On Interaction and External Fields in Relativistic Quantum Physics

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July 7, 2020

Abstract

The main subject of this thesis is the problem of introducing interactions into relativistic quantum mechanics. Two very different alternatives of approaching this issue are discussed.

The first approach considers a very recent relativistically invariant integral equation for multi-time wavefunctions by Matthias Lienert. This is an equation for fermions interacting directly along light-like configurations. Previous results on existence and uniqueness of solutions of a variant of this equation for scalar particles are extended to include more realistic types of interaction. Furthermore, a first result on existence and uniqueness of solutions of a variant of said equation for spin 1/2 particles is proven.

The second approach, as a first step to approach interactions, introduces external electrodynamic fields into otherwise free quantum field theory. In previous results candidates for the time evolution operator in this setting have been constructed; however, a phase that depends on the external field was left unidentified. This problem is partially addressed by a geometric construction of the missing phase in the scattering regime from an object fulfilling desirable properties, such as a certain causality condition. Secondly the particular approach to quantum field theory employed in this thesis is compared to the algebraic approach common in the setting of quantum field theory on curved spacetimes. The central objects of the two approaches and their natural equivalence classes are juxtaposed and theorems are proven how construct the central object of one approach from the central object of the other. Finally, a simple formula for the scattering operator in terms of the one particle scattering operator is given and proven to be well-defined.

Zusammenfassung

Das Hauptthema dieser Arbeit sind die Schwierigkeiten die dabei auftreten Wechselwirkungen in die relativistische Quantenmechanik einzuführen. Dabei werden zwei sehr unterschiedliche Herangehensweisen verfolgt.
In der Ersten wird eine kürzlich von Matthias Lienert vorgestellten relativistisch invariante Integralgleichung für Wellenfunktionen für mehrere Zeitkoordinaten thematisiert. Dies ist eine Gleichung für Fermionen welche direkt entlang lichtartiger Konfigurationen wechselwirken. Dabei werden bereits bestehende Resultate über Existenz und Eindeutigkeit von Lösungen von einer Variante dieser Gleichung für skalare Teilchen für realistischere Wechselwirkungen erweitert. Weiterhin wird ein erstes Resultat über Existenz und Eindeutigkeit von Lösungen für eine Variante der Gleichung für Teilchen mit Spin $1/2$ bewiesen.
In der zweiten Methode wird, als ersten Schritt in Richtung einer wechselwirkenden Theorie, ein externes elektromagnetisches Feld in eine ansonsten freie Quantenfeldtheorie eingeführt. In früheren Resultaten konstruierte Zeitentwicklungsoperatoren ließen dabei eine vom Feld abhängige Phase unbestimmt. Dieses Problem wird teilweise behoben, indem eine geometrische Konstruktion dieser Phase im Streuregime aus einem Objekt mit einigen wünschenswerten Eigenschaften, etwa eine Kausalitätsbedingung, angegeben wird. Anschließend wird die hier verwendete Formulierung der Quantenfeldtheorie mit dem in der Quantenfeldtheorie auf gekrümmter Raumzeit üblichen algebraischen Formulierung verglichen. Dabei werden die jeweils zentralen Objekte und deren natürlich auftretende Äquivalenzklassen betrachtet. Zusätzlich wird bewiesen wie das zentrale Objekt einer Formulierung aus dem der jeweils anderen hervorgeht. Schließlich wird eine einfache Formel für den Streuoperator als Funktion des Einteilchenstreuoperators angegeben und dessen Wohldefiniertheit gezeigt.

Notes on Style

In the following I will mostly be using personal pronouns denoting the plural such as “we”. This does not solely refer to my, the authors, person, nor does it refer to the author and his supervisors or other coauther of the preprints that led to this thesis. Instead it refers to you the reader and myself, as I want to follow the example of the textbooks I enjoyed reading as an undergraduate student which invited the reader to perform all of the announced tasks together with the text.

Furthermore, there will be many occasions to speak of individual terms that appear in previous equations or estimates. Often, such terms will reappear only once or twice and are not important enough to receive a name of their own. On such occations we will denote them by the number they were given initially enclosed in round brackets. If we want to emphasise the dependence of some of the variables appearing in the term we will be using standard notation for functions where the function name is the number of the term enclosed in round brackets.

Contents

Chapter 1

Introduction

The difficulties in writing down a mathematically consistent relativistically invariant interacting quantum mechanical theory in four dimensional spacetime have had a big impact on the development of theoretical physics of the past century and the question of whether these difficulties have been overcome today still remain subject of controversy. As an illustration see for example [47] for a book that disputes the existence of problems of the full theory and see e.g. [8] for a very modern book on the rigorous mathematical results on quantum field theory that barely moves past the free theory, which is not because of a fear of scaring off readers by too abstract or complicated mathematics. This controversy remains, despite the fact that the relativistic analog of Schrödingers equation, the Dirac equation

$$0 = (i\partial\!\!\!/ - m)\phi, \tag{1.1}$$

has been found as early as 1928[11]. Here and in the following a slashed four vector denotes

$$\not{A} := A_\alpha \gamma^\alpha, \tag{1.2}$$

where Einstein's summation convention is used. The matrices $\gamma^\alpha \in \mathbb{C}^{4\times 4}$ fulfil the anti-commutation relation

$$\forall \alpha, \beta \in \{0,1,2,3\} : \{\gamma^\alpha, \gamma^\beta\} := \gamma^\alpha\gamma^\beta + \gamma^\beta\gamma^\alpha = 2g^{\alpha\beta}, \tag{1.3}$$

where g is the Minkowski metric. We will work with the $+---$ metric signature and the Dirac representation of this algebra. Squared four dimensional objects always refer to the Minkowski square, meaning for all $a \in \mathbb{C}^4$, $a^2 := a^\alpha a_\alpha$. We will be working in units where $\hbar = 1 = c$. Two very important reasons for these difficulties are:

1. The spectrum of the Hamiltonian generating the time evolution of equation (1.1) is $\sigma(H^0) = \,]-\infty, -m] \cup [m, \infty[$.

2. Poincaré invariance of the equations of motion of a theory rules out mechanisms of interaction.

The first of these items is problematic, because a particle that couples to a radiation field but whose kinetic energy has no lower bound may continuously lose energy while radiating. That is, systems of such particles would not have a stable ground state. We will discuss two possible ways of omitting this problem.

1. In the first approach, we do not introduce a radiation field. We will stick as closely as possible to non-relativistic physics and introduce a wavefunction for $N \in \mathbb{N}$ particles. In order for this function to transform relativistically we will consider it a function of N spacetime points $x_i \in \mathbb{R}^4$. This *multi-time* formulation goes back to Dirac [10]. It heavily inspired works of Tomonaga [53] and Schwinger [49] and has inspired research since then [36, 48, 13, 45, 54]. See [42, 40, 27, 29, 26, 33, 25, 38] for mor recent work on the subject and [32] for an overview.

A natural way of introducing interaction in this setting would be to let the wavefunction solve N Dirac equations each minimally coupled to a potential; however, recent no-go theorems [41, 7] show that no such choice of potentials compatible with Poincaré invariance results in an interacting system of particles.
The approach we will follow in chapter 2 bypasses the main reason why the theorem in [7] is so powerful by introducing only a single integral equation for the wavefunction of the particles. The resulting equation

$$\psi(x_1,x_2) = \psi^{\text{free}}(x_1,x_2) + i\,\frac{e_1 e_2}{4\pi}\int d^4x_1'\,d^4x_2' \\ \times\, \mathcal{S}_1^{\text{ret}}(x_1-x_1')\mathcal{S}_2^{\text{ret}}(x_2-x_2')\,\gamma_1^{\mu}\gamma_{2,\mu}\delta((x_1'-x_2')^2)\psi(x_1',x_2'), \tag{1.4}$$

for whose relatives we will investigate existence of solutions seem to be hardly related to the theory of quantum electrodynamics; however, before coming to this conclusion one should compare this equation with equation (4) of [17] and take note of footnote 7 on the same page. The main difference is that Feynman only uses positive Fourier modes in time for his interaction term.
Despite this fact and the similarity between (1.4) and the Bethe-Salpeter equation, not much is known about the mathematical properties of equation (1.4), the main results about related equations are summarised in section 2.1.2.
2. Another approach to circumvent the problems associated with item 1 that results in quantum field theory is to fill up all negative energy states and use an antisymmetric description as called for by Paulis exclusion principle. This will prevent the instability described earlier, but renders many physically relevant quantities ill defined. The problems associated with finding a mathematically rigorous description of quantum field theory are quite serious. The basic motivation of this approach was to introduce otherwise missing ground states, yet prov-

ing existence of ground states even in simplified models such as quantum electrodynamics (QED) without photons has only been achieved in the recent past[20]. We will be content with working in external field QED, i.e. we will neglect all interaction between the fermions and describe the electrodynamic field as a smooth function. Even in this setting there are calssical theorems by Ruijsenaars[44] and Shale and Stinespring[51] that seem to prevent a dynamical mathematical description of the processes in question in the presence of magnetic fields. Recently, by the timescales of progress in this part of mathematical physics, this obstacle has been overcome by abandoning the restriction to work in a static Fock space [3, 5, 4]. These results form the basis upon which we will build our analysis in chapter 3.

Chapter 2

Direct Interaction in Relativistic Quantum Mechanics

As we have seen in the last chapter, having interaction mediated by potentials in a set of Dirac equations does not seem to be a viable option. One alternative approach to this problem is to reformulate Diracs equation as an integral equation and to introduce interaction analogous to how a potential would act in this formulation. This chapter is based on the preprints [30, 31], the results presented in this chapter are a result of the joint research of Matthias Lienert and the author of this thesis. While these results fall short of establishing the existence of a physically accurate alternative approach to a relativistic quantum mechanical theory, they do describe self consistent relativistic interacting quantum mechanical toy models living in three spacial and one temporal dimension.

For the benefit of the unfamiliar reader, we will first give a heuristic derivation of this type of equation, then briefly review the mathematical results that had been established in the past on this subject and finally discuss results appearing in [30, 31].

2.1 Overview

2.1.1 Derivation

We will now closely follow the heuristic derivation of [28] of an equation for a multi-time wave function for two particles that expresses direct interaction along light-like configurations. The derivation is organised as follows: We start out reformulating Diracs equation for a single particle as an integral equation. The reformulated version is then extended to two particles in an Poincaré invariant manner. Extending the equation is conveniently done in the framework of multi-time wavefunctions.

Diracs equation for one particle subject to an external potential V takes the form

$$i\partial_t \phi(t,\mathbf{x}) = \big(H^{\text{free}} + V(t,\mathbf{x})\big)\phi(t,\mathbf{x}), \tag{2.1}$$

here ϕ denotes the wavefunction in question, $\mathbf{x} \in \mathbb{R}^3, t \in \mathbb{R}$ and H^{free} is the Hamiltonian associated with a free Dirac particle. We denote by $\mathcal{S}^{ret}$ the retarded Green's function of the non interacting Dirac equation, that is S^{ret} satisfies

$$\big(i\partial_t - H^{\text{free}}\big)\mathcal{S}^{\text{ret}} = \delta^4, \tag{2.2}$$

$$\mathcal{S}^{\text{ret}}(t,\mathbf{x}) = 0 \quad \text{for } t < 0. \tag{2.3}$$

Then inverting the differential operator $i\partial_t - H^{\text{free}}$ in (2.1) results in

$$\phi(t,\mathbf{x}) = \phi^{\text{free}}(t,\mathbf{x}) + \int_{t^0}^{\infty} dt' \int d^3\mathbf{x}'\, \mathcal{S}^{\text{ret}}(t-t',\mathbf{x}-\mathbf{x}')V(t',\mathbf{x}')\phi(t',\mathbf{x}'), \tag{2.4}$$

where ϕ^{free} is the solution of the non interacting equation subject to the initial condition $\phi^{free}(t_0) = \phi_0$. Equations (2.4) and (2.1) subject to $\phi(t_0) = \phi_0$ yield equivalent descriptions, as can be verified directly: An action of $i\partial_t - H^{\text{free}}$ on (2.4) shows that a solution thereof also solves (2.1). Also the initial condition is fulfilled, as the integral term vanishes for $t = t_0$. Conversely equation (2.1) can be considered a free Dirac equation involving an inhomogeneous term of the form $V\phi$, whose solutions are known to be of the form (2.4). Executing the analogous procedure for the two particle Dirac equation

$$i\partial_t\phi(t,\mathbf{x}_1,\mathbf{x}_2) = \big(H_1^{\text{free}} + H_2^{\text{free}} + V(t,\mathbf{x}_1,\mathbf{x}_2)\big)\phi(t,\mathbf{x}_1,\mathbf{x}_2), \tag{2.5}$$

subject to the initial condition $\phi(t_0) = \phi_0$, results in the integral equation

$$\begin{aligned}\phi(t,\mathbf{x}_1,\mathbf{x}_2) = \phi^{\text{free}}(t,\mathbf{x}_1,\mathbf{x}_2) + \int_{t_0}^{\infty} dt' \int d^3\mathbf{x}_1'\, d^3\mathbf{x}_2'\, \mathcal{S}_1^{\text{ret}}(t-t',\mathbf{x}_1-\mathbf{x}_1') \\ \times \mathcal{S}_2^{\text{ret}}(t-t',\mathbf{x}_2-\mathbf{x}_2')V(t',\mathbf{x}_1',\mathbf{x}_2')\phi(t',\mathbf{x}_1',\mathbf{x}_2'),\end{aligned} \tag{2.6}$$

where now, $\phi^{\text{free}}(t)$ is a solution to the free Dirac equation for two particles subject to $\phi^{free}(t_0) = \phi_0$ and $\mathcal{S}_k^{ret}$ is the retarded Green's function of the free Dirac equation of particle number k. Here it is crucial to notice that the Green's function of the free two particle Dirac equation factorises into a product of two Green's functions of the Dirac equation for one particle.

Since equation (2.6) contains only one temporal variable, but six spatial ones, it is not obvious how it might be considered a relativistic equation at all. Now, we will generalise to two particles, but before we do so let us first rewrite equation (2.4) in a more suggestive way:

$$\psi(x) = \psi^{\text{free}}(x) + \int d^4x' \, \mathcal{S}^{\text{ret}}(x - x')V(x')\psi(x'), \tag{2.7}$$

where non bold letters denote elements of Minkowski spacetime and we replaced ϕ by ψ in order to make a visible switch to a relativistic notation. Furthermore we replaced the lower bound in the temporal integral domain by $-\infty$ in order to render the total domain of integral Poincaré invariant.
Equation (2.7) suggests to write down the following generalisation

$$\begin{aligned} \psi(x_1, x_2) = {} & \psi^{\text{free}}(x_1, x_2) \\ & + \int d^4x_1' \, d^4x_2' \, \mathcal{S}_1^{\text{ret}}(x_1 - x_1')\mathcal{S}_2^{\text{ret}}(x_2 - x_2')K(x_1', x_2')\psi(x_1', x_2'), \end{aligned} \tag{2.8}$$

where we integrate over all of $\mathbb{R}^8$ and ψ^{free} a solution of the free Dirac equation both in x_1 and x_2 and their respective spinor indices:

$$D_1\psi^{\text{free}}(x_1, x_2) = 0, \tag{2.9}$$

$$D_2\psi^{\text{free}}(x_1, x_2) = 0. \tag{2.10}$$

For the object K, called the "interaction kernel", the optimal choice is not yet known. However, for (2.8) to be Poincaré invariant, it should be invariant itself. A simple way to ensure this is to let it only depend directly the squared Minkowski distance $(x_1 - x_2)^2$. A choice that shows some reminiscence of Wheeler-Feynman electrodynamics is

$$K(x_1, x_2) = i \, \frac{e_1 e_2}{4\pi} \gamma_1^{\mu}\gamma_{2,\mu} \, \delta((x_1 - x_2)^2). \tag{2.11}$$

In an equation incorporating (2.11) the interaction between the particles happens along light-like distances. The constant in front of (2.11)

is fixed by the non-relativistic limit, recovering an equation very much like the Breit equation, see [28, section 3.6].
Summarising, we arrived at the equation

$$\psi(x_1,x_2) = \psi^{\text{free}}(x_1,x_2) + i\,\frac{e_1 e_2}{4\pi}\int d^4x_1'\,d^4x_2' \\ \times\, \mathcal{S}_1^{\text{ret}}(x_1-x_1')\mathcal{S}_2^{\text{ret}}(x_2-x_2')\,\gamma_1^{\mu}\gamma_{2,\mu}\delta((x_1'-x_2')^2)\psi(x_1',x_2'). \tag{2.12}$$

Despite the fact that the motivation for (2.12) holds for Dirac particles, it is also conceivable to replace ψ, $\mathcal{S}^{\text{ret}}$ and all the constant factor by quantities related to the Klein Gordon equation and arrive at

$$\psi(x_1,x_2) = \psi^{\text{free}}(x_1,x_2) + \frac{\lambda}{4\pi}\int d^4x_1'\,d^4x_2' \\ \times\, G_1^{\text{ret}}(x_1-x_1')G_2^{\text{ret}}(x_2-x_2')\,\delta((x_1'-x_2')^2)\psi(x_1',x_2'), \tag{2.13}$$

where ψ and ψ^{free} are no longer spinor valued, ψ^{free} is a solution to the free Klein Gordone equation in both x_1 and x_2,

$$(\Box_{x_1} + m_1^2)\psi = 0, \tag{2.14}$$

$$(\Box_{x_2} + m_2^2)\psi = 0 \tag{2.15}$$

and G^{ret} is the retarded Green's function of the Klein Gordon equation. In fact, most of the rigorous results about equations of a similar type as the ones motivated in this section are about the Klein Gordon version (2.13).

2.1.2 Previous Results on Directly Interacting Particles

In this section we summarise the most important existence results on equations of the type of (2.8). Because this line of work is still fairely young, it can still readily be summarised. The results are taken from [28] and [34]. The theorems are about the Klein-Gordon case, i.e. slightly different versions of equations of the type of (2.13). The necessary new notation is contained to within each of the theorems. Mentioned below are only the theorems that are about a four dimensional spacetime; however, there are also results concerning lower dimensions, the interested reader is refered to [28, 34]. The versions of equation (2.13) in the theorems are considerably modified:

(A) The spacetime of equation (2.13) is $\mathbb{R}^4$, i.e. Minkowski spacetime. All the rigorous results concerning vanishing curvature so far are about $\mathbb{R}^+ \times \mathbb{R}^3 =: \frac{1}{2}\mathbb{M}$. That is, there is a beginning in time. This modification has technical reasons. However, as current cosmological models of our universe do have a beginning in time this modification does not necessarily mean that the equation can no longer describe certain aspects of physics. As these cosmological models have nonzero curvature the authors of [34] have shown existence of solutions of versions of equation (2.13) on Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime. In section 2.3 and 2.2 we will also employ this simplification and show existence on this spacetime. This is not an attempt to treat general curved spacetimes, it is done as an act of consistency. We introduce a beginning in time and try to justify this by cosmological arguments and hence we treat a spacetime commonly used in cosmology.

(B) The interaction kernel K which we motivated to be proportional to $\delta((x_1 - x_2)^2)$ is replaced by various less singular objects. This modification is purely technical and we do not justify it. The previous results approach the singular K introduced in the last section to different degrees. In section 2.3 where we treat Dirac particles we will also use a rather soft interaction kernel. The new result about Klein Gordon particles presented in section 2.2 employs the fully singular $\delta((x_1 - x_2)^2)$ kernel.

Theorem 1 (Thm 3.4 ($d = 3$) of [35])**.** *Let $T > 0, \lambda \in \mathbb{C}$, for every bounded $K : \mathbb{R}^8 \to \mathbb{C}$ and every $\psi^{\text{free}} \in \mathcal{B} := L^\infty([0,T]^2, L^2(\mathbb{R}^6))$ the equation*

$$\begin{aligned}\psi(t_1,\mathbf{x}_1,t_2,\mathbf{x}_2) = \psi^{\text{free}}(t_1,\mathbf{x}_1,t_2,\mathbf{x}_2) + \frac{\lambda}{(4\pi)^2}\int d\mathbf{x}_1' d\mathbf{x}_2' \\ \times \frac{H(t_1 - |\mathbf{x}_1 - \mathbf{x}_1'|)}{|\mathbf{x}_1 - \mathbf{x}_1'|}\frac{H(t_2 - |\mathbf{x}_2 - \mathbf{x}_2'|)}{|\mathbf{x}_2 - \mathbf{x}_2'|} \\ \times K(t_1 - |\mathbf{x}_1 - \mathbf{x}_1'|, \mathbf{x}_1', t_2 - |\mathbf{x}_2 - \mathbf{x}_2'|, \mathbf{x}_2') \\ \times \psi(t_1 - |\mathbf{x}_1 - \mathbf{x}_1'|, \mathbf{x}_1', t_2 - |\mathbf{x}_2 - \mathbf{x}_2'|, \mathbf{x}_2')\end{aligned}$$

has a unique solution $\psi \in \mathcal{B}$.

Theorem 2 (Thm 3.5 of [35])**.** *Let $T > 0, \lambda \in \mathbb{C}$, for every bounded $f : \mathbb{R}^8 \to \mathbb{C}$ and every $\psi^{\text{free}} \in \mathcal{B} := L^\infty([0,T]^2, L^2(\mathbb{R}^6))$ the equation*

$$\begin{aligned}\psi(t_1,\mathbf{x}_1,t_2,\mathbf{x}_2) = \psi^{\text{free}}(t_1,\mathbf{x}_1,t_2,\mathbf{x}_2) + \frac{\lambda}{(4\pi)^2}\int d^3\mathbf{x}_1' d^3\mathbf{x}_2' \\ \times \frac{H(t_1 - |\mathbf{x}_1 - \mathbf{x}_1'|)}{|\mathbf{x}_1 - \mathbf{x}_1'|}\frac{H(t_2 - |\mathbf{x}_2 - \mathbf{x}_2'|)}{|\mathbf{x}_2 - \mathbf{x}_2'|} \\ \times \frac{f(t_1 - |\mathbf{x}_1 - \mathbf{x}_1'|, \mathbf{x}_1', t_2 - |\mathbf{x}_2 - \mathbf{x}_2'|, \mathbf{x}_2')}{|\mathbf{x}_1' - \mathbf{x}_1|} \\ \times \psi(t_1 - |\mathbf{x}_1 - \mathbf{x}_1'|, \mathbf{x}_1', t_2 - |\mathbf{x}_2 - \mathbf{x}_2'|, \mathbf{x}_2'),\end{aligned}$$

has a unique solution $\psi \in \mathcal{B}$.

The next results will be about the open FLRW spacetime. There are also results about the closed FLRW universe which we omit here, the reader is refered to [34, Thm 4.3]. We have to introduce some notation

before we can present them, in order to do so we follow [31, sec 3.3], also see section 2.2.3.3.
We consider particles on a flat (FLRW) spacetime which is described by the metric

$$ds^2 = a^2(\eta)\left(d\eta^2 - dr^2 - r^2 d\Omega^2\right), \tag{2.16}$$

where η denotes conformal time, $d\Omega$ denotes the surface measure on $\mathbb{S}^2$ and the scale function $a(\eta)$ is continuous with $a(0) = 0$ and $a(\eta) > 0$ for $\eta > 0$. In this spacetime the free wave equation takes the form

$$(\Box_g - R/6)\,\chi = 0, \tag{2.17}$$

where R denotes the Ricci scalar. The retarded and symmetric Greens functions of equation (2.17) are given by

$$\begin{aligned} G^{\text{ret}}_{\mathcal{M}}(x,x') &= \frac{1}{4\pi}\frac{1}{a(\eta)a(\eta')}\frac{\delta(\eta-\eta'-|\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|} \\ G^{sym}_{\mathcal{M}}(x,x') &= \frac{1}{4\pi}\frac{1}{a(\eta)a(\eta')}\delta((\eta-\eta')^2-|\mathbf{x}-\mathbf{x}'|^2). \end{aligned} \tag{2.18}$$

The generalization of (2.13) to FLRW spacetime is straightforward: ψ becomes a scalar function on $\mathcal{M}\times\mathcal{M}$, one exchanges the Minkowski spacetime volume element with

$$dV(x) = a^4(\eta)\,d\eta\,d^3\mathbf{x}, \tag{2.19}$$

the invariant 4-volume element on $\mathcal{M}$, and the Green's functions on $\frac{1}{2}\mathbb{M}$ get replaced with those on $\mathcal{M}$ as well. As in the Minkowski case, the interaction kernel is given by the symmetric Green's function. With this, the relevant integral equation becomes:

$$\begin{aligned} \psi(x,y) = \psi^{\text{free}}(x,y) + \lambda \int_{\mathcal{M}\times\mathcal{M}} dV(x)\,dV(y)\; G^{\text{ret}}_1(x,x')G^{\text{ret}}_2(y,y') \\ \times\, G^{sym}(x',y')\psi(x',y'). \end{aligned} \tag{2.20}$$

For regular and only weakly singular interaction kernels $K(x,y)$ instead of $G^{sym}(x',y')$, the problem of existence and uniqueness of solutions of this equation has been treated in [34]:

Theorem 3 (Thm 4.1 of [34]). *Let $T>0$, $\lambda \in \mathbb{C}$ and $\mathcal{B} := L^\infty([0,T]^2, L^2(\mathbb{R}^6))$. Furthermore, let $a : [0,\infty) \to [0,\infty)$ be a continuous function with $a(0)=0$ and $a(\eta)>0$ for $\eta>0$, and $\tilde{K} : ([0,\infty)\times\mathbb{R}^3)^2 \to \mathbb{C}$ be bounded. Then for every ψ^{free} with $a(\eta_1)a(\eta_2)\psi^{\text{free}} \in \mathcal{B}$, the respective integral equation on the 4−dimensional flat FLRW universe sith scale function $a(\eta)$:*

$$\begin{aligned}\psi(\eta_1,\mathbf{x}_1,\eta_2,\mathbf{x}_2) = \psi^{\text{free}}(\eta_1,\mathbf{x}_1,\eta_2,\mathbf{x}_2) &+ \frac{\lambda}{(4\pi)^2 a(\eta_1)a(\eta_2)} \\ &\times \int d\mathbf{x}_1' d\mathbf{x}_2' a^2(\eta_1 - |\mathbf{x}_1-\mathbf{x}_1'|)a^2(\eta_2-|\mathbf{x}_2-\mathbf{x}_2'|) \\ &\times \frac{H(\eta_1-|\mathbf{x}_1-\mathbf{x}_1'|)}{|\mathbf{x}_1-\mathbf{x}_1'|}\frac{H(\eta_2-|\mathbf{x}_2-\mathbf{x}_2'|)}{|\mathbf{x}_2-\mathbf{x}_2'|} \\ &\times \tilde{K}(\eta_1-|\mathbf{x}_1-\mathbf{x}_1'|,\mathbf{x}_1',\eta_2-|\mathbf{x}_2-\mathbf{x}_2'|,\mathbf{x}_2') \\ &\times \psi(\eta_1-|\mathbf{x}_1-\mathbf{x}_1'|,\mathbf{x}_1',\eta_2-|\mathbf{x}_2-\mathbf{x}_2'|,\mathbf{x}_2')\end{aligned} \tag{2.21}$$

has a unique solution ψ with $a(\eta_1)a(\eta_2)\psi \in \mathcal{B}$.

Theorem 4 (Thm 4.2 of [34]). *Let $f : ([0,\infty]\times\mathbb{R}^3)^2 \to \mathbb{C}$ be a bounded function. Then, under the same assumptions as in the last theorem but with*

$$\tilde{K}(\eta_1,\mathbf{x}_1,\eta_2,\mathbf{x}_2) = \frac{f(\eta_1,\mathbf{x}_1,\eta_1,\mathbf{x}_2)}{|\mathbf{x}_1-\mathbf{x}_2|}, \tag{2.22}$$

the integral equation (2.21) has a unique solution ψ with $a(\eta_1)a(\eta_2)\psi \in \mathcal{B}$.

2.2 Singular light cone interactions of scalar particles

In this section we prove existence and uniqueness of an equation of type (2.13), subject to the simplifying assumption (A). In contrast to the previous results of section 2.1.2 the equation will not be subject to the assumption (B). Additionally, we will extend the result to an arbitrary number of particles. This extension is not the only possible one; however, we choose the extension called most promising in [28]. In order to justify the cutoff in time, we extend the one-particle result to the FLRW spacetime, where the cutoff appears naturally.

2.2.1 Overview

This section is structured as follows. In subsec. 2.2.2 it is shown how to define the integral equation in a rigorous way (by using the delta distributions to eliminate certain integration variables). To this end, we again consider the equation on the Minkowski half-space (assuming a cutoff in time). Subsection 2.2.3 contains our main results: Thm. 5 contains explicit bounds for the integral operator in terms of a general weight function of a weighted L^∞ space. Thm. 6 shows that in the case of massless particles already an exponential weight function leads to the existence and uniqueness of solutions of the integral equation. Our main result is Thm. 7, an existence and uniqueness theorem for the full (massive) case. In that case, a different weight function growing like the exponential of a polynomial is used.

Subsection 2.2.3.2 deals with generalizing this existence and uniqueness theorem to N scalar particles; the corresponding theorem, Thm. 8, is a direct consequence of Thm. 7. To the best of the knowledge of the Matthias Lienert and the author, this is the first rigorous result about a multi-time integral equation for N-particles.

In Subsection 2.2.3.3 we show by considering a specific example (an open FLRW spacetime) that the cutoff in time can be achieved naturally for a cosmological spacetime with a Big Bang singularity, without breaking any spacetime symmetries. That is, we show the equivalent result of [34] for singular light cone interactions. The respective existence and uniqueness theorem is Thm. 10.
Subsection 2.2.4 contains the proofs.

2.2.2 Precise formulation of the problem

In the following, we show how to precisely define the integral equation (2.13) for the case of two scalar particles with masses m_1 and m_2 on the Minkowski half space $\frac{1}{2}\mathbb{M} = [0,\infty) \times \mathbb{R}^3$. Strictly speaking, to introduce a cutoff in time in this way breaks the Poincaré invariance of (2.13); however, we will give an argument for its use in Sec. 2.2.3.3. It is necessary to take special care of the definition of the integral equation as it contains certain combinations (convolutions and products) of distributions (the Green's functions). Our strategy is to consider the integral operator acting on test functions first where its action can be defined straightforwardly. Later it will be shown that it is bounded on test functions with respect to a suitably chosen weighted norm. This will make it possible to linearly extend the integral operator to the completion of test functions with respect to that norm.
The retarded Green's function of the Klein-Gordon equation is given by:

$$G^{\text{ret}}(x) = \frac{1}{4\pi|\mathbf{x}|}\delta(x^0 - |\mathbf{x}|) - \frac{m}{4\pi\sqrt{x^2}}H(x^0 - |\mathbf{x}|)\frac{J_1(m\sqrt{x^2})}{\sqrt{x^2}} \quad (2.23)$$

where H denotes the Heaviside function and J_1 is a Bessel function of the first kind. Then, with $K(x,y) = \frac{\lambda}{4\pi}\delta((x-y)^2)$, our integral equation (2.13) on $(\frac{1}{2}\mathbb{M})^2$ becomes:

$$\psi = \psi^{\text{free}} + A\psi \quad (2.24)$$

where $A = A_0 + A_1 + A_2 + A_{12}$ and

$$\begin{aligned}
(A_0\psi)(x,y) \ =& \frac{\lambda}{(4\pi)^3}\int_0^\infty dx'^0 \int_{\mathbb{R}^3} d^3\mathbf{x}' \int_0^\infty dy'^0 \int_{\mathbb{R}^3} \\
&\times \frac{\delta(x^0 - x'^0 - |\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|}\frac{\delta(y^0 - y'^0 - |\mathbf{y}-\mathbf{y}'|)}{|\mathbf{y}-\mathbf{y}'|} \\
&\times \delta((x'-y')^2)\psi(x',y'), \qquad (2.25)\\
(A_1\psi)(x,y) \ =& -\frac{\lambda\, m_1}{(4\pi)^3}\int_0^\infty dx'^0 \int d^3\mathbf{x}' \int_0^\infty dy'^0 \int d^3\mathbf{y}' \\
&\times H(x^0 - x'^0 - |\mathbf{x}-\mathbf{x}'|)\frac{J_1(m_1\sqrt{(x-x')^2})}{\sqrt{(x-x')^2}} \\
&\times \frac{\delta(y^0 - y'^0 - |\mathbf{y}-\mathbf{y}'|)}{|\mathbf{y}-\mathbf{y}'|}\delta((x'-y')^2)\psi(x',y') \qquad (2.26)\\
(A_2\psi)(x,y) \ =& -\frac{\lambda\, m_2}{(4\pi)^3}\int_0^\infty dx'^0 \int d^3\mathbf{x}' \int_0^\infty dy'^0 \int d^3\mathbf{y}' \\
&\times \frac{\delta(x^0 - x'^0 - |\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|}H(y^0 - y'^0 - |\mathbf{y}-\mathbf{y}'|) \\
&\times \frac{J_1(m_2\sqrt{(y-y')^2})}{\sqrt{(y-y')^2}}\delta((x'-y')^2)\psi(x',y') \qquad (2.27)\\
(A_{12}\psi)(x,y) \ =& \frac{\lambda\, m_1 m_2}{(4\pi)^3}\int_0^\infty dx'^0 \int d^3\mathbf{x}' \int_0^\infty dy'^0 \int d^3\mathbf{y}' \\
&\times H(x^0 - x'^0 - |\mathbf{x}-\mathbf{x}'|)\frac{J_1(m_1\sqrt{(x-x')^2})}{\sqrt{(x-x')^2}} \\
&\times H(y^0 - y'^0 - |\mathbf{y}-\mathbf{y}'|)\frac{J_1(m_2\sqrt{(y-y')^2})}{\sqrt{(y-y')^2}} \\
&\times \delta((x'-y')^2)\psi(x',y'). \qquad (2.28)
\end{aligned}$$

We now formally manipulate these informal expressions in such a way that the end results can be given a precise meaning on test functions. Let $\mathcal{S} = \mathcal{S}((\frac{1}{2}\mathbb{M})^2)$ denote the space of Schwartz functions on $(\frac{1}{2}\mathbb{M})^2$, and let $\psi \in \mathcal{S}$.

2.2.2.1 Definition of A_0.

We consider the massless term A_0 first which is also the most singular term. Using the δ-functions to eliminate the integration over x'^0 and y'^0 results in:

$$\begin{aligned}(A_0\psi)(x,y) =& \frac{\lambda}{(4\pi)^3}\int_{B_{x^0}(\mathbf{x})} d^3\mathbf{x}' \int_{B_{y^0}(\mathbf{y})} d^3\mathbf{y}' \\ &\times \frac{\delta((x^0-y^0-|\mathbf{x}'|+|\mathbf{y}'|)^2-|\mathbf{x}-\mathbf{y}+\mathbf{x}'-\mathbf{y}'|^2)}{|\mathbf{x}'||\mathbf{y}'|} \\ &\times \psi(x+x',y+y')\big|_{x'^0=-|\mathbf{x}'|,\,y'^0=-|\mathbf{y}'|},\end{aligned} \tag{2.29}$$

Note that the domain of integration has been reduced to a compact region whose size depends on x^0 and y^0. There is still one more δ-distribution left. We choose to use it to eliminate $|\mathbf{x}'| =: r$. It is convenient to introduce the vector

$$b = x - y - (-|\mathbf{y}'|, \mathbf{y}'). \tag{2.30}$$

Then the argument of the delta function can be written as:

$$(b^0 - |\mathbf{x}'|)^2 - |\mathbf{b} + \mathbf{x}'|^2. \tag{2.31}$$

This expression has a root in r for

$$r = r^* := \frac{1}{2}\frac{b^2}{b^0 + |\mathbf{b}|\cos\vartheta} \tag{2.32}$$

where ϑ is the angle between $\mathbf{b}$ and $\mathbf{x}'$. Of course, r^* inherits the restrictions of the range of r, thus is only a valid root for

$$0 < r^* < x^0. \tag{2.33}$$

The requirement $0 < r^*$ can be satisfied in two cases, either $b^2 > 0$ and $b^0 > 0$, or $b^2 < 0$ and $\cos\vartheta < -\frac{b^0}{|\mathbf{b}|}$. Using these restrictions, the condition $r^* < x^0$ can be converted into a restriction of the domain of integration in ϑ:

$$\begin{aligned}
& \frac{1}{2}\frac{b^2}{b^0 + |\mathbf{b}|\cos\vartheta} \;<\; x^0 \\
\Longleftrightarrow\quad & \operatorname{sgn}(b^2)b^2 \;<\; 2x^0\operatorname{sgn}(b^2)(b^0 + |\mathbf{b}|\cos\vartheta) \\
\Longleftrightarrow\quad & \frac{|b^2|}{2x^0|\mathbf{b}|} - \frac{\operatorname{sgn}(b^2)b^0}{|\mathbf{b}|} \;<\; \operatorname{sgn}(b^2)\cos\vartheta \\
\Longleftrightarrow\quad & \begin{cases} \cos\vartheta > \frac{b^2}{2x^0|\mathbf{b}|} - \frac{b^0}{|\mathbf{b}|}, & \text{for } b^2 > 0 \\ \cos\vartheta < \frac{b^2}{2x^0|\mathbf{b}|} - \frac{b^0}{|\mathbf{b}|}, & \text{for } b^2 < 0. \end{cases}
\end{aligned} \tag{2.34}$$

In case of $b^2 < 0$, the new restriction on $\cos\vartheta$ is stricter than $\cos\vartheta < -\frac{b^0}{|\mathbf{b}|}$; we thus use it to replace the latter. We evaluate the δ-function using spherical coordinates in $\mathbf{y}'$ and the usual rule

$$\delta(f(z)) = \sum_{z^*:f(z^*)=0} \frac{\delta(z - z^*)}{|f'(z^*)|}, \tag{2.35}$$

where $f(r) = (b^0 - r)^2 - (\mathbf{b} + x')^2 = -(r - r^*)2(b^0 + |\mathbf{b}|\cos\vartheta)$. The result is an expression for $A_0\psi$ which does not contain distributions anymore:

$$\begin{aligned}
(A_0\psi)(x,y) = \frac{\lambda}{(4\pi)^3}\int_{B_{y^0}(\mathbf{y})} d^3\mathbf{y}' \int_0^{2\pi} d\varphi \int_{-1}^{1} d\cos\vartheta\; \frac{|b^2|}{4(b^0 + |\mathbf{b}|\cos\vartheta)^2|\mathbf{y}'|} \\
\left(1_{b^2>0}1_{b^0>0}1_{\cos\vartheta > \frac{b^2}{2x^0|\mathbf{b}|} - \frac{b^0}{|\mathbf{b}|}} + 1_{b^2<0}1_{\cos\vartheta < \frac{b^2}{2x^0|\mathbf{b}|} - \frac{b^0}{|\mathbf{b}|}}\right)\psi(x+x', y+y'),
\end{aligned} \tag{2.36}$$

still subject to $x'^0 = -r^* = -|\mathbf{x}'|, y'^0 = -|\mathbf{y}'|$. The different cases for b have been implemented through the various indicator functions. Eq. (2.36) will serve as our *definition* of A_0 on test functions $\psi \in \mathcal{S}$.

2.2.2.2 Definition of A_1.

Next, we turn to the definition of A_1, starting from the informal expression (2.26). We first split up the δ-function of the interaction kernel according to (2.35). Then we use $\delta(y^0 - y'^0 - |\mathbf{y} - \mathbf{y}'|)$ to eliminate y'^0 $(= y^0 - |\mathbf{y} - \mathbf{y}'|)$. Note that the order of these two steps does not matter. This yields:

$$
\begin{aligned}
(A_1\psi)(x,y) = &-\frac{\lambda\, m_1}{2(4\pi)^3} \int_0^\infty dx'^0 \int d^3\mathbf{x}' \int d^3\mathbf{y}'\; H(x^0 - x'^0 - |\mathbf{x} - \mathbf{x}'|) \\
&\times \frac{J_1(m_1\sqrt{(x-x')^2})}{\sqrt{(x-x')^2}} \frac{H(y^0 - |\mathbf{y} - \mathbf{y}'|)}{|\mathbf{y} - \mathbf{y}'|} \frac{1}{|\mathbf{x}' - \mathbf{y}'|} \\
&\Big[\delta(x'^0 - y^0 + |\mathbf{y} - \mathbf{y}'| - |\mathbf{x}' - \mathbf{y}'|) + \delta(x'^0 - y^0 + |\mathbf{y} - \mathbf{y}'| + |\mathbf{x}' - \mathbf{y}'|)\Big] \\
&\times \psi(x', y^0 - |\mathbf{y} - \mathbf{y}'|, \mathbf{y}').
\end{aligned} \tag{2.37}
$$

Finally, we use the remaining δ-functions to eliminate x'^0. We obtain:

$$
\begin{aligned}
(A_1\psi)(x,y) = &-\frac{\lambda\, m_1}{2(4\pi)^3} \int d^3\mathbf{x}' \int d^3\mathbf{y}'\; \frac{H(y^0 - |\mathbf{y} - \mathbf{y}'|)}{|\mathbf{y} - \mathbf{y}'|} \frac{1}{|\mathbf{x}' - \mathbf{y}'|} \\
&\Bigg[H(x'^0) H(x^0 - x'^0 - |\mathbf{x} - \mathbf{x}'|) \\
&\quad \times \frac{J_1(m_1\sqrt{(x-x')^2})}{\sqrt{(x-x')^2}} \psi(x', y') \Bigg|_{\substack{y'^0 = y^0 - |\mathbf{y} - \mathbf{y}'|, \\ x'^0 = y^0 - |\mathbf{y} - \mathbf{y}'| + |\mathbf{x}' - \mathbf{y}'|}} \\
&+ H(x'^0) H(x^0 - x'^0 - |\mathbf{x} - \mathbf{x}'|) \\
&\times \frac{J_1(m_1\sqrt{(x-x')^2})}{\sqrt{(x-x')^2}} \psi(x', y') \Bigg|_{\substack{y'^0 = y^0 - |\mathbf{y} - \mathbf{y}'|, \\ x'^0 = y^0 - |\mathbf{y} - \mathbf{y}'| - |\mathbf{x}' - \mathbf{y}'|}} \Bigg].
\end{aligned} \tag{2.38}
$$

This expression is free of distributions, so it will serve as our definition of A_1 on test functions $\psi \in \mathcal{S}$. Note that the domain of integration is effectively finite due to the Heaviside functions.

2.2.2.3 Definition of A_2.

Starting from (2.27), the analogous steps as for A_1 yield:

$$(A_2\psi)(x,y) = -\frac{\lambda m_2}{2(4\pi)^3}\int d^3\mathbf{x}'\int d^3\mathbf{y}'\,\frac{H(x^0-|\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|}\frac{1}{|\mathbf{x}'-\mathbf{y}'|}$$
$$\Bigg[H(y'^0)H(y^0-y'^0-|\mathbf{y}-\mathbf{y}'|)$$
$$\times \frac{J_1(m_2\sqrt{(y-y')^2})}{\sqrt{(y-y')^2}}\psi(x',y')\Big|_{\substack{x'^0=x^0-|\mathbf{x}-\mathbf{x}'|,\\ y'^0=x^0-|\mathbf{x}-\mathbf{x}'|+|\mathbf{x}'-\mathbf{y}'|}}$$
$$+\; H(y'^0)H(y^0-y'^0-|\mathbf{y}-\mathbf{y}'|)$$
$$\times \frac{J_1(m_2\sqrt{(y-y')^2})}{\sqrt{(y-y')^2}}\psi(x',y')\Big|_{\substack{x'^0=x^0-|\mathbf{x}-\mathbf{x}'|,\\ y'^0=x^0-|\mathbf{x}-\mathbf{x}'|-|\mathbf{x}'-\mathbf{y}'|}}\Bigg]. \tag{2.39}$$

This serves as our definition of A_2 on test functions $\psi \in \mathcal{S}$.

2.2.2.4 Definition of A_{12}.

Here, we start with (2.28). We change variables $(\mathbf{x}',\mathbf{y}') \to (\mathbf{x}',\mathbf{z} = \mathbf{x}'-\mathbf{y}')$ (Jacobi determinant = 1), with the goal of using the remaining δ-function to eliminate $|\mathbf{z}| = |\mathbf{x}'-\mathbf{y}'|$ in mind. We find:

$$(A_{12}\psi)(x,y) = \frac{\lambda m_1 m_2}{(4\pi)^3}\int_0^\infty dx'^0\int d^3\mathbf{x}'\int_0^\infty dy'^0\int d^3\mathbf{z}\,H(x^0-x'^0-|\mathbf{x}-\mathbf{x}'|)$$
$$\times \frac{J_1(m_1\sqrt{(x-x')^2})}{\sqrt{(x-x')^2}}H(y^0-y'^0-|\mathbf{y}-\mathbf{x}'+\mathbf{z}|)$$
$$\times \frac{J_1(m_2\sqrt{(y-y')^2})}{\sqrt{(y-y')^2}}\delta((x'^0-y'^0)^2-|\mathbf{z}|^2)\psi(x',y')\Big|_{\mathbf{y}'=\mathbf{x}'-\mathbf{z}}. \tag{2.40}$$

■

Now we use spherical coordinates for $\mathbf{z}$ and eliminate $|\mathbf{z}|$ through the δ-function, using

$$\delta((x'^0 - y'^0)^2 - |\mathbf{z}|^2) = \frac{1}{2|\mathbf{z}|}\delta(|x^{0\prime} - y^{0\prime}| - |\mathbf{z}|). \tag{2.41}$$

This yields:

$$\begin{aligned}(A_{12}\psi)(x,y) &= \frac{\lambda\, m_1 m_2}{2(4\pi)^3}\int_0^\infty dx'^0 \int d^3\mathbf{x}' \int_0^\infty dy'^0 \int_0^{2\pi} d\varphi \int_0^\pi d\vartheta \\ &\times \cos(\vartheta)|x'^0 - y'^0| H(x^0 - x'^0 - |\mathbf{x}-\mathbf{x}'|)\frac{J_1(m_1\sqrt{(x-x')^2})}{\sqrt{(x-x')^2}} \\ &\times H(y^0 - y'^0 - |\mathbf{y}-\mathbf{x}'+\mathbf{z}|)\frac{J_1(m_2\sqrt{(y-y')^2})}{\sqrt{(y-y')^2}}\psi(x',y')\Big|_{\mathbf{y}'=\mathbf{x}'-\mathbf{z},\,|\mathbf{z}|=|x^{0\prime}-y^{0\prime}|}.\end{aligned} \tag{2.42}$$

The resulting expression does not contain distributions anymore and will serve as our definition of A_{12} on test functions $\psi \in \mathcal{S}$. Note that the domain of integration is again effectively finite.

2.2.2.5 Lifting of the integral operator from test functions to a suitable Banach space.

In order to prove the existence and uniqueness of solutions of the integral equation $\psi = \psi^{\text{free}} + A\psi$, we need to define the operator A not only on test functions but on a suitable Banach space which includes (at least) sufficiently many solutions ψ^{free} of the free multi-time Klein-Gordon equations, $(\Box_k + m_k^2)\psi^{free}(x_1, x_2) = 0,\ k = 1, 2$. We shall define this Banach space as the completion of $\mathcal{S} = \mathcal{S}((\frac{1}{2}\mathbb{M})^2)$ with respect to a suitable norm. A good choice which works well for the upcoming existence and uniqueness proofs is the class of weighted L^∞-norms

$$\|\psi\|_g := \operatorname*{ess\,sup}_{x,y\in\frac{1}{2}\mathbb{M}} \frac{|\psi(x,y)|}{g(x^0)g(y^0)}, \tag{2.43}$$

where $g : \mathbb{R}_0^+ \to \mathbb{R}^+$ is assumed to be a monotonically increasing function such that $1/g$ is bounded. Then our Banach space is given by the completion

$$\mathcal{B}_g = \overline{\mathcal{S}}^{\|\cdot\|_g}. \tag{2.44}$$

Our next goal is to find a weight function g such that the operator A is not only bounded but even defines a contraction on $\mathcal{B}_g$. By linear extension, it is sufficient to estimate $\|A\psi\|_g$ on test functions $\psi \in \mathcal{S}$.

Remarks:

1. We have attempted to use an $L^\infty_t L^2_{\mathbf{x}}$-based norm ($L^\infty$ in the times and L^2 in the space variables). However, we did not succeed with obtaining suitable estimates for that case. This might not be a problem in principle, but its treatment would require further technical innovation. More precisely, one would need to understand integral operators such as (2.36) whose kernel is in L^1 but not in L^2.

2. Nevertheless, our definition of $\mathcal{B}_g$ contains a large class of free solutions of the Klein-Gordon equation. As the Klein-Gordon equation preserves boundedness, all bounded initial data for $\psi^{\rm free}$ lead to a free solution $\psi^{\rm free} \in \mathcal{B}_g$ which can be used as an input to our integral equation.

2.2.3 Results

This section is structured as follows. Sec. 2.2.3.1 (which is about the two-particle case) contains the main results: the estimates of the integral operators as well as the theorems about existence and uniqueness of solutions. Sec. 2.2.3.2 extends these results to the N-particle case and in Sec. 2.2.3.3 we show that a curved spacetime with a Big Bang singularity can provide a natural reason for a cutoff in time.

2.2.3.1 The two-particle case

For $t \geqslant 0$, we define the functions:

$$\begin{aligned} g_0(t) &= g(t), \\ \text{and for } n \in \mathbb{N}: \quad g_n(t) &= \int_0^t dt'\, g_{n-1}(t'). \end{aligned} \tag{2.45}$$

Note that due to the properties of g, the functions g_n are monotonically increasing for all $n \in \mathbb{N}$; furthermore, by definition, they satisfy $g_n(0) = 0$.
Our first theorem gives explicit bounds for the operators A_0, A_1, A_2, A_{12} in terms of the functions g_n. The proof can be found in Sec. 2.2.4.1.

Theorem 5 (Bounds of the integral operators on $\mathcal{S}$.). *For all $\psi \in \mathcal{S}((\frac{1}{2}\mathbb{M})^2)$, the integral operators A_0, A_1, A_2, A_{12} satisfy the following bounds:*

$$\sup_{\psi \in \mathcal{S}((\frac{1}{2}\mathbb{M})^2)} \frac{\|A_0\psi\|_g}{\|\psi\|_g} \leqslant \frac{\lambda}{8\pi} \left(\sup_{t \geqslant 0} \frac{g_1(t)}{g(t)} \right)^2, \tag{2.46}$$

$$\begin{aligned} \sup_{\psi \in \mathcal{S}((\frac{1}{2}\mathbb{M})^2)} \frac{\|A_1\psi\|_g}{\|\psi\|_g} \leqslant \frac{\lambda\, m_1^2}{16\pi} \Bigg[& 3 \left(\sup_{t \geqslant 0} \frac{t g_1(t)}{g(t)} \right) \left(\sup_{t \geqslant 0} \frac{g_2(t)}{g(t)} \right) \\ & + 3 \left(\sup_{t \geqslant 0} \frac{g_1(t)}{g(t)} \right) \left(\sup_{t \geqslant 0} \frac{t g_2(t)}{g(t)} \right) \\ & + 2 \left(\sup_{t \geqslant 0} \frac{g_1(t)}{g(t)} \right) \left(\sup_{t \geqslant 0} \frac{g_3(t)}{g(t)} \right) \Bigg], \end{aligned} \tag{2.47}$$

$$\begin{aligned} \sup_{\psi \in \mathcal{S}((\frac{1}{2}\mathbb{M})^2)} \frac{\|A_2\psi\|_g}{\|\psi\|_g} \leqslant \frac{\lambda\, m_2^2}{16\pi} \Bigg[& 3 \left(\sup_{t \geqslant 0} \frac{t g_1(t)}{g(t)} \right) \left(\sup_{t \geqslant 0} \frac{g_2(t)}{g(t)} \right) \\ & + 3 \left(\sup_{t \geqslant 0} \frac{g_1(t)}{g(t)} \right) \left(\sup_{t \geqslant 0} \frac{t g_2(t)}{g(t)} \right) \end{aligned}$$

$$+ 2 \left(\sup_{t \geqslant 0} \frac{g_1(t)}{g(t)} \right) \left(\sup_{t \geqslant 0} \frac{g_3(t)}{g(t)} \right) \Bigg], \tag{2.48}$$

$$\sup_{\psi \in \mathcal{S}((\frac{1}{2}\mathbb{M})^2)} \frac{\|A_{12}\psi\|_g}{\|\psi\|_g} \leqslant \frac{\lambda\, m_1^2 m_2^2}{96\pi} \left[\left(\sup_{t \geqslant 0} \frac{t^2 g_2(t)}{g(t)} \right) \left(\sup_{t \geqslant 0} \frac{t g_1(t)}{g(t)} \right) \right.$$

$$\left. + \frac{1}{2} \left(\sup_{t \geqslant 0} \frac{t^2 g_3(t)}{g(t)} \right) \left(\sup_{t \geqslant 0} \frac{g_1(t)}{g(t)} \right) \right]. \tag{2.49}$$

In case these expressions are finite, A_0, A_1, A_2, A_{12} extend to linear operators on $\mathcal{B}_g$ with the same norms. Our next task is to find suitable weight functions g such that this is actually the case. We begin with the massless case where already an exponential weight function leads to an estimate which remains finite after taking the supremum. The massive case is treated subsequently; it is a little more difficult since all the estimates for the operators A_0, A_1, A_2, A_{12} have to be finite at the same time. This requires a different choice of weight function (see Thm. 7).

Theorem 6 (Bounds for A_0 and $g(t) = e^{\gamma t}$; existence of massless dynamics.)**.**
For any $\gamma > 0$, let $g(t) = e^{\gamma t}$. Then A_0 can be linearly extended to a bounded operator on $\mathcal{B}_g$ with norm

$$\|A_0\| \leqslant \frac{\lambda}{8\pi\gamma^2}. \tag{2.50}$$

Consequently, for all $\gamma > \sqrt{\frac{\lambda}{8\pi}}$, the integral equation $\psi = \psi^{\text{free}} + A_0\psi$ has a unique solution $\psi \in \mathcal{B}_g$ for every $\psi^{\text{free}} \in \mathcal{B}_g$.

Now we come to our main result.

Theorem 7 (Existence of dynamics in the massive case.)**.**
For any $\alpha > 0$, let

$$g(t) = (1 + \alpha t^2) e^{\alpha t^2/2}. \tag{2.51}$$

Then A_0, A_1, A_2 and A_{12} can be linearly extended to bounded operators on $\mathcal{B}_g$ with norms

$$\|A_0\| \leqslant \frac{\lambda}{32\pi}\frac{1}{\alpha}, \tag{2.52}$$
$$\|A_1\| \leqslant \frac{5\lambda\, m_1^2}{16\pi}\frac{1}{\alpha^2}, \tag{2.53}$$
$$\|A_2\| \leqslant \frac{5\lambda\, m_2^2}{16\pi}\frac{1}{\alpha^2}, \tag{2.54}$$
$$\|A_{12}\| \leqslant \frac{\lambda\, m_1^2 m_2^2}{80\pi}\frac{1}{\alpha^3}. \tag{2.55}$$

Consequently, for all $\alpha > 0$ with

$$\frac{\lambda}{8\pi\alpha}\left(\frac{1}{4} + \frac{5(m_1^2+m_2^2)}{2}\frac{1}{\alpha} + \frac{m_1^2\, m_2^2}{10}\frac{1}{\alpha^2}\right) < 1, \tag{2.56}$$

the integral equation $\psi = \psi^{\text{free}} + A\psi$ has a unique solution $\psi \in \mathcal{B}_g$ for every $\psi^{\text{free}} \in \mathcal{B}_g$.

The proof can be found in Sec. 2.2.4.3.

Remarks: 1. *Comparison of Thms. 6 and 7 in the massless case.* On the first glance, the result of Thm. 6 looks stronger in the sense that for $g(t) = e^{\gamma t}$, the estimate of $\|A_0\|$ goes with γ^{-2} while for $g(t) = (1+\alpha t^2)e^{\alpha t^2/2}$, the estimate of $\|A_0\|$ goes with α^{-1}. However, one should note that γ is the constant in front of t while α occurs in combination with t^2. Thus, if one wants to draw a comparison between these different cases at all, then it should be between γ and $\sqrt{\alpha}$. Of course, the main difference between the two theorems is the admitted growth rate of the solutions. In this regard, Thm. 6 contains the stronger statement.

2. A *physically realistic value of* λ is $\frac{1}{137}$, the value of the fine structure constant. In that case, α need not even be particularly large in order for condition (2.56) to be satisfied.

3. *Initial value problem.* By the integral equation (2.13), we obtain that the solution ψ satisfies $\psi(0, \mathbf{x}, 0, \mathbf{y}) = \psi^{\text{free}}(0, \mathbf{x}, 0, \mathbf{y})$. If ψ^{free} is a solution of the free multi-time Klein-Gordon equations, then it is itself determined by initial data at $x_1^0, x_2^0 = 0$. (As the Klein-Gordon equation is of second order in time, these initial data include data for $\partial_{x^0}\psi$, $\partial_{y^0}\psi$ and $\partial_{x^0}\partial_{y^0}\psi$, see [38, chap. 5].) Thus, we find that ψ is determined by these data at $x_1^0, x_2^0 = 0$ as well. Note that for later times, ψ and ψ^{free} do not, in general, coincide and consequently a similar statement does not hold.

4. *Finite propagation speed.* The theorem implies that $\psi = \sum_{k=0}^{\infty} A^k \psi^{\text{free}}$. As $(A\psi^{\text{free}})(x, y)$ involves only values of ψ^{free} in $\text{past}(x) \times \text{past}(y)$ where $\text{past}(x)$ denotes the causal past of $x \in \frac{1}{2}\mathbb{M}$ (see Eqs. (2.36), (2.38), (2.39), (2.42)), so do $A^k\psi^{\text{free}}$ for all $k \in \mathbb{N}$ and ψ. Therefore, we obtain: if the initial data for ψ^{free} at $x^0 = 0 = y^0$ are compactly supported in a region $R \subset (\{0\} \times \mathbb{R}^3)^2$, then for all Cauchy surfaces $\Sigma \subset \frac{1}{2}\mathbb{M}$, $\psi|_{\Sigma\times\Sigma}$ is supported in the causally grown set $\text{Gr}(R, \Sigma) = \left(\bigcup_{(x,y)\in R} \text{future}(x) \times \text{future}(y)\right) \cap (\Sigma \times \Sigma)$ where $\text{future}(x)$ stands for the causal future of $x \in \frac{1}{2}\mathbb{M}$.

5. *Square integrable solutions.* As a consequence of the previous item, compactly supported and bounded initial data for ψ^{free} lead to a compactly supported and bounded solution ψ. In particular, this implies that $\psi(x^0, \cdot, y^0)$ lies in $L^2(\mathbb{R}^6)$ for all times $x^0, y^0 \geqslant 0$.

2.2.3.2 The N-particle case

Here we extend Thm. 7 from two to $N \geqslant 3$ scalar particles. While there are different possibilities to generalize the two-particle integral equation (2.13), we focus on the one advocated in [28] as the most promising. For

$$\psi : \left(\tfrac{1}{2}\mathbb{M}\right)^N \to \mathbb{C}, \qquad (x_1, ..., x_N) \mapsto \psi(x_1, ..., x_N) \tag{2.57}$$

we consider the integral equation

$$\begin{aligned} \psi(x_1, ..., x_N) \;=\; & \psi^{\text{free}}(x_1, ..., x_N) + \frac{\lambda}{4\pi} \sum_{i,j=1,...,N;\, i<j} \\ & \times \int_{\frac{1}{2}\mathbb{M}} d^4x_i \int_{\frac{1}{2}\mathbb{M}} d^4x_j \; G_i^{\text{ret}}(x_i - x_i')G^{\text{ret}}(x_j - x_j') \\ & \times \delta((x_i' - x_j')^2)\psi(x_1, ..., x_i, ..., x_j, ..., x_N). \end{aligned} \tag{2.58}$$

Here, ψ^{free} is again a solution of the free Klein-Gordon equations $(\Box_k + m_k^2)\phi(x_k)$ in each spacetime variable and G_k^{ret} stands for the retarded Green's function of the operator $(\Box_k + m_k^2)$, $k = 1, 2, ..., N$.
Eq. (2.58) is written down in an informal way. To define a rigorous version, let $\psi \in \mathcal{S}\big((\tfrac{1}{2}\mathbb{M})^N\big)$ be a test function. Moreover, let $A^{(ij)}$ be the integral operator of the two-particle problem acting on the variables x_i and x_j instead of $x = x_1$ and $y = x_2$. We define

$$^{(N)}A \;=\; \sum_{i,j=1,...,N;\, i<j} A^{(ij)}. \tag{2.59}$$

As will be shown below, $^{(N)}A$ can be linearly extended to a bounded operator on the Banach space $^{(N)}\mathcal{B}_g$. That space is defined as the completion of $\mathcal{S}\big((\tfrac{1}{2}\mathbb{M})^N\big)$ with respect to the norm

$$\|\psi\|_g \;=\; \operatorname*{ess\,sup}_{x_1,...,x_N \in \frac{1}{2}\mathbb{M}} \frac{|\psi|(x_1, ..., x_N)}{g(x_1^0) \cdots g(x_N^0)}, \tag{2.60}$$

where the function g is defined as before.
Then we take the equation

$$^{(N)}A \;=\; \psi^{\text{free}} + {}^{(N)}A\psi, \tag{2.61}$$

to be the rigorous version of (2.58) on $^{(N)}\mathcal{B}_g$.
With these preparations, we are ready to formulate the N-particle existence and uniqueness theorem.

Theorem 8 (Existence of dynamics for N particles.)**.**
For any $\alpha > 0$, let $g(t) = (1+\alpha t^2)e^{\alpha t^2/2}$. Then the operator $^{(N)}A$ can be linearly extended to a bounded operator on $^{(N)}\mathcal{B}_g$ with norm

$$\|^{(N)}A\| \leqslant \frac{\lambda}{8\pi\alpha} \sum_{i,j=1,\dots,N;\, i<j} \left(\frac{1}{4} + \frac{5(m_i^2+m_j^2)}{2}\frac{1}{\alpha} + \frac{m_i^2\, m_j^2}{10}\frac{1}{\alpha^2} \right). \tag{2.62}$$

If α is such that this expression is strictly smaller than one, the integral equation (2.61) has a unique solution $\psi \in {}^{(N)}\mathcal{B}_g$ for every $\psi^{\text{free}} \in {}^{(N)}\mathcal{B}_g$.

The proof follows straightforwardly from that of Thm. 7 using

$$\|^{(N)}A\| \leqslant \sum_{i,j=1,\dots,N;\, i<j} \|A^{(ij)}\|_g. \tag{2.63}$$

For the norms of the operators $A^{(ij)}$, one can use the previous expressions as these operators act only as the identity on variables x_k with $k \notin \{i,j\}$.

Remark 9. *To the best of our knowledge, Thm. 8 is the first result about the existence and uniqueness of solutions of multi-time integral equations for N particles. While for the present contraction argument the generalization to N particles has been straightforward, this is not the case for other works. For example, the Volterra iterations used in [35] become increasingly complicated with increasing particle number*

N. For Dirac particles, a similar technique is used section 2.3. However, as the Dirac Green's functions contain distributional derivatives, one has to control weak derivatives of the solutions, and the number of such derivatives depends on N. That situation also does not allow for such a straightforward generalization to N particles as has been possible here.

2.2.3.3 On the possible origin of a cutoff in time

So far, we have assumed a cutoff in time. In the way this has been treated so far, this cutoff breaks the manifest Poincaré invariance of our integral equation. In this section, we demonstrate at a particular (simple and tractable) example that such a cutoff can arise naturally if the considered spacetime has a Big Bang singularity. Then the Big Bang defines the initial time. To consider a simple example is necessary as otherwise the Green's functions may not be known in detail, and in that case it would not be possible to explicitly define the integral operator, let alone to carry out an analysis of that operator comparable to the one before.
Our example consists of two massless scalar particles which, in absence of interactions, obey the conformally invariant wave equation on a curved spacetime $\mathcal{M}$ with metric g,

$$(\Box_g - R/6)\,\chi = 0, \tag{2.64}$$

where R denotes the Ricci scalar.
We consider these particles on a flat Friedman-Lemaître-Robertson-Walker (FLRW) spacetime which is described by the metric

$$ds^2 = a^2(\eta)\left(d\eta^2 - dr^2 - r^2 d\Omega^2\right), \tag{2.65}$$

where η denotes conformal time, $d\Omega$ denotes the surface measure on $\mathbb{S}^2$ and $a(\eta)$ is the so-called *scale function*, a continuous function with

$a(0) = 0$ and $a(\eta) > 0$ for $\eta > 0$. This form makes it obvious that the spacetime is conformally equivalent to a Minkowski half space $\frac{1}{2}\mathbb{M}$, with conformal factor $a(\eta)$.
In this case, it is well-known that the Green's functions of (2.64) on the flat FLRW spacetime $\mathcal{M}$ can be obtained from those of the usual wave equation on $\frac{1}{2}\mathbb{M}$ as follows (using coordinates $x = (\eta, \mathbf{x})$ and $x' = (\eta', \mathbf{x}')$ with $\eta, \eta' \in [0, \infty)$ and $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^3$):

$$G_{\mathcal{M}}(x, x') = \frac{1}{a(\eta)} \frac{1}{a(\eta')} G_{\frac{1}{2}\mathbb{M}}(x, x'), \tag{2.66}$$

which can be derived from the transformation behavior of the Ricci scalar

$$R_{\mathcal{M}_2} = a^{-2} R_{\mathcal{M}_1} - 6a^{-3} \Box_{\mathcal{M}_1} a \tag{2.67}$$

for general spacetimes $\mathcal{M}_2$ and $\mathcal{M}_1$ connected by a conformal transformation; see [34, 24] for more detailed explanations. Inserting the well-known expression for the retarded and symmetric Green's functions on $\frac{1}{2}\mathbb{M}$ (see (2.23)) yields:

$$\begin{aligned} G^{\text{ret}}_{\mathcal{M}}(x, x') &= \frac{1}{4\pi} \frac{1}{a(\eta)a(\eta')} \frac{\delta(\eta - \eta' - |\mathbf{x} - \mathbf{x}'|)}{|\mathbf{x} - \mathbf{x}'|} \\ G^{sym}_{\mathcal{M}}(x, x') &= \frac{1}{4\pi} \frac{1}{a(\eta)a(\eta')} \delta((\eta - \eta')^2 - |\mathbf{x} - \mathbf{x}'|^2). \end{aligned} \tag{2.68}$$

With this information, we are ready to write down the integral equation on $\mathcal{M}$. The generalization of (2.13) to curved spacetimes is straightforward: ψ becomes a scalar function on $\mathcal{M} \times \mathcal{M}$, one exchanges the Minkowski spacetime volume elements with the invariant 4-volume elements on $\mathcal{M}$, and the Green's functions on $\frac{1}{2}\mathbb{M}$ get replaced with those on $\mathcal{M}$ as well. As in the Minkowski case, the interaction kernel is given by the symmetric Green's function. With this, the relevant integral equation becomes:

$$\psi(x,y) = \psi^{\text{free}}(x,y) + \lambda \int_{\mathcal{M}\times\mathcal{M}} dV(x)dV(y)\; G_1^{\text{ret}}(x,x')G_2^{\text{ret}}(y,y') \times G^{\text{sym}}(x',y')\psi(x',y'), \tag{2.69}$$

For flat FLRW universes and scalar particles, we here extend [34] to the physically most interesting and mathematically challenging case $K(x,y) = G^{\text{sym}}(x,y)$.

We now formulate (2.69) explicitly. The spacetime volume element is given by:

$$dV(x) = a^4(\eta)\, d\eta\, d^3\mathbf{x}. \tag{2.70}$$

With this information, (2.69) becomes:

$$\begin{aligned}\psi(\eta_1,\mathbf{x}_1,\eta_2,\mathbf{x}_2) &= \psi^{\text{free}}(\eta_1,\mathbf{x}_1,\eta_2,\mathbf{x}_2) + \frac{\lambda}{(4\pi)^3}\frac{1}{a(\eta_1)a(\eta_2)}\\ &\int_0^{\eta_1} d\eta_1' \int d^3\mathbf{x}_1' \int_0^{\eta_2} d\eta_2' \int d^3\mathbf{x}_2' a^2(\eta_1')a^2(\eta_2')\\ &\times \frac{\delta(\eta_1-\eta_1'-|\mathbf{x}_1-\mathbf{x}_1'|)}{|\mathbf{x}_1-\mathbf{x}_1'|}\frac{\delta(\eta_2-\eta_2'-|\mathbf{x}_2-\mathbf{x}_2'|)}{|\mathbf{x}_2-\mathbf{x}_2'|}\\ &\times \delta((\eta_1'-\eta_2')^2 - |\mathbf{x}_1'-\mathbf{x}_2'|^2)\psi(\eta_1',\mathbf{x}_1',\eta_2',\mathbf{x}_2').\end{aligned} \tag{2.71}$$

Now let

$$\chi(\eta_1,\mathbf{x}_1,\eta_2) = a(\eta_1)a(\eta_2)\psi(\eta_1,\mathbf{x}_1,\eta_2). \tag{2.72}$$

and $\chi^{free}(\eta_1,\mathbf{x}_1,\eta_2) = a(\eta_1)a(\eta_2)\psi^{\text{free}}(\eta_1,\mathbf{x}_1,\eta_2)$. Then (2.71) is equivalent to:

$$\begin{aligned}\chi(\eta_1,\mathbf{x}_1,\eta_2,\mathbf{x}_2) &= \chi^{free}(\eta_1,\mathbf{x}_1,\eta_2,\mathbf{x}_2) + \frac{\lambda}{(4\pi)^3}\int_0^{\eta_1} d\eta_1' \int d^3\mathbf{x}_1' \\ &\times \int_0^{\eta_2} d\eta_2' \int d^3\mathbf{x}_2' \frac{\delta(\eta_1-\eta_1'-|\mathbf{x}_1-\mathbf{x}_1'|)}{|\mathbf{x}_1-\mathbf{x}_1'|}\frac{\delta(\eta_2-\eta_2'-|\mathbf{x}_2-\mathbf{x}_2'|)}{|\mathbf{x}_2-\mathbf{x}_2'|}\\ &\times a(\eta_1')a(\eta_2')\delta((\eta_1'-\eta_2')^2 - |\mathbf{x}_1'-\mathbf{x}_2'|^2)\chi(\eta_1',\mathbf{x}_1',\eta_2',\mathbf{x}_2').\end{aligned} \tag{2.73}$$

We can see that this equation has almost exactly the same form as the massless version of (2.13) on $\frac{1}{2}\mathbb{M}$ (see (2.25)). The only difference is the additional appearance of the factor $a(\eta_1')a(\eta_2')$ inside the integrals. Going through the same steps as for (2.36) before, (2.73) can be defined on test functions $\chi \in \mathcal{S}$ by

$$\chi \;=\; \chi^{free} + \widetilde{A}_0\chi, \tag{2.74}$$

where $\widetilde{A}_0$ is defined by (using coordinates $x = (\eta_1, \mathbf{x})$, $y = (\eta_2, \mathbf{y})$):

$$\begin{aligned}(\widetilde{A}_0\psi)(x,y) = &\frac{\lambda}{(4\pi)^3}\int_{B_{y^0}(\mathbf{y})} d^3\mathbf{y}' \int_0^{2\pi} d\varphi \int_{-1}^{1} d\cos\vartheta \, \frac{|b^2|}{4(b^0+|\mathbf{b}|\cos\vartheta)^2|\mathbf{y}'|}\\ &\times a(\eta_1+\eta_1')a(\eta_2+\eta_2')\psi(x+x',y+y')\\ &\times \left(1_{b^2>0}1_{b^0>0}1_{\cos\vartheta > \frac{b^2}{2x^0|\mathbf{b}|}-\frac{b^0}{|\mathbf{b}|}} + 1_{b^2<0}1_{\cos\vartheta < \frac{b^2}{2x^0|\mathbf{b}|}-\frac{b^0}{|\mathbf{b}|}}\right),\end{aligned} \tag{2.75}$$

with $\eta_1' = -r^* = -|\mathbf{x}'|, \eta_2' = -|\mathbf{y}'|$. (Here, b and r^* are defined as in (2.30) and (2.32), respectively).
Knowing precisely how our integral equation on the flat FLRW spacetime is to be understood, we can formulate the respective existence and uniqueness theorem:

Theorem 10 (Existence of dynamics for an open FLRW universe). *Let $a : [0,\infty) \to [0,\infty)$ be a continuous function with $a(0) = 0$ and $a(\eta) > 0$ for $\eta > 0$. Moreover, let*

$$g(t) = \exp\left(\gamma \int_0^t d\tau \, a(\tau)\right). \tag{2.76}$$

Then, the operator $\widetilde{A}_0$ satisfies the following estimate:

$$\sup_{\chi\in\mathcal{S}\left(([0,\infty)\times\mathbb{R}^3)^2\right)} \frac{\|\widetilde{A}_0\chi\|_g}{\|\chi\|_g} \;\leqslant\; \frac{\lambda}{8\pi\gamma^2}. \tag{2.77}$$

$\widetilde{A}_0$ can be extended to a linear operator on $\mathcal{B}_g$ which satisfies the same bound. Moreover, for $\gamma < \sqrt{\frac{\lambda}{8\pi}}$, the equation $\chi = \chi^{\rm free} + \widetilde{A}_0 \chi$ has a unique solution $\chi \in \mathcal{B}_g$ for every $\psi^{\rm free} \in \mathcal{B}_g$.

The proof can be found in Sec. 2.2.4.4.

Remarks:

1. *Manifest covariance.* The theorem shows the existence and uniqueness of solutions of the manifestly covariant integral equation (2.69). Our example of a particular FLRW spacetime thus achieves its goal of demonstrating that a cutoff in time can arise naturally in a cosmological context.

2. *Initial value problem.* As in the case of $\frac{1}{2}\mathbb{M}$, the solution χ satisfies $\chi(0, \mathbf{x}, 0, \mathbf{y}) = \chi^{\rm free}(0, \mathbf{x}, 0, \mathbf{y})$ where $\chi^{\rm free}$ is determined by the solution $\psi^{\rm free}$ of the free conformal wave equation (2.64) in both spacetime variables. Since $\psi^{\rm free}$ is determined by initial data at $\eta_1 = 0 = \eta_2$, so are $\chi^{\rm free}$ and χ.

3. *Behavior of ψ towards the Big Bang singularity.* While the transformed wave function χ remains bounded for $\eta_1, \eta_2 \to 0$, the physical wave function $\psi(\eta_1, \mathbf{x}, \eta_2, \mathbf{y}) = \frac{1}{a(\eta_1)a(\eta_2)} \chi(\eta_1, \mathbf{x}, \eta_2, \mathbf{y})$ diverges like $\frac{1}{a(\eta_1)a(\eta_2)}$. This is to be expected, as the Klein-Gordon equation has a preserved "energy" (given by a certain spatial integral) and as the volume in $\mathbf{x}, \mathbf{y}$ contracts to zero towards the Big Bang.

4. *N-particle generalization.* As shown in Sec. 2.2.3.2 for the Minkowski half-space, it would also be possible to directly extend Thm. 10 to N particles. To avoid duplication, we do not carry this out explicitly for the curved spacetime example here.

2.2.4 Proofs

2.2.4.1 Proof of Theorem 5

The proof is divided into the proofs of the estimates (2.46), (2.47), (2.48) and (2.49), respectively. Here, (2.46) is the most singular and difficult term which deserves the greatest attention.
Throughout this subsection, let $\psi \in \mathcal{S}((\frac{1}{2}\mathbb{M})^2)$.

2.2.4.1.1 Estimate of the massless term (2.46). We start with Eq. (2.36) and take the absolute value. Using, in addition, that

$$|\psi(x,y)| \;\leqslant\; \|\psi\|_g \, g(x^0)g(y^0) \tag{2.78}$$

leads us to:

$$\begin{aligned} |A_0\psi|(x,y) \leqslant & \frac{\lambda\|\psi\|_g}{4(4\pi)^3}\int_{B_{y^0}(\mathbf{y})} d^3\mathbf{y}' \int_0^{2\pi} d\varphi \int_{-1}^1 d\cos\vartheta \, \frac{|b^2|}{(b^0+|\mathbf{b}|\cos\vartheta)^2|\mathbf{y}'|} \\ & \times g(y^0-|\mathbf{y}'|)g\left(x^0-\frac{1}{2}\frac{b^2}{b^2+|\mathbf{b}|\cos\vartheta}\right) \\ & \times \left(1_{b^2>0}1_{b^0>0}1_{\cos\vartheta>\frac{b^2}{2x^0|\mathbf{b}|}-\frac{b^0}{|\mathbf{b}|}} + 1_{b^2<0}1_{\cos\vartheta<\frac{b^2}{2x^0|\mathbf{b}|}-\frac{b^0}{|\mathbf{b}|}}\right). \end{aligned} \tag{2.79}$$

Next, we observe that the fraction $\frac{|b^2|}{(b^0+|\mathbf{b}|\cos\vartheta)^2}$ is the derivative of the fraction which occurs in the argument of the second g-function. Introducing $u=\cos\vartheta$ allows us to rewrite (2.79) as

$$\begin{aligned} (2.79) = & \frac{\lambda\|\psi\|_g}{8(4\pi)^2}\int_{B_{y^0}(\mathbf{y})} d^3\mathbf{y}' \int_{-1}^1 du 2\,\mathrm{sgn}(b^2) \\ & \times \partial_u g_1\left(x^0-\frac{1}{2}\frac{b^2}{b^0+|\mathbf{b}|u}\right) g(y^0-|\mathbf{y}'|)\frac{1}{|\mathbf{b}||\mathbf{y}'|} \\ & \times \left(1_{b^2>0}1_{b^0>0}1_{u>\frac{b^2}{2x^0|\mathbf{b}|}-\frac{b^0}{|\mathbf{b}|}} + 1_{b^2<0}1_{u<\frac{b^2}{2x^0|\mathbf{b}|}-\frac{b^0}{|\mathbf{b}|}}\right) \end{aligned} \tag{2.80}$$

$$= \frac{\lambda\|\psi\|_g}{4(4\pi)^2}\int_{B_{y^0}(\mathbf{y})} d^3\mathbf{y}' \int_{-1}^{1} du\ \partial_u g_1\left(x^0 - \frac{1}{2}\frac{b^2}{b^0+|\mathbf{b}|u}\right) g(y^0 - |\mathbf{y}'|)$$
$$\times\left(1_{b^2>0}1_{b^0>0}1_{u>\frac{b^2}{2x^0|\mathbf{b}|}-\frac{b^0}{|\mathbf{b}|}} - 1_{b^2<0}1_{u<\frac{b^2}{2x^0|\mathbf{b}|}-\frac{b^0}{|\mathbf{b}|}}\right)\frac{1}{|\mathbf{b}||\mathbf{y}'|}. \quad (2.81)$$

This form allows for a direct integration with respect to u. Before we integrate, we check whether the conditions implicit in the characteristic functions can always be satisfied. (Otherwise, the respective term would not contribute any further and we could drop it.) Recall that $b = x - y - (-|\mathbf{y}'|, \mathbf{y}')$. First we check whether in the case $b^2 > 0, b^0 > 0$ it is true that $1 > \frac{b^2}{2x^0|\mathbf{b}|} - \frac{b^0}{|\mathbf{b}|}$ holds. (The comparison with 1 is due to the upper range for u.) We compute

$$\begin{aligned} 1 > \frac{b^2}{2x^0|\mathbf{b}|} - \frac{b^0}{|\mathbf{b}|} &\iff 2x^0|\mathbf{b}| + 2x^0b^0 > b^2 \\ &\iff 2x^0(b^0+|\mathbf{b}|) > (b^0+|\mathbf{b}|)(b^0-|\mathbf{b}|) \\ &\overset{b^2>0,b^0>0}{\iff} 2x^0 > b^0 - |\mathbf{b}| \\ &\iff x^0 + y^0 - |\mathbf{y}'| > -|\mathbf{b}|. \end{aligned} \quad (2.82)$$

Now because of $|\mathbf{y}'| < y^0$ we see that this inequality always holds true. Hence the respective term in (2.81) contributes without further restrictions.

Next, we turn to the case $b^2 < 0$. Here we check whether (or when) $-1 < \frac{b^2}{2x^0|\mathbf{b}|} - \frac{b^0}{|\mathbf{b}|}$ holds. (The comparison with -1 is due to the lower bound for u.) A similar calculation yields

$$-1 < \frac{b^2}{2x^0|\mathbf{b}|} - \frac{b^0}{|\mathbf{b}|} \iff -2x^0|\mathbf{b}| + 2x^0|\mathbf{b}| < b^2 \quad (2.83)$$
$$\iff 2x^0(b^0-|\mathbf{b}|) < (b^0-|\mathbf{b}|)(b^0+|\mathbf{b}|) \quad (2.84)$$
$$\overset{b^2<0}{\iff} 2x^0 > b^0 + |\mathbf{b}|. \quad (2.85)$$

This inequality need not always hold, as we can increase $|\mathbf{b}|$ with respect to b^0 as much as we like, e.g., by picking $|\mathbf{x}-\mathbf{y}|$ large. Therefore, in this case, the respective term is only sometimes nonzero. We make this clear by including the characteristic function $1_{2x^0>b^0+|\mathbf{b}|}$.
Taking these considerations into account, we now carry out the u-integration in (2.81):

$$\begin{aligned}|A_0\psi|(x,y) \;\leqslant\; & \frac{\lambda\|\psi\|_g}{4(4\pi)^2}\int_{B_{y^0}(\mathbf{y})} d^3\mathbf{y}'\; \frac{g(y^0-|\mathbf{y}'|)}{|\mathbf{b}||\mathbf{y}'|} \\ & \times \Bigg(1_{b^2>0,b^0>0}\Bigg[g_1\left(x^0-\frac{1}{2}\frac{b^2}{b^0+|\mathbf{b}|}\right) \\ & \quad - g_1\left(x^0-\frac{1}{2}\frac{b^2}{b^0+|\mathbf{b}|\max(-1,\frac{b^2}{2x^0|\mathbf{b}|}-\frac{b^0}{|\mathbf{b}|})}\right)\Bigg] \\ & - 1_{b^2<0}1_{2x^0>b^0+|\mathbf{b}|}\Bigg[g_1\left(x^0-\frac{1}{2}\frac{b^2}{b^0+|\mathbf{b}|\min(1,\frac{b^2}{2x^0|\mathbf{b}|}-\frac{b^0}{|\mathbf{b}|})}\right) \\ & \quad - g_1\left(x^0-\frac{1}{2}\frac{b^2}{b^0-|\mathbf{b}|}\right)\Bigg]\Bigg). \end{aligned} \tag{2.86}$$

The minima and maxima in this expression result from the indicator functions $1_{u>\frac{b^2}{2x^0|\mathbf{b}|}-\frac{b^0}{|\mathbf{b}|}}$ and $1_{u<\frac{b^2}{2x^0|\mathbf{b}|}-\frac{b^0}{|\mathbf{b}|}}$, respectively.
Our next step is to simplify the complicated fractions in (2.86) involving min and max. For the first one we use that $1/\max(a,b) = \min(1/a,1/b)$ whenever $a,b>0$ or $a,b<0$ holds. Therefore, we have:

$$\begin{aligned} & \frac{1}{2}\frac{b^2}{b^0+|\mathbf{b}|\max\left(-1,\frac{b^2}{2x^0|\mathbf{b}|}-\frac{b^0}{|\mathbf{b}|}\right)} = \frac{1}{2}\frac{b^2}{\max\left(b^0-|\mathbf{b}|,\frac{b^2}{2x^0}\right)} \\ & = \frac{1}{2}\min\left(\frac{b^2}{b^0-|\mathbf{b}|},2x^0\right) = \min\left(\frac{b^0+|\mathbf{b}|}{2},x^0\right). \end{aligned}$$

The fraction in (2.86) which contains a minimum can be simplified by observing that

$$b^0 + |\mathbf{b}| \min\left(1, \frac{b^2}{2x^0|\mathbf{b}|} - \frac{b^0}{|\mathbf{b}|}\right) = \min\left(b^0 + |\mathbf{b}|, \frac{b^2}{2x^0}\right) = \frac{b^2}{2x^0} \quad (2.87)$$

as the term contributes only for $b^2 < 0$ hence $b^0 + |\mathbf{b}| > 0 > b^2/(2x^0)$. Thus,

$$\frac{1}{2}\frac{b^2}{b^0 + |\mathbf{b}| \min(1, \frac{b^2}{2x^0|\mathbf{b}|} - \frac{b^0}{|\mathbf{b}|})} = x^0. \quad (2.88)$$

With these simplifications, we obtain (using $g_1(0) = 0$):

$$\begin{aligned}
|A_0\psi|(x,y) \leqslant{}& \frac{\lambda\|\psi\|_g}{4(4\pi)^2} \int_{B_{y^0}(\mathbf{y})} d^3\mathbf{y}' \, \frac{g(y^0 - |\mathbf{y}'|)}{|\mathbf{b}||\mathbf{y}'|} \\
&\times \left(1_{b^2>0,b^0>0}\left[g_1\left(x^0 - \frac{b^0 - |\mathbf{b}|}{2}\right) - g_1\left(x^0 - \min\left(\frac{b^0 + |\mathbf{b}|}{2}, x^0\right)\right)\right]\right. \\
&\left. -1_{b^2<0}\ 1_{2x^0>b^0+|\mathbf{b}|}\left[g_1\left(x^0 - x^0\right) - g_1\left(x^0 - \frac{b^0 + |\mathbf{b}|}{2}\right)\right]\right) \\
={}& \frac{\lambda\|\psi\|_g}{4(4\pi)^2} \int_{B_{y^0}(\mathbf{y})} d^3\mathbf{y}' \frac{g(y^0 - |\mathbf{y}'|)}{|\mathbf{b}||\mathbf{y}'|} 1_{b^2>0,b^0>0} \\
&\times g_1\left(\frac{x^0 + y^0 - |\mathbf{y}'| + |\mathbf{b}|}{2}\right) && (2.89) \\
&- \frac{\lambda\|\psi\|_g}{4(4\pi)^2} \int_{B_{y^0}(\mathbf{y})} d^3\mathbf{y}' \, \frac{g(y^0 - |\mathbf{y}'|)}{|\mathbf{b}||\mathbf{y}'|} 1_{b^2>0,b^0>0} \\
&\times g_1\left(\max\left(\frac{x^0 + y^0 - |\mathbf{y}'| - |\mathbf{b}|}{2}, 0\right)\right) && (2.90) \\
&+ \frac{\lambda\|\psi\|_g}{4(4\pi)^2} \int_{B_{y^0}(\mathbf{y})} d^3\mathbf{y}' \, \frac{g(y^0 - |\mathbf{y}'|)}{|\mathbf{b}||\mathbf{y}'|} 1_{b^2<0}\ 1_{x^0+y^0-|\mathbf{y}'|>|\mathbf{b}|} \\
&\times g_1\left(\frac{x^0 + y^0 - |\mathbf{y}'| - |\mathbf{b}|}{2}\right). && (2.91)
\end{aligned}$$

We now want to carry out as many of the remaining $\mathbf{y}'$-integrations as possible. In order to do so, we orient the coordinates such that $\mathbf{x}-\mathbf{y}$ is parallel to the $(\mathbf{y}')_3$ axis. Then the integrands in (2.89)-(2.91) are independent of the azimuthal angle φ of the respective spherical coordinate system (ρ, θ, φ) with standard conventions.
In order to perform the remaining angular and then the radial integral, we need to find out which boundaries for θ and r result from the characteristic functions. First we analyze for which arguments the maximum in (2.90) is greater than zero and therefore contributes to the integral (as $g_1(0) = 0$). We have:

$$\begin{aligned} &\frac{x^0 + y^0 - |\mathbf{y}'| - |\mathbf{b}|}{2} \;>\; 0 \\ &\iff\; (x^0+y^0-|\mathbf{y}'|)^2 > |\mathbf{x}-\mathbf{y}|^2+|\mathbf{y}'|^2+2|\mathbf{y}'||\mathbf{x}-\mathbf{y}|\cos\theta \\ &\iff\; \cos\theta < \frac{(x^0+y^0)^2}{2|\mathbf{y}'||\mathbf{x}-\mathbf{y}|} - \frac{|\mathbf{x}-\mathbf{y}|}{2|\mathbf{y}'|} - \frac{x^0+y^0}{|\mathbf{x}-\mathbf{y}|} =: P_{x,y}(|\mathbf{y}'|). \end{aligned} \tag{2.92}$$

This calculation also helps to reformulate the second indicator function $1_{b^2<0}\ 1_{x^0+y^0-|\mathbf{y}'|>|\mathbf{b}|}$ in (2.91) (for which we have $b^2 < 0$). The condition $b^0 > 0$ in (2.89) and (2.90) is readily seen to be equivalent to

$$|\mathbf{y}'| > y^0 - x^0. \tag{2.93}$$

In order to perform the θ-integral we have to translate $b^2 \gtrless 0$ into conditions on θ. We have:

$$\begin{aligned} b^2 > 0 \;&\iff\; (x^0 - y^0 + |\mathbf{y}'|)^2 \;>\; |\mathbf{x}-\mathbf{y}|^2 + |\mathbf{y}'|^2 + 2|\mathbf{y}'||\mathbf{x}-\mathbf{y}|\cos\theta \\ &\iff\; \cos\theta \;<\; \frac{(x-y)^2}{2|\mathbf{y}'||\mathbf{x}-\mathbf{y}|} + \frac{x^0-y^0}{|\mathbf{x}-\mathbf{y}|} := K_{x-y}(|\mathbf{y}'|). \end{aligned} \tag{2.94}$$

With these considerations, we have extracted relatively simple conditions on the boundaries of the integrals in spherical coordinates. However, if different restrictions of the boundaries conflict with each other, it may happen that for some parameter values the domain of integration is the empty set. We check whether this is so term by term, focusing on the θ-integration first. For term (2.89), θ needs to satisfy $-1 < \cos\theta < \min(1, K_{x-y}(|\mathbf{y}'|))$, so we need to check whether $-1 < K_{x-y}(|\mathbf{y}'|)$ holds. We have:

$$
\begin{aligned}
& -1 < K_{x-y}(|\mathbf{y}'|) \\
& \iff -2|\mathbf{y}'||\mathbf{x}-\mathbf{y}| < (x-y)^2 + 2|\mathbf{y}'|(x^0 - y^0) \\
& \iff 0 \;<\; (x-y)^2 + 2|\mathbf{y}'|(x^0 - y^0 + |\mathbf{x}-\mathbf{y}|) \\
& \iff \begin{cases} \frac{y^0 - x^0 + |\mathbf{x}-\mathbf{y}|}{2} < |\mathbf{y}'| & \text{for } |\mathbf{x}-\mathbf{y}| > y^0 - x^0 \\ \frac{y^0 - x^0 + |\mathbf{x}-\mathbf{y}|}{2} > |\mathbf{y}'| & \text{for } |\mathbf{x}-\mathbf{y}| < y^0 - x^0. \end{cases}
\end{aligned}
\tag{2.95}
$$

Together with (2.93), we obtain the condition $y^0 - x^0 < |\mathbf{y}'| < \frac{y^0 - x^0 + |\mathbf{x}-\mathbf{y}|}{2} < y^0 - x^0$ in the second case which means that there is no contribution to the integral. For $y^0 - x^0 = |\mathbf{x}-\mathbf{y}|$ we have $K_{x-y}(|\mathbf{y}'|) = -1$ so this case is also ruled out. So we focus on the first case,

$$
\frac{y^0 - x^0 + |\mathbf{x}-\mathbf{y}|}{2} < |\mathbf{y}'| \quad \text{and } |\mathbf{x}-\mathbf{y}| > y^0 - x^0, \tag{2.96}
$$

by including the characteristic function $1_{|\mathbf{x}-\mathbf{y}| > y^0 - x^0}$ in the integral. Next, we turn to the radial integral. By comparing its upper limit $|\mathbf{y}'| < y^0$ and lower limit $(y^0 - x^0 + |\mathbf{x}-\mathbf{y}|)/2$, we find that the integral can only be nonzero for

$$
y^0 + x^0 > |\mathbf{x}-\mathbf{y}|. \tag{2.97}
$$

For equality the integral vanishes, because the inegral domain, while not empty, is of measure zero. We make this clear by including the respective characteristic function.

2.2.4.1.1.1 Simplification of term (2.89). These considerations allow us to continue computing (2.89):

$$
\begin{aligned}
(2.89) = {} & \frac{\lambda\|\psi\|_g}{4(4\pi)^2} 1_{y^0+x^0>|\mathbf{x}-\mathbf{y}|} \int_{\max(0,y^0-x^0)}^{y^0} d\rho \int_0^{2\pi} d\varphi \; 1_{\frac{y^0-x^0+|\mathbf{x}-\mathbf{y}|}{2}<\rho} \\
& \times 1_{|\mathbf{x}-\mathbf{y}|>y^0-x^0} \int_{-1}^{\min(1,K_{x-y}(\rho))} d\cos\theta \; \frac{\rho\, g(y^0-\rho)}{\sqrt{|\mathbf{x}-\mathbf{y}|^2+\rho^2+2|\mathbf{x}-\mathbf{y}|\rho\cos\theta}} \\
& \times g_1\left(\frac{x^0+y^0-\rho+\sqrt{|\mathbf{x}-\mathbf{y}|^2+\rho^2+2\rho|\mathbf{x}-\mathbf{y}|\cos\theta}}{2}\right).
\end{aligned}
\tag{2.98}
$$

Now we carry out the φ-integration and use the same trick for the θ-integral as for the ϑ-integral in the $\mathbf{x}'$-integration earlier. Moreover, we absorb some of the restrictions of ρ into the limits of the integrals. This yields:

$$
\begin{aligned}
(2.89) = {} & \frac{\lambda\|\psi\|_g}{8(4\pi)} 1_{y^0+x^0>|\mathbf{x}-\mathbf{y}|>y^0-x^0} \int_{\max\left(0,y^0-x^0,\frac{y^0-x^0+|\mathbf{x}-\mathbf{y}|}{2}\right)}^{y^0} d\rho \\
& \int_{-1}^{\min(1,K_{x-y}(\rho))} dw \; \frac{2g(y^0-\rho)}{|\mathbf{x}-\mathbf{y}|} \\
& \times \partial_w g_2\left(\frac{x^0+y^0-\rho+\sqrt{|\mathbf{x}-\mathbf{y}|^2+\rho^2+2\rho|\mathbf{x}-\mathbf{y}|w}}{2}\right) \\
= {} & \frac{\lambda\|\psi\|_g}{4(4\pi)} 1_{x^0+y^0>|\mathbf{x}-\mathbf{y}|>y^0-x^0} \int_{\max\left(0,y^0-x^0,\frac{y^0-x^0+|\mathbf{x}-\mathbf{y}|}{2}\right)}^{y^0} d\rho \; \frac{g(y^0-\rho)}{|\mathbf{x}-\mathbf{y}|} \\
& \times \Bigg[g_2\left(\frac{x^0+y^0-\rho+\sqrt{|\mathbf{x}-\mathbf{y}|^2+\rho^2+2\rho|\mathbf{x}-\mathbf{y}|\min(1,K_{x-y}(\rho))}}{2}\right) \\
& - g_2\left(\frac{x^0+y^0-\rho+||\mathbf{x}-\mathbf{y}|-\rho|}{2}\right)\Bigg]
\end{aligned}
\tag{2.99}
$$

The square root can be simplified using the following identity:

$$\begin{aligned}&\sqrt{|\mathbf{x}-\mathbf{y}|^2+\rho^2+2\rho|\mathbf{x}-\mathbf{y}|K_{x-y}(\rho)}\\&=\sqrt{\rho^2+(x^0-y^0)^2+2\rho(x^0-y^0)}=|x^0-y^0+\rho|.\end{aligned}\tag{2.100}$$

Using this, we can effectively pull the minimum out of the square root. We obtain:

$$\begin{aligned}(2.89)=&\frac{\lambda\|\psi\|_g}{16\pi}1_{x^0+y^0>|\mathbf{x}-\mathbf{y}|>y^0-x^0}\int_{\max\left(0,y^0-x^0,\frac{y^0-x^0+|\mathbf{x}-\mathbf{y}|}{2}\right)}^{y^0}d\rho\,\frac{g(y^0-\rho)}{|\mathbf{x}-\mathbf{y}|}\\&\times\Bigg[g_2\left(\frac{x^0+y^0-\rho+\min(|\mathbf{x}-\mathbf{y}|+\rho,|x^0-y^0+\rho|)}{2}\right)\\&\qquad-g_2\left(\frac{x^0+y^0-\rho+||\mathbf{x}-\mathbf{y}|-\rho|}{2}\right)\Bigg].\end{aligned}\tag{2.101}$$

Next, we subdivide the conditions in the first indicator function into two cases, (a) $(x-y)^2\geqslant 0$ and (b) $(x-y)^2<0$. In case (a), the condition $|\mathbf{x}-\mathbf{y}|>y^0-x^0$ implies $x^0>y^0$. This, in turn, yields $\max\left(0,y^0-x^0,\frac{y^0-x^0+|\mathbf{x}-\mathbf{y}|}{2}\right)=0$. Moreover, the condition $x^0+y^0>|\mathbf{x}-\mathbf{y}|$ is automatically satisfied (note that $x^0,y^0>0$). In case (b), the condition $|\mathbf{x}-\mathbf{y}|>0$ is automatically satisfied. We find:

$$\begin{aligned}(2.89)=&\frac{\lambda\|\psi\|_g}{16\pi}1_{(x-y)^2\geqslant 0,x^0>y^0}\int_0^{y^0}d\rho\,\frac{g(y^0-\rho)}{|\mathbf{x}-\mathbf{y}|}\\&\times\left[g_2\left(\frac{x^0+y^0+|\mathbf{x}-\mathbf{y}|}{2}\right)-g_2\left(\frac{x^0+y^0-\rho+||\mathbf{x}-\mathbf{y}|-\rho|}{2}\right)\right]\\&+\frac{\lambda\|\psi\|_g}{16\pi}1_{(x-y)^2<0}\;1_{x^0+y^0>|\mathbf{x}-\mathbf{y}|}\int_{\frac{y^0-x^0+|\mathbf{x}-\mathbf{y}|}{2}}^{y^0}d\rho\,\frac{g(y^0-\rho)}{|\mathbf{x}-\mathbf{y}|}\\&\times\left[g_2\left(\frac{x^0+y^0-\rho+|x^0-y^0+\rho|}{2}\right)\right.\end{aligned}$$

$$
\begin{aligned}
&\qquad - g_2 \left(\frac{x^0 + y^0 - \rho + ||\mathbf{x}-\mathbf{y}| - \rho|}{2} \right) \Bigg] \\
&= \frac{\lambda \|\psi\|_g}{16\pi} 1_{(x-y)^2 \geqslant 0, x^0 > y^0} \int_0^{y^0} d\rho \, \frac{g(y^0 - \rho)}{|\mathbf{x}-\mathbf{y}|} \\
&\times \Bigg[g_2 \left(\frac{x^0 + y^0 + |\mathbf{x}-\mathbf{y}|}{2} \right) \\
&\qquad - g_2 \max \left(\frac{x^0 + y^0 - |\mathbf{x}-\mathbf{y}|}{2}, \frac{x^0 + y^0 + |\mathbf{x}-\mathbf{y}|}{2} - \rho \right) \Bigg] \\
&+ \frac{\lambda \|\psi\|_g}{16\pi} 1_{(x-y)^2 < 0} \; 1_{x^0 + y^0 > |\mathbf{x}-\mathbf{y}|} \int_{\frac{y^0 - x^0 + |\mathbf{x}-\mathbf{y}|}{2}}^{y^0} d\rho \, \frac{g(y^0 - \rho)}{|\mathbf{x}-\mathbf{y}|} \\
&\times \Bigg[g_2 \max \left(x^0, y^0 - \rho \right) \\
&\qquad - g_2 \max \left(\frac{x^0 + y^0 - |\mathbf{x}-\mathbf{y}|}{2}, \frac{x^0 + y^0 + |\mathbf{x}-\mathbf{y}|}{2} - \rho \right) \Bigg]. \qquad (2.102)
\end{aligned}
$$

Here and in the following we abbreviate $g_2(\max(\cdots))$ as $g_2 \max(\cdots)$, and similarly for the minimum. This ends the calculation of (2.89): we have arrived at an expression where no more exact calculations can be done and further estimates are needed.

2.2.4.1.1.2 Simplification of term (2.90). Next, we proceed with (2.90) in a similar fashion. In case the reader is not interested in the details of the calculation, the result can be found in (2.110).
The restrictions of the integration variables for (2.90) are the same as for (2.89), namely:

$$\cos\theta < K_{x-y}(|\mathbf{y}'|) \qquad \text{from (2.94)}, \tag{2.103}$$

$$\frac{y^0 - x^0 + |\mathbf{x} - \mathbf{y}|}{2} < |\mathbf{y}'| \qquad \text{from (2.96)} \tag{2.104}$$

$$y^0 - x^0 < |\mathbf{x} - \mathbf{y}| < y^0 + x^0 \qquad \text{from (2.96) and (2.97).} \tag{2.105}$$

The only difference is that from the maximum in (2.90), we obtain the additional restriction (2.92), i.e.

$$\cos\theta < P_{x,y}(|\mathbf{y}'|). \tag{2.106}$$

We need to check if there are new restrictions imposed by $P_{x,y}(|\mathbf{y}'|) > -1$. We compute

$$\begin{aligned} P_{x,y}(|\mathbf{y}'|) &> -1 \quad \Longleftrightarrow \\ \frac{(x^0+y^0)^2}{2|\mathbf{y}'||\mathbf{x}-\mathbf{y}|} - \frac{|\mathbf{x}-\mathbf{y}|}{2|\mathbf{y}'|} - \frac{x^0+y^0}{|\mathbf{x}-\mathbf{y}|} &> -1 \quad \Longleftrightarrow \\ |\mathbf{y}'| &< \frac{x^0+y^0+|\mathbf{x}-\mathbf{y}|}{2}; \end{aligned} \tag{2.107}$$

however, the last inequality is already ensured by (2.104) and $x^0 > 0$. In order to be able to evaluate (2.90) further, we next plug the condition $\cos\theta < P_{x,y}(|\mathbf{y}'|)$ into the expression for $|\mathbf{b}|$. This yields (recall that we use spherical variables for $|\mathbf{y}'|$):

$$\begin{aligned} |\mathbf{b}| &= \sqrt{|\mathbf{x}-\mathbf{y}|^2 + \rho^2 + 2\rho|\mathbf{x}-\mathbf{y}|\cos\theta} \\ &< \sqrt{|\mathbf{x}-\mathbf{y}|^2 + \rho^2 + 2\rho|\mathbf{x}-\mathbf{y}|P_{x,y}(\rho)} \\ &= \sqrt{\rho^2 - 2\rho(x^0+y^0) + (x^0+y^0)^2} = x^0 + y^0 - \rho. \end{aligned} \tag{2.108}$$

With this, we perform for (2.90) the analogous calculation to (2.98)–

(2.101). This yields:

$$
\begin{aligned}
(2.90) = {} & \frac{\lambda\|\psi\|_g}{16\pi} 1_{y^0-x^0<|\mathbf{x}-\mathbf{y}|<x^0+y^0} \int_{\max\left(0,y^0-x^0,\frac{y^0-x^0+|\mathbf{x}-\mathbf{y}|}{2}\right)}^{y^0} d\rho \frac{g(y^0-\rho)}{|\mathbf{x}-\mathbf{y}|} \\
& \times \left[g_2\left(\frac{x^0+y^0-\rho-\min(|\mathbf{x}-\mathbf{y}|+\rho,|x^0-y^0+\rho|,x^0+y^0-\rho)}{2}\right) \right. \\
& \qquad \left. -g_2\left(\frac{x^0+y^0-\rho-||\mathbf{x}-\mathbf{y}|-\rho|}{2}\right)\right] \\
= {} & \frac{\lambda\|\psi\|_g}{16\pi} 1_{(x-y)^2\geqslant 0, x^0>y^0} \int_0^{y^0} d\rho \, \frac{g(y^0-\rho)}{|\mathbf{x}-\mathbf{y}|} \\
& \times \left[g_2\left(\frac{x^0+y^0-|\mathbf{x}-\mathbf{y}|}{2}-\rho\right) - g_2\left(\frac{x^0+y^0-\rho-||\mathbf{x}-\mathbf{y}|-\rho|}{2}\right)\right] \\
& + \frac{\lambda\|\psi\|_g}{16\pi} 1_{(x-y)^2<0} \; 1_{x^0+y^0>|\mathbf{x}-\mathbf{y}|} \int_{\frac{y^0-x^0+|\mathbf{x}-\mathbf{y}|}{2}}^{y^0} d\rho \, \frac{g(y^0-\rho)}{|\mathbf{x}-\mathbf{y}|} \\
& \times \left[g_2\left(\frac{x^0+y^0-\rho-|x^0-y^0+\rho|}{2}\right) - g_2\left(\frac{x^0+y^0-\rho-||\mathbf{x}-\mathbf{y}|-\rho|}{2}\right)\right] \\
= {} & \frac{\lambda\|\psi\|_g}{16\pi} 1_{(x-y)^2\geqslant 0, x^0>y^0} \int_0^{y^0} d\rho \, \frac{g(y^0-\rho)}{|\mathbf{x}-\mathbf{y}|} \\
& \times \left[g_2\left(\frac{x^0+y^0-|\mathbf{x}-\mathbf{y}|}{2}-\rho\right) \right. \\
& \qquad \left. - g_2 \min\left(\frac{x^0+y^0-|\mathbf{x}-\mathbf{y}|}{2}, \frac{x^0+y^0+|\mathbf{x}-\mathbf{y}|}{2}-\rho\right)\right] \\
& + \frac{\lambda\|\psi\|_g}{16\pi} 1_{(x-y)^2<0} \; 1_{x^0+y^0>|\mathbf{x}-\mathbf{y}|} \int_{\frac{y^0-x^0+|\mathbf{x}-\mathbf{y}|}{2}}^{y^0} d\rho \, \frac{g(y^0-\rho)}{|\mathbf{x}-\mathbf{y}|} \\
& \times \left[g_2 \min\left(x^0, y^0-\rho\right) \right.
\end{aligned} \tag{2.109}
$$

$$- g_2 \mathrm{min}\left(\frac{x^0+y^0-|\mathbf{x}-\mathbf{y}|}{2}, \frac{x^0+y^0+|\mathbf{x}-\mathbf{y}|}{2}-\rho\right)\Bigg]. \tag{2.110}$$

This ends the calculation of (2.90).

2.2.4.1.1.3 Simplification of term (2.91). We next turn to (2.91). In case the reader is not interested in the details of the computation, the result can be found in (2.123). First we note that the restriction imposed by the first indicator function here is $\cos\theta > K_{x-y}(|\mathbf{y}'|)$ and the condition of the second indicator function is $\cos\theta < P_{x,y}(|\mathbf{y}'|)$. In order to to satisfy these conditions (and the restrictions of the regular range of integration) it is required that

$$\max(-1, K_{x-y}(|\mathbf{y}'|) < \cos\theta < \min(1, P_{x,y}(|\mathbf{y}'|)). \tag{2.111}$$

This leads us to ask which restrictions on $|\mathbf{y}'|$ are imposed by the conditions

$$K_{x-y}(|\mathbf{y}'|) \ < \ 1, \tag{2.112}$$
$$P_{x,y}(|\mathbf{y}'|) \ > \ -1, \tag{2.113}$$
$$K_{x-y}(|\mathbf{y}'|) \ < \ P_{x,y}(|\mathbf{y}'|). \tag{2.114}$$

These restrictions shall be computed next. With $|\mathbf{y}'| = \rho$, we find:

$$\begin{aligned} & K_{x-y}(|\mathbf{y}'|) < 1 \\ \iff & \frac{(x-y)^2}{2\rho|\mathbf{x}-\mathbf{y}|} + \frac{x^0-y^0}{|\mathbf{x}-\mathbf{y}|} \ < \ 1 \\ \iff & (x-y)^2 \ < \ 2\rho(y^0-x^0+|\mathbf{x}-\mathbf{y}|) \\ \iff & \begin{cases} \rho > \frac{y^0-x^0-|\mathbf{x}-\mathbf{y}|}{2} & \text{for } |\mathbf{x}-\mathbf{y}| > x^0-y^0, \\ \rho < \frac{y^0-x^0-|\mathbf{x}-\mathbf{y}|}{2} & \text{for } |\mathbf{x}-\mathbf{y}| < x^0-y^0. \end{cases} \end{aligned} \tag{2.115}$$

The second case in the last line is in conflict with $\rho > 0$, so we have to impose the first condition on (2.91). We continue with $P_{x,y}(\rho) > -1$.

$$\begin{aligned} & P_{x,y}(\rho) > -1 \\ \iff \quad & \frac{(x^0+y^0)^2}{2\rho|\mathbf{x}-\mathbf{y}|} - \frac{|\mathbf{x}-\mathbf{y}|}{2\rho} - \frac{x^0+y^0}{|\mathbf{x}-\mathbf{y}|} \; > \; -1 \\ \iff \quad & (x^0+y^0)^2 - |\mathbf{x}-\mathbf{y}|^2 \; > \; 2\rho(x^0+y^0-|\mathbf{x}-\mathbf{y}|) \\ \iff \quad & \begin{cases} \rho < \frac{x^0+y^0+|\mathbf{x}-\mathbf{y}|}{2} & \text{for } x^0+y^0 > |\mathbf{x}-\mathbf{y}|, \\ \rho > \frac{x^0+y^0+|\mathbf{x}-\mathbf{y}|}{2} & \text{for } x^0+y^0 < |\mathbf{x}-\mathbf{y}|. \end{cases} \end{aligned} \tag{2.116}$$

The second case is in conflict with $\rho < y^0$, so we implement indicator functions corresponding only to the first case in (2.91). The third condition $K_{x-y}(\rho) < P_{x,y}(\rho)$ in fact does not impose any additional conditions. This can be seen as follows:

$$\begin{aligned} & K_{x-y}(\rho) \; < \; P_{x,y}(\rho) \\ \iff \quad & \frac{(x-y)^2}{2\rho|\mathbf{x}-\mathbf{y}|} + \frac{x^0-y^0}{|\mathbf{x}-\mathbf{y}|} \; < \; \frac{(x^0+y^0)^2}{2\rho|\mathbf{x}-\mathbf{y}|} - \frac{|\mathbf{x}-\mathbf{y}|}{2\rho} - \frac{x^0+y^0}{|\mathbf{x}-\mathbf{y}|} \\ \iff \quad & -2x^0y^0 + 4\rho x^0 \; < \; 2x^0y^0 \\ \iff \quad & \rho < y^0, \end{aligned} \tag{2.117}$$

which always holds true.
Taking into account the computed restrictions, we arrive at:

$$\begin{aligned} (2.91) \overset{\cos\theta = w}{=} & \frac{\lambda\|\psi\|_g}{4(4\pi)^2} \int_0^{2\pi} d\varphi \int_0^{y^0} d\rho \int_{-1}^{1} dw \, 1_{K_{x-y}(\rho)<w<P_{x,y}(\rho)} \\ & \times 1_{\frac{y^0-x^0-|\mathbf{x}-\mathbf{y}|}{2}<\rho<\frac{x^0+y^0+|\mathbf{x}-\mathbf{y}|}{2}} \frac{g(y^0-\rho)\rho}{\sqrt{\rho^2+|\mathbf{x}-\mathbf{y}|^2+2\rho|\mathbf{x}-\mathbf{y}|w}} \\ & \times 1_{x^0-y^0<|\mathbf{x}-\mathbf{y}|<x^0+y^0} \, g_1\!\left(\frac{x^0+y^0-\sqrt{\rho^2+|\mathbf{x}-\mathbf{y}|^2+2\rho|\mathbf{x}-\mathbf{y}|w}}{2}\right) \end{aligned}$$

$$
\begin{aligned}
&= \frac{\lambda \|\psi\|_g 2\pi}{4(4\pi)^2} 1_{x^0-y^0<|\mathbf{x}-\mathbf{y}|<x^0+y^0} \int_{\max\left(0,\frac{y^0-x^0-|\mathbf{x}-\mathbf{y}|}{2}\right)}^{\min\left(y^0,\frac{x^0+y^0+|\mathbf{x}-\mathbf{y}|}{2}\right)} d\rho \\
&\quad \times \int_{\max(-1,K_{x-y}(\rho))}^{\min(1,P_{x,y}(\rho))} dw \, \frac{-2g(y^0-\rho)}{|\mathbf{x}-\mathbf{y}|} \\
&\quad \times \partial_w g_2\left(\frac{x^0+y^0-\sqrt{\rho^2+|\mathbf{x}-\mathbf{y}|^2+2\rho|\mathbf{x}-\mathbf{y}|w}}{2}\right) \\
&= \frac{\lambda \|\psi\|_g}{16\pi} 1_{x^0-y^0<|\mathbf{x}-\mathbf{y}|<x^0+y^0} \int_{\max\left(0,\frac{y^0-x^0-|\mathbf{x}-\mathbf{y}|}{2}\right)}^{\min\left(y^0,\frac{x^0+y^0+|\mathbf{x}-\mathbf{y}|}{2}\right)} d\rho \, \frac{g(y^0-\rho)}{|\mathbf{x}-\mathbf{y}|} \\
&\quad \times \left[g_2\left(\frac{x^0+y^0-\rho-\sqrt{\rho^2+|\mathbf{x}-\mathbf{y}|^2+2\rho|\mathbf{x}-\mathbf{y}|\max(-1,K_{x-y}(\rho))}}{2}\right)\right. \\
&\quad \left. -g_2\left(\frac{x^0+y^0-\rho-\sqrt{\rho^2+|\mathbf{x}-\mathbf{y}|^2+2\rho|\mathbf{x}-\mathbf{y}|\min(1,P_{x,y}(\rho))}}{2}\right)\right]. \qquad (2.118)
\end{aligned}
$$

At this point, the expressions look quite formidable. We can, however, achieve significant simplifications by inserting the functional form of $K_{x,y}(\rho)$ and $P_{x,y}(\rho)$ as in (2.108) and (2.100). This yields:

$$
\begin{aligned}
&(2.91) = \\
&\frac{\lambda \|\psi\|_g}{16\pi} 1_{x^0-y^0<|\mathbf{x}-\mathbf{y}|<x^0+y^0} \int_{\max\left(0,\frac{y^0-x^0-|\mathbf{x}-\mathbf{y}|}{2}\right)}^{\min\left(y^0,\frac{x^0+y^0+|\mathbf{x}-\mathbf{y}|}{2}\right)} d\rho \, \frac{g(y^0-\rho)}{|\mathbf{x}-\mathbf{y}|} \\
&\quad \times \left[g_2\left(\frac{x^0+y^0-\rho-\max(||\mathbf{x}-\mathbf{y}|-\rho|,|x^0-y^0+\rho|)}{2}\right)\right. \\
&\quad \left. -g_2\left(\frac{x^0+y^0-\rho-\min(|\mathbf{x}-\mathbf{y}|+\rho,x^0+y^0-\rho)}{2}\right)\right] \qquad (2.119)
\end{aligned}
$$

Now we simplify the arguments of the g_2-functions. For the first one, we have:

$$\begin{aligned}
&x^0+y^0-\rho-\max(||\mathbf{x}-\mathbf{y}|-\rho|,|x^0-y^0+\rho|)\\
&=x^0+y^0-\rho-\max(|\mathbf{x}-\mathbf{y}|-\rho,\rho-|\mathbf{x}-\mathbf{y}|,x^0-y^0+\rho,y^0-\rho-x^0)\\
&=\min\big(x^0+y^0-|\mathbf{x}-\mathbf{y}|,x^0+y^0+|\mathbf{x}-\mathbf{y}|-2\rho,2(y^0-\rho),2x^0\big)\,. \qquad (2.120)
\end{aligned}$$

For the second one we get

$$x^0+y^0-\rho-\min(|\mathbf{x}-\mathbf{y}|+\rho,x^0+y^0-\rho)=\max(x^0+y^0-|\mathbf{x}-\mathbf{y}|-2\rho,0). \qquad (2.121)$$

Using this in (2.119), we find:

$$\begin{aligned}
(2.91) &= \frac{\lambda\|\psi\|_g}{16\pi}1_{x^0-y^0<|\mathbf{x}-\mathbf{y}|<x^0+y^0}\int_{\max\left(0,\frac{y^0-x^0-|\mathbf{x}-\mathbf{y}|}{2}\right)}^{\min\left(y^0,\frac{x^0+y^0+|\mathbf{x}-\mathbf{y}|}{2}\right)}d\rho\,\frac{g(y^0-\rho)}{|\mathbf{x}-\mathbf{y}|}\\
&\times\left[g_2\min\left(\frac{x^0+y^0-|\mathbf{x}-\mathbf{y}|}{2},\frac{x^0+y^0+|\mathbf{x}-\mathbf{y}|}{2}-\rho,y^0-\rho,x^0\right)\right.\\
&\left.\quad-g_2\max\left(\frac{x^0+y^0-|\mathbf{x}-\mathbf{y}|}{2}-\rho,0\right)\right]. \qquad (2.122)
\end{aligned}$$

As in the consideration below (2.101), we split the expression into separate terms with $(x-y)^2 \gtrless 0$. Using $y^0 \gtrless x^0 + |\mathbf{x}-\mathbf{y}|$, we can simplify the expressions involving the minimum. This results in:

$$\begin{aligned}
(2.91) &= \frac{\lambda\|\psi\|_g}{16\pi}1_{(x-y)^2\geqslant 0,y^0>x^0}\int_{\frac{y^0-x^0-|\mathbf{x}-\mathbf{y}|}{2}}^{\frac{x^0+y^0+|\mathbf{x}-\mathbf{y}|}{2}}d\rho\,\frac{g(y^0-\rho)}{|\mathbf{x}-\mathbf{y}|}\\
&\times\left[g_2\min\left(\frac{x^0+y^0+|\mathbf{x}-\mathbf{y}|}{2}-\rho,x^0\right)-g_2\max\left(\frac{x^0+y^0-|\mathbf{x}-\mathbf{y}|}{2}-\rho,0\right)\right]\\
&+\frac{\lambda\|\psi\|_g}{16\pi}1_{(x-y)^2<0,|\mathbf{x}-\mathbf{y}|<x^0+y^0}\int_0^{y^0}d\rho\,\frac{g(y^0-\rho)}{|\mathbf{x}-\mathbf{y}|} \qquad (2.123)\\
&\times\left[g_2\min\left(\frac{x^0+y^0-|\mathbf{x}-\mathbf{y}|}{2},y^0-\rho\right)-g_2\max\left(\frac{x^0+y^0-|\mathbf{x}-\mathbf{y}|}{2}-\rho,0\right)\right].
\end{aligned}$$

This concludes the calculation of (2.91).

2.2.4.1.1.4 Summary of the first estimate. We have obtained the following bound for $|A_0\psi|(x,y)$:

$$
\begin{aligned}
&\frac{16\pi}{\lambda\|\psi\|_g}|A_0\psi|(x,y) \leqslant 1_{(x-y)^2\geqslant 0, x^0>y^0}\int_0^{y^0} d\rho\,\frac{g(y^0-\rho)}{|\mathbf{x}-\mathbf{y}|}\\
&\times\Bigg[g_2\left(\frac{x^0+y^0+|\mathbf{x}-\mathbf{y}|}{2}\right)\\
&\qquad -g_2\max\left(\frac{x^0+y^0-|\mathbf{x}-\mathbf{y}|}{2},\frac{x^0+y^0+|\mathbf{x}-\mathbf{y}|}{2}-\rho\right)\Bigg]\\
&+1_{(x-y)^2<0}\ 1_{x^0+y^0>|\mathbf{x}-\mathbf{y}|}\int_{\frac{y^0-x^0+|\mathbf{x}-\mathbf{y}|}{2}}^{y^0} d\rho\,\frac{g(y^0-\rho)}{|\mathbf{x}-\mathbf{y}|}\\
&\times\Big[g_2\max\big(x^0,y^0-\rho\big)\\
&\qquad -g_2\max\left(\frac{x^0+y^0-|\mathbf{x}-\mathbf{y}|}{2},\frac{x^0+y^0+|\mathbf{x}-\mathbf{y}|}{2}-\rho\right)\Bigg]\\
&+1_{(x-y)^2\geqslant 0, x^0>y^0}\int_0^{y^0} d\rho\,\frac{g(y^0-\rho)}{|\mathbf{x}-\mathbf{y}|}\\
&\times\Bigg[g_2\left(\frac{x^0+y^0-|\mathbf{x}-\mathbf{y}|}{2}-\rho\right)\\
&\qquad -g_2\min\left(\frac{x^0+y^0-|\mathbf{x}-\mathbf{y}|}{2},\frac{x^0+y^0+|\mathbf{x}-\mathbf{y}|}{2}-\rho\right)\Bigg]\\
&+1_{(x-y)^2<0}\ 1_{x^0+y^0>|\mathbf{x}-\mathbf{y}|}\int_{\frac{y^0-x^0+|\mathbf{x}-\mathbf{y}|}{2}}^{y^0} d\rho\,\frac{g(y^0-\rho)}{|\mathbf{x}-\mathbf{y}|}\\
&\times\Bigg[g_2\min\big(x^0,y^0-\rho\big)
\end{aligned}
$$

$$
\begin{aligned}
&\qquad - g_2 \min\left(\frac{x^0 + y^0 - |\mathbf{x}-\mathbf{y}|}{2}, \frac{x^0+y^0+ |\mathbf{x}-\mathbf{y}|}{2} - \rho\right) \Bigg] \\
&+ 1_{(x-y)^2 \geqslant 0, y^0 > x^0} \int_{\frac{y^0-x^0-|\mathbf{x}-\mathbf{y}|}{2}}^{\frac{x^0+y^0+|\mathbf{x}-\mathbf{y}|}{2}} d\rho \, \frac{g(y^0-\rho)}{|\mathbf{x}-\mathbf{y}|} \\
&\times \Bigg[g_2 \min\left(\frac{x^0+y^0+ |\mathbf{x}-\mathbf{y}|}{2} - \rho, x^0\right) \\
&\qquad - g_2 \max\left(\frac{x^0+y^0- |\mathbf{x}-\mathbf{y}|}{2} - \rho, 0\right) \Bigg] \\
&+ 1_{(x-y)^2 < 0, |\mathbf{x}-\mathbf{y}| < x^0+y^0} \int_0^{y^0} d\rho \, \frac{g(y^0-\rho)}{|\mathbf{x}-\mathbf{y}|} \\
&\times \Bigg[g_2 \min\left(\frac{x^0+y^0- |\mathbf{x}-\mathbf{y}|}{2}, y^0 - \rho\right) \\
&\qquad - g_2 \max\left(\frac{x^0+y^0- |\mathbf{x}-\mathbf{y}|}{2} - \rho, 0\right) \Bigg].
\end{aligned} \tag{2.124}
$$

In order to simplify the notation, we introduce the variables

$$\xi^+ := \frac{x^0 + y^0 + |\mathbf{x} - \mathbf{y}|}{2}, \tag{2.125}$$

$$\xi^- := \frac{x^0 + y^0 - |\mathbf{x} - \mathbf{y}|}{2}. \tag{2.126}$$

Moreover, we collect terms with the same indicator functions. This results in:

$$
\begin{aligned}
&\frac{16\pi}{\lambda \|\psi\|_g} |A_0 \psi|(x,y) \leqslant \\
&1_{(x-y)^2<0, \xi^->0} \int_0^{y^0} d\rho \frac{g(y^0-\rho)}{|\mathbf{x}-\mathbf{y}|} \Big[g_2 \min(\xi^-, y^0-\rho) - g_2 \max(\xi^- - \rho, 0)
\end{aligned}
$$

$$+ 1_{\frac{y^0-x^0+|\mathbf{x}-\mathbf{y}|}{2}<\rho}\big(g_2(x^0) + g_2(y^0-\rho) - g_2(\xi^-) - g_2(\xi^+-\rho)\big)\Big] \tag{2.127}$$

$$+ 1_{(x-y)^2\geqslant 0, x^0>y^0} \int_0^{y^0} d\rho\, \frac{g(y^0-\rho)}{|\mathbf{x}-\mathbf{y}|}\big[g_2(\xi^+) + g_2(\xi^- - \rho) - g_2(\xi^-) - g_2(\xi^+-\rho)\big] \tag{2.128}$$

$$+ 1_{(x-y)^2\geqslant 0, y^0>x^0} \int_{(y^0-x^0-|\mathbf{x}-\mathbf{y}|)/2}^{\xi^+} d\rho\, \frac{g(y^0-\rho)}{|\mathbf{x}-\mathbf{y}|} \times \big[g_2 \min(\xi^+-\rho, x^0) - g_2 \max(\xi^- - \rho, 0)\big]. \tag{2.129}$$

This estimate is an important stepping stone in the proof. Except for special weight functions, the resulting expressions are too complicated to be computed explicitly. We therefore continue with further estimates. The main difficulty in these estimates is that the $1/|\mathbf{x}-\mathbf{y}|$ singularity in the expressions needs to be compensated by the integrand and that this cancellation needs to be preserved by the respective estimate. Fortunately, the mean value theorem turns out suitable to provide such estimates.

2.2.4.1.1.5 Simplification of (2.127)-(2.129). First, we note that since g, g_1 and g_2 are monotonously increasing and since $\xi^- \leqslant \xi^+$, we have in (2.128):

$$g_2(\xi^- - \rho) - g_2(\xi^+ - \rho) \leqslant 0. \tag{2.130}$$

As the remaining terms in (2.128) still vanish in the limit $|\mathbf{x}-\mathbf{y}| \to 0$, we may replace this difference by zero to obtain a suitable estimate. Similarly, a brief calculations shows that we have $\xi^+ > y^0$ for $(x-y)^2 < 0$. It follows that:

$$g_2(y^0 - \rho) - g_2(\xi^+ - \rho) < 0. \tag{2.131}$$

We shall use this in (2.127).

Further simplifications can be obtained using the mean value theorem. We begin with the expression in the square brackets in (2.129). The mean value theorem then implies that there is a $\chi \in [\max(\xi^- - \rho, 0), \min(\xi^+ - \rho, x^0)]$ such that

$$\begin{aligned} & g_2 \min(\xi^+ - \rho, x^0) - g_2 \max(\xi^- - \rho, 0) \\ & \qquad = \big[\min(\xi^+ - \rho, x^0) - \max(\xi^- - \rho, 0) \big] g_1(\chi). \end{aligned} \tag{2.132}$$

Therefore, we have:

$$\begin{aligned} & g_2 \min(\xi^+ - \rho, x^0) - g_2 \max(\xi^- - \rho, 0) \\ & \quad \leqslant \ \min(\xi^+ - \xi^-, \xi^+ - \rho, x^0 - \xi^- + \rho, x^0)\, g_1 \min(\xi^+ - \rho, x^0) \\ & \quad \leqslant \ |\mathbf{x} - \mathbf{y}|\, g_1 \min(\xi^+ - \rho, x^0) \ \leqslant \ |\mathbf{x} - \mathbf{y}|\, g_1(x^0). \end{aligned} \tag{2.133}$$

Note that the factor $|\mathbf{x} - \mathbf{y}|$ exactly compensates the $1/|\mathbf{x} - \mathbf{y}|$ singularity. This is the main reason the mean value theorem is so useful here.

Analogously we find for the expression in the square bracket in the first line of (2.127):

$$\begin{aligned} & g_2 \min(\xi^-, y^0 - \rho) - g_2 \max(\xi^- - \rho, 0) \\ & \quad \leqslant \ \big[\min(\xi^-, y^0 - \rho) - \max(\xi^- - \rho, 0) \big] g_1 \min(\xi^-, y^0 - \rho) \\ & \quad = \ \min(\rho, \xi^-, y^0 - \xi^-, y^0 - \rho)\, g_1 \min(\xi^-, y^0 - \rho) \\ & \quad \leqslant \ (y^0 - \xi^-)\, g_1 \min(\xi^-, y^0 - \rho) \\ & \quad \leqslant \ |\mathbf{x} - \mathbf{y}|\, g_1 \min(\xi^-, y^0 - \rho), \end{aligned} \tag{2.134}$$

where we have used that the further restriction of that term, $(x-y)^2 < 0$, implies $|\mathbf{x} - \mathbf{y}| > |x^0 - y^0| \geqslant y^0 - x^0$.

With these considerations, we obtain a rougher but simpler estimate than (2.127)-(2.129):

$$\begin{aligned}
&\frac{16\pi}{\lambda\|\psi\|_g}|A_0\psi|(x,y) \\
&\leqslant\ 1_{(x-y)^2<0,\xi^->0}\int_0^{y^0} d\rho\ g(y^0-\rho)\Big[g_1\min(\xi^-,y^0-\rho) && (2.135)\\
&\quad +1_{\frac{y^0-x^0+|\mathbf{x}-\mathbf{y}|}{2}<\rho}\frac{g_2(x^0)-g_2(\xi^-)}{|\mathbf{x}-\mathbf{y}|}\Big] && (2.136)\\
&\quad +1_{(x-y)^2\geqslant 0,x^0>y^0}\frac{g_2(\xi^+)-g_2(\xi^-)}{|\mathbf{x}-\mathbf{y}|}\int_0^{y^0} d\rho\ g(y^0-\rho) && (2.137)\\
&\quad +1_{(x-y)^2\geqslant 0,y^0>x^0}\, g_1(x^0)\int_{\frac{y^0-x^0-|\mathbf{x}-\mathbf{y}|}{2}}^{\xi^+} d\rho\ g(y^0-\rho). && (2.138)
\end{aligned}$$

Next, we continue estimating these terms separately so that only expressions without integrals remain.

2.2.4.1.1.6 Further estimate of (2.135). Using the monotonicity of g_1 as well as $\min(\xi^-, y^0-\rho) \leqslant \xi^-$, we find:

$$\begin{aligned}
(2.135)\ &\leqslant\ 1_{(x-y)^2<0,\xi^->0}\, g_1(\xi^-)\int_0^{y^0} ds\ g(s)\\
&=\ 1_{(x-y)^2<0,\xi^->0}\, g_1(\xi^-)g_1(y^0). && (2.139)
\end{aligned}$$

For the constraints given by the indicator function, we have $\xi^- < x^0$. Thus:

$$(2.135) \leqslant 1_{(x-y)^2<0,\xi^->0}\, g_1(x^0)g_1(y^0). \qquad (2.140)$$

2.2.4.1.1.7 Further estimate of (2.136). We have:

$$\begin{aligned}
(2.136) &= 1_{(x-y)^2<0,\xi^->0} \frac{g_2(x^0)-g_2(\xi^-)}{|\mathbf{x}-\mathbf{y}|} \int_{\frac{y^0-x^0+|\mathbf{x}-\mathbf{y}|}{2}}^{y_0} d\rho\, g(y^0-\rho) \\
&= 1_{(x-y)^2<0,\xi^->0} \frac{g_2(x^0)-g_2(\xi^-)}{|\mathbf{x}-\mathbf{y}|} \int_0^{\xi^-} ds\, g(s) \\
&= 1_{(x-y)^2<0,\xi^->0} \frac{g_2(x^0)-g_2(\xi^-)}{|\mathbf{x}-\mathbf{y}|} \big[g_1(\xi^-) - \underbrace{g_1(0)}_{=0}\big]. \qquad (2.141)
\end{aligned}$$

Applying the mean value theorem to g_2 in the interval $[\xi^-, x^0]$ (note that here $\xi^- < x^0$), we obtain that:

$$(2.136) \leqslant 1_{(x-y)^2<0,\xi^->0} \frac{x^0-\xi^-}{|\mathbf{x}-\mathbf{y}|}\, g_1(x^0) g_1(\xi^-). \qquad (2.142)$$

Next, we use that $\frac{x^0-\xi^-}{|\mathbf{x}-\mathbf{y}|} = \frac{x^0-y^0+|\mathbf{x}-\mathbf{y}|}{2|\mathbf{x}-\mathbf{y}|} \leqslant 1$ as $|x^0-y^0| < |\mathbf{x}-\mathbf{y}|$. Thus:

$$(2.136) \leqslant 1_{(x-y)^2<0,\xi^->0}\, g_1(x^0) g_1(\xi^-). \qquad (2.143)$$

Using also that for the given constrains $\xi^- < y^0$, we finally obtain:

$$(2.136) \leqslant 1_{(x-y)^2<0,\xi^->0}\, g_1(x^0) g_1(y^0). \qquad (2.144)$$

2.2.4.1.1.8 Further estimate of (2.137). Here, we can directly carry out the remaining integral using the definition of g_1 as the integral of g:

$$(2.137) = 1_{(x-y)^2\geqslant 0,x^0>y^0} \frac{g_2(\xi^+)-g_2(\xi^-)}{|\mathbf{x}-\mathbf{y}|} g_1(y^0). \qquad (2.145)$$

Next, we apply the mean value theorem to g_2 in the interval $[\xi^-, \xi^+]$ noting that $\xi^+ - \xi^- = |\mathbf{x}-\mathbf{y}|$. This implies:

$$(2.137) \leqslant 1_{(x-y)^2>0,x^0>y^0}\, g_1(\xi^+) g_1(y^0). \qquad (2.146)$$

Next, we note that $(x-y)^2 \geqslant 0 \Leftrightarrow |x^0 - y^0| \geqslant |\mathbf{x}-\mathbf{y}|$. Together with $x^0 > y^0$, we obtain $x^0 \geqslant y^0 + |\mathbf{x}-\mathbf{y}|$ and therefore:

$$\xi^+ = \frac{x^0 + y^0 + |\mathbf{x}-\mathbf{y}|}{2} \leqslant x^0. \tag{2.147}$$

Thus, we obtain:

$$(2.137) \leqslant 1_{(x-y)^2 \geqslant 0, x^0 > y^0}\, g_1(x^0) g_1(y^0). \tag{2.148}$$

2.2.4.1.1.9 Further estimate of (2.138). Here, we carry out the remaining integral as well.

$$\begin{aligned}(2.138) &\leqslant 1_{(x-y)^2 > 0, y^0 > x^0}\, g_1(x^0)[g_1(\xi^+) - g_1((y^0 - x^0 - |\mathbf{x}-\mathbf{y}|)/2)] \\ &\leqslant 1_{(x-y)^2 > 0, y^0 > x^0}\, g_1(x^0) g_1(y^0).\end{aligned} \tag{2.149}$$

as $\xi^+ \leqslant y^0$.

2.2.4.1.1.10 Summary of the result. Gathering the terms (2.140), (2.144), (2.148) and (2.149) yields:

$$\begin{aligned}&\frac{16\pi}{\lambda \|\psi\|_g} |A_0 \psi|(x,y) \\ &\leqslant g_1(x^0) g_1(y^0) \left(2 \times 1_{(x-y)^2 < 0, \xi^- > 0} + 1_{(x-y)^2 \geqslant 0, x^0 > y^0} + 1_{(x-y)^2 \geqslant 0, y^0 > x^0}\right).\end{aligned} \tag{2.150}$$

Considering that the conditions in different indicator functions are mutually exclusive, we finally obtain:

$$\frac{16\pi}{\lambda \|\psi\|_g} |A_0 \psi|(x,y) \leqslant 2 g_1(x^0) g_1(y^0). \tag{2.151}$$

Dividing by $g(x^0)g(y^0)$, taking the supremum over $x, y \in \frac{1}{2}\mathbb{M}$ and factorizing into one-dimensional suprema finally yields the claim (2.46).

2.2.4.1.2 Estimate of the mixed terms (2.47) and (2.48). We focus on A_2 first, starting from its definition (2.39). We take the absolute value and make use of $|\psi(x,y)| \leqslant g(x^0)g(y^0)\,\|\psi\|_g$. Moreover, we use:

$$|J_1(t)/t| \leqslant \frac{1}{2}. \tag{2.152}$$

This yields:

$$\begin{aligned}
|A_2\psi|(x,y) \leqslant{}& \frac{\lambda\, m_2^2\,\|\psi\|_g}{4(4\pi)^3} \int d^3\mathbf{x}' \int d^3\mathbf{y}'\; \frac{H(x^0-|\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|}\,\frac{g(x^0-|\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}'-\mathbf{y}'|}\\
&\times \big[g(x^0-|\mathbf{x}-\mathbf{x}'|+|\mathbf{x}'-\mathbf{y}'|)H(x^0-|\mathbf{x}-\mathbf{x}'|+|\mathbf{x}'-\mathbf{y}'|)\\
&\quad\times H(y^0-x^0+|\mathbf{x}-\mathbf{x}'|-|\mathbf{x}'-\mathbf{y}'|-|\mathbf{y}-\mathbf{y}'|)\\
&\quad+g(x^0-|\mathbf{x}-\mathbf{x}'|-|\mathbf{x}'-\mathbf{y}'|)H(x^0-|\mathbf{x}-\mathbf{x}'|-|\mathbf{x}'-\mathbf{y}'|)\\
&\quad\times H(y^0-x^0+|\mathbf{x}-\mathbf{x}'|+|\mathbf{x}'-\mathbf{y}'|-|\mathbf{y}-\mathbf{y}'|)\big]\,.
\end{aligned} \tag{2.153}$$

As the remaining singularities are independent of each other for a suitable choice of integration variables (see below), we are left with an integrable function on a finite domain.

The next task is to bring the expressions into a simpler form. One possibility to do this is to use

$$\begin{aligned}
&H(y^0-x^0+|\mathbf{x}-\mathbf{x}'|+|\mathbf{x}'-\mathbf{y}'|-|\mathbf{y}-\mathbf{y}'|)\\
&\qquad\leqslant H(y^0-x^0+|\mathbf{x}-\mathbf{x}'|+|\mathbf{x}'-\mathbf{y}'|)
\end{aligned} \tag{2.154}$$

for the second Heaviside function in the second summand. The first Heaviside function in the first summand equals 1 anyway, as $|\mathbf{x}-\mathbf{x}'| < x^0$. We furthermore use

$$\begin{aligned}
&H(y^0-x^0+|\mathbf{x}-\mathbf{x}'|-|\mathbf{x}'-\mathbf{y}'|-|\mathbf{y}-\mathbf{y}'|)\\
&\qquad\leqslant H(y^0-x^0+|\mathbf{x}-\mathbf{x}'|-|\mathbf{x}'-\mathbf{y}'|),
\end{aligned} \tag{2.155}$$

as it simplifies the domain of integration. Overall, the domain of integration remains bounded. Introducing $\mathbf{z}_1 = \mathbf{x}-\mathbf{x}'$, $\mathbf{z}_2 = \mathbf{x}'-\mathbf{y}'$ (with

Jacobi determinant of modulus 1) and using spherical coordinates for $\mathbf{z}_2$, this leads to:

$$\begin{aligned}
&|A_2\psi|(x,y)\frac{4(4\pi)^3}{\lambda\, m_2^2\,\|\psi\|_g}\\
&\leqslant \int_{B_{x^0}(0)} d^3\mathbf{z}_1 4\pi \int_0^{\max(0,y^0-x^0+|\mathbf{z}_1|)} d^3\mathbf{z}_2 \frac{g(x^0-|\mathbf{z}_1|)g(x^0-|\mathbf{z}_1|+|\mathbf{z}_2|)}{|\mathbf{z}_1||\mathbf{z}_2|}|\mathbf{z}_2|^2\\
&+\int_{B_{x^0}(0)} d^3\mathbf{z}_1 4\pi \int_{\max(0,x^0-y^0-|\mathbf{z}_1|)}^{x^0-|\mathbf{z}_1|} d|\mathbf{z}_2| \frac{g(x^0-|\mathbf{z}_1|)g(x^0-|\mathbf{z}_1|-|\mathbf{z}_2|)}{|\mathbf{z}_1||\mathbf{z}_2|}|\mathbf{z}_2|^2.
\end{aligned} \tag{2.156}$$

Using spherical coordinates also for $\mathbf{z}_1$, this can be further simplified to:

$$\begin{aligned}
&|A_2\psi|(x,y)\frac{16\pi}{\lambda m_2^2\,\|\psi\|_g}\\
&\leqslant \int_0^{x^0} dr_1 \int_0^{\max(0,y^0-x^0+r_1)} dr_2\; r_1 r_2\, g(x^0-r_1)g(x^0-r_1+r_2) \qquad (2.157)\\
&+\int_0^{x^0} dr_1 \int_{\max(0,x^0-r_1-y^0)}^{x^0-r_1} dr_2\; r_1 r_2\; g(x^0-r_1)g(x^0-r_1-r_2). \qquad (2.158)
\end{aligned}$$

Our next task is to simplify the remaining integrals. We begin with making the change of variables $\rho = x^0 - r_1$:

$$\begin{aligned}
&|A_2\psi|(x,y)\frac{16\pi}{\lambda m_2^2\,\|\psi\|_g}\\
&\leqslant \int_0^{x^0} d\rho\; (x^0-\rho)g(\rho)\int_0^{\max(0,y^0-\rho)} dr_2\; r_2\, g(\rho+r_2)\\
&+\int_0^{x^0} d\rho\; (x^0-\rho)g(\rho)\int_{\max(0,\rho-y^0)}^{\rho} dr_2\; r_2\, g(\rho-r_2).
\end{aligned} \tag{2.159}$$

Now we consider the r_2-integral in both terms and integrate by parts. This yields:

$$\int_0^{\max(0,y^0-\rho)} dr_2\ r_2\, g(\rho+r_2) = \max(0,y^0-\rho)g_1(y^0) \\ -g_2(\max(\rho,y^0))+g_2(\rho), \qquad (2.160)$$

$$\int_{\max(0,\rho-y^0)}^{\rho} dr_2\ r_2\, g(\rho-r_2) = \max(0,\rho-y^0)g_1(y^0) \\ +g_2(\min(\rho,y^0)). \qquad (2.161)$$

We now use $-g_2(\max(\rho,y^0)) + g_2(\rho) \leqslant 0$ in the first term and then re-insert the resulting estimate into (2.159). Considering also $\max(0,y^0-\rho) + \max(0,\rho-y^0) = |y^0-\rho|$, this yields:

$$|A_2\psi|(x,y)\,\frac{16\pi}{\lambda\, m_2^2\, \|\psi\|_g} \\ \leqslant \int_0^{x^0} d\rho\ (x^0-\rho)g(\rho)\left[|y^0-\rho|g_1(y^0) + g_2(\min(\rho,y^0))\right] \qquad (2.162)$$

The first summand of (2.162) can be treated as follows. First we focus on whether $x^0 > y^0$ or $x^0 \leqslant y^0$. In the first case, we then differentiate between the cases $\rho < y^0$ and $\rho \geqslant y^0$ and split up the integrals accordingly. This yields:

$$\int_0^{x^0} d\rho\ (x^0-\rho)g(\rho)|y^0-\rho|g_1(y^0)$$

$$= \ g_1(y^0)\, 1_{x^0>y^0} \int_0^{y^0} d\rho\ (x^0-\rho)(y^0-\rho)g(\rho) \qquad (2.163)$$

$$-\ g_1(y^0)\, 1_{x^0>y^0} \int_{y^0}^{x^0} d\rho\ (x^0-\rho)(y^0-\rho)g(\rho) \qquad (2.164)$$

$$+\ g_1(y^0)\, 1_{x^0\leqslant y^0} \int_0^{x^0} d\rho\ (x^0-\rho)(y^0-\rho)g(\rho). \qquad (2.165)$$

We now calculate these terms separately using integration by parts. The first term yields:

$$(2.163) \;=\; g_1(y^0)1_{x^0>y^0}\left[(x^0-y^0)g_2(y^0)+2g_3(y^0)\right]. \tag{2.166}$$

We turn to (2.164):

$$\begin{aligned}(2.164) = -g_1(y^0)1_{x^0>y^0}\big[&(y^0-x^0)(g_2(x^0)\\ &+g_2(y^0))+2g_3(x^0)-2g_3(y^0)\big].\end{aligned} \tag{2.167}$$

The result of (2.165) is:

$$(2.165) \;=\; g_1(y^0)1_{x^0\leqslant y^0}\left[(y^0-x^0)g_2(x^0)+2g_3(x^0)\right]. \tag{2.168}$$

Gathering the terms (2.166), (2.167) and (2.168) yields:

$$\begin{aligned}
&|A_2\psi|(x,y)\frac{16\pi}{\lambda\, m_2^2\,\|\psi\|_g}\\
&\qquad\leqslant g_1(y^0)1_{x^0>y^0}\left[2(x^0-y^0)g_2(y^0)+4g_3(y^0)-2g_3(x^0)\right]\\
&\qquad\quad + g_1(y^0)|x^0-y^0|g_2(x^0)+2g_1(y^0)1_{x^0\leqslant y^0}g_3(x^0)\\
&\qquad\leqslant 2g_1(y^0)|x^0-y^0|g_2(x^0)+2g_1(y^0)g_3(x^0)1_{x^0<y^0}\\
&\qquad\quad + g_1(y^0)|x^0-y^0|g_2(x^0)+2g_1(y^0)g_3(x^0)1_{x^0\leqslant y^0}\\
&\qquad= 3g_1(y^0)|x^0-y^0|g_2(x^0)+2g_1(y^0)g_3(x^0)\\
&\qquad\leqslant 3(x^0+y^0)g_1(y^0)g_2(x^0)+2g_1(y^0)g_3(x^0).
\end{aligned} \tag{2.169}$$

In order to obtain $\|A_2\psi\|_g$, we divide by $g(x^0)g(y^0)$ and take the supremum over $x,y\in\frac{1}{2}\mathbb{M}$. This results in:

$$\begin{aligned}
&\sup_{\psi\in\mathcal{S}((\frac{1}{2}\mathbb{M})^2)}\frac{\|A_2\psi\|_g}{\|\psi\|_g} \qquad\qquad (2.170)\\
&\leqslant\;\frac{\lambda\,m_2^2}{16\pi}\left(3\sup_{x^0,y^0\geqslant 0}\frac{(x^0+y^0)g_2(x^0)\,g_1(y^0)}{g(x^0)g(y^0)}+2\sup_{x^0,y^0\geqslant 0}\frac{g_3(x^0)g_1(y^0)}{g(x^0)g(y^0)}\right).
\end{aligned}$$

After factorizing the two-dimensional suprema into one-dimensional ones, this exactly yields the claim, (2.48).
For the operator A_1, we find analogously:

$$\sup_{\psi\in\mathcal{S}((\frac{1}{2}\mathbb{M})^2)} \frac{\|A_1\psi\|_g}{\|\psi\|_g} \leqslant \frac{\lambda m_1^2}{16\pi}\left(3\sup_{x^0,y^0\geqslant 0}\frac{(x^0+y^0)g_1(x^0)g_2(y^0)}{g(x^0)g(y^0)}+2\sup_{x^0,y^0\geqslant 0}\frac{g_1(x^0)g_3(y^0)}{g(x^0)g(y^0)}\right). \quad (2.171)$$

which, after factorization into one-dimensional suprema, yields the claim (2.47).

2.2.4.1.3 Estimate of the mass-mass term (2.49). We begin with (2.42). Taking the absolute value and using $|\psi(x,y)| \leqslant \|\psi\|_g\, g(x^0)g(y^0)$ as well as $|J_1(t)/t| \leqslant \frac{1}{2}$ yields:

$$\begin{aligned} &|A_{12}\psi|(x,y)\\ &\leqslant \frac{\lambda\, m_1 m_2\, \|\psi\|_g}{8(4\pi)^3}\int_0^\infty dx'^0\int d^3\mathbf{x}'\int_0^\infty dy'^0\int_0^{2\pi}d\varphi\int_0^\pi d\vartheta\, \cos(\vartheta)|x'^0-y'^0|\\ &\qquad\times H(x^0-x'^0-|\mathbf{x}-\mathbf{x}'|)H(y^0-y'^0-|\mathbf{y}-\mathbf{x}'+\mathbf{z}|)\\ &\qquad\qquad\times g(x'^0)g(y'^0)\Big|_{|\mathbf{z}|=|x^{0\prime}-y^{0\prime}|}, \end{aligned} \quad (2.172)$$

where, we recall, $\mathbf{z}$ is the variable for which the spherical coordinates are used.
Next, we consider the ranges of integration which the Heaviside functions imply. $H(x^0 - x'^0 - |\mathbf{x} - \mathbf{x}'|)$ restricts the range of integration of $\mathbf{x}'$ to the ball $B_{x^0-x'^0}(\mathbf{x})$ and the range of the x'^0-integration to $(0, x^0)$. The range implied by the second Heaviside function is more complicated. We therefore use the estimate

$$H(y^0 - y'^0 - |\mathbf{y} - \mathbf{x}' + \mathbf{z}|) \leqslant H(y^0 - y'^0). \quad (2.173)$$

Then $y'^0 \in (0, y^0)$ and there is no further restriction for the angular variables. We obtain:

$$|A_{12}\psi|(x,y) \leqslant \frac{\lambda\, m_1 m_2\, \|\psi\|_g}{8(4\pi)^3} \int_0^{x^0} dx'^0 \int_{B_{x^0-x'^0}(\mathbf{x})} d^3\mathbf{x}' \int_0^{y^0} dy'^0 \int_0^{2\pi} d\varphi \int_0^{\pi} d\vartheta \\ \times \cos(\vartheta)|x'^0 - y'^0|\, g(x'^0)g(y'^0). \tag{2.174}$$

Performing the $\mathbf{x}'$-integration, as well as the angular integrals yields:

$$|A_{12}\psi|(x,y) \\ \leqslant \frac{\lambda\, m_1 m_2\, \|\psi\|_g}{96\pi} \int_0^{x^0} dx'^0 |x^0 - x'^0|^3 g(x'^0) \int_0^{y^0} dy'^0 |x'^0 - y'^0|\, g(y'^0). \tag{2.175}$$

Our next task is to estimate the term explicitly in terms of the functions g_n only. To do so, we use

$$|x'^0 - y'^0| \leqslant x'^0 + y'^0. \tag{2.176}$$

This yields:

$$|A_{12}\psi|(x,y) \\ \leqslant \frac{\lambda m_1 m_2 \|\psi\|_g}{96\pi} \int_0^{x^0} dx'^0 |x^0 - x'^0|^3 g(x'^0) \int_0^{y^0} dy'^0 (x'^0 + y'^0) g(y'^0). \tag{2.177}$$

Let

$$I(x^0, y^0) = \int_0^{x^0} dx'^0 |x^0 - x'^0|^3 g(x'^0) \int_0^{y^0} dy'^0\ (x'^0 + y'^0) g(y'^0) \tag{2.178}$$

and

$$L(x'^0, y^0) = \int_0^{y^0} dy'^0\ (x'^0 + y'^0) g(y'^0). \tag{2.179}$$

Integration by parts yields:

$$\begin{aligned} L(x'^0, y^0) =& x'^0 g_1(y^0) + y^0 g_1(y^0) - g_2(y^0) \\ \leqslant& x'^0 g_1(y^0) + y^0 g_1(y^0). \end{aligned} \tag{2.180}$$

Next, let

$$\begin{aligned} I_a(x^0) &= \int_0^{x^0} dx'^0 \; |x^0 - x'^0|^3 g(x'^0), \\ I_b(x^0) &= \int_0^{x^0} dx'^0 \; x'^0 |x^0 - x'^0|^3 g(x'^0). \end{aligned} \tag{2.181}$$

Then:

$$I(x^0, y^0) \leqslant I_a(x^0)\, y^0 g_1(y^0) + I_b(x^0)\, g_1(y^0). \tag{2.182}$$

We consider I_a first, using $(x^0 - x'^0)^2 \leqslant (x^0)^2$ and integrating by parts:

$$\begin{aligned} I_a(x^0) &\leqslant (x^0)^2 \int_0^{x^0} dx'^0 \; (x^0 - x'^0) g(x'^0) \qquad (2.183) \\ &= (x^0)^2 \left(\underbrace{(x^0 - x'^0) g_1(x'^0)|_{x'^0=0}^{x^0}}_{=0} + g_2(x^0) \right) = (x^0)^2 g_2(x^0). \end{aligned}$$

We turn to I_b, using $x'^0(x^0 - x'^0) \leqslant \frac{1}{4}(x^0)^2$ and integrating by parts twice. This results in:

$$I_b(x^0) \;\leqslant\; \frac{(x^0)^2}{4} \int_0^{x^0} dx'^0 \; (x^0 - x'^0)^2 g(x'^0) \;=\; \frac{(x^0)^2}{2}\, g_3(x^0). \tag{2.184}$$

Considering (2.182), we therefore obtain:

$$I(x^0, y^0) \;\leqslant\; (x^0)^2 g_2(x^0)\, y^0 g_1(y^0) + \frac{(x^0)^2}{2}\, g_3(x^0)\, g_1(y^0). \tag{2.185}$$

Returning to (2.177), we divide by $g(x^0)g(y^0)$ and take the supremum, with the result:

$$\sup_{\psi\in\mathcal{S}((\frac{1}{2}\mathbb{M})^2)} \frac{\|A_{12}\psi\|_g}{\|\psi\|_g} \leqslant \frac{\lambda\, m_1 m_2\, \|\psi\|_g}{96\pi} \Bigg[\sup_{x^0,y^0\geqslant 0} \frac{(x^0)^2 g_2(x^0)\, y^0 g_1(y^0)}{g(x^0)g(y^0)} + \frac{1}{2} \sup_{x^0,y^0\geqslant 0} \frac{(x^0)^2 g_3(x^0)\, g_1(y^0)}{g(x^0)g(y^0)} \Bigg]. \tag{2.186}$$

Factorizing the two-dimensional suprema into one-dimensional ones yields the claim, (2.49).

2.2.4.2 Proof of Theorem 6

Let $\psi \in \mathcal{S}$. It only remains to calculate the supremum in (2.46) for $g(t) = e^{\gamma t}$. We have:

$$g_1(t) = \frac{1}{\gamma}\left(e^{\gamma t} - 1\right) \tag{2.187}$$

and hence

$$\sup_{\psi\in\mathcal{S}((\frac{1}{2}\mathbb{M})^2)} \frac{\|A_0\psi\|_g}{\|\psi\|_g} \leqslant \frac{\lambda}{8\pi}\left(\sup_{t\geqslant 0} \frac{g_1(t)}{g(t)}\right)^2 = \frac{\lambda}{4\pi}\left(\sup_{t\geqslant 0} \frac{1}{\gamma}(1 - e^{-\gamma t})\right)^2 = \frac{\lambda}{8\pi\gamma^2}. \tag{2.188}$$

This shows that A_0 can be linearly extended to a bounded operator on $\mathcal{B}_g$ which satisfies the same estimate, (2.50). Moreover, for $\gamma > \sqrt{\frac{\lambda}{4\pi}}$, A_0 is a contraction and Banach's fixed point theorem implies the existence of a unique solution $\psi \in \mathcal{B}_g$ of the equation $\psi = \psi^{\text{free}} + A_0\psi$ for every $\psi^{\text{free}} \in \mathcal{B}_g$.

2.2.4.3 Proof of Theorem 7

Let again $\psi \in \mathcal{S}$. We need to calculate the suprema in (2.46) to (2.49) for $g(t) = (1 + \alpha t^2)e^{\alpha t^2/2}$. We first note:

$$\begin{aligned} g_1(t) &= te^{\alpha t^2/2}, \\ g_2(t) &= \frac{1}{\alpha}\left(e^{\alpha t^2/2} - 1\right), \\ g_3(t) &= \frac{1}{\alpha}\left[\sqrt{\frac{\pi}{2\alpha}}\,\mathrm{erfi}(\sqrt{\alpha/2}t) - t\right]. \end{aligned} \tag{2.189}$$

We can see that with each successive integration, the functions g_n grow slower as $t \to \infty$. Furthermore, the leading terms in g_n are inversely proportional to increasing powers of α. These two properties (and of course the fact that g_1, g_2, g_3 can be written down in terms of elementary functions) make this particular function $g(t)$ a suitable choice for the proof.

As we need to estimate the behavior of quotients like $g_3(t)/g(t)$ for $t \to \infty$, we look for a simpler estimate of g_3 in terms of exponential functions. We note:

$$\begin{aligned} g_3(t) &= \int_0^t dt'\,\frac{1}{\alpha}\left(e^{\alpha t'^2/2} - 1\right) \\ &\leqslant \frac{e^{\alpha t^2/2}}{\alpha} e^{-\alpha t^2/2}\sqrt{2/\alpha}\int_0^{\sqrt{\alpha/2}t} d\tau\, e^{\tau^2} \\ &= \frac{\sqrt{2}}{\alpha^{3/2}}\, e^{\alpha t^2/2}\, D(\sqrt{\alpha/2}\,t), \end{aligned} \tag{2.190}$$

where $D(t) = e^{-t^2}\int_0^t d\tau\, e^{\tau^2}$ denotes the Dawson function. Using the property $|tD(t)| < \frac{2}{3}$, we obtain:

$$tg_3(t) \leqslant \frac{4}{3}\frac{e^{\alpha t^2/2}}{\alpha^2}. \tag{2.191}$$

We are now well-equipped to calculate the suprema occurring in (2.46) to (2.49). Using

$$\sup_{t\geqslant 0}\frac{t^\beta}{1+t^2} = \begin{cases} 1 & \text{for } \beta=0 \\ \frac{1}{2} & \text{for } \beta=1 \\ 1 & \text{for } \beta=2 \end{cases} \tag{2.192}$$

we obtain:

$$\sup_{t\geqslant 0}\frac{g_1(t)}{g(t)} = \sup_{t\geqslant 0}\frac{t}{1+\alpha t^2} = \frac{1}{2}\frac{1}{\sqrt{\alpha}}, \tag{2.193}$$

$$\sup_{t\geqslant 0}\frac{tg_1(t)}{g(t)} = \sup_{t\geqslant 0}\frac{t^2}{1+\alpha t^2} = \frac{1}{\alpha}, \tag{2.194}$$

$$\sup_{t\geqslant 0}\frac{g_2(t)}{g(t)} \leqslant \sup_{t\geqslant 0}\frac{1}{\alpha}\frac{1}{1+\alpha t^2} = \frac{1}{\alpha}, \tag{2.195}$$

$$\sup_{t\geqslant 0}\frac{tg_2(t)}{g(t)} \leqslant \sup_{t\geqslant 0}\frac{1}{\alpha}\frac{t}{1+\alpha t^2} = \frac{1}{2}\frac{1}{\alpha^{3/2}}, \tag{2.196}$$

$$\sup_{t\geqslant 0}\frac{t^2g_2(t)}{g(t)} \leqslant \sup_{t\geqslant 0}\frac{1}{\alpha}\frac{t^2}{1+\alpha t^2} = \frac{1}{\alpha^2}. \tag{2.197}$$

Using, in addition, the property $|D(t)| < \frac{3}{5}$, we find:

$$\sup_{t\geqslant 0}\frac{g_3(t)}{g(t)} \leqslant \sup_{t\geqslant 0}\frac{\sqrt{2}}{\alpha^{3/2}}\frac{D(\sqrt{\alpha/2}t)}{1+\alpha t^2} = \frac{3\sqrt{2}}{5}\frac{1}{\alpha^{3/2}} < \frac{1}{\alpha^{3/2}}, \tag{2.198}$$

$$\sup_{t\geqslant 0}\frac{t^2g_3(t)}{g(t)} \leqslant \sup_{t\geqslant 0}\frac{4}{3}\frac{1}{\alpha^2}\frac{t}{1+\alpha t^2} = \frac{2}{3}\frac{1}{\alpha^{5/2}}. \tag{2.199}$$

In the last line, we have made use of (2.191).
With these results, we find for A_0:

$$(2.46) \leqslant \frac{\lambda}{8\pi}\left(\frac{1}{2}\frac{1}{\sqrt{\alpha}}\right)^2 = \frac{\lambda}{32\pi}\frac{1}{\alpha}. \tag{2.200}$$

This yields (2.52).
We continue with A_1.

$$(2.47) \leqslant \frac{\lambda m_1^2}{16\pi}\left[3\frac{1}{\alpha}\frac{1}{\alpha}+3\frac{1}{2}\frac{1}{\sqrt{\alpha}}\frac{1}{2}\frac{1}{\alpha^{3/2}}+2\frac{1}{2}\frac{1}{\sqrt{\alpha}}\frac{1}{\alpha^{3/2}}\right]$$
$$= \frac{\lambda m_1^2}{16\pi}\frac{19}{4}\frac{1}{\alpha^2} < \frac{\lambda m_1^2}{16\pi}\frac{5}{\alpha^2}. \tag{2.201}$$

This yields (2.53). Analogously, we obtain the estimate (2.54) for A_2.
Finally, for A_{12}, we have

$$(2.49) \leqslant \frac{\lambda m_1^2 m_2^2}{96\pi}\left[\frac{1}{\alpha^2}\frac{1}{\alpha}+\frac{1}{2}\frac{2}{3}\frac{1}{\alpha^{5/2}}\frac{1}{2}\frac{1}{\sqrt{\alpha}}\right]$$
$$= \frac{\lambda m_1^2 m_2^2}{96\pi}\frac{7}{6}\frac{1}{\alpha^3} < \frac{\lambda m_1^2 m_2^2}{80\pi}\frac{1}{\alpha^3}, \tag{2.202}$$

which yields (2.55).
Now, the estimates (2.52) to (2.55) show that the operators A_0, A_1, A_2 and A_{12} are bounded on test functions. Thus, they can be linearly extended to bounded operators on $\mathcal{B}_g$ with the same bounds.
The operator $A = A_0 + A_1 + A_2 + A_{12}$ then also defines a bounded linear operator on $\mathcal{B}_g$ with norm

$$\|A\| \leqslant \|A_0\| + \|A_1\| + \|A_2\| + \|A_{12}\|. \tag{2.203}$$

Using the previous results (2.52)- (2.55), we obtain:

$$\|A\| \leqslant \frac{\lambda}{8\pi\alpha}\left(\frac{1}{4}+\frac{5(m_1^2+m_2^2)}{2}\frac{1}{\alpha}+\frac{m_1^2 m_2^2}{10}\frac{1}{\alpha^2}\right). \tag{2.204}$$

If α is chosen such that this expression is strictly smaller than unity, A becomes a contraction and the existence and uniqueness of solutions of the equation $\psi = \psi^{\text{free}} + A\psi$ follows. This yields condition (2.56) and ends the proof.

2.2.4.4 Proof of Theorem 10

The proof can be reduced to the one for $\frac{1}{2}\mathbb{M}$. To do so, we take the absolute value of (2.75) and use $|\psi|(\eta_1, \mathbf{x}, \eta_2, \mathbf{y}) \leqslant g(\eta_1)g(\eta_2)\|\psi\|_g$. With

$$G(\eta) = a(\eta) \exp\left(\gamma \int_0^\eta d\eta' \; a(\eta')\right) \tag{2.205}$$

$$G_1(\eta) = \int_0^\eta d\eta' \; G(\eta) \tag{2.206}$$

we obtain the estimate

$$\begin{aligned}
&|\widetilde{A}_0\psi|(x,y) \\
&\leqslant \frac{\lambda\|\psi\|_g}{4(4\pi)^3}\int_{B_{\eta_2}(\mathbf{y})} d^3\mathbf{y}'\int_0^{2\pi} d\varphi \int_{-1}^{1} d\cos\vartheta \, \frac{|b^2|}{(b^0+|\mathbf{b}|\cos\vartheta)^2|\mathbf{y}'|} G(\eta_2-|\mathbf{y}'|) \\
&\times G\left(\eta_1 - \frac{1}{2}\frac{b^2}{b^2+|\mathbf{b}|\cos\vartheta}\right) \\
&\times \left(1_{b^2>0}1_{b^0>0}1_{\cos\vartheta>\frac{b^2}{2\eta_1^0|\mathbf{b}|}-\frac{b^0}{|\mathbf{b}|}} + 1_{b^2<0}1_{\cos\vartheta<\frac{b^2}{2\eta_1|\mathbf{b}|}-\frac{b^0}{|\mathbf{b}|}}\right).
\end{aligned} \tag{2.207}$$

This estimate is identical to (2.79) with the only difference that the function g is exchanged with G in the integral (but not in $\|\cdot\|_g$). Thus, going through the same steps as in Secs. 2.2.4.1, 2.2.4.3, we obtain:

$$\sup_{\psi\in\mathcal{S}(([0,\infty)\times\mathbb{R}^3)^2)} \frac{\|\widetilde{A}_0\psi\|_g}{\|\psi\|_g} \leqslant \frac{\lambda}{8\pi}\left(\sup_{t\geqslant 0}\frac{G_1(t)}{g(t)}\right)^2. \tag{2.208}$$

Now, recalling $g(t) = \exp\left(\gamma \int_0^t d\tau \, a(\tau)\right)$ we have

$$G_1(t) = \frac{1}{\gamma} g(t) \tag{2.209}$$

and it follows that

$$\sup_{\psi \in \mathcal{S}(([0,\infty)\times\mathbb{R}^3)^2)} \frac{\|\widetilde{A}_0\psi\|_g}{\|\psi\|_g} \leqslant \frac{\lambda}{8\pi\gamma^2}, \tag{2.210}$$

which yields (2.77). The rest of the claim follows as before.

2.3 Directly Interacting Dirac Particles

In this section we prove the existence and uniqueness of solutions of equation (2.8) for a class of kernels K subject to similar modifications (A) and (B). Similar to the results of the last chapter an analogous result is proven on FLRW spacetime to provide a justification for the cutoff in time. Furthermore, we show that the solutions are determined by Cauchy data at the initial time; however, no Cauchy problem is admissible at other times.

2.3.1 Introduction

In order to take a closer look at equation (2.8), we start with its constituents.
The Greens function of Diracs equation is given by

$$\mathcal{S}(x) = \overline{D}G(x), \tag{2.211}$$

where $\overline{D} = (-i\gamma^\mu\partial_\mu - m)$. This can be verified directly by computing $D\overline{D} = \Box + m^2$. The operator $\overline{D}$ will be refered to as the adjoint Dirac operator. Consequently, one has to define the integral operator in (2.8) on a function space where one can take certain weak derivatives. In contrast to most of non-relativistic physics, this also concerns the time derivatives here. The choice of function space can be a tricky issue, as the convergence of an iteration scheme (and of the Neumann

series, our strategy of proof) requires the integral operator to preserve the regularity, so that the regularity needs to be in harmony with the structure of the integral equation (see Sec. 2.3.2.2).
This section is structured as follows. In subsec. 2.3.2, we specify the integral equation (2.8) in detail. The difficulties with understanding the distributional derivatives are discussed and a suitable function space is identified. Subsec. 2.3.3 contains the main results of this section. In subsec. 2.3.3.1, we formulate an existence and uniqueness theorem (Thm. 14) for eq. (2.8) on $\frac{1}{2}\mathbb{M}$. It is shown that the relevant initial data are equivalent to Cauchy data at $t = 0$. In subsec. 2.3.3.2, we provide a physical justification for the cutoff at $t = 0$ by extending the results to a FLRW spacetime. In the massless case, we show that an existence and uniqueness theorem can be obtained from the one for $\frac{1}{2}\mathbb{M}$ via conformal invariance. The result, thm. 16, covers a fully relativistic interacting dynamics in 1+3 spacetime dimensions. The proofs are carried out in subsec. 2.3.4. Subsec.

2.3.2 Setting of the problem

2.3.2.1 Definition of the integral operator on test functions

In this section, we show how the integral operator in (2.8) can be defined rigorously on test functions. We consider the integral equation (2.8) on the Minkowski half space $\frac{1}{2}\mathbb{M}$ as we did in the last section. We focus on retarded Green's functions of the Dirac equation, $\mathcal{S}^{\text{ret}}(x) = \overline{D}G^{\text{ret}}(x)$ where $G^{\text{ret}}(x)$ is the retarded Green's function of the KG equation defined in equation (2.23). In order to define the meaning of the Green's functions as distributions, we introduce a suitable space of test functions:

$$\mathcal{D} = \mathcal{S}\big((\tfrac{1}{2}\mathbb{M})^2, \mathbb{C}^{16}\big), \tag{2.212}$$

the space of 16-component Schwarz functions on $(\frac{1}{2}\mathbb{M})^2$. For a smooth interaction kernel K and a test function $\psi \in \mathcal{D}$, we then understand

(2.8) by formally integrating by parts so that all partial derivatives act on $K\psi$:

$$\psi(x_1,x_2) = \psi^{\text{free}}(x_1,x_2) + \int_{\frac{1}{2}\mathbb{M}} d^4x_1' \int_{\frac{1}{2}\mathbb{M}} d^4x_2'\, G_1^{\text{ret}}(x_1-x_1') \tag{2.213}$$

$$\times\, G_2^{\text{ret}}(x_2-x_2')[D_1D_2(K\psi)](x_1',x_2')$$

$$+ \text{ boundary terms,} \tag{2.214}$$

where $D_k = (i\gamma_k^\mu \partial_{x_k^\mu} - m_k)$, $k=1,2$. The boundary terms result from the fact that $\psi(x_1,x_2) \neq 0$ for $x_1^0 = 0$ or $x_2^0 = 0$ and are given by:

$$\int_{\mathbb{R}^3} d^3\mathbf{x}_1' \int_{\mathbb{R}^3} d^3\mathbf{x}_2' i\gamma_1^0 G_1^{\text{ret}}(x_1-x_1') i\gamma_2^0 G_2^{\text{ret}}(x_2-x_2') \tag{2.215}$$

$$\times\ (K\psi)(x_1',x_2')|_{x_1^{0\prime}=0,\, x_2^{0\prime}=0}$$

$$+ \int_{\mathbb{R}^3} d^3\mathbf{x}_1' \int_{\frac{1}{2}\mathbb{M}} d^4x_2' i\gamma_1^0 G_1^{\text{ret}}(x_1-x_1') G_2^{\text{ret}}(x_2-x_2') \tag{2.216}$$

$$\times\ D_2(K\psi)(x_1',x_2')|_{x_1^{0\prime}=0}$$

$$+ \int_{\frac{1}{2}\mathbb{M}} d^4x_1' \int_{\mathbb{R}^3} d^3\mathbf{x}_2' G_1^{\text{ret}}(x_1-x_1') i\gamma_2^0 G_2^{\text{ret}}(x_2-x_2') \tag{2.217}$$

$$\times\ D_1(K\psi)(x_1',x_2')|_{x_2^{0\prime}=0}\,. \tag{2.218}$$

Now, G_k^{ret} still contains the δ-distribution. We use the latter to cancel the integrals over $x_k^{0\prime}$, $k=1,2$ in (2.214) in the following manner.

$$\frac{1}{4\pi}\int_{\frac{1}{2}\mathbb{M}} d^4x' \,\frac{\delta(x^0 - x^{0\prime} - |\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|} f(x') \tag{2.219}$$

$$= \frac{1}{4\pi}\int_{B_{x^0}(\mathbf{x})} d^3\mathbf{x}' \,\frac{1}{|\mathbf{x}-\mathbf{x}'|} f(x')|_{x^{0\prime}=x^0-|\mathbf{x}-\mathbf{x}'|}$$

$$= \frac{1}{4\pi}\int_{B_{x^0}(0)} d^3\mathbf{y}\, \frac{1}{|\mathbf{y}|} f(x+y)|_{y^0=-|\mathbf{y}|}. \tag{2.220}$$

Moreover,

$$
\begin{aligned}
& \frac{m}{4\pi}\int_{\frac{1}{2}\mathbb{M}} d^4x'\; H(x^0 - x^{0\prime} - |\mathbf{x}-\mathbf{x}'|)\frac{J_1(m\sqrt{(x-x')^2})}{\sqrt{(x-x')^2}} f(x') \\
= \;& \frac{m}{4\pi}\int_{[-x^0,\infty)\times\mathbb{R}^3} d^4y\; H(-y^0 - |\mathbf{y}|)\frac{J_1(m\sqrt{y^2})}{\sqrt{y^2}} f(x+y) \\
= \;& \frac{m}{4\pi}\int_{-x^0}^{0} dy^0 \int_{B_{|y^0|}(0)} d^3\mathbf{y}_k\; \frac{J_1(m\sqrt{y^2})}{\sqrt{y^2}} f(x+y). \qquad (2.221)
\end{aligned}
$$

For the boundary terms, we similarly use

$$
\frac{i\gamma^0}{4\pi}\int_{\mathbb{R}^3} d^3\mathbf{x}'\; \frac{\delta(x^0 - |\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|} f(0,\mathbf{x}') \qquad (2.222)
$$

$$
= \frac{i\gamma^0}{4\pi}\int_{\partial B_{x^0}(0)} d\sigma(\mathbf{y})\; \frac{f(0,\mathbf{x}+\mathbf{y})}{x^0} \qquad (2.223)
$$

as well as

$$
\begin{aligned}
& i\gamma^0\frac{m}{4\pi}\int_{\mathbb{R}^3} d^3\mathbf{x}'\; H(x^0 - x^{0\prime} - |\mathbf{x}-\mathbf{x}'|)\frac{J_1(m\sqrt{(x-x')^2})}{\sqrt{(x-x')^2}} f(x')|_{x^{0\prime}=0} \\
= \;& i\gamma^0\frac{m}{4\pi}\int_{B_{x^0}(0)} d^3\mathbf{y}\; \frac{J_1(m\sqrt{(x^0)^2-\mathbf{y}^2})}{\sqrt{(x^0)^2-\mathbf{y}^2}} f(0,\mathbf{x}+\mathbf{y}). \qquad (2.224)
\end{aligned}
$$

This yields the form of the integral equation which shall be the basis of our investigation:

$$
\psi(x_1,x_2) = \psi^{\text{free}}(x_1,x_2) + (A\psi)(x_1,x_2). \qquad (2.225)
$$

The operator A is first defined on test functions $\psi \in \mathcal{D}$ as

$$
A\psi \;=\; \prod_{j=1,2}\left(A_j^{(1)}(m) + A_j^{(2)}(m) + A_j^{(3)}(m) + A_j^{(4)}(m)\right) \qquad (2.226)
$$

where for $j = 1, 2,\ k = 1, 2, 3, 4$ the operator
$A_j^{(k)}(m) : \mathcal{D} \to C^\infty\big((\frac{1}{2}\mathbb{M})^2, \mathbb{C}^{16}\big)$ is defined by letting the respective operator $A^{(k)}(m)$, given below, act on the j-th spacetime-variable and spin index of $\psi(x_1, x_2),\ \psi \in \mathcal{D}$. [1]

$$\big(A^{(1)}(m)\,\phi\big)\,(x) \;=\; \frac{1}{4\pi}\int_{B_{x^0}(0)} d^3\mathbf{y}\ \frac{1}{|\mathbf{y}|}\phi(x+y)|_{y^0=-|\mathbf{y}|}, \tag{2.227}$$

$$\big(A^{(2)}(m)\,\phi\big)\,(x) \;=\; -\frac{m}{4\pi}\int_{-x^0}^{0} dy^0 \int_{B_{|y^0|}(0)} d^3\mathbf{y}\ \frac{J_1(m\sqrt{y^2})}{\sqrt{y^2}}\phi(x+y), \tag{2.228}$$

$$\big(A^{(3)}(m)\,\phi\big)\,(x) \;=\; \frac{i\gamma^0}{4\pi}\int_{\partial B_{x^0}(0)} d\sigma(\mathbf{y})\ \frac{\phi(0,\mathbf{x}+\mathbf{y})}{x^0}, \tag{2.229}$$

$$\begin{aligned}\big(A^{(4)}(m)\,\phi\big)\,(x) \;=\; & -i\gamma^0\frac{m}{4\pi}\int_{B_{x^0}(0)} d^3\mathbf{y}\ \frac{J_1(m\sqrt{(x^0)^2-\mathbf{y}^2})}{\sqrt{(x^0)^2-\mathbf{y}^2}} \\ & \times\,\phi(0,\mathbf{x}+\mathbf{y}),\end{aligned} \tag{2.230}$$

here $\phi \in \mathcal{S}(\frac{1}{2}\mathbb{M}, \mathbb{C}^4)$ and the dependence of $A_j^{(1)}$ and $A_j^{(3)}$ on m is only for notational convenience.
We now turn to the question of a suitable Banach space for Eq. (2.225).

2.3.2.2 Choice of Banach space

In order to prove the existence and uniqueness of solutions, we would like to demonstrate the convergence of the Neumann series. First of all, this requires to extend the integral operator A to an operator on a suitable Banach space $\mathcal{B}$. The behavior of solutions $\psi^{\mathrm{free}}(x_1, x_2)$ of the free Dirac equation in each spacetime variable x_1, x_2 suggests to

[1] We deliberately avoid using tensor products here, as the completion of an algebraic tensor product of Banach spaces depends sensitively on which completion is taken.

choose the Bochner space

$$\mathcal{B}_0 = L^\infty\left([0,\infty)^2_{(x_1^0,x_2^0)},\, L^2(\mathbb{R}^6,\mathbb{C}^{16})_{(\mathbf{x}_1,\mathbf{x}_2)}\right) \tag{2.231}$$

with norm

$$\|\psi\|_{\mathcal{B}_0} = \underset{x_1^0,x_2^0>0}{\operatorname{ess\,sup}}\, \|\psi(x_1^0,\cdot,x_2^0,\cdot)\|_{L^2}. \tag{2.232}$$

The reason for choosing $\mathcal{B}_0$ is that the spatial norm $\|\psi^{\text{free}}(x_1^0,\cdot,x_2^0,\cdot)\|_{L^2}$ of a solution of the free Dirac equations is constant in the two time variables x_1^0, x_2^0. A very similar space as $\mathcal{B}_0$ has been used for analyzing smoother versions of the KG case i.e. equation (2.13) in [35].
However, as (2.226) involves the Dirac operators D_1 and D_2, $\mathcal{B}_0$ is not sufficient for our problem. An appropriate Banach space $\mathcal{B}$ must allow us to take at least weak derivatives of ψ. The choice of $\mathcal{B}$ is a delicate matter. One can easily go wrong with demanding too much regularity, as we shall see next.

2.3.2.2.1 Possible problems with the choice of space. The problem can best be illustrated with an example which is structurally related to (2.8) but otherwise simpler. Consider the equation

$$f(t,x) = f^{\text{free}}(t,z) + \int_0^t dz'\, K(z,z')\partial_t f(t,z'), \tag{2.233}$$

where $f^{\text{free}}, f, K : \mathbb{R}^2 \to \mathbb{C}$ and f^{free} is given. Equation (2.233) is inspired by the term $A_1 D_1$ in (2.226).
We would like to set up an iteration scheme for (2.233). As we cannot integrate by parts to shift the t-derivative to K, we must demand at least weak differentiability of f with respect to t. This suggests using a Banach space such as $\mathcal{B} = H^1(\mathbb{R}^2)$. To prove that the integral operator in (2.233) maps $\mathcal{B}$ to $\mathcal{B}$ (the first step in every iteration scheme), we

then have to estimate the L^2-norm of

$$\partial_t \int_0^t dz'\, K(z,z')\partial_t f(t,z') = K(t,t)(\partial_t f)(t,t) + \int_0^t dz'\, K(z,z')\partial_t^2 f(t,z'). \tag{2.234}$$

This expression, however, contains $\partial_t^2 f$. For this to make sense, we must be allowed to take the second weak time derivative of f. This, in turn, requires to choose a different Sobolev space, such as $H^2(\mathbb{R}^2)$, and to estimate the L^2-norm of the second time derivative of the integral operator acting on f which involves $\partial_t^3 f$, and so on. One is thus led to a Sobolev space where all weak n-th time derivatives have to exist. Such infinite-order Sobolev spaces have, in fact, been investigated in [14]. However, it does not seem realistic to get an iteration to converge on these spaces. We therefore take a different approach.

2.3.2.2.2 A Banach space adapted to our integral equation. Considering the form of the integral operator A (2.226), one can see that it is sufficient that the derivatives $D_1\psi$, $D_2\psi$ and $D_1D_2\psi$ exist in a weak sense. As we want to prove later that A maps the Banach space to itself, we have to estimate, among other things, a suitable norm of $D_1(A\psi)$. If $\psi \in \mathcal{D}$ is a test function and K is smooth, we have

$$\begin{aligned} D_1(A\psi)(x_1,x_2) &= D_1 \int d^4x_1'\, d^4x_2'\, \mathcal{S}_1(x_1-x_2')\mathcal{S}_2(x_2-x_2') \\ &\qquad \times K(x_1',x_2')\psi(x_1',x_2') \\ &= \int d^4x_2'\, \mathcal{S}_2(x_2-x_2')K(x_1,x_2')\psi(x_1,x_2') \end{aligned} \tag{2.235}$$

where we have used $D_1\mathcal{S}_1(x_1-x_1') = \delta^{(4)}(x_1-x_1')$. The crucial point now is that (2.235) does not contain higher-order derivatives such as $D_1^2\psi$. The same holds true also for $D_2(A\psi)$ and $D_1D_2(A\psi)$. Thus, the problem of the toy example (2.233) is avoided.

Together with the previous considerations about $\mathcal{B}_0$ (2.231), we are led to define the Banach space $\mathcal{B}_g$ as the completion of $\mathcal{D}$ with respect to the following Sobolev-type norm:

$$\|\psi\|_g^2 = \operatorname*{ess\,sup}_{x_1^0, x_2^0 > 0} \frac{1}{g(x_1^0)g(x_2^0)} [\psi]^2(x_1^0, x_2^0) \tag{2.236}$$

where $g : [0, \infty[\to [0, \infty[$ is a monotonically increasing function which is such that the function $1/g$ is bounded. We admit such a weight factor with hindsight. As we shall see, a suitable choice of g will make a contraction mapping argument possible.
In (2.236) we use the notation

$$[\psi]^2(x_1^0, x_2^0) \;= \sum_{k=0}^{3} \|(\mathcal{D}_k\psi)(x_1^0, \cdot, x_2^0, \cdot)\|^2_{L^2(\mathbb{R}^6, \mathbb{C}^{16})} \tag{2.237}$$

with

$$\mathcal{D}_k = \begin{cases} 1, & k = 0 \\ D_1, & k = 1 \\ D_2, & k = 2 \\ D_1 D_2, & k = 3 \end{cases} \tag{2.238}$$

Remark 11. *One can see the purpose of integral equation (2.8) in determining an interacting correction to a solution ψ^{free} of the free multi-time Dirac equations $D_i\psi^{\text{free}} = 0,\ i = 1, 2$. Therefore, it is important to check that sufficiently many solutions of these free equations lie in $\mathcal{B}_g$. This is ensured by the following Lemma (see Sec. 2.3.4 for a proof).*

Lemma 12. *Let ψ^{free} be a solution of the free multi-time Dirac equations $D_i\psi^{\text{free}} = 0,\ i = 1, 2$ with initial data $\psi^{\text{free}}(0, \cdot, 0, \cdot) = \psi_0 \in C_c^\infty(\mathbb{R}^6, \mathbb{C}^{16})$. Furthermore, let $g : [0, \infty[\to]0, \infty[$ be a monotonically increasing function with $g(t) \to \infty$ for $t \to \infty$. Then ψ^{free} lies in $\mathcal{B}_g$.*

Given the definition of A on $\mathcal{D}$ as in Sec. 2.3.2.1, we shall now proceed with showing that A is bounded on this space. Furthermore, we show that for a suitable choice of the weight factor g in $\mathcal{B}_g$, we can achieve $\|A\| < 1$ on $\mathcal{D}$. This allows to extend A to a contraction on $\mathcal{B}_g$ so that the Neumann series $\psi = \sum_{k=0}^{\infty} A^k \psi^{\text{free}}$ yields the unique solution of $\psi = \psi^{\text{free}} + A\psi$.

2.3.3 Results

2.3.3.1 Results for a Minkowski half space

The core of our results is the following Lemma which allows us to control the growth of the spatial norm of ψ with the two time variables.

Lemma 13. *Let $\psi \in \mathcal{D}$, $\not{\partial}_k = \gamma_k^\mu \partial_{k,\mu},\ k = 1, 2$ and let $K \in C^2(\mathbb{R}^8, \mathbb{C})$ with*

$$\|K\| := \sup_{x_1, x_2 \in \frac{1}{2}\mathbb{M}} \max \big\{ |K(x_1, x_2)|, |\not{\partial}_1 K(x_1, x_2)|, \tag{2.239}$$

$$|\not{\partial}_2 K(x_1, x_2)|, |\not{\partial}_1 \not{\partial}_2 K(x_1, x_2)| \big\} < \infty. \tag{2.240}$$

Then we have:

$$[A\psi]^2(x_1^0, x_2^0) \ \leqslant\ \|K\|^2 \prod_{j=1,2} \big(\mathbb{1} + 8\mathcal{A}_j(m_j) \big) [\psi]^2(x_1^0, x_2^0), \tag{2.241}$$

where $\mathcal{A}_j(m) = \sum_{k=1}^4 \mathcal{A}_j^{(k)}(m)$ with $\mathcal{A}_j^{(k)}$ as defined in (2.271). The expression $[\psi]^2(x_1^0, x_2^0)$ is understood as a function in $C^\infty\big((\frac{1}{2}\mathbb{M})^2, \mathbb{R}_0^+\big)$ to which the operators in front of it are applied.

The proof can be found in Sec. 2.3.4.1.
Lemma 13 can now be used to identify (with some trial and error) a suitable weight factor g which allows us to extend A to a contraction on $\mathcal{B}_g$. Our main result is:

Theorem 14 (Existence and uniqueness of dynamics on a Minkowski half space.)**.** *Let* $0 < \|K\| < 1$*,* $\mu = \max\{m_1, m_2\}$ *and*

$$g(t) = \sqrt{1 + bt^8}\, \exp(bt^8/16), \tag{2.242}$$

$$b = \frac{\|K\|^4}{(1 - \|K\|)^4} \left(16 + \mu^4\right)^4. \tag{2.243}$$

Then for every $\psi^{\rm free} \in \mathcal{B}_g$*, the equation* $\psi = \psi^{\rm free} + A\psi$ *possesses a unique solution* $\psi \in \mathcal{B}_g$*.*

The proof is given in Sec. 2.3.4.2.

Remark 15. 1. *Note that Thm. 14 establishes the existence and uniqueness of a global-in-time solution. The non-Markovian nature of the dynamics makes it necessary to prove such a result directly instead of concatenating short-time solutions. The key step in our proof which makes the global-in-time result possible is the suitable choice of the weight factor* g*.*

2. *The main condition in Thm. 14 is* $\|K\| < 1$*. This means that the interaction must not be too strong (in a suitable sense). A condition of that kind is to be expected solely because of the contribution* $\|(D_1 D_2 (A\psi))(x_1^0, \cdot, x_2^0, \cdot)\|_{L^2} = \|K\psi(x_1^0, \cdot, x_2^0, \cdot)\|_{L^2}$ *to* $[A\psi](x_1^0, x_2^0)$*. Taking our strategy for setting up the Banach space for granted, we therefore think that one cannot avoid a condition on the interaction strength. Note that conditions on the interaction strength also occur at other places in quantum theory (albeit in a different sense). For example, the Dirac Hamiltonian plus a Coulomb potential is only self-adjoint if the prefactor of the latter is smaller than a certain value.*

3. *Cauchy problem. Thm. 14 shows that* $\psi^{\rm free}$ *uniquely determines the solution* ψ*. However, specifying a whole function in* $\mathcal{B}_g$

amounts to a lot of data. In case ψ^{free} is a solution of the free multi-time Dirac equations $D_1\psi^{\text{free}} = 0 = D_2\psi^{\text{free}}$ much less data are needed. ψ^{free} is then determined uniquely by Cauchy data, and hence ψ is as well. Furthermore, if ψ^{free} is differentiable, (2.8) yields

$$\psi(0, \mathbf{x}_1, 0, \mathbf{x}_2) \;=\; \psi^{\text{free}}(0, \mathbf{x}_1, 0, \mathbf{x}_2). \tag{2.244}$$

Thus, Cauchy data for ψ^{free} at $x_1^0 = x_2^0 = 0$ are also Cauchy data for ψ. The procedure works for arbitrary Cauchy data which are appropriate for the free multi-time Dirac equations. Note, however, that a Cauchy problem for ψ for times $x_1^0 = t_0 = x_2^0$ with $t_0 > 0$ is not possible. The reason is that $\psi(t_0, \mathbf{x}_1, t_0, \mathbf{x}_2) \neq \psi^{\text{free}}(t_0, \mathbf{x}_1, t_0, \mathbf{x}_2)$ in general (and contrary to (2.244) the pointwise evaluation may not make sense for ψ).

2.3.3.2 Results for a FLRW universe with a Big Bang singularity

This section is analogous to subsection 2.2.3.3, we show that a Big Bang singularity provides a natural and covariant justification for the cutoff at $t = 0$. As this justification is our main goal, we make the point at the example of a particular class of FLRW spacetimes and do not strive to treat more general spacetimes here. The reason for studying these FLRW spacetimes is that they are conformally equivalent to $\frac{1}{2}\mathbb{M}$ [23]. Together with the conformal invariance of the massless Dirac operator this allows for an efficient method of calculating the Green's functions which occur in the curved spacetime analog of the integral equation (2.8). By doing this, we show that the existence and uniqueness result on these spacetimes can be reduced to Thm. 14.

As shown in [34], Eq. (2.8) possesses a natural generalization to curved spacetimes $\mathcal{M}$,

$$\psi(x_1, x_2) = \psi^{\text{free}}(x_1, x_2) + \int dV(x_1') \int dV(x_2')\, \mathcal{S}_1(x_1, x_1')\mathcal{S}_2(x_2, x_2') \times K(x_1', x_2')\psi(x_1', x_2'). \tag{2.245}$$

Here, $dV(x)$ is the spacetime volume element, $\mathcal{S}_i$ are (retarded) Green's functions of the respective free Dirac equation, i.e.

$$D\mathcal{S}(x, x') = [-g(x)]^{-1/2}\,\delta^{(4)}(x, x'), \tag{2.246}$$

where $g(x)$ is the metric determinant, D the covariant Dirac operator on $\mathcal{M}$, and ψ a section of the tensor spinor bundle over $\mathcal{M} \times \mathcal{M}$.
In order to explicitly formulate (2.245), we need to know the detailed form of $\mathcal{S}$. Note that results for general classes of spacetimes showing that $\mathcal{S}$ is a bounded operator on a suitable function space are not sufficient to obtain a strong (global in time) existence and uniqueness result. We therefore focus on the case of a flat FLRW universe where it is easy to determine the Green's functions explicitly. In that case, the metric is given by

$$ds^2 = a^2(\eta)[d\eta^2 - d\mathbf{x}^2] \tag{2.247}$$

where, as before, η is conformal time and $a(\eta)$ denotes the *scale function*. The coordinate ranges are given by $\eta \in [0, \infty[$ and $\mathbf{x} \in \mathbb{R}^3$. For a FLRW universe with a Big Bang singularity, $a(\eta)$ is a continuous, monotonically increasing function of η with $a(\eta) = 0$, corresponding to the Big Bang singularity. The spacetime volume element reads

$$dV(x) = a^4(\eta)\, d\eta\, d^3\mathbf{x}. \tag{2.248}$$

The crucial point now is that according to (2.247) the spacetime is globally conformally equivalent to $\frac{1}{2}\mathbb{M}$, with conformal factor

$$\Omega(x) = a(\eta). \tag{2.249}$$

In addition, for $m = 0$, the Dirac equation is known to be conformally invariant (see e.g. [52, 39]). More accurately, consider two spacetimes $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ with metrics

$$\widetilde{g}_{ab} = \Omega^2 \, g_{ab}. \tag{2.250}$$

Then the massless Dirac operator D on $\mathcal{M}$ is related to the massless Dirac operator $\widetilde{D}$ on $\widetilde{\mathcal{M}}$ by (see [52, 15]):

$$\widetilde{D} = \Omega^{-5/2} \, D \, \Omega^{3/2}. \tag{2.251}$$

This implies the following transformation behavior of the Green's functions:

$$\widetilde{G}(x,x') = \Omega^{-3/2}(x) \, \Omega^{-3/2}(x') \, G(x,x'). \tag{2.252}$$

One can verify this easily using (2.251) and the definition of Green's functions on curved spacetimes (2.246).
Denoting the Green's functions of the Dirac operator on Minkowski spacetime by $G(x,x') = S(x-x')$ and using coordinates $\eta, \mathbf{x}$ we thus obtain the Green's functions $\widetilde{G}$ on flat FLRW spacetimes as:

$$\widetilde{G}(\eta, \mathbf{x}; \eta', \mathbf{x}') = a^{-3/2}(\eta) a^{-3/2}(\eta') \, \mathcal{S}(\eta - \eta', \mathbf{x} - \mathbf{x}'). \tag{2.253}$$

With this result, we can write out in detail the multi-time integral equation (2.245) for massless Dirac particles on flat FLRW spacetimes (using retarded Green's functions):

$$\begin{aligned} \psi(\eta_1, \mathbf{x}_1, \eta_2, \mathbf{x}_2) = & \psi^{\text{free}}(\eta_1, \mathbf{x}_1, \eta_2, \mathbf{x}_2) + a^{-3/2}(\eta_1) a^{-3/2}(\eta_2) \\ & \times \int_0^\infty d\eta_1' \int d^3\mathbf{x}_1' \int_0^\infty d\eta_2' \int d^3\mathbf{x}_2' \\ & \times a^{5/2}(\eta_1') a^{5/2}(\eta_2') \, \mathcal{S}_1^{\text{ret}}(\eta_1 - \eta_1', \mathbf{x}_1 - \mathbf{x}_1') \qquad (2.254) \\ & \times \mathcal{S}_2^{\text{ret}}(\eta_2 - \eta_2', \mathbf{x}_2 - \mathbf{x}_2') \, (K\psi)(\eta_1', \mathbf{x}_1', \eta_2', \mathbf{x}_2'). \qquad (2.255) \end{aligned}$$

Note that we can regard ψ as a map $\psi : (\frac{1}{2}\mathbb{M})^2 \to \mathbb{C}^{16}$ as the coordinates $\eta, \mathbf{x}$ cover the flat FLRW spacetime manifold globally.
It seems reasonable to allow for a singularity of the interaction kernel, i.e.

$$K(\eta_1, \mathbf{x}_1, \eta_2, \mathbf{x}_2) = a^{-\alpha}(\eta_1) a^{-\alpha}(\eta_1)\, \widetilde{K}(\eta_1, \mathbf{x}_1, \eta_2, \mathbf{x}_2). \tag{2.256}$$

Here, $\alpha \geqslant 0$. The singular behavior is motivated by that of the Green's functions of the conformal wave equation, see section 2.2.3.3. Recall from the introduction that the most natural interaction kernel on $\frac{1}{2}\mathbb{M}$ would be $K(x_1, x_2) \propto \delta((x_1 - x_2)_\mu (x_1 - x_2)^\mu)$ which is a Green's function of the wave equation – a concept that can be generalized to curved spacetimes using the conformal wave equation. Now, under conformal transformations, Green's functions of that equation transform as (2.66)

$$\widetilde{G}(x, x') = \Omega^{-1}(x)\, \Omega^{-1}(x')\, G(x, x'), \tag{2.257}$$

which corresponds to $\alpha = 1$ in (2.256).
Considering (2.256), our integral equation becomes:

$$\begin{aligned}
\psi(\eta_1, \mathbf{x}_1, \eta_2, \mathbf{x}_2) = \psi^{\text{free}}(\eta_1, \mathbf{x}_1, \eta_2, \mathbf{x}_2) &+ a^{-3/2}(\eta_1) a^{-3/2}(\eta_2) \int_0^\infty d\eta_1' \\
&\times \int d^3\mathbf{x}_1' \int_0^\infty d\eta_2' \int d^3\mathbf{x}_2' \\
&\times a^{5/2-\alpha}(\eta_1') a^{5/2-\alpha}(\eta_2')\, \mathcal{S}_1^{\text{ret}}(\eta_1 - \eta_1', \mathbf{x}_1 - \mathbf{x}_1') \\
&\times \mathcal{S}_2^{\text{ret}}(\eta_2 - \eta_2', \mathbf{x}_2 - \mathbf{x}_2')\, (\widetilde{K}\psi)(\eta_1', \mathbf{x}_1', \eta_2', \mathbf{x}_2').
\end{aligned} \tag{2.258}$$

Apart from the scale factors which produce a certain singularity of ψ for $\eta_1, \eta_2 \to 0$, this integral equation has the form of (2.8) on $\frac{1}{2}\mathbb{M}$. Indeed, we can use the transformation

$$\chi(\eta_1, \mathbf{x}_1, \eta_2, \mathbf{x}_2) = a^{3/2}(\eta_1) a^{3/2}(\eta_2)\, \psi(\eta_1, \mathbf{x}_1, \eta_2, \mathbf{x}_2) \tag{2.259}$$

to transform the two equations into each other. We arrive at the following result.

Theorem 16 (Existence and uniqueness of dynamics on a flat FLRW universe)**.** *Let,* $0 \leqslant \alpha \leqslant 1$ *and let* $a : [0, \infty) \to [0, \infty)$ *be a differentiable function with* $a(0) = 0$ *and* $a(\eta) > 0$ *for* $\eta > 0$*. Moreover, assume that* $\widetilde{K} \in C^2\left(([0,\infty) \times \mathbb{R}^3)^2, \mathbb{C}\right)$ *with*

$$\|a^{1-\alpha}(\eta_1)a^{1-\alpha}(\eta_2)\,\widetilde{K}\| < 1. \tag{2.260}$$

Then for every ψ^{free} *with* $a^{3/2}(\eta_1)a^{3/2}(\eta_2)\psi^{\mathrm{free}} \in \mathcal{B}_g$*, (2.258) has a unique solution* ψ *with* $a^{3/2}(\eta_1)a^{3/2}(\eta_2)\psi \in \mathcal{B}_g$ *(and with* g *as in Thm. 14).*

Proof. Multiplying (2.258) with $a^{3/2}(\eta_1)a^{3/2}(\eta_2)$ and using the relation yields

$$\begin{aligned}\chi(\eta_1, \mathbf{x}_1, \eta_2, \mathbf{x}_2) = &\chi^{\mathrm{free}}(\eta_1, \mathbf{x}_1, \eta_2, \mathbf{x}_2) + \int_0^\infty d\eta_1' \int d^3\mathbf{x}_1 \int_0^\infty d\eta_2' \\ &\times\ a^{1-\alpha}(\eta_1')a^{1-\alpha}(\eta_2') \\ &\times\ \mathcal{S}_1^{\mathrm{ret}}(\eta_1 - \eta_1', \mathbf{x}_1 - \mathbf{x}_1')\mathcal{S}_2^{\mathrm{ret}}(\eta_2 - \eta_2', \mathbf{x}_2 - \mathbf{x}_2') \\ &\times\ (\widetilde{K}\chi)(\eta_1', \mathbf{x}_1', \eta_2', \mathbf{x}_2').\end{aligned} \tag{2.261}$$

This equation has the form of (2.8) on $\frac{1}{2}\mathbb{M}$ with K replaced by $a^{1-\alpha}(\eta_1')a^{1-\alpha}(\eta_2')\widetilde{K}$. Thus, using the same distributional understanding of the Green's functions as before, thm. 14 yields the claim. □

Remark 17.
1. *Both* ψ^{free} *and* ψ *have a singularity proportional to* $a^{-3/2}(\eta_1)a^{-3/2}(\eta_2)$ *for* $\eta_1, \eta_2 \to 0$.
2. *For* $\alpha < 1$, $\widetilde{K}$ *has to compensate the singularities caused by* $a^{-3/2}(\eta_1)a^{-3/2}(\eta_2)$ *in order for (2.260) to hold. In the most natural case* $\alpha = 1$*, however,* $\widetilde{K}$ *only needs to satisfy* $\|\widetilde{K}\| < 1$*, i.e. the same condition as for* K *in Thm. 14.*

3. *Let* $\chi^{\text{free}} = a^{3/2}(\eta_1)a^{3/2}(\eta_2)\psi^{\text{free}}$ *be differentiable and let* χ *be the unique solution of* (2.261)*. Then, by* (2.261)*, we have:*

$$\chi^{\text{free}}(0, \mathbf{x}_1, 0, \mathbf{x}_2) = \chi(0, \mathbf{x}_1, 0, \mathbf{x}_2), \tag{2.262}$$

i.e. χ *satisfies a Cauchy problem "at the Big Bang".*

4. *Remarkably, Thm. 16 covers a class of manifestly covariant, interacting integral equations in 1+3 dimensions. Then the interaction kernel* $\widetilde{K}$ *has to be covariant as well. A class of examples (see also [34]) is given by* $\alpha = 1$ *and*

$$\widetilde{K}(x_1, x_2) = \begin{cases} f(d(x_1, x_2)) & \text{if } x_1, x_2 \text{ are time-like related} \\ 0 & \text{else,} \end{cases} \tag{2.263}$$

where $d(x_1, x_2)$ *denotes the geodesic distance of time-like separated the events* $x_1 = (\eta_1, \mathbf{x}_1)$ *and* $x_2 = (\eta_2, \mathbf{x}_2)$*, and* f *is an arbitrary smooth function which leads to* $\|\widetilde{K}\| < 1$*.*

2.3.4 Proofs

Proof of lemma 12. Consider a solution ψ of $D_i\psi^{\text{free}} = 0,\ i = 1, 2$ for compactly supported initial data at $x_1^0 = 0 = x_2^0$. As the Dirac equation has finite propagation speed, ψ^{free} is spatially compactly supported for all times. Without loss of generality we may assume $\|\psi^{\text{free}}(t_1, \cdot, t_2, \cdot)\|_{L^2(\mathbb{R}^6)} = 1$ for all times t_1, t_2, so it follows that also $[\psi^{\text{free}}](t_1, t_2) = 1$. In the following we will construct a sequence of test functions $(\psi_m)_{m\in\mathbb{N}}$ satisfying $\psi_m \xrightarrow[\|\cdot\|_g]{m\to\infty} \psi^{\text{free}}$. Let $\eta : \mathbb{R} \to \mathbb{R}$ be zero for arguments less than 0, be 1 for arugments greater than 1 and in between given by (see also Fig. 2.1)

$$\eta(t) = \exp\left(-\frac{1}{t}\exp\left(\frac{1}{t-1}\right)\right). \tag{2.264}$$

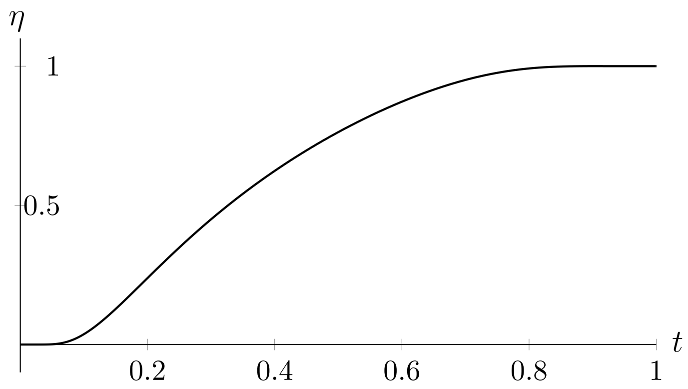

Figure 2.1: The function $\eta(t)$.

Note that η is smooth and monotonically increasing. Next, we define for every $m \in \mathbb{N}$

$$\psi_m^{\text{free}}(t_1, \mathbf{x}_1, t_2, \mathbf{x}_2) := e^{-(t_1-m)\eta(t_1-m)} e^{-(t_2-m)\eta(t_2-m)} \psi^{\text{free}}(t_1, \mathbf{x}_1, t_2, \mathbf{x}_2). \tag{2.265}$$

This function is smooth and decreases rapidly in all variables and thus lies in $\mathcal{D}$. Now we estimate $\|\psi^{\text{free}} - \psi_m\|_g$. Pick $m \in \mathbb{N}$. First consider $\|\psi^{\text{free}} - \psi_m\|_{L^2(\mathbb{R}^6)}(t_1, t_2)$. This function is identically zero for all $t_1 < m$ and $t_2 < m$, so we obtain the estimate

$$\begin{aligned}
&\sup_{t_1,t_2>0} \frac{1}{g(t_1)^2 g(t_2)^2} \|\psi^{\text{free}} - \psi_m\|^2_{L^2(\mathbb{R}^6)} \\
&= \sup_{t_1,t_2>0} \frac{1}{g(t_1)^2 g(t_2)^2} \left|1 - e^{-\eta(t_1-m)(t_1-m)} e^{-\eta(t_2-m)(t_2-m)}\right| \qquad (2.266) \\
&\leqslant \frac{1}{g(0)^2 g(m)^2}. \qquad (2.267)
\end{aligned}$$

For the other terms we use that ψ^{free} solves the free Dirac equation in each variable and that $\sup_{t>0} \partial_t e^{-\eta(t)t} =: \alpha < \infty$ is realized for some positive value of t. So we find for $i \in \{0,1\}$:

$$\begin{aligned}
&\sup_{t_1,t_2>0} \frac{1}{g(t_1)^2 g(t_2)^2} \|D_i(\psi^{\text{free}} - \psi_m)\|^2_{L^2(\mathbb{R}^6)}(t_1,t_2) \\
= &\sup_{t_1,t_2>0} \frac{1}{g(t_1)^2 g(t_2)^2} \qquad (2.268) \\
&\times \|\gamma_i^0 \psi^{\text{free}}(t_1,\cdot,t_2,\cdot) e^{-\eta(t_{3-i}-n)(t_{3-i}-n)} \partial_{t_i} e^{-\eta(t_i-n)(t_i-m)}\|^2_{L^2(\mathbb{R}^6)} \qquad (2.269) \\
\leqslant &\ \frac{\alpha}{g(0)^2 g(m)^2}. \qquad (2.270)
\end{aligned}$$

For the inequality it has been used that the factor with a derivative vanishes for $t_i < m$.
An analogous estimate repeated for the $D_1 D_2$-term yields

$$\sup_{t_1,t_2>0} \frac{1}{g(t_1)^2 g(t_2)^2} \|D_1 D_2(\psi^{\text{free}} - \psi_m)\|^2_{L^2(\mathbb{R}^6)}(t_1,t_2) \leqslant \frac{\alpha^2}{g(m)^4} \leqslant \frac{\alpha^2}{g(0)^2 g(m)^2}.$$

All in all, adding the estimates and taking the square root we find $\|\psi^{free} - \psi_n\|_g \leqslant \frac{1+\alpha}{g(0)g(n)}$, which together with the asymptotic behavior of g implies convergence. It follows that the free solution ψ^{free} can be approximated by Cauchy sequences in $\mathcal{D}$ and hence is contained in $\mathcal{B}_g$ which, we recall, has been defined as the completion of $\mathcal{D}$ with respect to $\|\cdot\|_g$. □

2.3.4.1 Proof of Lemma 13

Throughout the following subsections, let $\psi \in \mathcal{D}$ and $K : \mathbb{R}^8 \to \mathbb{C}$ be a smooth function. Furthermore define $\delta := 1 - \|K\|^2 > 0, \mu = \max\{m_1, m_2\}$ and let g be as in the statement of Thm. 14.
We begin with some lemmas which are useful for estimating $[A\psi]^2(x_1^0, x_2^0)$.

Lemma 18. *Let the following operators be defined on $C([0,\infty))$:*

$$
\begin{aligned}
\big(\mathcal{A}^{(1)}(m)f\big)(t) &= t\int_0^t d\rho\ (t-\rho)^2\, f(\rho),\\
\big(\mathcal{A}^{(2)}(m)f\big)(t) &= \frac{m^4t^4}{2^4\,3^2}\int_0^t d\rho\ (t-\rho)^3\, f(\rho),\\
\big(\mathcal{A}^{(3)}(m)f\big)(t) &= t^2\, f(0),\\
\big(\mathcal{A}^{(4)}(m)f\big)(t) &= \frac{m^4t^6}{2^2\,3^2}\, f(0).
\end{aligned}
\tag{2.271}
$$

Then, for $j=1,2$ and $k=1,2,3,4$, we define the operator $\mathcal{A}_j^{(k)}(m)$ acting on functions $\phi\in C([0,\infty)^2)$ by letting $\mathcal{A}^{(k)}(m)$ act on the j-th variable of $\phi(t_1,t_2)$. Then we have for all $\psi\in\mathcal{D}$, all $m_1,m_2\geqslant 0$ and all $k,l=1,2,3,4$:

$$
\left\| A_1^{(k)}(m_1)A_2^{(l)}(m_2)\psi(t_1,\cdot,t_2,\cdot)\right\|_{L^2}^2 \leqslant \mathcal{A}_j^{(k)}(m_1)\mathcal{A}_j^{(l)}(m_2)\,\|\psi(t_1,\cdot,t_2,\cdot)\|_{L^2}^2. \tag{2.272}
$$

Here, it is understood that the operators $\mathcal{A}_j^{(k)}$ are applied to the functions defined by the norms which follow them, e.g.
$\mathcal{A}_1^{(4)}(m_1)\,\|\psi(t_1,\cdot,t_2,\cdot)\|_{L^2}^2 = \frac{m_1^4t_1^6}{2^2\,3^2}\,\|\psi(0,\cdot,t_2,\cdot)\|_{L^2}^2$.

Proof. We prove (2.272) for $k=1$, $l=2$ and $k=3$, $l=4$. The remaining cases can be treated in the same way. We begin with $k=1$, $l=2$, using $|J_1(x)/x|\leqslant \frac{1}{2}$:

$$
\begin{aligned}
\|A_1^{(1)}(m_1)A_2^{(2)}(m_2)\,\psi(x_1^0,\cdot,x_2^0,\cdot)\|_{L^2}^2 &= \frac{m_2^2}{(4\pi)^4}\int_{\mathbb{R}^3\times\mathbb{R}^3} d^3\mathbf{x}_1\, d^3\mathbf{x}_2 \qquad (2.273)\\
\times\left|\int_{B_{x_1^0}(0)} d^3\mathbf{y}_1\int_{-x_2^0}^0 dy_2^0\int_{B_{|y_2^0|}(0)} d^3\mathbf{y}_2\ \frac{1}{|\mathbf{y}_1|}\frac{J_1(m_2\sqrt{y_2^2})}{\sqrt{y_2^2}}\psi(x_1+y_1,x_2+y_2)|_{y_1^0=-|\mathbf{y}_1|}\right|^2 &\\
\leqslant \frac{m_2^2}{(4\pi)^4}\int_{\mathbb{R}^3\times\mathbb{R}^3} d^3\mathbf{x}_1\, d^3\mathbf{x}_2 &
\end{aligned}
$$

$$
\begin{aligned}
&\times \left(\int_{B_{x_1^0}(0)} d^3\mathbf{y}_1 \int_{-x_2^0}^{0} dy_2^0 \int_{B_{|y_2^0|}(0)} d^3\mathbf{y}_2 \, \frac{1}{|\mathbf{y}_1|^2} \left| \frac{J_1(m_2\sqrt{y_2^2})}{\sqrt{y_2^2}} \right|^2 \right) \\
&\times \left(\int_{B_{x_1^0}(0)} d^3\mathbf{y}_1 \int_{-x_2^0}^{0} dy_2^0 \int_{B_{|y_2^0|}(0)} d^3\mathbf{y}_2 \, |\psi|^2(x_1+y_1, x_2+y_2)|_{y_1^0=-|\mathbf{y}_1|} \right) \\
\leqslant\ & \frac{m_2^2}{(4\pi)^4} \int_{\mathbb{R}^3\times\mathbb{R}^3} d^3\mathbf{x}_1 \, d^3\mathbf{x}_2 \; 4\pi x_1^0 \left(\frac{\pi m_2^2 (x_2^0)^4}{12} \right) \\
&\times \left(\int_{B_{x_1^0}(0)} d^3\mathbf{y}_1 \int_{-x_2^0}^{0} dy_2^0 \int_{B_{|y_2^0|}(0)} d^3\mathbf{y}_2 \, |\psi|^2(x_1+y_1, x_2+y_2)|_{y_1^0=-|\mathbf{y}_1|} \right) \\
\leqslant\ & \frac{m_2^4 \, x_1^0 \, (x_2^0)^4}{3\pi^2 \, 2^8} \int_{\mathbb{R}^3\times\mathbb{R}^3} d^3\mathbf{x}_1 \, d^3\mathbf{x}_2 \int_{B_{x_1^0}(0)} d^3\mathbf{y}_1 \int_{-x_2^0}^{0} dy_2^0 \int_{B_{|y_2^0|}(0)} d^3\mathbf{y}_2 \\
&\times |\psi|^2(x_1^0 - |\mathbf{y}_1|, \mathbf{x}_1+\mathbf{y}_1, x_2^0+y_2^0, \mathbf{x}_2+\mathbf{y}_2). \qquad (2.274)
\end{aligned}
$$

Exchanging the x and y integrals yields:

$$
\begin{aligned}
(2.274) \leqslant\ & \frac{m_2^4 \, x_1^0 \, (x_2^0)^4}{3\pi^2 \, 2^8} \int_{B_{x_1^0}(0)} d^3\mathbf{y}_1 \int_{-x_2^0}^{0} dy_2^0 \int_{B_{|y_2^0|}(0)} d^3\mathbf{y}_2 \\
& \qquad \times \|\psi(x_1^0 - |\mathbf{y}_1|, \cdot, x_2^0 + y_2^0, \cdot)\|_{L^2} \\
\leqslant\ & \frac{m_2^4 \, x_1^0 \, (x_2^0)^4}{3\pi^2 \, 2^8} \, 4\pi \int_0^{x_1^0} dr_1 \; r_1^2 \int_{-x_2^0}^{0} dy_2^0 \; \frac{4\pi}{3} |y_2^0|^3 \\
& \qquad \times \|\psi(x_1^0 - |\mathbf{y}_1|, \cdot, x_2^0 + y_2^0, \cdot)\|_{L^2} \\
\leqslant\ & \frac{m_2^4 \, x_1^0 \, (x_2^0)^4}{2^4 \, 3^2} \int_0^{x_1^0} d\rho_1 \; (x_1^0 - \rho_1)^2 \int_0^{x_2^0} d\rho_2 \; (x_2^0 - \rho_2)^3 \, \|\psi(\rho_1, \cdot, \rho_2, \cdot)\|_{L^2} \\
=\ & \mathcal{A}_1^{(1)}(m_1) \mathcal{A}_2^{(2)}(m_2) \, \|\psi(x_1^0, \cdot, x_2^0, \cdot)\|_{L^2}^2. \qquad (2.275)
\end{aligned}
$$

Next, we turn to the case $k = 3, l = 4$. Using that the modulus of the largest eigenvalue of γ^0 is 1, we obtain:

$$\begin{aligned}
&\|A_1^{(3)}(m_1)A_2^{(4)}(m_2)\,\psi(x_1^0,\cdot,x_2^0,\cdot)\|_{L^2}^2 \;\leqslant\; \frac{m_2^2}{(4\pi)^4(x_1^0)^2}\int_{\mathbb{R}^3\times\mathbb{R}^3} d^3\mathbf{x}_1\,d^3\mathbf{x}_1 \\
&\times\left|\int_{\partial B_{x_1^0}(0)} d\sigma(\mathbf{y}_1)\int_{B_{x_2^0}(0)} d^3\mathbf{y}_2\;\frac{J_1\left(m_2\sqrt{(x_2^0)^2-\mathbf{y}_2^2}\right)}{\sqrt{(x_2^0)^2-\mathbf{y}_2^2}}|\psi|(0,\mathbf{x}_1+\mathbf{y}_2,0,\mathbf{x}_2+\mathbf{y}_2)\right|^2 \\
&\leqslant \frac{m_2^4}{(4\pi)^4(x_1^0)^2}\int_{\mathbb{R}^3\times\mathbb{R}^3} d^3\mathbf{x}_1\,d^3\mathbf{x}_1 \\
&\times\left(\int_{\partial B_{x_1^0}(0)} d\sigma(\mathbf{y}_1)\int_{B_{x_2^0}(0)} d^3\mathbf{y}_2\left|\frac{J_1\left(m_2\sqrt{(x_2^0)^2-\mathbf{y}_2^2}\right)}{m_2\sqrt{(x_2^0)^2-\mathbf{y}_2^2}}\right|^2\right) \\
&\times\left(\int_{\partial B_{x_1^0}(0)} d\sigma(\mathbf{y}_1)\int_{\partial B_{x_2^0}(0)} d\sigma(\mathbf{y}_2)\;|\psi|^2(0,\mathbf{x}_1+\mathbf{y}_2,0,\mathbf{x}_2+\mathbf{y}_2)\right) \\
&= \frac{m_2^4}{(4\pi)^4(x_1^0)^2}\,4\pi(x_1^0)^2\,\frac{\pi(x_2^0)^3}{3}\int_{\mathbb{R}^3\times\mathbb{R}^3} d^3\mathbf{x}_1\,d^3\mathbf{x}_1\int_{\partial B_{x_1^0}(0)} d\sigma(\mathbf{y}_1) \\
&\quad\times\int_{B_{x_2^0}(0)} d^3\mathbf{y}_2\;|\psi|^2(0,\mathbf{x}_1+\mathbf{y}_2,0,\mathbf{x}_2+\mathbf{y}_2). \qquad (2.276)
\end{aligned}$$

Exchanging the order of the x and y integrals yields:

$$\begin{aligned}
(2.276) &= \frac{m_2^4}{3(4\pi)^3}\pi(x_2^0)^3\int_{\partial B_{x_1^0}(0)} d\sigma(\mathbf{y}_1)\int_{B_{x_2^0}(0)} d^3\mathbf{y}_2\;\|\psi(0,\cdot,0,\cdot)\|_{L^2}^2 \\
&= \frac{m_2^4\,(x_1^0)^2\,(x_2^0)^6}{2^2\,3^2}\;\|\psi(0,\cdot,0,\cdot)\|_{L^2}^2 \\
&= \mathcal{A}_1^{(3)}(m_1)\mathcal{A}_2^{(4)}(m_2)\;\|\psi(x_1^0,\cdot,x_2^0,\cdot)\|_{L^2}^2. \qquad (2.277)
\end{aligned}$$

□

Lemma 19. *For $j = 1, 2$ let $\mathcal{A}_j(m) = \sum_{k=1}^4 \mathcal{A}_j^{(k)}(m)$. Then the following estimates hold:*

$$\begin{aligned}
\|(A\psi)(x_1^0, \cdot, x_2^0, \cdot)\|_{L^2}^2 &\leqslant 64\, \|K\|^2 \mathcal{A}_1(m_1)\mathcal{A}_2(m_2)\, [\psi]^2(x_1^0, x_2^0), && (2.278)\\
\|(D_1(A\psi))(x_1^0, \cdot, x_2^0, \cdot)\|_{L^2}^2 &\leqslant 8\, \|K\|^2\, \mathcal{A}_2(m_2)\, [\psi](x_1^0, x_2^0), && (2.279)\\
\|(D_2(A\psi))(x_1^0, \cdot, x_2^0, \cdot)\|_{L^2}^2 &\leqslant 8\, \|K\|^2\, \mathcal{A}_1(m_1)\, [\psi](x_1^0, x_2^0), && (2.280)\\
\|(D_1 D_2(A\psi))(x_1^0, \cdot, x_2^0, \cdot)\|_{L^2}^2 &\leqslant \|K\|^2\, [\psi]^2(x_1^0, x_2^0), && (2.281)
\end{aligned}$$

where $[\psi]^2(x_1^0, x_2^0)$ is regarded as a function of x_1^0, x_2^0 to which the operators in front of it are applied.

Proof. We start with (2.278). Recalling (2.226), the expression $A\psi$ contains terms such as $D_1 D_2(K\psi)$ and $D_i(K\psi)$, $i = 1, 2$. Recalling also the definition of $\mathcal{D}_k$ (Eq. (2.238)), we have:

$$D_1 D_2(K\psi) = \sum_{k=0}^{3} (\nabla_{3-k} K)(\mathcal{D}_k \psi) \tag{2.282}$$

with

$$\nabla_k := \begin{cases} 1, & k = 0 \\ i\not{\partial}_1, & k = 1 \\ i\not{\partial}_2, & k = 2 \\ -\not{\partial}_1 \not{\partial}_2, & k = 3. \end{cases} \tag{2.283}$$

Hence, noting (2.240):

$$|D_1 D_2 \psi| \leqslant \|K\| \sum_{k=0}^{3} |\mathcal{D}_k \psi|. \tag{2.284}$$

Similarly, we find:

$$D_i(K\psi) \leqslant \|K\| \sum_{k=0}^{3} |\mathcal{D}_k \psi|, \quad i = 1, 2. \tag{2.285}$$

Considering the definition of $A_j^{(k)}(m)$, $j = 1, 2$, $k = 1, 2, 3, 4$ it follows that

$$|A\psi| \leqslant \|K\| \sum_{k=0}^{3} \prod_{j=1,2} \left[A_j(m_j)^{(1)} + A_j^{(2)}(m_j) + A_j^{(3)}(m_j) + A_j^{(4)}(m_j)\right] |\mathcal{D}_k\psi|. \quad (2.286)$$

In slight abuse of notation, we here use the same symbols for the operators $A_j^{(k)}(m)$ acting on functions with and without spin components. The idea now is to make use of lemma 18. In order to be able to apply the lemma, we first note that by Young's inequality for $a_1, ..., a_N \in \mathbb{R}$, we have $\left(\sum_{i=1}^N a_i\right)^2 \leqslant N \sum_{i=1}^N a_i^2$ and thus:

$$|A\psi(x_1, x_2)|^2 \leqslant 64 \|K\|^2 \sum_{i,j=1}^{4} \sum_{k=0}^{3} \left|A_1^{(i)}(m_1) A_2^{(j)}(m_2) \, |\mathcal{D}_k\psi|\right|^2. \quad (2.287)$$

Integrating over this expression and using lemma 18, we obtain:

$$\|(A\psi)(x_1^0, \cdot, x_2^0, \cdot)\|_{L^2}^2 \leqslant 64 \|K\|^2 \sum_{i,j=1}^{4} \sum_{k=0}^{3} \mathcal{A}_1^{(i)}(m_1) \mathcal{A}_2^{(j)}(m_2) \times \|(\mathcal{D}_k\psi)(x_1^0, \cdot, x_2^0, \cdot)\|_{L^2}^2. \quad (2.288)$$

Recalling the definition of $[\psi]^2(x_1^0, x_2^0)$, Eq. (2.237) yields (2.278).
Next, we turn to (2.279). We start from the initial form of the integral equation (2.8) and use that as a distributional identity on test functions $\psi \in \mathcal{D}$, we have $D_1 \mathcal{S}^{ret}(x_1 - x_1') = \delta^{(4)}(x_1 - x_1')$. Thus, we obtain:

$$(D_1 A\psi)(x_1, x_2) = \int_{\frac{1}{2}\mathbb{M}} d^4x_2' \; \mathcal{S}_2^{\text{ret}}(x_2 - x_2')(K\psi)(x_1, x_2'). \quad (2.289)$$

Proceeding similarly as for (2.226) we rewrite this as:

$$D_1(A\psi) = \Big(A_2^{(1)}(m_2)\, D_2 + A_2^{(2)}(m_2) D_2 + A_2^{(3)}(m_2) + A_2^{(4)}(m_2)\Big)(K\psi). \tag{2.290}$$

Considering the form of $A_j^{(k)}(m_j)$ this implies:

$$|D_1(A\psi)| \;\leqslant\; \|K\| \sum_{i=1}^{4} \sum_{k\in\{0,2\}} A_2^{(i)}(m_2)\, |\mathcal{D}_k \psi|. \tag{2.291}$$

We now square and use Young's inequality, finding:

$$|D_1(A\psi)|^2 \;\leqslant\; 8\, \|K\|^2 \sum_{i=1}^{4} \sum_{k\in\{0,2\}} A_2^{(i)}(m_2)\, |\mathcal{D}_k \psi|^2. \tag{2.292}$$

Integrating and using lemma 18 yields:

$$\begin{aligned} \|D_1(A\psi)(x_1^0,\cdot,x_2^0,\cdot)\|_{L^2}^2 \;\leqslant\; 8\, \|K\|^2 \sum_{i=1}^{4} \sum_{k\in\{0,2\}} \mathcal{A}_2^{(i)}(m_2) \\ \|(\mathcal{D}_k\psi)(x_1^0,\cdot,x_2^0,\cdot)\|_{L^2}^2. \end{aligned} \tag{2.293}$$

Adding the terms with $k = 1, 3$ and using the definition of $[\psi]^2(x_1^0, x_2^0)$ gives us (2.279).
The estimate (2.280) follows in an analogous way.
Finally, for (2.281) we also start from the initial integral equation (2.8) and use $D_i \mathcal{S}_i^{ret}(x_i - x_i') = \delta^{(4)}(x_i - x_i')$. This results in:

$$D_1 D_2 (A\psi) \;=\; K\psi. \tag{2.294}$$

Squaring and integrating gives us:

$$\begin{aligned} \|D_1 D_2(A\psi)(x_1^0,\cdot,x_2^0,\cdot)\|^2 \;&\leqslant\; \|K\|^2\, \|\psi(x_1^0,\cdot,x_2^0,\cdot)\|_{L^2}^2 \\ &\leqslant\; \|K\|^2\, [\psi]^2(x_1^0,x_2^0), \end{aligned} \tag{2.295}$$

which yields (2.281). □

These estimates are the core of:

Proof of Lemma 13: We use lemma 19 together with the definition of $[\psi]^2(x_1^0, x_2^0)$ to obtain:

$$[A\psi]^2(x_1^0, x_2^0) \leqslant (2.278) + (2.279) + (2.280) + (2.281). \tag{2.296}$$

Summarizing the operators into a product yields (2.241).

□

2.3.4.2 Proof of Theorem 14

In order to prove Thm. 14, we combine the previous estimates to show that $\|A\| < 1$, first on test functions $\psi \in \mathcal{D}$ and by linear extension also on the whole of $\mathcal{B}_g$. We start with Eq. (2.241) of lemma 13 using the definition of $\mathcal{A}_j$ for $j = 1, 2$, as well as the following estimate, valid for all $\psi \in \mathcal{D}, t_1, t_2 > 0$:

$$[\psi](t_1, t_2) = [\psi](t_1, t_2) \frac{g(t_1)g(t_2)}{g(t_1)g(t_2)} \leqslant \|\psi\|_g \, g(t_1)g(t_2). \tag{2.297}$$

Using this in (2.241) yields:

$$\|A\psi\|_g^2 \leqslant \sup_{x_1^0, x_2^0 > 0} \frac{1}{(g(x_1^0)g(x_2^0))^2} \|K\|^2 \prod_{j=1,2} (\mathbb{1} + 8\mathcal{A}_j(m_j)) \, [\psi]^2(x_1^0, x_2^0), \tag{2.298}$$

$$\leqslant \sup_{x_1^0, x_2^0 > 0} \frac{\|\psi\|^2}{(g(x_1^0)g(x_2^0))^2} \|K\|^2 \times \prod_{j=1,2} (\mathbb{1} + 8\mathcal{A}_j(m_j)) \, (g^2 \otimes g^2)(x_1^0, x_2^0), \tag{2.299}$$

$$\leqslant \|K\|^2 \, \|\psi\|_g^2 \left(\sup_{t>0} \frac{1}{g(t)^2} (\mathbb{1} + 8\mathcal{A}(\mu)) \, g^2(t) \right)^2, \tag{2.300}$$

where $\mu = \max\{m_1, m_2\}$ and $\mathcal{A}(\mu) = \sum_{k=1}^{4} \mathcal{A}^{(k)}(\mu)$ with $\mathcal{A}^{(k)}(\mu)$ as in (2.271).
Next, we shall estimate the term in the big round bracket. To this end, we first note some special properties of g^2, which motivated choosing g as in (2.242).

Lemma 20. *For all $t > 0$, we have*

$$\int_0^t d\tau\, g^2(\tau) = \frac{t}{1+bt^8}\, g^2(t). \tag{2.301}$$

Proof: Differentiating the right side of the equation and using the concrete function g^2 as in (2.242) shows that it is, indeed, the anti-derivative of g^2. Since this function vanishes at $t = 0$, the claim follows. □

Lemma 21. *For $c < 8$ we have*

$$\sup_{t>0} \frac{t^c}{1+bt^8} \;=\; \frac{c}{8}\, b^{-c/8} \left(\frac{8}{c} - 1\right)^{1-c/8}, \tag{2.302}$$

and furthermore for $c = 8$:

$$\sup_{t>0} \frac{t^8}{1+bt^8} \;=\; \frac{1}{b}. \tag{2.303}$$

Proof. To prove (2.302), considering the shape of the function $h(t) = t^c/(1+bt^8)$ we find that the supremum is in fact a maximum which is located at $t = b^{-1/8}\,(8/c-1)^{-1/8}$. Inserting this back into the function $h(t)$ yields (2.302). Equation (2.303) follows from $\frac{t^8}{1+bt^8} = \frac{1}{b}\frac{1}{1/(bt^8)+1} \leqslant \frac{1}{b}$. □

Proof of Thm. 14: Applying Lemma 20 to $\mathcal{A}(\mu)\, g^2$ yields:

$$
\begin{aligned}
\big(\mathcal{A}^{(1)}(\mu)\,g^2\big)(t) &= t\int_0^t d\rho\,(t-\rho)^2\,g^2(\rho) \;\leqslant\; t^3\int_0^t d\rho\,g^2(\rho) \\
&\qquad\qquad\qquad = \frac{t^4}{1+bt^8}\,g^2(t), \\
\big(\mathcal{A}^{(2)}(\mu)\,g^2\big)(t) &= \frac{\mu^4\,t^4}{2^4\,3^2}\int_0^t d\rho\,(t-\rho)^3\,g^2(\rho) \;\leqslant\; \frac{\mu^4\,t^8}{2^4\,3^2}\frac{g^2(t)}{1+bt^8}, \\
\big(\mathcal{A}^{(3)}(\mu)\,g^2\big)(t) &= t^2, \\
\big(\mathcal{A}^{(4)}(\mu)\,g^2\big)(t) &= \frac{\mu^4\,t^6}{2^2\,3^2}.
\end{aligned}
\tag{2.304}
$$

Multiplying with $1/g^2(t)$ and using Lemma 21 as well as $1/g(t)^2 \leqslant (1+bt^8)^{-1}$, we find:

$$
\begin{aligned}
g^{-2}(t)\,\big(\mathcal{A}^{(1)}(\mu)\,g^2\big)(t) &\leqslant \sqrt{2}b^{-\frac{1}{2}}, \\
g^{-2}(t)\,\big(\mathcal{A}^{(2)}(\mu)\,g^2\big)(t) &\leqslant \frac{\mu^4}{2^4\,3^2\,b}, \\
g^{-2}(t)\,\big(\mathcal{A}^{(3)}(\mu)\,g^2\big)(t) &\leqslant \frac{3^{3/4}}{2^2\,b^{1/4}}, \\
g^{-2}(t)\,\big(\mathcal{A}^{(4)}(\mu)\,g^2\big)(t) &\leqslant \frac{\mu^4}{2^4\,3^{5/4}}\,b^{-3/4}.
\end{aligned}
\tag{2.305}
$$

Using (2.300), we can employ these inequalities (whose right hand sides are inversely proportional to powers of b) to estimate the norm of A. According to (2.300), we have, first on $\mathcal{D}$ and by linear extension also on the whole of $\mathcal{B}_g$:

$$
\|A\| \;\leqslant\; \|K\|\,\sup_{t>0}\, g^{-2}(t)\big((\mathbb{1}+8\mathcal{A}(\mu))\,g^2\big)(t). \tag{2.306}
$$

Now we use (2.305) for the various contributions $A^{(k)}(\mu)$ to $\mathcal{A}(\mu) =$

$\sum_{k=1}^{4} A^{(k)}(\mu)$, finding:

$$\begin{aligned}\|A\| &\leqslant \|K\| + \frac{2^{3.5}\|K\|}{b^{1/2}} + \frac{\mu^4\|K\|}{18b} + \frac{3^{3/4}2\|K\|}{b^{1/4}} + \frac{\mu^4\|K\|}{2(3^5b^3)^{1/4}} \\ &\overset{b\geqslant 1}{\leqslant} \|K\| + \frac{\|K\|}{b^{1/4}}\left(2^{3.5} + \mu^4/18 + 3^{3/4}2 + \mu^4/(2\cdot 3^{5/4})\right) \qquad (2.307) \\ &< \|K\| + \frac{\|K\|}{b^{1/4}}(16+\mu^4). \qquad (2.308)\end{aligned}$$

Recalling that $b = \frac{\|K\|^4}{(1-\|K\|)^4}(16+\mu^4)^4$ (see (2.243)), we finally obtain that:

$$\|A\| < \|K\| + \frac{\|K\|}{b^{1/4}}(16+\mu^4) = \|K\| + 1 - \|K\| = 1. \qquad (2.309)$$

We have thus shown that A defines (by linear extension) a contraction on $\mathcal{B}_g$. Thus, the Neumann series $\psi = \sum_{k=0}^{\infty} A^k \psi^{\text{free}}$ yields the unique (global-in-time) solution of the equation $\psi = \psi^{\text{free}} + A\psi$. □

2.4 Summary and Conclusions

In this chapter we have extended the analysis of integral equations of the type of (2.12) and (2.13). The resulting degree of understanding is quite different in the two cases.

Extending previous work for Klein-Gordon particles [35, 34] to the Dirac case, we have established the existence of dynamics for a class of integral equations (2.8) which express direct interactions with time delay at the quantum level. To obtain this result, we have used both simplifying assumptions (A) of a cutoff of the spacetime before $t = 0$, and (B) of a smoother interaction kernel than is present in equation (2.12). While we have tried to justify assumption (A) by considering the equation on the FLRW universe that features a big bang, no such physical justification has been given for assumption (B).

In fact, assumption (B) consists of two parts here:
Firstly, we have assumed that K is complex-valued while it could be matrix-valued in the most general case. The reason for this assumption is that our proof requires the integral operator A to be a map from a certain Sobolev space onto itself in which weak derivatives with respect to the Dirac operators of the two particles can be taken. If K were matrix-valued, it would not commute with these Dirac operators in general. Then $A\psi$ would contain new types of weak derivatives which cannot be taken in the initial Sobolev space. As illustrated in Sec. 2.3.2.2, this creates a situation where more and more derivatives have to be controlled, possibly up to infinite order where the success of an iteration scheme seems unlikely. At present, we do not know how to deal with this issue. Improving on this point, however, defines an important task for future research, as e.g. electromagnetic interactions involve interaction kernels proportional to $\gamma_1^\mu \gamma_{2\mu}$ (see [28]).
Secondly, the physically most natural interaction kernel is given by a delta function along the light cone, $K(x_1, x_2) \propto \delta((x_1 - x_2)_\mu (x_1 - x_2)^\mu)$. In the Dirac case, the distributional derivatives make generalizations of results about more singular interaction kernels obtained in the KG case such as [35, 34] difficult, and we have not attempted it here. Another interesting question is whether the smallness condition on K can be alleviated such that arbitrarily peaked functions are admitted. This could allow taking a limit where K approaches the delta function along the light cone.
Improving upon any of these two points would be very desirable.
In the case of scalar particles, we have proved the existence and uniqueness of solutions of the fully singular scalar integral equation (2.13) and its N-particle generalization (2.58). Following previous works and the Dirac case, we have depended upon assumption (A), i.e. a cutoff in time; however, in contrast to those cases considering a more regular interaction kernel than what is present in (2.13), i.e. assumption (B) was not necessary. We have given the same justification for assump-

tion (A) as in the Dirac case and in [34] by extending the main part of our result to the FLRW spacetime.
We have worked with a weighted L^∞ norm both for time and space variables in the case of scalar particles, while it would be more natural to use a weighted $L^\infty L^2$ norm instead (L^∞ for the time variables and L^2 for the space variables). It would then be a challenging task to find the right inequalities to obtain similar estimates as we did. Moreover, one could also try to prove higher regularity not only in the sense of integrability but also differentiability. An interesting question, for example, is whether one can apply the Klein-Gordon operators $(\Box_k + m_k^2)$ to the solutions of (2.13) in a weak sense.
This work provides a rigorous proof of the existence of interacting relativistic quantum dynamics in 1+3 spacetime dimensions; in particular, this model does not suffer from ultraviolet divergences which are typically encountered in quantum field theoretic models. Of course, the model does not describe particle creation and annihilation and is therefore a toy model rather than an alternative to QFT. Nevertheless, one might find the fact that direct interactions, even singular ones along the light cone, can be made mathematically rigorous, remarkable. One might wonder whether in the long run the mechanism of interaction through multi-time integral equations and direct interactions could contribute to a rigorous formulation of quantum field theory.

Chapter 3

Quantum Field Theoretic Approach to Interactions

3.1 Motivation

In this chapter we will turn our attention to the more widely accepted way of introducing interaction into relativistic quantum mechanics, Quantum Field Theory. For the motivation and introduction we closely follow [6]. While the rigorous quantum field theoretic formulation of free relativistic fermions is textbook material [8] the introduction of interaction faces difficulties. In fact, introducing an external electromagnetic field acting on the fermions, while neglecting all interactions between the fermions already is a nontrivial matter. The completely satisfactory formulation of such an *external field quantum electrodynamics* (QED) is, to the best of the authors knowledge, still to be formulated, despite the combined effort of such bright minds

as those of Dirac[9], Feynman[16] and Schwinger[50]. In more recent years there have been several independend attempts to remedy the remaining difficulties such as the construction of the fermionic projector [18] and the construction of the geometric phase [37]. We will seek to extend the construction along the lines of [3, 5, 4], which will be motivated and summarised now.
When Dirac found the equation now bearing his name he recognised that the accessible range of kinetic energies to the particles described by it is $]-\infty, -m] \cup [m, \infty[$. So he was worried that particles coupled to an electromagnetic field might radiate and lower their kinetic energy without bound. Since particles capable of such behavior would not form stable matter he devised a way to introduce a stable ground state into the system. Instead of applying equation (1.1)

$$0 = (i\partial\!\!\!/ - m)\psi, \tag{1.1}$$

to a fixed finite number of electrons, he sought to apply it's evolution to an antisymmetric product

$$\Omega = \varphi_1 \wedge \varphi_2 \wedge \dots, \tag{3.1}$$

where $(\varphi_k)_{k\in\mathbb{N}}$ forms an ONB of the negative spectral subspace $\mathcal{H}^-$. Thus making use of Pauli's exclusion principle for fermions no particle is any longer able to lower it's kinetic energy. This *Dirac sea* Ω might be evolved by a one-particle evolution $U : \mathcal{H} \circlearrowleft$ according to

$$\mathcal{L}_U\Omega = U\varphi_1 \wedge U\varphi_2 \wedge \dots. \tag{3.2}$$

The vacuum Ω does most likely not coincide with the ground state of a full theory of QED. As we do not have such a theory at hand; however, one might hope that due to the homogeneity and isotropy of such a ground state the choice Ω is close enough as a first approximation in situations where the external field is large and the homogeneity and

isotropy of the true state effectively cancels the direct interactions between the electrons. Furthermore, expectation values with respect to Ω are used also in the calculations of QED showing agreement with experiments to a remarkable degree giving us all the more reason to use it as a starting point.
The first step towards a theory including interactions is to allow for an external field A to act upon the particles, changing (1.1) to

$$0 = (i\not{\partial} - \not{A} - m)\psi, \tag{3.3}$$

where we have set the electric charge of the electron to one. We might now imagine a field A that acts only during a brief period of time. Such a field could pull a particle of the Dirac Sea Ω, say φ_1, out of $\mathcal{H}^-$ and onto the surface $U^A\varphi_1 = \xi \in \mathcal{H}^+$. Such a field might not disturb the other wavefunctions much and the Dirac Sea after the action of the field might be represented by

$$\Psi = \xi \wedge \varphi_2 \wedge \varphi_3 \wedge \dots. \tag{3.4}$$

Now, since $\xi \in \mathcal{H}^+$ the corresponding particle behaves qualitatively differently compared to the remaining part of the Dirac sea which consists of wavefunctions taken from $\mathcal{H}^-$. This particle appears above the Dirac sea where it leaves behind a hole, the missing wavefunction $\varphi_1 \in \mathcal{H}^-$. These holes are also called positrons. Whenever the wavefunctions at greater depth are left relatively unperturbed by the action of the process, according to Dirac, we can switch to a leaner description. If Ω remains completely unchanged below a certain level, such as in our example, it suffices to follow the generated particles e.g. ξ and the created holes e.g. φ_1. If on the other hand all wavefunctions are affected one has to keep track of a net evolution of Ω as well. As in this description the particle number is not a constant, creation operators are introduced. These act as

$$a^*(\xi)\varphi_2 \wedge \varphi_3 \wedge \dots = \xi \wedge \varphi_2 \wedge \varphi_3 \wedge \dots. \tag{3.5}$$

The adjoint of the creation operator, called annihilation operator, is of equal importance. Using these equation (3.4) can be condensed to

$$\Psi = a^*(\xi)a(\varphi_1)\Omega. \tag{3.6}$$

Given a one-particle time evolution operator U^A, its *lift* $\tilde{U}^A$ acting on objects like the wedge product (3.1) needs to fulfil

$$\tilde{U}^A a^*(\psi) = a^*(U^A\psi)\tilde{U}^A. \tag{3.7}$$

Requirement (3.7) is enough to fix $\tilde{U}^A$ up to a phase. Now, still $a^*(\chi)\Omega$ behaves differently for $\chi \in \mathcal{H}^+$ compared to $\chi \in \mathcal{H}^-$, so in order to completely forget about the Dirac sea in the notation one performs the splitting

$$a^*(f) = b^*(f) + c^*(f) \quad, b^*(f) = a^*(P^+f), \quad c^*(f) = a^*(P^-f), \tag{3.8}$$

exploiting linearity of a^*. Now the space generated by elements of the form $b^*(f_1)b^*(f_2)\dots b^*(f_n)\Omega$ is called electron Fock space while the space generated by $c(f_1)c(f_2)\dots c(f_n)\Omega$ is called hole Fock space,

$$\mathcal{F}_e = \bigoplus_{k\in\mathbb{N}_0}(\mathcal{H}^+)^{\wedge k}, \quad \mathcal{F}_h = \bigoplus_{k\in\mathbb{N}_0}(\mathcal{H}^-)^{\wedge k}. \tag{3.9}$$

Here, the hole Fock space is generated by the annihilation operators of negative energy acting on the vacuum, which is another remnant of the fact that Ω is not a state devoid of particles. This can be hidden by one more change in notation

$$d^*(f) = c(f), \tag{3.10}$$

Since c is antilinear, but d^* is supposed to be linear, one replaces $\mathcal{H}^-$ by $\overline{\mathcal{H}^-}$ as the domain of definition of d^*. By $\overline{\mathcal{H}^-}$ we denote the space that is identical to $\mathcal{H}^-$ as a set, but differs in its definition of multiplication:

$$\mathbb{C} \times \overline{\mathcal{H}^-} \ni (\lambda, f) \mapsto \lambda^* f. \tag{3.11}$$

Resulting in a replacement of $\mathcal{F}_h$ by

$$\overline{\mathcal{F}}_h = \bigoplus_{k \in \mathbb{N}_0} \overline{\mathcal{H}^-}^{\wedge k}, \tag{3.12}$$

this results for the full space in

$$\mathcal{F} = \mathcal{F}_e \otimes \overline{\mathcal{F}}_h. \tag{3.13}$$

Using this we can represent Ω by $|0\rangle = 1 \otimes 1$ and Ψ by $b^*(\xi) d^*(\varphi_1)|0\rangle$. Forgetting about the structure of Ω leads to problems, as we will see now. Picking an ONB $(\varphi_n)_{n\in\mathbb{N}}$ of $\mathcal{H}^+$ and $(\varphi_{-n})_{n\in\mathbb{N}}$ of $\mathcal{H}^-$ we might express the probability of creation of at least one pair due to time evolution from t_0 to t_1 subject to an external potential A to first order in the expansion of the scalar product by

$$\sum_{k,n\in\mathbb{N}} |\langle \varphi_k, U^A(t_1, t_0)\varphi_{-n}\rangle|^2 = \|U^A_{+-}\|^2_{I_2}, \tag{3.14}$$

where $U_{\pm\mp} := P^\pm U P^\mp$ for any operator U and $\|\cdot\|_{I_2}$ is the Hilbert-Schmidt norm. The space of operators of type $\mathcal{H} \circlearrowleft$ induced by this norm is denoted by $I_2(\mathcal{H})$. Furthermore the orthogonal projectors onto the negative and positive energy subspaces of $\mathcal{H}$ will be denoted by

$$P^- : \mathcal{H} \to \mathcal{H}^- \tag{3.15}$$
$$P^+ := 1 - P^-, \tag{3.16}$$

respectively. As a probability the expression (3.14) needs to be bounded by one; however,this is not always the case.

Theorem 22 (Ruijsenaars [44])**.** *The right hand side of* (3.14) $< \infty$ *for all times* $t_0, t_1 \in \mathbb{R}$ *if and only if* $\vec{A} = 0$.

There is one more classical theorem to take note of in this context.

Theorem 23 (Shale-Stinespring [51]). *The one-particle operator U has a lift $\tilde{U} : \mathcal{F} \circlearrowleft$ satisfying (3.7) if and only if $U_{\pm}, U_{\mp} \in I_2(\mathcal{H})$.*

Combined these theorems imply that unless the in the light of Lorentz and gauge invariance highly artificial condition $\vec{A} = 0$ is fulfilled there is no representation of the time evolution operator subject to the field A into Fock space.
Stated in this way the last statement might nudge one into concluding that the Fock space representation is a dead end. In fact, the situation is more subtle. To get a heuristic idea, recall that positive and negative energy states differ in the direction of their spinors. Meaning that multiplying a negative energy with state e.g. γ matrices will in general result in a mixture of positive and negative energy states. Incidentally, this is exactly what happens in the Hamiltonian

$$H^A = \gamma^0(-i\vec{\gamma} \cdot \vec{\nabla} + m) + A_0 - \gamma^0\vec{\gamma} \cdot \vec{A}. \tag{3.17}$$

Since there are infinitely many particles in the wedge product of Ω and there is no mechanism of suppression for states of large momentum, the sum in (3.14) does not converge for $\vec{A} \neq 0$. However, as it turns out there is a $\tilde{U}^A(t_1, t_0) : \mathcal{F} \circlearrowleft$ whenever $\vec{A}(t_1) = 0 = \vec{A}(t_0)$. This implies that once the vector parts $\vec{A}$ vanishes, only finitely many pairs remain, justifying the term "virtual pairs" for the infinity of pairs that appear and vanish together with $\vec{A}$.
As we can read off of our construction of Fock space, this space consists of infinite wedge products that are sufficiently close to the initial state Ω. We just found out that, this space is not large enough to contain the state also at later times when the external field is nonzero, but that does not mean that we cannot find a mapping from the initial state to the later ones. It would be enough to adapt the choice of space at later times to the external field present at that time. These spaces

will in general not give a physically meaningful distinction between electrons and parts of the Dirac sea. Such a distinction may have to wait for a full interacting theory of QED, where it may be given in terms of ground states or states homogeneous and isotropic enough such that excitations above it behave effectively free. However, such a distinction is not necessary to answer physical questions such as which currents are induced by strong external fields or how Maxwell's equations are modified by those currents.
In the last decade progress has been made to construct the evolution operator of external field QED mapping states of one Fock space to another and to identify the remaining freedom of picking Fock spaces at each hypersurface or point in time. The results generalise theorems 22 and 23 in a way that exposes the gauge and relativistic invariance inherent to the problem, which is not apparent in the original versions. In order to state those theorems we need some mathematical notation which we introduce first.

3.2 Introduction

This section will first introduce some notation of [3, 5, 4, 6] in order to state the results of [4] and prepare for the sections to come. While doing so we will closely follow [6]. Throughout the whole chapter the class of four potentials we shall be interested in is

$$\mathcal{V} := C_c^\infty(\mathbb{R}^4, \mathbb{R}^4). \tag{3.18}$$

All of the results could be extended with a reasonable amount of additional work to slightly more general four-potentials, but not to physically realistic ones such as the Coulomb potential.
Any notion of time evolution in a relativistic setting needs to generalise the notion of simultaneity. For this reason we introduce Cauchy surfaces as follows.

Definition 24 (Cauchy surface, def 2.1 of [6]). *A Cauchy surface $\Sigma \subset \mathbb{R}^4$ is a smooth, 3-dimensional submanifold of $\mathbb{R}^4$ that fulfills the following three conditions:*

a) Every inextensible, two-sided, time- or light-like, continuous path in $\mathbb{R}^4$ intersects Σ in a unique point.

b) For every $x \in \Sigma$, the tangential space $T_x\Sigma$ is space-like.

c) The tangential spaces to Σ are bounded away from light-like directions in the following sense: The only light-like accumulation point of $\bigcup_{x\in\Sigma} T_x\Sigma$ is zero.

Definition 25 ($\mathcal{H}_\Sigma$, def 2.2 of [6]). *For every Cauchy surface Σ there is a parametrisation*

$$\Sigma = \{\pi_\Sigma(\vec{x}) := (t_\Sigma(\vec{x}), \vec{x}) \mid \vec{x} \in \mathbb{R}^3\}, \tag{3.19}$$

with a smooth function $t_\Sigma : \mathbb{R}^3 \to \mathbb{R}$. Agreeing with standard notation, $d^4x = dx^0dx^1dx^2dx^3$ denotes the standard volume form over $\mathbb{R}^4$, where the product of forms is the wedge product. By d^3x we denote the form $d^3x = dx^1dx^2dx^3$ both on $\mathbb{R}^4$ and on $\mathbb{R}^3$. When contracting a form ω with a vector v we will be denoting this by $i_v(\omega)$. We will keep writing $i_v(\omega)$ also for the spinor matrix valued vector $\gamma = (\gamma^0, \gamma^1, \gamma^2, \gamma^3) = \gamma^\mu e_\mu$:

$$i_\gamma(d^4x) = \gamma^\mu i_{e_\mu}(d^4x). \tag{3.20}$$

For any $x \in \Sigma$ restricting the 3-form $i_\gamma(d^4x)$ to the tangent space $T_x\Sigma$ results in

$$i_\gamma(d^4x) = \not{n}(x) i_n(d^4x) = \left(\gamma^0 - \sum_{\mu=1}^{3} \gamma^\mu \frac{\partial t_\Sigma(\vec{x})}{\partial x^\mu}\right) d^3x \tag{3.21}$$

Being able to write a Poincaré covariant measure on Cauchy sufaces we may introduce the scalar product

$$\phi, \psi \mapsto \int_\Sigma \overline{\phi}(x) i_\gamma(d^4\gamma)\psi(x) =: \langle \phi, \psi \rangle \tag{3.22}$$

and $\overline{\phi} = \phi^\dagger \gamma^0$. *with respect to this salar product we define* $\mathcal{H}_\Sigma = L^2(\Sigma, \mathbb{C}^4)$.

As is well known(e.g. [2, 5]) the Dirac equation coupled to an external potential, equation (3.3), has a one-particle evolution operator for each pair of Cauchy surfaces Σ, Σ'

$$U_{\Sigma',\Sigma} : \mathcal{H}_\Sigma \to \mathcal{H}_{\Sigma'}, \tag{3.23}$$

Using this covariant replacement of the standard Hilbert-space we repeat the Fock space construction of section 3.1 in a slightly more general fashion.

Definition 26 (Fock space of generalised polarisation, def 2.4 of [6])**.** *Let Pol*($\mathcal{H}_\Sigma$) *denote the set of all closed, linear subspaces* $V \subset \mathcal{H}$ *such that both* V *and* $V^\perp$ *are infinite dimensional. Any* $V \in$ *Pol*($\mathcal{H}_\Sigma$) *is called* polarisation *of* $\mathcal{H}_\Sigma$. *For* $V \in$ *Pol*($\mathcal{H}_\Sigma$), *let* $P^V_\Sigma : \mathcal{H}_\Sigma \to V$ *denote the orthogonal projection of* $\mathcal{H}_\Sigma$ *onto* V. *The Fock space corresponding to* V *on the Cauchy surface* Σ *is defined to be*

$$\mathcal{F}(V, \mathcal{H}_\Sigma) := \bigoplus_{c\in\mathbb{Z}} \mathcal{F}_c(V, \mathcal{H}_\Sigma), \quad \mathcal{F}_c(V, \mathcal{H}_\Sigma) := \bigoplus_{\substack{n,m\in\mathbb{N}_0 \\ c=m-n}} (V^\perp)^{\wedge n} \otimes \overline{V}^{\wedge m}, \tag{3.24}$$

where $\bigoplus$ *is the Hilbert space direct sum,* $\wedge$ *the antisymmetric tensor product of Hilbert spaces and* $\overline{V}$ *is the conjugate complex vector space of* V, *introduced in section 3.1.*

Pick Cauchy surfaces Σ, Σ' and polarisations $V \in \mathrm{Pol}(\mathcal{H}_\Sigma)$, $V' \in \mathrm{Pol}(\mathcal{H}_{\Sigma'})$ then we can give the analogue of the lift condition (3.7) in this setting: for all $\psi \in \mathcal{H}_\Sigma$

$$\tilde{U}^A_{V',\Sigma';V,\Sigma} a^*_\Sigma(\psi) = a^*_{\Sigma'}(U^A_{\Sigma',\Sigma}\psi)\tilde{U}^A_{V',\Sigma';V,\Sigma}, \qquad \text{(lift condition)}$$

holds, where $a^*_{\Sigma'}$ and a^*_Σ are the creation operator associated to $\mathcal{F}(V, \mathcal{H}_\Sigma)$ and $\mathcal{F}(V', \mathcal{H}_{\Sigma'})$ respectively.
The rephrasing of theorem 23 adapted to the more general notation we have developed now is

Theorem 27 ([4], also Cor 2.5 of [6])**.** *Let Σ, Σ' be Cauchy surfaces, $V \in Pol(\mathcal{H}_\Sigma)$, and $V' \in Pol(\mathcal{H}_{\Sigma'})$ be polarisations. Then the following two statements are equivalent:*

a) There is a unitary operator $\tilde{U}^A_{V',\Sigma';V,\Sigma} : \mathcal{F}(V, \Sigma) \to \mathcal{F}(V', \Sigma')$ that satisfies the (lift condition)

b) The operators $P^{V'\perp}_{\Sigma'} U^A_{\Sigma',\Sigma} P^V_\Sigma$ and $P^{V'}_{\Sigma'} U^A_{\Sigma',\Sigma} P^{V\perp}_\Sigma$ are Hilbert-Schmidt operators.

So the question given an initial state in an initial Fock space $\mathcal{F}(V, \Sigma)$, which Fock space we may pick at a final Cauchy surface Σ' such that there is a lift fulfilling the (lift condition) now becomes a question of polarisations. We know a priori that $U^A_{\Sigma',\Sigma}$ has a lift from $\mathcal{F}(V, \Sigma)$ to $\mathcal{F}(U^A_{\Sigma',\Sigma}V, \Sigma')$. Furthermore, we have a distinguished polarisation for very early times, namely the negative energy states with respect to the free Hamiltonian. Thus, we may characterize all relevant polarisation classes into the following equivalence classes

Definition 28 (polarisation classes, def 2.6 of [6])**.** *For a Cauchy surface Σ and a potential $A \in \mathcal{V}$ we define*

$$C_\Sigma(A) := \left\{ W \in Pol(\mathcal{H}_\Sigma) \mid W \approx U^A_{\Sigma,\Sigma_{\text{in}}} \mathcal{H}^-_{\Sigma_{\text{in}}} \right\}, \qquad (3.25)$$

where Σ_{in} is a Cauchy surface earlier than $\operatorname{supp} A$, *$\mathcal{H}^-_{\Sigma_{in}}$ is the subspace spanned by the wavefunctions in the negative spectrum of the free Dirac Hamiltonian. Furthermore, for $V, V' \in Pol(\mathcal{H}_\Sigma)$ we may write $V \approx V'$ whenever $P^V_\Sigma - P^{V'}_\Sigma$ is a Hilbert-Schmidt operator.*

Using this we immediately find

Corollary 29 (polarisation classes and lifts, cor 2.7 of [6])**.** *Let Σ, Σ' be Cauchy surfaces and $V \in C_\Sigma(A), V' \in Pol(\mathcal{H}_{\Sigma'})$ be polarisations. Then there is a unitary operator $\tilde{U}^A_{V',\Sigma';V,\Sigma} : \mathcal{F}(V,\Sigma) \to \mathcal{F}(V',\Sigma')$ satisfying the* (lift condition) *if and only if $V' \in C_{\Sigma'}(A)$.*

The definition 28 suggests a dependence of $C_\Sigma(A)$ on all of A as a function of time. As indicated in the last section, this is not the case.

Theorem 30 ($C_\Sigma(A)$ depends on $A|_{T\Sigma}$, thm. 1.5 of [4])**.** *Pick a Cauchy surface Σ and $A, A' \in \mathcal{V}$. Then we have*

$$C_\Sigma(A) = C_\Sigma(A') \iff A|_{T\Sigma} = A'|_{T\Sigma}, \tag{3.26}$$

where $A|_{T\Sigma} = A'|_{T\Sigma}$ means that for all $x \in \Sigma$ and $y \in T_x\Sigma$ the relation $A_\mu(x)y^\mu = A'_\mu(x)y^\mu$ holds.

This is a version of theorem 22 for general Cauchy surfaces. The next theorem shows how the polarisation classes change with the gauge and Lorentz transforms.

Theorem 31 (transformation of polarisation classes, thm. 1.6 of [4])**.** *Let $A \in \mathcal{V}$ be a four-potential and Σ a Cauchy surface.*

a) Let $(\mathfrak{s}, \Lambda) \in \mathbb{C}^{4\times 4} \times \mathrm{SO}^\uparrow(1,3)$ be an orthochronous Lorentz transform, i.e. the tuple fulfills $\Lambda^\mu_\sigma g_{\mu\nu} \Lambda^\nu_\tau = g_{\sigma,\tau}$ and $\Lambda^\mu_\nu \gamma^\nu = \mathfrak{s}^{-1}\gamma^\mu \mathfrak{s}$ and acts on wavefunctions as $\psi \mapsto \mathfrak{s}\psi(\Lambda^{-1}\cdot)$ (see sec. 2.3 of [5]). Then we have

$$V \in C_\Sigma(A) \iff (\mathfrak{s}, \Lambda)V \in C_{\Lambda\Sigma}(\Lambda A(\Lambda^{-1}\cdot)). \tag{3.27}$$

b) Let $A' = A + d\zeta$ be the gauge transformed potential, for some $\zeta \in C_c^\infty(\mathbb{R}^4, \mathbb{R})$. Then the gauge transformation acts on wavefunctions as $e^{-i\zeta} : \mathcal{H}_\Sigma \circlearrowleft, \psi \mapsto e^{-i\zeta}\psi$ and one obtains

$$V \in C_\Sigma(A) \iff e^{-i\zeta}V \in C_\Sigma(A + d\zeta). \tag{3.28}$$

Theorems 31, 30 and corollary 29 make clear in which way the original plan to work in a single Fock space was misguided and how it may be adapted to make it work.

When trying to construct an evolution operator from a Cauchy surface Σ to a second one Σ' subject to an external field A, one has to choose an initial polarisation $V \in C_\Sigma(A)$ and a final polarisation $V' \in C_{\Sigma'}(A)$. Then there is an evolution operator $\tilde{U}^A_{V',\Sigma';V,\Sigma}$, unique up to a phase. Picking polarisations is akin to picking a patch of coordinates on a nontrivial manifold, in the sense that there may not be a canonical choice and the choice is going to influence the representation of all the relevant objects. Nevertheless, one can obtain valuable information from calculations done with respect to one such choice and always transform the results to the representation induced by any other choice. Carrying out such a procedure will for every $\Phi \in \mathcal{F}(V, \Sigma)$ and $\Psi \in \mathcal{F}(V', \Sigma')$ result in finite transition probabilities $|\langle \Psi, \tilde{U}^A_{V',\Sigma',V,\Sigma}\Phi\rangle|^2$ without the need for renormalisation.

A key ingredient in the proofs of the last few theorems was a particular operator P^A_Σ which is Hilbert-Schmidt close to a projector. This operator is also examined in this thesis in section 3.5, see definition 80.

To make the discussion more concrete, we are going to introduce a particular representation of the Fock spaces and evolution operators discussed so far. This representation is heavily inspired by Diracs orignal idea discussed in section 3.1 and is usually referred to as inifinite wedge space. For further details please have a look at section 2 of [3]. The basic idea behind this representation is a generalisation of the following point of view of the scalar product of finitely many fermions but works for any Hilbert space $\mathcal{H}$. Pick $N \in \mathbb{N}$ and two states $\Lambda\Psi, \Lambda\Phi \in \mathcal{H}^{\wedge N}$ that can be written as the wedge product of N wave-functions

$$\Lambda\Psi = \psi_1 \wedge \dots \psi_N \tag{3.29}$$

$$\Lambda\Phi = \phi_1 \wedge \dots \phi_N, \tag{3.30}$$

then the standard scalar product $\langle\Psi, \Phi\rangle$ in $\mathcal{H}^{\wedge N}$ can be written as a determinant

$$\det_{\mathbb{R}^N}(\Psi^*\Phi), \tag{3.31}$$

where Φ and Ψ are interpreted to be linear maps of type $\mathbb{R}^N \to \mathcal{H}$

$$\forall k : \Psi : e_k \mapsto \psi_k \tag{3.32}$$

$$\forall k : \Phi : e_k \mapsto \phi_k \tag{3.33}$$

where $(e_k)_{k\in\{1,\dots,N\}}$ is an ONB of $\mathbb{R}^N$ and the star denotes the adjoint. Non product states first have to be decomposed into a sum of product states, then the determinant is continued to be linear in the left and right factor. Replacing $\mathbb{R}^N$ by an index space ℓ, e.g. $l^2(\mathbb{N})$ the space of square summable sequences, and using the Fredholm determinant this particular representation of the scalar product of $\mathcal{H}^N$ can be directly generalized to an infinite number of particles. Where now product states Φ can be thought of as maps

$$\Phi : \ell \to \mathcal{H}, e_k \mapsto \varphi_k. \tag{3.34}$$

Such maps will be called *Dirac seas*. The corresponding wedge product can be thought of as

$$\Lambda\Phi = \varphi_1 \wedge \varphi_2 \dots . \tag{3.35}$$

However, as operators on an infinite dimensional Vector space ℓ only have a Fredholm determinant if they are in the set $1 + I_1(\ell)$ (here I_1

denotes the space of operators with finite trace-norm), only Dirac seas $\Phi, \Psi : \ell \to \mathcal{H}$ satisfying

$$\Psi^*\Phi, \Psi^*\Psi, \Phi^*\Phi \in 1 + I_1(\ell) \tag{3.36}$$

have a scalar product. Starting with one particular infinite wedge product $\Lambda\Phi$ and collecting all infinite wedge products $\Lambda\Psi$ such that (3.36) is fulfilled and formal linear combinations thereof and taking the completion with respect to the pairing (3.31) results in what is referred to as infinite wedge space $\mathcal{F}_{\Lambda\Phi}$. For a rigorous construction please have a look at [3, section 2.1].
It is worth noticing here, that if one starts from some other inifinite wedge product $\Lambda\Psi$ such that $\Phi^*\Psi \in 1+I_1(\ell)$ to construct $\mathcal{F}_{\Lambda\Psi}$ one finds $\mathcal{F}_{\Lambda\Psi} = \mathcal{F}_{\Lambda\Phi}$, so there is no unique vacuum state in the infinite wedge space. Pick a second Hilbert space $\mathcal{H}'$ and some unitary operator $U : \mathcal{H} \to \mathcal{H}'$, which can be thought of as $U^A_{\Sigma',\Sigma}$. Next we define the operation form the left of U by

$$\mathcal{L}_U : \mathcal{F}_{\Lambda\Phi} \to \mathcal{F}_{\Lambda U\Phi}, \tag{3.37}$$

$$\mathcal{L}_U \Lambda\Psi = \Lambda U\Psi = (U\psi_1) \wedge (U\psi_2) \wedge \ldots, \tag{3.38}$$

where $\Psi : \ell \to \mathcal{H}$ satisfies $\Psi^*\Phi \in 1 + I_1(\ell)$. The operator $\mathcal{L}_U$ is a lift of U in the sense of the (lift condition) and maps one Fock space to another. The resulting target space $\mathcal{F}_{\Lambda U\Phi}$ is quite implicit and for $\mathcal{H} = \mathcal{H}_\Sigma$, $\mathcal{H}' = \mathcal{H}_{\Sigma'}$ and $U = U^A_{\Sigma',\Sigma}$ in general not identical to $\mathcal{F}_{\Lambda\Phi'}$ for some $\Phi' : \ell \to \mathcal{H}'$ even if range$(\Phi) \in C_\Sigma(A)$ and range$(\Phi') \in C_{\Sigma'}(A)$ hold. This shortcoming of the construction can be overcome by adding an additional operation from the right. Let $\Psi : \ell' \to \mathcal{H}_{\Sigma'}$ such that range$(\Psi) =$ range(Φ') holds, then there is a unitary $R : \ell' \to \ell$ such that $\Phi' = \Psi R$. Analogously to the action from the left we define one from the right:

$$\mathcal{R}_R : \mathcal{F}_{\Lambda\Psi} \to \mathcal{F}_{\Lambda\Phi'}, \tag{3.39}$$

$$\mathcal{R}_R \Lambda\Phi' = \Lambda(\Phi' R), \tag{3.40}$$

also here the Φ' generate the infinite wedge space $\mathcal{F}_{\wedge\Psi}$. The spaces $\mathcal{F}_{\wedge\Psi}$ and $\mathcal{F}_{\wedge\Psi R}$ only coincides if $\ell' = \ell$ and R has a determinant, i.e. $R \in 1 + I_1(\ell)$. The next theorem helps us to decide in which cases there is a unitary $R : \ell' \to \ell$ such that $\mathcal{F}_{\wedge U\Phi R} = \mathcal{F}_{\wedge\Phi'}$ holds.

Theorem 32 (thm. 23 for $\mathcal{L}$ and $\mathcal{R}$, [3, thm. 2.26], [6, thm 3.1]). *Let $\mathcal{H}, \ell, \mathcal{H}', \ell'$ be Hilbert spaces, $V \in \mathrm{Pol}(\mathcal{H})$ and $V' \in \mathrm{Pol}(\mathcal{H}')$ polarisations, $\Phi : \ell \to \mathcal{H}$ and $\Phi' : \ell' \to \mathcal{H}'$ be Dirac seas such that* $\mathrm{range}(\Phi) = V$ *and* $\mathrm{range}(\Phi') = V'$ *Then the following two statements are equivalent*

a) The off diagonal operators $P^{V'^{\perp}} U P^{V}$ and $P^{V'} U P^{V^{\perp}}$ are Hilbert-Schmidt operators.

b) There is a unitary $R : \ell \to \ell'$ such that $\mathcal{F}_{\wedge\Phi'} = \mathcal{F}_{\wedge U\Phi R}$.

So returning to $\Phi : \ell \to \mathcal{H}_\Sigma$, $\Phi' : \ell \to \mathcal{H}_{\Sigma'}$, with $\mathrm{range}(\Phi) \in C_\Sigma(A)$, $\mathrm{range}(\Phi') \in C_{\Sigma'}(A)$ for some Cauchy surfaces Σ, Σ' and some $A \in \mathcal{V}$, condition a) of the last theorem is satisfied so the existence of $R : \ell \to \ell$ such that the evolution operator

$$\tilde{U}^{A}_{V',\Sigma',V,\Sigma} : \mathcal{F}_{\wedge\Phi} \to \mathcal{F}_{\wedge\Phi'}, \quad \tilde{U}^{A}_{V',\Sigma',V,\Sigma} = \mathcal{L}_{U^{A}_{\Sigma',\Sigma}} \circ \mathcal{R}_R \tag{3.41}$$

is well-defined and unitary is ensured. The operators $\mathcal{R}_R$ are unique up to a phase, see [3, Cor. 2.28], as might have been expected since the (lift condition) allows for exactly this much freedom. A simple choice of ℓ, ℓ' that makes it possible to guess a choice for R is $\ell = \mathrm{range}(\Phi) \subseteq \mathcal{H}_\Sigma$, $\ell' = \mathrm{range}(\Phi') \subseteq \mathcal{H}_{\Sigma'}$. As discussed in section 3.1 one might hope that the motion of the electrons of very negative energy is irrelevant for understanding excitations at the surface. However, as we saw the motion of the electrons at great depth made it impossible to directly compare the time evolved states with the original ones. We can use R to revert this motion. So the idea is to pick $R' = (P^{V'} U P^{V})^{-1}$ whenever $P^{V'} U P^{V}$ is invertible, but this choice is not unitary. If

$P^{V'}UP^V$ is not invertible one can perform the construction outlined next in several steps and assemble a lift of the total operator U. This is possible, because of for two unitary operators U_1, U_2 and corresponing lifts $\tilde{U}_1, \tilde{U}_2$,

$$\tilde{U}_1\tilde{U}_2 \tag{3.42}$$

is a lift of U_1U_2. By virtue of the scalar product of two infinite wedge products being a determinant and the equation

$$\det((AR)^*BR) = \det(R^*A^*BR) = \det(RR^*A^*B) \tag{3.43}$$
$$= \det(RR^*)\det(A^*B) = \det(R^*R)\det(A^*B), \tag{3.44}$$

is also true for bounded operators A, B, R of apropriate type whenever R is invertible and R^*R, A^*B each have a determinant. So the operation from the right $\mathcal{R}_{R'}$ may still be defined and a posteriori be corrected by a factor of $\sqrt{\det(R'^*R')^{-1}}$ to turn it into a unitary operator. Using

$$1 = U^{A*}U^A = (P^V + P^{V^\perp})U^{A*}(P^{V'} + P^{V'^\perp})U^A(P^V + P^{V^\perp}), \tag{3.45}$$

and splitting the equation up according to the different initial and target spaces,this implies

$$P^V U^{A*} P^{V'} U^A P^V = 1 - P^V U^{A*} P^{V'^\perp} U^A P^V. \tag{3.46}$$

Polarisations V, V' belonging to the apropriate polarisation classes $V \in C_\Sigma(A)$, $V' \in C_{\Sigma'}(A)$ satisfy condition a) of theorem 32 and because the product of Hilbert-Schmidt operators is traceclass $R'^*R' \in 1 + I_1(V)$, i.e. has a determinant. Hence we may define

$$\tilde{U}^A_{V',\Sigma',V,\Sigma} : \mathcal{F}_{\wedge\Phi} \to \mathcal{F}_{\wedge\Phi'}, \tag{3.47}$$
$$\tilde{U}^A_{V',\Sigma',V,\Sigma} = \det|(P^{V'}U^A_{\Sigma',\Sigma}P^V)|\mathcal{R}_{(P^{V'}U^A_{\Sigma',\Sigma}P^V)} \circ \mathcal{L}_{U^A_{\Sigma',\Sigma}}. \tag{3.48}$$

Well-definedness can be checked directly, the operator

$$\Phi'^* U^A_{\Sigma',\Sigma} \Phi R' = P^{V'} U^A_{\Sigma',\Sigma} P^V R' = 1_{V'} \tag{3.49}$$

clearly has a determinant on V'. By similar calculations as (3.43) it follows that also all operators $\tilde{\Phi}' U^A_{\Sigma',\Sigma} \tilde{\Phi} R'$ with $\Lambda\tilde{\Phi}' \in \mathcal{F}_{\Lambda\Phi'}$, $\Lambda\tilde{\Phi} \in \mathcal{F}_{\Lambda\Phi}$ have a determinant.
So the construction of a evolution operator in external field QED was successful.

3.3 Geometric Construction of the Phase

In this chapter we perform a geometric construction of the phase of the evolution operator from an object c^+ having properties inspired by physical intuition. This object c^+ has not itself been constructed yet, but the author and his supervisors hope to construct it in the near future. We restrict ourselves to the scattering regime, because it reduces the available freedom. More precisely, in the scattering regime there is a canonical polarisation in the initial and final Hilbert-space. Because our notion of argument of a complex number and an invertible bounded operator is non standard, we introduce it next.

Definition 33 (polar decomposition and spectral projections)**.** *We denote by $\mathcal{H} = L^2(\mathbb{R}^3, \mathbb{C})$. For $X : \mathcal{H} \to \mathcal{H}$ bounded*

$$\mathrm{AG}(X) := X|X|^{-1}. \tag{3.50}$$

Furthermore, we define for any complex number $z \in \mathbb{C}\backslash\{0\}$

$$\mathrm{ag}(z) := \frac{z}{|z|}. \tag{3.51}$$

In abuse of notation we will define the expression

$$\partial_t \ln f(t) := \frac{\partial_t f(t)}{f(t)}, \tag{3.52}$$

for any differentiable $f : \mathbb{R} \to \mathbb{C}\backslash\{0\}$*, even if the expression* $\ln f(t)$ *cannot be interpreted as the principle branch of the logarithm.*
We also introduce $S^1 := \{z \in \mathbb{C} \mid |z| = 1\}$*. We will denote the space of bounded linear functions from one normed Vektorspace* V *into itself by* $\mathcal{B}(V)$*.*
Lastly we introduce the partial derivative in the direction of any four-potential F *of an operator valued function* $F : \mathcal{V} \to \mathcal{B}(\mathcal{F})$ *by*

$$\partial_F T(F) := \partial_\varepsilon T(\varepsilon F)|_{\varepsilon=0}, \tag{3.53}$$

where the limit is taken with respect to the operator norm topology.

Definition 34 (scattering operator and phases)**.** *We define for all* $A, B \in \mathcal{V}$

$$S_{A,B} := U^A_{\Sigma_{\text{in}},\Sigma_{\text{out}}} U^B_{\Sigma_{\text{out}},\Sigma_{\text{in}}}, \tag{3.54}$$

where Σ_{out} *and* Σ_{int} *are Cauchy-surfaces of Minkowski spacetime such that*

$$\forall (x, y) \in \operatorname{supp} A \cup \operatorname{supp} B \times \Sigma_{\text{in}} : (x - y)^2 \geqslant 0 \Rightarrow x^0 > y^0, \tag{3.55}$$
$$\forall (x, y) \in \operatorname{supp} A \cup \operatorname{supp} B \times \Sigma_{\text{out}} : (x - y)^2 \geqslant 0 \Rightarrow x^0 < y^0 \tag{3.56}$$

holds. Define for any a, b *elements of the same real or complex vectorspace the linesegment connecting them*

$$\overline{a\ b} := \{sa + (1 - s)b \mid s \in [0, 1]\}. \tag{3.57}$$

Let

$$\mathrm{dm} := \{(A, B) \in \mathcal{V}^2 \mid P^- S_{A,B} P^- \text{ and} \tag{3.58}$$
$$P^- S_{B,A} P^- : \mathcal{H}^- \circlearrowleft \text{ are invertible}\}, \tag{3.59}$$

we define

$$\operatorname{dom} \overline{S} := \{(A, B) \in \mathrm{dm} \mid \overline{A\ B} \times \overline{A\ B} \subseteq \mathrm{dm}\}. \tag{3.60}$$

Furthermore, we choose for all $A, B \in \operatorname{dom} \overline{S}$ the lift the lift discussed at the end of the last section

$$\overline{S}_{A,B} = \mathcal{R}_{\mathrm{AG}((P^- S_{A,B} P^-)^{-1})} \mathcal{L}_{S_{A,B}}. \tag{3.61}$$

For $(A, B), (B, C), (C, A) \in \operatorname{dom} \overline{S}$, we define the complex numbers

$$\gamma_{A,B,C} := \det_{\mathcal{H}^-}(P^- S_{A,B} P^- S_{B,C} P^- S_{C,A} P^-), \tag{3.62}$$

$$\Gamma_{A,B,C} := \mathrm{ag}(\gamma_{A,B,C}). \tag{3.63}$$

We will see in lemma 40 that $\gamma_{A,B,C} \neq 0$ and $P^- S_{A,B} P^- S_{B,C} P^- S_{C,A} \in 1 + I_1(\mathcal{H}^-)$, so that $\Gamma_{A,B,C}$ is well-defined. Next, we introduce for $A, B, C \in \mathcal{V}$ the function

$$c_A(F, G) := -i \partial_F \partial_G \Im \operatorname{tr}[P^- S_{A,A+F} P^+ S_{A,A+G} P^-]. \tag{3.64}$$

Finally, let a, b be two subsets of Minkowski spacetime, we say $a \prec b$ (in words: "a is causally prior to b") if and only if for all $(x, y) \in a \times b$

$$\big((x - y)^2 \geqslant 0 \wedge x \neq y\big) \Rightarrow x^0 < y^0 \tag{3.65}$$

holds. For $A, B \in \mathcal{V}$ in the expressions $a \prec A, A \prec a, A \prec B$ the four-potentials A, B are to be interpreted as $\operatorname{supp} A, \operatorname{supp} B$ respectively.

Lemma 35 (properties of $\operatorname{dom} \overline{S}$)**.** *The set $\operatorname{dom} \overline{S}$ has the following properties:*

1. *contains the diagonal: $\{(A, A) \mid A \in \mathcal{V}\} \subseteq \operatorname{dom} \overline{S}$.*
2. *openness: $\forall n \in \mathbb{N} : \big\{s \in \mathbb{R}^{2n} \mid \big(\sum_{k=1}^{n} s_k A_k, \sum_{k=n+1}^{2n} s_k A_k\big) \in \operatorname{dom} \overline{S}\big\}$ is an open subset of $\mathbb{R}^{2n}$ for all $A_1, \ldots A_{2n} \in \mathcal{V}$.*
3. *symmetry: $(A, A') \in \operatorname{dom} \overline{S} \iff (A', A) \in \operatorname{dom} \overline{S}$*
4. *star-shaped: $(A, tA) \in \operatorname{dom} \overline{S} \Rightarrow \forall s \in \overline{1 \quad t} : (A, sA) \in \operatorname{dom} \overline{S}$*

5. *well-defined-ness of* $\overline{S}$*:* $\operatorname{dom}\overline{S} \subseteq \{A, B \in \mathcal{V} \mid P^- S_{A,B} P^- : \mathcal{H}^- \circlearrowleft$ *is invertible*$\}$.

We will only prove openness, as the other properties follow directly from the definition (3.60). So pick $n \in \mathbb{N}$*,* $A_i \in \mathcal{V}$ *for* $i \in \mathbb{N}, i \leqslant 2n$ *and* $s \in \mathbb{R}^{2n}$ *such that* $\left(\sum_{k=1}^{n} s_k A_k, \sum_{k=n+1}^{2n} s_k A_k\right) \in \operatorname{dom}\overline{S}$*. We have to find a neighbourhood* $U \subseteq \mathbb{R}^{2n}$ *of* s *such that* $\{\left(\sum_{k=1}^{n} s'_k A_k, \sum_{k=n+1}^{2n} s'_k A_k\right) \mid s' \in U\} \subseteq \operatorname{dom}\overline{S}$ *holds. In doing so we have to ensure that the square*

$$\overline{\sum_{k=1}^{n} s'_k A_k \qquad \sum_{k=n+1}^{2n} s'_k A_k}^{\,2} \tag{3.66}$$

stays a subsets of dm *for all* $s' \in U$*. Now pick a metric* d *on* $\mathbb{R}^{2n}$ *and define*

$$r := \inf\left\{ d(s, s') \mid \overline{\sum_{k=1}^{n} s'_k A_k \qquad \sum_{k=n+1}^{2n} s'_k A_k}^{\,2} \cap \operatorname{dm}^c \neq \varnothing \right\}.$$

It cannot be the case that $r = 0$*, because the metric is continuous, the square compact in* $\mathbb{R}^{2n}$ *and the set of invertible bounded operators (defining* dm*) is open in the topology generated by the operator norm. If* $r = \infty$ *then* $U = \mathbb{R}^{2n}$ *will suffice. If* $r \in \mathbb{R}^+$ *then* $U = B_r(s, t)$ *the open ball of radius* r *around* s *works.*

3.3.1 Main Result of Construction

Definition 36 (causal splitting)**.** *We define a causal splitting as a function*

$$c^+ : \mathcal{V}^3 \to \mathbb{C}, \tag{3.67}$$

$$(A, F, G) \mapsto c^+_A(F, G), \tag{3.68}$$

such that c^+ restricted to any finite dimensional subspace is smooth in the first argument and linear in the second and third argument. Furthermore c^+ should satisfy

$$c_A(F,G) = c_A^+(F,G) - c_A^+(G,F), \tag{3.69}$$
$$\partial_H c_{A+H}^+(F,G) = \partial_G c_{A+G}^+(F,H), \tag{3.70}$$
$$\forall F \prec G : c_A^+(F,G) = 0. \tag{3.71}$$

Definition 37 (current)**.** *Given a lift $\hat{S}_{A,B}$ of the one-particle scattering operator $S_{A,B}$ for which the derivative in the following expression exists, we define the associated current by Bogolyubov's formula:*

$$j_A^{\hat{S}}(F) := i\partial_F \left\langle \Omega, \hat{S}_{A,A+F}\Omega \right\rangle. \tag{3.72}$$

Theorem 38 (existence of causal lift)**.** *Given a causal splitting c^+, there is a second quantised scattering operator $\tilde{S}$, lift of the one-particle scattering operator S with the following properties*

$$\forall A,B,C \in \mathcal{V} : \tilde{S}_{A,B}\tilde{S}_{B,C} = \tilde{S}_{A,C} \tag{3.73}$$
$$\forall F \prec G : \tilde{S}_{A,A+F} = \tilde{S}_{A+G,A+F+G} \tag{3.74}$$

and the associated current satisfies

$$\partial_G j_{A+G}^{\tilde{S}}(F) = \begin{cases} -2ic_A(F,G) & \text{for } G \prec F \\ 0 & \text{for } F \prec G \end{cases} \tag{3.75}$$

Remark 39. *One may wonder why we construct a lift with the properties (3.73) and (3.74). The project of finding a rigorous formulation of external field QED can be considered a success once a lift $\tilde{U}^A_{\Sigma',\Sigma}$ of the evolution from one cauchy surface to another $U_{\Sigma',\Sigma}$ has been constructed from which a current can be calculated that agrees with experiments to the degree that the approximations inherent to the model* *are applicable. In the light of this goal properties (3.73) and (3.74) should be judged. Property (3.73) is a basic requirement, any phase that does not fulfil it will not be directly generalisable to the the evolution between different Cauchy surfaces. In the proof of the theorem we will see that in order to satisfy it, it suffices to construct any global section, i.e. a lift of* $S_{0,A}$ *for any* $A \in \mathcal{V}$*. In order to see why (3.74) is a reasonable second requirement we may quickly go through its proof in the one-particle situation. Let* $A, F, G \in \mathcal{V}$ *such that* $F \prec G$*. We may then pick a Cauchy surface* Σ' *such that* $\operatorname{supp} F \prec \Sigma'$ *and* $\Sigma' \prec \operatorname{supp} G$*. This implies*

$$S_{A+G,A+G+F} = U^{A+G}_{\Sigma_{in},\Sigma_{out}} U^{A+G+F}_{\Sigma_{out},\Sigma_{in}} \tag{3.76}$$

$$= U^{A+G}_{\Sigma_{in},\Sigma'} U^{A+G}_{\Sigma',\Sigma_{out}} U^{A+G+F}_{\Sigma_{out},\Sigma'} U^{A+G+F}_{\Sigma',\Sigma_{in}} \tag{3.77}$$

$$\overset{*}{=} U^{A}_{\Sigma_{in},\Sigma'} U^{A+G}_{\Sigma',\Sigma_{out}} U^{A+G}_{\Sigma_{out},\Sigma'} U^{A+F}_{\Sigma',\Sigma_{in}} = U^{A}_{\Sigma_{in},\Sigma'} U^{A+F}_{\Sigma',\Sigma_{in}} \tag{3.78}$$

$$= U^{A}_{\Sigma_{in},\Sigma'} U^{A}_{\Sigma',\Sigma_{out}} U^{A}_{\Sigma_{out},\Sigma'} U^{A+F}_{\Sigma',\Sigma_{in}} = U^{A}_{\Sigma_{in},\Sigma_{out}} U^{A+F}_{\Sigma_{out},\Sigma_{in}} = S_{A,A+F}, \tag{3.79}$$

where for the marked equality we used the support properties of F *and* G *relative to* Σ' *and that* $S^{A}_{\Sigma',\Sigma}$ *only depends on values of* A *in the volume delimited by* Σ' *and* Σ*. For a lift of* $U^{A}_{\Sigma',\Sigma}$ *we would expect the last calculation to hold in the second quantised language as well. So property (3.74) is a way of incorporating attributes of the lift of the time evolution between different hypersurfaces without mentioning those hypersurfaces directly.*

3.3.2 Proofs

Since the phase of a lift relative to any other lift is fixed by a single matrix element, we may use the vacuum expectation values to characterise the phase of a lift. The function c captures the dependence of this object on variations of the external field, the connection between vacuum expectation values and c becomes clearer with the next lemma.

Lemma 40 (properties of Γ)**.** *The function Γ has the following properties for all $A, B, C, D \in \mathcal{V}$ such that the expressions occurring in each equation are well-defined:*

$$\gamma_{A,B,C} \neq 0 \tag{3.80}$$

$$\Gamma_{A,B,C} = \det_{\mathcal{H}^-}(P^- - P^- S_{A,C} P^+ S_{C,A} P^- - P^- S_{A,B} P^+ S_{B,C} P^- S_{C,A} P^-) \tag{3.81}$$

$$\Gamma^{-1}_{A,B,C} = \operatorname{ag}(\langle \Omega, \overline{S}_{A,B} \overline{S}_{B,C} \overline{S}_{C,A} \Omega \rangle) \tag{3.82}$$

$$\Gamma_{A,B,C} = \Gamma_{B,C,A} = \frac{1}{\Gamma_{B,A,C}} \tag{3.83}$$

$$\Gamma_{A,A,B} = 1 \tag{3.84}$$

$$\Gamma_{A,B,C} \Gamma_{B,A,D} \Gamma_{A,C,D} \Gamma_{C,B,D} = 1 \tag{3.85}$$

$$\Gamma_{A,B,C} = \Gamma_{D,B,C} \Gamma_{A,D,C} \Gamma_{A,B,D} \tag{3.86}$$

$$\overline{S}_{A,C} = \Gamma_{A,B,C} \overline{S}_{A,B} \overline{S}_{B,C} \tag{3.87}$$

$$c_A(B,C) = \partial_B \partial_C \ln \Gamma_{A,A+B,A+C}. \tag{3.88}$$

Proof. Pick $A, B, C \in \mathcal{V}$ such that $(X, Y) \in \operatorname{dom} \overline{S}$ for $X, Y \in \{A, B, C\}$. By definition γ is

$$\gamma_{A,B,C} = \det_{\mathcal{H}^-}(P^- S_{A,B} P^- S_{B,C} P^- S_{C,A} P^-). \tag{3.89}$$

The operator whose determinant we take in the last line is a product

$$P^- S_{A,B} P^- S_{B,C} P^- S_{C,A} P^- = P^- S_{A,B} P^- \quad P^- S_{B,C} P^- \quad P^- S_{C,A} P^-. \tag{3.90}$$

The three factors appearing in this product are all invertible due to the definition of $\operatorname{dom} \overline{S}$, therefore if the determinant exists we have

$\gamma_{A,B,C} \neq 0$. To see that it does exist, we reformulate

$$\gamma_{A,B,C} = \det_{\mathcal{H}^-}(P^- S_{A,B} P^- S_{B,C} P^- S_{C,A} P^-) \tag{3.91}$$
$$= \det_{\mathcal{H}^-}(P^- S_{A,C} P^- S_{C,A} P^- - P^- S_{A,B} P^+ S_{B,C} P^- S_{C,A} P^-) \tag{3.92}$$
$$= \det_{\mathcal{H}^-}(P^- - P^- S_{A,C} P^+ S_{C,A} P^- - P^- S_{A,B} P^+ S_{B,C} P^- S_{C,A} P^-), \tag{3.93}$$

now we know by theorem 22 of Ruisnaars that $P^+ S_{X,Y} P^-$ is a Hilbert-Schmidt operator for our setting, hence γ and also Γ are well-defined. Equation (3.93) also proves (3.81). Next we show (3.82). In the notation of the last section we have $\Omega = \bigwedge \Phi$ with the injection $\Phi : \mathcal{H}^- \hookrightarrow \mathcal{H}$. We begin by reformulating the right hand side of (3.82)

$$\langle \Omega, \overline{S}_{A,B} \overline{S}_{B,C} \overline{S}_{C,A} \Omega \rangle \tag{3.94}$$
$$= \Big\langle \bigwedge \Phi, \bigwedge \big(S_{A,B} S_{B,C} S_{C,A} \Phi \, \mathrm{AG}(P^- S_{C,A} P^-)^{-1} \\ \mathrm{AG}(P^- S_{B,C} P^-)^{-1} \, \mathrm{AG}(P^- S_{A,B} P^-)^{-1} \big) \Big\rangle$$
$$= \Big\langle \bigwedge \Phi, \bigwedge \big(\Phi \, \mathrm{AG}(P^- S_{C,A} P^-)^{-1} \\ \times \mathrm{AG}(P^- S_{B,C} P^-)^{-1} \, \mathrm{AG}(P^- S_{A,B} P^-)^{-1} \big) \Big\rangle \tag{3.95}$$
$$= \det_{\mathcal{H}^-} \Big((\Phi)^* \big[\Phi \, \mathrm{AG}(P^- S_{C,A} P^-)^{-1} \, \mathrm{AG}(P^- S_{B,C} P^-)^{-1} \\ \times \mathrm{AG}(P^- S_{A,B} P^-)^{-1} \big] \Big) \tag{3.96}$$
$$= \det_{\mathcal{H}^-} \big(\mathrm{AG}(P^- S_{C,A} P^-)^{-1} \, \mathrm{AG}(P^- S_{B,C} P^-)^{-1} \\ \times \mathrm{AG}(P^- S_{A,B} P^-)^{-1} \big) \tag{3.97}$$
$$= \frac{1}{\det_{\mathcal{H}^-} \mathrm{AG}(P^- S_{A,B} P^-) \, \mathrm{AG}(P^- S_{B,C} P^-) \, \mathrm{AG}(P^- S_{C,A} P^-)}. \tag{3.98}$$

We first note that $\det_{\mathcal{H}^-} |P^- S_{X,Y} P^-| \in \mathbb{R}^+$ for $X, Y \in \{A, B, C\}$. This is well-defined because

$$\begin{aligned}
\langle \Omega, \overline{S}_{X,Y}\Omega\rangle &= \langle \bigwedge \Phi, \bigwedge (S_{X,Y}\Phi\,\mathrm{AG}(P^- S_{X,Y} P^-)^{-1})\rangle && (3.99)\\
&= \det_{\mathcal{H}^-}\big(\Phi^* S_{X,Y}\Phi\,\mathrm{AG}(P^- S_{X,Y}P^-)^{-1}\big) && (3.100)\\
&= \det_{\mathcal{H}^-}\big(P^- S_{X,Y} P^-\,\mathrm{AG}(P^- S_{X,Y}P^-)^{-1}\big) && (3.101)\\
&= \det_{\mathcal{H}^-}\big(\mathrm{AG}(P^- S_{X,Y}P^-)^{-1} P^- S_{X,Y} P^-\big) && (3.102)\\
&= \det_{\mathcal{H}^-}\big(\mathrm{AG}(P^- S_{X,Y}P^-)^{-1}\,\mathrm{AG}(P^- S_{X,Y}P^-)|P^- S_{X,Y} P^-|\big) && (3.103)\\
&= \det_{\mathcal{H}^-}|P^- S_{X,Y} P^-| && (3.104)
\end{aligned}$$

holds. Moreover this determinant does not vanish, since the $P^- S_{X,Y} P^-$ is invertible. Also clearly the eigenvalues are positive since $|P^- S_{X,Y} P^-|$ is an absolute value. We continue with the result of (3.98). Thus, we find

$$\begin{aligned}
&\langle \Omega, \overline{S}_{A,B}\overline{S}_{B,C}\overline{S}_{C,A}\Omega\rangle^{-1} && (3.105)\\
&= \det_{\mathcal{H}^-}\big(\mathrm{AG}(P^- S_{A,B}P^-)\,\mathrm{AG}(P^- S_{B,C}P^-)\,\mathrm{AG}(P^- S_{C,A}P^-)\big) && (3.106)\\
&= \det_{\mathcal{H}^-}\big(\mathrm{AG}(P^- S_{A,B}P^-)\,\mathrm{AG}(P^- S_{B,C}P^-)P^- S_{C,A}P^- \\
&\qquad\qquad \times |P^- S_{C,A}P^-|^{-1}\big) && (3.107)\\
&= \det_{\mathcal{H}^-}\big(\mathrm{AG}(P^- S_{A,B}P^-)\,\mathrm{AG}(P^- S_{B,C}P^-)P^- S_{C,A}P^-\big) \\
&\qquad\qquad \times \det_{\mathcal{H}^-}|P^- S_{C,A}P^-|^{-1} && (3.108)\\
&= \det_{\mathcal{H}^-}\big(P^- S_{C,A}P^-\,\mathrm{AG}(P^- S_{A,B}P^-)\,\mathrm{AG}(P^- S_{B,C}P^-)\big) \\
&\qquad\qquad \times \det_{\mathcal{H}^-}|P^- S_{C,A}P^-|^{-1} && (3.109)\\
&= \frac{\det_{\mathcal{H}^-}(P^- S_{A,B}P^- P^- S_{B,C}P^- P^- S_{C,A}P^-)}{\det_{\mathcal{H}^-}|P^- S_{A,B}P^-|\cdot\det_{\mathcal{H}^-}|P^- S_{B,C}P^-|\cdot\det_{\mathcal{H}^-}|P^- S_{C,A}P^-|}. && (3.110)
\end{aligned}$$

Now since the denominator of this fraction is real we can use (3.63) to identity

$$\mathrm{ag}(\langle \Omega, \overline{S}_{A,B}\overline{S}_{B,C}\overline{S}_{C,A}\Omega\rangle) = \Gamma^{-1}_{A,B,C}, \tag{3.111}$$

which proves (3.82).
For the first equality in (3.83) we use $\det(X(1+Y)X^{-1}) = \det(1+Y)$ for any Y trace-class and X bounded and invertible. So we can cyclicly permute the factors $P^- S_{X,Y} P^-$ in the determinant and find

$$\begin{aligned}\Gamma_{A,B,C} &= \mathrm{ag}(\det_{\mathcal{H}^-} P^- S_{A,B} P^- S_{B,C} P^- S_{C,A} P^-)\\ &= \mathrm{ag}(\det_{\mathcal{H}^-} P^- S_{C,A} P^- S_{A,B} P^- S_{B,C} P^-) = \Gamma_{C,A,B}.\end{aligned}$$

For the second equality of (3.83) we use (3.63) to represent both $\Gamma_{A,B,C}$ and $\Gamma_{B,A,C}$. Using this and the manipulations of the determinant we already employed, we arrive at

$$\Gamma_{A,B,C}\Gamma_{B,A,C} \tag{3.112}$$
$$= \mathrm{ag}(\det_{\mathcal{H}^-}(P^- S_{A,B} P^- S_{B,C} P^- S_{C,A} P^-)) \tag{3.113}$$
$$\times\, \mathrm{ag}(\det_{\mathcal{H}^-}(P^- S_{B,A} P^- S_{A,C} P^- S_{C,B} P^-)) \tag{3.114}$$
$$= \mathrm{ag}(\det_{\mathcal{H}^-}(P^- S_{A,B} P^- S_{B,C} P^- S_{C,A} P^-)) \tag{3.115}$$
$$\times\, (\mathrm{ag}(\det_{\mathcal{H}^-}(P^- S_{B,C} P^- S_{C,A} P^- S_{A,B} P^-)))^* \tag{3.116}$$
$$= \mathrm{ag}(\det_{\mathcal{H}^-}(P^- S_{A,B} P^- S_{B,C} P^- S_{C,A} P^-)) \tag{3.117}$$
$$\times\, (\mathrm{ag}(\det_{\mathcal{H}^-}(P^- S_{A,B} P^- S_{B,C} P^- S_{C,A} P^-)))^* \tag{3.118}$$
$$= |\,\mathrm{ag}(\det_{\mathcal{H}^-}(P^- S_{A,B} P^- S_{B,C} P^- S_{C,A} P^-))|^2 = 1, \tag{3.119}$$

which proves (3.83).
Next, using (3.62) inserting twice the same argument yields

$$\gamma_{A,A,C} = \det_{\mathcal{H}^-} P^- S_{A,C} P^- S_{C,A} P^- = \det_{\mathcal{H}^-}(P^- S_{C,A} P^-)^* P^- S_{C,A} P^- \in \mathbb{R}^+, \tag{3.120}$$

hence (3.84) follows.
For proving (3.85) we will use the definition of Γ directly and repeatedly use that we can cyclicly permute operator groups of the form $P^- S_{X,Y} P^-$ for $X, Y \in \{A, B, C, D\}$ in the determinant, i.e.

$$\det P^- S_{X,Y} P^- O = \det O P^- S_{X,Y} P^-, \tag{$\circlearrowleft$}$$

whenever O has a determinant. This is possible, because $P^- S_{X,Y} P^-$ is bounded and invertible. Furthermore we will use that

$$\det O_1 O_2 = \det O_1 \det O_2 \tag{$\leftrightarrow$}$$

holds whenever both O_1 and O_2 have a determinant. Moreover for any $(P^- S_{X,Y} P^-)^* P^- S_{X,Y} P^-$ is the modulus squared of an invertible operator and hence its determinant is positive which means that

$$\operatorname{ag} \det (P^- S_{X,Y} P^-)^* P^- S_{X,Y} P^- = 1. \tag{ag$|\ |$}$$

These three rules will be repeatedly used. We calculate

$$\Gamma_{A,B,C} \Gamma_{B,A,D} \Gamma_{A,C,D} \Gamma_{C,B,D} \tag{3.121}$$
$$= \operatorname{ag} \det_{\mathcal{H}^-} P^- S_{A,B} P^- S_{B,C} P^- S_{C,A} P^-$$
$$\times \operatorname{ag} \det_{\mathcal{H}^-} P^- S_{B,A} P^- S_{A,D} P^- S_{D,B} P^- \quad \Gamma_{A,C,D} \Gamma_{C,B,D} \tag{3.122}$$
$$\overset{(\circlearrowleft)}{=} \operatorname{ag} \det_{\mathcal{H}^-} P^- S_{A,D} P^- S_{D,B} P^- S_{B,A} P^-$$
$$\times \operatorname{ag} \det_{\mathcal{H}^-} P^- S_{A,B} P^- S_{B,C} P^- S_{C,A} P^- \quad \Gamma_{A,C,D} \Gamma_{C,B,D} \tag{3.123}$$
$$\overset{(\leftrightarrow)}{=} \operatorname{ag} \det_{\mathcal{H}^-} \big(P^- S_{A,D} P^- S_{D,B} \big[P^- S_{B,A} P^- S_{A,B} P^- \big]$$
$$\times S_{B,C} P^- S_{C,A} P^- \big) \, \Gamma_{A,C,D} \Gamma_{C,B,D} \tag{3.124}$$
$$\overset{(\circlearrowleft)}{=} \operatorname{ag} \det_{\mathcal{H}^-} P^- S_{B,C} P^- S_{C,A} P^- S_{A,D} P^- S_{D,B} \big[P^- S_{B,A} P^- S_{A,B} P^- \big]$$

$$\times \Gamma_{A,C,D}\Gamma_{C,B,D} \tag{3.125}$$

$$\overset{(\leftrightarrow)}{=} \operatorname{ag}\det_{\mathcal{H}^-} P^- S_{B,C} P^- S_{C,A} P^- S_{A,D} P^- S_{D,B} P^- \\ \times \operatorname{ag}\det_{\mathcal{H}^-} P^- S_{B,A} P^- S_{A,B} P^- \quad \Gamma_{A,C,D}\Gamma_{C,B,D} \tag{3.126}$$

$$\overset{(\operatorname{ag}|\,|)}{=} \operatorname{ag}\det_{\mathcal{H}^-} P^- S_{B,C} P^- S_{C,A} P^- S_{A,D} P^- S_{D,B} P^- \quad \Gamma_{A,C,D}\Gamma_{C,B,D} \tag{3.127}$$

$$\overset{(\circlearrowleft)}{=} \operatorname{ag}\det_{\mathcal{H}^-} P^- S_{A,D} P^- S_{D,B} P^- S_{B,C} P^- S_{C,A} P^- \quad \Gamma_{A,C,D}\Gamma_{C,B,D} \tag{3.128}$$

$$= \operatorname{ag}\det_{\mathcal{H}^-} P^- S_{A,C} P^- S_{C,D} P^- S_{D,A} P^- \\ \times \operatorname{ag}\det_{\mathcal{H}^-} P^- S_{A,D} P^- S_{D,B} P^- S_{B,C} P^- S_{C,A} P^- \quad \Gamma_{C,B,D} \tag{3.129}$$

$$\overset{(\leftrightarrow)}{=} \operatorname{ag}\det_{\mathcal{H}^-} \big(P^- S_{A,C} P^- S_{C,D} P^- \big[P^- S_{D,A} P^- P^- S_{A,D} P^-\big] \\ \times S_{D,B} P^- S_{B,C} P^- S_{C,A} P^-\big) \quad \Gamma_{C,B,D} \tag{3.130}$$

$$\overset{(\circlearrowleft)}{=} \operatorname{ag}\det_{\mathcal{H}^-} \big(P^- S_{D,B} P^- S_{B,C} P^- S_{C,A} P^- S_{A,C} P^- S_{C,D} P^- \\ \times \big[P^- S_{D,A} P^- P^- S_{A,D} P^-\big]\big) \quad \Gamma_{C,B,D} \tag{3.131}$$

$$\overset{(\leftrightarrow)}{=} \operatorname{ag}\det_{\mathcal{H}^-} P^- S_{D,B} P^- S_{B,C} P^- S_{C,A} P^- S_{A,C} P^- S_{C,D} P^- \\ \times \operatorname{ag}\det_{\mathcal{H}^-} P^- S_{D,A} P^- P^- S_{A,D} P^- \quad \Gamma_{C,B,D} \tag{3.132}$$

$$\overset{(\operatorname{ag}|\,|)}{=} \operatorname{ag}\det_{\mathcal{H}^-} \big(P^- S_{D,B} P^- S_{B,C} P^- \big[P^- S_{C,A} P^- S_{A,C} P^-\big] \\ \times P^- S_{C,D} P^-\big)\Gamma_{C,B,D} \tag{3.133}$$

$$\overset{(\circlearrowleft)}{=} \operatorname{ag}\det_{\mathcal{H}^-} P^- S_{C,D} P^- S_{D,B} P^- S_{B,C} P^- \big[P^- S_{C,A} P^- S_{A,C} P^-\big] \\ \times \Gamma_{C,B,D} \tag{3.134}$$

$$\overset{(\leftrightarrow)}{=} \operatorname{ag}\det_{\mathcal{H}^-} P^- S_{C,D} P^- S_{D,B} P^- S_{B,C} P^- \\ \times \operatorname{ag}\det P^- S_{C,A} P^- S_{A,C} P^- \quad \Gamma_{C,B,D} \tag{3.135}$$

$$\overset{(\operatorname{ag}|\ |)}{=} \operatorname{ag}\det_{\mathcal{H}^-} P^- S_{C,D} P^- S_{D,B} P^- S_{B,C} P^- \quad \Gamma_{C,B,D} \tag{3.136}$$

$$= \operatorname{ag}\det_{\mathcal{H}^-} P^- S_{C,D} P^- S_{D,B} P^- S_{B,C} P^- \\ \times \operatorname{ag}\det_{\mathcal{H}^-} P^- S_{C,B} P^- S_{B,D} P^- S_{D,C} P^- \tag{3.137}$$

$$= |\operatorname{ag}\det_{\mathcal{H}^-} P^- S_{C,D} P^- S_{D,B} P^- S_{B,C} P^-|^2 = 1. \tag{3.138}$$

Equation (3.86) is a direct consequence of (3.85) and (3.83).
For (3.87) we realise that according to [3] that two lifts can only differ by a phase, that is

$$\overline{S}_{A,C} = \xi \overline{S}_{A,B} \overline{S}_{B,C} \tag{3.139}$$

for some $\xi \in \mathbb{C}, |\xi| = 1$.
In order to identify ξ we recognise that $\overline{S}_{X,Y} = \overline{S}_{Y,X}^{-1}$ for four potentials X, Y and find

$$1\xi^{-1} = \overline{S}_{A,B} \overline{S}_{B,C} \overline{S}_{C,A}. \tag{3.140}$$

Now we take the vacuum expectation value on both sides of this equation and use (3.82) to find

$$\xi^{-1} = \langle \Omega, \overline{S}_{A,B} \overline{S}_{B,C} \overline{S}_{C,A} \Omega \rangle = \Gamma_{A,B,C}^{-1}. \tag{3.141}$$

Finally we prove (3.88). We start from the right hand side of this equation and work our way towards the left hand side of it. In the following calculation we will repeatedly make use of the fact that $(P^- S_{A,A+B} P^- S_{A+B,A} P^-)$ is the absolute value squared of an invertible operator and has a determinant, which is therefore positive. For the marked equality we will use that for a differentiable function $z : \mathbb{R} \to \mathbb{C}$

at points t where $z(t) \in \mathbb{R}^+$ holds, we have

$$(z/|z|)'(t) = \frac{z'}{|z|}(t) + \frac{-z}{|z|^2}\frac{z'z^* + z^{*\prime}z}{2|z|}(t) = \frac{z'}{2|z|}(t) - \frac{z^2 z^{*\prime}}{2|z|^3}(t)$$
$$= i(\Im(z'))/z(t). \quad (3.142)$$

Furthermore, we will use the following expressions for the derivative of the determinant which holds for all operator valued functions on the reals $M : \mathbb{R} \to 1 + I_1(\mathcal{H})$ such that M is invertible for all $t \in \mathbb{R}$

$$\partial_\varepsilon \det M(\varepsilon)|_{\varepsilon=0} = \det M(0) \operatorname{tr}(M^{-1}(0)\partial_\varepsilon M(\varepsilon)|_\varepsilon), \quad (3.143)$$

likewise we need the following expression for the derivative of M^{-1} for $M : \mathbb{R} \to (\mathcal{H} \to \mathcal{H})$ such that $M(t)$ is invertible and bounded for every $t \in \mathbb{R}$

$$\partial_\varepsilon M^{-1}(\varepsilon)|_{\varepsilon=0} = -M^{-1}(0)\partial_\varepsilon M(\varepsilon)|_{\varepsilon=0} M^{-1}(0). \quad (3.144)$$

The handling of derivatives of operators in the following calculation is justified by the chapter on regularity 4.1. We compute

$$\partial_B \partial_C \ln \Gamma_{A,A+B,A+C} \quad (3.145)$$
$$\overset{(3.63)}{=} \partial_B \partial_C \ln \operatorname{ag}(\det_{\mathcal{H}^-}(P^- S_{A,A+B} P^- S_{A+B,A+C} P^- S_{A+C,A} P^-)) \quad (3.146)$$
$$= \partial_B \frac{\partial_C \operatorname{ag}(\det_{\mathcal{H}^-}(P^- S_{A,A+B} P^- S_{A+B,A+C} P^- S_{A+C,A} P^-))}{\operatorname{ag}(\det_{\mathcal{H}^-}(P^- S_{A,A+B} P^- S_{A+B,A} P^-))} \quad (3.147)$$
$$= \partial_B \partial_C \operatorname{ag}(\det_{\mathcal{H}^-}(P^- S_{A,A+B} P^- S_{A+B,A+C} P^- S_{A+C,A} P^-)) \quad (3.148)$$
$$\overset{*}{=} i\partial_B \frac{\Im \partial_C \det_{\mathcal{H}^-}(P^- S_{A,A+B} P^- S_{A+B,A+C} P^- S_{A+C,A} P^-)}{\det_{\mathcal{H}^-}(P^- S_{A,A+B} P^- S_{A+B,A} P^-)} \quad (3.149)$$
$$= i\partial_B \Big[\frac{\det_{\mathcal{H}^-}(P^- S_{A,A+B} P^- S_{A+B,A} P^-)}{\det_{\mathcal{H}^-}(P^- S_{A,A+B} P^- S_{A+B,A} P^-)}$$
$$\times \Im \operatorname{tr}((P^- S_{A,A+B} P^- S_{A+B,A} P^-)^{-1}$$
$$\times \partial_C P^- S_{A,A+B} P^- S_{A+B,A+C} P^- S_{A+C,A} P^-) \Big] \quad (3.150)$$

The fraction in front of the trace equals 1. As a next step we replace the second but last projector $P^- = 1 - P^+$, the resulting first summand vanishes, because the dependence on C cancels. This results in

$$(3.150) = -i\partial_B \Im \operatorname{tr}((P^- S_{A,A+B} P^- S_{A+B,A} P^-)^{-1} \times \partial_C P^- S_{A,A+B} P^- S_{A+B,A+C} P^+ S_{A+C,A} P^-). \quad (3.151)$$

Now, because $P^+ P^- = 0$ only one summand of the product rule survives:

$$(3.151) = -i\partial_B \Im \operatorname{tr}((P^- S_{A,A+B} P^- S_{A+B,A} P^-)^{-1} \times \partial_C P^- S_{A,A+B} P^- S_{A+B,A} P^+ S_{A+C,A} P^-). \quad (3.152)$$

Next we use $(MN)^{-1} = N^{-1} M^{-1}$ for invertible operators M and N for the first factor in the trace and cancel as much as possible of the second factor:

$$\begin{aligned}
(3.152) &= -i\partial_B \Im \operatorname{tr}((P^- S_{A+B,A} P^-)^{-1} P^- S_{A+B,A} \\
&\qquad \times P^+ \partial_C S_{A+C,A} P^-) && (3.153) \\
&= -i \Im \operatorname{tr}(\partial_B [(P^- S_{A+B,A} P^-)^{-1} P^- S_{A+B,A} \\
&\qquad \times P^+ \partial_C S_{A+C,A} P^-]) && (3.154) \\
&= -i \Im \operatorname{tr}(\partial_B P^- S_{A+B,A} P^+ \partial_C S_{A+C,A} P^-) && (3.155) \\
&= -i \Im \operatorname{tr}(\partial_B P^- S_{A,A+B} P^+ \partial_C S_{A,A+C} P^-) && (3.156) \\
&= -i \partial_B \partial_C \Im \operatorname{tr}(P^- S_{A,A+B} P^+ S_{A,A+C} P^-) && (3.157)
\end{aligned}$$

which proves the claim.

□

In order to construct the lift announced in theorem 38, we first construct a reference lift $\hat{S}$, that is well-defined on all of $\mathcal{V}$. Afterwards we will study the dependence of the relative phase between this global lift $\hat{S}_{0,A}$ and a local lift given by $\hat{S}_{0,B} \overline{S}_{B,A}$ for $B - A$ small. By exploiting

properties of this phase and the causal splitting c^+ we will construct a global lift that has the desired properties.
Since $\mathcal{V}$ is star shaped, we may reach any four-potential A from 0 through the straight line $\{tA \mid t \in [0,1]\}$.

Definition 41 (ratio of lifts)**.** *For any $A, B \in \mathcal{V}$ and any two lifts $S'_{A,B}, S''_{A,B}$ of the one particle scattering operator $S_{A,B}$ we define the ratio*

$$\frac{S'_{A,B}}{S''_{A,B}} \in S^1 \tag{3.158}$$

to be the unique complex number $z \in S^1$ such that

$$z\, S''_{A,B} = S'_{A,B} \tag{3.159}$$

holds.

Theorem 42 (existence of global lift)**.** *There is a unique map $\hat{S}_{0,\cdot} : \mathcal{V} \to U(\mathcal{F})$ which maps $A \in \mathcal{V}$ to a lift of $S_{0,A}$ and solves the parallel transport differential equation*

$$A, B \in \mathcal{V} \textit{ linearly dependent} \Rightarrow \partial_B \frac{\hat{S}_{0,A+B}}{\hat{S}_{0,A}\overline{S}_{A,A+B}} = 0, \tag{3.160}$$

subject to the initial condition $\hat{S}_{0,0} = 1$.

The proof of theorem 42 is divided into two lemmas due to its length. We will introduce the integral flow ϕ_A associated with the differential equation (3.160) for some $A \in \mathcal{V}$. We will then study the properties of ϕ_A in the two lemmas and finally construct $\hat{S}_{0,A} := \phi_A(0,1)$. In the first lemma we will establish the existence of a local solution. The solution will be constructed along the line $\overline{0\ \ A}$. In the second lemma we patch local solutions together to a global one.

Lemma 43 (ϕ local existence and uniqueness)**.** *There is a unique* $\phi_A : \{(t,s) \in \mathbb{R}^2 \mid (tA, sA) \in \operatorname{dom} \overline{S}\} \to U(\mathcal{F})$ *for every* $A \in \mathcal{V}$ *satisfying*

$$\forall (t,s) \in \operatorname{dom} \phi_A : \phi_A(t,s) \text{ is a lift of } S_{tA,sA} \tag{3.161}$$

$$\forall (t,s),(s,l),(l,t) \in \operatorname{dom} \phi_A : \phi_A(t,s)\phi_A(s,l) = \phi_A(t,l) \tag{3.162}$$

$$\forall t \in \mathbb{R} : \phi_A(t,t) = 1 \tag{3.163}$$

$$\forall s \in \mathbb{R} : \partial_t \left. \frac{\phi_A(s,t)}{\overline{S}_{sA,tA}} \right|_{t=s} = 0. \tag{3.164}$$

Proof. We first define the phase

$$z : \{(A,B) \in \operatorname{dom} \overline{S} \mid A, B \text{ linearly dependent}\} \to S^1 \tag{3.165}$$

by the differential equation

$$\frac{d}{dx} \ln z(tA, xA) = - \left. \left(\frac{d}{dy} \ln \Gamma_{tA,xA,yA} \right) \right|_{y=x} \tag{3.166}$$

and the initial condition

$$z(A,A) = 1 \tag{3.167}$$

for any $A \in \mathcal{V}$. The phase z takes the form

$$z(tA, xA) = \exp\left(- \int_t^x dx' \left. \left(\frac{d}{dx'} \ln \Gamma_{tA,yA,x'A} \right) \right|_{y=x'} \right). \tag{3.168}$$

Please note that both differential equation and initial condition are invariant under rescaling of the potential A, so z is well-defined. We will now construct a local solution to (3.160) and define ϕ_A using this solution. Pick $A \in \mathcal{V}$ the expression

$$\hat{S}_{0,sA} = \hat{S}_{0,A} \overline{S}_{A,sA} z(A, sA) \tag{3.169}$$

solves (3.160) locally. Local here means that s is close enough to 1 such that $(A, sA) \in \operatorname{dom} \overline{S}$. Calculating the argument of the derivative of (3.160) we find:

$$\frac{\hat{S}_{0,(s+\varepsilon)A}}{\hat{S}_{0,sA}\overline{S}_{sA,(s+\varepsilon)A}} = \frac{\hat{S}_{0,A}\overline{S}_{A,(s+\varepsilon)A}z(A,(s+\varepsilon)A)}{\hat{S}_{0,A}\overline{S}_{A,sA}\overline{S}_{sA,(s+\varepsilon)}z(A,sA)} \tag{3.170}$$

$$\overset{(3.86)}{=} \frac{\hat{S}_{0,A}\overline{S}_{A,sA}\overline{S}_{sA,(s+\varepsilon)}\Gamma_{A,sA,(s+\varepsilon)A}z(A,(s+\varepsilon)A)}{\hat{S}_{0,A}\overline{S}_{A,sA}\overline{S}_{sA,(s+\varepsilon)}z(A,sA)} \tag{3.171}$$

$$= \frac{\Gamma_{A,sA,(s+\varepsilon)A}z(A,(s+\varepsilon)A)}{z(A,sA)} \tag{3.172}$$

Now we take the derivative with respect to ε at $\varepsilon = 0$, cancel the factor that does not depend on ε and relabel $s = x$ to obtain

$$0 = \left(\frac{d}{dy}(\Gamma_{A,xA,yA}\ z(A,yA))\right)\bigg|_{y=x} \tag{3.173}$$

$$\iff \frac{d}{dx}\ln z(tA,xA) = \left(-\frac{d}{dy}\ln \Gamma_{tA,xA,yA}\right)\bigg|_{y=x}, \tag{3.174}$$

which is exactly the defining differential equation of z. The initial condition of z equation (3.167) is necessary to match the initial condition in (3.169) for $s = 1$. The connection to ϕ from the statement of the lemma can now be made. We define

$$\phi_A(t,s) := z(tA,sA)\overline{S}_{tA,sA}, \tag{3.175}$$

for $(tA, sA) \in \operatorname{dom} \overline{S}$. Since $\overline{S}$ is a lift of S, we see that (3.161) holds. Equation (3.163) follows from (3.167) and $\overline{S}_{tA,tA} = 1$ for general $t \in \mathbb{R}$. Equation (3.164) follows by plugging in (3.175) and using the defining differential equation for z (3.166) as well as its initial condition (3.167) and the property (3.83) of Γ:

$$\partial_s \left. \frac{\phi_A(t,s)}{\overline{S}_{tA,sA}} \right|_{s=t} = \partial_s \left. \frac{z(tA,sA)\overline{S}_{tA,sA}}{\overline{S}_{tA,sA}} \right|_{s=t} \tag{3.176}$$

$$= \partial_s\, z(tA,sA)|_{t=s} = z(tA,tA)\left(\frac{d}{ds} \ln \Gamma_{tA,tA,sA}\right)\bigg|_{s=t} = 0. \tag{3.177}$$

It remains to see that (3.162), i.e. that

$$\phi_A(t,s)\phi_A(s,l) = \phi_A(t,l) \tag{3.178}$$

holds for $(tA,sA),(sA,lA),(tA,lA) \in \operatorname{dom}\overline{S}$. In order to do so we plug in the definition (3.175) of ϕ_A and obtain

$$\phi_A(t,s)\phi_A(s,l) = \phi_A(t,l) \tag{3.179}$$

$$\iff z(tA,sA)z(sA,lA)\overline{S}_{tA,sA}\overline{S}_{sA,lA} = z(tA,lA)\overline{S}_{tA,lA} \tag{3.180}$$

$$\iff z(tA,sA)z(sA,lA)\overline{S}_{tA,sA}\overline{S}_{sA,lA} \tag{3.181}$$

$$= z(tA,lA)\overline{S}_{tA,sA}\overline{S}_{sA,lA}\Gamma_{tA,sA,lA} \tag{3.182}$$

$$\iff z(tA,sA)z(sA,lA)z(tA,lA)^{-1} = \Gamma_{tA,sA,lA}. \tag{3.183}$$

In order to check the validity of the last equality we plug in the integral formula (3.168) for z, we also abbreviate $\frac{d}{dx} = \partial_x$

$$z(tA,sA)z(sA,lA)z(tA,lA)^{-1} \tag{3.184}$$

$$= e^{-\int_t^s dx'\,(\partial_{x'} \ln \Gamma_{tA,yA,x'A})|_{y=x'} - \int_s^l dx'\,(\partial_{x'} \ln \Gamma_{sA,yA,x'A})|_{y=x'}} \tag{3.185}$$

$$\times e^{+\int_t^l dx'\,(\partial_{x'} \ln \Gamma_{tA,yA,x'A})|_{y=x'}} \tag{3.186}$$

$$= e^{-\int_l^s dx'\,(\partial_{x'} \ln \Gamma_{tA,yA,x'A})|_{y=x'} - \int_s^l dx'\,(\partial_{x'} \ln \Gamma_{sA,yA,x'A})|_{y=x'}} \tag{3.187}$$

$$\overset{(3.86)}{=} e^{-\int_l^s dx'\,(\partial_{x'} \ln \Gamma_{sA,yA,x'A})|_{y=x'} - \int_l^s dx'\,(\partial_{x'} \ln \Gamma_{tA,sA,x'A})|_{y=x'}} \tag{3.188}$$

$$\times e^{-\int_l^s dx' \, (\partial_{x'} \ln \Gamma_{tA,yA,sA})\big|_{y=x'} - \int_s^l dx' \, (\partial_{x'} \ln \Gamma_{sA,yA,x'A})\big|_{y=x'}} \tag{3.189}$$

$$= e^{-\int_l^s dx' \, (\partial_{x'} \ln \Gamma_{tA,sA,x'A})\big|_{y=x'}} \tag{3.190}$$

$$= e^{-\int_l^s dx' \partial_{x'} \ln \Gamma_{tA,sA,x'A}} \tag{3.191}$$

$$\stackrel{(3.84)}{=} \Gamma_{tA,sA,lA}, \tag{3.192}$$

which proves the validity of the consistency relation (3.178).
In order to prove uniqueness we pick $A \in \mathcal{V}$ and assume there is ϕ' also defined on $\operatorname{dom}\phi_A$ and satisfies (3.161) to (3.164). Then we may use (3.161) to conclude that for any $(t,s) \in \operatorname{dom}\phi_A$ there is $\gamma(t,s) \in S^1$ such that

$$\phi_A(t,s) = \phi'(t,s)\gamma(t,s) \tag{3.193}$$

holds true. Picking l such that $(t,s),(s,l),(t,l) \in \operatorname{dom}\phi_A$ and using (3.162) we find

$$\phi'(t,s)\gamma(t,s) = \phi_A(t,s) = \phi_A(t,l)\phi_A(l,s) \tag{3.194}$$

$$= \gamma(t,l)\phi'(t,l)\gamma(l,s)\phi'(l,s) = \gamma(t,l)\gamma(l,s)\phi'(t,s), \tag{3.195}$$

hence we have

$$\gamma(t,s) = \gamma(t,l)\gamma(l,s). \tag{3.196}$$

From property (3.163) we find

$$\gamma(t,t) = 1, \tag{3.197}$$

for any t. Using equation (3.164) we conclude that

$$0 = \partial_t \left. \frac{\phi'(s,t)}{\overline{S}_{sA,tA}} \right|_{t=s} = \partial_t \left. \frac{\phi_A(s,t)\gamma(s,t)}{\overline{S}_{sA,tA}} \right|_{t=s} \tag{3.198}$$

$$= \partial_t \, \gamma(s,t) \left. \frac{\phi_A(s,t)}{\overline{S}_{sA,tA}} \right|_{t=s} = \partial_t \, \gamma(s,t)|_{t=s} + \partial_t \left. \frac{\phi_A(s,t)}{\overline{S}_{sA,tA}} \right|_{t=s} \tag{3.199}$$

$$= \partial_t \, \gamma(s,t)|_{t=s}. \tag{3.200}$$

Finally we find for general $(s,t) \in \operatorname{dom} \phi_A$:

$$\partial_x \gamma(s,x)|_{x=t} = \partial_x(\gamma(s,t)\gamma(t,x))|_{x=t} = \gamma(s,t)\partial_x\gamma(t,x)|_{x=t} = 0. \tag{3.201}$$

So $\gamma(t,s) = 1$ everywhere. We conclude $\phi_A = \phi'$.

□

Lemma 44 (ϕ global existence and uniqueness)**.** *For any $A \in \mathcal{V}$ the map ϕ_A constructed in lemma 43 can be uniquely extended to all of $\mathbb{R}^2$ keeping its defining properties*

$$\forall (t,s) \in \mathbb{R}^2 : \phi_A(t,s) \textit{ is a lift of } S_{tA,sA} \tag{3.202}$$

$$\forall (t,s),(s,l),(l,t) \in \mathbb{R}^2 : \phi_A(t,s)\phi_A(s,l) = \phi_A(t,l) \tag{3.203}$$

$$\forall t \in \mathbb{R} : \phi_A(t,t) = 1 \tag{3.204}$$

$$\forall s \in \mathbb{R} : \partial_t \left.\frac{\phi_A(s,t)}{\overline{S}_{sA,tA}}\right|_{t=s} = 0. \tag{3.205}$$

Proof. Pick $A \in \mathcal{V}$. For $x \in \mathbb{R}$ we define the set

$$U_x := \{y \in \mathbb{R} \mid (xA, yA) \in \operatorname{dom} \overline{S}\}, \tag{3.206}$$

which according to properties 2 and 4 of lemma 35 is an open interval and fulfills that $\bigcup_{x\in\mathbb{R}} U_x \times U_x$ is an open neighbourhood of the diagonal $\{(x,x) \mid x \in \mathbb{R}\}$. Therefore ϕ_A is defined for arguments that are close enough to each other. Since properties (3.205) and (3.204) only concern the behavior of ϕ_A at the diagonal any extension fulfils them. We pick a sequence $(x_k)_{k\in\mathbb{N}} \subset \mathbb{R}$ such that

$$\bigcup_{k\in\mathbb{N}_0} U_{x_k} = \mathbb{R} \tag{3.207}$$

holds and

$$\forall n \in \mathbb{N}_0 : \bigcup_{k=0}^{n} U_{x_k} =: \operatorname{dom}_n \tag{3.208}$$

is an open interval. Please note that such a sequence always exists. We are going to prove that for any $n \in \mathbb{N}_0$ There is a function $\psi_n : \mathrm{dom}_n \times \mathrm{dom}_n \to U(\mathcal{F})$, which satisfies the conditions

$$\forall (t,s) \in \mathrm{dom}_n \times \mathrm{dom}_n : \psi_n(t,s) \text{ is a lift of } S_{tA,sA} \tag{3.209}$$

$$\forall s,k,l \in \mathrm{dom}_n : \psi_n(k,s)\psi_n(s,l) = \psi_n(k,l) \tag{3.210}$$

$$\forall x,y \in \mathrm{dom}_n : (xA,yA) \in \mathrm{dom}\,\overline{S} \Rightarrow \psi_n(x,y) = \phi_A(x,y) \tag{3.211}$$

and is the unique function to do so, i.e. any other function $\tilde{\psi}_n$ fulfilling properties (3.209)-(3.211) possibly being defined on a larger domain coincides with ψ_n on $\mathrm{dom}_n \times \mathrm{dom}_n$.
We start with $\psi_0 = \phi_A$ restricted to $U_{x_0} \times U_{x_0}$. This function is a restriction of ϕ_A and because of lemma 43 it fulfils all of the required properties directly.
For the induction step we pick $t \in \mathrm{dom}_n \cap U_{x_{n+1}}$ and define ψ_{n+1} on the domain $\mathrm{dom}_{n+1} \times \mathrm{dom}_{n+1}$ by

$$\psi_{n+1}(x,y) := \begin{cases} \psi_n(x,y) & \text{for } x,y \in \mathrm{dom}_n \\ \phi_A(x,y) & \text{for } x,y \in U_{x_{n+1}} \\ \psi_n(x,t)\phi_A(t,y) & \text{for } x \in \mathrm{dom}_n, y \in U_{x_{n+1}} \\ \phi_A(x,t)\psi_n(t,y) & \text{for } y \in \mathrm{dom}_n, x \in U_{x_{n+1}}. \end{cases} \tag{3.212}$$

In order to complete the induction step we have to show that ψ_{n+1} is well-defined and fulfils properties (3.209)-(3.211) with n replaced by $n+1$ and is the unique function to do so.
To see that ψ_{n+1} is well-defined we have to check that the cases in the definition agree when they overlap.

1. If we have $x,y \in \mathrm{dom}_n \cap U_{x_{n+1}}$ all four cases overlap; however, the alternative definitions all equal $\phi_A(x,y)$:

$$\begin{aligned} \psi_n(x,y) &\overset{(3.211)}{=} \phi_A(x,y) \overset{(3.162)}{=} \phi_A(x,t)\phi_A(t,y) \\ &\overset{(3.211)}{=} \begin{cases} \psi_n(x,t)\phi_n(t,y) \\ \phi_A(x,t)\psi_n(t,y). \end{cases} \end{aligned} \tag{3.213}$$

2. Furthermore, if we have $x \in \mathrm{dom}_n$, $y \in \mathrm{dom}_n \cap U_{x_{n+1}}$ cases one and three overlap. Here both alternatives are equal to $\psi_n(x, y)$, since $x, y \in \mathrm{dom}_n$ and we obtain:

$$\psi_n(x,y) \overset{(3.210)}{=} \psi_n(x,t)\psi_n(t,y) \overset{(3.211)}{=} \psi_n(x,t)\phi_A(t,y). \quad (3.214)$$

3. Additionally, if $y \in \mathrm{dom}_n$, $x \in \mathrm{dom}_n \cap U_{x_{n+1}}$ cases one and four overlap. Here they are equal to $\psi_n(x, y)$, since $x, y \in \mathrm{dom}_n$ a quick calculation yields:

$$\psi_n(x,y) \overset{(3.210)}{=} \psi_n(x,t)\psi_n(t,y) \overset{(3.211)}{=} \phi_A(x,t)\psi_n(t,y). \quad (3.215)$$

4. Moreover, if we have $y \in U_{x_{n+1}}$, $x \in \mathrm{dom}_n \cap U_z$ cases two and three overlap. Here both candidate definitions are equal to $\phi_A(x, y)$, since $x, t \in U_z$ we arrive at:

$$\phi_A(x,y) \overset{(3.162)}{=} \phi_A(x,t)\phi_A(t,y) \overset{(3.211)}{=} \psi_n(x,t)\phi_A(t,y). \quad (3.216)$$

5. Also, if we have $x \in U_{x_{n+1}}$, $y \in \mathrm{dom}_n \cap U_{x_{n+1}}$ cases two and four overlap. In this case both alternatives are equal to $\phi_A(x, y)$, since $y, t \in U_{x_{n+1}}$ we get:

$$\phi_A(x,y) \overset{(3.162)}{=} \phi_A(x,t)\phi_A(t,y) \overset{(3.211)}{=} \phi_A(x,t)\psi_n(t,y). \quad (3.217)$$

We proceed to show the induction claim, starting with $(3.209)_{n+1}$. By the induction hypothesis we know that $\psi_n(x, y)$ as well as $\phi_A(x, y)$ are lifts of $S_{xA,yA}$ for any (x, y) in their domain of definition. Therefore we have for $x, y \in \mathrm{dom}_n \cup U_{x_{n+1}}$

$$\psi_{n+1}(x,y) = \begin{cases} \psi_n(x,y) & \text{for } x,y \in \mathrm{dom}_n, \\ \phi_A(x,y) & \text{for } x,y \in U_{x_{n+1}}, \\ \psi_n(x,t)\phi_A(t,y) & \text{for } x \in \mathrm{dom}_n, y \in U_{x_{n+1}}, \\ \phi_A(x,t)\psi_n(t,y) & \text{for } y \in \mathrm{dom}_n, x \in U_{x_{n+1}}, \end{cases} \quad (3.212)$$

where each of the lines is a lift of $S_{xA,yA}$ whenever the expression is defined.
Equation $(3.210)_{n+1}$ we will again show in a case by case manner depending on the s, k and l:

1. $s, k, l \in \mathrm{dom}_n$: $(3.210)_{n+1}$ follows directly from the induction hypothesis;

2. $s, k \in \mathrm{dom}_n$ and $l \in U_{x_{n+1}}$:
$$\begin{aligned}\psi_{n+1}(s,k)\psi_{n+1}(k,l) &= \psi_n(s,k)\psi_n(k,t)\phi_A(t,l)\\ &\overset{(3.210)}{=} \psi_n(s,t)\phi_A(t,l) = \psi_{n+1}(s,l),\end{aligned}\tag{3.218}$$

3. $s, l \in \mathrm{dom}_n$ and $k \in U_{x_{n+1}}$:
$$\begin{aligned}\psi_{n+1}(s,k)\psi_{n+1}(k,l) &= \psi_n(s,t)\phi_A(t,k)\phi_A(t,k)\psi_n(t,l)\\ &\overset{(3.163),(3.162)}{=} \psi_n(s,t)\psi_n(t,l) \overset{(3.210)}{=} \psi_n(s,l) = \psi_{n+1}(s,l),\end{aligned}$$

4. $s \in \mathrm{dom}_n$ and $k, l \in U_{x_{n+1}}$:
$$\begin{aligned}\psi_{n+1}(s,k)\psi_{n+1}(k,l) &= \psi_n(s,t)\phi_A(t,k)\phi_A(k,l)\\ &\overset{(3.162)}{=} \psi_n(s,t)\phi_A(t,l) = \psi_{n+1}(s,l),\end{aligned}$$

5. $k, l \in \mathrm{dom}_n$ and $s \in U_{x_{n+1}}$:
$$\begin{aligned}\psi_{n+1}(s,k)\psi_{n+1}(k,l) &= \phi_A(s,t)\psi_n(t,k)\psi_n(k,l)\\ &\overset{(3.210)}{=} \phi_A(s,t)\psi_n(t,l) = \psi_{n+1}(s,l),\end{aligned}$$

6. $k \in \mathrm{dom}_n$ and $s, l \in U_{x_{n+1}}$:
$$\begin{aligned}\psi_{n+1}(s,k)\psi_{n+1}(k,l) &= \phi_A(s,t)\psi_n(t,k)\psi_n(k,t)\phi_A(t,l)\\ &\overset{(3.210)}{=} \phi_A(s,t)\psi(t,t)\phi_A(t,l) \overset{(3.211),(3.163)}{=} \phi_A(s,t)\phi_A(t,l)\\ &\overset{(3.162)}{=} \phi_A(s,l) = \psi_{n+1}(s,l),\end{aligned}$$

7. $l \in \mathrm{dom}_n$ and $s, k \in U_{x_{n+1}}$:

$$\begin{aligned}\psi_{n+1}(s,k)\psi_{n+1}(k,l) &= \phi_A(s,k)\phi_A(k,t)\psi_n(t,l) \\ &\overset{(3.162)}{=} \phi_A(s,t)\psi_n(t,l) = \psi_{n+1}(s,l),\end{aligned}$$

8. and if $s, k, l \in U_z$:

$$\begin{aligned}\psi_{n+1}(s,k)\psi_{n+1}(k,l) &= \phi_A(s,k)\phi_A(k,l) \\ &\overset{(3.162)}{=} \phi_A(s,l) = \psi_{n+1}(s,l).\end{aligned}$$

To see $(3.211)_{n+1}$, i.e. that ψ_{n+1} is coincides with ϕ_A where both functions are defined pick $x, y \in \mathrm{dom}_{n+1}$ such that $(xA, yA) \in \mathrm{dom}\,\overline{S})$. Recall the definition of ψ_{n+1}

$$\psi_{n+1}(x,y) = \begin{cases} \psi_n(x,y) & \text{for } x, y \in \mathrm{dom}_n, \\ \phi_A(x,y) & \text{for } x, y \in U_{x_{n+1}}, \\ \psi_n(x,t)\phi_A(t,y) & \text{for } x \in \mathrm{dom}_n, y \in U_{x_{n+1}}, \\ \phi_A(x,t)\psi_n(t,y) & \text{for } y \in \mathrm{dom}_n, x \in U_{x_{n+1}}. \end{cases} \tag{3.212}$$

Therefore if $x, y \in \mathrm{dom}_n$ we may use the induction hypothesis directly and if $x, y \in U_{x_{n+1}}$ we also arrived a the claim we want to prove. Excluding these cases, we are left with rows number three and four of this definition with the restriction

3. $x \in \mathrm{dom}_n \setminus U_{x_{n+1}}, y \in U_{x_{n+1}} \setminus \mathrm{dom}_n$ or

4. $y \in \mathrm{dom}_n \setminus U_{x_{n+1}}, x \in U_{x_{n+1}} \setminus \mathrm{dom}_n$,

respectively. Because t satisfies $t \in \mathrm{dom}_n \cap U_{x_{n+1}}$, we have in both cases $t \in \overline{x\,y}$. By using property 4 of lemma 35 we infer from $(xA, yA) \in \mathrm{dom}\,\overline{S}$ that in both cases $(xA, tA), (tA, yA) \in \mathrm{dom}\,\overline{S}$ also holds. Hence we may apply the induction hypothesis $(3.211)_n$.

It remains to show uniqueness. So let $\tilde{\psi}_{n+1}$ be defined on $\mathrm{dom}_{n+1} \times \mathrm{dom}_{n+1}$ fulfil

$$\forall (t,s) \in \mathrm{dom}_{n+1} \times \mathrm{dom}_{n+1} : \tilde{\psi}(t,s) \text{ is a lift of } S_{tA,sA}, \quad (3.209_{\tilde{\psi}})$$

$$\forall s,k,l \in \mathbb{R} : \tilde{\psi}(k,s)\tilde{\psi}(s,l) = \tilde{\psi}(k,l), \quad (3.210_{\tilde{\psi}})$$

$$\forall (x,y) \in \mathrm{dom}_{n+1} : (xA, yA) \in \mathrm{dom}\,\overline{S} \Rightarrow \tilde{\psi}(x,y) = \phi_A(x,y). \quad (3.211_{\tilde{\psi}})$$

Now pick $x, y \in \mathrm{dom}_{n+1}$. We proceed in a case by case manner

1. If $x, y \in \mathrm{dom}_n$ holds, then $\psi_{n+1}(x,y) = \tilde{\psi}_{n+1}(x,y)$ follows directly from the induction hypothesis.

2. Similarly if $x, y \in U_{x_{n+1}}$ holds, we have

$$\psi_{n+1}(x,y) = \phi_A(x,y) = \tilde{\psi}_{n+1}(x,y). \tag{3.219}$$

3. Additionally, if $x \in \mathrm{dom}_n, y \in U_{x_{n+1}}$ holds, then

$$\psi_{n+1}(x,y) \overset{(3.212)}{=} \psi_n(x,t)\phi_A(t,y) \tag{3.220}$$

$$\overset{t \in \mathrm{dom}_n \cap U_{x_{n+1}}}{=} \tilde{\psi}_{n+1}(x,t)\tilde{\psi}_{n+1}(t,y) \overset{(3.210_{\tilde{\psi}})}{=} \tilde{\psi}_{n+1}(x,y) \tag{3.221}$$

is satisfied.

4. Conversely, if $y \in \mathrm{dom}_n, x \in U_{x_{n+1}}$ holds, we may use the same calculation to obtain

$$\psi_{n+1}(x,y) \overset{(3.212)}{=} \phi_A(x,t)\psi_n(t,y) \tag{3.222}$$

$$\overset{t \in \mathrm{dom}_n \cap U_{x_{n+1}}}{=} \tilde{\psi}_{n+1}(x,t)\tilde{\psi}_{n+1}(t,y) \overset{(3.210_{\tilde{\psi}})}{=} \tilde{\psi}_{n+1}(x,y). \tag{3.223}$$

Now we have established a unique extension ψ_n of ϕ_A fulfilling properties (3.209)-(3.211).

We know that for each $n \in \mathbb{N}$ the function $\psi_{n+1} : \mathrm{dom}^2_{n+1} \to U(\mathcal{F})$ is an extention of $\psi_{n+1} : \mathrm{dom}^2_n \to U(\mathcal{F})$. Furthermore, the sets dom_n cover $\mathbb{R}$ according to equation (3.207). Consequently there is a unique common extension, by small abuse of notation again called $\phi_A : \mathbb{R}^2 \to U(\mathcal{F})$, of all ψ_n. This function fulfills the claim (3.202)-(3.205), because any $t, l, s \in \mathbb{R}$ are contained in some dom_n.

□

Lemma 44 enables us to define a global lift.

Definition 45 (global lift)**.** *For any $A \in \mathcal{V}$ we define*

$$\hat{S}_{0,A} := \phi_A(0,1). \tag{3.224}$$

Using lemma 44 we are now in a position to prove theorem 42.

proof of theorem 42. The operator $\hat{S}$ fulfils the claimed differential equation (3.160) due to the global multiplication property (3.203) and the differential equation (3.205). Its uniqueness is inherited from the uniqueness of ϕ_A for $A \in \mathcal{V}$ from lemma 44. □

Definition 46 (relative phase)**.** *Let $(A, B) \in \mathrm{dom}\,\overline{S}$, we define $z(A, B) \in S^1$ by*

$$z(A,B) := \frac{\hat{S}_{0,B}}{\hat{S}_{0,A}\overline{S}_{A,B}}. \tag{3.225}$$

Please note that for such A, B the lift $\bar{S}_{A,B}$ is well-defined. This means that the product in the denominator is a lift of $S_{0,B}$ and according to definition 45 the ratio is well-defined.

Remark 47. *The function z defined here is an extension of the funtion z appearing locally in the the proof of lemma 43, cf. formula* (3.165). *Please note that z is smooth when restricted to $\mathcal{W}^2 \cap \mathrm{dom}\,\overline{S}$ for any finite dimensional subspace $\mathcal{W} \subseteq \mathcal{V}$, since $\hat{S}$ is smooth as a solution to* *a differential equation with smooth initial conditions. The parameter $\overline{S}$ appearing in the defining differential equation of $\hat{S}$ is smooth since it is directly constructed in terms of the one particle scattering operator which is smooth due to chapter 4.1 in the appendix.*

Lemma 48 (properties of the relative phase)**.** *For all $(A, F), (F, G), (G, A) \in \operatorname{dom} \overline{S}$, as well as or all $H, K \in \mathcal{V}$, we have*

$$z(A, F) = z(F, A)^{-1} \tag{3.226}$$

$$z(F, A) z(A, G) z(G, F) = \Gamma_{F,A,G} \tag{3.227}$$

$$\partial_H \partial_K \ln z(A + H, A + K) = c_A(H, K). \tag{3.228}$$

Proof. Pick $A, F, G \in \mathcal{V}$ as in the lemma. We start off by analysing

$$\hat{S}_{0,F} \overline{S}_{F,G} \overset{(3.225)}{=} z(A, F) \hat{S}_{0,A} \overline{S}_{A,F} \overline{S}_{F,G} \tag{3.229}$$

$$\overset{(3.87)}{=} z(A, F) \Gamma^{-1}_{A,F,G} \hat{S}_{0,A} \overline{S}_{A,G}. \tag{3.230}$$

Exchanging A and F in this equation yields

$$\hat{S}_{0,A} \overline{S}_{A,G} = z(F, A) \Gamma^{-1}_{F,A,G} \hat{S}_{0,F} \overline{S}_{F,G}. \tag{3.231}$$

This is equivalent to

$$\hat{S}_{0,F} \overline{S}_{F,G} = z(F, A)^{-1} \Gamma_{F,A,G} \hat{S}_{0,A} \overline{S}_{A,G} \ . \tag{3.232}$$

Comparing the last equation with formula (3.230) and taking the permutation properties (3.83) of Γ into account this implies that

$$z(A, F) = z(F, A)^{-1} \tag{3.233}$$

holds true. Equation (3.230) solved for $\hat{S}_{0,A} \overline{S}_{A,G}$ also gives us

$$\hat{S}_{0,G} \overset{(3.225)}{=} z(A, G) \hat{S}_{0,A} \overline{S}_{A,G} \tag{3.234}$$

$$\overset{(3.230)}{=} z(A, G) z(A, F)^{-1} \Gamma_{A,F,G} \hat{S}_{0,F} \overline{S}_{F,G}. \tag{3.235}$$

The latter equation compared with

$$\hat{S}_{0,G} \overset{(3.225)}{=} z(F,G)\hat{S}_{0,F}\overline{S}_{F,G}, \tag{3.236}$$

yields a direct connection between Γ and z:

$$\frac{z(A,G)}{z(A,F)}\Gamma_{A,F,G} = z(F,G), \tag{3.237}$$

which we rewrite using the antisymmetry (3.226) of z as

$$\Gamma_{A,F,G} = z(F,G)z(A,F)z(G,A). \tag{3.238}$$

Finally, in this equation, we substitute $F = A + \varepsilon_1 H$ as well as $G = A + \varepsilon_2 K$, where $\varepsilon_1, \varepsilon_2$ is small enough so that z and Γ are still well-defined. Then we take the second logarithmic derivative to find

$$\begin{aligned}\partial_{\varepsilon_1}\partial_{\varepsilon_2} \ln z(A+\varepsilon_1 H, A+\varepsilon_2 K) &= \partial_{\varepsilon_1}\partial_{\varepsilon_2} \ln \Gamma_{A,A+\varepsilon_1 H, A+\varepsilon_2 K} \\ &\overset{(3.88)}{=} c_A(H,K).\end{aligned} \tag{3.239}$$

□

So we find that c_A is the second mixed logarithmic derivative of z. In the following we will characterise z more thoroughly by c and c^+.

Definition 49 (p-forms of four potentials, phase integral)**.** *For $p \in \mathbb{N}$, we introduce the set Ω^p of p-forms to consist of all maps $\omega : \mathcal{V}\times\mathcal{V}^p \to \mathbb{C}$ such that ω is linear and antisymmetric in its p last arguments and smooth in its first argument when restricted to any finite dimensional subspace of $\mathcal{V}$.*
Additionally, we define the 1-form $\chi \in \Omega^1$ by

$$\chi_A(B) := \partial_B \ln z(A, A+B) \tag{3.240}$$

for all $A, B \in \mathcal{V}$. Furthermore, for $p \in \mathbb{N}$ and any differential form $\omega \in \Omega^p$, we define its exterior derivative, $d\omega \in \Omega^{p+1}$ by

$$(d\omega)_A(B_1, \ldots, B_{p+1}) := \sum_{k=1}^{p+1} (-1)^{k+1} \partial_{B_k} \omega_{A+B_k}(B_1, \ldots, \widehat{B_k}, \ldots, B_{p+1}), \tag{3.241}$$

for $A, B_1, \ldots, B_{p+1} \in \mathcal{V}$, where the notation $\widehat{B_k}$ denotes that B_k is dropped as an argument.

Lemma 50 (connection between c and the relative phase)**.** *The differential form χ fulfils*

$$(d\chi)_A(F, G) = 2c_A(F, G) \tag{3.242}$$

for all $A, F, G \in \mathcal{V}$.

Proof. Pick $A, F, G \in \mathcal{V}$, we calculate

$$\begin{aligned}(d\chi)_A(F, G) = \partial_F \partial_G \ln z(A + F, A + F + G)& \\ - \partial_F \partial_G \ln z(A + G, A + F + G)& \qquad (3.243)\\ = \partial_F \partial_G (\ln z(A, A + F + G) + \ln z(A + F, A + G))& \qquad (3.244)\\ - \partial_F \partial_G (\ln z(A, A + F + G) + \ln z(A + G, A + F))& \qquad (3.245)\\ \overset{(3.226)}{=} 2\partial_F \partial_G \ln z(A + F, A + G) \overset{(3.228)}{=} 2c_A(F, G).& \qquad (3.246)\end{aligned}$$

□

Now since $dc = 0$, we might use Poincaré's lemma as a method independent of z to construct a differential form ω such that $d\omega = c$. In order to execute this plan, we first need to prove Poincaré's lemma for our setting:

Lemma 51 (Poincaré)**.** *Let $\omega \in \Omega^p$ for $p \in \mathbb{N}$ be closed, i.e. $d\omega = 0$. Then ω is also exact, more precisely we have*

$$\omega = d \int_0^1 \iota_t^* i_X f^* \omega dt, \tag{3.247}$$

where X, ι_t for $t \in \mathbb{R}$ and f are given by

$$\begin{aligned}
&X : \mathbb{R} \times \mathcal{V} \to \mathbb{R} \times \mathcal{V}, && (3.248)\\
&\qquad (t, B) \mapsto (1, 0) && (3.249)\\
&\forall t \in \mathbb{R} : \iota_t : \mathcal{V} \to \mathbb{R} \times \mathcal{V}, && (3.250)\\
&\qquad B \mapsto (t, B) && (3.251)\\
&f : \mathbb{R} \times \mathcal{V} \mapsto \mathcal{V}, && (3.252)\\
&\qquad (t, B) \mapsto tB && (3.253)\\
&i_W : \Omega^p \to \Omega^{p-1}, && (3.254)\\
&\qquad \omega \mapsto ((A; Y_1, \ldots, Y_{p-1}) \mapsto \omega_A(W, Y_1, \ldots, Y_{p-1})) && (3.255)
\end{aligned}$$

For a proof see section 4.2 of the appendix. This lemma gives the next definition meaning.

Definition 52 (antiderivative of a closed p form)**.** *For a closed exterior form $\omega \in \Omega^p$ we define the form $\Pi[\omega]$*

$$\Omega^{p-1} \ni \Pi[\omega] := \int_0^1 \iota_t^* i_X f^* \omega dt. \tag{3.256}$$

For $A, B_1, \ldots, B_{p-1} \in \mathcal{V}$ it takes the form

$$\Pi[\omega]_A(B_1, \ldots, B_{p-1}) = \int_0^1 t^{p-1} \omega_{tA}(A, B_1, \ldots, B_{p-1}) dt. \tag{3.257}$$

By lemma 51 we know $d\Pi[\omega] = \omega$ if $d\omega = 0$.

Now we found two one forms each produces c when the exterior derivative is taken. The next lemma informs us about their relationship.

Lemma 53 (inversion of lemma 50)**.** *The following equality holds*

$$\chi = 2\Pi[c]. \tag{3.258}$$

Proof. By definition 52 of Π and lemma 50 we have $d(\chi - 2\Pi[c]) = 0$. Hence, by the Poincaré lemma 51, we know that there is $v : \mathcal{V} \to \mathbb{R}$ such that

$$dv = \chi - 2\Pi[c] \tag{3.259}$$

holds. Using the definition 46 of z, the parallel transport equation (3.160) translates into the following ODE for z:

$$\partial_B \ln z(0, B) = 0, \quad \partial_\varepsilon \ln z(A, (1+\varepsilon)A)|_{\varepsilon=0} = 0 \tag{3.260}$$

for all $A, B \in \mathcal{V}$. Therefore have

$$\chi_0(B) = 0 = \Pi[c]_0(B), \quad \chi_A(A) = 0 = \Pi[c]_A(A), \tag{3.261}$$

which implies

$$\partial_\varepsilon v_{\varepsilon A}|_{\varepsilon=0} = 0, \quad \partial_\varepsilon v_{A+\varepsilon A}|_{\varepsilon=0} = 0. \tag{3.262}$$

In conclusion, v is constant. □

From this point on we will assume the existence of a function c^+ fulfilling (3.69),(3.70) and (3.71). Recall property (3.70):

$$\forall A, F, G, H : \partial_H c^+_{A+H}(F, G) = \partial_G c^+_{A+G}(F, H). \tag{3.263}$$

For a fixed $F \in \mathcal{V}$, this condition can be read as $d(c^+_\cdot(F, \cdot)) = 0$. As a consequence we can apply Poincaré's lemma 51 to define a one form.

Definition 54 (integral of the causal splitting)**.** *For any $A, F \in \mathcal{V}$, we define*

$$\beta_A(F) := 2\Pi[c^+_\cdot(F, \cdot)]_A. \tag{3.264}$$

Lemma 55 (relation between the integral of the causal splitting and the phase integral). *The following two equations hold:*

$$d\beta = -2c, \tag{3.265}$$
$$d(\beta + \chi) = 0. \tag{3.266}$$

Proof. We start with the exterior derivative of β. Pick $A, F, G \in \mathcal{V}$:

$$d\beta_A(F, G) = \partial_F \beta_{A+F}(G) - \partial_G \beta_{A+G}(F) \tag{3.267}$$
$$= d\Big(2\Pi[c_\cdot^+(G, \cdot)]\Big)_A(F) - d\Big(2\Pi[c_\cdot^+(F, \cdot)]\Big)_A(G) \tag{3.268}$$
$$= 2c_A^+(G, F) - 2c_A^+(F, G) \overset{(3.69)}{=} -2c_A(F, G). \tag{3.269}$$

This proves the first equality. The second equality follows directly by $d\chi = 2c$. □

Definition 56 (corrected lift). *Since $\beta + \chi$ is closed, we may use lemma 51 again to define the phase*

$$\alpha := \Pi[\beta + \chi]. \tag{3.270}$$

Furthermore, for all $A, B \in \mathcal{V}$ we define the corrected second quantised scattering operator

$$\tilde{S}_{0,A} := e^{-\alpha_A} \hat{S}_{0,A}, \tag{3.271}$$
$$\tilde{S}_{A,B} := \tilde{S}_{0,A}^{-1} \tilde{S}_{0,B}. \tag{3.272}$$

Using this definition one immediately gets:

Corollary 57 (group structure of the corrected lift). *We have $\tilde{S}_{A,B}\tilde{S}_{B,C} = \tilde{S}_{A,C}$ for all $A, B, C \in \mathcal{V}$.*

Theorem 58 (causality of the corrected lift). *The corrected second quantised scattering operator fulfils the following causality condition for all $A, F, G \in \mathcal{V}$ such that $F \prec G$:*

$$\tilde{S}_{A,A+F} = \tilde{S}_{A+G,A+G+F}. \tag{3.273}$$

Proof. Let $A, F, G \in \mathcal{V}$ such that $F \prec G$. For the first quantised scattering operator we have

$$S_{A+G,A+G+F} = S_{A,A+F}, \tag{3.274}$$

which we proved in remark 39. So that by definition of $\overline{S}$ we obtain

$$\overline{S}_{A+G,A+G+F} = \overline{S}_{A,A+F}. \tag{3.275}$$

Therefore any lift this equality is true up to a phase, meaning that

$$f(A,F,G) := \frac{\tilde{S}_{A+G,A+G+F}}{\tilde{S}_{A,A+F}} \tag{3.276}$$

is well-defined. We see immediately

$$f(A,0,G) = 1 = f(A,F,0). \tag{3.277}$$

Pick $F_1, F_2 \prec G_1, G_2$. We abbreviate $F = F_1 + F_2, G = G_1 + G_2$ and we calculate

$$f(A,F,G) = \frac{\tilde{S}_{A+G,A+F+G}}{\tilde{S}_{A,A+F}} \tag{3.278}$$

$$= \frac{\tilde{S}_{A+G,A+F+G}}{\tilde{S}_{A+G_1,A+G_1+F}} \frac{\tilde{S}_{A+G_1,A+G_1+F}}{\tilde{S}_{A,A+F}} \tag{3.279}$$

$$= \frac{\tilde{S}_{A+G,A+G+F_1}\tilde{S}_{A+G+F_1,A+F+G}}{\tilde{S}_{A+G_1,A+F_1+G_1}\tilde{S}_{A+G_1+F_1,A+G_1+F}} \frac{\tilde{S}_{A+G_1,A+G_1+F}}{\tilde{S}_{A,A+F}} \tag{3.280}$$

$$= \frac{\tilde{S}_{A+G,A+G+F_1}}{\tilde{S}_{A+G_1,A+F_1+G_1}} \frac{\tilde{S}_{A+G+F_1,A+F+G}}{\tilde{S}_{A+G_1+F_1,A+G_1+F}} f(A,G_1,F_1+F_2) \tag{3.281}$$

$$= f(A+G_1,F_1,G_2)f(A+G_1+F_1,G_2,F_2)f(A,G_1,F_1+F_2). \tag{3.282}$$

Taking the mixed logarithmic derivative we find:

$$\partial_{F_2}\partial_{G_2}\ln f(A, F_1 + F_2, G_1 + G_2) = \partial_{F_2}\partial_{G_2}\ln f(A + F_1 + G_1, F_2, G_2). \tag{3.283}$$

Next we pick $F_2 = \alpha_1 F_1$ and $G_2 = \alpha_2 G_1$ for $\alpha_1, \alpha_2 \in \mathbb{R}^+$ small enough so that $(A + (1+\alpha_1)F_1 + (1+\alpha_2)G_1, A + F_1 + G_1), (A + (1+\alpha_1)F_1 + (1+\alpha_2)G_1, A + F_1 + (1+\alpha_2)G_1), (A + (1+\alpha_1)F_1 + (1+\alpha_2)G_1, A + (1+\alpha_1)F_1 + G_1) \in \operatorname{dom}\overline{S}$ holds. We abbreviate $A' = A + G_1 + F_1$, use the definition of z (3.225) and compute

$$\begin{aligned}
&f(A', F_2, G_2)\\
&\overset{(3.271)}{=} \exp(-\alpha_{A'+F_2+G_2} + \alpha_{A'+G_2} + \alpha_{A'+F_2} - \alpha_{A'})\\
&\qquad\qquad \times \frac{\hat{S}^{-1}_{0,A'+G_2}\hat{S}_{0,A'+G_2+F_2}}{\hat{S}^{-1}_{0,A'}\hat{S}_{0,A'+F_2}} &(3.284)\\
&\overset{(3.225)}{=} \exp(-\alpha_{A'+F_2+G_2} + \alpha_{A'+G_2} + \alpha_{A'+F_2} - \alpha_{A'})\\
&\qquad\qquad \times \frac{z(A'+G_2, A'+G_2+F_2)}{z(A', A'+F_2)}\frac{\overline{S}_{A'+G_2,A'+G_2+F_2}}{\overline{S}_{A',A'+F_2}} &(3.285)\\
&\overset{F_2\leq G_2}{=} \exp(-\alpha_{A'+F_2+G_2} + \alpha_{A'+G_2} + \alpha_{A'+F_2} - \alpha_{A'})\\
&\qquad\qquad \times \frac{z(A'+G_2, A'+G_2+F_2)}{z(A', A'+F_2)} &(3.286)
\end{aligned}$$

Most of the factors do not depend on both F_2 and G_2, so taking the mixed logarithmic derivative things simplify:

$$\begin{aligned}
&\partial_{G_2}\partial_{F_2}\ln f(A', F_2, G_2) =\\
&\partial_{G_2}\partial_{F_2}(-\alpha_{A'+F_2+G_2} + \ln z(A'+G_2, A'+G_2+F_2)) &(3.287)\\
&\overset{(3.270),(3.240)}{=} \partial_{G_2}(-\beta_{A'+G_2}(F_2) - \chi_{A'+G_2}(F_2) + \chi_{A'+G_2}(F_2)) &(3.288)\\
&\overset{(3.264)}{=} -2c^+_{A'}(F_2, G_2) \overset{F_2<G_2,(3.71)}{=} 0. &(3.289)
\end{aligned}$$

So by (3.283) we also have

$$\partial_{F_2}\partial_{G_2}\ln f(A,F_1+F_2,G_1+G_2)=0 \tag{3.290}$$
$$=\partial_{\alpha_1}\partial_{\alpha_2}\ln f(A,F_1(1+\alpha_1),G_1(1+\alpha_2)). \tag{3.291}$$

Using this then we can integrate and obtain

$$0=\int_{-1}^{0}d\alpha_1\int_{-1}^{0}d\alpha_2\partial_{\alpha_1}\partial_{\alpha_2}\ln f(A,F_1(1+\alpha_1),G_1(1+\alpha_2)) \tag{3.292}$$
$$=\ln f(A,F_1,G_1)-\ln f(A,0,G_1)-\ln f(A,F_1,0) \tag{3.293}$$
$$+\ln f(A,0,0)$$
$$\overset{(3.277)}{=}\ln f(A,F_1,G_1). \tag{3.294}$$

Recalling equation (3.276), the definition of f, this ends our proof. □

Next, we investigate the current associated with $\tilde{S}$.

Theorem 59 (evaluation of the current of the corrected lift)**.** *For general $A,F\in\mathcal{V}$ we have*

$$j_A^{\tilde{S}}(F)=-i\beta_A(F). \tag{3.295}$$

So in particular for $G\in\mathcal{V}$

$$\partial_G j_{A+G}^{\tilde{S}}(F)=-2ic_A^{+}(F,G). \tag{3.296}$$

holds.

Proof. Pick $A,F\in\mathcal{V}$ as in the theorem. We calculate

$$i\partial_F\ln\left\langle\Omega,\tilde{S}_{A,A+F}\Omega\right\rangle \tag{3.297}$$
$$\overset{(3.271)}{=} i\partial_F\left(-\alpha_{A+F}-\alpha_A+\ln\left\langle\Omega,\hat{S}_{0,A}^{-1}\hat{S}_{0,A+F}\Omega\right\rangle\right) \tag{3.298}$$
$$\overset{(3.225)}{=} i\partial_F\left(-\alpha_{A+F}+\ln z(A,A+F)+\ln\left\langle\Omega,\overline{S}_{A,A+F}\Omega\right\rangle\right) \tag{3.299}$$

The last summand vanishes, as can be seen by the following calculation

$$\partial_F \ln \left\langle \Omega, \overline{S}_{A,A+F}\Omega \right\rangle \tag{3.300}$$

$$= i\partial_F \ln \det_{\mathcal{H}^-}(P^- S_{A,A+F} P^- \,\mathrm{AG}(P^- S_{A,A+F} P^-)^{-1}) \tag{3.301}$$

$$\overset{(3.50)}{=} i\partial_F \ln \det_{\mathcal{H}^-} |P^- S_{A,A+F} P^-| \tag{3.302}$$

$$= \frac{i}{2}\partial_F \ln \det_{\mathcal{H}^-}((P^- S_{A,A+F} P^-)^* P^- S_{A,A+F} P^-) \tag{3.303}$$

$$= \frac{i}{2}\partial_F \det_{\mathcal{H}^-}(P^- S_{A+F,A} P^- S_{A,A+F} P^-) \tag{3.304}$$

$$= \frac{i}{2}\operatorname{tr}(\partial_F P^- S_{A+F,A} P^- S_{A,A+F} P^-) \tag{3.305}$$

$$= \frac{i}{2}\operatorname{tr}(\partial_F P^- S_{A,A+F} P^- + \partial_F P^- S_{A+F,A} P^-) = 0 \tag{3.306}$$

where we made use of (3.143). Theorem 89 serves to justify the necessary regularity. So we are left with

$$j_A(F) = i\partial_F(-\alpha_{A+F} + \ln z(A, A+F)) \tag{3.307}$$

$$= i(-\beta_A(F) - \chi_A(F) + \chi_A(F)) = -i\beta_A(F). \tag{3.308}$$

Finally by taking the derivative with respect to $G \in \mathcal{V}$ and using the definition of β we find

$$\partial_G j_{A+G}(F) = -2ic_A^+(F, G). \tag{3.309}$$

□

proof of theorem 38. The operator *tildeS* constructed in this section fulfills properties (3.73) and (3.74) by corollary 57 and theorem 58. The characterisation of the current (3.75) follows form theorem 59 and the properties of c^+, (3.69) to (3.71). □

3.4 Analyticity of the Scattering Operator

In this chapter we will present a relatively simple formula for the second quantised scattering operator in terms of the one-particle scattering operator. This formula is valid for small external fields, where "small" will be made precise later. The formula has implications for the analyticity of the second quantised scattering operator for general external fields, which we will also present. Also this chapter is concerned with the scattering regime, hence we will be working with a Hilbert space $\mathcal{H}$ which can be thought $\mathcal{H}_\Sigma$, where Σ is a Cauchy surface prior to any external field we are going to consider in this section, and the standard polarisation and its projectors $P^\pm$. The shorthand defined next will turn out to be useful. Let $B : \mathcal{H} \to \mathcal{H}$ be linear and bounded, then we introduce

$$B_{\#,\tilde{\#}} = P^{\#} B P^{\tilde{\#}}, \tag{3.310}$$

where $\#, \tilde{\#} \in \{+,-\}$ holds. Recall the definition of Fock space in this setting

$$\mathcal{F} := \bigoplus_{m,p=0}^{\infty} \left(\mathcal{H}^+\right)^{\wedge m} \otimes \left(\overline{\mathcal{H}^-}\right)^{\wedge p}, \tag{3.311}$$

where $\bigoplus$ is the Hilbert space direct sum. We denote the sectors of Fock space of fixed particle numbers by $\mathcal{F}_{m,p}$. The element of $\mathcal{F}_{0,0}$ of norm 1 will be denoted by Ω.
The annihilation operator a acts on an arbitrary sector of Fock space $\mathcal{F}_{m,p}$, for any $m, p \in \mathbb{N}_0$ with the operator type

$$a : \overline{\mathcal{H}} \times \mathcal{F}_{m,p} \to \mathcal{F}_{m-1,p} \oplus \mathcal{F}_{m,p+1}, \tag{3.312}$$

where the second argument is usually not included in the parenthesis, as is common practice for bounded operators on a Hilbert space. We start out by defining a on elements of $\{\bigwedge_{l=1}^{m} \varphi_l \otimes \bigwedge_{c=1}^{p} \phi_c \mid \forall c : \varphi_c \in \mathcal{H}^+, \phi_c \in \mathcal{H}^-\}$ which spans a dense subset of $\mathcal{F}_{m,p}$, then one continues this operator uniquely by linearity and finally by the bounded linear extension theorem to all of $\mathcal{F}_{m,p}$ and then again by linearity to all of $\overline{\mathcal{H}} \otimes \mathcal{F}_{m,p}$.

$$a(\phi) \bigwedge_{l=1}^{m} \varphi_l \otimes \bigwedge_{c=1}^{p} \phi_c \tag{3.313}$$

$$= \sum_{k=1}^{m} (-1)^{1+k} \langle P^+\phi, \varphi_k \rangle \bigwedge_{\substack{l=1 \\ l \neq k}}^{m} \varphi_l \otimes \bigwedge_{c=1}^{p} \phi_c + \bigwedge_{l=1}^{m} \varphi_l \otimes P^-\phi \wedge \bigwedge_{c=1}^{p} \phi_c \tag{3.314}$$

where $\langle , \rangle$ denotes that the scalar product of $\mathcal{H}$. The first summand on the right hand side is taken to vanish for $m = 0$. The operator norm of a is given by

$$\|a(\phi)\| = \|\phi\|. \tag{3.315}$$

The operators a and its adjoint a^* fulfil the canonical anticommutation relations:

$$\forall \phi, \psi : \{a(\phi), a^*(\psi)\} = a(\phi)a^*(\psi) + a^*(\psi)a(\phi) = \langle \phi, \psi \rangle \tag{3.316}$$

$$\forall \phi, \psi : \{a^*(\phi), a^*(\psi)\} = \{a(\phi), a(\psi)\} = 0. \tag{3.317}$$

Now for the construction of the second quantised S-matrix please recall the lift condition

$$\forall \phi \in \mathcal{H} : \quad \tilde{S}^A \circ a(\phi) = a\left(S^A \phi\right) \circ \tilde{S}^A, \tag{lift condition}$$

which is to be satisfied by any lift $\tilde{S}^A$ of the one-particle scattering matrix S^A.

In the appendix we carry out an explicit, albiet heuristic, construction of a power series expression of a lift of S^A in chapter 4.3 that culminates in the formula which will be directly verified in this chapter.
In order to state this formula, we have to introduce some more notation.

3.4.1 Differential second quantisation

Let $B : \mathcal{H} \to \mathcal{H}$ be a bounded operator on $\mathcal{H}$, such that iB is self adjoint and B_{+-} is a Hilbert-Schmidt operator. We would like to construct a version $\mathrm{d}\Gamma(B)$ of B that acts on Fock space and also is skew adjoint. The proof of the skew adjointness of $\mathrm{d}\Gamma(B)$ is a bit lengthy, as is typical of such proofs. Even though one might speed it up a little by using the tools available e.g. in [8] the author is of the opinion that it is instructive to give a direct proof. In this section we will associate with every set $C \subset \mathbb{N}$ such that $|C| < \infty$ the sequence $(C_k)_{1\leqslant k\leqslant |C|}$ such that

$$\forall 1 \leqslant k \leqslant |C| : C_k \in C \tag{3.318}$$

$$\forall 1 \leqslant k < l \leqslant |C| : C_k < C_l \tag{3.319}$$

hold. This notation is confined to within the proofs of this section. The strategy of this section is to construct an operator in two steps that is essentially self adjoint on the domain of the Fock space of finitely many particles, a dense subset of Fock space. It is denoted by

Definition 60.

$$\mathcal{F}' := \bigoplus_{m,p=0}^{\infty} \mathcal{F}_{m,p}, \tag{3.320}$$

where $\bigoplus$ *refers to the algebraic direct sum. Furthermore, we define*

$$\mathcal{F}^0 \subset \mathcal{F}' \tag{3.321}$$

such that for each element $\alpha \in \mathcal{F}^0$ there is a basis an ONB $(\tilde{\varphi}_k)_{k\in\mathbb{N}}$ of $\mathcal{H}^+$ and an ONB $(\tilde{\varphi}_{-k})_{k\in\mathbb{N}}$ of $\mathcal{H}^-$ such that

$$\alpha \in \operatorname{span}\Bigg\{ \prod_{k=1}^{m} a^*(\tilde{\varphi}_{L_k}) \prod_{c=1}^{p} a(\tilde{\varphi}_{-C_c})\Omega \tag{3.322}$$

$$\mid m, p \in \mathbb{N}, (L_k)_k, (C_c)_c \subset \mathbb{N}, |L| = m, |C| = p \Bigg\} \tag{3.323}$$

holds.

Constructing $\mathrm{d}\Gamma$ piecewise turns out to be advantageous.

Definition 61. *We define the following operators of type $\mathcal{F}^0 \to \mathcal{F}$*

$$\mathrm{d}\Gamma(B_{++}) := \sum_{n\in\mathbb{N}} a^*(B_{++}\varphi_n)a(\varphi_n) \tag{3.324}$$

$$\mathrm{d}\Gamma(B_{--}) := -\sum_{n\in\mathbb{N}} a(\varphi_{-n})a^*(B_{--}\varphi_{-n}) \tag{3.325}$$

$$\mathrm{d}\Gamma(B_{-+}) := \sum_{n\in\mathbb{N}} a^*(B_{-+}\varphi_n)a(\varphi_n) \tag{3.326}$$

where the sum converges in the strong operator topology and $(\varphi_n)_n$, $(\varphi_{-n})_n$ are arbitrary ONBs of $\mathcal{H}^+$ and $\mathcal{H}^-$.

Lemma 62. *The operators $\mathrm{d}\Gamma(B_{++}), \mathrm{d}\Gamma(B_{--})$ and $\mathrm{d}\Gamma(B_{-+})$ restricted to $\mathcal{F}^0_{m,p}$ have the following type*

$$\mathrm{d}\Gamma(B_{++})|_{\mathcal{F}^0_{m,p}} : \quad \mathcal{F}^0_{m,p} \to \mathcal{F}_{m,p} \tag{3.327}$$

$$\mathrm{d}\Gamma(B_{--})|_{\mathcal{F}^0_{m,p}} : \quad \mathcal{F}^0_{m,p} \to \mathcal{F}_{m,p} \tag{3.328}$$

$$\mathrm{d}\Gamma(B_{-+})|_{\mathcal{F}^0_{m,p}} : \quad \mathcal{F}^0_{m,p} \to \mathcal{F}_{m-1,p-1} \tag{3.329}$$

and fulfil the following bounds for all m, p

$$\|d\Gamma(B_{++})|_{\mathcal{F}^0_{m,p}}\| \leqslant (m+4)\|B_{++}\| \tag{3.330}$$

$$\|d\Gamma(B_{--})|_{\mathcal{F}^0_{m,p}}\| \leqslant (p+4)\|B_{--}\| \tag{3.331}$$

$$\|d\Gamma(B_{-+})|_{\mathcal{F}^0_{m,p}}\| \leqslant 2\|B_{-+}\|_{I_2}, \tag{3.332}$$

Moreover the operator $d\Gamma(B_{-+})|_{\mathcal{F}^0_{m,p}}$ *also assumes the following form*

$$d\Gamma(B_{-+})|_{\mathcal{F}^0_{m,p}} = -\sum_{k\in\mathbb{N}} a^*(\varphi_{-k})a(B_{+-}\varphi_{-k})|_{\mathcal{F}^0_{m,p}}. \tag{3.333}$$

The equality of operators can then be continued to all of $\mathcal{F}$.

Proof. Pick $\alpha \in \mathcal{F}^0_{m,p}$ for $m, p \in \mathbb{N}_0$, α can be expressed in terms of some ONB $(\tilde{\varphi}_k)_{k\in\mathbb{N}}$ of $\mathcal{H}^+$ and $(\tilde{\varphi}_{-k})_{k\in\mathbb{N}}$ of $\mathcal{H}^-$

$$\alpha = \sum_{\substack{L,C\subset\mathbb{N}\\|L|=m,|C|=p}} \alpha_{L,C} \prod_{l=1}^{m} a^*(\tilde{\varphi}_{L_l}) \prod_{c=1}^{p} a(\tilde{\varphi}_{-C_c})\Omega. \tag{3.334}$$

In this expansion only finitely many coefficients $\alpha_{\cdot,\cdot}$ are nonzero. Our operators all map the vacuum onto the zero vector, so commuting them through the products of of creation and annihilation operators in the expansion of α we can make the action of them more explicit:

$$d\Gamma(B_{++})\alpha = \sum_{\substack{L,C\subset\mathbb{N}\\|L|=m,|C|=p}} \alpha_{L,C} \sum_{b=1}^{m} \prod_{l=1}^{b-1} a^*(\tilde{\varphi}_{L_l}) \sum_{n\in\mathbb{N}} a^*(B_{++}\varphi_n)\langle\varphi_n, \tilde{\varphi}_{L_b}\rangle$$

$$\times \prod_{l=b+1}^{m} a^*(\tilde{\varphi}_l) \prod_{c=1}^{p} a(\tilde{\varphi}_{-C_c})\Omega \tag{3.335}$$

$$= \sum_{\substack{L,C\subset\mathbb{N}\\|L|=m,|C|=p}} \alpha_{L,C} \sum_{b=1}^{m} \prod_{l=1}^{b-1} a^*(\tilde{\varphi}_{L_l}) a^*(B_{++}\tilde{\varphi}_{L_b}) \prod_{l=b+1}^{m} a^*(\tilde{\varphi}_l) \prod_{c=1}^{p} a(\tilde{\varphi}_{-C_c})\Omega. \tag{3.336}$$

We notice, that $\mathrm{d}\Gamma(B_{++})\alpha \in \mathcal{F}_{m,p}$ holds. What is left to show for the first operator is therefore its norm. For estimating this we see that B_{++} in the last line can be replaced by

$$B^{L}_{L_b} := \left(1 - \sum_{\substack{l=1\\ l\neq b}}^{m} |\tilde{\varphi}_{L_l}\rangle\langle\tilde{\varphi}_{L_l}|\right) B_{++}, \tag{3.337}$$

due to the antisymmetry of fermions. Expanding

$$\begin{aligned}
&\|\mathrm{d}\Gamma(B_{++})\alpha\|^2 = \langle \mathrm{d}\Gamma(B_{++})\alpha, \mathrm{d}\Gamma(B_{++})\alpha\rangle \\
&= \sum_{\substack{L,C,L',C'\subset\mathbb{N}\\ |L'|=|L|=m,|C'|=|C|=p}} \overline{\alpha_{L,C}}\alpha_{L',C'} \sum_{b,b'=1}^{m} \left\langle \prod_{l=1}^{b-1} a^*(\tilde{\varphi}_{L_l}) \;\; a^*(B^{L}_{L_b}\tilde{\varphi}_{L_b}) \right. \\
&\prod_{l=b+1}^{m} a^*(\tilde{\varphi}_{L_l}) \prod_{c=1}^{p} a(\tilde{\varphi}_{-C_c})\Omega, \prod_{l=1}^{b'-1} a^*(\tilde{\varphi}_{L'_l}) \;\; a^*(B^{L'}_{L'_{b'}}\tilde{\varphi}_{L'_b}) \\
&\left. \prod_{l=b'+1}^{m} a^*(\tilde{\varphi}_{L'_l}) \prod_{c=1}^{p} a(\tilde{\varphi}_{-C'_c})\Omega \right\rangle
\end{aligned} \tag{3.338}$$

we see that in fact C and C' need to agree, because we can just commute the corresponding annihilation operators from one end of the scalar product to the other. Furthermore only a single wavefunction on each side of the scalar product is modified, this implies that in order for the scalar product not to vanish $|L \cap L'| \geqslant m - 2$ has to hold. For the case $L \neq L'$ we split up the sum over sets into the sum over a new L such that $|L| = m-2$ holds and an additional sum over four indices $n_1 < n_2, p_1 < p_2$. The double sum over b, b' only has contribution where $b = n_1$ or $b = 2_2$ and $b' = p_1$ or $b' = p_2$ are selected. Because each factor in the first half is orthogonal to each other factor in this half and analogously for the second half, this will result in a sum of eight terms. In the case $L' = L$ the full sum contributes, yielding

$$
\begin{aligned}
&\|\mathrm{d}\Gamma(B_{++})\alpha\|^2 = \\
&= \sum_{\substack{L,C\subset\mathbb{N}\\ |C|=p\\ |L|=m-2}} \sum_{\substack{n_1<n_2,p_1<p_2\in\mathbb{N}\setminus L\\ \{n_1,n_2\}\neq\{p_1,p_2\}}} \overline{\alpha_{L\cup\{n_1,n_2\},C}}\alpha_{L\cup\{p_1,p_2\},C} \begin{bmatrix} n_1\\ n_2\end{bmatrix} \begin{bmatrix} p_1\\ p_2\end{bmatrix} \\
&\times \Big(\langle\tilde\varphi_{n_1},\tilde\varphi_{p_1}\rangle\langle B_{n_2}^{L\cup\{n_1,n_2\}}\tilde\varphi_{n_2}, B_{p_2}^{L\cup\{p_1,p_2\}}\tilde\varphi_{p_2}\rangle && (3.339)\\
&\qquad - \langle\tilde\varphi_{n_1}, B_{p_2}^{L\cup\{p_1,p_2\}}\tilde\varphi_{p_2}\rangle\langle B_{n_2}^{L\cup\{n_1,n_2\}}\tilde\varphi_{n_2},\tilde\varphi_{p_1}\rangle && (3.340)\\
&+ \langle B_{n_1}^{L\cup\{n_1,n_2\}}\tilde\varphi_{n_1},\tilde\varphi_{p_1}\rangle\langle\tilde\varphi_{n_2}, B_{p_2}^{L\cup\{p_1,p_2\}}\tilde\varphi_{p_2}\rangle && (3.341)\\
&\qquad - \langle B_{n_1}^{L\cup\{n_1,n_2\}}\tilde\varphi_{n_1}, B_{p_2}^{L\cup\{p_1,p_2\}}\tilde\varphi_{p_2}\rangle\langle\tilde\varphi_{n_2},\tilde\varphi_{p_1}\rangle && (3.342)\\
&+ \langle\tilde\varphi_{n_1}, B_{p_1}^{L\cup\{p_1,p_2\}}\tilde\varphi_{p_1}\rangle\langle B_{n_2}^{L\cup\{n_1,n_2\}}\tilde\varphi_{n_2},\tilde\varphi_{p_2}\rangle && (3.343)\\
&\qquad - \langle\tilde\varphi_{n_1},\tilde\varphi_{p_2}\rangle\langle B_{n_2}^{L\cup\{n_1,n_2\}}\tilde\varphi_{n_2}, B_{p_1}^{L\cup\{p_1,p_2\}}\tilde\varphi_{p_1}\rangle && (3.344)\\
&+ \langle B_{n_1}^{L\cup\{n_1,n_2\}}\tilde\varphi_{n_1}, B_{p_1}^{L\cup\{p_1,p_2\}}\tilde\varphi_{p_1}\rangle\langle\tilde\varphi_{n_2},\tilde\varphi_{p_2}\rangle && (3.345)\\
&\qquad - \langle B_{n_1}^{L\cup\{n_1,n_1\}}\tilde\varphi_{n_1},\tilde\varphi_{p_2}\rangle\langle\tilde\varphi_{n_2}, B_{p_1}^{L\cup\{p_1,p_2\}}\tilde\varphi_{p_1}\rangle\Big) && (3.346)\\
&+ \sum_{\substack{L,C\subset\mathbb{N}\\ |L|=m,|C|=p}} |\alpha_{L,C}|^2 \sum_{b,b'=1}^{m} \Big\langle \prod_{l=1}^{b-1} a^*(\tilde\varphi_{L_l})\ \ a^*(B_{L_b}^{L}\tilde\varphi_{L_b}) && (3.347)\\
&\prod_{l=b+1}^{m} a^*(\tilde\varphi_{L_l})\Omega, \prod_{l=1}^{b'-1} a^*(\tilde\varphi_{L_l})\ \ a^*(B_{L_{b'}}^{L}\tilde\varphi_{L_b}) \prod_{l=b'+1}^{m} a^*(\tilde\varphi_{L_l}))\Omega\Big\rangle,
\end{aligned}
$$

where

$$\begin{bmatrix} n_1\\ n_2\end{bmatrix} := (-1)^{|\{l\in L|l<n_1\}|+|\{l\in L|l<n_2\}|} \tag{3.348}$$

keeps track of the number of anti commutations. This is non standard notation but it is meant to keep the notation as compact as possible and its use is contained to this section.

Due to the antisymmetry each summand containing a factor without an occurrence of the B operator are only nonzero if $n_1 = p_1$ or $n_2 = p_2$. Each factor containing exactly one occurrence of B obtains a similar restriction. So we can split the block of terms into one corresponding to the two cases just mentioned.

$$
\begin{aligned}
&\|\mathrm{d}\Gamma(B_{++})\alpha\|^2 = \\
&= \sum_{\substack{L,C\subset\mathbb{N}\\ |C|=p\\ |L|=m-2}} \sum_{\substack{n_1<n_2, p_1<p_2\in\mathbb{N}\setminus L\\ \{n_1,n_2\}\neq\{p_1,p_2\}}} \overline{\alpha_{L\cup\{n_1,n_2\},C}}\alpha_{L\cup\{p_1,p_2\},C} \begin{bmatrix} n_1\\ n_2\end{bmatrix} \begin{bmatrix} p_1\\ p_2\end{bmatrix} \\
&\times \Big(-\langle\tilde{\varphi}_{n_1}, B^{L\cup\{p_1,p_2\}}_{p_2}\tilde{\varphi}_{p_2}\rangle\langle B^{L\cup\{n_1,n_2\}}_{n_2}\tilde{\varphi}_{n_2},\tilde{\varphi}_{p_1}\rangle 1_{n_1\neq p_1} \qquad (3.349)\\
&+ \langle B^{L\cup\{n_1,n_2\}}_{n_1}\tilde{\varphi}_{n_1},\tilde{\varphi}_{p_1}\rangle\langle\tilde{\varphi}_{n_2}, B^{L\cup\{p_1,p_2\}}_{p_2}\tilde{\varphi}_{p_2}\rangle 1_{n_2\neq p_1}\\
&+ \langle\tilde{\varphi}_{n_1}, B^{L\cup\{p_1,p_2\}}_{p_1}\tilde{\varphi}_{p_1}\rangle\langle B^{L\cup\{n_1,n_2\}}_{n_2}\tilde{\varphi}_{n_2},\tilde{\varphi}_{p_2}\rangle 1_{n_1\neq p_2}\\
&- \langle B^{L\cup\{n_1,n_2\}}_{n_1}\tilde{\varphi}_{n_1},\tilde{\varphi}_{p_2}\rangle\langle\tilde{\varphi}_{n_2}, B^{L\cup\{p_1,p_2\}}_{p_1}\tilde{\varphi}_{p_1}\rangle 1_{n_2\neq p_2}\Big)\\
&+ \sum_{\substack{L,C\subset\mathbb{N}\\ |C|=p\\ |L|=m-1}} \sum_{n\neq p\in\mathbb{N}\setminus L} \overline{\alpha_{L\cup\{n\},C}}\alpha_{L\cup\{p\},C} \begin{bmatrix} n\\ p\end{bmatrix} \langle B^{L\cup\{n\}}_n\tilde{\varphi}_n, B^{L\cup\{p\}}_p\tilde{\varphi}_p\rangle \qquad (3.350)\\
&+ \sum_{\substack{L,C\subset\mathbb{N}\\ |L|=m,|C|=p}} |\alpha_{L,C}|^2 \sum_{b,b'=1}^{m} \Big\langle \prod_{l=1}^{b-1} a^*(\tilde{\varphi}_{L_l}) \;\; a^*(B^L_{L_b}\tilde{\varphi}_{L_b}) \qquad (3.351)\\
&\prod_{l=b+1}^{m} a^*(\tilde{\varphi}_{L_l})\Omega, \prod_{l=1}^{b'-1} a^*(\tilde{\varphi}_{L_l}) \;\; a^*(B^L_{L_{b'}}\tilde{\varphi}_{L_b}) \prod_{l=b'+1}^{m} a^*(\tilde{\varphi}_{L_l}))\Omega \Big\rangle,
\end{aligned}
$$

where we also summarised the terms of the second block. the restrictions $n_1 < n_2$ and $p_1 < p_2$ have the effect that the negative terms sum

up to just one term without restrictions, while the positive terms add up to two such terms.
For the term (3.350) we add and subtract the term where $n = p$. The enlarged sum can then be reformulated

$$\begin{aligned}
&\sum_{\substack{L,C\subset\mathbb{N}\\|L|=m-1\\|C|=p}}\sum_{n,n'\in\mathbb{N}\setminus L}\overline{\alpha_{L\cup\{n\},C}}\alpha_{L\cup\{n'\},C}\left\langle B_n^{L\cup\{n\}}\tilde{\varphi}_n, B_{n'}^{L\cup\{n'\}}\tilde{\varphi}_{n'}\right\rangle\begin{bmatrix}n\\n'\end{bmatrix}\\
&=\sum_{\substack{L,C\subset\mathbb{N}\\|L|=m-1,|C|=p}}\left\|\sum_{n\in\mathbb{N}\setminus L}\alpha_{L\cup\{n\},C}B_n^{L\cup\{n\}}\tilde{\varphi}_n\begin{bmatrix}n\\0\end{bmatrix}\right\|^2\\
&=\sum_{\substack{L,C\subset\mathbb{N}\\|L|=m-1\\|C|=p}}\left\|\left(1-\sum_{l\in L}|\tilde{\varphi}_l\rangle\langle\tilde{\varphi}_l|\right)B_{++}\sum_{n\in\mathbb{N}\setminus L}\alpha_{L\cup\{n\},C}\tilde{\varphi}_n\begin{bmatrix}n\\0\end{bmatrix}\right\|^2 \qquad (3.352)
\end{aligned}$$

Now the operator product inside the norm has operator norm $\|B_{++}\|$ and so we can estimate the whole object by

$$(3.352) \leqslant \|\alpha\|^2\|B_{++}\|^2. \qquad (3.353)$$

We need to estimate the term we added to complete the norm square in (3.350), this is done as follows

$$\begin{aligned}
&\sum_{\substack{L,C\subset\mathbb{N}\\|L|=m-1,|C|=p}}\sum_{n\in\mathbb{N}\setminus L}|\alpha_{L\cup\{n\},C}|^2\|B_n^{L\cup\{n\}}\tilde{\varphi}_n\|^2\\
&\leqslant\sum_{\substack{L,C\subset\mathbb{N}\\|L|=m,|C|=p}}\|B_{++}\|^2|\alpha_{L,C}|^2=\|\alpha\|^2\|B_{++}\|^2. \qquad (3.354)
\end{aligned}$$

For (3.349) and the following 3 lines we notice that we may replace all one particle operators with B_{++}, since the projector acts as the identity in these cases. Subsequently, the two terms of equal sign are identical except for the extra condition on the sum, resulting in

$$
\begin{aligned}
&(3.349) = \\
&\sum_{\substack{L,C\subset\mathbb{N}\\|C|=p\\|L|=m-2}} \sum_{\substack{n_1<n_2,p_1<p_2\in\mathbb{N}\setminus L\\ \{n_1,n_2\}\neq\{p_1,p_2\}}} \overline{\alpha_{L\cup\{n_1,n_2\},C}}\alpha_{L\cup\{p_1,p_2\},C} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \quad (3.355)\\
&\times \Big(\langle B_{++}\tilde{\varphi}_{n_1}, \tilde{\varphi}_{p_1}\rangle\langle\tilde{\varphi}_{n_2}, B_{++}\tilde{\varphi}_{p_2}\rangle(1_{n_1\neq p_2} + 1_{n_2\neq p_1}) \quad (3.356)\\
&- \langle\tilde{\varphi}_{n_1}, B_{++}\tilde{\varphi}_{p_2}\rangle\langle B_{++}\tilde{\varphi}_{n_2}, \tilde{\varphi}_{p_1}\rangle(1_{n_1\neq p_1} + 1_{n_2\neq p_2})\Big). \quad (3.357)
\end{aligned}
$$

Next, we are going to repeatedly add and subtract terms, such that we may factorise the n and the p sums. In order to do so we impose the condition $\{n_1, n_2\} \neq \{p_1, p_2\}$ in the sum by the factor $1 - \delta_{n_1,p_1}\delta_{n_2,p_2}$. Similarly we rewrite the other conditions in the following way

$$
\begin{aligned}
1_{n_1\neq p_2} + 1_{n_2\neq p_1} &= 2 - \delta_{n_1,p_2} - \delta_{n_2,p_1}, \quad (3.358)\\
1_{n_1\neq p_1} + 1_{n_2\neq p_2} &= 2 - \delta_{n_1,p_1} - \delta_{n_2,p_2}. \quad (3.359)
\end{aligned}
$$

These are to be multiplied by $1 - \delta_{n_1,p_1}\delta_{n_2,p_2}$ resulting in the two expressions

$$
\begin{aligned}
2 - \delta_{n_1,p_2} - \delta_{n_2,p_1} - 2\delta_{n_1,p_1}\delta_{n_2,p_2} \quad (3.360)\\
2 - \delta_{n_1,p_1} - \delta_{n_2,p_2}, \quad (3.361)
\end{aligned}
$$

wehre the upper expression yields the restrictions on the sum of over $\langle B_{++}\tilde{\varphi}_{n_1}, \tilde{\varphi}_{p_1}\rangle\langle\tilde{\varphi}_{n_2}, B_{++}\tilde{\varphi}_{p_2}\rangle$ and the lower expression analogously for $\langle\tilde{\varphi}_{n_1}, B_{++}\tilde{\varphi}_{p_2}\rangle\langle B_{++}\tilde{\varphi}_{n_2}, \tilde{\varphi}_{p_1}\rangle$. For the term without further restrictions we may add the sum of the terms (3.356) and (3.357), the rest is treated separately.

The terms are all estimated after rewriting the scalar products as a single sum of two scalar products in $\mathcal{H}^+ \otimes \mathcal{H}^+$.

2 :

$$2\sum_{\substack{L,C\subset\mathbb{N}\\|C|=p\\|L|=m-2}}\sum_{\substack{n_1<n_2\in\mathbb{N}\setminus L\\p_1<p_2\in\mathbb{N}\setminus L}}\overline{\alpha_{L\cup\{n_1,n_2\},C}}\alpha_{L\cup\{p_1,p_2\},C}\begin{bmatrix}n_1\\n_2\end{bmatrix}\begin{bmatrix}p_1\\p_2\end{bmatrix} \tag{3.362}$$

$$\times\Big(\langle B_{++}\tilde{\varphi}_{n_1},\tilde{\varphi}_{p_1}\rangle\langle\tilde{\varphi}_{n_2},B_{++}\tilde{\varphi}_{p_2}\rangle-\langle\tilde{\varphi}_{n_1},B_{++}\tilde{\varphi}_{p_2}\rangle\langle B_{++}\tilde{\varphi}_{n_2},\tilde{\varphi}_{p_1}\rangle\Big) \tag{3.363}$$

$$=2\sum_{\substack{L,C\subset\mathbb{N}\\|C|=p\\|L|=m-2}}\Big\langle\sum_{n_1<n_2\in\mathbb{N}\setminus L}\begin{bmatrix}n_1\\n_2\end{bmatrix}\alpha_{L\cup\{n_1,n_2\},C}B_{++}\tilde{\varphi}_{n_1}\otimes\tilde{\varphi}_{n_2}, \tag{3.364}$$

$$\sum_{p_1<p_2\in\mathbb{N}\setminus L}\begin{bmatrix}p_1\\p_2\end{bmatrix}\alpha_{L\cup\{p_1,p_2\},C}\tilde{\varphi}_{p_1}\otimes B_{++}\tilde{\varphi}_{p_2}\Big\rangle \tag{3.365}$$

$$-2\sum_{\substack{L,C\subset\mathbb{N}\\|C|=p\\|L|=m-2}}\Big\langle\sum_{n_1<n_2\in\mathbb{N}\setminus L}\begin{bmatrix}n_1\\n_2\end{bmatrix}\alpha_{L\cup\{n_1,n_2\},C}\tilde{\varphi}_{n_1}\otimes B_{++}\tilde{\varphi}_{n_2}, \tag{3.366}$$

$$\sum_{p_1<p_2\in\mathbb{N}\setminus L}\begin{bmatrix}p_1\\p_2\end{bmatrix}\alpha_{L\cup\{p_1,p_2\},C}B_{++}\tilde{\varphi}_{p_2}\otimes\tilde{\varphi}_{p_1}\Big\rangle \tag{3.367}$$

$$\leqslant 4\|B_{++}\|^2\|\alpha\|^2 \tag{3.368}$$

$\boldsymbol{\delta_{n_1,p_1}}$ **:**

$$\sum_{\substack{L,C\subset\mathbb{N}\\|C|=p\\|L|=m-2}}\sum_{\substack{n_1<n_2\in\mathbb{N}\setminus L\\p_1<p_2\in\mathbb{N}\setminus L}}\overline{\alpha_{L\cup\{n_1,n_2\},C}}\alpha_{L\cup\{p_1,p_2\},C}\begin{bmatrix}n_1\\n_2\end{bmatrix}\begin{bmatrix}p_1\\p_2\end{bmatrix}\delta_{n_1,p_1} \tag{3.369}$$

$$\times\langle\tilde{\varphi}_{n_1},B_{++}\tilde{\varphi}_{p_2}\rangle\langle B_{++}\tilde{\varphi}_{n_2},\tilde{\varphi}_{p_1}\rangle \tag{3.370}$$

$$=\sum_{\substack{L,C\subset\mathbb{N}\\|C|=p\\|L|=m-2}}\sum_{n,p,l\in\mathbb{N}\setminus L}\overline{\alpha_{L\cup\{l,n\},C}}\alpha_{L\cup\{l,p\},C}\begin{bmatrix}0\\n\end{bmatrix}\begin{bmatrix}0\\p\end{bmatrix} \tag{3.371}$$

$$\times\langle\tilde{\varphi}_l,B_{++}\tilde{\varphi}_p\rangle\langle B_{++}\tilde{\varphi}_n,\tilde{\varphi}_l\rangle \tag{3.372}$$

$$=\sum_{\substack{L,C\subset\mathbb{N}\\|C|=p\\|L|=m-2}}\sum_{l\in\mathbb{N}\setminus L}\left\langle\tilde{\varphi}_l,B_{++}\sum_{p\in\mathbb{N}\setminus L}\alpha_{L\cup\{l,p\},C}\begin{bmatrix}0\\p\end{bmatrix}\tilde{\varphi}_p\right\rangle \tag{3.373}$$

$$\times\left\langle B_{++}\sum_{n\in\mathbb{N}\setminus L}\alpha_{L\cup\{l,n\},C}\begin{bmatrix}0\\n\end{bmatrix}\tilde{\varphi}_n,\tilde{\varphi}_l\right\rangle \tag{3.374}$$

$$\leqslant\|B_{++}\|^2\|\alpha\|^2 \tag{3.375}$$

$\boldsymbol{-\delta_{n_1,p_2}-\delta_{n_2,p_1}}$ **:**

$$-\sum_{\substack{L,C\subset\mathbb{N}\\|C|=p\\|L|=m-2}}\sum_{\substack{n_1<n_2\in\mathbb{N}\setminus L\\p_1<p_2\in\mathbb{N}\setminus L}}\overline{\alpha_{L\cup\{n_1,n_2\},C}}\alpha_{L\cup\{p_1,p_2\},C}\begin{bmatrix}n_1\\n_2\end{bmatrix}\begin{bmatrix}p_1\\p_2\end{bmatrix}(\delta_{n_1,p_2}+\delta_{n_2,p_1}) \tag{3.376}$$

$$\times\langle B_{++}\tilde{\varphi}_{n_1},\tilde{\varphi}_{p_1}\rangle\langle\tilde{\varphi}_{n_2},B_{++}\tilde{\varphi}_{p_2}\rangle \tag{3.377}$$

$$=-\sum_{\substack{L,C\subset\mathbb{N}\\|C|=p\\|L|=m-2}}\sum_{n,p,l\in\mathbb{N}\setminus L}\overline{\alpha_{L\cup\{l,n\},C}}\alpha_{L\cup\{l,p\},C}\begin{bmatrix}0\\n\end{bmatrix}\begin{bmatrix}0\\p\end{bmatrix} \tag{3.378}$$

$$\times\Big(1_{p<l<n}\langle B_{++}\tilde{\varphi}_l,\tilde{\varphi}_p\rangle\langle\tilde{\varphi}_n,B_{++}\tilde{\varphi}_l\rangle \tag{3.379}$$

$$+1_{n<l<p}\langle B_{++}\tilde{\varphi}_n,\tilde{\varphi}_l\rangle\langle\tilde{\varphi}_l,B_{++}\tilde{\varphi}_p\rangle\Big) \tag{3.380}$$

$$=-\sum_{\substack{L,C\subset\mathbb{N}\\|C|=p\\|L|=m-2}}\sum_{l\in\mathbb{N}\setminus L} \tag{3.381}$$

$$\left[\left\langle B_{++}\tilde{\varphi}_l, \sum_{\substack{p\in\mathbb{N}\setminus L\\ p<l}} \begin{bmatrix} 0 \\ p \end{bmatrix} \alpha_{L\cup\{l,p\},C}\tilde{\varphi}_p \right\rangle \left\langle \sum_{\substack{n\in\mathbb{N}\setminus L\\ l<n}} \begin{bmatrix} 0 \\ n \end{bmatrix} \alpha_{L\cup\{l,n\},C}\tilde{\varphi}_n, B_{++}\tilde{\varphi}_l \right\rangle \right. \tag{3.382}$$

$$\left. + \left\langle B_{++}\tilde{\varphi}_l, \sum_{\substack{p\in\mathbb{N}\setminus L\\ p>l}} \begin{bmatrix} 0 \\ p \end{bmatrix} \alpha_{L\cup\{l,p\},C}\tilde{\varphi}_p \right\rangle \left\langle \sum_{\substack{n\in\mathbb{N}\setminus L\\ l>n}} \begin{bmatrix} 0 \\ n \end{bmatrix} \alpha_{L\cup\{l,n\},C}\tilde{\varphi}_n, B_{++}\tilde{\varphi}_l \right\rangle \right] \tag{3.383}$$

$$\leqslant 2\|B_{++}\|^2\|\alpha\|^2 \tag{3.384}$$

$\boldsymbol{\delta_{n_2,p_2}}$ **:** completely analogous to δ_{n_1,p_1} one arrives at

$$\leqslant \|B_{++}\|^2\|\alpha\|^2. \tag{3.385}$$

$\boldsymbol{\delta_{n_1,p_1}\delta_{n_2,p_2}}$ **:**

$$-2\sum_{\substack{L,C\subset\mathbb{N}\\ |C|=p\\ |L|=m-2}} \sum_{\substack{n_1<n_2\in\mathbb{N}\setminus L\\ p_1<p_2\in\mathbb{N}\setminus L}} \overline{\alpha_{L\cup\{n_1,n_2\},C}}\alpha_{L\cup\{p_1,p_2\},C} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \delta_{n_1,p_1}\delta_{n_2,p_2} \tag{3.386}$$

$$\times \langle B_{++}\tilde{\varphi}_{n_1}, \tilde{\varphi}_{p_1}\rangle\langle\tilde{\varphi}_{n_2}, B_{++}\tilde{\varphi}_{p_2}\rangle \tag{3.387}$$

$$= -2\sum_{\substack{L,C\subset\mathbb{N}\\ |C|=p\\ |L|=m-2}} \sum_{n<p\in\mathbb{N}\setminus L} |\alpha_{L\cup\{n,p\},C}|^2 \langle B_{++}\tilde{\varphi}_n, \tilde{\varphi}_n\rangle\langle\tilde{\varphi}_p, B_{++}\tilde{\varphi}_p\rangle \tag{3.388}$$

$$\leqslant 2\|B_{++}\|^2\|\alpha\|^2 \tag{3.389}$$

So all together we have

$$(3.349) \leqslant 10\|B_{++}\|^2\|\alpha\|^2 \tag{3.390}$$

What remains is term (3.351), for this term there are two cases. If $b = b'$ then the scalar product is equal to $\langle B^L_{L_b}\tilde{\varphi}_b, B^L_{L_b}\tilde{\varphi}_b\rangle$. If $b \neq b'$ the scalar product is, up to a sign, equal to $\langle B^L_{L_b}\tilde{\varphi}_b, \tilde{\varphi}_b\rangle\langle\tilde{\varphi}_{b'}, B^L_{L_{b'}}\tilde{\varphi}_{b'}\rangle$.

However both of these terms can be estimated by $\|B_{++}\|^2$. So all m^2 summands of this sum contribute $\|B_{++}\|^2$. Overall this estimate yields

$$\begin{aligned}\|\mathrm{d}\Gamma(B_{++})\alpha\|^2 &\leqslant (3.353) + (3.354) + (3.390) + \|\alpha\|^2 m^2 \|B_{++}\|^2 \\ &= \|\alpha\|^2(12 + m^2)\|B_{++}\|^2.\end{aligned}$$

For convenience of notation the estimate can be weakened to

$$\|\mathrm{d}\Gamma(B_{++})\alpha\| \leqslant (m+4)\|B_{++}\|\|\alpha\|. \tag{3.391}$$

A completely analogous argument works for $\mathrm{d}\Gamma(B_{--})$. So lets move on to $\mathrm{d}\Gamma(B_{-+})$. Applying it to the same $\alpha \in \mathcal{F}^0_{m,p}$ again we permute all the operators to the right, where they annihilate the vacuum. The remaining terms are

$$\begin{aligned}&\sum_{n\in\mathbb{N}} a^*(B_{-+}\varphi_n)a(\varphi_n) \sum_{\substack{L,C\subset\mathbb{N}\\|L|=m,|C|=p}} \alpha_{L,C} \prod_{l=1}^{m} a^*(\tilde{\varphi}_{L_l}) \prod_{c=1}^{p} a(\tilde{\varphi}_{-C_c})\Omega \\ &= \sum_{n\in\mathbb{N}} \sum_{\substack{L,C\subset\mathbb{N}\\|L|=m,|C|=p}} \alpha_{L,C} \sum_{b=1}^{m}\sum_{d=1}^{p} (-1)^{m-1+b+d} \langle\varphi_n, \tilde{\varphi}_{L_b}\rangle\langle\tilde{\varphi}_{-C_d}, B_{-+}\varphi_n\rangle \\ &\qquad\qquad \times \prod_{\substack{l=1\\l\neq b}}^{m} a^*(\tilde{\varphi}_{L_l}) \prod_{\substack{c=1\\c\neq d}}^{p} a(\tilde{\varphi}_{-C_c})\Omega.\end{aligned} \tag{3.392}$$

From here we can eliminate the sum over n, and reintroduce a sum over the ONB of $\mathcal{H}^-$ to arrive at the expression for $\mathrm{d}\Gamma(B_{-+})$:

$$= \sum_{\substack{L,C\subset\mathbb{N}\\|L|=m,|C|=p}} \alpha_{L,C} \sum_{b=1}^{m}\sum_{d=1}^{p}(-1)^{m+b+d}\langle B_{+-}\tilde{\varphi}_{-C_d},\tilde{\varphi}_{L_b}\rangle \times \prod_{\substack{l=1\\l\neq b}}^{m} a^*(\tilde{\varphi}_{L_l}) \prod_{\substack{c=1\\c\neq d}}^{p} a(\tilde{\varphi}_{-C_c})\Omega \quad (3.393)$$

$$= \sum_{\substack{L,C\subset\mathbb{N}\\|L|=m,|C|=p}} \alpha_{L,C} \sum_{b=1}^{m}\sum_{d=1}^{p}\sum_{k\in\mathbb{N}}(-1)^{m+b+d}\langle \tilde{\varphi}_{-C_d},\varphi_{-k}\rangle\langle B_{+-}\varphi_{-k},\tilde{\varphi}_{L_b}\rangle \times \prod_{\substack{l=1\\l\neq b}}^{m} a^*(\tilde{\varphi}_{L_l}) \prod_{\substack{c=1\\c\neq d}}^{p} a(\tilde{\varphi}_{-C_c})\Omega \quad (3.394)$$

$$= -\sum_{k\in\mathbb{N}} a^*(\varphi_{-k})a(B_{+-}\varphi_{-k}) \sum_{\substack{L,C\subset\mathbb{N}\\|L|=m,|C|=p}} \alpha_{L,C} \prod_{l=1}^{m} a^*(\tilde{\varphi}_{L_l}) \prod_{c=1}^{p} a(\tilde{\varphi}_{-C_c})\Omega. \quad (3.395)$$

For the estimate of the operator norm we continue with expression (3.393). By counting the remaining creation and annihilation operators we immediately see that $\mathrm{d}\Gamma(B_{-+})\alpha \in \mathcal{F}_{m-1,p-1}$. We take the norm squared of the expression and notice that the scalar product is only not zero in cases where $|L\backslash L'| \leqslant 1$ and $|C\backslash C'| \leqslant 1$. Furthermore, whenever $L = L'$ holds, the two sums of $1 \leqslant b, b' \leqslant m$ collapses to a single sum over this range and analogously for $C = C'$ and d, d'. In case $L \neq L'$ no sum over b or b' remains for the same reason. Hence we arrive at

$$\|\mathrm{d}\Gamma(B_{-+})\alpha\|^2 \leqslant \sum_{\substack{C,L\subset\mathbb{N}\\|L|=m-1\\|C|=p-1}} \sum_{\substack{n_1,n_2\in\mathbb{N}\backslash L\\l_1,l_2\in\mathbb{N}\backslash C}} |\alpha_{L\cup\{n_2\},C\cup\{l_2\}}||\alpha_{L\cup\{n_1\},C\cup\{l_1\}}| \quad (3.396)$$

$$\Bigg[\delta_{n_1,n_2}\delta_{l_1,l_2}\sum_{b\in L\cup\{n_1\}}\sum_{d\in C\cup\{l_1\}}|\langle B_{+-}\tilde{\varphi}_{-d},\tilde{\varphi}_b\rangle|^2 \tag{3.397}$$

$$+(1-\delta_{n_1,n_2})\delta_{l_1,l_2}\sum_{d\in C\cup\{l_1\}}|\langle B_{+-}\tilde{\varphi}_{-d},\tilde{\varphi}_{n_1}\rangle||\langle\tilde{\varphi}_{n_2},B_{+-}\tilde{\varphi}_{-d}\rangle| \tag{3.398}$$

$$+\delta_{n_1,n_2}(1-\delta_{l_1,l_2})\sum_{b\in L\cup\{n_1\}}|\langle B_{+-}\tilde{\varphi}_{-l_1},\tilde{\varphi}_b\rangle||\langle\tilde{\varphi}_b,B_{+-}\tilde{\varphi}_{-l_2}\rangle| \tag{3.399}$$

$$+(1-\delta_{n_1,n_2})(1-\delta_{l_1,l_2})|\langle B_{+-}\tilde{\varphi}_{-C_{l_1}},\tilde{\varphi}_{n_1}\rangle||\langle\tilde{\varphi}_{n_2},B_{+-}\tilde{\varphi}_{-C_{l_2}}\rangle|\Bigg]. \tag{3.400}$$

In the next step we split the sum into the four already indicated, estimate the terms $(1-\delta)\leqslant 1$ and eliminate sums with the remaining Kronecker deltas. For the first term, we subsequently enlarge the sum over part of the Basis $\tilde{\varphi}$ in the scalar product to the sum over all basis elements, yielding

$$\|\mathrm{d}\Gamma(B_{-+})\alpha\|^2\leqslant\|\alpha\|^2\sum_{b\in\mathbb{N}}\sum_{c\in\mathbb{N}}|\langle B_{+-}\tilde{\varphi}_{-c},\tilde{\varphi}_b\rangle|^2 \tag{3.401}$$

$$+\sum_{\substack{C,L\subset\mathbb{N}\\|L|=m-1\\|C|=p}}\sum_{n_1,n_2\in\mathbb{N}\setminus L}|\alpha_{L\cup\{n_2\},C}||\alpha_{L\cup\{n_1\},C}| \tag{3.402}$$

$$\times\sum_{d\in C}|\langle B_{+-}\tilde{\varphi}_{-d},\tilde{\varphi}_{n_1}\rangle||\langle\tilde{\varphi}_{n_2},B_{+-}\tilde{\varphi}_{-d}\rangle| \tag{3.403}$$

$$+\sum_{\substack{C,L\subset\mathbb{N}\\|L|=m\\|C|=p-1}}\sum_{l_1,l_2\in\mathbb{N}\setminus C}|\alpha_{L,C\cup\{l_2\}}||\alpha_{L,C\cup\{l_1\}}| \tag{3.404}$$

$$\times\sum_{b\in L}|\langle B_{+-}\tilde{\varphi}_{-l_1},\tilde{\varphi}_b\rangle||\langle\tilde{\varphi}_b,B_{+-}\tilde{\varphi}_{-l_2}\rangle| \tag{3.405}$$

$$+\sum_{\substack{C,L\subset\mathbb{N}\\|L|=m-1\\|C|=p-1}}\sum_{\substack{n_1,n_2\in\mathbb{N}\setminus L\\l_1,l_2\in\mathbb{N}\setminus C}}|\alpha_{L\cup\{n_2\},C\cup\{l_2\}}||\alpha_{L\cup\{n_1\},C\cup\{l_1\}}| \tag{3.406}$$

$$\times |\langle B_{+-}\tilde{\varphi}_{-l_1}, \tilde{\varphi}_{n_1}\rangle||\langle\tilde{\varphi}_{n_2}, B_{+-}\tilde{\varphi}_{-l_2}\rangle|. \tag{3.407}$$

Now we identity the sums over l_1, l_2 and n_1, n_2 as scalar products in tensor products of $l^2(\mathbb{N})$ and apply the Cauchy-Schwarz inequality. Additionally, we identify $\sum_{b\in\mathbb{N}}\sum_{c\in\mathbb{N}} |\langle B_{+-}\tilde{\varphi}_{-c}, \tilde{\varphi}_b\rangle|^2 = \|B_{+-}\|^2_{I_2}$. This results in

$$\|\mathrm{d}\Gamma(B_{-+})\alpha\|^2 \leqslant \|\alpha\|^2\|B_{-+}\|^2_{I_2} \tag{3.408}$$

$$+ \sum_{\substack{C,L\subset\mathbb{N}\\|L|=m-1\\|C|=p}}\sum_{n\in\mathbb{N}\setminus L} |\alpha_{L\cup\{n\},C}|^2 \sum_{d\in C}\sum_{u\in\mathbb{N}\setminus L} |\langle B_{+-}\tilde{\varphi}_{-d}, \tilde{\varphi}_u\rangle|^2 \tag{3.409}$$

$$+ \sum_{\substack{C,L\subset\mathbb{N}\\|L|=m\\|C|=p-1}}\sum_{l\in\mathbb{N}\setminus C} |\alpha_{L,C\cup\{l\}}|^2 \sum_{b\in L}\sum_{u\in\mathbb{N}\setminus C} |\langle B_{+-}\tilde{\varphi}_{-u}, \tilde{\varphi}_b\rangle|^2 \tag{3.410}$$

$$+ \sum_{\substack{C,L\subset\mathbb{N}\\|L|=m-1\\|C|=p-1}}\sum_{\substack{n\in\mathbb{N}\setminus L\\l\in\mathbb{N}\setminus C}} |\alpha_{L\cup\{n\},C\cup\{l\}}|^2 \sum_{\substack{u\in\mathbb{N}\setminus L\\d\in\mathbb{N}\setminus C}} |\langle B_{+-}\tilde{\varphi}_{-d}, \tilde{\varphi}_u\rangle|^2 \tag{3.411}$$

$$\leqslant 4\|\alpha\|^2\|B_{-+}\|^2_{I_2}. \tag{3.412}$$

□

Corollary 63. *The operators* $\mathrm{d}\Gamma(B_{--})$ *and* $\mathrm{d}\Gamma(B_{++})$ *can be extended by continuity on* $\mathcal{F}^0_{m,p}$ *to unbounded operators on all of* $\mathcal{F}'$. *The operator* $\mathrm{d}\Gamma(B_{-+})$ *can be continuously extended to all of* $\mathcal{F}$.

Lemma 64. *The operator* $(\mathrm{d}\Gamma(B_{-+}))^*$ *acts on elements of* $\mathcal{F}^0$ *as*

$$-\sum_{n\in\mathbb{N}} a^*(B_{+-}\varphi_{-n})a(\varphi_{-n}) =: -\mathrm{d}\Gamma(B_{+-}). \tag{3.413}$$

So also $\mathrm{d}\Gamma(B_{+-}) : \mathcal{F}^0 \to \mathcal{F}$ *can be extended continuously to all of* $\mathcal{F}$. *Moreover* $\mathrm{d}\Gamma(B_{-+}) + \mathrm{d}\Gamma(B_{+-})$ *is skew-adjoint.*

Proof. Pick $\beta, \alpha \in \mathcal{F}^0$. We use the form (3.333) to obtain

$$\begin{aligned}\langle \beta, \mathrm{d}\Gamma(B_{-+})\alpha\rangle &= \left\langle \beta, -\sum_{n\in\mathbb{N}} a^*(\varphi_{-n})a(B_{+-}\varphi_{-n})\alpha \right\rangle \\ &= -\sum_{n\in\mathbb{N}}\langle \beta, a^*(\varphi_{-n})a(B_{+-}\varphi_{-n})\alpha\rangle = -\sum_{n\in\mathbb{N}}\langle a^*(B_{+-}\varphi_{-n})a(\varphi_{-n})\beta, \alpha\rangle \\ &= \left\langle -\sum_{n\in\mathbb{N}} a^*(B_{+-}\varphi_{-n})a(\varphi_{-n})\beta, \alpha \right\rangle \end{aligned} \tag{3.414}$$

So we see that $\mathrm{d}\Gamma(B_{+-})$ and $\mathrm{d}\Gamma(B_{-+})^*$ agree on $\mathcal{F}^0$ which is dense. So they are the same bounded and continuous operator on all of Fock space. $\square$

Definition 65. *We define the set*

$$\mathfrak{B} := \{B : \mathcal{H} \circlearrowleft |\text{linear}, \|B\| + \|B_{+-}\|_{I_2} + \|B_{-+}\|_{I_2} \in \mathbb{R}, B^* = -B\} \tag{3.415}$$

and the operator

$$\mathrm{d}\Gamma(B) := \mathrm{d}\Gamma(B_{++}) + \mathrm{d}\Gamma(B_{+-}) + \mathrm{d}\Gamma(B_{-+}) + \mathrm{d}\Gamma(B_{--}). \tag{3.416}$$

Furhtermore, we endow $\mathfrak{B}$ with the topology induced by the norm $B \mapsto \|B\| + \|B_{+-}\|_{I_2} + \|B_{-+}\|_{I_2}$.

Lemma 66. *The operator $\mathrm{d}\Gamma(B)$ is skew symmetric and real linear in its argument $B \in \mathfrak{B}$. Moreover, for each $m, p \in \mathbb{N}$ the functional $\mathrm{d}\Gamma(\cdot)|_{\mathcal{F}'_{m,p}}$ is bounded and hence continuous as a map from $\mathfrak{B}$ to the set of bounded linear operators of type $\mathcal{F}'_{m,p} \to \mathcal{F}'_{m-1,p-1} \oplus \mathcal{F}'_{m,p} \oplus \mathcal{F}'_{m+1,p+1}$.*

Proof. Since the sum of skew symmetric operators is skew symmetric, it suffices to show skew symmetry of $\mathrm{d}\Gamma(B_{++})$ and $\mathrm{d}\Gamma(B_{--})$. Moreover since both of these operators are extended versions of operators of the same name of type $\mathcal{F}^0 \to \mathcal{F}$ it suffices to show skew symmetry on this domain. We will only do the calculation for $\mathrm{d}\Gamma(B_{++})$, the other calculation is analogous. Pick $\beta, \alpha \in \mathcal{F}^0$ and basis $\tilde{\varphi}$, φ' such that β, α are expressible with finite sums over elements of the generating sets with respect to their respective basis. We caculate

$$
\begin{aligned}
\langle \beta, \mathrm{d}\Gamma(B_{++})\alpha \rangle = \sum_{L,L',C,C' \subset \mathbb{N}} \overline{\beta}_{L',C'} \alpha_{L,C} \Bigg\langle \prod_{l=1}^{|L'|} a^*(\tilde{\varphi}_{L'_l}) \prod_{c=1}^{|C'|} a(\tilde{\varphi}_{-C'_c})\Omega, \\
\sum_{n \in \mathbb{N}} a^*(B_{++}\varphi_n) a(\varphi_n) \prod_{l=1}^{|L|} a^*(\varphi'_{L_l}) \prod_{c=1}^{|C|} a(\varphi'_{-C_c})\Omega \Bigg\rangle \\
= \sum_{L,L',C,C' \subset \mathbb{N}} \overline{\beta}_{L',C'} \alpha_{L,C} \sum_{n \in \mathbb{N}} \Bigg\langle \prod_{l=1}^{|L'|} a^*(\tilde{\varphi}_{L'_l}) \prod_{c=1}^{|C'|} a(\tilde{\varphi}_{-C'_c})\Omega, \\
a^*(B_{++}\varphi_n) a(\varphi_n) \prod_{l=1}^{|L|} a^*(\varphi'_{L_l}) \prod_{c=1}^{|C|} a(\varphi'_{-C_c})\Omega \Bigg\rangle \\
= \sum_{L,L',C,C' \subset \mathbb{N}} \overline{\beta}_{L',C'} \alpha_{L,C} \sum_{n \in \mathbb{N}} \Bigg\langle a^*(\varphi_n) a(B_{++}\varphi_n) \prod_{l=1}^{|L'|} a^*(\tilde{\varphi}_{L'_l}) \prod_{c=1}^{|C'|} a(\tilde{\varphi}_{-C'_c})\Omega, \\
\prod_{l=1}^{|L|} a^*(\varphi'_{L_l}) \prod_{c=1}^{|C|} a(\varphi'_{-C_c})\Omega \Bigg\rangle .
\end{aligned}
$$

In the next step we perform the standard anticommutations to move the operator B_{++} from the annihilation operator to the creation operator:

$$
\sum_{L,C \subset \mathbb{N}} \beta_{L,C} \sum_{n \in \mathbb{N}} a^*(\varphi_n) a(B_{++}\varphi_n) \prod_{l=1}^{|L|} a^*(\tilde{\varphi}_{L_l}) \prod_{c=1}^{|C|} a(\tilde{\varphi}_{-C_c})\Omega \tag{3.417}
$$

$$= \sum_{L,C\subset\mathbb{N}} \beta_{L,C} \sum_{b=1}^{|L|} \prod_{l=1}^{b-1} a^*(\tilde{\varphi}_{L_l})(-1)a^*(B_{++}\tilde{\varphi}_{L_b}) \tag{3.418}$$

$$\times \prod_{l=b+1}^{|L|} a^*(\tilde{\varphi}_{L_l}) \prod_{c=1}^{|C|} a(\tilde{\varphi}_{-C_c})\Omega \tag{3.419}$$

$$= \sum_{L,C\subset\mathbb{N}} \beta_{L,C}(-1) \sum_{k\in\mathbb{N}} a^*(B_{++}\varphi_k)a(\varphi_k) \prod_{l=1}^{|L|} a^*(\tilde{\varphi_{L_l}}) \prod_{c=1}^{|C|} a(\tilde{\varphi}_{-C_c})\Omega \tag{3.420}$$

$$= -\sum_{k\in\mathbb{N}} a^*(B_{++}\varphi_k)a(\varphi_k)\beta, \tag{3.421}$$

where we used $B_{++}^* = -B_{++}$. This yields

$$\langle \beta, \mathrm{d}\Gamma(B_{++})\alpha \rangle = -\langle \mathrm{d}\Gamma(B_{++})\beta, \alpha \rangle . \tag{3.422}$$

Real linearity follows directly from the definition of $\mathrm{d}\Gamma$ on $\mathcal{F}^0$ and hence by extention on all of $\mathcal{F}'$. Continuity of the restriction to any $\mathcal{F}'_{m,p}$ follows directly from the forms of the bounds of lemma 62. $\square$

Now we would like to define $e^{\mathrm{d}\Gamma(B)}$, in order to do so, we will show that $\mathrm{d}\Gamma(B)$ is essentially skew-adjoint. One way of doing so is by Nelson's analytic vector theorem.

Theorem 67 (Nelson's analytic vector theorem)**.** *Let C be a symmetric operator on a Hilbert space $\mathcal{H}$. If* $\mathrm{dom}(C)$ *contains a total set $S \subset \bigcap_{n=1}^{\infty} \mathrm{dom}(C^n)$ of analytic vectors, then C is essentially self adjoint. A vector $\phi \in \bigcap_{n=1}^{\infty} \mathrm{dom}(C^n)$ is called analytic if there is $t > 0$ such that $\sum_{k=0}^{\infty} \frac{\|C^n\phi\|}{n!} t^n < \infty$ holds. A set S is said to be total if $\overline{span(S)} = \mathcal{H}$*

For a proof see e.g. [43].

Lemma 68. *For any $\alpha \in \mathcal{F}', t > 0$ and $B \in \mathfrak{B}$ the operator* $\mathrm{d}\Gamma(B) : \mathcal{F}' \to \mathcal{F}$ *satisfies*

$$\sum_{k=0}^{\infty} \frac{\|\mathrm{d}\Gamma(B)^k \alpha\|}{k!} t^k < \infty. \tag{3.423}$$

Proof. By definition of $\mathcal{F}'$ there are $m, p \in \mathbb{N}$ such that $\alpha \in \bigoplus_{l=0}^{m} \bigoplus_{c=0}^{p} \mathcal{F}_{l,p}$. Fix $t > 0$. We dissect α into its parts of fixed particle numbers:

$$\sum_{k=0}^{\infty} \frac{\|\mathrm{d}\Gamma(B)^k \alpha\|}{k!} t^k \leqslant \sum_{l=0}^{m} \sum_{c=0}^{p} \sum_{k=0}^{\infty} \frac{\|\mathrm{d}\Gamma(B)^k \alpha_{l,c}\|}{k!} t^k. \tag{3.424}$$

Using the following abbreviations

$$\Gamma_{-1} := \mathrm{d}\Gamma(B_{-+}) \tag{3.425}$$

$$\Gamma_0 := \mathrm{d}\Gamma(B_{++}) + \mathrm{d}\Gamma(B_{--}) \tag{3.426}$$

$$\Gamma_{+1} := \mathrm{d}\Gamma(B_{+-}) \tag{3.427}$$

$$\beta := \max\{\|B_{++}\| + \|B_{--}\|, \|B_{-+}, \|_{I_2}\} \tag{3.428}$$

we estimate

$$\|\mathrm{d}\Gamma(B)^k \alpha_{l,c}\| \leqslant \sum_{x \in \{-1,0,+1\}^k} \left\| \prod_{b=1}^{k} \Gamma_{x_b} \alpha_{l,c} \right\|$$

$$\leqslant \sum_{x \in \{-1,0,+1\}^k} \prod_{b=1}^{k} \left\| \Gamma_{x_b}|_{\mathcal{F}_{l+\sum_{d=1}^{b-1} x_d, c+\sum_{d=1}^{b-1} x_d}} \right\| \|\alpha_{l,c}\| \tag{3.429}$$

$$\leqslant 3^k \|\alpha\| \max_{x \in \{-1,0,+1\}^k} \prod_{b=1}^{k} \left\| \Gamma_{x_b}|_{\mathcal{F}_{l+\sum_{d=1}^{b-1} x_d, c+\sum_{d=1}^{b-1} x_d}} \right\|. \tag{3.430}$$

At this point the factors only depend on the number of particles the Fock space vector attains as we act on it with the operators $\Gamma_{\#}$ for

$\# \in \{-1, 0, 1\}$. As these bounds increase with the particle number we can restrict the set $\{-1, 0, +1\}$ in the last line to $\{0, +1\}$. We notice that the bound in (3.430) will only increase if we exchange each pair $x_{i+1} = 1, x_i = 0$ by the pair $x_i = 1, x_{i+1} = 0$ so that the norm of the operator that acts like a particle number operator is taken after the particle number is increased. Therefore we for each fixed $\sum_{b=1}^{k} x_b = d$ we can estimate the maximum by lemma 62 to be $\beta^k(c+l+8+2d)^{k-d}$, which we bound by $(2\beta)^k(c/2 + l/2 + 4 + d)^{k-d}$. The Faktor constant in d will be omitted for the maximization problem. For maximising

$$(c/2 + l/2 + 4 + d)^{k-d} \tag{3.431}$$

we treat d as a continuous variable take the derivative and set it to zero. From the form of the function to be maximised it is clear that it is equal to 1 for $d = k$ and at $d = -c/2 - l/2 - 3$, it is be bigger in between. We abbreviate $y = c/2 + l/2 + 4$.

$$\begin{aligned}
0 &= (y + d)^{k-d}(-\ln(y + d) + \frac{k - d}{y + d}) && (3.432)\\
\iff \frac{k - d}{y + d} &= \ln(y + d) && (3.433)\\
\iff \frac{k + y}{y + d} - 1 &= -1 + \ln(e(y + d)) && (3.434)\\
\iff e(k + y) &= e(y + d)\ln(e(y + d)) && (3.435)\\
\iff e(k + y) &= \ln(e(y + d))e^{\ln(e(y+d))} && (3.436)\\
\iff W_0(e(k + y)) &= \ln(e(y + d)) && (3.437)\\
\iff e^{W_0(e(k+y))-1} - y &= d, && (3.438)
\end{aligned}$$

where we made use of the Lambert W function, which is the inverse function of $x \mapsto xe^x$ and has multiple branches; however as $e(y+d) > 0$ W_0 is the only real branch which is applicable here, it corresponds to the inverse of $x \mapsto xe^x$ for $x > -1$. From the form of the maximising value we see, that it is always bigger than $-y$. Plugging this back onto our function we find its maximum

$$\begin{aligned}
&\max_{d\in]-y,\infty[}(y+d)^{k-d} = e^{(W_0(e(k+y))-1)(k+y)-(W_0(e(k+y))-1)e^{W_0(e(k+y))-1}} \\
&= e^{-(k+y)+(k+y)W_0(e(k+y))+e^{W_0(e(k+y))-1}-((k+y)e)/e} \\
&= e^{-2(k+y)+(k+y)W_0((k+y)e)+\frac{e(k+y)}{eW_0((k+y)e)}} \\
&= e^{(k+y)(-2+W_0((k+y)e)+W_0((k+y)e)^{-1})}, \qquad (3.439)
\end{aligned}$$

where we repeatedly used $W_0(x)e^{W_0(x)} = x$. Putting things together we find

$$\|\Gamma(B)^k\alpha_{l,c}\| \leqslant (6\beta)^k\|\alpha\|e^{(k+y)(-2+W_0((k+y)e)+W_0((k+y)e)^{-1})}. \qquad (3.440)$$

Dividing this by $k!$ and using the lower bound given by Sterling's formula we would like to prove that

$$\sum_{k=1}^{\infty}(6\beta t)^k e^{k(1-\ln(k))-\frac{1}{2}\ln(k)+(k+y)(-2+W_0((k+y)e)+W_0((k+y)e)^{-1})} < \infty \qquad (3.441)$$

holds, where we neglected constant factors and the summand $k = 0$ which do not matter for the task at hand. Next we are going to use an inequality about the growth of W_0 proven in [21]. For any $x \geqslant e$

$$W_0(x) \leqslant \ln(x) - \ln(\ln(x)) + \frac{e}{e-1}\frac{\ln(\ln(x))}{\ln(x)} \qquad (3.442)$$

holds true. Plugging this into our sum the exponent is bounded from above by

$$
\begin{aligned}
&k(1-\ln(k)) - \frac{1}{2}\ln(k) + (k+y)\Big[-1+\ln(k+y)-\ln(1+\ln(k+y)) \\
&+\frac{e}{e-1}\frac{\ln(1+\ln(k+y))}{1+\ln(k+y)} + W_0((k+y)e)^{-1}\Big] \\
&= -y + k\ln\left(1+\frac{y}{k}\right) + y\ln(k+y) - \frac{1}{2}\ln(k) + \\
&(k+y)\left[\ln(1+\ln(k+y))\frac{1-(e-1)\ln(k+y)}{(e-1)(1+\ln(k+y))} + W_0((k+y)e)^{-1}\right] \\
&\leqslant y\ln(k+y) - \frac{1}{2}\ln(k) + (k+y)W_0((k+y)e)^{-1} + \qquad (3.443) \\
&(k+y)\ln(1+\ln(k+y))\frac{1-(e-1)\ln(k+y)}{(e-1)(1+\ln(k+y))}.
\end{aligned}
$$

Now it is important to notice that the only remaining term that grows faster than linearly in magnitude is the last summand. This term; however, is negative for large k, as the fraction converges to -1 for large k, while the double logarithm in front grows without bounds. So there is a k^* big enough such that for all $k > k^*$ (3.443) is smaller than $-k(\ln(6\beta t)+1)$, proving that (3.441) in fact holds.

□

Theorem 69. *The operator* $\mathrm{d}\Gamma(B) : \mathcal{F}' \to \mathcal{F}$ *is essentially skew adjoint and hence by Stones theorem generates a strongly continuous unitary group* $\left(e^{t\ \widehat{\mathrm{d}\Gamma(B)}}\right)_t$*, where* $\widehat{\mathrm{d}\Gamma(B)}$ *is the closure of* $\mathrm{d}\Gamma(B)$*.*

Proof. In order to apply Nelson's analytic vector theorem we pick $S = \mathcal{F}'$. Pick $\alpha \in \mathcal{F}'$. We need to show that there is $t > 0$ such that

$$\sum_{k=0}^{\infty} \frac{\|\mathrm{d}\Gamma(B)^k \alpha\|}{k!} t^k < \infty \qquad (3.444)$$

holds. This is guaranteed by the last lemma. □

Lastly in this chapter, we will investigate the commutation properties of $\mathrm{d}\Gamma(B)$ with general creation and annihilation operators. These properties are the reason we are interested in this operator, they will prove to be very useful in the next chapter.

Theorem 70. *For $\psi \in \mathcal{H}$ and $\alpha \in \mathcal{F}'$ we have*

$$[\mathrm{d}\Gamma(B), a^{\#}(\psi)]\alpha = a^{\#}(B\psi)\alpha, \tag{3.445}$$

where $a^{\#}$ can be either a or a^*.

Proof. Because $\mathrm{d}\Gamma(B)$ is defined as the extension of an operator on $\mathcal{F}^0$ it suffices to show the desired identity on this space. We will do the case $a(\psi)$, the other case is completely analogous. As a first step we decompose $\mathrm{d}\Gamma(B)$ into its four parts

$$[\mathrm{d}\Gamma(B), a(\psi)] = [\mathrm{d}\Gamma(B_{++}) + \mathrm{d}\Gamma(B_{-+}) + \mathrm{d}\Gamma(B_{-+}) + \mathrm{d}\Gamma(B_{--}), a(\psi)]\,, \tag{3.446}$$

each of those parts is evaluated directly. We begin with the B_{++} part, this can be expressed as

$$[\mathrm{d}\Gamma(B_{++}), a(\psi)] \tag{3.447}$$

$$= \sum_{n\in\mathbb{N}} a^*(B_{++}\varphi_n)a(\varphi_n)a(\psi) - \sum_{n\in\mathbb{N}} a(\psi)a^*(B_{++}\varphi_n)a(\varphi_n) \tag{3.448}$$

$$= \sum_{n\in\mathbb{N}} [-\langle\psi, B_{++}\varphi_n\rangle a(\varphi_n) + a(\psi)a^*(B_{++}\varphi_n)a(\varphi_n)] \tag{3.449}$$

$$- \sum_{n\in\mathbb{N}} a(\psi)a^*(B_{++}\varphi_n)a(\varphi_n). \tag{3.450}$$

Let $\alpha \in \mathcal{F}^0$. Now applying the expression in the last two lines to α, considering each $\alpha_{m,p} \in \mathcal{F}_{m,p}$ separately and commuting the annihilation operators in (3.449) and (3.450) to the right, the sums over n will be absolutely convergent. hence we may split the firs sum into two and observe the cancellation between the last two terms. Continuing we find

$$[\mathrm{d}\Gamma(B_{++}), a(\psi)]\,\alpha = -\sum_{n\in\mathbb{N}}\langle\psi, B_{++}\varphi_n\rangle a(\varphi_n)\alpha \tag{3.451}$$

$$= -a\left(\sum_{n\in\mathbb{N}}\langle B_{++}\varphi_n, \psi\rangle\varphi_n\right)\alpha = a\left(\sum_{n\in\mathbb{N}}\langle\varphi_n, B_{++}\psi\rangle\varphi_n\right)\alpha \tag{3.452}$$

$$= a(B_{++}\psi)\alpha, \tag{3.453}$$

where we used $B^* = -B^*$. The final extension of this equation to all $\alpha \in \mathcal{F}'$ happens via the continuous linear extension theorem on $\mathcal{F}_{m,p}$ for each $m, p \in \mathbb{N}$. The proof in all seven other cases are completely analogous, except that the off diagonal terms switch. More precisely, from the excatly analogous calculation it follows that

$$\left[\mathrm{d}\Gamma(B_{-+}), a^{\#}(\psi)\right]\alpha = a^{\#}\left(B_{+-}\psi\right)\alpha \tag{3.454}$$

and

$$\left[\mathrm{d}\Gamma(B_{+-}), a^{\#}(\psi)\right]\alpha = a^{\#}\left(B_{-+}\psi\right)\alpha \tag{3.455}$$

hold. Putting things together again we obtain

$$[\mathrm{d}\Gamma(B_{++}), a(\psi)] + [\mathrm{d}\Gamma(B_{-+}), a(\psi)] \tag{3.456}$$

$$+ [\mathrm{d}\Gamma(B_{+-}), a(\psi)] + [\mathrm{d}\Gamma(B_{--}), a(\psi)] = \tag{3.457}$$

$$a(B_{++}\psi) + a(B_{+-}\psi) + a(B_{-+}\psi) + a(B_{--}\psi) \iff \tag{3.458}$$

$$[\mathrm{d}\Gamma(B), a(\psi)] = a(B\psi) \tag{3.459}$$

on all of $\mathcal{F}'$.

□

3.4.2 Presentation and Proof of the Formula

In this chapter we verify the formula for the S-matrix directly. For a heuristic derivation see section 4.3 of the appendix.

Theorem 71 (Analyticity for small A)**.** *Let $A \in \mathcal{V}$ be such that the one particle scattering operator S^A fulfills*

$$\|\mathbb{1} - S^A\| < 1. \tag{3.460}$$

Then the operator

$$\tilde{S}^A = e^{\mathrm{d}\Gamma(\ln(S^A))} \tag{3.461}$$

is a lift of S^A, where the logarithm is defined by its taylor series around the identity.

Proof. In order to establish this theorem we need to verify that the expression given in equation (3.461) for the scattering operator is a well-defined object and fulfils the (lift condition).
Well-definedness is established, by theorem 69, because for unitary S^A with $\|1 - S^A\| < 1$ the power series of the logarithm converges and fulfils

$$\|\ln(S^A)\| = \|\ln(1-(1-S^A))\| = \left\| -\sum_{k=1}^{\infty} \frac{(1-S^A)^k}{k} \right\| \tag{3.462}$$

$$\leqslant \sum_{k=1}^{\infty} \frac{\|1-S^A\|^k}{k} = -\ln(1-\|1-S^A\|) \tag{3.463}$$

implying that the power series of the logarithm around the identity is a well-defined map from the one particle operators of norm less than one to the bounded one particle operators. Moreover this operator fulfils $[\ln(S^A)]^* = \ln(S^A)^* = \ln(S^A)^{-1} = -\ln(S^A)$, so $\mathrm{d}\Gamma(\ln S^A)$ is a well-defined unbounded operator that is essentially skew adjoint on the finite particle sector of Fock space $\mathcal{F}'$. Finally the off diagonal Hilbert-Schmidt norm can also be controlled by the same norm of S^A:

$$\|P^+ \ln(S^A)P^-\|_{I_2} = \|P^+ \ln(1-(1-S^A))P^-\|_{I_2} \tag{3.464}$$

$$= \left\| -P^+ \sum_{k=1}^{\infty} \frac{(1-S^A)^k}{k} P^- \right\|_{I_2} \tag{3.465}$$

$$\leqslant \|S^A_{+-}\|_{I_2} \sum_{k=1}^{\infty} \max\{\|1-S^A\|, \|P^- - S^A_{--}\|\}^{k-1} \tag{3.466}$$

$$= \frac{\|S^A_{+-}\|_{I_2}}{1-\|1-S^A\|}. \tag{3.467}$$

Let $\varphi \in \mathcal{H}$ and $\alpha \in \mathcal{F}'$, for any $k \in \mathbb{N}_0$ we see applying the commutation relation of $\mathrm{d}\Gamma$:

$$\begin{aligned}
&\mathrm{d}\Gamma(\ln U) \sum_{l=0}^{k} \binom{k}{l} a\left((\ln U)^l \varphi\right) (\mathrm{d}\Gamma(\ln U))^{k-l} \alpha \\
&\qquad = \sum_{l=0}^{k} \binom{k}{l} a\left((\ln U)^{l+1} \varphi\right) (\mathrm{d}\Gamma(\ln U))^{k-l} \alpha \\
&\qquad + \sum_{l=0}^{k} \binom{k}{l} a\left((\ln U)^{l} \varphi\right) (\mathrm{d}\Gamma(\ln U))^{k-l+1} \alpha \\
&= \sum_{b=0}^{k+1} \left(\binom{k}{b-1} + \binom{k}{b} \right) a\left((\ln U)^b \varphi\right) (\mathrm{d}\Gamma(\ln U))^{k+1-b} \alpha \\
&\qquad = \sum_{b=0}^{k+1} \binom{k+1}{b} a\left((\ln U)^b \varphi\right) (\mathrm{d}\Gamma(\ln U))^{k+1-b} \alpha,
\end{aligned}$$

so we see that for $k \in \mathbb{N}_0$

$$(\mathrm{d}\Gamma(\ln U))^k a(\varphi)\alpha = \sum_{b=0}^{k} \binom{k}{b} a\left((\ln U)^b \varphi\right) (\mathrm{d}\Gamma(\ln U))^{k-b} \alpha \tag{3.468}$$

holds. Using what we just obtained, we conclude

$$
\begin{aligned}
e^{\mathrm{d}\Gamma(\ln U)}a(\varphi)\alpha &= \sum_{k=0}^{\infty}\frac{1}{k!}\left(\mathrm{d}\Gamma(\ln U)\right)^k a(\varphi)\alpha \\
&= \sum_{k=0}^{\infty}\frac{1}{k!}\sum_{b=0}^{k}\binom{k}{b}a\left((\ln U)^b\varphi\right)\left(\mathrm{d}\Gamma(\ln U)\right)^{k-b}\alpha \\
&\overset{*}{=} \sum_{c=0}^{\infty}\sum_{l=0}^{\infty}\frac{1}{c!l!}a\left((\ln U)^c\varphi\right)\left(\mathrm{d}\Gamma(\ln U)\right)^{l}\alpha \\
&= a\left(e^{\ln U}\varphi\right)e^{\mathrm{d}\Gamma(\ln U)}\alpha = a\left(U\varphi\right)e^{\mathrm{d}\Gamma(\ln U)}\alpha.
\end{aligned}
$$

For the marked equality changing oder of summation is justified, because by the bounds $\|a((\ln U)^c\varphi)\| \leqslant \|\ln U\|^c$ and lemma 68 the sum obtained by changing the order of summands converges absolutely. Clearly multiplying the second quantised operator by an additional phase as in (71) does not influence this calculation. So (lift condition) holds when applied to any $\alpha \in \mathcal{F}'$ and can be continued to all of $\mathcal{F}$ by continuity of $\tilde{S}$. □

The last theorem can be restated as for A small enough there is a power series of operators on $\mathcal{F}$ that converges against a lift of S^A. Power series in A is used here in the sense that it is of the form $\sum_{k\in\mathbb{N}_0} T_k(A)$, where $T_K(A)$ is homogeneous in A of degree k. The next Theorem establishes such a power series for all $A \in \mathcal{V}$.

Theorem 72 (Analyticity for all A)**.** *Let $A \in \mathcal{V}$ and S^A the corresponding one-particle scattering operator. There is a lift $\tilde{S}^A$ of S^A that fulfills*

$$\tilde{S}^A = \sum_{k\in\mathbb{N}_0} T_k(A), \tag{3.469}$$

where $T_k(A)$ are unbounded operators defined on $\mathcal{F}'$ that are homogeneous of degree k in A. This series is strongly convergent on $\mathcal{F}'$.

Proof. Pick $A \in \mathcal{V}$. Recall the definition 34 of the one-particle scattering operator S^A

$$S^A = U^0_{\Sigma_{\mathrm{in}},\Sigma_{\mathrm{out}}} U^A_{\Sigma_{\mathrm{out}},\Sigma_{\mathrm{in}}}. \tag{3.470}$$

Pick $N \in \mathbb{N}$ such that

$$S^A_k = U^{(k-1)A/N}_{\Sigma_{\mathrm{in}},\Sigma_{\mathrm{out}}} U^{Ak/N}_{\Sigma_{\mathrm{out}},\Sigma_{\mathrm{in}}} \tag{3.471}$$

fulfills

$$\|1 - S^A_k\| < 1 \tag{3.472}$$

for all $0 < k < N$. Then by the theorem 71

$$\tilde{S}^A_k = e^{\mathrm{d}\Gamma(\ln(S^A_k))} \tag{3.473}$$

is a lift of S^A_k for each k, so the product

$$\tilde{S}^A = \prod_{k=1}^{N} \tilde{S}^A_k \tag{3.474}$$

is a lift of S^A. Pick $\alpha \in \mathcal{F}^0$, next we show that $\tilde{S}^A\alpha$ is given by a convergent power series and hence by linear extension it is also convergent on $\mathcal{F}'$, which finishes the proof. Pick the basis $(\varphi_k)_{k\in\mathbb{N}}$ and $(\varphi_{-k})_{k\in\mathbb{N}}$ of $\mathcal{H}^+$ and $\mathcal{H}^-$ respectively such that α is a finite linear combination of the form

$$\alpha = \sum_{\substack{L,C\subset\mathbb{N}\\ |L|=m,|C|=p}} \alpha_{L,C} \prod_{l=1}^{m} a^*(\varphi_{L_l}) \prod_{k=1}^{p} a(\varphi_{-C_k})\Omega. \tag{3.475}$$

We calculate

$$\tilde{S}^A\alpha = \prod_{k=1}^{N} \tilde{S}^A_k\alpha = \prod_{k=1}^{N-1} \tilde{S}^A_k \sum_{k_N\in\mathbb{N}_0} \frac{1}{k_N!}\left(\mathrm{d}\Gamma(\ln S^A_N)\right)^{k_N} \alpha \tag{3.476}$$

$$= \prod_{k=1}^{N-2} \tilde{S}^A_k \sum_{k_N\in\mathbb{N}_0} \tilde{S}^A_{N-1}\frac{1}{k_N!}\left(\mathrm{d}\Gamma(\ln S^A_N)\right)^{k_N} \alpha, \tag{3.477}$$

where by continuity of the unitary $\tilde{S}_l^A$ we may pull them into the sum, and expand the exponential, since inside the sum its argument is again in $\mathcal{F}'$. We may continue this process by induction. Since all of these sums are norm convergent, we may forget about the order in which they are to be carried out in our notation:

$$\tilde{S}^A\alpha = \sum_{k_1,\dots,k_N\in\mathbb{N}_0} \prod_{l=1}^{N} \frac{1}{k_l!}\Big(\mathrm{d}\Gamma(\ln S_l^A)\Big)^{k_l} \quad \alpha. \tag{3.478}$$

Since $\alpha \in \mathcal{F}'$, there is a maximal number of particles of each summand. This implies that $d\Gamma$ is continuous as a function of $\ln S_l^A$. The logarithm of S_l^A is given by a norm convergent series in A,

$$\ln S_l^A = -\sum_{k=1}^{\infty}\frac{1}{k}(1-S_l^A)^k = \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\Big(\sum_{c\in\mathbb{N}_0}\frac{1}{c!}Z_{l,c}^A\Big)^k \tag{3.479}$$

$$= \sum_{\substack{k\in\mathbb{N}\\ c\in\mathbb{N}_0^k}}\frac{(-1)^{k+1}}{k\ c!}\prod_{v=1}^{k} Z_{l,c_v}^A. \tag{3.480}$$

This may be plugged into the expression for $\tilde{S}^A$ and the sum may be pulled out of $\mathrm{d}\Gamma$ due to linearity and continuity

$$\tilde{S}^A\alpha = \sum_{k\in\mathbb{N}_0^N}\frac{1}{k!}\prod_{l=1}^{N}\left(\sum_{\substack{b\in\mathbb{N}\\ c\in\mathbb{N}_0^b}}\frac{(-1)^{b+1}}{b\ c!}\mathrm{d}\Gamma\Big(\prod_{v=1}^{b} Z_{l,c_v}^A\Big)\right)^{k_l}\alpha. \tag{3.481}$$

Since all of these sums are absolutely convergent, we may pull out the innermost sums as well and finally change the order of summation according to the degree of homogeneity in A of the summands. The result is a strongly convergent power series on $\mathcal{F}'$ as claimed in the theorem.

$\square$

3.5 The Relationship Between Hadamard States and Admissible Polarisation Classes

This section compares central objects of two different types of approaches to quantum field theory (QFT): The study of Hadamard states and admissible polarisation classes related to sections 3.1 and 3.2 as well as [3, 5, 4] and future work generalising section 3.3 to finite times. These approaches have different scopes and pursue different motivations, which makes a direct comparison difficult. Nevertheless, the basic protagonists the two approaches share much more similarity than is apparent at first glance. It is the purpose of this section to highlight these common features.

In subsection 3.5.1 we give the definition of a Hadamard state, briefly motivate it's usage and give its explicit form in the case of flat spacetime subject to an external field as computed in the physics literature [46].

In subsection 3.5.2 we briefly describe why in the approach of admissible polarisation classes one only keeps track of the time evolution of the projector up to an error that is a Hilbert-Schmidt operator. Furthermore, we will give a class of candidate I_2-almost projectors that have a simple time evolution.

In section 3.5.3 we find that each Hadamard state corresponds to an I_2-almost projector in a natural way.

We choose a fixed four-potential $A \in \mathcal{V}$ and Σ_{in} denotes a Cauchy surface earlier than the support of A.

For brevity of notation we denote the minimally coupled differential by $\nabla_\alpha = \partial_\alpha + iA_\alpha$.

3.5.1 Hadamard States

In the algebraic approach to QFT one puts less emphasise on the Hilbert space than is commonly done in non relativistic physics because it is not a relativistically invariant object. Instead of introducing a sequence of Fock spaces as was motivated in sections 3.1 and 3.2, one focuses on the algebra of operators that are chosen to do the bookkeeping of statistical outcomes of measurements.
To infer predictions, some part of the necessary computation can be conducted on the level of this algebra. However, eventually, expectation values are to be computed in a certain representation, usually found by the GNS construction with respect to a certain state. This choice has to be made on physical grounds. Hadamard states are often thought to be physically sensible states because they have positive energy in a certain sense.
In order to introduce Hadamard states first we have to define the notion of wavefront set, which itself needs some preliminaries. For the introduction of these concepts we follow Hömander [22, Chapter 8].
Throughout this section we will consider a fixed but arbitrary four-potential $A \in \mathcal{V}$. We begin by introducing the singular support of a distribution

Definition 73. *Let for $n, m \in \mathbb{N}$, $v \in (C^\infty(\mathbb{R}^n, \mathbb{C}^m))'$ the singular support of v is defined to be the subset of points $x \in \mathbb{R}^n$ such that there is no neighbourhood U of x such that there is a smooth function $\phi_{x,v} \in C^\infty(\mathbb{R}^n, \mathbb{C}^m)$ such that v acts on test functions $\varphi \in C_c^\infty(U, \mathbb{C}^m)$ as*

$$v(\varphi) = \int \phi_{x,v}^\dagger(x)\varphi(x)dx. \tag{3.482}$$

The singular support contains all the points of a distribution such that the distribution does not act like a smooth function at that point. The wavefront set which we are about to introduce gives additional directional information of where these singularities propagate. We incorporate this information by the Fourier transform in the following definition.

Definition 74. *Let for $n, m \in \mathbb{N}$, $v \in (C^\infty(\mathbb{R}^n, \mathbb{C}^m))'$, we denote by $\Xi(v) \subset \mathbb{R}^n \backslash \{0\}$ the set of all η such that there is no cone $V_\eta \subset \mathbb{R}^n$, neighbourhood of η, such that for all $a \in \mathbb{N}$ there is a $C_a > 0$ such that for all $\xi \in V_\eta$ we have*

$$|\hat{v}(\xi)| \leqslant \frac{C_a}{1 + |\xi|^a}. \tag{3.483}$$

Furthermore for each $x \in \mathbb{R}^n$ we define

$$\Xi_x(v) := \bigcap_{\substack{\phi \in C_c^\infty(\mathbb{R}^n) \\ x \in \operatorname{supp}(\phi)}} \Xi(v\phi), \tag{3.484}$$

Where $v\phi$ is the pointwise multiplication of a distribution and a scalar test function, which acts as $v\phi : C^\infty(\mathbb{R}^n, \mathbb{C}^m) \ni \psi \mapsto v(\psi\phi)$.
Analogously for a tempered distribution that takes two arguments $v : \mathcal{S}(\mathbb{R}^4) \times \mathcal{S}(\mathbb{R}^4) \to \mathbb{C}$ we define

$$\Xi_{x,y}(v) := \bigcap_{\substack{\phi \\ (x,y) \in \operatorname{supp}(\phi)}} \Xi(v\phi), \tag{3.485}$$

where now $\phi_{x,y} = \psi_x \chi_y \in C_c^\infty(\mathbb{R}^8)$ with $\psi_x, \chi_y \in C_c^\infty(\mathbb{R}^4)$ and $x \in \operatorname{supp}(\psi_x), y \in \operatorname{supp}(\chi_y)$.

We have collected the tools to introduce the wavefront set and the notion of Hadamard states.

Definition 75 (Definition 3.1 of [55])**.** *Let for $n, m \in \mathbb{N}$, $v \in (\mathcal{S}(\mathbb{R}^n, \mathbb{C}^m))'$ be a tempered distribution. The wavefront set $\mathrm{WF}(v)$ of the distribution v is defined as*

$$\mathrm{WF}(v) := \{(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n \mid \xi \in \Xi_x(v)\}. \tag{3.486}$$

Analogously for a tempered distribution $v : \mathcal{S}(\mathbb{R}^4, \mathbb{C}^4) \times \mathcal{S}(\mathbb{R}^4, \mathbb{C}^4) \to \mathbb{C}$ *we define the wavefront set as*

$$\mathrm{WF}(v) := \left\{ (x, \xi; y, \xi') \in \mathbb{R}^{16} \mid (\xi, \xi') \in \Xi_{x,y}(v) \right\}. \tag{3.487}$$

Definition 76. *A map* $H : \mathcal{S}(\mathbb{R}^4, \mathbb{C}^4) \times \mathcal{S}(\mathbb{R}^4, \mathbb{C}^4) \to \mathbb{C}$*, is called Hadamard state if it fulfils for all* $f, g \in \mathcal{S}(\mathbb{R}^4, \mathbb{C}^4)$*:*

$$H(Df, g) = 0 \tag{3.488}$$
$$H(f, g) + H(g, f) = iS_{prop}(f, g) \tag{3.489}$$
$$\overline{H(f, g)} = H(\overline{f}, \overline{g}) \tag{3.490}$$
$$WF(H) \subset C_+, \tag{3.491}$$

where $S_{prop}(f, g)$ *is the propagator of the Dirac equation, and* $C_+ := \{(x, y; k_1, -k_1) \in \mathbb{R}^{16} \mid (x; k_1) \approx (y; k_2), k_1^2 \geqslant 0, k_1^0 > 0\}$ *and* $(x; k_1) \approx (y; k_2)$ *holds whenever* $(x - y)^2 = 0$ *and* $(y - x) \parallel k_1$*.*

It is in the sense of the fourth condition that Hadamard states are of positive energy.
In the scenario of Minkowski spacetime in an external field Dirac [12] already studied the Hadamard states, although that name was not established at the time. More recently the subject has attracted considerable attention. The Hadamard states were computed in [55, 46]. They are given in terms of the Klein-Gordon operator corresponding to Dirac's equation:

Definition 77. *The Klein-Gordon operator corresponding to the Dirac equation in an external field* (3.3) *reads*

$$P : C^\infty(\mathbb{R}^4, \mathbb{C}^4) \to C^\infty(\mathbb{R}^4, \mathbb{C}^4) \tag{3.492}$$
$$P = (i\nabla\!\!\!/ - m)(-i\nabla\!\!\!/ - m) = \nabla_\alpha \nabla^\alpha + \frac{i}{2}\gamma^\alpha \gamma^\beta F_{\alpha,\beta} + m^2, \tag{3.493}$$

where $F_{\alpha,\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$ is the field strength tensor of the electromagnetic field. Furthermore we define for $f \in \mathcal{C}^\infty(\mathbb{R}^4 \times \mathbb{R}^4, \mathbb{C}^{16})$ the differential operator

$$\overleftarrow{\nabla}^* f(x,x') = \left(\frac{\partial}{\partial y^\alpha} - iA_\alpha(y)\right) f(x,y)\gamma^\alpha. \tag{3.494}$$

For the special case of a Dirac field in Minkowski space-time Zahn [46] gave a more explicit form of the Hadamard states $H \in (\mathcal{S}(\mathcal{M}, \mathbb{C}^4) \times \mathcal{S}(\mathcal{M}, \mathbb{C}^4))'$ on which we base our analysis below. According to this, H acts for $f_1, f_2 \in C_c^\infty(\mathcal{M}) \otimes \mathbb{C}^4$ as

$$H(f_1, f_2) = \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^4} d^4x \overline{f_1}(x) \int_{\mathbb{R}^4} d^4y \; h_\varepsilon(x,y) f_2(y), \tag{3.495}$$

where h_ε is of the form

$$h_\varepsilon(x,y) =$$

$$\frac{-1}{2(2\pi)^2}(-i\nabla\!\!\!/ + i\overleftarrow{\nabla}^* - 2m)\left[\frac{e^{-i(x-y)^\alpha \int_0^1 ds A_\alpha(xs+(1-s)y)}}{(y-x-i\varepsilon e_0)^2}\right] \tag{3.496}$$

$$+\frac{-1}{2(2\pi)^2}(-i\nabla\!\!\!/ + i\overleftarrow{\nabla}^* - 2m)\Big[V(x,y)\ln(-(y-x-i\varepsilon e_0)^2)\Big] \tag{3.497}$$

$$+ B(x,y), \tag{3.498}$$

where $V, B : \mathbb{R}^4 \to \mathbb{C}^4$ are smooth functions, B is completely arbitrary, whereas V is fixed by the external potential. The expansion

$$V^N(x,y) := \sum_{k=1}^N \frac{1}{4^k k!(k-1)!} V_k(x,y)\big((x-y)^2\big)^{k-1}, \tag{3.499}$$

is an asymptotic expansion for V for $N \to \infty$, in the sense that $(V - V^N)(x,y)\ln(-(x-y)^2)$ as a function of x and y is in $C^{N-2}(\mathbb{R}^{4+4})$

and $(V - V^N)(x,y) = \mathcal{O}\left(((x-y)^2)^{N-2}\right)$. The functions V_k fulfil the recursive set of partial differential differential equations

$$(x-y)^\alpha(\partial_{x,\alpha} + iA_\alpha(x))V_n(x,y) + nV_n(x,y) = -nPV_{n-1}(x,y), \tag{3.500}$$

where $V_0(x,y) = e^{-i(x-y)^\alpha \int_0^1 ds A_\alpha(xs+(1-s)y)}$.
For the rest of this paper, we assume that for any $A \in C_c^\infty(\mathbb{R}^4)$ there are H, $(h_\varepsilon)_{\varepsilon>0}$ and V fulfilling all of the conditions described in this subsection.

3.5.2 Projectors for Polarisation Classes

Polarisation classes as introduced in definition 28 can be represented by a projector onto a polarisation of that class. Alternatively it can also be represented by an operator that is Hilbert-Schmidt close to such a projector, these will be called I_2-almost projectors.
We follow [4] in introducing the necessary notation to do so. We start out by characterising the polarisation classes for free motion.

Definition 78 (kernel fo P^-)**.** *The projector $P_\Sigma^{\mathcal{H}^-}$ has the well known representation as the weak limit of the integral operator with the kernel[4]*

$$p_\varepsilon^-(x,y) = -\frac{m^2}{4\pi^2}(i\partial\!\!\!/_x + m)\frac{K_1(m\sqrt{-(y-x-i\varepsilon e_0)^2})}{m\sqrt{-(y-x-i\varepsilon e_0)^2}}, \tag{3.501}$$

where the square is a Minkowski square, the square root denotes its principle value and K_1 is a modified Bessel function. By weak limit we mean

$$\langle \phi, P_\Sigma^{\mathcal{H}^-}\psi\rangle = \lim_{\varepsilon\searrow 0}\int_{\Sigma\times\Sigma} \overline{\phi}(x) i_\gamma(d^4x) p_\varepsilon^-(x,y) i_\gamma(d^4y)\psi(y), \tag{3.502}$$

for general $\phi, \psi \in \mathcal{H}_\Sigma$.

Remark 79. *By inserting the expansion of K_1 in terms of a Laurent series and a logarithm, [1] one obtains:*

$$K_1(\xi) = \frac{1}{\xi} - \frac{\xi}{4}\sum_{k=0}^{\infty}\left(2\psi(k+1) + \frac{1}{k+1} + 2\ln 2 - 2\ln\xi\right)\frac{\left(\frac{\xi^2}{4}\right)^k}{k!^2(k+1)} \tag{3.503}$$

$$:= \frac{1}{\xi} + \xi Q_1(\xi^2)\ln\xi + \xi Q_2(\xi^2) =: \frac{1}{\xi} + \xi Q_3(\xi). \tag{3.504}$$

It is not obvious from the equation (3.500) but well known that the vacuum of Minkowski spacetime does indeed correspond to a Hadamard state subject to vanishing four potential. In fact, the Hadamard states were constructed to agree with the Minkowski vacuum up to smooth terms.

Because all we are interested in is representations of equivalence classes, we are content with finding objects that differ from a Projector onto $U^A_{\Sigma,\Sigma_{\mathrm{in}}}\mathcal{H}^-_{\Sigma_{in}}$ by a Hilbert-Schmidt operator. Therefore we need not keep track of the exact evolution of the projectors, but define a whole class of admissible ones.

Definition 80 (kernel of P^λ, I_2-almost projectors)**.** *The set $\mathcal{G}^A$ denotes the set of all functions $\lambda^A \in C^\infty(\mathbb{R}^4 \times \mathbb{R}^4, \mathbb{R})$ that satisfy*

- *i) There is a compact set $K \subset \mathbb{R}^4$ such that $\operatorname{supp}\lambda \subseteq K \times \mathbb{R}^4 \cup \mathbb{R}^4 \times K$.*
- *ii) λ satisfies $\forall x \in \mathbb{R}^4 : \lambda(x,x) = 0$.*
- *iii) On the diagonal the first derivatives fulfil*

$$\forall x, y \in \mathbb{R}^4 : \partial_x\lambda(x,y)|_{y=x} = -\partial_y\lambda(x,y)|_{y=x} = A(x). \tag{3.505}$$

We futhermore define a corresponding I_2-almost projector P^λ:

$$\langle\phi, P^\lambda_\Sigma\psi\rangle = \lim_{\varepsilon\searrow 0}\langle\phi, P^{A,\varepsilon}_\Sigma\psi\rangle, \tag{3.506}$$

$$\langle\phi, P^{\lambda,\varepsilon}_\Sigma\psi\rangle := \int_{\Sigma\times\Sigma}\overline{\phi}(x)i_\gamma(d^4x)\,\overbrace{e^{-i\lambda(x,y)}p^-_\varepsilon(y-x)}^{=:p^\lambda_\varepsilon(x,y)}\,i_\gamma(d^4y)\psi(y), \tag{3.507}$$

for general $\phi,\psi\in\mathcal{H}_\Sigma$.

Remark 81 (theorem 2.8 and 1.5 of [4])**.** *P^λ_Σ and $P^\lambda_{\Sigma'}$ are equivalent if transported appropriately by time evolution operators:*

$$P^\lambda_\Sigma - U^A_{\Sigma,\Sigma'}P^\lambda_{\Sigma'}U_{\Sigma',\Sigma}\in I_2(\mathcal{H}_\Sigma). \tag{3.508}$$

Also for four-potentials $A,B\in\mathcal{V}$ the corresponding projectors are equivalent if and only if the four-potentials projected onto the hypersurface agree:

$$P^{\lambda^A}_\Sigma - P^{\lambda^B}_\Sigma\in I_2(\mathcal{H}_\Sigma)\iff\forall x\in\Sigma\ \forall z\in T_x\Sigma: z^\alpha(A_\alpha(x)-B_\alpha(x))=0. \tag{3.509}$$

Taking into account the freedom within each classification the notions Hadamard state and projectors of polarisation classes are extremely close. This is the topic of the next section.

3.5.3 Comparing Hadamard States and P^λ_Σ

The following theorems is the basis of our comparison between Hadamard states and I_2- almost Projectors.

Theorem 82. *Given a four-potential $A\in\mathcal{V}$, and a Hadamard state H of the form (3.495) to (3.498), there is a family of smooth functions*

$(w_\varepsilon)_{\varepsilon>0}$ *in* $C^\infty(\mathbb{R}^4 \times \mathbb{R}^4, \mathbb{C}^{4\times 4})$*, such that for any Cauchy surface* Σ *there is an operator acting as*

$$\mathcal{H}_\Sigma \ni \psi \mapsto \tilde{P}\psi = \lim_{\varepsilon\to 0} \int_\Sigma (h_\varepsilon - w_\varepsilon)(\cdot, y) i_\gamma(d^4y)\psi(y). \tag{3.510}$$

This operator $\tilde{P}$ *is bounded on* $\mathcal{H}_\Sigma$ *and fulfils* $P^{\lambda^A} - \tilde{P} \in I_2(\mathcal{H}_\Sigma)$ *for any* $\lambda^A \in \mathcal{G}^A$ *. Additionally, the pointwise limit of* $(w_\varepsilon)_\varepsilon$ *for* $\varepsilon \to 0$ *is smooth.*

Theorem 83. *Given a four-potential* $A \in \mathcal{V}$ *and a* $\lambda^A \in \mathcal{G}^A$*, there is a family of functions* $(w_\varepsilon)_\varepsilon > 0$ *in* $C^\infty(\mathbb{R}^4 \times \mathbb{R}^4, \mathbb{C}^{4\times 4})$ *such that for all Cauchy surfaces* Σ *the restriction* $w_\varepsilon|_{\Sigma\times\Sigma}$ *can be split*

$$w_\varepsilon|_{\Sigma\times\Sigma} = w_{1,\varepsilon} + w_{2,\varepsilon}, \tag{3.511}$$

such that $w_{1,\varepsilon}$ *has a* $L^2(\Sigma\times\Sigma, \mathbb{C}^{4\times 4})$ *limit for* $\varepsilon \to 0$ *and* $w_{2,\varepsilon}$ *has a smooth pointwise limit and is continuous as a function of type* $\Sigma \times \Sigma \times [0,1] \to \mathbb{C}^{4\times 4}$*. Furthermore,* $\tilde{H}$ *acting on test functions* $f_1, f_2 \in C^\infty(\mathbb{R}^4, \mathbb{C}^4)$ *as*

$$\tilde{H}(f_1, f_2) = \lim_{\varepsilon\to 0} \int_{\mathbb{R}^4\times\mathbb{R}^4} \overline{f_1}(x) \left(p_\varepsilon^{\lambda^A}(x,y) - w_\varepsilon(x,y)\right) f_2(y) d^4x d^4y \tag{3.512}$$

is a Hadamard state of the form (3.495) *to* (3.498)*.*

The proofs are to be found in section 4.4 of the appendix.

Remark 84. *In the approach of polarisation classes, two projectors represent the same polarisation class if and only if their difference is a Hilbert-Schmidt operator. Because of this fact, it is enough to keep track of the time evolution of a projector only up to changes by a Hilbert-Schmidt operator, i.e. it is enough to find the time evolution of an* I_2*-almost projector representing the correct polarisation class.*

Keeping this in mind, theorem 82 states that, up to a C^∞ correction, the integral kernel of the Hadamard state is the integral kernel of a I_2-almost projector. Theorem 83 states that, up to $L^2 + C^\infty$ corrections, the integral kernel of a I_2-almost projector is a Hadamard state.
The C^∞ freedom is due to the definition of a Hadamard state. The L^2 freedom originates from the definition of I_2-almost projector.
In this sense, the relevant difference in singularity structure between a Hadamard state and an I_2-almost projector is given by terms (4.273) to (4.278) of the appendix.

3.6 Summary and Conclusions

In the third chapter of this thesis we worked on a quantum field theoretic formulation of electromagnetic interactions. While this approach is more conventional than what was presented in chapter 2 that does not at all imply that the general theory has been worked out on a mathematically rigorous level. So much so that our work on external field QED, i.e. neglecting all interaction between particles, can be regarded as at the frontier of our present understanding. The chapter started with a short summary of the approach to construct a lift of the one-particle time evolution operator where we mentioned the shortcoming of the present method, not uniquely identifying the phase of this operator. Subsequently, we gave a geometric construction of said phase in the scattering regime from an object c^+ very closely related to the current induced by an external field. If such an object were identified the residual freedom might be reduced to an irrelevant constant phase and a single number related to the charge of the electron. Furthermore we showed that there is a lift of the one-particle scattering operator that is analytic in the external field and gave a compact explicit fomula for weak fields. Furthermore, we saw that this implies that the scattering operator can be seen as a power series in the external field and that this also holds true for arbitrary four-potentials $A \in \mathcal{V}$. Finally we gave theorems on how I_2-almost projectors representing the state of the fermion field are related to the approach we are following here compare to the concept of Hadamard state popular in the algebraic formulation of QFT.

I hope that in the near future the construction of c^+ will succed, so that one can further analyse a self consisten model. In such a model one feeds the current generated by the fermion field induced by the action of an electrodynamic field into Maxwells equations and acts with the resulting fields again on the fermion field. Such a model would incorporate a mean field interaction between the fermions and would thus be a further step in the direction of a fully interacting theory.

Chapter 4

Appendix

4.1 Regularity of the One Particle Scattering Operator

In this section we analyse the construction of the one particle scattering operator S_A carried out in [3] and answer the question whether operators like

$$P^+ \partial_B S_A^* S_{A+B} P^- \tag{4.1}$$

are Hilbert-Schmidt operators. This is important for the geometric construction carried out in chapter 3.3.
Since this section is heavily inspired by [3], we need to introduce some notation from this paper.

Definition 85. *Let $A \in \mathcal{V}$, we define the the integral operator Q^A :*

$\mathcal{H}\circlearrowleft$ by giving its integral kernel, which is also denoted by Q^A:

$$\mathbb{R}^3 \times \mathbb{R}^3 \ni (p,q) \mapsto Q^A(p,q) := \frac{Z^A_{+-}(p,q) - Z^A_{-+}(p,q)}{i(E(p)+E(q))} \tag{4.2}$$

$$\text{with } Z^A_{\pm\mp}(p,q) := P_{\pm}(p) Z^A(p-q) P_{\mp}(q), \tag{4.3}$$

$$Z^A = -ie\gamma^0\gamma^\alpha \hat{A}_\alpha, \tag{4.4}$$

$$\hat{A}_\mu := \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} A_\mu(x) e^{-ipx} d^3x, \tag{4.5}$$

$$\text{and } E(p) := \sqrt{m^2 + |p|^2}. \tag{4.6}$$

Fact 86. *Plase recall that for general $A, F \in \mathcal{V}$ and $t_0, t_1 \in \mathbb{R}$ we have the well known equations for the one-particle time evolution operators*

$$U^A(t_1,t_0) = U^0(t_1,t_0) + \int_{t_0}^{t_1} dt\ U^0(t_1,t) Z^A(t) U^A(t,t_0) \tag{4.7}$$

$$U^{A+F}(t_1,t_0) = U^A(t_1,t_0) + \int_{t_0}^{t_1} dt\ U^A(t_1,t) Z^F(t) U^{A+F}(t,t_0). \tag{4.8}$$

Definition 87. *For any $A \in \mathcal{V}$, we introduce the integral operator ${Q'}^A : \mathcal{H}\circlearrowleft$ by it's kernel*

$$\mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \ni (t,p,q) \mapsto {Q'}^A(t,p,q) = \partial_t Q^A(t,p,q), \tag{4.9}$$

where the time dependence is due to the time dependence of the four-potential A. The following notion of even and odd part of an arbitrary bounded linear operator $T : \mathcal{F}\circlearrowleft$ on Fock space will come in handy:

$$T_{\text{odd}} := P^+ T P^- + P^- T P^+ \tag{4.10}$$

$$T_{\text{ev}} := P^+ T P^+ + P^- T P^-. \tag{4.11}$$

Additionally, we define the norm

$$T : \mathcal{H}\circlearrowleft \ \|T\|_{\text{op}+I_2} = \|T\| + \|T_{\text{odd}}\|_{I_2}, \tag{4.12}$$

where $\|\cdot\|$ *is the operator norm and* $\|\cdot\|_{I_2}$ *is the Hilbert-Schmidt norm and the space*

$$I_2^{\text{odd}} := \{T : \mathcal{F} \to \mathcal{F} \mid \|T\| < \infty, \|T_{\text{odd}}\|_{I_2} < \infty\}. \tag{4.13}$$

Lemma 88. *The space* I_2^{odd} *equipped with the norm* $\|\cdot\|_{\text{op}+I_2}$ *is a Banach space.*

Proof. Let $(T_n)_{n\in\mathbb{N}} \subset I_2^{\text{odd}}$ be a Cauchy sequence with respect to $\|\cdot\|_{\text{op}+I_2}$. Then it follows directly that $(T_n)_{n\in\mathbb{N}}$ is also a Cauchy sequence with respect to $\|\cdot\|$ and $(T_{n,\text{odd}})_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to $\|\cdot\|_{I_2}$. Since the space of bounded operators equipped with $\|\cdot\|$ and the space of Hilbert-Schmidt oeprators equipped with $\|\cdot\|_{I_2}$ both are complete we have

$$T_n \xrightarrow[\|\cdot\|]{n\to\infty} T^1 \tag{4.14}$$

$$T_{n,\text{odd}} \xrightarrow[\|\cdot\|_{I_2}]{n\to\infty} T^2 \tag{4.15}$$

for some bounded operator T^1 and some Hilbert-Schmidt operator T^2. Now because the Hilbert-Schmidt norm fulfills

$$\|T\| \leqslant \|T\|_{I_2}, \tag{4.16}$$

we obtain directly

$$T_{n,\text{odd}} \xrightarrow[\|\cdot\|]{n\to\infty} T^2, \tag{4.17}$$

hence $T^1_{\text{odd}} = T^2$. Therefore, $T^1 \in I_2^{\text{odd}}$ holds. Finally, since $\|\cdot\|_{\text{op}+I_2} = \|\cdot\| + \|\cdot_{\text{odd}}\|_{I_2}$ is true, we find

$$T_n \xrightarrow[\|\cdot\|_{\text{op}+I_2}]{n\to\infty} T^1, \tag{4.18}$$

proving completeness. □

For the following theorem and lemma we are going to make use of the following shorthand notation of [3]. For operator valued maps $T_1, T_2 : \mathbb{R}^2 \to \mathcal{B}(\mathcal{H})$ we define for $t_1, t_0 \in \mathbb{R}$

$$T_1 T_2 := \int_{t_0}^{t_1} dt \; T_1(t_1, t) T_2(t, t_0), \tag{4.19}$$

as a map of the same type as T_1 and T_2 whenever this is well-defined. Furthermore, for operator valued functions $W_1, W_2 : \mathbb{R} \to \mathcal{B}(\mathcal{H})$ we define

$$T_1 W_1 \; (t', t) := T_1(t', t) C_1(t) \tag{4.20}$$

$$W_1 T_1 \; (t', t) := W(t_1) T_1(t', t) \tag{4.21}$$

$$W_1 W_2 \; (t) := W_1(t) W_2(t), \tag{4.22}$$

as maps of the same type as T_1, T_1 and C_1 respectively.
Pick $k \in \mathbb{N}$, $A, H_b \in \mathcal{V}$ for $b \leqslant k$ and t_1, t_0 such that t_1 is later and t_0 is earlier than the support of A and all H_b. Whenever the shorthand (4.20) and (4.21) is used without specific arguments, by convention $t' = t_1, t = t_0$. We abbreviate

$$H := \sum_{b=1}^{k} H_b, \quad B := A + H. \tag{4.23}$$

We introduce

$$R^B(t', t) := (1 - Q^B) U^B (1 + Q^B)(t', t), \tag{4.24}$$

for general $t', t \in \mathbb{R}$. Because of the choice of t_1, t_0 we have

$$R^B(t_1, t_0) = (1 - Q^B) U^B (1 + Q^B) = U^B(t_1, t_0), \tag{4.25}$$

because $B = 0$ both at t_1 and t_0.

So it suffices to study the family of operators R^B. As shown in the proof of [3, lemma 3.5] R^B for $B \in \mathcal{V}$ is the limit in the sense of the operator norm of the sequence

$$R_0^B := 0, \quad R_{n+1}^B := U^0 \mathsf{F}^B R_n^B + U^0 + \mathsf{G}^B, \tag{4.26}$$

where F and G are given By

$$\mathsf{F}^B :=(-Q'^B + Z_{\mathrm{ev}}^B - Q^B Z^B)(1 + Q^B), \tag{4.27}$$

$$\mathsf{G}^B := -U^0 Q^B Q^B \tag{4.28}$$

$$+U^0(-Q'^B + Z_{\mathrm{ev}} - Q^B Z^B) Q^B Q^B U^B (1 + Q^B). \tag{4.29}$$

Finally we introduce the auxiliary norms for operators T and W depending on one and two scalar variables respectively.

$$\|T\|_{\mathrm{op}+I_2,\gamma} := \sup_{t\in[t_1,t_0]} e^{-\gamma(t-t_0)} \|T(t)\| + \sup_{t\in[t_1,t_0]} e^{-\gamma(t-t_0)} \|T_{\mathrm{odd}}(t)\|_{I_2} \tag{4.30}$$

$$\|T\|_\gamma := \sup_{t\in[t_1,t_0]} e^{-\gamma(t-t_0)} \|T(t)\| \tag{4.31}$$

$$\|W\|_0 := \sup_{t,t'\in[t_1,t_0]} \|W(t,t')\| \tag{4.32}$$

$$\|T\|_{I_2,\gamma} := \sup_{t\in[t_1,t_0]} e^{-\gamma(t-t_0)} \|T(t)\|_{I_2}, \tag{4.33}$$

$$\|W\|_{I_2,0} := \sup_{t,t'\in[t_1,t_0]} \|W(t,t')\|_{I_2}, \tag{4.34}$$

for $\gamma \geqslant 0$.
Now we have collected enough tools to proof

Theorem 89 (Smoothness of S)**.** *Let $n \in \mathbb{N}$, $A, H_k \in \mathcal{V}$ for $k \leqslant n$, pick t_1 after* $\operatorname{supp} A \cup \bigcup_{k\leqslant n} \operatorname{supp} H_k$ *and t_0 before* $\operatorname{supp} A \cup \bigcup_{k\leqslant n} \operatorname{supp} H_k$ *then the derivative*

$$\partial_{H_1} \dots \partial_{H_k} U^{A+\sum_{b=1}^k H_b}(t_1, t_0) \tag{4.35}$$

exists with respect to the topology induced by the norm $\|\cdot\|_{\mathrm{op}+I_2}$.

Proof. We will follow the corresponding proof in [3]. In the proof of the Grönwall lemma in [3, equation (3.42)] we also have that the recursive equation

$$R_n^B = U^0 \mathsf{F}_{\mathrm{ev}}^B R_{n-1}^B + U^0 \mathsf{F}_{\mathrm{odd}}^B U^0 \mathsf{F}^B R_{n-2}^B \tag{4.36}$$
$$+ U^0 \mathsf{F}_{\mathrm{odd}}^B \mathsf{G}^B + U^0 F_{\mathrm{odd}}^B U^0 + U^0 + \mathsf{G}^B \tag{4.37}$$

is fulfilled by the same sequence of operators for $n \geqslant 2$. Furthermore, we introduce the notation

$$[k] = \{l \in \mathbb{N} \mid l \leqslant k\} \tag{4.38}$$
$$\forall u \subseteq [k] : \partial_u = \prod_{k \in u} \partial_{H_k}, \tag{4.39}$$
$$\Delta^n = R_{n+1}^B - R_n^B, \tag{4.40}$$

where the product of derivatives is to be understood as the mixed derivative with respect to all the factors and we use the derivative defined in (3.53).
Hence we have for such n:

$$\Delta_n = U^0 \mathsf{F}_{\mathrm{ev}}^B \Delta_{n-1} + U^0 \mathsf{F}_{\mathrm{odd}}^B U^0 \mathsf{F}^B \Delta_{n-2}.$$

Abbreviating $U^0 \mathsf{F}_{\mathrm{ev}}^B =: a, \quad U^0 \mathsf{F}_{\mathrm{odd}}^B U^0 \mathsf{F}^B := b$, we obtain

$$\Delta_n = a\Delta_{n-1} + b\Delta_{n-2}. \tag{4.41}$$

we estimate for any set $u \subset [k]$:

$$\sup_{p\subseteq u}\|\partial_p\Delta^n_{\text{odd}}(\cdot,t_0)\|_{I_2,\gamma} \tag{4.42}$$

$$\leqslant \sup_{p\subseteq u}\sup_{t\in[t_0,t_1]} e^{-\gamma(t-t_0)}\left\|\sum_{w\subseteq p}(\partial_{p\setminus w}a\ \partial_w\Delta^{n-1}_{\text{odd}})(t,t_0)\right\|_{I_2} \tag{4.43}$$

$$+\sup_{p\subseteq u}\sup_{t\in[t_0,t_1]} e^{-\gamma(t-t_0)}\left\|P^+\sum_{w\subseteq p}(\partial_{p\setminus w}b\ \partial_w\Delta^{n-2})(t,t_0)P^-\right\|_{I_2} \tag{4.44}$$

$$+\sup_{p\subseteq u}\sup_{t\in[t_0,t_1]} e^{-\gamma(t-t_0)}\left\|P^-\sum_{w\subseteq p}(\partial_{p\setminus w}b\ \partial_w\Delta^{n-2})(t,t_0)P^+\right\|_{I_2} \tag{4.45}$$

$$\leqslant \sup_{p\subseteq u}\sup_{t\in[t_0,t_1]} e^{-\gamma(t-t_0)}\sum_{w\subseteq p}\|(\partial_{p\setminus w}a\ \partial_w\Delta^{n-1}_{\text{odd}})(t,t_0)\|_{I_2} \tag{4.46}$$

$$+2\sup_{p\subseteq u}\sup_{t\in[t_0,t_1]} e^{-\gamma(t-t_0)}\sum_{w\subseteq p}\|(\partial_{p\setminus w}b\ \partial_w\Delta^{n-2})(t,t_0)\|_{I_2} \tag{4.47}$$

$$\leqslant \sup_{p\subseteq u}\sup_{t\in[t_0,t_1]} e^{-\gamma(t-t_0)}\sum_{w\subseteq p}\int_{t_0}^t dt'\|\partial_{p\setminus w}a(t,t')\ \partial_w\Delta^{n-1}_{\text{odd}}(t',t_0)\|_{I_2} \tag{4.48}$$

$$+2\sup_{p\subseteq u}\sup_{t\in[t_0,t_1]} e^{-\gamma(t-t_0)}\sum_{w\subseteq p}\int_{t_0}^t dt'\|\partial_{p\setminus w}b(t,t')\ \partial_w\Delta^{n-2}(t',t_0)\|_{I_2} \tag{4.49}$$

$$\leqslant \sup_{p\subseteq u}\sup_{t\in[t_0,t_1]} e^{-\gamma(t-t_0)}\sum_{w\subseteq p}\int_{t_0}^t dt'\|\partial_{p\setminus w}a\|_0\|\partial_w\Delta^{n-1}_{\text{odd}}(t',t_0)\|_{I_2} \tag{4.50}$$

$$+2\sup_{p\subseteq u}\sup_{t\in[t_0,t_1]} e^{-\gamma(t-t_0)}\sum_{w\subseteq p}\int_{t_0}^t dt'\|\partial_{p\setminus w}b(t,t')\|_{I_2}\|\partial_w\Delta^{n-2}(t',t_0)\| \tag{4.51}$$

$$\leqslant \sup_{p\subseteq u}\sup_{t\in[t_0,t_1]} e^{-\gamma t}\sum_{w\subseteq p}\int_{t_0}^t dt' e^{\gamma t'}\|\partial_{p\setminus w}a\|_0\|\partial_w\Delta^{n-1}_{\text{odd}}(\cdot,t_0)\|_{I_2,\gamma} \tag{4.52}$$

$$+2\sup_{p\subseteq u}\sup_{t\in[t_0,t_1]} e^{-\gamma t}\sum_{w\subseteq p}\int_{t_0}^t dt' e^{\gamma t'}\|\partial_{p\setminus w}b\|_{I_2,0}\|\partial_w\Delta^{n-2}(\cdot,t_0)\|_{\gamma} \tag{4.53}$$

$$\leqslant \frac{1}{\gamma} \sup_{p \subseteq u} \sum_{w \subseteq p} \|\partial_{p \setminus w} a\|_0 \|\partial_w \Delta^{n-1}_{\text{odd}}(\cdot, t_0)\|_{I_2,\gamma} \tag{4.54}$$

$$+ \frac{2}{\gamma} \sup_{p \subseteq u} \sum_{w \subseteq p} \|\partial_{u \setminus w} b\|_{I_2,0} \|\partial_w \Delta^{n-2}(\cdot, t_0)\|_\gamma \tag{4.55}$$

$$\leqslant \frac{2^{|u|}}{\gamma} \sup_{u' \subseteq u} \|\partial_{u'} a\|_0 \sup_{p \subseteq u} \|\partial_p \Delta^{n-1}_{\text{odd}}(\cdot, t_0)\|_{I_2,\gamma} \tag{4.56}$$

$$+ \frac{2^{|u|+1}}{\gamma} \sup_{u' \subseteq u} \|\partial_{u'} b\|_{I_2,0} \sup_{p \subseteq u} \|\partial_p \Delta^{n-2}(\cdot, t_0)\|_\gamma \tag{4.57}$$

Similarly we compute the operator norm:

$$\sup_{p \subseteq u} \|\partial_p \Delta^n(\cdot, t_0)\|_\gamma \tag{4.58}$$

$$\leqslant \sup_{t \in [t_0, t_1]} e^{-\gamma(t-t_0)} \sup_{p \subseteq u} \sum_{w \subseteq p} \|(\partial_{p \setminus w} a \ \partial_w \Delta^{n-1})(t, t_0)\| \tag{4.59}$$

$$+ \sup_{t \in [t_0, t_1]} e^{-\gamma(t-t_0)} \sup_{p \subseteq u} \sum_{w \subseteq p} \|(\partial_{p \setminus w} b \ \partial_w \Delta^{n-2})(t, t_0)\| \tag{4.60}$$

$$\leqslant \sup_{t \in [t_0, t_1]} e^{-\gamma(t-t_0)} \sup_{p \subseteq u} \sum_{w \subseteq p} \int_{t_0}^{t} dt' \|\partial_{p \setminus w} a(t, t') \ \partial_w \Delta^{n-1}(t', t_0)\| \tag{4.61}$$

$$+ \sup_{t \in [t_0, t_1]} e^{-\gamma(t-t_0)} \sup_{p \subseteq u} \sum_{w \subseteq u} \int_{t_0}^{t} dt' \|\partial_{p \setminus w} b(t, t') \ \partial_w \Delta^{n-2}(t', t_0)\| \tag{4.62}$$

$$\leqslant \sup_{t \in [t_0, t_1]} e^{-\gamma(t-t_0)} \sup_{p \subseteq u} \sum_{w \subseteq p} \int_{t_0}^{t} dt' \|\partial_{p \setminus w} a(t, t')\| \|\partial_w \Delta^{n-1}(t', t_0)\| \tag{4.63}$$

$$+ \sup_{t \in [t_0, t_1]} e^{-\gamma(t-t_0)} \sup_{p \subseteq u} \sum_{w \subseteq p} \int_{t_0}^{t} dt' \|\partial_{p \setminus w} b(t, t')\| \|\partial_w \Delta^{n-2}(t', t_0)\| \tag{4.64}$$

$$\leqslant \sup_{t \in [t_0, t_1]} e^{-\gamma t} \sup_{p \subseteq u} \sum_{w \subseteq p} \int_{t_0}^{t} dt' e^{\gamma t'} \|\partial_{p \setminus w} a\|_0 \|\partial_w \Delta^{n-1}(\cdot, t_0)\|_\gamma \tag{4.65}$$

$$+ \sup_{t \in [t_0, t_1]} e^{-\gamma t} \sup_{p \subseteq u} \sum_{w \subseteq p} \int_{t_0}^{t} dt' e^{\gamma t} \|\partial_{u \backslash p} b\|_0 \|\partial_w \Delta^{n-2}(\cdot, t_0)\|_\gamma \tag{4.66}$$

$$\leqslant \frac{1}{\gamma} \sup_{p \subseteq u} \sum_{w \subseteq p} \|\partial_{p \backslash w} a\|_0 \|\partial_w \Delta^{n-1}(\cdot, t_0)\|_\gamma \tag{4.67}$$

$$+ \frac{1}{\gamma} \sup_{p \subseteq u} \sum_{w \subseteq p} \|\partial_{p \backslash w} b\|_0 \|\partial_w \Delta^{n-2}(\cdot, t_0)\|_\gamma \tag{4.68}$$

$$\leqslant \frac{2^{|u|}}{\gamma} \sup_{u' \subseteq u} \|\partial_{u'} a\|_0 \sup_{w \subseteq u} \|\partial_w \Delta^{n-1}(\cdot, t_0)\|_\gamma \tag{4.69}$$

$$+ \frac{2^{|u|}}{\gamma} \sup_{u' \subseteq u} \|\partial_{u'} b\|_0 \sup_{w \subseteq u} \|\partial_w \Delta^{n-2}(\cdot, t_0)\|_\gamma \tag{4.70}$$

We can summarise the last calculations more briefly using the abbreviation

$$\alpha = \frac{2^{|u|+1}}{\gamma} \sup_{u' \subseteq u} \left\{ \|\partial_{u'} a\|_0, \|\partial_{u'} b\|_{I_2, \infty}, \|\partial_{u'} b\|_0 \right\}. \tag{4.71}$$

Here α is finite. This can be seen as follows: firstly, $\partial_u b = U^0 \partial_u \mathsf{F}^B_{\text{odd}} U^0 \mathsf{F}^B$ vanishes if $|u| \geqslant 7$ because the factors Q'^B, Q^B and Z^B are all linear in B and the longest product of such operators appearing in b has six factors, analogously all derivatives $\partial_u a = 0$ for $|u| \geqslant 4$. Secondly, each of the operators Q'^C, Q^C and Z^C are bounded for every $C \in \mathcal{V}$, hence the polynomials a and b of these operators are also bounded. This shows finiteness of the two operator norms appearing in the expression for α. For the Hilbert-Schmidt norm we see that $\partial_u b$ is always a sum of terms where each term has a factor $U^0 \partial_p \mathsf{F}^B_{\text{odd}} U^0$ with $p \subseteq u$. This factor has finite Hilbert-Schmidt norm due to the I_2 estimate lemma 90.

We can thus summarise the last two calculations

$$\begin{pmatrix} \sup_{p\subseteq u} \|\partial_p \Delta^n\|_{\mathrm{op}+I_2,\gamma} \\ \sup_{p\subseteq u} \|\partial_p \Delta^{n-1}\|_{\mathrm{op}+I_2,\gamma} \end{pmatrix} \tag{4.72}$$

$$\leqslant \begin{pmatrix} \alpha & \alpha \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \sup_{p\subseteq u} \|\partial_p \Delta^{n-1}\|_{\mathrm{op}+I_2,\gamma} \\ \sup_{p\subseteq u} \|\partial_p \Delta^{n-2}\|_{\mathrm{op}+I_2,\gamma} \end{pmatrix} \tag{4.73}$$

$$\leqslant \begin{pmatrix} \alpha & \alpha \\ 1 & 0 \end{pmatrix}^{n-1} \begin{pmatrix} \sup_{p\subseteq u} \|\partial_p \Delta^{1}\|_{\mathrm{op}+I_2,\gamma} \\ \sup_{p\subseteq u} \|\partial_p \Delta^{0}\|_{\mathrm{op}+I_2,\gamma} \end{pmatrix}. \tag{4.74}$$

This matrix can be diagonalised, it's eigenvalues are

$$\lambda_\pm = \frac{\alpha}{2}\left(1 \pm \sqrt{1+\frac{4}{\alpha}}\right). \tag{4.75}$$

The larger eigenvalue λ_+ is less than 1 if and only if $0 < \alpha < 0.5$ holds true, as can be seen from a quick calculation:

$$\frac{\alpha}{2}\left(1 + \sqrt{1+\frac{4}{\alpha}}\right) < 1 \tag{4.76}$$

$$\iff \sqrt{1+\frac{4}{\alpha}} < \frac{2}{\alpha} - 1. \tag{4.77}$$

If $\alpha \geqslant \frac{1}{2}$ or $\alpha < 0$ this inequality is not satisfied, otherwise we may square both sides to find

$$1 + \frac{4}{\alpha} < (2/\alpha - 1)^2 = 4/\alpha^2 - 4/\alpha + 1 \tag{4.78}$$

$$\iff \alpha < \frac{1}{2}. \tag{4.79}$$

So we conclude that for γ large enough the right hand side of (4.74) tends to zero as $c\lambda_+^n$ for $n \to \infty$, with

$$c = \sqrt{\sup_{p\subseteq u} \|\partial_p \Delta^1\|^2_{\mathrm{op}+I_2,\gamma} + \sup_{p\subseteq u} \|\partial_p \Delta^0\|^2_{\mathrm{op}+I_2,\gamma}}. \tag{4.80}$$

Concerning the norms of $\partial_p \Delta^1$ and $\partial_p \Delta^0$: the operator norms of these terms are finite, since they are polynomials of bounded operators linear in the external potential. The Hilbert-Schmidt norm of the odd part of these terms can be bounded using lemma 90.
That is, we have

$$\sup_{p \subseteq u} \|\partial_p \Delta^n\|_{\mathrm{op}+I_2,\gamma} \leqslant \lambda_+^n c \xrightarrow{n\to\infty} 0. \tag{4.81}$$

For $2 \leqslant m \leqslant n$ we obtain

$$\sup_{p \subseteq u} \|\partial_p R_n^B - \partial_p R_m^B\|_{\mathrm{op}+I_2,\gamma} \leqslant \sum_{k=m}^{n-1} \sup_{p \subseteq u} \|\partial_p \Delta^k\|_{\mathrm{op}+I_2,\gamma} \tag{4.82}$$

$$\leqslant \sum_{k=m}^{\infty} \lambda_+^k c = \frac{\lambda_+^m}{1-\lambda_+} c \xrightarrow{m\to\infty} 0 \tag{4.83}$$

since the the norms $\|\cdot\|_{\mathrm{op}+I_2}$ and $\|\cdot\|_{\mathrm{op}+I_2,\gamma}$ are equivalent, we have just proven that $\partial_{[k]} R_m^B$ is a Cauchy sequence with respect to the norm $\|\cdot\|_{\mathrm{op}+I_2}$ and hence convergence by lemma 88.

□

The following lemma is a necessary ingredient for theorem 89. Morally, it has already been proven in [3, Lemma 3.7]; however, as that paper was not concerned with multiple four-potentials the lemma was not formulated general enough for our needs here. So we restate it and show how to modify the original proof.

Lemma 90 (I_2 estimates)**.** *Let $k \in \mathbb{N}$ and $A, H_b \in \mathcal{V}$ for $b \leqslant k$. Using the abbreviations introduced in (4.38) and (4.39) we have for any $u \subset [k]$ the following bounds:*

$$\|\partial_u U^0 \mathsf{F}_{\mathrm{odd}}^{A+\sum_{b=1}^k H_b} U^0\|_{I_2,0} < \infty \tag{4.84}$$

$$\|\partial_u \mathsf{G}^{A+\sum_{b=1}^k H_b}\|_{I_2,0} < \infty. \tag{4.85}$$

Proof. For $B \in \mathcal{V}$ recall

$$\mathsf{F}^B_{\mathrm{odd}} := ((-Q'^B + Z^B_{\mathrm{ev}} - Q^B Z^B)(1 + Q^B))_{\mathrm{odd}} \tag{4.86}$$

$$= -Q'^B + Z^B_{\mathrm{ev}} Q^B - Q^B Z^B_{\mathrm{ev}} - Q^B Z^B_{\mathrm{odd}} Q^B, \tag{4.87}$$

$$\mathsf{G}^B := -U^0 Q^B Q^B \tag{4.88}$$

$$+ U^0(-Q'^B + Z_{\mathrm{ev}} - Q^B Z^B) Q^B Q^B U^B (1 + Q^B). \tag{4.89}$$

Pick $k \in \mathbb{N}$ and $A, B, H_b \in \mathcal{V}$ for $b \leqslant k$.
According to [3, lemma 3.7] the operators $U^0 Z^B_{\mathrm{ev}} Q^B U^0$, $U^0 Q^B Z^B_{\mathrm{ev}} U^0$, $U^0 Q'^B U^0$, $Q^B Q^B, Q'^B Q^B$ and $Q^B Z^B Q^B$ are Hilbert-Schmidt operators. Additionally, their Hilbert-Schmidt norm is uniformly bounded in time and F and G fulfil the following norm bound:

$$\|U^0 \mathsf{F}^B_{\mathrm{odd}} U^0\|_{I_2,0} < \infty \text{ and } \|\mathsf{G}^B\|_{I_2,0} < \infty. \tag{4.90}$$

In fact, the proof given in [3] also workes for non identical four-potentials $A, B, C \in \mathcal{V}$ proving

$$\|U^0 Z^A_{\mathrm{ev}} Q^B U^0\|_{I_2,0} < \infty, \tag{4.91}$$

$$\|U^0 Q^A Z^B_{\mathrm{ev}} U^0\|_{I_2,0} < \infty, \tag{4.92}$$

$$\|U^0 Q'^B U^0\|_{I_2,0} < \infty, \tag{4.93}$$

$$\|Q^B Q^A\|_{I_2,0} < \infty, \tag{4.94}$$

$$\|Q'^B Q^A\|_{I_2,0} < \infty, \tag{4.95}$$

$$\|Q^A Z^B Q^C\|_{I_2,0} < \infty \tag{4.96}$$

and therefore also

$$\|\partial_u U^0 \mathsf{F}^{A+\sum_{b=0}^k H_b}_{\mathrm{odd}} U^0\|_{I_2,0} < \infty \text{ and } \|\partial_u \mathsf{G}^{A+\sum_{b=0}^k H_b}\|_{I_2,0} < \infty \tag{4.97}$$

for any $u \subseteq [k]$.

For the benefit of the reader, we will reproduce the proof of the estimate

$$\|\partial_u \mathsf{G}^{A+\sum_{b=0}^{k} H_b}\|_{I_2,0} < \infty, \tag{4.98}$$

to make clear the structure of the entire proof.
The operator $\mathsf{G}^{A+\sum_{b=0}^{k} H_b}$ consists of two summands. Each summand is a product of operators with operator norm uniformly bounded in time and containing a factor of $Q^{A+\sum_{b=0}^{k} H_b} Q^{A+\sum_{b=0}^{k} H_b}$. All the other factors contributing to G stay bounded when differentiated and the map $Q : \mathcal{V} \to I_2$ is linear, so the bound

$$\|Q^B Q^D\|_{I_2,0} < \infty \tag{4.99}$$

for general $B, D \in \mathcal{V}$ will suffice to prove (4.98).
Pick $B, D \in \mathcal{V}$, we estimate

$$\sup_{t\in\mathbb{R}} \|Q^B(t) Q^D(t)\|_{I_2} \tag{4.100}$$

$$\leqslant \sum_{\mu,\nu=0}^{3} \Big(\sup_{p,k,q\in\mathbb{R}^3} |P_+(p)\gamma^0\gamma^\mu P_-(k)\gamma^0\gamma^\nu P_+(q)| \tag{4.101}$$

$$+ \sup_{p,k,q\in\mathbb{R}^3} |P_-(p)\gamma^0\gamma^\mu P_+(k)\gamma^0\gamma^\nu P_-(q)| \Big) \tag{4.102}$$

$$\sup_{t\in\mathbb{R}} \left\| \int_{\mathbb{R}^3} dk \frac{\hat{B}_\mu(t, p-k)\hat{D}_\nu(t, k-q)|}{(E(p)+E(k))(E(k)+E(q))} \right\|_{I_2,(p,q)}, \tag{4.103}$$

where the index in the norm indicates with respect to which variables the integral of the norm is to be performed. The prefactor is finite since $P_\pm(p) : \mathbb{C}^4 \to \mathbb{C}^4$ is a projector for any $p \in \mathbb{R}^3$. Abbreviating

$$\tilde{c} := \sum_{\mu,\nu=0}^{3} (\sup_{p,k,q\in\mathbb{R}^3} |P_+(p)\gamma^0\gamma^\mu P_-(k)\gamma^0\gamma^\nu P_+(q)| \tag{4.104}$$

$$+ \sup_{p,k,q\in\mathbb{R}^3} |P_-(p)\gamma^0\gamma^\mu P_+(k)\gamma^0\gamma^\nu P_-(q)|), \tag{4.105}$$

and using the integral estimate lemma [3, lemma 3.8 (iii)] we find

$$(4.100) \leqslant \tilde{c}\; C_{8,\text{of [3]}} \sum_{\mu,\nu=0}^{3} \sup_{t\in\mathbb{R}} \|\hat{B}_\mu(t)\|_{I_1} \|\hat{D}_\nu(t)\|_{I_2}. \tag{4.106}$$

Because of $B, D \in C_c^\infty(\mathbb{R}^4)$, we have $\hat{B}(t), \hat{D}(t)$ are analytic functions decaying faster than any negative power at anfinity for any t, so (4.106) is finite.
Also in order to proof the first estimate in (4.84) the proof of [3, lemma 3.7] can be followed almost verbatim. First one dissects F_{odd} into the sum U^0(4.87)U^0, next the summands have to be bounded individually. This is achieved by repeating the proof of the partial integration lemma [3, lemma 3.6], estimates of the form of (4.100)-(4.103) and making use of the integral estimate lemma [3, lemma 3.8].

□

Theorem 91 (Properties of Derivatives of S)**.** *Let* $A, H \in \mathcal{V}$*, pick* t_1 *after* supp $A \cup$ supp H *and* t_0 *before* supp $A \cup$ supp H*, let* $T_1 \in I_2(\mathcal{F})$ *then the following equalities are satisfied:*

$$\begin{aligned} &\partial_H \operatorname{tr}(T_1 P^{\pm} U^A(t_0,t_1) U^{A+H}(t_1,t_0) P^{\mp}) \\ &\quad = \operatorname{tr}(T_1 P^{\pm} U^A(t_0,t_1) \partial_H U^{A+H}(t_1,t_0) P^{\mp}) \end{aligned} \tag{4.107}$$

Proof. Let $A, H \in \mathcal{V}$ and $t_0, t_1 \in \mathbb{R}$ and T_1 be as in the theorem. The proof of the two equalities is analogous, so we only explicitly prove the first one. The trace is linear, so we have

$$\begin{aligned} &\Bigg| \operatorname{tr}\left(T_1 P^+ U^A(t_0,t_1) \frac{1}{\varepsilon} (U^{A+\varepsilon H}(t_1,t_0) - U^A(t_1,t_0)) P^- \right) \\ &\qquad - \operatorname{tr}(T_1 P^+ U^A(t_0,t_1) \partial_H U^{A+H}(t_1,t_0) P^-) \Bigg| \end{aligned} \tag{4.108}$$

$$\leqslant \|T_1\|_{I_2} \Bigg\| P^+ U^A(t_0,t_1) \frac{1}{\varepsilon} \big(U^{A+\varepsilon H}(t_1,t_0) - U^A(t_1,t_0) \big) P^-$$

$$- P^+U^A(t_0,t_1)\partial_H U^{A+H}(t_1,t_0)P^-\Big\|_{I_2} \tag{4.109}$$

For the first summand we insert the identity in the form $P^+ + P^-$ and obtain

$$P^+U^A(t_0,t_1)\frac{1}{\varepsilon}\left(U^{A+\varepsilon H}(t_1,t_0) - U^A(t_1,t_0)\right)P^- \tag{4.110}$$

$$= P^+U^A(t_0,t_1)P^+\frac{1}{\varepsilon}\left(U^{A+\varepsilon H}(t_1,t_0) - U^A(t_1,t_0)\right)P^- \tag{4.111}$$

$$+P^+U^A(t_0,t_1)P^-\frac{1}{\varepsilon}\left(U^{A+\varepsilon H}(t_1,t_0) - U^A(t_1,t_0)\right)P^-. \tag{4.112}$$

Analogously for the second summand. Now because of the Smoothness of S theorem 89 we know that

$$P^-\frac{1}{\varepsilon}\left(U^{A+\varepsilon H}(t_1,t_0) - U^A(t_1,t_0)\right)P^- \xrightarrow[\|\cdot\|]{\varepsilon\to 0} P^-\partial_H U^{A+H}(t_1,t_0)P^- \tag{4.113}$$

$$P^+\frac{1}{\varepsilon}\left(U^{A+\varepsilon H}(t_1,t_0) - U^A(t_1,t_0)\right)P^- \xrightarrow[\|\cdot\|_{I_2}]{\varepsilon\to 0} P^+\partial_H U^{A+H}(t_1,t_0)P^- \tag{4.114}$$

holds true. Hence we find in total

$$\frac{(4.109)}{\|T_1\|_{I_2}} \tag{4.115}$$

$$\leqslant \left\|P^+U^A(t_0,t_1)\right\|\left\|P^+\frac{1}{\varepsilon}\big(U^{A+\varepsilon H}(t_1,t_0) - U^A(t_1,t_0)\big)P^- - P^+\partial_H U^{A+H}(t_1,t_0)P^-\right\|_{I_2} \tag{4.116}$$

$$+ \left\|P^+U^A(t_0,t_1)P^-\right\|_{I_2}\left\|\frac{1}{\varepsilon}\big(U^{A+\varepsilon H}(t_1,t_0) - U^A(t_1,t_0)\big)P^- - \partial_H U^{A+H}(t_1,t_0)P^-\right\| \xrightarrow{\varepsilon\to 0} 0. \tag{4.117}$$

□

4.2 Lemma of Poincaré in infinite dimensions

In this section we give prove of the Poincaré's lemma in infinite dimensions used in section 3.3. First recall the lemma itself.

Lemma 92 (Poincaré)**.** *Let $\omega \in \Omega^p(\mathcal{V})$ for $p \in \mathbb{N}$ be closed, i.e. $d\omega = 0$. Then ω is also exact, more precisely we have*

$$\omega = d \int_0^1 \iota_t^* i_X f^* \omega dt, \tag{4.118}$$

where X, ι_t for $t \in \mathbb{R}$ and f are given by

$$\begin{aligned} X : \mathbb{R} \times \mathcal{V} &\to \mathbb{R} \times \mathcal{V}, && (4.119)\\ (t, B) &\mapsto (1, 0) && (4.120)\\ \forall t \in \mathbb{R} : \iota_t : \mathcal{V} &\to \mathbb{R} \times \mathcal{V}, && (4.121)\\ B &\mapsto (t, B) && (4.122)\\ f : \mathbb{R} \times \mathcal{V} &\mapsto \mathcal{V}, && (4.123)\\ (t, B) &\mapsto tB && (4.124)\\ i_X : \Omega^p(\mathcal{V}) &\to \Omega^{p-1}(\mathcal{V}), && (4.125)\\ \omega &\mapsto ((A; Y_1, \ldots, Y_{p-1}) \mapsto \omega_A(X, Y_1, \ldots, Y_{p-1})) && (4.126) \end{aligned}$$

Proof. Pick some $\omega \in \Omega^p(\mathcal{V})$. We will first show the more general formula

$$f_b^* \omega - f_a^* \omega = d \int_a^b \iota_t^* i_X f^* \omega \, dt + \int_a^b \iota_t^* i_X f^* d\omega dt, \tag{4.127}$$

where f_t is defined as

$$\forall t \in \mathbb{R} : f_t := f(t, \cdot). \tag{4.128}$$

The lemma follows then by $b = 1, a = 0$, $f_1^*\omega = \omega$, $f_0^*\omega = 0$ and $d\omega = 0$ for a closed ω. We begin by rewriting the right hand side of (4.127):

$$\begin{aligned} d\int_a^b \iota_t^* i_X f^*\omega \; dt + \int_a^b \iota_t^* i_X f^* d\omega dt \\ = \int_a^b (d\iota_t^* i_X f^*\omega + \iota_t^* i_X f^* d\omega) dt. \end{aligned} \tag{4.129}$$

Next we look at both of these terms separately. Let therefore $p \in \mathbb{N}$, $t, s_k \in \mathbb{R}$ and $A, B_k \in \mathcal{V}$ for each $p + 1 \geqslant k \in \mathbb{N}$. First, we calculate $d\iota_t^* i_X f^*\omega$:

$$\begin{aligned} &(f^*\omega)_{(t,A)}((s_1, B_1), \ldots, (s_p, B_p)) \\ &\qquad = \omega_{tA}(s_1 A + tB_1, \ldots, s_p A + tB_p) \end{aligned} \tag{4.130}$$

$$\begin{aligned} &\Rightarrow (i_X f^*\omega)_{(t,A)}((s_1, B_1), \ldots, (s_{p-1}, B_{p-1})) \\ &\qquad = \omega_{tA}(A, s_1 A + tB_1, \ldots, s_{p-1} A + tB_{p-1}) \end{aligned} \tag{4.131}$$

$$\Rightarrow (\iota_t^* i_X f^*\omega)_A(B_1, \ldots, B_{p-1}) = t^{p-1}\omega_{tA}(A, B_1, \ldots, B_{p-1}) \tag{4.132}$$

$$\Rightarrow (d\iota_t^* i_X f^*\omega)_A(B_1, \ldots, B_p)$$

$$= \partial_\varepsilon|_{\varepsilon=0} \sum_{k=1}^{p} (-1)^{k+1} t^{p-1} \omega_{tA+\varepsilon tB_k}(A, B_1, \ldots, \widehat{B_k}, \ldots, B_p) \tag{4.133}$$

$$+ \partial_\varepsilon|_{\varepsilon=0} \sum_{k=1}^{p} (-1)^{k+1} t^{p-1} \omega_{tA}(A + \varepsilon B_k, B_1, \ldots, \widehat{B_k}, \ldots, B_p) \tag{4.134}$$

$$\begin{aligned} = \partial_\varepsilon|_{\varepsilon=0} \sum_{k=1}^{p} t^p (-1)^{k+1} \omega_{tA+\varepsilon B_k}(A, B_1, \ldots, \widehat{B_k}, \ldots, B_p) \\ + pt^{p-1}\omega_{tA}(B_1, \ldots, B_p). \end{aligned} \tag{4.135}$$

Now, we calculate $\iota_t^* i_X f^* d\omega$:

$$(d\omega)_A(B_1, \cdots, B_{p+1}) = \partial_\varepsilon|_{\varepsilon=0} \sum_{k=1}^{p+1} (-1)^{k+1} \omega_{A+\varepsilon B_k}(B_1, \ldots, \widehat{B_k}, \ldots, B_{p+1}) \tag{4.136}$$

$$(f^* d\omega)(t, A)((s_1, B_1), \ldots, (s_{p+1}, B_{p+1})) \tag{4.137}$$

$$= (d\omega)_{tA}(s_1 A + tB_1, \ldots, s_{p+1} A + tB_{p+1})$$

$$= \partial_\varepsilon|_{\varepsilon=0} \sum_{k=1}^{p+1} (-1)^{k+1} \tag{4.138}$$

$$\times\, \omega_{tA+\varepsilon(s_k A + tB_k)}(s_1 A + tB_1, \ldots, s_k \widehat{A + t} B_k, \ldots, s_p A + tB_p)$$

$$(i_X f^* d\omega)_{(t,A)}((s_1, B_1), \ldots, (s_p, B_p)) \tag{4.139}$$

$$= \partial_\varepsilon|_{\varepsilon=0} \omega_{(t+\varepsilon)A}(s_1 A + tB_1, \ldots, s_p A + tB_p)$$

$$+ \partial_\varepsilon|_{\varepsilon=0} \sum_{k=1}^{p} (-1)^k$$

$$\times\, \omega_{tA+\varepsilon(s_k A + tB_k)}(A, s_1 A + tB_1, \ldots, s_k \widehat{A + t} B_k, \ldots, s_p A + tB_p)$$

$$= t^p \partial_\varepsilon|_{\varepsilon=0} \omega_{(t+\varepsilon)A}(B_1, \ldots, B_p)) \tag{4.140}$$

$$+ \sum_{k=1}^{p} s_k t^{p-1} (-1)^{k+1} \partial_\varepsilon|_{\varepsilon=0} \omega_{(t+\varepsilon)A}(A, B_1, \ldots, \widehat{B_k}, \ldots, B_p) \tag{4.141}$$

$$+ \partial_\varepsilon|_{\varepsilon=0} \sum_{k=1}^{p} (-1)^k t^{p-1} (\omega_{(t+s_k\varepsilon)A}(A, B_1, \ldots, \widehat{B_k}, \ldots, B_p) \tag{4.142}$$

$$+ \omega_{tA+\varepsilon t B_k}(A, B_1, \ldots, \widehat{B_k}, \ldots, B_p)) \tag{4.143}$$

$$= t^p \partial_\varepsilon|_{\varepsilon=0} \Big(\omega_{(t+\varepsilon)A}(B_1, \ldots, B_p) \tag{4.144}$$

$$+\sum_{k=1}^{p}(-1)^k\omega_{tA+\varepsilon B_k}(A,B_1,\dots,\widehat{B_k},\dots,B_p)\Big)$$

$$(\iota_t^* i_X f^* d\omega)_A(B_1,\dots,B_p) = t^p\partial_\varepsilon|_{\varepsilon=0}\Big(\omega_{(t+\varepsilon)A}(B_1,\dots,B_p) \tag{4.145}$$

$$+\sum_{k=1}^{p}(-1)^k\omega_{tA+\varepsilon B_k}(A,B_1,\dots,\widehat{B_k},\dots,B_p)\Big)$$

Adding (4.135) and (4.145) we find for (4.129):

$$\int_a^b (d\iota_t^* i_X f^*\omega + \iota_t^* i_X f^* d\omega)dt = \tag{4.146}$$

$$\int_a^b \Big(t^p\partial_\varepsilon|_{\varepsilon=0}\omega_{(t+\varepsilon)A}(B_1,\dots,B_p) + pt^{p-1}\omega_{tA}(B_1,\dots,B_p)\Big)dt \tag{4.147}$$

$$= \int_a^b \frac{d}{dt}(t^p\omega_{tA}(B_1,\dots,B_p))dt = \int_a^b \frac{d}{dt}(f_t^*\omega)_A(B_1,\dots,B_p)dt \tag{4.148}$$

$$= (f_b^*\omega)_A(B_1,\dots,B_p) - (f_a^*\omega)_A(B_1,\dots,B_p). \tag{4.149}$$

□

4.3 Heuristic Construction of S-Matrix expression

This section is dedicated to the heuristic construction of the expression for the scattering operator stated in theorem 71.

We start from the power series of the one-particle scattering operator S^A:

$$S^A = \sum_{k=0}^{\infty}\frac{1}{k!}Z_k(A), \tag{4.150}$$

where $Z_k(A)$ are bounded operators on $\mathcal{H}$, which are homogeneous of degree k in A. Our strategy in this chapter is to try an analogous formal power series ansatz for the second quantised scattering operator $\tilde{S}^A$

$$\tilde{S}^A = \sum_{k=0}^{\infty} \frac{1}{k!} T_k(A). \tag{4.151}$$

Here T_k are assumed to be homogeneous of degree k in A; however, they will only turn out to be bounded on fixed particle number subspaces $\mathcal{F}_{m,p}$ of Fock space. We will identify operators T_k such that (4.151) holds up to a global phase. In order to fully characterise $\tilde{S}^A$ it is enough to characterise all of the T_k operators. Using the (lift condition) one can derive commutation relations for the operators T_k by plugging in (4.150) and (4.151) into the (lift condition) and its adjoint and collecting all terms with the same degree of homogeneity. They are given by

$$\left[T_m(A), a^{\#}(\phi)\right] = \sum_{j=1}^{m} \binom{m}{j} a^{\#}\left(Z_j(A)\phi\right) T_{m-j}(A), \tag{4.152}$$

where $a^{\#}$ is either a or a^*. In the following we will derive a recursive equation for the coefficients of the expansion of the second quantized scattering operator. The starting point of this derivation is the commutator of T_m, equation (4.152).

4.3.1 Guessing Equations

Looking at equation (4.152) for a while, one comes to the conclusion that if one replaces T_m by

$$T_m - \frac{1}{2} \sum_{k=1}^{m-1} \binom{m}{k} T_k T_{m-k}, \tag{4.153}$$

no T_k with $k > m-2$ will occur on the right hand side of the resulting equation. So if one subtracts the right polynomial in T_k for suitable k one might achieve a commutator which contains only the creation respectively annihilation operator concatenated with some one particle operator.
So having this in Mind we start with the ansatz

$$\Xi_m := \sum_{g=2}^{m} \sum_{\substack{b \in \mathbb{N}^g \\ |b|=m}} c_b \prod_{k=1}^{g} T_{b_k}. \tag{4.154}$$

Now in order to show that T_m and Ξ_m agree up to operators which have a commutation relation of the form (3.445), we calculate $\left[T_m - \Xi_m, a^{\#}(\varphi)\right]$ for arbitrary $\varphi \in \mathcal{H}$ and try to choose the coefficients c_b of (4.154) such that all contributions vanish which do not have the form $a^{\#}\left(\prod_k Z_{\alpha_k}\varphi\right)$ for any suitable $(\alpha_k)_k \subset \mathbb{N}$. If one does so, one is led to a system of equations of which the first few are written down to give an overview of its structure. The objects α_k, β_l in the system of equations can be any natural Number for any $k, l \in \mathbb{N}$.

$$\begin{aligned}
0 &= c_{\alpha_1,\beta_1} + c_{\beta_1,\alpha_1} + \binom{\alpha_1+\beta_1}{\alpha_1} \\
0 &= c_{\alpha_1,\alpha_2,\beta_1} + c_{\beta_1,\alpha_1,\alpha_2} + c_{\alpha_2,\alpha_1,\beta_1} + \binom{\alpha_2+\beta_1}{\alpha_2} c_{\alpha_1,\alpha_2+\beta_1} \\
&\qquad + \binom{\alpha_1+\beta_1}{\alpha_1} c_{\alpha_1+\beta_1,\alpha_2} \\
0 &= c_{\alpha_1,\alpha_2,\alpha_3,\beta_1} + c_{\alpha_1,\alpha_2,\beta_1,\alpha_3} + c_{\alpha_1,\beta_1,\alpha_2,\alpha_3} + c_{\beta_1,\alpha_1,\alpha_2,\alpha_3} \\
&+ \binom{\alpha_1+\beta_1}{\beta_1} c_{\alpha_1+\beta_1,\alpha_2,\alpha_3} + \binom{\alpha_2+\beta_1}{\beta_1} c_{\alpha_1,\alpha_2+\beta_1,\alpha_3} \\
&\qquad + \binom{\alpha_3+\beta_1}{\beta_1} c_{\alpha_1,\alpha_2,\alpha_3+\beta_1}
\end{aligned}$$

$$
\begin{aligned}
0 &= c_{\alpha_1,\alpha_2,\beta_1,\beta_2} + c_{\alpha_1,\beta_1,\alpha_2,\beta_2} + c_{\beta_1,\alpha_1,\alpha_2,\beta_2} + c_{\alpha_1,\beta_1,\beta_2,\alpha_2}\\
&+ c_{\beta_1,\alpha_1,\beta_2,\alpha_2} + c_{\beta_1,\beta_2,\alpha_1,\alpha_2} + \binom{\alpha_1+\beta_1}{\alpha_1}\big(c_{\alpha_1+\beta_1,\alpha_2,\beta_2}\\
&+ c_{\alpha_1+\beta_1,\beta_2,\alpha_2}\big) + \binom{\alpha_1+\beta_2}{c}_{\beta_1,\alpha_1+\beta_2,\alpha_1}\\
&+ \binom{\alpha_2+\beta_1}{\alpha_2} c_{\alpha_1,\alpha_2+\beta_1,\beta_2} + \binom{\alpha_2+\beta_2}{\alpha_2}\big(c_{\alpha_1,\beta_1,\alpha_2+\beta_2}\\
&+ c_{\beta_1,\alpha_1,\alpha_2+\beta_2}\big) + \binom{\alpha_1+\beta_1}{\alpha_1}\binom{\alpha_2+\beta_2}{\alpha_2} c_{\alpha_1+\beta_1,\alpha_2+\beta_2}\\
0 &= c_{\alpha_1,\beta_1,\beta_2,\beta_3,\beta_4} + c_{\beta_1,\alpha_1,\beta_2,\beta_3,\beta_4} + c_{\beta_1,\beta_2,\alpha_1,\beta_3,\beta_4}\\
&\qquad + c_{\beta_1,\beta_2,\beta_3,\alpha_1,\beta_4} + c_{\beta_1,\beta_2,\beta_3,\beta_4,\alpha_1}\\
&+ \binom{\alpha_1+\beta_1}{\alpha_1} c_{\alpha_1+\beta_1,\beta_2,\beta_3,\beta_4} + \binom{\alpha_1+\beta_2}{\alpha_1} c_{\beta_1,\alpha_1+\beta_2,\beta_3,\beta_4}\\
&+ \binom{\alpha_1+\beta_3}{\alpha_1} c_{\beta_1,\beta_2,\alpha_1+\beta_3,\beta_4} + \binom{\alpha_1+\beta_4}{\alpha_1} c_{\beta_1,\beta_2,\beta_3,\alpha_1+\beta_4}\\
0 &= c_{\alpha_1,\alpha_2,\beta_1,\beta_2,\beta_3} + c_{\alpha_1,\beta_1,\alpha_2,\beta_2,\beta_3} + c_{\beta_1,\alpha_1,\alpha_2,\beta_2,\beta_3}\\
&+ c_{\alpha_1,\beta_1,\beta_2,\alpha_2,\beta_3} + c_{\beta_1,\alpha_1,\beta_2,\alpha_2,\beta_3} + c_{\beta_1,\beta_2,\alpha_1,\alpha_2,\beta_3}\\
&+ c_{\alpha_1,\beta_1,\beta_2,\beta_3,\alpha_2} + c_{\beta_1,\alpha_1,\beta_2,\beta_3,\alpha_2} + c_{\beta_1,\beta_2,\alpha_1,\beta_3,\alpha_2}\\
&+ c_{\beta_1,\beta_2,\beta_3,\alpha_1,\alpha_2} + \binom{\alpha_1+\beta_1}{\beta_1}\big(c_{\alpha_1+\beta_1,\alpha_2,\beta_2,\beta_3}\\
&+ c_{\alpha_1+\beta_1,\beta_2,\alpha_2,\beta_3} + c_{\alpha_1+\beta_1,\beta_2,\beta_3,\alpha_2}\big)\\
&+ \binom{\alpha_2+\beta_1}{\beta_1} c_{\alpha_1,\alpha_2+\beta_1,\beta_2,\beta_3}\\
&+ \binom{\alpha_2+\beta_2}{\beta_2}\big(c_{\beta_1,\alpha_1,\alpha_2+\beta_2,\beta_3} + c_{\alpha_1,\beta_1,\alpha_2+\beta_2,\beta_3}\big)\\
&+ \binom{\alpha_1+\beta_2}{\beta_2}\big(c_{\beta_1,\alpha_1+\beta_2,\alpha_2,\beta_3} + c_{\beta_1,\alpha_1+\beta_2,\beta_3,\alpha_2}\big)
\end{aligned}
$$

$$
\begin{aligned}
&+ \binom{\alpha_2+\beta_3}{\beta_3}(c_{\alpha_1,\beta_1,\beta_2,\alpha_2+\beta_3} + c_{\beta_1,\alpha_1,\beta_2,\alpha_2+\beta_3} \\
&+ c_{\beta_1,\beta_2,\alpha_1,\alpha_2+\beta_3}) + \binom{\alpha_1+\beta_3}{\beta_3} c_{\beta_1,\beta_2,\alpha_1+\beta_3,\alpha_2} \\
&+ \binom{\alpha_1+\beta_1}{\alpha_1}\binom{\alpha_2+\beta_2}{\alpha_2} c_{\alpha_1+\beta_1,\alpha_2+\beta_2,\beta_3} \\
&+ \binom{\alpha_1+\beta_2}{\alpha_1}\binom{\alpha_2+\beta_3}{\alpha_2} c_{\beta_1,\alpha_1+\beta_2,\alpha_2+\beta_3} \\
&+ \binom{\alpha_1+\beta_1}{\alpha_1}\binom{\alpha_2+\beta_3}{\alpha_2} c_{\alpha_1+\beta_1,\beta_2,\alpha_2+\beta_3} \\
&\qquad\qquad\vdots
\end{aligned}
$$

Solving the first few equations and plugging the solution into the consecutive equations one can see that at least the first few equations are solved by

$$c_{\alpha_1,\dots,\alpha_k} = \frac{(-1)^k}{k}\binom{\sum_{l=1}^k \alpha_l}{\alpha_1\ \alpha_2\ \cdots\ \alpha_k}, \tag{4.155}$$

where the last factor is the multinomial coefficient of the indices $\alpha_1, \dots, \alpha_k \in \mathbb{N}$.

4.3.2 Recursive equation for Coefficients of the second quantised scattering operator

We are going to use the following definition of binomial coefficients:

Definition 93. *For $a \in \mathbb{C}, b \in \mathbb{Z}$ we define*

$$\binom{a}{b} := \begin{cases} \prod_{l=0}^{b-1} \frac{a-l}{l+1} & \textit{for } b \geqslant 0 \\ 0 & \textit{otherwise.} \end{cases} \tag{4.156}$$

Defining the binomial coefficient for negative lower index to be zero has the merit, that one can extend the range of validity of many rules and sums involving binomial coefficients, also one does not have to worry about the range of summation in many cases.
The coefficients which we have already guessed result in the following

Conjecture 94. *For any $n \in \mathbb{N}$ the n-th expansion coefficient of the second quantised scattering operator T_n is given by*

$$\begin{aligned} T_n = &\sum_{g=2}^{n} \sum_{\substack{\vec{b} \in \mathbb{N}^g \\ |\vec{b}|=n}} \frac{(-1)^g}{g} \binom{n}{\vec{b}} \prod_{l=1}^{g} T_{b_l} + C_n \mathbb{1}_{\mathcal{F}} \\ &+ d\Gamma\left(\sum_{g=1}^{n} \sum_{\substack{\vec{b} \in \mathbb{N}^g \\ |\vec{b}|=n}} \frac{(-1)^{g+1}}{g} \binom{n}{\vec{b}} \prod_{l=1}^{g} Z_{b_l} \right), \end{aligned} \tag{4.157}$$

for some $C_n \in \mathbb{C}$ which depends on the external field A. The last summand will henceforth be abbreviated by Γ_n.

Motivation: We compute the commutator of the difference between T_n and the first summand of (4.157) with the creation and annihilation operator of an element of the basis of $\mathcal{H}$. This will turn out to be exactly equal to the corresponding commutator of the second summand of (4.157), since two operators on Fock space only have the same commutator with general creation and annihilation operators if they agree up to multiples of the identity this will conclude the motivation of this conjecture.
In order to simplify the notation as much as possible, I will denote by $a^{\#}z$ either $a(z(\varphi_p))$ or $a^*(z(\varphi_p))$ for any one particle operator z and any element φ_p of the orthonormal basis $(\varphi_p)_{p\in\mathbb{Z}\setminus\{0\}}$ of $\mathcal{H}$. (We need not decide between creation and annihilation operator, since the expressions all agree)

In order to organize the bookkeeping of all the summands which arise from iteratively making use of the commutation rule (4.152) we organize them by the looking at a spanning set of the possible terms that arise our choice is

$$\left\{ a^{\#} \prod_{k=1}^{m_1} Z_{\alpha_k} \prod_{k=1}^{m_2} T_{\beta_k} \middle| m_1 \in \mathbb{N}, m_2 \in \mathbb{N}_0, \alpha \in \mathbb{N}^{m_1}, \beta \in \mathbb{N}^{m_2}, |\alpha| + |\beta| = n \right\}. \tag{4.158}$$

As a first step of computing the commutator in question we look at the summand corresponding to a fixed value of the summation index g of

$$-\sum_{g=1}^{n} \sum_{\substack{\vec{b} \in \mathbb{N}^g \\ |\vec{b}|=n}} \frac{(-1)^g}{g} \binom{n}{\vec{b}} \prod_{l=1}^{g} T_{b_l}. \tag{4.159}$$

We need to bring this object into the form of a sum of terms which are multiples of elements of the set (4.158). This we will commit ourselves to for the next few pages. First we apply the product rule for the commutator:

$$\begin{aligned} &\left[\sum_{\substack{\vec{l} \in \mathbb{N}^g \\ |\vec{l}|=n}} \frac{(-1)^g}{g} \binom{n}{\vec{l}} \prod_{k=1}^{g} T_{l_k}, a^{\#} \right] \\ &= \sum_{\substack{\vec{l} \in \mathbb{N}^g \\ |\vec{l}|=n}} \frac{(-1)^g}{g} \binom{n}{\vec{l}} \sum_{\tilde{k}=1}^{g} \prod_{j=1}^{\tilde{k}-1} T_{l_j} \left[T_{l_{\tilde{k}}}, a^{\#} \right] \prod_{j=\tilde{k}+1}^{g} T_{l_j} \end{aligned}$$

$$= \sum_{\substack{\vec{l}\in\mathbb{N}^g \\ |\vec{l}|=n}} \frac{(-1)^g}{g} \binom{n}{\vec{l}} \sum_{\tilde{k}=1}^{g} \prod_{j=1}^{\tilde{k}-1} T_{l_j} \sum_{\sigma_{\tilde{k}}=1}^{l_{\tilde{k}}} \binom{l_{\tilde{k}}}{\sigma_{\tilde{k}}} a^{\#} Z_f T_{l_{\tilde{k}}-\sigma_{\tilde{k}}} \prod_{j=\tilde{k}+1}^{g} T_{l_j},$$

in the second step we used (4.152). Now we commute all the T_ls to the left of $a^{\#}$ to its right:

$$= \sum_{\substack{\vec{l}\in\mathbb{N}^g \\ |l|=n}} \frac{(-1)^g}{g} \binom{n}{\vec{l}} \sum_{\tilde{k}=1}^{g} \sum_{\substack{\forall 1\leqslant j<\tilde{k} \\ 0\leqslant\sigma_j\leqslant l_j}} \sum_{\sigma_{\tilde{k}}=1}^{l_{\tilde{k}}} \prod_{j=1}^{\tilde{k}} \binom{l_j}{\sigma_j} a^{\#} \prod_{j=1}^{\tilde{k}} Z_{\sigma_j} \prod_{j=1}^{\tilde{k}} T_{l_j-\sigma_j} \prod_{j=\tilde{k}+1}^{g} T_{l_j}. \tag{4.160}$$

At this point we notice that the multinomial coefficient can be combined with all the binomial coefficients to form a single multinomial coefficient of degree $g+\tilde{k}$. Incidentally this is also the amount of Z operators plus the amount of T operators in each product. Moreover the indices of the multinomial index agree with the indices of the Z and T operators in the product. Because of this, we see that if we fix an element of the spanning set (4.158) $a^{\#}\prod_{k=1}^{m_1} Z_{\alpha_k}\prod_{k=1}^{m_2} T_{\beta_k}$, each summand of (4.160) which contributes to this element, has the prefactor

$$\frac{(-1)^g}{g} \binom{n}{\alpha_1 \ \cdots \alpha_{m_1} \ \beta_1 \cdots \beta_{m_2}} \tag{4.161}$$

no matter which summation index $l \in \mathbb{N}^g$ it corresponds to. In order to do the matching one may ignore the indices σ_j and $l_j-\sigma_j$ which vanish, because the corresponding operators Z_0 and T_0 are equal to the identity operator on $\mathcal{H}$ respectively Fock space.

Since we know that

$$\left[d\Gamma\left(\sum_{g=1}^{n} \sum_{\substack{\vec{b}\in\mathbb{N}^g \\ |\vec{b}|=n}} \frac{(-1)^g}{g} \binom{n}{\vec{b}} \prod_{l=1}^{g} Z_{b_l} \right), a^{\#} \right] = a^{\#} \sum_{g=1}^{n} \sum_{\substack{\vec{b}\in\mathbb{N}^g \\ |\vec{b}|=n}} \frac{(-1)^g}{g} \binom{n}{\vec{b}} \prod_{l=1}^{g} Z_{b_l}$$

holds, all that is left to show is that

$$\left[-\sum_{g=1}^{n} \sum_{\substack{\vec{b}\in\mathbb{N}^g \\ |\vec{b}|=n}} \frac{(-1)^g}{g} \binom{n}{\vec{b}} \prod_{l=1}^{g} T_{b_l}, a^{\#} \right] = a^{\#} \sum_{g=1}^{n} \sum_{\substack{\vec{b}\in\mathbb{N}^g \\ |\vec{b}|=n}} \frac{(-1)^{g+1}}{g} \binom{n}{\vec{b}} \prod_{l=1}^{g} Z_{b_l} \tag{4.162}$$

also holds. For which we need to count the summands which are multiples of each element of (4.158) corresponding to each g in (4.159). So let us fix some element $K(m_1, m_2)$ of (4.158) corresponding to some $m_1 \in \mathbb{N}, m_2 \in \mathbb{N}_0, \alpha \in \mathbb{N}^{m_1}$ and $\beta \in \mathbb{N}^{m_2}$. Rephrasing this problem, we can ask which products

$$\prod_{l=1}^{g} T_{\gamma_l} \tag{4.163}$$

for suitable g and $(\gamma_l)_l$ produces, when commuted with a creation or annihilation operator, multiples of $K(m_1, m_2)$? We will call this number of total contributions weighted with the factor $-\frac{(-1)^g}{g}$ borrowed from (4.159) $\#K(m_1, m_2)$. Looking at the commutation relations (4.152) we split the set of indices $\{\gamma_1 \ldots \gamma_g\}$ into three sets A, B

and C, where the commutation relation has to be used in such a way, that

$$\begin{aligned} \forall k : \gamma_k \in A &\iff \exists j \leqslant m_1 : \gamma_k = \alpha_j, \\ \wedge \forall k : \gamma_k \in B &\iff \exists j \leqslant m_2 : \gamma_k = \beta_j \\ \wedge \forall k : \gamma_k \in C &\iff \exists j \leqslant m_1, l \leqslant m_2 : \gamma_k = \alpha_j + \beta_l \end{aligned}$$

holds. Unfortunately not every splitting corresponds to a contribution and not every order of multiplication of a legal splitting corresponds to a contribution either. However $\prod_j T_{\alpha_j} \prod_j T_{\beta_j}$ gives a contribution and it is in fact the longes product that does. We may apply the commutation relations backwards to obtain any shorter valid combination and hence all combinations. Transforming the commutation rule for T_k read from right to left into a game results in the following rules. Starting from the string

$$A_1 A_2 \ldots A_{m_1} B_1 B_2 \ldots B_{m_2}, \tag{4.164}$$

representing the longes product, where here and in the following A's represent operators T_k which will turn into Z_k by the commutation rule, B's represent operators T_k which will stay T_k after commutation and C's represent operators T_k which will produce both a Z_l in the creation/annihilation operator and a T_{k-l} behind that operator. The indices are merely there to keep track of which operator moved where. So the game consists in the answering how many strings can we produce by applying the following rules to the initial string?

1. You may replace any occurrence of $A_k B_j$ by $B_j A_k$ for any j and k.

2. You may replace any occurrence of $A_k B_j$ by $C_{k,j}$ for any j and k.

Where we have to count the number of times we applied the second rule, or equivalently the number $\#C$ of C's in the resulting string, because the summation index g in (4.159) corresponds to $m_1 + m_2 - \#C$.
Fix $\#C \in \{0, \ldots, \min(m_1, m_2)\}$. A valid string has $m_1 + m_2 - \#C$ characters, because the number of its Cs is $\#C$, its number of As is $m_1 - \#C$ and its number of Bs is $m_2 - \#C$. Ignoring the labelling of the As, Bs and Cs there are $\binom{m_1+m_2-\#C}{\#C\ (m_1-\#C)\ (m_2-\#C)}$ such strings. Now if we consider one such string without labelling, e.g.

$$CAABACCBBACBBABBBB, \tag{4.165}$$

there is only one correct labelling to be restored, namely the one where each A and the first index of any C receive increasing labels from left to right and analogously for B and the second index of any C, resulting for our example in

$$C_{1,1}A_2A_3B_2A_4C_{5,3}C_{6,4}B_5B_6A_7C_{8,7}B_8B_9A_9B_{10}B_{11}B_{12}B_{13}. \tag{4.166}$$

So any unlabelled string corresponds to exactly one labelled string which in turn corresponds to exactly one choice of operator product $\prod T$. So returning to our Operators, we found the number $\#K(m_1, m_2)$ it is

$$\#K(m_1, m_2) = -\sum_{g=\max(m_1,m_2)}^{m_1+m_2} \frac{(-1)^g}{g} \binom{g}{(m_1+m_2-g)\ (g-m_1)\ (g-m_2)}, \tag{4.167}$$

where the total minus sign comes from the total minus sign in front of (4.162) with respect to (4.157).
Now since we introduced the slightly non-standard definition of binomial coefficients used in [19] we can make use of the rules for summing

binomial coefficients derived there. As a first step to evaluate (4.167) we split the trinomial coefficient into binomial ones and make use of the absorption identity

$$\forall a \in \mathbb{C}\ \forall b \in \mathbb{Z} : b\binom{a}{b} = a\binom{a-1}{b-1} \qquad \text{(absorption)}$$

for $m_2, m_1 \neq 0$ as follows

$$\begin{aligned}
&\#K(m_1, m_2)\\
&= -\sum_{g=\max(m_1,m_2)}^{m_1+m_2} \frac{(-1)^g}{g}\binom{g}{(m_1+m_2-g)\ (g-m_1)\ (g-m_2)}\\
&= -\sum_{g=\max(m_1,m_2)}^{m_1+m_2} \frac{(-1)^g}{g}\binom{g}{m_2}\binom{m_2}{g-m_1}\\
&\overset{\text{(absorption)}}{=} -\sum_{g=\max(m_1,m_2)}^{m_1+m_2} \frac{(-1)^g}{m_2}\binom{g-1}{m_2-1}\binom{m_2}{g-m_1}\\
&= \frac{-1}{m_2}\sum_{g=\max(m_1,m_2)}^{m_1+m_2} (-1)^g\binom{g-1}{m_2-1}\binom{m_2}{g-m_1}\\
&\overset{m_1>0}{=} \frac{-1}{m_2}\sum_{g\in\mathbb{Z}}(-1)^g\binom{m_2}{g-m_1}\binom{g-1}{m_2-1}\\
&\overset{*}{=} \frac{-1}{m_2}(-1)^{m_2-m_1}\binom{m_1-1}{-1} = 0,
\end{aligned}$$

where for the second but last equality $m_1 > 0$ is needed for the $g = 0$ summand not to contribute and for the marked equality we used summation rule (5.24) of [19]. So all the coefficients vanish that fulfil $m_1, m_2 \neq 0$. The sum for the remaining cases is readily computed, since there is just one summand. Summarising we find

$$\#K(m_1,m_2)=\delta_{m_2,0}\frac{(-1)^{1+m_1}}{m_1}+\delta_{m_1,0}\frac{(-1)^{1+m_2}}{m_2},$$

where the second summand can be ignored, since terms with $m_1=0$ are irrelevant for our considerations.
So the left hand side of (4.162) can be evaluated

$$\left[-\sum_{g=1}^{n}\sum_{\substack{\vec{b}\in\mathbb{N}^g\\|\vec{b}|=n}}\frac{(-1)^g}{g}\binom{n}{\vec{b}}\prod_{l=1}^{g}T_{b_l},a^{\#}\right]$$
$$=\sum_{g=1}^{n}\sum_{\substack{\vec{b}\in\mathbb{N}^g\\|b|=n}}\frac{(-1)^{g+1}}{g}\binom{n}{\vec{b}}a^{\#}\prod_{l=1}^{g}Z_{b_l},$$

which is exactly equal to the right hand side of (4.162). This ends the motivation of the conjecture.

4.3.3 Solution to Recursive Equation

So we found a recursive equation for the T_ns, now we need to solve it. In order to do so we need the following lemma about combinatorial distributions

Lemma 95. *For any $g\in\mathbb{N},k\in\mathbb{N}$*

$$\sum_{\substack{\vec{g}\in\mathbb{N}^g\\|\vec{g}|=k}}\binom{k}{\vec{g}}=\sum_{l=0}^{g}(-1)^l(g-l)^k\binom{g}{l}\tag{4.168}$$

holds. The reader interested in terminology may be eager to know, that the right hand side is equal to $g!$ times the Stirling number of the second kind $\left\{\begin{matrix}k\\g\end{matrix}\right\}$.

Proof: We would like to apply the multinomial theorem but there are all the summands missing where at least one of the entries of $\vec{g}$ is zero, so we add an appropriate expression of zero. We also give the expression in question a name, since we will later on arrive at a recursive expression.

$$\begin{aligned} F(g,k) &:= \sum_{\substack{\vec{g}\in\mathbb{N}^g \\ |\vec{g}|=k}} \binom{k}{\vec{g}} = \sum_{\substack{\vec{g}\in\mathbb{N}_0^g \\ |\vec{g}|=k}} \binom{k}{\vec{g}} - \sum_{\substack{\vec{g}\in\mathbb{N}_0^g \\ |\vec{g}|=k \\ \exists l:g_l=0}} \binom{k}{\vec{g}} \\ &= g^k - \sum_{\substack{\vec{g}\in\mathbb{N}_0^g \\ |\vec{g}|=k \\ \exists l:g_l=0}} \binom{k}{\vec{g}} = g^k - \sum_{n=1}^{g-1} \sum_{\substack{\vec{g}\in\mathbb{N}_0^g \\ |\vec{g}|=k}} \binom{k}{\vec{g}} 1_{\exists! i_1\ldots i_n:(\forall l\neq b:i_l\neq i_b)\wedge\forall l:g_{i_l}=0} \end{aligned} \tag{4.169}$$

where in the last line the indicator function is to enforce there being exactly n different indices i_l for which $g_{i_l} = 0$ holds. Now since it does not matter which entries of the vector vanish because the multinomial coefficient is symmetric and its value is identical to the corresponding multinomial coefficient where the vanishing entries are omitted, we can further simplify the sum:

$$F(g,k) = g^k - \sum_{n=1}^{g-1} \binom{g}{n} \sum_{\substack{\vec{g}\in\mathbb{N}^{g-n} \\ |\vec{g}|=k}} \binom{k}{\vec{g}}$$

The inner sum turns out to be $F(g-n,k)$, so we found the recursive relation for F:

$$F(g,k) = g^k - \sum_{n=1}^{g-1} \binom{g}{n} F(g-n,k) = g^k - \sum_{n=1}^{g-1} \binom{g}{n} F(n,k), \tag{4.170}$$

where for the last equality we used the symmetry of binomial coefficients. By iteratively applying this equation, we find the following formula, which we will now prove by induction

$$\begin{aligned}\forall d \in \mathbb{N}_0 : F(g,k) &= \sum_{l=0}^{d} (-1)^l (g-l)^k \binom{g}{l} \\ &\quad + (-1)^{d+1} \sum_{n=1}^{g-d-1} \binom{n+d-1}{d} \binom{g}{n+d} F(g-d-n,k). \end{aligned} \tag{4.171}$$

We already showed the start of the induction, so what's left is the induction step. Before we do so the following remark is in order: We are only interested in the case $d = g$ and the formula seems meaningless for $d > g$; however, the additional summands in the left sum vanish, where as the right sum is empty for these values of d since the upper bound of the summation index is lower than its lower bound.
For the induction step, pick $d \in \mathbb{N}_0$, use (4.171) and pull the first summand out of the second sum, on this summand we apply the recursive relation (4.170) resulting in

$$\begin{aligned}F(g,k) &= \sum_{l=0}^{d} (-1)^l (g-l)^k \binom{g}{l} \\ &\quad + (-1)^{d+1} \sum_{n=2}^{g-d-1} \binom{n+d-1}{d} \binom{g}{n+d} F(g-d-n,k) \\ &\quad + (-1)^{d+1} \binom{d}{d} \binom{g}{d+1} F(g-d-1,k) \\ &\overset{(4.170)}{=} \sum_{l=0}^{d+1} (-1)^l (g-l)^k \binom{g}{l}\end{aligned}$$

$$+ (-1)^{d+1} \sum_{n=2}^{g-d-1} \binom{n+d-1}{d} \binom{g}{n+d} F(g-d-n,k)$$
$$- (-1)^{d+1} \binom{g}{d+1} \sum_{n=1}^{g-d-2} \binom{g-d-1}{n} F(g-d-1-n,k)$$
$$= \sum_{l=0}^{d+1} (-1)^l (g-l)^k \binom{g}{l}$$
$$+ (-1)^{d+1} \sum_{n=1}^{g-d-2} \binom{n+d}{d} \binom{g}{n+d+1} F(g-d-1-n,k)$$
$$- (-1)^{d+1} \binom{g}{d+1} \sum_{n=1}^{g-d-2} \binom{g-d-1}{n} F(g-d-1-n,k). \quad (4.172)$$

After the index shift we can combine the last two sums.

$$F(g,k) = \sum_{l=0}^{d+1} (-1)^l (g-l)^k \binom{g}{l}$$
$$+ \sum_{n=1}^{g-d-2} \left[\binom{g}{d+1} \binom{g-d-1}{n} - \binom{n+d}{d} \binom{g}{n+d+1} \right]$$
$$(-1)^{d+2} F(g-d-1-n,k). \quad (4.173)$$

In order to combine the two binomials we reassemble $\binom{g}{d+1}\binom{g-d-1}{n}$ into $\binom{g}{n+d+1}\binom{n+d+1}{d+1}$, which can be seen to be possible by representing everything in terms of factorials. This results in

$$F(g,k) = \sum_{l=0}^{d+1} (-1)^l (g-l)^k \binom{g}{l}$$
$$+(-1)^{d+2} \sum_{n=1}^{g-d-2} \left[\binom{n+d+1}{d+1} - \binom{n+d}{d} \right] \binom{g}{n+d+1} F(g-d-1-n,k)$$

$$= \sum_{l=0}^{d+1} (-1)^l (g-l)^k \binom{g}{l}$$

$$+ (-1)^{d+2} \sum_{n=1}^{g-d-2} \binom{n+d}{d+1} \binom{g}{n+d+1} F(g-d-1-n,k), \quad (4.174)$$

where we used the addition formula for binomials:

$$\forall n \in \mathbb{C} \forall k \in \mathbb{Z} : \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}. \quad (4.175)$$

This concludes the proof by induction. By setting $d = g$ in equation (4.171) we arrive at the desired result. □

Using the previous lemma, we are able to show the next

Lemma 96. *For any $k \in \mathbb{N} \backslash \{1\}$ the following equation holds*

$$\sum_{g=1}^{k} \frac{(-1)^g}{g} \sum_{\substack{\vec{g} \in \mathbb{N}^g \\ |\vec{g}|=k}} \binom{k}{\vec{g}} = 0. \quad (4.176)$$

Proof: Let $k \in \mathbb{N} \backslash \{1\}$, as a first step we apply lemma 95. We change the order of summation, use (absorption), extend the range of summation and shift summation index to arrive at

$$\sum_{g=1}^{k} \frac{(-1)^g}{g} \sum_{l=0}^{g} (-1)^l (g-l)^k \binom{g}{l} = \sum_{g=1}^{k} \frac{1}{g} \sum_{l=0}^{g} (-1)^{g-l} (g-l)^k \binom{g}{g-l}$$

$$= \sum_{g=1}^{k} \sum_{p=0}^{g} (-1)^p p^k \frac{1}{g} \binom{g}{p} = \sum_{g=1}^{k} \sum_{p=0}^{g} (-1)^p p^k \frac{1}{p} \binom{g-1}{p-1}$$

$$= \sum_{g=1}^{k} \sum_{p \in \mathbb{Z}} (-1)^p p^{k-1} \binom{g-1}{p-1} = \sum_{p \in \mathbb{Z}} (-1)^p p^{k-1} \sum_{g=1}^{k} \binom{g-1}{p-1}$$

$$= \sum_{p\in\mathbb{Z}} (-1)^p p^{k-1} \sum_{g=0}^{k-1} \binom{g}{p-1}. \quad (4.177)$$

Now we use equation (5.10) of [19]:

$$\forall m, n \in \mathbb{N}_0 : \sum_{k=0}^{n} \binom{k}{m} = \binom{n+1}{m+1}, \qquad \text{(upper summation)}$$

which can for example be proven by induction on n.
We furthermore rewrite the power of the summation index p in terms of the derivative of an exponential and change order summation and differentiation. This results in

$$\begin{aligned}
\sum_{g=1}^{k} \frac{(-1)^g}{g} \sum_{l=0}^{g} (-1)^l (g-l)^k \binom{g}{l} &= \sum_{p\in\mathbb{Z}} (-1)^p p^{k-1} \binom{k}{p} \\
&= \sum_{p=0}^{k} (-1)^p \left.\frac{\partial^{k-1}}{\partial\alpha^{k-1}} e^{\alpha p}\right|_{\alpha=0} \binom{k}{p} = \left.\frac{\partial^{k-1}}{\partial\alpha^{k-1}} \sum_{p=0}^{k} (-1)^p e^{\alpha p} \binom{k}{p}\right|_{\alpha=0} \\
&= \left.\frac{\partial^{k-1}}{\partial\alpha^{k-1}} (1-e^{\alpha p})^k\right|_{\alpha=0} = (-1)^k \left.\frac{\partial^{k-1}}{\partial\alpha^{k-1}} \left(\sum_{l=1}^{\infty} \frac{(\alpha p)^l}{l!}\right)^k\right|_{\alpha=0} \\
&= (-1)^k \left.\frac{\partial^{k-1}}{\partial\alpha^{k-1}} ((\alpha p)^k + \mathcal{O}((\alpha p)^{k+1}))\right|_{\alpha=0} = 0.
\end{aligned}$$

□

We are now in a position to state the solution to the recursive equation (4.157) and motivate that it is in fact a solution.

Conjecture 97. *For $n \in \mathbb{N}$ the solution of the recursive equation (4.157) solely in terms of Γ_a and C_a is given by*

$$T_n = \sum_{g=1}^{n} \sum_{\substack{\vec{b}\in\mathbb{N}^g \\ |\vec{b}|=n}} \sum_{\vec{d}\in\{0,1\}^g} \frac{1}{g!}\binom{n}{\vec{b}} \prod_{l=1}^{g} F_{b_l,d_l}, \tag{4.178}$$

where F is given by

$$F_{a,b} = \begin{cases} \Gamma_a & \textit{for } b=0 \\ C_a & \textit{for } b=1 \end{cases}. \tag{4.179}$$

For the readers convenience we remind her, that Γ_a and the constants C_n are defined in conjecture 94.

Motivation: The structure of this proof will be induction over n. For $n = 1$ the whole expression on the right hand side collapses to $C_1 + \Gamma_1$, which we already know to be equal to T_1. For arbitrary $n \in \mathbb{N}$ we apply for the induction step the recursive equation (4.157) once and use the induction hypothesis for all $k \leqslant n$ and thereby arrive at the rather convoluted expression

$$\begin{aligned} T_{n+1} &\overset{(4.157)}{=} \Gamma_{n+1} + C_{n+1} + \sum_{g=2}^{n+1} \sum_{\substack{\vec{b}\in\mathbb{N}^g \\ |\vec{b}|=n+1}} \frac{(-1)^g}{g}\binom{n+1}{\vec{b}} \prod_{l=1}^{g} T_{b_l} \\ &\overset{\text{induction hyp}}{=} \Gamma_{n+1} + C_{n+1} + \sum_{g=2}^{n+1} \sum_{\substack{\vec{b}\in\mathbb{N}^g \\ |\vec{b}|=n+1}} \frac{(-1)^g}{g}\binom{n+1}{\vec{b}} \prod_{l=1}^{g} \\ &\qquad \sum_{g_l=1}^{b_l} \sum_{\substack{\vec{c}_l\in\mathbb{N}^{g_l} \\ |\vec{c}_l|=b_l}} \sum_{\vec{e}_l\in\{0,1\}^{g_l}} \frac{1}{g_l!}\binom{b_l}{\vec{c}_l} \prod_{k=1}^{g_l} F_{c_{l,k},e_{l,k}}. \end{aligned} \tag{4.180}$$

If we were to count the contributions of this sum to a specific product $\prod F_{c_j,e_j}$ for some choice of $(c_j)_j, (e_j)_j$ we would first recognize that

all the multinomial factors in (4.180) combine to a single one whose indices are given by the first indices of all the F factors involved. Other than this factor each contribution adds $\frac{(-1)^g}{g}\prod_{l=1}^{g}\frac{1}{g_l!}$ to the sum. So we need to keep track of how many contributions there are and which distributions of g_l they belong to.
Fix some product $\prod F := \prod_{j=1}^{\tilde{g}} F_{\tilde{b}_j,\tilde{d}_j}$. In the sum (4.180) we pick some initial short product of length g and split each factor into g_l pieces to arrive at one of length $\tilde{g}$ if the product is to contribute to $\prod F$. So clearly $\sum_{l=1}^{g} g_l = \tilde{g}$ holds for any contribution to $\prod F$. The reverse is also true, for any g and $g_1,\ldots,g_g \in \mathbb{N}$ such that $\sum_{l=1}^{g} g_l = \tilde{g}$ holds the corresponding expression in (4.180) contributes to $\prod F$. Furthermore $\prod F$ and g, $g_1,\ldots g_g$ is enough to uniquely determine the summand of (4.180) the contribution belongs to. For an illustration of this splitting see

$$\underbrace{\underbrace{F^1_{3,1}F^2_{2,0}F^3_{7,1}}_{g_1=3}\ \underbrace{F^4_{5,0}}_{g_2=1}\ \underbrace{F^5_{4,1}F^6_{2,1}}_{g_3=2}\ \underbrace{F^7_{1,1}F^8_{3,0}F^9_{4,1}}_{g_4=3}\ \underbrace{F^{10}_{4,1}F^{11}_{1,0}}_{g_5=2}}_{g=5}$$

$$g_1+g_2+g_3+g_4+g_5 = 11 = \tilde{g}.$$

We recognize that the sum we are about to perform is by no means unique for each order of n but only depends on the number of appearing factors and the number of splittings performed on them. By the preceding argument we need

$$\sum_{g=2}^{\tilde{g}} \frac{(-1)^g}{g} \sum_{\substack{\vec{g}\in\mathbb{N}^g \\ |\vec{g}|=\tilde{g}}} \prod_{l=1}^{g} \frac{1}{g_l!} = \frac{1}{\tilde{g}!} \tag{4.181}$$

to hold for $\tilde{g} > 1$, in order to find agreement with the proposed solution (4.179). Now proving (4.181) is done by realizing, that one can include the right hand side into the sum as the $g = 1$ summand, dividing the equation by $\tilde{g}!$ and using lemma 96 with $k = \tilde{g}$. The remaining case, $\tilde{g} = 1$, can directly be read off of (4.180). This ends the motivation of this conjecture.

Conjecture 98. *For $n \in \mathbb{N}$, T_n can be written as*

$$\frac{1}{n!}T_n = \sum_{\substack{1 \leqslant c+g \leqslant n \\ c,g \in \mathbb{N}_0}} \sum_{\substack{\vec{g} \in \mathbb{N}^g \\ \vec{c} \in \mathbb{N}^c \\ |\vec{c}|+|\vec{g}|=n}} \frac{1}{c!g!} \prod_{l=1}^{c} \frac{1}{c_l!} C_{c_l} \prod_{l=1}^{g} \frac{1}{g_l!} \Gamma_{g_l}. \tag{4.182}$$

Please note that for ease of notation we defined $\mathbb{N}^0 := \{1\}$.

Motivation: By an argument completely analogous to the combinatorial argument in the motivation of conjecture (94) we see that we can disentangle the Fs in (4.178) into Γs and Cs if we multiply by a factor of $\binom{c+g}{c}$ where c is the number of Cs and g is the number of Γs giving

$$T_n = \sum_{\substack{1 \leqslant c+g \leqslant n \\ c,g \in \mathbb{N}_0}} \sum_{\substack{\vec{g} \in \mathbb{N}^g \\ \vec{c} \in \mathbb{N}^c \\ |\vec{c}|+|\vec{g}|=n}} \binom{c+g}{c} \frac{1}{(c+g)!} \binom{n}{\vec{g} \oplus \vec{c}} \prod_{l=1}^{c} C_{c_l} \prod_{l=1}^{g} \Gamma_{g_l}, \tag{4.183}$$

which directly reduces to the equation we wanted to prove, by plugging in the multinomials in terms of factorials.

Conjecture 99. *As a formal power series, the second quantized scattering operator can be written in the form*

$$S = e^{\sum_{l \in \mathbb{N}} \frac{C_l}{l!}} e^{\sum_{l \in \mathbb{N}} \frac{\Gamma_l}{l!}}. \tag{4.184}$$

Motivation: We plug conjecture 98 into the defining Series for the T_ns giving

$$
\begin{aligned}
S &= \sum_{n\in\mathbb{N}_0} \frac{1}{n!} T_n && (4.185)\\
&= \mathbb{1}_{\mathcal{F}} + \sum_{n\in\mathbb{N}} \sum_{\substack{1\leqslant c+g\leqslant n\\ c,g\in\mathbb{N}_0}} \sum_{\substack{\vec{g}\in\mathbb{N}^g\\ \vec{c}\in\mathbb{N}^c\\ |\vec{c}|+|\vec{g}|=n}} \frac{1}{c!g!} \prod_{l=1}^{c} \frac{1}{c_l!} C_{c_l} \prod_{l=1}^{g} \frac{1}{g_l!} \Gamma_{g_l} && (4.186)\\
&= \mathbb{1}_{\mathcal{F}} + \sum_{\substack{1\leqslant c+g\\ c,g\in\mathbb{N}_0}} \sum_{\substack{\vec{g}\in\mathbb{N}^g\\ \vec{c}\in\mathbb{N}^c}} \frac{1}{c!g!} \prod_{l=1}^{c} \frac{1}{c_l!} C_{c_l} \prod_{l=1}^{g} \frac{1}{g_l!} \Gamma_{g_l} && (4.187)\\
&= \sum_{c,g\in\mathbb{N}_0} \sum_{\substack{\vec{g}\in\mathbb{N}^g\\ \vec{c}\in\mathbb{N}^c}} \frac{1}{c!g!} \prod_{l=1}^{c} \frac{1}{c_l!} C_{c_l} \prod_{l=1}^{g} \frac{1}{g_l!} \Gamma_{g_l} && (4.188)\\
&= \sum_{c\in\mathbb{N}_0} \frac{1}{c!} \sum_{\vec{c}\in\mathbb{N}^c} \prod_{l=1}^{c} \frac{1}{c_l!} C_{c_l} \sum_{g\in\mathbb{N}_0} \frac{1}{g!} \sum_{\vec{g}\in\mathbb{N}^g} \prod_{l=1}^{g} \frac{1}{g_l!} \Gamma_{g_l} && (4.189)\\
&= \sum_{c\in\mathbb{N}_0} \frac{1}{c!} \prod_{l=1}^{c} \sum_{k\in\mathbb{N}} \frac{1}{k!} C_k \sum_{g\in\mathbb{N}_0} \frac{1}{g!} \prod_{l=1}^{g} \sum_{b\in\mathbb{N}} \frac{1}{b!} \Gamma_b && (4.190)\\
&= \sum_{c\in\mathbb{N}_0} \frac{1}{c!} \left(\sum_{k\in\mathbb{N}} \frac{1}{k!} C_k \right)^c \sum_{g\in\mathbb{N}_0} \frac{1}{g!} \left(\sum_{b\in\mathbb{N}} \frac{1}{b!} \Gamma_b \right)^g && (4.191)\\
&= e^{\sum_{l\in\mathbb{N}} \frac{1}{l!} C_l} e^{\sum_{l\in\mathbb{N}} \frac{1}{l!} \Gamma_l}. && (4.192)
\end{aligned}
$$

Conjecture 100. *For A such that*

$$\|\mathbb{1} - U^A\| < 1. \tag{4.193}$$

The second quantized scattering operator fulfils

$$S = e^{\sum_{n\in\mathbb{N}} \frac{C_n}{n!}} e^{\mathrm{d}\Gamma(\ln(U))} \tag{4.194}$$

where C_n must be imaginary for any $n \in \mathbb{N}$ in order to satisfy unitarity.

Motivation: First the remark about $C_n \in i\mathbb{R}$ for any n is a direct consequence of the second factor of (4.194) begin unitary. This in turn follows directly from $\mathrm{d}\Gamma^*(K) = -\mathrm{d}\Gamma(K)$ for any K in the domain of $\mathrm{d}\Gamma$. That $\ln U$ is in the domain of $\mathrm{d}\Gamma$ follows from $(\ln U)^* = \ln U^* = \ln U^{-1} = -\ln U$ and $\|U - \mathbb{1}\| < 1$.

We are going to change the sum in the second exponential of (4.184), so let's take a closer look at that: by exchanging summation we can step by step simplify

$$\begin{aligned}
\sum_{l\in\mathbb{N}} \frac{\Gamma_l}{l!} &= \sum_{n\in\mathbb{N}} \frac{1}{n!} \mathrm{d}\Gamma \left(\sum_{g=1}^{n} \sum_{\substack{\vec{b}\in\mathbb{N}^g \\ |\vec{b}|=n}} \frac{(-1)^{g+1}}{g} \binom{n}{\vec{b}} \prod_{l=1}^{g} Z_{b_l} \right) \\
&= \mathrm{d}\Gamma \left(\sum_{n\in\mathbb{N}} \frac{1}{n!} \sum_{g=1}^{n} \sum_{\substack{\vec{b}\in\mathbb{N}^g \\ |\vec{b}|=n}} \frac{(-1)^{g+1}}{g} \binom{n}{\vec{b}} \prod_{l=1}^{g} Z_{b_l} \right) \\
&= \mathrm{d}\Gamma \left(\sum_{n\in\mathbb{N}} \sum_{g=1}^{n} \sum_{\substack{\vec{b}\in\mathbb{N}^g \\ |\vec{b}|=n}} \frac{(-1)^{g+1}}{g} \prod_{l=1}^{g} \frac{Z_{b_l}}{b_l!} \right) \\
&= \mathrm{d}\Gamma \left(\sum_{g\in\mathbb{N}} \sum_{\vec{b}\in\mathbb{N}^g} \frac{(-1)^{g+1}}{g} \prod_{l=1}^{g} \frac{Z_{b_l}}{b_l!} \right) \\
&= \mathrm{d}\Gamma \left(\sum_{g\in\mathbb{N}} \frac{(-1)^{g+1}}{g} \prod_{l=1}^{g} \left(\sum_{b_l\in\mathbb{N}} \frac{Z_{b_l}}{b_l!} \right) \right) \\
&= \mathrm{d}\Gamma \left(\sum_{g\in\mathbb{N}} \frac{(-1)^{g+1}}{g} \left(\sum_{b\in\mathbb{N}} \frac{Z_b}{b!} \right)^{g} \right)
\end{aligned}$$

$$= \mathrm{d}\Gamma\left(\sum_{g\in\mathbb{N}}\frac{(-1)^{g+1}}{g}(U-\mathbb{1})^g\right) = \mathrm{d}\Gamma\left(-\sum_{g\in\mathbb{N}}\frac{1}{g}(\mathbb{1}-U)^g\right)$$
$$= \mathrm{d}\Gamma\left(\ln\left(\mathbb{1}-(\mathbb{1}-U)\right)\right) = \mathrm{d}\Gamma\left(\ln\left(U\right)\right). \quad (4.195)$$

The last conjecture is proven directly in section 3.4.2

4.4 Proofs of Section 3.5

We first break down the problem of the proof of theorem 82 to a problem on a bounded domain. Then we go on to show that the relevant functions are locally bounded so that an application of Lebesgue dominated convergence yields our result. The same estimates we need for theorem 82 will be used again for the proof of theorem 83.

Definition 101. *Let $A \in \mathcal{V}$, Σ be a Cauchy surface and $\delta > 0$, we introduce the sets*

$$J_1 := \{(x,y)\in\mathbb{R}^{4+4} \mid (x-y)^2 \geqslant -\delta^2/4 \wedge \overline{x\ y} \cap \mathrm{supp}(A) \neq \varnothing\} \quad (4.196)$$

$$J_2 := \{(x,y)\in\mathbb{R}^{4+4} \mid (x-y)^2 \geqslant -\delta^2 \wedge \overline{x\ y} \cap \big(B_{\delta/2}(0)+\mathrm{supp}(A)\big) \neq \varnothing\} \quad (4.197)$$

$$J_3 := \{(x,y)\in\Sigma\times\Sigma \mid (x-y)^2 \geqslant -\delta^2 \wedge \overline{x\ y} \cap \big(B_{\delta/2}(0)+\mathrm{supp}(A)\big) \neq \varnothing\}, \quad (4.198)$$

where the sum of two sets is defined as $S_1 + S_2 := \{s_1 + s_2 \mid s_1 \in S_1, s_2 \in S_2\}$. Please recall that the line segment between x and y by is denoted by $\overline{x\ y}$.

Definition 102. *For ease of comparison, we define for a four-potential $A \in \mathcal{V}$, $\tilde{\lambda} : \mathbb{R}^{4+4} \to \mathbb{C}$:*

$$\tilde{\lambda}(x,y) = (x-y)^\alpha \int_0^1 ds\ A_\alpha(xs + (1-s)y), \quad (4.199)$$

and furthermore introduce the abbreviation $G(x,y) := e^{-i\tilde{\lambda}(x,y)}$.

Remark 103. *The function* $\tilde{\lambda} \notin \mathcal{G}^A$, *because it does not satisfy one of the technical conditions in definition 80, there is not compact* $K \subset \mathbb{R}^4$ *such that* $\operatorname{supp}(\tilde{\lambda}) \subset K \times \mathbb{R}^4 \cup \mathbb{R}^4 \times K$ *holds. However, the other conditions are all fulfilled. We need to introduce it nonetheless, because it is present in the representation for the Hadamard states.*

Lemma 104. *For any* $A \in \mathcal{V}$ *and any Hadamard state* H *of the form* (3.495) *to* (3.498) *and any* $\lambda^A \in \mathcal{G}^A$, $\delta > 0$, *there is a smooth family of functions* $w_\varepsilon \in C^\infty(\mathbb{R}^4 \times \mathbb{R}^4, \mathbb{C}^{4\times 4})$ *with* $\varepsilon \in [0,1]$ *and a smooth function* $w \in C^\infty(\mathbb{R}^4 \times \mathbb{R}^4, \mathbb{C}^{4\times 4})$ *obeying*

$$\int_{\mathbb{R}^4\times\mathbb{R}^4} \overline{f_1}(x) w_\varepsilon(x,y) f_2(y) dx dy = \int_{\mathbb{R}^4\times\mathbb{R}^4} \overline{f_1}(x)(h_\varepsilon - p_\varepsilon^\lambda)(x,y) f_2(y) dx dy \tag{4.200}$$

$$\int_{\mathbb{R}^4\times\mathbb{R}^4} \overline{f_1}(x) w(x,y) f_2(y) dx dy$$
$$= \lim_{\epsilon\to 0} \int_{\mathbb{R}^4\times\mathbb{R}^4} \overline{f_1}(x)(h_\epsilon(x,y) - p_\epsilon^\lambda(x,y)) f_2(y) dx dy, \tag{4.201}$$

for all test functions f_1, f_2 *such that* $\operatorname{supp}(f_1) \times \operatorname{supp}(f_2) \subset J_2^c$ *Additionally* $\lim_{\varepsilon\to 0} w_\varepsilon = w$ *pointwise and in addition is continuous as a function of type* $\mathbb{R}^{4+4} \times [0,1] \to \mathbb{C}^{4\times 4}$.

Proof. Let $A \in \mathcal{V}, \varepsilon, \delta > 0$. Let H be a Hadamard state acting as

$$H(f_1, f_2) = \lim_{\varepsilon\searrow 0} \int_{\mathbb{R}^4} d^4x \overline{f_1}(x) \int_{\mathbb{R}^4} d^4y \; h_\varepsilon(x,y) f_2(y). \tag{4.202}$$

be of the form of equation (3.496) to (3.498). Pick a function $\lambda \in \mathcal{G}^A$. We may choose $w' \in C^\infty(\mathbb{R}^4 \times \mathbb{R}^4, \mathbb{C}^{4\times 4})$ such that the following conditions is fulfilled:

For all test functions $f_1, f_2 \in C_c^\infty(\mathbb{R}^4, \mathbb{C}^4)$ such that $\mathrm{supp}(f_1)\times\mathrm{supp}(f_2) \subset J_1^c$ we have

$$\int_{\mathbb{R}^4\times\mathbb{R}^4} \overline{f_1}(x)w'(x,y)f_2(y)dxdy \tag{4.203}$$

$$= \lim_{\epsilon\to 0}\int_{\mathbb{R}^4\times\mathbb{R}^4} \overline{f_1}(x)(h_\epsilon(x,y) - p_\epsilon^\lambda(x,y))f_2(y)dxdy. \tag{4.204}$$

To understand why, we consider two cases regarding the support of the testfunctions f_1, f_2,

1. $(x,y) \in \mathrm{supp}(f_1) \times \mathrm{supp}(f_2)$ implies $\overline{x\ y} \cap \mathrm{supp}(A) = \varnothing$.
2. $(x,y) \in \mathrm{supp}(f_1) \times \mathrm{supp}(f_2)$ implies $(x-y)^2 < -\delta^2/4$.

Regarding 1: In the representation of the Hadamard state H, equation (3.500), can be explicitly solved recursively (See [2][lemma 2.2.2]. The factor V_0 was already given, for $k \geqslant 1$ the recursion is given by

$$V_k(x,y) = -kG(x,y)\int_0^1 ds s^{k-1}G(x+s(y-x),x)PV_{k-1}(x,x+s(y-x)). \tag{4.205}$$

Recall that P depends on the four potential in a local manner. Here we observe that, if the support of the external field A does not intersect the straight line connecting x and y, the function $V(x,y)$ in the expression for $h_\varepsilon(x,y)$ can be calculated to be

$$V(x,y) = \sum_{k=0}^\infty \frac{((y-x)^2m^2/4)^k}{k!(k+1)!}, \tag{4.206}$$

which equals the logarithmic part of p_ε^-, corresponding to $Q_1(\xi)$ in (3.503).

This shows that in this case $p_\varepsilon^-(x,y)$ agrees with $h_\varepsilon(x,y)$ up to smooth terms in case 1.
Regarding 2: we notice that the only points (x,y) in the singular support of h_ε and p_ε^λ need to fulfil $(y-x)^2 = 0$. This implies that for functions $f_1, f_2 \in C_c^\infty(\mathbb{R}^4, \mathbb{C}^4)$ of case 2 these operators act as integral operators with smooth kernel.
In fact, as w' is smooth, condition (4.203) specifies the values of w' uniquely for arguments $(x,y) \in \mathbb{R}^{4+4}$ in the complement of J_1. Analogously, we define a smooth function $w'_\varepsilon \in C^\infty(\mathbb{R}^4 \times \mathbb{R}^4, \mathbb{C}^{4\times 4})$ for every $\varepsilon > 0$ fulfilling

$$\int_{\mathbb{R}^4\times\mathbb{R}^4} \overline{f_1}(x) w'_\varepsilon(x,y) f_2(y) dxdy = \int_{\mathbb{R}^4\times\mathbb{R}^4} \overline{f_1}(x)(h_\varepsilon - p_\varepsilon^\lambda)(x,y) f_2(y) dxdy \tag{4.207}$$

for test functions $f_1, f_2 \in C_c^\infty(\mathbb{R}^4, \mathbb{C}^4)$ such that $\mathrm{supp}(f_1) \times \mathrm{supp}(f_2) \subset J_1^c$. Next, we pick a function $\chi \in C^\infty(\mathbb{R}^{4+4})$ such that

$$\chi|_{J_2^c} = 1, \quad \chi|_{J_1} = 0 \tag{4.208}$$

hold. Now, we define

$$w := \chi w', \quad w_\varepsilon := \chi w'_\varepsilon, \tag{4.209}$$

where w', w'_ε are extended by the zero function inside J_1.
The functions w, w_ε are now uniquely fixed in all of $\mathbb{R}^{4+4}$ and fulfil (4.203) and (4.207), respectively for test functions f_1, f_2 such that $\mathrm{supp}(f_1) \times \mathrm{supp}(f_2) \subset J_2^c$ holds. Observing the exact form of h_ε, (3.496) to (3.498) and p_ε^λ, (3.507), we notice that in fact $w_\varepsilon(x,y) \xrightarrow{\varepsilon\to 0} w(x,y)$ for all $x, y \in J_1^c$, because of the cutoff function χ this also holds for $(x,y) \in J_1$. Moreover, we notice from the explicit form of h_ε and p_ε^λ that, $w_{\cdot_3}(\cdot_1, \cdot_2)$ is even continuous as a function of type $\mathbb{R}^4 \times \mathbb{R}^4 \times [0,1] \to \mathbb{C}^{4\times 4}$. $\square$

Definition 105. *Let Σ be a Cauchy surface and $\delta > 0$. We define*

$$\mathfrak{g}_1 : \Sigma \times \Sigma \times]0,1[\to \mathbb{C}^{4\times 4} \tag{4.210}$$

$$(x, y, \varepsilon) \mapsto e^{-i\tilde{\lambda}(x,y)} \frac{m^2}{4\pi^2} (i\partial\!\!\!/_x + m) \frac{1}{m^2(y - x - i\varepsilon e_0)^2} - (3.496)(x, y) \tag{4.211}$$

and denoting $\mathrm{dom}(g_2) := \Sigma \times \Sigma \times [0,1] \backslash \{(x, x, 0) \mid x \in \Sigma\}$*.*

$$\mathfrak{g}_2 : \mathrm{dom}(g_2) \ni (x, y, \varepsilon) \mapsto \frac{1}{(y - x - i\varepsilon e_0)^2} \tag{4.212}$$

$$\times \Bigg(A\!\!\!/(x) + A\!\!\!/(y) - 2\int_0^1 \mathrm{d}s A\!\!\!/(sx + (1-s)y) \tag{4.213}$$

$$+ (y - x)^\alpha \int_0^1 \mathrm{d}s (1 - 2s)(\partial\!\!\!/ A_\alpha)(sx + (1-s)y) \Bigg). \tag{4.214}$$

Lemma 106. *Let Σ be a Cauchy surface and $\delta > 0$. We have $|\mathfrak{g}_1| = |\mathfrak{g}_2|$. There is a constant $M_1 \in \mathbb{R}$, such that*

$$\forall (x, y, \varepsilon) \in J_3 \times]0,1[: |\mathfrak{g}_1(x, y, \varepsilon)| \leqslant M_1. \tag{4.215}$$

Furthermore, there is a function $\mathfrak{g}_3 \in C(\Sigma \times \Sigma, \mathbb{R}^+)$ such that

$$\forall (x, y, \varepsilon) \in \mathrm{dom}(g_2) : |\mathfrak{g}_2(x, y, \varepsilon)| \leqslant \mathfrak{g}_3(x, y) \tag{4.216}$$

holds.

Proof. Pick a Cauchy surface Σ, a four-potential $A \in \mathcal{V}$ and $(x, y, \varepsilon) \in$

$\mathrm{dom}(g_2)$. A direct calculation yields

$$
\begin{aligned}
& e^{-i\tilde{\lambda}(x,y)} \frac{m^2}{4\pi^2}(i\partial\!\!\!/_x + m)\frac{1}{m^2(y-x-i\varepsilon e_0)^2} - (3.496)(x,y) && (4.217)\\
=& \frac{G(x,y)}{4\pi^2}\Bigg((i\partial\!\!\!/_x + m)\frac{1}{(y-x-i\varepsilon e_0)^2} \\
& \qquad - \frac{1}{G(x,y)}(i\nabla\!\!\!\!/\,/2 - i\nabla\!\!\!\!/^{\,*}/2 + m)\frac{G(x,y)}{(y-x-i\varepsilon e_0)^2}\Bigg) && (4.218)\\
=& \frac{iG(x,y)}{4\pi^2}\Bigg(\partial\!\!\!/_x \frac{1}{(y-x-i\varepsilon e_0)^2} \\
& \qquad - \frac{1}{2G(x,y)}(\partial\!\!\!/_x + iA\!\!\!/(x) - \partial\!\!\!/_y + iA\!\!\!/(y))\frac{G(x,y)}{(y-x-i\varepsilon e_0)^2}\Bigg) \\
& && (4.219)\\
=& \frac{iG(x,y)}{4\pi^2}\Bigg(-i\frac{A\!\!\!/(x) + A\!\!\!/(y)}{2(y-x-i\varepsilon e_0)^2} + \partial\!\!\!/_x \frac{1}{(y-x-i\varepsilon e_0)^2} \\
& - \frac{1}{2}(\partial\!\!\!/_x - \partial\!\!\!/_y)\frac{1}{(y-x-i\varepsilon e_0)^2} - \frac{1}{2G(x,y)}\frac{(\partial\!\!\!/_x - \partial\!\!\!/_y)G(x,y)}{(y-x-i\varepsilon e_0)^2}\Bigg) && (4.220)\\
=& \frac{iG(x,y)}{4\pi^2}\Bigg(-i\frac{A\!\!\!/(x) + A\!\!\!/(y)}{2(y-x-i\varepsilon e_0)^2} + \frac{1}{2}(\partial\!\!\!/_x + \partial\!\!\!/_y)\frac{1}{(y-x-i\varepsilon e_0)^2} \\
& \qquad + \frac{1}{2G(x,y)}\frac{(-\partial\!\!\!/_x + \partial\!\!\!/_y)G(x,y)}{(y-x-i\varepsilon e_0)^2}\Bigg) && (4.221)\\
=& \frac{-iG(x,y)}{8\pi^2}\frac{1}{(y-x-i\varepsilon e_0)^2}\left(iA\!\!\!/(x) + iA\!\!\!/(y) + G(x,y)^{-1}\left(\partial\!\!\!/_x - \partial\!\!\!/_y\right)G(x,y)\right) \\
=& \frac{G(x,y)}{8\pi^2}\frac{1}{(y-x-i\varepsilon e_0)^2}\Bigg(A\!\!\!/(x) + A\!\!\!/(y) - 2\int_0^1 \mathrm{d}s A\!\!\!/(sx + (1-s)y)
\end{aligned}
$$

$$+(x-y)^\alpha \int_0^1 \mathrm{d}s(1-2s)(\not{\partial} A_\alpha)(sx+(1-s)y)\Bigg). \tag{4.222}$$

Now using Taylor's series for A around $(x+y)/2$ reveals

$$\not{A}(x)+\not{A}(y)-2\int_0^1 \mathrm{d}s\not{A}(sx+(1-s)y) = \frac{(x-y)^\alpha}{6}(x-y)^\beta(\partial_\alpha\partial_\beta\not{A})((x+y)/2)+\mathcal{O}(\|x-y\|^3) \tag{4.223}$$

$$(x-y)^\alpha\int_0^1 \mathrm{d}s(1-2s)(\not{\partial} A_\alpha)(sx+(1-s)y) = \tag{4.224}$$
$$-\frac{(x-y)^\alpha}{12}(x-y)^\beta(\not{\partial}\partial_\beta A_\alpha)((x+y)/2)+\mathcal{O}(\|x-y\|^3).$$

Because A is smooth and J_3 is a compact region the terms (4.223) and (4.224) are bounded in this region. Because of their behaviour close to $y=x$ the function is a bounded and point-wise an upper bound of the absolute value of

$$\left| e^{-i\tilde{\lambda}(x,y)}\frac{m^2}{4\pi^2}(i\not{\partial}_x+m)\frac{1}{m^2(y-x-i\varepsilon e_0)^2}-(3.496)(x,y)\right| \tag{4.225}$$
$$\leqslant \frac{\|(4.223)\|(x,y)+\|(4.224)\|(x,y)}{8\pi^2}\frac{1}{|(y-x)^2|}. \tag{4.226}$$

Also from the Taylor's expansion (4.223) and (4.224) directly follows that the right hand side of (4.226) is continuous. □

Lemma 107. *For any Cauchy surface Σ, four-potential $A \in \mathcal{V}$ and $\lambda^A \in \mathcal{G}^A$, the function*

$$\Sigma\times\Sigma\times]0,1[\to\mathbb{C}^{4\times 4} \tag{4.227}$$
$$(x,y,\varepsilon)\mapsto\left(e^{-i\tilde{\lambda}(x,y)}-e^{-i\lambda^A(x,y)}\right)\frac{1}{4\pi^2}(i\not{\partial}_x+m)\frac{1}{(y-x-i\varepsilon e_0)^2} \tag{4.228}$$

is bounded by

$$\left|\left(e^{-i\tilde{\lambda}(x,y)} - e^{-i\lambda^A(x,y)}\right) \frac{1}{4\pi^2}(i\partial\!\!\!/_x + m)\frac{1}{(y-x-i\varepsilon e_0)^2}\right| \tag{4.229}$$

$$\leqslant \frac{\left|e^{-i\tilde{\lambda}(x,y)} - e^{-i\lambda^A(x,y)}\right|}{4\pi^2}\left(\eta(y-x) + \frac{\|y-x\|}{((y-x)^2)^2} + \frac{m}{|(y-x)^2|}\right) \tag{4.230}$$

$$:= M_2(x,y), \tag{4.231}$$

where η is given by

$$\eta(y-x) := \frac{1}{(-(y-x)^2\varepsilon^{*-0.5} + \varepsilon^{*1.5})^2 + \varepsilon^*(y^0-x^0)^2} \tag{4.232}$$

and ε^* is given by

$$\varepsilon^* := \frac{1}{\sqrt{6}}\sqrt{-\beta - 2\alpha + \sqrt{(2\alpha+\beta)^2 + 12\alpha^2}}, \tag{4.233}$$

with $\alpha := -(y-x)^2$ and $(y^0-x^0)^2$. Furthermore $M_2 \in L^2_{\text{loc}}(\Sigma \times \Sigma)$.

Proof. Pick $A \in \mathcal{V}$, Cauchy surface Σ and $\lambda^A \in \mathcal{G}^A$. By expanding $\tilde{\lambda}(x,y) - \lambda^A(x,y)$ in a Taylor series around $\frac{y+x}{2}$ we see directly that

$$e^{-i\tilde{\lambda}(x,y)} - e^{-i\lambda^A(x,y)} = \mathcal{O}(\|x-y\|^2). \tag{4.234}$$

In order to show that (4.230) has a square integrable upper bound, we consider each term separately. First, we look at the mass term. It obeys

$$\left\|\frac{m}{(y-x-i\varepsilon e_0)^2}\right\| = \frac{m}{\sqrt{((y-x)^2-\varepsilon^2)^2 + \varepsilon^2(y^0-x^0)^2}} < \frac{m}{|(y-x)^2|}, \tag{4.235}$$

since $(y-x)^2 < 0$ for $x, y \in \Sigma$, hence this term is bounded once multiplied with the difference of exponentials. The term with the derivative will be split into two:

$$\left\| i\partial\!\!\!/_x \frac{1}{(y-x-i\varepsilon e_0)^2} \right\| \tag{4.236}$$

$$\leqslant \frac{\|y\!\!\!/ - x\!\!\!/ - i\varepsilon e\!\!\!/_0\|}{|(y-x-i\varepsilon e_0)^2|^2} \leqslant \frac{\varepsilon + \|x-y\|}{((y-x)^2-\varepsilon^2)^2 + \varepsilon^2(y^0-x^0)^2} \tag{4.237}$$

$$< \frac{1}{(-(y-x)^2\varepsilon^{-0.5}+\varepsilon^{1.5})^2 + \varepsilon(y^0-x^0)^2} + \frac{\|y-x\|}{((y-x)^2)^2}, \tag{4.238}$$

we notice at this point, that the second term becomes locally square integrable in $\Sigma \times \Sigma$ once multiplied with a function of type $\mathcal{O}(\|x-y\|^2)$ close to $x = y$. Indeed, the first term also has this property, in order to deduce this more readily, we will maximise this term now. Considering the limits $\varepsilon \to 0$ and $\varepsilon \to \infty$ we see that this term has for arbitrary $(x, y \in \Sigma)$ a maximum in the interval $]0, \infty[$. Abbreviating $-(y-x)^2 := \alpha$, $(y^0-x^0)^2 := \beta$, this value of ε fulfills

$$\partial_\varepsilon(\varepsilon^{-0.5}\alpha + \varepsilon^{1.5})^2 + \beta = 0 \tag{4.239}$$

$$\iff 3\varepsilon^4 + \varepsilon^2(2\alpha+\beta) - \alpha^2 = 0 \tag{4.240}$$

$$\iff \varepsilon^2 = \frac{-\beta - 2\alpha + \sqrt{(2\alpha+\beta)^2 + 12\alpha^2}}{6} \tag{4.241}$$

$$\iff \varepsilon = \varepsilon^* := \frac{1}{\sqrt{6}}\sqrt{-\beta - 2\alpha + \sqrt{(2\alpha+\beta)^2 + 12\alpha^2}}. \tag{4.242}$$

So we can find an upper bound on the first summand of (4.238) by replacing ε by what we just found. Now because all the terms in the resulting expression

$$\frac{1}{(-(y-x)^2\varepsilon^{*-0.5} + \varepsilon^{*1.5})^2 + \varepsilon^*(y^0-x^0)^2} =: \eta(y-x) \tag{4.243}$$

are positive, we did not introduce any new singularities. Also the expression is homogenous of degree -3:

$$\eta(\delta(y-x)) = \delta^{-3}\eta(y-x), \tag{4.244}$$

so we can conclude that also this term as well as the second term in (4.238), are of type $\mathcal{O}(\|x-y\|^{-3})$ and therefore locally square integrable once multiplied by the difference of exponentials in (4.234). So summarising (4.229) can be bounded by

$$\|(4.229)(x,y)\| \leqslant \frac{\left|e^{-i\lambda^A(x,y)} - e^{-i\tilde{\lambda}(x,y)}\right|}{4\pi^2}\left(\eta(y-x) + \frac{\|y-x\|}{((y-x)^2)^2} + \frac{m}{|(y-x)^2|}\right), \tag{4.245}$$

which is locally square integrable in on $\Sigma \times \Sigma$. $\square$

Lemma 108. *For any Cauchy surface Σ, four-potential $A \in \mathcal{V}$, the function*

$$\Sigma \times \Sigma \times]0,1[\to \mathbb{C}^{4\times 4} \tag{4.246}$$

$$(x,y,\varepsilon) \mapsto (i\partial\!\!\!/_x + m)\ln(-m^2(y-x-i\varepsilon e_0)^2) \tag{4.247}$$

has the locally in $\Sigma \times \Sigma$ square integrable bound $M_3 + M_4$. More explicitly,

$$\left|\partial\!\!\!/_x \ln(-(y-x-i\varepsilon e_0)^2)\right| \leqslant \tag{4.248}$$

$$\frac{2\|y-x\|}{|(y-x)^2|} + \frac{2}{\sqrt{4|(y-x)^2| + (y^0-x^0)^2}} := M_3(x,y), \tag{4.249}$$

and

$$\|\ln(-m^2(y-x-i\varepsilon e_0)^2)\| \leqslant |\ln(-m^2(y-x)^2)| + \pi/2 := M_4(x,y). \tag{4.250}$$

Proof. Pick $A \in C_c^\infty(\mathbb{R}^4)$, a Cauchy surface Σ and $f \in C(\Sigma \times \Sigma, \mathbb{C}^{4\times 4})$. The terms containing $\partial\!\!\!/ \ln(-(y-x-i\varepsilon e_0)^2)$ are bounded as follows

$$\begin{aligned}\|i\partial_\alpha \ln(-m^2(y-x-i\varepsilon e_0)^2)\| &= 2\left\|\frac{y^\alpha - x^\alpha - i\varepsilon e_0^\alpha}{-(y-x-i\varepsilon e_0)^2}\right\| &(4.251)\\ &\leqslant \frac{2\|y-x\|}{|(y-x)^2|} + \frac{2\varepsilon}{|(y-x-i\varepsilon e_0)^2|} &(4.252)\\ = \frac{2\|y-x\|}{|(y-x)^2|} &+ \frac{2}{\sqrt{(-(y-x)^2/\varepsilon+\varepsilon)^2 + (y^0-x^0)^2}} &(4.253)\\ &\leqslant \frac{2\|y-x\|}{|(y-x)^2|} + \frac{2}{\sqrt{4|(y-x)^2| + (y^0-x^0)^2}}, &(4.254)\end{aligned}$$

where inequality in (4.254) we maximised the expression with respect to $\varepsilon \in]0,\infty[$. The terms containing the logarithm without any derivative are directly bounded by

$$|\ln(-m^2(y-x-i\varepsilon e_0)^2)| \leqslant |\ln(-m^2(y-x)^2)| + \pi/2.$$

□

Lemma 109. *For every four-potential $A \in \mathcal{V}$ and every Hadamard state H of the form (3.495) to (3.498) the smooth function $w \in C^\infty(\mathbb{R}^4 \times \mathbb{R}^4, \mathbb{C}^{4\times 4})$ and family $(w_\varepsilon)_{\varepsilon\in]0,1[} \subset C^\infty(\mathbb{R}^4\times\mathbb{R}^4, \mathbb{C}^{4\times 4})$ provided by lemma 104 also satisfy for any Cauchy surface Σ and any $\lambda^A \in \mathcal{G}^A$:*

$$h_\varepsilon^A - w_\varepsilon - p_\varepsilon^{\lambda^A}\Big|_{\Sigma\times\Sigma} \in L^2(\Sigma\times\Sigma) \qquad (4.255)$$

and the $L^2(\Sigma\times\Sigma)$ limit $\lim_{\varepsilon\to 0} h_\varepsilon^A - w_\varepsilon - p_\varepsilon^{\lambda^A}\Big|_{\Sigma\times\Sigma}$ exists.

Proof. Pick a Cauchy surface Σ, a four-potential $A \in C_c^\infty(\mathbb{R}^4)$, a Hadamard state of the form (3.495) to (3.498), $\lambda^A \in \mathcal{G}^A$ and for

$\varepsilon \in]0,1[$ smooth functions $w_\varepsilon \in C^\infty(\mathbb{R}^{4+4}, \mathbb{C}^{4\times 4})$ according to lemma 104. Our aim is to show that $h_\varepsilon - w_\varepsilon - p_\varepsilon^{\lambda^A}\Big|_{\Sigma\times\Sigma}$ converges in the sense of $L^2(\Sigma\times\Sigma)$ in the limit $\varepsilon \to 0$. According to lemma 104 we have that $h_\varepsilon - w_\varepsilon - p_\varepsilon^{\lambda^A}|_{\Sigma\times\Sigma}$ vanishes outside the set J_3 which is bounded, independent of ε and of finite measure. Taking the exact form of h_ε and $p_\varepsilon^{\lambda^A}$ into account, one notices that the function $h_\varepsilon - w_\varepsilon - p_\varepsilon^{\lambda^A}|_{\Sigma\times\Sigma}(x,y)$ converges point-wise to a function defined on $\Sigma\times\Sigma\backslash\{(x,x) \mid x \in \Sigma\}$. In order to show that the convergence also holds in the sense of $L^2(\Sigma\times\Sigma)$ we would like to use dominated convergence. Hence we should find a function $M \in L^2(\Sigma\times\Sigma)$ such that

$$\left|(h_\varepsilon - w_\varepsilon - p_\varepsilon^{\lambda^A})(x,y)\right| \leqslant |M|(x,y) \tag{4.256}$$

holds almost for almost all $x, y \in \Sigma$ and all ε small enough. We may pick $M|_{J_3^c} = 0$, of course. So we only have to pick M for arguments inside J_3. Now because w_ε is continuous as a function from $\mathbb{R}^{4+4} \times [0,1] \to \mathbb{C}^{4\times 4}$ and hence also when restricted to $J_3 \times [0,1]$. The set $J_3 \times [0,1]$ is compact which implies that there is a constant M_5 such that

$$\forall (x,y,\varepsilon) \in J_3 \times [0,1] : |w_\varepsilon(x,y)| \leqslant M_5 \tag{4.257}$$

holds. So by the triangular inequality we only need to bound $h_\varepsilon - p_\varepsilon^{\lambda^A}$ We now dissect h_ε as well as $p_\varepsilon^{\lambda^A}$ each into three pieces

$$h_\varepsilon(x,y) = \frac{-1}{2(2\pi)^2}(-i\nabla\!\!\!/ + i\nabla\!\!\!/^* - 2m)\frac{G(x,y)}{(y-x-i\varepsilon e_0)^2} \tag{4.258}$$

$$+\frac{-1}{2(2\pi)^2}(-i\nabla\!\!\!/ + i\nabla\!\!\!/^* - 2m)V(x,y)\ln(-(y-x-i\varepsilon e_0)^2) \tag{4.259}$$

$$+B(x,y), \tag{4.260}$$

$$p_\varepsilon^{\lambda^A}(x,y) = e^{-i\lambda^A(x,y)}\frac{m^2}{4\pi^2}(i\partial\!\!\!/ + m)\frac{1}{m^2(y-x-i\varepsilon e_0)^2} \tag{4.261}$$

$$-\frac{m^2}{4\pi^2}(i\partial\!\!\!/ + m)Q_1(-m^2(y-x-i\varepsilon e_0)^2)\ln(-m^2(y-x-i\varepsilon e_0)^2) \tag{4.262}$$

$$-\frac{m^2}{4\pi^2}(i\partial\!\!\!/ + m)Q_2(-m^2(y-x-i\varepsilon e_0)^2). \tag{4.263}$$

Let us first consider the most singular terms of $h_\varepsilon - w_\varepsilon - p_\varepsilon^{\lambda^A}|_{\Sigma\times\Sigma}$, namely (4.258) – (4.261). In order to better compare these terms, we will add 0 in the form of (4.261) – (4.261) where we replace λ^A by $\tilde{\lambda}$. By lemma 106 the term

$$(4.258)(x,y) - G(x,y)\frac{m^2}{4\pi^2}(i\partial\!\!\!/ + m)\frac{1}{m^2(y-x-i\varepsilon e_0)^2} \tag{4.264}$$

is uniformly bounded in J_3.
Next we find a square integrable upper bound on

$$e^{-i\tilde{\lambda}(x,y)}\frac{m^2}{4\pi^2}(i\partial\!\!\!/_x + m)\frac{1}{m^2(y-x-i\varepsilon e_0)^2} - (4.261)(x,y). \tag{4.265}$$

We rewrite this term as

$$e^{-i\tilde{\lambda}(x,y)}\frac{m^2}{4\pi^2}(i\partial\!\!\!/_x + m)\frac{1}{m^2(y-x-i\varepsilon e_0)^2} - (4.261)(x,y) = \tag{4.266}$$

$$\left(e^{-i\tilde{\lambda}(x,y)} - e^{-i\lambda(x,y)}\right)\frac{1}{4\pi^2}(i\partial\!\!\!/_x + m)\frac{1}{(y-x-i\varepsilon e_0)^2}, \tag{4.267}$$

according to lemma 107 this has the upper bound $M_2 \in L^2_{\text{loc}}(\Sigma\times\Sigma)$, which is independend of ε. The term (4.262) – (4.259) are bounded by lemma 108. Thus we obtain for M for $(x,y) \in J_3$:

$$
\begin{aligned}
M(x,y) &:= M_5 + M_1 + M_2(x,y)\\
&+ \frac{1}{4\pi^2}\left(4|V(x,y)| + m^2 \sup_{\varepsilon\in[0,1]} |Q_1(-m^2(y-x-i\varepsilon e_0)^2)|\right) M_3(x,y)\\
&+ \frac{1}{4\pi^2}\Big(|(i\nabla\!\!\!/ - i\nabla\!\!\!/^* - 2m)V(x,y)|/2\\
&\qquad + m^2 \sup_{\varepsilon\in[0,1]} |(i\partial\!\!\!/ + m)Q_1(-m^2(y-x-i\varepsilon e_0))|\Big) M_4(x,y)\\
&+ |B(x,y)| + \frac{m^2}{4\pi^2} \sup_{\varepsilon\in[0,1]} |(i\partial\!\!\!/_x + m)Q_2(-m^2(y-x-i\varepsilon e_0)^2)|
\end{aligned}
$$

The argument of $h_\varepsilon - w_\varepsilon - p^\lambda_\varepsilon|_{\Sigma\times\Sigma}(x,y)$ converging to $h_0 - w - p^\lambda_0|_{\Sigma\times\Sigma}(x,y)$ in the sense of $L^2(\Sigma\times\Sigma)$ is completed by applying dominated convergence theorem.
As a final point: the construction seems to depend on the function λ^A we chose, however for any different $\lambda'^A \in \mathcal{G}^A$ we have due to remark 81

$$
P^\lambda - P^{\lambda'} \in I_2(\mathcal{H}_\Sigma), \tag{4.268}
$$

therefore also the limit $\lim_{\varepsilon\to 0} h^A_\varepsilon - w_\varepsilon - p^{\lambda'^A}_\varepsilon$ exists in the sense of L^2. □

Proof of theorem 82: We pick for a four-potential $A \in \mathcal{V}$ a Hadamard state H of the form (3.495) to (3.498) and a $\lambda^A \in \mathcal{G}^A$. Then we pick $w_\varepsilon, w \in C^\infty_c(\mathbb{R}^{4+4}, \mathbb{C}^{4\times 4})$ for all $\varepsilon \in [0,1]$ according to lemma 104. Because of lemma 109, we have that

$$
h^A_\varepsilon - w_\varepsilon - p^{\lambda^A}_\varepsilon \in L^2(\Sigma\times\Sigma) \tag{4.269}
$$

holds. We can define the operator $Z \in I_2(\mathcal{H}_\Sigma)$ for any Cauchy surface Σ as

$$\mathcal{H}_\Sigma \ni \psi \mapsto Z\psi := \lim_{\varepsilon\to 0}\int_\Sigma (h^A_\varepsilon - w_\varepsilon - p^{\lambda^A}_\varepsilon)(\cdot, y) i_\gamma(d^4y)\psi(y) \tag{4.270}$$

and

$$\tilde{P}_\Sigma := Z + P^{\lambda^A}_\Sigma. \tag{4.271}$$

Because of (3.509) we have for any other $\lambda' \in \mathcal{G}^A$ that $\tilde{P}_\Sigma - P^{\lambda'} \in I_2(\mathcal{H}_\Sigma)$.

□

Proof of theorem 83. Pick a four-potential $A \in \mathcal{V}$, a $\lambda^A \in \mathcal{G}^A$ and a Hadamard state of the form (3.495) to (3.498). Next, we pick w_ε according to lemma 104; however, we will denote it by R_ε.
Define for each $\varepsilon \in]0,1]$ the function w_ε as

$$w_\varepsilon(x,y) := \tag{4.272}$$

$$-\frac{G(x,y) - e^{-i\lambda^A(x,y)}}{(2\pi)^2}(i\not{\partial}_x + m)\frac{1}{(y-x-i\varepsilon e_0)^2} \tag{4.273}$$

$$+\frac{G(x,y)}{2(2\pi)^2(y-x-i\varepsilon e_0)^2}\Bigg(\not{A}(x) + \not{A}(y) - 2\int_0^1 ds \not{A}(sx + (1-s)y) \tag{4.274}$$

$$+(x-y)^\alpha \int_0^1 ds (1-2s)(\not{\partial} A_\alpha)(sx+(1-s)y)\Bigg) \tag{4.275}$$

$$+\frac{1}{2(2\pi)^2}(-i\not{\nabla} + i\not{\nabla}^* - 2m)V(x,y)\ln(-(y-x-i\varepsilon e_0)^2) \tag{4.276}$$

$$-\frac{m^2 e^{-i\lambda^A(x,y)}}{4\pi^2}(i\not{\partial}_x + m)\Bigg(Q_2(m\sqrt{-(y-x-i\varepsilon e_0)^2} \tag{4.277}$$

$$+ Q_1(m\sqrt{-(y-x-i\varepsilon e_0)^2}\ln(-m^2(y-x-i\varepsilon e_0)^2)\Big) \tag{4.278}$$

$$+ R_\varepsilon(x,y). \tag{4.279}$$

We further define the distribution H^{λ^A} by its action on test functions $f_1, f_2 \in C_c^\infty(\mathbb{R}^4, \mathbb{C}^4)$

$$\tilde{H}(f_1, f_2) := \lim_{\varepsilon\to 0}\int_{\mathbb{R}^4}\int_{\mathbb{R}^4}\overline{f}_1(x)(p_\varepsilon^{\lambda^A}(x,y) - w_\varepsilon(x,y))f_2(y)d^4x d^4y. \tag{4.280}$$

First, we verify that H^{λ^A} is indeed a Hadamard state. Pick $\varepsilon > 0$, we find

$$\begin{aligned}
&p_\varepsilon^{\lambda^A}(x,y) - w_\varepsilon(x,y)\\
&= -e^{-i\lambda^A(x,y)}\frac{m^2}{4\pi^2}(i\partial\!\!\!/_x + m)\frac{K_1(m\sqrt{-(y-x-i\varepsilon e_0)^2})}{m\sqrt{-(y-x-i\varepsilon e_0)^2}} - w_\varepsilon(x,y)\\
&= -e^{-i\lambda^A(x,y)}\frac{m^2}{4\pi^2}(i\partial\!\!\!/_x + m)\Bigg(\frac{-1}{m^2(y-x-i\varepsilon e_0)^2}\\
&+ Q_1(-m^2(y-x-i\varepsilon e_0)^2)\ln(m\sqrt{-(y-x-i\varepsilon e_0)^2})\\
&+ Q_2(-m^2(y-x-i\varepsilon e_0)^2)\Bigg)\\
&- w_\varepsilon(x,y).
\end{aligned}$$

Inserting $w_\varepsilon(x,y)$ we find that the most divergent terms proportional to $e^{-i\lambda^A}$, as well as the terms involving Q_1 and Q_2 cancel. This results in

$$
\begin{aligned}
&p_\varepsilon^{\lambda^A}(x,y) - w_\varepsilon(x,y) \\
&= \frac{G(x,y)}{(2\pi)^2}(i\partial\!\!\!/_x + m)\frac{1}{(y-x-i\varepsilon e_0)^2} \\
&+ \frac{G(x,y)}{2(2\pi)^2(y-x-i\varepsilon e_0)^2}\Bigg(A\!\!\!/(x) + A\!\!\!/(y) - 2\int_0^1 ds A\!\!\!/(sx+(1-s)y) \\
&+(x-y)^\alpha \int_0^1 ds(1-2s)(\partial\!\!\!/ A)(sx+(1-s)y)\Bigg) \\
&- \frac{1}{2(2\pi)^2}(-i\nabla\!\!\!\!/ + i\nabla\!\!\!\!/^* - 2m)V(x,y)\ln(-(y-x-i\varepsilon e_0)^2) \\
&- R_\varepsilon(x,y).
\end{aligned}
$$

Now the terms involving $G(x,y) = e^{-i(x-y)^\alpha \int_0^1 ds A_\alpha(xs+(1-s)y)}$ can be summarised

$$
\begin{aligned}
&\frac{G(x,y)}{(2\pi)^2}(i\partial\!\!\!/_x + m)\frac{1}{(y-x-i\varepsilon e_0)^2} \\
&+ \frac{G(x,y)}{2(2\pi)^2(y-x-i\varepsilon e_0)^2}\Bigg(A\!\!\!/(x) + A\!\!\!/(y) - 2\int_0^1 ds A\!\!\!/(sx+(1-s)y) \\
&+(x-y)^\alpha \int_0^1 ds(1-2s)(\partial\!\!\!/ A)(sx+(1-s)y)\Bigg) \\
&= \frac{1}{2(2\pi)^2}(i\nabla\!\!\!\!/ - i\nabla\!\!\!\!/^* + 2m)\frac{G(x,y)}{(y-x-i\varepsilon e_0)^2},
\end{aligned}
$$

so overall we find

$$
\begin{aligned}
&p_\varepsilon^{\lambda^A}(x,y) - w_\varepsilon(x,y) = \\
&\frac{-1}{2(2\pi)^2}(-i\nabla\!\!\!/ + i\nabla\!\!\!/^* - 2m)\frac{G(x,y)}{(y-x-i\varepsilon e_0)^2} \\
&- \frac{1}{2(2\pi)^2}(-i\nabla\!\!\!/ + i\nabla\!\!\!/^* - 2m)V(x,y)\ln(-(y-x-i\varepsilon e_0)^2) \\
&- R_\varepsilon(x,y).
\end{aligned}
$$

Because the term in the last line is smooth, this means that indeed $p_\varepsilon^{\lambda^A}(x,y) - w_\varepsilon(x,y)$ is the integration kernel of a Hadamard state. Furthermore, for a given Cauchy surface Σ the function $w_\varepsilon - R_\varepsilon$ has a limit in the sense of $L^2_{\mathrm{loc}}(\Sigma \times \Sigma)$, because of Lebesgue dominated convergence and lemmata 106, 107 and 108. That the term $R_\varepsilon|_{\Sigma\times\Sigma}$ has a smooth pointwise limit follows from lemma 104. Finally outside of the diagonal $\{(x,x) \mid x \in \Sigma\}$ w_ε is smooth also in the limit $\varepsilon \to 0$. Since the set J_3 has a positive distance form the diagonal we may pick a function $\mathfrak{R}_\varepsilon \in C^\infty(\Sigma \times \Sigma \to \mathbb{C}^{4\times 4})$ such that $\mathfrak{R}_\varepsilon|_{J_3^c} = w_\varepsilon - R_\varepsilon$. Since J_3 is of finite measure we then have

$$
w_\varepsilon = \overbrace{w_\varepsilon - R_\varepsilon - \mathfrak{R}_\varepsilon}^{\exists L^2-\lim} + \overbrace{R_\varepsilon + \mathfrak{R}_\varepsilon}^{\exists \text{pointwise-} \lim \in C^\infty} . \tag{4.281}
$$

□

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