Let $(\varphi_l)_{l \in \mathbb{Z}/\{0\}} =: B \subset \mathcal{H}$, $(\varphi_l)_{l \in \mathbb{N}} =: B_+ \subset \mathcal{H}^+$ and $(\varphi_k)_{k \in -\mathbb{N}} =: B_- \subset \mathcal{H}^-$ be an ONB of all of, respectively positive respectively negative part of the Hilbert space. We investigate linear operators on Fockspace which fulfill the following commutation relations

$$[T_m(A), a(\phi)] = a\left(Z_m(A)\phi\right) + 1_{[2,\infty[}(m) \sum_{j=1}^{m-1} \binom{m}{j} a\left(Z_j(A)\phi\right) \circ T_{m-j}(A), \quad (1)$$

$$[T_m(A), a^*(\phi)] = a^*\left(Z_m(A)\phi\right) + 1_{[2,\infty[}(m) \sum_{j=1}^{m-1} \binom{m}{j} a^*\left(Z_j(A)\phi\right) \circ T_{m-j}(A), . \quad (2)$$

We also know that the following equation is fulfilled for all $n \in \mathbb{N}$, due to the unitarity of the scattering operator.

$$\sum_{k=0}^{n} \binom{n}{k} T_k^* T_{n-k} = 0 = \sum_{k=0}^{n} \binom{n}{k} T_k T_{n-k}^*. \quad (3)$$

In this documentwe guess the explicit form of the first few of these operators directly and normalize such that they have the correct vacuum expectation value. The correct vacuum expectation value is inferred by other arguments not presented here.

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The following construction will be occurring at every order, so we introduce it now. Denote by Q the following set $Q := \{f : \mathcal{H} \to \mathcal{H} \text{ linear} \mid i \cdot f \text{ is selfadjoint}\}$.

Definition 0.1. *Let G be the following function*

$$G : \quad Q \to (\mathcal{F} \to \mathcal{F}) \qquad\qquad \text{(Def G)}$$

$$f \mapsto \sum_{n \in \mathbb{N}} a^*(f\varphi_n) a(\varphi) - \sum_{n \in -\mathbb{N}} a(\varphi_n) a^*(f\varphi_n).$$

Lemma 0.2. *Every image of the map G has vanishing vacuum expectation value and fulflls the following commutation relation.*

$$\forall \alpha \in \mathcal{H}, \forall f \in Q : \left[G(f), a^\#(\alpha)\right] = a^\# \left(G(f)\alpha\right), \qquad (4)$$

where $a^\#$ is either a or a^*.

Proof: The vanishing expectation value of $G(f)$ can be read off directly from (Def G). The rest of the proof consists of short calculations, which uses the anti-commutation relations of the creation and annihilation operators. Let $f \in Q, \alpha \in \mathcal{H}$ be given. We divide the proof into two cases.

Case $a^\# = a$: In this case we get

$$\left[G(f), a^\#(\alpha)\right] = \left[\sum_{n \in \mathbb{N}} a^*(f\varphi_n) a(\varphi) - \sum_{n \in -\mathbb{N}} a(\varphi_n) a^*(f\varphi_n), a(\alpha)\right]$$

$$= \left[\sum_{n \in \mathbb{N}} a^*(f\varphi_n) a(\varphi), a(\alpha)\right] - \left[\sum_{n \in -\mathbb{N}} a(\varphi_n) a^*(f\varphi_n), a(\alpha)\right]$$

$$= -\sum_{n \in \mathbb{N}} \{a^*(f\varphi_n), a(\alpha)\} a(\varphi) - \sum_{n \in -\mathbb{N}} a(\varphi_n) \{a^*(f\varphi_n), a(\alpha)\}$$

$$= -\sum_{n \in \mathbb{Z}} \langle \alpha, f\varphi_n \rangle a(\varphi_n) \overset{f=-f^*}{=} \sum_{n \in \mathbb{Z}} \langle f\alpha, \varphi_n \rangle a(\varphi_n) = a\left(\sum_{n \in \mathbb{Z}} \varphi_n \langle \varphi_n, f\alpha \rangle\right)$$

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$$= a(f\alpha).$$

Case $a^{\#} = a^*$: In this case we get

$$\left[G(f), a^{\#}(\alpha)\right] = \left[\sum_{n\in\mathbb{N}} a^*(f\varphi_n)a(\varphi) - \sum_{n\in-\mathbb{N}} a(\varphi_n)a^*(f\varphi_n), a^*(\alpha)\right]$$

$$= \left[\sum_{n\in\mathbb{N}} a^*(f\varphi_n)a(\varphi), a^*(\alpha)\right] - \left[\sum_{n\in-\mathbb{N}} a(\varphi_n)a^*(f\varphi_n), a^*(\alpha)\right]$$

$$= \sum_{n\in\mathbb{N}} a^*(f\varphi_n)\{a(\varphi_n), a^*(\alpha)\} + \sum_{n\in-\mathbb{N}} \{a(\varphi_n), a^*(\alpha)\} a^*(f\varphi_n)$$

$$= \sum_{n\in\mathbb{Z}} \langle f\alpha, \varphi_n\rangle a^*(\varphi_n) = a^*\left(\sum_{n\in\mathbb{Z}} \varphi_n \langle\varphi_n, f\alpha\rangle\right)$$

$$= a^*(f\alpha),$$

which ends the proof.

1. T_1

The linearisation of the scattering operator T_1 is supposed to fulfil (1) and (2) for $n = 1$, which coincides with (4) for $f = Z_1$. This operator is antiselfadjoint, so it is in the domain of definition of G.

2. T_2

The commutation relations for the second order can be put into the form

$$\forall \alpha \in \mathcal{H} \left[T_2, a^{\#}(\alpha)\right] = a^{\#}(Z_2\alpha) + 2a^{\#}(Z_1\alpha) T_1, \tag{5}$$

where $a^{\#}$ is either a or a^*. We compute the corresponding commutation relation for $G(Z_2 - Z_1 Z_1) + T_1 T_1$, let $a^{\#}$ be either a or a^* and $\alpha \in \mathcal{H}$ be given. The anti selfadjointness of $Z_2 - Z_1 Z_1$ can be directly checked using (3). We get

$$\left[G(Z_2 - Z_1 Z_1) + T_1 T_1, a^{\#}(\alpha)\right] = \left[G(Z_2 - Z_1 Z_1), a^{\#}(\alpha)\right] + T_1\left[T_1, a^{\#}(\alpha)\right] + \left[T_1, a^{\#}(\alpha)\right] T_1$$

$$= a^{\#}((Z_2 - Z_1 Z_1)\alpha) + T_1 a^{\#}(Z_1\alpha) + a^{\#}(Z_1\alpha) T_1$$

$$= a^{\#}(Z_2\alpha) - a^{\#}(Z_1 Z_1\alpha) + \left[T_1, a^{\#}(Z_1\alpha)\right] + a^{\#}(Z_1\alpha) T_1 + a^{\#}(Z_1\alpha) T_1$$

$$= a^{\#}(Z_2\alpha) + 2a^{\#}(Z_1\alpha) T_1.$$

So we found the operator T_2 up to a multiple of the identity. This ambiguity is fixed by the condition

$$\langle\Omega, T_2\Omega\rangle = \langle T_2\rangle, \tag{6}$$

where $\langle T_2\rangle$ is fixed by other conditions. For our candidate we get the following vacuum expectation value

$$\langle\Omega, G(Z_2 - Z_1 Z_1) + T_1 T_1\Omega\rangle = \langle\Omega, T_1 T_1\Omega\rangle$$

$$= -\langle\Omega, \sum_{n\in\mathbb{N}}\sum_{k\in-\mathbb{N}} a^*\left(Z_{-+\atop 1}\varphi_n\right) a(\varphi_n)a(\varphi_k)a^*\left(Z_{+-\atop 1}\varphi_k\right)\Omega\rangle$$

$$= \langle\Omega, \sum_{k\in-\mathbb{N}} a^*\left(Z_{-+\atop 1} Z_{+-\atop 1}\varphi_k\right) = \text{tr}\left(Z_{-+\atop 1} Z_{+-\atop 1}\right).$$

This leads us to the correct representation of T_2

$$T_2 = G(Z_2 - Z_1 Z_1) + T_1 T_1 - \text{tr}\left(Z_{\underset{-+}{1}} Z_{\underset{+-}{1}}\right) + \langle T_2 \rangle. \tag{7}$$

3. T_3

The commutation relations for the third order can be put into the form

$$\forall \alpha \in \mathcal{H} \left[T_3, a^{\#}(\alpha)\right] = a^{\#}(Z_3 \alpha) + 3a^{\#}(Z_1 \alpha) T_2 + 3a^{\#}(Z_2 \alpha) T_1, \tag{8}$$

We will divide the calculation into parts, since it gets quite lengthy otherwise. Our initial guess for this operator is

$$T_3' = G\left(Z_3 - \frac{3}{2} Z_2 Z_1 - \frac{3}{2} Z_1 Z_2 + 2 Z_1 Z_1 Z_1\right) + \frac{3}{2} T_2 T_1 + \frac{3}{2} T_1 T_2 - 2 T_1 T_1 T_1. \tag{9}$$

We begin by checking that the argument of G is anti selfadjoint.

$$\left(Z_3 - \frac{3}{2} Z_2 Z_1 - \frac{3}{2} Z_1 Z_2 + 2 Z_1 Z_1 Z_1\right)^* = Z_3^* - \frac{3}{2} Z_2^* Z_1^* - \frac{3}{2} Z_1^* Z_2^* + 2 Z_1^* Z_1^* Z_1^*$$

$$\overset{(3)}{=} (-Z_3 - 3 Z_2^* Z_1 - 3 Z_1^* Z_2) + \frac{3}{2}(-Z_2 + 2 Z_1 Z_1) Z_1 + \frac{3}{2} Z_1 (-Z_2 + 2 Z_1 Z_1) - 2 Z_1 Z_1 Z_1$$

$$\overset{(3)}{=} -Z_3 - 3(-Z_2 + 2 Z_1 Z_1) Z_1 + 3 Z_1 Z_2 - \frac{3}{2} Z_2 Z_1 - \frac{3}{2} Z_1 Z_2 + 4 Z_1 Z_1 Z_1$$

$$= -Z_3 + \frac{3}{2} Z_2 Z_1 + \frac{3}{2} Z_1 Z_2 - 2 Z_1 Z_1 Z_1 = -\left(Z_3 - \frac{3}{2} Z_2 Z_1 - \frac{3}{2} Z_1 Z_2 + 2 Z_1 Z_1 Z_1\right)$$

holds. Since we already know the commutation relation for G the next step is computing the commutation relations for the terms containing T_2. Let $\alpha \in \mathcal{H}$ be given and $a^{\#}$ be a or a^*, the commutation relation is

$$\left[\frac{3}{2} T_2 T_1 + \frac{3}{2} T_1 T_2, a^{\#}(\alpha)\right] = \left[\frac{3}{2} T_2 T_1, a^{\#}(\alpha)\right] + \left[\frac{3}{2} T_1 T_2, a^{\#}(\alpha)\right]$$

$$= \frac{3}{2} T_2 \left[T_1, a^{\#}(\alpha)\right] + \frac{3}{2} \left[T_2, a^{\#}(\alpha)\right] T_1 + \frac{3}{2} T_1 \left[T_2, a^{\#}(\alpha)\right] + \frac{3}{2} \left[T_1, a^{\#}(\alpha)\right] T_2$$

$$= \frac{3}{2} T_2 a^{\#}(Z_1 \alpha) + \frac{3}{2} \left(a^{\#}(Z_2 \alpha) + 2 a^{\#}(Z_1 \alpha) T_1\right) T_1 +$$

$$\frac{3}{2} T_1 \left(a^{\#}(Z_2 \alpha) + 2 a^{\#}(Z_1 \alpha) T_1\right) + \frac{3}{2} a^{\#}(Z_1 \alpha) T_2$$

$$= \frac{3}{2} a^{\#}(Z_1 \alpha) T_2 + \frac{3}{2} a^{\#}(Z_2 Z_1 \alpha) + 3 a^{\#}(Z_1 Z_1 \alpha) T_1 + \frac{3}{2} a^{\#}(Z_2 \alpha) T_1 + 3 a^{\#}(Z_1 \alpha) T_1 T_1 +$$

$$\frac{3}{2} a^{\#}(Z_2 \alpha) T_1 + \frac{3}{2} a^{\#}(Z_1 Z_2 \alpha) + 3 a^{\#}(Z_1 \alpha) T_1 T_1 + 3 a^{\#}(Z_1 Z_1 \alpha) T_1 + \frac{3}{2} a^{\#}(Z_1 \alpha) T_2$$

$$= 3 a^{\#}(Z_1 \alpha) T_2 + 3 a^{\#}(Z_2 \alpha) T_1 + \frac{3}{2} a^{\#}(Z_1 Z_2 \alpha) + \frac{3}{2} a^{\#}(Z_2 Z_1 \alpha) \tag{$T_2 T_1$}$$

$$+ 6 a^{\#}(Z_1 Z_1 \alpha) T_1 + 6 a^{\#}(Z_1 \alpha) T_1 T_1.$$

The terms in the last line do not correspond to G, nor to the original commutator of T_3, so we hope they cancel with terms of the $T_1 T_1 T_1$ summand of T_3'. Let $\alpha \in \mathcal{H}$

be given and $a^\#$ be a or a^*, indeed

$$-2\left[T_1T_1T_1, a^\#(\alpha)\right] = -2T_1T_1\left[T_1, a^\#(\alpha)\right] - 2T_1\left[T_1, a^\#(\alpha)\right]T_1 - 2\left[T_1, a^\#(\alpha)\right]T_1T_1$$

$$= -2T_1T_1a^\#(Z_1\alpha) - 2T_1a^\#(Z_1\alpha)T_1 - 2a^\#(T_1\alpha)T_1T_1$$

$$= -2T_1a^\#(Z_1\alpha)T_1 - 2T_1a^\#(Z_1Z_1\alpha) - 2a^\#(Z_1\alpha)T_1T_1 - 2a^\#(Z_1Z_1\alpha)T_1 - 2a^\#(T_1\alpha)T_1T_1$$

$$= -2a^\#(Z_1\alpha)T_1T_1 - 2a^\#(Z_1Z_1\alpha)T_1 - 2a^\#(Z_1Z_1\alpha)T_1 - 2a^\#(Z_1Z_1Z_1\alpha)$$

$$- 2a^\#(Z_1\alpha)T_1T_1 - 2a^\#(Z_1Z_1\alpha)T_1 - 2a^\#(T_1\alpha)T_1T_1$$

$$= -6a^\#(Z_1\alpha)T_1T_1 - 6a^\#(Z_1Z_1\alpha)T_1 - 2a^\#(Z_1Z_1Z_1\alpha). \tag{$T_1T_1T_1$}$$

holds. Adding up equations (T_2T_1), $(T_1T_1T_1)$ and the commutator of $G\left(Z_3 - \frac{3}{2}Z_2Z_1 - \frac{3}{2}Z_1Z_2 + 2Z_1Z_1Z_1\right)$ results exactly in the right hand side of (8). What about the vacuum expectation value? The abstract argument should also hold for T_1^3, but $\langle\Omega, T_1^3\Omega\rangle \neq 0$ holds! To obtain T_3 we need to adjust the vacuum expectation value. In order to do so, we first compute the vacuum expectation value of T_3':

$$\langle\Omega, T_3'\Omega\rangle = \frac{3}{2}\langle\Omega, T_2T_1\Omega\rangle + \frac{3}{2}\langle\Omega T_1T_2\Omega\rangle - 2\langle\Omega.T_1T_1T_1\Omega\rangle$$

$$= \frac{3}{2}\langle\Omega, G\left(Z_2 - Z_1Z_1\right)T_1\Omega\rangle + \frac{3}{2}\langle\Omega, T_1G\left(Z_2 - Z_1Z_1\right)\Omega\rangle + \langle\Omega, T_1T_1T_1\Omega\rangle$$

$$= -\frac{3}{2}\sum_{k\in\mathbb{N}}\sum_{l\in-\mathbb{N}}\langle\Omega, a^*\left(Z_2 - Z_1Z_1\varphi_k\right)a(\varphi_k)a(\varphi_l)a^*\left(Z_1\varphi_l\right)\Omega\rangle$$

$$- \frac{3}{2}\sum_{k\in\mathbb{N}}\sum_{l\in-\mathbb{N}}\langle\Omega, a^*\left(Z_1\varphi_k\right)a(\varphi_k)a(\varphi_l)a^*\left(Z_2 - Z_1Z_1\varphi_l\right)\Omega\rangle$$

$$- \sum_{k\in\mathbb{N}}\sum_{l\in-\mathbb{N}}\sum_{c\in\mathbb{N}}\langle\Omega, a^*\left(Z_1\varphi_k\right)a(\varphi_k)\left(a^*\left(Z_1\varphi_c\right)a(\varphi_c) - a(\varphi_{-c})a^*\left(Z_1\varphi_{-c}\right)\right)a(\varphi_l)a^*\left(Z_1\varphi_l\right)\Omega\rangle$$

$$= -\frac{3}{2}\operatorname{tr}\left(Z_1Z_1Z_{\underset{+-}{1}}\right) - \frac{3}{2}\operatorname{tr}\left(Z_{\underset{-+}{1}}Z_1Z_1\right) + \operatorname{tr}\left(Z_{\underset{-+}{2}}Z_{\underset{+-}{1}}\right) + \operatorname{tr}\left(Z_{\underset{-+}{1}}Z_{\underset{+-}{2}}\right)$$

$$- \sum_{k\in\mathbb{N}}\sum_{l\in-\mathbb{N}}\langle\Omega, a(\varphi_k)a^*\left(Z_1Z_{\underset{-+}{1}}\varphi_k\right)a(\varphi_l)a^*\left(Z_1\varphi_l\right)\Omega\rangle$$

$$+ \sum_{k\in\mathbb{N}}\sum_{l\in-\mathbb{N}}\langle\Omega, a^*\left(Z_1\varphi_k\right)a(\varphi_k)a^*\left(Z_1Z_{\underset{+-}{1}}\varphi_l\right)a(\varphi_l)\Omega\rangle$$

$$= -\frac{3}{2}\operatorname{tr}\left(Z_1Z_1Z_{\underset{+-}{1}}\right) - \frac{3}{2}\operatorname{tr}\left(Z_{\underset{-+}{1}}Z_1Z_1\right) + \operatorname{tr}\left(Z_{\underset{-+}{2}}Z_{\underset{+-}{1}}\right) + \operatorname{tr}\left(Z_{\underset{-+}{1}}Z_{\underset{+-}{2}}\right)$$

$$- \operatorname{tr}\left(Z_{\underset{+-}{1}}Z_{\underset{--}{1}}Z_{\underset{-+}{1}}\right) + \operatorname{tr}\left(Z_{\underset{-+}{1}}Z_{\underset{++}{1}}Z_{\underset{+-}{1}}\right)$$

$$= -4\operatorname{tr}\left(Z_{\underset{+-}{1}}Z_{\underset{--}{1}}Z_{\underset{-+}{1}}\right) - 2\operatorname{tr}\left(Z_{\underset{-+}{1}}Z_{\underset{++}{1}}Z_{\underset{+-}{1}}\right) + \operatorname{tr}\left(Z_{\underset{-+}{2}}Z_{\underset{+-}{1}}\right) + \operatorname{tr}\left(Z_{\underset{-+}{1}}Z_{\underset{+-}{2}}\right)$$

4. T_4

Let $b\in\mathbb{R}$ be arbitrary we find that there is a $C\in\mathbb{C}$ such that T_4 is given by

$$\begin{aligned}T_4 :=\ & 2T_1T_3 + 2T_3T_1 + 3T_2T_2 - bT_1T_1T_2 - bT_2T_1T_1 - 2(6-b)T_1T_2T_1 \\ & + 6T_1T_1T_1T_1 + G\left(Z_4 - 2Z_1Z_3 - 2Z_3Z_1 - 3Z_2Z_2\right. \\ & \left. + bZ_1^2Z_2 + 2(6-b)Z_1Z_2Z_1 + bZ_2Z_1^2 - 6Z_1^4\right) + C.\end{aligned} \tag{10}$$

This can be seen as follows. We know the commutator of T_4 to be

$$\left[T_4, a^\#(\alpha)\right] = a^\#(Z_4\alpha) + 4a^\#(Z_1\alpha) T_3 + 6a^\#(Z_2\alpha) T_2 + 4a^\#(Z_3\alpha) T_1. \quad (11)$$

We would like to subtract from T_4 linear combinations of T_3T_1, T_1T_3, T_2T_2, $T_2T_1^2$, $T_1T_2T_1$, $T_1^2T_2$, T_1^4 such that the remainder on the right hand only involves a sum of creation/annihilation operators. We group the first three summands together with binomial coefficients, since those are needed to produce an anti selfadjoint one-particle operator inside the creation/annihilation operator on the right hand side of the commutation relation. So we compute the commutation relation of these summands. These can be found to be

$$\left[2T_1T_3 + 2T_3T_1 + 3T_2T_2, a^\#(\alpha)\right] = 6a^\#(Z_2\alpha) T_2 + 4a^\#(Z_3\alpha) T_1 + 4a^\#(Z_1\alpha) T_3$$
$$+ 2a^\#(Z_1Z_3\alpha) + 2a^\#(Z_3Z_1\alpha) + 3a^\#(Z_2Z_2\alpha)$$
$$+ 12a^\#(Z_1\alpha) T_2T_1 + 12a^\#(Z_1\alpha) T_1T_2 + 12a^\#(Z_1Z_1\alpha) T_2 + 12a^\#(Z_2\alpha) T_1T_1$$
$$+ 12a^\#(Z_1Z_2\alpha) T_1 + 12a^\#(Z_2Z_1\alpha) T_1 + 12a^\#(Z_1Z_1\alpha) T_1T_1, \quad (12)$$

$$\left[T_1^2T_2, a^\#(\alpha)\right] = a^\#(Z_1^2Z_2\alpha) + a^\#(Z_1^2\alpha) T_2 + 2a^\#(Z_1Z_2\alpha) T_1 + 2a^\#(Z_1^3\alpha) T_1$$
$$+ 4a^\#(Z_1^2\alpha) T_1^2 + 2a^\#(Z_1\alpha) T_1T_2 + a^\#(Z_2\alpha) T_1^2 + 2a^\#(Z_1\alpha) T_1^3, \quad (13)$$

$$\left[T_1T_2T_1, a^\#(\alpha)\right] = a^\#(Z_1Z_2Z_1\alpha)$$
$$+ a^\#(Z_1^2\alpha) T_2 + a^\#(Z_1Z_2\alpha) T_1 + a^\#(Z_2Z_1\alpha) T_1 + 2a^\#(Z_1^3\alpha) T_1 \quad (14)$$
$$+ a^\#(Z_1\alpha) T_1T_2 + a^\#(Z_1\alpha) T_2T_1 + a^\#(Z_2\alpha) T_1^2 + 4a^\#(Z_1^2\alpha) T_1^2 + 2a^\#(Z_1\alpha) T_1^3,$$

$$\left[T_2T_1^2, a^\#(\alpha)\right] = a^\#(Z_2Z_1^2\alpha) + a^\#(Z_1^2\alpha) T_2 + 2a^\#(Z_2Z_1\alpha) T_1 + 2a^\#(Z_1^3\alpha) T_1$$
$$+ 2a^\#(Z_1\alpha) T_2T_1 + a^\#(Z_2\alpha) T_1^2 + 4a^\#(Z_1^2\alpha) T_1^2 + 2a^\#(Z_1\alpha) T_1^3, \quad (15)$$

$$\left[T_1^4, a^\#(\alpha)\right] = a^\#(Z_1^4\alpha) + 4a^\#(Z_1^3\alpha) T_1 + 6a^\#(Z_1^2\alpha) T_1^2 + 4a^\#(Z_1\alpha) T_1^3. \quad (16)$$

Demanding the equation

$$\left[T_4 + 2T_1T_3 + 2T_3T_1 + 3T_2T_2 + c_1T_1^2T_2 \right.$$
$$\left. + c_2T_1T_2T_1 + c_3T_2T_1^2 + c_4T_1^4, a^\#(\alpha)\right] \overset{!}{=} a^\#(F\alpha), \quad (17)$$

where $c_1, c_2, c_3, c_4 \in \mathbb{R}$ and F is an anti selfadjoint one-particle operator, we arrive at (10).