I begin with a collection of definitions and formulas useful in this setting.

1. basic definitions

Definition 1.1. *Throuought this document the letters* A, B, C, G *and* F *with or without indices represent four-potentials, elements of* $C_c^\infty(\mathbb{R}^4, \mathbb{R}^4)$*. Furthermore we assume that for some fixed compact* $K \subseteq \mathbb{R}^4$ *all the appearing four-potentials are supported in* K*.*

Definition 1.2. *Likewise the greek letter* Σ *with or without indices and with or without (multiple)* $'$*s attached to it represent spacelike hypersurfaces of Minkowski spacetime.*

Definition 1.3. *Furthermore*

$$\forall \Sigma : \mathcal{H}_\Sigma := L^2(\Sigma, \mathbb{C}^4, i_\gamma(d^4x)). \tag{1}$$

Definition 1.4. *We denote the one-particle Dirac time evolution operator by*

$$U^A_{\Sigma,\Sigma'} : \mathcal{H}_\Sigma \to \mathcal{H}_{\Sigma'}. \tag{2}$$

Definition 1.5. *For initial polarization* $V_{\Sigma_0} \subseteq \mathcal{H}_{\Sigma_0}$ *with* Σ_0 *before* K *and* $\Phi_{\Sigma_0} : \ell_2 \to \mathcal{H}_{\Sigma_0}$ *with range*$\Phi_{\Sigma_0} = V_{\Sigma_0}$*, we define*

$$\forall \Sigma, \forall A : C_\Sigma(A) := \{V^A_\Sigma \mid V^A_\Sigma \approx U^A_{\Sigma,\Sigma_0} V_{\Sigma_0}\}. \tag{3}$$

It turns out (ref) that $C_\Sigma(A)$ *only depends on the projection* $\Sigma \ni x \mapsto x_\alpha A^\alpha(x)$*, furthermore we define*

$$\mathcal{F}^A_\Sigma := \mathcal{F}(U^A_{\Sigma,\Sigma_0} \Phi_{\Sigma_0}). \tag{4}$$

Definition 1.6. *We call* $\tilde{U}^A_{\Sigma',\Sigma} : \mathcal{F}^A_\Sigma \to \mathcal{F}^{A'}_{\Sigma'}$ *unitary such that* $\forall A, A', \Sigma, \Sigma', \Sigma''$*:*

$$\tilde{U}^A_{\Sigma'',\Sigma'} \tilde{U}^A_{\Sigma',\Sigma} = \tilde{U}^A_{\Sigma'',\Sigma} \tag{func}$$

$$germ_{Vol\Sigma',\Sigma}(A) = germ_{Vol\Sigma',\Sigma}(A') \Rightarrow \tilde{U}^A_{\Sigma',\Sigma} = \tilde{U}^{A'}_{\Sigma',\Sigma} \tag{loc}$$

$$\tag{reg}$$

a lift of $U^A_{\Sigma',\Sigma}$*. Here the germ of two functions* A *and* A' *are equal iff*

$$\exists U \supseteq \Sigma, U open : A|_U = A'|_U . \tag{5}$$

Definition 1.7. *For* Σ_{in} *before* K *and* Σ_{out} *after* K *we define*

$${}_A\tilde{S}_{A+F} := \tilde{U}^A_{\Sigma_{in},\Sigma_{out}} \tilde{U}^{A+F}_{\Sigma_{out},\Sigma_{in}} : \mathcal{F}^A_{\Sigma_{in}} \circlearrowleft \tag{6}$$

Definition 1.8. *We denote for a fixed* $U^A_{\Sigma_{in},\Sigma_0} \Phi_{\Sigma_0} \sim \Psi_{in} : \ell^2 \to \mathcal{H}_{in}$ *by*

$$\mathbb{1}_{in} : \mathcal{H}_{in} \circlearrowleft, \tag{7}$$

the identity on that space and denote by

$$\overline{\mathbb{1}}_{in} : \mathcal{F}(\Psi_{in}) \to \mathcal{F}(U^A_{\Sigma_{in},\Sigma_0} \Phi_{\Sigma_0}) \tag{8}$$

its lift.

Definition 1.9. *We also introduce the projector notation:*

$${}_AS_{F--} := \Psi^*_{in}\ {}_AS_F \Psi_{in} : \ell^2 \circlearrowleft . \tag{9}$$

Definition 1.10. *and the only (partial) lift which can naturally be written down*

$$ {}_A\overline{S}_F := \mathcal{L}_{{}_AS_F}\mathcal{R}_{{}_FS_A}\frac{1}{\sqrt{\det {}_AS_{F--}\ {}_FS_{A--}}} : \mathcal{F}(\Psi_{in}) \circlearrowleft, \tag{10} $$

where the last factor is just for normalisation i.e. to make this operator unitary. Please note that this is not a lift in the sense of Definition 1.6 since it may not fulfill any of the conditions.

Definition 1.11. *We can make a connection between the proper lift and (10) by*

$$ {}_A\hat{S}_F := \mathbb{1}^*_{in}\,{}_A\tilde{S}_F\mathbb{1}_{in} : \mathcal{F}(\Psi_{in}) \circlearrowleft . \tag{11} $$

Now $\overline{S}$ and $\hat{S}$ agree up to a phase (ref), so we define

$$ {}_Az_F \circ {}_A\overline{S}_F := {}_A\hat{S}_F. \tag{12} $$

There is yet another phase which characterizes the deficiency of (10), namely

Definition 1.12.

$$ \forall A, B, C : \Gamma^{-1}_{A,B,C} \circ \mathbb{1} = {}_A\overline{S}_B\ \ {}_B\overline{S}_C\ \ {}_C\overline{S}_A. \tag{13} $$

We also introduce a notation for general complex numbers

Definition 1.13.

$$ \forall z \in \mathbb{C}\backslash\{0\} : arg(z) := \frac{z}{|z|}. \tag{14} $$

2. useful formulas

Lemma 1. *It is true that*

$$ \forall F \prec G : {}_A\tilde{S}_{A+F+G} = {}_A\tilde{S}_{A+G}\,{}_A\tilde{S}_{A+F} \tag{temporal separation} $$

holds.

Lemma 2. *It is true that*

$$ \forall F \prec G : {}_{A+G}z_{A+F+G} = {}_Az_{A+F} \tag{15} $$

holds.

Lemma 3. *There are more ways to conveniently express $\Gamma_{A,B,C}$ for all A, B and C, namely*

$$ \Gamma_{A,B,C} = {}_Az_B\ \ {}_Bz_C\ \ {}_Cz_A, \tag{16} $$

and

$$ \Gamma^{-1}_{A,B,C} = argdet({}_AS_{B--}\ \ {}_BS_{C--}\ \ {}_CS_{A--}). \tag{17} $$

Furthermore it is true that

$$ \Gamma_{A,B,C} = \Gamma_{B,C,A} = \Gamma^{-1}_{C,B,A} \tag{18} $$

holds. Furthermore, the tetrahedron rule holds for all A, B, C and D

$$ \Gamma_{B,C,D} = \Gamma_{A,C,D}\Gamma_{B,A,D}\Gamma_{B,C,A}. \tag{19} $$

Lemma 4. *For the case $F \prec G$ one can find a relation between z and Γ involving just one instance of each object:*

$$ \partial_F\partial_G \ln {}_Az_{A+F+G} = -\partial_F\partial_G \ln \Gamma_{A,A+G,A+F+G} \tag{20} $$

Finally we present a compact formula that connects derivatives of the current to Γ:

Lemma 5. *In the case $F \prec G$,*

$$\partial_F j_{A+F}(G) = i\partial_F \partial_G \ln \frac{\Gamma_{A,A+F,A+F+G}}{\Gamma_{A,A+G,A+F+G}} \tag{21}$$

holds.

Lemma 5 makes it easy to see that the derivative of the current is antisymmetric with respect to F and G.
There is another simplification of the derivative of the current that follows from the symmetries of Γ.

Lemma 6. *For $F \prec G$ we can simplify the result of lemma 5 to*

$$\partial_F j_{A+F}(G) = -2i\partial_F \partial_G \ln \Gamma_{A,A+G,A+F} = 2i\partial_F \partial_G \ln {}_A z_{A+F+G}. \tag{22}$$

Theorem 2.1. *For $F \prec G$, the derivative of the current can more explicitly be expressed as*

$$\partial_F j_{A+F}(G) = -2\Im \operatorname{tr}\left[(\partial_G\, {}_A S_{A+G})_{-+} (\partial_F\, {}_A S_{A+F})_{+-} \right]. \tag{23}$$

Markus: todo: understand better why the off-diagonal parts plus all of their derivatives are Hilbert-Schmidt operators

3. proofs of useful formulas

Proof of lemma 1: Let F and G be such that $F \prec G$ holds. Then choose Σ such that Σ is before $\operatorname{supp} G$ but after $\operatorname{supp} F$. Then it follows that

$$\begin{aligned}
&{}_A\tilde{S}_{A+F+G} = {}_A\tilde{S}_{A+G}\, {}_A\tilde{S}_{A+F} \\
&\iff \tilde{U}^{A}_{\Sigma_{\text{in}},\Sigma_{\text{out}}} \tilde{U}^{A+F+G}_{\Sigma_{\text{out}},\Sigma_{\text{in}}} = \tilde{U}^{A}_{\Sigma_{\text{in}},\Sigma_{\text{out}}} \tilde{U}^{A+G}_{\Sigma_{\text{out}},\Sigma_{\text{in}}} \tilde{U}^{A}_{\Sigma_{\text{in}},\Sigma_{\text{out}}} \tilde{U}^{A+F}_{\Sigma_{\text{out}},\Sigma_{\text{in}}} \\
&\iff \tilde{U}^{A+F+G}_{\Sigma_{\text{out}},\Sigma_{\text{in}}} = \tilde{U}^{A+G}_{\Sigma_{\text{out}},\Sigma_{\text{in}}} \tilde{U}^{A}_{\Sigma_{\text{in}},\Sigma_{\text{out}}} \tilde{U}^{A+F}_{\Sigma_{\text{out}},\Sigma_{\text{in}}} \\
&\overset{\text{(func)}}{\iff} \tilde{U}^{A+F+G}_{\Sigma_{\text{out}},\Sigma} \tilde{U}^{A+F+G}_{\Sigma,\Sigma_{\text{in}}} = \tilde{U}^{A+G}_{\Sigma_{\text{out}},\Sigma} \tilde{U}^{A+G}_{\Sigma,\Sigma_{\text{in}}} \tilde{U}^{A}_{\Sigma_{\text{in}},\Sigma} \tilde{U}^{A}_{\Sigma,\Sigma_{\text{out}}} \tilde{U}^{A+F}_{\Sigma_{\text{out}},\Sigma} \tilde{U}^{A+F}_{\Sigma,\Sigma_{\text{in}}} \\
&\overset{\text{(loc)}}{\iff} \tilde{U}^{A+G}_{\Sigma_{\text{out}},\Sigma} \tilde{U}^{A+F}_{\Sigma,\Sigma_{\text{in}}} = \tilde{U}^{A+G}_{\Sigma_{\text{out}},\Sigma} \tilde{U}^{A}_{\Sigma,\Sigma_{\text{in}}} \tilde{U}^{A}_{\Sigma_{\text{in}},\Sigma} \tilde{U}^{A}_{\Sigma,\Sigma_{\text{out}}} \tilde{U}^{A}_{\Sigma_{\text{out}},\Sigma} \tilde{U}^{A+F}_{\Sigma,\Sigma_{\text{in}}} \\
&\iff \tilde{U}^{A+G}_{\Sigma_{\text{out}},\Sigma} \tilde{U}^{A+F}_{\Sigma,\Sigma_{\text{in}}} = \tilde{U}^{A+G}_{\Sigma_{\text{out}},\Sigma} \tilde{U}^{A+F}_{\Sigma,\Sigma_{\text{in}}}.
\end{aligned}$$

□

Now we prove lemma 2. Let $F \prec G$. Using definition (12), as well as lemma 1 which holds for $\hat{S}$ as well (simply by inserting the proper identities) we compute

$$\begin{aligned}
&{}_{A+G}z_{A+F+G} \circ {}_{A+G}\overline{S}_{A+F+G} = {}_{A+G}\hat{S}_{A+F+G} = {}_{A+G}\hat{S}_A\, {}_A\hat{S}_{A+F+G} \\
&\overset{F\prec G}{=} {}_{A+G}\hat{S}_A\ \ {}_A\hat{S}_{A+G}\ \ {}_A\hat{S}_{A+F} = {}_A\hat{S}_{A+F} = {}_A z_{A+F} \circ {}_A\overline{S}_{A+F}.
\end{aligned}$$

Now by evaluation ${}_{A+F+G}\overline{S}_{A+G}\ \ {}_A\overline{S}_{A+F}$ we find the relation between the appearing phases. By a computation analogous to the one we just did for the one-particle scattering operator we see that the left-operation part of this operator is just the identity. The right-operation may still contribute a determinant; however, since $\overline{S}$ is unitary the determinant may only be a phase. Therefore we see that

$$\begin{aligned}
&{}_{A+G}z_{A+F+G}\ \ {}_{A+F}z_A = \operatorname{argdet}(({}_{A+F}S_A)_{--}({}_{A+G}S_{A+F+G})_{--}) \\
&= \operatorname{argdet}(({}_{A+F}S_A)_{--}({}_{A+G}S_A\ \ {}_AS_{A+F+G})_{--}) = \operatorname{argdet}(({}_{A+F}S_A)_{--}({}_AS_{A+F})_{--}) \\
&= \operatorname{argdet}(({}_{A+F}S_A)_{--}({}_{A+F}S_A)^*_{--}) = 1
\end{aligned}$$

holds. □

Now for lemma 3. Formula (17) can be seen from the definition of Γ by taking the vacuum expectation value. Formula (18) can directly be seen form the definition of Γ. We prove (16), by observing that

$$ {}_A\tilde{S}_C = {}_A\tilde{S}_B \ {}_B\tilde{S}_C \tag{24} $$

holds, therefore it also holds for $\hat{S}$. Inserting definitions (12) and (13) yields

$$ {}_Az_C \circ {}_A\overline{S}_C = {}_Az_B \ {}_Bz_C \circ {}_A\overline{S}_B \ {}_B\overline{S}_C \tag{25} $$

and

$$ {}_Az_C \ {}_Bz_A \ {}_Cz_B \circ \mathbb{1} = {}_A\overline{S}_B \ {}_B\overline{S}_C \ {}_C\overline{S}_A = \Gamma^{-1}_{A,B,C}. \tag{26} $$

Rearranging yields (16). For the tetrahedron rule we simply insert (16) into the right hand side and get

$$ \begin{aligned} &\Gamma_{A,C,D}\Gamma_{B,A,D}\Gamma_{B,C,A} = {}_Az_C \ {}_Cz_D \ {}_Dz_A \ {}_Bz_A \ {}_Az_D \ {}_Dz_B \ {}_Bz_C \ {}_Cz_A \ {}_Az_B \\ &= {}_Cz_D \ {}_Dz_B \ {}_Bz_C = \Gamma_{B,C,D}. \end{aligned} $$

□

Now to prove lemma 4. Let again $F \prec G$ be true. By adding terms which vanish after splitting products into sums in the logarithm and application of derivatives we obtain

$$ \partial_F\partial_G \ln {}_Az_{A+F+G} = -\partial_F\partial_G \ln {}_{A+F+G}z_A \ {}_Az_{A+G} \ {}_Az_{A+F}. $$

Modifying the last factor by (15) yields

$$ \partial_F\partial_G \ln {}_Az_{A+F+G} = -\partial_F\partial_G \ln {}_{A+F+G}z_A \ {}_Az_{A+G} \ {}_{A+G}z_{A+F+G} = -\partial_F\partial_G \ln \Gamma_{A,A+G,A+F+G}. $$

□

We come to the proof of lemma 5. Let $F \prec G$ be true. we start with the definition of the current:

$$ j_A(G) = i\partial_G \left\langle \bigwedge \Phi, {}_A\tilde{S}_{A+G} \bigwedge \Phi \right\rangle $$

Now we take the derivative of this expression and insert the definition of z

$$ \partial_F j_{A+F}(G) = i\partial_F\partial_G \ {}_{A+F}z_{A+F+G} \left\langle \bigwedge \Phi, {}_{A+F}\overline{S}_{A+F+G} \bigwedge \Phi \right\rangle. $$

As the expression which we take derivatives of is equal to 1 at $G = 0$ and the linearisation of the logarithm around 1 is the identity we can safely insert a logarithm, yielding

$$ \partial_F j_{A+F}(G) = i\partial_F\partial_G \ln {}_{A+F}z_{A+F+G} \left\langle \bigwedge \Phi, {}_{A+F}\overline{S}_{A+F+G} \bigwedge \Phi \right\rangle. $$

This will greatly simplify the upcoming calculations. Next we insert the relation between $\Gamma_{A+F,A+F+G,A}$ and z using ${}_Az_{A+F}$ with respect to G vanishes, (16) giving

$$ \partial_F j_{A+F}(G) = i\partial_F\partial_G \ln {}_Az_{A+F+G}\Gamma_{A+F,A+F+G,A} \left\langle \bigwedge \Phi, {}_{A+F}\overline{S}_{A+F+G} \bigwedge \Phi \right\rangle. $$

Now we insert the identity twice inside the scalar product

$$ \begin{aligned} &\partial_F j_{A+F}(G) = i\partial_F\partial_G \ln {}_Az_{A+F+G}\Gamma_{A+F,A+F+G,A} \\ &\left\langle \bigwedge \Phi, {}_{A+F}\overline{S}_A \ {}_A\overline{S}_{A+F} \ {}_{A+F}\overline{S}_{A+F+G} \ {}_{A+F+G}\overline{S}_A \ {}_A\overline{S}_{A+F+G} \bigwedge \Phi \right\rangle. \end{aligned} \tag{27} $$

The central three occurrence of $\overline{S}$ give $\Gamma^{-1}_{A,A+F,A+F+G}$ cancelling exactly the gamma factor in front after cyclic permutation. As a next step we evaluate the scalar product. Since the operators $\overline{S}$ are unitary this yields the argument of a determinant:

$$\begin{aligned}
&\left\langle \bigwedge \Phi, {}_{A+F}\overline{S}_A \ \ {}_A\overline{S}_{A+F+G} \bigwedge \Phi \right\rangle \\
&= \left\langle \bigwedge \Phi, \mathcal{L}_{{}_{A+F}S_A} \mathcal{R}_{({}_AS_{A+F})_{--}} \mathcal{L}_{{}_AS_{A+F+G}} \mathcal{R}_{({}_{A+F+G}S_A)_{--}} \bigwedge \Phi \right\rangle \frac{1}{N} \\
&= \left\langle \bigwedge \Phi, \mathcal{L}_{{}_{A+F}S_A \ \ {}_AS_{A+F+G}} \mathcal{R}_{({}_{A+F+G}S_A)_{--} \ \ ({}_AS_{A+F})_{--}} \bigwedge \Phi \right\rangle \frac{1}{N} \\
&= \left\langle \bigwedge \Phi, \mathcal{L}_{{}_{A+F}S_{A+F+G}} \bigwedge \Phi ({}_{A+F+G}S_A)_{--} \ \ ({}_AS_{A+F})_{--} \right\rangle \frac{1}{N} \\
&= \operatorname{argdet}(({}_{A+F}S_{A+F+G})_{--}({}_{A+F+G}S_A)_{--} \ \ ({}_AS_{A+F})_{--}),
\end{aligned}$$

which is, by (17), given by

$$\left\langle \bigwedge \Phi, {}_{A+F}\overline{S}_A \ \ {}_A\overline{S}_{A+F+G} \bigwedge \Phi \right\rangle = \Gamma_{A,A+F,A+F+G}. \tag{28}$$

Taking all of this together yields

$$\partial_F j_{A+F}(G) = i\partial_F \partial_G \ln {}_A z_{A+F+G} \Gamma_{A,A+F,A+F+G}.$$

Now we replace the appearance of z using lemma 4, giving

$$\partial_F j_{A+F}(G) = i\partial_F \partial_G \ln \frac{\Gamma_{A,A+F,A+F+G}}{\Gamma_{A,A+G,A+F+G}}. \tag{29}$$

□

Now for lemma 6. We will show the first equality, the second follows by lemma 4. For this proof we abbreviate, for variables $a, b, c, \overline{a}, \overline{b}, \overline{b} \in \mathbb{R}$, $x := ((a, \overline{a}), (b, \overline{b}), (c, \overline{c}))$

$$f(x) := f((a, \overline{a}), (b, \overline{b}), (c, \overline{c})) := \ln \Gamma_{A+a\cdot F+\overline{a}\cdot G, A+b\cdot F+\overline{b}\cdot G, A+c\cdot F+\overline{c}\cdot G}. \tag{30}$$

Now we are interested in

$$\begin{aligned}
\partial_F \partial_G \ln \Gamma_{A,A+F,A+F+G} &= \partial_\varepsilon \partial_\delta f((0,0),(\varepsilon,0),(\varepsilon,\delta)) \\
&= (\partial_b + \partial_c)\partial_{\overline{c}} \, f(x)|_{x=0} = \partial_b \partial_{\overline{c}} \, f(x)|_{x=0} \, .
\end{aligned} \tag{31}$$

The last equality holds due to $f((a, \overline{a}), (b, \overline{b}), (c, \overline{c})) = -f((b, \overline{b}), (a, \overline{a}), (c, \overline{c}))$, which implies

$$\partial_c \partial_{\overline{c}} f(x)|_{x=0} = -\partial_c \partial_{\overline{c}} f(x)|_{x=0} = 0. \tag{32}$$

For the very same reason we conclude

$$\begin{aligned}
&- \partial_F \partial_G \ln \Gamma_{A,A+G,A+F+G} = -\partial_\varepsilon \partial_\delta f((0,0),(0,\delta),(\varepsilon,\delta)) \\
&= -\partial_c(\partial_{\overline{b}} + \partial_{\overline{c}}) f(x)|_{x=0} = -\partial_c \partial_{\overline{b}} f(x)|_{x=0} = \partial_b \partial_{\overline{c}} f(x)|_{x=0},
\end{aligned}$$

where the last equality follows again from the antisymmetry of f. We conclude

$$- \partial_F \partial_G \ln \Gamma_{A,A+G,A+F+G} = \partial_F \partial_G \ln \Gamma_{A,A+F,A+F+G}. \tag{33}$$

and thereby the lemma. □

We will now continue with the proof of theorem 2.1. The major part of this proof is contained in the following auxiliary

Lemma 7. *For operators*

$$\begin{aligned}
A &: \mathbb{R} \to (\mathcal{H} \to \mathcal{H}) \\
B &: \mathbb{R}^2 \to (\mathcal{H} \to \mathcal{H}) \, ,
\end{aligned}$$

such that $A(\varepsilon), B(\varepsilon,\delta) \in I_1$*,* $A^* = A, B(\varepsilon,0)^* = B^*(\varepsilon,0)$ *and* $A(0) = 0 = B(\varepsilon,0)$ *for all* $\varepsilon,\delta \in \mathbb{R}$ *hold,*

$$\partial_\delta \partial_\varepsilon \ln \mathit{argdet}\,[1 + A(\varepsilon) + B(\varepsilon,\delta))] = i\Im \operatorname{tr} D_1 D_2 B(x)|_{x=0} \tag{34}$$

is true.

Now follows a corollary which we will not prove. Since most of the time we work with the arg of complex numbers it is worth noting that

Corollary 3.1.

$$\forall z : \mathbb{R} \to \mathbb{C}\backslash\{0\} : \left(\frac{z}{|z|}\right)' = i\frac{z}{|z|}\Im\frac{z'}{z} \tag{35}$$

holds.

Proof of the lemma: We use corollary 3.1 to find

$$\begin{aligned}
&\partial_\varepsilon \ln \operatorname{argdet}\,[1 + A(\varepsilon) + B(\varepsilon,\delta))] = \frac{\partial_\varepsilon \operatorname{argdet}[1 + A(\varepsilon) + B(\varepsilon,\delta)]}{\operatorname{argdet}[1 + A(0) + B(0,\delta)} \\
&= i\frac{\operatorname{argdet}[1 + B(0,\delta)]}{\operatorname{argdet}[1 + B(0,\delta)]}\Im\left[\frac{\operatorname{argdet}[1 + B(0,\delta)]^* \partial_\varepsilon \operatorname{argdet}[1 + A(\varepsilon) + B(\varepsilon,\delta)]}{\operatorname{argdet}[1 + B(0,\delta)]^2}\right] \\
&= i\Im\left[\frac{\operatorname{argdet}[1 + B(0,\delta)]\partial_\varepsilon \operatorname{argdet}[1 + A(\varepsilon) + B(\varepsilon,\delta)]}{\operatorname{argdet}[1 + B(0,\delta)]^2}\right] \\
&= i\Im\left[\frac{\partial_\varepsilon \operatorname{argdet}[1 + A(\varepsilon) + B(\varepsilon,\delta)]}{\operatorname{argdet}[1 + B(0,\delta)]}\right].
\end{aligned}$$

Now we use that the linearisation of the determinant around the identity equal to the trace is. This yields

$$\partial_\varepsilon \ln \operatorname{argdet}\,[1 + A(\varepsilon) + B(\varepsilon,\delta))] = i\Im\left[\frac{\operatorname{tr}[\partial_\varepsilon A(\varepsilon) + \partial_\varepsilon B(\varepsilon,\delta)]}{\operatorname{argdet}[1 + B(0,\delta)]}\right]. \tag{36}$$

Inserting the second derivative simplifies the expression after one recognizes that the second summand inside the imaginary part is real, since its a product of derivatives of selfadjoint traceclass operators. The corresponding calculation is

$$\begin{aligned}
&\partial_\delta\partial_\varepsilon \ln \operatorname{argdet}\,[1 + A(\varepsilon) + B(\varepsilon,\delta))] \\
&= i\Im\left[\operatorname{tr}\partial_\delta\partial_\varepsilon B(\varepsilon,\delta) - \frac{\operatorname{tr}[\partial_\varepsilon A(\varepsilon) + \partial_\varepsilon B(\varepsilon,0)]}{\det(1 + B(0,0))}\operatorname{tr}[\partial_\delta B(0,\delta)]\right] \\
&= i\Im\left[\operatorname{tr}\partial_\delta\partial_\varepsilon B(\varepsilon,\delta) - \operatorname{tr}[\partial_\varepsilon A(\varepsilon)]\operatorname{tr}[\partial_\delta B(0,\delta)]\right] \\
&= i\Im\left[\operatorname{tr}\partial_\delta\partial_\varepsilon B(\varepsilon,\delta)\right],
\end{aligned}$$

where we used that $B(0,\delta)$ is selfadjoint and $B(\varepsilon,0) = 0$. This concludes the proof of the lemma.

So we start out with the most recent result about $\partial_F j_{A+F}(G)$, use lemma 3 and manipulate the appearing projections to bring it in a form that more explicitly has a determinant:

$$\begin{aligned}
&\partial_F\partial_G \ln \Gamma_{A,A+G,A+F+G} = \partial_F\partial_G \ln \operatorname{argdet}\left({}_A S_{A+G}\right)_{--}\left({}_{A+G}S_{A+F}\right)_{--}\left({}_{A+F}S_A\right)_{--} \\
&= \partial_F\partial_G \ln \operatorname{argdet}\left[\left({}_A S_{A+F}\right)_{--}\left({}_{A+F}S_A\right)_{--} - \left({}_A S_{A+G}\right)_{-+}\left({}_{A+G}S_{A+F}\right)_{+-}\left({}_{A+F}S_A\right)_{--}\right] \\
&= \partial_F\partial_G \ln \operatorname{argdet}\left[\mathbb{1}_{--} - \left({}_A S_{A+F}\right)_{-+}\left({}_{A+F}S_A\right)_{+-} - \left({}_A S_{A+G}\right)_{-+}\left({}_{A+G}S_{A+F}\right)_{+-}\left({}_{A+F}S_A\right)_{--}\right].
\end{aligned}$$

As we now take the derivative of a trace-class perturbation of the identity we can see that 1. this expression is well-defined, since the off diagonal components of the scattering matrix and its derivatives are Hilbert-Schmidt, and 2. we can use the just derived lemma. This results in

$$\begin{aligned}&\partial_F\partial_G \ln \Gamma_{A,A+G,A+F+G} = -i\partial_F\partial_G \Im \operatorname{tr} \left({}_A S_{A+G}\right)_{-+} \left({}_{A+G} S_{A+F}\right)_{+-} \left({}_{A+F} S_A\right)_{--}\\ &= -i\Im \operatorname{tr} \left(\partial_G\, {}_A S_{A+G}\right)_{-+} \left(\partial_F\, {}_A S_{A+F}\right)_{+-},\end{aligned}$$

where the last equality follows by acknowledging that terms vanish if they contain a factor of $\mathbb{1}_{+-}$. The theorem follows by inserting lemma 6. $\square$