

# The Relationship Between Hadamard States, Admissible Polarisation Classes and the Fermionic Projector

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## Abstract

to be written

## 1 Introduction

This paper compares central objects of three different approaches to quantum field theory (QFT). These approaches have different scopes and pursue different ideas, so a comparison of the theories as a whole is difficult. Since they all study different aspects of QFT; however, one can make out objects in each of these theories which closely resemble one another. The first class of objects of interest are Hadamard states which appear in the algebraic approach to QFT[5]. The second class of objects are the (almost) projectors  $P_\Sigma^\lambda$  which are closely linked to polarisation classes of the vacuum of external field quantum electrodynamics (QED). Polarisation classes play a central role in the analysis of external field QED[2, 4, 3]. The third class of objects are the continuum limit of the Fermionic Projectors[?]. In the following sections we will very briefly summarise the role of each of the targeted objects in their respective approach to QFT. In these sections we want to focus on the similar physical interpretation of these objects. A reader familiar with some of these theories may skip the respective sections. Afterwards we shift our attention to the similarities between the mathematical structure inherent in these objects. Throughout the paper  $\Sigma, \Sigma', \Sigma''$  denote arbitrary Cauchy surfaces, while  $A \in C_c^\infty(\mathbb{R}^4, \mathbb{R}^4)$  is a four potential and  $\Sigma_{\text{in}}$  denotes a Cauchy surface earlier than  $\text{supp}(A)$ .

$$(i\partial - m - A)\psi = 0 \tag{1}$$

$$U_{\Sigma', \Sigma} : \mathcal{H}_\Sigma \rightarrow \mathcal{H}_{\Sigma'}, \tag{2}$$

## 2 Hadamard States

In the algebraic approach to QFT one puts less emphasise on the Hilbert space as is commonly done in non relativistic physics, because once one talks about quantum fields in a relativistic setting the specific representation of operators by operators on a Hilbert

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space is no longer canonical. In stead one tries to focus on the operators that represent the physical measurements one wants to perform later on. One can construct an algebra which contains these operators and do most of the analysis on this algebra itself; however, many construction common to that analysis already make use of special properties of special states. The so called Hadamard states (We will introduce them shortly) often play a role in constructing operators that represent e.g. objects like the energy momentum tensor of a quantum field. Additionally at some point in the physical analysis one would like to produce some numbers which can be compared to experiments. In order to do so, one has to pick a representation. The GNS construction is very helpful in this regard, since it reduces the construction of a representation to picking one state to build a Hilbert space around. This choice has to be made on physical grounds. The Hadamard states are usually interpreted as states of positive energy. This interpretation has to be taken with a grain of salt. Even in Minkowski spacetime with a non vanishing external electromagnetic field but no interaction the splitting of the spectrum into positive and negative part becomes somewhat arbitrary, since no particular splitting is Lorentz invariant[?]. Instead one could interpret picking a Hadamard state as making a choice of which states one calls positive energy states. This may become clearer in section 3. Hadamard states are singled out by their dynamical properties. The definition of Hadamard states is a technical one, so we will omit the general case and focus on the condition for Fermions in flat Minkowski spacetime subject to an external potential. This scenario was already considered by Dirac [?]. The condition in this case reads as follows:

**Definition 1.** *A state  $\omega$  is called Hadamard if its two point function  $w \in (C_c^\infty(\mathcal{M}) \times C_c^\infty(\mathcal{M}))'$  acts for  $f_1, f_2 \in C_c^\infty(\mathcal{M})$  as*

$$w(f_1, f_2) = \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^4} d^4x \bar{f}_1(x) \int_{\mathbb{R}^4} d^4y b_\varepsilon(x, y) f_2(y) \quad (3)$$

with  $b_\varepsilon$  being of the form

$$b_\varepsilon(x, y) = -i(i\cancel{\partial}_x - \cancel{A}(x) - m) \left[ \frac{e^{-i(x-y)^\alpha} \int_0^1 ds A_\alpha(xs + (1-s)y)}{(x - y - i\varepsilon e_0)^2} + V(x, y) \ln((x - y - i\varepsilon e_0)^2) \right] + W(x, y). \quad (4)$$

In this equation  $V, W : \mathbb{R}^4 \rightarrow \mathbb{C}^4$  are smooth functions,  $W$  is completely arbitrary and varies for different Hadamard states, whereas  $V$  is fixed by the external potential. For  $(x - y)^2$  small enough the series

$$V(x, y) = \sum_{k=0}^{\infty} \frac{1}{2^{k-1} k!} V_k(x, y) (x - y)^{2k} \quad (5)$$

converges.[?] The functions  $V_k$  fulfil a recursive set of partial differential equations

$$\begin{aligned} & ((\partial_x^\alpha - iA^\alpha(x))(\partial_{x,\alpha} - iA_\alpha(x)) + m^2) V_{n-1}(x, y) \\ & + (x - y)^\alpha (\partial_{x,\alpha} - iA_\alpha(x)) V_n(x, y) + (2n + 2) V_n(x, y) = 0, \end{aligned} \quad (6)$$

where  $V_{-1}(x, y) = e^{-i(x-y)^\alpha} \int_0^1 ds A_\alpha(xs + (1-s)y)$ .

### 3 Projectors for Polarisation Classes

The concept of polarisation classes arises naturally in the study of QED in external electromagnetic fields. It does need some machinery to be introduced and related to more familiar objects which we are going to introduce first.

We start out by characterising the polarisation classes for free motion.

**Definition 2.** *The projector  $P_\Sigma^{\mathcal{H}^-}$  has the well known representation as the weak limit of the integral operator with the kernel[3]*

$$p_\varepsilon^-(x, y) = -\frac{m^2}{4\pi^2}(i\cancel{\partial}_x + m) \frac{K_1(m\sqrt{-(y-x-i\varepsilon e_0)^2})}{m\sqrt{-(y-x-i\varepsilon e_0)^2}}, \quad (7)$$

where the square is a Minkowski square and the square root denotes its principle value. By weak limit we mean

$$\langle \phi, P_\Sigma^{\mathcal{H}^-} \psi \rangle = \lim_{\varepsilon \searrow 0} \int_{\Sigma \times \Sigma} \bar{\phi}(x) i_\gamma(d^4 x) p_\varepsilon^-(x, y) i_\gamma(d^4 y) \psi(y), \quad (8)$$

for general  $\phi, \psi \in \mathcal{H}_\Sigma$ .

**Remark 1.** *By inserting the expansion of  $K_1$  in terms of a Laurent series and a logarithm, [1] one obtains:*

$$K_1(\xi) = \frac{1}{\xi} - \frac{\xi}{4} \sum_{k=0}^{\infty} \left( 2\psi(k+1) + \frac{1}{k+1} - 2 \ln \frac{\xi}{2} \right) \frac{\left(\frac{\xi^2}{4}\right)^k}{k!^2(k+1)}. \quad (9)$$

It is not obvious from the equation (6) but in fact well known that the vacuum of Minkowski spacetime does indeed correspond to a Hadamard state subject to vanishing four potential. In fact, the Hadamard states were constructed to agree with the Minkowski vacuum up to smooth terms.

**Definition 3.** *Let  $\text{Pol}(\mathcal{H}_\Sigma)$  denote the set of all closed, linear subspaces  $V \subset \mathcal{H}$  such that both  $V$  and  $V^\perp$  are infinite dimensional. Any  $V \in \text{Pol}(\mathcal{H}_\Sigma)$  is called polarisation of  $\mathcal{H}_\Sigma$ . For  $V \in \text{Pol}(\mathcal{H}_\Sigma)$ , let  $P_\Sigma^V \rightarrow V$  denote the orthogonal projection of  $\mathcal{H}_\Sigma$  onto  $V$ . The Fock space corresponding to  $V$  on the Cauchy surface  $\Sigma$  is defined to be*

$$\mathcal{F}(V, \mathcal{H}_\Sigma) := \bigoplus_{c \in \mathbb{Z}} \mathcal{F}_c(V, \mathcal{H}_\Sigma), \quad \mathcal{F}_c(V, \mathcal{H}_\Sigma) := \bigoplus_{\substack{n, m \in \mathbb{N}_0 \\ c = m - n}} (V^\perp)^{\wedge n} \otimes \bar{V}^{\wedge m}, \quad (10)$$

where  $\bigoplus$  is the Hilbert space direct sum,  $\wedge$  the antisymmetric tensor product of Hilbert spaces and  $\bar{V}$  is the conjugate complex vector space of  $V$ , which coincides with  $V$  as a set, has the same vector space operations as  $V$  except for scalar multiplication, which is defined by  $(z, \psi) \mapsto z^* \psi$  for  $z \in \mathbb{C}, \psi \in V$ .

**Remark 2.** *Given two polarisations  $V, W \in \text{Pol}(\mathcal{H}_\Sigma)$ , the two Fockspaces  $\mathcal{F}(V, \mathcal{H}_\Sigma)$  and  $\mathcal{F}(W, \mathcal{H}_\Sigma)$  are canonically unitary equivalent if **does also  $\iff$  hold??**  $P_\Sigma^V - P_\Sigma^W \in I_2(\mathcal{H}_\Sigma)$  by the theorem of Shale and Stinespring [?].*

Remark 2 gives us a natural limit with respect to which it is useful to analyse the regularity of Projectors  $P_\Sigma^A$ . We will therefore be content with the following equivalence class.

**Definition 4.** For  $V, W \in \text{Pol}(\mathcal{H}_\Sigma)$  we write

$$V \approx W \iff P_\Sigma^V - P_\Sigma^W \in I_2(\mathcal{H}_\Sigma), \quad (11)$$

$$C_\Sigma(A) := [U_{\Sigma\Sigma_{in}}^A \mathcal{H}_{\Sigma_{in}}^-]_{\approx}. \quad (12)$$

The equivalence class  $C_\Sigma(A)$  transforms naturally with respect to gauge and Lorentz transforms[3]. Now for hyperplanes  $\Sigma \cap \text{supp}(A) \neq \emptyset$  the operator  $P_\Sigma^{\mathcal{H}^-}$  does not represent  $C_\Sigma(A)$ . Because all we are interested in is representations of equivalence classes, we are content with finding objects that differ from a Projector onto  $U_{\Sigma, \Sigma_{in}}^A \mathcal{H}_{\Sigma_{in}}^-$  by a Hilbert-Schmidt operator. Therefore we need not keep track of the exact evolution of the projection operators, but define a whole class of admissible ones.

**Definition 5.** For any function  $\lambda \in C^\infty(\mathbb{R}^4 \times \mathbb{R}^4, \mathbb{R})$  fulfilling

- i) There is a compact set  $K \subset \mathbb{R}^4$  such that  $\text{supp } \lambda \subseteq K \times \mathbb{R}^4 \cup \mathbb{R}^4 \times K$ .
- ii)  $\lambda$  satisfies  $\forall x \in \mathbb{R}^4 \lambda(x, x) = 0$ .
- iii) On the diagonal the first derivatives fulfil

$$\forall x, y \in \mathbb{R}^4 : \partial_x \lambda(x, y) = -\partial_y \lambda(x, y) = A(x), \quad (13)$$

we define a corresponding (quasi) projector  $P^\lambda$ :

$$\langle \phi, P_\Sigma^\lambda \psi \rangle = \lim_{\epsilon \searrow 0} \langle \phi, P_\Sigma^{A, \epsilon} \psi \rangle, \quad (14)$$

$$\langle \phi, P_\Sigma^{A, \epsilon} \psi \rangle := \int_{\Sigma \times \Sigma} \bar{\phi}(x) i_\gamma(d^4 x) e^{-i\lambda(x, y)} p_\epsilon^-(y - x) i_\gamma(d^4 y) \psi(y), \quad (15)$$

for general  $\phi, \psi \in \mathcal{H}_\Sigma$ .

**Remark 3.**  $P_\Sigma^\lambda$  and  $P_{\Sigma'}^\lambda$  are equivalent if transported appropriately by time evolution operators[3, theorem 2.8]:

$$P_\Sigma^\lambda - U_{\Sigma, \Sigma'}^A P_{\Sigma'}^\lambda U_{\Sigma', \Sigma} \in I_2(\mathcal{H}_\Sigma). \quad (16)$$

Also for four-potentials  $A, B \in C_c^\infty(\mathbb{R}^4)$  the corresponding projectors are typically inequivalent[3, theorem 1.5]

$$P_\Sigma^{A^*} - P_\Sigma^{A^B} \in I_2(\mathcal{H}_\Sigma) \iff \forall x \in \Sigma \forall z \in T_x \Sigma : z^\alpha (A_\alpha(x) - B_\alpha(x)) = 0. \quad (17)$$

Now the notion of Hadamard state and projectors of polarisation classes are extremely close.

**Theorem 1.** Given a global Hadamard state  $H$  there is a smooth function  $w \in C^\infty(\mathbb{R}^4 \times \mathbb{R}^4)$  such that for any hypersurface  $\Sigma$ ,  $H - w$  can be interpreted as linear bounded map:  $\mathcal{H}_\Sigma \rightarrow \mathcal{H}_\Sigma$ . Furthermore for each  $\lambda$  fulfilling i) – iii),  $H - w$  and  $P^\lambda$  yield equivalent descriptions:

$$H - w - P^\lambda \in I_2(\mathcal{H}_\Sigma). \quad (18)$$

## 4 The Fermionic Projector

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