The Relationship Between Hadamard States, the Fermionic Projector, and Admissible Polarisation Classes

D.-A. Deckert*, Felix Finster† and Markus Nöth‡

October 31, 2019

Abstract

This paper compares central objects of three different types of approaches to quantum field theory (QFT): The study of Hadamard states, the Fermionic Projector, and admissible polarisation classes. These approaches have different scopes and pursue different motivations, which makes a direct comparison difficult. Nevertheless, the basic protagonists in all three approaches share much more similarity than is apparent at first glance. It is the purpose of this work to highlight these common features. Since they all study different aspects of QFT; however, one can make out objects in each of these theories which closely resemble one another.

1 Introduction

The first class of objects of interest are Hadamard states which appear in the algebraic approach to QFT [8].
The third class of objects are the continuum limit of the Fermionic Projectors [7].
The second class of objects are the (I_2-almost) projectors P^λ_Σ which are closely linked to polarisation classes of the vacuum of external field quantum electrodynamics (QED) [3, 5, 4].
In section 2 we give the definition of a Hadamard state, briefly motivate it's usage and give its explicit form in the case of flat spacetime subject to an external field as computed in the physics literature [10].
In section 3 ...
In section 4 we briefly describe why in the approach of admissible polarisation classes one only keeps track of the time evolution of the projector up to an error that is a Hilbert-Schmidt operator. Furthermore we will find a class of candidate I_2-almost projectors that have a simple time evolution.
In section 5...comparison fermionic projector and hadamard state
In section 6 we find that each Hadamard state corresponds to an I_2-almost projector in a natural way.
is that specific enough?
Throughout the paper $\Sigma, \Sigma', \Sigma''$ denote arbitrary Cauchy surfaces, while for the sake of simplicity we choose $A \in C_c^\infty(\mathbb{R}^4, \mathbb{R}^4)$ to be a four-potential and Σ_{in} denotes a Cauchy surface earlier than the support of (A).

*deckert@math.lmu.de
Mathematisches Institut der Ludwig-Maximilians-Universität München, Theresienstr. 39, 80333 München, Germany
†felix.finster@mathematik.uni-regensburg.de
Universität Regensburg, Universitätsstraße 31, 93053 Regensburg, Germany
‡noeth@math.lmu.de
Mathematisches Institut der Ludwig-Maximilians-Universität München, Theresienstr. 39, 80333 München, Germany

In this paper we will focus on a system of Dirac fields subject to an external electromagnetic four-potential A in flat Minkowski spacetime. We choose the sign convention $\eta = \mathrm{diag}(+1, -1, -1, -1)$. We denote the minimally coupled differential by $\nabla_\alpha = \partial_\alpha + iA_\alpha$ and make use of the Feynman slash notation $\nabla\!\!\!/ =: \nabla_\alpha \gamma^\alpha$ with γ^α fulfilling $\{\gamma^\alpha, \gamma^\beta\} := \gamma^\alpha\gamma^\beta + \gamma^\beta\gamma^\alpha = 2\eta^{\alpha,\beta}$ each field solving Dirac's equation

$$(i\nabla\!\!\!/ - m)\psi =: D\psi = 0 \tag{1}$$

and collectively constituting the free vacuum prior to the support of said external field.For a 4-spinor $\psi \in \mathbb{C}^4$ (viewed as a column vector), $\overline{\psi}$ stands for the row vector $\psi^*\gamma^0$, where * denotes hermitian conjugation.

macht das überhaupt Sinn so wenn ich von Dirac Felder spreche? Bei operatoren ist ja der Zustand nicht der Zustand des Feldes, sondern der Zustand ist extra.

2 Hadamard States

In the algebraic approach to QFT one puts less emphasise on the Hilbert space than is commonly done in non relativistic physics because it is not a relativistically invariant object. Instead one focuses on the algebra of operators that are chosen to do the bookkeeping of statistical outcomes of measurements.
To infer predictions, some part of the necessary computation can be conducted on the level of this algebra. However, eventually, expectation values are to be computed in a certain representation, usually found by the GNS construction with respect to a certain state. This choice has to be made on physical grounds. Hadamard states are often thought to be physically sensible states because they have positive energy in a certain sense.
In order to introduce Hadamard states we have to first define the notion of wavefront set, which itself needs some preliminaries. For the introduction of these concepts we follow Hömander [9, Chapter 8].
We begin by introducing the singular support of a distribution

Definition 1. *Let for $n, m \in \mathbb{N}$, $v \in (C^\infty(\mathbb{R}^n, \mathbb{C}^m))'$ the singular support of v is defined to be the subset of points $x \in \mathbb{R}^n$ such that there is no neighbourhood U of x such that there is a smooth function $\phi_{x,v} \in C^\infty(\mathbb{R}^n, \mathbb{C}^m)$ such that v acts on test functions $\varphi \in C_c^\infty(U, \mathbb{C}^m)$ as*

$$v(\varphi) = \int \phi^\dagger_{x,v}(x)\varphi(x)dx. \tag{2}$$

The singular support contains all the points of a distribution such that the distribution does not act like a smooth function at that point. The wavefront set which we are about to introduce gives an additional directional information of where these singularities propagate. We incorporate this information by the Fourier transform in the following definition.

Definition 2. *Let for $n, m \in \mathbb{N}$, $v \in (C^\infty(\mathbb{R}^n, \mathbb{C}^m))'$, we denote by $\Xi(v) \subset \mathbb{R}^n \backslash \{0\}$ the set of all η such that there is no cone $V \subset \mathbb{R}^n$, neighbourhood of η, such that for all $a \in \mathbb{N}$ there are $C_a > 0$ such that for all $\xi \in V$ we have*

$$|\hat{v}(\xi)| \leq \frac{C_a}{1 + |\xi|^a}. \tag{3}$$

Furthermore for each $x \in \mathbb{R}^n$ we define

$$\Xi_x(v) := \bigcap_{\substack{\phi \in C_c^\infty(\mathbb{R}^n) \\ x \in \mathrm{supp}(\phi)}} \Xi(v\phi), \tag{4}$$

Where $v\phi$ is the pointwise multiplication of a distribution and a scalar test function, which acts as $v\phi : C^\infty(\mathbb{R}^n, \mathbb{C}^m) \ni \psi \mapsto v(\psi\phi)$.

We have collected the tools to introduce the wavefront set and the notion of Hadamard states.

Definition 3. *Let for $n, m \in \mathbb{N}$, $v \in (\mathcal{S}(\mathbb{R}^n, \mathbb{C}^m))'$ be a tempered distribution. The wavefront set $\mathrm{WF}(v)$ of the distribution v is defined as*

$$\mathrm{WF}(v) := \{(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n \mid \xi \in \Xi_x(v)\}\,. \tag{5}$$

soll ich das schon alles direkt für Distributionen die zwei Funktionen schlucken hinschreiben?

Definition 4. *A map $H : C_c^\infty(\mathbb{R}^4, \mathbb{C}^4) \times C_c^\infty(\mathbb{R}^4, \mathbb{C}^4) \to \mathbb{C}$, is called Hadamard state if it fulfils for all $f, g \in C_c^\infty(\mathbb{R}^4, \mathbb{C}^4)$:*

$$H(Df, g) = 0 \tag{6}$$
$$H(f, g) + H(g, f) = iS(f, g) \tag{7}$$
$$\overline{H(f, g)} = H(\overline{f}, \overline{g}) \tag{8}$$
$$WF(H) \subset C_+, \tag{9}$$

where $S(f, g)$ is the propagator of the Dirac equation, and $C_+ := \{(x, y; k_1, -k_2) \in \mathbb{R}^{16} \mid (x; k_1) \approx (y; k_2), k_1^2 \geq 0, k_1^0 > 0\}$ and $(x; k_1) \approx (y; k_2)$ holds whenever $(x - y)^2 = 0$ and $(y - x) \parallel k_1 = k_2$.

It is in the sense of the fourth condition that Hadamard states are of positive energy.
In the scenario of Minkowski spacetime in an external field Dirac [6] already studied the Hadamard states, although that name was not established at the time. More recently the subject has attracted considerable attention [11, 10], which computed the Hadamard states. They are given in terms of the Klein-Gordon operator corresponding to Dirac's equation:

Definition 5. *The Klein-Gordon operator corresponding to the Dirac equation (1) reads*

$$P : C^\infty(\mathbb{R}^4, \mathbb{C}^4) \to C^\infty(\mathbb{R}^4, \mathbb{C}^4) \tag{10}$$
$$P = (i\nabla\!\!\!/ - m)(-i\nabla\!\!\!/ - m) = \nabla_\alpha \nabla^\alpha + \frac{i}{2}\gamma^\alpha\gamma^\beta F_{\alpha,\beta} + m^2, \tag{11}$$

where $F_{\alpha,\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$ is the field strength tensor of the electromagnetic field. Furthermore we define for $f \in \mathcal{C}_c^\infty(\mathbb{R}^4 \times \mathbb{R}^4, \mathbb{C}^{16})$ the differential operator

$$\nabla\!\!\!/^* f(x, x') = \left(\frac{\partial}{\partial y^\alpha} - iA_\alpha(y)\right) f(x, y)\gamma^\alpha. \tag{12}$$

For the special case of a Dirac field in Minkowski space-time Zahn [10] gave a more explicit form of the Hadamard states $H \in (C_c^\infty(\mathcal{M}) \times C_c^\infty(\mathcal{M}))'$ on which we base our analysis below. According to this, H acts for $f_1, f_2 \in C_c^\infty(\mathcal{M}) \otimes \mathbb{C}^4$ as

$$H(f_1, f_2) = \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^4} d^4x \overline{f_1}(x) \int_{\mathbb{R}^4} d^4y\ h_\varepsilon(x, y) f_2(y), \tag{13}$$

where h_ε is of the form

$$
\begin{aligned}
h_\varepsilon(x,y) = \\
&\frac{-1}{2(2\pi)^2}(-i\nabla\!\!\!/ + i\nabla\!\!\!/^* - 2m)\Big[\frac{e^{-i(x-y)^\alpha \int_0^1 ds A_\alpha(xs+(1-s)y)}}{(y-x-i\varepsilon e_0)^2}\Big] && (14)\\
&+\frac{-1}{2(2\pi)^2}(-i\nabla\!\!\!/ + i\nabla\!\!\!/^* - 2m)\Big[V(x,y)\ln(-(y-x-i\varepsilon e_0)^2)\Big] && (15)\\
&+ B(x,y), && (16)
\end{aligned}
$$

where $V, B : \mathbb{R}^4 \to \mathbb{C}^4$ are smooth functions, B is completely arbitrary, whereas V is fixed by the external potential. The expansion

$$
V^N(x,y) := \sum_{k=1}^{N} \frac{1}{4^k k!(k-1)!} V_k(x,y)(x-y)^{2(k-1)}, \tag{17}
$$

is an asymptotic expansion for V for $k \to \infty$, in the sense that $(V - V^N)(x,y)\ln(-(x-y)^2)$ as a function of x and y is in $C^{N-2}(\mathbb{R}^{4+4})$ and $V - V^N = \mathcal{O}\left(\left((x-y)^2\right)^{N-2}\right)$. The functions V_k fulfil a recursive set of partial differential differential equations

$$
(x-y)^\alpha(\partial_{x,\alpha} + iA_\alpha(x))V_n(x,y) + nV_n(x,y) = -nPV_{n-1}(x,y), \tag{18}
$$

where $V_0(x,y) = e^{-i(x-y)^\alpha \int_0^1 ds A_\alpha(xs+(1-s)y)}$.
For the rest of this paper, we assume that for any $A \in C_c^\infty(\mathbb{R}^4)$ there are H, $(h_\varepsilon)_{\varepsilon>0}$ and V fulfilling all of the conditions described in this paragraph.

3 The Fermionic Projector

4 Projectors for Polarisation Classes

The concept of polarisation classes arises naturally in the study of QED in external electromagnetic fields. It does need some machinery to be introduced and related to more familiar objects which we are going to introduce first. In doing so we follow [4]

Definition 6. *We define a Cauchy surface Σ in $\mathbb{R}^4$ to be a smooth, 3-dimensional submanifold of $\mathbb{R}^4$ that fulfills the following three conditions:*

a) Every inextensible, two-sided, time- or light-like, continuous path in $\mathbb{R}^4$ intersects Σ in a unique point.

b) For every $x \in \Sigma$, the tangential space $T_x\Sigma$ is space-like.

c) The tangential spaces to Σ are bounded away from light-like directions in the following sense: The only light-like accumulation point of $\bigcup_{x\in\Sigma} T_x\Sigma$ is zero.

In coordinates, every Cauchy surface Σ can be parametrised as

$$
\Sigma = \{\pi_\Sigma(\vec{x}) := (t_\Sigma(\vec{x}), \vec{x}) \mid \vec{x} \in \mathbb{R}^3\} \tag{19}
$$

with a smooth function $t_\Sigma : \mathbb{R}^3 \to \mathbb{R}$. For convenience and without restricting generality of our results we keep a global constant

$$
0 < V_{\max} < 1 \tag{20}
$$

fixed and work only with Cauchy surfaces Σ such that

$$\sup_{\vec{x}\in\mathbb{R}^3} |\mathrm{grad}\ t_\Sigma(\vec{x})| < V_{\max}. \tag{21}$$

The standard volume form over $\mathbb{R}^4$ is denoted by $d^4x = dx^0dx^1dx^2dx^3$; the product of forms is understood as wedge product. The symbol d^3x mean the 3-form $d^3x = dx^1dx^2dx^3$ on $\mathbb{R}^4$ and on $\mathbb{R}^3$ respectively. Contraction of a form ω with a vector v is denoted by $i_v(\omega)$. The notation $i_v(\omega)$ is also used for the spinor matrix valued vector $\gamma = (\gamma^0, \gamma^1, \gamma^2, \gamma^3) = \gamma^\mu e_\mu$:

$$i_\gamma(d^4x) = \gamma^\mu i_{e_\mu}(d^4x). \tag{22}$$

For $x \in \Sigma$ the restriction of the spinor matrix valued 3-form $i_\gamma(d^4x)$ to the tangential space $T_x\Sigma$ is given by

$$i_\gamma(d^4x) = \not{n}(x) i_n(d^4x) = \left(\gamma^0 - \sum_{\mu=1}^{3} \gamma^\mu \frac{\partial t_\Sigma(\vec{x})}{\partial x^\mu}\right) d^3x =: \Gamma(\vec{x})d^3x \tag{23}$$

As a consequence of (21), there is a positive constant $\Gamma_{\max} = \Gamma_{\max}(V_{\max})$ such that

$$\|\Gamma(\vec{x}) \leq \Gamma_{\max}, \quad \forall \vec{x} \in \mathbb{R}^3. \tag{24}$$

As is well known equation (1) gives us a one-particle time evolution operator for each pair of Cauchy surfaces Σ, Σ'

$$U_{\Sigma',\Sigma} : \mathcal{H}_\Sigma \to \mathcal{H}_{\Sigma'}, \tag{25}$$

where $\mathcal{H}_\Sigma$ is the space of square integrable functions on the Cauchy surface Σ. The scalar product on $\mathcal{H}_\Sigma$ is given by

$$\phi, \psi \mapsto \int_\Sigma \overline{\phi}(x) i_\gamma(d^4\gamma)\psi(x). \tag{26}$$

We start out by characterising the polarisation classes for free motion.

Definition 7. *The projector $P_\Sigma^{\mathcal{H}^-}$ has the well known representation as the weak limit of the integral operator with the kernel[4]*

$$p_\varepsilon^-(x,y) = -\frac{m^2}{4\pi^2}(i\not{\partial}_x + m)\frac{K_1(m\sqrt{-(y-x-i\varepsilon e_0)^2})}{m\sqrt{-(y-x-i\varepsilon e_0)^2}}, \tag{27}$$

where the square is a Minkowski square and the square root denotes its principle value. By weak limit we mean

$$\langle \phi, P_\Sigma^{\mathcal{H}^-}\psi\rangle = \lim_{\varepsilon\searrow 0}\int_{\Sigma\times\Sigma} \overline{\phi}(x) i_\gamma(d^4x) p_\varepsilon^-(x,y) i_\gamma(d^4y)\psi(y), \tag{28}$$

for general $\phi, \psi \in \mathcal{H}_\Sigma$.

Remark 1. *By inserting the expansion of K_1 in terms of a Laurent series and a logarithm, [1] one obtains:*

$$K_1(\xi) = \frac{1}{\xi} - \frac{\xi}{4}\sum_{k=0}^{\infty}\left(2\psi(k+1) + \frac{1}{k+1} + 2\ln 2 - 2\ln\xi\right)\frac{\left(\frac{\xi^2}{4}\right)^k}{k!^2(k+1)} \tag{29}$$

$$:= \frac{1}{\xi} + \xi Q_1(\xi^2)\ln\xi + \xi Q_2(\xi^2) =: \frac{1}{\xi} + \xi Q_3(\xi). \tag{30}$$

It is not obvious from the equation (18) but well known that the vacuum of Minkowski spacetime does indeed correspond to a Hadamard state subject to vanishing four potential. In fact, the Hadamard states were constructed to agree with the Minkowski vacuum up to smooth terms.

Definition 8. *Let $Pol(\mathcal{H}_\Sigma)$ denote the set of all closed, linear subspaces $V \subset \mathcal{H}$ such that both V and $V^\perp$ are infinite dimensional. Any $V \in Pol(\mathcal{H}_\Sigma)$ is called* polarisation *of $\mathcal{H}_\Sigma$. For $V \in Pol(\mathcal{H}_\Sigma)$, let $P^V_\Sigma : \mathcal{H}_\Sigma \to V$ denote the orthogonal projection of $\mathcal{H}_\Sigma$ onto V. The Fock space corresponding to V on the Cauchy surface Σ is defined to be*

$$\mathcal{F}(V, \mathcal{H}_\Sigma) := \bigoplus_{c\in\mathbb{Z}} \mathcal{F}_c(V, \mathcal{H}_\Sigma), \quad \mathcal{F}_c(V, \mathcal{H}_\Sigma) := \bigoplus_{\substack{n,m\in\mathbb{N}_0\\ c=m-n}} (V^\perp)^{\wedge n} \otimes \overline{V}^{\wedge m}, \tag{31}$$

where $\bigoplus$ is the Hilbert space direct sum, $\wedge$ the antisymmetric tensor product of Hilbert spaces and $\overline{V}$ is the conjugate complex vector space of V, which coincides with V as a set, has the same vector space operations as V except for scalar multiplication, which is defined by $(z, \psi) \mapsto z^*\psi$ for $z \in \mathbb{C}, \psi \in V$.

Remark 2. *Given two polarisations $V, W \in Pol(\mathcal{H}_\Sigma)$, for two Fockspaces $\mathcal{F}(V, \mathcal{H}_\Sigma)$ and $\mathcal{F}(W, \mathcal{H}_\Sigma)$ there is a unitary operator $U : \mathcal{F}(V, \mathcal{H}_\Sigma) \to \mathcal{F}(W, \mathcal{H}_\Sigma$ if and only if $P^V_\Sigma - P^W_\Sigma \in I_2(\mathcal{H}_\Sigma)$ by the theorem of Shale and Stinespring [**?**].*

Remark 2 gives us a natural limit with respect to which it is useful to analyse the regularity of Projectors P^A_Σ. We will therefore be content with the following equivalence class.

Definition 9. *For $V, W \in Pol(\mathcal{H}_\Sigma)$ we write*

$$V \approx W \iff P^V_\Sigma - P^W_\Sigma \in I_2(\mathcal{H}_\Sigma), \tag{32}$$

$$C_\Sigma(A) := [U^A_{\Sigma\Sigma_{in}}\mathcal{H}^-_{\Sigma_{in}}]_\approx. \tag{33}$$

The equivalence class $C_\Sigma(A)$ transforms naturally with respect to gauge and Lorentz transforms[4]. Now for hyperplanes $\Sigma \cap \operatorname{supp}(A) \neq \emptyset$ the operator $P^{\mathcal{H}^-}_\Sigma$ does not represent $C_\Sigma(A)$. Because all we are interested in is representations of equivalence classes, we are content with finding objects that differ from a Projector onto $U^A_{\Sigma,\Sigma_{\mathrm{in}}}\mathcal{H}^-_{\Sigma_in}$ by a Hilbert-Schmidt operator. Therefore we need not keep track of the exact evolution of the projection operators, but define a whole class of admissible ones.

Definition 10. *The set $\mathcal{G}^A$ denotes the set of all functions $\lambda^A \in C^\infty(\mathbb{R}^4 \times \mathbb{R}^4, \mathbb{R})$ that satisfy*

i) There is a compact set $K \subset \mathbb{R}^4$ such that $\operatorname{supp}\lambda \subseteq K \times \mathbb{R}^4 \cup \mathbb{R}^4 \times K$.

ii) λ satisfies $\forall x \in \mathbb{R}^4 : \lambda(x, x) = 0$.

iii) On the diagonal the first derivatives fulfil

$$\forall x, y \in \mathbb{R}^4 : \partial_x\lambda(x, y)|_{y=x} = -\partial_y\lambda(x, y)|_{y=x} = A(x). \tag{34}$$

We futhermore define a corresponding (quasi) projector P^λ:

$$\langle \phi, P_\Sigma^\lambda \psi \rangle = \lim_{\varepsilon \searrow 0} \langle \phi, P_\Sigma^{A,\varepsilon} \psi \rangle, \tag{35}$$

$$\langle \phi, P_\Sigma^{\lambda,\varepsilon} \psi \rangle := \int_{\Sigma\times\Sigma} \overline{\phi}(x) i_\gamma(d^4x) \overbrace{e^{-i\lambda(x,y)} p_\varepsilon^-(y-x)}^{=:p_\varepsilon^\lambda(x,y)} i_\gamma(d^4y)\psi(y), \tag{36}$$

for general $\phi, \psi \in \mathcal{H}_\Sigma$.

Remark 3. *P_Σ^λ and $P_{\Sigma'}^\lambda$ are equivalent if transported appropriately by time evolution operators[4, theorem 2.8]:*

$$P_\Sigma^\lambda - U_{\Sigma,\Sigma'}^A P_{\Sigma'}^\lambda U_{\Sigma',\Sigma} \in I_2(\mathcal{H}_\Sigma). \tag{37}$$

Also for four-potentials $A, B \in C_c^\infty(\mathbb{R}^4)$ the corresponding projectors are equivalent if and only if the four potentials projected onto the hypersurface agree[4, theorem 1.5]:

$$P_\Sigma^{\lambda^A} - P_\Sigma^{\lambda^B} \in I_2(\mathcal{H}_\Sigma) \iff \forall x \in \Sigma\ \forall z \in T_x\Sigma : z^\alpha(A_\alpha(x) - B_\alpha(x)) = 0. \tag{38}$$

Taking into account the freedom within each classification the notions Hadamard state and projectors of polarisation classes are extremely close. This is the topic of the next section.

5 Comparison Between Hadamard States and the Fermionic Projector

6 Comparison Between Hadamard States and P_Σ^λ

The following theorem is the basis of our comparison between Hadamard states and I_2-almost Projectors. We first discuss its consequences and postpone the proof until the appendix

Theorem 1. *Given as four-potential $A \in C_c^\infty(\mathbb{R}^4)$, and a Hadamard state H of the form (13)-(16), there is a family of smooth functions $(w_\varepsilon)_{\varepsilon>0}$ in $C^\infty(\mathbb{R}^4 \times \mathbb{R}^4, \mathbb{C}^{4\times 4})$, such that for any Cauchy surface Σ and $\tilde{P}$ acting as*

$$\mathcal{H}_\Sigma \ni \psi \mapsto \tilde{P}\psi = \lim_{\varepsilon\to 0} \int_\Sigma (h_\varepsilon - w_\varepsilon)(\cdot, y) i_\gamma(d^4y)\psi(y), \tag{39}$$

is a bounded operator on $\mathcal{H}_\Sigma$ that fulfils $P^{\lambda^A} - \tilde{P} \in I_2(\mathcal{H}_\Sigma)$ for any $\lambda^A \in \mathcal{G}^A$. Additionally, the pointwise limit of $(w_\varepsilon)_\varepsilon$ for $\varepsilon \to 0$ is smooth.

Theorem 2. *Given a four-potential $A \in C_c^\infty(\mathbb{R}^4)$ and a $\lambda^A \in \mathcal{G}^A$, there is a family of functions $(w_\varepsilon)_\varepsilon$ in $C^\infty(\mathbb{R}^4 \times \mathbb{R}^4, \mathbb{C}^{4\times 4})$ such that for all Cauchy surfaces Σ the $L^2(\Sigma \times \Sigma) + C^\infty(\mathbb{R}^{4+4})$limit for $\varepsilon \to 0$ exists. Furthermore, the $\tilde{H}$ acting on test functions $f_1, f_2 \in C^\infty(\mathbb{R}^4, \mathbb{C}^4)$ as*

is being in $L^2(A) + C^\infty(B)$ even a sensible statement?

$$\tilde{H}(f_1, f_2) = \lim_{\varepsilon\to 0} \int_{\mathbb{R}^4\times\mathbb{R}^4} \overline{f_1}(x) \left(p_\varepsilon^{\lambda^A}(x,y) - w_\varepsilon(x,y) \right) f_2(y) d^4x d^4y \tag{40}$$

is a Hadamard state of the form (13)-(16).

Remark 4. *In the approach of polarisation classes, two projectors represent the same polarisation class if and only if their difference is a Hilbert-Schmidt operator. Because of this fact, it is enough to keep track of the time evolution of a projector only up to changes by a Hilbert-Schmidt operator, i.e. it is enough to find the time evolution of an I_2-almost projector representing the correct polarisation class.*
Keeping this in mind, theorem 1 states that, up to a C^∞ correction, the integral kernel of the Hadamard state is the integral kernel of a I_2-almost projector. Theorem 2 states that, up to $L^2 + C^\infty$ corrections, the integral kernel of a I_2-almost projector is a Hadamard state.
The C^∞ freedom is due to the definition of a Hadamard state. The L^2 freedom originates from the definition of I_2-almost projector.
In this sense, the relevant singularity structure of a Hadamard state to become a I_2-almost projector is given by terms (111) - (115) only

7 Appendix: Proof of theorem 1 and 2

We first break down the problem of the proof of theorem 1 to a problem on a bounded domain. Then we go on to show that the relevant functions are locally bounded so that an application of Lebesgue dominated convergence yields our result. The same estimates we need for theorem 1 will be used again for the proof of theorem 2.

Definition 11. *Let $A \in C_c^\infty(\mathbb{R}^4)$ and Σ be a Cauchy surface, we introduce the sets*

$$J_1 := \{(x,y) \in \mathbb{R}^{4+4} \mid (x-y)^2 \geq -\delta^2/4 \wedge \overline{x\ y} \cap \operatorname{supp}(A) \neq \emptyset\} \tag{41}$$
$$J_2 := \{(x,y) \in \mathbb{R}^{4+4} \mid (x-y)^2 \geq -\delta^2 \wedge \overline{x\ y} \cap \big(B_{\delta/2}(0) + \operatorname{supp}(A)\big) \neq \emptyset\} \tag{42}$$
$$J_3 := \{(x,y) \in \Sigma \times \Sigma \mid (x-y)^2 \geq -\delta^2 \wedge \overline{x\ y} \cap \big(B_{\delta/2}(0) + \operatorname{supp}(A)\big) \neq \emptyset\}, \tag{43}$$

where the sum of two sets is defined as $S_1 + S_2 := \{s_1 + s_2 \mid s_1 \in S_1, s_2 \in S_2\}$ and we denote the line segment between x and y by $\overline{x\ y}$.

Definition 12. *For ease of comparison, we define for a four-potential $A \in C_c^\infty(\mathbb{R}^4)$ $\tilde{\lambda} : \mathbb{R}^{4+4} \to \mathbb{C}$:*

$$\tilde{\lambda}(x,y) = (x-y)^\alpha \int_0^1 ds A_\alpha(xs + (1-s)y), \tag{44}$$

and furthermore introduce the abbreviation $G(x,y) := e^{-i\tilde{\lambda}(x,y)}$.

Remark 5. *The function $\tilde{\lambda} \notin \mathcal{G}^A$, because it does not satisfy one of the technical conditions in definition 10, there is not compact $K \subset \mathbb{R}^4$ such that $\operatorname{supp}(\tilde{\lambda}) \subset K \times \mathbb{R}^4 \cup \mathbb{R}^4 \times K$ holds. However, the other conditions are all fulfilled. We need to introduce it nonetheless, because it is present in the representation for the Hadamard states.*

Lemma 1. *For any $A \in C_c^\infty(\mathbb{R}^4)$ and any Hadamard state H of the form (13) and (14)-(16) and any $\lambda^A \in \mathcal{G}^A$, there is a smooth family of functions $w_\varepsilon \in C^\infty(\mathbb{R}^4 \times \mathbb{R}^4, \mathbb{C}^{4\times 4})$ with $\varepsilon \in [0,1]$ and a smooth function $w \in C^\infty(\mathbb{R}^4 \times \mathbb{R}^4, \mathbb{C}^{4\times 4})$ obeying*

$$\int_{\mathbb{R}^4\times\mathbb{R}^4} \overline{f_1}(x) w_\varepsilon(x,y) f_2(y) dx dy = \int_{\mathbb{R}^4\times\mathbb{R}^4} \overline{f_1}(x)(h_\varepsilon - p_\varepsilon^\lambda)(x,y) f_2(y) dx dy \tag{45}$$
$$\int_{\mathbb{R}^4\times\mathbb{R}^4} \overline{f_1}(x) w(x,y) f_2(y) dx dy = \lim_{\epsilon\to 0} \int_{\mathbb{R}^4\times\mathbb{R}^4} \overline{f_1}(x)(h_\epsilon(x,y) - p_\epsilon^\lambda(x,y)) f_2(y) dx dy, \tag{46}$$

for all test functions f_1, f_2 *such that* $\mathrm{supp}(f_1) \times \mathrm{supp}(f_2) \subset J_2^c$
Additionally $\lim_{\varepsilon \to 0} w_\varepsilon = w$ *pointwise and in addition is continuous as a function of type* $\mathbb{R}^{4+4} \times [0,1] \to \mathbb{C}^{4\times 4}$.

Proof. Let $A \in C_c^\infty(\mathbb{R}^4), \varepsilon > 0$. Let H be a Hadamard state acting as

$$H(f_1, f_2) = \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^4} d^4x \overline{f_1}(x) \int_{\mathbb{R}^4} d^4y \; h_\varepsilon(x,y) f_2(y). \tag{47}$$

be of the form of equation (14)-(16). Pick a function $\lambda \in \mathcal{G}^A$.
We may choose $w' \in C^\infty(\mathbb{R}^4 \times \mathbb{R}^4, \mathbb{C}^{4\times 4})$ such that the following conditions is fulfilled: For all test functions $f_1, f_2 \in C_c^\infty(\mathbb{R}^4, \mathbb{C}^4)$ such that $\mathrm{supp}(f_1) \times \mathrm{supp}(f_2) \subset J_1^c$ we have

$$\int_{\mathbb{R}^4 \times \mathbb{R}^4} \overline{f_1}(x) w'(x,y) f_2(y) dx dy = \lim_{\epsilon \to 0} \int_{\mathbb{R}^4 \times \mathbb{R}^4} \overline{f_1}(x) (h_\epsilon(x,y) - p_\epsilon^\lambda(x,y)) f_2(y) dx dy. \tag{48}$$

To understand why, we consider two cases regarding the support of the testfunctions f_1, f_2,

1. $(x,y) \in \mathrm{supp}(f_1) \times \mathrm{supp}(f_2)$ implies $(x-y)^2 < -\delta^2/4$.
2. $(x,y) \in \mathrm{supp}(f_1) \times \mathrm{supp}(f_2)$ implies $\overline{x_y} \cup \mathrm{supp}(A) = \emptyset$.

Regarding 1: In the representation of the Hadamard state H, equation (18), can be explicitly solved recursively (See [2][lemma 2.2.2]. The factor V_0 was already given, for $k \geq 1$ the recursion is given by

$$V_k(x,y) = -kG(x,y) \int_0^1 ds s^{k-1} G(x + s(y-x), x) P V_{k-1}(x, x + s(y-x)). \tag{49}$$

Recall that P depends on the four potential in a local manner. Here we observe that, if the support of the external field A does not intersect the straight line connecting x and y, the function $V(x,y)$ in the expression for $h_\varepsilon(x,y)$ can be calculated to be

$$V(x,y) = \sum_{k=0}^\infty \frac{((y-x)^2 m^2/4)^k}{k!(k+1)!}, \tag{50}$$

which equals the logarithmic part of p_ε^-, corresponding to $Q_1(\xi)$ in (29).
This shows that in this case $p_\varepsilon^-(x,y)$ agrees with $h_\varepsilon(x,y)$ up to smooth terms in case 1.
Regarding 2: we notice that the only points (x,y) in the singular support of both h_ε and p_ε^λ need to fulfil $(y-x)^2 = 0$, if we further demand that both x and y belong to the same spacelike hypersurface $x = y$ follows. Now pick some $\delta > 0$. This implies that for functions $f_1, f_2 \in C_c^\infty(\mathbb{R}^4, \mathbb{C}^4)$ such that $(x,y) \in \mathrm{supp}(f_1) \times \mathrm{supp}(f_2)$ implies $(x-y)^2 \geq -\delta^2$ these operators act as integral operators with smooth kernel.
In fact, as w' is smooth, condition (48) specifies the values of w' uniquely for arguments $(x,y) \in \mathbb{R}^{4+4}$ in the complement of J_1. Analogously, we define a smooth function $w'_\varepsilon \in C^\infty(\mathbb{R}^4 \times \mathbb{R}^4, \mathbb{C}^{4\times 4})$ for every $\varepsilon > 0$ fulfilling

$$\int_{\mathbb{R}^4 \times \mathbb{R}^4} \overline{f_1}(x) w'_\varepsilon(x,y) f_2(y) dx dy = \int_{\mathbb{R}^4 \times \mathbb{R}^4} \overline{f_1}(x) (h_\varepsilon - p_\varepsilon^\lambda)(x,y) f_2(y) dx dy \tag{51}$$

for test functions $f_1, f_2 \in C_c^\infty(\mathbb{R}^4, \mathbb{C}^4)$ such that $\mathrm{supp}(f_1) \times \mathrm{supp}(f_2) \subset J_1$. Next, we pick a function $\chi \in C^\infty(\mathbb{R}^{4+4})$ such that

$$\chi|_{J_2^c} = 1, \quad \chi|_{J_1} = 0 \tag{52}$$

hold. Now, we define

$$w := \chi w', \quad w_\varepsilon := \chi w'_\varepsilon, \tag{53}$$

where w', w'_ε are extended by the zero function inside J_1.
The functions w, w_ε are now uniquely fixed in all of $\mathbb{R}^{4+4}$ and fulfil (48) and (51), respectively for test functions f_1, f_2 such that $\mathrm{supp}(f_1) \times \mathrm{supp}(f_2) \subset J_2^c$ holds. Observing the exact form of h_ε, (14)-(16) and p_ε^λ, (36), we notice that in fact $w_\varepsilon(x,y) \xrightarrow{\varepsilon\to 0} w(x,y)$ for all $x, y \in J_1^c$, because of the cutoff function χ this also holds for $(x,y) \in J_1$. Moreover, we notice from the explicit form of h_ε and p_ε^λ that, $w_{\cdot 3}(\cdot_1, \cdot 2)$ is even continuous as a function of type $\mathbb{R}^4 \times \mathbb{R}^4 \times [0,1] \to \mathbb{C}^{4\cdot 4}$. □

Lemma 2. *we define*

$$g_1 : J_3 \times]0,1[\to \mathbb{C}^{4\times 4} \tag{54}$$

$$(x,y,\varepsilon) \mapsto e^{-i\tilde{\lambda}(x,y)} \frac{m^2}{4\pi^2}(i\partial\!\!\!/_x + m)\frac{1}{m^2(y-x-i\varepsilon e_0)^2} - \mathrm{Term}_{(14)}(x,y). \tag{55}$$

There is a constant $M_1 \in \mathbb{R}$, such that

$$\forall \varepsilon, x, y |g_1(x,y,\varepsilon)| \leq M_1. \tag{56}$$

Also define

$$g_1 : \mathbb{R}^{4+4} \times [0,1] \ni (x,y) \mapsto \frac{1}{(y-x-i\varepsilon e_0)^2}\left(A\!\!\!/(x) + A\!\!\!/(y) - 2\int_0^1 \mathrm{d}s A\!\!\!/(sx+(1-s)y)\right. \tag{57}$$

$$\left. +(y-x)^\alpha \int_0^1 \mathrm{d}s(1-2s)(\partial\!\!\!/ A_\alpha)(sx+(1-s)y)\right). \tag{58}$$

There is a function $g_3 \in C(\mathbb{R}^{4+4}, \mathbb{R}^+)$ such that

$$\forall \varepsilon \in [0,1] : |g_2(x,y,\varepsilon)| \leq g_3(x,y) \tag{59}$$

holds.

Proof. Pick a Cauchy surface Σ, a four-potential $A \in C_c^\infty(\mathbb{R}^4$. A direct calculation yields

$$
\begin{aligned}
&e^{-i\tilde{\lambda}(x,y)}\frac{m^2}{4\pi^2}(i\partial\!\!\!/_x + m)\frac{1}{m^2(y-x-i\varepsilon e_0)^2} - \mathrm{Term}_{(14)}(x,y) \qquad (60)\\
=&\frac{G(x,y)}{4\pi^2}\left((i\partial\!\!\!/_x+m)\frac{1}{(y-x-i\varepsilon e_0)^2} - \frac{1}{G(x,y)}(i\nabla\!\!\!\!/\,/2 - i\nabla\!\!\!\!/^*/2+m)\frac{G(x,y)}{(y-x-i\varepsilon e_0)^2}\right)\\
=&\frac{iG(x,y)}{4\pi^2}\left(\partial\!\!\!/_x\frac{1}{(y-x-i\varepsilon e_0)^2} - \frac{1}{2G(x,y)}(\partial\!\!\!/_x + iA\!\!\!/(x) - \partial\!\!\!/_y + iA\!\!\!/(y))\frac{G(x,y)}{(y-x-i\varepsilon e_0)^2}\right)\\
=&\frac{iG(x,y)}{4\pi^2}\left(-i\frac{A\!\!\!/(x)+A\!\!\!/(y)}{2(y-x-i\varepsilon e_0)^2} + \partial\!\!\!/_x\frac{1}{(y-x-i\varepsilon e_0)^2} - \frac{1}{2}(\partial\!\!\!/_x - \partial\!\!\!/_y)\frac{1}{(y-x-i\varepsilon e_0)^2}\right.\\
&\left.-\frac{1}{2G(x,y)}\frac{(\partial\!\!\!/_x-\partial\!\!\!/_y)G(x,y)}{(y-x-i\varepsilon e_0)^2}\right)\\
=&\frac{iG(x,y)}{4\pi^2}\left(-i\frac{A\!\!\!/(x)+A\!\!\!/(y)}{2(y-x-i\varepsilon e_0)^2} + \frac{1}{2}(\partial\!\!\!/_x + \partial\!\!\!/_y)\frac{1}{(y-x-i\varepsilon e_0)^2} + \frac{1}{2G(x,y)}\frac{(-\partial\!\!\!/_x+\partial\!\!\!/_y)G(x,y)}{(y-x-i\varepsilon e_0)^2}\right)\\
=&\frac{-iG(x,y)}{8\pi^2}\frac{1}{(y-x-i\varepsilon e_0)^2}\left(iA\!\!\!/(x)+iA\!\!\!/(y) + G(x,y)^{-1}\left(\partial\!\!\!/_x - \partial\!\!\!/_y\right)G(x,y)\right)\\
=&\frac{G(x,y)}{8\pi^2}\frac{1}{(y-x-i\varepsilon e_0)^2}\left(A\!\!\!/(x)+A\!\!\!/(y) - 2\int_0^1 \mathrm{d}s A\!\!\!/(sx+(1-s)y)\right.\\
&\left.+(y-x)^\alpha\int_0^1 \mathrm{d}s(1-2s)(\partial\!\!\!/ A_\alpha)(sx+(1-s)y)\right). \qquad (61)
\end{aligned}
$$

Now using Taylor's series for A around x, y as well as $(x+y)/2$ reveals

$$
\begin{aligned}
&A\!\!\!/(x)+A\!\!\!/(y) - 2\int_0^1 \mathrm{d}s A\!\!\!/(sx+(1-s)y) =\\
&\qquad\qquad \frac{(x-y)^\alpha}{12}(x-y)^\beta(\partial_\alpha\partial_\beta A\!\!\!/)((x+y)/2) + \mathcal{O}(\|x-y\|^3) \qquad (62)\\
&(x-y)^\alpha\int_0^1 \mathrm{d}s(1-2s)(\partial\!\!\!/ A_\alpha)(sx+(1-s)y) =\\
&\qquad\qquad -\frac{(x-y)^\alpha}{2}(x-y)^\beta(\partial\!\!\!/\partial_\beta A_\alpha)((x+y)/2) + \mathcal{O}(\|x-y\|^3). \qquad (63)
\end{aligned}
$$

Because A is smooth and J_3 is a compact region the terms (62) and (63) are bounded in this region. Because of their behaviour close to $y = x$ the function is a bounded and point-wise an upper bound of the absolute value of

$$
\begin{aligned}
&\left|e^{-i\tilde{\lambda}(x,y)}\frac{m^2}{4\pi^2}(i\partial\!\!\!/_x+m)\frac{1}{m^2(y-x-i\varepsilon e_0)^2} - \mathrm{Term}_{(14)}(x,y)\right| \qquad (64)\\
&\leq \frac{\|\,\mathrm{Term}_{(62)}\,\|(x,y) + \|\,\mathrm{Term}_{(63)}\,\|(x,y)}{8\pi^2}\frac{1}{|(y-x)^2|}. \qquad (65)
\end{aligned}
$$

Also from the Taylor's expansion (62) and (63) directly follows that the right hand side of (65) is continuous. ☐

Lemma 3. *For any Cauchy surface Σ, four-potential $A \in C_c^\infty(\mathbb{R}^4)$ and $\lambda^A \in \mathcal{G}^A$, the function*

$$
\begin{aligned}
&\Sigma\times\Sigma\times]0,1[\to \mathbb{C}^{4\times 4} \qquad (66)\\
&(x,y,\varepsilon)\mapsto \left(e^{-i\tilde{\lambda}(x,y)} - e^{-i\lambda^A(x,y)}\right)\frac{1}{4\pi^2}(i\partial\!\!\!/_x+m)\frac{1}{(y-x-i\varepsilon e_0)^2} \qquad (67)
\end{aligned}
$$

is bounded by

$$\left|\left(e^{-i\tilde{\lambda}(x,y)} - e^{-i\lambda^A(x,y)}\right)\frac{1}{4\pi^2}(i\partial\!\!\!/_x + m)\frac{1}{(y-x-i\varepsilon e_0)^2}\right| \tag{68}$$

$$\leq \frac{\left|e^{-i\tilde{\lambda}(x,y)} - e^{-i\lambda^A(x,y)}\right|}{4\pi^2}\left(\eta(y-x) + \frac{\|y-x\|}{((y-x)^2)^2} + \frac{m}{|(y-x)^2|}\right) := M_2(x,y), \tag{69}$$

where η is given by

$$\eta(y-x) := \frac{1}{(-(y-x)^2\varepsilon^{*-0.5} + \varepsilon^{*1.5})^2 + \varepsilon^*(y^0-x^0)^2} \tag{70}$$

and ε^* is given by

$$\varepsilon^* := \frac{1}{\sqrt{6}}\sqrt{-\beta - 2\alpha + \sqrt{(2\alpha+\beta)^2 + 12\alpha^2}}. \tag{71}$$

Furthermore $M_2 \in L^2_{\text{loc}}(\Sigma\times\Sigma)$.

Proof. Pick $A \in C_c^\infty(\mathbb{R}^4)$, Cauchy surface Σ and $\lambda^A \in \mathcal{G}^A$. By expanding $\tilde{\lambda}(x,y) - \lambda^A(x,y)$ in a Taylor series around $\frac{y+x}{2}$ we see directly that

$$e^{-i\tilde{\lambda}(x,y)} - e^{-i\lambda^A(x,y)} = \mathcal{O}(\|x-y\|^2). \tag{72}$$

In order to show that (88) has a square integrable upper bound, we consider each term separately. First, we look at the mass term. It obeys

$$\left\|\frac{m}{(y-x-i\varepsilon e_0)^2}\right\| = \frac{m}{\sqrt{((y-x)^2-\varepsilon^2)^2 + \varepsilon^2(y^0-x^0)^2}} < \frac{m}{|(y-x)^2|}, \tag{73}$$

since $(y-x)^2 < 0$ for $x, y \in \Sigma$. The term with the derivative will be split into two:

$$\left\|i\partial\!\!\!/_x\frac{1}{(y-x-i\varepsilon e_0)^2}\right\| \leq \frac{\|y\!\!\!/ - x\!\!\!/ - i\varepsilon e\!\!\!/_0\|}{|(y-x-i\varepsilon e_0)^2|^2} \leq \frac{\varepsilon + \|x-y\|}{((y-x)^2-\varepsilon^2)^2 + \varepsilon^2(y^0-x^0)^2} \tag{74}$$

$$< \frac{1}{(-(y-x)^2\varepsilon^{-0.5} + \varepsilon^{1.5})^2 + \varepsilon(y^0-x^0)^2} + \frac{\|y-x\|}{((y-x)^2)^2}, \tag{75}$$

we notice at this point, that the second term becomes locally square integrable in $\Sigma\times\Sigma$ once multiplied with a function of type $\mathcal{O}(\|x-y\|^2)$ close to $x = y$. Indeed, the first term also has this property, in order to deduce this more readily, we will maximise this term now. Considering the limits $\varepsilon \to 0$ and $\varepsilon \to \infty$ we see that this term has for arbitrary $(x, y \in \Sigma)$ a maximum in the interval $]0,\infty[$. Abbreviating $-(y-x)^2 := \alpha$, $(y^0-x^0)^2 := \beta$, this value of ε fulfills

$$\partial_\varepsilon(\varepsilon^{-0.5}\alpha + \varepsilon^{1.5})^2 + \beta = 0 \tag{76}$$

$$\iff 3\varepsilon^4 + \varepsilon^2(2\alpha+\beta) - \alpha^2 = 0 \tag{77}$$

$$\iff \varepsilon^2 = \frac{-\beta - 2\alpha + \sqrt{(2\alpha+\beta)^2 + 12\alpha^2}}{6} \tag{78}$$

$$\iff \varepsilon = \varepsilon^* := \frac{1}{\sqrt{6}}\sqrt{-\beta - 2\alpha + \sqrt{(2\alpha+\beta)^2 + 12\alpha^2}}. \tag{79}$$

So we can find an upper bound on the first summand of (75) by replacing ε by what we just found. Now because all the terms in the resulting expression

$$\frac{1}{(-(y-x)^2\varepsilon^{*-0.5}+\varepsilon^{*1.5})^2+\varepsilon^*(y^0-x^0)^2} =: \eta(y-x) \tag{80}$$

are positive, we did not introduce any new singularities. Also the expression is homogenous of degree -3:

$$\eta(\delta(y-x)) = \delta^{-3}\eta(y-x), \tag{81}$$

so we can conclude that also this term as well as the second term in (75), are of type $\mathcal{O}(\|x-y\|^{-3})$ and therefore locally square integrable once multiplied by the difference of exponentials in (88). So summarising (75) can be bounded by

$$\begin{aligned}&\|\operatorname{Term}_{(75)}(x,y)\| \leq \\ &\qquad \frac{\left|e^{-i\lambda^A(x,y)}-e^{-i\tilde{\lambda}(x,y)}\right|}{4\pi^2}\left(\eta(y-x)+\frac{\|y-x\|}{((y-x)^2)^2}+\frac{m}{|(y-x)^2|}\right),\end{aligned} \tag{82}$$

which is locally square integrable in on $\Sigma\times\Sigma$. $\square$

Lemma 4. *For any Cauchy surface Σ, four-potential $A \in C_c^\infty(\mathbb{R}^4)$, the function*

$$\Sigma\times\Sigma\times]0,1[\to \mathbb{C}^{4\times 4} \tag{83}$$

$$(x,y,\varepsilon)\mapsto (i\partial\!\!\!/_x+m)\ln(-m^2(y-x-i\varepsilon e_0)^2) \tag{84}$$

has the locally in $\Sigma\times\Sigma$ square integrable bound

$$\left|(i\partial\!\!\!/_x+m)\ln(-(y-x-i\varepsilon e_0)^2)\right| \leq \tag{85}$$

$$\frac{2\|y-x\|}{|(y-x)^2|}+\frac{2}{\sqrt{4|(y-x)^2|+(y^0-x^0)^2}} := M_3(x,y), \tag{86}$$

while the function

$$\Sigma\times\Sigma\times]0,1[\to \mathbb{C}^{4\times 4} \tag{87}$$

$$(x,y,\varepsilon)\mapsto \ln(-m^2(y-x-i\varepsilon e_0)^2) \tag{88}$$

has the L^2_{loc} bound

$$\|\ln(-m^2(y-x-i\varepsilon e_0)^2)\| \leq |\ln(-(y-x)^2)|+\pi/2 := M_4(x,y). \tag{89}$$

Proof. Pick $A\in C_c^\infty(\mathbb{R}^4)$, a Cauchy surface Σ and $f\in C(\Sigma\times\Sigma,\mathbb{C}^{4\times 4})$. The terms containing $\partial\!\!\!/\ln(-(y-x-i\varepsilon e_0)^2)$ are bounded as follows

$$\|i\partial_\alpha\ln(-(y-x-i\varepsilon e_0)^2)\| = 2\left\|\frac{y^\alpha-x^\alpha-i\varepsilon e_0^\alpha}{-(y-x-i\varepsilon e_0)^2}\right\| \tag{90}$$

$$\leq \frac{2\|y-x\|}{|(y-x)^2|}+\frac{2\varepsilon}{|(y-x-i\varepsilon e_0)^2|} = \frac{2\|y-x\|}{|(y-x)^2|}+\frac{2}{\sqrt{(-(y-x)^2/\varepsilon+\varepsilon)^2+(y^0-x^0)^2}} \tag{91}$$

$$\leq \frac{2\|y-x\|}{|(y-x)^2|}+\frac{2}{\sqrt{4|(y-x)^2|+(y^0-x^0)^2}}, \tag{92}$$

where inequality in (92) we maximised the expression with respect to $\varepsilon \in]0,\infty[$. The terms containing the logarithm without any derivative are directly bounded by

$$\begin{aligned} &|\ln(-(y-x-i\varepsilon e_0)^2)| \\ \leq\ &|\ln(-(y-x)^2)| + \pi/2 := \mathrm{Term}_{(93)}\,. \end{aligned} \tag{93}$$

□

Lemma 5. *For every four-potential $A \in C_c^\infty(\mathbb{R}^4)$ and every Hadamard state H of the form (13)-(16) the smooth function $w \in C^\infty(\mathbb{R}^4 \times \mathbb{R}^4, \mathbb{C}^{4\times 4})$ and family $(w_\varepsilon)_{\varepsilon\in]0,1[} \subset C^\infty(\mathbb{R}^4 \times \mathbb{R}^4, \mathbb{C}^{4\times 4})$ of lemma 1 also satisfy for any Cauchy surface Σ and any $\lambda^A \in \mathcal{G}^A$:*

$$h_\varepsilon^A - w_\varepsilon - p_\varepsilon^{\lambda^A}\Big|_{\Sigma\times\Sigma} \in L^2(\Sigma\times\Sigma) \tag{94}$$

and the $L^2(\Sigma\times\Sigma)$ limit $\lim_{\varepsilon\to 0} h_\varepsilon^A - w_\varepsilon - p_\varepsilon^{\lambda^A}\Big|_{\Sigma\times\Sigma}$ exists.

Proof of lemma 5: Pick a Cauchy surface Σ, a four-potential $A \in C_c^\infty(\mathbb{R}^4)$, a Hadamard state of the form (13) -(16), $\lambda^A \in \mathcal{G}^A$ and for $\varepsilon \in]0,1[$ smooth functions $w_\varepsilon \in C^\infty(\mathbb{R}^{4+4}, \mathbb{C}^{4\times 4})$ according to lemma 1. Our aim is to show that $h_\varepsilon - w_\varepsilon - p_\varepsilon^{\lambda^A}\Big|_{\Sigma\times\Sigma}$ converges in the sense of $L^2(\Sigma\times\Sigma)$ in the limit $\varepsilon\to 0$. According to lemma 1 we have that $h_\varepsilon - w_\varepsilon - p_\varepsilon^{\lambda^A}|_{\Sigma\times\Sigma}$ vanishes outside the set J_3 which is bounded, independent of ε and of finite measure. Taking the exact form of h_ε and $p_\varepsilon^{\lambda^A}$ into account, one notices that the function $h_\varepsilon - w_\varepsilon - p_\varepsilon^{\lambda^A}|_{\Sigma\times\Sigma}(x,y)$ converges point-wise to a function defined on $\Sigma\times\Sigma\backslash\{(x,x) \mid x\in\Sigma\}$. In order to show that the convergence also holds in the sense of $L^2(\Sigma\times\Sigma)$ we would like to use dominated convergence. Hence we should find a function $M \in L^2(\Sigma\times\Sigma)$ such that

$$\left|(h_\varepsilon - w_\varepsilon - p_\varepsilon^{\lambda^A})(x,y)\right| \leq |M|(x,y) \tag{95}$$

holds almost for almost all $x,y \in \Sigma$ and all ε small enough. We may pick $M|_{J_3^c} = 0$, of course. So we only have to pick M for arguments inside J_3. Now because w_ε is continuous as a function from $\mathbb{R}^{4+4}\times[0,1] \to \mathbb{C}^{4\times 4}$ and hence also when restricted to $J_3\times[0,1]$. The set $J_3\times[0,1]$ is compact which implies that there is a constant M_5 such that

$$\forall (x,y,\varepsilon) \in J_3\times[0,1] : |w_\varepsilon(x,y)| \leq M_5 \tag{96}$$

holds. So by the triangular inequality we only need to bound $h_\varepsilon - p_\varepsilon^{\lambda^A}$
We now dissect h_ε as well as $p_\varepsilon^{\lambda^A}$ each into three pieces

$$h_\varepsilon(x,y) = \frac{-1}{2(2\pi)^2}(-i\nabla\!\!\!/ + i\nabla\!\!\!/^* - 2m)\frac{G(x,y)}{(y-x-i\varepsilon e_0)^2} \tag{97}$$

$$+\frac{-1}{2(2\pi)^2}(-i\nabla\!\!\!/ + i\nabla\!\!\!/^* - 2m)V(x,y)\ln(-(y-x-i\varepsilon e_0)^2) \tag{98}$$

$$+B(x,y), \tag{99}$$

$$p_\varepsilon^{\lambda^A}(x,y) = e^{-i\lambda^A(x,y)}\frac{m^2}{4\pi^2}(i\partial\!\!\!/ + m)\frac{1}{m^2(y-x-i\varepsilon e_0)^2} \tag{100}$$

$$-\frac{m^2}{4\pi^2}(i\partial\!\!\!/ + m)Q_1(-m^2(y-x-i\varepsilon e_0)^2)\ln(-m^2(y-x-i\varepsilon e_0)^2) \tag{101}$$

$$-\frac{m^2}{4\pi^2}(i\partial\!\!\!/ + m)Q_2(-m^2(y-x-i\varepsilon e_0)^2). \tag{102}$$

Let us first consider the most singular terms of $h_\varepsilon - w_\varepsilon - p_\varepsilon^{\lambda^A}|_{\Sigma\times\Sigma}$, namely $\mathrm{Term}_{(97)} - \mathrm{Term}_{(100)}$. In order to better compare these terms, we will add 0 in the form of $\mathrm{Term}_{(100)} - \mathrm{Term}_{(100)}$ where we replace λ^A by $\tilde{\lambda}$. By lemma 2 the term

$$\mathrm{Term}_{(97)} - G(x,y)\frac{m^2}{4\pi^2}(i\partial\!\!\!/ + m)\frac{1}{m^2(y-x-i\varepsilon e_0)^2} \tag{103}$$

is uniformly bounded in J_3.
Next we find a square integrable upper bound on

$$e^{-i\tilde{\lambda}(x,y)}\frac{m^2}{4\pi^2}(i\partial\!\!\!/_x + m)\frac{1}{m^2(y-x-i\varepsilon e_0)^2} - \mathrm{Term}_{(100)}(x,y). \tag{104}$$

We rewrite this term as

$$e^{-i\tilde{\lambda}(x,y)}\frac{m^2}{4\pi^2}(i\partial\!\!\!/_x + m)\frac{1}{m^2(y-x-i\varepsilon e_0)^2} - \mathrm{Term}_{(100)}(x,y) = \tag{105}$$

$$\left(e^{-i\tilde{\lambda}(x,y)} - e^{-i\lambda(x,y)}\right)\frac{1}{4\pi^2}(i\partial\!\!\!/_x + m)\frac{1}{(y-x-i\varepsilon e_0)^2}, \tag{106}$$

according to lemma 3 this has the upper bound $M_2 \in L^2_{\mathrm{loc}}(\Sigma\times\Sigma$, which is independend of ε. The term $\mathrm{Term}_{(101)} - \mathrm{Term}_{(98)}$ are bounded by lemma 4. Thus we obtain for M for $(x,y)\in J_3$:

$$\begin{aligned}
&M(x,y) := M_5 + M_1 + M_2(x,y)\\
&+\frac{1}{4\pi^2}\left(4|V(x,y)| + m^2\sup_{\varepsilon\in[0,1]}|Q_1(-m^2(y-x-i\varepsilon e_0)^2)|\right)M_3(x,y)\\
&+\frac{1}{4\pi^2}\left(|(i\nabla\!\!\!\!/ - i\nabla\!\!\!\!/^* - 2m)V(x,y)|/2 + m^2\sup_{\varepsilon\in[0,1]}|(i\partial\!\!\!/ + m)Q_1(-m^2(y-x-i\varepsilon e_0))|\right)M_4(x,y)\\
&+|B(x,y)| + \frac{m^2}{4\pi^2}\sup_{\varepsilon\in[0,1]}|(i\partial\!\!\!/_x + m)Q_2(-m^2(y-x-i\varepsilon e_0)^2)| + \sup_{\varepsilon\in]0,1]}|w_\varepsilon|(x,y)
\end{aligned}$$

The argument of $h_\varepsilon - w_\varepsilon - p_\varepsilon^\lambda|_{\Sigma\times\Sigma}(x,y)$ converging to $h_0 - w - p_0^\lambda|_{\Sigma\times\Sigma}(x,y)$ in the sense of $L^2(\Sigma\times\Sigma)$ is completed by applying dominated convergence theorem.
As a final point: the construction seems to depend on the function λ^A we chose, however for any different $\lambda'^A \in \mathcal{G}^A$ we have due to remark 3

$$P^\lambda - P^{\lambda'} \in I_2(\mathcal{H}_\Sigma), \tag{107}$$

therefore also the limit $\lim_{\varepsilon\to 0} h_\varepsilon^A - w_\varepsilon - p_\varepsilon^{\lambda'^A}$ exists in the sense of L^2.

□

Proof of theorem 1: We pick for a four-potential $A \in C_c^\infty(\mathbb{R}^4)$ a Hadamard state H of the form (13) -(16) and a $\lambda^A \in \mathcal{G}^A$. Then we pick $w_\varepsilon, w \in C_c^\infty(\mathbb{R}^{4+4}, \mathbb{C}^{4\times 4})$ for all $\varepsilon\in[0,1]$ according to lemma 1. Because of lemma 5, we have that

$$h_\varepsilon^A - w_\varepsilon - p_\varepsilon^{\lambda^A} \in L^2(\Sigma\times\Sigma) \tag{108}$$

holds. We can define the operator $Z \in I_2(\mathcal{H}_\Sigma)$ for any Cauchy surface Σ as

$$\mathcal{H}_\Sigma \ni \psi \mapsto Z\psi := \lim_{\varepsilon\to 0}\int_\Sigma (h^A_\varepsilon - w_\varepsilon - p^{\lambda^A}_\varepsilon)(\cdot,y) i_\gamma(d^4y)\psi(y) \tag{109}$$

and

$$\tilde{P}_\Sigma := Z + P^{\lambda^A}_\Sigma. \tag{110}$$

Because of (38) we have for any other $\lambda' \in \mathcal{G}^A$ that $\tilde{P}_\Sigma - P^{\lambda'} \in I_2(\mathcal{H}_\Sigma)$.

□

Proof of theorem 2. Pick a four-potential $A \in C^\infty_c(\mathbb{R}^4)$ and a $\lambda^A \in \mathcal{G}^A$. Also pick some Hadamard state of the form (13)-(16). Next, we pick w_ε according to lemma 1; however, due to notational reasons we will denote it by R_ε.
Define for each $\varepsilon \in]0,1]$ the function w_ε as

$$w_\varepsilon(x,y) := -\frac{e^{-i(x-y)^\alpha \int_0^1 ds A_\alpha(xs+(1-s)y)} - e^{-i\lambda^A(x,y)}}{(2\pi)^2}(i\partial\!\!\!/_x + m)\frac{1}{(y-x-i\varepsilon e_0)^2} \tag{111}$$

$$+\frac{e^{-i(x-y)^\alpha \int_0^1 ds A_\alpha(xs+(1-s)y)}}{2(2\pi)^2(y-x-i\varepsilon e_0)^2}\Bigg(A\!\!\!/(x) + A\!\!\!/(y) - 2\int_0^1 ds A\!\!\!/(sx+(1-s)y) \tag{112}$$

$$+(x-y)^\alpha \int_0^1 ds(1-2s)(\partial\!\!\!/ A_\alpha)(sx+(1-s)y)\Bigg) \tag{113}$$

$$+\frac{1}{2(2\pi)^2}(-i\nabla\!\!\!\!/ + i\nabla\!\!\!\!/^* - 2m)V(x,y)\ln(-(y-x-i\varepsilon e_0)^2) \tag{114}$$

$$-\frac{m^2 e^{-i\lambda^A(x,y)}}{4\pi^2}(i\partial\!\!\!/_x + m)\Big(Q_1(m\sqrt{-(y-x-i\varepsilon e_0)^2}\ln(-m^2(y-x-i\varepsilon e_0)^2) \tag{115}$$

$$+Q_2(m\sqrt{-(y-x-i\varepsilon e_0)^2}\Big) + R_\varepsilon(x,y). \tag{116}$$

We further define the distribution H^{λ^A} by its action on test functions $f_1, f_2 \in C^\infty_c(\mathbb{R}^4, \mathbb{C}^4)$

$$\tilde{H}(f_1,f_2) := \lim_{\varepsilon\to 0}\int_{\mathbb{R}^4}\int_{\mathbb{R}^4}\overline{f}_1(x)(p^{\lambda^A}_\varepsilon(x,y) - w_\varepsilon(x,y))f_2(y)d^4x d^4y, \tag{117}$$

with w_ε given by the terms from (111) until (115). First, we verify that H^{λ^A} is indeed a Hadamard state. Pick $\varepsilon > 0$, we find

$$p^{\lambda^A}_\varepsilon(x,y) - w_\varepsilon(x,y) = -e^{-i\lambda^A(x,y)}\frac{m^2}{4\pi^2}(i\partial\!\!\!/_x + m)\frac{K_1(m\sqrt{-(y-x-i\varepsilon e_0)^2})}{m\sqrt{-(y-x-i\varepsilon e_0)^2}} - w_\varepsilon(x,y)$$

$$= -e^{-i\lambda^A(x,y)}\frac{m^2}{4\pi^2}(i\partial\!\!\!/_x + m)\Bigg(\frac{-1}{m^2(y-x-i\varepsilon e_0)^2}$$

$$+Q_1(-m^2(y-x-i\varepsilon e_0)^2)\ln(m\sqrt{-(y-x-i\varepsilon e_0)^2}) + Q_2(-m^2(y-x-i\varepsilon e_0)^2)\Bigg) - w_\varepsilon(x,y).$$

Inserting $w_\varepsilon(x,y)$ we find that the most divergent terms proportional to $e^{-i\lambda^A}$, as well as the terms involving Q_1 and Q_2 cancel. This results in

$$
\begin{aligned}
&p_\varepsilon^{\lambda A}(x,y) - w_\varepsilon(x,y) \\
&= \frac{e^{-i(x-y)^\alpha \int_0^1 ds A_\alpha(xs+(1-s)y)}}{(2\pi)^2}(i\partial\!\!\!/_x + m)\frac{1}{(y-x-i\varepsilon e_0)^2} \\
&+ \frac{e^{-i(x-y)^\alpha \int_0^1 ds A_\alpha(xs+(1-s)y)}}{2(2\pi)^2(y-x-i\varepsilon e_0)^2}\Bigg(A\!\!\!/(x) + A\!\!\!/(y) - 2\int_0^1 ds A\!\!\!/(sx+(1-s)y) \\
&+(x-y)^\alpha \int_0^1 ds(1-2s)(\partial\!\!\!/ A)(sx+(1-s)y)\Bigg) \\
&- \frac{1}{2(2\pi)^2}(-i\nabla\!\!\!\!/ + i\nabla\!\!\!\!/^* - 2m)V(x,y)\ln(-(y-x-i\varepsilon e_0)^2) \\
&- R_\varepsilon(x,y).
\end{aligned}
$$

Now the terms involving $e^{-i(x-y)^2 \int_0^1 ds A_\alpha(xs+(1-s)y)}$ can be summarised

$$
\begin{aligned}
&\frac{e^{-i(x-y)^\alpha \int_0^1 ds A_\alpha(xs+(1-s)y)}}{(2\pi)^2}(i\partial\!\!\!/_x + m)\frac{1}{(y-x-i\varepsilon e_0)^2} \\
&+ \frac{e^{-i(x-y)^\alpha \int_0^1 ds A_\alpha(xs+(1-s)y)}}{2(2\pi)^2(y-x-i\varepsilon e_0)^2}\Bigg(A\!\!\!/(x) + A\!\!\!/(y) - 2\int_0^1 ds A\!\!\!/(sx+(1-s)y) \\
&+(x-y)^\alpha \int_0^1 ds(1-2s)(\partial\!\!\!/ A)(sx+(1-s)y)\Bigg) \\
&= \frac{1}{2(2\pi)^2}(i\nabla\!\!\!\!/ - i\nabla\!\!\!\!/^* + 2m)\frac{e^{-i(x-y)^\alpha \int_0^1 ds A_\alpha(xs+(1-s)y)}}{(y-x-i\varepsilon e_0)^2},
\end{aligned}
$$

so overall we find

$$
\begin{aligned}
&p_\varepsilon^{\lambda A}(x,y) - w_\varepsilon(x,y) = \\
&\frac{-1}{2(2\pi)^2}(-i\nabla\!\!\!\!/ + i\nabla\!\!\!\!/^* - 2m)\frac{e^{-i(x-y)^\alpha \int_0^1 ds A_\alpha(xs+(1-s)y)}}{(y-x-i\varepsilon e_0)^2} \\
&- \frac{1}{2(2\pi)^2}(-i\nabla\!\!\!\!/ + i\nabla\!\!\!\!/^* - 2m)V(x,y)\ln(-(y-x-i\varepsilon e_0)^2) \\
&- R_\varepsilon(x,y).
\end{aligned}
$$

Because the term in the last line is smooth, this means that indeed $p_\varepsilon^{\lambda A}(x,y) - w_\varepsilon(x,y)$ is the integration kernel of a Hadamard state. Furthermore, for a given Cauchy surface Σ the function $w_\varepsilon - R_\varepsilon$ has a limit in the sense of $L^2_{\text{loc}}(\Sigma \times \Sigma)$, because of Lebesgue dominated convergence and lemmata 2, 3 and 4. That the term R_ε converges in $C^\infty(\mathbb{R}^{4+4}, \mathbb{C}^{4\cdot 4})$ follows from lemma 1.

□

References

[1] Milton Abramowitz and Irene A Stegun. *Handbook of mathematical functions: with formulas, graphs, and mathematical tables*, volume 55. Courier Corporation, 1965.

[2] Christian Bär, Nicolas Ginoux, and Frank Pfäffle. *Wave equations on Lorentzian manifolds and quantization*, volume 3. European Mathematical Society, 2007.

[3] D-A Deckert, D Dürr, F Merkl, and M Schottenloher. Time-evolution of the external field problem in quantum electrodynamics. *Journal of Mathematical Physics*, 51(12):122301, 2010.

[4] D-A Deckert and F Merkl. External field qed on cauchy surfaces for varying electromagnetic fields. *Communications in Mathematical Physics*, 345(3):973–1017, 2016.

[5] D-A Deckert and Franz Merkl. Dirac equation with external potential and initial data on cauchy surfaces(ivp1). *Journal of Mathematical Physics*, 55(12):122305, 2014.

[6] Paul AM Dirac. Discussion of the infinite distribution of electrons in the theory of the positron. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 30, pages 150–163. Cambridge University Press, 1934.

[7] Felix Finster. *The principle of the fermionic projector*, volume 35. American Mathematical Soc., 2006.

[8] Stephen A Fulling, Mark Sweeny, and Robert M Wald. Singularity structure of the two-point function in quantum field theory in curved spacetime. *Communications in Mathematical Physics*, 63(3):257–264, 1978.

[9] Lars Hörmander. The analysis of linear partial differential operators. i. distribution theory and fourier analysis. reprint of the second (1990) edition, 2003.

[10] Jan Schlemmer and Jochen Zahn. The current density in quantum electrodynamics in external potentials. *Annals of Physics*, 359:31–45, 2015.

[11] Jochen Zahn. The renormalized locally covariant dirac field. *Reviews in Mathematical Physics*, 26(01):1330012, 2014.