What we know about $T1$

$T1$ is an unbounded linear map from the Fock space of QED onto itself. To be more specific, let $\mathcal{H}_+$ denote the Hilbert space of electron wavefunctions and $\mathcal{H}_-$ denote the Hilbert space of positron wavefunctions. Then

$$T1 : \mathcal{F}\left(\mathcal{H}_+\right) \otimes \mathcal{F}\left(\mathcal{H}_-\right) \to \mathcal{F}\left(\mathcal{H}_+\right) \otimes \mathcal{F}\left(\mathcal{H}_-\right), \tag{1}$$

but by far not all matrix elements are nonzero, expressed in the fixed particle subspaces we already see more structure:

$$T1 : \mathcal{H}_+^{\otimes n} \otimes \mathcal{H}_-^{\otimes p} \to \left(\mathcal{H}_+^{\otimes n-1} \otimes \mathcal{H}_-^{\otimes p-1}\right) \oplus \left(\mathcal{H}_+^{\otimes n} \otimes \mathcal{H}_-^{\otimes p}\right) \oplus \left(\mathcal{H}_+^{\otimes n+1} \otimes \mathcal{H}_-^{\otimes p+1}\right). \tag{2}$$

So restricted to the vacuum sector this simplifies, the part from the vacuum sector is set to zero, because the vacuum polarization current should be zero.

$$T1 : \mathbb{C} \to \left(\mathcal{H}_+ \otimes \mathcal{H}_-\right) \tag{3}$$

We will construct the finite operator norm of $T1$ restricted to an arbitrary fixed particle sector, beginning with the vacuum sector. Let the vector potential of the electromagnetic field A be

$$A \in C_c^\infty\left(\mathbb{R}^4\right). \tag{4}$$

Then one obtains in physicists notation:

$$T1\Omega = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 k}{\sqrt{2k^0}} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{\sqrt{2p^0}} \int_{\mathbb{R}^4} \mathrm{d}^4 x \sum_{r,s=\pm 1} b_s^\dagger(k) d_r^\dagger(p) \bar{u_s}(k) \gamma^\mu v_r(p) \Omega e^{ix_\alpha(k^\alpha + p^\alpha)} A_\mu(x), \tag{5}$$

Where the γ^μ are the dirac matrices fulfilling the anticommutation relation

$$\{\gamma^\alpha \gamma^\beta\} = 2\eta^{\alpha\beta} \tag{6}$$

The u, v are given by:

$$u_s(p) = \begin{pmatrix} \sqrt{p \cdot \sigma}\xi_s \\ \sqrt{p \cdot \bar{\sigma}}\xi_s \end{pmatrix} \qquad v_s(p) = \begin{pmatrix} \sqrt{p \cdot \sigma}\xi_{-s} \\ -\sqrt{p \cdot \bar{\sigma}}\xi_{-s} \end{pmatrix}, \tag{7}$$

with

$$\sigma = (\mathbb{1}, \vec{\sigma}), \qquad \bar{\sigma} = (\mathbb{1}, -\vec{\sigma}) \tag{8}$$

$$\xi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \xi_{-1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{9}$$

Where $\vec{\sigma}$ is the vector containing the Pauli matrices. The creation and annihilation operator fulfill the following identity:

$$\{b_s(p), b_r^\dagger(p')\} = \delta^3(p - p') = \{d_s(p), d_r^\dagger(p')\} \tag{10}$$

Useful identities for sums of u and v are:

$$\sum_s u_s(p)\bar{u}_s(p) = \not{p} + m \qquad \sum_s v_s(p)\bar{v}_s(p) = \not{p} - m \tag{11}$$

For the norm of the image of the vacuum we therefore arrive at:

$$
\begin{aligned}
\|T1\Omega\|^2 = & \frac{1}{(2\pi)^6}\int_{\mathbb{R}^3}\frac{\mathrm{d}^3k}{\sqrt{2k^0}}\int_{\mathbb{R}^3}\frac{\mathrm{d}^3p}{\sqrt{2p^0}}\int_{\mathbb{R}^3}\frac{\mathrm{d}^3k'}{\sqrt{2k'^0}}\int_{\mathbb{R}^3}\frac{\mathrm{d}^3p'}{\sqrt{2p'^0}}\int_{\mathbb{R}^4}\mathrm{d}^4x\int_{\mathbb{R}^4}\mathrm{d}^4y\sum_{r,s,r',s'=\pm1} \\
& \Omega^\dagger d_{r'}(p')b_{s'}(k')\bar{v}_{r'}(p')\gamma^\epsilon u_{s'}(k')b_s^\dagger(k)d_r^\dagger(p)\bar{u}_s(k)\gamma^\mu v_r(p)\Omega \\
& e^{ix_\alpha(k^\alpha+p^\alpha)}e^{-iy_\beta(k'^\beta+p'^\beta)}A_\mu(x)A_\epsilon(y) \\
= & \frac{1}{(2\pi)^6}\int_{\mathbb{R}^3}\frac{\mathrm{d}^3k}{\sqrt{2k^0}}\int_{\mathbb{R}^3}\frac{\mathrm{d}^3p}{\sqrt{2p^0}}\int_{\mathbb{R}^3}\frac{\mathrm{d}^3k'}{\sqrt{2k'^0}}\int_{\mathbb{R}^3}\frac{\mathrm{d}^3p'}{\sqrt{2p'^0}}\int_{\mathbb{R}^4}\mathrm{d}^4x\int_{\mathbb{R}^4}\mathrm{d}^4y\sum_{r,s,r',s'=\pm1} \\
& \Omega^\dagger\left\{d_{r'}(p')\left\{b_{s'}(k'),b_s^\dagger(k)\right\},d_r^\dagger(p)\right\}\bar{v}_{r'}(p')\gamma^\epsilon u_{s'}(k')\bar{u}_s(k)\gamma^\mu v_r(p)\Omega \\
& e^{ix_\alpha(k^\alpha+p^\alpha)}e^{-iy_\beta(k'^\beta+p'^\beta)}A_\mu(x)A_\epsilon(y) \\
= & \frac{1}{(2\pi)^6}\int_{\mathbb{R}^3}\frac{\mathrm{d}^3k}{2k^0}\int_{\mathbb{R}^3}\frac{\mathrm{d}^3p}{2p^0}\int_{\mathbb{R}^4}\mathrm{d}^4x\int_{\mathbb{R}^4}\mathrm{d}^4y\sum_{r,s=\pm1}\bar{v}_r(p)\gamma^\epsilon u_s(k)\bar{u}_s(k)\gamma^\mu v_r(p) \\
& e^{ix_\alpha(k^\alpha+p^\alpha)}e^{-iy_\beta(k^\beta+p^\beta)}A_\mu(x)A_\epsilon(y) \\
= & \frac{1}{(2\pi)^6}\int_{\mathbb{R}^3}\frac{\mathrm{d}^3k}{2k^0}\int_{\mathbb{R}^3}\frac{\mathrm{d}^3p}{2p^0}\int_{\mathbb{R}^4}\mathrm{d}^4x\int_{\mathbb{R}^4}\mathrm{d}^4y\sum_{r,s=\pm1}\sum_{a=1}^4\langle e_a|\,v_r(p)\bar{v}_r(p)\gamma^\epsilon u_s(k)\bar{u}_s(k)\gamma^\mu\,|e_a\rangle \\
& e^{ix_\alpha(k^\alpha+p^\alpha)}e^{-iy_\beta(k^\beta+p^\beta)}A_\mu(x)A_\epsilon(y) \\
= & \frac{1}{(2\pi)^6}\int_{\mathbb{R}^3}\frac{\mathrm{d}^3k}{2k^0}\int_{\mathbb{R}^3}\frac{\mathrm{d}^3p}{2p^0}\int_{\mathbb{R}^4}\mathrm{d}^4x\int_{\mathbb{R}^4}\mathrm{d}^4y\;\operatorname{tr}\left[(\not{p}-m)\gamma^\epsilon(\not{k}+m)\gamma^\mu\right] \\
& e^{ix_\alpha(k^\alpha+p^\alpha)}e^{-iy_\beta(k^\beta+p^\beta)}A_\mu(x)A_\epsilon(y) \\
= & \frac{1}{(2\pi)^2}\int_{\mathbb{R}^3}\frac{\mathrm{d}^3k}{2k^0}\int_{\mathbb{R}^3}\frac{\mathrm{d}^3p}{2p^0}\left(p_\zeta k_\omega\operatorname{tr}\left[\gamma^\zeta\gamma^\epsilon\gamma^\omega\gamma^\mu\right]-m^2\operatorname{tr}\left[\gamma^\epsilon\gamma^\mu\right]\right) \\
& \hat{A}_\mu(k+p)\hat{A}_\epsilon^*(k+p) \\
= & \frac{1}{(2\pi)^2}\int_{\mathbb{R}^3}\frac{\mathrm{d}^3k}{2k^0}\int_{\mathbb{R}^3}\frac{\mathrm{d}^3p}{2p^0}4\left(p_\zeta k_\omega(\eta^{\zeta,\epsilon}\eta^{\omega,\mu}-\eta^{\zeta,\omega}\eta^{\epsilon,\mu}+\eta^{\zeta,\mu}\eta^{\epsilon,\omega})-m^2\eta^{\epsilon,\mu}\right) \\
& \hat{A}_\mu(k+p)\hat{A}_\epsilon^*(k+p) \\
= & \frac{1}{(2\pi)^2}\int_{\mathbb{R}^3}\frac{\mathrm{d}^3k}{k^0}\int_{\mathbb{R}^3}\frac{\mathrm{d}^3p}{p^0}\Big(k^\mu\hat{A}_\mu(k+p)p^\epsilon\hat{A}_\epsilon^*(k+p)-p^\alpha k_\alpha\hat{A}^\mu(k+p)\hat{A}_\mu^*(k+p) \\
& +k^\mu\hat{A}_\mu^*(k+p)p^\epsilon\hat{A}_\epsilon(k+p)-m^2\hat{A}^\mu(k+p)\hat{A}_\mu^*(k+p)\Big) \\
\overset{k^2=p^2=m^2}{=} & \frac{1}{(2\pi)^2}\int_{\mathbb{R}^3}\frac{\mathrm{d}^3k}{k^0}\int_{\mathbb{R}^3}\frac{\mathrm{d}^3p}{p^0}\Big(k^\mu\hat{A}_\mu(k+p)p^\epsilon\hat{A}_\epsilon^*(k+p) \\
& +k^\mu\hat{A}_\mu^*(k+p)p^\epsilon\hat{A}_\epsilon(k+p)-\frac{1}{2}(k+p)^2\hat{A}^\mu(k+p)\hat{A}_\mu^*(k+p)\Big) \\
= & \frac{1}{(2\pi)^2}\int_{\mathbb{R}^3}\frac{\mathrm{d}^3k}{k^0}\int_{\mathbb{R}^3}\frac{\mathrm{d}^3p}{p^0}\Big(k^\mu\hat{A}_\mu(k+p)p^\epsilon\hat{A}_\epsilon^*(k+p) \\
& +k^\mu\hat{A}_\mu^*(k+p)p^\epsilon\hat{A}_\epsilon(k+p)+\frac{1}{2}\hat{A}_\mu^*(k+p)\widehat{\Box A}^\mu(k+p)\Big) \qquad (12)
\end{aligned}
$$

Up until this point the calculation is exact. If one wants to obtain a nice looking expression for this bound one can add the terms $k^\alpha \hat{A}_\alpha k^\beta \hat{A}^*_\beta + p^\alpha \hat{A}_\alpha p^\beta \hat{A}^*_\beta$. These terms are clearly positive. One then ends up with:

$$\begin{aligned}
\|T1\Omega\|^2 \leq \frac{1}{(2\pi)^2} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 k}{k^0} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{p^0} \Big(&(k^\mu + p^\mu)\hat{A}^*_\mu(k+p)(k^\epsilon + p^\epsilon)\hat{A}_\epsilon(k+p) \\
&+ \frac{1}{2}\hat{A}^*_\mu(k+p)\widehat{\Box A}^\mu(k+p) \Big) \\
= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 k}{k^0} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{p^0} \Big(&\widehat{\partial^\alpha A_\alpha}^*(k+p)\widehat{\partial^\beta A_\beta}(k+p) \\
&+ \frac{1}{2}\hat{A}^*_\mu(k+p)\widehat{\Box A}^\mu(k+p) \Big) \\
= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 k}{k^0} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{p^0} \Big(&\left\|\widehat{\partial^\beta A_\beta}(k+p)\right\|^2 + \frac{1}{2}\hat{A}^*_\mu(k+p)\widehat{\Box A}^\mu(k+p) \Big)
\end{aligned} \tag{13}$$

Taking $T1 : \mathbb{C} \to \mathcal{H}_+ \otimes \mathcal{H}_-$ for granted, we would like to define $T1$ on all of Fockspace. In order to do this, we introduce the "restriction of proper lifting" for the annihilation operator:

$$\forall \phi \in \mathcal{H} : \quad a\,(U\phi) \circ \tilde{U} = \tilde{U} \circ a(\phi) \tag{14}$$

Which is equivalent to the commutativity of the following diagram.

$$\begin{array}{ccc}
\mathcal{F} & \xrightarrow{\tilde{U}^A} & \mathcal{F} \\
\uparrow a & & \uparrow a \\
\mathcal{H} \otimes \mathcal{F} & \xrightarrow{U^A \otimes \tilde{U}^A} & \mathcal{H} \otimes \mathcal{F}
\end{array} \tag{15}$$

The respective condition for the creation operator, which can easily be derived from (14) is:

$$\forall \phi \in \mathcal{H} : \quad a^*\left(U^A\phi\right) \circ \tilde{U}^A = \tilde{U}^A \circ a^*(\phi) \tag{16}$$

Expanding U^A and $\tilde{U}^A$ in a powerseries, one obtains the following commutation relations for the coefficients of said expansion:

$$U^A = \mathbb{1}_{\mathcal{H}} + \sum_{l=1}^{\infty} \tag{17}$$