1. Does the image of a general element not specify the operator uniquely?

Let $\text{モ} \in \mathcal{H}$ and $f \in \mathcal{C}_c^\infty\left(\mathbb{R}^4, \mathbb{C}^4\right)$. Let $H : \mathcal{F} \to \mathcal{F}$ be a linear Fock space valued operator on Fock space. Assume it has zero vacuum expectation value and fulfils the commutation relations

$$[H, a(\text{モ})] = a\left(N\, \text{モ}\right), \tag{1}$$

$$[H, a^*(\text{モ})] = a^*\left(N\, \text{モ}\right). \tag{2}$$

Let $(\text{マ}_l)_{l\in\mathbb{N}} =: B \subset \mathcal{H}$, $(\varphi_l)_{l\in\mathbb{N}} =: B_+ \subset \mathcal{H}^+$ and $(y_k)_{k\in\mathbb{N}} =: B_- \subset \mathcal{H}^-$ be Schauder basis of all of, respectively positive respectively negative part of the Hilbert space. We introduce for a more compact notation

$$\begin{aligned}
\forall n, l \in \mathbb{N} : \Delta_{n,l} &: \left(\mathcal{F}\left(\mathcal{H}\right) \to \mathcal{F}\left(\mathcal{H}\right)\right) \to \left(\mathcal{F}\left(\mathcal{H}\right) \to \mathcal{F}\left(\mathcal{H}\right)\right) \\
F &\mapsto \begin{cases} F & \text{if } n = l \\ \mathbb{1} & \text{otherwise}, \end{cases} \\
\forall n, l \in \mathbb{N} : \Delta_{n,l} &: \left(\mathcal{H} \to \mathcal{H}\right) \to \left(\mathcal{H} \to \mathcal{H}\right) \\
F &\mapsto \begin{cases} F & \text{if } n = l \\ \mathbb{1} & \text{otherwise}. \end{cases}
\end{aligned}$$

Let $m, p \in \mathbb{N}$. We would like to compute $H\alpha$ for any element α of the canonical Schauder basis of the fixed m-p particle Sector of Fockspace. In order to do so we first compute the image of the vacuum. It holds that

$$\begin{aligned}
&\langle \prod_{l=1}^{m} a^*(\varphi_l) \prod_{k=1}^{p} a(y_k)\Omega, H\Omega\rangle = (-1)^{m+p} \langle \Omega, \prod_{k=1}^{p} a^*(y_k) \prod_{l=1}^{m} a(\varphi_l) H\Omega\rangle \\
&= (-1)^{m+p+1} \delta_{m,1} \langle \Omega, \prod_{k=1}^{p} a^*(y_k) \prod_{l=1}^{m-1} a(\varphi_l) a\left(N\varphi_M\right)\Omega\rangle \\
&= (-1)^p \delta_{m,1}\delta_{p,1} \langle \Omega, \prod_{k=1}^{p} a^*(y_k) a\left(N_{-+}\varphi_1\right)\Omega\rangle = -\delta_{m,1}\delta_{p,1}\left\langle y_1, N_{-+}\varphi_1\right\rangle.
\end{aligned}$$

So we conclude that

$$H\Omega\rangle = -\sum_{y\in B^-}\sum_{\varphi\in B^+}\left\langle y, N_{-+}\varphi\right\rangle a^*(\varphi)a(y)\Omega\rangle \tag{3}$$

holds. We can now compute the image of a general Element of the canonical basis of the m-p particle sector. It is

$$\begin{aligned}
&H \prod_{l=1}^{m} a^*(\varphi_l) \prod_{k=1}^{p} a(y_k)\Omega\rangle \\
&= \sum_{b=1}^{m+p} \prod_{l=1}^{m} a^*\left(\Delta_{l,b}\left(N\right)\varphi_l\right) \prod_{k=1}^{p} a\left(\Delta_{m+k,b}\left(N\right) y_k\right)\Omega\rangle + \prod_{l=1}^{m} a^*(\varphi_l) \prod_{k=1}^{p} a(y_k) H\Omega\rangle \\
&= \sum_{b=1}^{m+p} \prod_{l=1}^{m} a^*\left(\Delta_{l,b}\left(N\right)\varphi_l\right) \prod_{k=1}^{p} a\left(\Delta_{m+k,b}\left(N_{--}\right) y_k\right)\Omega\rangle
\end{aligned}$$

$$
-\prod_{l=1}^{m} a^*(\varphi_l) \prod_{k=1}^{p} a(y_k) \sum_{y\in B^-} \sum_{\varphi\in B^+} \langle y, N_{-+}\varphi\rangle\, a^*(\varphi)a(y)\Omega\rangle
$$

$$
= \sum_{b=1}^{m+p} \prod_{l=1}^{m} a^*\left(\Delta_{l,b}\left(N_{++}\right)\varphi_l\right) \prod_{k=1}^{p} a\left(\Delta_{m+k,b}\left(N_{--}\right)y_k\right)\Omega\rangle \tag{H1}
$$

$$
- \sum_{\varphi\in B^+} a^*(\varphi)a\left(N_{-+}\varphi\right) \prod_{l=1}^{m} a^*(\varphi_l) \prod_{k=1}^{p} a(y_k)\Omega\rangle \tag{H2}
$$

$$
+ \sum_{b=1}^{m} \sum_{c=1}^{p} (-1)^{m-b+c} \left\langle y_c, N_{\substack{2\\-+}}\varphi_b\right\rangle \prod_{\substack{l=1\\ l\neq b}}^{m} a^*(\varphi_l) \prod_{\substack{k=1\\ k\neq c}}^{p} a(y_k)\Omega\rangle \tag{H3}
$$

We define a second operator on Fock space $G : \mathcal{F} \to \mathcal{F}$ by

$$
G := \sum_{\phi\in B^+} a^*\left(N\phi\right)a(\phi) + \sum_{\text{マ}\in B} a\left(P_- N\,\text{マ}\right)a^*(\text{マ}). \tag{4}
$$

You can easily see that

$$
\left(\sum_{\phi\in B^+} a^*\left(N\phi\right)a(\phi) + \sum_{\text{マ}\in B} a\left(P_- N\,\text{マ}\right)a^*(\text{マ})\right) \prod_{l=1}^{m} a^*(\varphi_l) \prod_{k=1}^{p} a(y_k)\Omega\rangle
$$

$$
= \sum_{b=1}^{m} \prod_{l=1}^{m} a^*\left(\Delta_{b,l}\left(N\right)\varphi_l\right) \prod_{k=1}^{p} a(y_k)\Omega\rangle + \sum_{b=1}^{p} \prod_{l=1}^{m} a^*(\varphi_l) \prod_{k=1}^{p} a\left(\Delta_{k,b}\left(N_{--}\right)y_k\right)\Omega\rangle
$$

$$
- \sum_{\varphi\in B^+} a^*(\varphi)a\left(N_{-+}\varphi\right) \prod_{l=1}^{m} a^*(\varphi_l) \prod_{k=1}^{p} a(y_k)\Omega\rangle
$$

$$
= \sum_{b=1}^{m} \prod_{l=1}^{m} a^*\left(\Delta_{b,l}\left(N_{++}\right)\varphi_l\right) \prod_{k=1}^{p} a(y_k)\Omega\rangle + \sum_{b=1}^{p} \prod_{l=1}^{m} a^*(\varphi_l) \prod_{k=1}^{p} a\left(\Delta_{k,b}\left(N_{--}\right)y_k\right)\Omega\rangle \tag{G1}
$$

$$
- \sum_{\varphi\in B^+} a^*(\varphi)a\left(N_{-+}\varphi\right) \prod_{l=1}^{m} a^*(\varphi_l) \prod_{k=1}^{p} a(y_k)\Omega\rangle \tag{G2}
$$

$$
+ \sum_{b=1}^{m} \sum_{c=1}^{p} (-1)^{m-b+c} \left\langle y_c, N_{-+}\varphi_b\right\rangle \prod_{\substack{l=1\\ l\neq b}}^{m} a^*(\varphi_l) \prod_{\substack{k=1\\ k\neq c}}^{p} a(y_k)\Omega\rangle \tag{G3}
$$

holds. Now the lines (G1)-(G3) are identical to (H1)-(H3). Which leads me to wanting to conclude that

$$
H = G \tag{5}
$$

holds. However we can compute the commutation relations analogous to (1) and (2). They are

$$
\left[\sum_{\phi\in B^+} a^*\left(N\phi\right)a(\phi) + \sum_{\text{マ}\in B} a\left(P_- N\,\text{マ}\right)a^*(\text{マ}), a(\text{モ})\right]
$$

$$
= -a\left(P_+ N^*\,\text{モ}\right) + a\left(P_- N\,\text{モ}\right) \overset{!}{=} a\left(N\,\text{モ}\right),
$$

$$\left[\sum_{\phi\in B^+} a^*\left(N\phi\right)a(\phi)+\sum_{\text{マ}\in B} a\left(P_- N\,\text{マ}\right)a^*(\text{マ}), a^*(\text{モ})\right]$$
$$= a^*\left(NP_+\,\text{モ}\right)-a^*\left(N^*P_-\,\text{モ}\right)\stackrel{!}{=} a^*\left(N\,\text{モ}\right).$$

These constraints on N seem to appear out of the structure of Fock space.