Calculation for Generating Function

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Yesterday we arrived at the following expression for the generating function of the relevant part of c^+

$$G = \int_{\left(\mathbb{R}^+\right)^{2n}} dy \; i\left(\frac{\pi}{\sum_{j=1}^{2n} y_j}\right)^{d/2} \exp\left(\sum_{j=1}^{2n} y_j(q_j^2 - m^2) - \frac{\left(\sum_{j=1}^{2n} y_j q_j + i/2\xi\right)^2}{\sum_{j=1}^{2n} y_j}\right). \tag{1}$$

For this expression we can further treat $\sum_{j=1}^{2n} y_j = t$ as an independent variable and perform its integral. For this (and the next few steps) we introduce the following abbreviations

$$t := \sum_{j=1}^{2n} y_j \tag{2}$$

$$z_j := y_j/t \tag{3}$$

$$\overline{q} := \sum_{j=1}^{2n} z_j q_j \tag{4}$$

$$\overline{q^2} := \sum_{j=1}^{2n} z_j q_j^2 \tag{5}$$

$$\lambda := \sqrt{m^2 - \overline{q^2} + \overline{q}^2}. \tag{6}$$

For the following calculations we need $\mathrm{Re}(\lambda), -\mathrm{Re}(\xi^2) \geqslant 0$. We then arrive at

$$G = \frac{i}{\Gamma(2n)} E_z\left[\int_{\mathbb{R}^+} dt \; t^{2n-1}\left(\frac{\pi}{t}\right)^{d/2} \exp\left(-t(m^2 - \overline{q^2} + \overline{q}^2) - i\xi\overline{q} + \frac{\xi^2}{4t}\right)\right]. \tag{7}$$

This can be brought into a known integral expression for the modified Bessel function.

$$\begin{aligned} G &= \frac{i\pi^{d/2}}{\Gamma(2n)} E_z\left[e^{-i\xi\overline{q}} \lambda^{d-4n} \int_{\mathbb{R}^+} \frac{d\tau}{\tau^{d/2-2n+1}} e^{-\tau - \frac{-\xi^2\lambda^2}{4\tau}}\right] & (8) \\ &= \frac{i\pi^{d/2}}{\Gamma(2n)} E_z\left[e^{-i\xi\overline{q}} \lambda^{d-4n} 2(\lambda\sqrt{-\xi^2}/2)^{2n-d/2} K_{d/2-2n}(\lambda\sqrt{-\xi^2})\right] & (9) \\ &\overset{K_\nu(z)=K_{-\nu}(z)}{=} \frac{i\pi^{d/2}}{\Gamma(2n)} 2^{1-2n+d/2} E_z\left[e^{-i\xi\overline{q}}\left(\frac{\sqrt{-\xi^2}}{\lambda}\right)^{2n-d/2} K_{2n-d/2}(\lambda\sqrt{-\xi^2})\right]. & (10) \end{aligned}$$

In the limit $\xi^2 \to 0$ one should use the asymptotic behavior of K according to

$$K_\nu(z) \approx \frac{\Gamma(\nu)}{2}(z/2)^{-\nu} \quad \text{for } \mathrm{Re}(\nu) > 0, \text{ and } z \to 0 \tag{11}$$

to recover the case we discussed the last time.
More precisly, we use the series expression for $\nu \in \mathbb{N}_0$:

$$K_\nu(z) = \frac{1}{2}(z/2)^{-\nu} \sum_{k=0}^{\nu-1} \frac{(\nu-k-1)!}{k!}(-z^2/4)^k + (-1)^{\nu+1}\ln(z/2)I_\nu(z) \tag{12}$$

$$+(-1)^\nu \frac{1}{2}(z/2)^\nu \sum_{k=0}^{\infty}(\psi(k+1)+\psi(\nu+k+1))\frac{(z^2/4)^k}{k!(\nu+k)!} \tag{13}$$

$$= \frac{1}{2}(z/2)^{-\nu} \sum_{k=0}^{\nu-1} \frac{(\nu-k-1)!}{k!}(-z^2/4)^k \tag{14}$$

$$+(-1)^\nu \frac{1}{2}(z/2)^\nu \sum_{k=0}^{\infty}(\psi(k+1)+\psi(\nu+k+1)-\ln(z^2/4))\frac{(z^2/4)^k}{k!(\nu+k)!}. \tag{15}$$

Plugging this into the expression for G we find

$$G = \frac{i\pi^{d/2}}{\Gamma(2n)} E_z\Bigg[e^{-i\xi\overline{q}} 2^{d-4n}\lambda^{d-4n} \sum_{k=0}^{2n-d/2-1} \frac{(2n-d/2-k-1)!}{k!}(\lambda^2\xi^2/4)^k \tag{16}$$

$$+e^{-i\xi\overline{q}}(\xi^2)^{2n-d/2}\sum_{k=0}^{\infty}(\psi(k+1)+\psi(2n-d/2+k+1)-\ln(-\lambda^2\xi^2/4))\frac{(-\lambda^2\xi^2/4)^k}{k!(2n-d/2+k)!}\Bigg]. \tag{17}$$

After taking at most $2n$ derivatives with respect to ξ at $\xi = 0$ we find that for $n = 2, d = 4$ the second term diverges logarithmically.