We begin with a bit of notation:

Definition 0.1. *We will be working in the spaces*

$$\mathcal{D} := C_c^\infty(\mathbb{R}^4, \mathbb{R}^4), \tag{1}$$

$$\mathcal{D}' := \mathcal{D}'(B_1(0) \cap \mathit{Causal}) := \\ \{L : \mathcal{D} \to \mathbb{C} \mid L \text{ is linear and bounded} \wedge \operatorname{supp} L \subseteq B_1(0) \cap \mathit{Causal}\}. \tag{2}$$

For $n \in \mathbb{N}_0$ we furthermore introduce

$$\mathcal{D}^n := (\mathcal{D}, \|\cdot\| := \sum_{|\alpha| \leqslant n} \|D^\alpha \cdot \|_\infty), \tag{3}$$

$$\mathcal{D}^{n\prime} := (\mathcal{D}', \|\cdot\|_n', \tag{4}$$

and for $k \in \mathbb{N}_0$ we introduce

$$\mathcal{D}_k := \{F \in \mathcal{D} \mid \forall \alpha, |\alpha| \leqslant k : D^\alpha F(0) = 0\} \tag{5}$$

and $\mathcal{D}^n, \mathcal{D}^{n\prime}$ are analogously defined.

We now choose fixed (but arbitrary) functions $\chi \in C^\infty(\mathbb{R}^4, \mathbb{R})$ and $\eta \in C_c^\infty(\mathbb{R}^4, \mathbb{R})$, which fulfil

$$\begin{aligned} &\operatorname{supp} \eta \subseteq B_1(0) \wedge \forall x \in B_{1/2}(0) : \eta(x) = 1 \\ &\forall \lambda \in \mathbb{R}, \forall x \in \mathbb{R}^4 : \chi(\lambda x) = \chi(x) \wedge (x^2 > 0 \wedge x^0 > 0 \Rightarrow \chi(x) = 1) \\ &\forall x \in \mathbb{R}^4 : \big(2(x^0)^2 - \vec{x}^2 \leqslant 0 \Rightarrow \chi(x) = 0\big) \wedge (x^0 \leqslant 0 \Rightarrow \chi(x) = 0). \end{aligned}$$

Furthermore we define for any $\varepsilon > 0$, $\eta_\varepsilon : x \mapsto \eta\left(\frac{x}{\varepsilon}\right)$ and the splitting of test-functions:

Definition 0.2. *for any $k, m, n \in \mathbb{N}_0, \varepsilon > 0$*

$$\begin{aligned} \mathit{Split}_{k,\varepsilon} : \mathcal{D}_k^m &\to \mathcal{D}^n \\ F &\mapsto \chi(1 - \eta_\varepsilon)F. \end{aligned} \tag{6}$$

The main result of this document is

Theorem 0.3. *For all $k, m, n \in \mathbb{N}_0$ such that $n + 1 \leqslant k \leqslant m$ is fulfilled, $\mathit{split}_{k,\varepsilon}$ has for $\varepsilon \to 0$ the Cauchy-property in the topology induced by $\|\cdot\|_{\mathcal{D}_k^m \to \mathcal{D}^n}$, meaning that*

$$\forall \delta > 0, \exists E > 0 : \forall \varepsilon, \tilde{\varepsilon} < E : \sup_{\substack{\|F\|_m \leqslant 1 \\ F \in \mathcal{D}_k^m}} \|\mathit{Split}_{k,\varepsilon}[F] - \mathit{Split}_{k,\tilde{\varepsilon}}[F]\|_n < \delta \tag{7}$$

is fulfilled. Therefore the operator

$$\mathit{split}_k := \lim_{\varepsilon \to 0} \mathit{split}_{k,\varepsilon} : \mathcal{D}_k^m \to \hat{\mathcal{D}}^n \tag{8}$$

exists and is bounded, where we denoted the completion of $\mathcal{D}$ by $\hat{\mathcal{D}}$.

Proof of theorem 0.3: Let $k, m, n \in \mathbb{N}_0$ be such that $n + 1 \leqslant k \leqslant m$ holds. Let $\delta > 0$, $F \in \mathcal{D}_k^m$ and $\tilde{\varepsilon} \leqslant \varepsilon < E$. We will choose E in hindsight, but independent of ε and $\tilde{\varepsilon}$. We would like to estimate the left hand side of (7). There are constants $C_1 > 0$ depending on n such that for all multiindices $|\alpha| \leqslant n$

$$D^\alpha[\chi(\eta_{\tilde{\varepsilon}} - \eta_\varepsilon)F < C_1 \sum_{\substack{\beta,\gamma,\xi \in \mathbb{N}_0^4 \\ \beta+\gamma+\xi=\alpha}} \left|D^\beta \chi D^\gamma(\eta_{\tilde{\varepsilon}} - \eta_\varepsilon) D^\xi F\right| \tag{9}$$

holds. We continue by considering each factor separately. For the test function we can estimate for all $|\xi| \leqslant m$ by Taylors theorem, note that $F \in \mathcal{D}_k^m$, for more than one dimension

$$|D^\xi F(x)| = \left|(k+1-|\tau|) \sum_{|\tau|=k-|\xi|} \frac{x^\tau}{\tau!} \int_0^1 (1-s)^{k-|\xi|} D^{\tau+|\xi|} F(sx) \mathrm{d}s\right| \\ \leqslant \sum_{|\tau|=k-|\xi|} \left|\frac{x^\tau}{\tau!}\right| \left\|D^{\alpha+\xi} F\right\|_\infty,$$

therefore we find for all $x \in \mathbb{R}^4$:

$$|D^\xi F(x)| \leqslant C_2 \|x\|^{k-|\xi|} \|F\|_k, \tag{10}$$

for some constant $C_2 > 0$ depending on n. In order to estimate χ we exploit homogeneity. We find for $\lambda > 0$, $x \in \mathbb{R}^4$

$$D^\beta \chi(x) = D^\beta \chi(\lambda x) = \lambda^{|\beta|} D_x^\beta \chi(\lambda x).$$

Since the derivatives of χ are continuous their restriction to the unit sphere is bounded. By letting $\lambda = \|x\|$ we arrive at

$$\left|D^\beta \chi(x)\right| \leqslant C_3 \|x\|^{-|\beta|}, \tag{11}$$

for a constant $C_3 > 0$ depending on n and χ. For the estimate of η we split the term up in the case $\gamma = 0$ and its opposite yielding for $x \in \mathbb{R}^4$

$$\begin{aligned} |D^\gamma(\eta_\varepsilon - \eta_{\tilde\varepsilon})|(x) &= \left|D^\gamma\left(\eta\left(\frac{x}{\tilde\varepsilon}\right) - \eta\left(\frac{x}{\varepsilon}\right)\right)\right| \\ &\leqslant \begin{cases} \mathbb{1}_{B_\varepsilon(0)\backslash B_{\tilde\varepsilon}(0)}(x) & \text{for } \gamma = 0 \\ C_4[\tilde\varepsilon^{-|\gamma|}\mathbb{1}_{B_{\tilde\varepsilon}(0)\backslash B_{\tilde\varepsilon/2}(0)} + \varepsilon^{-|\gamma|}\mathbb{1}_{B_\varepsilon(0)\backslash B_{\varepsilon/2}(0)}] & \text{for } \gamma \neq 0, \end{cases} \end{aligned} \tag{12}$$

for some constant $C_4 > 0$ depending on η. We will now estimate (9), it suffices to pick $x \in B_1(0)$. We split the term for $\alpha, \gamma, \beta, \xi \in \mathbb{N}^4$ conditional on γ as follows

$$\begin{aligned} &|D^\alpha[\chi(\eta_{\tilde\varepsilon} - \eta_\varepsilon]F](x)| \\ &\leqslant C_1 \left| \sum_{\substack{\beta,\xi\in\mathbb{N}_0^4 \\ \beta+\xi=\alpha}} D^\beta\chi(x)(\eta_{\tilde\varepsilon} - \eta_\varepsilon)(x) D^\xi F(x) \right| && \text{(A)} \\ &+ C_1 \left| \sum_{\substack{\beta,\gamma,\xi\in\mathbb{N}_0^4 \\ \beta+\gamma+\xi=\alpha, \gamma\neq 0}} D^\beta\chi(x) D^\gamma(\eta_{\tilde\varepsilon} - \eta_\varepsilon)(x) D^\xi F(x) \right| && \text{(B)} \end{aligned}$$

Let for term (A), $l \in \mathbb{N}_0$ such that $\|x\| \in]2^{-(l+1)}, 2^{-l}] \subseteq B_\varepsilon(0)$ holds. Using estimates (10), (11) and (12) we find

$$\begin{aligned} \text{(A)} &\leqslant C_1 \mathbb{1}_{B_\varepsilon(0)\backslash B_{\tilde\varepsilon}(0)}(x) \sum_{\substack{\beta,\xi\in\mathbb{N}_0^4 \\ \beta+\xi=\alpha}} \left|D^\beta\chi(x)\right| \left|D^\xi F(x)\right| \\ &\leqslant C_1C_2C_3\|F\|_k \sum_{\substack{\beta,\xi\in\mathbb{N}_0^4 \\ \beta+\xi=\alpha}} 2^{(l+1)|\beta|} 2^{-l(k-|\xi|)} = C_1C_2C_3C_5/2\|F\|_k 2^{-l(k-|\alpha|} \\ &\leqslant C_1C_2C_3C_5/2\|F\|_k 2^{-l} \leqslant C_1C_2C_3C_5\|F\|_k\|x\| \leqslant C_1C_2C_3C_5\|F\|_k\varepsilon, \end{aligned}$$

where the n dependent constant C_5 was introduced and $k \geqslant n+1 \geqslant 1+|\alpha|$ was exploited. Now for the second Term, we split (B) again into two terms, (Ba) containing η_ε and (Bb) other containing $\eta_{\tilde{\varepsilon}}$. The estimate goes as follows

$$\begin{aligned}
\text{(Ba)} \leqslant C_1C_2C_3C_4\|F\|_k & \sum_{\substack{\beta,\gamma,\xi\in\mathbb{N}_0^4 \\ \beta+\gamma+\xi=\alpha,\gamma\neq 0}} \|x\|^{-|\beta|}\varepsilon^{-\gamma}\mathbb{1}_{B_\varepsilon(0)\setminus B_{\varepsilon/2}(0)}\|x\|^{k-|\xi|} \\
\leqslant C_1C_2C_3C_4\|F\|_k & \sum_{\substack{\beta,\gamma,\xi\in\mathbb{N}_0^4 \\ \beta+\gamma+\xi=\alpha,\gamma\neq 0}} \varepsilon^{-|\beta|}2^{|\beta|}\varepsilon^{-\gamma}\mathbb{1}_{B_\varepsilon(0)\setminus B_{\varepsilon/2}(0)}(x)\varepsilon^{k-|\xi|} \\
& \leqslant C_1C_2C_3C_4C_6\|F\|_k\varepsilon^{k-|\alpha|} \leqslant C_1C_2C_3C_4C_6\|F\|_k\varepsilon,
\end{aligned}$$

with some n dependent C_6. The very same estimate with ε replaced by $\tilde{\varepsilon}$ holds for (Bb). Taking all this together and choosing

$$E \leqslant \frac{\delta}{C_1C_2C_3C_4(C_5+2C_6)} \tag{13}$$

yields the claim. □