Begin der S-Matrixreihe nach "Zusätzliche Rechtfertigung Heavysidefunktionen durch aufteilen des Impulsintegrals zu ersetzen"

Seien $t, t_0 \in \mathbb{R}, y \in \mathbb{R}^4$

$$\begin{aligned}
\phi_t(y) = \phi_{t_0}(y) - i\int_{t_0}^{t} \mathrm{d}s L_s\phi_s(y) &= \phi_{t_0}(y) - i\int_{t_0}^{t} \mathrm{d}s\mathcal{F}_{0\Sigma_s}(v_s \not{n}_s \not{A})\mathcal{F}_{\Sigma_s 0}\phi_s(y) \\
&= \phi_{t_0}(y) - i\int_{t_0}^{t} \mathrm{d}s\mathcal{F}_{0\mathcal{M}}\mathcal{F}_{\mathcal{M}\Sigma_s}(v_s \not{n}_s \not{A})\mathcal{F}_{\Sigma_s 0}\phi_s(y) \\
&= \phi_{t_0}(y) - i\int_{t_0}^{t} \mathrm{d}s\frac{1}{(2\pi)^{1.5}m}\int_{\mathcal{M}} e^{-ipy} i_p(\mathrm{d}^4p)\mathcal{F}_{\mathcal{M}\Sigma_s}(v_s \not{n}_s \not{A})\mathcal{F}_{\Sigma_s 0}\phi_s(y) \\
&= \phi_{t_0}(y) - i\int_{t_0}^{t} \mathrm{d}s\frac{1}{(2\pi)^{3}m}\int_{\mathcal{M}} e^{-ipy} i_p(\mathrm{d}^4p)\frac{\not{p}+m}{2m}\int_{\Sigma_s} e^{ipx} i_\gamma(\mathrm{d}^4x) v_s(x)\not{n}_s(x)\not{A}(x)\phi_s(x) \\
&= \phi_{t_0}(y) - i\int_{t_0}^{t} \mathrm{d}s\int_{\Sigma_s}\int_{\mathcal{M}} \frac{\not{p}+m}{2m^2} e^{ip(x-y)} i_p(\mathrm{d}^4p)\frac{i_\gamma(\mathrm{d}^4x)}{(2\pi)^3} v_s(x)\not{n}_s(x)\not{A}(x)\phi_s(x) \\
&\overset{\text{verwende Impulsumformung durch Residuensatz}}{=} \\
\phi_{t_0}(y) - i\int_{t_0}^{t} \mathrm{d}s\int_{\Sigma_s}\frac{1}{2\pi i}&\left(\int_{\mathbb{R}^4 - i\epsilon e_0} - \int_{\mathbb{R}^4 + i\epsilon e_0}\right)(\not{p}-m)^{-1} e^{ip(x-y)}\mathrm{d}^4p\frac{i_\gamma(\mathrm{d}^4x)}{(2\pi)^3} v_s(x)\not{n}_s(x)\not{A}(x)\phi_s(x)
\end{aligned} \tag{1}$$

Nun spezialisiere ich mich auf Gleichzeitigkeitsflächen in den üblichen Koordinatensystemen: $v_s = 1, \not{n}_s = \gamma^0 e_0, i_\gamma(\mathrm{d}^4x) = \gamma^0 d^3x$

$$\begin{aligned}
\phi_t(y) &= \phi_{t_0}(y) - \int_{[t_0,t]\times\mathbb{R}^3}\frac{\mathrm{d}^4x}{(2\pi)^4}\left(\int_{\mathbb{R}^4 - i\epsilon e_0} - \int_{\mathbb{R}^4 + i\epsilon e_0}\right)(\not{p}-m)^{-1} e^{ip^\alpha(x-y)_\alpha}\mathrm{d}^4p\not{A}(x)\phi_{x^0}(x) \\
&\overset{\text{iterating}}{=} \phi_{t_0}(y) - \int_{[t_0,t]\times\mathbb{R}^3}\frac{\mathrm{d}^4x}{(2\pi)^4}\left(\int_{\mathbb{R}^4 - i\epsilon e_0} - \int_{\mathbb{R}^4 + i\epsilon e_0}\right)(\not{p}-m)^{-1} e^{ip^\beta(x-y)_\beta}\mathrm{d}^4p\not{A}(x)\Bigg\{\phi_{t_0}(x) - \int_{[t_0,x^0]\times\mathbb{R}^3} \\
&\qquad \frac{\mathrm{d}^4z}{(2\pi)^4}\left(\int_{\mathbb{R}^4 - i\epsilon e_0} - \int_{\mathbb{R}^4 + i\epsilon e_0}\right)(\not{q}-m)^{-1} e^{iq^\alpha(z-x)_\alpha}\mathrm{d}^4q\not{A}(z)\phi_{z^0}(z)\Bigg\} \\
&= \phi_{t_0}(y) - \int_{[t_0,t]\times\mathbb{R}^3}\frac{\mathrm{d}^4x}{(2\pi)^4}\left(\int_{\mathbb{R}^4 - i\epsilon e_0} - \int_{\mathbb{R}^4 + i\epsilon e_0}\right)\mathrm{d}^4p(\not{p}-m)^{-1} e^{ip^\alpha(x-y)_\alpha}\not{A}(x)\phi_{t_0}(x) \\
&+ (-1)^2\int_{[t_0,t]\times\mathbb{R}^3}\frac{\mathrm{d}^4x}{(2\pi)^4}\left(\int_{\mathbb{R}^4 - i\epsilon e_0} - \int_{\mathbb{R}^4 + i\epsilon e_0}\right)\mathrm{d}^4p(\not{p}-m)^{-1} e^{ip^\beta(x-y)_\beta}\not{A}(x)\int_{[t_0,x^0]\times\mathbb{R}^3}\frac{\mathrm{d}^4z}{(2\pi)^4} \\
&\qquad \left(\int_{\mathbb{R}^4 - i\epsilon e_0} - \int_{\mathbb{R}^4 + i\epsilon e_0}\right)\mathrm{d}^4q(\not{q}-m)^{-1} e^{iq^\alpha(z-x)_\alpha}\not{A}(z)\phi_{t_0}(z) + \dots
\end{aligned} \tag{2}$$

Nun lassen wir $t \to \infty, t_0 \to -\infty$ gehen, und betrachten nur den Term 2. Ordnung:

$$
\begin{aligned}
&\int_{\mathbb{R}^4}\frac{\mathrm{d}^4x}{(2\pi)^4}\left(\int_{\mathbb{R}^4-i\epsilon e_0}-\int_{\mathbb{R}^4+i\epsilon e_0}\right)\mathrm{d}^4p(\not{p}-m)^{-1}e^{ip^\beta(x-y)_\beta}\not{A}(x)\int_{[-\infty,x^0]\times\mathbb{R}^3}\frac{\mathrm{d}^4z}{(2\pi)^4}\\
&\qquad\left(\int_{\mathbb{R}^4-i\epsilon e_0}-\int_{\mathbb{R}^4+i\epsilon e_0}\right)\mathrm{d}^4q(\not{q}-m)^{-1}e^{iq^\alpha(z-x)_\alpha}\not{A}(z)\phi_{t_0}(z)\\
&\overset{\text{u=z-x}}{=}\int_{\mathbb{R}^4}\frac{\mathrm{d}^4x}{(2\pi)^4}\left(\int_{\mathbb{R}^4-i\epsilon e_0}-\int_{\mathbb{R}^4+i\epsilon e_0}\right)\mathrm{d}^4p(\not{p}-m)^{-1}e^{ip^\beta(x-y)_\beta}\not{A}(x)\\
&\left(\int_{\mathbb{R}^4-i\epsilon e_0}-\int_{\mathbb{R}^4+i\epsilon e_0}\right)\mathrm{d}^4q(\not{q}-m)^{-1}\int_{\mathbb{R}^4}\frac{\mathrm{d}^4u}{(2\pi)^4}e^{iq^\alpha u_\alpha}\not{A}(u+x)\phi_{t_0}(u+x)\theta(-u^0)\\
&=\int_{\mathbb{R}^4}\frac{\mathrm{d}^4x}{(2\pi)^4}\left(\int_{\mathbb{R}^4-i\epsilon e_0}-\int_{\mathbb{R}^4+i\epsilon e_0}\right)\mathrm{d}^4p(\not{p}-m)^{-1}e^{ip^\beta(x-y)_\beta}\not{A}(x)\\
&\left(\int_{\mathbb{R}^4-i\epsilon e_0}-\int_{\mathbb{R}^4+i\epsilon e_0}\right)\mathrm{d}^4q(\not{q}-m)^{-1}\frac{1}{(2\pi)^2}\mathcal{F}\left(\not{A}(\cdot+x)\phi_{t_0}(\cdot+x)\theta(-\cdot^0)\right)(q)
\end{aligned}
\tag{3}
$$

Wobei $\mathcal{F}$ die Fouriertransformation bezeichnet.
Überprüfe das Verhalten der Rücktransformierten von $(\not{q}-m)^{-1}\frac{1}{(2\pi)^2}\mathcal{F}\left(\not{A}(\cdot+x)\phi_{t_0}(\cdot+x)\theta(-\cdot^0)\right)(q)$:(ignoriere erst probleme von fehlender Fouriertransformation)

$$
\begin{aligned}
&\left\|\mathcal{F}^{-1}\left((\not{\ }-m)^{-1}\frac{1}{(2\pi)^2}\mathcal{F}\left(\not{A}(\cdot\cdot+x)\phi_{t_0}(\cdot\cdot+x)\theta(-\cdot\cdot^0)\right)\right)(y)\right\|\\
&\qquad=\left\|\mathcal{F}^{-1}\left((\not{\ }-m)^{-1}\right)\frac{1}{(2\pi)^2}*\not{A}(\cdot+x)\phi_{t_0}(\cdot+x)\theta(-\cdot^0)(y)\right\|
\end{aligned}
\tag{4}
$$

please note at this point that $(\not{\ }-m)^{-1}$ is an analytic function on either cone $\mathbb{R}^4-i$ future. Therefore $(\not{\ }-m)^{-1}$ is the boundary value of an analytic function on a cone in the sense of theorem IX.16 of Simon and Reed.

$$
\begin{aligned}
&= \left\| \int_{\mathbb{R}^-\times\mathbb{R}^3} \mathrm{d}^4 z \not{A}(z+x)\phi_{t_0}(z+x)\theta(-z^0) \int_{\mathbb{R}^4} \mathrm{d}^4 q \lim_{\epsilon\to 0} (\not{q} - i\epsilon \not{e}_0 - m)^{-1} e^{-iq^\alpha (y-z)_\alpha} \right\| \\
&\overset{\text{not valid...}}{=} \left\| \int_{\mathbb{R}^-\times\mathbb{R}^3} \mathrm{d}^4 z \not{A}(z+x)\phi_{t_0}(z+x)\theta(-z^0) \lim_{\epsilon\to 0} \int_{\mathbb{R}^4} \mathrm{d}^4 q (\not{q} - i\epsilon \not{e}_0 - m)^{-1} e^{-iq^\alpha (y-z)_\alpha} \right\| \\
&\overset{\text{analytic integrand}}{=} \left\| \int_{\mathbb{R}^-\times\mathbb{R}^3} \mathrm{d}^4 z \not{A}(z+x)\phi_{t_0}(z+x)\theta(-z^0) \lim_{\epsilon\to 0} \int_{\mathbb{R}^4 - i\kappa e_0} \mathrm{d}^4 q (\not{q} - i\epsilon \not{e}_0 - m)^{-1} e^{-iq^\alpha (y-z)_\alpha} \right\| \\
&\overset{\text{probably valid}}{=} \left\| \int_{\mathbb{R}^-\times\mathbb{R}^3} \mathrm{d}^4 z \not{A}(z+x)\phi_{t_0}(z+x)\theta(-z^0) \int_{\mathbb{R}^4 - i\kappa e_0} \mathrm{d}^4 q (\not{q} - m)^{-1} e^{-iq^\alpha (y-z)_\alpha} \right\| \\
&\leq e^{-\kappa y^0} \int_{\mathbb{R}^-\times\mathbb{R}^3} \mathrm{d}^4 z \left| \not{A}(z+x)\phi_{t_0}(z+x)\theta(-z^0) \right| \left\| \int_{\mathbb{R}^4} \mathrm{d}^4 q (\not{q} - i\kappa \not{e}_0 - m)^{-1} e^{-iq^\alpha (y-z)_\alpha} \right\| \\
&\qquad\overset{\text{partial integration}}{=} e^{-\kappa y^0} \int_{\mathbb{R}^-\times\mathbb{R}^3} \mathrm{d}^4 z \left| \not{A}(z+x)\phi_{t_0}(z+x)\theta(-z^0) \right| \frac{4!}{(y^0 - z^0)^4} \\
&\qquad\qquad \left\| \int_{\mathbb{R}^4} \mathrm{d}^4 q (\not{q} - i\kappa \not{e}_0 - m)^{-5} e^{-iq^\alpha (y-z)_\alpha} \right\| \\
&\leq e^{-\kappa y^0} \int_{\mathbb{R}^-\times\mathbb{R}^3} \mathrm{d}^4 z \left| \not{A}(z+x)\phi_{t_0}(z+x)\theta(-z^0) \right| \frac{4!}{(y^0 - z^0)^4} \int_{\mathbb{R}^4} \mathrm{d}^4 q \left\| (\not{q} - i\kappa \not{e}_0 - m)^{-5} \right\| \\
&\leq e^{-\kappa y^0} \int_{\mathbb{R}^-\times\mathbb{R}^3} \mathrm{d}^4 z \left| \not{A}(z+x)\phi_{t_0}(z+x)\theta(-z^0) \right| \frac{4!}{(y^0 - z^0)^4} \int_{\mathbb{R}^4} \mathrm{d}^4 q \left\| (\not{q} - i\kappa \not{e}_0 - m)^{-5} \right\|
\end{aligned}
\tag{5}
$$

We estimate the last integrand separately:

$$
\begin{aligned}
\left\| (\not{q} - i\kappa \not{e}_0 - m)^{-5} \right\| &\leq \left\| (\not{q} - i\kappa \not{e}_0 - m)^{-1} \right\|^5 = \left\| \frac{\not{q} - i\kappa \not{e}_0 + m}{q^2 - 2i\kappa q^0 - \kappa^2 - m^2} \right\|^5 \\
&\leq \left(\frac{\|\not{q}\| + \kappa + m}{|q^2 - 2i\kappa q^0 - \kappa^2 - m^2|} \right)^5 \leq \left(\frac{|q| + \kappa + m}{\sqrt{(q^2 - \kappa^2 - m^2)^2 + 4\kappa^2 (q^0)^2}} \right)^5 \\
&\overset{\lambda\kappa = \mathrm{q}}{=} \left(\frac{|\lambda| + 1 + \frac{m}{\kappa}}{\sqrt{(\kappa\lambda^2 - \kappa - \frac{m^2}{\kappa})^2 + 4\kappa^2(\lambda^0)^2}} \right)^5 = \left(\frac{|\lambda| + 1 + \frac{m}{\kappa}}{\sqrt{\kappa^2(\lambda^2 - 1)^2 + \frac{m^4}{\kappa^2} - 2m^2(\lambda^2 - 1) + 4\kappa^2(\lambda^0)^2}} \right)^5 \\
&= \left(\frac{|\lambda| + 1 + \frac{m}{\kappa}}{\kappa\sqrt{(\lambda^2 - 1)^2 + \frac{m^4}{\kappa^4} - 2m^2\frac{(\lambda^2 - 1)}{\kappa^2} + 4(\lambda^0)^2}} \right)^5 \\
&\leq \left(\frac{|\lambda| + 2}{\kappa\sqrt{(\lambda^2 - 1)^2 + 4(\lambda^0)^2 - (1 + 2|\lambda - 1|)}} \right)^5
\end{aligned}
\tag{6}
$$

Where the last inequality holds only for κ large enough. For $y^0 > 0$, we can let the first factor tend to zero, the problem is however, that the behaviour of the last integral for large κ is not easily shown to be nice.