\ln U^{A+\varepsilon X}} = -e^{-G\ln U^A}j_A^0(X) + G(\partial_\varepsilon\left(U^{A+\varepsilon X}\right)^{-1}U^A)e^{-G\ln U^A}, \quad (5)$$

with

$$j_A^0(X) := \sum_{l\in\mathbb{N}_0}\frac{(-1)^{l+1}}{(l+2)!}\left(\mathrm{tr}\,P_-\ln U^A P_+\left[\ln U^A,\partial_\varepsilon\ln U^{A+\varepsilon X}\right]_l\right.$$
$$\left. - \mathrm{tr}\,P_+\ln U^A P_-\left[\ln U^A,\partial_\varepsilon\ln U^{A+\varepsilon X}\right]_l\right). \quad (6)$$

Proof: We start out by employing Duhamel's and Hadamard's formulas. These are

> ref!! + restrictions, something better than this

and

$$\partial_\alpha\,e^{Y+\alpha X}\big|_{\alpha=0} = \int_0^1 dt\,e^{(1-t)Y}Xe^{tY} \qquad \text{(Duhamel's formula)}$$

and

$$e^X Y e^{-X} = \sum_{k=0}^\infty \frac{1}{k!}[X,Y]_k. \qquad \text{(Hadamard's formula)}$$

So one gets

$$\partial_\varepsilon|_{\varepsilon=0} e^{G \ln U^{A+\varepsilon X}} = \int_0^1 dz\, e^{(1-z)G \ln U^A} \partial_\varepsilon|_{\varepsilon=0} G \ln U^{A+\varepsilon X} e^{zG \ln U^A} \tag{7}$$

$$= e^{G \ln U^A} \int_0^1 dz \sum_{l \in \mathbb{N}_0} \frac{1}{l!} \left[-zG \ln U^A, \partial_\varepsilon|_{\varepsilon=0} G \ln U^{A+\varepsilon X} \right]_l$$

$$= e^{G \ln U^A} \int_0^1 dz \sum_{l \in \mathbb{N}_0} \frac{(-z)^l}{l!} \partial_\varepsilon|_{\varepsilon=0} \left[G \ln U^A, G \ln U^{A+\varepsilon X} \right]_l.$$

At this point we see that for $l = 0$ the summand vanishes. For all other values of l we use lemma 1.3, yielding

$$\partial_\varepsilon|_{\varepsilon=0} e^{G \ln U^{A+\varepsilon X}} = e^{G \ln U^A} \int_0^1 dz \sum_{l \in \mathbb{N}} \frac{(-z)^l}{l!} \partial_\varepsilon|_{\varepsilon=0} \left(G \left(\left[\ln U^A, \ln U^{A+\varepsilon X} \right] \right) \right.$$

$$+ \operatorname{tr} P_- \ln U^A P_+ \left[\ln U^A, \partial_\varepsilon|_{\varepsilon=0} \ln U^{A+\varepsilon X} \right]_{l-1}$$

$$\left. - \operatorname{tr} P_+ \ln U^A P_- \left[\ln U^A, \partial_\varepsilon|_{\varepsilon=0} \ln U^{A+\varepsilon X} \right]_{l-1} \right). \tag{8}$$

The last two terms together result in the first term of (4) after performing the integration and shifting the summation index. For the first term we will use linearity and continuity of G and use the same identities backwards to give [continuity of G!!]

$$e^{G \ln U^A} \int_0^1 dz \sum_{l \in \mathbb{N}} \frac{(-z)^l}{l!} \partial_\varepsilon|_{\varepsilon=0} G \left(\left[\ln U^A, \ln U^{A+\varepsilon X} \right] \right)$$

$$= e^{G \ln U^A} G \left(\int_0^1 dz \sum_{l \in \mathbb{N}} \frac{1}{l!} \left[-z \ln U^A, \partial_\varepsilon|_{\varepsilon=0} \ln U^{A+\varepsilon X} \right] \right)$$

$$= e^{G \ln U^A} G \left(e^{-\ln U^A} \int_0^1 dz\, e^{\ln U^A} e^{-z \ln U^A} \partial_\varepsilon|_{\varepsilon=0} \ln U^{A+\varepsilon X} e^{z \ln U^A} \right)$$

$$= e^{G \ln U^A} G \left(e^{\ln (U^A)^{-1}} \partial_\varepsilon|_{\varepsilon=0} e^{\ln U^{A+\varepsilon X}} \right)$$

$$= e^{G \ln U^A} G \left(\left(U^A \right)^{-1} \partial_\varepsilon|_{\varepsilon=0} U^{A+\varepsilon X} \right). \tag{9}$$

Putting things together results in the first equality we wanted to prove. For the second one the computation is completely analogous, except for after applying Duhamel's formula as in (7) we substitute $u = 1 - z$. The minus sign in front of the first term then arises by the chain rule, where as the second term does not share the sign change with the first, since we have to revert the use of the chain rule in the second half of the calculation when we apply (Duhamel's formula) backwards. $\square$

Definition 1.5. *We use Bogoliubov's formula to define the vacuum expectation value of the current* [ref!!]

$$j_A(F) = i\partial_\varepsilon \left\langle \Omega, S^{A*} S^{A+\varepsilon F} \Omega \right\rangle \Big|_{\varepsilon=0}. \tag{10}$$

Theorem 1.6. *The vacuum expectation value of the current of the scattering operator takes the form*

$$j_A(F) = -\partial_\varepsilon \, \varphi(A + \varepsilon F)\big|_{\varepsilon=0}$$

$$-2 \int_0^1 dz (1-z) \Im \operatorname{tr} \left(P_+ \ln U^A P_- e^{-z \ln U^A} \partial_\varepsilon \ln U^{A+\varepsilon F}\big|_{\varepsilon=0} e^{z \ln U^A} \right)$$

$$= -\partial_\varepsilon \, \varphi(A + \varepsilon F)\big|_{\varepsilon=0}$$

$$-2\Im \sum_{k=0}^\infty \frac{(-1)^k}{(k+2)!} \operatorname{tr} \left(P_+ \ln U^A P_- \left[\ln U^A, \partial_\varepsilon \ln U^{A+\varepsilon F}\big|_{\varepsilon=0} \right]_k \right)$$

Proof: By theorem **??** and abbreviating $\varphi(A) = \sum_{n \in \mathbb{N}} \frac{C_n(A)}{n!}$ we see that the current can be written in the form

$$j_A(F) = i\partial_\varepsilon \, \langle \Omega, S^{A*} S^{A+\varepsilon F} \Omega \rangle \Big|_{\varepsilon=0}$$

$$= i\partial_\varepsilon \, \langle \Omega, e^{-i\varphi(A)} e^{-G(\ln(U^A))} e^{i\varphi(A+\varepsilon F)} e^{G(\ln(U^{A+\varepsilon F}))} \Omega \rangle \Big|_{\varepsilon=0}$$

$$= -\partial_\varepsilon \, \varphi(A + \varepsilon F)\big|_{\varepsilon=0} + i\langle \Omega, \partial_\varepsilon \, e^{-G(\ln(U^A))} e^{G(\ln(U^{A+\varepsilon F}))} \Omega \rangle \Big|_{\varepsilon=0}, \quad (11)$$

so the first summand works out just as claimed. For the second summand we employ lemma 1.4 and note that the vacuum expectation value of G vanishes no matter its argument.

$$i\langle \Omega, \partial_\varepsilon \, e^{-G(\ln(U^A))} e^{G(\ln(U^{A+\varepsilon F}))} \Omega \rangle \Big|_{\varepsilon=0}$$

$$= -i\partial_\varepsilon \sum_{k=0}^\infty \frac{(-1)^k}{(k+2)!} \operatorname{tr} \left(P_- \ln U^A P_+ [\ln U^A, \ln U^{A+\varepsilon F}]_k \right) \Big|_{\varepsilon=0}$$

$$+ i\partial_\varepsilon \sum_{k=0}^\infty \frac{(-1)^k}{(k+2)!} \operatorname{tr} \left(P_+ \ln U^A P_- [\ln U^A, \ln U^{A+\varepsilon F}]_k \right) \Big|_{\varepsilon=0}$$

$$(12)$$

In order to apply Hadamard's formula once again in the opposite direction, we introduce two auxiliary integrals. The second term then becomes

$$i\partial_\varepsilon \sum_{k=0}^\infty \frac{(-1)^k}{(k+2)!} \operatorname{tr} \left(P_+ \ln U^A P_- [\ln U^A, \ln U^{A+\varepsilon F}]_k \right) \Big|_{\varepsilon=0}$$

$$= i\partial_\varepsilon \sum_{k=0}^\infty \frac{(-1)^k}{k!} \int_0^1 dt \int_0^1 s^k t^{k+1} \operatorname{tr} \left(P_+ \ln U^A P_- [\ln U^A, \ln U^{A+\varepsilon F}]_k \right) \Big|_{\varepsilon=0}$$

$$= i\partial_\varepsilon \sum_{k=0}^\infty \frac{1}{k!} \int_0^1 dt \int_0^1 ds \, t \operatorname{tr} \left(P_+ \ln U^A P_- [-ts \ln U^A, \ln U^{A+\varepsilon F}]_k \right) \Big|_{\varepsilon=0}$$

$$= i\partial_\varepsilon \sum_{k=0}^\infty \frac{1}{k!} \int_0^1 dz \int_z^1 ds \, \operatorname{tr} \left(P_+ \ln U^A P_- [-z \ln U^A, \ln U^{A+\varepsilon F}]_k \right) \Big|_{\varepsilon=0}$$

$$= i\partial_\varepsilon \sum_{k=0}^{\infty} \frac{1}{k!} \int_0^1 dz(1-z)\,\mathrm{tr}\left(P_+ \ln U^A P_- [-z\ln U^A, \ln U^{A+\varepsilon F}]_k\right)\Bigg|_{\varepsilon=0}$$

$$= i\partial_\varepsilon \int_0^1 dz(1-z)\,\mathrm{tr}\left(P_+ \ln U^A P_- \sum_{k=0}^{\infty} \frac{1}{k!}[-z\ln U^A, \ln U^{A+\varepsilon F}]_k\right)\Bigg|_{\varepsilon=0}$$

$$\overset{\text{(Hadamard's formula)}}{=} i\partial_\varepsilon \int_0^1 dz(1-z)\,\mathrm{tr}\left(P_+ \ln U^A P_- e^{-z\ln U^A} \ln U^{A+\varepsilon F} e^{z\ln U^A}\right)\Bigg|_{\varepsilon=0}$$

$$= i\int_0^1 dz(1-z)\,\mathrm{tr}\left(P_+ \ln U^A P_- e^{-z\ln U^A} \partial_\varepsilon \ln U^{A+\varepsilon F}\big|_{\varepsilon=0} e^{z\ln U^A}\right). \quad (13)$$

The calculation for the first term of (12) is identical. At this point we notice that (13) and the term where the projectors are exchanged are complex conjugates of one another. So summarising we find

$$j_A(F) = -\partial_\varepsilon\,\varphi(A+\varepsilon F)|_{\varepsilon=0}$$
$$- 2\int_0^1 dz(1-z)\Im\,\mathrm{tr}\left(P_+ \ln U^A P_- e^{-z\ln U^A} \partial_\varepsilon \ln U^{A+\varepsilon F}\big|_{\varepsilon=0} e^{z\ln U^A}\right).$$

$$\square$$

Theorem 1.7. *Independent of the phase that is used to correct the scattering operator the following formula holds true for any four potentials A, F, H, with A small enough so that the relevant series converge.*

$$\partial_\varepsilon|_{\varepsilon=0}(j_{A+\varepsilon H}(F) - j_{A+\varepsilon F}(H)) =$$
$$2\Im\,\mathrm{tr}\left(P_+ \left(U^A\right)^{-1} \partial_\varepsilon|_{\varepsilon=0} U^{A+\varepsilon F} P_- \left(U^A\right)^{-1} \partial_\delta|_{\delta=0} U^{A+\delta H}\right) \quad (14)$$

Proof: We compute $\partial_\varepsilon|_{\varepsilon=0} j_{A+\varepsilon F}(H)$.

$$- i\partial_\varepsilon|_{\varepsilon=0} j_{A+\varepsilon H}(F) =$$
$$\partial_\varepsilon|_{\varepsilon=0}\partial_\delta|_{\delta=0}\langle\Omega, e^{i\varphi(A+\varepsilon H+\delta F)-i\varphi(A+\varepsilon H)} e^{-G\ln U^{A+\varepsilon H}} e^{G\ln U^{A+\varepsilon H+\delta F}}\Omega\rangle$$

We first act with the derivative with respect to H, fixing F.

$$- i\partial_\varepsilon|_{\varepsilon=0} j_{A+\varepsilon H}(F) =$$
$$\partial_\delta|_{\delta=0} i(\partial_\varepsilon|_{\varepsilon=0}\varphi(A+\varepsilon H+\delta F) - \partial_\varepsilon|_{\varepsilon=0}\varphi(A+\varepsilon H))e^{i\varphi(A+\delta F)-i\varphi(A)}$$
$$\langle\Omega, e^{-G\ln U^A} e^{G\ln U^{A+\delta F}}\Omega\rangle$$
$$+ \partial_\delta|_{\delta=0} e^{i\varphi(A+\delta F)-i\varphi(A)}\langle\Omega\partial_\varepsilon|_{\varepsilon=0} e^{-G\ln U^{A+\varepsilon H}} e^{G\ln U^{A+\delta F}}\Omega\rangle$$
$$+ \partial_\delta|_{\delta=0} e^{i\varphi(A+\delta F)-i\varphi(A)}\langle\Omega e^{-G\ln U^A} \partial_\varepsilon|_{\varepsilon=0} e^{G\ln U^{A+\varepsilon H+\delta F}}\Omega\rangle$$

In computing further one can notice a few cancellations. For the first summand the first factor vanishes if δ is set to zero, so the only the first summand in the product rule will not vanish. For the second and third summand we will use lemma 1.4, giving

$$-i\partial_\varepsilon|_{\varepsilon=0}j_{A+\varepsilon H}(F) =$$
$$\partial_\delta|_{\delta=0}i\partial_\varepsilon|_{\varepsilon=0}\varphi(A+\varepsilon H+\delta F)$$
$$-\partial_\delta|_{\delta=0}e^{i\varphi(A+\delta F)-i\varphi(A)}j_A^0(H)\langle\Omega, e^{-G\ln U^A}e^{G\ln U^{A+\delta F}}\Omega\rangle$$
$$+\partial_\delta|_{\delta=0}e^{i\varphi(A+\delta F)-i\varphi(A)}\langle\Omega, G\left(\partial_\varepsilon|_{\varepsilon=0}\left(U^{A+\varepsilon H}\right)^{-1}U^A\right)e^{-G\ln U^A}e^{G\ln U^{A+\delta F}}\Omega\rangle$$
$$+\partial_\delta|_{\delta=0}e^{i\varphi(A+\delta F)-i\varphi(A)}j_{A+\delta F}^0(H)\langle\Omega, e^{-G\ln U^A}e^{G\ln U^{A+\delta F}}\Omega\rangle$$
$$+\partial_\delta|_{\delta=0}e^{i\varphi(A+\delta F)-i\varphi(A)}\langle\Omega, e^{-G\ln U^A}e^{G\ln U^{A+\delta F}}G\left(\left(U^{A+\delta F}\right)^{-1}\partial_\varepsilon|_{\varepsilon=0}U^{A+\varepsilon H+\delta F}\right)\Omega\rangle.$$

Now there are a few further simplifications to appreciate: since $\langle\Omega, G\Omega\rangle = 0$, in the third and last summand only the derivatives with respect to δ which produce by lemma 1.4 another factor of G will contribute to the sum. For the other summands except for the first we can spot the appearance of j^0. Respecting all this results in

$$-i\partial_\varepsilon|_{\varepsilon=0}j_{A+\varepsilon H}(F) = i\partial_\delta|_{\delta=0}\partial_\varepsilon|_{\varepsilon=0}\varphi(A+\varepsilon H+\delta F)$$
$$-i\partial_\delta|_{\delta=0}\varphi(A+\delta F)j_A^0(H) - j_A^0(H)j_A^0(F)$$
$$+\langle\Omega, G\left(\partial_\varepsilon|_{\varepsilon=0}\left(U^{A+\varepsilon H}\right)^{-1}U^A\right)G\left(\left(U^A\right)^{-1}\partial_\delta|_{\delta=0}U^{A+\delta F}\right)\Omega\rangle$$
$$+i\partial_\delta|_{\delta=0}\varphi(A+\delta F)j_A^0(H) + \partial_\delta|_{\delta=0}j_{A+\delta F}^0(H) + j_A^0(H)j_A^0(F)$$
$$+\langle\Omega, G\left(\left(U^A\right)^{-1}\partial_\delta|_{\delta=0}U^{A+\delta F}\right)G\left(\left(U^A\right)^{-1}\partial_\varepsilon|_{\varepsilon=0}U^{A+\varepsilon H}\right)\Omega\rangle.$$

A few more terms cancel in the second and fourth line, also since $\partial_\varepsilon|_{\varepsilon=0}\left(U^{A+\varepsilon H}\right)^{-1}U^{A+\varepsilon H} = 0$ we can combine the two products of G into a commutator:

$$-i\partial_\varepsilon|_{\varepsilon=0}j_{A+\varepsilon H}(F) = i\partial_\delta|_{\delta=0}\partial_\varepsilon|_{\varepsilon=0}\varphi(A+\varepsilon H+\delta F)$$
$$+\partial_\delta|_{\delta=0}j_{A+\delta F}^0(H)$$
$$+\langle\Omega, \left[G\left(\left(U^A\right)^{-1}\partial_\delta|_{\delta=0}U^{A+\delta F}\right), G\left(\left(U^A\right)^{-1}\partial_\varepsilon|_{\varepsilon=0}U^{A+\varepsilon H}\right)\right]\Omega\rangle.$$

So we can once again apply lemma 1.1, which results in exactly right hand side of the equation we claimed to produce in the statement of this theorem. So all that is left is to recognise that one can combine the first two summands into $-i\partial_\varepsilon j_{A+\varepsilon H}(F)$, which is a direct consequence of theorem 1.6. $\qquad\square$

1.1. Quantitative Estimates

Since we do not only want to give an expression for the time evolution operator, but also give bounds on the numerical errors which are due to truncate the occurring series we need to look at these series a little closer. The series involve powers of the second quantisation operator G, so we start by examining these in greater depth. In order to do so we define an object closely related to G.

Todo: probably this part cannot be made rigorous. Decide whether to keep it as heuristics

Definition 1.8.

$$L : \{M \subset B(\mathcal{H}) \mid |M| < \infty\} \times \{M \subset B(\mathcal{H}) \mid |M| < \infty\} \to B(\mathcal{F})$$

$$L(\{A_1, \ldots, A_c\}, \{B_1, \ldots, B_m\}) := \prod_{l=1}^{m} a(\varphi_{-k_l})$$

$$\prod_{l=1}^{c} a^*(A_l \varphi_{n_l}) \prod_{l=1}^{m} a^*(B_l \varphi_{-k_l}) \prod_{l=1}^{c} a(\varphi_{n_l}), \quad (15)$$

where for notational reasons we chose to list the occurring one-particle operators in a specific order; however, the order does not matter, since commutation of the relevant creation and annihilation operators always results an overall factor of one.

Since this operator L occurs when computing powers of G we compute is product with some G with the following

Lemma 1.9. *For any $a, b, \in \mathbb{N}_0$ and apropriate one particle operators A_k, B_l, C for $1 \leqslant k \leqslant a$, $1 \leqslant l \leqslant b$ we have the following equality*

$$L\left(\bigcup_{l=1}^{a}\{A_l\}; \bigcup_{l=1}^{b}\{B_l\}\right) G(C) = \tag{16}$$

$$(-1)^{a+b} L\left(\bigcup_{l=1}^{a}\{A_l\} \cup \{C\}; \bigcup_{l=1}^{b}\{B_l\}\right) \tag{17}$$

$$+ (-1)^{a+b+1} L\left(\bigcup_{l=1}^{a}\{A_l\}; \bigcup_{l=1}^{b}\{B_l\} \cup \{C\}\right) \tag{18}$$

$$+ \sum_{f=1}^{a} L\left(\bigcup_{\substack{l=1 \\ l \neq f}}^{a}\{A_l\} \cup \{A_f P_+ C\}; \bigcup_{l=1}^{b}\{B_l\}\right) \tag{19}$$

$$+ \sum_{f=1}^{a} L\left(\bigcup_{\substack{l=1 \\ f \neq l}}^{a}\{A_l\} \cup \{-C P_- A_f\}; \bigcup_{l=1}^{b}\{B_l\}\right) \tag{20}$$

$$- \sum_{f=1}^{a} L\left(\bigcup_{\substack{l=1 \\ f \neq l}}^{a}\{A_l\}; \bigcup_{l=1}^{b}\{B_l\} \cup \{A_f P_+ C\}\right) \tag{21}$$

$$+ \sum_{f=1}^{b} L\left(\bigcup_{l=1}^{a}\{A_l\}; \bigcup_{\substack{l=1 \\ l \neq f}}^{b}\{B_l\} \cup \{-C P_- B_f\}\right) \tag{22}$$

$$+ (-1)^{a+b+1} \sum_{f=1}^{a} \mathrm{tr}\left(P_+ C P_- A_f\right) L\left(\bigcup_{\substack{l=1 \\ l \neq f}}^{a}\{A_l\}; \bigcup_{l=1}^{b}\{B_l\}\right) \tag{23}$$

$$+ (-1)^{a+b+1} \sum_{\substack{f_1, f_2=1 \\ f_1 \neq f_2}}^{a} L\left(\bigcup_{\substack{l=1 \\ l \neq f_1, f_2}}^{a}\{A_l\} \cup \{-A_{f_2} P_+ C P_- A_{f_1}\}; \bigcup_{l=1}^{b}\{B_l\}\right) \tag{24}$$

$$+ (-1)^{a+b+1} \sum_{f=1}^{b} \sum_{g=1}^{a} L\left(\bigcup_{\substack{l=1 \\ l \neq g}}^{a}\{A_l\}; \bigcup_{\substack{l=1 \\ l \neq f}}^{b}\{B_l\} \cup \{-A_g P_+ C P_- B_f\}\right). \tag{25}$$

Proof: The proof of this equality is a rather long calculation, where (15) is used repeatedly. We break up the calculation into several parts. Let us start with

$$
L\left(\bigcup_{l=1}^{a}\{A_l\};\bigcup_{l=1}^{b}\{B_l\}\right)L(C;) =
$$

$$
\prod_{l=1}^{b}a(\varphi_{-k_l})\prod_{l=1}^{a}a^*(A_l\varphi_{n_l})\prod_{l=1}^{b}a^*(B_l\varphi_{-k_l})\prod_{l=1}^{a}a(\varphi_{n_l})a^*(C\varphi_m)a(\varphi_m). \quad (26)
$$

We (anti)commute the creation operator involving C to its place at the end of the second product, after that the term will be normally ordered and can be rephrased in terms of Ls. During the commutation the creation operator in question can be picked up by any of the annihilation operators in the rightmost product. For each term where that happens we can perform the sum over the basis of $\mathcal{H}^-$ related to the annihilation operator whose anticommutator triggered. After this sum the corresponding term is also normally ordered and can be rephrased in terms of an L after some reshuffling which may only produce signs. So performing these steps we get

$$
L\left(\bigcup_{l=1}^{a}\{A_l\};\bigcup_{l=1}^{b}\{B_l\}\right)L(C;) =
$$

$$
\sum_{f=a}^{1}(-1)^{a-f}\prod_{l=1}^{b}a(\varphi_{-k_l})\prod_{l=1}^{f-1}a^*(A_l\varphi_{n_l})a^*(A_fP_+C\varphi_m)
$$

$$
\prod_{l=f+1}^{a}a^*(A_l\varphi_{n_l})\prod_{l=1}^{b}a^*(B_l\varphi_{-k_l})\prod_{\substack{l=1\\l\neq f}}^{a}a(\varphi_{n_l})a(\varphi_m)
$$

$$
+ L\left(\bigcup_{l=1}^{a}\{A_l\}\cup\{C\};\bigcup_{l=1}^{b}\{B_l\}\right)
$$

$$
= \sum_{f=1}^{a}L\left(\bigcup_{\substack{l=1\\l\neq f}}^{a}\{A_l\}\cup\{A_fP_+C\};\bigcup_{l=1}^{b}\{B_l\}\right)
$$

$$
+ L\left(\bigcup_{l=1}^{a}\{A_l\}\cup\{C\};\bigcup_{l=1}^{b}\{B_l\}\right). \quad (27)
$$

Now the remaining case is more laborious, that is why we will split off and treat some of the appearing terms separately. We start off analogous to before

$$
L\left(\bigcup_{l=1}^{a}\{A_l\};\bigcup_{l=1}^{b}\{B_l\}\right)L(;C) =
$$

$$
\prod_{l=1}^{b}a(\varphi_{-k_l})\prod_{l=1}^{a}a^*(A_l\varphi_{n_l})\prod_{l=1}^{b}a^*(B_l\varphi_{-k_l})\prod_{l=1}^{a}a(\varphi_{n_l})a(\varphi_{-m})a^*(C\varphi_{-m}). \quad (28)
$$

This time we need to (anti)commute the rightmost annihilation operator all the way to the end of the first product and the creation operator to the end of the second but last product. So there will be several qualitatively different terms. From the first step alone we get

$$
L\left(\bigcup_{l=1}^{a}\{A_l\};\bigcup_{l=1}^{b}\{B_l\}\right)L(;C) =
$$

$$
(-1)^a\sum_{f=b}^{1}(-1)^{b-f}\prod_{l=1}^{b}a(\varphi_{-k_l})\prod_{l=1}^{a}a^*(A_l\varphi_{n_l})\prod_{\substack{l=1\\l\neq f}}^{b}a^*(B_l\varphi_{-k_l})
$$

$$
\prod_{l=1}^{a}a(\varphi_{n_l})a^*(CP_-B_f\varphi_{-k_f}) \tag{29}
$$

$$
+(-1)^{a+b}\sum_{f=a}^{1}(-1)^{b-f}\prod_{l=1}^{b}a(\varphi_{-k_l})\prod_{\substack{l=1\\l\neq f}}^{a}a^*(A_l\varphi_{n_l})
$$

$$
\prod_{l=1}^{b}a^*(B_l\varphi_{-k_l})\prod_{l=1}^{a}a(\varphi_{n_l})a^*(CP_-\varphi_{n_f}) \tag{30}
$$

$$
+(-1)^b\prod_{l=1}^{b}a(\varphi_{-k_l})a(\varphi_{-m})\prod_{l=1}^{a}a^*(A_l\varphi_{n_l})
$$

$$
\prod_{l=1}^{b}a^*(B_l\varphi_{-k_l})\prod_{l=1}^{a}a(\varphi_{n_l})a^*(C\varphi_{-m}). \tag{31}
$$

We will discuss terms (29), (30) and (31) separately. In Term (29) we need to commute the last creation operator into its place in the third product, it can be picked up by one of the annihilation operators of the last product, but after performing the sum over the corresponding basis the resulting term can be rephrased in terms of an L operator by commuting only creation operators of the second and third product. Performing these steps yields the identity

$$
(29) = \sum_{f=1}^{b}L\left(\bigcup_{l=1}^{a}\{A_l\};\bigcup_{\substack{l=1\\l\neq f}}^{b}\{B_l\}\cup\{CP_-B_f\}\right)
$$

$$
+(-1)^{a+b+1}\sum_{f=1}^{b}\sum_{g=1}^{a}L\left(\bigcup_{\substack{l=1\\l\neq g}}^{a}\{A_l\};\bigcup_{\substack{l=1\\l\neq f}}^{b}\{B_l\}\{A_gP_+CP_-B_f\}\right). \tag{32}
$$

For (30) the last creation operator needs to be commuted to the end of the second product. It can be picked up by one of the annihilation operators of the last product, but here we have to distinguish between two cases. If the index of this annihilation operator equals f the resulting commutator will be $\operatorname{tr}P_+CP_-A_f$ otherwise one can again perform the sum over the corresponding index and express the whole Product in terms of an L operator. All this results in

$$(30) = \sum_{f=1}^{a} L\left(\bigcup_{\substack{l=1\\l\neq f}}^{a}\{A_l\} \cup \{CP_-A_f\}; \bigcup_{l=1}^{b}\{B_l\}\right)$$

$$+ (-1)^{a+b} \sum_{f=1}^{a} L\left(\bigcup_{\substack{l=1\\l\neq f}}^{a}\{A_l\}; \bigcup_{l=1}^{b}\{B_l\}\right) \mathrm{tr}(P_+CP_-A_f)$$

$$+ (-1)^{a+b+1} \sum_{\substack{f_1,f_2=1\\f_1\neq f_2}}^{a} L\left(\bigcup_{\substack{l=1\\l\neq f_1,f_2}}^{a}\{A_l\} \cup \{A_{f_2}P_+CP_-A_{f_1}\}; \bigcup_{l=1}^{b}\{B_l\}\right). \quad (33)$$

For (31) the procedure is basically the same as for (29), it results in

$$(31) = (-1)^{a+b} L\left(\bigcup_{l=1}^{a}\{A_l\}; \bigcup_{l=1}^{b}\{B_l\}\right)$$

$$+ \sum_{f=1}^{a} L\left(\bigcup_{\substack{l=1\\l\neq f}}^{a}\{A_l\} \cup \{CP_-A_f\}; \bigcup_{l=1}^{b}\{B_l\} \cup \{A_fP_+C\}\right). \quad (34)$$

Putting the results of the calculation together results in the claimed equation, after pulling in some factors of -1 into L. $\qquad\square$

We carry on with defining the important quantities for powers of G. First we introduce for each $k \in \mathbb{N}$ a linear bounded operator on $\mathcal{H}$, X_k which fulfils $\mathrm{tr}\, P_+X_kP_-X_k < \infty \wedge \mathrm{tr}\, P_-X_kP_+X_k < \infty$.

Definition 1.10. *Let*

$$Y := \{X_k \mid k \in \mathbb{N}\}.$$

Let for $n \in \mathbb{N}$

$$\langle n \rangle := \{X_l \mid l \in \mathbb{N}, l \leqslant n\}.$$

Definition 1.11. *Let for* $b \subset Y$, *such that* $|b| < \infty$

$$f_b : \{l \in \mathbb{N} \mid l \leqslant |b|\} \to b$$
$$\forall k < |b| : f_b(k) = X_l \wedge f_b(k+1) = X_m \to l < m \quad (35)$$

Definition 1.12. *For any set* b, *we denote by* $S(b)$ *the symmetric group (group of permutations) over* b.

Definition 1.13. *Let for* $b \subset Y$, *such that* $|b| < \infty$ *and* $\sigma_b \in S(b)$

$$VZ_{\sigma_b}^b : \{k \in \mathbb{N} \mid k < |b|\} \to \{-1, 1\}$$
$$VZ_{\sigma_b}^b(k) := sgn\big[f_b^{-1}(\sigma_b(f_b(k+1))) - f_b^{-1}(\sigma_b(f_b(k)))\big]$$

In what is to follow the order of one particle operators will be changed in all possible ways, to keep track of this by use of a compact notation we introduce

Definition 1.14.

$$W : \{(b, \sigma_b) \mid b \subseteq Y \wedge |b| < \infty \wedge \sigma_b \in S(b)\} \to B(\mathcal{H})$$

$$W(b, \sigma_b) := \left(\prod_{k=1}^{|b|-1} \sigma_b(f_b(k)) P_{VZ^b_{\sigma_b}(k)} VZ^b_{\sigma_b}(k) \right) \sigma_b(f_b(|b|))$$

Definition 1.15. *Let l be any finite subset of Y. Denote by X^l_{max} the operator $X_k \in l$ such that for any $X_c \in l$ the relation $k \geqslant c$ is fulfilled. Furthermore define*

$$PT : \{T \subset \mathcal{P}(Y) \mid |T| < \infty, \forall b \in T : |b| < \infty\} \to \mathbb{C}$$

$$\text{for: } T = \varnothing : PT(T) = 1, \text{ otherwise:}$$

$$PT(T) = \sum_{\substack{\forall l \in T: \\ \sigma_l \in S(l \setminus \{X^l_{max}\})}} \prod_{l \in T} \text{tr}[P_+ X^l_{max} P_- W(l, \sigma_l)]$$

There is one more function left to define

Definition 1.16.

$$Op : \{R \in \mathcal{P}(Y) \mid |R| < \infty\} \times \{D \subset \mathcal{P}(Y) \mid |T| < \infty\} \to \mathcal{B}(\mathcal{F})$$

$$Op(R, D) = \sum_{\substack{\forall l \in D: \\ \sigma_l \in S(l)}} \sum_{a \subseteq R \cup \bigcup_{l \in D} \{W(l, \sigma_l)\}} L(a, a^c)(-1)^{|a| + \frac{(|R|+|D|)(|R|+|D|+1)}{2}}$$

Now we are able to state the main theorem which will help us do quantitative estimates.

Theorem 1.17. *Let $n \in \mathbb{N}$, $X_1, \dots, X_n \in Y$ then the following equation holds*

$$\prod_{k=1}^{n} G(X_k) = \sum_{\substack{\langle n \rangle = \dot{\cup}_{l \in T} l \dot{\cup} \dot{\cup}_{l \in D} l \dot{\cup} R \\ \forall l \in T \cup D : |l| \geqslant 2}} PT(T) Op(R, D), \tag{36}$$

where the abbreviation $\langle n \rangle := \{X_k \mid k \leqslant n\}$ was used.

Proof: The proof will be by induction on n. Since the formula in the claim reduces to 1 for $n = 0$ we will not spend any more time on the start of the induction. The general strategy of the proof is to break up the right hand side of (36) for $n+1$ into small pieces and show for each piece that it corresponds to one of the contributions of lemma 1.9, while also each term in this lemma is represented by one of the terms obtained by breaking up (36).

As a first step we break the right hand side of (36) into three pieces separated by in which set X_{n+1} ends up in :

$$\sum_{\substack{\langle n+1 \rangle = \dot{\cup}_{l \in T} l \dot{\cup} \dot{\cup}_{l \in D} l \dot{\cup} R \\ \forall l \in T \cup D : |l| \geqslant 2}} PT(T) Op(R, D) =$$

$$\sum_{\substack{\langle n+1 \rangle = \dot{\cup}_{l \in T} l \dot{\cup} \dot{\cup}_{l \in D} l \dot{\cup} R \\ \exists l \in T : X_{n+1} \in l \\ \forall l \in T \cup D : |l| \geqslant 2}} PT(T) Op(R, D) \tag{37}$$

$$+ \sum_{\substack{\langle n+1 \rangle = \dot{\cup}_{l \in T} l \dot{\cup} \dot{\cup}_{l \in D} l \dot{\cup} R \\ \exists l \in D : X_{n+1} \in l \\ \forall l \in T \cup D : |l| \geqslant 2}} PT(T) Op(R, D) \tag{38}$$

$$+ \sum_{\substack{\langle n+1\rangle = \biguplus_{l\in T} l \uplus \biguplus_{l\in D} l \uplus R \\ X_{n+1}\in R \\ \forall l\in T\cup D:|l|\geqslant 2}} \mathrm{PT}(T)\mathrm{Op}(R,D), \tag{39}$$

We will discuss each term separately. For term (37) the term containing X_{n+1} is in one of the elements l' of T, but each such element has to have more than one element. So if we were to sum over the partitions of $\langle n\rangle$ instead, the rest of $l'\backslash\{X_{n+1}\}$ is either an element of D or, if it contains only one element, of R. Picking D instead of T is at this stage an arbitrary choice, but this choice leads to the terms of lemma 1.9. Alls this means that one correct rewriting of term (37) is

$$(37) = \sum_{\substack{\langle n\rangle = \biguplus_{l\in T} l \uplus \biguplus_{l\in D} l \uplus R \\ \forall l\in D\cup T:|l|>2}} \sum_{b\in D\cup\{\{r\}|r\in R\}}$$

$$\mathrm{PT}(T\cup\{\{X_{n+1}\cup f\})\mathrm{Op}(R\backslash b, D\backslash\{b\}). \tag{40}$$

Next we pull one factor and the corresponding sum out of PT and write out Op. Then we see that the sums over permutations can be merged into one. There we take the convention that for any set f such that $|f|=1$ holds, we define σ_f to be the identity on that set.

This results in

$$(40) = \sum_{\substack{\langle n\rangle = \biguplus_{l\in T} l \uplus \biguplus_{l\in D} l \uplus R \\ \forall l\in D\cup T:|l|>2}} \sum_{b\in D\cup\{\{r\}|r\in R\}} \sum_{\sigma_b\in S(b)}$$

$$\mathrm{tr}[P_+ X_{n+1} P_- W(b,\sigma_b)]\mathrm{PT}(T) \sum_{\substack{\forall l\in D\backslash\{b\} \\ \sigma_l\in S(l)}}$$

$$\sum_{a\subseteq R\backslash b \cup \bigcup_{l\in D\backslash\{b\}}\{W(l,\sigma_l)\}} L(a,a^c)(-1)^{|a|+\frac{(|R|+|D|-1)(|R|+|D|)}{2}}$$

$$= \sum_{\substack{\langle n\rangle = \biguplus_{l\in T} l \uplus \biguplus_{l\in D} l \uplus R \\ \forall l\in D\cup T:|l|>2}} \sum_{\substack{\forall l\in D \\ \sigma_l\in S(l)}} \sum_{b\in D\cup\{\{r\}|r\in R\}} \mathrm{PT}(T)$$

$$\sum_{a\subseteq R\backslash b \cup \bigcup_{l\in D\backslash\{b\}}\{W(l,\sigma_l)\}} L(a,a^c)(-1)^{|a|+\frac{(|R|+|D|-1)(|R|+|D|)}{2}}$$

$$\mathrm{tr}[P_+ X_{n+1} P_- W(b,\sigma_b)]$$

$$= \sum_{\substack{\langle n\rangle = \biguplus_{l\in T} l \uplus \biguplus_{l\in D} l \uplus R \\ \forall l\in D\cup T:|l|>2}} \sum_{\substack{\forall l\in D \\ \sigma_l\in S(l)}} \mathrm{PT}(T) \sum_{a\subseteq R\cup\bigcup_{l\in D}\{W(l,\sigma_l)\}}$$

$$\sum_{b\in D\cup\{\{r\}|r\in R\}} \mathbb{1}_{W(b,\sigma_b)\in a} L(a\backslash\{b\}, a^c)(-1)^{|a|+1}$$

$$(-1)^{\frac{(|R|+|D|-1)(|R|+|D|)}{2}} \mathrm{tr}[P_+ X_{n+1} P_- W(b,\sigma_b)]$$

$$= \sum_{\substack{\langle n\rangle = \biguplus_{l\in T} l \uplus \biguplus_{l\in D} l \uplus R \\ \forall l\in D\cup T:|l|>2}} \mathrm{PT}(T) \sum_{\substack{\forall l\in D \\ \sigma_l\in S(l)}} \sum_{a\subseteq R\cup\bigcup_{l\in D}\{W(l,\sigma_l)\}}$$

$$\sum_{b \in a} L(a \backslash \{b\}, a^c)(-1)^{|a| + \frac{(|R|+|D|+1)(|R|+|D|)}{2}}$$

$$(-1)^{1+|R|+|D|} \operatorname{tr}[P_+ X_{n+1} P_- W(b, \sigma_b)]$$

$$= \sum_{\substack{\langle n \rangle = \biguplus_{l \in T} l \uplus \biguplus_{l \in D} l \uplus R \\ \forall l \in D \cup T : |l| > 2}} \operatorname{PT}(T) \sum_{\substack{\forall l \in D \\ \sigma_l \in S(l)}} \sum_{a \subseteq R \cup \bigcup_{l \in D}\{W(l, \sigma_l)\}}$$

$$(-1)^{|a| + \frac{(|R|+|D|+1)(|R|+|D|)}{2}} (23)_{L(a, a^c)G(X_{n+1})}, \qquad (41)$$

where the notation in the last line is to be taken as "apply Lemma 1.9 apply it to $L(a, a^c)G(X_{n+1})$ and pick only term (23)". We will use this abbreviating notation also for the next terms.

The next term is (38). Here we need a few more notational conventions. For any set $b \subseteq \langle n \rangle$ and corresponding permutation $\sigma_b \in S(b)$, we denote by the same symbol σ_b the continuation of σ_b to $b \cup \{X_{n+1}\}$, where for this continuation X_{n+1} is a fixed point. Furthermore we define for any set $b \subseteq \langle n \rangle$, σ_c^b by

$$\sigma_c^b \in S(b \cup \{X_{n+1}\}),$$
$$\forall k \leqslant |b| : \sigma_c^b(f_{b \cup \{X_{n+1}\}}(k)) = f_{b \cup \{X_{n+1}\}}(k+1)$$
$$\sigma_c^b(X_{n+1}) = f_b(1). \qquad (42)$$

Finally we define for sets $b_1, b_2 \subseteq \langle n \rangle, b_1 \cap b_2 = \varnothing$ and corresponding permutations $\sigma_{b_1} \in S(b_1), \sigma_{b_2} \in S(b_2)$ the permutation σ_{b_1,b_2}^{n+1} by

$$M_{b_1,b_2}^{n+1} := b_1 \cup b_2 \cup \{X_{n+1}\}$$
$$\sigma_{b_1,b_2}^{n+1} \in S(M_{b_1,b_2}^{n+1})$$
$$\forall 1 \leqslant k \leqslant |b_1| : \sigma_{b_1,b_2}^{n+1}(f_{M_{b_1,b_2}^{n+1}}(k)) = \sigma_{b_1}(f_{b_1}(k))$$
$$\sigma_{b_1,b_2}^{n+1}(f_{M_{b_1,b_2}^{n+1}}(|b_1|+1)) = X_{n+1}$$
$$\forall |b_1| + 2 \leqslant k \leqslant |b_1| + |b_2| + 1 :$$
$$\sigma_{b_1,b_2}^{n+1}(f_{M_{b_1,b_2}^{n+1}}(k)) = \sigma_{b_2}(f_{b_2}(k - |b_1| - 1)) \qquad (43)$$

The beginning of the treatment of term (38) is analogous to (37), we rewrite the partition of $\langle n+1 \rangle$ into one of $\langle n \rangle$ with an additional sum over where the other operators packed to together with X_{n+1} come from. This splits into three parts, either X_{n+1} is put at the beginning of the compound operator, or its put at the end of the compound object, or to the left as well as to the right are operators with smaller index. Since the overall sign is decided by how often the operator index rises or falls, this separation into cases is helpful. The last case we then rewrite as picking two sets of operators, one of which will be in front of X_{n+1} and the other one behind this operator.

The calculation is as follows

$$(38) = \sum_{\substack{\langle n+1\rangle = \uplus_{l\in T}\, l \,\uplus\, \uplus_{l\in D}\, l \,\uplus\, R \\ \exists l\in D: X_{n+1}\in l \\ \forall l\in T\cup D: |l|\geqslant 2}} \mathrm{PT}(T)\,\mathrm{Op}(R,D)$$

$$= \sum_{\substack{\langle n\rangle = \uplus_{l\in T}\, l \,\uplus\, \uplus_{l\in D}\, l \,\uplus\, R \\ \forall l\in D\cup T: |l|\geqslant 2}} \mathrm{PT}(T) \sum_{b\in D\cup\{\{r\}|r\in R\}} \mathrm{Op}(R\backslash b,\, D\cup\{b\cup\{X_{n+1}\}\backslash\{b\}\})$$

$$= \sum_{\substack{\langle n\rangle = \uplus_{l\in T}\, l \,\uplus\, \uplus_{l\in D}\, l \,\uplus\, R \\ \forall l\in D\cup T: |l|\geqslant 2}} \mathrm{PT}(T) \sum_{b\in D\cup\{\{r\}|r\in R\}} \sum_{\substack{\forall l\in D\cup\{b\cup\{X_{n+1}\}\} \\ \sigma_l\in S(l)}}$$

$$\sum_{a\subseteq R\backslash b\cup\bigcup_{l\in D\cup\{b\cup\{X_{n+1}\}\}\backslash\{b\}}\{W(l,\sigma_l)\}} L(a,a^c)(-1)^{|a|+\frac{(|R|+|D|)(|R|+|D|+1)}{2}}$$

$$= \sum_{\substack{\langle n\rangle = \uplus_{l\in T}\, l \,\uplus\, \uplus_{l\in D}\, l \,\uplus\, R \\ \forall l\in D\cup T: |l|\geqslant 2}} \mathrm{PT}(T) \sum_{b\in D\cup\{\{r\}|r\in R\}} \sum_{\substack{\forall l\in D \\ \sigma_l\in S(l)}} \Bigg[$$

$$\sum_{a\subseteq R\backslash b\cup\bigcup_{l\in D\backslash\{b\}}\{W(l,\sigma_l)\}\cup\{W(b\cup\{X_{n+1}\},\sigma_b)\}} L(a,a^c)(-1)^{|a|+\frac{(|R|+|D|)(|R|+|D|+1)}{2}} \tag{44}$$

$$+ \sum_{a\subseteq R\backslash b\cup\bigcup_{l\in D\backslash\{b\}}\{W(l,\sigma_l)\}\cup\{W(b\cup\{X_{n+1}\},\sigma_c^b\circ\sigma_b)\}} L(a,a^c)(-1)^{|a|+\frac{(|R|+|D|)(|R|+|D|+1)}{2}} \Bigg] \tag{45}$$

$$+ \sum_{\substack{\langle n\rangle = \uplus_{l\in T}\, l \,\uplus\, \uplus_{l\in \overline{D}}\, l \,\uplus\, \overline{R} \\ \forall l\in \overline{D}\cup T: |l|\geqslant 2}} \mathrm{PT}(T) \sum_{\substack{b_1,b_2\in \overline{D}\cup\{\{r\}|r\in \overline{R}\} \\ b_1\neq b_2}} \sum_{\substack{\forall l\in \overline{D} \\ \sigma_l\in S(l)}}$$

$$\sum_{a\subseteq \tilde{R}\cup\bigcup_{l\in \tilde{D}}\{W(l,\sigma_l)\}\cup\{W(b_1\cup\{X_{n+1}\}\cup b_2,\sigma_{b_1,b_2}^{n+1})\}} L(a,a^c)(-1)^{|a|+\frac{(|\overline{R}|+|\overline{D}|-1)(|\overline{R}|+|\overline{D}|)}{2}} \tag{46}$$

where $\tilde{R} = \overline{R}\backslash(b_1\cup b_2)$ and $\tilde{D} := \overline{D}\cup\{b_1\cup\{X_{n+1}\}\cup b_2\}\backslash\{b_1,b_2\}$. For the term (46) we had to reshuffle the outermost sum a bit. For each term in the original sum where X_{n+1} is neither the first nor the last factor in its product (we will call the set of factors in front of X_{n+1} α and the factors behind it β) there is a different splitting of $\langle n\rangle$ into $\overline{R}$ and $\overline{D}$ such that α and β are separate elements of $\overline{D}\cup\{\{r\}\mid r\in \overline{R}\}$. So we replace the original sum over D and R into one of $\overline{D}$ and $\overline{R}$. Since this is a one to one correspondence and the sum is finite this is always possible. The exponent of the sign also changes for this reason, since $|R|+|D| = |\overline{R}|+|\overline{D}|-1$ holds. Continuing with (44) the next steps are similar to the last steps in treating (37). They are

$$(44) = \sum_{\substack{\langle n\rangle = \uplus_{l\in T}\, l \,\uplus\, \uplus_{l\in D}\, l \,\uplus\, R \\ \forall l\in D\cup T: |l|\geqslant 2}} \mathrm{PT}(T) \sum_{b\in D\cup\{\{r\}|r\in R\}} \sum_{\substack{\forall l\in D \\ \sigma_l\in S(l)}}$$

$$\sum_{a\subseteq R\backslash b\cup\bigcup_{l\in D\backslash\{b\}}\{W(l,\sigma_l)\}\cup\{W(b\cup\{X_{n+1}\},\sigma_b)\}} L(a,a^c)(-1)^{|a|+\frac{(|R|+|D|)(|R|+|D|+1)}{2}}$$

$$= \sum_{\substack{\langle n\rangle = \uplus_{l\in T}\, l \,\uplus\, \uplus_{l\in D}\, l \,\uplus\, R \\ \forall l\in D\cup T: |l|\geqslant 2}} \mathrm{PT}(T) \sum_{b\in D\cup\{\{r\}|r\in R\}} \sum_{\substack{\forall l\in D \\ \sigma_l\in S(l)}}$$

$$\sum_{a\subseteq R\cup\bigcup_{l\in D}\{W(l,\sigma_l)\}}(-1)^{|a|+\frac{(|R|+|D|)(|R|+|D|+1)}{2}}\mathbb{1}_{W(b,\sigma_b)\in a}$$

$$\Big[L(a\backslash\{W(b,\sigma_b)\}\cup\{W(b\cup\{X_{n+1}\},\sigma_b)\},a^c)$$
$$-L(a\backslash\{W(b,\sigma_b)\},a^c\cup\{W(b\cup\{X_{n+1}\},\sigma_b)\})\Big]$$

$$=\sum_{\substack{\langle n\rangle=\biguplus_{l\in T}l\uplus\biguplus_{l\in D}l\uplus R\\\forall l\in D\cup T:|l|\geqslant 2}}\mathrm{PT}(T)\sum_{\substack{\forall l\in D\\\sigma_l\in S(l)}}$$

$$\sum_{a\subseteq R\cup\bigcup_{l\in D}\{W(l,\sigma_l)\}}(-1)^{|a|+\frac{(|R|+|D|)(|R|+|D|+1)}{2}}\sum_{W(b,\sigma_b)\in a}$$

$$\Big[L(a\backslash\{W(b,\sigma_b)\}\cup\{W(b\cup\{X_{n+1}\},\sigma_b)\},a^c)$$
$$-L(a\backslash\{W(b,\sigma_b)\},a^c\cup\{W(b\cup\{X_{n+1}\},\sigma_b)\})\Big]$$

$$=\sum_{\substack{\langle n\rangle=\biguplus_{l\in T}l\uplus\biguplus_{l\in D}l\uplus R\\\forall l\in D\cup T:|l|\geqslant 2}}\mathrm{PT}(T)\sum_{\substack{\forall l\in D\\\sigma_l\in S(l)}}\sum_{a\subseteq R\cup\bigcup_{l\in D}\{W(l,\sigma_l)\}}$$

$$(-1)^{|a|+\frac{(|R|+|D|)(|R|+|D|+1)}{2}}((19)+(21))_{L(a,a^c)G(X_{n+1})}.\tag{47}$$

Almost the same procedure applies to (45). It yields

$$(45)=\sum_{\substack{\langle n\rangle=\biguplus_{l\in T}l\uplus\biguplus_{l\in D}l\uplus R\\\forall l\in D\cup T:|l|\geqslant 2}}\mathrm{PT}(T)\sum_{b\in D\cup\{\{r\}|r\in R\}}\sum_{\substack{\forall l\in D\\\sigma_l\in S(l)}}$$

$$\sum_{a\subseteq R\backslash b\cup\bigcup_{l\in D\backslash\{b\}}\{W(l,\sigma_l)\}\cup\{W(b\cup\{X_{n+1}\},\sigma_c^b\circ\sigma_b)\}}L(a,a^c)(-1)^{|a|+\frac{(|R|+|D|)(|R|+|D|+1)}{2}}$$

$$=\sum_{\substack{\langle n\rangle=\biguplus_{l\in T}l\uplus\biguplus_{l\in D}l\uplus R\\\forall l\in D\cup T:|l|\geqslant 2}}\mathrm{PT}(T)\sum_{b\in D\cup\{\{r\}|r\in R\}}\sum_{\substack{\forall l\in D\\\sigma_l\in S(l)}}$$

$$\sum_{a\subseteq R\cup\bigcup_{l\in D}\{W(l,\sigma_l)\}}(-1)^{|a|+\frac{(|R|+|D|)(|R|+|D|+1)}{2}}$$

$$\Big[\mathbb{1}_{W(b,\sigma_c^b\circ\sigma_b)\in a}L(a\backslash\{W(b,\sigma_b)\}\cup\{W(b\cup\{X_{n+1}\},\sigma_c^b\circ\sigma_b)\},a^c)$$
$$+\mathbb{1}_{W(b,\sigma_c^b\circ\sigma_b)\in a^c}L(a\backslash\{W(b,\sigma_b)\},a^c\cup\{W(b\cup\{X_{n+1}\},\sigma_c^b\circ\sigma_b)\})\Big]$$

$$=\sum_{\substack{\langle n\rangle=\biguplus_{l\in T}l\uplus\biguplus_{l\in D}l\uplus R\\\forall l\in D\cup T:|l|\geqslant 2}}\mathrm{PT}(T)\sum_{\substack{\forall l\in D\\\sigma_l\in S(l)}}$$

$$\sum_{a\subseteq R\cup\bigcup_{l\in D}\{W(l,\sigma_l)\}}(-1)^{|a|+\frac{(|R|+|D|)(|R|+|D|+1)}{2}}$$

$$\Big[\sum_{W(b,\sigma_b)\in a}L(a\backslash\{W(b,\sigma_b)\}\cup\{W(b\cup\{X_{n+1}\},\sigma_c^b\circ\sigma_b)\},a^c)$$
$$+\sum_{W(b,\sigma_b)\in a^c}L(a,a^c\backslash\{W(b,\sigma_b)\}\cup\{W(b\cup\{X_{n+1}\},\sigma_c^b\circ\sigma_b)\})\Big]$$

$$=\sum_{\substack{\langle n\rangle=\biguplus_{l\in T}l\uplus\biguplus_{l\in D}l\uplus R\\\forall l\in D\cup T:|l|\geqslant 2}}\mathrm{PT}(T)\sum_{\substack{\forall l\in D\\\sigma_l\in S(l)}}\sum_{a\subseteq R\cup\bigcup_{l\in D}\{W(l,\sigma_l)\}}$$

$$(-1)^{|a|+\frac{(|R|+|D|)(|R|+|D|+1)}{2}}((20)+(22))_{L(a,a^c)}G(X_{n+1}). \tag{48}$$

Also for (46) the procedure is almost the same. We bring the sums into a form such that one can read off the terms generated by the induction. We begin be renaming the sets which we had to change by resumming back to the names of the original sets.

$$
(46) = \sum_{\substack{\langle n\rangle=\biguplus_{l\in T}l\,\uplus\,\biguplus_{l\in\overline{D}}l\,\uplus\,\overline{R}\\ \forall l\in\overline{D}\cup T:|l|\geqslant 2}} \mathrm{PT}(T) \sum_{\substack{b_1,b_2\in\overline{D}\cup\{\{r\}|r\in\overline{R}\}\\ b_1\neq b_2}} \sum_{\substack{\forall l\in\overline{D}\\ \sigma_l\in S(l)}}
$$

$$
\sum_{a\subseteq\tilde{R}\cup\bigcup_{l\in\tilde{D}}\{W(l,\sigma_l)\}\cup\{W(b_1\cup\{X_{n+1}\}\cup b_2,\sigma_{b_1,b_2}^{n+1})\}} L(a,a^c)(-1)^{|a|+\frac{(|\overline{R}|+|\overline{D}|-1)(|\overline{R}|+|\overline{D}|)}{2}}
$$

$$
= \sum_{\substack{\langle n\rangle=\biguplus_{l\in T}l\,\uplus\,\biguplus_{l\in D}l\,\uplus\,R\\ \forall l\in D\cup T:|l|\geqslant 2}} \mathrm{PT}(T) \sum_{\substack{b_1,b_2\in D\cup\{\{r\}|r\in R\}\\ b_1\neq b_2}} \sum_{\substack{\forall l\in D\\ \sigma_l\in S(l)}}
$$

$$
\sum_{a\subseteq R\cup\bigcup_{l\in D}\{W(l,\sigma_l)\}\cup\{W(b_1\cup\{X_{n+1}\}\cup b_2,\sigma_{b_1,b_2}^{n+1})\}} L(a,a^c)(-1)^{|a|+\frac{(|R|+|D|-1)(|R|+|D|)}{2}}
$$

$$
= \sum_{\substack{\langle n\rangle=\biguplus_{l\in T}l\,\uplus\,\biguplus_{l\in D}l\,\uplus\,R\\ \forall l\in D\cup T:|l|\geqslant 2}} \mathrm{PT}(T) \sum_{\substack{b_1,b_2\in D\cup\{\{r\}|r\in R\}\\ b_1\neq b_2}} \sum_{\substack{\forall l\in D\\ \sigma_l\in S(l)}} \sum_{a\subseteq R\cup\bigcup_{l\in D}\{W(l,\sigma_l)\}}
$$

$$
(-1)^{|R|+|D|+\frac{(|R|+|D|)(|R|+|D|+1)}{2}}\mathbb{1}_{W(b_1,\sigma_1)\in a}
$$

$$
\Big[+(-1)^{|a|+1}\mathbb{1}_{W(b_2,\sigma_2)\in a}L\Big(a\backslash\{W(b_1,\sigma_1),W(b_2,\sigma_2)\}\cup
$$

$$
\cup\{W(b_1\cup\{X_{n+1}\}\cup b_2,\sigma_{b_1,b_2}^{n+1})\},a^c\Big)
$$

$$
+(-1)^{|a|+1}\mathbb{1}_{W(b_2,\sigma_2)\in a^c}L\Big(a\backslash\{W(b_1,\sigma_1)\},a^c\backslash\{W(b_2,\sigma_2)\}\cup
$$

$$
\cup\{W(b_1\cup\{X_{n+1}\}\cup f_2,\sigma_{b_1,b_2}^{n+1})\}\Big)\Big]
$$

$$
= \sum_{\substack{\langle n\rangle=\biguplus_{l\in T}l\,\uplus\,\biguplus_{l\in D}l\,\uplus\,R\\ \forall l\in D\cup T:|l|\geqslant 2}} \mathrm{PT}(T) \sum_{\substack{\forall l\in D\\ \sigma_l\in S(l)}} \sum_{a\subseteq R\cup\bigcup_{l\in D}\{W(l,\sigma_l)\}}
$$

$$
(-1)^{|R|+|D|+\frac{(|R|+|D|)(|R|+|D|+1)}{2}}
$$

$$
\Big[(-1)^{|a|+1}\sum_{\substack{b_1,b_2\in a\\ b_1\neq b_2}} L\Big(a\backslash\{W(b_1,\sigma_1),W(b_2,\sigma_2)\}\cup
$$

$$
\cup\{W(b_1\cup\{X_{n+1}\}\cup b_2,\sigma_{b_1,b_2}^{n+1})\},a^c\Big)
$$

$$
+(-1)^{|a|+1}\sum_{b_1\in a,b_2\in a^c} L\Big(a\backslash\{W(b_1,\sigma_1)\},a^c\backslash\{W(b_2,\sigma_2)\}\cup
$$

$$
\cup\{W(b_1\cup\{X_{n+1}\}\cup f_2,\sigma_{b_1,b_2}^{n+1})\}\Big)\Big]
$$

$$
= \sum_{\substack{\langle n\rangle=\biguplus_{l\in T}l\,\uplus\,\biguplus_{l\in D}l\,\uplus\,R\\ \forall l\in D\cup T:|l|\geqslant 2}} \mathrm{PT}(T) \sum_{\substack{\forall l\in D\\ \sigma_l\in S(l)}} \sum_{a\subseteq R\cup\bigcup_{l\in D}\{W(l,\sigma_l)\}}
$$

$$
(-1)^{|a|+\frac{(|R|+|D|)(|R|+|D|+1)}{2}}((24)+(25))_{L(a,a^c)}G(X_{n+1})
$$

Lastly we will discuss term (39); luckily, this term is less involved than the other two. The general procedure; however, stays the same. First we reformulate the partition of $\langle n+1 \rangle$ into one of $\langle n \rangle$, where the terms acquire modifications. Secondly we massage these terms until the involved sums look exactly like the one in our induction hypothesis (36) and realise that the terms are produced by lemma 1.9. For term (39) this results in

$$
(39) = \sum_{\substack{\langle n+1 \rangle = \biguplus_{l \in T} l \,\uplus\, \biguplus_{l \in D} l \,\uplus\, R \\ X_{n+1} \in R \\ \forall l \in T \cup D : |l| \geqslant 2}} \mathrm{PT}(T)\mathrm{Op}(R, D)
$$

$$
= \sum_{\substack{\langle n \rangle = \biguplus_{l \in T} l \,\uplus\, \biguplus_{l \in D} l \,\uplus\, R \\ \forall l \in T \cup D : |l| \geqslant 2}} \mathrm{PT}(T)\mathrm{Op}(R \cup \{X_{n+1}\}, D)
$$

$$
= \sum_{\substack{\langle n \rangle = \biguplus_{l \in T} l \,\uplus\, \biguplus_{l \in D} l \,\uplus\, R \\ \forall l \in T \cup D : |l| \geqslant 2}} \mathrm{PT}(T) \sum_{\substack{\forall l \in D: \\ \sigma_l \in S(l)}} \sum_{a \subseteq R \cup \{X_{n+1}\} \cup \bigcup_{l \in D} \{W(l, \sigma_l)\}}
$$

$$
L(a, a^c)(-1)^{|a| + \frac{(|R|+1+|D|)(|R|+|D|+2)}{2}}
$$

$$
= \sum_{\substack{\langle n \rangle = \biguplus_{l \in T} l \,\uplus\, \biguplus_{l \in D} l \,\uplus\, R \\ \forall l \in T \cup D : |l| \geqslant 2}} \mathrm{PT}(T) \sum_{\substack{\forall l \in D: \\ \sigma_l \in S(l)}} \sum_{a \subseteq R \cup \bigcup_{l \in D} \{W(l, \sigma_l)\}} (-1)^{|a| + \frac{(|R|+1+|D|)(|R|+|D|)}{2}}
$$

$$
\Big(-L(a \cup \{X_{n+1}\}, a^c) + L(a, a^c \cup \{X_{n+1}\}) \Big)(-1)^{|R|+|D|+1}
$$

$$
= \sum_{\substack{\langle n \rangle = \biguplus_{l \in T} l \,\uplus\, \biguplus_{l \in D} l \,\uplus\, R \\ \forall l \in T \cup D : |l| \geqslant 2}} \mathrm{PT}(T) \sum_{\substack{\forall l \in D: \\ \sigma_l \in S(l)}} \sum_{a \subseteq R \cup \bigcup_{l \in D} \{W(l, \sigma_l)\}} (-1)^{|a| + \frac{(|R|+1+|D|)(|R|+|D|)}{2}}
$$

$$
\Big(L(a \cup \{X_{n+1}\}, a^c)(-1)^{|R|+|D|} + L(a, a^c \cup \{X_{n+1}\})(-1)^{|R|+|D|+1} \Big)
$$

$$
= \sum_{\substack{\langle n \rangle = \biguplus_{l \in T} l \,\uplus\, \biguplus_{l \in D} l \,\uplus\, R \\ \forall l \in T \cup D : |l| \geqslant 2}} \mathrm{PT}(T) \sum_{\substack{\forall l \in D: \\ \sigma_l \in S(l)}} \sum_{a \subseteq R \cup \bigcup_{l \in D} \{W(l, \sigma_l)\}} (-1)^{|a| + \frac{(|R|+1+|D|)(|R|+|D|)}{2}}
$$

$$
\Big((17) + (18) \Big)_{L(a, a^c)G(X_{n+1})}.
$$

Summarising we showed

$$
\sum_{\substack{\langle n+1 \rangle = \biguplus_{l \in T} l \,\uplus\, \biguplus_{l \in D} l \,\uplus\, R \\ \forall l \in T \cup D : |l| \geqslant 2}} \mathrm{PT}(T)\mathrm{Op}(R, D)
$$

$$
= \sum_{\substack{\langle n \rangle = \biguplus_{l \in T} l \,\uplus\, \biguplus_{l \in D} l \,\uplus\, R \\ \forall l \in D \cup T : |l| > 2}} \mathrm{PT}(T) \sum_{\substack{\forall l \in D \\ \sigma_l \in S(l)}} \sum_{a \subseteq R \cup \bigcup_{l \in D} \{W(l, \sigma_l)\}}
$$

$$
(-1)^{|a| + \frac{(|R|+|D|+1)(|R|+|D|)}{2}}
$$

$$
\Big((23) + (19) + (21) + (20) + (22)
$$

$$
+ (24) + (25) + (17) + (18) \Big)_{L(a, a^c)G(X_{n+1})}
$$

$$
= \sum_{\substack{\langle n \rangle = \biguplus_{l \in T} l \,\uplus\, \biguplus_{l \in D} l \,\uplus\, R \\ \forall l \in D \cup T : |l| > 2}} \mathrm{PT}(T) \sum_{\substack{\forall l \in D \\ \sigma_l \in S(l)}} \sum_{a \subseteq R \cup \bigcup_{l \in D} \{W(l, \sigma_l)\}}
$$

$$
(-1)^{|a| + \frac{(|R|+|D|+1)(|R|+|D|)}{2}} L(a, a^c)G(X_{n+1})
$$

$$= \sum_{\substack{\langle n \rangle = \biguplus_{l \in T} l \uplus \biguplus_{l \in D} l \uplus R \\ \forall l \in D \cup T : |l| > 2}} \mathrm{PT}(T) \mathrm{Op}(R, D) G(X_{n+1})$$

$$= \prod_{l=1}^{n} G(X_l) \quad G(X_{n+1}),$$

which ends our proof by induction. $\qquad\qquad\qquad\square$