The Relationship Between Hadamard States, the Fermionic Projector, and Admissible Polarisation Classes

D.-A. Deckert[*], Felix Finster[†] and Markus Nöth[‡]

October 22, 2019

Abstract

This paper compares central objects of three different types of approaches to quantum field theory (QFT): The study of Hadamard states, the Fermionic Projector, and admissible polarisation classes. These approaches have different scopes and pursue different motivations, which makes a direct comparison difficult. Nevertheless, the basic protagonists in all three approaches share much more similarity than is apparent at first glance. It is the purpose of this work to highlight these common features. Since they all study different aspects of QFT; however, one can make out objects in each of these theories which closely resemble one another.

1 Introduction

The first class of objects of interest are Hadamard states which appear in the algebraic approach to QFT [8].

The third class of objects are the continuum limit of the Fermionic Projectors [7].

The second class of objects are the (I_2-almost) projectors P_Σ^λ which are closely linked to polarisation classes of the vacuum of external field quantum electrodynamics (QED) [3, 5, 4].

In section 2 we give the definition of a Hadamard state, briefly motivate it's usage and give its explicit form in the case of flat spacetime subject to an external field as computed in the physics literature [10].

In section 3 ...

In section 4 we briefly describe why in the approach of admissible polarisation classes one only keeps track of the time evolution of the projector up to an error that is a Hilbert-Schmidt operator. Furthermore we will find a class of candidate I_2-almost projectors that have a simple time evolution.

In section 5...comparison fermionic projector and hadamard state

In section 6 we find that each Hadamard state corresponds to an I_2-almost projector in a natural way.

Throughout the paper $\Sigma, \Sigma', \Sigma''$ denote arbitrary Cauchy surfaces, while for the sake of simplicity we choose $A \in C_c^\infty(\mathbb{R}^4, \mathbb{R}^4)$ to be a four-potential and Σ_{in} denotes a Cauchy surface earlier than the support of (A).

is that specific enough?

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[*]deckert@math.lmu.de
Mathematisches Institut der Ludwig-Maximilians-Universität München, Theresienstr. 39, 80333 München, Germany

[†]felix.finster@mathematik.uni-regensburg.de
Universität Regensburg, Universitätsstraße 31, 93053 Regensburg, Germany

[‡]noeth@math.lmu.de
Mathematisches Institut der Ludwig-Maximilians-Universität München, Theresienstr. 39, 80333 München, Germany

In this paper we will focus on a system of Dirac fields subject to an external electromagnetic four-potential A in flat Minkowski spacetime. We choose the sign convention $\eta = \mathrm{diag}(+1, -1, -1, -1)$. We denote the minimally coupled differential by $\nabla_\alpha = \partial_\alpha + iA_\alpha$ and make use of the Feynman slash notation $\slashed{\nabla} =: \nabla_\alpha \gamma^\alpha$ with γ^α fulfilling $\{\gamma^\alpha, \gamma^\beta\} := \gamma^\alpha \gamma^\beta + \gamma^\beta \gamma^\alpha = 2\eta^{\alpha,\beta}$ each field solving Dirac's equation

$$(i\slashed{\nabla} - m)\psi =: D\psi = 0 \tag{1}$$

and collectively constituting the free vacuum prior to the support of said external field. For a 4-spinor $\psi \in \mathbb{C}^4$ (viewed as a column vector), $\overline{\psi}$ stands for the row vector $\psi^* \gamma^0$, where $*$ denotes hermitian conjugation.

> macht das überhaupt Sinn so wenn ich von Dirac Felder spreche? Bei operatoren ist ja der Zustand nicht der Zustand des Feldes, sondern der Zustand ist extra.

2 Hadamard States

In the algebraic approach to QFT one puts less emphasise on the Hilbert space than is commonly done in non relativistic physics because it is not a relativistically invariant object. Instead one focuses on the algebra of operators that are chosen to do the bookkeeping of statistical outcomes of measurements.

To infer predictions, some part of the necessary computation can be conducted on the level of this algebra. However, eventually, expectation values are to be computed in a certain representation, usually found by the GNS construction with respect to a certain state. This choice has to be made on physical grounds. Hadamard states are often thought to be physically sensible states because they have positive energy in a certain sense.

In order to introduce Hadamard states we have to first define the notion of wavefront set, which itself needs some preliminaries. For the introduction of these concepts we follow Hömander [9, Chapter 8].

We begin by introducing the singular support of a distribution

Definition 1. *Let for $n, m \in \mathbb{N}$, $v \in (C^\infty(\mathbb{R}^n, \mathbb{C}^m))'$ the singular support of v is defined to be the subset of points $x \in \mathbb{R}^n$ such that there is no neighbourhood U of x such that there is a smooth function $\phi_{x,v} \in C^\infty(\mathbb{R}^n, \mathbb{C}^m)$ such that v acts on test functions $\varphi \in C_c^\infty(U, \mathbb{C}^m)$ as*

$$v(\varphi) = \int \phi_{x,v}^\dagger(x)\varphi(x)dx. \tag{2}$$

The singular support contains all the points of a distribution such that the distribution does not act like a smooth function at that point. The wavefront set which we are about to introduce gives an additional directional information of where these singularities propagate. We incorporate this information by the Fourier transform in the following definition.

Definition 2. *Let for $n, m \in \mathbb{N}$, $v \in (C^\infty(\mathbb{R}^n, \mathbb{C}^m))'$, we denote by $\Xi(v) \subset \mathbb{R}^n \backslash \{0\}$ the set of all η such that there is no cone $V \subset \mathbb{R}^n$, neighbourhood of η, such that for all $a \in \mathbb{N}$ there are $C_a > 0$ such that for all $\xi \in V$ we have*

$$|\hat{v}(\xi)| \leq \frac{C_a}{1 + |\xi|^a}. \tag{3}$$

Furthermore for each $x \in \mathbb{R}^n$ we define

$$\Xi_x(v) := \bigcap_{\substack{\phi \in C_c^\infty(\mathbb{R}^n) \\ x \in \mathrm{supp}(\phi)}} \Xi(v\phi), \tag{4}$$

Where $v\phi$ is the pointwise multiplication of a distribution and a scalar test function, which acts as $v\phi : C^\infty(\mathbb{R}^n, \mathbb{C}^m) \ni \psi \mapsto v(\psi\phi)$.

We have collected the tools to introduce the wavefront set and the notion of Hadamard states.

Definition 3. *Let for $n, m \in \mathbb{N}$, $v \in (\mathcal{S}(\mathbb{R}^n, \mathbb{C}^m))'$ be a tempered distribution. The wavefront set $\mathrm{WF}(v)$ of the distribution v is defined as*

$$\mathrm{WF}(v) := \{(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n \mid \xi \in \Xi_x(v)\}. \tag{5}$$

Definition 4. *A map $H : C_c^\infty(\mathbb{R}^4, \mathbb{C}^4) \times C_c^\infty(\mathbb{R}^4, \mathbb{C}^4) \to \mathbb{C}$, is called Hadamard state if it fulfils for all $f, g \in C_c^\infty(\mathbb{R}^4, \mathbb{C}^4)$:*

$$H(Df, g) = 0 \tag{6}$$

$$H(f, g) + H(g, f) = iS(f, g) \tag{7}$$

$$\overline{H(f, g)} = H(\overline{f}, \overline{g}) \tag{8}$$

$$WF(H) \subset C_+, \tag{9}$$

where $S(f, g)$ is the propagator of the Dirac equation, and $C_+ := \{(x, y; k_1, -k_2) \in \mathbb{R}^{16} \mid (x; k_1) \approx (y; k_2), k_1^2 \geq 0, k_1^0 > 0\}$ and $(x; k_1) \approx (y; k_2)$ holds whenever $(x - y)^2 = 0$ and $(y - x) \parallel k_1 = k_2$.

It is in the sense of the fourth condition that Hadamard states are of positive energy. In the scenario of Minkowski spacetime in an external field Dirac [6] already studied the Hadamard states, although that name was not established at the time. More recently the subject has attracted considerable attention [11, 10], which computed the Hadamard states. They are given in terms of the Klein-Gordon operator corresponding to Dirac's equation:

Definition 5. *The Klein-Gordon operator corresponding to the Dirac equation (1) reads*

$$P : C^\infty(\mathbb{R}^4, \mathbb{C}^4) \to C^\infty(\mathbb{R}^4, \mathbb{C}^4) \tag{10}$$

$$P = (i\slashed{\nabla} - m)(-i\slashed{\nabla} - m) = \nabla_\alpha \nabla^\alpha + \frac{i}{2}\gamma^\alpha \gamma^\beta F_{\alpha,\beta} + m^2, \tag{11}$$

where $F_{\alpha,\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$ is the field strength tensor of the electromagnetic field. Furthermore we define for $f \in \mathcal{C}_c^\infty(\mathbb{R}^4 \times \mathbb{R}^4, \mathbb{C}^{16})$ the differential operator

$$\slashed{\nabla}^* f(x, x') = \left(\frac{\partial}{\partial y^\alpha} - iA_\alpha(y)\right) f(x, y)\gamma^\alpha. \tag{12}$$

For the special case of a Dirac field in Minkowski space-time Zahn [10] gave a more explicit form of the Hadamard states $H \in (C_c^\infty(\mathcal{M}) \times C_c^\infty(\mathcal{M}))'$ on which we base our analysis below. According to this, H acts for $f_1, f_2 \in C_c^\infty(\mathcal{M}) \otimes \mathbb{C}^4$ as

$$H(f_1, f_2) = \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^4} d^4x \overline{f_1}(x) \int_{\mathbb{R}^4} d^4y \; h_\varepsilon(x, y) f_2(y), \tag{13}$$

where h_ε is of the form

$$h_\varepsilon(x, y) =$$

$$\frac{-1}{2(2\pi)^2}(-i\overleftarrow{\nabla} + i\overrightarrow{\nabla}^* - 2m)\left[\frac{e^{-i(x-y)^\alpha \int_0^1 ds A_\alpha(xs+(1-s)y)}}{(y - x - i\varepsilon e_0)^2}\right] \tag{14}$$

$$+\frac{-1}{2(2\pi)^2}(-i\overleftarrow{\nabla} + i\overrightarrow{\nabla}^* - 2m)\left[V(x, y)\ln(-(y - x - i\varepsilon e_0)^2)\right] \tag{15}$$

$$+ B(x, y), \tag{16}$$

where $V, B : \mathbb{R}^4 \to \mathbb{C}^4$ are smooth functions, B is completely arbitrary, whereas V is fixed by the external potential. The expansion

$$V^N(x, y) := \sum_{k=1}^{N} \frac{1}{4^k k!(k-1)!} V_k(x, y)(x - y)^{2(k-1)}, \tag{17}$$

is an asymptotic expansion for V for $k \to \infty$, in the sense that $(V - V^N)(x, y)\ln(-(x-y)^2)$ as a function of x and y is in $C^{N-2}(\mathbb{R}^{4+4})$ and $V - V^N = \mathcal{O}\left(\left((x-y)^2\right)^{N-2}\right)$. The functions V_k fulfil a recursive set of partial differential differential equations

$$(x - y)^\alpha(\partial_{x,\alpha} + iA_\alpha(x))V_n(x, y) + nV_n(x, y) = -nPV_{n-1}(x, y), \tag{18}$$

where $V_0(x, y) = e^{-i(x-y)^\alpha \int_0^1 ds A_\alpha(xs+(1-s)y)}$.

For the rest of this paper, we assume that for any $A \in C_c^\infty(\mathbb{R}^4)$ there are $H, (h_\varepsilon)_{\varepsilon>0}$ and V fulfilling all of the conditions described in this paragraph.

3 The Fermionic Projector

4 Projectors for Polarisation Classes

The concept of polarisation classes arises naturally in the study of QED in external electromagnetic fields. It does need some machinery to be introduced and related to more familiar objects which we are going to introduce first. In doing so we follow [4]

Definition 6. *We define a Cauchy surface Σ in $\mathbb{R}^4$ to be a smooth, 3-dimensional submanifold of $\mathbb{R}^4$ that fulfills the following three conditions:*

a) *Every inextensible, two-sided, time- or light-like, continuous path in $\mathbb{R}^4$ intersects Σ in a unique point.*

b) *For every $x \in \Sigma$, the tangential space $T_x\Sigma$ is space-like.*

c) *The tangential spaces to Σ are bounded away from light-like directions in the following sense: The only light-like accumulation point of $\bigcup_{x\in\Sigma} T_x\Sigma$ is zero.*

In coordinates, every Cauchy surface Σ can be parametrised as

$$\Sigma = \{\pi_\Sigma(\vec{x}) := (t_\Sigma(\vec{x}), \vec{x}) \mid \vec{x} \in \mathbb{R}^3\} \tag{19}$$

with a smooth function $t_\Sigma : \mathbb{R}^3 \to \mathbb{R}$. For convenience and without restricting generality of our results we keep a global constant

$$0 < V_{\max} < 1 \tag{20}$$

fixed and work only with Cauchy surfaces Σ such that

$$\sup_{\vec{x} \in \mathbb{R}^3} |\text{grad } t_\Sigma(\vec{x})| < V_{\max}. \tag{21}$$

The standard volume form over $\mathbb{R}^4$ is denoted by $d^4x = dx^0 dx^1 dx^2 dx^3$; the product of forms is understood as wedge product. The symbol d^3x mean the 3-form $d^3x = dx^1 dx^2 dx^3$ on $\mathbb{R}^4$ and on $\mathbb{R}^3$ respectively. Contraction of a form ω with a vector v is denoted by $i_v(\omega)$. The notation $i_v(\omega)$ is also used for the spinor matrix valued vector $\gamma = (\gamma^0, \gamma^1, \gamma^2, \gamma^3) = \gamma^\mu e_\mu$:

$$i_\gamma(d^4x) = \gamma^\mu i_{e_\mu}(d^4x). \tag{22}$$

For $x \in \Sigma$ the restriction of the spinor matrix valued 3-form $i_\gamma(d^4x)$ to the tangential space $T_x\Sigma$ is given by

$$i_\gamma(d^4x) = \slashed{n}(x) i_n(d^4x) = \left(\gamma^0 - \sum_{\mu=1}^{3} \gamma^\mu \frac{\partial t_\Sigma(\vec{x})}{\partial x^\mu} \right) d^3x =: \Gamma(\vec{x}) d^3x \tag{23}$$

As a consequence of (21), there is a positive constant $\Gamma_{\max} = \Gamma_{\max}(V_{\max})$ such that

$$\|\Gamma(\vec{x}) \le \Gamma_{\max}, \quad \forall \vec{x} \in \mathbb{R}^3. \tag{24}$$

As is well known equation (1) gives us a one-particle time evolution operator for each pair of Cauchy surfaces Σ, Σ'

$$U_{\Sigma',\Sigma} : \mathcal{H}_\Sigma \to \mathcal{H}_{\Sigma'}, \tag{25}$$

where $\mathcal{H}_\Sigma$ is the space of square integrable functions on the Cauchy surface Σ. The scalar product on $\mathcal{H}_\Sigma$ is given by

$$\phi, \psi \mapsto \int_\Sigma \overline{\phi}(x) i_\gamma(d^4\gamma) \psi(x). \tag{26}$$

We start out by characterising the polarisation classes for free motion.

Definition 7. *The projector $P_\Sigma^{\mathcal{H}^-}$ has the well known representation as the weak limit of the integral operator with the kernel[4]*

$$p_\varepsilon^-(x,y) = -\frac{m^2}{4\pi^2} (i\slashed{\partial}_x + m) \frac{K_1(m\sqrt{-(y-x-i\varepsilon e_0)^2})}{m\sqrt{-(y-x-i\varepsilon e_0)^2}}, \tag{27}$$

where the square is a Minkowski square and the square root denotes its principle value. By weak limit we mean

$$\langle \phi, P_\Sigma^{\mathcal{H}^-} \psi \rangle = \lim_{\varepsilon \searrow 0} \int_{\Sigma \times \Sigma} \overline{\phi}(x) i_\gamma(d^4x) p_\varepsilon^-(x,y) i_\gamma(d^4y) \psi(y), \tag{28}$$

for general $\phi, \psi \in \mathcal{H}_\Sigma$.

Remark 1. *By inserting the expansion of K_1 in terms of a Laurent series and a logarithm, [1] one obtains:*

$$K_1(\xi) = \frac{1}{\xi} - \frac{\xi}{4} \sum_{k=0}^{\infty} \left(2\psi(k+1) + \frac{1}{k+1} + 2\ln 2 - 2\ln \xi \right) \frac{\left(\frac{\xi^2}{4}\right)^k}{k!^2(k+1)} \tag{29}$$

$$:= \frac{1}{\xi} + \xi Q_1(\xi^2)\ln \xi + \xi Q_2(\xi^2) =: \frac{1}{\xi} + \xi Q_3(\xi). \tag{30}$$

It is not obvious from the equation (18) *but well known that the vacuum of Minkowski spacetime does indeed correspond to a Hadamard state subject to vanishing four potential. In fact, the Hadamard states were constructed to agree with the Minkowski vacuum up to smooth terms.*

Definition 8. *Let $\mathrm{Pol}(\mathcal{H}_\Sigma)$ denote the set of all closed, linear subspaces $V \subset \mathcal{H}$ such that both V and $V^\perp$ are infinite dimensional. Any $V \in \mathrm{Pol}(\mathcal{H}_\Sigma)$ is called* polarisation *of $\mathcal{H}_\Sigma$. For $V \in \mathrm{Pol}(\mathcal{H}_\Sigma)$, let $P_\Sigma^V : \mathcal{H}_\Sigma \to V$ denote the orthogonal projection of $\mathcal{H}_\Sigma$ onto V. The Fock space corresponding to V on the Cauchy surface Σ is defined to be*

$$\mathcal{F}(V, \mathcal{H}_\Sigma) := \bigoplus_{c \in \mathbb{Z}} \mathcal{F}_c(V, \mathcal{H}_\Sigma), \quad \mathcal{F}_c(V, \mathcal{H}_\Sigma) := \bigoplus_{\substack{n,m \in \mathbb{N}_0 \\ c = m-n}} (V^\perp)^{\wedge n} \otimes \overline{V}^{\wedge m}, \tag{31}$$

where $\bigoplus$ is the Hilbert space direct sum, $\wedge$ the antisymmetric tensor product of Hilbert spaces and $\overline{V}$ is the conjugate complex vector space of V, which coincides with V as a set, has the same vector space operations as V except for scalar multiplication, which is defined by $(z, \psi) \mapsto z^*\psi$ for $z \in \mathbb{C}, \psi \in V$.

Remark 2. *Given two polarisations $V, W \in \mathrm{Pol}(\mathcal{H}_\Sigma)$, for two Fockspaces $\mathcal{F}(V, \mathcal{H}_\Sigma)$ and $\mathcal{F}(W, \mathcal{H}_\Sigma)$ there is a unitary operator $U : \mathcal{F}(V, \mathcal{H}_\Sigma) \to \mathcal{F}(W, \mathcal{H}_\Sigma$ if and only if $P_\Sigma^V - P_\Sigma^W \in I_2(\mathcal{H}_\Sigma)$ by the theorem of Shale and Stinespring [?].*

Remark 2 gives us a natural limit with respect to which it is useful to analyse the regularity of Projectors P_Σ^A. We will therefore be content with the following equivalence class.

Definition 9. *For $V, W \in \mathrm{Pol}(\mathcal{H}_\Sigma)$ we write*

$$V \approx W \iff P_\Sigma^V - P_\Sigma^W \in I_2(\mathcal{H}_\Sigma), \tag{32}$$

$$C_\Sigma(A) := [U_{\Sigma\Sigma_{in}}^A \mathcal{H}_{\Sigma_{in}}^-]_\approx. \tag{33}$$

The equivalence class $C_\Sigma(A)$ transforms naturally with respect to gauge and Lorentz transforms[4]. Now for hyperplanes $\Sigma \cap \mathrm{supp}(A) \neq \emptyset$ the operator $P_\Sigma^{\mathcal{H}^-}$ does not represent $C_\Sigma(A)$. Because all we are interested in is representations of equivalence classes, we are content with finding objects that differ from a Projector onto $U_{\Sigma,\Sigma_{in}}^A \mathcal{H}_{\Sigma_i n}^-$ by a Hilbert-Schmidt operator. Therefore we need not keep track of the exact evolution of the projection operators, but define a whole class of admissible ones.

Definition 10. *For any function $\lambda \in C^\infty(\mathbb{R}^4 \times \mathbb{R}^4, \mathbb{R})$ fulfilling*

   i) There is a compact set $K \subset \mathbb{R}^4$ such that $\mathrm{supp}\,\lambda \subseteq K \times \mathbb{R}^4 \cup \mathbb{R}^4 \times K$.

   ii) λ satisfies $\forall x \in \mathbb{R}^4 : \lambda(x, x) = 0$.

   iii) On the diagonal the first derivatives fulfil

$$\forall x, y \in \mathbb{R}^4 : \partial_x \lambda(x, y)|_{y=x} = -\partial_y \lambda(x, y)|_{y=x} = A(x), \tag{34}$$

we define a corresponding (quasi) projector P^λ:

$$\langle \phi, P_\Sigma^\lambda \psi \rangle = \lim_{\varepsilon \searrow 0} \langle \phi, P_\Sigma^{A,\varepsilon} \psi \rangle, \tag{35}$$

$$\langle \phi, P_\Sigma^{\lambda,\varepsilon} \psi \rangle := \int_{\Sigma \times \Sigma} \overline{\phi}(x) i_\gamma(d^4 x) \overbrace{e^{-i\lambda(x,y)} p_\varepsilon^-(y-x)}^{=:p_\varepsilon^\lambda(x,y)} i_\gamma(d^4 y)\psi(y), \tag{36}$$

for general $\phi, \psi \in \mathcal{H}_\Sigma$.

Remark 3. P_Σ^λ and $P_{\Sigma'}^\lambda$ are equivalent if transported appropriately by time evolution operators[4, theorem 2.8]:

$$P_\Sigma^\lambda - U_{\Sigma,\Sigma'}^A P_{\Sigma'}^\lambda U_{\Sigma',\Sigma} \in I_2(\mathcal{H}_\Sigma). \tag{37}$$

Also for four-potentials $A, B \in C_c^\infty(\mathbb{R}^4)$ the corresponding projectors are equivalent if and only if the four potentials projected onto the hypersurface agree[4, theorem 1.5]:

$$P_\Sigma^{\lambda^A} - P_\Sigma^{\lambda^B} \in I_2(\mathcal{H}_\Sigma) \iff \forall x \in \Sigma \; \forall z \in T_x\Sigma : z^\alpha(A_\alpha(x) - B_\alpha(x)) = 0. \tag{38}$$

Taking into account the freedom within each classification the notions Hadamard state and projectors of polarisation classes are extremely close. This is the topic of the next section.

5 Comparison Between Hadamard States and the Fermionic Projector

6 Comparison Between Hadamard States and P_Σ^λ

The following theorem is the basis of our comparison between Hadamard states and I_2- almost Projectors. We first discuss its consequences and postpone the proof until the appendix

Theorem 1. Given as four-potential $A \in C_c^\infty(\mathbb{R}^4)$ a Hadamard state H of the form (13) ~~and~~ (14)-(16) ~~and (14)-(16)~~, there is a family of smooth functions $(w_\varepsilon)_{\varepsilon > 0} \subset C^\infty(\mathbb{R}^4 \times \mathbb{R}^4, \mathbb{C}^{4\times 4})$ and a limiting ~~function~~ function $w \in C^\infty(\mathbb{R}^4 \times \mathbb{R}^4, \mathbb{C}^{4\times 4})$ such that $\tilde{P}$ defined by

$$\mathcal{H}_\Sigma \ni \psi \mapsto \tilde{P}\psi = \lim_{\varepsilon \to 0} \int_\Sigma (h_\varepsilon - w_\varepsilon)(\cdot, y) i_\gamma(d^4 y)\psi(y), \tag{39}$$

where Σ is a Cauchy surface, is bounded and fulfils $P^{\lambda^A} - \tilde{P} \in I_2(\mathcal{H}_\Sigma)$ for any λ^A fulfilling $i) - iii)$. Additionally $\lim_{\varepsilon \to 0} w_\varepsilon(x,y) = w(x,y)$ holds for any $x, y \in \mathbb{R}^4$.

Theorem 2. Given a four-potential $A \in C_c^\infty(\mathbb{R}^4)$ and a λ^A fulfilling $i) - iii)$, there is a family of functions $w_\varepsilon \in C^\infty(\mathbb{R}^4 \times \mathbb{R}^4, \mathbb{C}^{4\times 4})$ such that for all Cauchy surfaces Σ the

Theorem 2. *Given a four-potential $A \in C_c^\infty(\mathbb{R}^4)$ and a λ^A fulfilling i) – iii), there is a family of functions $w_\varepsilon \in C^\infty(\mathbb{R}^4 \times \mathbb{R}^4, \mathbb{C}^{4\times4})$ such that for all Cauchy surfaces Σ the*

$L_{loc}^2(\Sigma \times \Sigma)$ limit for $\varepsilon \to 0$ of w_ε exists. The family w_ε fulfills

$$w_\varepsilon(x,y) = -\frac{e^{-i(x-y)^\alpha \int_0^1 ds A_\alpha(xs+(1-s)y)} - e^{-i\lambda^A(x,y)}}{(2\pi)^2}(i\partial_x + m)\frac{1}{(y-x-i\varepsilon e_0)^2} \tag{40}$$

$$+\frac{e^{-i(x-y)^\alpha \int_0^1 ds A_\alpha(xs+(1-s)y)}}{2(2\pi)^2(y-x-i\varepsilon e_0)^2}\left(\slashed{A}(x) + \slashed{A}(y) - 2\int_0^1 ds \slashed{A}(sx+(1-s)y)\right. \tag{41}$$

$$\left.+(x-y)^\alpha \int_0^1 ds(1-2s)(\slashed{\partial}A)(sx+(1-s)y)\right) \tag{42}$$

$$+\frac{1}{2(2\pi)^2}(-i\slashed{\nabla} + i\slashed{\nabla}^* - 2m)V(x,y)\ln(-(y-x-i\varepsilon e_0)^2) \tag{43}$$

$$-\frac{m^2 e^{-i\lambda^A(x,y)}}{4\pi^2}(i\partial_x + m)\left(Q_1(m\sqrt{-(y-x-i\varepsilon e_0)^2}\ln(-m^2(y-x-i\varepsilon e_0)^2))\right) \tag{44}$$

~~holds and the object~~ H^{λ^A} *acting on test functions $f_1, f_2 \in C^\infty(\mathbb{R}^4, \mathbb{C}^4)$ as*

$$H^{\lambda^A}(f_1, f_2) = \lim_{\varepsilon \to 0}\int_{\mathbb{R}^{4+4}}\overline{f_1}(x)\left(p_\varepsilon^{\lambda^A}(x,y) - w_\varepsilon(x,y)\right)f_2(y)d^4xd^4y \tag{45}$$

is a Hadamard state given by (13) ~~and (14)~~-(16).

7 Appendix: Proof of theorem 1 and 2

We first break down the problem of the proof of theorem 1 to a problem on a bounded domain. Then we go on to show that the relevant functions are locally bounded so that an application of Lebesgue dominated convergence yields our result. The same estimates we need for theorem 1 will be used again for the proof of theorem 2.

Definition 11. *Let $A \in C_c^\infty(\mathbb{R}^4)$ and Σ be a Cauchy surface, we introduce the sets*

$$J_1 := \{(x,y) \in \mathbb{R}^{4+4} \mid (x-y)^2 \geq -\delta^2/4 \wedge \overline{x\,y} \cap \text{supp}(A) \neq \emptyset\} \tag{46}$$

$$J_2 := \{(x,y) \in \mathbb{R}^{4+4} \mid (x-y)^2 \geq -\delta^2 \wedge \overline{x\,y} \cap (B_{\delta/2}(0) + \text{supp}(A)) \neq \emptyset\} \tag{47}$$

$$J_3 := \{(x,y) \in \Sigma \times \Sigma \mid (x-y)^2 \geq -\delta^2 \wedge \overline{x\,y} \cap (B_{\delta/2}(0) + \text{supp}(A)) \neq \emptyset\}, \tag{48}$$

where the sum of two sets is defined as $S_1 + S_2 := \{s_1 + s_2 \mid s_1 \in S_1, s_2 \in S_2\}$.

Lemma 1. *For any $A \in C_c^\infty(\mathbb{R}^4)$ and any Hadamard state H of the form* (13) ~~and~~ ~~(14)~~(16) *and any λ^A satisfying definition 10, there is a smooth family of functions $w_\varepsilon \in C^\infty(\mathbb{R}^4 \times \mathbb{R}^4, \mathbb{C}^{4\times4})$ with $\varepsilon \in [0,1]$ and a smooth function $w \in C^\infty(\mathbb{R}^4 \times \mathbb{R}^4, \mathbb{C}^{4\times4})$ obeying*

$$\int_{\mathbb{R}^4\times\mathbb{R}^4}\overline{f_1}(x)w(x,y)f_2(y)dxdy = \lim_{\epsilon \to 0}\int_{\mathbb{R}^4\times\mathbb{R}^4}\overline{f_1}(x)(h_\epsilon(x,y) - p_\epsilon^\lambda(x,y))f_2(y)dxdy, \tag{49}$$

$$\int_{\mathbb{R}^4\times\mathbb{R}^4}\overline{f_1}(x)w_\varepsilon(x,y)f_2(y)dxdy = \int_{\mathbb{R}^4\times\mathbb{R}^4}\overline{f_1}(x)(h_\varepsilon - p_\varepsilon^\lambda)(x,y)f_2(y)dxdy \tag{50}$$

for all test functions f_1, f_2 such that $\text{supp}(f_1) \times \text{supp}(f_2) \subset J_2^c$
Additionally w_ε converges point-wise to w.

Proof. Let $A \in C_c^\infty(\mathbb{R}^4), \varepsilon > 0$. Let H be a Hadamard state acting as

$$H(f_1, f_2) = \lim_{\varepsilon \searrow 0}\int_{\mathbb{R}^4}d^4x\overline{f_1}(x)\int_{\mathbb{R}^4}d^4y\ h_\varepsilon(x,y)f_2(y). \tag{51}$$

be of the form of equation (14)-(16). Pick a function λ according to definition 10. In the representation of the Hadamard state H, equation (18), can be explicitly solved recursively (See [2][lemma 2.2.2]) The factor V_0 was already given, for $k \geq 1$ the recursion is given by

$$V_k(x,y) = -kG(x,y)\int_0^1 ds\, s^{k-1} G(x+s(y-x),x)PV_{k-1}(x,x+s(y-x)). \qquad (52)$$

One can read off of this, that if the support of the external field A does not intersect the line connecting x and y, the function $V(x,y)$ multiplying the logarithm in the expression for $h_\varepsilon(x,y)$ can be calculated to be

$$V(x,y) = \sum_{k=0}^{\infty} \frac{((y-x)^2 m^2/4)^k}{k!(k+1)!}, \qquad (53)$$

wich is exactly equal to the logarithmic part of p_ε^-, corresponding to $Q_1(\xi)$ in (29). This shows that in this case $p_\varepsilon^-(x,y)$ agrees with $h_\varepsilon(x,y)$ up to smooth terms, when acting as

$$\lim_{\epsilon\to 0}\int_{\mathbb{R}^4\times\mathbb{R}^4} \overline{f_1}(x) h_\epsilon(x,y) f_2(y)\, dx\, dy \qquad (54)$$

$$= \lim_{\epsilon\to 0}\int_{\mathbb{R}^4\times\mathbb{R}^4} \overline{f_1}(x) p_\varepsilon^-(y-x) f_2(y)\, dx\, dy \qquad (55)$$

on functions $f_1, f_2 \in C_c^\infty(\mathbb{R}^4, \mathbb{C}^4)$ such that there are no $x \in \mathrm{supp}(f_1), y \in \mathrm{supp}(f_2)$ such that $\overline{x\,y} \cap \mathrm{supp}(A) \neq \emptyset$. Here, we denote the line segment between x and y by $\overline{x\,y}$. Furthermore we can read off of both h_ε and p_ε^λ that the only points (x,y) in the singular support of the distributions in (55) and (54) need to fulfil $(y-x)^2 = 0$. If we further demand that both x and y belong to the same spacelike hypersurface $x = y$ follows. Now pick some $\delta > 0$. This implies that for functions $f_1, f_2 \in C_c^\infty(\mathbb{R}^4, \mathbb{C}^4)$ such that $(x,y) \in \mathrm{supp}(f_1)\times\mathrm{supp}(f_2)$ implies $(x-y)^2 \geq -\delta^2$, the distributions (55) and (54) act like integration against a smooth function.

So summarising we may choose $w \in C^\infty(\mathbb{R}^4\times\mathbb{R}^4, \mathbb{C}^{4\times 4})$ such that the following conditions is fulfilled:

For all test functions $f_1, f_2 \in C_c^\infty(\mathbb{R}^4, \mathbb{C}^4)$ such that $\mathrm{supp}(f_1)\times\mathrm{supp}(f_2) \subset J_1^c$ we have

$$\int_{\mathbb{R}^4\times\mathbb{R}^4}\overline{f_1}(x) w(x,y) f_2(y)\, dx\, dy = \lim_{\epsilon\to 0}\int_{\mathbb{R}^4\times\mathbb{R}^4}\overline{f_1}(x)(h_\epsilon(x,y) - p_\epsilon^\lambda(x,y)) f_2(y)\, dx\, dy. \qquad (56)$$

In fact, this condition specifies the values of w^λ uniquely for arguments $(x,y) \in \mathbb{R}^{4+4}$ in the complement of J_1. Analogously, we define a smooth function $w_\varepsilon \in C^\infty(\mathbb{R}^4\times\mathbb{R}^4, \mathbb{C}^{4\times 4})$ for every $\varepsilon > 0$ fulfilling

$$\int_{\mathbb{R}^4\times\mathbb{R}^4}\overline{f_1}(x) w_\varepsilon(x,y) f_2(y)\, dx\, dy = \int_{\mathbb{R}^4\times\mathbb{R}^4}\overline{f_1}(x)(h_\varepsilon - p_\varepsilon^\lambda)(x,y) f_2(y)\, dx\, dy \qquad (57)$$

for test functions $f_1, f_2 \in C_c^\infty(\mathbb{R}^4, \mathbb{C}^4)$ fulfilling the same condition on their support. Next, we pick a function $\chi \in C^\infty(\mathbb{R}^{4+4})$ such that

$$\chi|_{J_2^c} = 1, \quad \chi|_{J_1} = 0 \qquad (58)$$
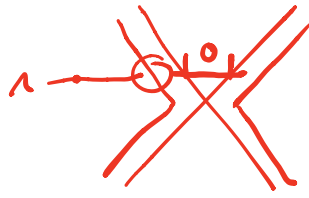
hold. Now we define

$$w := \chi w', \quad w_\varepsilon := \chi w'_\varepsilon. \tag{59}$$

w, w_ε

These functions are now uniquely fixed in all of $\mathbb{R}^{4+4}$ and fulfil (56), ~~respectively~~ (57), *and respectively.*
for test functions f_1, f_2 such that $\text{supp}(f_1) \times \text{supp}(f_2) \subset J_2^c$ holds. From the exact form
of h_ε, (14)-(16), and p_ε^λ, (36), we can read off that in fact $w_\varepsilon(x,y) \xrightarrow{\varepsilon \to 0} w(x,y)$ for all
$x, y \in J_1^c$, because of the cutoff function χ this also holds for $(x,y) \in J_1$. $\qquad\square$

observe

Definition 12. *For ease of comparison, we define for a four-potential $A \in C_c^\infty(\mathbb{R}^4)$
$\tilde{\lambda} : \mathbb{R}^{4+4} \to \mathbb{C}$:*

$$\tilde{\lambda}(x,y) = (x-y)^\alpha \int_0^1 ds\, A_\alpha(xs + (1-s)y), \tag{60}$$

and furthermore introduce the abbreviation $G(x,y) := e^{-i\tilde{\lambda}(x,y)}$.

Lemma 2. *The functions*

$$J_3 \times\,]0, 1[\, \to \mathbb{C}^{4\times 4} \tag{61}$$

$$f: (x, y, \varepsilon) \mapsto e^{-i\tilde{\lambda}(x,y)} \frac{m^2}{4\pi^2}(i\!\not{\partial}_x + m) \frac{1}{m^2(y - x - i\varepsilon e_0)^2} - (14)(x,y), \tag{62}$$

is ~~uniformly bounded in ε~~ by some constant $M_1 \in \mathbb{R}$. Also the function

$$\mathbb{R}^{4+4} \times [0,1] \ni (x,y) \mapsto \frac{1}{(y-x-i\varepsilon e_0)^2} \left(\not{A}(x) + \not{A}(y) - 2 \int_0^1 ds\, \not{A}(sx + (1-s)y) \right) \tag{63}$$

$$+ (y-x)^\alpha \int_0^1 ds(1-2s)(\not{\partial} A_\alpha)(sx + (1-s)y) \Big) \tag{64}$$

u *has a bound that is uniform in ε in the limit $x \to y$.*

Proof. Pick a Cauchy surface Σ, a four-potential $A \in C_c^\infty(\mathbb{R}^4)$. A direct calculation yields

$$\not{k}\, e^{-i\tilde{\lambda}(x,y)} \frac{m^2}{4\pi^2}(i\!\not{\partial}_x + m) \frac{1}{m^2(y-x-i\varepsilon e_0)^2} - (14)(x,y) \tag{65}$$

$$= \frac{G(x,y)}{4\pi^2} \left((i\!\not{\partial}_x + m) \frac{1}{(y-x-i\varepsilon e_0)^2} - \frac{1}{G(x,y)}(i\!\not{\nabla}/2 - i\!\not{\nabla}^*/2 + m) \frac{G(x,y)}{(y-x-i\varepsilon e_0)^2} \right)$$

$$= \frac{iG(x,y)}{4\pi^2} \left(\not{\partial}_x \frac{1}{(y-x-i\varepsilon e_0)^2} - \frac{1}{2G(x,y)}(\not{\partial}_x + i\!\not{A}(x) - \not{\partial}_y + i\!\not{A}(y)) \frac{G(x,y)}{(y-x-i\varepsilon e_0)^2} \right)$$

$$= \frac{iG(x,y)}{4\pi^2} \left(-i\frac{\not{A}(x) + \not{A}(y)}{2(y-x-i\varepsilon e_0)^2} + \not{\partial}_x \frac{1}{(y-x-i\varepsilon e_0)^2} - \frac{1}{2}(\not{\partial}_x - \not{\partial}_y) \frac{1}{(y-x-i\varepsilon e_0)^2} \right.$$

$$\not{k} \left. - \frac{1}{2G(x,y)} \frac{(\not{\partial}_x - \not{\partial}_y)G(x,y)}{(y-x-i\varepsilon e_0)^2} \right)$$

$$= \frac{iG(x,y)}{4\pi^2} \left(-i\frac{\not{A}(x) + \not{A}(y)}{2(y-x-i\varepsilon e_0)^2} + \frac{1}{2}(\not{\partial}_x + \not{\partial}_y) \frac{1}{(y-x-i\varepsilon e_0)^2} + \frac{1}{2G(x,y)} \frac{(-\not{\partial}_x + \not{\partial}_y)G(x,y)}{(y-x-i\varepsilon e_0)^2} \right)$$

$$= \frac{-iG(x,y)}{8\pi^2} \frac{1}{(y-x-i\varepsilon e_0)^2} \left(i\!\not{A}(x) + i\!\not{A}(y) + G(x,y)^{-1}\left(\not{\partial}_x - \not{\partial}_y\right)G(x,y) \right)$$

$$= \frac{G(x,y)}{8\pi^2} \frac{1}{(y-x-i\varepsilon e_0)^2} \left(\not{A}(x) + \not{A}(y) - 2\int_0^1 ds\, \not{A}(sx + (1-s)y) \right.$$

$$\not{k} \left. + (y-x)^\alpha \int_0^1 ds(1-2s)(\not{\partial} A_\alpha)(sx + (1-s)y) \right). \tag{66}$$

Now using Taylor's series for A around x, y as well as $(x+y)/2$ reveals

$$\slashed{A}(x) + \slashed{A}(y) - 2\int_0^1 \mathrm{d}s\, \slashed{A}(sx + (1-s)y) =$$

$$\frac{(x-y)^\alpha}{12}(x-y)^\beta(\partial_\alpha\partial_\beta\slashed{A})((x+y)/2) + \mathcal{O}(\|x-y\|^3) \qquad (67)$$

$$(x-y)^\alpha \int_0^1 \mathrm{d}s(1-2s)(\slashed{\partial}A_\alpha)(sx+(1-s)y) =$$

$$-\frac{(x-y)^\alpha}{2}(x-y)^\beta(\slashed{\partial}\partial_\beta A_\alpha)((x+y)/2) + \mathcal{O}(\|x-y\|^3). \qquad (68)$$

Because A is smooth and J_3 is a compact region the terms (67) and (68) are bounded in this region. Because of their behaviour close to $y = x$ the function

$$\frac{\|(67)\|(x,y) + \|(68)\|(x,y)}{8\pi^2}\frac{1}{|(y-x)^2|} \qquad (69)$$

is a bounded and point-wise an upper bound of the absolute value of $e^{-i\tilde\lambda(x,y)}\frac{m^2}{4\pi^2}(i\slashed{\partial}_x + m)\frac{1}{m^2(y-x-i\varepsilon e_0)^2} - (14)(x,y)$. $\qquad\square$

Lemma 3. *For any Cauchy surface Σ, four-potential $A \in C_c^\infty(\mathbb{R}^4)$ and λ fulfilling definition 10 the function* $\qquad$ $\lambda^A \in \mathcal{G}(\cdot)$

$$\Sigma \times \Sigma \times]0,1[\to \mathbb{C}^{4\times 4} \qquad (70)$$

$$(x,y,\varepsilon) \mapsto \left(e^{-i\tilde\lambda(x,y)} - e^{-i\lambda(x,y)}\right)\frac{1}{4\pi^2}(i\slashed{\partial}_x + m)\frac{1}{(y-x-i\varepsilon e_0)^2} \qquad (71)$$

~~has the locally in $\Sigma \times \Sigma$ square integrable bound~~ is bounded by

$$\left|\left(e^{-i\tilde\lambda(x,y)} - e^{-i\lambda(x,y)}\right)\frac{1}{4\pi^2}(i\slashed{\partial}_x + m)\frac{1}{(y-x-i\varepsilon e_0)^2}\right| \overset{\color{red}\le}{} \qquad (72)$$

$$\frac{\left|e^{-i\lambda(x,y)} - e^{-i\lambda(x,y)}\right|}{4\pi^2}\left(\eta(y-x) + \frac{\|y-x\|}{((y-x)^2)^2} + \frac{m}{|(y-x)^2|}\right) := M_2(x,y), \qquad (73)$$

where η is given by

$$\eta(y-x) := \frac{1}{(-(y-x)^2\varepsilon^{*-0.5} + \varepsilon^{*1.5})^2 + \varepsilon^*(y^0 - x^0)^2} \qquad (74)$$

and ε^* is given by

$$\varepsilon^* := \frac{1}{\sqrt{6}}\sqrt{-\beta - 2\alpha + \sqrt{(2\alpha+\beta)^2 + 12\alpha^2}}. \qquad (75)$$

Furthermore, $M_2 \in L^2_{loc}(\Sigma \times \Sigma)$.

Proof. Pick $A \in C_c^\infty(\mathbb{R}^4)$, Cauchy surface Σ and λ according to definition 10. By expanding $\tilde\lambda(x,y) - \lambda(x,y)$ in a Taylor series around $\frac{y+x}{2}$ we see directly ~~that~~

$$e^{-i\tilde\lambda(x,y)} - e^{-i\lambda(x,y)} = \mathcal{O}(\|x-y\|^2). \qquad (76)$$

In order to show that (92) has a square integrable upper bound, we consider each term separately. First look at the mass term. It obeys

$$\left\|\frac{m}{(y-x-i\varepsilon e_0)^2}\right\| = \frac{m}{\sqrt{((y-x)^2 - \varepsilon^2)^2 + \varepsilon^2(y^0 - x^0)^2}} < \frac{m}{|(y-x)^2|}, \qquad (77)$$

since $(y-x)^2 < 0$ for $x, y \in \Sigma$. The term with the derivative will be split into two:

$$\left\| i\partial\!\!\!/_x \frac{1}{(y-x-i\varepsilon e_0)^2} \right\| \leq \frac{\|y\!\!\!/ - x\!\!\!/ - i\varepsilon e\!\!\!/_0\|}{|(y-x-i\varepsilon e_0)^2|^2} \leq \frac{\varepsilon + \|x-y\|}{((y-x)^2 - \varepsilon^2)^2 + \varepsilon^2(y^0-x^0)^2} \tag{78}$$

$$< \frac{1}{(-(y-x)^2\varepsilon^{-0.5} + \varepsilon^{1.5})^2 + \varepsilon(y^0-x^0)^2} + \frac{\|y-x\|}{((y-x)^2)^2}, \tag{79}$$

we read off here, that the second term becomes locally square integrable in $\Sigma \times \Sigma$ once multiplied with a function of type $\mathcal{O}(\|x-y\|^2)$ close to $x=y$. Indeed, the first term also has this property, in order to deduce this more readily, we will maximise this term now. Considering the limits $\varepsilon \to 0$ and $\varepsilon \to \infty$ we see that this term has for arbitrary $(x, y \in \Sigma)$ a maximum in the interval $]0, \infty[$. Abbreviating $-(y-x)^2 := \alpha, (y^0-x^0)^2 := \beta$, this value of ε fulfills

$$\partial_\varepsilon(\varepsilon^{-0.5}\alpha + \varepsilon^{1.5})^2 + \beta = 0 \tag{80}$$

$$\iff 3\varepsilon^4 + \varepsilon^2(2\alpha + \beta) - \alpha^2 = 0 \tag{81}$$

$$\iff \varepsilon^2 = \frac{-\beta - 2\alpha + \sqrt{(2\alpha+\beta)^2 + 12\alpha^2}}{6} \tag{82}$$

$$\iff \varepsilon = \varepsilon^* := \frac{1}{\sqrt{6}}\sqrt{-\beta - 2\alpha + \sqrt{(2\alpha+\beta)^2 + 12\alpha^2}}. \tag{83}$$

So we can find an upper bound on the first summand of (79) by replacing ε by what we just found. Now because all the terms in the resulting expression

$$\frac{1}{(-(y-x)^2\varepsilon^{*-0.5} + \varepsilon^{*1.5})^2 + \varepsilon^*(y^0-x^0)^2} =: \eta(y-x) \tag{84}$$

are positive, we did not introduce any new singularities. Also the expression is homogenous of degree -3:

$$\eta(\delta(y-x)) = \delta^{-3}\eta(y-x), \tag{85}$$

so we can conclude that also this term as well as the second term in (79), are of type $\mathcal{O}(\|x-y\|^{-3})$ and therefore locally square integrable once multiplied by the difference of exponentials in (92). So summarising (79) can be bounded by

$$\|(79)(x,y)\| \leq$$

$$\frac{\left| e^{-i\lambda(x,y)} - e^{-i\bar{\lambda}(x,y)} \right|}{4\pi^2}\left(\eta(y-x) + \frac{\|y-x\|}{((y-x)^2)^2} + \frac{m}{|(y-x)^2|} \right), \tag{86}$$

which is locally square integrable in on $\Sigma \times \Sigma$. $\qquad\square$

(handwritten annotations: "right-hand side of (86)"; "$\|(79)(x,y)\| \leq$" circled; "Term $RG :=$" underlined with arrow)

Lemma 4. *For any Cauchy surface Σ, four-potential $A \in C_c^\infty(\mathbb{R}^4)$, the function*

$$\Sigma \times \Sigma \times]0,1[\to \mathbb{C}^{4\times 4} \tag{87}$$

$$(x, y, \varepsilon) \mapsto (i\partial\!\!\!/_x + m)\ln(-m^2(y-x-i\varepsilon e_0)^2) \tag{88}$$

has the locally in $\Sigma \times \Sigma$ square integrable bound

$$\left| (i\partial\!\!\!/_x + m)\ln(-(y-x-i\varepsilon e_0)^2) \right| \leq \tag{89}$$

$$\frac{2\|y-x\|}{|(y-x)^2|} + \frac{2}{\sqrt{4|(y-x)^2| + (y^0-x^0)^2}} := M_3(x, y), \tag{90}$$

while the function

$$\Sigma \times \Sigma \times]0,1[\to \mathbb{C}^{4\times4} \tag{91}$$

$$(x,y,\varepsilon) \mapsto \ln(-m^2(y-x-i\varepsilon e_0)^2) \tag{92}$$

has the square integrable bound

$$\| \ln(-m^2(y-x-i\varepsilon e_0)^2)\| \le |\ln(-(y-x)^2)| + \pi/2 := M_4(x,y). \tag{93}$$

Proof. Pick $A \in C_c^\infty(\mathbb{R}^4)$, a Cauchy surface Σ and $f \in C(\Sigma \times \Sigma, \mathbb{C}^{4\times4})$. The terms containing $\partial \ln(-(y-x-i\varepsilon e_0)^2)$ are bound as follows

$$\|i\partial_\alpha \ln(-(y-x-i\varepsilon e_0)^2)\| = 2\left\|\frac{y^\alpha - x^\alpha - i\varepsilon e_0^\alpha}{-(y-x-i\varepsilon e_0)^2}\right\| \tag{94}$$

$$\le \frac{2\|y-x\|}{|(y-x)^2|} + \frac{2\varepsilon}{|(y-x-i\varepsilon e_0)^2|} = \frac{2\|y-x\|}{|(y-x)^2|} + \frac{2}{\sqrt{(-(y-x)^2/\varepsilon+\varepsilon)^2 + (y^0-x^0)^2}} \tag{95}$$

$$\le \frac{2\|y-x\|}{|(y-x)^2|} + \frac{2}{\sqrt{4|(y-x)^2| + (y^0-x^0)^2}}, \tag{96}$$

where for the marked inequality we maximised the expression with respect to $\varepsilon \in]0,\infty[$. The terms containing the logarithm without any derivative are directly bound by

$$|\ln(-(y-x-i\varepsilon e_0)^2)|$$
$$\le |\ln(-(y-x)^2)| + \pi/2. \tag{97}$$

$\square$

Lemma 5. *For every four-potential $A \in C_c^\infty(\mathbb{R}^4)$ and every Hadamard state H of the form (13) and (14)-(16) the smooth function $w \in C^\infty(\mathbb{R}^4 \times \mathbb{R}^4, \mathbb{C}^{4\times4})$ and family $(w_\varepsilon)_{\varepsilon\in]0,1[}$ in $C^\infty(\mathbb{R}^4 \times \mathbb{R}^4, \mathbb{C}^{4\times4})$ of lemma 1 also fulfill for any Cauchy surface Σ and any λ^A fulfilling $i) - iii)$:*

$$\left.h_\varepsilon^A - w_\varepsilon - p_\varepsilon^{\lambda^A}\right|_{\Sigma\times\Sigma} \in L^2(\Sigma \times \Sigma) \tag{98}$$

and the $L^2(\Sigma \times \Sigma)$ limit $\lim_{\varepsilon\to0} h_\varepsilon^A - w_\varepsilon - p_\varepsilon^{\lambda^A}\big|_{\Sigma\times\Sigma}$ exists.

Proof of lemma 5: Pick a Cauchy surface Σ, a four-potential $A \in C_c^\infty(\mathbb{R}^4)$, a Hadamard state of the form (13) and (14)-(16), λ according to definition 10 and for $\varepsilon \in]0,1[$ smooth functions $w_\varepsilon \in C^\infty(\mathbb{R}^{4+4}, \mathbb{C}^{4\times4})$ according to lemma 1. Our aim is to show that $h_\varepsilon - w_\varepsilon - p_\varepsilon^\lambda\big|_{\Sigma\times\Sigma}$ converges in the sense of $L^2(\Sigma \times \Sigma)$ in the limit $\varepsilon \to 0$. According to lemma 1 we have that $h_\varepsilon - w_\varepsilon - p_\varepsilon^\lambda|_{\Sigma\times\Sigma}$ vanishes outside the set J_3 which is bounded, independent of ε and of finite measure. Taking the exact form of h_ε and p_ε^λ into account, one notices that the function $h_\varepsilon - w_\varepsilon - p_\varepsilon^\lambda|_{\Sigma\times\Sigma}(x,y)$ converges point-wise to a function defined on $\Sigma \times \Sigma \setminus \{(x,x) \mid x \in \Sigma\}$. In order to show that the convergence also holds in the sense of $L^2(\Sigma \times \Sigma)$ we would like to use dominated convergence. Hence, we should find a function $M \in L^2(\Sigma \times \Sigma)$ such that

$$\left|h_\varepsilon - w_\varepsilon - p_\varepsilon^\lambda\right|_{\Sigma\times\Sigma}(x,y)\Big| \le |M|(x,y) \tag{99}$$

$$\exists\ M \in L^2(\Sigma \times \Sigma)\ \forall A \in \varepsilon_0, \mathcal{J}_3:\ \|_{(x,y)}(\omega_\varepsilon^2(x,\cdot))| \le |\eta(x,y)|$$
$$\underbrace{\qquad}_{\mathcal{J}_3}$$

holds almost everywhere for all ε small enough. We may pick $M|_{J_3^c} = 0$, of course. So we only have to pick M for arguments inside J_3. We now dissect h_ε as well as p_ε^λ each into three pieces

$$h_\varepsilon(x,y) = \frac{-1}{2(2\pi)^2}(-i\slashed{\nabla} + i\slashed{\nabla}^* - 2m)\frac{G(x,y)}{(y-x-i\varepsilon e_0)^2} \tag{100}$$

$$+\frac{-1}{2(2\pi)^2}(-i\slashed{\nabla} + i\slashed{\nabla}^* - 2m)V(x,y)\ln(-(y-x-i\varepsilon e_0)^2) \tag{101}$$

$$+B(x,y), \tag{102}$$

$$p_\varepsilon^\lambda(x,y) = e^{-i\lambda(x,y)}\frac{m^2}{4\pi^2}(i\slashed{\partial}+m)\frac{1}{m^2(y-x-i\varepsilon e_0)^2} \tag{103}$$

$$-\frac{m^2}{4\pi^2}(i\slashed{\partial}+m)Q_1(-m^2(y-x-i\varepsilon e_0)^2)\ln(-m^2(y-x-i\varepsilon e_0)^2) \tag{104}$$

$$-\frac{m^2}{4\pi^2}(i\slashed{\partial}+m)Q_2(-m^2(y-x-i\varepsilon e_0)^2). \tag{105}$$

Let us first consider the most singular terms of $h_\varepsilon - w_\varepsilon - p_\varepsilon^\lambda|_{\Sigma\times\Sigma}$, namely (100)-(103). In order to better compare these terms, we will add 0 in the form of $(103) - (103)$ where we replace λ by $\tilde\lambda$. By lemma 2 the term (100)- $G(x,y)\frac{m^2}{4\pi^2}(i\slashed{\partial}+m)\frac{1}{m^2(y-x-i\varepsilon e_0)^2}$ is uniformly bounded in J_3.

Next we find a square integrable upper bound on $e^{-i\tilde\lambda(x,y)}\frac{m^2}{4\pi^2}(i\slashed{\partial}_x + m)\frac{1}{m^2(y-x-i\varepsilon e_0)^2} -$ $(103)(x,y)$. We rewrite this term as

$$e^{-i\tilde\lambda(x,y)}\frac{m^2}{4\pi^2}(i\slashed{\partial}_x + m)\frac{1}{m^2(y-x-i\varepsilon e_0)^2} - (103)(x,y) = \tag{106}$$

$$\left(e^{-i\tilde\lambda(x,y)} - e^{-i\lambda(x,y)}\right)\frac{1}{4\pi^2}(i\slashed{\partial}_x + m)\frac{1}{(y-x-i\varepsilon e_0)^2}, \tag{107}$$

which according to lemma 3 has the upper bound M_2, which is independend of ε and locally square integrable over $\Sigma \times \Sigma$.

The term (104) -(101) are bound by lemma 4. Thus we obtain for M for $(x,y) \in J_3$:

$$M(x,y) := M_1 + M_2(x,y) + \frac{1}{4\pi^2}\left(4|V(x,y)| + m^2\sup_{\varepsilon\in[0,1]}|Q_1(-m^2(y-x-i\varepsilon e_0)^2)|\right)M_3(x,y)$$

$$+\frac{1}{4\pi^2}\left(|(i\slashed{\nabla} - i\slashed{\nabla}^* - 2m)V(x,y)|/2 + m^2\sup_{\varepsilon\in[0,1]}|(i\slashed{\partial}+m)Q_1(-m^2(y-x-i\varepsilon e_0))|\right)M_4(x,y)$$

$$+|B(x,y)| + \frac{m^2}{4\pi^2}\sup_{\varepsilon\in[0,1]}|(i\slashed{\partial}_x+m)Q_2(-m^2(y-x-i\varepsilon e_0)^2)| + \sup_{\varepsilon\in]0,1]}|w_\varepsilon|(x,y)$$

The argument of $h_\varepsilon - w_\varepsilon - p_\varepsilon^\lambda|_{\Sigma\times\Sigma}(x,y)$ converging to $h_0 - w - p_0^\lambda|_{\Sigma\times\Sigma}(x,y)$ in the sense of $L^2(\Sigma \times \Sigma)$ is completed by applying dominated convergence theorem.

As a final point: the construction seems to depend on the function λ^A we chose, however for any different λ'^A we have due to remark 3

$$P^\lambda - P^{\lambda'} \in I_2(\mathcal{H}_\Sigma), \tag{108}$$

therefore also the limit $\lim_{\varepsilon \to 0} h_\varepsilon^A - w_\varepsilon - p_\varepsilon^{\lambda'^A}$ exists in the sense of L^2.

$\square$

Proof of theorem 1: We pick for a four-potential $A \in C_c^\infty(\mathbb{R}^4)$ a Hadamard state H of the form (13) and (14)-(16) and a λ^A according to definition 10. Then we pick $w_\varepsilon, w \in C_c^\infty(\mathbb{R}^{4+4}, \mathbb{C}^{4 \times 4})$ for all $\varepsilon \in]0, 1[$ according to lemma 1. Because of lemma 5, we have that

$$h_\varepsilon^A - w_\varepsilon - p_\varepsilon^{\lambda^A} \in L^2(\Sigma \times \Sigma) \tag{109}$$

holds, we can define the operator $Z \in I_2(\mathcal{H}_\Sigma)$ for any Cauchy surface Σ as

$$\mathcal{H}_\Sigma \ni \psi \mapsto Z\psi := \lim_{\varepsilon \to 0} \int_\Sigma (h_\varepsilon^A - w_\varepsilon - p_\varepsilon^{\lambda^A})(\cdot, y) i_\gamma(d^4 y)\psi(y) \tag{110}$$

and

$$\tilde{P}_\Sigma := Z + P_\Sigma^{\lambda^A}. \tag{111}$$

Because of (38) we have for any other λ' fulfilling definition 10 that $\tilde{P}_\Sigma - P^{\lambda'} \in I_2(\mathcal{H}_\Sigma)$.

$\square$

Proof of theorem 2. Pick a four-potential $A \in C_c^\infty(\mathbb{R}^4)$ and a λ^A according to definition 10. We define the distribution H^{λ^A} by its action on test functions $f_1, f_2 \in C_c^\infty(\mathbb{R}^4, \mathbb{C}^4)$

$$H^{\lambda^A}(f_1, f_2) := \lim_{\varepsilon \to 0} \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} \overline{f}_1(x)(p_\varepsilon^{\lambda^A}(x, y) - w_\varepsilon(x, y))f_2(y) d^4 x d^4 y, \tag{112}$$

with w_ε given by the terms from (40) until (44). In a first check, we verify that indeed H^{λ^A} is a Hadamard state. Pick $\varepsilon > 0$, we find

$$p_\varepsilon^{\lambda^A}(x, y) - w_\varepsilon(x, y) = -e^{-i\lambda^A(x,y)} \frac{m^2}{4\pi^2} (i\slashed{\partial}_x + m) \frac{K_1(m\sqrt{-(y - x - i\varepsilon e_0)^2})}{m\sqrt{-(y - x - i\varepsilon e_0)^2}} - w_\varepsilon(x, y)$$

$$= -e^{-i\lambda^A(x,y)} \frac{m^2}{4\pi^2} (i\slashed{\partial}_x + m) \left(\frac{-1}{m^2(y - x - i\varepsilon e_0)^2} \right.$$

$$+ Q_1(-m^2(y - x - i\varepsilon e_0)^2) \ln(m\sqrt{-(y - x - i\varepsilon e_0)^2}) + Q_2(-m^2(y - x - i\varepsilon e_0)^2) \Big) - w_\varepsilon(x, y).$$

Inserting $w_\varepsilon(x, y)$ we find that the most divergent terms proportional to $e^{-i\lambda^A}$, as well as the terms involving Q_1 cancel. This results in

$$p_\varepsilon^{\lambda^A}(x, y) - w_\varepsilon(x, y)$$

$$= \frac{e^{-i(x-y)^\alpha \int_0^1 ds A_\alpha(xs + (1-s)y)}}{(2\pi)^2} (i\slashed{\partial}_x + m) \frac{1}{(y - x - i\varepsilon e_0)^2}$$

$$+ \frac{e^{-i(x-y)^\alpha \int_0^1 ds A_\alpha(xs + (1-s)y)}}{2(2\pi)^2 (y - x - i\varepsilon e_0)^2} \left(\slashed{A}(x) + \slashed{A}(y) - 2\int_0^1 ds \slashed{A}(sx + (1-s)y) \right.$$

$$+ (x - y)^\alpha \int_0^1 ds(1 - 2s)(\slashed{\partial}A)(sx + (1-s)y) \Big)$$

$$- \frac{1}{2(2\pi)^2} (-i\slashed{\nabla} + i\slashed{\nabla}^* - 2m)V(x, y) \ln(-(y - x - i\varepsilon e_0)^2)$$

$$- e^{-i\lambda^A(x,y)} \frac{m^2}{4\pi^2} (i\slashed{\partial}_x + m) Q_2(-m^2(y - x - i\varepsilon e_0)^2).$$

Now the terms involving $e^{-i(x-y)^2 \int_0^1 ds A_\alpha(xs+(1-s)y)}$ can be summarized

$$\frac{e^{-i(x-y)^\alpha \int_0^1 ds A_\alpha(xs+(1-s)y)}}{(2\pi)^2}(i\partial\!\!\!/_x + m)\frac{1}{(y-x-i\varepsilon e_0)^2}$$

$$+ \frac{e^{-i(x-y)^\alpha \int_0^1 ds A_\alpha(xs+(1-s)y)}}{2(2\pi)^2(y-x-i\varepsilon e_0)^2}\left(A\!\!\!/(x) + A\!\!\!/(y) - 2\int_0^1 ds A\!\!\!/(sx+(1-s)y)\right.$$

$$\left. + (x-y)^\alpha \int_0^1 ds(1-2s)(\partial\!\!\!/A)(sx+(1-s)y)\right)$$

$$= \frac{1}{2(2\pi)^2}(i\nabla\!\!\!/ - i\nabla\!\!\!/^* + 2m)\frac{e^{-i(x-y)^\alpha \int_0^1 ds A_\alpha(xs+(1-s)y)}}{(y-x-i\varepsilon e_0)^2},$$

so overall we find

$$p_\varepsilon^{\lambda^A}(x,y) - w_\varepsilon(x,y) =$$

$$\frac{-1}{2(2\pi)^2}(-i\nabla\!\!\!/ + i\nabla\!\!\!/^* - 2m)\frac{e^{-i(x-y)^\alpha \int_0^1 ds A_\alpha(xs+(1-s)y)}}{(y-x-i\varepsilon e_0)^2}$$

$$- \frac{1}{2(2\pi)^2}(-i\nabla\!\!\!/ + i\nabla\!\!\!/^* - 2m)V(x,y)\ln(-(y-x-i\varepsilon e_0)^2)$$

$$- e^{-i\lambda^A(x,y)}\frac{m^2}{4\pi^2}(i\partial\!\!\!/_x + m)Q_2(-m^2(y-x-i\varepsilon e_0)^2).$$

Because the term in the last line is smooth, this means that indeed $p_\varepsilon^{\lambda^A}(x,y) - w_\varepsilon(x,y)$ is the integration kernel of a Hadamard state. The claim that for a given Cauchy surface Σ the function w_ε has a limit in the sense of $L^2_{\text{loc}}(\Sigma \times \Sigma)$ follows from Lebesgue dominated convergence and lemmata 2, 3 and 4.

□

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