The Connection Between Multinomial Coefficients and Sterling Numbers of the Second Kind

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Abstract. The connection between Sterling numbers of the second kind and the multinomial coefficients is highlighted in this paper. This is motivated by the combinatorial interpretation of these objects and also proven by explicit calculation.

1. INTRODUCTION AND NOTATION.

I would like to discuss the relationship between the multinomial coefficients and the Sterling numbers of the second kind. For some reason this relationship is very rarely discussed in textbooks about combinatorics. In fact the only reference I am aware of is the handbook of mathematical functions [?][p 823], where the equation in question is marked corrected but is incorrect.

Binomial coefficients.

The multinomial coefficients are usually introduced in terms of binomial coefficients, which can be defined in different ways, the reader and I follow the convention of the book "Concrete Mathematics"[?].

Definition (binomial coefficient). For $a \in \mathbb{C}, b \in \mathbb{Z}$ we define

$$\binom{a}{b} := \begin{cases} \frac{1}{b!}\prod_{l=0}^{b-1}(a-l) & \text{for } b \geq 0 \\ 0 & \text{else.} \end{cases} \tag{1}$$

By defining the coefficients for negative lower index to be zero, we do not have to worry about boundary conditions in many formulas. The combinatorial interpretation of the binomial coefficient $\binom{a}{b}$ for $a, b \in \mathbb{N}$ is the number of ways to choose b elements of a set of a elements.

Multinomial coefficients

The multinomial coefficients are a product of binomial coefficients.

Definition (multinomial coefficient). For $g \in \mathbb{N}$ and $a \in \mathbb{N}, \vec{b} \in \mathbb{N}^g$ with $\sum_{k=1}^{g} b_k = a$ we define

$$\binom{a}{\vec{b}} := \prod_{k=1}^{g-1}\binom{a-\sum_{l=1}^{k-1} b_k}{b_k}, \tag{2}$$

where $\sum_{l=1}^{0} f(l) := 0$ holds for any summand f by convention.

The combinatorial interpretation of $\binom{a}{\vec{b}}$ for $g \in \mathbb{N}$ and $a \in \mathbb{N}, \vec{b} \in \mathbb{N}^g$ is the number of ways to partition a set of a elements into g distinct sets, where the j-th set has b_j elements.

The multinomial coefficients are applied e.g. in the well known multinomial theorem[?][p 823]: