Electron-Positron Pair Creation in External Fields

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Abstract

In this project we investigate the phenomenon of creation of matter-antimatter pairs of particles, more precisely electron-positron pairs, out of the vacuum subject to strong external electromagnetic fields. Although this phenomenon was predicted already in 1929 and was observed in many experiments, its rigorous mathematical description still lies at the frontier of human understanding of nature. Dirac introduced the heuristic description of the vacuum of quantum electrodynamics (QED) as a homogeneous sea of particles. Although the picture of pair creation as lifting a particle out of the Dirac sea, leaving a hole in the sea, is very explanatory the mathematical formulation of the time evolution of the Dirac sea faces many mathematical challenges. A straightforward interaction of sea particles and the radiation field is ill-defined and physically important quantities such as the total charge current density are badly divergent due to the infinitely many occupied states in the

sea. Nevertheless, in the last century physicists and mathematicians have developed strong methods called "perturbative renormalisation theory" that allow at least to treat the scattering regime perturbatively. Non-perturbative methods have to be developed in order to give an adequate theoretical description of upcoming next-generation experiments allowing a study of the time evolution. Our endeavour focuses on the so-called *external field model of QED* in which one neglects the interaction between the sea particles and only allows an interaction with a prescribed electromagnetic field. It is the goal of this project to develop the necessary non-perturbative methods in this model to give a rigorous construction of the scattering and the time evolution operator.
Keywords: Second Quantised Dirac Equation, non-perturbative QED, external-field QED, Scattering Operator

Contents

Chapter 1

Introduction

Todo: Historische Einleitung durch Anfänge relativistischer Quantenphysik, zweitquantisierung Ruijsnaars Resultat und ivp0. Falls möglich Verbindung zur Physikliteratur. Falls möglich Resultat zur Bestimmung der Phase und Analytizität.

Chapter 2

Nonperturbative discussion of the Scattering Operator

As we have seen in the last chapter, straightforwardly lifting the one particle dynamics to Fock Space leads to difficulties whenever the vector part of the four potential is nonzero. Clearly this is quite devastating for the approach, but even more the result does not respect gauge symmetry, a symmetry of the physical system. This fact tells us, that our description of the physical system as an element of Fock space needs extra restraints, which are purely artefacts of our particular treatment.
Inspired by this, we take a closer look at the construction of Fock space, we closely follow [**?**].

Chapter 3

Axiomatic Construction of Scattering Operator

Todo: Starte von vorne, mache dies klar. Nehme 1-Teilchen stuff und Axiome an, versuche Wohldefiniertheit zu zeigen. Motiviere Axiome durch Eigenschaften der 1-Teilchen Operatoren. Schreibe Induktionsschema auf.

In order to be able to state our main conjecture (3.6.1) precisely I will need to introduce the one-particle dynamics for electrons and their scattering operator, see section 3.1 below, and then move on to the second quantised dynamics and its corresponding scattering operator in section 3.2. Second quantization is the canonical method of turning a one-particle theory into one of an arbitrary and possibly changing number of particles. The informal series expansion of the one-particle scattering operator U is derived from Dirac's equation of motion for the electron. In section 3.1.1 the convergence of this expansion is shown. The informal expansion of the second quantized scattering operator S is then derived from U by second quantisation in section 3.2. At this point I have gathered enough tools to present the main conjecture 3.6.1 in section 3.6. After the main conjecture is known, I present several of my own results in sections 3.6.2, 3.6.3 and 3.6.4 about the first order, the second order and all other odd orders. These results give an intuition on how to control the convergence of the informal expansion of the scattering operator S.

3.1 Defining One-Particle Scattering-Matrix

In order to introduce the one-particle dynamics I introduce Diracs equation (3.1) and reformulate it in integral form in equation (3.7). By iterating this equation we will naturally be led to the informal series expansion of the scattering operator equation (3.12), whose convergence is discussed in the next section.
Throughout this thesis I will consider four-potentials A, F or G to be smooth functions in $C_c^\infty(\mathbb{R}^4) \otimes \mathbb{C}^4$, where the index c denotes that the elements have compact support. Also throughout this thesis I will denote by A, F and G some arbitrary but fixed four-potentials. The Dirac equation for a wave function $\phi \in L^2(\mathbb{R}^3) \otimes \mathbb{C}^4$ is

$$0 = (i\partial\!\!\!/ - eA\!\!\!/ - m\mathbb{1})\phi, \tag{3.1}$$

where m is the mass of the electron, $\mathbb{1} : \mathbb{C}^4 \to \mathbb{C}^4$ is the identity and crossed out letters mean that their four-index is contracted with Dirac matrices

$$A\!\!\!/ := A_\alpha \gamma^\alpha, \tag{3.2}$$

where Einstein's summation convention is used. These matrices fulfil the anti-commutation relation

$$\forall \alpha, \beta \in \{0,1,2,3\} : \{\gamma^\alpha, \gamma^\beta\} := \gamma^\alpha\gamma^\beta + \gamma^\beta\gamma^\alpha = g^{\alpha\beta}, \tag{3.3}$$

where g is the Minkowski metric. I work with the $+---$ metric signature and the Dirac representation of this algebra. Squared four

dimensional objects always refer to the Minkowski square, meaning for all $a \in \mathbb{C}^4$, $a^2 := a^\alpha a_\alpha$.
In order to define Lorentz invariant measures for four dimensional integrals I employ the same notation as in [**?**]. The standard volume form over $\mathbb{R}^4$ is denoted by $\mathrm{d}^4x = \mathrm{d}x^0\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3$, the product of forms is understood as the wedge product. The symbol d^3x means the 3-form $\mathrm{d}^3x = \mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3$ on $\mathbb{R}^4$. Contraction of a form ω with a vector v is denoted by $\mathfrak{i}_v(\omega)$. The notation $\mathfrak{i}_v(\omega)$ is also used for the spinor matrix valued vector $\gamma = (\gamma^0, \gamma^1, \gamma^2, \gamma^3) = \gamma^\alpha e_\alpha$:

$$\mathfrak{i}_\gamma(\mathrm{d}^4x) := \gamma^\alpha \mathfrak{i}_{e_\alpha}(\mathrm{d}^4x), \tag{3.4}$$

with $(e_\alpha)_\alpha$ being the canonical basis of $\mathbb{C}^4$. Let $\mathcal{C}_A$ be the space of solutions to (3.1) which have compact support on any spacelike hyperplane Σ. Let ϕ, ψ be in $\mathcal{C}_\mathcal{A}$, the scalar product $\langle \cdot, \cdot \rangle$ of elements of $\mathcal{C}_A$ is defined as

$$\langle \phi, \psi \rangle := \int_\Sigma \overline{\phi(x)} \mathfrak{i}_\gamma(\mathrm{d}^4x)\psi(x) =: \int_\Sigma \phi^\dagger(x)\gamma^0 \mathfrak{i}_\gamma(\mathrm{d}^4x)\psi(x). \tag{3.5}$$

Furthermore define $\mathcal{H}$ to be $\mathcal{H} := \overline{\mathcal{C}_A}^{\langle \cdot, \cdot \rangle}$. The mas-shell $\mathcal{M} \subset \mathbb{R}^4$ is given by

$$\mathcal{M} = \{p \in \mathbb{R}^4 \mid p^2 = m^2\}. \tag{3.6}$$

The subset $\mathcal{M}^+$ of $\mathcal{M}$ is defined to be $\mathcal{M}^+ := \{p \in \mathcal{M} \mid p^0 > 0\}$. The image of $\mathcal{H}$ by the projector $1_{\mathcal{M}^+}$, given in momentum space representation, is denoted by $\mathcal{H}^+$ and its orthogonal complement by $\mathcal{H}^-$. I introduce a family of Cauchy hypersurfaces $(\Sigma_t)_{t\in\mathbb{R}}$ governed by a family of normal vector fields $(v_t n|_{\Sigma_t})$, where $n : \mathbb{R}^4 \times \mathbb{R} \to \mathbb{R}^4$ and $v : \mathbb{R}^4 \times \mathbb{R} \to \mathbb{R}$ are smooth functions. For $x \in \Sigma_t$ the vector $n_t(x)$ denotes the future directed unit-normal vector to Σ_t at x and $v_t(x)$ the corresponding normal velocity of the flow of the Cauchy surfaces. Now we have the tools to recast the Dirac equation into an integral version which will allow me to define the scattering operator. Let $\psi \in$

$\mathcal{C}_A$, for any $t \in \mathbb{R}$ I denote by ϕ_t the solution to the free Dirac equation, that is equation (3.1) with $A = 0$, with $\psi|_{\Sigma_t}$ as initial condition on Σ_t. Let $t_0 \in \mathbb{R}$ have some fixed value, equation (3.1) can be reformulated, c.f. theorem 2.23 of [?], as

$$\phi_t(y) = \phi_{t_0}(y) - i \int_{t_0}^{t} \mathrm{d}s \int_{\Sigma_s} \int_{\mathcal{M}} \frac{\not{p} + m}{2m^2} e^{ip(x-y)} \mathfrak{i}_p(\mathrm{d}^4 p) \frac{\mathfrak{i}_\gamma(\mathrm{d}^4 x)}{(2\pi)^3} \\ v_s(x) \not{n}_s(x) \not{A}(x) \phi_s(x), \quad (3.7)$$

which holds for any $t \in \mathbb{R}$. Employing the following rewriting of integrals

$$\int_{\mathcal{M}} \frac{\not{p} + m}{2m^2} f(p) \mathfrak{i}_p(\mathrm{d}^4 p) = \frac{1}{2\pi i} \left(\int_{\mathbb{R}^4 - i\epsilon e_0} - \int_{\mathbb{R}^4 + i\epsilon e_0} \right) (\not{p} - m)^{-1} f(p) \mathrm{d}^4 p, \quad (3.8)$$

which is due to the theorem of residues, equation (3.7) assumes the form

$$\phi_t(y) = \phi_{t_0}(y) - \int_{[t_0,t]\times\mathbb{R}^3} \left(\int_{\mathbb{R}^4 - i\epsilon e_0} - \int_{\mathbb{R}^4 + i\epsilon e_0} \right) \\ (\not{p} - m)^{-1} e^{ip(x-y)} \mathrm{d}^4 p \frac{\mathrm{d}^4 x}{(2\pi)^4} \not{A}(x) \phi_s(x). \quad (3.9)$$

In the last expression I picked all hypersurfaces Σ_s to be equal time hyperplanes such that $v_s = 1$ and $\not{n}_s = \gamma^0 e_0$. We identify the advanced and retarded Greens functions of the Dirac equation:

$$\Delta^{\pm}(x) := \frac{-1}{(2\pi)^4} \int_{\mathbb{R}^4 \pm i\varepsilon e_0} \frac{\not{p} + m}{p^2 - m^2} e^{-ipx} d^4 p, \quad (3.10)$$

yielding

$$\phi_t(y) = \phi_{t_0}(y) + \int_{[t_0,t]\times\mathbb{R}^3} (\Delta^- - \Delta^+)(y - x) \mathrm{d}^4 x \not{A}(x) \phi_s(x). \quad (3.11)$$

Iteratingequation (3.11) and picking t in the future of $\operatorname{supp} A$ and t_0 in the past of it, denoting them by $\pm\infty$ since their exact value is no longer important, the following series expansion is obtained informally

$$\phi_\infty(y) = U^A \phi_{-\infty} := \sum_{k=0}^{\infty} Z_k(A)\phi_{-\infty}, \tag{3.12}$$

with $Z_0 = \mathbb{1}$, the identity on $\mathbb{C}^4$, and where for arbitrary $\phi \in \mathcal{H}$, Z_k is defined as

$$Z_k(A)\phi(y) := \int_{\mathbb{R}^4} (\Delta^- - \Delta^+)(y - x_1) \mathrm{d}^4 x_1 \not{A}(x_1)$$
$$\prod_{l=2}^{k} \left[\int_{[-\infty, x^0_{l-1}] \times \mathbb{R}^3} (\Delta^- - \Delta^+)(x_{l-1} - x_l) \not{A}(x_l) \mathrm{d}^4 x_l \right] \phi(x_k).$$

Now since the integration variables are time ordered and $\operatorname{supp} \Delta^\pm \subseteq \mathrm{Cau}^\pm$ in every one but the first factor the contribution of Δ^- vanishes. Therefore we can simply drop it. Furthermore we may continue the integration domain to all of $\mathbb{R}^4$, since there Δ^+ gives no contribution, giving

Todo: führe Cau als kausale Menge ein

$$Z_k(A)\phi(y) = (-1)^{k-1} \int_{\mathbb{R}^4} \mathrm{d}^4 x_1 (\Delta^- - \Delta^+)(y - x_1) \not{A}(x_1)$$
$$\prod_{l=2}^{k} \left[\int_{\mathbb{R}^4} \mathrm{d}^4 x_l \Delta^+(x_{l-1} - x_l) \not{A}(x_l) \right] \phi(x_k). \tag{3.13}$$

This is convenient, because we may now use the spacetime integration with the exponential factor of the definition of Δ^- as a Fourier transform acting on the four-potentials and the wave function. Undoing the substitutions again for the first factor and executing the just mentioned Fourier transforms using the convolution theorem inductively results in

$$
\begin{aligned}
Z_k(A)\phi(y) = -i \int_{\mathcal{M}} \frac{\mathfrak{i}_p(\mathrm{d}^4 p_1)}{(2\pi)^3} \frac{\not{p}_1 + m}{2m} e^{-ip_1 y} \\
\prod_{l=2}^{k} \left[\int_{\mathbb{R}^4 + i\epsilon e_0} \frac{\mathrm{d}^4 p_l}{(2\pi)^4} \not{A}(p_{l-1} - p_l)(\not{p}_l - m)^{-1} \right] \\
\int_{\mathcal{M}} \mathfrak{i}_p(\mathrm{d}^4 p_{k+1}) \not{A}(p_k - p_{k+1}) \hat{\phi}(p_{k+1}).
\end{aligned} \tag{3.14}
$$

Due to the representation (3.13) one may also represent Z_k in terms of The operators

$$
\Delta^0 := \Delta^+ - \Delta^- \tag{3.15}
$$

$$
L_A^{\pm,0} := \Delta^{\pm,0} * \not{A} \tag{3.16}
$$

in this manner

$$
Z_k(A)\phi(y) = (-1)^k L_A^0 \left(L_A^{+\,k-1}(\phi) \right)(y), \tag{3.17}
$$

where the upper right index for an operator means iterative application of said operator.

3.1.1 Well-definedness of U

I will outline in this section how to prove that the informally inferred series expansion of U in (3.12) is well-defined, i.e. that the series converges. In doing so it is crucial to find appropriate bounds on the summands of said series. The domain of integration of the temporal variables in the iterated form of equation (3.7) is a simplex. The volume of this simplex is related to the volume of the cube by the factor $n!$, using this one usually introduces the time ordering Operator and the factor of $\frac{1}{n!}$. This line of argument has been translated into

the momentum space, which might turn out to be more convenient for proving the main conjecture.
Using Parsevals theorem one translates the operators Z_k into momentum space, then one applies standard approximation techniques and the theorem of Paley and Wiener and Youngs inequality for convolution operators. Next one minimizes with respect to the arbitrary ϵ in the equation (3.14), which can be done due to the rules for changing the contour of integration of analytic functions. The estimate is valid only for $k > 1$, it is given by

$$\|Z_k(A)\| \leq \left(\frac{1}{\sqrt{2}}\right)^{k-1} \frac{(ea)^{k-1}}{(k-1)^{k-1}} \frac{C_N^k}{\pi^{4k+2}8} f^{k-1} g, \tag{3.18}$$

where $C_N > 0$ is a constant obtained by application of the theorem of Paley and Wiener (it can for example be found in [?]). In order to simplify the notation I used $a := \operatorname{diam}(\operatorname{supp}(A))$, $f := \left\| \frac{1}{(1+|\cdot|)^N} \right\|_{\mathcal{L}^1(\mathbb{R}^4, \mathrm{d}^4 x)}$, $g := \left\| \frac{1}{(1+|\cdot|)^N} \right\|_{\mathcal{L}^1(\mathcal{M}, \mathrm{i}_p \mathrm{d}^4 p)}$ and e being Euler's number. By $\mathcal{L}^1(E, \mathrm{d}\mu)$ I denote the space of functions with domain of definition E which are integrable with respect to the measure $\mathrm{d}\mu$, i.e.

$$\mathcal{L}^1(E, \mathrm{d}\mu) := \{\psi : E \to \mathbb{C} \mid \int_E \|\psi(x)\| \mathrm{d}\mu(x) < \infty\}. \tag{3.19}$$

For the operator norm of $Z_1(A)$ the bound

$$\|Z_1(A)\| \leq \left\| \|\not{A}\|_{spec} \right\|_{\mathcal{L}^1(\mathcal{M})} \tag{3.20}$$

can be found more easily. It is finite, because in position space A is compactly supported, which means that at infinity its Fourier transform falls off faster than any polynomial. Some lengthy calculations and the use of the well known bound on the factorial $n! \leq \sqrt{2\pi n} \left(\frac{n}{e}\right)^n e^{\frac{1}{12n}}$

result in the following bound for the series representing the operator U

$$\|U^A\| = \left\|\sum_{k=0}^{\infty} Z_k(A)\right\| \leq 1+\Big\| \|\not{A}\|_{spec}\Big\|_{\mathcal{L}^1(\mathcal{M})}+fg\frac{aC_N^2}{\pi^{\frac{19}{2}}4}e^{\frac{aC_N f}{\pi^4\sqrt{2}}+\frac{1}{12}} < \infty. \tag{3.21}$$

The series representing U^A therefore converges, so it gives rise to a well defined operator.

3.2 Construction of the Second Quantised Scattering-Matrix

The main objective of my thesis is to do the analogous proof of section 3.1.1 in the second quantised case, i.e. to prove conjecture 3.6.1. For doing so we have gathered a lot of tools from the one-particle theory. In the following I outline how the construction of the second quantised scattering operator is to be carried out, we will naturally be led to an informal power series representation for the scattering operator S. This time the construction is more delicate, so I will consider different kinds of terms of the expansion using different techniques. I will first consider all odd orders in the expansion in section 3.6.2, then mention additional results about the first order in section 3.6.3 and move on to the second order in section 3.6.4. The control of the orders greater than two are outstanding and forms the main part of the work in this project. In section 3.6 below I will give arguments why the necessary control for the convergence can be achieved.

First I fix some more notation. Using the space of solutions of the Dirac equation $\mathcal{H}$ one constructs Fock space in the following way

$$\mathcal{F} := \bigoplus_{m,p=0}^{\infty} \left(\mathcal{H}^{+}\right)^{\Lambda m} \otimes \left(\overline{\mathcal{H}^{-}}\right)^{\Lambda p}, \tag{3.22}$$

where the bar denotes complex conjugation and Λ in the exponent denotes that only elements which are antisymmetric with respect to permutations are allowed. The Factor $(\mathcal{H}^{\pm})^0$ is understood as $\mathbb{C}$. I will denote the sectors of Fock space of fixed particle numbers by $\mathcal{F}_{m,p} := (\mathcal{H}^{+})^{\Lambda m} \otimes \left(\overline{\mathcal{H}^{-}}\right)^{\Lambda p}$. The element of $\mathcal{F}_{0,0}$ of norm 1 will be denoted by Ω. The annihilation operator a acts on an arbitrary sector of Fock space $\mathcal{F}_{m,p}$, for any $m, p \in \mathbb{N}_0$ as

$$\begin{aligned} a :& \mathcal{H} \otimes \mathcal{F}_{m,p} \to \mathcal{F}_{m-1,p} \oplus \mathcal{F}_{m,p-1} \\ & \phi \otimes \alpha \mapsto \langle \mathrm{P}_{+}\phi(x), \alpha(x, \cdot, \dots) \rangle_x + \langle \mathrm{P}_{-}\phi(x), \alpha(\cdot, \dots, \cdot, x) \rangle_x, \end{aligned} \tag{3.23}$$

where $\langle , \rangle_x$ denotes that the scalar product of $\mathcal{H}$ is to be taken with respect to x and $\mathrm{P}_{\pm}$ denotes the projector onto $\mathcal{H}^{+}$ and $\mathcal{H}^{-}$ respectively. The vacuum sector is mapped to the zero element of Fock space.

Now we turn to the construction of the S-matrix, the second quantised analogue of U. This construction is carried out axiomatically. The first axiom makes sure that the following diagram, and the analogue for the adjoint of the annihilation operator commute.

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{S^A} & \mathcal{F} \\ \uparrow a & & \uparrow a \\ \mathcal{H} \otimes \mathcal{F} & \xrightarrow{U^A \otimes S^A} & \mathcal{H} \otimes \mathcal{F} \end{array} \tag{3.24}$$

Axiom 1. *The S operator fulfils the "lift condition".*

$$\forall \phi \in \mathcal{H} : \quad S^A \circ a(\phi) = a\left(U^A \phi\right) \circ S^A, \tag{lift condition}$$

$$\forall \phi \in \mathcal{H} : \quad S^A \circ a^*(\phi) = a^*\left(U^A \phi\right) \circ S^A, \tag{adjoint lift condition}$$

where a^* is the adjoint of the annihilation operator, the creation operator.

The scattering operator is then expanded in an informal power series

$$S^A = \mathbb{1}_{\mathcal{F}} + \sum_{l=1}^{\infty} \frac{1}{l!} T_l(A). \tag{3.25}$$

In order to fully characterise S^A it is enough to characterise all of the T_l operators. For $k \in \mathbb{N}$ the operators $T_k(A)$ are also defined for k non-identical arguments by homogeneity of $T_k(A)$ to be symmetric in its arguments. For ease of notation we define $T_0 := \mathbb{1}_{\mathcal{F}}$. Using the (lift condition) one can easily derive commutation relations for the operators T_m, which are given by

$$\left[T_m(A), a^{\#}(\phi)\right] = \sum_{j=1}^{m} \binom{m}{j} a^{\#}\left(Z_j(A)\phi\right) T_{m-j}(A), \tag{3.26}$$

where $a^{\#}$ is either a or a^*. The matrix elements of the expansion coefficients T_l of (3.25) can therefore be constructed from the matrix elements of the lower expansion coefficients T_k with $k < l$ and the vacuum expectation value of T_l. As will be shown in section 3.6.2, the vacuum expectation value of all odd orders can naturally be chosen to zero, due to charge conjugation symmetry. I will be using the method of Eppstein and Glaser (see [?, ?]) to find the vacuum expectation value of the even orders.
Besides the scattering operator I will also need the expansion coefficients of its adjoint.

$$\left(S^A\right)^* = \mathbb{1}_{\mathcal{F}} + \sum_{l=1}^{\infty} \frac{1}{l!} \tilde{T}_l(A) \tag{3.27}$$

Since the scattering operator has to be unitary, it is not difficult to find the following expression for the coefficients of its adjoint

$$\forall m > 0 : \quad \sum_{k=0}^{m} \binom{m}{k} T_{m-k}(A)\tilde{T}_k(A) = 0. \tag{3.28}$$

Thus to find the adjoint coefficient of order n, it suffices to know the coefficients of S itself up to order n.

3.3 Construction of Recursive Equation for T_m

In the following I derive a recursive equation for the coefficients of the expansion of the second quantized scattering operator. The starting point of this derivation is the commutator of T_m, equation (3.26).

3.3.1 Heuristics

Why at this point one might suspect that such a representation exists is, because looking at equation (3.26) for a while, one comes to the conclusion that if one replaces T_m by

$$T_m - \frac{1}{2} \sum_{k=1}^{m-1} \binom{m}{k} T_k T_{m-k}, \tag{3.29}$$

no T_k with $k > m-2$ will occur on the right hand side of the resulting equation. So if one subtracts the right polynomial in T_k for suitable k one might achieve a commutator which contains only the creation respectively annihilation operator concatenated with some one particle operator. From our treatment of T_1 we know which operators have such commutation relations.

Todo: place proper reference to definition of G operator

So having this in Mind we start with the ansatz

$$\Gamma_m := \sum_{g=2}^{m} \sum_{\substack{b\in\mathbb{N}^g \\ |b|=m}} c_b \prod_{k=1}^{g} T_{b_k}. \tag{3.30}$$

Now in order to show that T_m and Γ_m agree up to operators which have a commutation relation of the form (3.115), we calculate $\left[T_m - \Gamma_m, a^{\#}(\varphi_n)\right]$ for arbitrary $n \in \mathbb{Z}$ and try to choose the coefficients c_b of (3.30) such that all contributions vanish which do not have the form $a^{\#}\left(\prod_k Z_{\alpha_k}\right)$ for any suitable $(\alpha_k)_k \subset \mathbb{N}$. If one does so, one is led to a system of equations of which I wrote down a few to give an overview of its structure. The objects α_k, β_l in the system of equations can be any natural Number for any $k, l \in \mathbb{N}$.

$$\begin{aligned}
0 &= c_{\alpha_1,\beta_1} + c_{\beta_1,\alpha_1} + \binom{\alpha_1+\beta_1}{\alpha_1} \\
0 &= c_{\alpha_1,\alpha_2,\beta_1} + c_{\beta_1,\alpha_1,\alpha_2} + c_{\alpha_2,\alpha_1,\beta_1} + \binom{\alpha_2+\beta_1}{\alpha_2} c_{\alpha_1,\alpha_2+\beta_1} \\
&\qquad + \binom{\alpha_1+\beta_1}{\alpha_1} c_{\alpha_1+\beta_1,\alpha_2} \\
0 &= c_{\alpha_1,\alpha_2,\alpha_3,\beta_1} + c_{\alpha_1,\alpha_2,\beta_1,\alpha_3} + c_{\alpha_1,\beta_1,\alpha_2,\alpha_3} + c_{\beta_1,\alpha_1,\alpha_2,\alpha_3} \\
&+ \binom{\alpha_1+\beta_1}{\beta_1} c_{\alpha_1+\beta_1,\alpha_2,\alpha_3} + \binom{\alpha_2+\beta_1}{\beta_1} c_{\alpha_1,\alpha_2+\beta_1,\alpha_3} \\
&\qquad + \binom{\alpha_3+\beta_1}{\beta_1} c_{\alpha_1,\alpha_2,\alpha_3+\beta_1} \\
0 &= c_{\alpha_1,\alpha_2,\beta_1,\beta_2} + c_{\alpha_1,\beta_1,\alpha_2,\beta_2} + c_{\beta_1,\alpha_1,\alpha_2,\beta_2} + c_{\alpha_1,\beta_1,\beta_2,\alpha_2} \\
&+ c_{\beta_1,\alpha_1,\beta_2,\alpha_2} + c_{\beta_1,\beta_2,\alpha_1,\alpha_2} + \binom{\alpha_1+\beta_1}{\alpha_1} (c_{\alpha_1+\beta_1,\alpha_2,\beta_2} \\
&+ c_{\alpha_1+\beta_1,\beta_2,\alpha_2}) + \binom{\alpha_1+\beta_2}{c}_{\beta_1,\alpha_1+\beta_2,\alpha_1}
\end{aligned}$$

$$
\begin{aligned}
&+\binom{\alpha_2+\beta_1}{\alpha_2}c_{\alpha_1,\alpha_2+\beta_1,\beta_2}+\binom{\alpha_2+\beta_2}{\alpha_2}(c_{\alpha_1,\beta_1,\alpha_2+\beta_2}\\
&+c_{\beta_1,\alpha_1,\alpha_2+\beta_2})+\binom{\alpha_1+\beta_1}{\alpha_1}\binom{\alpha_2+\beta_2}{\alpha_2}c_{\alpha_1+\beta_1,\alpha_2+\beta_2}\\
&0=c_{\alpha_1,\beta_1,\beta_2,\beta_3,\beta_4}+c_{\beta_1,\alpha_1,\beta_2,\beta_3,\beta_4}+c_{\beta_1,\beta_2,\alpha_1,\beta_3,\beta_4}\\
&\qquad+c_{\beta_1,\beta_2,\beta_3,\alpha_1,\beta_4}+c_{\beta_1,\beta_2,\beta_3,\beta_4,\alpha_1}\\
&+\binom{\alpha_1+\beta_1}{\alpha_1}c_{\alpha_1+\beta_1,\beta_2,\beta_3,\beta_4}+\binom{\alpha_1+\beta_2}{\alpha_1}c_{\beta_1,\alpha_1+\beta_2,\beta_3,\beta_4}\\
&+\binom{\alpha_1+\beta_3}{\alpha_1}c_{\beta_1,\beta_2,\alpha_1+\beta_3,\beta_4}+\binom{\alpha_1+\beta_4}{\alpha_1}c_{\beta_1,\beta_2,\beta_3,\alpha_1+\beta_4}\\
&0=c_{\alpha_1,\alpha_2,\beta_1,\beta_2,\beta_3}+c_{\alpha_1,\beta_1,\alpha_2,\beta_2,\beta_3}+c_{\beta_1,\alpha_1,\alpha_2,\beta_2,\beta_3}\\
&+c_{\alpha_1,\beta_1,\beta_2,\alpha_2,\beta_3}+c_{\beta_1,\alpha_1,\beta_2,\alpha_2,\beta_3}+c_{\beta_1,\beta_2,\alpha_1,\alpha_2,\beta_3}\\
&+c_{\alpha_1,\beta_1,\beta_2,\beta_3,\alpha_2}+c_{\beta_1,\alpha_1,\beta_2,\beta_3,\alpha_2}+c_{\beta_1,\beta_2,\alpha_1,\beta_3,\alpha_2}\\
&+c_{\beta_1,\beta_2,\beta_3,\alpha_1,\alpha_2}+\binom{\alpha_1+\beta_1}{\beta_1}(c_{\alpha_1+\beta_1,\alpha_2,\beta_2,\beta_3}\\
&+c_{\alpha_1+\beta_1,\beta_2,\alpha_2,\beta_3}+c_{\alpha_1+\beta_1,\beta_2,\beta_3,\alpha_2})\\
&+\binom{\alpha_2+\beta_1}{\beta_1}c_{\alpha_1,\alpha_2+\beta_1,\beta_2,\beta_3}\\
&+\binom{\alpha_2+\beta_2}{\beta_2}(c_{\beta_1,\alpha_1,\alpha_2+\beta_2,\beta_3}+c_{\alpha_1,\beta_1,\alpha_2+\beta_2,\beta_3})\\
&+\binom{\alpha_1+\beta_2}{\beta_2}(c_{\beta_1,\alpha_1+\beta_2,\alpha_2,\beta_3}+c_{\beta_1,\alpha_1+\beta_2,\beta_3,\alpha_2})\\
&+\binom{\alpha_2+\beta_3}{\beta_3}(c_{\alpha_1,\beta_1,\beta_2,\alpha_2+\beta_3}+c_{\beta_1,\alpha_1,\beta_2,\alpha_2+\beta_3}\\
&+c_{\beta_1,\beta_2,\alpha_1,\alpha_2+\beta_3})+\binom{\alpha_1+\beta_3}{\beta_3}c_{\beta_1,\beta_2,\alpha_1+\beta_3,\alpha_2}\\
&+\binom{\alpha_1+\beta_1}{\alpha_1}\binom{\alpha_2+\beta_2}{\alpha_2}c_{\alpha_1+\beta_1,\alpha_2+\beta_2,\beta_3}
\end{aligned}
$$

$$
\begin{aligned}
&+\binom{\alpha_1+\beta_2}{\alpha_1}\binom{\alpha_2+\beta_3}{\alpha_2}c_{\beta_1,\alpha_1+\beta_2,\alpha_2+\beta_3}\\
&+\binom{\alpha_1+\beta_1}{\alpha_1}\binom{\alpha_2+\beta_3}{\alpha_2}c_{\alpha_1+\beta_1,\beta_2,\alpha_2+\beta_3}\\
&\qquad\qquad\vdots
\end{aligned}
$$

Solving the first few equations and plugging the solution into the consecutive equations one can see that at least the first equations are solved by

$$c_{\alpha_1,\dots,\alpha_k} = \frac{(-1)^k}{k}\binom{\sum_{l=1}^k \alpha_l}{\alpha_1\ \alpha_2\ \cdots\ \alpha_k}, \tag{3.31}$$

where the last factor is the multinomial coefficient of the indices $\alpha_1,\dots,\alpha_k \in \mathbb{N}$.

3.3.2 Theorem and Proof

The above considerations lead us to the following

Theorem 3.3.1. *For any $n \in \mathbb{N}$ the nth expansion coefficient of the second quantized scattering operator T_n is given by*

$$
\begin{aligned}
T_n = &\sum_{g=2}^{n}\sum_{\substack{\vec{b}\in\mathbb{N}^g\\|\vec{b}|=n}}\frac{(-1)^g}{g}\binom{n}{\vec{b}}\prod_{l=1}^{g}T_{b_l} + C_n\mathbb{1}_{\mathcal{F}}\\
&+G\left(\sum_{g=1}^{n}\sum_{\substack{\vec{b}\in\mathbb{N}^g\\|\vec{b}|=n}}\frac{(-1)^{g+1}}{g}\binom{n}{\vec{b}}\prod_{l=1}^{g}Z_{b_l}\right),
\end{aligned}\tag{3.32}
$$

for some $C_n \in \mathbb{C}$ which depends on the external field A. The last summand will henceforth be abbreviated by G_n.

Proof: The way we will prove this is to compute the commutator of the difference between T_n and the first summand of (3.32) with the creation and annihilation operator of an element of the basis of $\mathcal{H}$. This will turn out to be exactly equal to the corresponding commutator of the second summand of (3.32), since two operators on Fockspace only have the same commutator with general creation and annihilation operators if they agree up to multiples of the identity this will conclude our proof.
In order to simplify the notation as much as possible, I will denote by $a^{\#}z$ either $a(z(\varphi_p))$ or $a^*(z(\varphi_p))$ for any one particle operator z and any element φ_p of the orthonormal basis $(\varphi_p)_{p\in\mathbb{Z}}$ of $\mathcal{H}$. (We need not decide between creation and annihilation operator, since the expressions all agree)
In order to organize the bookkeeping of all the summands which arise from iteratively making use of the commutation rule (3.26) we organize them by the looking at a spanning set of the possible terms that arise my choice is

$$\left\{ a^{\#} \prod_{k=1}^{m_1} Z_{\alpha_k} \prod_{k=1}^{m_2} T_{\beta_k} \middle| m_1 \in \mathbb{N}, m_2 \in \mathbb{N}_0, \alpha \in \mathbb{N}^{m_1}, \beta \in \mathbb{N}^{m_2}, |\alpha| + |\beta| = n \right\}. \tag{3.33}$$

As a first step of computing the commutator in question we look at the summand corresponding to a fixed value of the summation index g of

$$-\sum_{g=1}^{n} \sum_{\substack{\vec{b}\in\mathbb{N}^g \\ |\vec{b}|=n}} \frac{(-1)^g}{g} \binom{n}{\vec{b}} \prod_{l=1}^{g} T_{b_l}. \tag{3.34}$$

We need to bring this object into the form of a sum of terms which are multiples of elements of the set (3.33). This we will commit ourselves to for the next few pages. First we apply the product rule for the commutator:

$$\left[\sum_{\substack{\vec{l}\in\mathbb{N}^g\\|\vec{l}|=n}} \frac{(-1)^g}{g}\binom{n}{\vec{l}}\prod_{k=1}^{g} T_{l_k}, a^{\#}\right]$$

$$= \sum_{\substack{\vec{l}\in\mathbb{N}^g\\|\vec{l}|=n}} \frac{(-1)^g}{g}\binom{n}{\vec{l}}\sum_{\tilde{k}=1}^{g}\prod_{j=1}^{\tilde{k}-1} T_{l_j}\left[T_{l_{\tilde{k}}}, a^{\#}\right]\prod_{j=\tilde{k}+1}^{g} T_{l_j}$$

$$= \sum_{\substack{\vec{l}\in\mathbb{N}^g\\|\vec{l}|=n}} \frac{(-1)^g}{g}\binom{n}{\vec{l}}\sum_{\tilde{k}=1}^{g}\prod_{j=1}^{\tilde{k}-1} T_{l_j}\sum_{\sigma_{\tilde{k}}=1}^{l_{\tilde{k}}}\binom{l_{\tilde{k}}}{\sigma_{\tilde{k}}} a^{\#} Z_f T_{l_{\tilde{k}}-\sigma_{\tilde{k}}}\prod_{j=\tilde{k}+1}^{g} T_{l_j},$$

in the second step we used (3.26). Now we commute all the T_ls to the left of $a^{\#}$ to its right:

$$= \sum_{\substack{\vec{l}\in\mathbb{N}^g\\|l|=n}} \frac{(-1)^g}{g}\binom{n}{\vec{l}}\sum_{\tilde{k}=1}^{g}\sum_{\substack{\forall 1\le j<\tilde{k}\\0\le\sigma_j\le l_j}}\sum_{\sigma_{\tilde{k}}=1}^{l_{\tilde{k}}}\prod_{j=1}^{\tilde{k}}\binom{l_j}{\sigma_j} a^{\#}\prod_{j=1}^{\tilde{k}} Z_{\sigma_j}\prod_{j=1}^{\tilde{k}} T_{l_j-\sigma_j}\prod_{j=\tilde{k}+1}^{g} T_{l_j}. \tag{3.35}$$

At this point we notice that the multinomial coefficient can be combined with all the binomial coefficients to form a single multinomial coefficient of degree $g+\tilde{k}$. Incidentally this is also the amount of Z operators plus the amount of T operators in each product. Moreover the indices of the Multinomial index agree with the indices of the Z and T operators in the product. Because of this, we see that if we fix an element of the spanning set (3.33) $a^{\#}\prod_{k=1}^{m_1} Z_{\alpha_k}\prod_{k=1}^{m_2} T_{\beta_k}$, each summand of (3.35) which is a multiple of this element, has the prefactor

$$\frac{(-1)^g}{g}\binom{n}{\alpha_1 \;\cdots \alpha_{m_1}\; \beta_1 \cdots \beta_{m_2}} \tag{3.36}$$

no matter which summation index $l \in \mathbb{N}^g$ it corresponds to. In order to do the matching one may ignore the indices σ_j and $l_j - \sigma_j$ which vanish, because the corresponding operators Z_0 and T_0 are equal to the identity operator on $\mathcal{H}$ respectively Fockspace.
Since we know that

$$\left[G\left(\sum_{g=1}^{n}\sum_{\substack{\vec{b}\in\mathbb{N}^g\\|\vec{b}|=n}}\frac{(-1)^g}{g}\binom{n}{\vec{b}}\prod_{l=1}^{g}Z_{b_l}\right), a^{\#}\right]$$
$$= a^{\#}\sum_{g=1}^{n}\sum_{\substack{\vec{b}\in\mathbb{N}^g\\|\vec{b}|=n}}\frac{(-1)^g}{g}\binom{n}{\vec{b}}\prod_{l=1}^{g}Z_{b_l}$$

holds, all that is left to show is that

$$\left[-\sum_{g=1}^{n}\sum_{\substack{\vec{b}\in\mathbb{N}^g\\|\vec{b}|=n}}\frac{(-1)^g}{g}\binom{n}{\vec{b}}\prod_{l=1}^{g}T_{b_l}, a^{\#}\right] \tag{3.37}$$
$$= a^{\#}\sum_{g=1}^{n}\sum_{\substack{\vec{b}\in\mathbb{N}^g\\|\vec{b}|=n}}\frac{(-1)^{g+1}}{g}\binom{n}{\vec{b}}\prod_{l=1}^{g}Z_{b_l}$$

holds. For which we need to count the summands which are multiples of each element of (3.33) corresponding to each g in (3.34). So let us fix some element $K(m_1, m_2)$ of (3.33) corresponding to some $m_1 \in$

$\mathbb{N}, m_2 \in \mathbb{N}_0, \alpha \in \mathbb{N}^{m_1}$ and $\beta \in \mathbb{N}^{m_2}$. Rephrasing this problem, we can ask which products

$$\prod_{l=1}^{g} T_{\gamma_l} \tag{3.38}$$

for suitable g and $(\gamma_l)_l$ produce multiples of $K(m_1, m_2)$? We will call this number of total contributions weighted with the factor $-\frac{(-1)^g}{g}$ borrowed from (3.34) $\#K(m_1, m_2)$. Looking at the commutation relations (3.26) we split the set of indices $\{\gamma_1 \dots \gamma_g\}$ into three sets A, B and C, where the commutation relation has to be used in such a way, that

$$\begin{aligned}
\forall k : \gamma_k \in A &\iff \exists j \leq m_1 : \gamma_k = \alpha_j, \\
\wedge \forall k : \gamma_k \in B &\iff \exists j \leq m_2 : \gamma_k = \beta_j \\
\wedge \forall k : \gamma_k \in C &\iff \exists j \leq m_1, l \leq m_2 : \gamma_k = \alpha_j + \beta_l
\end{aligned}$$

holds. Unfortunately not any splitting corresponds to a contribution and not any order of multiplication of a legal splitting corresponds to a contribution either. However we can be sure that $\prod_j T_{\alpha_j} \prod_j T_{\beta_j}$ gives a contribution and we may apply the commutation relations backwards to obtain any valid combination. This results in the following game: Starting from the string

$$A_1 A_2 \dots A_{m_1} B_1 B_2 \dots B_{m_2}, \tag{3.39}$$

how many strings can we produce by applying the following rules?

1. You may replace any occurrence of $A_k B_j$ by $B_j A_k$ for any j and k.

2. You may replace any occurrence of $A_k B_j$ by $C_{k,j}$ for any j and k.

Where we have to count the number of times we applied the second rule, or equivalently the number $\#C$ of C's in the resulting string, because the summation index g in (3.34) corresponds to $m_1+m_2-\#C$. Fix $\#C \in \{0,\ldots,\min(m_1,m_2)\}$. A valid string has $m_1+m_2-\#C$ characters, because the number of its Cs is $\#C$, its number of As is $m_1-\#C$ and its number of Bs is $m_2-\#C$. Ignoring the labelling of the As, Bs and Cs there are $\binom{m_1+m_2-\#C}{\#C\ (m_1-\#C)\ (m_2-\#C)}$ such strings. Now if we consider one such string without labelling, e.g.

$$CAABACCBBACBBABBBB, \tag{3.40}$$

there is only one correct labelling to be restored, namely the one where each (first index of any) C and A receive increasing labels from left to right and analogously for (the second index of any) C and B, resulting in

$$C_{1,1}A_2A_3B_2A_4C_{5,3}C_{6,4}B_5B_6A_7C_{8,7}B_8B_9A_9B_{10}B_{11}B_{12}B_{13}. \tag{3.41}$$

So any unlabelled string corresponds to exactly one labelled string which in turn corresponds to exactly one choice of operator product $\prod T$. So returning to our Operators, we found the number $\#K(m_1,m_2)$ it is

$$\#K(m_1,m_2) = -\sum_{g=\max(m_1,m_2)}^{m_1+m_2} \frac{(-1)^g}{g}\binom{g}{(m_1+m_2-g)\ (g-m_1)\ (g-m_2)}, \tag{3.42}$$

where the total minus sign comes from the total minus sign in front of (3.37) with respect to (3.32).

Now a quick route to evaluate this sum requires us to slightly generalize the standard definition of binomial coefficient to the one in [?]:

Definition 3.3.2. *For $a \in \mathbb{C}, b \in \mathbb{Z}$ we define*

$$\binom{a}{b} := \begin{cases} \prod_{l=0}^{b-1} \frac{a-l}{l+1} & \text{for } b \geq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{3.43}$$

Defining the binomial coefficient for negative lower index to be zero has the merit, that one can extend the range of validity of many rules and sums involving binomial coefficients, also one does not have to worry about the range of summation in many cases.
As a first step to evaluate (3.42) we split the trinomial coefficient into binomial ones and make use of the absorption identity

$$\forall a \in \mathbb{C}\ \forall b \in \mathbb{Z} : b\binom{a}{b} = a\binom{a-1}{b-1} \tag{absorption}$$

for $m_2, m_1 \neq 0$ as follows

$$\begin{aligned}
&\#K(m_1, m_2) \\
&= -\sum_{g=\max(m_1,m_2)}^{m_1+m_2} \frac{(-1)^g}{g} \binom{g}{(m_1+m_2-g)\ (g-m_1)\ (g-m_2)} \\
&= -\sum_{g=\max(m_1,m_2)}^{m_1+m_2} \frac{(-1)^g}{g} \binom{g}{m_2} \binom{m_2}{g-m_1} \\
&\overset{\text{(absorption)}}{=} -\sum_{g=\max(m_1,m_2)}^{m_1+m_2} \frac{(-1)^g}{m_2} \binom{g-1}{m_2-1} \binom{m_2}{g-m_1} \\
&= \frac{-1}{m_2} \sum_{g=\max(m_1,m_2)}^{m_1+m_2} (-1)^g \binom{g-1}{m_2-1} \binom{m_2}{g-m_1} \\
&\overset{m_1 \geq 0}{=} \frac{-1}{m_2} \sum_{g \in \mathbb{Z}} (-1)^g \binom{m_2}{g-m_1} \binom{g-1}{m_2-1}
\end{aligned}$$

$$\overset{*}{=} \frac{-1}{m_2}(-1)^{m_2-m_1}\binom{m_1-1}{-1} = 0,$$

where for the marked equality we used summation rule (5.24) of [**?**]. So all the coefficients vanish that fulfil $m_1, m_2 \neq 0$. The sum for the remaining cases is trivial, since there is just one summand. Summarising we find

$$\#K(m_1, m_2) = \delta_{m_2,0}\frac{(-1)^{1+m_1}}{m_1} + \delta_{m_1,0}\frac{(-1)^{1+m_2}}{m_2},$$

where the second summand can be ignored, since terms with $m_1 = 0$ are irrelevant for our considerations.
So the left hand side of (3.37) can be evaluated

$$\begin{aligned}&\left[-\sum_{g=1}^{n}\sum_{\substack{\vec{b}\in\mathbb{N}^g\\|\vec{b}|=n}}\frac{(-1)^g}{g}\binom{n}{\vec{b}}\prod_{l=1}^{g}T_{b_l}, a^{\#}\right]\\ &= \sum_{g=1}^{n}\sum_{\substack{\vec{b}\in\mathbb{N}^g\\|b|=n}}\frac{(-1)^{g+1}}{g}\binom{n}{\vec{b}}a^{\#}\prod_{l=1}^{g}Z_{b_l},\end{aligned}$$

which is exactly equal to the right hand side of (3.37). □

3.4 Solution to Recursive Equation

So we found a recursive equation for the T_ns, now we need to solve it. In order to do so we need the following lemma about combinatorial distributions

Lemma 3.4.1. *For any $g \in \mathbb{N}, k \in \mathbb{N}$*

$$\sum_{\substack{\vec{g}\in\mathbb{N}^g \\ |\vec{g}|=k}} \binom{k}{\vec{g}} = \sum_{l=0}^{g} (-1)^l (g-l)^k \binom{g}{l} \tag{3.44}$$

holds. The reader interested in terminology may be eager to know, that the right hand side is equal to $g!$ times the Stirling number of the second kind $\left\{ {k \atop g} \right\}$.

Proof: We would like to apply the multinomial theorem but there are all the summands missing where at least one of the entries of $\vec{g}$ is zero, so we add an appropriate expression of zero. We also give the expression in question a name, since we will later on arrive at a recursive expression.

$$\begin{aligned} F(g,k) &:= \sum_{\substack{\vec{g}\in\mathbb{N}^g \\ |\vec{g}|=k}} \binom{k}{\vec{g}} = \sum_{\substack{\vec{g}\in\mathbb{N}_0^g \\ |\vec{g}|=k}} \binom{k}{\vec{g}} - \sum_{\substack{\vec{g}\in\mathbb{N}_0^g \\ |\vec{g}|=k \\ \exists l: g_l=0}} \binom{k}{\vec{g}} \\ &= g^k - \sum_{\substack{\vec{g}\in\mathbb{N}_0^g \\ |\vec{g}|=k \\ \exists l: g_l=0}} \binom{k}{\vec{g}} = g^k - \sum_{n=1}^{g-1} \sum_{\substack{\vec{g}\in\mathbb{N}_0^g \\ |\vec{g}|=k}} \binom{k}{\vec{g}} 1_{\exists! i_1 \ldots i_n : \forall i_l \neq i_k \wedge \forall l : g_{i_l}=0} \end{aligned} \tag{3.45}$$

where in the last line the indicator function is to enforce there being exactly n different indices i_l for which $g_{i_l} = 0$ holds. Now since it does not matter which entries of the vector vanish because the multinomial coefficient is symmetric and its value is identical to the corresponding multinomial coefficient where the vanishing entries are omitted, we can further simplify the sum:

$$F(g,k) = g^k - \sum_{n=1}^{g-1} \binom{g}{n} \sum_{\substack{\vec{g}\in\mathbb{N}^n \\ |\vec{g}|=k}} \binom{k}{\vec{g}}$$

The inner sum turns out to be $F(g-n,k)$, so we found the recursive relation for F:

$$F(g,k) = g^k - \sum_{n=1}^{g-1} \binom{g}{n} F(n,k) = g^k - \sum_{n=1}^{g-1} \binom{g}{n} F(g-n,k), \quad (3.46)$$

where for the last equality we used the symmetry of binomial coefficients. By iteratively applying this equation, we find the following formula, which we will now prove by induction

$$\begin{aligned} \forall d \in \mathbb{N}_0 : F(g,k) &= \sum_{l=0}^{d} (-1)^l (g-l)^k \binom{g}{l} \\ &+ (-1)^{d+1} \sum_{n=1}^{g-d-1} \binom{n+d-1}{d} \binom{g}{n+d} F(g-d-n,k). \quad (3.47) \end{aligned}$$

We already showed the start of the induction, so what's left is the induction step. Before we do so the following remark is in order: We are only interested in the case $d = g$ and the formula seems meaningless for $d > g$; however, the additional summands in the left sum vanish, where as the right sum is empty for these values of d since the upper bound of the summation index is lower than its lower bound.
For the induction step, pick $d \in \mathbb{N}_0$, use (3.47) and pull the first summand out of the second sum, on this summand we apply the recursive relation (3.46) resulting in

$$\begin{aligned}
F(g,k) = &\sum_{l=0}^{d}(-1)^l(g-l)^k\binom{g}{l} \\
&+ (-1)^{d+1}\sum_{n=2}^{g-d-1}\binom{n+d-1}{d}\binom{g}{n+d}F(g-d-n,k) \\
&+ (-1)^{d+1}\binom{d}{d}\binom{g}{d+1}F(g-d-1,k) \\
\overset{(3.46)}{=} &\sum_{l=0}^{d+1}(-1)^l(g-l)^k\binom{g}{l} \\
&+ (-1)^{d+1}\sum_{n=2}^{g-d-1}\binom{n+d-1}{d}\binom{g}{n+d}F(g-d-n,k) \\
&- (-1)^{d+1}\binom{g}{d+1}\sum_{n=1}^{g-d-2}\binom{g-d-1}{n}F(g-d-1-n,k) \\
= &\sum_{l=0}^{d+1}(-1)^l(g-l)^k\binom{g}{l} \\
&+ (-1)^{d+1}\sum_{n=1}^{g-d-2}\binom{n+d}{d}\binom{g}{n+d+1}F(g-d-1-n,k) \\
&- (-1)^{d+1}\binom{g}{d+1}\sum_{n=1}^{g-d-2}\binom{g-d-1}{n}F(g-d-1-n,k). \quad (3.48)
\end{aligned}$$

After the index shift we can combine the last two sums.

$$F(g,k) = \sum_{l=0}^{d+1}(-1)^l(g-l)^k\binom{g}{l}$$

$$+\sum_{n=1}^{g-d-2}\left[\binom{g}{d+1}\binom{g-d-1}{n}-\binom{n+d}{d}\binom{g}{n+d+1}\right]$$
$$(-1)^{d+2}F(g-d-1-n,k). \quad (3.49)$$

In order to combine the two binomials we reassemble $\binom{g}{d+1}\binom{g-d-1}{n}$ into $\binom{g}{n+d+1}\binom{n+d+1}{d+1}$, which can be seen to be possible by representing everything in terms of factorials. This results in

$$F(g,k)=\sum_{l=0}^{d+1}(-1)^l(g-l)^k\binom{g}{l}$$
$$+(-1)^{d+2}\sum_{n=1}^{g-d-2}\left[\binom{n+d+1}{d+1}-\binom{n+d}{d}\right]\binom{g}{n+d+1}F(g-d-1-n,k)$$
$$=\sum_{l=0}^{d+1}(-1)^l(g-l)^k\binom{g}{l}$$
$$+(-1)^{d+2}\sum_{n=1}^{g-d-2}\binom{n+d}{d+1}\binom{g}{n+d+1}F(g-d-1-n,k), \quad (3.50)$$

where we used the addition formula for binomials:

$$\forall n\in\mathbb{C}\forall k\in\mathbb{Z}:\binom{n}{k}=\binom{n-1}{k}+\binom{n-1}{k-1}. \quad (3.51)$$

This concludes the proof by induction. By setting $d=g$ in equation (3.47) we arrive at the desired result. $\square$

Using the previous lemma, we are able to show the next

Lemma 3.4.2. *For any $k\in\mathbb{N}\backslash\{1\}$ the following equation holds*

$$\sum_{g=1}^{k}\frac{(-1)^g}{g}\sum_{\substack{\vec{g}\in\mathbb{N}^g\\|\vec{g}|=k}}\binom{k}{\vec{g}}=0. \quad (3.52)$$

Proof: Let $k \in \mathbb{N}\backslash\{1\}$, as a first step we apply lemma 3.4.1. We change the order of summation, use (absorption), extend the range of summation and shift summation index to arrive at

$$\begin{aligned}
\sum_{g=1}^{k} \frac{(-1)^g}{g} \sum_{l=0}^{g} (-1)^l (g-l)^k \binom{g}{l} &= \sum_{g=1}^{k} \frac{1}{g} \sum_{l=0}^{g} (-1)^{g-l} (g-l)^k \binom{g}{g-l} \\
= \sum_{g=1}^{k} \sum_{p=0}^{g} (-1)^p p^k \frac{1}{g} \binom{g}{p} &= \sum_{g=1}^{k} \sum_{p=0}^{g} (-1)^p p^k \frac{1}{p} \binom{g-1}{p-1} \\
= \sum_{g=1}^{k} \sum_{p\in\mathbb{Z}} (-1)^p p^{k-1} \binom{g-1}{p-1} &= \sum_{p\in\mathbb{Z}} (-1)^p p^{k-1} \sum_{g=1}^{k} \binom{g-1}{p-1} \\
&= \sum_{p\in\mathbb{Z}} (-1)^p p^{k-1} \sum_{g=0}^{k-1} \binom{g}{p-1}. \qquad (3.53)
\end{aligned}$$

Now we use equation (5.10) of [**?**]:

$$\forall m, n \in \mathbb{N}_0 : \sum_{k=0}^{n} \binom{k}{m} = \binom{n+1}{m+1}, \qquad \text{(upper summation)}$$

which can for example be proven by induction on n.
We furthermore rewrite the power of the summation index p in terms of the derivative of an exponential and change order summation and differentiation. This results in

$$\begin{aligned}
\sum_{g=1}^{k} \frac{(-1)^g}{g} \sum_{l=0}^{g} (-1)^l (g-l)^k \binom{g}{l} &= \sum_{p\in\mathbb{Z}} (-1)^p p^{k-1} \binom{k}{p} \\
= \sum_{p=0}^{k} (-1)^p \left.\frac{\partial^{k-1}}{\partial\alpha^{k-1}} e^{\alpha p}\right|_{\alpha=0} \binom{k}{p} &= \left.\frac{\partial^{k-1}}{\partial\alpha^{k-1}} \sum_{p=0}^{k} (-1)^p e^{\alpha p} \binom{k}{p}\right|_{\alpha=0}
\end{aligned}$$

$$= \frac{\partial^{k-1}}{\partial\alpha^{k-1}} (1 - e^{\alpha p})^k \bigg|_{\alpha=0} = (-1)^k \frac{\partial^{k-1}}{\partial\alpha^{k-1}} \left(\sum_{l=1}^{\infty} \frac{(\alpha p)^l}{l!} \right)^k \Bigg|_{\alpha=0}$$
$$= (-1)^k \frac{\partial^{k-1}}{\partial\alpha^{k-1}} ((\alpha p)^k + \mathcal{O}((\alpha p)^{k+1})) \bigg|_{\alpha=0} = 0.$$

□

I am now in a position to state the solution to the recursive equation (3.32) and have us prove together that it is in fact a solution.

Theorem 3.4.3. *For $n \in \mathbb{N}$ the solution of the recursive equation (3.32) solely in terms of G_a and C_a is given by*

$$T_n = \sum_{g=1}^{n} \sum_{\substack{\vec{b} \in \mathbb{N}^g \\ |\vec{b}|=n}} \sum_{\vec{d} \in \{0,1\}^g} \frac{1}{g!} \binom{n}{\vec{b}} \prod_{l=1}^{g} F_{b_l, d_l}, \tag{3.54}$$

where F is given by

$$F_{a,b} = \begin{cases} G_a & \text{for } b = 0 \\ C_a & \text{for } b = 1 \end{cases}. \tag{3.55}$$

For the readers convenience we remind her, that G_a is defined by (3.32). Whereas the constants C_n depend on the vacuum expectation value of T_n as well as on the products on the right hand side of (3.54) and are yet either to be found in terms of or eliminated in favour of $\langle T_n \rangle$.

Proof: The structure of this proof will be induction over n. For $n = 1$ the whole expression on the right hand side collapses to $C_1 + G_1$, which we already know to be equal to T_1. For arbitrary $n + 1 \in \mathbb{N} \setminus \{1\}$ we apply the recursive equation (3.32) once and use the induction hypothesis for all $k \leq n$ and thereby arrive at the rather convoluted expression

$$
\begin{aligned}
T_{n+1} &\overset{(3.32)}{=} G_{n+1} + C_{n+1} + \sum_{g=2}^{n+1} \sum_{\substack{\vec{b}\in\mathbb{N}^g \\ |\vec{b}|=n+1}} \frac{(-1)^g}{g} \binom{n+1}{\vec{b}} \prod_{l=1}^{g} T_{b_l} \\
&\overset{\text{induction hyp}}{=} G_{n+1} + C_{n+1} + \sum_{g=2}^{n+1} \sum_{\substack{\vec{b}\in\mathbb{N}^g \\ |\vec{b}|=n+1}} \frac{(-1)^g}{g} \binom{n+1}{\vec{b}} \prod_{l=1}^{g} \\
&\qquad \sum_{g_l=1}^{b_l} \sum_{\substack{\vec{c}_l\in\mathbb{N}^{g_l} \\ |\vec{c}_l|=b_l}} \sum_{\vec{e}_l\in\{0,1\}^{g_l}} \frac{1}{g_l!} \binom{b_l}{\vec{c}_l} \prod_{k=1}^{g_l} F_{c_{l,k},e_{l,k}}. \qquad (3.56)
\end{aligned}
$$

If we were to count the contributions of this sum to a specific product $\prod F_{c_j,e_j}$ for some choice of $(c_j)_j, (e_j)_j$ we would first recognize that all the multinomial factors in (3.56) combine to a single one whose indices are given by the first indices of all the F factors involved. Other than this factor each contribution adds $\frac{(-1)^g}{g} \prod_{l=1}^{g} \frac{1}{g_l!}$ to the sum. So we need to keep track of how many contributions there are and which distributions of g_l they belong to.

Fix some product $\prod F := \prod_{j=1}^{\tilde{g}} F_{\tilde{b}_j,\tilde{d}_j}$. In the sum (3.56) we pick some initial short product of length g and split each factor into g_l pieces to arrive at one of length $\tilde{g}$ if the product is to contribute to $\prod F$. So clearly $\sum_{l=1}^{g} g_l = \tilde{g}$ holds for any contribution to $\prod F$. The reverse is also true, for any g and $g_1, \dots, g_g \in \mathbb{N}$ such that $\sum_{l=1}^{g} g_l = \tilde{g}$ holds the corresponding expression in (3.56) contributes to $\prod F$. Furthermore $\prod F$ and $g, g_1, \dots g_g$ is enough to uniquely determine the summand of (3.56) the contribution belongs to. For an illustration of this splitting see

$$\underbrace{\underbrace{F^1_{3,1}F^2_{2,0}F^3_{7,1}}_{g_1=3}\underbrace{F^4_{5,0}}_{g_2=1}\underbrace{F^5_{4,1}F^6_{2,1}}_{g_3=2}\underbrace{F^7_{1,1}F^8_{3,0}F^9_{4,1}}_{g_4=3}\underbrace{F^{10}_{4,1}F^{11}_{1,0}}_{g_5=2}}_{g=5}$$

$$g_1 + g_2 + g_3 + g_4 + g_5 = 11 = \tilde{g},$$

where I labelled the factors in the upper right index for the readers convenience. We recognize that the sum we are about to perform is by no means unique for each order of n but only depends on the number of appearing factors and the number of splittings performed on them. By the preceding argument we need

$$\sum_{g=2}^{\tilde{g}} \frac{(-1)^g}{g} \sum_{\substack{\vec{g}\in\mathbb{N}^g \\ |\vec{g}|=\tilde{g}}} \prod_{l=1}^{g} \frac{1}{g_l!} = \frac{1}{\tilde{g}!} \tag{3.57}$$

to hold for $\tilde{g} > 1$, in order to find agreement with the proposed solution (3.55). Now proving (3.57) is done by realizing, that one can include the right hand side into the sum as the $g = 1$ summand, dividing the equation by $\tilde{g}!$ and using lemma 3.4.2 with $k = \tilde{g}$. The remaining case, $\tilde{g} = 1$, can directly be read off of (3.56). □

Corollary 3.4.4. *For $n \in \mathbb{N}$, T_n can be written as*

$$\frac{1}{n!}T_n = \sum_{\substack{1\le c+g\le n \\ c,g\in\mathbb{N}_0}} \sum_{\substack{\vec{g}\in\mathbb{N}^g \\ \vec{c}\in\mathbb{N}^c \\ |\vec{c}|+|\vec{g}|=n}} \frac{1}{c!g!} \prod_{l=1}^{c} \frac{1}{c_l!} C_{c_l} \prod_{l=1}^{g} \frac{1}{g_l!} G_{g_l}. \tag{3.58}$$

Please note that for ease of notation we defined $\mathbb{N}^0 := \{1\}$.

Proof: By an argument completely analogous to the combinatorial argument in the proof of theorem (3.3.1) we see that we can disentangle the Fs in (3.54) into Gs and Cs if we multiply by a factor of $\binom{c+g}{c}$ where c is the number of Cs and g is the number of Gs giving

$$T_n = \sum_{\substack{1\le c+g\le n\\ c,g\in\mathbb{N}_0}} \sum_{\substack{\vec{g}\in\mathbb{N}^g\\ \vec{c}\in\mathbb{N}^c\\ |\vec{c}|+|\vec{g}|=n}} \binom{c+g}{c} \frac{1}{(c+g)!} \binom{n}{\vec{g}\oplus\vec{c}} \prod_{l=1}^{c} C_{c_l} \prod_{l=1}^{g} G_{g_l}, \quad (3.59)$$

which directly reduces to the equation we wanted to prove, by plugging in the multinomials in terms of factorials.

Corollary 3.4.5. *As a formal power series, the second quantized scattering operator can be written in the form*

$$S = e^{\sum_{l\in\mathbb{N}} \frac{C_l}{l!}} e^{\sum_{l\in\mathbb{N}} \frac{G_l}{l!}}, \quad (3.60)$$

which the author finds quite amusing.

Proof: We plug corollary 3.4.4 into the defining Series for the T_ns giving

$$\begin{aligned} S &= \sum_{n\in\mathbb{N}_0} \frac{1}{n!} T_n \\ &= \mathbb{1}_{\mathcal{F}} + \sum_{n\in\mathbb{N}} \sum_{\substack{1\le c+g\le n\\ c,g\in\mathbb{N}_0}} \sum_{\substack{\vec{g}\in\mathbb{N}^g\\ \vec{c}\in\mathbb{N}^c\\ |\vec{c}|+|\vec{g}|=n}} \frac{1}{c!g!} \prod_{l=1}^{c} \frac{1}{c_l!} C_{c_l} \prod_{l=1}^{g} \frac{1}{g_l!} G_{g_l} \\ &= \mathbb{1}_{\mathcal{F}} + \sum_{\substack{1\le c+g\\ c,g\in\mathbb{N}_0}} \sum_{\substack{\vec{g}\in\mathbb{N}^g\\ \vec{c}\in\mathbb{N}^c}} \frac{1}{c!g!} \prod_{l=1}^{c} \frac{1}{c_l!} C_{c_l} \prod_{l=1}^{g} \frac{1}{g_l!} G_{g_l} \\ &= \sum_{c,g\in\mathbb{N}_0} \sum_{\substack{\vec{g}\in\mathbb{N}^g\\ \vec{c}\in\mathbb{N}^c}} \frac{1}{c!g!} \prod_{l=1}^{c} \frac{1}{c_l!} C_{c_l} \prod_{l=1}^{g} \frac{1}{g_l!} G_{g_l} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{c\in\mathbb{N}_0} \frac{1}{c!} \sum_{\vec{c}\in\mathbb{N}^c} \prod_{l=1}^{c} \frac{1}{c_l!} C_{c_l} \sum_{g\in\mathbb{N}_0} \frac{1}{g!} \sum_{\vec{g}\in\mathbb{N}^g} \prod_{l=1}^{g} \frac{1}{g_l!} G_{g_l} \\
&= \sum_{c\in\mathbb{N}_0} \frac{1}{c!} \prod_{l=1}^{c} \sum_{k\in\mathbb{N}} \frac{1}{k!} C_k \sum_{g\in\mathbb{N}_0} \frac{1}{g!} \prod_{l=1}^{g} \sum_{b\in\mathbb{N}} \frac{1}{b!} G_b \\
&= \sum_{c\in\mathbb{N}_0} \frac{1}{c!} \left(\sum_{k\in\mathbb{N}} \frac{1}{k!} C_k \right)^c \sum_{g\in\mathbb{N}_0} \frac{1}{g!} \left(\sum_{b\in\mathbb{N}} \frac{1}{b!} G_b \right)^g \\
&= e^{\sum_{l\in\mathbb{N}} \frac{1}{l!} C_l} e^{\sum_{l\in\mathbb{N}} \frac{1}{l!} G_l}.
\end{aligned} \tag{3.61}
$$

$\square$

Theorem 3.4.6. *The second quantized scattering operator fulfils*

$$S = e^{\sum_{n\in\mathbb{N}} \frac{C_n}{n!}} e^{G(\ln(U))} \tag{3.62}$$

for external potentials small enough, where C_n must be imaginary for any $n \in \mathbb{N}$ in order to satisfy unitarity. For smallness the following condition is sufficient:

$$\|\mathbb{1} - U^A\| < 1. \tag{3.63}$$

Proof: Due to the last few lemmas and theorems this proof has become much less difficult. First the remark about $C_n \in i\mathbb{R}$ for any n is a direct consequence of the second factor of (3.62) begin unitary. This in turn follows directly from $G^*(K) = -G(K)$ for any K in the domain of G. That $\ln U$ is in the domain of G follows from $(\ln U)^* = \ln U^* = \ln U^{-1} = -\ln U$ and $\|U - \mathbb{1}\| < 1$.
We are going to change the sum in the second exponential of (3.60), so let's take a closer look at that: by exchanging summation we can step by step simplify

$$
\begin{aligned}
\sum_{l\in\mathbb{N}} \frac{G_l}{l!} = \sum_{n\in\mathbb{N}} \frac{1}{n!} G\left(\sum_{g=1}^{n} \sum_{\substack{\vec{b}\in\mathbb{N}^g \\ |\vec{b}|=n}} \frac{(-1)^{g+1}}{g} \binom{n}{\vec{b}} \prod_{l=1}^{g} Z_{b_l} \right) \\
= G\left(\sum_{n\in\mathbb{N}} \frac{1}{n!} \sum_{g=1}^{n} \sum_{\substack{\vec{b}\in\mathbb{N}^g \\ |\vec{b}|=n}} \frac{(-1)^{g+1}}{g} \binom{n}{\vec{b}} \prod_{l=1}^{g} Z_{b_l} \right) \\
= G\left(\sum_{n\in\mathbb{N}} \sum_{g=1}^{n} \sum_{\substack{\vec{b}\in\mathbb{N}^g \\ |\vec{b}|=n}} \frac{(-1)^{g+1}}{g} \prod_{l=1}^{g} \frac{Z_{b_l}}{b_l!} \right) \\
= G\left(\sum_{g\in\mathbb{N}} \sum_{\vec{b}\in\mathbb{N}^g} \frac{(-1)^{g+1}}{g} \prod_{l=1}^{g} \frac{Z_{b_l}}{b_l!} \right) \\
= G\left(\sum_{g\in\mathbb{N}} \frac{(-1)^{g+1}}{g} \prod_{l=1}^{g} \left(\sum_{b_l\in\mathbb{N}} \frac{Z_{b_l}}{b_l!} \right) \right) \\
= G\left(\sum_{g\in\mathbb{N}} \frac{(-1)^{g+1}}{g} \left(\sum_{b\in\mathbb{N}} \frac{Z_b}{b!} \right)^g \right) \\
= G\left(\sum_{g\in\mathbb{N}} \frac{(-1)^{g+1}}{g} (U - \mathbb{1})^g \right) = G\left(-\sum_{g\in\mathbb{N}} \frac{1}{g} (\mathbb{1} - U)^g \right) \\
= G\left(\ln\left(\mathbb{1} - (\mathbb{1} - U)\right)\right) = G\left(\ln\left(U\right)\right). \quad (3.64)
\end{aligned}
$$

□

Remark 3.4.7. $\sum_{n\in\mathbb{N}} \frac{C_n}{n!}$ *will henceforth be abbreviated by* $i\varphi$.

Remark 3.4.8. *If one starts with the right hand side of* (3.4.6) *as the definition of some operator S, all it takes to verify that it fulfils* (lift condition) *and* (adjoint lift condition) *is the following (in comparison with its derivation) short calculation:*
Let $\varphi \in \mathcal{H}$, for any $k \in \mathbb{N}_0$ we see applying the commutation relation of G:

$$G(\ln U) \sum_{l=0}^{k} \binom{k}{l} a^{\#}\left((\ln U)^{l}\, \varphi\right)(G(\ln U))^{k-l} =$$

$$\sum_{l=0}^{k} \binom{k}{l} a^{\#}\left((\ln U)^{l+1}\, \varphi\right)(G(\ln U))^{k-l} + \sum_{l=0}^{k} \binom{k}{l} a^{\#}\left((\ln U)^{l}\, \varphi\right)(G(\ln U))^{k-l+1}$$

$$= \sum_{b=0}^{k+1} \left(\binom{k}{b-1} + \binom{k}{b}\right) a^{\#}\left((\ln U)^{b}\varphi\right)(G(\ln U))^{k+1-b}$$

$$= \sum_{b=0}^{k+1} \binom{k+1}{b} a^{\#}\left((\ln U)^{b}\varphi\right)(G(\ln U))^{k+1-b},$$

so we see that for $k \in \mathbb{N}_0$

$$(G(\ln U))^{k}\, a^{\#}(\varphi) = \sum_{b=0}^{k} \binom{k}{b} a^{\#}\left((\ln U)^{b}\varphi\right)(G(\ln U))^{k-b} \tag{3.65}$$

holds. Using that, we conclude

$$e^{G(\ln U)} a^{\#}(\varphi) = \sum_{k=0}^{\infty} \frac{1}{k!} (G(\ln U))^{k}\, a^{\#}(\varphi)$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{b=0}^{k} \binom{k}{b} a^{\#}\left((\ln U)^{b}\varphi\right)(G(\ln U))^{k-b}$$

$$= \sum_{c=0}^{\infty} \sum_{l=0}^{\infty} \frac{1}{c!l!} a^{\#}\left((\ln U)^{c}\varphi\right)(G(\ln U))^{l}$$

$$= a^{\#}\left(e^{\ln U}\varphi\right)e^{G(\ln U)} = a^{\#}\left(U\varphi\right)e^{G(\ln U)}.$$

Clearly multiplying the second quantised operator by an additional phase as in (3.4.6) does not influence this calculation.

As a preparation for calculating the vacuum polarisation current we proof the following

Lemma 3.4.9. *Let $P_k, P_l \in Q$ then the following holds*

$$[G(P_k), G(P_l)] = \operatorname{tr}\left(P_{\substack{k\\-+}} P_{\substack{l\\+-}}\right) - \operatorname{tr}\left(P_{\substack{l\\-+}} P_{\substack{k\\+-}}\right) + G\left([P_k, P_l]\right). \quad (3.66)$$

For a proof of this lemma let $P_k, P_l \in Q$, we compute

$$[G(P_k), G(P_l)] \overset{(3.115)}{=}$$
$$= \sum_{n,b\in\mathbb{N}} \left[a^*\left(P_k\varphi_n\right)a(\varphi_n), a^*\left(P_l\varphi_b\right)a(\varphi_n)\right] - \sum_{-b,n\in\mathbb{N}} \left[a^*\left(P_k\varphi_n\right)a(\varphi_n), a(\varphi_b)a^*\left(P_l\varphi_b\right.\right.$$
$$- \sum_{-n,b\in\mathbb{N}} \left[a(\varphi_n)a^*\left(P_k\varphi_n\right), a^*\left(P_l\varphi_b\right)a(\varphi_b)\right] + \sum_{n,b\in\mathbb{N}} \left[a(\varphi_n)a^*\left(P_k\varphi_n\right), a(\varphi_b)a^*\left(P_l\varphi_b\right.\right.$$
$$= \sum_{n,b\in\mathbb{N}} \left(a^*\left(P_k\varphi_n\right)\langle\varphi_n, P_l\varphi_b\rangle a(\varphi_b) - a^*\left(P_l\varphi_b\right)\langle\varphi_b, P_k\varphi_n\rangle a(\varphi_n)\right)$$
$$- \sum_{-b,n\in\mathbb{N}} \left(-\langle\varphi_b, P_k\varphi_n\rangle a(\varphi_n)a^*\left(P_l\varphi_b\right) + a(\varphi_b)a^*\left(P_k\varphi_n\right)\langle\varphi_n, P_l\varphi_b\rangle\right)$$
$$- \sum_{-n,b\in\mathbb{N}} \left(-\langle\varphi_n, P_l\varphi_b\rangle a^*\left(P_k\varphi_n\right)a(\varphi_b) + a^*\left(P_l\varphi_b\right)a(\varphi_n)\langle\varphi_b, P_k\varphi_n\rangle\right)$$
$$+ \sum_{n,b\in-\mathbb{N}} \left(a(\varphi_n)\langle\varphi_b, P_k\varphi_n\rangle a^*\left(P_l\varphi_b\right) - a(\varphi_b)\langle\varphi_n, P_l\varphi_b\rangle a^*\left(P_k\varphi_n\right)\right)$$
$$= \sum_{b\in\mathbb{N}} a^*\left(P_k P_{\substack{l\\++}}\varphi_b\right)a(\varphi_b) - \sum_{n\in\mathbb{N}} a^*\left(P_l P_{\substack{k\\++}}\varphi_n\right)a(\varphi_n)$$
$$+ \sum_{n\in\mathbb{N}} a(\varphi_n)a^*\left(P_l P_{\substack{k\\-+}}\varphi_n\right) - \sum_{-b\in\mathbb{N}} a(\varphi_b)a^*\left(P_k P_{\substack{l\\+-}}\varphi_b\right)$$

$$
\begin{aligned}
&+\sum_{b\in\mathbb{N}} a^*\left(P_k P_{\substack{l\\-+}}\varphi_b\right)a(\varphi_b) - \sum_{-n\in\mathbb{N}} a^*\left(P_l P_{\substack{k\\+-}}\varphi_n\right)a(\varphi_n)\\
&+\sum_{-n\in\mathbb{N}} a(\varphi_n)a^*\left(P_l P_{\substack{k\\--}}\varphi_n\right) - \sum_{-b\in\mathbb{N}} a(\varphi_b)a^*\left(P_k P_{\substack{l\\--}}\varphi_b\right)\\
=&\sum_{n\in\mathbb{N}} a^*\left(P_k P_l\varphi_n\right)a(\varphi_n) - \sum_{n\in\mathbb{N}} a^*\left(P_l P_{\substack{k\\++}}\varphi_n\right)a(\varphi_n)\\
&+\operatorname{tr}\left(P_{\substack{l\\+-}}P_{\substack{k\\-+}}\right) - \sum_{n\in\mathbb{N}} a^*\left(P_l P_{\substack{k\\-+}}\varphi_n\right)a(\varphi_n)\\
&-\operatorname{tr}\left(P_{\substack{l\\-+}}P_{\substack{k\\+-}}\right) + \sum_{-b\in\mathbb{N}} a(\varphi_b)a^*\left(P_l P_{\substack{k\\+-}}\varphi_b\right)\\
&+\sum_{-b\in\mathbb{N}} a(\varphi_b)a^*\left(P_l P_{\substack{k\\--}}\varphi_b\right) - \sum_{-b\in\mathbb{N}} a(\varphi_b)a^*\left(P_k P_l\varphi_b\right)\\
&=\operatorname{tr}\left(P_{\substack{l\\+-}}P_{\substack{k\\-+}}\right) - \operatorname{tr}\left(P_{\substack{l\\-+}}P_{\substack{k\\+-}}\right)\\
&+\sum_{n\in\mathbb{N}} a^*\left(\left[P_k,P_l\right]\varphi_n\right)a(\varphi_n) + \sum_{-b\in\mathbb{N}} a(\varphi_b)a^*\left(\left[P_l,P_k\right]\varphi_b\right)\\
&=\operatorname{tr}\left(P_{\substack{l\\+-}}P_{\substack{k\\-+}}\right) - \operatorname{tr}\left(P_{\substack{l\\-+}}P_{\substack{k\\+-}}\right) + G\left(\left[P_k,P_l\right]\right)
\end{aligned}
$$

□

Definition 3.4.10. *For $k\in\mathbb{N}_0$, $X,Y\in\mathcal{B}(\mathcal{H})$ the nested commutator $[X,Y]_k$ is defined inductively as*

$$
\begin{aligned}
[X,Y]_0 &:= Y\\
[X,Y]_{k+1} &:= [X,[X,Y]_k] \quad \forall k\in\mathbb{N}_0.
\end{aligned}
$$

Lemma 3.4.11. *For $m\in\mathbb{N}$ and $B,C\in Q$ the following holds*

$$
\begin{aligned}
\left[G(B),G(C)\right]_m = \operatorname{tr}\left(P_-BP_+[B,C]_{m-1}\right) - \operatorname{tr}\left(P_+BP_-[B,C]_{m-1}\right)&\\
+\,G\left([B,C]_m\right).& \quad (3.67)
\end{aligned}
$$

Proof: Proof by Induction is the first thing that comes to mind, looking at the claim. Indeed, $m = 1$ is the consequence of the lemma 3.4.9. For m general we have

$$
\begin{aligned}
&[G(B), G(C)]_{m+1} = [G(B), [G(B), G(C)]_m] \\
&\overset{\text{ind.hyp.}}{=} [G(B), \operatorname{tr}(P_- B P_+ [B, C]_{m-1}) - \operatorname{tr}(P_+ B P_- [B, C]_{m-1}) + G([B, C]_m)] \\
&\qquad = [G(B), G([B, C]_m)] \\
&\overset{\text{lemma 3.4.9}}{=} \operatorname{tr}(P_- B P_+ [B, C]_m) - \operatorname{tr}(P_+ B P_- [B, C]_m) + G([B, [B, C]_m]) \\
&= \operatorname{tr}(P_- B P_+ [B, C]_m) - \operatorname{tr}(P_+ B P_- [B, C]_m) + G([B, C]_{m+1}) \qquad (3.68)
\end{aligned}
$$

□

Lemma 3.4.12. *For external potentials A, X small enough the derivatives of the scattering operator can be computed to fulfil*

$$\partial_\varepsilon|_{\varepsilon=0} e^{G \ln U^{A+\varepsilon X}} = e^{G \ln U^A} j^0_A(X) + e^{G \ln U^A} G((U^A)^{-1} \partial_\varepsilon U^{A+\varepsilon X}) \tag{3.69}$$

$$\partial_\varepsilon|_{\varepsilon=0} e^{-G \ln U^{A+\varepsilon X}} = -e^{-G \ln U^A} j^0_A(X) + G(\partial_\varepsilon \left(U^{A+\varepsilon X}\right)^{-1} U^A) e^{-G \ln U^A}, \tag{3.70}$$

with

$$
\begin{aligned}
j^0_A(X) := \sum_{l \in \mathbb{N}_0} \frac{(-1)^{l+1}}{(l+2)!} \Big(&\operatorname{tr} P_- \ln U^A P_+ \left[\ln U^A, \partial_\varepsilon \ln U^{A+\varepsilon X}\right]_l \\
&- \operatorname{tr} P_+ \ln U^A P_- \left[\ln U^A, \partial_\varepsilon \ln U^{A+\varepsilon X}\right]_l \Big). \qquad (3.71)
\end{aligned}
$$

Proof: We start out by employing Duhamel's and Hadamard's for-

ref!! + restrictions, something better than this

s. These are

$$\partial_\alpha \, e^{Y+\alpha X}\Big|_{\alpha=0} = \int_0^1 dt e^{(1-t)Y} X e^{tY} \qquad \text{(Duhamel's formula)}$$

and

$$e^X Y e^{-X} = \sum_{k=0}^{\infty} \frac{1}{k!}[X,Y]_k. \qquad \text{(Hadamard's formula)}$$

So one gets

$$\begin{aligned}
\partial_\varepsilon|_{\varepsilon=0} e^{G \ln U^{A+\varepsilon X}} &= \int_0^1 dz e^{(1-z)G\ln U^A} \partial_\varepsilon|_{\varepsilon=0} G \ln U^{A+\varepsilon X} e^{zG\ln U^A} \qquad (3.72)\\
&= e^{G\ln U^A} \int_0^1 dz \sum_{l\in\mathbb{N}_0} \frac{1}{l!} \left[-zG\ln U^A, \partial_\varepsilon|_{\varepsilon=0} G\ln U^{A+\varepsilon X}\right]_l \\
&= e^{G\ln U^A} \int_0^1 dz \sum_{l\in\mathbb{N}_0} \frac{(-z)^l}{l!} \partial_\varepsilon|_{\varepsilon=0} \left[G\ln U^A, G\ln U^{A+\varepsilon X}\right]_l .
\end{aligned}$$

At this point we see that for $l = 0$ the summand vanishes. For all other values of l we use lemma 3.4.11, yielding

$$\begin{aligned}
\partial_\varepsilon|_{\varepsilon=0} e^{G\ln U^{A+\varepsilon X}} = e^{G\ln U^A} \int_0^1 dz \sum_{l\in\mathbb{N}} \frac{(-z)^l}{l!} \partial_\varepsilon|_{\varepsilon=0} \Big(G\left(\left[\ln U^A, \ln U^{A+\varepsilon X}\right]\right) \\
+ \operatorname{tr} P_- \ln U^A P_+ \left[\ln U^A, \partial_\varepsilon|_{\varepsilon=0} \ln U^{A+\varepsilon X}\right]_{l-1} \\
- \operatorname{tr} P_+ \ln U^A P_- \left[\ln U^A, \partial_\varepsilon|_{\varepsilon=0} \ln U^{A+\varepsilon X}\right]_{l-1}\Big) . \qquad (3.73)
\end{aligned}$$

The last two terms together result in the first term of (3.69) after performing the integration and shifting the summation index. For the first term we will use linearity and continuity of G and use the same identities backwards to give

contiuity of G!!

$$
\begin{aligned}
&e^{G \ln U^A} \int_0^1 dz \sum_{l \in \mathbb{N}} \frac{(-z)^l}{l!} \partial_\varepsilon|_{\varepsilon=0} G\left(\left[\ln U^A, \ln U^{A+\varepsilon X}\right]\right) \\
&\quad = e^{G \ln U^A} G\left(\int_0^1 dz \sum_{l \in \mathbb{N}} \frac{1}{l!}\left[-z \ln U^A, \partial_\varepsilon|_{\varepsilon=0} \ln U^{A+\varepsilon X}\right]\right) \\
&\quad = e^{G \ln U^A} G\left(e^{-\ln U^A} \int_0^1 dz e^{\ln U^A} e^{-z \ln U^A} \partial_\varepsilon|_{\varepsilon=0} \ln U^{A+\varepsilon X} e^{z \ln U^A}\right) \\
&\quad = e^{G \ln U^A} G\left(e^{\ln\left(U^A\right)^{-1}} \partial_\varepsilon|_{\varepsilon=0} e^{\ln U^{A+\varepsilon X}}\right) \\
&\quad = e^{G \ln U^A} G\left(\left(U^A\right)^{-1} \partial_\varepsilon|_{\varepsilon=0} U^{A+\varepsilon X}\right). \qquad (3.74)
\end{aligned}
$$

Putting things together results in the first equality we wanted to prove. For the second one the computation is completely analogous, except for after applying Duhamel's formula as in (3.72) we substitute $u = 1-z$. The minus sign in front of the first term then arises by the chain rule, where as the second term does not share the sign change with the first, since we have to revert the use of the chain rule in the second half of the calculation when we apply (Duhamel's formula) backwards. □

Definition 3.4.13. *We use Bogoliubov's formula to define the vacuum* ref!! *[redacted]tation value of the current*

$$
j_A(F) = i\partial_\varepsilon \left.\langle \Omega, S^{A^*} S^{A+\varepsilon F} \Omega \rangle\right|_{\varepsilon=0}. \qquad (3.75)
$$

Theorem 3.4.14. *The vacuum expectation value of the current of the* *scattering operator takes the form*

$$
\begin{aligned}
& j_A(F) = -\partial_\varepsilon\, \varphi(A+\varepsilon F)|_{\varepsilon=0} \\
& -2\int_0^1 dz(1-z)\Im\,\mathrm{tr}\left(P_+ \ln U^A P_- e^{-z\ln U^A}\partial_\varepsilon \ln U^{A+\varepsilon F}\big|_{\varepsilon=0} e^{z\ln U^A}\right) \\
& \qquad\qquad = -\partial_\varepsilon\, \varphi(A+\varepsilon F)|_{\varepsilon=0} \\
& -2\Im\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+2)!}\,\mathrm{tr}\left(P_+ \ln U^A P_- \left[\ln U^A, \partial_\varepsilon \ln U^{A+\varepsilon F}\big|_{\varepsilon=0}\right]_k\right)
\end{aligned}
$$

Proof: By theorem 3.4.6 and abbreviating $\varphi(A) = \sum_{n\in\mathbb{N}} \frac{C_n(A)}{n!}$we see that the current can be written in the form

$$
\begin{aligned}
j_A(F) &= i\partial_\varepsilon \left.\langle \Omega, S^{A^*} S^{A+\varepsilon F}\Omega\rangle\right|_{\varepsilon=0} \\
&= i\partial_\varepsilon \left.\langle \Omega, e^{-i\varphi(A)} e^{-G(\ln(U^A))} e^{i\varphi(A+\varepsilon F)} e^{G(\ln(U^{A+\varepsilon F}))}\Omega\rangle\right|_{\varepsilon=0} \\
&= -\partial_\varepsilon\, \varphi(A+\varepsilon F)|_{\varepsilon=0} + i\langle \Omega, \partial_\varepsilon\, e^{-G(\ln(U^A))} e^{G(\ln(U^{A+\varepsilon F}))}\Omega\rangle\Big|_{\varepsilon=0},
\end{aligned} \tag{3.76}
$$

so the first summand works out just as claimed. For the second summand we employ lemma 3.4.12 and note that the vacuum expectation value of G vanishes no matter its argument.

$$
\begin{aligned}
& i\langle \Omega, \partial_\varepsilon\, e^{-G(\ln(U^A))} e^{G(\ln(U^{A+\varepsilon F}))}\Omega\rangle\Big|_{\varepsilon=0} \\
&\quad = -i\partial_\varepsilon \sum_{k=0}^{\infty}\frac{(-1)^k}{(k+2)!}\,\mathrm{tr}\left(P_- \ln U^A P_+ [\ln U^A, \ln U^{A+\varepsilon F}]_k\right)\Big|_{\varepsilon=0} \\
&\quad\ + i\partial_\varepsilon \sum_{k=0}^{\infty}\frac{(-1)^k}{(k+2)!}\,\mathrm{tr}\left(P_+ \ln U^A P_- [\ln U^A, \ln U^{A+\varepsilon F}]_k\right)\Big|_{\varepsilon=0}
\end{aligned} \tag{3.77}
$$

In order to apply Hadamard's formula once again in the opposite direction, we introduce two auxiliary integrals. The second term then becomes

$$
\begin{aligned}
& i\partial_\varepsilon \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+2)!} \operatorname{tr}\left(P_+ \ln U^A P_- [\ln U^A, \ln U^{A+\varepsilon F}]_k\right)\Big|_{\varepsilon=0} \\
&= i\partial_\varepsilon \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_0^1 dt \int_0^1 s^k t^{k+1} \operatorname{tr}\left(P_+ \ln U^A P_- [\ln U^A, \ln U^{A+\varepsilon F}]_k\right)\Big|_{\varepsilon=0} \\
&= i\partial_\varepsilon \sum_{k=0}^{\infty} \frac{1}{k!} \int_0^1 dt \int_0^1 ds\, t \operatorname{tr}\left(P_+ \ln U^A P_- [-ts \ln U^A, \ln U^{A+\varepsilon F}]_k\right)\Big|_{\varepsilon=0} \\
&= i\partial_\varepsilon \sum_{k=0}^{\infty} \frac{1}{k!} \int_0^1 dz \int_z^1 ds \;\operatorname{tr}\left(P_+ \ln U^A P_- [-z \ln U^A, \ln U^{A+\varepsilon F}]_k\right)\Big|_{\varepsilon=0} \\
&= i\partial_\varepsilon \sum_{k=0}^{\infty} \frac{1}{k!} \int_0^1 dz(1-z) \operatorname{tr}\left(P_+ \ln U^A P_- [-z \ln U^A, \ln U^{A+\varepsilon F}]_k\right)\Big|_{\varepsilon=0} \\
&= i\partial_\varepsilon \int_0^1 dz(1-z) \operatorname{tr}\left(P_+ \ln U^A P_- \sum_{k=0}^{\infty} \frac{1}{k!} [-z \ln U^A, \ln U^{A+\varepsilon F}]_k\right)\Big|_{\varepsilon=0} \\
&\overset{\text{(Hadamard's formula)}}{=} i\partial_\varepsilon \int_0^1 dz(1-z) \operatorname{tr}\left(P_+ \ln U^A P_- e^{-z \ln U^A} \ln U^{A+\varepsilon F} e^{z \ln U^A}\right)\Big|_{\varepsilon=0} \\
&= i \int_0^1 dz(1-z) \operatorname{tr}\left(P_+ \ln U^A P_- e^{-z \ln U^A} \partial_\varepsilon \ln U^{A+\varepsilon F}\big|_{\varepsilon=0} e^{z \ln U^A}\right).
\end{aligned}
\tag{3.78}
$$

The calculation for the first term of (3.77) is identical. At this point we notice that (3.78) and the term where the projectors are exchanged are complex conjugates of one another. So summarising we find

$$\begin{aligned}
&j_A(F) = -\partial_\varepsilon\, \varphi(A+\varepsilon F)|_{\varepsilon=0} \\
&- 2\int_0^1 dz(1-z)\Im\operatorname{tr}\left(P_+ \ln U^A P_- e^{-z\ln U^A}\partial_\varepsilon \ln U^{A+\varepsilon F}\big|_{\varepsilon=0} e^{z\ln U^A}\right).
\end{aligned}$$

□

Theorem 3.4.15. *Independent of the phase that is used to correct the scattering operator the following formula holds true for any four potentials A, F, H, with A small enough so that the relevant series converge.*

$$\begin{aligned}
&\partial_\varepsilon|_{\varepsilon=0}(j_{A+\varepsilon H}(F) - j_{A+\varepsilon F}(H)) = \\
&\qquad 2\Im\operatorname{tr}\left(P_+ \left(U^A\right)^{-1}\partial_\varepsilon|_{\varepsilon=0}U^{A+\varepsilon F}P_- \left(U^A\right)^{-1}\partial_\delta|_{\delta=0}U^{A+\delta H}\right)
\end{aligned} \tag{3.79}$$

Proof: We compute $\partial_\varepsilon|_{\varepsilon=0}j_{A+\varepsilon F}(H)$.

$$\begin{aligned}
&-i\partial_\varepsilon|_{\varepsilon=0}j_{A+\varepsilon H}(F) = \\
&\partial_\varepsilon|_{\varepsilon=0}\partial_\delta|_{\delta=0}\langle\Omega, e^{i\varphi(A+\varepsilon H+\delta F)-i\varphi(A+\varepsilon H)}e^{-G\ln U^{A+\varepsilon H}}e^{G\ln U^{A+\varepsilon H+\delta F}}\Omega\rangle
\end{aligned}$$

We first act with the derivative with respect to H, fixing F.

$$\begin{aligned}
&-i\partial_\varepsilon|_{\varepsilon=0}j_{A+\varepsilon H}(F) = \\
&\partial_\delta|_{\delta=0}i(\partial_\varepsilon|_{\varepsilon=0}\varphi(A+\varepsilon H+\delta F) - \partial_\varepsilon|_{\varepsilon=0}\varphi(A+\varepsilon H))e^{i\varphi(A+\delta F)-i\varphi(A)} \\
&\langle\Omega, e^{-G\ln U^A}e^{G\ln U^{A+\delta F}}\Omega\rangle \\
&+\partial_\delta|_{\delta=0}e^{i\varphi(A+\delta F)-i\varphi(A)}\langle\Omega\partial_\varepsilon|_{\varepsilon=0}e^{-G\ln U^{A+\varepsilon H}}e^{G\ln U^{A+\delta F}}\Omega\rangle \\
&+\partial_\delta|_{\delta=0}e^{i\varphi(A+\delta F)-i\varphi(A)}\langle\Omega e^{-G\ln U^A}\partial_\varepsilon|_{\varepsilon=0}e^{G\ln U^{A+\varepsilon H+\delta F}}\Omega\rangle
\end{aligned}$$

In computing further one can notice a few cancellations. For the first summand the first factor vanishes if δ is set to zero, so the only the first summand in the product rule will not vanish. For the second and third summand we will use lemma 3.4.12, giving

$$
\begin{aligned}
&-i\partial_\varepsilon|_{\varepsilon=0}j_{A+\varepsilon H}(F)=\\
&\partial_\delta|_{\delta=0}i\partial_\varepsilon|_{\varepsilon=0}\varphi(A+\varepsilon H+\delta F)\\
&-\partial_\delta|_{\delta=0}e^{i\varphi(A+\delta F)-i\varphi(A)}j^0_A(H)\langle\Omega,e^{-G\ln U^A}e^{G\ln U^{A+\delta F}}\Omega\rangle\\
&+\partial_\delta|_{\delta=0}e^{i\varphi(A+\delta F)-i\varphi(A)}\langle\Omega,G\left(\partial_\varepsilon|_{\varepsilon=0}\left(U^{A+\varepsilon H}\right)^{-1}U^A\right)e^{-G\ln U^A}e^{G\ln U^{A+\delta F}}\Omega\rangle\\
&+\partial_\delta|_{\delta=0}e^{i\varphi(A+\delta F)-i\varphi(A)}j^0_{A+\delta F}(H)\langle\Omega,e^{-G\ln U^A}e^{G\ln U^{A+\delta F}}\Omega\rangle\\
&+\partial_\delta|_{\delta=0}e^{i\varphi(A+\delta F)-i\varphi(A)}\langle\Omega,e^{-G\ln U^A}e^{G\ln U^{A+\delta F}}G\left(\left(U^{A+\delta F}\right)^{-1}\partial_\varepsilon|_{\varepsilon=0}U^{A+\varepsilon H+\delta F}\right)\Omega\rangle.
\end{aligned}
$$

Now there are a few further simplifications to appreciate: since $\langle\Omega,G\Omega\rangle=0$, in the third and last summand only the derivatives with respect to δ which produce by lemma 3.4.12 another factor of G will contribute to the sum. For the other summands except for the first we can spot the appearance of j^0. Respecting all this results in

$$
\begin{aligned}
&-i\partial_\varepsilon|_{\varepsilon=0}j_{A+\varepsilon H}(F)=i\partial_\delta|_{\delta=0}\partial_\varepsilon|_{\varepsilon=0}\varphi(A+\varepsilon H+\delta F)\\
&-i\partial_\delta|_{\delta=0}\varphi(A+\delta F)j^0_A(H)-j^0_A(H)j^0_A(F)\\
&+\langle\Omega,G\left(\partial_\varepsilon|_{\varepsilon=0}\left(U^{A+\varepsilon H}\right)^{-1}U^A\right)G\left(\left(U^A\right)^{-1}\partial_\delta|_{\delta=0}U^{A+\delta F}\right)\Omega\rangle\\
&+i\partial_\delta|_{\delta=0}\varphi(A+\delta F)j^0_A(H)+\partial_\delta|_{\delta=0}j^0_{A+\delta F}(H)+j^0_A(H)j^0_A(F)\\
&+\langle\Omega,G\left(\left(U^A\right)^{-1}\partial_\delta|_{\delta=0}U^{A+\delta F}\right)G\left(\left(U^A\right)^{-1}\partial_\varepsilon|_{\varepsilon=0}U^{A+\varepsilon H}\right)\Omega\rangle.
\end{aligned}
$$

A few more terms cancel in the second and fourth line, also since $\partial_\varepsilon|_{\varepsilon=0}\left(U^{A+\varepsilon H}\right)^{-1}U^{A+\varepsilon H}=0$ we can combine the two products of G into a commutator:

$$
\begin{aligned}
&- i\partial_\varepsilon|_{\varepsilon=0} j_{A+\varepsilon H}(F) = i\partial_\delta|_{\delta=0}\partial_\varepsilon|_{\varepsilon=0}\varphi(A+\varepsilon H+\delta F)\\
&+ \partial_\delta|_{\delta=0} j^0_{A+\delta F}(H)\\
&+ \langle\Omega, \left[G\left(\left(U^A\right)^{-1}\partial_\delta|_{\delta=0}U^{A+\delta F}\right), G\left(\left(U^A\right)^{-1}\partial_\varepsilon|_{\varepsilon=0}U^{A+\varepsilon H}\right)\right]\Omega\rangle.
\end{aligned}
$$

So we can once again apply lemma 3.4.9, which results in exactly right hand side of the equation we claimed to produce in the statement of this theorem. So all that is left is to recognise that one can combine the first two summands into $-i\partial_\varepsilon j_{A+\varepsilon H}(F)$, which is a direct consequence of theorem 3.4.14. □

3.5 Quantitative Estimates

Since we do not only want to give an expression for the time evolution operator, but also give bounds on the numerical errors which are due to truncate the occurring series we need to look at these series a little closer. The series involve powers of the second quantisation operator G, so we start by examining these in greater depth. In order to do so we define an object closely related to G.

Definition 3.5.1.

$$
L : \{M \subset B(\mathcal{H}) \mid |M| < \infty\} \times \{M \subset B(\mathcal{H}) \mid |M| < \infty\} \to B(\mathcal{F})
$$

$$
\begin{aligned}
L(\{A_1,\dots,A_c\},\{B_1,\dots,B_m\}) := \prod_{l=1}^{m} a(\varphi_{-k_l})\\
\prod_{l=1}^{c} a^*(A_l\varphi_{n_l}) \prod_{l=1}^{m} a^*(B_l\varphi_{-k_l}) \prod_{l=1}^{c} a(\varphi_{n_l}),
\end{aligned}
\tag{3.80}
$$

where for notational reasons we chose to list the occurring one-particle operators in a specific order; however, the order does not matter, since *commutation of the relevant creation and annihilation operators always results an overall factor of one.*

Since this operator L occurs when computing powers of G we compute is product with some G with the following

Lemma 3.5.2. *For any $a, b, \in \mathbb{N}_0$ and apropriate one particle operators A_k, B_l, C for $1 \le k \le a$, $1 \le l \le b$ we have the following equality*

$$L\Big(\bigcup_{l=1}^{a}\{A_l\};\bigcup_{l=1}^{b}\{B_l\}\Big)G(C) = \tag{3.81}$$

$$(-1)^{a+b}L\Big(\bigcup_{l=1}^{a}\{A_l\}\cup\{C\};\bigcup_{l=1}^{b}\{B_l\}\Big) \tag{3.82}$$

$$+(-1)^{a+b+1}L\Big(\bigcup_{l=1}^{a}\{A_l\};\bigcup_{l=1}^{b}\{B_l\}\cup\{C\}\Big) \tag{3.83}$$

$$+\sum_{f=1}^{a}L\Big(\bigcup_{\substack{l=1\\ l\neq f}}^{a}\{A_l\}\cup\{A_f P_+ C\};\bigcup_{l=1}^{b}\{B_l\}\Big) \tag{3.84}$$

$$+\sum_{f=1}^{a}L\Big(\bigcup_{\substack{l=1\\ f\neq l}}^{a}\{A_l\}\cup\{-CP_- A_f\};\bigcup_{l=1}^{b}\{B_l\}\Big) \tag{3.85}$$

$$-\sum_{f=1}^{a}L\Big(\bigcup_{\substack{l=1\\ f\neq l}}^{a}\{A_l\};\bigcup_{l=1}^{b}\{B_l\}\cup\{A_f P_+ C\}\Big) \tag{3.86}$$

$$+\sum_{f=1}^{b}L\Big(\bigcup_{l=1}^{a}\{A_l\};\bigcup_{\substack{l=1\\ l\neq f}}^{b}\{B_l\}\cup\{-CP_- B_f\}\Big) \tag{3.87}$$

$$+ (-1)^{a+b+1} \sum_{f=1}^{a} \operatorname{tr}\Big(P_+ C P_- A_f\Big) L\Big(\bigcup_{\substack{l=1\\ l\neq f}}^{a}\{A_l\};\bigcup_{l=1}^{b}\{B_l\}\Big) \tag{3.88}$$

$$+ (-1)^{a+b+1} \sum_{\substack{f_1,f_2=1\\ f_1\neq f_2}}^{a} L\Big(\bigcup_{\substack{l=1\\ l\neq f_1,f_2}}^{a}\{A_l\}\cup\{-A_{f_2}P_+CP_-A_{f_1}\};\bigcup_{l=1}^{b}\{B_l\}\Big) \tag{3.89}$$

$$+ (-1)^{a+b+1} \sum_{f=1}^{b}\sum_{g=1}^{a} L\Big(\bigcup_{\substack{l=1\\ l\neq g}}^{a}\{A_l\};\bigcup_{\substack{l=1\\ l\neq f}}^{b}\{B_l\}\cup\{-A_gP_+CP_-B_f\}\Big). \tag{3.90}$$

Proof: The proof of this equality is a rather long calculation, where (3.80) is used repeatedly. We break up the calculation into several parts. Let us start with

$$L\left(\bigcup_{l=1}^{a}\{A_l\};\bigcup_{l=1}^{b}\{B_l\}\right) L(C;) = \\ \prod_{l=1}^{b} a(\varphi_{-k_l}) \prod_{l=1}^{a} a^*(A_l\varphi_{n_l}) \prod_{l=1}^{b} a^*(B_l\varphi_{-k_l}) \prod_{l=1}^{a} a(\varphi_{n_l}) a^*(C\varphi_m) a(\varphi_m). \tag{3.91}$$

We (anti)commute the creation operator involving C to its place at the end of the second product, after that the term will be normally ordered and can be rephrased in terms of Ls. During the commutation the creation operator in question can be picked up by any of the annihilation operators in the rightmost product. For each term where that happens we can perform the sum over the basis of $\mathcal{H}^-$ related to the annihilation operator whose anticommutator triggered. After this sum the corresponding term is also normally ordered and can be rephrased in terms of an L after some reshuffling which may only produce signs. So performing these steps we get

$$\begin{aligned}
&L\left(\bigcup_{l=1}^{a}\{A_l\};\bigcup_{l=1}^{b}\{B_l\}\right)L(C;)= \\
&\qquad \sum_{f=a}^{1}(-1)^{a-f}\prod_{l=1}^{b}a(\varphi_{-k_l})\prod_{l=1}^{f-1}a^*(A_l\varphi_{n_l})a^*(A_fP_+C\varphi_m) \\
&\qquad\quad \prod_{l=f+1}^{a}a^*(A_l\varphi_{n_l})\prod_{l=1}^{b}a^*(B_l\varphi_{-k_l})\prod_{\substack{l=1\\l\neq f}}^{a}a(\varphi_{n_l})a(\varphi_m) \\
&\qquad\qquad + L\left(\bigcup_{l=1}^{a}\{A_l\}\cup\{C\};\bigcup_{l=1}^{b}\{B_l\}\right) \\
&\qquad = \sum_{f=1}^{a}L\left(\bigcup_{\substack{l=1\\l\neq f}}^{a}\{A_l\}\cup\{A_fP_+C\};\bigcup_{l=1}^{b}\{B_l\}\right) \\
&\qquad\qquad + L\left(\bigcup_{l=1}^{a}\{A_l\}\cup\{C\};\bigcup_{l=1}^{b}\{B_l\}\right). \qquad (3.92)
\end{aligned}$$

Now the remaining case is more laborious, that is why we will split off and treat some of the appearing terms separately. We start off analogous to before

$$L\left(\bigcup_{l=1}^{a}\{A_l\};\bigcup_{l=1}^{b}\{B_l\}\right)L(;C)=$$

$$\prod_{l=1}^{b} a(\varphi_{-k_l}) \prod_{l=1}^{a} a^*(A_l\varphi_{n_l}) \prod_{l=1}^{b} a^*(B_l\varphi_{-k_l}) \prod_{l=1}^{a} a(\varphi_{n_l}) a(\varphi_{-m}) a^*(C\varphi_{-m}). \tag{3.93}$$

This time we need to (anti)commute the rightmost annihilation operator all the way to the end of the first product and the creation operator to the end of the second but last product. So there will be several qualitatively different terms. From the first step alone we get

$$L\left(\bigcup_{l=1}^{a}\{A_l\};\bigcup_{l=1}^{b}\{B_l\}\right) L(;C) =$$

$$(-1)^a \sum_{f=b}^{1} (-1)^{b-f} \prod_{l=1}^{b} a(\varphi_{-k_l}) \prod_{l=1}^{a} a^*(A_l\varphi_{n_l}) \prod_{\substack{l=1\\ l\neq f}}^{b} a^*(B_l\varphi_{-k_l}) \prod_{l=1}^{a} a(\varphi_{n_l}) a^*(CP_- B_f\varphi_{-k_f}) \tag{3.94}$$

$$+(-1)^{a+b} \sum_{f=a}^{1} (-1)^{b-f} \prod_{l=1}^{b} a(\varphi_{-k_l}) \prod_{\substack{l=1\\ l\neq f}}^{a} a^*(A_l\varphi_{n_l}) \prod_{l=1}^{b} a^*(B_l\varphi_{-k_l}) \prod_{l=1}^{a} a(\varphi_{n_l}) a^*(CP_-\varphi_{n_f}) \tag{3.95}$$

$$+(-1)^b \prod_{l=1}^{b} a(\varphi_{-k_l}) a(\varphi_{-m}) \prod_{l=1}^{a} a^*(A_l\varphi_{n_l}) \prod_{l=1}^{b} a^*(B_l\varphi_{-k_l}) \prod_{l=1}^{a} a(\varphi_{n_l}) a^*(C\varphi_{-m}). \tag{3.96}$$

We will discuss terms (3.94), (3.95) and (3.96) separately. In Term (3.94) we need to commute the last creation operator into its place in the third product, it can be picked up by one of the annihilation operators of the last product, but after performing the sum over the corresponding basis the resulting term can be rephrased in terms of an L operator by commuting only creation operators of the second and third product. Performing these steps yields the identity

$$\begin{aligned}(3.94) = &\sum_{f=1}^{b} L\left(\bigcup_{l=1}^{a}\{A_l\};\bigcup_{\substack{l=1\\l\neq f}}^{b}\{B_l\}\cup\{CP_-B_f\}\right)\\ &+(-1)^{a+b+1}\sum_{f=1}^{b}\sum_{g=1}^{a} L\left(\bigcup_{\substack{l=1\\l\neq g}}^{a}\{A_l\};\bigcup_{\substack{l=1\\l\neq f}}^{b}\{B_l\}\{A_gP_+CP_-B_f\}\right). \quad (3.97)\end{aligned}$$

For (3.95) the last creation operator needs to be commuted to the end of the second product. It can be picked up by one of the annihilation operators of the last product, but here we have to distinguish between two cases. If the index of this annihilation operator equals f the resulting commutator will be $\operatorname{tr} P_+CP_-A_f$ otherwise one can again perform the sum over the corresponding index and express the whole Product in terms of an L operator. All this results in

$$\begin{aligned}(3.95) = &\sum_{f=1}^{a} L\left(\bigcup_{\substack{l=1\\l\neq f}}^{a}\{A_l\}\cup\{CP_-A_f\};\bigcup_{l=1}^{b}\{B_l\}\right)\\ &+(-1)^{a+b}\sum_{f=1}^{a} L\left(\bigcup_{\substack{l=1\\l\neq f}}^{a}\{A_l\};\bigcup_{l=1}^{b}\{B_l\}\right)\operatorname{tr}(P_+CP_-A_f)\end{aligned}$$

$$+(-1)^{a+b+1}\sum_{\substack{f_1,f_2=1\\f_1\neq f_2}}^{a} L\left(\bigcup_{\substack{l=1\\l\neq f_1,f_2}}^{a}\{A_l\}\cup\{A_{f_2}P_+CP_-A_{f_1}\};\bigcup_{l=1}^{b}\{B_l\}\right). \tag{3.98}$$

For (3.96) the procedure is basically the same as for (3.94), it results in

$$\begin{aligned}(3.96) = (-1)^{a+b}L&\left(\bigcup_{l=1}^{a}\{A_l\};\bigcup_{l=1}^{b}\{B_l\}\right)\\ &+\sum_{f=1}^{a} L\left(\bigcup_{\substack{l=1\\l\neq f}}^{a}\{A_l\}\cup\{CP_-A_f\};\bigcup_{l=1}^{b}\{B_l\}\cup\{A_fP_+C\}\right).\end{aligned} \tag{3.99}$$

Putting the results of the calculation together results in the claimed equation, after pulling in some factors of -1 into L. □

We carry on with defining the important quantities for powers of G. First we introduce for each $k\in\mathbb{N}$ a linear bounded operator on $\mathcal{H}$, X_k which fulfils $\operatorname{tr} P_+X_kP_-X_k<\infty \wedge \operatorname{tr} P_-X_kP_+X_k<\infty$.

Definition 3.5.3. *Let*

$$Y := \{X_k \mid k\in\mathbb{N}\}.$$

Let for $n\in\mathbb{N}$

$$\langle n\rangle := \{X_l \mid l\in\mathbb{N}, l\le n\}.$$

Definition 3.5.4. *Let for* $b\subset Y$*, such that* $|b|<\infty$

$$\begin{aligned} f_b : \{l\in\mathbb{N}\mid l\le |b|\}\to b\\ \forall k<|b| : f_b(k)=X_l\wedge f_b(k+1)=X_m \to l<m\end{aligned} \tag{3.100}$$

Definition 3.5.5. *We denote for any set b by $S(b)$ the symmetric group (group of permutations) over b.*

Definition 3.5.6. *Let for $b \subset Y$, such that $|b| < \infty$ and $\sigma_b \in S(b)$*

$$VZ^{l}_{\sigma_b} : \{k \in \mathbb{N} \mid k < |b|\} \to \{-1, 1\}$$
$$VZ^{l}_{\sigma_b}(k) := sgn\big[f_b^{-1}(\sigma_b(f_b(k+1))) - f_b^{-1}(\sigma_b(f_b(k)))\big]$$

In what is to follow the order of one particle operators will be changed in all possible ways, to keep track of this by use of a compact notation we introduce

Definition 3.5.7.

$$W : \{(b, \sigma_b) \mid b \subseteq Y \wedge |b| < \infty \wedge \sigma_b \in S(b)\} \to B(\mathcal{H})$$
$$W(b, \sigma_b) := \left(\prod_{k=1}^{|b|-1} \sigma_b(f_b(k)) P_{VZ^{b}_{\sigma_b}(k)} VZ^{b}_{\sigma_b}(k) \right) \sigma_b(f_b(|b|))$$

Definition 3.5.8. *Let l be any finite subset of Y. Denote by X^l_{max} the operator $X_k \in l$ such that for any $X_c \in l$ the relation $k \geq c$ is fulfilled. Furthermore define*

$$PT : \{T \subset \mathcal{P}(Y) \mid |T| < \infty\} \to \mathbb{C}$$
$$\textit{for: } T = \emptyset : PT(T) = 1, \ \textit{otherwise:}$$
$$PT(T) = \sum_{\substack{\forall l \in T: \\ \sigma_l \in S(l \setminus \{X^l_{max}\})}} \prod_{l \in T} \mathrm{tr}[P_+ X^l_{max} P_- W(l, \sigma_l)]$$

There is one more function left to define

Definition 3.5.9.

$$Op : \{R \in \mathcal{P}(Y) \mid |R| < \infty\} \times \{D \subset \mathcal{P}(Y) \mid |T| < \infty\} \to \mathcal{B}(\mathcal{F})$$
$$Op(R, D) = \sum_{\substack{\forall l \in D: \\ \sigma_l \in S(l)}} \sum_{a \subseteq R \cup \bigcup_{l \in D} \{W(l, \sigma_l)\}} L(a, a^c)(-1)^{|a| + \frac{(|R|+|D|)(|R|+|D|+1)}{2}}$$

Now we are able to state the main theorem which will help us do quantitative estimates.

Theorem 3.5.10. *Let $n \in \mathbb{N}$, $X_1, \dots, X_n \in Y$ then the following equation holds*

$$\prod_{k=1}^{n} G(X_k) = \sum_{\substack{\langle n\rangle = \dot\bigcup_{l\in T} l \,\dot\cup\, \dot\bigcup_{l\in D} l \,\dot\cup\, R \\ \forall l\in T\cup D: |l|\geq 2}} PT(T)Op(R,D), \tag{3.101}$$

where the abbreviation $\langle n\rangle := \{X_k \mid k \leq n\}$ was used.

Proof: The proof will be by induction on n. Since the formula in the claim reduces to 1 for $n = 0$ we will not spend any more time on the start of the induction. The general strategy of the proof is to break up the right hand side of (3.101) for $n+1$ into small pieces and show for each piece that it corresponds to one of the contributions of lemma 3.5.2, while also each term in this lemma is represented by one of the terms obtained by breaking up (3.101).
As a first step we break the right hand side of (3.101) into three pieces separated by in which set X_{n+1} ends up in :

$$\sum_{\substack{\langle n+1\rangle = \dot\bigcup_{l\in T} l \,\dot\cup\, \dot\bigcup_{l\in D} l \,\dot\cup\, R \\ \forall l\in T\cup D: |l|\geq 2}} \mathrm{PT}(T)\mathrm{Op}(R,D) =$$

$$\sum_{\substack{\langle n+1\rangle = \dot\bigcup_{l\in T} l \,\dot\cup\, \dot\bigcup_{l\in D} l \,\dot\cup\, R \\ \exists l\in T: X_{n+1}\in l \\ \forall l\in T\cup D: |l|\geq 2}} \mathrm{PT}(T)\mathrm{Op}(R,D) \tag{3.102}$$

$$+ \sum_{\substack{\langle n+1\rangle = \dot\bigcup_{l\in T} l \,\dot\cup\, \dot\bigcup_{l\in D} l \,\dot\cup\, R \\ \exists l\in D: X_{n+1}\in l \\ \forall l\in T\cup D: |l|\geq 2}} \mathrm{PT}(T)\mathrm{Op}(R,D) \tag{3.103}$$

$$+ \sum_{\substack{\langle n+1\rangle = \dot\bigcup_{l\in T} l \,\dot\cup\, \dot\bigcup_{l\in D} l \,\dot\cup\, R \\ X_{n+1}\in R \\ \forall l\in T\cup D: |l|\geq 2}} \mathrm{PT}(T)\mathrm{Op}(R, D), \tag{3.104}$$

We will discuss each term separately. For term (3.102) the term containing X_{n+1} is in one of the elements l' of T, but each such element has to have more than one element. So if we were to sum over the partitions of $\langle n\rangle$ instead, the rest of $l'\backslash\{X_{n+1}\}$ is either an element of D or, if it contains only one element, of R. Picking D instead of T is at this stage an arbitrary choice, but this choice leads to the terms of lemma 3.5.2. Alls this means that one correct rewriting of term (3.102) is

$$(3.102) = \sum_{\substack{\langle n\rangle = \dot\bigcup_{l\in T} l \,\dot\cup\, \dot\bigcup_{l\in D} l \,\dot\cup\, R \\ \forall l\in D\cup T: |l|>2}} \sum_{b\in D\cup\{\{r\}|r\in R\}} \mathrm{PT}(T\cup\{\{X_{n+1}\cup f\}) \mathrm{Op}(R\backslash b, D\backslash\{b\}). \tag{3.105}$$

Next we pull one factor and the corresponding sum out of PT and write out Op. Then we see that the sums over permutations can be merged into one. There we take the convention that for any set f such that $|f| = 1$ holds, we define σ_f to be the identity on that set.
This results in

$$(3.105) = \sum_{\substack{\langle n\rangle = \dot\bigcup_{l\in T} l \,\dot\cup\, \dot\bigcup_{l\in D} l \,\dot\cup\, R \\ \forall l\in D\cup T: |l|>2}} \sum_{b\in D\cup\{\{r\}|r\in R\}} \sum_{\sigma_b\in S(b)} \mathrm{tr}[P_+ X_{n+1} P_- W(b,\sigma_b)] \mathrm{PT}(T) \sum_{\substack{\forall l\in D\backslash\{b\} \\ \sigma_l\in S(l)}}$$

$$\begin{aligned}
&\sum_{a\subseteq R\backslash b\cup\bigcup_{l\in D\backslash\{b\}}\{W(l,\sigma_l)\}} L(a,a^c)(-1)^{|a|+\frac{(|R|+|D|-1)(|R|+|D|)}{2}} \\
&\quad= \sum_{\substack{\langle n\rangle=\dot\bigcup_{l\in T} l\,\dot\cup\,\dot\bigcup_{l\in D} l\,\dot\cup\, R\\ \forall l\in D\cup T:|l|>2}} \sum_{\substack{\forall l\in D\\ \sigma_l\in S(l)}} \sum_{b\in D\cup\{\{r\}|r\in R\}} \mathrm{PT}(T) \\
&\sum_{a\subseteq R\backslash b\cup\bigcup_{l\in D\backslash\{b\}}\{W(l,\sigma_l)\}} L(a,a^c)(-1)^{|a|+\frac{(|R|+|D|-1)(|R|+|D|)}{2}} \\
&\qquad\qquad \mathrm{tr}[P_+X_{n+1}P_-W(b,\sigma_b)] \\
&= \sum_{\substack{\langle n\rangle=\dot\bigcup_{l\in T} l\,\dot\cup\,\dot\bigcup_{l\in D} l\,\dot\cup\, R\\ \forall l\in D\cup T:|l|>2}} \sum_{\substack{\forall l\in D\\ \sigma_l\in S(l)}} \mathrm{PT}(T) \sum_{a\subseteq R\cup\bigcup_{l\in D}\{W(l,\sigma_l)\}} \\
&\qquad \sum_{b\in D\cup\{\{r\}|r\in R\}} \mathbb{1}_{W(b,\sigma_b)\in a} L(a\backslash\{b\},a^c)(-1)^{|a|+1} \\
&\qquad (-1)^{\frac{(|R|+|D|-1)(|R|+|D|)}{2}}\,\mathrm{tr}[P_+X_{n+1}P_-W(b,\sigma_b)] \\
&= \sum_{\substack{\langle n\rangle=\dot\bigcup_{l\in T} l\,\dot\cup\,\dot\bigcup_{l\in D} l\,\dot\cup\, R\\ \forall l\in D\cup T:|l|>2}} \mathrm{PT}(T) \sum_{\substack{\forall l\in D\\ \sigma_l\in S(l)}} \sum_{a\subseteq R\cup\bigcup_{l\in D}\{W(l,\sigma_l)\}} \\
&\qquad \sum_{b\in a} L(a\backslash\{b\},a^c)(-1)^{|a|+\frac{(|R|+|D|+1)(|R|+|D|)}{2}} \\
&\qquad (-1)^{1+|R|+|D|}\,\mathrm{tr}[P_+X_{n+1}P_-W(b,\sigma_b)] \\
&= \sum_{\substack{\langle n\rangle=\dot\bigcup_{l\in T} l\,\dot\cup\,\dot\bigcup_{l\in D} l\,\dot\cup\, R\\ \forall l\in D\cup T:|l|>2}} \mathrm{PT}(T) \sum_{\substack{\forall l\in D\\ \sigma_l\in S(l)}} \sum_{a\subseteq R\cup\bigcup_{l\in D}\{W(l,\sigma_l)\}} \\
&\qquad (-1)^{|a|+\frac{(|R|+|D|+1)(|R|+|D|)}{2}} (3.88)_{L(a,a^c)G(X_{n+1})},
\end{aligned} \tag{3.106}$$

where the notation in the last line is to be taken as "apply Lemma 3.5.2 apply it to $L(a,a^c)G(X_{n+1})$ and pick only term (3.88)". We will use this abbreviating notation also for the next terms.
The next term is (3.103). Here we need a few more notational conven-

tions. For any set $b \subseteq \langle n \rangle$ and corresponding permutation $\sigma_b \in S(b)$, we denote by the same symbol σ_b the continuation of σ_b to $b \cup \{X_{n+1}\}$, where for this continuation X_{n+1} is a fixed point. Furthermore we define for any set $b \subseteq \langle n \rangle$, σ_c^b by

$$\begin{aligned} \sigma_c^b &\in S(b \cup \{X_{n+1}\}), \\ \forall k \leq |b| : \sigma_c^b(f_{b \cup \{X_{n+1}\}}(k)) &= f_{b \cup \{X_{n+1}\}}(k+1) \\ \sigma_c^b(X_{n+1}) &= f_b(1). \end{aligned} \tag{3.107}$$

Finally we define for sets $b_1, b_2 \subseteq \langle n \rangle, b_1 \cap b_2 = \emptyset$ and corresponding permutations $\sigma_{b_1} \in S(b_1), \sigma_{b_2} \in S(b_2)$ the permutation σ_{b_1,b_2}^{n+1} by

$$\begin{aligned} M_{b_1,b_2}^{n+1} &:= b_1 \cup b_2 \cup \{X_{n+1}\} \\ \sigma_{b_1,b_2}^{n+1} &\in S(M_{b_1,b_2}^{n+1}) \\ \forall 1 \leq k \leq |b_1| : \sigma_{b_1,b_2}^{n+1}(f_{M_{b_1,b_2}^{n+1}}(k)) &= \sigma_{b_1}(f_{b_1}(k)) \\ \sigma_{b_1,b_2}^{n+1}(f_{M_{b_1,b_2}^{n+1}}(|b_1|+1)) &= X_{n+1} \\ \forall |b_1| + 2 \leq k \leq |b_1| + |b_2| + 1 : \\ \sigma_{b_1,b_2}^{n+1}(f_{M_{b_1,b_2}^{n+1}}(k)) &= \sigma_{b_2}(f_{b_2}(k - |b_1| - 1)) \end{aligned} \tag{3.108}$$

The beginning of the treatment of term (3.103) is analogous to (3.102), we rewrite the partition of $\langle n+1 \rangle$ into one of $\langle n \rangle$ with an additional sum over where the other operators packed to together with X_{n+1} come from. This splits into three parts, either X_{n+1} is put at the beginning of the compound operator, or its put at the end of the compound object, or to the left as well as to the right are operators with smaller index. Since the overall sign is decided by how often the operator index rises or falls, this separation into cases is helpful. The last case we then rewrite as picking two sets of operators, one of which will be in front of X_{n+1} and the other one behind this operator.

The calculation is as follows

$$
\begin{aligned}
(3.103) = &\sum_{\substack{\langle n+1\rangle=\dot\bigcup_{l\in T} l\,\dot\cup\,\dot\bigcup_{l\in D} l\,\dot\cup\, R\\ \exists l\in D: X_{n+1}\in l\\ \forall l\in T\cup D:|l|\geq 2}} \mathrm{PT}(T)\mathrm{Op}(R,D)\\
= &\sum_{\substack{\langle n\rangle=\dot\bigcup_{l\in T} l\,\dot\cup\,\dot\bigcup_{l\in D} l\,\dot\cup\, R\\ \forall l\in D\cup T:|l|\geq 2}} \mathrm{PT}(T)\sum_{b\in D\cup\{\{r\}|r\in R\}} \mathrm{Op}(R\backslash b, D\cup\{b\cup\{X_{n+1}\}\backslash\{b\}\})\\
= &\sum_{\substack{\langle n\rangle=\dot\bigcup_{l\in T} l\,\dot\cup\,\dot\bigcup_{l\in D} l\,\dot\cup\, R\\ \forall l\in D\cup T:|l|\geq 2}} \mathrm{PT}(T)\sum_{b\in D\cup\{\{r\}|r\in R\}}\sum_{\substack{\forall l\in D\cup\{b\cup\{X_{n+1}\}\}\\ \sigma_l\in S(l)}}\\
&\sum_{a\subseteq R\backslash b\cup\bigcup_{l\in D\cup\{b\cup\{X_{n+1}\}\}\backslash\{b\}}\{W(l,\sigma_l)\}} L(a,a^c)(-1)^{|a|+\frac{(|R|+|D|)(|R|+|D|+1)}{2}}\\
= &\sum_{\substack{\langle n\rangle=\dot\bigcup_{l\in T} l\,\dot\cup\,\dot\bigcup_{l\in D} l\,\dot\cup\, R\\ \forall l\in D\cup T:|l|\geq 2}} \mathrm{PT}(T)\sum_{b\in D\cup\{\{r\}|r\in R\}}\sum_{\substack{\forall l\in D\\ \sigma_l\in S(l)}}\Bigg[\\
&\sum_{a\subseteq R\backslash b\cup\bigcup_{l\in D\backslash\{b\}}\{W(l,\sigma_l)\}\cup\{W(b\cup\{X_{n+1}\},\sigma_b)\}} L(a,a^c)(-1)^{|a|+\frac{(|R|+|D|)(|R|+|D|+1)}{2}}
\end{aligned}
\tag{3.109}
$$

$$
+\sum_{a\subseteq R\backslash b\cup\bigcup_{l\in D\backslash\{b\}}\{W(l,\sigma_l)\}\cup\{W(b\cup\{X_{n+1}\},\sigma_c^b\circ\sigma_b)\}} L(a,a^c)(-1)^{|a|+\frac{(|R|+|D|)(|R|+|D|+1)}{2}}\Bigg]
\tag{3.110}
$$

$$
\begin{aligned}
+ &\sum_{\substack{\langle n\rangle=\dot\bigcup_{l\in T} l\,\dot\cup\,\dot\bigcup_{l\in \overline{D}} l\,\dot\cup\, \overline{R}\\ \forall l\in \overline{D}\cup T:|l|\geq 2}} \mathrm{PT}(T)\sum_{\substack{b_1,b_2\in\overline{D}\cup\{\{r\}|r\in\overline{R}\}\\ b_1\neq b_2}}\sum_{\substack{\forall l\in\overline{D}\\ \sigma_l\in S(l)}}\\
&\sum_{a\subseteq\tilde{R}\cup\bigcup_{l\in\tilde{D}}\{W(l,\sigma_l)\}\cup\{W(b_1\cup\{X_{n+1}\}\cup b_2,\sigma^{n+1}_{b_1,b_2})\}} L(a,a^c)(-1)^{|a|+\frac{(|\overline{R}|+|\overline{D}|-1)(|\overline{R}|+|\overline{D}|)}{2}}
\end{aligned}
\tag{3.111}
$$

where $\tilde{R} = \overline{R} \backslash (b_1 \cup b_2)$ and $\tilde{D} := \overline{D} \cup \{b_1 \cup \{X_{n+1}\} \cup b_2\} \backslash \{b_1, b_2\}$. For the term (3.111) we had to reshuffle the outermost sum a bit. For each term in the original sum where X_{n+1} is neither the first nor the last factor in its product (we will call the set of factors in front of X_{n+1} α and the factors behind it β) there is a different splitting of $\langle n \rangle$ into $\overline{R}$ and $\overline{D}$ such that α and β are separate elements of $\overline{D} \cup \{\{r\} \mid r \in \overline{R}\}$. So we replace the original sum over D and R into one of $\overline{D}$ and $\overline{R}$. Since this is a one to one correspondence and the sum is finite this is always possible. The exponent of the sign also changes for this reason, since $|R| + |D| = |\overline{R}| + |\overline{D}| - 1$ holds. Continuing with (3.109) the next steps are similar to the last steps in treating (3.102). They are

$$
\begin{aligned}
(3.109) = & \sum_{\substack{\langle n \rangle = \dot{\bigcup}_{l \in T} l \,\dot{\cup}\, \dot{\bigcup}_{l \in D} l \,\dot{\cup}\, R \\ \forall l \in D \cup T : |l| \geq 2}} \mathrm{PT}(T) \sum_{b \in D \cup \{\{r\} | r \in R\}} \sum_{\substack{\forall l \in D \\ \sigma_l \in S(l)}} \\
& \sum_{a \subseteq R \backslash b \cup \bigcup_{l \in D \backslash \{b\}} \{W(l,\sigma_l)\} \cup \{W(b \cup \{X_{n+1}\}, \sigma_b)\}} L(a, a^c)(-1)^{|a| + \frac{(|R|+|D|)(|R|+|D|+1)}{2}} \\
= & \sum_{\substack{\langle n \rangle = \dot{\bigcup}_{l \in T} l \,\dot{\cup}\, \dot{\bigcup}_{l \in D} l \,\dot{\cup}\, R \\ \forall l \in D \cup T : |l| \geq 2}} \mathrm{PT}(T) \sum_{b \in D \cup \{\{r\} | r \in R\}} \sum_{\substack{\forall l \in D \\ \sigma_l \in S(l)}} \\
& \sum_{a \subseteq R \cup \bigcup_{l \in D} \{W(l, \sigma_l)\}} (-1)^{|a| + \frac{(|R|+|D|)(|R|+|D|+1)}{2}} \mathbb{1}_{W(b, \sigma_b) \in a} \\
& \big[L(a \backslash \{W(b, \sigma_b)\} \cup \{W(b \cup \{X_{n+1}\}, \sigma_b)\}, a^c) \\
& - L(a \backslash \{W(b, \sigma_b)\}, a^c \cup \{W(b \cup \{X_{n+1}\}, \sigma_b)\}) \big] \\
= & \sum_{\substack{\langle n \rangle = \dot{\bigcup}_{l \in T} l \,\dot{\cup}\, \dot{\bigcup}_{l \in D} l \,\dot{\cup}\, R \\ \forall l \in D \cup T : |l| \geq 2}} \mathrm{PT}(T) \sum_{\substack{\forall l \in D \\ \sigma_l \in S(l)}}
\end{aligned}
$$

$$\sum_{a\subseteq R\cup\bigcup_{l\in D}\{W(l,\sigma_l)\}} (-1)^{|a|+\frac{(|R|+|D|)(|R|+|D|+1)}{2}} \sum_{W(b,\sigma_b)\in a}$$
$$\big[L(a\backslash\{W(b,\sigma_b)\}\cup\{W(b\cup\{X_{n+1}\},\sigma_b)\},a^c)$$
$$-L(a\backslash\{W(b,\sigma_b)\},a^c\cup\{W(b\cup\{X_{n+1}\},\sigma_b)\})\big]$$
$$=\sum_{\substack{\langle n\rangle=\dot\bigcup_{l\in T} l\,\dot\cup\,\dot\bigcup_{l\in D} l\,\dot\cup\, R\\ \forall l\in D\cup T:|l|\geq 2}} \mathrm{PT}(T) \sum_{\substack{\forall l\in D\\ \sigma_l\in S(l)}} \sum_{a\subseteq R\cup\bigcup_{l\in D}\{W(l,\sigma_l)\}}$$
$$(-1)^{|a|+\frac{(|R|+|D|)(|R|+|D|+1)}{2}}((3.84)+(3.86))_{L(a,a^c)G(X_{n+1})}. \tag{3.112}$$

Almost the same procedure applies to (3.110). It yields

$$(3.110)=\sum_{\substack{\langle n\rangle=\dot\bigcup_{l\in T} l\,\dot\cup\,\dot\bigcup_{l\in D} l\,\dot\cup\, R\\ \forall l\in D\cup T:|l|\geq 2}} \mathrm{PT}(T) \sum_{b\in D\cup\{\{r\}|r\in R\}} \sum_{\substack{\forall l\in D\\ \sigma_l\in S(l)}}$$
$$\sum_{a\subseteq R\backslash b\cup\bigcup_{l\in D\backslash\{b\}}\{W(l,\sigma_l)\}\cup\{W(b\cup\{X_{n+1}\},\sigma_c^b\circ\sigma_b)\}} L(a,a^c)(-1)^{|a|+\frac{(|R|+|D|)(|R|+|D|+1)}{2}}$$
$$=\sum_{\substack{\langle n\rangle=\dot\bigcup_{l\in T} l\,\dot\cup\,\dot\bigcup_{l\in D} l\,\dot\cup\, R\\ \forall l\in D\cup T:|l|\geq 2}} \mathrm{PT}(T) \sum_{b\in D\cup\{\{r\}|r\in R\}} \sum_{\substack{\forall l\in D\\ \sigma_l\in S(l)}}$$
$$\sum_{a\subseteq R\cup\bigcup_{l\in D}\{W(l,\sigma_l)\}} (-1)^{|a|+\frac{(|R|+|D|)(|R|+|D|+1)}{2}}$$
$$\big[\mathbb{1}_{W(b,\sigma_c^b\circ\sigma_b)\in a} L(a\backslash\{W(b,\sigma_b)\}\cup\{W(b\cup\{X_{n+1}\},\sigma_c^b\circ\sigma_b)\},a^c)$$
$$+\mathbb{1}_{W(b,\sigma_c^b\circ\sigma_b)\in a^c} L(a\backslash\{W(b,\sigma_b)\},a^c\cup\{W(b\cup\{X_{n+1}\},\sigma_c^b\circ\sigma_b)\})\big]$$
$$=\sum_{\substack{\langle n\rangle=\dot\bigcup_{l\in T} l\,\dot\cup\,\dot\bigcup_{l\in D} l\,\dot\cup\, R\\ \forall l\in D\cup T:|l|\geq 2}} \mathrm{PT}(T) \sum_{\substack{\forall l\in D\\ \sigma_l\in S(l)}}$$
$$\sum_{a\subseteq R\cup\bigcup_{l\in D}\{W(l,\sigma_l)\}} (-1)^{|a|+\frac{(|R|+|D|)(|R|+|D|+1)}{2}}$$

$$
\Big[\sum_{W(b,\sigma_b)\in a} L(a\backslash\{W(b,\sigma_b)\}\cup\{W(b\cup\{X_{n+1}\},\sigma_c^b\circ\sigma_b)\},a^c)
$$

$$
+\sum_{W(b,\sigma_b)\in a^c} L(a,a^c\backslash\{W(b,\sigma_b)\}\cup\{W(b\cup\{X_{n+1}\},\sigma_c^b\circ\sigma_b)\})\Big]
$$

$$
=\sum_{\substack{\langle n\rangle=\dot\bigcup_{l\in T} l\,\dot\cup\,\dot\bigcup_{l\in D} l\,\dot\cup\, R\\ \forall l\in D\cup T:|l|\geq 2}} \mathrm{PT}(T)\sum_{\substack{\forall l\in D\\ \sigma_l\in S(l)}}\sum_{a\subseteq R\cup\bigcup_{l\in D}\{W(l,\sigma_l)\}}
$$

$$
(-1)^{|a|+\frac{(|R|+|D|)(|R|+|D|+1)}{2}}((3.85)+(3.87))_{L(a,a^c)G(X_{n+1})}. \tag{3.113}
$$

Also for (3.111) the procedure is almost the same. We bring the sums into a form such that one can read off the terms generated by the induction. We begin be renaming the sets which we had to change by resumming back to the names of the original sets.

$$
(3.111)=\sum_{\substack{\langle n\rangle=\dot\bigcup_{l\in T} l\,\dot\cup\,\dot\bigcup_{l\in \overline{D}} l\,\dot\cup\, \overline{R}\\ \forall l\in \overline{D}\cup T:|l|\geq 2}} \mathrm{PT}(T)\sum_{\substack{b_1,b_2\in\overline{D}\cup\{\{r\}|r\in\overline{R}\}\\ b_1\neq b_2}}\sum_{\substack{\forall l\in \overline{D}\\ \sigma_l\in S(l)}}
$$

$$
\sum_{a\subseteq\tilde{R}\cup\bigcup_{l\in\tilde{D}}\{W(l,\sigma_l)\}\cup\{W(b_1\cup\{X_{n+1}\}\cup b_2,\sigma_{b_1,b_2}^{n+1})\}} L(a,a^c)(-1)^{|a|+\frac{(|\overline{R}|+|\overline{D}|-1)(|\overline{R}|+|\overline{D}|)}{2}}
$$

$$
=\sum_{\substack{\langle n\rangle=\dot\bigcup_{l\in T} l\,\dot\cup\,\dot\bigcup_{l\in D} l\,\dot\cup\, R\\ \forall l\in D\cup T:|l|\geq 2}} \mathrm{PT}(T)\sum_{\substack{b_1,b_2\in D\cup\{\{r\}|r\in R\}\\ b_1\neq b_2}}\sum_{\substack{\forall l\in D\\ \sigma_l\in S(l)}}
$$

$$
\sum_{a\subseteq R\cup\bigcup_{l\in D}\{W(l,\sigma_l)\}\cup\{W(b_1\cup\{X_{n+1}\}\cup b_2,\sigma_{b_1,b_2}^{n+1})\}} L(a,a^c)(-1)^{|a|+\frac{(|R|+|D|-1)(|R|+|D|)}{2}}
$$

$$
=\sum_{\substack{\langle n\rangle=\dot\bigcup_{l\in T} l\,\dot\cup\,\dot\bigcup_{l\in D} l\,\dot\cup\, R\\ \forall l\in D\cup T:|l|\geq 2}} \mathrm{PT}(T)\sum_{\substack{b_1,b_2\in D\cup\{\{r\}|r\in R\}\\ b_1\neq b_2}}\sum_{\substack{\forall l\in D\\ \sigma_l\in S(l)}}\sum_{a\subseteq R\cup\bigcup_{l\in D}\{W(l,\sigma_l)\}}
$$

$$
(-1)^{|R|+|D|+\frac{(|R|+|D|)(|R|+|D|+1)}{2}}\mathbb{1}_{W(b_1,\sigma_1)\in a}
$$

$$
\begin{aligned}
&\Big[+(-1)^{|a|+1}\mathbb{1}_{W(b_2,\sigma_2)\in a}L\Big(a\backslash\{W(b_1,\sigma_1),W(b_2,\sigma_2)\}\cup\\
&\cup\{W(b_1\cup\{X_{n+1}\}\cup b_2,\sigma^{n+1}_{b_1,b_2})\},a^c\Big)\\
&+(-1)^{|a|+1}\mathbb{1}_{W(b_2,\sigma_2)\in a^c}L\Big(a\backslash\{W(b_1,\sigma_1)\},a^c\backslash\{W(b_2,\sigma_2)\}\cup\\
&\cup\{W(b_1\cup\{X_{n+1}\}\cup f_2,\sigma^{n+1}_{b_1,b_2})\}\Big)\Big]\\
&=\sum_{\substack{\langle n\rangle=\dot\bigcup_{l\in T}l\,\dot\cup\,\dot\bigcup_{l\in D}l\,\dot\cup\,R\\ \forall l\in D\cup T:|l|\geq 2}}\mathrm{PT}(T)\sum_{\substack{\forall l\in D\\ \sigma_l\in S(l)}}\sum_{a\subseteq R\cup\bigcup_{l\in D}\{W(l,\sigma_l)\}}\\
&(-1)^{|R|+|D|+\frac{(|R|+|D|)(|R|+|D|+1)}{2}}\\
&\Big[(-1)^{|a|+1}\sum_{\substack{b_1,b_2\in a\\ b_1\neq b_2}}L\Big(a\backslash\{W(b_1,\sigma_1),W(b_2,\sigma_2)\}\cup\\
&\cup\{W(b_1\cup\{X_{n+1}\}\cup b_2,\sigma^{n+1}_{b_1,b_2})\},a^c\Big)\\
&+(-1)^{|a|+1}\sum_{b_1\in a,b_2\in a^c}L\Big(a\backslash\{W(b_1,\sigma_1)\},a^c\backslash\{W(b_2,\sigma_2)\}\cup\\
&\cup\{W(b_1\cup\{X_{n+1}\}\cup f_2,\sigma^{n+1}_{b_1,b_2})\}\Big)\Big]\\
&=\sum_{\substack{\langle n\rangle=\dot\bigcup_{l\in T}l\,\dot\cup\,\dot\bigcup_{l\in D}l\,\dot\cup\,R\\ \forall l\in D\cup T:|l|\geq 2}}\mathrm{PT}(T)\sum_{\substack{\forall l\in D\\ \sigma_l\in S(l)}}\sum_{a\subseteq R\cup\bigcup_{l\in D}\{W(l,\sigma_l)\}}\\
&(-1)^{|a|+\frac{(|R|+|D|)(|R|+|D|+1)}{2}}((3.89)+(3.90))_{L(a,a^c)G(X_{n+1})}
\end{aligned}
$$

Lastly we will discuss term (3.104); luckily, this term is less involved than the other two. The general procedure; however, stays the same. First we reformulate the partition of $\langle n+1\rangle$ into one of $\langle n\rangle$, where the terms acquire modifications. Secondly we massage these terms until the involved sums look exactly like the one in our induction hypothesis (3.101) and realise that the terms are produced by lemma 3.5.2. For term (3.104) this results in

$$(3.104) = \sum_{\substack{\langle n+1\rangle = \dot\bigcup_{l\in T} l \,\dot\cup\, \dot\bigcup_{l\in D} l \,\dot\cup\, R \\ X_{n+1}\in R \\ \forall l\in T\cup D: |l|\geq 2}} \mathrm{PT}(T)\mathrm{Op}(R,D)$$

$$= \sum_{\substack{\langle n\rangle = \dot\bigcup_{l\in T} l \,\dot\cup\, \dot\bigcup_{l\in D} l \,\dot\cup\, R \\ \forall l\in T\cup D: |l|\geq 2}} \mathrm{PT}(T)\mathrm{Op}(R\cup\{X_{n+1}\},D)$$

$$= \sum_{\substack{\langle n\rangle = \dot\bigcup_{l\in T} l \,\dot\cup\, \dot\bigcup_{l\in D} l \,\dot\cup\, R \\ \forall l\in T\cup D: |l|\geq 2}} \mathrm{PT}(T) \sum_{\substack{\forall l\in D: \\ \sigma_l\in S(l)}} \sum_{a\subseteq R\cup\{X_{n+1}\}\cup\bigcup_{l\in D}\{W(l,\sigma_l)\}}$$

$$L(a,a^c)(-1)^{|a|+\frac{(|R|+1+|D|)(|R|+|D|+2)}{2}}$$

$$= \sum_{\substack{\langle n\rangle = \dot\bigcup_{l\in T} l \,\dot\cup\, \dot\bigcup_{l\in D} l \,\dot\cup\, R \\ \forall l\in T\cup D: |l|\geq 2}} \mathrm{PT}(T) \sum_{\substack{\forall l\in D: \\ \sigma_l\in S(l)}} \sum_{a\subseteq R\cup\bigcup_{l\in D}\{W(l,\sigma_l)\}} (-1)^{|a|+\frac{(|R|+1+|D|)(|R|+|D|)}{2}}$$

$$\Big(-L(a\cup\{X_{n+1}\},a^c)+L(a,a^c\cup\{X_{n+1}\})\Big)(-1)^{|R|+|D|+1}$$

$$= \sum_{\substack{\langle n\rangle = \dot\bigcup_{l\in T} l \,\dot\cup\, \dot\bigcup_{l\in D} l \,\dot\cup\, R \\ \forall l\in T\cup D: |l|\geq 2}} \mathrm{PT}(T) \sum_{\substack{\forall l\in D: \\ \sigma_l\in S(l)}} \sum_{a\subseteq R\cup\bigcup_{l\in D}\{W(l,\sigma_l)\}} (-1)^{|a|+\frac{(|R|+1+|D|)(|R|+|D|)}{2}}$$

$$\Big(L(a\cup\{X_{n+1}\},a^c)(-1)^{|R|+|D|}+L(a,a^c\cup\{X_{n+1}\})(-1)^{|R|+|D|+1}\Big)$$

$$= \sum_{\substack{\langle n\rangle = \dot\bigcup_{l\in T} l \,\dot\cup\, \dot\bigcup_{l\in D} l \,\dot\cup\, R \\ \forall l\in T\cup D: |l|\geq 2}} \mathrm{PT}(T) \sum_{\substack{\forall l\in D: \\ \sigma_l\in S(l)}} \sum_{a\subseteq R\cup\bigcup_{l\in D}\{W(l,\sigma_l)\}} (-1)^{|a|+\frac{(|R|+1+|D|)(|R|+|D|)}{2}}$$

$$\Big((3.82)+(3.83)\Big)_{L(a,a^c)G(X_{n+1})}. \qquad \blacksquare$$

Summarising we showed

$$\sum_{\substack{\langle n+1\rangle = \dot\bigcup_{l\in T} l \,\dot\cup\, \dot\bigcup_{l\in D} l \,\dot\cup\, R \\ \forall l\in T\cup D: |l|\geq 2}} \mathrm{PT}(T)\mathrm{Op}(R,D)$$

$$
\begin{aligned}
&= \sum_{\substack{\langle n\rangle=\dot\bigcup_{l\in T} l\,\dot\cup\,\dot\bigcup_{l\in D} l\,\dot\cup\, R\\ \forall l\in D\cup T:|l|>2}} \mathrm{PT}(T) \sum_{\substack{\forall l\in D\\ \sigma_l\in S(l)}} \sum_{a\subseteq R\cup\bigcup_{l\in D}\{W(l,\sigma_l)\}} \\
&(-1)^{|a|+\frac{(|R|+|D|+1)(|R|+|D|)}{2}} \\
&\big((3.88)+(3.84)+(3.86)+(3.85)+(3.87) \\
&+(3.89)+(3.90)+(3.82)+(3.83)\big)_{L(a,a^c)G(X_{n+1})} \\
&= \sum_{\substack{\langle n\rangle=\dot\bigcup_{l\in T} l\,\dot\cup\,\dot\bigcup_{l\in D} l\,\dot\cup\, R\\ \forall l\in D\cup T:|l|>2}} \mathrm{PT}(T) \sum_{\substack{\forall l\in D\\ \sigma_l\in S(l)}} \sum_{a\subseteq R\cup\bigcup_{l\in D}\{W(l,\sigma_l)\}} \\
&(-1)^{|a|+\frac{(|R|+|D|+1)(|R|+|D|)}{2}} L(a,a^c)G(X_{n+1}) \\
&= \sum_{\substack{\langle n\rangle=\dot\bigcup_{l\in T} l\,\dot\cup\,\dot\bigcup_{l\in D} l\,\dot\cup\, R\\ \forall l\in D\cup T:|l|>2}} \mathrm{PT}(T)\mathrm{Op}(R,D)G(X_{n+1}) \\
&= \prod_{l=1}^{n} G(X_l) \quad G(X_{n+1}),
\end{aligned}
$$

which ends our proof by induction. □

3.6 Main Conjecture

Now I can state the primary objective of my thesis in terms of the following

Conjecture 3.6.1. *For all smooth four-potentials $A \in C_c^\infty(\mathbb{R}^4)\otimes\mathbb{C}^4$, $e\in\mathbb{R}$, and for all $\psi,\phi\in\mathcal{F}$ the following limit exists*

$$\lim_{n\to\infty}\left\langle \psi, \sum_{k=0}^{n}\frac{e^k}{k!}T_k(A)\phi\right\rangle . \tag{3.114}$$

Such a uniform convergence would be optimal. In case, it can not be achieved, a weaker form of this conjecture in which $|e|$ has to be chosen sufficiently small and the possible scattering states Ψ, Φ have to be restricted to a certain regularity would still be physically interesting. The main difficulty in proving this theorem is the large number of possible summands in the determinant-like structure of the term of n-th order. I am optimistic about finding the proof of conjecture 3.6.1 for several reasons:

1. For the summand involving T_n one gets a factor of $\frac{1}{n!}$ from the simplex. In the expression for T_n there are n time integrals, and in the integrand the temporal variables are ordered. Since there are $n!$ possible orderings each particular order contributes only one part in $n!$. This argument can be made precise and has been translated into momentum space, where it was already used to estimate the one-particle scattering operator, see section 3.1.1.

2. The operators T_n posses the property called "charge conservation", i.e. T_n maps any element of the b, p particle sector of Fock space to c, o particle sectors fulfilling $c - o = b - p$. Hence many possible transitions are forbidden by the structure of the operators T_n.

3. The iterative character of the operators T_n illustrated by equations (**??**) suggests that the control of T_1 and T_2, discussed in sections 3.6.3 and 3.6.4, is sufficient to also control the n-th order. This behavior is also suggested by the renormalisability of QED (see [**?**, Chapter 4.3]) which states that only finite many types of renormalisations are needed.

4. Many of the remaining possible transitions are forbidden by the antisymmetry of the fermionic Fock space.

After a successful proof of the main conjecture this method can be generalised in a canonical manner to yield a direct construction of a more general time evolution operator, as was mentioned in the introduction this is especially desirable in the non-perturbative regime of QED. In the rest of this section I will present the results about T_n for $n = 1$, $n = 2$, and all other odd n.

3.6.1 Explicit Representations

I introduce the operator G as follows. I denote by Q the following set $Q := \{f : \mathcal{H} \to \mathcal{H} \text{ linear} \mid i \cdot f \text{ is selfadjoint}\}$.

Definition 3.6.2. *Let then G be the following function*

$$\begin{aligned} G: \quad & Q \to (\mathcal{F} \to \mathcal{F}) \\ & f \mapsto \sum_{n \in \mathbb{N}} a^*(f\varphi_n) a(\varphi) - \sum_{n \in -\mathbb{N}} a(\varphi_n) a^*(f\varphi_n). \end{aligned} \tag{Def G}$$

Lemma 1. *For any $q \in Q$ the operator $G(q)$ fulfils the commutation relation*

$$\forall n \in \mathbb{Z} : \left[G(q), a^{\#}(\varphi_n)\right] = a^{\#}(q(\varphi_n)). \tag{3.115}$$

The proof of this lemma follows by direct calculation and exploitation of the commutation relations of a and a^*.

Todo: vielleicht Beweis einfügen

The first expansion coefficient of the scattering operator, T_1, is then given by

$$T_1(A) = G(Z_1(A)), \tag{3.116}$$

given $\langle T_2 \rangle \in \mathbb{C}$, the second order by

$$T_2 = G(Z_2 - Z_1 Z_1) + T_1 T_1 - \operatorname{tr}\left(Z_{\underset{-+}{1}} Z_{\underset{+-}{1}}\right) + \langle T_2 \rangle, \tag{3.117}$$

and the third order by

$$T_3 = G\left(Z_3 - \frac{3}{2} Z_2 Z_1 - \frac{3}{2} Z_1 Z_2 + 2 Z_1 Z_1 Z_1\right) + \frac{3}{2} T_2 T_1 + \frac{3}{2} T_1 T_2 - 2 T_1 T_1 T_1. \tag{3.118}$$

Let $b \in \mathbb{R}$ be arbitrary, there is a $C \in \mathbb{C}$ such that T_4 is given by

$$\begin{aligned} T_4 &:= 2T_1T_3 + 2T_3T_1 + 3T_2T_2 - bT_1T_1T_2 - bT_2T_1T_1 - 2(6-b)T_1T_2T_1 \\ &+ 6T_1T_1T_1T_1 + G\left(Z_4 - 2Z_1Z_3 - 2Z_3Z_1 - 3Z_2Z_2\right. \\ &\left.+bZ_1^2Z_2 + 2(6-b)Z_1Z_2Z_1 + bZ_2Z_1^2 - 6Z_1^4\right) + C. \end{aligned} \tag{3.119}$$

Todo: habe noch keinen guten Kandidaten für T_n.

These expressions can easily be verified by means of the commutation rules (??).

3.6.2 Results About All Odd Orders

In order to show that any serious candidate for the construction of the scattering-matrix fulfils $\langle \Omega, T_{2n+1}\Omega\rangle = 0$ for any $n \in \mathbb{N}_0$, I also lift the charge conjugation operator to Fock space.

3.6.2.1 Lifting the Charge Conjugation Operator

I will define the second quantised charge conjugation operator $\mathfrak{C}$ on all of Fock space analogously to the way I am currently in the process of defining the second quantised S-matrix operator. The operator $\mathfrak{C} : \mathcal{F} \to \mathcal{F}$ is defined to be the linear bounded operator on Fock space fulfilling the "lift condition"

$$\forall \phi \in \mathcal{H}: \quad \begin{aligned} a\left(C\phi\right) \circ \mathfrak{C} &= \mathfrak{C} \circ a^*(\phi), \\ a^*\left(C\phi\right) \circ \mathfrak{C} &= \mathfrak{C} \circ a(\phi), \end{aligned} \tag{3.120}$$

where C is the charge conjugation operator on the one particle Hilbert space. The operator $\mathfrak{C}$ is furthermore defined to fulfil

$$\mathfrak{C}\Omega = \Omega. \tag{3.121}$$

Lemma 2. *Properties of* $\mathfrak{C}$*:*
The lifted operator $\mathfrak{C}$ *has the following important properties.*

$$\mathfrak{C}\mathfrak{C} = \mathbb{1} \tag{3.122}$$

$$\mathfrak{C}^*\mathfrak{C} = \mathbb{1} \tag{3.123}$$

The proof of this lemma consists of fairly lengthy but straightforward computations.

3.6.2.2 Commutation of Charge Conjugation and Scattering Operators

I first introduce another operator and use it to find the commutation properties of the charge conjugation operator with the scattering operator. Consider the commutating diagram in the one-particle picture.

$$\begin{tikzcd} \mathcal{H} \arrow[r, "U^A"] \arrow[d, "C"] & \mathcal{H} \arrow[d, "C"] \\ \overline{\mathcal{H}} \arrow[r, "U^{-A}"] & \overline{\mathcal{H}}. \end{tikzcd} \tag{3.124}$$

Inspired by this diagram I introduce for each four potential A the one particle operator $K : \mathcal{H} \to \mathcal{H}$ with $K = U^A C = C U^{-A}$. It is easy to see that K is unitary and $P_- K P_+$and $P_+ K P_-$ are Hilbert-Schmidt operators, due to the analogous property of the one particle scattering Operator, for more details see [**?**]. This means that K has a second quantised analogue $\tilde{K}$ that is unique up to a phase. The operator is then defined as follows

$$\tilde{K} : \mathcal{F}_{\mathcal{H}^+ \oplus \overline{\mathcal{H}^-}} \to \mathcal{F}_{\overline{\mathcal{H}}^+ \oplus \mathcal{H}^-} \tag{3.125}$$

$$\forall \psi \in \mathcal{H} : \quad \tilde{K} a^\#(\psi) = a^\#(K\psi)\tilde{K}, \tag{3.126}$$

where $a^\#$ can be either a or a^*.

Axiom 2. *The two unknown phases between $\tilde{K}$ and $S^A\mathfrak{C}$ and $\mathfrak{C}S^{-A}$ agree, i.e.*

$$\exists\phi[A] \in \mathbb{R} : \mathfrak{C}S^A = e^{i\phi[A]}\tilde{K} = S^{-A}\mathfrak{C}. \tag{3.127}$$

I have now collected enough tools to prove the following

Lemma 3. *It follows from axiom 2 that for all four potentials A*

$$\forall n \in \mathbb{N}_0 : \langle \Omega, T_{2n+1}(A)\Omega \rangle = 0 \tag{3.128}$$

holds. I.e. the vacuum expectation value of all odd expansion coefficients of (3.25) *vanishes.*

The proof of lemma 3 uses homogeneity of degree $2n+1$ of T_{2n+1}, and the properties of operator $\mathfrak{C}$.

3.6.3 Explicit Bound of the First Order

The bound of $T_1(A)$ on a sector of arbitrary but fixed particle number of Fock space $\mathcal{F}_{m,p}$ for any $m, p \in \mathbb{N}_0$ can be found to be

$$\left\| T_1(A)\Big|_{\mathcal{F}_{m,p}} \right\| \leq \sqrt{mp\alpha + (m\beta + p\gamma)^2 + (m+1)(p+1)\delta}, \tag{3.129}$$

for some positive numbers $\alpha, \beta, \gamma, \delta \in \mathbb{R}^+$. This bound is found by exploiting the commutation properties of T_1 and the determinant like structure of the scalar product of Fock space.

3.6.4 Results about the Second Order

Historically it was found that it is notoriously difficult to give a mathematically well defined description of T_2. This can now be achieved by means of the method of Epstein und Glaser [?]. Knowing the explicit form of T_2, (3.117) all that is left to define this operator is to find its vacuum expectation value. This is achieved by

Axiom 3. *Any disturbance of the electromagnetic field should not influence the behaviour of the system previous to its existence. More precisely, the second quantised scattering-matrix should fulfil*

$$\left(S^f\right)^{-1} S^{f+g} = \left(S^0\right)^{-1} S^g, \tag{causality}$$

for any four potentials f and g such that the support of f is not earlier than the support of g. That is, (causality) *should hold whenever*

$$\mathrm{supp}(f) \succ \mathrm{supp}(g) :\iff \nexists p \in \mathrm{supp}(f)\ \exists l \in \mathrm{supp}(g) : (p-l)^2 \geq 0 \wedge p^0 \leq l^0 \tag{3.130}$$

is fulfilled.

Equation (causality) also holds when I choose slightly different functions. Let $\varepsilon, \delta \in \mathbb{R}$, and let g, f be such that (causality) is satisfied then also

$$\left(S^{\varepsilon f}\right)^{-1} S^{\varepsilon f+\delta g} = \left(S^0\right)^{-1} S^{\delta g} \tag{3.131}$$

holds. Expanding equation (3.131) differentiating with respect to ε and δ once, one gets

$$0 = \tilde{T}_1(f)T_1(g) + T_2(f,g) =: A_1(f,g). \tag{3.132}$$

Exchanging f and g in equations (3.130) and (3.131) and taking the same derivatives, one gets

$$0 = \tilde{T}_1(g)T_1(f) + T_2(f,g) =: R_1(f,g). \tag{3.133}$$

I now extent the domain of A_1 and R_1 to all possible sets of two four-potentials and define another operator valued distribution by

$$D_1(f,g) := A(f,g) - R(f,g) = \tilde{T}_1(f)T_1(g) - \tilde{T}_1(g)T_1(f). \tag{3.134}$$

It can be inferred from above that $D_1(f,g)$ is zero if $f \succ g$ and $f \prec g$ are both true. Thus to obtain T_2, I first compute D_1 using only T_1 and

$\tilde{T}_1$, then I decompose D_1 into parts fulfilling the support properties of A_1 and R_1. Finally I subtract from the obtained operator $A_1(f,g)$ the expression $\tilde{T}_1(f)T_1(g)$. I will only work with vacuum expectation values, since it is easier and suffices to define T_2 uniquely.
Using $\tilde{T}_1 = -T_1$, and the closed expression (3.116) for T_1 and the commutation relations of the annihilation and creation operators one obtains

$$\langle \Omega, D_1(f,g)\Omega\rangle = -\operatorname{tr}\left(P_- Z_1(f) P_+ Z_1(g) P_-\right) + \operatorname{tr}\left(P_- Z_1(g) P_+ Z(f) P_-\right). \tag{3.135}$$

Expressing the traces in terms of integrals, using equation (3.14) together with a lengthy calculation reveals that

$$\begin{aligned}\langle \Omega, D_1(f,g)\Omega\rangle = -\frac{2\pi m^2}{3}\int_{\substack{k\in\mathbb{R}^4, k\in\text{Future}\\ k^2>4m^2}} \sqrt{1-\frac{4m^2}{k^2}}(k^2+2m^2) \\ (g^{\alpha\beta}-\frac{k^\alpha k^\beta}{k^2})\left(f_\alpha(k)g_\beta(-k) - f_\alpha(-k)g_\beta(k)\right)\ \mathrm{d}^4k \\ = -\frac{8\pi m^4}{3}\int_{k\in\mathbb{R}^4} d^{\alpha\beta}(k) f_\alpha(k) g_\beta(-k)\ \mathrm{d}^4k,\end{aligned} \tag{3.136}$$

holds, where d is given by

$$d^{\alpha\beta}(k) := I\left(\frac{k^2}{4m^2}\right) 1_{k^2>4m^2}(k)[\theta(k_0)-\theta(-k_0)]\left(g^{\alpha\beta}-\frac{k^\alpha k^\beta}{k^2}\right) \tag{3.137}$$

and I is given by

$$I(\kappa) := \sqrt{1-\frac{1}{\kappa}}\left(\kappa+\frac{1}{2}\right). \tag{3.138}$$

By $\text{Causal}_\pm \subset \mathbb{R}^4$ I denote the set such that all its elements fulfil $\zeta \in \text{Causal} \Rightarrow \zeta^2 \geq 0 \wedge \zeta^0 \in \mathbb{R}^\pm$. Now, to split up the distribution the following theorem comes in handy; it can be found as Theorem IX.16 in [?].

Theorem 3.6.3. *Paley-Wiener theorem for causal distributions:*

(A) *Let* $T \in \mathcal{S}'(\mathbb{R}^4)$ *with* $\operatorname{supp}(T) \subseteq Causal_\pm$ *and let* $\hat{T}$ *denote its Fourier transform. Then the following is true:*

(i) $\hat{T}(l + i\eta)$ *is analytic for* $l, \eta \in \mathbb{R}^4$ *and* $\eta^2 > 0 \in Causal^\circ_\pm$ *and* $\hat{T}$ *is the boundary value in the sense of* $\mathcal{S}'$*.*

(ii) *There is a polynomial* P *and an* $n \in \mathbb{N}$ *such that*

$$\left|\hat{T}(l + i\eta)\right| \leq |P(l + i\eta)| \left(1 + dist(\eta, \partial Causal_\pm)^{-n}\right). \tag{3.139}$$

(B) *Let* $\hat{F}(l + i\eta)$ *be analytic for* $l \in \mathbb{R}^4$ *and* $\eta \in Causal^\circ_\pm$ *and let* $\hat{F}$ *fulfil:*

(i) *For all* $\eta_0 \in Causal^\circ_\pm$ *there is a polynomial* P_{η_0} *such that for all* $l \in \mathbb{R}^4$ *and* $\eta \in Causal^\circ_\pm$

$$|\hat{F}(l + i(\eta + \eta_0))| \leq |P_{\eta_0}(l, \eta)|. \tag{3.140}$$

(ii) *There is an* $n \in \mathbb{N}$ *such that for all* $\eta_0 \in Causal^\circ_\pm$ *there is a polynomial* Q_{η_0} *with*

$$\forall \varepsilon > 0 : |\hat{F}(l + i\varepsilon\eta_0)| \leq \frac{|Q_{\eta_0}(l)|}{\varepsilon^n}. \tag{3.141}$$

Then there is a $T \in \mathcal{S}'$ *with* $\operatorname{supp} T \subset Causal_\pm$ *such that* T *is the boundary value of* $\hat{F}(l + i\eta)$ *in the sense of* $\mathcal{S}'$*, the relation between* $\hat{F}$ *and* T *being*

$$\hat{F}(l + i\eta) = \frac{1}{(2\pi)^2} \int \mathrm{d}^4x e^{-\eta x} e^{ilx} T(x) \tag{3.142}$$

for all $l \in \mathbb{R}^4$*,* $\eta \in Causal^\circ_\pm$ *and* $x \in \operatorname{supp}(T)$*.*

As an ansatz for the splitting I take

$$\hat{D}^{\alpha\beta}_{\pm} : \mathbb{R}^4 + i \cdot \mathrm{Causal}_{\pm} \to \mathbb{C}, \quad k \mapsto (g^{\alpha\beta} - \frac{k^\alpha k^\beta}{k^2}) J\left(\frac{k^2}{4m^2}\right), \tag{3.143}$$

where

$$J : \mathbb{C}\backslash\mathbb{R}^+_0 \to \mathbb{C}, \quad J(\kappa) := \frac{\kappa^2}{2\pi i}\int_1^\infty \mathrm{d}s \sqrt{1-\frac{1}{s}}\frac{s+\frac{1}{2}}{s^2(s-\kappa)} \tag{3.144}$$

and $\sqrt{\cdot}$ denotes the principal value of the square root with its branch cut at $\mathbb{R}^-_0$. Therefore J is well defined on its domain. Furthermore, $k = l + i\,\varepsilon\,\eta$ with $l \in \mathbb{R}^4$ and $\eta \in \mathrm{Causal}_{\pm}$ implies:

$$k^2 \in \mathbb{R} \Rightarrow k^2 = l^2 - \eta^2 + i\,\varepsilon l^\alpha \eta_\alpha \in \mathbb{R} \Rightarrow \left(l \perp \eta \wedge \eta^2 > 0 \Rightarrow l^2 \leq 0 \Rightarrow k^2 < 0\right). \tag{3.145}$$

Hence the argument of the square root $1 - \frac{1}{s}$ stays away from the branch cut and the denominator is never zero, therefore the integral on the right-hand side of equation (3.144) exists. Furthermore, $D^{\alpha\beta}_{\pm}(k)$ is holomorphic on its domain.

It can be shown using standard techniques of complex analysis that

$$d^{\alpha\beta}(l) = \lim_{\varepsilon\searrow 0}\left(D^{\alpha\beta}_{+}(l+i\varepsilon\eta) - D^{\alpha\beta}_{-}(l-i\varepsilon\eta)\right) \tag{3.146}$$

holds for almost all $l \in \mathbb{R}^4$.

Using similar techniques and Euler substitutions one finds the boundary value of $\hat{D}^{\alpha\beta}_{\pm}$. For almost all $l \in \mathbb{R}^4$ and $\eta \in \mathrm{Causal}_{\pm}$ it holds that

$$
\begin{aligned}
&\lim_{\varepsilon \to 0} \hat{D}_{\pm}^{\alpha\beta}(l + i\varepsilon\eta) = \\
&\quad \left(g^{\alpha\beta} - \frac{l^\alpha l^\beta}{l^2}\right)\left[\pm \mathbb{1}_{l^2 > 4m^2} \operatorname{sgn}(l^0) \frac{1}{2}\sqrt{1 - \frac{4m^2}{l^2}}\left(\frac{l^2}{4m^2} + \frac{1}{2}\right)\right. \\
&\left. + \frac{1}{2\pi i}\left(1 + \frac{5}{3}\frac{l^2}{4m^2} - \left(1 + \frac{l^2}{2m^2}\right)\sqrt{1 - \frac{4m^2}{l^2}} \arctan\left(\sqrt{\frac{l^2}{4m^2 - l^2}}\right)\right)\right].
\end{aligned}
\tag{3.147}
$$

This is not true for the arguments fulfilling $l^2 = 4m^2$; however, this is irrelevant since $\hat{D}_{\pm}$ is to be understood as a distribution which means that changes on sets of Lebesgue measure zero are of no concern.
In order not to convolute notation too much we define $\lim_{\varepsilon \to 0} \hat{D}_{\pm}^{\alpha\beta}(l + i\varepsilon\eta) =: \hat{D}_{\pm}^{\alpha\beta}(l)$, that is we extend the holomorphic functions $\hat{D}_{\pm}^{\alpha\beta}$ to the boundary of their domain.
Furthermore we will make use of the abbreviations

$$
\mathcal{K}^{\alpha\beta}(k) := \left(g^{\alpha\beta} - \frac{k^\alpha k^\beta}{k^2}\right) \tag{3.148}
$$

$$
\mathcal{R}(k) := \mathbb{1}_{k^2 > 4m^2}\sqrt{1 - \frac{4m^2}{k^2}}\left(\frac{k^2}{4m^2} + \frac{1}{2}\right) \tag{3.149}
$$

$$
\begin{aligned}
\tau(k) := \frac{1}{\pi i}&\left(1 + \frac{5}{3}\frac{k^2}{4m^2} - \left(1 + \frac{k^2}{2m^2}\right) \cdot \right. \\
&\left. \cdot \sqrt{1 - \frac{4m^2}{k^2}} \arctan\left(\sqrt{\frac{k^2}{4m^2 - k^2}}\right)\right).
\end{aligned}
\tag{3.150}
$$

Theorem 3.6.3 guarantees us that $\operatorname{supp} D_{\pm}^{\alpha\beta} \subset \mathrm{Causal}_{\pm}$ holds. We use this property to compare the support properties of $\operatorname{supp} D_{\pm}^{\alpha\beta}$ with the ones of A_1 and R_1 using the following short calculation: Substituting $\operatorname{supp} D_{+}^{\alpha\beta}$ for $d^{\alpha\beta}$ in (3.136) we see that

$$\langle\Omega, D_+(f,g)\Omega\rangle := -\frac{8\pi m^4}{3}\int_{\mathbb{R}^4} D_+^{\alpha\beta}(k) f_\alpha(k) g_\beta(-k) d^4k = \\ -\frac{2m^4}{3\pi}\int_{\mathbb{R}^4}\int_{\mathbb{R}^4} D^{\alpha\beta}(z-y) f_\alpha(y) g_\beta(z) d^4z d^4y \quad (3.151)$$

holds. So we see that D_+ vanishes whenever the support of g is earlier than the support of f or they lie acausally or a mixture of these conditions. That is it vanishes whenever $\operatorname{supp} f \succ \operatorname{supp} g$ holds, which is exactly the support property of A_1. Since the analogous treatment holds for D_- and R_1 and we know

$$\langle\Omega, D\Omega\rangle = \langle\Omega, A_1\Omega\rangle - \langle\Omega, R_1\Omega\rangle = \langle\Omega, D_+\Omega\rangle - \langle\Omega, D_-\Omega\rangle,$$

we identify $\langle\Omega, A_1\Omega\rangle = \langle\Omega, D_+\Omega\rangle$ and $\langle\Omega, R_1\Omega\rangle = \langle\Omega, D_-\Omega\rangle$. Going back to the definition of A_1 we can now find $\langle\Omega, T_2\Omega\rangle$. We reuse the calculation resulting in (3.136) to find

$$\langle\Omega, T_2(f,g)\Omega\rangle = \langle\Omega, A_1(f,g)\Omega\rangle - \langle\Omega, T_1^\dagger(f) T_1^\dagger(g)\Omega\rangle \\ = -\frac{8\pi m^4}{3}\int_{\mathbb{R}^4} D_+^{\alpha\beta}(k) f_\alpha(k) g_\beta(-k) d^4k - \frac{8\pi m^4}{3}\int_{\{k\in\mathbb{R}^4 | k^0>0, k^2>4m^2\}} \\ \sqrt{1-\frac{4m^2}{k^2}}\left(\frac{k^2}{4m^2}+\frac{1}{2}\right)\left(g^{\alpha\beta}-\frac{k^\alpha k^\beta}{k^2}\right) f_\alpha(-k) g_\beta(k) d^4k \\ = -\frac{8\pi m^4}{3}\int_{\mathbb{R}^4} f_\alpha(k) g_\beta(-k)\mathcal{K}^{\alpha\beta}(k)\left[\mathcal{R}(k)\left\{\frac{1}{2}\operatorname{sgn}\left(k^0\right) + 1_{k^0<0}\right\}\right. \\ \left. +\frac{1}{2}\tau(k)\right] = \int_{\mathbb{R}^4} f_\alpha(k) g_\beta(-k) t_2^{\alpha\beta}(k) d^4k, \quad (3.152)$$

with

$$t_2^{\alpha\beta}(k) := -\frac{4\pi m^4}{3}\mathcal{K}^{\alpha\beta}(k)\left[\mathcal{R}(k)+\tau(k)\right] \quad (3.153)$$

Chapter 4

Mathematical Justification

Appendix A

One Particle S-Matrix; Explicit Bounds

We find the estimates of Z_k by using (3.14). Let $\psi \in \mathcal{H}$ arbitrary, Σ be an arbitrary spacelike hypersurface in Minkowski space,

$$
\begin{aligned}
\langle\psi|\ Z_k\phi(y)\rangle = \int_\Sigma \bar{\psi}(y)\ \ i_\gamma(\mathrm{d}^4 y)(-i)\int_{\mathcal{M}} \frac{i_p(\mathrm{d}^4 p_1)}{(2\pi)^3}\frac{\not{p}_1 + m}{2m}e^{-ip_1 y} \\
\prod_{l=2}^{k}\left[\int_{\mathbb{R}^4+i\epsilon e_0}\frac{\mathrm{d}^4 p_l}{(2\pi)^4}\not{A}(p_{l-1}-p_l)(\not{p}_l - m)^{-1}\right] \\
\int_{\mathcal{M}} i_p(\mathrm{d}^4 p_{k+1})\not{A}(p_k - p_{k+1})\hat{\phi}(p_{k+1}) = -i\int_{\mathcal{M}}\frac{i_p(\mathrm{d}^4 p_1)}{(2\pi)^{\frac{3}{2}}}\bar{\psi}(p_1)\ \ \frac{\not{p}_1 + m}{2m} \\
\prod_{l=2}^{k}\left[\int_{\mathbb{R}^4+i\epsilon e_0}\frac{\mathrm{d}^4 p_l}{(2\pi)^4}\not{A}(p_{l-1}-p_l)(\not{p}_l - m)^{-1}\right]
\end{aligned}
$$

$$\int_{\mathcal{M}} i_p(\mathrm{d}^4 p_{k+1}) \not{A}(p_k - p_{k+1}) \hat{\phi}(p_{k+1}) = -i \int_{\mathcal{M}} \frac{i_p(\mathrm{d}^4 p_1)}{(2\pi)^{\frac{3}{2}}} \overline{\frac{\not{p}_1 + m}{2m} \psi(p_1)}$$
$$\prod_{l=2}^{k} \left[\int_{\mathbb{R}^4 + i\epsilon e_0} \frac{\mathrm{d}^4 p_l}{(2\pi)^4} \not{A}(p_{l-1} - p_l)(\not{p}_l - m)^{-1} \right]$$
$$\int_{\mathcal{M}} i_p(\mathrm{d}^4 p_{k+1}) \not{A}(p_k - p_{k+1}) \hat{\phi}(p_{k+1}) = -i \int_{\mathcal{M}} \frac{i_p(\mathrm{d}^4 p_1)}{(2\pi)^{\frac{3}{2}}} \overline{\psi}(p_1)$$
$$\prod_{l=2}^{k} \left[\int_{\mathbb{R}^4 + i\epsilon e_0} \frac{\mathrm{d}^4 p_l}{(2\pi)^4} \not{A}(p_{l-1} - p_l)(\not{p}_l - m)^{-1} \right]$$
$$\int_{\mathcal{M}} i_p(\mathrm{d}^4 p_{k+1}) \not{A}(p_k - p_{k+1}) \hat{\phi}(p_{k+1})$$

We therefore find for the operator norm of Z_k:

$$\|Z_k\| = \sup_{\psi,\phi \in \mathcal{H}} \frac{|\langle \psi |\, Z_k \phi(y) \rangle|}{\|\psi\| \|\phi\|} = \sup_{\psi,\phi \in \mathcal{H}} \left| \int_{\mathcal{M}} \frac{i_p(\mathrm{d}^4 p_1)}{(2\pi)^{\frac{3}{2}}} \frac{\overline{\psi}(p_1)}{\|\psi\|} \right.$$
$$\prod_{l=2}^{k} \left[\int_{\mathbb{R}^4 + i\epsilon e_0} \frac{\mathrm{d}^4 p_l}{(2\pi)^4} \not{A}(p_{l-1} - p_l)(\not{p}_l - m)^{-1} \right]$$
$$\left. \int_{\mathcal{M}} i_p(\mathrm{d}^4 p_{k+1}) \not{A}(p_k - p_{k+1}) \frac{\hat{\phi}(p_{k+1})}{\|\phi\|} \right|$$
$$\overset{\text{C.S.I.}}{\leq} \sup_{\phi \in \mathcal{H}} \int_{\mathcal{M}} \frac{i_p(\mathrm{d}^4 p_1)}{(2\pi)^3} \left| \prod_{l=2}^{k} \left[\int_{\mathbb{R}^4 + i\epsilon e_0} \frac{\mathrm{d}^4 p_l}{(2\pi)^4} \not{A}(p_{l-1} - p_l)(\not{p}_l - m)^{-1} \right] \right.$$
$$\left. \int_{\mathcal{M}} i_p(\mathrm{d}^4 p_{k+1}) \not{A}(p_k - p_{k+1}) \frac{\hat{\phi}(p_{k+1})}{\|\phi\|} \right|^2$$
$$\leq \sup_{\phi \in \mathcal{H}} \int_{\mathcal{M}} \frac{i_p(\mathrm{d}^4 p_1)}{(2\pi)^3} \left(\prod_{l=2}^{k} \left[\int_{\mathbb{R}^4 + i\epsilon e_0} \frac{\mathrm{d}^4 p_l}{(2\pi)^4} \|\not{A}(p_{l-1} - p_l)\|_{\mathrm{spec}} \right. \right.$$

$$
\|(\not{p}_l - m)^{-1}\|_{\text{spec}}\Big] \int_{\mathcal{M}} i_p(\mathrm{d}^4 p_{k+1}) \|\not{A}(p_k - p_{k+1})\|_{\text{spec}} \frac{\|\hat{\phi}(p_{k+1})\|}{\|\phi\|}\Bigg)^2
$$

$$
\leq \sup_{\lambda\in\mathbb{R}^4+i\epsilon e_0} \|(\not{\lambda}_l - m)^{-1}\|^{k-1}_{\text{spec}} \sup_{\phi\in\mathcal{H}} \int_{\mathcal{M}} \frac{i_p(\mathrm{d}^4 p_1)}{(2\pi)^3} \Bigg(\prod_{l=2}^{k} \Big[\int_{\mathbb{R}^4+i\epsilon e_0} \frac{\mathrm{d}^4 p_l}{(2\pi)^4}
$$

$$
\|\not{A}(p_{l-1} - p_l)\|_{\text{spec}}\Big] \int_{\mathcal{M}} i_p(\mathrm{d}^4 p_{k+1}) \|\not{A}(p_k - p_{k+1})\|_{\text{spec}} \frac{\|\hat{\phi}(p_{k+1})\|}{\|\phi\|}\Bigg)^2
$$

$$
\overset{\text{section A.1}}{\leq} \left(\frac{2}{\epsilon}\right)^{k-1} \sup_{\phi\in\mathcal{H}} \int_{\mathcal{M}} \frac{i_p(\mathrm{d}^4 p_1)}{(2\pi)^3} \Bigg(\prod_{l=2}^{k} \Big[\int_{\mathbb{R}^4+i\epsilon e_0} \frac{\mathrm{d}^4 p_l}{(2\pi)^4}
$$

$$
\|\not{A}(p_{l-1} - p_l)\|_{\text{spec}}\Big] \int_{\mathcal{M}} i_p(\mathrm{d}^4 p_{k+1}) \|\not{A}(p_k - p_{k+1})\|_{\text{spec}} \frac{\|\hat{\phi}(p_{k+1})\|}{\|\phi\|}\Bigg)^2 . \tag{A.1}
$$

Where we assumed ε to be large enough so that the estimate in section A.1 holds. The following estimation is only valid for $k > 1$. We now apply the theorem of Parley and Wiener (e.g. [?]) to all occurrences of $\not{A}$. Since A is compactly supported in Minkowski spacetime its Fourier transform fulfills:

$$
\forall N \in \mathbb{N} : \exists C_N \in \mathbb{R} : \forall p \in \mathbb{C}^4 \|\hat{\not{A}}\|(p) \leq \frac{C_N 8\pi}{1 + |p|^N} e^{\frac{1}{2}|\Im p| \mathrm{diam}(A)}, \tag{A.2}
$$

where $\mathrm{diam}(A)$ is the diameter of the support of A in Minkowski spacetime and the constant in the numerator was slightly modified to simplify our notation.

$$
\leq \left(\frac{C_N}{\epsilon}\right)^{k-1} e^{\epsilon \mathrm{diam}(\mathrm{supp}(A))} \frac{1}{\pi^2} \sup_{\phi\in\mathcal{H}} \int_{\mathcal{M}} i_p(\mathrm{d}^4 p_1)
$$

$$
\Bigg(\prod_{l=2}^{k} \Big[\int_{\mathbb{R}^4+i\epsilon e_0} \mathrm{d}^4 p_l \frac{1}{(1 + |p_{l-1} - p_l|)^N}\Big] \int_{\mathcal{M}} i_p(\mathrm{d}^4 p_{k+1}) \frac{1}{(1 + |p_k - p_{k+1}|)^N} \frac{\|\hat{\phi}(p_{k+1})\|}{\|\phi\|}\Bigg)^2
$$

$$= \left(\frac{C_N}{\epsilon}\right)^{k-1} e^{\epsilon \operatorname{diam}(\operatorname{supp}(A))} \frac{1}{\pi^2}$$
$$\sup_{\phi \in \mathcal{H}} \left\| \mathop{\Huge *}_{\substack{l=2 \\ \mathbb{R}^4}}^{k} \left[\frac{1}{(1+|\cdot|)^N}, \frac{1}{(1+|\cdot|)^N} \overset{\mathcal{M}}{*} \frac{\|\hat{\phi}(\cdot)\|}{\|\phi\|} \right] \right\|_{\mathcal{L}^2(\mathcal{M})} \tag{A.3}$$

We are going to use Young's inequality for convolution operators acting $L^2(\mathcal{M}) \to L^2(\mathcal{M})$, the appropriate lemma is found in appendix A.2

$$\overset{\text{Young Inequ. Raum?!}}{\leq} \left(\frac{1}{\sqrt{2}\epsilon}\right)^{k-1} e^{\epsilon \operatorname{diam}(\operatorname{supp}(A))} \frac{C_N^k}{\pi^{4k-1}} \left\| \frac{1}{(1+|\cdot|)^N} \right\|_{\mathcal{L}(\mathbb{R}^4)}^{k-1}$$
$$\left\| \frac{1}{(1+|\cdot|)^N} \right\|_{\mathcal{L}(\mathcal{M})} \sup_{\phi \in \mathcal{H}} \left\| \frac{\|\hat{\phi}(\cdot)\|}{\|\phi\|} \right\|_{\mathcal{L}^2(\mathcal{M})}$$
$$= \left(\frac{1}{\sqrt{2}\epsilon}\right)^{k-1} e^{\epsilon \operatorname{diam}(\operatorname{supp}(A))} \frac{C_N^k}{\pi^{4k-1}} \left\| \frac{1}{(1+|\cdot|)^N} \right\|_{\mathcal{L}(\mathbb{R}^4)}^{k-1} \left\| \frac{1}{(1+|\cdot|)^N} \right\|_{\mathcal{L}(\mathcal{M})} \tag{A.4}$$

Where C_N is the constant obtained by application of the theorem of Parley an Wiener, ϵ is still an arbitrary positive number. This is why we now optimise over this parameter. In order to simplify the notation we define $a := \operatorname{diam}(\operatorname{supp}(A))$, $b := k-1$, $f := \left\| \frac{1}{(1+|\cdot|)^N} \right\|_{\mathcal{L}(\mathbb{R}^4)}$, $g := \left\| \frac{1}{(1+|\cdot|)^N} \right\|_{\mathcal{L}(\mathcal{M})}$.

$$h : \mathbb{R}^+ \to \mathbb{R}, \qquad \epsilon \mapsto \frac{e^{a\epsilon}}{\epsilon^b} \tag{A.5}$$

h is a smooth positive function which diverges at zero and at infinity, so it must attain a minimum somewhere in between. We find this minimum by elementary calculus:

$$h'(\epsilon) \stackrel{!}{=} 0 \iff -b\frac{e^{a\epsilon}}{\epsilon^{b+1}} + a\frac{e^{a\epsilon}}{\epsilon^{b}} = 0 \iff -b + a\epsilon = 0 \iff \epsilon = \frac{b}{a} \tag{A.6}$$

Therefore the value of the minimum is:

$$\inf_{\epsilon\in\mathbb{R}^+} h(\epsilon) = h(\frac{b}{a}) = \frac{e^b}{\left(\frac{b}{a}\right)^b} = \frac{(ae)^b}{b^b} \tag{A.7}$$

Which means for the operator norm of Z_k, $k > 1$:

$$\|Z_k\| \leq \left(\frac{1}{\sqrt{2}}\right)^{k-1} \frac{(ea)^{k-1}}{(k-1)^{k-1}} \frac{C_N^k}{\pi^{4k-1}} f^{k-1} g \tag{A.8}$$

This means that we can find the operator norm of the S operator, once we have read off the operator norm of Z_1. In order to do so, we start at the end of (A.1) and use the Young inequality right away to find:

$$\|Z_1\| \leq \left\| \|\not{A}\|_{spec} \right\|_{\mathcal{L}^1(\mathcal{M})} \tag{A.9}$$

Which is finite, because A is compactly supported, which means that its Fouriertransform falls off at infinity faster than any polynomial. We will be using the well known upper bound for the factorial of an arbitrary number:

$$n! \leq \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{1}{12n}} \tag{A.10}$$

We will employ the abbreviation $w = \frac{aC_N f}{\pi^4\sqrt{2}}$

$$
\begin{aligned}
\|S\| &= \left\|\sum_{k=0}^{\infty} Z_k\right\| \leq \sum_{k=0}^{\infty} \|Z_k\| \leq 1+\big\|\|\not{A}\|_{spec}\big\|_{\mathcal{L}^1(\mathcal{M})}+\sum_{k=2}^{\infty}\left(\frac{1}{\sqrt{2}}\right)^{k-1}\frac{(ea)^{k-1}}{(k-1)^{k-1}}\frac{C}{\pi^{4k}} \\
&= 1+\big\|\|\not{A}\|_{spec}\big\|_{\mathcal{L}^1(\mathcal{M})}+g\frac{C_N}{\pi^3}\sum_{k=2}^{\infty}\frac{(we)^{k-1}}{(k-1)^{k-1}} = 1+\big\|\|\not{A}\|_{spec}\big\|_{\mathcal{L}^1(\mathcal{M})}+g\frac{C_N}{\pi^3}\sum_{k=1}^{\infty}\frac{w^k}{\left(\frac{k}{e}\right)^k} \\
&\overset{(A.10)}{\leq} 1+\big\|\|\not{A}\|_{spec}\big\|_{\mathcal{L}^1(\mathcal{M})}+g\frac{C_N}{\pi^3}\sum_{k=1}^{\infty}\frac{w^k}{k!}e^{\frac{1}{12k}}\sqrt{2\pi k} \\
&\overset{e^{\frac{1}{12k}}\leq\sqrt{k}e^{\frac{1}{12}}}{\leq} 1+\big\|\|\not{A}\|_{spec}\big\|_{\mathcal{L}^1(\mathcal{M})}+e^{\frac{1}{12}}g\frac{C_N\sqrt{2}}{\pi^{\frac{5}{2}}}\sum_{k=1}^{\infty}\frac{w^k}{k!}k \\
&= 1+\big\|\|\not{A}\|_{spec}\big\|_{\mathcal{L}^1(\mathcal{M})}+e^{\frac{1}{12}}g\frac{wC_N\sqrt{2}}{\pi^{\frac{5}{2}}}\sum_{l=0}^{\infty}\frac{w^l}{l!} = 1+\big\|\|\not{A}\|_{spec}\big\|_{\mathcal{L}^1(\mathcal{M})}+e^{\frac{1}{12}}g\frac{wC_N\sqrt{2}}{\pi^{\frac{5}{2}}}e^{w} \\
&= 1+\big\|\|\not{A}\|_{spec}\big\|_{\mathcal{L}^1(\mathcal{M})}+2\pi^{\frac{3}{2}}agfC_N^2e^{\frac{aC_Nf}{\pi^4\sqrt{2}}+\frac{1}{12}} \\
&= 1+\big\|\|\not{A}\|_{spec}\big\|_{\mathcal{L}^1(\mathcal{M})}+\left\|\frac{1}{(1+|\cdot|)^N}\right\|_{\mathcal{L}(\mathbb{R}^4)}\left\|\frac{1}{(1+|\cdot|)^N}\right\|_{\mathcal{L}(\mathcal{M})}2\pi^{\frac{3}{2}}\operatorname{diam}(\operatorname{supp}(A))C_N^2 \\
&\qquad e^{\frac{\operatorname{diam}(\operatorname{supp}(A))C_N\left\|\frac{1}{(1+|\cdot|)^N}\right\|_{\mathcal{L}(\mathbb{R}^4)}}{\pi^4\sqrt{2}}+\frac{1}{12}} < \infty
\end{aligned} \tag{A.11}
$$

A.1 Bound on $\|(\not{\lambda}-m)^{-1}\|_{\mathbf{spec}}$

In this section we will find an upper bound on the supremum over all $\lambda \in \mathbb{R}^4 + i\varepsilon e_0$ of

$$\|(\not{\lambda}-m)^{-1}\|_{\text{spec}} = \left\|\frac{\not{\lambda}+m}{\lambda^2-m^2}\right\|_{\text{spec}}. \tag{A.12}$$

In order to do so, we will find a lower bound on the inverse of the expression in question. To simplify the notation call $\left(\Re\lambda^0\right)^2 = x \geq 0$

and write out $\Im\lambda = \varepsilon e_0$ explicitly. Since the problem is symmetric in λ^0 this suffices. Furthermore, since nothing depends on the direction of $\vec{\lambda}$, the problem is really just two-dimensional. Therefore we define $r := \|\vec{\lambda}\|^2 > 0$ and will only speak of these quantities from now on. The object to minimize is

$$f_0(x,r) := \frac{\sqrt{(x-r-\varepsilon^2-m^2)^2+4\varepsilon^2 x}}{\sqrt{x+\varepsilon^2+r}+m}. \tag{A.13}$$

We continue with the triangular inequality in the denominator and the concavity of the square root in the numerator giving.

$$\begin{aligned} f_0(x,r) \geq f_1(x,r) &:= \frac{\frac{1}{\sqrt{2}}\left|x-r-\varepsilon^2-m^2\right| + \frac{1}{\sqrt{2}}2\varepsilon\sqrt{x}}{\sqrt{x}+\sqrt{r}+m+\varepsilon} \\ &= \frac{1}{\sqrt{2}}\frac{\left|x-r-\varepsilon^2-m^2\right|+2\varepsilon\sqrt{x}}{\sqrt{x}+\sqrt{r}+m+\varepsilon}. \end{aligned} \tag{A.14}$$

In order to find the minimum of this expression we will use the following strategy. First we find stationary points in $M^+ := \{(x,r) \in \mathbb{R}^{+2} \mid x > r+\varepsilon^2+m^2\}$ and $M^+ := \{(x,r) \in \mathbb{R}^{+2} \mid x < r+\varepsilon^2+m^2\}$, since there may be Minima on the boundary between these sets we also minimize f_1 in $M^0 := \{(x,r) \in \mathbb{R}^{+2} \mid x = r+\varepsilon^2+m^2\}$. Finally, since there might be no minimum, we find estimates on the boundary of $M^+ \cup M^- \cup M^0$.

case a) $x > r+\varepsilon^2+m^2$:

The gradient of f_1 is

$$\sqrt{2}(\sqrt{x}+\sqrt{r}+m+\varepsilon)^2\nabla f_1(x,r) = \begin{pmatrix} \frac{1}{2}\sqrt{x}+\sqrt{r}+m+\varepsilon+\varepsilon\frac{m+\sqrt{r}}{\sqrt{x}}+\frac{r+m^2+3\varepsilon^2}{2\sqrt{x}} \\ -\sqrt{x}-\frac{\sqrt{r}}{2}-m-\varepsilon-\frac{x-\varepsilon^2-m^2}{2\sqrt{r}}-\varepsilon\sqrt{\frac{x}{r}} \end{pmatrix},$$

since the first element of this vector is always positive, there are no stationary points in this case.

case b) $x < r + \varepsilon^2 + m^2$:

Here the gradient takes the form

$$\sqrt{2}(\sqrt{x}+\sqrt{r}+m+\varepsilon)^2\nabla f_1(x,r) =$$
$$\begin{pmatrix} \frac{-\sqrt{x}}{2} - \sqrt{r} - m - \varepsilon + \varepsilon\frac{\sqrt{r}+m}{\sqrt{x}} - \frac{m^2-\varepsilon^2+r}{2\sqrt{x}} \\ +\sqrt{x} + \frac{\sqrt{r}}{2} + m + \varepsilon - \frac{m^2+\varepsilon^2-x}{2\sqrt{r}} - \varepsilon\sqrt{\frac{x}{r}} \end{pmatrix}$$
$$= \begin{pmatrix} \frac{-1}{\sqrt{x}}\left(\frac{x}{2}+\frac{r}{2}+\sqrt{xr}-\varepsilon(\sqrt{r}+m)+\sqrt{x}(m+\varepsilon)+\frac{m^2-\varepsilon^2}{2}\right) \\ \frac{1}{\sqrt{r}}\left(\frac{x}{2}+\frac{r}{2}+\sqrt{xr}+\sqrt{r}(m+\varepsilon)-\varepsilon\sqrt{x}-\frac{m^2+\varepsilon^2}{2}\right) \end{pmatrix},$$

we can read off the relation

$$\sqrt{x^*} = \sqrt{r^*} + \frac{m}{2}\frac{1-\frac{m}{\varepsilon}}{1+\frac{m}{2\varepsilon}} =: \sqrt{r} + \frac{m}{2}c, \tag{A.15}$$

which holds for stationary points (x^*, r^*) and use it to solve for them. If we want to make sure that the stationary point stays within M^- we have to ensure that $x^* < r^* + m^2 + \varepsilon^2$ for $(x^*, r^*$ being a solution to $\nabla f_1(x,r) = 0$. This results in the condition

$$r < \frac{1}{m^2}\left[\frac{\varepsilon^2 + m^2(1-\frac{1}{4}c^2)}{c}\right]^2 = \frac{\varepsilon^4}{m^2} + \mathcal{O}(\varepsilon^2).$$

Since for the estimation of the one particle scattering matrix we are interested in the regime where ε, this is the relevant estimation. We will shortly see that $r^* = \mathcal{O}(\varepsilon^2)$, therefore we need not worry about the stationary point being outside of M^- for ε large. Indeed, plugging the relation (A.15) into $\nabla f_1(x^*, r^*) \stackrel{!}{=} 0$ and solving for r^* we find

$$\sqrt{r^*} = -\frac{m}{4}(c+1) + \frac{1}{2}\sqrt{\varepsilon^2 + \frac{\varepsilon c m}{4} + \frac{m^2}{4}(5+2c)}.$$

One can immediately see that the right hand side is actually positive once one has restored the summand $\frac{m^2}{4}(c+1)^2$ in the discriminant. By substituting Taylor's where appropriate we find for x^*, r^*:

$$\begin{aligned}
\sqrt{r^*} &= \frac{\varepsilon}{2} - \frac{3}{8}m + \frac{m^2}{\varepsilon}\frac{91}{128} + \mathcal{O}\left(\frac{1}{\varepsilon}\right) \\
\sqrt{x^*} &= \frac{\varepsilon}{2} + \frac{1}{8}m - \frac{m^2}{\varepsilon}\frac{5}{128} + \mathcal{O}\left(\frac{1}{\varepsilon}\right) \\
x^* &= \frac{\varepsilon^2}{4} + \varepsilon m\frac{1}{8} - m^2\frac{3}{128} + \mathcal{O}\left(\frac{1}{\varepsilon}\right) \\
r^* &= \frac{\varepsilon^2}{4} - \varepsilon m\frac{3}{8} + m^2\frac{109}{128} + \mathcal{O}\left(\frac{1}{\varepsilon}\right),
\end{aligned}$$

yielding for the stationary point

$$f_1(x^*, r^*) = \frac{\varepsilon}{\sqrt{2}} + \frac{m}{4\sqrt{2}} + \mathcal{O}\left(\frac{1}{\varepsilon}\right). \tag{A.16}$$

case c) $x = r + \varepsilon^2 + m^2$:

Plugging this into f_1 gives us

$$f_1(r+\varepsilon^2+m^2, r) =: f_2(r) = \sqrt{2}\varepsilon\frac{\sqrt{r+\varepsilon^2+m^2}}{\sqrt{r+\varepsilon^2+m^2}+\sqrt{r}+m+\varepsilon}$$

to minimise. The derivative of this function is given by

$$\begin{aligned}
&\sqrt{2}(\sqrt{r+\varepsilon^2+m^2}+\sqrt{r}+m+\varepsilon)^2 f_2'(r) \\
&\qquad\qquad = \varepsilon\left(\frac{\sqrt{r}+m+\varepsilon}{\sqrt{r+m^2+\varepsilon^2}} - \sqrt{1+\frac{m^2+\varepsilon^2}{r}}\right).
\end{aligned}$$

From the derivative we can read off that the function has a minimum at $\sqrt{r^*} = \varepsilon\frac{1+\frac{m^2}{\varepsilon^2}}{1+\frac{m}{\varepsilon}}$. We estimate this value, its square and the minimum to be

$$\begin{aligned}\sqrt{r^*} &= \varepsilon - m + \mathcal{O}\left(\frac{1}{\varepsilon}\right)\\ r^* &= \varepsilon^2 - 2\varepsilon m + \mathcal{O}\left(1\right)\\ f_2(r^*) &= (2-\sqrt{2})\varepsilon - (3-2\sqrt{2})m + \mathcal{O}\left(\frac{1}{\varepsilon}\right).\end{aligned} \tag{A.17}$$

case d) $x = 0$:

In this case f_1 simplifies to

$$f_1(0,r) =: f_3(r) = \frac{1}{\sqrt{2}}\frac{r+\varepsilon^2+m^2}{\sqrt{r}+m+\varepsilon},$$

its derivative is given by

$$2\sqrt{2}(\sqrt{r}+m+\varepsilon)^2\sqrt{r}f_3'(r) = r + 2\sqrt{r}(m+\varepsilon) - \varepsilon^2 - m^2.$$

So we read off that f_3 has a minimum at $\sqrt{r^*} = -m - \varepsilon + \sqrt{2}\sqrt{\varepsilon^2+\varepsilon m+m^2}$. The same approximations as above yield

$$\begin{aligned}\sqrt{r^*} &= \varepsilon(\sqrt{2}-1) - \frac{2-\sqrt{2}}{2}m + \mathcal{O}\left(\frac{1}{\varepsilon}\right)\\ r^* &= \varepsilon^2(3-2\sqrt{2}) - (3\sqrt{2}-4)\varepsilon m + \mathcal{O}(1)\\ f_3(r^*) &= \varepsilon(2+\sqrt{2}) - m\left(\frac{3}{2\sqrt{2}} - 1\right) + \mathcal{O}\left(\frac{1}{\varepsilon}\right).\end{aligned} \tag{A.18}$$

case e) $x \to \infty$:

In this case f_1 diverges to $+\infty$.

case f) $r = 0$:

In this case f_1 reduces to

$$f_1(x,0) := f_4(x) = \frac{1}{\sqrt{2}} \frac{|x - \varepsilon^2 - m^2| + 2\varepsilon\sqrt{x}}{\sqrt{x} + m + \varepsilon},$$

for $x \neq \varepsilon^2 + m^2$ its derivative is given by

$$\begin{aligned}\sqrt{2x}(\sqrt{x} + m + \varepsilon)^2 f_4'(x) \\ &= \frac{1}{2}\mathrm{sgn}(x - \varepsilon^2 - m^2)(x + \varepsilon^2 + m^2) \\ &\quad + \sqrt{x}(m+\varepsilon)\mathrm{sgn}(x - \varepsilon^2 - m^2) + \varepsilon m + \varepsilon^2.\end{aligned}$$

For ε large this function has a minimum at the kink and a maximum between 0 and $\varepsilon^2 + m^2$. So we take note of the minimum at the kink and the minimum for $x \to 0$. These values are

$$f_4(0) = \frac{\varepsilon}{\sqrt{2}} \frac{1 + \frac{m^2}{\varepsilon^2}}{1 + \frac{m}{\varepsilon}} = \frac{\varepsilon}{\sqrt{2}} - \frac{m}{\sqrt{2}} + \mathcal{O}\left(\frac{1}{\varepsilon}\right) \tag{A.19}$$

$$f_4(\varepsilon^2 + m^2) = \frac{\sqrt{2}\varepsilon}{1 + \frac{1+\frac{m}{\varepsilon}}{\sqrt{1+\frac{m^2}{\varepsilon^2}}}} = \frac{\varepsilon}{\sqrt{2}} - \frac{m}{2\sqrt{2}} + \mathcal{O}\left(\frac{1}{\varepsilon}\right) \tag{A.20}$$

case g) $r \to \infty$:

In this case holds $f_1 \to \infty$.

case h) simultaneous limits $x, r \to \infty$. The non trivial limits $\sqrt{x} = \sqrt{r} + c'$ for $c' \in \mathbb{R}$ and $x - r = c''$ for $c'' \in \mathbb{R}$ all give limits equal to or greater than $\frac{\varepsilon}{\sqrt{2}}$.

Therefore the global minimum is the minimum of case c), which is $(2 - \sqrt{2})\varepsilon - (3 - 2\sqrt{2})m + \mathcal{O}\left(\frac{1}{\varepsilon}\right)$. So for ε large enough relative to m

we found the lower bound $\frac{\varepsilon}{2}$ of f_1. So overall

$$\sup_{\lambda \in \mathbb{R}^4 + \varepsilon i e_0} \left\| (\not{\lambda} - m)^{-1} \right\|_{spec} \leq \frac{2}{\varepsilon} \tag{A.21}$$

holds.

A.2 Young's Inequality on $L^2(\mathcal{M})$

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