The Relationship Between Hadamard States, the Fermionic Projector, and Admissible Polarisation Classes

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Abstract

This paper compares central objects of three different types of approaches to quantum field theory (QFT): The study of Hadamard states, the Fermionic Projector, and admissible polarisation classes. These approaches have different scopes and pursue different motivations, which makes a direct comparison difficult. Nevertheless, the basic protagonists in all three approaches share much more similarity than is apparent at first glance. It is the purpose of this work to highlight these common features. Since they all study different aspects of QFT; however, one can make out objects in each of these theories which closely resemble one another.

1 Introduction

The first class of objects of interest are Hadamard states which appear in the algebraic approach to QFT [7].
The third class of objects are the continuum limit of the Fermionic Projectors [6].
The second class of objects are the (I_2-almost) projectors P^{λ}_{Σ} which are closely linked to polarisation classes of the vacuum of external field quantum electrodynamics (QED) [3, 5, 4].
In section 2 we give the definition of a Hadamard state, briefly motivate it's usage and give its explicit form in the case of flat spacetime subject to an external field as computed in the physics literature [9].
In section 3 ...
In section 4 we briefly describe why in the approach of admissible polarisation classes one only keeps track of the time evolution of the projector up to an error that is a Hilbert-Schmidt operator. Furthermore we will find a class of candidate I_2-almost projectors that have a simple time evolution.
In section 5...comparison fermionic projector and hadamard state
In section 6 we find that each Hadamard state corresponds to an I_2-almost projector in a natural way.

is that specific enough?

Throughout the paper $\Sigma, \Sigma', \Sigma''$ denote arbitrary Cauchy surfaces, while for the sake of simplicity we choose $A \in C_c^\infty(\mathbb{R}^4, \mathbb{R}^4)$ to be a four-potential and Σ_{in} denotes a Cauchy surface earlier than the support of (A).

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In this paper we will focus on a system of Dirac fields subject to an external electromagnetic four-potential A in flat Minkowski spacetime. We choose the sign convention $\eta = \mathrm{diag}(+1, -1, -1, -1)$. We denote the minimally coupled differential by $\nabla_\alpha = \partial_\alpha + iA_\alpha$ and make use of the Feynman slash notation $\not\nabla =: \nabla_\alpha \gamma^\alpha$ with γ^α fulfilling $\{\gamma^\alpha, \gamma^\beta\} := \gamma^\alpha\gamma^\beta + \gamma^\beta\gamma^\alpha = 2\eta^{\alpha,\beta}$ each field solving Dirac's equation

$$(i\not\nabla - m)\psi =: D\psi = 0 \tag{1}$$

and collectively constituting the free vacuum prior to the support of said external field. For a 4-spinor $\psi \in \mathbb{C}^4$ (viewed as a column vector), $\overline{\psi}$ stands for the row vector $\psi^*\gamma^0$, where * denotes hermitian conjugation.

macht das überhaupt Sinn so wenn ich von Dirac Felder spreche? Bei operatoren ist ja der Zustand nicht der Zustand des Feldes, sondern der Zustand ist extra.

2 Hadamard States

In the algebraic approach to QFT one puts less emphasise on the Hilbert space than is commonly done in non relativistic physics because it is not a relativistically invariant object. Instead one focuses on the algebra of operators that are chosen to do the bookkeeping of statistical outcomes of measurements.
To infer predictions, some part of the necessary computation can be conducted on the level of this algebra. However, eventually, expectation values are to be computed in a certain representation, usually found by the GNS construction with respect to a certain state. This choice has to be made on physical grounds. Hadamard states are often thought to be physically sensible states because they have positive energy in a certain sense.
In order to introduce Hadamard states we have to first define the notion of wavefront set, which itself needs some preliminaries. For the introduction of these concepts we follow Hömander [8, Chapter 8].
We begin by introducing the singular support of a distribution

Definition 1. *Let for $n, m \in \mathbb{N}$, $v \in (C^\infty(\mathbb{R}^n, \mathbb{C}^m))'$ the singular support of v is defined to be the subset of points $x \in \mathbb{R}^n$ such that there is no neighbourhood U of x such that there is a smooth function $\phi_{x,v} \in C^\infty(\mathbb{R}^n, \mathbb{C}^m)$ such that v acts on test functions $\varphi \in C_c^\infty(U, \mathbb{C}^m)$ as*

$$v(\varphi) = \int \phi^\dagger_{x,v}(x)\varphi(x)dx. \tag{2}$$

The singular support contains all the points of a distribution such that the distribution does not act like a smooth function at that point. The wavefront set which we are about to introduce gives an additional directional information of where these singularities propagate. We incorporate this information by the Fourier transform in the following definition.

Definition 2. *Let for $n, m \in \mathbb{N}$, $v \in (C^\infty(\mathbb{R}^n, \mathbb{C}^m))'$, we denote by $\Xi(v) \subset \mathbb{R}^n \backslash \{0\}$ the set of all η such that there is no cone $V \subset \mathbb{R}^n$, neighbourhood of η, such that for all $a \in \mathbb{N}$ there are $C_a > 0$ such that for all $\xi \in V$ we have*

$$|\hat{v}(\xi)| \leq \frac{C_a}{1 + |\xi|^a}. \tag{3}$$

Furthermore for each $x \in \mathbb{R}^n$ we define

$$\Xi_x(v) := \bigcap_{\substack{\phi \in C_c^\infty(\mathbb{R}^n) \\ x \in \mathrm{supp}(\phi)}} \Xi(v\phi), \tag{4}$$

Where $v\phi$ is the pointwise multiplication of a distribution and a scalar test function, which acts as $v\phi : C^\infty(\mathbb{R}^n, \mathbb{C}^m) \ni \psi \mapsto v(\psi\phi)$.

We have collected the tools to introduce the wavefront set and the notion of Hadamard states.

Definition 3. *Let for $n, m \in \mathbb{N}$, $v \in (\mathcal{S}(\mathbb{R}^n, \mathbb{C}^m))'$ be a tempered distribution. The wavefront set $\mathrm{WF}(v)$ of the distribution v is defined as*

$$\mathrm{WF}(v) := \{(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n \mid \xi \in \Xi_x(v)\}\,. \tag{5}$$

soll ich das schon alles direkt für Distributionen die zwei Funktionen schlucken hinschreiben?

Definition 4. *A map $H : C_c^\infty(\mathbb{R}^4, \mathbb{C}^4) \times C_c^\infty(\mathbb{R}^4, \mathbb{C}^4) \to \mathbb{C}$, is called Hadamard state if it fulfils for all $f, g \in C_c^\infty(\mathbb{R}^4, \mathbb{C}^4)$:*

$$H(Df, g) = 0 \tag{6}$$

$$H(f, g) + H(g, f) = iS(f, g) \tag{7}$$

$$\overline{H(f, g)} = H(\overline{f}, \overline{g}) \tag{8}$$

$$WF(H) \subset C_+, \tag{9}$$

where $S(f, g)$ is the propagator of the Dirac equation, and $C_+ := \{(x, y; k_1, -k_2) \in \mathbb{R}^{16} \mid (x; k_1) \approx (y; k_2), k_1^2 \geq 0, k_1^0 > 0\}$ and $(x; k_1) \approx (y; k_2)$ holds whenever $(x - y)^2 = 0$ and $(y - x) \parallel k_1 = k_2$.

It is in the sense of the fourth condition that Hadamard states are of positive energy.
In the scenario of Minkowski spacetime in an external field Dirac [**?**] already studied the Hadamard states, although that name was not established at the time. More recently the subject has attracted considerable attention [10, 9], which computed the Hadamard states. They are given in terms of the Klein-Gordon operator corresponding to Dirac's equation:

Definition 5. *The Klein-Gordon operator corresponding to the Dirac equation* (1) *reads*

$$P : C^\infty(\mathbb{R}^4, \mathbb{C}^4) \to C^\infty(\mathbb{R}^4, \mathbb{C}^4) \tag{10}$$

$$P = (i\nabla\!\!\!/ - m)(-i\nabla\!\!\!/ - m) = \nabla_\alpha \nabla^\alpha + \frac{i}{2}\gamma^\alpha\gamma^\beta F_{\alpha,\beta} + m^2, \tag{11}$$

where $F_{\alpha,\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$ is the field strength tensor of the electromagnetic field. Furthermore we define for $f \in C_c^\infty(\mathbb{R}^4 \times \mathbb{R}^4, \mathbb{C}^{16})$ the differential operator

$$\nabla\!\!\!/^* f(x, x') = \left(\frac{\partial}{\partial y^\alpha} - iA_\alpha(y)\right) f(x, y)\gamma^\alpha. \tag{12}$$

For the special case of a Dirac field in Minkowski space-time Zahn [9] gave a more explicit form of the Hadamard states $H \in (C_c^\infty(\mathcal{M}) \times C_c^\infty(\mathcal{M}))'$ on which we base our analysis below. According to this, H acts for $f_1, f_2 \in C_c^\infty(\mathcal{M}) \otimes \mathbb{C}^4$ as

$$\lim_{x \to y} \left\| \frac{C(x,y)}{\|x-y\|} \right\| < \infty$$

$$H(f_1, f_2) = \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^4} d^4x \overline{f_1}(x) \int_{\mathbb{R}^4} d^4y \; h_\varepsilon(x, y) f_2(y), \tag{13}$$

where h_ε is of the form

$$\frac{C(x,y)}{(x-y-i\varepsilon e_0)^2}$$

$$h_\varepsilon(x, y) = \frac{-1}{2(2\pi)^2}(-i\nabla\!\!\!/ + i\nabla\!\!\!/^* - 2m)\Big[\frac{e^{-i(x-y)^\alpha \int_0^1 ds A_\alpha(xs + (1-s)y)}}{(y - x - i\varepsilon e_0)^2}$$
$$+ V(x, y)\ln(-(y - x - i\varepsilon e_0)^2)\Big] + B(x, y), \tag{14}$$

$\varepsilon \to 0$ im $L^?$-Sinne

$\ln(-(y-x)^2)$

where $V, B : \mathbb{R}^4 \to \mathbb{C}^4$ are smooth functions, B is completely arbitrary, whereas V is fixed by the external potential. The expansion

$$V^N(x,y) := \sum_{k=1}^{N} \frac{1}{4^k k!(k-1)!} V_k(x,y)(x-y)^{2(k-1)}, \tag{15}$$

is an asymptotic expansion for V for $k \to \infty$, in the sense that $(V - V^N)(x,y)\ln(-(x-y)^2)$ as a function of x and y is in $C^{N-2}(\mathbb{R}^{4+4})$ and $V - V^N = \mathcal{O}\left(\left((x-y)^2\right)^{N-2}\right)$. The functions V_k fulfil a recursive set of partial differential differential equations

$$(x-y)^\alpha(\partial_{x,\alpha} + iA_\alpha(x))V_n(x,y) + nV_n(x,y) = -nPV_{n-1}(x,y), \tag{16}$$

where $V_0(x,y) = e^{-i(x-y)^\alpha \int_0^1 ds A_\alpha(xs+(1-s)y)}$.

3 The Fermionic Projector

4 Projectors for Polarisation Classes

The concept of polarisation classes arises naturally in the study of QED in external electromagnetic fields. It does need some machinery to be introduced and related to more familiar objects which we are going to introduce first. In doing so we follow [4]

Definition 6. *We define a Cauchy surface Σ in $\mathbb{R}^4$ to be a smooth, 3-dimensional submanifold of $\mathbb{R}^4$ that fulfills the following three conditions:*

a) Every inextensible, two-sided, time- or light-like, continuous path in $\mathbb{R}^4$ intersects Σ in a unique point.

b) For every $x \in \Sigma$, the tangential space $T_x\Sigma$ is space-like.

c) The tangential spaces to Σ are bounded away from light-like directions in the following sense: The only light-like accumulation point of $\bigcup_{x\in\Sigma} T_x\Sigma$ is zero.

In coordinates, every Cauchy surface Σ can be parametrised as

$$\Sigma = \{\pi_\Sigma(\vec{x}) := (t_\Sigma(\vec{x}), \vec{x}) \mid \vec{x} \in \mathbb{R}^3\} \tag{17}$$

with a smooth function $t_\Sigma : \mathbb{R}^3 \to \mathbb{R}$. For convenience and without restricting generality of our results we keep a global constant

$$0 < V_{\max} < 1 \tag{18}$$

fixed and work only with Cauchy surfaces Σ such that

$$\sup_{\vec{x}\in\mathbb{R}^3} |\text{grad } t_\Sigma(\vec{x})| < V_{\max}. \tag{19}$$

The standard volume form over $\mathbb{R}^4$ is denoted by $d^4x = dx^0dx^1dx^2dx^3$; the product of forms is understood as wedge product. The symbol d^3x mean the 3-form $d^3x = dx^1dx^2dx^3$ on $\mathbb{R}^4$ and on $\mathbb{R}^3$ respectively. Contraction of a form ω with a vector v is denoted by $i_v(\omega)$. The notation $i_v(\omega)$ is also used for the spinor matrix valued vector $\gamma = (\gamma^0, \gamma^1, \gamma^2, \gamma^3) = \gamma^\mu e_\mu$:

$$i_\gamma(d^4x) = \gamma^\mu i_{e_\mu}(d^4x). \tag{20}$$

For $x \in \Sigma$ the restriction of the spinor matrix valued 3-form $i_\gamma(d^4x)$ to the tangential space $T_x\Sigma$ is given by

$$i_\gamma(d^4x) = \not{n}(x) i_n(d^4x) = \left(\gamma^0 - \sum_{\mu=1}^{3} \gamma^\mu \frac{\partial t_\Sigma(\vec{x})}{\partial x^\mu}\right) d^3x =: \Gamma(\vec{x}) d^3x \tag{21}$$

As a consequence of (19), there is a positive constant $\Gamma_{\max} = \Gamma_{\max}(V_{\max})$ such that

$$\|\Gamma(\vec{x}) \leq \Gamma_{\max}, \quad \forall \vec{x} \in \mathbb{R}^3. \tag{22}$$

As is well known equation (1) gives us a one-particle time evolution operator for each pair of Cauchy surfaces Σ, Σ'

$$U_{\Sigma',\Sigma} : \mathcal{H}_\Sigma \to \mathcal{H}_{\Sigma'}, \tag{23}$$

where $\mathcal{H}_\Sigma$ is the space of square integrable functions on the Cauchy surface Σ. The scalar product on $\mathcal{H}_\Sigma$ is given by

$$\phi, \psi \mapsto \int_\Sigma \overline{\phi}(x) i_\gamma(d^4\gamma)\psi(x). \tag{24}$$

We start out by characterising the polarisation classes for free motion.

Definition 7. *The projector $P_\Sigma^{\mathcal{H}^-}$ has the well known representation as the weak limit of the integral operator with the kernel[4]*

$$p_\varepsilon^-(x,y) = -\frac{m^2}{4\pi^2}(i\not{\partial}_x + m)\frac{K_1(m\sqrt{-(y-x-i\varepsilon e_0)^2})}{m\sqrt{-(y-x-i\varepsilon e_0)^2}}, \tag{25}$$

where the square is a Minkowski square and the square root denotes its principle value. By weak limit we mean

$$\langle \phi, P_\Sigma^{\mathcal{H}^-}\psi\rangle = \lim_{\varepsilon \searrow 0} \int_{\Sigma\times\Sigma} \overline{\phi}(x) i_\gamma(d^4x) p_\varepsilon^-(x,y) i_\gamma(d^4y)\psi(y), \tag{26}$$

for general $\phi, \psi \in \mathcal{H}_\Sigma$.

Remark 1. *By inserting the expansion of K_1 in terms of a Laurent series and a logarithm, [1] one obtains:*

$$K_1(\xi) = \frac{1}{\xi} - \frac{\xi}{4}\sum_{k=0}^{\infty}\left(2\psi(k+1) + \frac{1}{k+1} + 2\ln 2 - 2\ln\xi\right)\frac{\left(\frac{\xi^2}{4}\right)^k}{k!^2(k+1)} \tag{27}$$

$$:= \frac{1}{\xi} + \xi Q_1(\xi)\ln\xi + \xi Q_2 =: \frac{1}{\xi} + \xi Q_3(\xi). \tag{28}$$

It is not obvious from the equation (16) but well known that the vacuum of Minkowski spacetime does indeed correspond to a Hadamard state subject to vanishing four potential. In fact, the Hadamard states were constructed to agree with the Minkowski vacuum up to smooth terms.

Definition 8. *Let $Pol(\mathcal{H}_\Sigma)$ denote the set of all closed, linear subspaces $V \subset \mathcal{H}$ such that both V and $V^\perp$ are infinite dimensional. Any $V \in Pol(\mathcal{H}_\Sigma)$ is called* polarisation *of $\mathcal{H}_\Sigma$. For $V \in Pol(\mathcal{H}_\Sigma)$, let $P^V_\Sigma : \mathcal{H}_\Sigma \to V$ denote the orthogonal projection of $\mathcal{H}_\Sigma$ onto V. The Fock space corresponding to V on the Cauchy surface Σ is defined to be*

$$\mathcal{F}(V, \mathcal{H}_\Sigma) := \bigoplus_{c\in\mathbb{Z}} \mathcal{F}_c(V, \mathcal{H}_\Sigma), \quad \mathcal{F}_c(V, \mathcal{H}_\Sigma) := \bigoplus_{\substack{n,m\in\mathbb{N}_0 \\ c=m-n}} (V^\perp)^{\wedge n} \otimes \overline{V}^{\wedge m}, \tag{29}$$

where $\bigoplus$ is the Hilbert space direct sum, $\wedge$ the antisymmetric tensor product of Hilbert spaces and $\overline{V}$ is the conjugate complex vector space of V, which coincides with V as a set, has the same vector space operations as V except for scalar multiplication, which is defined by $(z, \psi) \mapsto z^*\psi$ for $z \in \mathbb{C}, \psi \in V$.

Remark 2. *Given two polarisations $V, W \in Pol(\mathcal{H}_\Sigma)$, for two Fockspaces $\mathcal{F}(V, \mathcal{H}_\Sigma)$ and $\mathcal{F}(W, \mathcal{H}_\Sigma)$ there is a unitary operator $U : \mathcal{F}(V, \mathcal{H}_\Sigma) \to \mathcal{F}(W, \mathcal{H}_\Sigma$ if and only if $P^V_\Sigma - P^W_\Sigma \in I_2(\mathcal{H}_\Sigma)$ by the theorem of Shale and Stinespring [**?**].*

Remark 2 gives us a natural limit with respect to which it is useful to analyse the regularity of Projectors P^A_Σ. We will therefore be content with the following equivalence class.

Definition 9. *For $V, W \in Pol(\mathcal{H}_\Sigma)$ we write*

$$V \approx W \iff P^V_\Sigma - P^W_\Sigma \in I_2(\mathcal{H}_\Sigma), \tag{30}$$

$$C_\Sigma(A) := [U^A_{\Sigma\Sigma_{in}} \mathcal{H}^-_{\Sigma_{in}}]_\approx. \tag{31}$$

The equivalence class $C_\Sigma(A)$ transforms naturally with respect to gauge and Lorentz transforms[4]. Now for hyperplanes $\Sigma \cap \operatorname{supp}(A) \neq \emptyset$ the operator $P^{\mathcal{H}^-}_\Sigma$ does not represent $C_\Sigma(A)$. Because all we are interested in is representations of equivalence classes, we are content with finding objects that differ from a Projector onto $U^A_{\Sigma,\Sigma_{\text{in}}} \mathcal{H}^-_{\Sigma_{in}}$ by a Hilbert-Schmidt operator. Therefore we need not keep track of the exact evolution of the projection operators, but define a whole class of admissible ones.

Definition 10. *For any function $\lambda \in C^\infty(\mathbb{R}^4 \times \mathbb{R}^4, \mathbb{R})$ fulfilling*

i) There is a compact set $K \subset \mathbb{R}^4$ such that $\operatorname{supp} \lambda \subseteq K \times \mathbb{R}^4 \cup \mathbb{R}^4 \times K$.

ii) λ satisfies $\forall x \in \mathbb{R}^4 : \lambda(x, x) = 0$.

iii) On the diagonal the first derivatives fulfil

$$\forall x, y \in \mathbb{R}^4 : \partial_x \lambda(x, y)|_{y=x} = -\partial_y \lambda(x, y)|_{y=x} = A(x), \tag{32}$$

we define a corresponding (quasi) projector P^λ:

$$\langle \phi, P^\lambda_\Sigma \psi\rangle = \lim_{\varepsilon \searrow 0} \langle \phi, P^{A,\varepsilon}_\Sigma \psi\rangle, \tag{33}$$

$$\langle \phi, P^{\lambda,\varepsilon}_\Sigma \psi\rangle := \int_{\Sigma\times\Sigma} \overline{\phi}(x) i_\gamma(d^4x) \overbrace{e^{-i\lambda(x,y)} p^-_\varepsilon (y - x)}^{=:p^\lambda_\varepsilon(x,y)} i_\gamma(d^4y)\psi(y), \tag{34}$$

for general $\phi, \psi \in \mathcal{H}_\Sigma$.

Remark 3. *P^λ_Σ and $P^\lambda_{\Sigma'}$ are equivalent if transported appropriately by time evolution operators[4, theorem 2.8]:*

$$P^\lambda_\Sigma - U^A_{\Sigma,\Sigma'} P^\lambda_{\Sigma'} U_{\Sigma',\Sigma} \in I_2(\mathcal{H}_\Sigma). \tag{35}$$

Also for four-potentials $A, B \in C^\infty_c(\mathbb{R}^4)$ the corresponding projectors are equivalent if and only if the four potentials projected onto the hypersurface agree[4, theorem 1.5]:

$$P^{\lambda^A}_\Sigma - P^{\lambda^B}_\Sigma \in I_2(\mathcal{H}_\Sigma) \iff \forall x \in \Sigma\ \forall z \in T_x\Sigma : z^\alpha(A_\alpha(x) - B_\alpha(x)) = 0. \tag{36}$$

Taking into account the freedom within each classification the notions Hadamard state and projectors of polarisation classes are extremely close. This is the topic of the next section.

5 Comparison Between Hadamard States and the Fermionic Projector

6 Comparison Between Hadamard States and P^λ_Σ

The following theorem is the basis of our comparison between Hadamard states and I_2-almost Projectors. We first discuss its consequences and postpone the proof until the appendix

Theorem 1. *Given as four-potential $A \in C^\infty_c(\mathbb{R}^4)$ and a λ^A fulfilling $i) - iii)$ and a Hadamard state H of the form (13) and (14), there is a family of smooth functions $(w_\varepsilon)_{\varepsilon>0} \subset C^\infty(\mathbb{R}^4 \times \mathbb{R}^4, \mathbb{C}^{4\times4})$ and a limiting function function $w \in C^\infty(\mathbb{R}^4 \times \mathbb{R}^4, \mathbb{C}^{4\times4})$ such that $\tilde{P}$ defined by*

$$\mathcal{H}_\Sigma \ni \psi \mapsto \tilde{P}\psi = \lim_{\varepsilon\to0} \int_\Sigma (h_\varepsilon - w_\varepsilon)(\cdot, y) i_\gamma(d^4y)\psi(y), \tag{37}$$

where Σ is a Cauchy surface, is bounded and fulfils $P^{\lambda^A} - \tilde{P} \in I_2(\mathcal{H}_\Sigma)$. Additionally $\lim_{\varepsilon\to0} w_\varepsilon(x,y) = w(x,y)$ holds for any $x, y \in \mathbb{R}^4$.

Theorem 2. *Given a four-potential $A \in C^\infty_c(\mathbb{R}^4)$ and a λ^A fulfilling $i) - iii)$ and a Hadamard state H of the form (13) and (14), there is a family of smooth functions $(w_\varepsilon)_{\varepsilon>0} \subset C^\infty(\mathbb{R}^4 \times \mathbb{R}^4, \mathbb{C}^{4\times4})$ and a function $w \subset C^\infty(\mathbb{R}^4 \times \mathbb{R}^4, \mathbb{C}^{4\times4})$ such that for all $x, y \in \mathbb{R}^4$*

Somehow this direction doesn't look like much of a result. the trouble is that the fractions just aren't equal, they're just close enough to be square integrable

mention $w \in L^2_{loc}(\Sigma \times \Sigma)$ for Σ arbitrary message?!

$$w(x,y) = \lim_{\varepsilon\to0} w_\varepsilon(x,y) + \frac{1}{2(2\pi)^2}(-i\nabla\!\!\!/ + i\nabla\!\!\!/^* - 2m) \tag{38}$$

$$\left(\frac{e^{-i(x-y)^\alpha \int_0^1 ds A_\alpha(xs+(1-s)y)}}{(y-x-i\varepsilon e_0)^2} + V(x,y)\ln(-(y-x-i\varepsilon e_))^2 \right) \tag{39}$$

$$- e^{-i\lambda^A(x,y)}(i\partial\!\!\!/ + m) \tag{40}$$

$$\left[\left(\frac{-1}{m^2(y-x-i\varepsilon e_0)^2} + Q_1(m\sqrt{-(y-x-i\varepsilon e_0)^2}) \right) \frac{\ln(-(y-x-i\varepsilon e_0)^2)m^2}{4\pi^2} \right] \tag{41}$$

and the object H^{λ^A} acting on test functions $f_1, f_2 \in C^\infty(\mathbb{R}^4, \mathbb{C}^4)$ as

$$H^{\lambda^A}(f_1, f_2) = \lim_{\varepsilon\to0} \int_{\mathbb{R}^{4+4}} \overline{f_1}(x) \left(p^{\lambda^A}_\varepsilon(x,y) - w_\varepsilon(x,y) \right) f_2(y) d^4x d^4y \tag{42}$$

is a Hadamard state given by (13) and (14).

7 Appendix: Proof of theorem 1 and 2

The proof requires the following lemma.

Lemma 1. *For every four-potential $A \in C_c^\infty(\mathbb{R}^4)$, for every λ^A fulfilling $i) - iii)$ and every Hadamard state H of the form (13) and (14) there is a smooth function $w \in C^\infty(\mathbb{R}^4 \times \mathbb{R}^4, \mathbb{C}^{4\times 4})$ and a sequence $w_\varepsilon \in C^\infty(\mathbb{R}^4 \times \mathbb{R}^4, \mathbb{C}^{4\times 4})$ converging to w in the limit $\varepsilon \to 0$, such that for any hypersurface Σ, one has for every $\varepsilon > 0$:*

Pointwise

$+\zeta_\varepsilon$ $L^2(\Sigma\times\Sigma)$-limit

$$h_\varepsilon^A - w_\varepsilon - p_\varepsilon^{\lambda^A}\Big|_{\Sigma\times\Sigma} \in L^2(\Sigma \times \Sigma) \tag{43}$$

and $\lim_{\varepsilon \to 0} h_\varepsilon^A - w_\varepsilon - p_\varepsilon^{\lambda^A}\Big|_{\Sigma\times\Sigma}$ ~~$\in L^2(\Sigma \times \Sigma)$~~ exists.

Proof of lemma 1: Let $A \in C_c^\infty(\mathbb{R}^4), \varepsilon > 0$. Let H be a Hadamard state acting as

$$H(f_1, f_2) = \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^4} d^4x \overline{f_1}(x) \int_{\mathbb{R}^4} d^4y \; h_\varepsilon(x,y) f_2(y). \tag{44}$$

be of the form of equation (14). Pick a function λ according to definition 10. For ease of comparison we also choose $\tilde{\lambda} : \mathbb{R}^4 \to \mathbb{C}$

$$\tilde{\lambda}(x,y) = (x-y)^\alpha \int_0^1 ds A_\alpha(xs + (1-s)y), \tag{45}$$

and furthermore introduce the abbreviation $G(x,y) := e^{-i\tilde{\lambda}(x,y)}$.
In the representation of the Hadamard state H, equation (16), can be explicitly solved recursively (See [2][lemma 2.2.2]) The factor V_0 was already given, for $k \geq 1$ the recursion is given by

$$V_k(x,y) = -kG(x,y) \int_0^1 ds s^{k-1} G(x + s(y-x), x) P V_{k-1}(x, x + s(y-x)). \tag{46}$$

One can read off of this, that if the support of the external field A does not intersect the line connecting x and y, the function $V(x,y)$ multiplying the logarithm in the expression for $h_\varepsilon(x,y)$ can be calculated to be

$$V(x,y) = \sum_{k=0}^{\infty} \frac{((y-x)^2 m^2/4)^k}{k!(k+1)!}, \tag{47}$$

wich is exactly equal to the logarithmic part of p_ε^-, corresponding to $Q_1(\xi)$ in (27). This shows that in this case $p_\varepsilon^-(x,y)$ agrees with $h_\varepsilon(x,y)$ up to smooth terms, when acting as

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}^4 \times \mathbb{R}^4} \overline{f_1}(x) h_\epsilon(x,y) f_2(y) dx dy \tag{48}$$

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}^4 \times \mathbb{R}^4} \overline{f_1}(x) p_\epsilon^-(y-x) f_2(y) dx dy \tag{49}$$

on functions $f_1, f_2 \in C_c^\infty(\mathbb{R}^4, \mathbb{C}^4)$ such that there are no $x \in \operatorname{supp}(f_1), y \in \operatorname{supp}(f_2)$ such that $\overline{x\,y} \cap \operatorname{supp}(A) \neq \emptyset$. Here we denote the line segment between x and y by $\overline{x\,y}$.
Furthermore we can read off of both h_ε and p_ε^λ that the only points (x,y) in the singular support of the distributions in (49) and (48) need to fulfil $(y-x)^2 = 0$, if we further demand that both x and y belong to the same spacelike hypersurface $x = y$ follows. Now pick some $\delta > 0$. This implies that for for functions $f_1, f_2 \in C_c^\infty(\mathbb{R}^4, \mathbb{C}^4)$ such that

$(x,y) \in \operatorname{supp}(f_1) \times \operatorname{supp}(f_2)$ implies $(x-y)^2 \geq -\delta^2$ the distributions (49) and (48) act like integration against a smooth function.
So summarising we may choose $w \in C^\infty(\mathbb{R}^4 \times \mathbb{R}^4, \mathbb{C}^{4\times 4})$ such that the following two conditions are fulfilled.
1). For all $f_1, f_2 \in C_c^\infty(\mathbb{R}^4, \mathbb{C}^4)$, such that for all pairs $(x,y) \in \operatorname{supp}(f_1) \times \operatorname{supp}(f_2)$ implies $\overline{x\ y} \cap \operatorname{supp}(A) = \emptyset$, we have

$$\int_{\mathbb{R}^4\times\mathbb{R}^4} \overline{f_1}(x) w(x,y)) f_2(y) dx dy = \lim_{\epsilon\to 0} \int_{\mathbb{R}^4\times\mathbb{R}^4} \overline{f_1}(x)(h_\epsilon(x,y) - p_\epsilon^-(y-x)) f_2(y) dx dy. \quad (50)$$

2). For all test functions $f_1, f_2 \in C_c^\infty(\mathbb{R}^4, \mathbb{C}^4)$ such that $(x,y) \in \operatorname{supp}(f_1) \times \operatorname{supp}(f_2)$ implies $(x-y)^2 < -\delta^2$ we have

$$\int_{\mathbb{R}^4\times\mathbb{R}^4} \overline{f_1}(x) w(x,y) f_2(y) dx dy = \lim_{\epsilon\to 0} \int_{\mathbb{R}^4\times\mathbb{R}^4} \overline{f_1}(x)(h_\epsilon(x,y) - p_\epsilon^\lambda(x,y)) f_2(y) dx dy. \quad (51)$$

In fact conditions 1) and 2) specify the values of w uniquely for arguments $(x,y) \in \mathbb{R}^{4+4}$ in the complement of the set

$$\tilde{J}_\delta := \{(x,y) \in \mathbb{R}^{4+4} \mid (x-y)^2 \geq -\delta^2 \wedge \overline{x\ y} \cap \operatorname{supp}(A) \neq \emptyset\}. \quad (52)$$

Using this, we define a smooth function $w_\varepsilon \in C^\infty(\mathbb{R}^4 \times \mathbb{R}^4, \mathbb{C}^{4\times 4})$ for every $\varepsilon > 0$ fulfilling

$$\int_{\mathbb{R}^4\times\mathbb{R}^4} \overline{f_1}(x) w_\varepsilon(x,y) f_2(y) dx dy = \int_{\mathbb{R}^4\times\mathbb{R}^4} \overline{f_1}(x)(h_\varepsilon - p_\varepsilon^\lambda)(x,y) f_2(y) dx dy \quad (53)$$

both for test functions $f_1, f_2 \in C_c^\infty(\mathbb{R}^4, \mathbb{C}^4)$ satisfying the requirement of 1) or the requirement of 2). This does not fix $w_\varepsilon(x,y)$ in regions where $(x-y)^2 > -\delta^2$ and $\overline{xy} \cap \operatorname{supp}(A) \neq \emptyset$; however, there are choices that agree with w in the limit $\varepsilon \to 0$. We pick such w_ε.
What follows is the estimate of $\|h_\varepsilon - w_\varepsilon - p_\varepsilon^\lambda\|_{L_2(\Sigma\times\Sigma)}$: So pick $\varepsilon > 0$ and a hypersurface Σ. We begin by noticing that according to our choice of w_ε we have that $h_\varepsilon - w_\varepsilon - p_\varepsilon^\lambda|_{\Sigma\times\Sigma}$ vanishes outside the set

$$J_\delta := \{(x,y) \in \Sigma \times \Sigma \mid (x-y)^2 \geq -\delta^2 \wedge \overline{x\ y} \cap \operatorname{supp}(A) \neq \emptyset\} \quad (54)$$

which is bounded and therefore of finite measure.
This directly leads us to the following estimate

$$\begin{aligned} \|h_\varepsilon - w_\varepsilon - p_\varepsilon^\lambda\|_{L^2(\Sigma\times\Sigma)} &= \|h_\varepsilon - w_\varepsilon - p_\varepsilon^\lambda\|_{L^2(J_\delta)} && (55)\\ &\leq \|h_\varepsilon - w_\varepsilon - p_\varepsilon^{\tilde{\lambda}}\|_{L^2(J_\delta)} + \|p_\varepsilon^\lambda - p_\varepsilon^{\tilde{\lambda}}\|_{L^2(J_\delta)} && (56)\end{aligned}$$

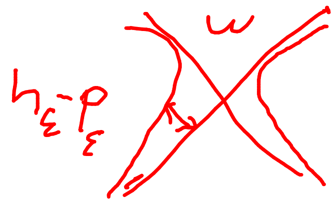
Pick $(x,y) \in J_\delta$. The relevant terms of the first summand in (55) are estimated as follows

argument zu Punktweise Konvergenz noch hinzufügen!

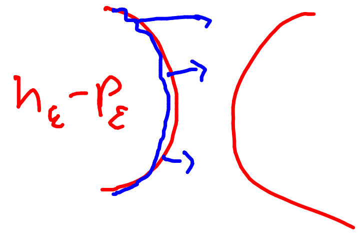
Argument der Konvergenz von $h_\varepsilon - P_\varepsilon$ im L^2 Sinne mit dominierter Konvergenz aufschreiben!

$$\tilde{J}_\delta = \{ (x,y) \in \mathbb{R}^{4+4} \mid (x-y)^2 \geq -\delta^2 \wedge \overline{xy} \cap \operatorname{supp}(A) \neq \emptyset \}$$

$$w_\varepsilon : \tilde{J}_\delta^{\,c} \to \mathbb{C}^{16} \quad \text{glatt}$$



$$w_\varepsilon : \tilde{J}_{\delta/2} \to \mathbb{C}^{16}$$
$$x \mapsto 0$$



$$\chi\big|_{\tilde{J}^c_\delta} = 1 \qquad \chi\big|_{\tilde{J}_{\delta/2}} = 0 \qquad \chi \in C^\infty(\mathbb{R}^{4+4})$$

$$A, B \subset \mathbb{R}^8 \quad \bar{A} = A, \ \bar{B} = B \Rightarrow \exists\, \psi_{A,B}^{C^\infty} : \ \psi_A\big|_A = 1$$
$$A \cap B = \emptyset \qquad\qquad \psi_A\big|_B = 0$$

glatte Version von Urysohn's Lemma | und umgekehrt

$$
\begin{aligned}
&p_\varepsilon^{\tilde\lambda}(x,y)-h_\varepsilon(x,y)-B(x,y)+G(x,y)\frac{m^2}{4\pi^2}(-\not\partial_x+m)Q(m\sqrt{-(y-x-i\varepsilon e_0)^2})\\
&-\frac{-i\not\nabla+i\not\nabla^*-2m}{8\pi^2}V(x,y)\ln(-(y-x-i\varepsilon e_0)^2) \qquad (57)\\
=&\frac{G(x,y)}{4\pi^2}\left((i\not\partial_x+m)\frac{1}{(y-x-i\varepsilon e_0)^2}-\frac{1}{G(x,y)}(i\not\nabla/2-i\not\nabla^*/2+m)\frac{G(x,y)}{(y-x-i\varepsilon e_0)^2}\right) \qquad (58)\\
&=\frac{iG(x,y)}{4\pi^2}\left(\not\partial_x\frac{1}{(y-x-i\varepsilon e_0)^2}-\frac{1}{2G(x,y)}(\not\partial_x+i\not A(x)-\not\partial_y+i\not A(y))\frac{G(x,y)}{(y-x-i\varepsilon e_0)^2}\right)\\
&=\frac{iG(x,y)}{4\pi^2}\left(-i\frac{\not A(x)+\not A(y)}{2(y-x-i\varepsilon e_0)^2}+\not\partial_x\frac{1}{(y-x-i\varepsilon e_0)^2}-\frac{1}{2}(\not\partial_x-\not\partial_y)\frac{1}{(y-x-i\varepsilon e_0)^2}\right.\\
&\left.-\frac{1}{2G(x,y)}\frac{(\not\partial_x-\not\partial_y)G(x,y)}{(y-x-i\varepsilon e_0)^2}\right)\\
&=\frac{iG(x,y)}{4\pi^2}\left(-i\frac{\not A(x)+\not A(y)}{2(y-x-i\varepsilon e_0)^2}+\frac{1}{2}(\not\partial_x+\not\partial_y)\frac{1}{(y-x-i\varepsilon e_0)^2}+\frac{1}{2G(x,y)}\frac{(-\not\partial_x+\not\partial_y)G(x,y)}{(y-x-i\varepsilon e_0)^2}\right)\\
&=\frac{-iG(x,y)}{8\pi^2}\frac{1}{(y-x-i\varepsilon e_0)^2}\left(i\not A(x)+i\not A(y)+G(x,y)^{-1}\left(\not\partial_x-\not\partial_y\right)G(x,y)\right)\\
&=\frac{G(x,y)}{8\pi^2}\frac{1}{(y-x-i\varepsilon e_0)^2}\left(\not A(x)+\not A(y)-2\int_0^1\mathrm{d}s\not A(sx+(1-s)y)\right.\\
&\left.+(y-x)^\alpha\int_0^1\mathrm{d}s(1-2s)(\not\partial A_\alpha)(sx+(1-s)y)\right).
\end{aligned}
$$

Now using Taylor's series for A around x, y as well as $(x+y)/2$ reveals

$$
\not A(x)+\not A(y)-2\int_0^1\mathrm{d}s\not A(sx+(1-s)y)=\frac{(x-y)^\alpha}{4}(x-y)^\beta(\partial_\alpha\partial_\beta\not A)((x+y)/2) +\mathcal{O}(\|x-y\|^2) \qquad (59)
$$

$$
(x-y)^\alpha\int_0^1\mathrm{d}s(1-2s)(\not\partial A_\alpha)(sx+(1-s)y)=-\frac{(x-y)^\alpha}{2}(x-y)^\beta(\not\partial\partial_\beta A_\alpha)((x+y)/2) +\mathcal{O}(\|x-y\|^3). \qquad (60)
$$

Because the remaining terms are all locally square integrable, it follows that $h_\varepsilon-w_\varepsilon-p_\varepsilon^{\tilde\lambda}$ is square integrable in J_δ. Moreover, because there is $C>0$ upper bound on $\frac{(58)+(59)}{8\pi^2(y-x-i\varepsilon e_0)^2}$, this estimate remains finite in the limit $\varepsilon\to 0$.
Regarding the second summand of (55) we notice that this can be bounded by

$$
\|P_\Sigma^{\tilde\lambda}-P^{\hat\lambda_\Sigma}\|_{I_2(\mathcal{H}_\Sigma,\mathcal{H})\Sigma)} \qquad (61)
$$

for any $\hat\lambda$ that agrees with $\tilde\lambda$ on the set J_δ. This quantity is finite because of (36) if $\hat\lambda$ fulfils the conditions in definition 10. Now $\tilde\lambda$ already fulfils conditions ii) and iii). So if we can find compact sets $K\subset K'\subset\mathbb{R}^4$ such that $J_\delta\subset K\times\mathbb{R}^4\cup\mathbb{R}^4\times K$ we can construct $\hat\lambda$ by multiplying $\tilde\lambda$ with a function $\chi\in C_c^\infty(\mathbb{R}^4)$ that is 1 on $K\times\mathbb{R}^4\cup\mathbb{R}^4\times K$ and zero outside $K'\times\mathbb{R}^4\cup\mathbb{R}^4\times K'$.

We use the parametrisation $\Sigma = \{(t_\Sigma(\vec{x}), \vec{x}) \in \mathbb{R}^4\}$. Pick $(x, y) \in J_\delta$, the parametrisation and the mean value theorem imply

$$(x-y)^2 = (x^0 - y^0)^2 - |\vec{x} - \vec{y}|^2 = |\vec{x} - \vec{y}|^2 (|\mathrm{grad} t_\Sigma(\xi)|^2 - 1) \tag{62}$$

for some vector $\xi \in \overline{\vec{x}\,\vec{y}}$. This means we can find an upper bound on the spacial distance of x and y:

$$(x-y)^2 > -\delta^2 \iff |\vec{x} - \vec{y}|^2 < \frac{\delta^2}{1 - |\mathrm{grad} t_\Sigma(\xi)|^2} < \frac{\delta^2}{1 - V_{\max}^2}. \tag{63}$$

But that means that

$$K := \mathrm{A} + \left\{ (0, \vec{x}) \in \mathbb{R}^4 \mid |\vec{x}| < \frac{\delta}{\sqrt{1 - V_{\max}^2}} \right\} \tag{64}$$

and any larger but compact set K' work.

$\square$

Proof of theorem 1: We pick for a four-potential $A \in C_c^\infty(\mathbb{R}^4)$ a Hadamard state H of the form (13) and (14) and a λ^A according to definition 10. Then we pick $w_\varepsilon, w \in C_c^\infty(\mathbb{R}^{4+4}, \mathbb{C}^{4\times 4})$ for all $\varepsilon > 0$ according to lemma 1. Because of

$$h_\varepsilon^A - w_\varepsilon - p_\varepsilon^{\lambda^A} \in L^2(\Sigma \times \Sigma) \tag{65}$$

holds, we can define the operator $\tilde{P}_\Sigma - P_\Sigma^{\lambda^A} \in I_2(\mathcal{H}_\Sigma)$ for any Cauchy surface Σ as

$$\mathcal{H}_\Sigma \ni \psi \mapsto (\tilde{P}_\Sigma - P_\Sigma^{\lambda^A})\psi := \lim_{\varepsilon \to 0} \int_\Sigma (h_\varepsilon^A - w_\varepsilon - p_\varepsilon^{\lambda^A})(\cdot, y) i_\gamma(d^4y)\psi(y) \tag{66}$$

and

$$\tilde{P}_\Sigma := \tilde{P}_\Sigma - P_\Sigma^{\lambda^A} + P_\Sigma^{\lambda^A}. \tag{67}$$

$\square$

Proof of theorem 2. Pick a four-potential $A \in C_c^\infty(\mathbb{R}^4)$, a Hadamard state H of the form (13) and (14) and a λ^A according to definition 10. We define the distribution H^{λ^A} by its action on test functions $f_1, f_2 \in C_c^\infty(\mathbb{R}^4, \mathbb{C}^4)$

$$H^{\lambda^A}(f_1, f_2) := \lim_{\varepsilon \to 0} \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} \overline{f}_1(x) h_\varepsilon^\lambda(x, y) f_2(y) d^4x d^4y. \tag{68}$$

As the regularised integration kernel $h_\varepsilon^{\lambda^A}$ we pick

$$h_\varepsilon^{\lambda^A}(x, y) := p_\varepsilon^{\lambda^A}(x, y) - (p_\varepsilon^{\lambda^A} - h_\varepsilon(x, y)), \tag{69}$$

where h_ε defines H by (13) and (14). Looking at the series representation of $p_\varepsilon^{\lambda^A}(x, y)$ and $h_\varepsilon(x, y)$ we can directly confirm (38)-(41).

$\square$

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