The Phase of the Second Quantised Time Evolution Operator

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Abstract

abstract to be written

1 Introduction

We follow the necessary definitions in [**?**].

is das ok hier ivp2 so ausgiebig zu zitieren? Mir scheint aber, die ganzen Sachen müssen eben genau nochmal eingeführt werden.

Definition 1. *For a Cauchy surface Σ, we define $\mathcal{H}_\Sigma$ to be the Hilbert space of $\mathcal{C}^4$ valued, square integrable functions on Σ. Furthermore, let $\mathrm{Pol}(\mathcal{H}_\Sigma)$ denote the set of all closed, linear subspaces $V \subset \mathcal{H}_\Sigma$ such that both V and $V^\perp$ are infinite dimensional. Any $V \in \mathrm{Pol}(\mathcal{H}_\Sigma)$ is called a polarisation of $\mathcal{H}$. For $V \in \mathrm{Pol}$, let $P^V_\Sigma : \mathcal{H}_\Sigma \to V$ denote the orthogonal projection of $\mathcal{H}_\Sigma$ onto V.*
The Fock space corresponding to polarisation V on Cauchy surface Σ is then defined by

$$\mathcal{F}(V, \mathcal{H}_\Sigma) := \bigoplus_{c\in\mathbb{Z}} \mathcal{F}_c(V, \mathcal{H}_\Sigma), \quad \mathcal{F}_c(V, \mathcal{H}_\Sigma) := \bigoplus_{\substack{n,m\in\mathbb{N}_0\\ c=m-n}} \left(V^\perp\right)^{\wedge n} \otimes \overline{V}^{\wedge m}, \tag{1}$$

where $\bigoplus$ denotes the Hilbert space direct sum, $\wedge$ the antisymmetric tensor product of Hilbert spaces, and $\overline{V}$ the conjugate complex vector space of V, which coincides with V as a set and has the same vector space operations as V with the exception of the scalar multiplication, which is replaced by $(z, \psi) \mapsto z^*\psi$ for $z \in \mathbb{C}), \psi \in V$.

Each polarisation V splits the Hilbert space $\mathcal{H}_\Sigma$ into a direct sum, i.e., $\mathcal{H}_\Sigma = V^\perp \oplus V$. The "standard“ polarisation $\mathcal{H}^+_\Sigma$ and $\mathcal{H}^-_\Sigma$ are determined by the orthogonal projectors P^+_Σ and P^-_Σ onto the free positive and negative energy Dirac solutions, respectively, restricted to Σ:

$$\mathcal{H}^+_\Sigma := P^+_\Sigma \mathcal{H} = (1 - P^-_\Sigma)\mathcal{H}_\Sigma, \quad \mathcal{H}^-_\Sigma := P^-_\Sigma \mathcal{H}_\Sigma. \tag{2}$$

Given two Cauchy surfaces Σ, Σ' and two polarisations $V \in \mathrm{Pol}(\mathcal{H}_\Sigma)$ and $W \in \mathrm{Pol}(\Sigma_{\Sigma'})$ a sensible lift of the one particle Dirac evolution $U^A_{\Sigma'\Sigma} : \mathcal{H} \to \mathcal{H}_\Sigma$ should be given by a unitary operator $\tilde{U}^A_{\Sigma',\Sigma} : \mathcal{F}(V, \mathcal{H}_\Sigma) \to \mathcal{F}(W, \mathcal{H}_{\Sigma'})$ that fulfils

decide upon how much of this is actually useful for the reader. We restrict ourselves to the scattering setting anyhow.

$$\tilde{U}^A_{\Sigma',\Sigma} \psi_{V,\Sigma}(f) (\tilde{U}^A_{\Sigma',\Sigma})^{-1} = \psi_{W,\Sigma'}(U^A_{\Sigma',\Sigma} f), \quad \forall f \in \mathcal{H}_\Sigma. \tag{3}$$

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Here, $\psi_{V,\Sigma}$ denotes the Dirac field operator corresponding to Fock space $\mathcal{F}(V,\Sigma)$, i.e., → Alg. Pol. lassen def... was lassen

$$\psi_{V,\Sigma}(f) := b_\Sigma\big(P_\Sigma^{V^\perp} f\big) + d_\Sigma^*\big(P_\Sigma^V f\big), \quad \forall f \in \mathcal{H}_\Sigma, \tag{4}$$

where b_Σ, d_Σ^* denote the annihilation and creation operators on the $V^\perp$ and $\overline{V}$ sectors of $\mathcal{F}_c(V,\mathcal{H}_\Sigma)$, respectively. Note that $P_\Sigma^{V^\perp} : \mathcal{H}_\Sigma \to \overline{V}$ is anti-linear; thus, $\psi_{V,\Sigma}(f)$ is anti-linear in its argument f. The condition under which such a lift $\tilde{U}^A_{\Sigma',\Sigma}$ exists can be inferred from a straight-forward application of Shale and Stinespring's well-known theorem [**?**]

Theorem 1 (Shale-Stinespring)**.** *The following statements are equivalent:*

(α) *There is a unitary operator $\tilde{U}^A_{\Sigma\Sigma'} : \mathcal{F}(V,\mathcal{H}_\Sigma) \to \mathcal{F}(W,\mathcal{H}_{\Sigma'})$ which fulfils (3).*

(β) *The off-diagonals $P_{\Sigma'}^{W^\perp} U^A_{\Sigma'\Sigma} P_\Sigma^V$ and $P_{\Sigma'}^W U^A_{\Sigma'\Sigma}$ are Hilbert-Schmidt operators.*

Please note that condition (β) for fixed polarisations V, W and general external field A is not always satisfied; see e.g. [**?**]. However, when carefully adapting the choices of polarisation V to $A|_\Sigma$ and W to $A|_{\Sigma'}$ one can always fulfil condition (β) and therefore construct a lift $\tilde{U}^A_{\Sigma'\Sigma}$, see [**?**, **?**, **?**].
Furthermore condition (3) does not fix the phase of the lift $\tilde{U}^A_{\Sigma'\Sigma}$. Considering Bogolyubov's formula

$$j^\mu(x) = i\tilde{U}^A_{\Sigma_{\text{in}},\Sigma_{\text{out}}} \frac{\delta \tilde{U}^A_{\Sigma_{\text{out}},\Sigma_{\text{in}}}}{\delta A_\mu(x)}, \tag{5}$$

where $\Sigma_{\text{in}}, \Sigma_{\text{out}}$ are Cauchy-surfaces in the remote past and future, respectively. The current operator thus depends in a rather sensitive way on the phase of $\tilde{U}^A$. Since the current is experimentally accessible we would like to fix the phase by additional physical constraints. This paper is a step this direction.
The set of four-potentials we will be concerned with is defined as follows

Definition 2. *We define the set $\mathcal{A}$ of four potentials*

Versuche C_c^∞ statt $A|_\Sigma$ $\forall$.

$$\mathcal{A} := \{A \in C_c^\infty(\mathbb{R}^4,\mathbb{R}^4) \mid (x,y) \in \operatorname{supp} \vec{A} \times \Sigma_{\text{in}} : (x-y)^2 \geq 0 \Rightarrow x^0 > y^0\}, \tag{6}$$

$A|_\Sigma$

where Σ_{in} is a fixed Cauchy-surface and $\vec{A}$ is the spacial part of the four-potential.

We introduce the standard polarisation in the remote past

$$P^- := P^-_{\Sigma_{\text{in}}}, \quad P^+ = 1 - P^-, \tag{7}$$

given by the negative respectively positive energy subspaces of $\mathcal{H}_{\Sigma_{\text{in}}}$.

Definition 3. *We define for all four potentials $A, B \in \mathcal{A}$*

$$S_{A,B} := U^A_{\Sigma_{\text{in}},\Sigma_{\text{out}}} U^B_{\Sigma_{\text{out}},\Sigma_{\text{in}}}, \tag{8}$$

where Σ_{out} is such that

$$\forall\ (x,y) \in \operatorname{supp} \vec{A} \cup \operatorname{supp} \vec{B} \times \Sigma_{\text{out}} : (x-y)^2 \geq 0 \Rightarrow x^0 < y^0 \tag{9}$$

holds. We choose for all $S_{A,B}$ with $\|\mathbb{1} - S_{A,B}\| < 1$ the lift

$$\overline{S}_{A,B} = \mathcal{R}_{\text{ARG}((P^- S_{A,B} P^-)^{-1})} \mathcal{L}_{S_{A,B}}, \tag{10}$$

$X : \mathcal{H} \to \mathcal{H}$, bdd.

nach oben schieben

where

$$\mathrm{ARG}(X) := X|X|^{-1}. \tag{11}$$

Furthermore, we define for any complex number $z \in \mathbb{C}\backslash\{0\}$ erkläre warum nicht org

$$\mathrm{ag}(z) := \frac{z}{|z|} \tag{12}$$

and for $A, B, C \in \mathcal{A}$ such that $\|\mathbb{1} - S_{X,Y}\| < 1$ for $(X, Y) \in \{A, B, C\}$. The complex numbers

eine andere Version als Def. verwenden

$$\gamma_{A,B,C} := \det_{\mathcal{H}^-}(P^- - P^-S_{A,C}P^+S_{C,A}P^- - P^-S_{A,B}P^+S_{B,C}P^-S_{C,A}P^-), \tag{13}$$

$$\Gamma_{A,B,C} := \mathrm{ag}(\gamma_{A,B,C}), \tag{14}$$

where Ω is a vacuum vector corresponding to the standard polarisation on $\mathcal{H}_{\Sigma_{\text{in}}}$. We will see in lemma 1 that $\gamma_{A,B,C} \neq 0$, so that $\Gamma_{A,B,C}$ is well-defined. Lastly we introduce the partial derivative in the direction of any four-potential F by

def p-Formen.

$$\partial_F T(F) := \partial_\varepsilon T(\varepsilon F)|_{\varepsilon=0} \tag{15}$$

def Ableitungen von S-operaturen mittels

and for $A, B, C \in \mathcal{A}$ the function

$$c_A(F, G) := -i\partial_F\partial_G \Im\,\mathrm{tr}[P^-S_{A,A+F}P^+S_{A,A+G}P^-]. \tag{16}$$

$U^{A+B}_{\Sigma_{out}\Sigma_{in}} \cong U^{A}_{\Sigma_{out}\Sigma_{in}} + \int dt\, U^{A}_{\Sigma_{out}\Sigma_t} H_D U^{A+B}_{\Sigma_t \Sigma_{in}}$

$S_{A,B+\varepsilon C}$ ...

2 Main Result

Definition 4. *We define a causal splitting as a function*

$$c^+ : \mathcal{A}^3 \to \mathbb{C}, \tag{17}$$

$$(A, F, G) \mapsto c^+_A(F, G), \tag{18}$$

smooth in the first and linear in the second and third argument, satisfying

$\forall B \subset A,\ \dim B \in \mathbb{N}$ subspace $c^+|_B$ glatt

$$c_A(F, G) = c^+_A(F, G) - c^+_A(G, F), \tag{19}$$

$$\partial_H c^+_{A+H}(F, G) = \partial_G c^+_{A+G}(F, H), \tag{20}$$

$$\forall F \prec G : c^+_A(F, G) = 0. \tag{21}$$

definiere Lift.

Definition 5. *Given a lift $\hat{S}_{A,B}$ of the one-particle scattering operator $S_{A,B}$ we define the associated current by Bogolyubov's formula:*

ist nur definiert, wenn (22) Sinn macht

$$j^{\hat{S}}_A(F) := i\partial_F \left\langle \Omega, \hat{S}_{A,A+F}\Omega \right\rangle. \tag{22}$$

Theorem 2. *Given a causal splitting c^+, there is a second quantised scattering operator $\tilde{S}$, lift of the one-particle scattering operator S with the following properties*

$$\forall A, B, C \in \mathcal{A} : \tilde{S}_{A,B}\tilde{S}_{B,C} = \tilde{S}_{A,C} \tag{23}$$

$$\forall F \prec G : \tilde{S}_{A,A+F} = \tilde{S}_{A+G,A+F+G} \tag{24}$$

and the associated current satisfies

$$\partial_G j^{\tilde{S}}_{A+G}(F) = \begin{cases} -2ic_A(F, G) & \text{for } G \prec F \\ 0 & \text{otherwise.} \end{cases} \tag{25}$$

3 Proofs

Since the phase of a lift relative to any other lift is fixed by a single matrix element, we may use the vacuum expectation values to characterise the phase of a lift. The function c captures the dependence of this object on variation of the external fields, the connection between vacuum expectation values and c becomes clearer with the next lemma.

Lemma 1. *The function Γ has the following properties for all $A, B, C \in \mathcal{A}$ such that the expressions occurring in each equation are well defined:*

$$\gamma_{A,B,C} \neq 0 \tag{26}$$

$$\Gamma_{A,B,C} = \operatorname{ag}(\det_{\mathcal{H}^-}(P^- S_{A,B} P^- S_{B,C} P^- S_{C,A} P^-)) \tag{27}$$

$$\Gamma^{-1}_{A,B,C} = \operatorname{ag}(\langle \Omega, \overline{S}_{A,B} \overline{S}_{B,C} \overline{S}_{C,A} \Omega \rangle) \tag{28}$$

$$\Gamma_{A,B,C} = \Gamma_{B,C,A} = \frac{1}{\Gamma_{B,A,C}} \tag{29}$$

$$\Gamma_{A,A,B} = 1 \tag{30}$$

$$\Gamma_{A,B,C} \Gamma_{B,A,D} \Gamma_{A,C,D} \Gamma_{C,B,D} = 1 \quad \Leftarrow \text{is that necessary?}$$

vllt trotzdem zeigen

$$\overline{S}_{A,C} = \Gamma_{A,B,C} \overline{S}_{A,B} \overline{S}_{B,C} \tag{31}$$

$$c_A(B, C) = \partial_B \partial_C \ln \Gamma_{A,A+B,A+C}. \tag{32}$$

Proof. Pick $A, B, C \in \mathcal{A}$ such that $\|1 - S_{X,Y}\| < 1$ for $X, Y \in \{A, B, C\}$. We begin by reformulating

$$\begin{aligned} \gamma_{A,B,C} &= \det_{\mathcal{H}^-}(P^- - P^- S_{A,C} P^+ S_{C,A} P^- - P^- S_{A,B} P^+ S_{B,C} P^- S_{C,A} P^-) && (33)\\ &= \det_{\mathcal{H}^-}(P^- S_{A,C} P^- S_{C,A} P^- - P^- S_{A,B} P^+ S_{B,C} P^- S_{C,A} P^-) && (34)\\ &= \det_{\mathcal{H}^-}(P^- S_{A,B} P^- S_{B,C} P^- S_{C,A} P^-), && (35) \end{aligned}$$

now the operator whose determinant we take in the last line is a product

$$P^- S_{A,B} P^- S_{B,C} P^- S_{C,A} P^- = P^- S_{A,B} P^- \quad P^- S_{B,C} P^- \quad P^- S_{C,A} P^-. \tag{36}$$

The three factors appearing in this product are all invertible, hence the product is also invertible as operators of type $\mathcal{H}^- \to \mathcal{H}^-$ because of the conditions of $\{A, B, C\}$ imply that $\|P^- - P^- S_{X,Y} P^-\| < 1$ which means that the Von Neumann series of the inverse converges, therefore $\gamma_{A,B,C} \neq 0$. Equation (35) also proves (27). Next we show (28). Borrowing notation from [?, section 2] to identify $\Omega = \bigwedge \Phi$ with the injection $\Phi : \mathcal{H}^- \hookrightarrow \mathcal{H}$ and $\bigwedge$ is used to construct the infinite wedge spaces that are the perspective of Fock space introduced in [?]. We begin by reformulating the right hand side of (28)

$$\begin{aligned} &\langle \Omega, \overline{S}_{A,B} \overline{S}_{B,C} \overline{S}_{C,A} \Omega \rangle && (37)\\ &= \langle \bigwedge \Phi, \bigwedge (S_{A,B} S_{B,C} S_{C,A} \Phi \operatorname{ARG}(P^- S_{C,A} P^-)^{-1} && \cancel{(38)}\\ &\qquad \operatorname{ARG}(P^- S_{B,C} P^-)^{-1} \operatorname{ARG}(P^- S_{A,B} P^-)^{-1}) \rangle && (39)\\ &= \langle \bigwedge \Phi, \bigwedge (\Phi \operatorname{ARG}(P^- S_{C,A} P^-)^{-1} \operatorname{ARG}(P^- S_{B,C} P^-)^{-1} \operatorname{ARG}(P^- S_{A,B} P^-)^{-1}) \rangle && (40)\\ &= \det_{\mathcal{H}^-} \left((\Phi)^* \left[\Phi \operatorname{ARG}(P^- S_{C,A} P^-)^{-1} \operatorname{ARG}(P^- S_{B,C} P^-)^{-1} \operatorname{ARG}(P^- S_{A,B} P^-)^{-1} \right] \right) && (41)\\ &= \det_{\mathcal{H}^-} \left(\operatorname{ARG}(P^- S_{C,A} P^-)^{-1} \operatorname{ARG}(P^- S_{B,C} P^-)^{-1} \operatorname{ARG}(P^- S_{A,B} P^-)^{-1} \right) && (42)\\ &= \frac{1}{\det_{\mathcal{H}^-} (\operatorname{ARG}(P^- S_{A,B} P^-) \operatorname{ARG}(P^- S_{B,C} P^-) \operatorname{ARG}(P^- S_{C,A} P^-))}. && (43) \end{aligned}$$

eine Zeile?

We first note that $\det_{\mathcal{H}^-} |P^- S_{X,Y} P^-| \in \mathbb{R}^+$ for $X, Y \in \{A, B, C\}$. This is well defined because

$$\langle \Omega, \overline{S}_{X,Y} \Omega \rangle = \langle \bigwedge \Phi, \bigwedge (S_{X,Y} \Phi \operatorname{ARG}(P^- S_{X,Y} P^-)^{-1}) \rangle \tag{44}$$

$$= \det_{\mathcal{H}^-} \left(\Phi^* S_{X,Y} \Phi \operatorname{ARG}(P^- S_{X,Y} P^-)^{-1} \right) = \det_{\mathcal{H}^-} \left(P^- S_{X,Y} P^- \operatorname{ARG}(P^- S_{X,Y} P^-)^{-1} \right) \tag{45}$$

$$= \det_{\mathcal{H}^-} \left(\operatorname{ARG}(P^- S_{X,Y} P^-)^{-1} P^- S_{X,Y} P^- \right) \tag{46}$$

$$= \det_{\mathcal{H}^-} \left(\operatorname{ARG}(P^- S_{X,Y} P^-)^{-1} \operatorname{ARG}(P^- S_{X,Y} P^-) |P^- S_{X,Y} P^-| \right) = \det_{\mathcal{H}^-} |P^- S_{X,Y} P^-| \tag{47}$$

holds. Moreover this determinant does not vanish, since the $P^- S_{X,Y} P^-$ is invertible. Also clearly the eigenvalues are positive since $|P^- S_{X,Y} P^-|$ is an absolute value. We continue with the result of (43). Thus, we find

$$\langle \Omega, \overline{S}_{A,B} \overline{S}_{B,C} \overline{S}_{C,A} \Omega \rangle^{-1} = \det_{\mathcal{H}^-} \left(\operatorname{ARG}(P^- S_{A,B} P^-) \operatorname{ARG}(P^- S_{B,C} P^-) \operatorname{ARG}(P^- S_{C,A} P^-) \right) \tag{48}$$

$$= \det_{\mathcal{H}^-} \left(\operatorname{ARG}(P^- S_{A,B} P^-) \operatorname{ARG}(P^- S_{B,C} P^-) P^- S_{C,A} P^- |P^- S_{C,A} P^-|^{-1} \right) \tag{49}$$

$$= \det_{\mathcal{H}^-} \left(\operatorname{ARG}(P^- S_{A,B} P^-) \operatorname{ARG}(P^- S_{B,C} P^-) P^- S_{C,A} P^- \right) \det_{\mathcal{H}^-} |P^- S_{C,A} P^-|^{-1} \tag{50}$$

$$= \det_{\mathcal{H}^-} \left(P^- S_{C,A} P^- \operatorname{ARG}(P^- S_{A,B} P^-) \operatorname{ARG}(P^- S_{B,C} P^-) \right) \det_{\mathcal{H}^-} |P^- S_{C,A} P^-|^{-1} \tag{51}$$

$$= \frac{\det_{\mathcal{H}^-} \left(P^- S_{A,B} P^- P^- S_{B,C} P^- P^- S_{C,A} P^- \right)}{\det_{\mathcal{H}^-} |P^- S_{A,B} P^-| \det_{\mathcal{H}^-} |P^- S_{B,C} P^-| \det_{\mathcal{H}^-} |P^- S_{C,A} P^-|}. \tag{52}$$

Versuche mit Abständen, \cdot multiplikation zu trennen.

Now since the denominator of this fraction is real we can use (27) to identity

$$\operatorname{ag}(\langle \Omega, \overline{S}_{A,B} \overline{S}_{B,C} \overline{S}_{C,A} \Omega \rangle) = \Gamma_{A,B,C}^{-1}, \tag{53}$$

which proves (28).
For the first equality in (29) we use $\det X(1+Y)X^{-1} = \det(1+Y)$ for any Y trace-class and X bounded and ~~boundedly~~ invertible. So we can cyclicly permute the factors $P^- S_{X,Y} P^-$ in the determinant and find

$$\begin{aligned} \Gamma_{A,B,C} &= \operatorname{ag}(\det_{\mathcal{H}^-} P^- S_{A,B} P^- S_{B,C} P^- S_{C,A} P^-) \\ &= \operatorname{ag}(\det_{\mathcal{H}^-} P^- S_{C,A} P^- S_{A,B} P^- S_{B,C} P^-) = \Gamma_{C,A,B}. \end{aligned}$$

For the second equality of (29) we use (27) to represent both $\Gamma_{A,B,C}$ and $\Gamma_{B,A,C}$. Using this and the manipulations of the determinant we already employed, we arrive at

$$
\begin{aligned}
&\Gamma_{A,B,C}\Gamma_{B,A,C} && (54)\\
&= \operatorname{ag}(\det_{\mathcal{H}^-}(P^- S_{A,B} P^- S_{B,C} P^- S_{C,A} P^-)) && (55)\\
&\quad \times \operatorname{ag}(\det_{\mathcal{H}^-}(P^- S_{B,A} P^- S_{A,C} P^- S_{C,B} P^-)) && (56)\\
&= \operatorname{ag}(\det_{\mathcal{H}^-}(P^- S_{A,B} P^- S_{B,C} P^- S_{C,A} P^-)) && (57)\\
&\quad \times (\operatorname{ag}(\det_{\mathcal{H}^-}(P^- S_{B,C} P^- S_{C,A} P^- S_{A,B} P^-)))^* && (58)\\
&= \operatorname{ag}(\det_{\mathcal{H}^-}(P^- S_{A,B} P^- S_{B,C} P^- S_{C,A} P^-)) && (59)\\
&\quad \times (\operatorname{ag}(\det_{\mathcal{H}^-}(P^- S_{A,B} P^- S_{B,C} P^- S_{C,A} P^-)))^* && (60)\\
&= |\operatorname{ag}(\det_{\mathcal{H}^-}(P^- S_{A,B} P^- S_{B,C} P^- S_{C,A} P^-))|^2 = 1, && (61)
\end{aligned}
$$

which proves (29).
Next, (30) follows directly from the right side of equation (48). det $MM^* \geq 0$, $\Rightarrow$ $\Gamma = 1$
For (31) we realise that according to [?] that two lifts can only differ by a phase, that is

$$\overline{S}_{A,C} = \alpha \overline{S}_{A,B} \overline{S}_{B,C} \tag{62}$$

for some $\alpha \in \mathbb{C}, |\alpha| = 1$.
In order to identify α we recognise that $\overline{S}_{X,Y} = \overline{S}^{-1}_{Y,X}$ for four potentials X, Y and find

$$\mathbb{1}\alpha^{-1} = \overline{S}_{A,B} \overline{S}_{B,C} \overline{S}_{C,A}. \tag{63}$$

Now we take the vacuum expectation value on both sides of this equation and use (28) to find

$$\alpha^{-1} = \langle \Omega, \overline{S}_{A,B} \overline{S}_{B,C} \overline{S}_{C,A} \Omega \rangle = \Gamma^{-1}_{A,B,C}. \tag{64}$$

Finally we prove (32). We start from the right hand side of this equation and work our way towards the left hand side of it. In the following calculation we will repeatedly make use of the fact that $(P^- S_{A,A+B} P^- S_{A+B,A} P^-)$ is the absolute value squared of an invertible operator and has a determinant, which is therefore positive. For the marked equality we will use that for a differentiable function $z : \mathbb{R} \to \mathbb{C}$ at points t where $z(t) \in \mathbb{R}^+$ holds, we have

$$(z/|z|)'(t) = z'/(2|z|)(t) - z^2 z^{*\prime}/(2|z|^3)(t) = i(\Im(z'))/z(t). \tag{65}$$

hinschreiben

Furthermore, we will use the following expressions for derivative of operators:

$$\partial_\varepsilon \det M(\varepsilon)|_{\varepsilon=0} = \det M(0) \operatorname{tr}(M^{-1}(0)\partial_\varepsilon M(\varepsilon)|_\varepsilon) \tag{66}$$

$$\partial_\varepsilon M^{-1}(\varepsilon)|_{\varepsilon=0} = -M^{-1}(0)\partial_\varepsilon M(\varepsilon)|_{\varepsilon=0} M^{-1}(0) \tag{67}$$

quantifiziere M

$|z| = \sqrt{zz^*}$

$$\frac{z'}{|z|} + z \cdot \frac{-1}{|z|^2} \frac{z'z^* + z^{*\prime}z}{2|z|}$$

We compute

$$
\begin{aligned}
\partial_B\partial_C \ln \Gamma_{A,A+B,A+C} &\overset{(27)}{=} \partial_B\partial_C \ln \operatorname{ag}(\det_{\mathcal{H}^-}(P^-S_{A,A+B}P^-S_{A+B,A+C}P^-S_{A+C,A}P^-)) && (68)\\
&= \partial_B \frac{\partial_C \operatorname{ag}(\det_{\mathcal{H}^-}(P^-S_{A,A+B}P^-S_{A+B,A+C}P^-S_{A+C,A}P^-))}{\operatorname{ag}(\det_{\mathcal{H}^-}(P^-S_{A,A+B}P^-S_{A+B,A}P^-))} && (69)\\
&= \partial_B\partial_C \operatorname{ag}(\det_{\mathcal{H}^-}(P^-S_{A,A+B}P^-S_{A+B,A+C}P^-S_{A+C,A}P^-)) && (70)\\
&\overset{*}{=} i\partial_B \frac{\Im\partial_C \det_{\mathcal{H}^-}(P^-S_{A,A+B}P^-S_{A+B,A+C}P^-S_{A+C,A}P^-)}{\det_{\mathcal{H}^-}(P^-S_{A,A+B}P^-S_{A+B,A}P^-)} && (71)\\
&= i\partial_B \Big[\frac{\det_{\mathcal{H}^-}(P^-S_{A,A+B}P^-S_{A+B,A}P^-)}{\det_{\mathcal{H}^-}(P^-S_{A,A+B}P^-S_{A+B,A}P^-)} && \cancel{(72)}\\
&\quad\times \Im \operatorname{tr}((P^-S_{A,A+B}P^-S_{A+B,A}P^-)^{-1}\partial_C P^-S_{A,A+B}P^-S_{A+B,A+C}P^-S_{A+C,A}P^-)\Big] && (73)\\
&= i\partial_B \Im \operatorname{tr}((P^-S_{A+B,A}P^-)^{-1}P^-\partial_C S_{A+B,A+C}P^- + \underline{\partial_C P^-S_{A+C,A}} P^-) && (74)\\
&= i\partial_B \Im \operatorname{tr}((P^-S_{A+B,A}P^-)^{-1}P^-\partial_C S_{A+B,A+C}P^-) && (75)\\
&= i\Im \operatorname{tr}(-\partial_B P^-S_{A+B,A}P^-\partial_C S_{A,A+C}P^- + P^-\partial_B\partial_C S_{A+B,A+C}P^-) && (76)\\
&= i\Im \operatorname{tr}(\partial_B P^-S_{A+B,A}P^+\partial_C S_{A,A+C}P^-) && (77)\\
&= -i\partial_B\partial_C \Im \operatorname{tr}(P^-S_{A,A+B}P^+S_{A,A+C}P^-), && (78)
\end{aligned}
$$

mehr erklären.

Extra Schritt

$$\frac{\operatorname{tr}(A_\varepsilon) - \operatorname{tr}(A)}{\varepsilon} = \operatorname{tr}\left(\tfrac{1}{\varepsilon}(A_\varepsilon - A)\right)$$

which proves the claim.

$$= \operatorname{tr}\left(\tfrac{1}{\varepsilon}[A_\varepsilon - A] - D_\varepsilon A_\varepsilon\right) + \operatorname{tr}(D_\varepsilon A_\varepsilon)$$

anomaly in thm 2.

mit Dyson Gl. beschreiben

in den Apendix

□

In order to construct the ~~desired~~ lift, we first construct a reference lift $\hat{S}$ that is well defined for any $A \in \mathcal{A}$. Afterwards we will study the dependence of the relative phase between this global lift $\hat{S}_{0,A}$ and a local lift given by $\hat{S}_{0,B}\overline{S}_{B,A}$ for $B - A$ small ~~as a multiplication operator on one-particle wave functions~~. By exploiting properties of this phase and c^+ we will ~~be able to~~ construct a global lift that has the desired properties. Since $\mathcal{A}$ is star shaped, we may reach any four-potential A from 0 through the straight line $\{tA \mid t \in [0,1]\}$.

Definition 6. *For any $A, B \in \mathcal{A}$ and any two lifts $S'_{A,B}, S''_{A,B}$ of the one particle scattering operator $S_{A,B}$ we define*

$$\frac{S'_{A,B}}{S''_{A,B}} \tag{79}$$

to be the unique complex number $z \in S^1$ such that

$$\frac{S'_{A,B}}{S''_{A,B}} S''_{A,B} = S'_{A,B} \tag{80}$$

holds. Furthermore, for any $A \in \mathcal{A}$ we define the lift $\hat{S}_{0,A}$ as the unique solution of the differential equation

$$A, B \in \mathcal{A} \text{ linearly dependent} \Rightarrow \partial_B \frac{\hat{S}_{0,A+B}}{\hat{S}_{0,A}\overline{S}_{A,A+B}} = 0, \tag{81}$$

subject to the boundary condition $\hat{S}_{0,0} = \mathbb{1}$.

Remark 1. *The lift $\hat{S}_{0,A}$ can also be calculated differently: pick $N \in \mathbb{N}$ and a series $(\delta_k)_{k\in\mathbb{N}} \subset \mathbb{R}^+$ such that $\|\mathbb{1} - S_{\sum_{k=1}^{n-1}\delta_k A, \sum_{k=1}^{n}\delta_k A}\| < 1$ holds true for all k and $\sum_{k=1}^{N}\delta_k = 1$. Then* *is there some kind of direct correspondence between $\|1-S_{0,\delta_n A}\|$ and $\|1-S_{\sum_{k=1}^{n-1}\delta_k A, \sum_{k=1}^{n}\delta_k A}\|$??*

$$P^- S_{\sum_{k=1}^{n-1}\delta_k A, \sum_{k=1}^{n}\delta_k A} P^- \tag{82}$$

is invertible. Now,

$$\hat{S}_{0,A} = \prod_{n=0}^{N} \overline{S}_{\sum_{k=1}^{n-1}\delta_k A, \sum_{k=1}^{n}\delta_k A}. \tag{83}$$

Before proving this claim, we notice the following property of $\overline{S}$: For any $A \in \mathcal{A}$ and $\alpha, \beta, \gamma > 0$ such that all factors exist the following identity holds

$$\overline{S}_{\alpha A, \gamma A} = \overline{S}_{\alpha A, \beta A} \overline{S}_{\beta A, \gamma A}. \tag{84}$$

This property is by (31) equivalent to

$$\Gamma_{\alpha A, \beta A, \gamma A} = 1. \tag{85}$$

This claim can be reduced to the one where $\alpha = 1$ by the following renaming scheme

$$\alpha A =: \tilde{A} \tag{86}$$

$$\beta/\alpha =: \tilde{\beta} \tag{87}$$

$$\gamma/\alpha =: \tilde{\gamma}. \tag{88}$$

In fact, the claim holds, as the following calculation shows:

$$\ln\Gamma_{A,\beta A,\gamma A} = \int_1^\beta d\beta' \partial_{\beta'} \ln \Gamma_{A,\beta' A,\gamma A} + \overbrace{\ln \Gamma_{A,A,\gamma A}}^{\overset{(30)}{=}0} \tag{89}$$

$$= \int_1^\beta d\beta' \left(\int_1^\gamma d\gamma' \partial_{\gamma'} \partial_{\beta'} \ln \Gamma_{A,\beta' A,\gamma' A} + \partial_{\beta'} \overbrace{\ln \Gamma_{A,\beta' A, A}}^{\overset{(30)}{=}0} \right) \tag{90}$$

$$= \int_1^\beta d\beta' \int_1^\gamma d\gamma' c_A((\beta'-1)A, (\gamma'-1)A) \tag{91}$$

$$= \int_1^\beta d\beta' \int_1^\gamma d\gamma' (\beta'-1)(\gamma'-1) \overbrace{c_A(A,A)}^{\overset{(29),(32)}{=}0} = 0, \tag{92}$$

where we have used various properties of lemma 1.
Because of (85) we see that (83) is actually independent of the choice of sequence $(\delta_k)_k$. The right side of equation (83) satisfies the initial condition of the ordinary differential equation, so we only need to check the differential equation itself. Next we reformulate the ordinary differential equation (81), pick $B = tA$ then we have

$$0 = \partial_B \frac{\hat{S}_{0,A+B}}{\hat{S}_{0,A}\overline{S}_{A,A+B}} = \partial_\varepsilon \frac{\hat{S}_{0,A(1+\varepsilon t)}}{\hat{S}_{0,A}\overline{S}_{A,A(1+\varepsilon t)}} = t\partial_\varepsilon \frac{\hat{S}_{0,A(1+\varepsilon)}}{\hat{S}_{0,A}\overline{S}_{A,A(1+\varepsilon)}} \tag{93}$$

What is more, we can pick $(\delta_k)_k$ such that $\sum_{k=1}^{N}\delta_k = 1$ and $\delta_{N+1} = \varepsilon$ then we have

$$\hat{S}_{0,A(1+\varepsilon)} = \prod_{n=0}^{N+1} \overline{S}_{\sum_{k=1}^{n-1}\delta_k A, \sum_{k=1}^{n}\delta_k A} = \prod_{n=0}^{N} \overline{S}_{\sum_{k=1}^{n-1}\delta_k A, \sum_{k=1}^{n}\delta_k A}\ \overline{S}_{A,A(1+\varepsilon)} \tag{94}$$

$$= \hat{S}_{0,A}\overline{S}_{A,A+B}, \tag{95}$$

or in other words

$$\frac{\hat{S}_{0,A+B}}{\hat{S}_{0,A}\overline{S}_{A,A+B}} = 1. \tag{96}$$

If $A = 0$ holds, we see directly that

$$\frac{\hat{S}_{0,B}}{\hat{S}_{0,0}\overline{S}_{0,B}} = 1, \tag{97}$$

so in both cases the ordinary differential equation is satisfied.

Definition 7. *Let $A, B \in \mathcal{A}$ such that $\|1 - S_{A,B}\| < 1$ holds. We define $\theta_{A,B} \in [-\pi, \pi[$ by*

$$e^{i\theta_{A,B}} := \frac{\hat{S}_{0,B}}{\hat{S}_{0,A}\overline{S}_{A,B}}. \tag{98}$$

Lemma 2. *For all $A, F, G \in \mathcal{A}$ such that $\|1 - S_{A,F}\| < 1, \|1 - S_{F,G}\| < 1, \|1 - S_{A,G}\| < 1$ hold, as well as for all $H, K \in \mathcal{A}$, we have*

$$\theta_{A,F} = -\theta_{F,A} \tag{99}$$

$$e^{i(\theta_{F,A}+\theta_{A,G}+\theta_{G,F})} = \Gamma_{F,A,G} \tag{100}$$

$$i\partial_{\varepsilon_1}\partial_{\varepsilon_2}\theta_{A+\varepsilon_1 H, A+\varepsilon_2 K} = c_A(H, K). \tag{101}$$

Proof. Pick $A, F, G \in \mathcal{A}$ as in the lemma. We start off by analysing

$$\hat{S}_{0,F}\overline{S}_{F,G} \overset{(98)}{=} e^{i\theta_{A,F}}\hat{S}_{0,A}\overline{S}_{A,F}\overline{S}_{F,G} \tag{102}$$

$$\overset{(31)}{=} e^{i\theta_{A,F}}\Gamma^{-1}_{A,F,G}\hat{S}_{0,A}\overline{S}_{A,G}. \tag{103}$$

Exchanging A and F in this equation yields

$$\hat{S}_{0,A}\overline{S}_{A,G} = e^{i\theta_{F,A}}\Gamma^{-1}_{F,A,G}\hat{S}_{0,F}\overline{S}_{F,G}. \tag{104}$$

This is equivalent to

$$\hat{S}_{0,F}\overline{S}_{F,G} = e^{-i\theta_{F,A}}\Gamma_{F,A,G}\hat{S}_{0,A}\overline{S}_{A,G} \ , \tag{105}$$

taking (29) into account this means that

$$\theta_{A,F} = -\theta_{F,A} \tag{106}$$

holds true. Equation (103) solved for $\hat{S}_{0,A}\overline{S}_{A,G}$ also gives us

$$\hat{S}_{0,G} \overset{(98)}{=} e^{i\theta_{A,G}}\hat{S}_{0,A}\overline{S}_{A,G} \overset{(103)}{=} e^{i\theta_{A,G}}e^{-i\theta_{A,F}}\Gamma_{A,F,G}\hat{S}_{0,F}\overline{S}_{F,G}. \tag{107}$$

The latter equation compared with

$$\hat{S}_{0,G} \overset{(98)}{=} e^{i\theta_{F,G}}\hat{S}_{0,F}\overline{S}_{F,G}, \tag{108}$$

yields a direct connection between Γ and θ:

$$e^{i\theta_{A,G}-i\theta_{A,F}}\Gamma_{A,F,G} = e^{i\theta_{F,G}}, \tag{109}$$

or by (99)

$$\Gamma_{A,F,G} = e^{i\theta_{F,G}+i\theta_{A,F}+i\theta_{G,A}}. \tag{110}$$

Finally, in this equation we replace $F = A + \varepsilon_1 H$ as well as $G = A + \varepsilon_2 K$, where $\varepsilon_1, \varepsilon_2$ is small enough so that θ and Γ are still well defined. Then we take the logarithm and derivatives to find

$$i\partial_{\varepsilon_1}\partial_{\varepsilon_2}\theta_{A+\varepsilon_1 H, A+\varepsilon_2 K} = \partial_{\varepsilon_1}\partial_{\varepsilon_2} \ln \Gamma_{A, A+\varepsilon_1 H, A+\varepsilon_2 K} \overset{(32)}{=} c_A(H, K). \tag{111}$$

□

So we find that c is the mixed derivative of θ. In the following we will characterise θ more thoroughly by c and c^+.

Definition 8. *We introduce for $p \in \mathbb{N}$*

$$\begin{aligned}\Omega^p := \{\omega : \mathcal{A}^{p+1} \to \mathbb{C} \mid \forall \sigma \in S^p, \alpha \in \mathbb{R}, A_1, \ldots, A_{p+1} \in \mathcal{A} :\\ \omega(A_1, \alpha A_{\sigma(1)}, \ldots, A_{\sigma(p)}) = \operatorname{sgn}(\sigma)\alpha\omega(A_1, A_2, \ldots, A_2)\},\end{aligned} \tag{112}$$

where $\operatorname{sgn}(\sigma)$ is the sign of the permutation σ. We define the one form $\chi \in \Omega^1(\mathcal{A})$ by

$$\chi_A(B) := \partial_B \theta_{A, A+B} \tag{113}$$

for all $A, B \in \mathcal{A}$. Furthermore for a differential form $\omega \in \Omega^p(\mathcal{A})$ for some $p \in \mathbb{N}$ we define the exterior derivative of ω, $d\omega \in \Omega^{p+1}(\mathcal{A})$ by

$$(d\omega)_A(B_1, \ldots, B_{p+1}) := \sum_{k=1}^{p+1} (-1)^{k+1} \partial_{B_k} \omega_{A+B_k}(B_1, \ldots, \cancel{B_k}, \ldots, B_{p+1}), \tag{114}$$

for general $A, B_1, \ldots, B_{p+1} \in \mathcal{A}$, where the notation $\cancel{B_k}$ denotes that B_k is not to be inserted as an argument.

Lemma 3. *The differential form χ fulfils*

$$(d\chi)_A(F, G) = -2ic_A(F, G) \tag{115}$$

for all $A, F, G \in \mathcal{A}$.

Proof. Pick $A, F, G \in \mathcal{A}$, we calculate

$$(d\omega)_A(F, G) = \partial_F \partial_G \theta_{A+F, A+F+G} - \partial_F \partial_G \theta_{A+G, A+F+G} \tag{116}$$

$$= \partial_F \partial_G (\theta_{A, A+F+G} + \theta_{A+F, A+G}) - \partial_F \partial_G (\theta_{A, A+F+G} + \theta_{A+G, A+F}) \tag{117}$$

$$\overset{(99)}{=} 2\partial_F \partial_G \theta_{A+F, A+G} \overset{(101)}{=} -2ic_A(F, G). \tag{118}$$

□

Now since $dc = 0$, we might use Poincaré's lemma as a method independent of θ to construct a differential form ω such that $d\omega = c$. In order to execute this plan, we first need to prove Poincaré's lemma for our setting:

Lemma 4 (Poincaré)**.** *Let $\omega \in \Omega^p(\mathcal{A})$ for $p \in \mathbb{R}$ be closed, i.e. $d\omega = 0$. Then ω is also exact, more precisely we have*

$$\omega = d \int_0^1 \iota_t^* i_X f^* \omega dt, \tag{119}$$

where X, ι_t for $t \in \mathbb{R}$ and f are given by

$$X : \mathbb{R} \times \mathcal{A} \to \mathbb{R} \times \mathcal{A}, \tag{120}$$
$$(t, B) \mapsto (1, 0) \tag{121}$$
$$\forall t \in \mathbb{R} : \iota_t : \mathcal{A} \to \mathbb{R} \times \mathcal{A}, \tag{122}$$
$$B \mapsto (t, B) \tag{123}$$
$$f : \mathbb{R} \times \mathcal{A} \mapsto \mathcal{A}, \tag{124}$$
$$(t, B) \mapsto tB \tag{125}$$
$$\forall t \in \mathbb{R} : f_t := f(t, \cdot). \tag{126}$$

Proof. Pick some $\omega \in \Omega^p(\mathcal{A})$. We will first show the more general formula

$$f_b^*\omega - f_a^*\omega = d\int_a^b \iota_t^* i_X f^*\omega \; dt + \int_a^b \iota_t^* i_X f^* d\omega dt. \tag{127}$$

The lemma follows then by $b = 1, a = 0$, $f_1^*\omega = \omega, f_0^*\omega = 0$ and $d\omega = 0$ for a closed ω. We begin by rewriting the right hand side of (127):

$$d\int_a^b \iota_t^* i_X f^*\omega \; dt + \int_a^b \iota_t^* i_X f^* d\omega dt = \int_a^b (d\iota_t^* i_X f^*\omega + \iota_t^* i_X f^* d\omega) dt. \tag{128}$$

Next we look at both of these terms separately. Let therefore $p \in \mathbb{N}$, $t, s_k \in \mathbb{R}$ and $A, B_k \in \mathcal{A}$ for each $p + 1 \geqslant k \in \mathbb{N}$. First, we calculate $d\iota_t^* i_X f^*\omega$

$$(f^*\omega)_{(t,A)}((s_1, B_1), \ldots, (s_p, B_p)) = \omega_{tA}(s_1 A + tB_1, \ldots, s_p A + tB_p) \tag{129}$$

$$(i_X f^*\omega)_{(t,A)}((s_1, B_1), \ldots, (s_{p-1}, B_{p-1})) = \omega_{tA}(A, s_1 A + tB_1, \ldots, s_{p-1} A + tB_{p-1} \tag{130}$$

$$(\iota_t^* i_X f^*\omega)_A(B_1, \ldots, B_{p-1}) = t^{p-1}\omega_{tA}(A, B_1, \ldots, B_{p-1}) \tag{131}$$

$$\begin{aligned}(d\iota_t^* i_X f^*\omega)_A(B_1, \ldots, B_p) &= \partial_\varepsilon|_{\varepsilon=0} \sum_{k=1}^p (-1)^{k+1} t^{p-1} \omega_{tA+\varepsilon t B_k}(A, B_1, \ldots, \cancel{B_k}, \ldots, B_p) \\ &+ \partial_\varepsilon|_{\varepsilon=0} \sum_{k=1}^p (-1)^{k+1} t^{p-1} \omega_{tA}(A + \varepsilon B_k, B_1, \ldots, \cancel{B_k}, \ldots, B_p) \qquad (132) \\ &= \partial_\varepsilon|_{\varepsilon=0} \sum_{k=1}^p t^p (-1)^{k+1} \omega_{tA+\varepsilon B_k}(A, B_1, \ldots, \cancel{B_k}, \ldots, B_p) + pt^{p-1}\omega_{tA}(B_1, \ldots, B_p). \qquad (133)\end{aligned}$$

Now, we calculate $\iota_t^* i_X f^* d\omega$:

$$(d\omega)_A(B_1, \cdots, B_{p+1}) = \partial_\varepsilon|_{\varepsilon=0} \sum_{k=1}^{p+1} (-1)^{k+1} \omega_{A+\varepsilon B_k}(B_1, \ldots, \cancel{B_k}, \ldots, B_{p+1}) \tag{134}$$

$$(f^* d\omega)(t, A)((s_1, B_1), \ldots, (s_{p+1}, B_{p+1})) = (d\omega)_{tA}(s_1 A + tB_1, \ldots, s_{p+1} A + tB_{p+1}) \tag{135}$$

$$= \partial_\varepsilon|_{\varepsilon=0} \sum_{k=1}^{p+1} (-1)^{k+1} \omega_{tA+\varepsilon(s_k A + tB_k)}(s_1 A + tB_1, \ldots, \cancel{s_k A + tB_k}, \ldots, s_p A + tB_p) \tag{136}$$

$$(i_X f^* d\omega)_{(t,A)}((s_1, B_1), \ldots, (s_p, B_p)) = \partial_\varepsilon|_{\varepsilon=0} \omega_{(t+\varepsilon)A}(s_1 A + tB_1, \ldots, s_p A + tB_p) \tag{137}$$

$$+ \partial_\varepsilon|_{\varepsilon=0} \sum_{k=1}^{p} (-1)^k \omega_{tA+\varepsilon(s_k A + tB_k)}(A, s_1 A + tB_1, \ldots, \cancel{s_k A + tB_k}, \ldots, s_p A + tB_p) \tag{138}$$

$$= t^p \partial_\varepsilon|_{\varepsilon=0} \omega_{(t+\varepsilon)A}(B_1, \ldots, , \cancel{B_k}, \ldots, B_p)) \tag{139}$$

$$+ \sum_{k=1}^{p} s_k t^{p-1} (-1)^{k+1} \partial_\varepsilon|_{\varepsilon=0} \omega_{(t+\varepsilon)A}(A, B_1, \ldots, B_p) \tag{140}$$

$$+ \partial_\varepsilon|_{\varepsilon=0} \sum_{k=1}^{p} (-1)^k t^{p-1} (\omega_{(t+s_k\varepsilon)A}(A, B_1, \ldots, \cancel{B_k}, \ldots, B_p) \tag{141}$$

$$+ \omega_{tA+\varepsilon tB_k}(A, B_1, \ldots, \cancel{B_k}, \ldots, B_p)) \tag{142}$$

$$= t^p \partial_\varepsilon|_{\varepsilon=0} \left(\omega_{(t+\varepsilon)A}(B_1, \ldots, B_p) + \sum_{k=1}^{p} (-1)^k \omega_{tA+\varepsilon B_k}(A, B_1, \ldots, \cancel{B_k}, \ldots, B_p) \right) \tag{143}$$

$$(\iota_t^* i_X f^* d\omega)_A(B_1, \ldots, B_p) = t^p \partial_\varepsilon|_{\varepsilon=0} \Big(\omega_{(t+\varepsilon)A}(B_1, \ldots, B_p) \tag{144}$$

$$+ \sum_{k=1}^{p} (-1)^k \omega_{tA+\varepsilon B_k}(A, B_1, \ldots, \cancel{B_k}, \ldots, B_p) \Big) \tag{145}$$

Adding (133) and (145) we find for (128):

$$\int_a^b (d\iota_t^* i_X f^* \omega + \iota_t^* i_X f^* d\omega) dt = \tag{146}$$

$$\int_a^b \Big(t^p \partial_\varepsilon|_{\varepsilon=0} \omega_{(t+\varepsilon)A}(B_1, \ldots, B_p) + p t^{p-1} \omega_{tA}(B_1, \ldots, B_p) \Big) dt \tag{147}$$

$$= \int_a^b \frac{d}{dt} (t^p \omega_{tA}(B_1, \ldots, B_p)) dt = \int_a^b \frac{d}{dt} (f_t^* \omega)_A(B_1, \ldots, B_p)) dt \tag{148}$$

$$= (f_b^* \omega)_A(B_1, \ldots, B_p) - (f_a^* \omega)_A(B_1, \ldots, B_p). \tag{149}$$

□

Definition 9. *For a closed exterior form $\omega \in \Omega^p(\mathcal{A})$ we define the form $\prod[\omega]$*

$$\prod[\omega] := \int_0^1 \iota_t^* i_X f^* \omega dt. \tag{150}$$

For $A, B_1, \ldots, B_{p-1} \in \mathcal{A}$ it takes the form

$$\prod[\omega]_A(B_1, \ldots, B_p) = \int_0^1 t^{p-1} \omega_{tA}(A, B_1, \ldots, B_{p-1}) dt. \tag{151}$$

By lemma 4 we know $d\prod[\omega] = \omega$ if $d\omega = 0$.

Now we found two one forms each produces c when the exterior derivative is taken. The next lemma informs us about their relationship.

Lemma 5. *The following equality holds*

$$\chi = -2i \prod[c]. \tag{152}$$

Proof. We have $d(\chi + 2i \prod[c]) = 0$ so by lemma 4 we know that there is $v : \mathcal{A} \to \mathbb{R}$ such that

$$dv = \chi + 2i \prod[c] \tag{153}$$

holds. Now (81) translates into the following ODE for θ:

$$\partial_B \theta_{0,B} = 0, \quad \partial_B \theta_{A,A+B}|_{A=B} = 0 \tag{154}$$

for all $A, B \in \mathcal{A}$. This means that

$$\chi_0(B) = 0 = \prod[c]_0(B), \quad \chi_{A,A} = 0 = \prod[c]_A(A) \tag{155}$$

hold. This implies

$$\partial_\varepsilon v_{\varepsilon A} = 0, \quad \partial_\varepsilon v_{A+\varepsilon A} = 0, \tag{156}$$

which means that v is constant. $\square$

From this point on we will assume the existence of a function c^+ fulfilling (19),(20) and (21). Recall equation (20):

$$\forall A, F, G, H : \partial_H c^+_{A+H}(F, G) = \partial_G c^+_{A+G}(F, H). \tag{157}$$

For a fixed $F \in \mathcal{A}$, this condition can be read as $d(c^+_\cdot(F, \cdot)) = 0$. As a consequence we can apply lemma 4 to define a one form.

Definition 10. *For any $F \in \mathcal{A}$, we define*

$$\beta_A(F) := 2i \prod[c^+_\cdot(F, \cdot)]_A. \tag{158}$$

Lemma 6. *The following two equations hold:*

$$d\beta = -2ic \tag{159}$$

$$d(\beta - \chi) = 0. \tag{160}$$

Proof. We start with the exterior derivative of β. Pick $A, F, G \in \mathcal{A}$:

$$d\beta_A(F, G) = \partial_F \beta_{A+F}(G) - \partial_G \beta_{A+G}(F) \tag{161}$$

$$= d\Big(\prod[c^+_\cdot(G, \cdot)]\Big)_A(F) - d\Big(\prod[c^+_\cdot(F, \cdot)]\Big)_A(G) \tag{162}$$

$$= 2ic^+_A(G, F) - 2ic^+_A(F, G) \overset{(19)}{=} -2ic_A(F, G). \tag{163}$$

This proves the first equality. The second equality follows directly by $d\chi = -2ic$. $\square$

Definition 11. *Since $\beta - \chi$ is closed, we may use lem 4 again to define the phase*

$$\alpha := \prod[\beta - \chi]. \tag{164}$$

Furthermore, for all $A, B \in \mathcal{A}$ we define the corrected second quantised scattering operator

$$\tilde{S}_{0,A} := e^{i\alpha_A} \hat{S}_{0,A} \tag{165}$$

$$\tilde{S}_{A,B} := \tilde{S}^{-1}_{0,A} \tilde{S}_{0,B}. \tag{166}$$

Corollary 1. *We have* $\tilde{S}_{A,B}\tilde{S}_{B,C} = \tilde{S}_{A,C}$ *for all* $A, B, C \in \mathcal{A}$.

Theorem 3. *The corrected second quantised scattering operator fulfils the following causality condition for all* $A, F, G \in \mathcal{A}$ *such that* $F \prec G$*:*

$$\tilde{S}_{A,A+F} = \tilde{S}_{A+G,A+G+F}. \tag{167}$$

Proof. Let $A, F, G \in \mathcal{A}$ such that $F \prec G$ We note that for the first quantised scattering operator we have

$$S_{A+G,A+G+F} = S_{A,A+F}, \tag{168}$$

so by definition of $\overline{S}$ we also have

$$\overline{S}_{A+G,A+G+F} = \overline{S}_{A,A+F}. \tag{169}$$

So for any lift this equality is true up to a phase, meaning that

$$f(A, F, G) := \frac{\tilde{S}_{A+G,A+G+F}}{\tilde{S}_{A,A+F}} \tag{170}$$

is well defined. We see immediately

$$f(A, 0, G) = 1 = F(A, F, 0). \tag{171}$$

Pick $F_1, F_2 \prec G_1, G_2$. We abbreviate $F = F_1 + F_2, G = G_1 + G_2$ and we calculate

$$\begin{aligned}
f(A, F, G) &= \frac{\tilde{S}_{A+G,A+F+G}}{\tilde{S}_{A,A+F}} && (172)\\
&= \frac{\tilde{S}_{A+G,A+F+G}}{\tilde{S}_{A+G_1,A+G_1+F}} \frac{\tilde{S}_{A+G_1,A+G_1+F}}{\tilde{S}_{A,A+F}} && (173)\\
&= \frac{\tilde{S}_{A+G,A+G+F_1}\tilde{S}_{A+G+F_1,A+F+G}}{\tilde{S}_{A+G_1,A+F_1+G_1}\tilde{S}_{A+G_1+F_1,A+G_1+F}} \frac{\tilde{S}_{A+G_1,A+G_1+F}}{\tilde{S}_{A,A+F}} && (174)\\
&= \frac{\tilde{S}_{A+G,A+G+F_1}}{\tilde{S}_{A+G_1,A+F_1+G_1}} \frac{\tilde{S}_{A+G+F_1,A+F+G}}{\tilde{S}_{A+G_1+F_1,A+G_1+F}} f(A, G_1, F_1 + F_2) && (175)\\
&= f(A + G_1, F_1, G_2) f(A + G_1 + F_1, G_2, F_2) f(A, G_1, F_1 + F_2). && (176)
\end{aligned}$$

Taking the logarithm and differentiating we find:

$$\partial_{F_2}\partial_{G_2} \ln f(A, F_1 + F_2, G_1 + G_2) = \partial_{F_2}\partial_{G_2} \ln f(A + F_1 + G_1, F_2, G_2). \tag{177}$$

Next we pick $F_2 = \alpha_1 F_1$ and $G_2 = \alpha_2 G_1$ for $\alpha_1, \alpha_2 \in \mathbb{R}^+$ small enough so that

$$\begin{aligned}
\|1 - S_{A+F+G,A+F_1+G_1}\| &< 1 && (178)\\
\|1 - S_{A+F+G,A+F_1+G}\| &< 1 && (179)\\
\|1 - S_{A+F+G,A+F+G_1}\| &< 1 && (180)
\end{aligned}$$

hold. We abbreviate $A' = A + G_1 + F_1$, use (98) and compute

$$f(A', F_2, G_2) = \frac{e^{i\alpha_{A'+F_2+G_2} + i\theta_{A',A'+F_2+G_2} - i\alpha_{A'+G_2} - i\theta_{A',A'+G_2}}}{e^{i\alpha_{A'+F_2} + i\theta_{A',A'+F_2} - i\alpha_{A'} - i\theta_{A',A'}}} \frac{\overline{S}_{A'+G_2,A'}\overline{S}_{A',A'+F_2+G_2}}{\overline{S}_{A',A'}\overline{S}_{A',A'+F_2}}. \tag{181}$$

The second factor in this product can be simplified significantly:

$$\frac{\overline{S}_{A'+G_2,A'}\overline{S}_{A',A'+F_2+G_2}}{\overline{S}_{A',A'}\overline{S}_{A',A'+F_2}} = \frac{\overline{S}_{A'+G_2,A'}\overline{S}_{A',A'+F_2+G_2}}{\overline{S}_{A',A'+F_2}} \tag{182}$$

$$\overset{(31)}{=} \Gamma^{-1}_{A'+G_2,A',A'+F_2+G_2}\frac{\overline{S}_{A'+G_2,A'+F_2+G_2}}{\overline{S}_{A',A'+F_2}} \tag{183}$$

$$\overset{(169)}{=} \Gamma_{A',A'+G_2,A'+F_2+G_2} \overset{(100)}{=} e^{i\theta_{A',A'+G_2}+i\theta_{A'+G_2,A'+G_2+F_2}+i\theta_{A'+F_2+G_2,A'}}. \tag{184}$$

So in total we find

$$f(A',F_2,G_2) = \frac{e^{i\alpha_{A'+F_2+G_2}+i\theta_{A',A'+F_2+G_2}-i\alpha_{A'+G_2}-i\theta_{A',A'+G_2}}}{e^{i\alpha_{A'+F_2}+i\theta_{A',A'+F_2}-i\alpha_{A'}-i\theta_{A',A'}}}\times \tag{185}$$

$$e^{i\theta_{A',A'+G_2}+i\theta_{A'+G_2,A'+G_2+F_2}+i\theta_{A'+F_2+G_2,A'}} \tag{186}$$

$$= \exp(i\alpha_{A'+F_2+G_2} - i\alpha_{A'+G_2} - i\alpha_{A'+F_2} + i\alpha_{A'} + i\theta_{A'+G_2,A'+G_2+F_2} - i\theta_{A',A'+F_2}). \tag{187}$$

Most of the terms in the exponent do not depend on F_2 and G_2, so taking the mixed logarithmic derivative things simplify:

$$\partial_{G_2}\partial_{F_2}\ln f(A',F_2,G_2) = i\partial_{G_2}\partial_{F_2}(\alpha_{A'+F_2+G_2} + \theta_{A'+G_2,A'+G_2+F_2}) \tag{188}$$

$$\overset{(164),(113)}{=} i\partial_{G_2}(\beta_{A'+G_2}(F_2) - \chi_{A'+G_2}(F_2) + \chi_{A'+G_2}(F_2)) \tag{189}$$

$$\overset{(159)}{=} -2c^+_{A'}(F_2,G_2) \overset{F_2<G_2,(21)}{=} 0. \tag{190}$$

So by (177) we also have

$$\partial_{F_2}\partial_{G_2}\ln f(A,F_1+F_2,G_1+G_2) = 0 = \partial_{\alpha_1}\partial_{\alpha_2}\ln f(A,F_1(1+\alpha_1),G_1(1+\alpha_2)) \tag{191}$$

But then we can integrate and obtain

$$0 = \int_{-1}^{0} d\alpha_1 \int_{-1}^{0} d\alpha_2 \partial_{\alpha_1}\partial_{\alpha_2}\ln f(A,F_1(1+\alpha_1),G_1(1+\alpha_2)) \tag{192}$$

$$= \ln f(A,F_1,G_1) - \ln f(A,0,G_1) - \ln f(A,F_1,0) + \ln f(A,0,0) \tag{193}$$

$$\overset{(171)}{=} \ln f(A,F_1,G_1). \tag{194}$$

remembering equation (170), the definition of f, this ends our proof. □

Using $\tilde{S}$ we introduce the current associated to it.

Definition 12. *Let $A,F \in \mathcal{A}$, define*

$$j_A(F) := i\partial_F \left\langle \Omega, \tilde{S}_{A,A+F}\Omega \right\rangle = i\partial_F \ln \left\langle \Omega, \tilde{S}_{A,A+F}\Omega \right\rangle. \tag{195}$$

Theorem 4. *For general $A,F \in \mathcal{A}$ we have*

$$j_A(F) = -\beta_A(F). \tag{196}$$

So in particular for $G \in \mathcal{A}$

$$\partial_G j_{A+G}(F) = -2ic^+_A(F,G). \tag{197}$$

holds.

Proof. Pick $A, F \in \mathcal{A}$ as in the theorem. We calculate

$$i\partial_F \ln \left\langle \Omega, \tilde{S}_{A,A+F}\Omega \right\rangle \tag{198}$$

$$= i\partial_F \left(i\alpha_{A+F} - i\alpha_A + \ln \left\langle \Omega, \hat{S}_{0,A}^{-1}\hat{S}_{0,A+F}\Omega \right\rangle \right) \tag{199}$$

$$= i\partial_F \left(i\alpha_{A+F} + i\theta_{A,A+F} + \ln \langle \Omega, \overline{S}_{A,A+F}\Omega \rangle \right) \tag{200}$$

The last summand vanishes, as can be seen by the following calculation

$$\partial_F \ln \langle \Omega, \overline{S}_{A,A+F}\Omega \rangle = i\partial_F \ln \det_{\mathcal{H}^-}(P^- S_{A,A+F} P^- \operatorname{ARG}(P^- S_{A,A+F} P^-)^{-1}) \tag{201}$$

$$= i\partial_F \ln \det_{\mathcal{H}^-} |P^- S_{A,A+F} P^-| = \frac{i}{2}\partial_F \ln \det_{\mathcal{H}^-}((P^- S_{A,A+F} P^-)^* P^- S_{A,A+F} P^-) \tag{202}$$

$$= \frac{i}{2}\partial_F \det(P^- S_{A+F,A} P^- S_{A,A+F} P^-) = \frac{i}{2} \operatorname{tr}(\partial_F P^- S_{A+F,A} P^- S_{A,A+F} P^-) \tag{203}$$

$$= \frac{i}{2} \operatorname{tr}(\partial_F P^- S_{A,A+F} P^- + \partial_F P^- S_{A+F,A} P^-) = 0 \tag{204}$$

where we made use of (66). So we are left with

$$j_A(F) = -\partial_F(\alpha_{A+F} + \theta_{A,A+F}) = -(\beta_A(F) - \chi_A(F) + \chi_A(F)) = -\beta_A(F). \tag{205}$$

Finally by taking the derivative with respect to $G \in \mathcal{A}$ and using the definition of β we find

$$\partial_G j_{A+G}(F) = -2ic_A^+(F, G). \tag{206}$$

□