

1. The Relationship between T_n^\dagger and T_n

We know for any four-vectorfield $A \in C_c^\infty(\mathbb{R}^4, \mathbb{R}^4)$ by unitarity of the scattering matrix that

$$\tilde{S}^{A*} \tilde{S}^A = 1 \quad (1)$$

holds. By expanding the S matrix and its adjoint in a power series in A one obtains

$$1 = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{1}{k!l!} T_l^\dagger(A) T_k(A). \quad (2)$$

By resummation follows

$$1 = \sum_{s=0}^{\infty} \frac{1}{s!} \sum_{l=0}^s \binom{s}{l} T_l^\dagger(A) T_{s-l}(A). \quad (3)$$

By scaling it follows from homogeneity of degree n of T_n and T_n^\dagger that

$$\forall s \in \mathbb{N} : 0 = \sum_{l=0}^s \binom{s}{l} T_l^\dagger(A) T_{s-l}(A) \quad (4)$$

holds. Let $n \in \mathbb{N}$. This means we can solve for T_n^\dagger yielding

$$T_n^\dagger(A) = - \sum_{l=0}^{n-1} \binom{n}{l} T_l^\dagger(A) T_{n-l}(A). \quad (5)$$

To get an explicit expression for T_n for different arguments we use the symmetry and multilinearity of T_n . Let A_1, \dots, A_n be four-vectorfields. Thereby it follows that

$$\begin{aligned} T_n^\dagger(A_1, \dots, A_n) &= \frac{1}{n!} \left(\prod_{l=1}^n \partial_{\varepsilon_l} \right) T_n^\dagger \left(\sum_{k=1}^n \varepsilon_k A_k \right) \Big|_{\varepsilon_1=\dots=\varepsilon_n=0} \\ &= -\frac{1}{n!} \left(\prod_{c=1}^n \partial_{\varepsilon_c} \right) \sum_{l=0}^{n-1} \binom{n}{l} T_l^\dagger \left(\sum_{k=1}^n \varepsilon_k A_k \right) T_{n-l} \left(\sum_{k=1}^n \varepsilon_k A_k \right) \Big|_{\varepsilon_1=\dots=\varepsilon_n=0} \end{aligned}$$

Due to linearity of T_k in each of its k arguments it can at most be linearised k times with respect to different four-potentials without vanishing. This means however, that in each summand each factor needs to receive exactly as many derivatives as it has arguments. This simplifies the expression considerably. We directly plug in the relationship between T_n for one and n arguments finding

$$\begin{aligned} T_n^\dagger(A_1, \dots, A_n) &= -\frac{1}{n!} \sum_{B \subseteq \{A_1, \dots, A_n\}} |B|! |B^c|! \binom{n}{|B|} T_{|B|}^\dagger(B) T_{|B^c|}(B^c) \\ &= - \sum_{B \subseteq \{A_1, \dots, A_n\}} T_{|B|}^\dagger(B) T_{|B^c|}(B^c). \end{aligned}$$

This equation is solved by

$$T_n^\dagger(A_1, \dots, A_n) = \sum_{m \in \mathcal{T}(\{A_1, \dots, A_n\})} (-1)^{|m|} \prod_{l=1}^{|m|} T_{|m_l|}(m_l), \quad (6)$$

where \mathcal{T} is given by

$$\mathcal{T}(\{A_1, \dots, A_n\}) := \bigcup_{1 \leq k \leq n} \left\{ (m_1, \dots, m_k) \mid \right. \\ \left. \forall l \in \{1, \dots, n\} : m_l \subseteq \{A_1, \dots, A_n\}, \bigcup_{1 \leq l \leq k} m_l = \{A_1, \dots, A_n\} \right\}$$

which best checked by directly plugging it in.