What we know about $T1$

$T1$ is an unbounded linear map from the Fock space of QED onto itself. To be more specific, let $\mathcal{H}_+$ denote the Hilbert space of electron wavefunctions and $\mathcal{H}_-$ denote the Hilbert space of positron wavefunctions. Then

$$T1 : \mathcal{F}(\mathcal{H}_+) \otimes \mathcal{F}(\mathcal{H}_-) \to \mathcal{F}(\mathcal{H}_+) \otimes \mathcal{F}(\mathcal{H}_-), \tag{1}$$

but by far not all matrix elements are nonzero, expressed in the fixed particle subspaces we already see more structure:

$$T1 : \mathcal{H}_+^{\otimes n} \otimes \mathcal{H}_-^{\otimes p} \to \left(\mathcal{H}_+^{\otimes n-1} \otimes \mathcal{H}_-^{\otimes p-1}\right) \oplus \left(\mathcal{H}_+^{\otimes n} \otimes \mathcal{H}_-^{\otimes p}\right) \oplus \left(\mathcal{H}_+^{\otimes n+1} \otimes \mathcal{H}_-^{\otimes p+1}\right). \tag{2}$$

So restricted to the vacuum sector this simplifies, the part from the vacuum sector is set to zero, because the vacuum polarization current should be zero.

$$T1 : \mathbb{C} \to (\mathcal{H}_+ \otimes \mathcal{H}_-) \tag{3}$$

We will construct the finite operator norm of $T1$ restricted to an arbitrary fixed particle sector, beginning with the vacuum sector. Let the vector potential of the electromagnetic field A be

$$A \in C_c^\infty\left(\mathbb{R}^4\right). \tag{4}$$

Then one obtains in physicists notation:

$$T1\Omega = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 k}{\sqrt{2k^0}} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{\sqrt{2p^0}} \int_{\mathbb{R}^4} \mathrm{d}^4 x \sum_{r,s=\pm 1} b_s^\dagger(k) d_r^\dagger(p) \bar{u_s}(k) \gamma^\mu v_r(p) \Omega e^{i x_\alpha (k^\alpha + p^\alpha)} A_\mu(x), \tag{5}$$

Where the γ^μ are the dirac matrices fulfilling the anticommutation relation

$$\{\gamma^\alpha \gamma^\beta\} = 2\eta^{\alpha\beta} \tag{6}$$

The u, v are given by:

$$u_s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi_s \\ \sqrt{p\cdot\bar{\sigma}}\xi_s \end{pmatrix} \qquad v_s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi_{-s} \\ -\sqrt{p\cdot\bar{\sigma}}\xi_{-s} \end{pmatrix}, \tag{7}$$

with

$$\sigma = (\mathbb{1}, \vec{\sigma}), \qquad \bar{\sigma} = (\mathbb{1}, -\vec{\sigma}) \tag{8}$$

$$\xi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \xi_{-1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{9}$$

Where $\vec{\sigma}$ is the vector containing the Pauli matrices. The creation and annihilation operator fulfill the following identity:

$$\{b_s(p), b_r^\dagger(p')\} = \delta^3(p - p') = \{d_s(p), d_r^\dagger(p')\} \tag{10}$$

Useful identities for sums of u and v are:

$$\sum_s u_s(p)\bar{u}_s(p) = \not{p} + m \qquad \sum_s v_s(p)\bar{v}_s(p) = \not{p} - m \tag{11}$$

For the norm of the image of the vacuum we therefore arrive at:

$$\begin{aligned}
\|T1\Omega\|^2 = \frac{1}{(2\pi)^6} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3k}{\sqrt{2k^0}} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3p}{\sqrt{2p^0}} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3k'}{\sqrt{2k'^0}} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3p'}{\sqrt{2p'^0}} \int_{\mathbb{R}^4} \mathrm{d}^4x \int_{\mathbb{R}^4} \mathrm{d}^4y \sum_{r,s,r',s'=\pm 1} \\
\Omega^\dagger d_{r'}(p') b_{s'}(k') \bar{v}_{r'}(p') \gamma^\epsilon u_{s'}(k') b_s^\dagger(k) d_r^\dagger(p) \bar{u}_s(k) \gamma^\mu v_r(p) \Omega \\
e^{ix_\alpha(k^\alpha+p^\alpha)} e^{-iy_\beta(k'^\beta+p'^\beta)} A_\mu(x) A_\epsilon(y) \\
= \frac{1}{(2\pi)^6} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3k}{\sqrt{2k^0}} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3p}{\sqrt{2p^0}} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3k'}{\sqrt{2k'^0}} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3p'}{\sqrt{2p'^0}} \int_{\mathbb{R}^4} \mathrm{d}^4x \int_{\mathbb{R}^4} \mathrm{d}^4y \sum_{r,s,r',s'=\pm 1} \\
\Omega^\dagger \left\{ d_{r'}(p') \left\{ b_{s'}(k'), b_s^\dagger(k) \right\}, d_r^\dagger(p) \right\} \bar{v}_{r'}(p') \gamma^\epsilon u_{s'}(k') \bar{u}_s(k) \gamma^\mu v_r(p) \Omega \\
e^{ix_\alpha(k^\alpha+p^\alpha)} e^{-iy_\beta(k'^\beta+p'^\beta)} A_\mu(x) A_\epsilon(y) \\
= \frac{1}{(2\pi)^6} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3k}{2k^0} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3p}{2p^0} \int_{\mathbb{R}^4} \mathrm{d}^4x \int_{\mathbb{R}^4} \mathrm{d}^4y \sum_{r,s=\pm 1} \bar{v}_r(p) \gamma^\epsilon u_s(k) \bar{u}_s(k) \gamma^\mu v_r(p) \\
e^{ix_\alpha(k^\alpha+p^\alpha)} e^{-iy_\beta(k^\beta+p^\beta)} A_\mu(x) A_\epsilon(y) \\
= \frac{1}{(2\pi)^6} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3k}{2k^0} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3p}{2p^0} \int_{\mathbb{R}^4} \mathrm{d}^4x \int_{\mathbb{R}^4} \mathrm{d}^4y \sum_{r,s=\pm 1} \sum_{a=1}^{4} \langle e_a | v_r(p) \bar{v}_r(p) \gamma^\epsilon u_s(k) \bar{u}_s(k) \gamma^\mu | e_a \rangle \\
e^{ix_\alpha(k^\alpha+p^\alpha)} e^{-iy_\beta(k^\beta+p^\beta)} A_\mu(x) A_\epsilon(y) \\
= \frac{1}{(2\pi)^6} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3k}{2k^0} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3p}{2p^0} \int_{\mathbb{R}^4} \mathrm{d}^4x \int_{\mathbb{R}^4} \mathrm{d}^4y \ \mathrm{tr} \left[(\not{p} - m) \gamma^\epsilon (\not{k} + m) \gamma^\mu \right] \\
e^{ix_\alpha(k^\alpha+p^\alpha)} e^{-iy_\beta(k^\beta+p^\beta)} A_\mu(x) A_\epsilon(y) \\
= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3k}{2k^0} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3p}{2p^0} \left(p_\zeta k_\omega \mathrm{tr} \left[\gamma^\zeta \gamma^\epsilon \gamma^\omega \gamma^\mu \right] - m^2 \mathrm{tr} \left[\gamma^\epsilon \gamma^\mu \right] \right) \\
\hat{A}_\mu(k+p) \hat{A}_\epsilon^*(k+p) \\
= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3k}{2k^0} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3p}{2p^0} 4 \left(p_\zeta k_\omega (\eta^{\zeta,\epsilon} \eta^{\omega,\mu} - \eta^{\zeta,\omega} \eta^{\epsilon,\mu} + \eta^{\zeta,\mu} \eta^{\epsilon,\omega}) - m^2 \eta^{\epsilon,\mu} \right) \\
\hat{A}_\mu(k+p) \hat{A}_\epsilon^*(k+p) \\
= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3k}{k^0} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3p}{p^0} \Big(k^\mu \hat{A}_\mu(k+p) p^\epsilon \hat{A}_\epsilon^*(k+p) - p^\alpha k_\alpha \hat{A}^\mu(k+p) \hat{A}_\mu^*(k+p) \\
+ k^\mu \hat{A}_\mu^*(k+p) p^\epsilon \hat{A}_\epsilon(k+p) - m^2 \hat{A}^\mu(k+p) \hat{A}_\mu^*(k+p) \Big) \\
\overset{k^2=p^2=m^2}{=} \frac{1}{(2\pi)^2} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3k}{k^0} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3p}{p^0} \Big(k^\mu \hat{A}_\mu(k+p) p^\epsilon \hat{A}_\epsilon^*(k+p) \\
+ k^\mu \hat{A}_\mu^*(k+p) p^\epsilon \hat{A}_\epsilon(k+p) - \frac{1}{2} (k+p)^2 \hat{A}^\mu(k+p) \hat{A}_\mu^*(k+p) \Big) \\
= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3k}{k^0} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3p}{p^0} \Big(k^\mu \hat{A}_\mu(k+p) p^\epsilon \hat{A}_\epsilon^*(k+p) \\
+ k^\mu \hat{A}_\mu^*(k+p) p^\epsilon \hat{A}_\epsilon(k+p) + \frac{1}{2} \hat{A}_\mu^*(k+p) \widehat{\Box A}^\mu(k+p) \Big) \qquad (12)
\end{aligned}$$

Up until this point the calculation is exact. If one wants to obtain a nice looking expression for this bound one can add the terms $k^\alpha \hat{A}_\alpha k^\beta \hat{A}^*_\beta + p^\alpha \hat{A}_\alpha p^\beta \hat{A}^*_\beta$. These terms are clearly positive. One then ends up with:

$$\begin{aligned}
\|T1\Omega\|^2 \leq \frac{1}{(2\pi)^2} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3k}{k^0} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3p}{p^0} \Big(& (k^\mu + p^\mu)\hat{A}^*_\mu(k+p)(k^\epsilon + p^\epsilon)\hat{A}_\epsilon(k+p) \\
& + \frac{1}{2}\hat{A}^*_\mu(k+p)\widehat{\Box A}^\mu(k+p)\Big) \\
= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3k}{k^0} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3p}{p^0} \Big(& \widehat{\partial^\alpha A_\alpha}^*(k+p)\widehat{\partial^\beta A_\beta}(k+p) \\
& + \frac{1}{2}\hat{A}^*_\mu(k+p)\widehat{\Box A}^\mu(k+p)\Big) \\
= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3k}{k^0} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3p}{p^0} & \left(\left\| \widehat{\partial^\beta A_\beta}(k+p) \right\|^2 + \frac{1}{2}\hat{A}^*_\mu(k+p)\widehat{\Box A}^\mu(k+p)\right)
\end{aligned} \tag{13}$$

Taking $T1 : \mathbb{C} \to \mathcal{H}_+ \otimes \mathcal{H}_-$ for granted, we would like to define $T1$ on all of Fockspace. In order to do this, we introduce the "restriction of proper lifting" for the annihilation operator:

$$\forall \phi \in \mathcal{H}: \quad a\,(U\phi) \circ \tilde{U} = \tilde{U} \circ a(\phi) \tag{14}$$

Which is equivalent to the commutativity of the following diagram.

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\tilde{U}^A"] & \mathcal{F} \\
\mathcal{H} \otimes \mathcal{F} \arrow[u, "a"] \arrow[r, "U^A \otimes \tilde{U}^A"] & \mathcal{H} \otimes \mathcal{F} \arrow[u, "a"]
\end{tikzcd} \tag{15}$$

The respective condition for the creation operator, which can easily be derived from (14) is:

$$\forall \phi \in \mathcal{H}: \quad a^*\left(U^A\phi\right) \circ \tilde{U}^A = \tilde{U}^A \circ a^*(\phi) \tag{16}$$

Expanding U^A and $\tilde{U}^A$ in a powerseries, one obtains the following commutation relations for the coefficients of said expansion:

$$U^A \qquad = \qquad \mathbb{1}_\mathcal{H} \qquad + \qquad \sum_{l=1}^{\infty} \tag{17}$$