1 General Preliminaries

We will define T_k as the expansion coefficients of the second quantised S-matrix S, which is an operator on Fockspace. If we denote the Hilbert space of square integrable positive, resp. negative energy solutions(find proper reference) of the Dirac equation with external potential (1) by $\mathcal{H}^{\pm}$ then Fockspace is given by:

$$i\partial\!\!\!/\psi = (A\!\!\!/ + m)\psi \tag{1}$$

$$\mathcal{F} := \bigoplus_{m,p=0}^{\infty} \left(\mathcal{H}^+\right)^{\Lambda m} \otimes \left(\mathcal{H}^-\right)^{\Lambda p} \tag{2}$$

We will denote the fixed particles sectors of Fockspace by $\mathcal{F}_{m,p} := \left(\mathcal{H}^+\right)^{\Lambda m} \otimes \left(\mathcal{H}^-\right)^{\Lambda p}$ We employ the following property as defining the second quantised S-matrix:

$$\forall \phi \in \mathcal{H} : \quad a\left(U\phi\right) \circ S = S \circ a(\phi), \tag{3}$$

Where a is the annihilation operator of the Fockspace $\mathcal{F}$ over the m, p particle sectors, $\mathcal{F}_{m,p} :=$. This property is called the "Lift condition". Which is equivalent to the commutativity of the following diagram.

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{S^A} & \mathcal{F} \\ \uparrow a & & \uparrow a \\ \mathcal{H} \otimes \mathcal{F} & \xrightarrow{U^A \otimes S^A} & \mathcal{H} \otimes \mathcal{F} \end{array} \tag{4}$$

The respective condition for the creation operator, which can easily be derived from (3) is:

$$\forall \phi \in \mathcal{H} : \quad a^*\left(U^A\phi\right) \circ S^A = S^A \circ a^*(\phi) \tag{5}$$

Expanding U^A and S^A in a powerseries and sorting the resulting series in powers of A, one obtains commutation relations for the coefficients of said expansion:

$$\begin{aligned} U^A &= \mathbb{1}_{\mathcal{H}} + \sum_{l=1}^{\infty} \frac{1}{l!} Z_l(A) \\ S^A &= \mathbb{1}_{\mathcal{F}} + \sum_{l=1}^{\infty} \frac{1}{l!} T_l(A) \end{aligned} \tag{6}$$

$$
\begin{aligned}
&a\left(U\phi\right)\circ\tilde{U}=\tilde{U}\circ a(\phi)\\
&\iff a\left(\left[\mathbb{1}_{\mathcal{H}}+\sum_{l=1}^{\infty}\frac{1}{l!}Z_l(A)\right]\phi\right)\circ\left(\mathbb{1}_{\mathcal{F}}+\sum_{l=1}^{\infty}\frac{1}{l!}T_l(A)\right)=\\
&\qquad\left(\mathbb{1}_{\mathcal{F}}+\sum_{l=1}^{\infty}\frac{1}{l!}T_l(A)\right)\circ a(\phi)\\
&\iff a(\phi)+\sum_{l=1}^{\infty}\frac{1}{l!}a\left(Z_l(A)\phi\right)+a(\phi)\circ\sum_{l=1}^{\infty}\frac{1}{l!}T_l(A)+\sum_{l=1}^{\infty}\frac{1}{l!}a\left(Z_l(A)\phi\right)\circ\sum_{k=1}^{\infty}\frac{1}{k!}T_k(A)=\\
&\qquad a(\phi)+\sum_{l=1}^{\infty}\frac{1}{l!}T_l(A)\circ a(\phi)\\
&\iff \sum_{l=1}^{\infty}\frac{1}{l!}a\left(Z_l(A)\phi\right)+\sum_{l=1}^{\infty}\frac{1}{l!}a\left(Z_l(A)\phi\right)\circ\sum_{k=1}^{\infty}\frac{1}{k!}T_k(A)=\\
&\qquad\sum_{l=1}^{\infty}\frac{1}{l!}T_l(A)\circ a(\phi)-a(\phi)\circ\sum_{l=1}^{\infty}\frac{1}{l!}T_l(A)\\
&\iff \sum_{l=1}^{\infty}\frac{1}{l!}a\left(Z_l(A)\phi\right)+\sum_{m=2}^{\infty}\sum_{\substack{k+l=m\\ l,k\geq 1}}\frac{1}{l!k!}a\left(Z_l(A)\phi\right)\circ T_k(A)=\sum_{l=1}^{\infty}\frac{1}{l!}\left[T_l(A),a(\phi)\right]\\
&\iff \sum_{m=1}^{\infty}\frac{1}{m!}a\left(Z_m(A)\phi\right)+\sum_{m=2}^{\infty}\sum_{j=1}^{m-1}\frac{1}{j!(m-j)!)}a\left(Z_j(A)\phi\right)\circ T_{m-j}(A)=\sum_{l=1}^{\infty}\frac{1}{l!}\left[T_l(A),a(\phi)\right]\\
&\iff \sum_{m=1}^{\infty}\left(\frac{1}{m!}a\left(Z_m(A)\phi\right)+\mathbb{1}_{[2,\infty[}(m)\sum_{j=1}^{m-1}\frac{1}{j!(m-j)!}a\left(Z_j(A)\phi\right)\circ T_{m-j}(A)\right)=\\
&\qquad\sum_{m=1}^{\infty}\frac{1}{m!}\left[T_m(A),a(\phi)\right]
\end{aligned}
\tag{7}
$$

Since the last line of (7) needs to hold for every smooth compactly supported A, the equality must hold for all the sets of terms of equal power of A separately, hence we get:

$$
\left[T_m(A),a(\phi)\right]=a\left(Z_m(A)\phi\right)+\mathbb{1}_{[2,\infty[}(m)\sum_{j=1}^{m-1}\binom{m}{j}a\left(Z_j(A)\phi\right)\circ T_{m-j}(A) \tag{8}
$$

By the very same computation we get the corresponding expression for the creation operator:

$$
\left[T_m(A),a^*(\phi)\right]=a^*\left(Z_m(A)\phi\right)+\mathbb{1}_{[2,\infty[}(m)\sum_{j=1}^{m-1}\binom{m}{j}a^*\left(Z_j(A)\phi\right)\circ T_{m-j}(A) \tag{9}
$$

Equations (8) and (9) enable us to extend the definition of T_k to any fixed particle sector once we have defined its action on the vacuum vector of Fockspace.

2 Defining T_1

In addition to (8) and (9) we define $\langle\Omega|\,T_1(A)\Omega\rangle=0$ for any external field A. We will show in the following that the resulting Fockspace operator is well defined, bounded on

every fixed particle sector, but unbounded on all of Fockspace.
First of all we find the image of $\mathcal{F}_{m,p}$ under $T1$. For $(\varphi_l)_{l\in\mathbb{N}}$, respectively $(f_l)_{l\in\mathbb{N}}$ being an ONB for $\mathcal{H}^{\pm}$ the set

$$\left\{ \prod_{l=1}^{m} a^*(\varphi_{\omega(l)}) \prod_{b=1}^{p} a(f_{\sigma(b)})\Omega \middle| \omega, \sigma : \mathbb{N} \to \mathbb{N}, \text{bijective} \right\} \tag{10}$$

form an ONB of $\mathcal{F}_{m,p}$. First of all, we begin by finding the image of the vacuum under T_1. An arbitrary matrix element has the form:

$$\begin{aligned}
&\left\langle \prod_{l=1}^{m} a^*(\varphi_{\omega(l)}) \prod_{b=1}^{p} a(f_{\sigma(b)})\Omega \middle| T_1\ \Omega \right\rangle = \langle \Omega | \prod_{b=1}^{p} a^*(f_{\sigma(b)}) \prod_{l=1}^{m} a(\varphi_{\omega(l)}) T_1\ \Omega\rangle \overset{(8),(9)}{=} \\
&- \langle \Omega | \, \delta_{p,1}\delta_{m,1} a^*(f_{\sigma(1)}) a(Z_1(A)\varphi_{\omega(1)}) \, |\Omega\rangle = -\delta_{p,1}\delta_{m,1} \langle \Omega | \, a^*(f_{\sigma(1)}) a(Z_{1,-+}(A)\varphi_{\omega(1)}) \ \Omega\rangle \\
&\qquad = -\delta_{p,1}\delta_{m,1} \left\langle Z_{1,-+}(A)\varphi_{\omega(1)} \middle| \ f_{\sigma(1)} \right\rangle
\end{aligned} \tag{11}$$

This image is well defined because

$$\begin{aligned}
\| \, |T_1(A)\Omega\rangle \, \|^2 = &\sum_{\substack{\varphi\in \mathrm{ONB}(\mathcal{H}^+) \\ f\in\mathrm{ONB}(\mathcal{H}^-)}} |\langle Z_{1,-+}(A)\varphi | \ f\rangle|^2 = \|Z_{1,-+}\|^2_{\mathrm{HS}} = \|\not{A}(\cdot_1 - \cdot_2)\|^2_{\mathcal{L}^2(\mathcal{M}_-\times\mathcal{M}_+)} \\
&\int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{E_p} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 k}{E_k} \left\| \not{A}(-E_p - E_k, p_1 - k_1, p_2 - k_2, p_3 - k_3) \right\|^2 < \infty
\end{aligned} \tag{12}$$

We will be using (11) to find the image of an arbitrary element of (10):

$$T_1(A)\prod_{l=1}^{m} a^*(\varphi_{\omega(l)})\prod_{b=1}^{p} a(f_{\sigma(b)})\Omega \overset{(8),(9)}{=} \sum_{c=1}^{m}\left[\prod_{l=1}^{c-1} a^*(\varphi_{\omega(l)})\right] a^*\left(Z_1(A)\varphi_{\omega(c)}\right)\prod_{k=c+1}^{m} a^*(\varphi_{\omega(k)})\prod_{b=1}^{p} a(f_{\sigma(b)})\Omega$$

$$+\sum_{e=1}^{p}\prod_{l=1}^{m} a^*(\varphi_{\omega(l)})\left[\prod_{b=1}^{e-1} a(f_{\sigma(b)})\right] a\left(Z_1(A)f_{\sigma(e)}\right)\prod_{k=e+1}^{p} a(f_{\sigma(k)})\Omega+\prod_{l=1}^{m} a^*(\varphi_{\omega(l)})\prod_{b=1}^{p} a(f_{\sigma(b)})T_1(A)\Omega$$

$$\overset{(11)}{=}\sum_{c=1}^{m}\left[\prod_{l=1}^{c-1} a^*(\varphi_{\omega(l)})\right] a^*\left(Z_1(A)\varphi_{\omega(c)}\right)\prod_{k=c+1}^{m} a^*(\varphi_{\omega(k)})\prod_{b=1}^{p} a(f_{\sigma(b)})\Omega+$$

$$\sum_{e=1}^{p}\prod_{l=1}^{m} a^*(\varphi_{\omega(l)})\left[\prod_{b=1}^{e-1} a(f_{\sigma(b)})\right] a\left(Z_1(A)f_{\sigma(e)}\right)\prod_{k=e+1}^{p} a(f_{\sigma(k)})\Omega+$$

$$\sum_{\substack{\varphi\in\mathrm{ONB}(\mathcal{H}^+)\\ f\in\mathrm{ONB}(\mathcal{H}^-)}}\langle Z_{1,-+}(A)\varphi|\, f\rangle\prod_{l=1}^{m} a^*(\varphi_{\omega(l)})\left[\prod_{b=1}^{p} a(f_{\sigma(b)})\right] a^*(\varphi)a(f)\Omega$$

$$=\sum_{c=1}^{m}\left[\prod_{l=1}^{c-1} a^*(\varphi_{\omega(l)})\right] a^*\left(Z_{1,++}(A)\varphi_{\omega(c)}\right)\prod_{k=c+1}^{m} a^*(\varphi_{\omega(k)})\prod_{b=1}^{p} a(f_{\sigma(b)})\Omega+$$

$$\sum_{e=1}^{p}\prod_{l=1}^{m} a^*(\varphi_{\omega(l)})\left[\prod_{b=1}^{e-1} a(f_{\sigma(b)})\right] a\left(Z_{1,--}(A)f_{\sigma(e)}\right)\prod_{k=e+1}^{p} a(f_{\sigma(k)})\Omega+$$

$$\sum_{j=1}^{p}\sum_{c=1}^{m}(-1)^{m-c+j-1}\left\langle f_{\sigma(j)}\middle|\, Z_{1,-+}(A)\varphi_{\omega(c)}\right\rangle\prod_{\substack{l=1\\ l\neq c}}^{m} a^*(\varphi_{\omega(l)})\prod_{\substack{b=1\\ b\neq j}}^{p} a(f_{\sigma(b)})\Omega+$$

$$(-1)^{p+1}\sum_{\substack{\varphi\in\mathrm{ONB}(\mathcal{H}^+)\\ f\in\mathrm{ONB}(\mathcal{H}^-)}}\langle Z_{1,-+}(A)\varphi|\, f\rangle\left[\prod_{l=1}^{m} a^*(\varphi_{\omega(l)})\right] a^*(\varphi)\left[\prod_{b=1}^{p} a(f_{\sigma(b)})\right] a(f)\Omega \quad (13)$$

In the last line we can now read off the particle sectors to which $\prod_{l=1}^{m} a^*(\varphi_{\omega(l)})\prod_{b=1}^{p} a(f_{\sigma(b)})\Omega$ gets mapped. The first two terms did not change their particle number, the third term lost one particle and one hole in the sea, the fourth term gained a particle and a hole.

$$T_1 : \mathcal{F}_{m,p} \to \mathcal{F}_{m-1,p-1}\oplus\mathcal{F}_{m,p}\oplus\mathcal{F}_{m+1,p+1} \quad (14)$$

The extension of T_1 to all fixed particle sectors via(13) results in a well defined operator. What is left to show is that this operator is bounded if restricted to a particular fixed particle sector $\mathcal{F}_{m,p}$, but unbounded in general.
T_1 maps $\mathcal{F}_{m,p}$ to the direct sum of three distinct vectors spaces, so to find the norm of T_1 we have to compare these cases:

Case 1: $\mathcal{F}_{m,p} \to \mathcal{F}_{m,p}$

Let $\alpha, \beta \in \mathcal{F}_{m,p}$, $(\varphi_l)_{l\in\mathbb{N}}$ be an ONB of $\mathcal{H}^+$ and $(f_l)_{l\in\mathbb{N}}$ be an ONB of $\mathcal{H}^-$. Then α and β can be written as:

$$
\begin{aligned}
\alpha &= \sum_{\substack{(i_1,\dots i_m)\in\mathbb{N}^m \\ (k_1,\dots k_p)\in\mathbb{N}^p}} \alpha_{i_1,\dots,i_m,k_1,\dots,k_p} \prod_{l=1}^{m} a^*(\varphi_{i_l}) \prod_{d=1}^{p} a(f_{k_d})\Omega \\
\beta &= \sum_{\substack{(q_1,\dots q_m)\in\mathbb{N}^m \\ (r_1,\dots r_p)\in\mathbb{N}^p}} \beta_{q_1,\dots,q_m,r_1,\dots,r_p} \prod_{l=1}^{m} a^*(\varphi_{q_l}) \prod_{d=1}^{p} a(f_{r_d})\Omega
\end{aligned}
\tag{15}
$$

Where $\alpha_{i_1,\dots,i_m,k_1,\dots,k_p}$ and $\beta_{q_1,\dots,q_m,r_1,\dots,r_p}$ are complex numbers square sumable over their indices. Also, they obey the antisymmetry characteristic for fermions, i.e. for any $\omega \in S_m$, $\sigma \in S_p$: $\alpha_{\omega(i_1),\dots,\omega(i_m),\sigma(k_1),\dots,\sigma(k_p)} = \alpha_{i_1,\dots,i_m,k_1,\dots,k_p}(-1)^\omega(-1)^\sigma$, where $(-1)^\omega$, respectively $(-1)^\sigma$ designates the sign of the permutation.

We need to find a bound for the matrix element $\langle\beta\ , T_1(A)\ \alpha\rangle$:

$$\langle\beta\ , T_1(A)\ \alpha\rangle \overset{(13)}{=} \sum_{\substack{(i_1,\dots i_m)\in\mathbb{N}^m\\(k_1,\dots k_p)\in\mathbb{N}^p}} \sum_{\substack{(q_1,\dots q_m)\in\mathbb{N}^m\\(r_1,\dots r_p)\in\mathbb{N}^p}} \sum_{\pi\in S_m}\sum_{\sigma\in S_p} \beta^*_{q_1,\dots,q_m,r_1,\dots,r_p}\alpha_{i_1,\dots,i_m,k_1,\dots,k_p}(-1)^\omega(-1)^\sigma$$

$$\left[\sum_{c=1}^m \left\langle\varphi_{q_{\pi(c)}}, Z_{1,++}(A)\varphi_{i_c}\right\rangle \prod_{\substack{l=1\\l\neq c}}^m \left\langle\varphi_{q_{\pi(l)}},\varphi_{i_l}\right\rangle \prod_{d=1}^p\left\langle f_{k_d}, f_{r_{\sigma(d)}}\right\rangle + \right.$$

$$\left.\sum_{c=1}^p \left\langle f_{r_{\sigma(c)}}, Z_{1,--}(A)f_{k_c}\right\rangle \prod_{l=1}^m \left\langle\varphi_{q_{\pi(l)}},\varphi_{i_l}\right\rangle \prod_{\substack{d=1\\d\neq c}}^p\left\langle f_{k_d}, f_{r_{\sigma(d)}}\right\rangle\right]$$

$$= \sum_{\substack{(i_1,\dots i_m)\in\mathbb{N}^m\\(k_1,\dots k_p)\in\mathbb{N}^p}} \sum_{\substack{(q_1,\dots q_m)\in\mathbb{N}^m\\(r_1,\dots r_p)\in\mathbb{N}^p}} \sum_{\pi\in S_m}\sum_{\sigma\in S_p} \beta^*_{q_1,\dots,q_m,r_1,\dots,r_p}\alpha_{i_1,\dots,i_m,k_1,\dots,k_p}(-1)^\omega(-1)^\sigma$$

$$\left[\sum_{c=1}^m \left\langle\varphi_{q_{\pi(c)}}, Z_{1,++}(A)\varphi_{i_c}\right\rangle \prod_{\substack{l=1\\l\neq c}}^m \delta_{q_{\pi(l)},i_l} \prod_{d=1}^p \delta_{k_{\sigma(d)},r_d} + \sum_{c=1}^p \left\langle f_{r_{\sigma(c)}}, Z_{1,--}(A)f_{k_c}\right\rangle \prod_{l=1}^m \delta_{q_{\pi(l)},i_l} \prod_{\substack{d=1\\d\neq c}}^p \delta_{k_d,r_{\sigma(d)}}\right]$$

$$= \sum_{\substack{(q_1,\dots q_m)\in\mathbb{N}^m\\(r_1,\dots r_p)\in\mathbb{N}^p}} \sum_{\pi\in S_m}\sum_{\sigma\in S_p}(-1)^\omega(-1)^\sigma$$

$$\left[\sum_{c=1}^m \sum_{\substack{i_c\in\mathbb{N}\\(k_1,\dots k_p)\in\mathbb{N}^p}} \beta^*_{q_1,\dots,q_m,r_1,\dots,r_p}\alpha_{q_{\pi(1)},\dots q_{\pi(c-1)},i_c,q_{\pi(c+1)}\dots,q_{\pi(m)},r_{\sigma(1)},\dots,r_{\sigma(p)}} \left\langle\varphi_{q_{\pi(c)}}, Z_{1,++}(A)\varphi_{i_c}\right\rangle + \right.$$

$$\left.\sum_{c=1}^p \sum_{\substack{(i_1,\dots,i_m)\in\mathbb{N}^m\\k_c\in\mathbb{N}}} \beta^*_{q_1,\dots,q_m,r_1,\dots,r_p}\alpha_{q_{\pi(1)},\dots,q_{\pi(m)},r_{\sigma(1)},\dots r_{\sigma(c-1)},k_c,r_{\sigma(c+1)}\dots,r_{\sigma(p)}} \left\langle f_{r_{\sigma(c)}}, Z_{1,--}(A)f_{k_c}\right\rangle\right]$$

$$= \sum_{\substack{(q_1,\dots q_m)\in\mathbb{N}^m\\(r_1,\dots r_p)\in\mathbb{N}^p}} \sum_{\pi\in S_m}\sum_{\sigma\in S_p}(-1)^\omega(-1)^\sigma$$

$$\left[\sum_{\substack{i_1\in\mathbb{N}\\(k_1,\dots k_p)\in\mathbb{N}^p}} m\beta^*_{q_1,\dots,q_m,r_1,\dots,r_p}\alpha_{i_1,q_{\pi(2)}\dots,q_{\pi(m)},r_{\sigma(1)},\dots,r_{\sigma(p)}} \left\langle\varphi_{q_{\pi(1)}}, Z_{1,++}(A)\varphi_{i_1}\right\rangle + \right.$$

$$\left.\sum_{\substack{(i_1,\dots i_m)\in\mathbb{N}\\k_1\in\mathbb{N}}} p\beta^*_{q_1,\dots,q_m,r_1,\dots,r_p}\alpha_{q_{\pi(1)},\dots,q_{\pi(m)},k_1,r_{\sigma(2)},\dots,r_{\sigma(p)}} \left\langle f_{r_{\sigma(1)}}, Z_{1,--}(A)f_{k_1}\right\rangle\right] \quad (16)$$

$$
\begin{aligned}
&\langle \beta\ , T_1(A)\ \alpha\rangle = \\
&\quad\Big[m\beta^*_{q_1,\dots,q_m,r_1,\dots,r_p}\alpha_{q_{\pi(1)},\dots q_{\pi(c-1)},q_{i_c},q_{\pi(c+1)}\dots,q_{\pi(m)},r_{\sigma(1)},\dots,r_{\sigma(p)}}\left\langle \varphi_{q_{\pi(c)}}, Z_{1,++}(A)\varphi_{i_c}\right\rangle + \\
&\quad\ p\beta^*_{q_1,\dots,q_m,r_1,\dots,r_p}\alpha_{q_{\pi(1)},\dots,q_{\pi(m)},r_{\sigma(1)},\dots r_{\sigma(c-1)},k_c,r_{\sigma(c+1)}\dots,r_{\sigma(p)}}\left\langle f_{r_{\sigma(c)}}, Z_{1,--}(A)f_{k_c}\right\rangle\Big] \\
&\quad = \sum_{\substack{(i_1,\dots i_m)\in\mathbb{N}^m\\(k_1,\dots k_p)\in\mathbb{N}^p}}\ \sum_{\substack{(q_1,\dots q_m)\in\mathbb{N}^m\\(r_1,\dots r_p)\in\mathbb{N}^p}}\ \sum_{\pi\in S_m}\sum_{\sigma\in S_p}\beta^*_{q_1,\dots,q_m,r_1,\dots,r_p}\alpha_{i_1,\dots,i_m,k_1,\dots,k_p}(-1)^\omega(-1)^\sigma \\
&\left[m\left\langle \varphi_{q_{\pi(1)}}, Z_{1,++}(A)\varphi_{i_1}\right\rangle \prod_{l=2}^m \delta_{q_{\pi(l)},i_l}\prod_{d=1}^p\delta_{k_{\sigma(d)},r_d} + p\left\langle f_{r_{\sigma(1)}}, Z_{1,--}(A)f_{k_1}\right\rangle\prod_{l=1}^m\delta_{q_{\pi(l)},i_l}\prod_{d=2}^p\delta_{k_d,r_{\sigma(d)}}\right] \\
&\quad = \langle \beta\ , mZ_{1,++}\bigotimes_{l=2}^m \mathbb{1}\otimes\bigotimes_{d=1}^p\mathbb{1} + p\bigotimes_{l=1}^m\mathbb{1}\otimes Z_{1,--}\otimes\bigotimes_{d=2}^p\mathbb{1}\ \alpha\rangle
\end{aligned} \tag{17}
$$

This means for the norm of the operator in **Case 1**:

$$
\|T_1(A)\|_{\mathcal{F}_{m,p}\to\mathcal{F}_{m,p}} \leq m\|Z_{1,++}\| + p\|Z_{1,--}\| \tag{18}
$$

Case 2: $\mathcal{F}_{m,p}\to\mathcal{F}_{m+1,p+1}$

Let $\alpha,\in\mathcal{F}_{m,p}$,$\beta,\in\mathcal{F}_{m+1,p+1}$, $(\varphi_l)_{l\in\mathbb{N}}$ and $(f_l)_{l\in\mathbb{N}}$ analogous to **case 1** . Then α and β can be written as:

$$
\begin{aligned}
\alpha &= \sum_{\substack{(i_1,\dots i_m)\in\mathbb{N}^m\\(k_1,\dots k_p)\in\mathbb{N}^p}}\alpha_{i_1,\dots,i_m,k_1,\dots,k_p}\prod_{l=1}^m a^*(\varphi_{i_l})\prod_{d=1}^p a(f_{k_d})\Omega \\
\beta &= \sum_{\substack{(q_1,\dots q_{m+1})\in\mathbb{N}^{m+1}\\(r_1,\dots r_{p+1})\in\mathbb{N}^{p+1}}}\beta_{q_1,\dots,q_{m+1},r_1,\dots,r_{p+1}}\prod_{l=1}^{m+1} a^*(\varphi_{q_l})\prod_{d=1}^{p+1} a(f_{r_d})\Omega
\end{aligned} \tag{19}
$$

So for an arbitrary matrix element we get:

$$\langle\beta, T_1(A)\alpha\rangle \overset{(13)}{=} \sum_{\substack{(q_1,\dots q_{m+1})\in\mathbb{N}^{m+1}\\(r_1,\dots r_{p+1})\in\mathbb{N}^{p+1}}} \sum_{\substack{(i_1,\dots i_m)\in\mathbb{N}^m\\(k_1,\dots k_p)\in\mathbb{N}^p}} \alpha_{i_1,\dots,i_m,k_1,\dots,k_p}\beta^*_{q_1,\dots,q_{m+1},r_1,\dots,r_{p+1}}$$

$$\left\langle \prod_{l=1}^{m+1} a^*(\varphi_{q_l}) \prod_{d=1}^{p+1} a(f_{r_d})\Omega \right| (-1)^p \sum_{\substack{\varphi\in\mathrm{ONB}(\mathcal{H}^+)\\f\in\mathrm{ONB}(\mathcal{H}^-)}} \langle Z_{1,-+}(A)\varphi|\, f\rangle \left[\prod_{l=1}^{m} a^*(\varphi_{i_l})\right] a^*(\varphi) \left[\prod_{b=1}^{p} a(f_{k_b})\right] a(f)\Omega \left.\vphantom{\prod_{l=1}^{m+1}}\right\rangle$$

$$= \sum_{\substack{(q_1,\dots q_{m+1})\in\mathbb{N}^{m+1}\\(r_1,\dots r_{p+1})\in\mathbb{N}^{p+1}}} \sum_{\substack{(i_1,\dots i_m)\in\mathbb{N}^m\\(k_1,\dots k_p)\in\mathbb{N}^p}} \sum_{\substack{\varphi\in\mathrm{ONB}(\mathcal{H}^+)\\f\in\mathrm{ONB}(\mathcal{H}^-)}} (-1)^p \langle Z_{1,-+}(A)\varphi|\, f\rangle\, \alpha_{i_1,\dots,i_m,k_1,\dots,k_p}\beta^*_{q_1,\dots,q_{m+1},r_1,\dots,r_{p+1}}$$

$$\left\langle \prod_{l=1}^{m+1} a^*(\varphi_{q_l}) \prod_{d=1}^{p+1} a(f_{r_d})\Omega \right| \left[\prod_{l=1}^{m} a^*(\varphi_{i_l})\right] a^*(\varphi) \left[\prod_{b=1}^{p} a(f_{k_b})\right] a(f)\Omega \left.\vphantom{\prod_{l=1}^{m+1}}\right\rangle \quad (20)$$

By integrating φ and f into the $(\varphi_l)_{l\in\mathbb{N}}$ respectively $(f_l)_{l\in\mathbb{N}}$ we can simplify the notation.

$$\langle\beta, T_1(A)\alpha\rangle = \sum_{\substack{(q_1,\dots q_{m+1})\in\mathbb{N}^{m+1}\\(r_1,\dots r_{p+1})\in\mathbb{N}^{p+1}}} \sum_{\substack{(i_1,\dots i_{m+1})\in\mathbb{N}^{m+1}\\(k_1,\dots k_{p+1})\in\mathbb{N}^{p+1}}} \overbrace{(-1)^p \left\langle Z_{1,-+}(A)\varphi_{i_{m+1}}\right| \left. f_{k_{p+1}}\right\rangle \alpha_{i_1,\dots,i_m,k_1,\dots,k_p}}^{\tilde{\alpha}_{i_1,\dots,i_{m+1},k_1,\dots,k_{p+1}}}$$

$$\beta^*_{q_1,\dots,q_{m+1},r_1,\dots,r_{p+1}} \sum_{\pi\in S_{m+1}} \sum_{\sigma\in S_{p+1}} (-1)^\pi(-1)^\sigma \prod_{l=1}^{m+1} \left\langle \varphi_{q_l}, \varphi_{i_{\pi(l)}}\right\rangle \prod_{e=1}^{p+1} \left\langle f_{k_{\sigma(e)}}, f_{r_e}\right\rangle = \langle\beta, \tilde{\alpha}\rangle \quad (21)$$

We focus our attention on $\tilde{\alpha}$.

$$\langle\tilde{\alpha}, \tilde{\alpha}\rangle = \sum_{\substack{(q_1,\dots q_{m+1})\in\mathbb{N}^{m+1}\\(r_1,\dots r_{p+1})\in\mathbb{N}^{p+1}}} \sum_{\substack{(i_1,\dots i_{m+1})\in\mathbb{N}^{m+1}\\(k_1,\dots k_{p+1})\in\mathbb{N}^{p+1}}} \tilde{\alpha}^*_{q_1,\dots,q_{m+1},r_1,\dots r_{p+1}}\tilde{\alpha}_{i_1,\dots,i_{m+1},k_1,\dots k_{p+1}}$$

$$\sum_{\pi\in S_{m+1}} \sum_{\sigma\in S_{p+1}} (-1)^\pi(-1)^\sigma \prod_{l=1}^{m+1} \left\langle \varphi_{q_l}, \varphi_{i_{\pi(l)}}\right\rangle \prod_{e=1}^{p+1} \left\langle f_{k_{\sigma(e)}}, f_{r_e}\right\rangle$$

$$= \sum_{\substack{(i_1,\dots i_{m+1})\in\mathbb{N}^{m+1}\\(k_1,\dots k_{p+1})\in\mathbb{N}^{p+1}}} \sum_{\pi\in S_{m+1}} \sum_{\sigma\in S_{p+1}} (-1)^\pi(-1)^\sigma \tilde{\alpha}^*_{i_{\pi(1)},\dots,i_{\pi(m+1)},k_{\sigma(1)},\dots k_{\sigma(p+1)}}\tilde{\alpha}_{i_1,\dots,i_{m+1},k_1,\dots k_{p+1}}$$

$$= (m+1)!(p+1)!\, \langle P_{\text{antisymm}}\tilde{\alpha}_{\cdot,\dots,\cdot,\dots}, \tilde{\alpha}_{\cdot,\dots,\cdot,\dots}\rangle_{\ell^2} \quad (22)$$

Where the scalar product in the last line is in the Hilbertspace of the coefficients $\ell^2(\mathbb{N}^{m+1+p+1})$. We defined the orthogonal projector onto the antisymmetric subspace:

$$P_{\text{antisymm}} : \ell^2(\mathbb{N}^{m+1+p+1}) \to \ell^2(\mathbb{N}^{m+1+p+1})$$
$$\beta \mapsto \frac{1}{(m+1)!(p+1)!} \sum_{\pi\in S_{m+1}} \sum_{\sigma\in S_{p+1}} (-1)^\pi(-1)^\sigma \beta_{\pi(\cdot),\dots\pi(\cdot),\sigma(\cdot),\dots\sigma(\cdot)} \quad (23)$$

So as an estimation on the norm of $\tilde{\alpha}$ we get:

$$\langle\tilde{\alpha},\tilde{\alpha}\rangle = (m+1)!(p+1)!\,\langle P_{\text{antisymm}}\tilde{\alpha}_{\cdot,\ldots,\cdot,\ldots},\tilde{\alpha}_{\cdot,\ldots,\cdot,\ldots}\rangle_{\ell^2} \leq (m+1)!(p+1)!\,\langle\tilde{\alpha}_{\cdot,\ldots,\cdot,\ldots},\tilde{\alpha}_{\cdot,\ldots,\cdot,\ldots}\rangle_{\ell^2} =$$

$$(m+1)!(p+1)! \sum_{\substack{(i_1,\ldots i_{m+1})\in\mathbb{N}^{m+1}\\(k_1,\ldots k_{p+1})\in\mathbb{N}^{p+1}}} \tilde{\alpha}^*_{i_1,\ldots i_{m+1},k_1,\ldots k_{p+1}}\tilde{\alpha}_{i_1,\ldots i_{m+1},k_1,\ldots k_{p+1}} =$$

$$(m+1)!(p+1)! \sum_{\substack{(i_1,\ldots i_{m+1})\in\mathbb{N}^{m+1}\\(k_1,\ldots k_{p+1})\in\mathbb{N}^{p+1}}} \alpha^*_{i_1,\ldots i_{m+1},k_1,\ldots k_{p+1}}\alpha_{i_1,\ldots i_m,k_1,\ldots k_p}\left|\left\langle Z_{1,-+}(A)\varphi_{i_{m+1}},f_{p+1}\right\rangle\right|^2 =$$

$$m!p! \sum_{\substack{(i_1,\ldots i_m)\in\mathbb{N}^m\\(k_1,\ldots k_p)\in\mathbb{N}^p}} \alpha^*_{i_1,\ldots i_{m+1},k_1,\ldots k_{p+1}}\alpha_{i_1,\ldots i_m,k_1,\ldots k_p}(m+1)(p+1)\sum_{c,d=1}^{\infty}|\langle Z_{1,-+}(A)\varphi_c,f_d\rangle|^2 =$$

$$\sum_{\substack{(i_1,\ldots i_m)\in\mathbb{N}^m\\(k_1,\ldots k_p)\in\mathbb{N}^p}}\sum_{\pi\in S_m}\sum_{\sigma\in S_p}(-1)^\pi(-1)^\sigma\alpha^*_{i_{\pi(1)},\ldots i_{\pi(m+1)},k_{(\sigma(1)},\ldots k_{\sigma(p+1)}}\alpha_{i_1,\ldots i_m,k_1,\ldots k_p}(m+1)(p+1)\|Z_{1,-+}(A)\|^2_{\text{HS}} =$$

$$\langle\alpha,\alpha\rangle\,(m+1)(p+1)\|Z_{1,-+}(A)\|^2_{\text{HS}} \quad (24)$$

Which means for the operator norm of T_1:

$$\|T_1(A)\|_{\mathcal{F}_{m,p}\to\mathcal{F}_{m+1,p+1}} \leq \sqrt{(m+1)(p+1)}\|Z_{1,-+}(A)\|_{\text{HS}} \quad (25)$$

Case 3: $\mathcal{F}_{m,p}\to\mathcal{F}_{m-1,p-1}$

Since T_1 is anti selfadjoint (i times the generator of a unitary evolution) it follows directly that:

$$\|T_1(A)\|_{\mathcal{F}_{m,p}\to\mathcal{F}_{m-1,p-1}} \leq \sqrt{mp}\|Z_{1,+-}(A)\|_{\text{HS}} \quad (26)$$

T_1 overall

We find for T_1 on a fixed particle sector the operator norm to be:

$$\|T_1(A)\|_{\mathcal{F}_{m,p}\to\mathcal{F}_{m-1,p-1}\oplus\mathcal{F}_{m,p}\oplus\mathcal{F}_{m+1,p+1}} \leq \sqrt{mp\|Z_{1,+-}(A)\|^2_{\text{HS}} + (m\|Z_{1,++}\| + p\|Z_{1,--}\|)^2 + (m+1)(p+1)\|Z_{1,-+}(A)\|^2_{\text{HS}}} \quad (27)$$