I begin with a collection of definitions and formulas useful in this setting.

1. basic definitions

Definition 1.1. *Throuought this document the letters* A, G *and* F *with or without indices represent four-potentials, elements of* $C_c^\infty(\mathbb{R}^4, \mathbb{R}^4)$*. Furthermore we assume that for some fixed compact* $K \subseteq \mathbb{R}^4$ *all the appearing four-potentials are supported in* K*.*

Definition 1.2. *Likewise the greek letter* Σ *with or without indices and with or without (multiple)* $'$*s attached to it represent spacelike hypersurfaces of Minkowski spacetime.*

Definition 1.3. *Furthermore*

$$\forall \Sigma : \mathcal{H}_\Sigma := L^2(\Sigma, \mathbb{C}^4, i_\gamma(d^4x)). \tag{1}$$

Definition 1.4. *We denote the one-particle Dirac time evolution operator by*

$$U^A_{\Sigma,\Sigma'} : \mathcal{H}_\Sigma \to \mathcal{H}_{\Sigma'}. \tag{2}$$

Definition 1.5. *For initial polarization* $V_{\Sigma_0} \subseteq \mathcal{H}_{\Sigma_0}$ *with* Σ_0 *before* K *and* $\Phi_{\Sigma_0} : \ell_2 \to \mathcal{H}_{\Sigma_0}$ *with range*$\Phi_{\Sigma_0} = V_{\Sigma_0}$*, we define*

$$\forall \Sigma, \forall A : C_\Sigma(A) := \{V^A_\Sigma \mid V^A_\Sigma \approx U^A_{\Sigma,\Sigma_0} V_{\Sigma_0}\}. \tag{3}$$

It turns out (ref) that $C_\Sigma(A)$ *only depends on the projection* $\Sigma \ni x \mapsto x_\alpha A^\alpha(x)$*, furthermore we define*

$$\mathcal{F}^A_\Sigma := \mathcal{F}(U^A_{\Sigma,\Sigma_0} \Phi_{\Sigma_0}). \tag{4}$$

Definition 1.6. *We call* $\tilde{U}^A_{\Sigma',\Sigma} : \mathcal{F}^A_\Sigma \to \mathcal{F}^{A'}_{\Sigma'}$ *unitary such that* $\forall A, A', \Sigma, \Sigma', \Sigma''$*:*

$$\tilde{U}^A_{\Sigma'',\Sigma'} \tilde{U}^A_{\Sigma',\Sigma} = \tilde{U}^A_{\Sigma'',\Sigma} \tag{func}$$

$$germ_{Vol\Sigma',\Sigma}(A) = germ_{Vol\Sigma',\Sigma}(A') \Rightarrow \tilde{U}^A_{\Sigma',\Sigma} = \tilde{U}^{A'}_{\Sigma',\Sigma} \tag{loc}$$

$$\tag{reg}$$

a lift of $U^A_{\Sigma',\Sigma}$*. Here the germ of two functions* A *and* A' *are equal iff*

$$\exists U \supseteq \Sigma, U open : A|_U = A'|_U . \tag{5}$$

Definition 1.7. *For* Σ_{in} *before* K *and* Σ_{out} *after* K *we define*

$${}_A\tilde{S}_{A+F} := \tilde{U}^A_{\Sigma_{in},\Sigma_{out}} \tilde{U}^{A+F}_{\Sigma_{out},\Sigma_{in}} : \mathcal{F}^A_{\Sigma_{in}} \circlearrowleft \tag{6}$$

Definition 1.8. *We denote for a fixed* $U^A_{\Sigma_{in},\Sigma_0}\Phi_{\Sigma_0} \sim \Psi_{in} : \ell^2 \to \mathcal{H}_{in}$ *by*

$$\mathbb{1}_{in} : \mathcal{H}_{in} \circlearrowleft, \tag{7}$$

the identity on that space and denote by

$$\overline{\mathbb{1}}_{in} : \mathcal{F}(\Psi_{in}) \to \mathcal{F}(U^A_{\Sigma_{in},\Sigma_0}\Phi_{\Sigma_0}) \tag{8}$$

its lift.

Definition 1.9. *We also introduce the projector notation:*

$${}_AS_{F--} := \Psi^*_{in}\ {}_AS_F \Psi_{in} : \ell^2 \circlearrowleft . \tag{9}$$

Definition 1.10. *and the only (partial) lift which can naturally be written down*

$$
{}_A\overline{S}_F := \mathcal{L}_{{}_AS_F}\mathcal{R}_{{}_FS_A}\frac{1}{\sqrt{\det {}_AS_{F--}\ {}_FS_{A--}}} : \mathcal{F}(\Psi_{in}) \circlearrowleft, \tag{10}
$$

where the last factor is just for normalisation i.e. to make this operator unitary. Please note that this is not a lift in the sense of Definition 1.6 since it may not fulfill any of the conditions.

Definition 1.11. *We can make a connection between the proper lift and (10) by*

$$
{}_A\hat{S}_F := \mathbb{1}^*_{in}\, {}_A\tilde{S}_F \mathbb{1}_{in} : \mathcal{F}(\Psi_{in}) \circlearrowleft . \tag{11}
$$

Now $\overline{S}$ and $\hat{S}$ agree up to a phase (ref), so we define

$$
{}_Az_F \circ {}_A\overline{S}_F := {}_A\hat{S}_F. \tag{12}
$$

There is yet another phase which characterizes the deficiency of (10), namely

Definition 1.12.

$$
\forall A, B, C : \Gamma^{-1}_{A,B,C} \circ \mathbb{1} = {}_A\overline{S}_B\ {}_B\overline{S}_C\ {}_C\overline{S}_A. \tag{13}
$$

We also introduce a notation for general complex numbers

Definition 1.13.

$$
\forall z \in \mathbb{C}\backslash\{0\} : arg(z) := \frac{z}{|z|}. \tag{14}
$$

We introduce the Greens functions of the Dirac equation.

Definition 1.14. *For $x \in \mathbb{R}^4$ we define*

$$
\Delta^{\pm}(x) : = \frac{-1}{(2\pi)^4}\int_{\mathbb{R}^4 \pm i\varepsilon e_0} \frac{\not{p} + m}{p^2 - m^2} e^{-ipx} d^4p \tag{15}
$$

$$
= \pm \frac{i\not{\partial} + m}{2\pi}\Theta(\pm x^0)\left[\delta(x^2) - \Theta(x^2)\frac{m}{2\sqrt{x^2}}J_1(m\sqrt{x^2})\right]. \tag{16}
$$

This function is the retarded(advanced) Greens function of the Dirac equation. (for a proof see e.g. Scharf)
The difference between the two Greens functions is denoted by $\Delta^0 = \Delta^+ - \Delta^-$.

Definition and Lemma 1.1. *Furthermore we define components of the single particle time evolution operator. For $\Psi \in C^\infty(\mathbb{R}^4, \mathbb{C}^4)$, define*

$$
L^{\pm,0}_A\Psi := \Delta^{\pm,0} * (\not{A}\Psi), \tag{17}
$$

$$
\Omega^{\pm}_A := \sum_{k=0}^{\infty} {L^{\pm}_A}^k = \left(1 - L^{\pm}_A\right)^{-1} \tag{18}
$$

$$
S_A := \left(\Omega^-_A\right)^{-1}\Omega^+_A = (1 - L^-_A)\sum_{k=0}^{\infty} {L^+_A}^k = 1 + L^0_A\Omega^+_A \tag{19}
$$

$$
S^{-1}_A = 1 - L^0_A\Omega^-_A. \tag{20}
$$

2. useful formulas

Lemma 1. *It is true that*

$$\forall F \prec G : {}_A\tilde{S}_{A+F+G} = {}_A\tilde{S}_{A+G}\ {}_A\tilde{S}_{A+F} \qquad \text{(temporal separation)}$$

holds.

Lemma 2. *It is true that*

$$\forall F \prec G : {}_{A+G}z_{A+F+G} = {}_A z_{A+F} \tag{21}$$

holds.

Lemma 3. *There are more ways to conveniently express $\Gamma_{A,B,C}$ for all A, B and C, namely*

$$\Gamma_{A,B,C} = {}_A z_B\ \ {}_B z_C\ \ {}_C z_A, \tag{22}$$

and

$$\Gamma^{-1}_{A,B,C} = argdet({}_A S_{B--}\ \ {}_B S_{C--}\ \ {}_C S_{A--}). \tag{23}$$

Furthermore it is true that

$$\Gamma_{A,B,C} = \Gamma_{B,C,A} = \Gamma^{-1}_{C,B,A} \tag{24}$$

holds. Furthermore, the tetrahedron rule holds for all A, B, C and D

$$\Gamma_{B,C,D} = \Gamma_{A,C,D}\Gamma_{B,A,D}\Gamma_{B,C,A}. \tag{25}$$

Lemma 4. *For the case $F \prec G$ one can find a relation between z and Γ involving just one instance of each object:*

$$\partial_F\partial_G \ln {}_A z_{A+F+G} = -\partial_F\partial_G \ln \Gamma_{A,A+G,A+F+G} \tag{26}$$

Finally we present a compact formula that connects derivatives of the current to Γ:

Lemma 5. *In the case $F \prec G$,*

$$\partial_F j_{A+F}(G) = i\partial_F\partial_G \ln \frac{\Gamma_{A,A+F,A+F+G}}{\Gamma_{A,A+G,A+F+G}} \tag{27}$$

holds.

Lemma 5 makes it easy to see that the derivative of the current is antisymmetric with respect to F and G.
There is another simplification of the derivative of the current that follows from the symmetries of Γ.

Lemma 6. *For $F \prec G$ we can simplify the result of lemma 5 to*

$$\partial_F j_{A+F}(G) = -2i\partial_F\partial_G \ln \Gamma_{A,A+G,A+F} = 2i\partial_F\partial_G \ln {}_A z_{A+F+G}. \tag{28}$$

Theorem 2.1. *For $F \prec G$, the derivative of the current can more explicitly be expressed as*

$$\partial_F j_{A+F}(G) = -2\Im \operatorname{tr}\left[\left(\partial_G\ {}_A S_{A+G}\right)_{-+}\left(\partial_F\ {}_A S_{A+F}\right)_{+-}\right]. \tag{29}$$

Markus: todo: understand better why the off-diagonal parts plus all of their derivatives are Hilbert-Schmidt operators

Next we want an explicit formula for the nth derivative of the current, since this will be handy in a perturbative expansion of the scattering operator. For this we need a formula for the nth derivative of S_A.

Lemma 7. *Using definition/lemma 1.1 we find for $n \in \mathbb{N}$*

$$\left(\prod_{k=1}^{n} \partial_{F_k}\right) S_{A+\sum_{k=1}^{n} F_k} = \sum_{l=1}^{n} \quad L^0_{F_l}\Omega^+_A \sum_{\sigma\in S(\{1,\ldots,n\}\setminus\{l\})} \prod_{\substack{k=1\\k\neq l}}^{n} \left(L^+_{F_{\sigma(k)}}\Omega^+_A\right) + L^0_A\Omega^+_A \sum_{\sigma\in S(\{1,\ldots,n\})} \prod_{k=1}^{n} \left(L^+_{F_{\sigma(k)}}\Omega^+_A\right). \tag{30}$$

$$\left(\prod_{k=1}^{n} \partial_{F_k}\right) S^{-1}_{A+\sum_{k=1}^{n} F_k} = -\sum_{l=1}^{n} L^0_{F_l}\Omega^-_A \sum_{\sigma\in S(\{1,\ldots,n\}\setminus\{l\})} \prod_{\substack{k=1\\k\neq l}}^{n} \left(L^-_{F_{\sigma(k)}}\Omega^-_A\right) - L^0_A\Omega^-_A \sum_{\sigma\in S(\{1,\ldots,n\})} \prod_{k=1}^{n} \left(L^-_{F_{\sigma(k)}}\Omega^-_A\right). \tag{31}$$

Using this lemma and lemma 2.1 we find the nth derivative of the current.

Theorem 2.2. *Using the abbreviations $X_a := X\setminus\{a\}$ and $X^a := X \cup \{a\}$ for any set X and element a, we find for the current*

$$\left(\prod_{k=1}^{n+1} \partial_{F_k}\right) j_{A+\sum_{k=1}^{n+1} F_k}(F_0) = -2 \sum_{\substack{B,C,D,E\subseteq\{1,\ldots,n\}\\ B\dot\cup C\dot\cup D\dot\cup E=\{1,\ldots,n\}}} \Im \operatorname{tr}$$
$$\Bigg[\Bigg\{\left(\delta_{B,\varnothing} - \sum_{l\in B} L^0_{F_l}\Omega^-_A \sum_{\sigma\in S(B_l)} \prod_{b\in B_l} \left(L^-_{F_{\sigma(b)}}\Omega^-_A\right) - L^0_A\Omega^-_A \sum_{\sigma\in S(B)} \prod_{b\in B} \left(L^-_{F_{\sigma(b)}}\Omega^-_A\right)\right)$$
$$\left(\sum_{l\in C^0} L^0_{F_l}\Omega^+_A \sum_{\sigma\in S(C^0_l)} \prod_{b\in C^0_l} \left(L^+_{F_{\sigma(b)}}\Omega^+_A\right) + L^0_A\Omega^+_A \sum_{\sigma\in S(C^0)} \prod_{b\in C^0} \left(L^+_{F_{\sigma(b)}}\Omega^+_A\right)\right)\Bigg\}_{-+}$$
$$\Bigg\{\left(\delta_{D,\varnothing} - \sum_{l\in D} L^0_{F_l}\Omega^-_A \sum_{\sigma\in S(D_l)} \prod_{b\in D_l} \left(L^-_{F_{\sigma(b)}}\Omega^-_A\right) - L^0_A\Omega^-_A \sum_{\sigma\in S(D)} \prod_{b\in D} \left(L^-_{F_{\sigma(b)}}\Omega^-_A\right)\right)$$
$$\left(\sum_{l\in E^{n+1}} L^0_{F_l}\Omega^+_A \sum_{\sigma\in S\left(E^{n+1}_l\right)} \prod_{b\in E^{n+1}_l} \left(L^+_{F_{\sigma(b)}}\Omega^+_A\right) + L^0_A\Omega^+_A \sum_{\sigma\in S(E^{n+1})} \prod_{b\in E^{n+1}} \left(L^+_{F_{\sigma(b)}}\Omega^+_A\right)\right)\Bigg\}_{+-}\Bigg]$$

Setting $A = 0$ this expression simplifies somewhat

Corollary 2.3. *Using the same abbreviations as in theorem 2.2 for $A = 0$ the derivatives of the current are given by*

$$\left(\prod_{k=1}^{n+1} \partial_{F_k}\right) j_{\sum_{k=1}^{n+1} F_k}(F_0) = -2 \sum_{\substack{B,C,D,E\subseteq\{1,\ldots,n\}\\ B\dot\cup C\dot\cup D\dot\cup E=\{1,\ldots,n\}}} \Im \operatorname{tr}$$
$$\Bigg[\Bigg\{\left(\delta_{B,\varnothing} - \sum_{l\in B} L^0_{F_l} \sum_{\sigma\in S(B_l)} \prod_{b\in B_l} L^-_{F_{\sigma(b)}}\right)\left(\sum_{l\in C^0} L^0_{F_l} \sum_{\sigma\in S(C^0_l)} \prod_{b\in C^0_l} L^+_{F_{\sigma(b)}}\right)\Bigg\}_{-+}$$

$$\left\{\left(\delta_{D,\varnothing}-\sum_{l\in D}L^0_{F_l}\sum_{\sigma\in S(D_l)}\prod_{b\in D_l}L^-_{F_{\sigma(b)}}\right)\left(\sum_{l\in E^{n+1}}L^0_{F_l}\sum_{\sigma\in S\left(E^{n+1}_l\right)}\prod_{b\in E^{n+1}_l}L^+_{F_{\sigma(b)}}\right)\right\}_{+-}\Bigg]$$

3. proofs of useful formulas

Proof of lemma 1: Let F and G be such that $F \prec G$ holds. Then choose Σ such that Σ is before $\operatorname{supp} G$ but after $\operatorname{supp} F$. Then it follows that

$$\begin{aligned}
&{}_A\tilde{S}_{A+F+G} = {}_A\tilde{S}_{A+G}\,{}_A\tilde{S}_{A+F}\\
\iff\ &\tilde{U}^A_{\Sigma_{\text{in}},\Sigma_{\text{out}}}\tilde{U}^{A+F+G}_{\Sigma_{\text{out}},\Sigma_{\text{in}}} = \tilde{U}^A_{\Sigma_{\text{in}},\Sigma_{\text{out}}}\tilde{U}^{A+G}_{\Sigma_{\text{out}},\Sigma_{\text{in}}}\tilde{U}^A_{\Sigma_{\text{in}},\Sigma_{\text{out}}}\tilde{U}^{A+F}_{\Sigma_{\text{out}},\Sigma_{\text{in}}}\\
\iff\ &\tilde{U}^{A+F+G}_{\Sigma_{\text{out}},\Sigma_{\text{in}}} = \tilde{U}^{A+G}_{\Sigma_{\text{out}},\Sigma_{\text{in}}}\tilde{U}^A_{\Sigma_{\text{in}},\Sigma_{\text{out}}}\tilde{U}^{A+F}_{\Sigma_{\text{out}},\Sigma_{\text{in}}}\\
\overset{\text{(func)}}{\iff}\ &\tilde{U}^{A+F+G}_{\Sigma_{\text{out}},\Sigma}\tilde{U}^{A+F+G}_{\Sigma,\Sigma_{\text{in}}} = \tilde{U}^{A+G}_{\Sigma_{\text{out}},\Sigma}\tilde{U}^{A+G}_{\Sigma,\Sigma_{\text{in}}}\tilde{U}^A_{\Sigma_{\text{in}},\Sigma}\tilde{U}^A_{\Sigma,\Sigma_{\text{out}}}\tilde{U}^{A+F}_{\Sigma_{\text{out}},\Sigma}\tilde{U}^{A+F}_{\Sigma,\Sigma_{\text{in}}}\\
\overset{\text{(loc)}}{\iff}\ &\tilde{U}^{A+G}_{\Sigma_{\text{out}},\Sigma}\tilde{U}^{A+F}_{\Sigma,\Sigma_{\text{in}}} = \tilde{U}^{A+G}_{\Sigma_{\text{out}},\Sigma}\tilde{U}^A_{\Sigma,\Sigma_{\text{in}}}\tilde{U}^A_{\Sigma_{\text{in}},\Sigma}\tilde{U}^A_{\Sigma,\Sigma_{\text{out}}}\tilde{U}^A_{\Sigma_{\text{out}},\Sigma}\tilde{U}^{A+F}_{\Sigma,\Sigma_{\text{in}}}\\
\iff\ &\tilde{U}^{A+G}_{\Sigma_{\text{out}},\Sigma}\tilde{U}^{A+F}_{\Sigma,\Sigma_{\text{in}}} = \tilde{U}^{A+G}_{\Sigma_{\text{out}},\Sigma}\tilde{U}^{A+F}_{\Sigma,\Sigma_{\text{in}}}.
\end{aligned}$$

□

Now we prove lemma 2. Let $F \prec G$. Using definition (12), as well as lemma 1 which holds for $\hat{S}$ as well (simply by inserting the proper identities) we compute

$$\begin{aligned}
&{}_{A+G}z_{A+F+G}\circ{}_{A+G}\overline{S}_{A+F+G} = {}_{A+G}\hat{S}_{A+F+G} = {}_{A+G}\hat{S}_A\,{}_A\hat{S}_{A+F+G}\\
&\overset{F\prec G}{=} {}_{A+G}\hat{S}_A\ \ {}_A\hat{S}_{A+G}\ \ {}_A\hat{S}_{A+F} = {}_A\hat{S}_{A+F} = {}_Az_{A+F}\circ{}_A\overline{S}_{A+F}.
\end{aligned}$$

Now by evaluation ${}_{A+F+G}\overline{S}_{A+G}\ \ {}_A\overline{S}_{A+F}$ we find the relation between the appearing phases. By a computation analogous to the one we just did for the one-particle scattering operator we see that the left-operation part of this operator is just the identity. The right-operation may still contribute a determinant; however, since $\overline{S}$ is unitary the determinant may only be a phase. Therefore we see that

$$\begin{aligned}
&{}_{A+G}z_{A+F+G}\ \ {}_{A+F}z_A = \operatorname{argdet}(({}_{A+F}S_A)_{--}({}_{A+G}S_{A+F+G})_{--})\\
&= \operatorname{argdet}(({}_{A+F}S_A)_{--}({}_{A+G}S_A\ \ {}_AS_{A+F+G})_{--}) = \operatorname{argdet}(({}_{A+F}S_A)_{--}({}_AS_{A+F})_{--})\\
&= \operatorname{argdet}(({}_{A+F}S_A)_{--}({}_{A+F}S_A)^*_{--}) = 1
\end{aligned}$$

holds. □

Now for lemma 3. Formula (23) can be seen from the definition of Γ by taking the vacuum expectation value. Formula (24) can directly be seen form the definition of Γ. We prove (22), by observing that

$$ {}_A\tilde{S}_C = {}_A\tilde{S}_B\ \ {}_B\tilde{S}_C \tag{32}$$

holds, therefore it also holds for $\hat{S}$. Inserting definitions (12) and (13) yields

$$ {}_Az_C\circ{}_A\overline{S}_C = {}_Az_B\ \ {}_Bz_C\circ{}_A\overline{S}_B\ \ {}_B\overline{S}_C \tag{33}$$

and

$$ {}_Az_C\ \ {}_Bz_A\ \ {}_Cz_B\circ\mathbb{1} = {}_A\overline{S}_B\ \ {}_B\overline{S}_C\ \ {}_C\overline{S}_A = \Gamma^{-1}_{A,B,C}. \tag{34}$$

Rearranging yields (22). For the tetrahedron rule we simply insert (22) into the right hand side and get

$$\begin{aligned}&\Gamma_{A,C,D}\Gamma_{B,A,D}\Gamma_{B,C,A} = {}_A z_C \;\; {}_C z_D \;\; {}_D z_A \;\; {}_B z_A \;\; {}_A z_D \;\; {}_D z_B \;\; {}_B z_C \;\; {}_C z_A \;\; {}_A z_B \\ &= {}_C z_D \;\; {}_D z_B \;\; {}_B z_C = \Gamma_{B,C,D}.\end{aligned}$$

□

Now to prove lemma 4. Let again $F \prec G$ be true. By adding terms which vanish after splitting products into sums in the logarithm and application of derivatives we obtain

$$\partial_F \partial_G \ln {}_A z_{A+F+G} = -\partial_F \partial_G \ln {}_{A+F+G} z_A \;\; {}_A z_{A+G} \;\; {}_A z_{A+F}.$$

Modifying the last factor by (21) yields

$$\partial_F \partial_G \ln {}_A z_{A+F+G} = -\partial_F \partial_G \ln {}_{A+F+G} z_A \;\; {}_A z_{A+G} \;\; {}_{A+G} z_{A+F+G} = -\partial_F \partial_G \ln \Gamma_{A,A+G,A+F+G}.$$

□

We come to the proof of lemma 5. Let $F \prec G$ be true. we start with the definition of the current:

$$j_A(G) = i\partial_G \left\langle \bigwedge \Phi, {}_A\tilde{S}_{A+G} \bigwedge \Phi \right\rangle$$

Now we take the derivative of this expression and insert the definition of z

$$\partial_F j_{A+F}(G) = i\partial_F \partial_G \; {}_{A+F} z_{A+F+G} \left\langle \bigwedge \Phi, {}_{A+F}\overline{S}_{A+F+G} \bigwedge \Phi \right\rangle.$$

As the expression which we take derivatives of is equal to 1 at $G = 0$ and the linearisation of the logarithm around 1 is the identity we can safely insert a logarithm, yielding

$$\partial_F j_{A+F}(G) = i\partial_F \partial_G \ln {}_{A+F} z_{A+F+G} \left\langle \bigwedge \Phi, {}_{A+F}\overline{S}_{A+F+G} \bigwedge \Phi \right\rangle.$$

This will greatly simplify the upcoming calculations. Next we insert the relation between $\Gamma_{A+F,A+F+G,A}$ and z using ${}_A z_{A+F}$ with respect to G vanishes, (22) giving

$$\partial_F j_{A+F}(G) = i\partial_F \partial_G \ln {}_A z_{A+F+G} \Gamma_{A+F,A+F+G,A} \left\langle \bigwedge \Phi, {}_{A+F}\overline{S}_{A+F+G} \bigwedge \Phi \right\rangle.$$

Now we insert the identity twice inside the scalar product

$$\begin{aligned}&\partial_F j_{A+F}(G) = i\partial_F \partial_G \ln {}_A z_{A+F+G} \Gamma_{A+F,A+F+G,A} \\ &\left\langle \bigwedge \Phi, {}_{A+F}\overline{S}_A \;\; {}_A\overline{S}_{A+F} \;\; {}_{A+F}\overline{S}_{A+F+G} \;\; {}_{A+F+G}\overline{S}_A \;\; {}_A\overline{S}_{A+F+G} \bigwedge \Phi \right\rangle. \qquad (35)\end{aligned}$$

The central three occurrence of $\overline{S}$ give $\Gamma^{-1}_{A,A+F,A+F+G}$ cancelling exactly the gamma factor in front after cyclic permutation. As a next step we evaluate the scalar product. Since the operators $\overline{S}$ are unitary this yields the argument of a determinant:

$$\begin{aligned}&\left\langle \bigwedge \Phi, {}_{A+F}\overline{S}_A \;\; {}_A\overline{S}_{A+F+G} \bigwedge \Phi \right\rangle \\ &= \left\langle \bigwedge \Phi, \mathcal{L}_{{}_{A+F}S_A} \mathcal{R}_{({}_A S_{A+F})_{--}} \mathcal{L}_{{}_A S_{A+F+G}} \mathcal{R}_{({}_{A+F+G} S_A)_{--}} \bigwedge \Phi \right\rangle \frac{1}{N} \\ &= \left\langle \bigwedge \Phi, \mathcal{L}_{{}_{A+F}S_A \;\; {}_A S_{A+F+G}} \mathcal{R}_{({}_{A+F+G} S_A)_{--} \;\; ({}_A S_{A+F})_{--}} \bigwedge \Phi \right\rangle \frac{1}{N} \\ &= \left\langle \bigwedge \Phi, \mathcal{L}_{{}_{A+F}S_{A+F+G}} \bigwedge \Phi ({}_{A+F+G} S_A)_{--} \;\; ({}_A S_{A+F})_{--} \right\rangle \frac{1}{N} \\ &= \operatorname{argdet}(({}_{A+F} S_{A+F+G})_{--} ({}_{A+F+G} S_A)_{--} \;\; ({}_A S_{A+F})_{--}),\end{aligned}$$

which is, by (23), given by

$$\left\langle \bigwedge \Phi, {}_{A+F}\overline{S}_A \ {}_A\overline{S}_{A+F+G} \bigwedge \Phi \right\rangle = \Gamma_{A,A+F,A+F+G}. \tag{36}$$

Taking all of this together yields

$$\partial_F j_{A+F}(G) = i\partial_F\partial_G \ln {}_A z_{A+F+G}\Gamma_{A,A+F,A+F+G}.$$

Now we replace the appearance of z using lemma 4, giving

$$\partial_F j_{A+F}(G) = i\partial_F\partial_G \ln \frac{\Gamma_{A,A+F,A+F+G}}{\Gamma_{A,A+G,A+F+G}}. \tag{37}$$

□

Now for lemma 6. We will show the first equality, the second follows by lemma 4. For this proof we abbreviate, for variables $a, b, c, \overline{a}, \overline{b}, \overline{b} \in \mathbb{R}$, $x := ((a, \overline{a}), (b, \overline{b}), (c, \overline{c}))$

$$f(x) := f((a, \overline{a}), (b, \overline{b}), (c, \overline{c})) := \ln \Gamma_{A+a\cdot F+\overline{a}\cdot G, A+b\cdot F+\overline{b}\cdot G, A+c\cdot F+\overline{c}\cdot G}. \tag{38}$$

Now we are interested in

$$\begin{aligned} &\partial_F\partial_G \ln \Gamma_{A,A+F,A+F+G} = \partial_\varepsilon\partial_\delta f((0,0),(\varepsilon,0),(\varepsilon,\delta)) \\ &= (\partial_b + \partial_c)\partial_{\overline{c}} \left. f(x)\right|_{x=0} = \partial_b\partial_{\overline{c}} \left. f(x)\right|_{x=0}. \end{aligned} \tag{39}$$

The last equality holds due to $f((a, \overline{a}), (b, \overline{b}), (c, \overline{c})) = -f((b, \overline{b}), (a, \overline{a}), (c, \overline{c}))$, which implies

$$\partial_c\partial_{\overline{c}} f(x)|_{x=0} = -\partial_c\partial_{\overline{c}} f(x)|_{x=0} = 0. \tag{40}$$

For the very same reason we conclude

$$\begin{aligned} &-\partial_F\partial_G \ln \Gamma_{A,A+G,A+F+G} = -\partial_\varepsilon\partial_\delta f((0,0),(0,\delta),(\varepsilon,\delta)) \\ &= -\partial_c(\partial_{\overline{b}} + \partial_{\overline{c}}) f(x)|_{x=0} = -\partial_c\partial_{\overline{b}} f(x)|_{x=0} = \partial_b\partial_{\overline{c}} f(x)|_{x=0}, \end{aligned}$$

where the last equality follows again from the antisymmetry of f. We conclude

$$-\partial_F\partial_G \ln \Gamma_{A,A+G,A+F+G} = \partial_F\partial_G \ln \Gamma_{A,A+F,A+F+G}. \tag{41}$$

and thereby the lemma. □

We will now continue with the proof of theorem 2.1. The major part of this proof is contained in the following auxiliary

Lemma 8. *For operators*

$$\begin{aligned} A &: \mathbb{R} \to (\mathcal{H} \to \mathcal{H}) \\ B &: \mathbb{R}^2 \to (\mathcal{H} \to \mathcal{H}), \end{aligned}$$

such that $A(\varepsilon), B(\varepsilon, \delta) \in I_1$, $A^* = A, B(\varepsilon, 0) = B^*(\varepsilon, 0)$ *and* $A(0) = 0 = B(\varepsilon, 0)$ *for all* $\varepsilon, \delta \in \mathbb{R}$ *hold,*

$$\partial_\delta\partial_\varepsilon \ln \mathit{argdet}[1 + A(\varepsilon) + B(\varepsilon, \delta))] = i\Im \operatorname{tr} D_1 D_2 B(x)|_{x=0} \tag{42}$$

is true.

We mention a corollary which we will not prove. Since most of the time we work with the arg of complex numbers it is worth noting that

Corollary 3.1.

$$\forall z : \mathbb{R} \to \mathbb{C}\backslash\{0\} : \left(\frac{z}{|z|}\right)' = i\frac{z}{|z|}\Im\frac{z'}{z} \tag{43}$$

holds.
Proof of the lemma 8 We use corollary 3.1 to find

$$\begin{aligned}\partial_\varepsilon \ln \operatorname{argdet}\left[1+A(\varepsilon)+B(\varepsilon,\delta)\right)] &= \frac{\partial_\varepsilon \operatorname{argdet}[1+A(\varepsilon)+B(\varepsilon,\delta)]}{\operatorname{argdet}[1+A(0)+B(0,\delta)}\\ &= i\frac{\operatorname{argdet}[1+B(0,\delta)]}{\operatorname{argdet}[1+B(0,\delta)]}\Im\left[\frac{\partial_\varepsilon \det[1+A(\varepsilon)+B(\varepsilon,\delta)]}{\det[1+B(0,\delta)]}\right]\\ &= i\Im\left[\frac{\partial_\varepsilon \det[1+A(\varepsilon)+B(\varepsilon,\delta)]}{\det[1+B(0,\delta)]}\right].\end{aligned}$$

Now we use that the linearisation of the determinant around the identity equal to the trace is. This yields

$$\partial_\varepsilon \ln \operatorname{argdet}\left[1+A(\varepsilon)+B(\varepsilon,\delta)\right)] = i\Im\left[\frac{\operatorname{tr}[\partial_\varepsilon A(\varepsilon)+\partial_\varepsilon B(\varepsilon,\delta)]}{\det[1+B(0,\delta)]}\right]. \tag{44}$$

Inserting the second derivative simplifies the expression after one recognizes that the second summand inside the imaginary part is real, since its a product of derivatives of selfadjoint traceclass operators. The corresponding calculation is

$$\begin{aligned}&\partial_\delta\partial_\varepsilon \ln \operatorname{argdet}\left[1+A(\varepsilon)+B(\varepsilon,\delta)\right)]\\ &= i\Im\left[\operatorname{tr}\partial_\delta\partial_\varepsilon B(\varepsilon,\delta) - \frac{\operatorname{tr}[\partial_\varepsilon A(\varepsilon)+\partial_\varepsilon B(\varepsilon,0)]}{\det(1+B(0,0))^2}\operatorname{tr}[\partial_\delta B(0,\delta)]\right]\\ &= i\Im\left[\operatorname{tr}\partial_\delta\partial_\varepsilon B(\varepsilon,\delta) - \operatorname{tr}[\partial_\varepsilon A(\varepsilon)]\operatorname{tr}[\partial_\delta B(0,\delta)]\right]\\ &= i\Im\left[\operatorname{tr}\partial_\delta\partial_\varepsilon B(\varepsilon,\delta)\right],\end{aligned}$$

where we used that $B(0,\delta)$ is selfadjoint and $B(\varepsilon,0)=0$. This concludes the proof of the lemma.
So we start out with the most recent result about $\partial_F j_{A+F}(G)$, use lemma 3 and manipulate the appearing projections to bring it in a form that more explicitly has a determinant:

$$\begin{aligned}\partial_F\partial_G \ln\Gamma_{A,A+G,A+F+G} &= \partial_F\partial_G \ln \operatorname{argdet}\left({}_AS_{A+G}\right)_{--}\left({}_{A+G}S_{A+F}\right)_{--}\left({}_{A+F}S_A\right)_{--}\\ &= \partial_F\partial_G \ln \operatorname{argdet}\left[\left({}_AS_{A+F}\right)_{--}\left({}_{A+F}S_A\right)_{--} - \left({}_AS_{A+G}\right)_{-+}\left({}_{A+G}S_{A+F}\right)_{+-}\left({}_{A+F}S_A\right)_{--}\right]\\ &= \partial_F\partial_G \ln \operatorname{argdet}\left[\mathbb{1}_{--} - \left({}_AS_{A+F}\right)_{-+}\left({}_{A+F}S_A\right)_{+-} - \left({}_AS_{A+G}\right)_{-+}\left({}_{A+G}S_{A+F}\right)_{+-}\left({}_{A+F}S_A\right)_{--}\right].\end{aligned}$$

As we now take the derivative of a trace-class perturbation of the identity we can see that 1. this expression is well-defined, since the off diagonal components of the scattering matrix and its derivatives are Hilbert-Schmidt, and 2. we can use the lemma we just proved. This results in

$$\begin{aligned}\partial_F\partial_G \ln\Gamma_{A,A+G,A+F+G} &= -i\partial_F\partial_G\Im\operatorname{tr}\left({}_AS_{A+G}\right)_{-+}\left({}_{A+G}S_{A+F}\right)_{+-}\left({}_{A+F}S_A\right)_{--}\\ &= -i\Im\operatorname{tr}\left(\partial_G\, {}_AS_{A+G}\right)_{-+}\left(\partial_F\, {}_AS_{A+F}\right)_{+-},\end{aligned}$$

where the last equality follows by acknowledging that terms vanish if they contain a factor of $\mathbb{1}_{+-}$. The theorem follows by inserting lemma 6. $\square$

Proof of lemma 7. First we plug in the definition of S and use the product rule, resulting in

$$\left(\prod_{k=1}^{n}\partial_{F_k}\right)S_{A+\sum_{k=1}^{n}F_k}=\sum_{l=1}^{n}L^0_{F_l}\left(\prod_{\substack{k=1\\k\neq l}}^{n}\partial_{F_k}\right)\Omega^+_{A+\sum_{\substack{k=1\\k\neq l}}^{n}F_k} \tag{45}$$

$$+L^0_A\left(\prod_{k=1}^{n}\partial_{F_k}\right)\Omega^+_{A+\sum_{k=1}^{n}F_k}, \tag{46}$$

Now since $\Omega^\pm_A=(1-L^\pm_A)^{-1}$ holds we have $\partial_F\Omega^\pm_{A+F}=\Omega^\pm_A L^\pm_F\Omega^\pm_A$. Applying this n times results in a sum over all permutations of the pertubative fields we are taking derivatives with respect to. Yielding

$$\left(\prod_{k=1}^{n}\partial_{F_k}\right)S_{A+\sum_{k=1}^{n}F_k}=\sum_{l=1}^{n}L^0_{F_l}\Omega^+_A\sum_{\sigma\in S(\{1,\dots,n\}\setminus\{l\})}\prod_{\substack{k=1\\k\neq l}}^{n}\left(L^+_{F_{\sigma(k)}}\Omega^+_A\right)$$

$$+L^0_A\Omega^+_A\sum_{\sigma\in S(\{1,\dots,n\})}\prod_{k=1}^{n}\left(L^+_{F_{\sigma(k)}}\Omega^+_A\right). \tag{47}$$

Analogously for S^{-1} we find

$$\left(\prod_{k=1}^{n}\partial_{F_k}\right)S^{-1}_{A+\sum_{k=1}^{n}F_k}=-\sum_{l=1}^{n}L^0_{F_l}\Omega^-_A\sum_{\sigma\in S(\{1,\dots,n\}\setminus\{l\})}\prod_{\substack{k=1\\k\neq l}}^{n}\left(L^-_{F_{\sigma(k)}}\Omega^-_A\right)$$

$$-L^0_A\Omega^-_A\sum_{\sigma\in S(\{1,\dots,n\})}\prod_{k=1}^{n}\left(L^-_{F_{\sigma(k)}}\Omega^-_A\right). \tag{48}$$

□

Proof of theorem 2.2 : Lemma 7 and theorem 2.1 already provides most of the proof of this theorem. We use the abbreviations

$$\tilde{A}:=A+\sum_{k=1}^{n}F_k$$

$$\tilde{A}^0:=A+\sum_{k=0}^{n}F_k$$

$$\tilde{A}^{n+1}:=A+\sum_{k=1}^{n+1}F_k.$$

Starting with this theorem we have for any $n\in\mathbb{N}$

$$\left(\prod_{k=1}^{n+1}\partial_{F_k}\right)j_{\tilde{A}^{n+1}}(F_0)=-2\left(\prod_{k=1}^{n}\partial_{F_k}\right)\Im\operatorname{tr}\left[\left(\partial_{F_0}\,{}_{\tilde{A}}S_{\tilde{A}^0}\right)_{-+}\left(\partial_{F_{n+1}}\,{}_{\tilde{A}}S_{\tilde{A}^{n+1}}\right)_{+-}\right]. \tag{49}$$

Using definition/lemma 1.1 we can expand the right hand side a bit further to

$$=-2\left(\prod_{k=1}^{n}\partial_{F_k}\right)\Im\operatorname{tr}\left[\left(\left(1-L^0_{\tilde{A}}\Omega^-_{\tilde{A}}\right)\partial_{F_0}\left(1+L^0_{\tilde{A}^{n+1}}\Omega^+_{\tilde{A}^{n+1}}\right)\right)_{-+}\right.$$

$$\left.\left(\left(1-L^0_{\tilde{A}}\Omega^-_{\tilde{A}}\right)\partial_{F_{n+1}}\left(1+L^0_{\tilde{A}^0}\Omega^+_{\tilde{A}^0}\right)\right)_{+-}\right].$$

Now we need to distribute n derivatives over four terms. The possibilities of doing so are found by the analogous problem of distributing n labelled balls over four labelled boxes. The product rule for non commuting products tells us that we have to sum over these possibilities. We realise this sum by a sum over all possible distributions of the elements of $\{1,\ldots,n\}$ into four disjoint sets B, C, D, E whose union is again $\{1,\ldots,n\}$. This yields for the right hand side

$$= -2 \sum_{\substack{B,C,D,E\subseteq\{1,\ldots,n\}\\ B\dot\cup C\dot\cup D\dot\cup E=\{1,\ldots,n\}}} \Im \operatorname{tr}\left[\left(\left(\prod_{l\in B}\partial_{F_l}\right)\left(1-L^0_{\tilde A}\Omega^-_{\tilde A}\right)\left(\prod_{l\in C}\partial_{F_l}\right)\partial_{F_0}\left(1+L^0_{\tilde A^{n+1}}\Omega^+_{\tilde A^{n+1}}\right)\right)_{-+}\right.$$
$$\left.\left(\left(\prod_{l\in D}\partial_{F_l}\right)\left(1-L^0_{\tilde A}\Omega^-_{\tilde A}\right)\left(\prod_{l\in E}\partial_{F_l}\right)\partial_{F_{n+1}}\left(1+L^0_{\tilde A^0}\Omega^+_{\tilde A^0}\right)\right)_{+-}\right].$$

Applying the derivatives to the brackets and using once more the product rule yields the theorem. In doing so one should not forget that the L operators are linear, therefore yield zero when linearised twice with respect to different fields. $\square$