Theorem 0.1. *Let $\mathcal{M}$ denote the 4-dimensional mass shell and $\mathcal{H}_{\mathcal{M}}$ (in the notation of Deckert, Merkl 2014) be ...* **(insert lengthy definition)** *. We are interested in calculating the trace of an integral operator. Let therefore $K : \mathcal{H} \to \mathcal{H}$ be an operator acting as*

$$\forall \xi \in \mathcal{H} : \widehat{K\xi}(l) = \int_{\mathcal{M}} i_p \left(\mathrm{d}^4 p\right) K(l,p) \hat{\xi}(p) \tag{1}$$

for some nice integral kernel K.
Then the trace of the operator K is given by:

$$\operatorname{tr} K = \int_{\mathcal{M}} i_p \left(\mathrm{d}^4 p\right) \operatorname{tr}_{\mathcal{D}_p} K(p,p). \tag{2}$$

Proof: As a first step, we choose $(\varphi_{n,k})_{k \in \{1,2\}, n \in \mathbb{Z} \setminus \{0\}}$ be an ONB of $\mathcal{H}$ such that for each $p \in \mathcal{M}$ there is an ONB of $\mathcal{D}_p$, denoted by $\{e_1^p, e_2^p\}$ such that

$$\forall n \in \mathbb{Z} \setminus \{0\} : \langle \varphi_{n,1}(p), e_1^p \rangle = \langle \varphi_{n,2}(p), e_2^p \rangle$$

holds, where $(\varphi_n)_{n \in \mathbb{Z} \setminus \{0\}}$ is an ONB of $L^2\left(\mathcal{M}, i_p\left(\mathrm{d}^4 p\right)\right) =: \mathcal{H}'$. The sum representing the trace of the operator in question can be reordered in this manner

$$\operatorname{tr} K = \sum_{k \in \{1,2\}, n \in \mathbb{Z} \setminus \{0\}} \langle \varphi_{k,n}, K \varphi_{k,n} \rangle = \sum_{n \in \mathbb{Z} \setminus \{0\}} \langle \varphi_n, \operatorname{tr}_{\mathcal{D}.} (K) \varphi_n \rangle .$$

For the next step we will be using the bra ket notation of Dirac. The identity on $\mathcal{H}'$ can be written as

$$1 = \sum_{n \in \mathbb{Z} \setminus \{0\}} |\varphi_n\rangle \langle \varphi_n| .$$

Now for any function $g :\in \mathcal{H}'$ and any $p \in \mathcal{M}$, one gets

$$g(p) = (1 \cdot g)(p) = \sum_{n \in \mathbb{Z} \setminus \{0\}} \left(|\varphi_n\rangle \langle \varphi_n, g \rangle \right)(p) = \sum_{n \in \mathbb{Z} \setminus \{0\}} \varphi_n(p) \langle \varphi_n, g \rangle .$$

This implies

$$\sum_{n \in \mathbb{Z} \setminus \{0\}} \int_{\mathcal{M}} i_l \left(\mathrm{d}^4 l\right) \varphi_n^*(l) \operatorname{tr}_{\mathcal{D}.}(K(l,p)) \varphi_n(p) = \operatorname{tr}_{\mathcal{D}.}(K(p,p)),$$

which in turn means for the trace

$$\begin{aligned} \operatorname{tr} K = \sum_{n \in \mathbb{Z} \setminus \{0\}} \int_{\mathcal{M}} i_p \left(\mathrm{d}^4 p\right) \int_{\mathcal{M}} i_l \left(\mathrm{d}^4 l\right) \varphi_n^*(p) \operatorname{tr}_{\mathcal{D}.}(K(p,l)) \varphi_n(l) \\ = \int_{\mathcal{M}} i_p \left(\mathrm{d}^4 p\right) \operatorname{tr}_{\mathcal{D}.}(K(p,p)), \end{aligned}$$

where the niceness of K was used for interchanging the sum and the integrals. $\square$