1 $\frac{1}{(z+i\varepsilon e_0)^2}$:

Let $f \in C_c^\infty(\mathbb{R}^4, \mathbb{C}^4)$. We are interested in

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^4} d^4z \frac{f(z^0, \vec{z})}{(z+i\varepsilon e_0)^2} = \lim_{\varepsilon\to 0} \int_{\{\vec{z}\in\mathbb{R}^3|\vec{z}\neq 0\}} d^3z \int_{\mathbb{R}} dz^0 \frac{f(z^0, \vec{z})}{(z+i\varepsilon e_0)^2}. \tag{1}$$

Now if we exclude the points in $\mathbb{R}^4$ where the two poles of the integrand coincide we are able to use

$$\frac{1}{(z+i\varepsilon e_0)^2} = \frac{1}{(z^0+i\varepsilon-|\vec{z}|)(z^0+i\varepsilon+|\vec{z}|)} = \frac{1}{2|\vec{z}|(z^0+i\varepsilon-|\vec{z}|)} - \frac{1}{2|\vec{z}|(z^0+i\varepsilon+|\vec{z}|)}. \tag{2}$$

For one of these terms and fixed $\varepsilon > 0, \vec{z} \in \mathbb{R}^3$ we obtain

$$\int dz^0 \frac{f(z^0, \vec{z})}{z^0+i\varepsilon-|\vec{z}|} = \int du \frac{f(u+|\vec{z}|, \vec{z})}{u+i\varepsilon} = \int du \frac{uf(u+|\vec{z}|, \vec{z})}{u^2+\varepsilon^2} - i\varepsilon \int du \frac{f(u+|\vec{z}|, \vec{z})}{u^2+\varepsilon^2} \tag{3}$$

$$= \int_0^\infty du\, (f(u+|\vec{z}|, \vec{z}) - f(-u+|\vec{z}|, \vec{z})) \frac{u}{u^2+\varepsilon^2} - i \int du \frac{f(\varepsilon u+|\vec{z}|, \vec{z})}{u^2+1}. \tag{4}$$

Now in order to pull the limit inside the first integral in (1), we recognize, that for ε small enough the inner integral can be bounded by

$$\int_0^\infty du\, |f(u+|\vec{z}|, \vec{z}) - f(-u+|\vec{z}|, \vec{z})| \frac{1}{u} + \pi \sup_{u\in[-1,1]} |f(u+|\vec{z}|, \vec{z})|, \tag{5}$$

which is independend of ε and nevertheless integrable in $\vec{z}$. We can also Pull the limit into the inner integral to obtain:

$$\lim_{\varepsilon\to 0} \int dz^0 \frac{f(z^0, \vec{z})}{z^0+i\varepsilon-|\vec{z}|} = \mathcal{P} \int du \frac{f(u+|\vec{z}|, \vec{z})}{u} - i\pi f(|\vec{z}|, \vec{z}) \tag{6}$$

So in total we obtain

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^4} d^4z \frac{f(z^0, \vec{z})}{(z+i\varepsilon e_0)^2} = \int_{\mathbb{R}^3} d^3z \frac{1}{2|\vec{z}|} \left(\mathcal{P} \int du \frac{f(u+|\vec{z}|, \vec{z}) - f(u-|\vec{z}|, \vec{z})}{u} - i\pi(f(|\vec{z}|, \vec{z}) - f(-|\vec{z}|, \vec{z})) \right) \tag{7}$$

2 $\frac{1}{(z+i\varepsilon e_0)^4}$:

In this case we follow the analogous strategy as for the first case; however, we will have to be more careful with the singularity at $\vec{z} = 0$. Let $f \in C_c^\infty(\mathbb{R}^4, \mathbb{C})$. As before we have

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^4} d^4z \frac{f(z^0, \vec{z})}{(z+i\varepsilon e_0)^4} = \lim_{\varepsilon\to 0} \int_{\mathbb{R}^3} d^3z \int_{\mathbb{R}} dz^0 \frac{f(z^0, \vec{z})}{((z^0)^2+2i\varepsilon-\varepsilon^2-\vec{z}^2)^2}. \tag{8}$$

Similarly as before we use partial fractions to brake the integral up into parts.

$$\frac{1}{((z^0)^2+2i\varepsilon-\varepsilon^2-\vec{z}^2)^2}=\frac{1}{(z^0+i\varepsilon-|\vec{z}|)^2(z^0+i\varepsilon+|\vec{z}|)^2)}=\frac{1}{(z^0-a)^2(z^0-b)^2)}\bigg|_{a=\dots,b=\dots}= \tag{9}$$

$$\partial_a\partial_b\frac{1}{(z^0-a)(z^0-b)}\bigg|_{a=\dots,b=\dots}=\partial_a\partial_b\left(\frac{1}{(a-b)(z^0-a)}+\frac{1}{(b-a)(z^0-b)}\right)_{a=\dots,b=\dots}= \tag{10}$$

$$\frac{1}{(a-b)^2(z^0-a)^2}-\frac{2}{(a-b)^3(z^0-a)}+\frac{1}{(b-a)^2(z^0-b)^2}-\frac{2}{(b-a)^3(z^0-b)}\bigg|_{a=\dots,b=\dots}= \tag{11}$$

$$\frac{1}{4|\vec{z}|^2(z^0+i\varepsilon-|\vec{z}|)^2}-\frac{1}{4|\vec{z}|^3(z^0+i\varepsilon-|\vec{z}|)}+\frac{1}{4|\vec{z}|^2(z^0+i\varepsilon+|\vec{z}|)^2}+\frac{1}{4|\vec{z}|^3(z^0+i\varepsilon+|\vec{z}|)} \tag{12}$$

The first and third term can be treated analogously to before. We explicitly deal with the first, the third one can be dealt with analogously

$$\int dz^0\frac{f(z^0,\vec{z})}{(z^0+i\varepsilon-|\vec{z}|)^2}=\int dz^0 f(z^0,\vec{z})\partial_0\frac{-1}{z^0+i\varepsilon-|\vec{z}|}=\int dz^0\frac{\partial_0 f(z^0,\vec{z})}{z^0+i\varepsilon-|\vec{z}|}= \tag{13}$$

$$\int_0^\infty du(\partial_0 f(u+|\vec{z}|,\vec{z})-\partial_0 f(-u+|\vec{z}|,\vec{z}))\frac{u}{u^2+\varepsilon^2}-i\int du\frac{\partial_0 f(\varepsilon u+|\vec{z}|,\vec{z})}{u^2+\varepsilon^2} \tag{14}$$

This term is not divergent close to $\vec{z}=0$, which means that the $\frac{1}{|\vec{z}|^2}$ singularity, which is locally integrable, is the only one for these two terms. The other two terms are individually more singular, but can be dealt with together:

$$\int dz^0 f(z^0,\vec{z})\left(\frac{1}{z^0+i\varepsilon+|\vec{z}|}-\frac{1}{z^0+i\varepsilon-|\vec{z}|}\right)= \tag{15}$$

$$\int_0^\infty du\left(f(u-|\vec{z}|,\vec{z})-f(-u-|\vec{z}|,\vec{z})-(f(u+|\vec{z}|,\vec{z})-f(-u+|\vec{z}|,\vec{z}))\right)\frac{u}{u^2+\varepsilon^2} \tag{16}$$

$$-i\int du\frac{f(\varepsilon u-|\vec{z}|,\vec{z})-f(\varepsilon u+|\vec{z}|,\vec{z})}{u^2+1}. \tag{17}$$

In the last two lines we can see that the integral is $\mathcal{O}(|\vec{z}|)$ close to $\vec{z}=0$. So in fact the term is integrable even with the $\frac{1}{|\vec{z}|^3}$ factor, once can also find an integrable upper bound analogous to (5). Due to these bounds we can apply Lebegues dominated convergence and pull the limit inside the integral, to obtain

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^4}d^4z\frac{f(z^0,\vec{z})}{(z+i\varepsilon e_0)^4}=\int d^3z\left[\frac{1}{4|\vec{z}|^2}\mathcal{P}\int du\frac{\partial_0 f(u+|\vec{z}|,\vec{z})}{u}-\frac{i\pi}{4|\vec{z}|^2}\partial_0 f(|\vec{z}|,\vec{z})\right. \tag{18}$$

$$+\frac{1}{4|\vec{z}|^2}\mathcal{P}\int du\frac{\partial_0 f(u-|\vec{z}|,\vec{z})}{u}-\frac{i\pi}{4|\vec{z}|^2}\partial_0 f(-|\vec{z}|,\vec{z}) \tag{19}$$

$$\left.+\frac{1}{4|\vec{z}|^3}\mathcal{P}\int du\frac{f(u-|\vec{z}|,\vec{z})-f(u+|\vec{z}|,\vec{z})}{u}-\frac{i\pi}{4|\vec{z}|^3}(f(-|\vec{z}|,\vec{z})-f(|\vec{z}|,\vec{z}))\right]. \tag{20}$$