Electron-Positron Pair Creation in External Fields

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Abstract

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Chapter 1

Introduction

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Chapter 2

Direct Interaction in Relativistic Quantum Mechanics

As we have seen in the last chapter, having interaction mediated by potentials in a Dirac equation does not seem to be a viable option. One alternative approach to this problem is to reformulate Diracs equation as an integral equation and to introduce interaction afterwards. For the benefit of the unfamiliar reader, we will first follow the heuristic derivation of this type of equation in [29], then briefly review the mathematical results that had been established in the past and finally discuss new results for which the author is at least partially to blame. This chapter is based on the preprints [31, 32]

2.1 Overview

2.1.1 Derivation

We now follow the heuristic derivation of [29] of an equation for a multi-time wave function for two particles that expresses direct interaction along light-like configurations. This type of equation will then keep us occupied for the rest of this chapter. The derivatin is organised as follows: We start out reformulating Diracs equation for a single particle as an integral equation. The reformulated version is then extended to two particles in an Poincaré invariant manner. Extending the equation is conveniently done in the framework of multi-time wavefunctions.

Diracs equation for one particle subject to an external potential V takes the form

$$i\partial_t\phi(t,\mathbf{x}) = \big(H^{\mathrm{free}} + V(t,\mathbf{x})\big)\phi(t,\mathbf{x}), \tag{2.1}$$

here ϕ denotes the wavefunction in question, $\mathbf{x} \in \mathbb{R}^3, t \in \mathbb{R}$ and H^{free} is the Hamiltonian associated with a free Dirac particle. We denote by S^{ret} the retarded Green's function of the non interacting Dirac equation, that is S^{ret} satisfies

$$\big(i\partial_t - H^{\mathrm{free}}\big)S^{\mathrm{ret}} = \delta^4, \tag{2.2}$$

$$S^{\mathrm{ret}}(t,\mathbf{x}) = 0 \quad \text{for } t < 0. \tag{2.3}$$

Then inverting the differential operator $i\partial_t - H^{\mathrm{free}}$ in (2.1) results in

$$\phi(t,\mathbf{x}) = \phi^{\mathrm{free}}(t,\mathbf{x}) + \int_{t^0}^{\infty} dt' \int d^3\mathbf{x}'\, S^{\mathrm{ret}}(t-t',\mathbf{x}-\mathbf{x}')V(t',\mathbf{x}')\phi(t',\mathbf{x}'), \tag{2.4}$$

where ϕ^{free} is the solution of the non interacting equation subject to the initial condition $\phi^{\mathrm{free}}(t_0) = \phi_0$. Equations (2.4) and (2.1) subject to $\phi(t_0) = \phi_0$ yield equivalent descriptions, as can be verified directly: An action of $i\partial_t - H^{\text{free}}$ on (2.4) shows that a solution thereof also solves (2.1). Also the initial condition is fulfilled, as the integral term vanishes for $t = t_0$. Conversely equation (2.1) can be considered a free Dirac equation involving an inhomogeneous term of the form $V\phi$, whose solutions are known to be of the form (2.4). Executing the analogous procedure for the two particle Dirac equation

$$i\partial_t \phi(t, \mathbf{x}_1, \mathbf{x}_2) = \big(H_1^{\text{free}} + H_2^{\text{free}} + V(t, \mathbf{x}_1, \mathbf{x}_2)\big)\phi(t, \mathbf{x}_1, \mathbf{x}_2), \qquad (2.5)$$

subject to the initial condition $\phi(t_0) = \phi_0$, results in the integral equation

$$\begin{aligned}\phi(t, \mathbf{x}_1, \mathbf{x}_2) = \phi^{\text{free}}(t, \mathbf{x}_1, \mathbf{x}_2) + \int_{t_0}^{\infty} dt' \int d^3\mathbf{x}_1'\, d^3\mathbf{x}_2'\, S_1^{\text{ret}}(t - t', \mathbf{x}_1 - \mathbf{x}_1') \\ \times S_2^{\text{ret}}(t - t', \mathbf{x}_2 - \mathbf{x}_2') V(t', \mathbf{x}_1', \mathbf{x}_2')\phi(t', \mathbf{x}_1', \mathbf{x}_2'), \qquad (2.6)\end{aligned}$$

where now, $\phi^{\text{free}}(t)$ is a solution to the free Dirac equation for two particles subject to $\phi^{\text{free}}(t_0) = \phi_0$ and S_k^{ret} is the retarded Green's function of the free Dirac equation of particle number k. Here it is crucial to notice that The Green's function of the free two particle Dirac equation factorises into a product of two Green's functions of the Dirac equation for one particle.
Since equation (2.6) contains only one temporal variable, but six spatial ones, it is not obvious how it might be considered a relativistic equation at all. Now, we will generalise to two particles, but before we do so let us first rewrite equation (2.4) in a more suggestive way:

$$\psi(x) = \psi^{\text{free}}(x) + \int d^4x'\, S^{\text{ret}}(x - x')V(x')\psi(x'), \qquad (2.7)$$

where non bold letters denote elements of Minkowski spacetime and we replaced ϕ by ψ in order to make a visible switch to a relativistic

notation. Furthermore we replaced the lower bound in the temporal integral domain by $-\infty$ in order to render the total domain of integral Poincaré invariant.
Equation (2.7) suggests to write down the following generalisation

$$\begin{aligned}\psi(x_1, x_2) &= \psi^{\text{free}}(x_1, x_2) \\ &\quad + \int d^4x_1'\, d^4x_2'\, S_1^{\text{ret}}(x_1 - x_1')S_2^{\text{ret}}(x_2 - x_2')K(x_1', x_2')\psi(x_1', x_2').\end{aligned} \tag{2.8}$$

we integrate over all of $\mathbb{R}^8$ and ψ^{free} a solution of the free Dirac equation both in x_1 and x_2 and their respective spinor indices:

$$D_1\psi^{\text{free}}(x_1, x_2) = 0, \tag{2.9}$$

$$D_2\psi^{\text{free}}(x_1, x_2) = 0. \tag{2.10}$$

For the object K, called the "interaction kernel“, the optimal choice is not yet known. However, for (2.8) to be Poincaré invariant, it should be invariant itself. A simple way to ensure this is to let it only depend directly the squared Minkowski distance $(x_1 - x_2)^2$. A choice that shows some resemblence of Wheeler-Feynman electrodynamics is

$$K(x_1, x_2) = i\, \frac{e_1e_2}{4\pi} \gamma_1^\mu \gamma_{2,\mu}\, \delta((x_1 - x_2)^2). \tag{2.11}$$

In an equation incorporating (2.11) the interaction between the particles happens along light-like distances. The constant in front of (2.11) is fixed by the non-relativistic limit, recovering an equation very much like the Breit equation, see [**?**, section 3.6].
Summarising, we arrived at the equation

$$\begin{aligned}\psi(x_1, x_2) &= \psi^{\text{free}}(x_1, x_2) + i\, \frac{e_1e_2}{4\pi} \int d^4x_1'\, d^4x_2' \\ &\quad \times S_1^{\text{ret}}(x_1 - x_1')S_2^{\text{ret}}(x_2 - x_2')\, \gamma_1^\mu \gamma_{2,\mu} \delta((x_1' - x_2')^2)\psi(x_1', x_2').\end{aligned} \tag{2.12}$$

Despite the fact that the motivation for (2.12) holds for Dirac particles, it is also conceivable to replace ψ, S^{ret} and all the constant factor by quantities related to the Klein Gordon equation and arrive at

$$\psi(x_1, x_2) = \psi^{\mathrm{free}}(x_1, x_2) + \lambda \int d^4x_1'\, d^4x_2' \tag{2.13}$$
$$\times\, G_1^{\mathrm{ret}}(x_1 - x_1')G_2^{\mathrm{ret}}(x_2 - x_2')\, \delta((x_1' - x_2')^2)\psi(x_1', x_2'),$$

where ψ and ψ^{free} are no longer spinor valued, ψ^{free} is a solution to the free Klein Gordone equation in both x_1 and x_2,

$$(\Box_{x_1} + m_1^2)\psi = 0, \tag{2.14}$$
$$(\Box_{x_2} + m_2^2)\psi = 0 \tag{2.15}$$

and G^{ret} is the retarded Green's function of the Klein Gordon equation. In fact, most of the rigorous results about equations of a similar type as the ones motivated in this chapter are about the Klein Gordon version (2.13).

2.1.2 Previous Results on Directly Interacting Particles

In this section we summarise the most important existence results on equations of the type of (2.8). Because this line of work is still fairely young, it can still readily be summarised. The results are taken from [**?**] and [35]. The theorems are about the Klein-Gordon case, i.e. slightly different versions of equations of the type of (2.13). I tried to contain the necessary notation to within each of the theorems. Mentioned below are only the theorems that are about a four dimensional spacetime; however, there are also results concerning lower dimensions, the interested reader is refered to [29, 35]. The versions of equation (2.13) in the theorems are considerably modified:

- The spacetime of equation (2.13) is $\mathbb{R}^4$, i.e. Minkowski spacetime. All the rigorous results concerning vanishing curvature so far are about $\mathbb{R}^+ \times \mathbb{R}^3 =: \frac{1}{2}\mathbb{M}$. That is, there is a beginning in time. This modification has technical reasons. However, as current cosmological models of our universe do have a beginning in time this modification does not necessarily mean that the equation can no longer describe certain aspects of physics. As these cosmological models have nonzero curature the authors of [35] have shown existence of solutions of versions of equation (2.13) on Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime. In section 2.2 and 2.3 we will also employ this simplification and show existence on this spacetime. This is not an attempt to treat general curved spacetimes, it is done as an act of consistency. We introduce a beginning in time and try to justify this by cosmological arguments and hence we treat a spacetime commonly used in cosmology.

- The interaction kernel K which we motivated to be proportional to $\delta((x_1 - x_2)^2)$ is replaced by various less singular objects. This modification is purely technical and we do not justify it. The previous results approach the singular K introduced in the last section to different degrees. In section 2.2 where we treat Dirac particles we will also use a rather soft interaction kernel. The new result about Klein Gordon particles presented in section 2.3 employs the fully singular $\delta((x_1 - x_2)^2)$ kernel.

Theorem 1 (Thm 3.4 ($d = 3$) of [37]). *Let $T > 0, \lambda \in \mathbb{C}$, for every bounded $K : \mathbb{R}^8 \to \mathbb{C}$ and every $\psi^{\text{free}} \in \mathcal{B}_3 := L^\infty([0,T]^2, L^2(\mathbb{R}^6))$ the equation*

$$\begin{aligned}\psi(t_1,\mathbf{x}_1,t_2,\mathbf{x}_2) &= \psi^{\text{free}}(t_1,\mathbf{x}_1,t_2,\mathbf{x}_2) + \frac{\lambda}{(4\pi)^2}\int d\mathbf{x}_1' d\mathbf{x}_2' \\ &\times \frac{H(t_1-|\mathbf{x}_1-\mathbf{x}_1'|)}{|\mathbf{x}_1-\mathbf{x}_1'|}\frac{H(t_2-|\mathbf{x}_2-\mathbf{x}_2'|)}{|\mathbf{x}_2-\mathbf{x}_2'|} \\ &\times K(t_1-|\mathbf{x}_1-\mathbf{x}_1'|,\mathbf{x}_1',t_2-|\mathbf{x}_2-\mathbf{x}_2'|,\mathbf{x}_2') \\ &\times \psi(t_1-|\mathbf{x}_1-\mathbf{x}_1'|,\mathbf{x}_1',t_2-|\mathbf{x}_2-\mathbf{x}_2'|,\mathbf{x}_2')\end{aligned}$$

has a unique solution $\psi \in \mathcal{B}_3$.

Theorem 2 (Thm 3.5 of [37])**.** *Let* $T > 0, \lambda \in \mathbb{C}$, *for every bounded* $f : \mathbb{R}^8 \to \mathbb{C}$ *and every* $\psi^{\text{free}} \in \mathcal{B}_3 := L^\infty([0,T]^2, L^2(\mathbb{R}^6))$ *the equation*

$$\begin{aligned}\psi(t_1,\mathbf{x}_1,t_2,\mathbf{x}_2) &= \psi^{\text{free}}(t_1,\mathbf{x}_1,t_2,\mathbf{x}_2) + \frac{\lambda}{(4\pi)^2}\int d^3\mathbf{x}_1' d^3\mathbf{x}_2' \\ &\times \frac{H(t_1-|\mathbf{x}_1-\mathbf{x}_1'|)}{|\mathbf{x}_1-\mathbf{x}_1'|}\frac{H(t_2-|\mathbf{x}_2-\mathbf{x}_2'|)}{|\mathbf{x}_2-\mathbf{x}_2'|} \\ &\times \frac{f(t_1-|\mathbf{x}_1-\mathbf{x}_1'|,\mathbf{x}_1',t_2-|\mathbf{x}_2-\mathbf{x}_2'|,\mathbf{x}_2')}{|\mathbf{x}_1'-\mathbf{x}_1|} \\ &\times \psi(t_1-|\mathbf{x}_1-\mathbf{x}_1'|,\mathbf{x}_1',t_2-|\mathbf{x}_2-\mathbf{x}_2'|,\mathbf{x}_2'),\end{aligned}$$

has a unique solution $\psi \in \mathcal{B}_3$.

The next results will be about the open FLRW spacetime. There are also results about the closed FLRW universe which we omit here, the reader is refered to [35, Thm 4.3]. We have to introduce some notation before we can present them, in order to do so we follow [32, sec 3.3]. We consider particles on a flat (FLRW) spacetime which is described by the metric

$$ds^2 = a^2(\eta)\left(d\eta^2 - dr^2 - r^2 d\Omega^2\right), \tag{2.16}$$

where η denotes conformal time, $d\Omega$ denotes the surface measure on $\mathbb{S}^2$ and $a(\eta)$ is the so-called *scale function*, a continuous function with

$a(0) = 0$ and $a(\eta) > 0$ for $\eta > 0$. This form makes it obvious that the spacetime is conformally equivalent to a Minkowski half space $\frac{1}{2}\mathbb{M}$, with conformal factor $a(\eta)$.
In this spacetime the free wave equation takes the form

$$(\Box_g - \xi R)\,\chi = 0, \tag{2.17}$$

where R denotes the Ricci scalar and in 1+3 dimensions $\xi = \frac{1}{6}$.
In this case, it is well-known that the Green's functions of (2.177) on the flat FLRW spacetime $\mathcal{M}$ can be obtained from those of the usual wave equation on $\frac{1}{2}\mathbb{M}$ as follows (using coordinates $x = (\eta, \mathbf{x})$ and $x' = (\eta', \mathbf{x}')$ with $\eta, \eta' \in [0, \infty)$ and $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^3$; see [35] for a more detailed explanation):

$$G_{\mathcal{M}}(x, x') \;=\; \frac{1}{a(\eta)}\frac{1}{a(\eta')}G_{\frac{1}{2}\mathbb{M}}(x, x'). \tag{2.18}$$

Inserting the well-known expression for the retarded and symmetric Green's functions on $\frac{1}{2}\mathbb{M}$ (see (**??**) below) yields:

$$\begin{aligned} G^{\mathrm{ret}}_{\mathcal{M}}(x, x') &= \frac{1}{4\pi}\frac{1}{a(\eta)a(\eta')}\frac{\delta(\eta - \eta' - |\mathbf{x} - \mathbf{x}'|)}{|\mathbf{x} - \mathbf{x}'|} \\ G^{sym}_{\mathcal{M}}(x, x') &= \frac{1}{4\pi}\frac{1}{a(\eta)a(\eta')}\delta((\eta - \eta')^2 - |\mathbf{x} - \mathbf{x}'|^2). \end{aligned} \tag{2.19}$$

With this information, we are ready to write down the integral equation on $\mathcal{M}$. The generalization of (2.13) to curved spacetimes is straightforward: ψ becomes a scalar function on $\mathcal{M} \times \mathcal{M}$, one exchanges the Minkowski spacetime volume element with

$$dV(x) = a^4(\eta)\,d\eta\,d^3\mathbf{x}, \tag{2.20}$$

the invariant 4-volume element on $\mathcal{M}$, and the Green's functions on $\frac{1}{2}\mathbb{M}$ get replaced with those on $\mathcal{M}$ as well. As in the Minkowski case, the interaction kernel is given by the symmetric Green's function. With this, the relevant integral equation becomes:

$$\psi(x,y) = \psi^{\text{free}}(x,y) + \lambda \int_{\mathcal{M}\times\mathcal{M}} dV(x)\, dV(y)\ G_1^{\text{ret}}(x,x') G_2^{\text{ret}}(y,y') \times G^{\text{sym}}(x',y')\psi(x',y'). \tag{2.21}$$

For regular and only weakly singular interaction kernels $K(x,y)$ instead of $G^{sym}(x',y')$, the problem of existence and uniqueness of solutions of this equation has been treated in [35]:

when copying KG results here om[redacted]t this part here!

Theorem 3 (Thm 4.1 of [35]). *Let $T > 0$, $\lambda \in \mathbb{C}$ and $\mathcal{B}_3 := L^\infty([0,T]^2, L^2(\mathbb{R}^6))$. Furthermore, let $a : [0,\infty) \to [0,\infty)$ be a continuous function with $a(0) = 0$ and $a(\eta) > 0$ for $\eta > 0$, and $\tilde{K} : ([0,\infty) \times \mathbb{R}^3)^2 \to \mathbb{C}$ be bounded. Then for every ψ^{free} with $a(\eta_1)a(\eta_2)\psi^{\text{free}} \in \mathcal{B}_3$, the respective integral equation on the 4−dimensional flat FLRW universe sith scale function $a(\eta)$:*

$$\begin{aligned}\psi(\eta_1,\mathbf{x}_1,\eta_2,\mathbf{x}_2) = \psi^{\text{free}}(\eta_1,\mathbf{x}_1,\eta_2,\mathbf{x}_2) &+ \frac{\lambda}{(4\pi)^2 a(\eta_1)a(\eta_2)} \\ &\times \int d\mathbf{x}_1' d\mathbf{x}_2' a^2(\eta_1 - |\mathbf{x}_1 - \mathbf{x}_1'|) a^2(\eta_2 - |\mathbf{x}_2 - \mathbf{x}_2'|) \\ &\times \frac{H(\eta_1 - |\mathbf{x}_1 - \mathbf{x}_1'|)}{|\mathbf{x}_1 - \mathbf{x}_1'|} \frac{H(\eta_2 - |\mathbf{x}_2 - \mathbf{x}_2'|)}{|\mathbf{x}_2 - \mathbf{x}_2'|} \\ &\times \tilde{K}(\eta_1 - |\mathbf{x}_1 - \mathbf{x}_1'|, \mathbf{x}_1', \eta_2 - |\mathbf{x}_2 - \mathbf{x}_2'|, \mathbf{x}_2') \\ &\times \psi(\eta_1 - |\mathbf{x}_1 - \mathbf{x}_1'|, \mathbf{x}_1', \eta_2 - |\mathbf{x}_2 - \mathbf{x}_2'|, \mathbf{x}_2')\end{aligned} \tag{2.22}$$

has a unique solution ψ with $a(\eta_1)a(\eta_2)\psi \in \mathcal{B}_3$.

Theorem 4 (Thm 4.2 of [35]). *Let $f : ([0,\infty] \times \mathbb{R}^3)^2 \to \mathbb{C}$ be a bounded function. Then, under the same assumptions as in the last theorem but with*

$$\tilde{K}(\eta_1,\mathbf{x}_1,\eta_2,\mathbf{x}_2) = \frac{f(\eta_1,\mathbf{x}_1,\eta_1,\mathbf{x}_2)}{|\mathbf{x}_1 - \mathbf{x}_2|}, \tag{2.23}$$

the integral equation (2.22) has a unique solution ψ with $a(\eta_1)a(\eta_2)\psi \in \mathcal{B}_3$.

2.2 Directly Interacting Dirac Particles

2.2.1 Introduction

The Dirac equation is perhaps the most important equation in relativistic quantum theory, thus it may seem surprising that no completely satisfactory mathematical mechanism of interaction has been found for it. Usually, interactions between many particles are implemented in one of the following ways: (a) adding a potential to the free Hamiltonian, (b) using a second quantized electromagnetic field which mediates the interaction. Both approaches face difficulties. Approach (a) corresponds to postulating the equation

$$i\partial_t\varphi(t,\mathbf{x}_1,\mathbf{x}_2) = \left(H_1^{\mathrm{D}irac} + H_2^{\mathrm{D}irac} + V(t,\mathbf{x}_1,\mathbf{x}_2)\right)\varphi(t,\mathbf{x}_1,\mathbf{x}_2), \quad (2.24)$$

where V is a potential and $H_k^{\mathrm{D}irac}$ the Dirac Hamiltonian acting on the variables of the k-th particle. Under appropriate circumstances, it is clear that (2.24) defines an interacting dynamics (see e.g. [11] and references therein). However, (2.24) is not Lorentz invariant.

Approach (b), on the other hand, easily leads to a Lorentz invariant dynamics. However, one encounters difficulties with ultraviolet divergences. These difficulties have led to the situation that, great efforts notwithstanding, it has so far only been possible to rigorously define a Lorentz invariant dynamics for toy models in 1+1 and 1+2 spacetime dimensions (see e.g. [52, 17, 23]). In 1+3 dimensions, it has been an open problem to prove the existence of the dynamics for any interacting and completely relativistic model.

In this paper, we pursue a new approach to defining interacting dynamics, neither via potentials nor via second quantized fields, but rather through *direct interactions with time delay*, and prove the existence of dynamics for the simple case of two Dirac particles in 1+3 dimensions. The key innovation is to make use of *multi-time wave functions*. This concept goes back to Dirac [13], played an important role in the works of Tomonaga [54] and Schwinger [51], has been studied by different authors over the years [20, 39, 50, 14, 49, 55] and has recently undergone considerable developments [45, 46, 44, 28, 30, 27, 9, 34, 36, 26, 41]; an overview can be found in [33]. For two Dirac particles in Minkowski spacetime $\mathbb{M}$, a multi-time wave function is a map

$$\psi : \mathbb{M} \times \mathbb{M} \to \mathbb{C}^4 \otimes \mathbb{C}^4 \cong \mathbb{C}^{16}, \quad (x_1, x_2) \mapsto \psi(x_1, x_2). \tag{2.25}$$

ψ can be considered a generalization of the single-time wave function φ in the Schrödinger picture, as in Eq. (2.24). The relation of ψ to φ is straightforwardly given by

$$\varphi(t, \mathbf{x}_1, \mathbf{x}_2) = \psi((t, \mathbf{x}_1), (t, \mathbf{x}_2)). \tag{2.26}$$

Contrary to the single-time wave function φ (which refers to a frame), ψ is a manifestly covariant object. Under a Poincaré transformation (a, Λ), ψ transforms as

$$\psi'(x_1, x_2) = S[\Lambda] \otimes S[\Lambda] \psi(\Lambda^{-1}(x_1 - a), \Lambda^{-1}(x_2 - a)), \tag{2.27}$$

where $S[\Lambda]$ are the matrices appearing in the spinor representation of the Lorentz group.

For the present purposes, it is crucial that ψ is defined on general space-time configurations $(x_1, x_2) \in \mathbb{M} \times \mathbb{M}$, not only on equal-time configurations as φ. By relating configurations (x_1, x_2) with different time coordinates $x_1^0 \neq x_2^0$ one can express *interactions with a time delay*. It has been pointed out in [29] that in this way, *direct relativistic interactions* (unmediated by fields) can be expressed at the quantum level. In particular, it becomes possible to formulate a quantum analog of direct interactions along light cones, such as in the Wheeler-Feynman formulation of classical electrodynamics [56, 57], using values of $\psi(x_1, x_2)$ with $(x_1 - x_2)_\mu (x_1 - x_2)^\mu = 0$. This is not directly feasible using just φ. We thus note that *new kinds of interacting quantum dynamics can be defined using a multi-time wave function.*

An interesting class of such dynamics has recently been suggested in [29] and has been subsequently analyzed rigorously in [37, 35]: *multi-time integral equations.* But why study integral equations instead of PDEs? To answer this question, note that the initial value problem $\varphi(0, \mathbf{x}_1, \mathbf{x}_2) = \psi_0(\mathbf{x}_1, \mathbf{x}_2)$ of the single-time Schrödinger equation (2.24) can equivalently be formulated as the following integral equation:

$$\begin{aligned}\varphi(t, \mathbf{x}_1, \mathbf{x}_2) =& \varphi^{free}(t, \mathbf{x}_1, \mathbf{x}_2) + \int_0^\infty dt' \int d^3\mathbf{x}_1'\, d^3\mathbf{x}_2' \\ & \times \gamma_1^0 S_1^{\text{ret}}(t - t', \mathbf{x}_1 - \mathbf{x}_1') \gamma_2^0 S_2^{\text{ret}}(t - t', \mathbf{x}_2 - \mathbf{x}_2') \\ & \times V(t', \mathbf{x}_1', \mathbf{x}_2') \varphi(t', \mathbf{x}_1', \mathbf{x}_2'), \end{aligned} \tag{2.28}$$

where φ^{free} is the solution of the same initial value problem of the free equation ((2.24) with $V = 0$) and S_k^{ret} is the retarded Green's function of the k-th Dirac operator.

Now, contrary to the PDE (2.24), the integral equation (2.28) possesses a straightforward manifestly covariant generalization in terms of a multi-time wave function, namely:

$$\begin{aligned}\psi(x_1, x_2) = \psi^{\text{free}}(x_1, x_2) + \int d^4x_1'\, d^4x_2'\, S_1(x_1 - x_2') S_2(x_2 - x_2') \\ \times K(x_1', x_2') \psi(x_1', x_2'), \end{aligned} \tag{2.29}$$

where ψ^{free} is a solution of the equations $D_1 \psi^{\text{free}} = 0$, $D_2 \psi^{\text{free}} = 0$, $D_k = (i\gamma_k^\mu \partial_{k,\mu} - m_k)$ and S_1, S_2 are (retarded or other) Green's functions of D_1, D_2, respectively. $K(x_1, x_2)$ denotes the so-called *interaction kernel*, a Poincaré invariant function (or distribution) which

generalizes the potential in Eq. (2.28). The crucial point is that (2.129) incorporates interactions with time delay which cannot be expressed through a PDE. It has been demonstrated in [29] that for $K(x_1, x_2) \propto \delta((x_1 - x_2)_\mu (x_1 - x_2)^\mu)$, the Dirac delta distribution along the light cone, one re-obtains (2.24) with $V(t, \mathbf{x}_1, \mathbf{x}_2) \propto \frac{1}{|\mathbf{x}_1 - \mathbf{x}_2|}$ if one neglects the time delay of the interaction. Thus, (2.129) constitutes a natural generalization of (2.28).
Further support for considering the integral equation (2.129) comes from the fact that the Bethe-Salpeter (BS) equation of QFT [48], which is usually considered an effective equation for a bound state, has a similar form as (2.129). That being said, there are also significant physical and mathematical differences between the two equations (see [29, sec. 3.3]).

2.2.1.1 Previous results.

To the best of our knowledge, the first results about the existence and uniqueness of dynamics for Eq. (2.129) have been obtained in [37], for the case of a Minkowski half-space and Klein-Gordon (KG) particles. A "Minkowski half-space" means to use $\frac{1}{2}\mathbb{M} \times \frac{1}{2}\mathbb{M}$ with $\frac{1}{2}\mathbb{M} = [0, \infty) \times \mathbb{R}^3$, i.e. Minkowski spacetime cut off before $t = 0$, as the domain of integration in (2.129). The KG case refers to replacing S_1, S_2 with (retarded) Green's functions of the KG equation and ψ^{free} with a solution of $(\Box_k + m_k^2)\psi^{\text{free}} = 0$, $k = 1, 2$. The main result in [37] was to show that for every ψ^{free} which is L^2 in the spatial directions and L^∞ in the time directions there is a unique solution ψ with the same properties. In addition, at $t_1 = t_2 = 0$, ψ^{free} and ψ agree so that one actually has a Cauchy problem at the initial time. In order to obtain that result, the interaction kernel was assumed to be either bounded or to just have a $1/|\mathbf{x}_1 - \mathbf{x}_2|$ singularity. In 1+3 dimensions, only the massless case was treated. The proof was based on exploiting a Volterra property which appears for retarded Green's

functions and $\frac{1}{2}\mathbb{M}$, i.e. the time integrations in (2.129) reach only from 0 to x_1^0 or x_2^0 (given by the time arguments of ψ on the left hand side). This allowed an effective iteration scheme for Eq. (2.129), leading to a global existence and uniqueness result for a formidable-looking non-Markovian (history dependent) type of dynamics.
The cutoff of spacetime at $t = 0$ was introduced in [37] to obtain the Volterra property. While such a cutoff destroys Lorentz invariance, there could be physical justification for a beginning in time which is compatible with relativity. Such a justification has been provided in [**?**]. There, the integral equation was extended to curved spacetimes and analyzed in more detail for certain spacetimes which feature a Big Bang singularity, Friedman-Lemaître-Robertson-Walker (FLRW) spacetimes. The Big Bang then provides a natural cutoff in the cosmological time. In this way, the existence of certain classes of fully covariant dynamics for massless KG particles was demonstrated.

2.2.1.2 Goal of the paper.

Here we would like to extend the previous results to the case of Dirac instead of KG particles. This is desirable as the Dirac equation describes actual elementary particles (fermions) while the KG equation is usually considered only a toy equation as its currents do not have the right properties to play the role of a probability current. Mathematically, the Dirac case is more challenging than the KG case as contrary to the latter, the Dirac Green's functions contain distributional derivatives. A Green's function of the Dirac equation is given by acting with the adjoint Dirac operator $\overline{D} = (-i\gamma^\mu \partial_\mu - m)$ on a Green's function $G(x)$ of the KG equation, i.e.

$$S(x) = \overline{D}G(x). \tag{2.30}$$

Consequently, one has to define the integral operator in (2.129) on a function space where one can take certain weak derivatives. In contrast to most of non-relativistic physics, this also concerns the time derivatives here. The choice of function space can be a tricky issue, as the convergence of an iteration scheme (and of the Neumann series, our strategy of proof) requires the integral operator to preserve the regularity, so that the regularity needs to be in harmony with the structure of the integral equation (see Sec. 2.2.2.2).

2.2.1.3 Further motivation.

1. It is quite challenging to set up an interacting dynamics for multi-time wave functions. The issue here is not only Lorentz invariance but rather the mere compatibility of the time evolutions in the various time coordinates. A no-go theorem [45, 9] for example rules out interaction potentials (which could be Poincaré invariant functions in the multi-time approach). Thus, interaction is more difficult to achieve for multi-time than for single-time wave functions. So far, the only rigorous, interacting and Lorentz invariant multi-time models for Dirac particles have been constructed in 1+1 spacetime dimensions [28, 30] (see, however, [14, 49, 55] for non-rigorous Lorentz invariant models in 1+3 dimensions and [41, chap. 3] for a not fully Lorentz invariant but rigorous model in 1+3 dimensions). Considering these difficulties, the multi-time aspect of our model is interesting in its own right.

2. Eq. (2.129) defines, in the case of retarded Green's functions, a new class of Volterra-type equations which may be interesting also for researchers specializing in integral equations. It provides a reason why a multi-dimensional Volterra-type equation would be relevant for physics, and shows which properties to expect for applications.

2.2.1.4 Overview.

The paper is structured as follows. In Sec. 2.2.2, we specify the integral equation (2.129) in detail. The difficulties with understanding the distributional derivatives are discussed and a suitable function space is identified. Sec. 2.3.3 contains our main results. In Sec. 2.2.3.1, we formulate an existence and uniqueness theorem (Thm. 8) for Eq. (2.129) on $\frac{1}{2}\mathbb{M}$. It is shown that the relevant initial data are equivalent to Cauchy data at $t = 0$. In Sec. 2.2.3.2, we provide a physical justification for the cutoff at $t = 0$ by extending the results to a FLRW spacetime. In the massless case, we show that an existence and uniqueness theorem can be obtained from the one for $\frac{1}{2}\mathbb{M}$ via conformal invariance. The result, Thm. 10, covers a fully relativistic interacting dynamics in 1+3 spacetime dimensions. The proofs are carried out in Sec. 2.3.4. Sec. 2.2.5 contains a discussion and an outlook on future research.

2.2.2 Setting of the problem

2.2.2.1 Definition of the integral operator on test functions

In this section, we show how the integral operator in (2.129) can be defined rigorously on test functions. We consider the integral equation (2.129) on the Minkowski half space $\frac{1}{2}\mathbb{M} := [0, \infty) \times \mathbb{R}^3$ equipped with the metric $g = \mathrm{diag}(1, -1, -1, -1)$. We focus on retarded Green's functions of the Dirac equation, $S^{\mathrm{ret}}(x) = \overline{D}G^{\mathrm{ret}}(x)$ where $G^{\mathrm{ret}}(x)$ is the retarded Green's function of the KG equation. Explicitly,

$$G^{\mathrm{ret}}(x) = \frac{1}{4\pi}\frac{\delta(x^0 - |\mathbf{x}|)}{|\mathbf{x}|} - \frac{m}{4\pi}H(x^0 - |\mathbf{x}|)\frac{J_1(m\sqrt{x^2})}{\sqrt{x^2}} \tag{2.31}$$

where H denotes the Heaviside function, J_1 a Bessel function and $x^2 = (x^0)^2 - |\mathbf{x}|^2$.

In order to define the meaning of the Green's functions as distributions, we introduce a suitable space of test functions:

$$\mathcal{D} = \mathcal{S}\big((\tfrac{1}{2}\mathbb{M})^2, \mathbb{C}^{16}\big), \tag{2.32}$$

the space of 16-component Schwarz functions on $(\frac{1}{2}\mathbb{M})^2$. For a smooth interaction kernel K and a test function $\psi \in \mathcal{D}$, we then understand (2.129) by formally integrating by parts so that all partial derivatives act on $K\psi$:

$$\psi(x_1, x_2) = \psi^{\text{free}}(x_1, x_2) + \int_{\frac{1}{2}\mathbb{M}} d^4x_1' \int_{\frac{1}{2}\mathbb{M}} d^4x_2' \, G_1^{\text{ret}}(x_1 - x_1') \tag{2.33}$$

$$\times\, G_2^{\text{ret}}(x_2 - x_2')[D_1 D_2(K\psi)](x_1', x_2')$$

$$+ \text{ boundary terms}, \tag{2.34}$$

where $D_k = (i\gamma_k^\mu \partial_{x_k^\mu} - m_k)$, $k = 1, 2$. The boundary terms result from the fact that $\psi(x_1, x_2) \neq 0$ for $x_1^0 = 0$ or $x_2^0 = 0$ and are given by:

$$\int_{\mathbb{R}^3} d^3\mathbf{x}_1' \int_{\mathbb{R}^3} d^3\mathbf{x}_2' i\gamma_1^0 G_1^{\text{ret}}(x_1 - x_1') i\gamma_2^0 G_2^{\text{ret}}(x_2 - x_2') \tag{2.35}$$

$$\times\ (K\psi)(x_1', x_2')|_{x_1^{0\prime}=0,\, x_2^{0\prime}=0}$$

$$+ \int_{\mathbb{R}^3} d^3\mathbf{x}_1' \int_{\frac{1}{2}\mathbb{M}} d^4x_2' i\gamma_1^0 G_1^{\text{ret}}(x_1 - x_1') G_2^{\text{ret}}(x_2 - x_2') \tag{2.36}$$

$$\times\ D_2(K\psi)(x_1', x_2')|_{x_1^{0\prime}=0}$$

$$+ \int_{\frac{1}{2}\mathbb{M}} d^4x_1' \int_{\mathbb{R}^3} d^3\mathbf{x}_2' G_1^{\text{ret}}(x_1 - x_1') i\gamma_2^0 G_2^{\text{ret}}(x_2 - x_2') \tag{2.37}$$

$$\times\ D_1(K\psi)(x_1', x_2')|_{x_2^{0\prime}=0}\,. \tag{2.38}$$

Now, G_k^{ret} still contains the δ-distribution. We use the latter to cancel the integrals over $x_k^{0\prime}$, $k=1,2$ in (2.34) in the following manner.

$$\begin{aligned} &\frac{1}{4\pi}\int_{\frac{1}{2}\mathbb{M}} d^4x' \, \frac{\delta(x^0 - x^{0\prime} - |\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|} f(x') \qquad (2.39) \\ =\ &\frac{1}{4\pi}\int_{B_{x^0}(\mathbf{x})} d^3\mathbf{x}' \, \frac{1}{|\mathbf{x}-\mathbf{x}'|} f(x')|_{x^{0\prime}=x^0-|\mathbf{x}-\mathbf{x}'|} \\ =\ &\frac{1}{4\pi}\int_{B_{x^0}(0)} d^3\mathbf{y} \, \frac{1}{|\mathbf{y}|} f(x+y)|_{y^0=-|\mathbf{y}|}. \qquad (2.40) \end{aligned}$$

Moreover,

$$\begin{aligned} &\frac{m}{4\pi}\int_{\frac{1}{2}\mathbb{M}} d^4x' \; H(x^0 - x^{0\prime} - |\mathbf{x}-\mathbf{x}'|) \frac{J_1(m\sqrt{(x-x')^2})}{\sqrt{(x-x')^2}} f(x') \\ =\ &\frac{m}{4\pi}\int_{[-x^0,\infty)\times\mathbb{R}^3} d^4y \; H(-y^0 - |\mathbf{y}|) \frac{J_1(m\sqrt{y^2})}{\sqrt{y^2}} f(x+y) \\ =\ &\frac{m}{4\pi}\int_{-x^0}^{0} dy^0 \int_{B_{|y^0|}(0)} d^3\mathbf{y}_k \, \frac{J_1(m\sqrt{y^2})}{\sqrt{y^2}} f(x+y). \qquad (2.41) \end{aligned}$$

For the boundary terms, we similarly use

$$\begin{aligned} &\frac{i\gamma^0}{4\pi}\int_{\mathbb{R}^3} d^3\mathbf{x}' \, \frac{\delta(x^0 - |\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|} f(0,\mathbf{x}') \qquad (2.42) \\ &\qquad = \frac{i\gamma^0}{4\pi}\int_{\partial B_{x^0}(0)} d\sigma(\mathbf{y}) \; \frac{f(0,\mathbf{x}+\mathbf{y})}{x^0} \qquad (2.43) \end{aligned}$$

as well as

$$\begin{aligned} &i\gamma^0 \frac{m}{4\pi}\int_{\mathbb{R}^3} d^3\mathbf{x}' \; H(x^0 - x^{0\prime} - |\mathbf{x}-\mathbf{x}'|) \frac{J_1(m\sqrt{(x-x')^2})}{\sqrt{(x-x')^2}} f(x')|_{x^{0\prime}=0} \\ &= i\gamma^0 \frac{m}{4\pi}\int_{B_{x^0}(0)} d^3\mathbf{y} \; \frac{J_1(m\sqrt{(x^0)^2-\mathbf{y}^2})}{\sqrt{(x^0)^2-\mathbf{y}^2}} f(0,\mathbf{x}+\mathbf{y}). \qquad (2.44) \end{aligned}$$

This yields the form of the integral equation which shall be the basis of our investigation:

$$\psi(x_1, x_2) = \psi^{\text{free}}(x_1, x_2) + (A\psi)(x_1, x_2). \tag{2.45}$$

The operator A is first defined on test functions $\psi \in \mathcal{D}$ as

$$A\psi = \prod_{j=1,2} \Big(A_j^{(1)}(m) + A_j^{(2)}(m) + A_j^{(3)}(m) + A_j^{(4)}(m) \Big) \tag{2.46}$$

where for $j = 1, 2$, $k = 1, 2, 3, 4$ the operator $A_j^{(k)}(m) : \mathcal{D} \to C^\infty\big((\frac{1}{2}\mathbb{M})^2, \mathbb{C}^{16}\big)$ is defined by letting the respective operator $A^{(k)}(m)$, given below, act on the j-th 4-variable and spin index of $\psi(x_1, x_2)$, $\psi \in \mathcal{D}$. [1]

$$\begin{aligned}
\big(A^{(1)}(m)\,\psi\big)(x) &= \frac{1}{4\pi}\int_{B_{x^0}(0)} d^3\mathbf{y}\ \frac{1}{|\mathbf{y}|}\psi(x+y)|_{y^0=-|\mathbf{y}|}, && (2.47)\\
\big(A^{(2)}(m)\,\psi\big)(x) &= -\frac{m}{4\pi}\int_{-x^0}^{0} dy^0 \int_{B_{|y^0|}(0)} d^3\mathbf{y}\ \frac{J_1(m\sqrt{y^2})}{\sqrt{y^2}}\psi(x+y), && (2.48)\\
\big(A^{(3)}(m)\,\psi\big)(x) &= \frac{i\gamma^0}{4\pi}\int_{\partial B_{x^0}(0)} d\sigma(\mathbf{y})\ \frac{\psi(0,\mathbf{x}+\mathbf{y})}{x^0}, && (2.49)\\
\big(A^{(4)}(m)\,\psi\big)(x) &= -i\gamma^0\frac{m}{4\pi}\int_{B_{x^0}(0)} d^3\mathbf{y}\ \frac{J_1(m\sqrt{(x^0)^2-\mathbf{y}^2})}{\sqrt{(x^0)^2-\mathbf{y}^2}} \\
&\qquad \times \psi(0,\mathbf{x}+\mathbf{y}). && (2.50)
\end{aligned}$$

Here, the dependence of $A_j^{(1)}$ and $A_j^{(3)}$ on m is only for notational convenience.

We now turn to the question of a suitable Banach space for Eq. (2.45).

[1] We deliberately avoid using tensor products here, as the tensor product of Banach spaces is an ambiguous notion.

2.2.2.2 Choice of Banach space

In order to prove the existence and uniqueness of solutions, we would like to demonstrate the convergence of the Neumann series. First of all, this requires to extend the integral operator A to an operator on a suitable Banach space $\mathcal{B}$. The behavior of solutions $\psi^{\rm free}(x_1, x_2)$ of the free Dirac equation in each spacetime variable x_1, x_2 suggests to choose the Bochner space

$$\mathcal{B}_0 = L^\infty \left([0,\infty)^2_{(x_1^0, x_2^0)},\ L^2(\mathbb{R}^6, \mathbb{C}^{16})_{(\mathbf{x}_1, \mathbf{x}_2)} \right) \tag{2.51}$$

with norm

$$\|\psi\|_{\mathcal{B}_0} = \operatorname*{ess\,sup}_{x_1^0, x_2^0 > 0} \|\psi(x_1^0, \cdot, x_2^0, \cdot)\|_{L^2}. \tag{2.52}$$

The reason for choosing $\mathcal{B}_0$ is that the spatial norm $\|\psi^{free}(x_1^0, \cdot, x_2^0, \cdot)\|_{L^2}$ of a solution of the free Dirac equations is constant in the two time variables x_1^0, x_2^0. A very similar space as $\mathcal{B}_0$ has been used for analyzing (2.129) in the KG case [37].
However, as (2.46) involves the Dirac operators D_1, D_2, $\mathcal{B}_0$ is not sufficient for our problem. An appropriate Banach space $\mathcal{B}$ must allow us to take at least weak derivatives of ψ. The choice of $\mathcal{B}$ is a delicate matter. One can easily go wrong with demanding too much regularity, as we shall illustrate now.

2.2.2.2.1 Possible problems with the choice of space. The problem can best be illustrated with an example which is structurally related to (2.129) but otherwise simpler. Consider the equation

$$f(t,x) = f^{\rm free}(t,z) + \int_0^t dz'\, K(z,z') \partial_t f(t,z'), \tag{2.53}$$

where $f^{\rm free}, f, K : \mathbb{R}^2 \to \mathbb{C}$ and $f^{\rm free}$ is given. (2.53) is inspired by the term $A_1 D_1$ in (2.46).

We would like to set up an iteration scheme for (2.53). As we cannot integrate by parts to shift the t-derivative to K, we must demand at least weak differentiability of f with respect to t. This suggests using a Banach space such as $\mathcal{B} = H^1(\mathbb{R}^2)$. To prove that the integral operator in (2.53) maps $\mathcal{B}$ to $\mathcal{B}$ (the first step in every iteration scheme), we then have to estimate the L^2-norm of

$$\partial_t \int_0^t dz'\, K(z,z')\partial_t f(t,z') = K(t,t)(\partial_t f)(t,t) + \int_0^t dz'\, K(z,z')\partial_t^2 f(t,z'). \tag{2.54}$$

This expression, however, contains $\partial_t^2 f$. For this to make sense, we must be allowed to take the second weak time derivative of f. This, in turn, requires to choose a different Sobolev space, such as $H^2(\mathbb{R}^2)$, and to estimate the L^2-norm of the second time derivative of the integral operator acting on f which involves $\partial_t^3 f$, and so on. One is thus led to a Sobolev space where all weak n-th time derivatives have to exist. Such infinite-order Sobolev spaces have, in fact, been investigated in [15]. However, it does not seem realistic to get an iteration to converge on these spaces. We therefore take a different approach.

2.2.2.2.2 A Banach space adapted to our integral equation. Considering the form of the integral operator A (2.46), one can see that it is sufficient that the derivatives $D_1\psi$, $D_2\psi$ and $D_1 D_2\psi$ exist in a weak sense. As we want to prove later that A maps the Banach space to itself, we have to estimate, among other things, a suitable norm of $D_1(A\psi)$. If $\psi \in \mathcal{D}$ is a test function and K is smooth, we have

$$\begin{aligned} D_1(A\psi)(x_1,x_2) &= D_1 \int d^4x_1'\, d^4x_2'\, S_1(x_1 - x_2')S_2(x_2 - x_2') \\ &\qquad \times K(x_1',x_2')\psi(x_1',x_2') \\ &= \int d^4x_2'\, S_2(x_2 - x_2')K(x_1,x_2')\psi(x_1,x_2') \end{aligned} \tag{2.55}$$

where we have used $D_1 S_1(x_1 - x_1') = \delta^{(4)}(x_1 - x_1')$. The crucial point now is that (2.55) does not contain higher-order derivatives such as $D_1^2\psi$. The same holds true also for $D_2(A\psi)$ and $D_1 D_2(A\psi)$. Thus, the problem of the toy example (2.53) is avoided.
Together with the previous considerations about $\mathcal{B}_0$ (2.51), we are led to define the Banach space $\mathcal{B}_g$ as the completion of $\mathcal{D}$ with respect to the following Sobolev-type norm:

$$\|\psi\|_g^2 = \operatorname*{ess\,sup}_{x_1^0, x_2^0 > 0} \frac{1}{g(x_1^0) g(x_2^0)} [\psi]^2(x_1^0, x_2^0) \tag{2.56}$$

where $g : [0, \infty) \to (0, \infty)$ is a monotonically increasing function which is such that the function $1/g$ is bounded. We admit such a weight factor with hindsight. As we shall see, a suitable choice of g will make a contraction mapping argument possible.
In (2.56) we use the notation

$$[\psi]^2(x_1^0, x_2^0) = \sum_{k=0}^{3} \|(\mathcal{D}_k \psi)(x_1^0, \cdot, x_2^0, \cdot)\|^2_{L^2(\mathbb{R}^6, \mathbb{C}^{16})} \tag{2.57}$$

with

$$\mathcal{D}_k = \begin{cases} 1, & k = 0 \\ D_1, & k = 1 \\ D_2, & k = 2 \\ D_1 D_2, & k = 3 \end{cases} \tag{2.58}$$

Remark 5. *One can see the purpose of integral equation (2.129) in determining an interacting correction to a solution ψ^{free} of the free multi-time Dirac equations $D_i \psi^{\text{free}} = 0$, $i = 1, 2$. Therefore, it is important to check that sufficiently many solutions of these free equations lie in $\mathcal{B}_g$. This is ensured by the following Lemma (see Sec. 2.2.4.1 for a proof).*

Lemma 6. *Let ψ^{free} be a solution of the free multi-time Dirac equations $D_i\psi^{\text{free}} = 0,\ i = 1,2$ with initial data $\psi^{\text{free}}(0,\cdot,0,\cdot) = \psi_0 \in C_c^\infty(\mathbb{R}^6,\mathbb{C}^{16})$. Furthermore, let $g : [0,\infty) \to (0,\infty)$ be a monotonically increasing function with $g(t) \to \infty$ for $t \to \infty$ and $g(0) = 1$. Then ψ^{free} lies in $\mathcal{B}_g$.*

Given the definition of A on $\mathcal{D}$ as in Sec. 2.2.2.1, we shall now proceed with showing that A is bounded on this space. Furthermore, we show that for a suitable choice of the weight factor g in $\mathcal{B}_g$, we can achieve $\|A\| < 1$ on $\mathcal{D}$. This allows to extend A to a contraction on $\mathcal{B}_g$ so that the Neumann series $\psi = \sum_{k=0}^{\infty} A^k\psi^{\text{free}}$ yields the unique solution of $\psi = \psi^{\text{free}} + A\psi$.

2.2.3 Results

2.2.3.1 Results for a Minkowski half space

The core of our results is the following Lemma which allows us to control the growth of the spatial norm of ψ with the two time variables.

Lemma 7. *Let $\psi \in \mathcal{D}$, $\partial\!\!\!/_k = \gamma_k^\mu \partial_{k,\mu},\ k = 1,2$ and let $K \in C^2(\mathbb{R}^8,\mathbb{C})$ with*

$$\|K\| := \sup_{x_1,x_2 \in \frac{1}{2}\mathbb{M}} \max\Big\{|K(x_1,x_2)|, |\partial\!\!\!/_1 K(x_1,x_2)|, \tag{2.59}$$

$$|\partial\!\!\!/_2 K(x_1,x_2)|, |\partial\!\!\!/_1\partial\!\!\!/_2 K(x_1,x_2)|\Big\} < \infty. \tag{2.60}$$

Then we have:

$$[A\psi]^2(x_1^0,x_2^0) \ \leqslant\ \|K\|^2 \prod_{j=1,2} \big(\mathbb{1} + 8\mathcal{A}_j(m_j)\big)\,[\psi]^2(x_1^0,x_2^0), \tag{2.61}$$

where $\mathcal{A}_j(m) = \sum_{k=1}^4 \mathcal{A}_j^{(k)}(m)$ with $\mathcal{A}_j^{(k)}$ as defined in (2.91). The expression $[\psi]^2(x_1^0,x_2^0)$ is understood as a function in $C^\infty\big((\frac{1}{2}\mathbb{M})^2,\mathbb{R}_0^+\big)$ to which the operators in front of it are applied.

The proof can be found in Sec. 2.2.4.2.
Lemma 7 can now be used to identify (with some trial and error) a suitable weight factor g which allows us to extend A to a contraction on $\mathcal{B}_g$. Our main result is:

Theorem 8 (Existence and uniqueness of dynamics on a Minkowski half space.)**.** *Let* $0 < \|K\| < 1$, $\mu = \max\{m_1, m_2\}$ *and*

$$g(t) = \sqrt{1 + bt^8}\, \exp(bt^8/16), \tag{2.62}$$

$$b = \frac{\|K\|^4}{(1 - \|K\|)^4} \left(6 + \mu^4\right)^4. \tag{2.63}$$

Then for every $\psi^{\text{free}} \in \mathcal{B}_g$*, the equation* $\psi = \psi^{\text{free}} + A\psi$ *possesses a unique solution* $\psi \in \mathcal{B}_g$*.*

The proof is given in Sec. 2.2.4.3.

Remark 9. 1. *Note that Thm. 8 establishes the existence and uniqueness of a global-in-time solution. The non-Markovian nature of the dynamics makes it necessary to prove such a result directly instead of concatenating short-time solutions. The key step in our proof which makes the global-in-time result possible is the suitable choice of the weight factor* g*.*

2. *The main condition in Thm. 8 is* $\|K\| < 1$*. This means that the interaction must not be too strong (in a suitable sense). A condition of that kind is to be expected solely because of the contribution* $\|(D_1 D_2(A\psi))(x_1^0, \cdot, x_2^0, \cdot)\|_{L^2} = \|K\psi(x_1^0, \cdot, x_2^0, \cdot)\|_{L^2}$ *to* $[A\psi](x_1^0, x_2^0)$*. Taking our strategy for setting up the Banach space for granted, we therefore think that one cannot avoid a condition on the interaction strength. Note that conditions on the interaction strength also occur at other places in quantum theory (albeit in a different sense). For example, the Dirac Hamiltonian plus*

a Coulomb potential is only self-adjoint if the prefactor of the latter is smaller than a certain value.

3. *Cauchy problem. Thm. 8 shows that ψ^{free} uniquely determines the solution ψ. However, specifying a whole function in $\mathcal{B}_g$ amounts to a lot of data. In case ψ^{free} is a solution of the free multi-time Dirac equations $D_1\psi^{\text{free}} = 0 = D_2\psi^{\text{free}}$ much less data are needed. ψ^{free} is then determined uniquely by Cauchy data, and hence ψ is as well. Furthermore, if ψ^{free} is differentiable,* (2.129) *yields*

$$\psi(0, \mathbf{x}_1, 0, \mathbf{x}_2) \;=\; \psi^{\text{free}}(0, \mathbf{x}_1, 0, \mathbf{x}_2). \tag{2.64}$$

Thus, Cauchy data for ψ^{free} at $x_1^0 = x_2^0 = 0$ are also Cauchy data for ψ. The procedure works for arbitrary Cauchy data which are appropriate for the free multi-time Dirac equations. Note, however, that a Cauchy problem for ψ for times $x_1^0 = t_0 = x_2^0$ with $t_0 > 0$ is not possible. The reason is that $\psi(t_0, \mathbf{x}_1, t_0, \mathbf{x}_2) \neq \psi^{\text{free}}(t_0, \mathbf{x}_1, t_0, \mathbf{x}_2)$ in general (and contrary to (2.64)*) the pointwise evaluation may not make sense for ψ).*

2.2.3.2 Results for a FLRW universe with a Big Bang singularity

In this section we show that a Big Bang singularity provides a natural and covariant justification for the cutoff at $t = 0$. As this justification is our main goal, we make the point at the example of a particular class of Friedman-Lemaître-Robertson-Walker (FLRW) spacetimes and do not strive to treat more general spacetimes here. The reason for studying these FLRW spacetimes is that they are conformally equivalent to $\frac{1}{2}\mathbb{M}$ [22]. Together with the conformal invariance of the massless Dirac operator this allows for an efficient method of calculating the Green's functions which occur in the curved spacetime analog of the integral equation (2.129). By doing this, we show that the existence and uniqueness result on these spacetimes can be reduced to Thm. 8. As shown in [35], Eq. (2.129) possesses a natural generalization to curved spacetimes $\mathcal{M}$,

$$\psi(x_1, x_2) = \psi^{\text{free}}(x_1, x_2) + \int dV(x_1') \int dV(x_2')\, G_1(x_1, x_1') G_2(x_2, x_2') \\ \times K(x_1', x_2') \psi(x_1', x_2'). \tag{2.65}$$

Here, $dV(x)$ is the spacetime volume element, S_i are (retarded) Green's functions of the respective free wave equation, i.e.

$$DG(x, x') = [-g(x)]^{-1/2}\, \delta^{(4)}(x, x'), \tag{2.66}$$

where $g(x)$ is the metric determinant, D the covariant Dirac operator on $\mathcal{M}$, and ψ a section of the tensor spinor bundle over $\mathcal{M} \times \mathcal{M}$.
In order to explicitly formulate (2.181), we need to know the detailed form of S^{ret}. Note that results for general classes of spacetimes showing that S^{ret} is a bounded operator on a suitable function space are not sufficient to obtain a strong (global in time) existence and uniqueness result. We therefore focus on the case of a flat FLRW universe where it is easy to determine the Green's functions explicitly. In that case, the metric is given by

$$ds^2 = a^2(\eta)[d\eta^2 - d\mathbf{x}^2] \tag{2.67}$$

where η is cosmological time and $a(\eta)$ denotes the so-called *scale factor*. The coordinate ranges are given by $\eta \in [0, \infty)$ and $\mathbf{x} \in \mathbb{R}^3$. For a FLRW universe with a Big Bang singularity, $a(\eta)$ is a continuous, monotonically increasing function of η with $a(\eta) = 0$, corresponding to the Big Bang singularity. The spacetime volume element reads

$$dV(x) = a^4(\eta)\, d\eta\, d^3\mathbf{x}. \tag{2.68}$$

The crucial point now is that according to (2.67) the spacetime is globally conformally equivalent to $\frac{1}{2}\mathbb{M}$, with conformal factor

$$\Omega(x) = a(\eta). \tag{2.69}$$

In addition, for $m = 0$, the Dirac equation is known to be conformally invariant (see e.g. [43]). More accurately, consider two spacetimes $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ with metrics

$$\widetilde{g}_{ab} = \Omega^2 \, g_{ab}. \tag{2.70}$$

Then the massless Dirac operator D on $\mathcal{M}$ is related to the massless Dirac operator $\widetilde{D}$ on $\widetilde{\mathcal{M}}$ by (see [1]):

$$\widetilde{D} = \Omega^{-5/2} \, D \, \Omega^{3/2}. \tag{2.71}$$

This implies the following transformation behavior of the Green's functions:

$$\widetilde{G}(x,x') = \Omega^{-3/2}(x) \, \Omega^{-3/2}(x') \, G(x,x'). \tag{2.72}$$

One can verify this easily using (2.71) and the definition of Green's functions on curved spacetimes (2.66).
Denoting the Green's functions of the Dirac operator on Minkowski spacetime by $G(x,x') = S(x-x')$ and using coordinates $\eta, \mathbf{x}$ we thus obtain the Green's functions $\widetilde{G}$ on flat FLRW spacetimes as:

$$\widetilde{G}(\eta,\mathbf{x};\eta',\mathbf{x}') = a^{-3/2}(\eta) a^{-3/2}(\eta') \, S(\eta-\eta',\mathbf{x}-\mathbf{x}'). \tag{2.73}$$

With this result, we can write out in detail the multi-time integral equation (2.181) for massless Dirac particles on flat FLRW spacetimes (using retarded Green's functions):

$$\begin{aligned}\psi(\eta_1,\mathbf{x}_1,\eta_2,\mathbf{x}_2) =& \psi^{\text{free}}(\eta_1,\mathbf{x}_1,\eta_2,\mathbf{x}_2) + a^{-3/2}(\eta_1)a^{-3/2}(\eta_2)\\ &\times \int_0^\infty d\eta_1' \int d^3\mathbf{x}_1' \int_0^\infty d\eta_2' \int d^3\mathbf{x}_2'\\ &\times a^{5/2}(\eta_1')a^{5/2}(\eta_2')\, S_1^{\text{ret}}(\eta_1-\eta_1',\mathbf{x}_1-\mathbf{x}_1') \qquad (2.74)\\ &\times S_2^{\text{ret}}(\eta_2-\eta_2',\mathbf{x}_2-\mathbf{x}_2')\,(K\psi)(\eta_1',\mathbf{x}_1',\eta_2',\mathbf{x}_2'). \qquad (2.75)\end{aligned}$$

Note that we can regard ψ as a map $\psi : (\frac{1}{2}\mathbb{M})^2 \to \mathbb{C}^{16}$ as the coordinates $\eta, \mathbf{x}$ cover the flat FLRW spacetime manifold globally.
It seems reasonable to allow for a singularity of the interaction kernel, i.e.

$$K(\eta_1,\mathbf{x}_1,\eta_2,\mathbf{x}_2) = a^{-\alpha}(\eta_1)a^{-\alpha}(\eta_1)\,\widetilde{K}(\eta_1,\mathbf{x}_1,\eta_2,\mathbf{x}_2). \qquad (2.76)$$

Here, $\alpha \geqslant 0$. The singular behavior is motivated by that of the Green's functions of the conformal wave equation [2]. Recall from the introduction that the most natural interaction kernel on $\frac{1}{2}\mathbb{M}$ would be $K(x_1,x_2) \propto \delta((x_1-x_2)_\mu(x_1-x_2)^\mu)$ which is a Green's function of the wave equation – a concept that can be generalized to curved spacetimes using the conformal wave equation. Now, under conformal transformations, Green's functions of that equation transform as [24]

$$\widetilde{G}(x,x') = \Omega^{-1}(x)\,\Omega^{-1}(x')\,G(x,x'), \qquad (2.77)$$

which corresponds to $\alpha = 1$ in (2.76).

[2] The conformal wave equation reads $(\Box - R/6)\phi = 0$ where $\Box$ is the d'Alembertian and R the Ricci scalar of the respective spacetime.

Considering (2.76), our integral equation becomes:

$$\begin{aligned}\psi(\eta_1,\mathbf{x}_1,\eta_2,\mathbf{x}_2) = \psi^{\text{free}}(\eta_1,\mathbf{x}_1,\eta_2,\mathbf{x}_2) + a^{-3/2}(\eta_1)a^{-3/2}(\eta_2)\int_0^\infty d\eta_1'\\ \times \int d^3\mathbf{x}_1' \int_0^\infty d\eta_2' \int d^3\mathbf{x}_2'\\ \times a^{5/2-\alpha}(\eta_1')a^{5/2-\alpha}(\eta_2')\, S_1^{\text{ret}}(\eta_1-\eta_1',\mathbf{x}_1-\mathbf{x}_1')\\ \times S_2^{\text{ret}}(\eta_2-\eta_2',\mathbf{x}_2-\mathbf{x}_2')\,(\widetilde{K}\psi)(\eta_1',\mathbf{x}_1',\eta_2',\mathbf{x}_2'). \quad (2.78)\end{aligned}$$

Apart from the scale factors which produce a certain singularity of ψ for $\eta_1,\eta_2 \to 0$, this integral equation has the form of (2.129) on $\frac{1}{2}\mathbb{M}$. Indeed, we can use the transformation

$$\chi(\eta_1,\mathbf{x}_1,\eta_2,\mathbf{x}_2) = a^{3/2}(\eta_1)a^{3/2}(\eta_2)\,\psi(\eta_1,\mathbf{x}_1,\eta_2,\mathbf{x}_2) \quad (2.79)$$

to transform the two equations into each other. We arrive at the following result.

Theorem 10 (Existence and uniqueness of dynamics on a flat FLRW universe)**.** *Let, $0 \leqslant \alpha \leqslant 1$ and let $a : [0,\infty) \to [0,\infty)$ be a differentiable function with $a(0) = 0$ and $a(\eta) > 0$ for $\eta > 0$. Moreover, assume that $\widetilde{K} \in C^2\left(([0,\infty)\times\mathbb{R}^3)^2,\mathbb{C}\right)$ with*

$$\|a^{1-\alpha}(\eta_1)a^{1-\alpha}(\eta_2)\,\widetilde{K}\| < 1. \quad (2.80)$$

Then for every ψ^{free} with $a^{3/2}(\eta_1)a^{3/2}(\eta_2)\psi^{\text{free}} \in \mathcal{B}_g$, (2.183) has a unique solution ψ with $a^{3/2}(\eta_1)a^{3/2}(\eta_2)\psi \in \mathcal{B}_g$ (and with g as in Thm. 8).

Proof. Multiplying (2.183) with $a^{3/2}(\eta_1)a^{3/2}(\eta_2)$ and using the relation yields

$$\begin{aligned}\chi(\eta_1,\mathbf{x}_1,\eta_2,\mathbf{x}_2) =& \chi^{\text{free}}(\eta_1,\mathbf{x}_1,\eta_2,\mathbf{x}_2) + \int_0^\infty d\eta_1' \int d^3\mathbf{x}_1 \int_0^\infty d\eta_2' \\ &\times\; a^{1-\alpha}(\eta_1')a^{1-\alpha}(\eta_2') \\ &\times\; S_1^{\text{ret}}(\eta_1-\eta_1',\mathbf{x}_1-\mathbf{x}_1')S_2^{\text{ret}}(\eta_2-\eta_2',\mathbf{x}_2-\mathbf{x}_2') \\ &\times (\widetilde{K}\chi)(\eta_1',\mathbf{x}_1',\eta_2',\mathbf{x}_2'). \end{aligned} \tag{2.81}$$

This equation has the form of (2.129) on $\frac{1}{2}\mathbb{M}$ with K replaced by $a^{1-\alpha}(\eta_1')a^{1-\alpha}(\eta_2')\widetilde{K}$. Thus, using the same distributional understanding of the Green's functions as before, Thm. 8 yields the claim. □ □

Remark 11. *1. Both ψ^{free} and ψ have a singularity proportional to $a^{-3/2}(\eta_1)a^{-3/2}(\eta_2)$ for $\eta_1,\eta_2 \to 0$.*

2. For $\alpha < 1$, $\widetilde{K}$ has to compensate the singularities caused by $a^{-3/2}(\eta_1)a^{-3/2}(\eta_2)$ in order for (2.80) to hold. In the most natural case $\alpha = 1$, however, $\widetilde{K}$ only needs to satisfy $\|\widetilde{K}\| < 1$, i.e. the same condition as for K in Thm. 8.

3. Let $\chi^{\text{free}} = a^{3/2}(\eta_1)a^{3/2}(\eta_2)\psi^{\text{free}}$ be differentiable and let χ be the unique solution of (2.81). Then, by (2.81), we have:

$$\chi^{\text{free}}(0,\mathbf{x}_1,0,\mathbf{x}_2) = \chi(0,\mathbf{x}_1,0,\mathbf{x}_2), \tag{2.82}$$

i.e. χ satisfies a Cauchy problem "at the Big Bang".

4. Remarkably, Thm. 10 covers a class of manifestly covariant, interacting integral equations in 1+3 dimensions. Then the interaction kernel $\widetilde{K}$ has to be covariant as well. A class of examples (see also [35]) is given by $\alpha = 1$ and

$$\widetilde{K}(x_1,x_2) = \begin{cases} f(d(x_1,x_2)) & \text{if } x_1,x_2 \text{ are time-like related} \\ 0 & \text{else,} \end{cases} \tag{2.83}$$

where $d(x_1, x_2) = (|\eta_1 - \eta_2| - |\mathbf{x}_1 - \mathbf{x}_2|) \int_0^1 d\tau\, a(\tau\eta_1 + (1-\tau)\eta_2)$ *denotes the time-like distance of the spacetime points* $x_1 = (\eta_1, \mathbf{x}_1)$ *and* $x_2 = (\eta_2, \mathbf{x}_2)$*, and* f *is an arbitrary smooth function which leads to* $\|\widetilde{K}\| < 1$*.*

2.2.4 Proofs

2.2.4.1 Proof of lemma 6

Consider a solution ψ of $D_i\psi^{\text{free}} = 0,\ i = 1, 2$ for compactly supported initial data at $x_1^0 = 0 = x_2^0$. As the Dirac equation has finite propagation speed, ψ^{free} is spatially compactly supported for all times. Without loss of generality we may assume $\|\psi^{free}(t_1, \cdot, t_2, \cdot)\|_{L^2(\mathbb{R}^6)} = 1$ for all times t_1, t_2, so it follows that also $[\psi^{free}](t_1, t_2) = 1$. In the following we will construct a sequence of test functions $(\psi_m)_{m\in\mathbb{N}}$ satisfying $\psi_m \xrightarrow{\|\cdot\|_g,\ m\to\infty} \psi^{free}$. Let $\eta : \mathbb{R} \to \mathbb{R}$ be zero for arguments less than 0 and greater than 1 and in between given by (see also Fig. 2.1)

$$\eta(t) = \exp\left(-\frac{1}{t}\exp\left(\frac{1}{t-1}\right)\right). \tag{2.84}$$

Note that η is smooth and monotonically increasing. Next, we define for every $m \in \mathbb{N}$

$$\psi_m^{free}(t_1, \mathbf{x}_1, t_1, \mathbf{x}_2) \ := \ e^{-\eta(t_1-m)(t_1-m)} e^{-\eta(t_2-1)(t_2-m)} \psi^{free}(t_1, \mathbf{x}_1, t_2, \mathbf{x}_2). \tag{2.85}$$

This function is smooth and decreases rapidly in all variables and thus lies in $\mathcal{D}$. Now we estimate $\|\psi^{free} - \psi_m\|_g$. Pick $m \in \mathbb{N}$. First consider $\|\psi^{free} - \psi_m\|_{L^2(\mathbb{R}^6)}(t_1, t_2)$. This function is identically zero for all $t_1 < n$

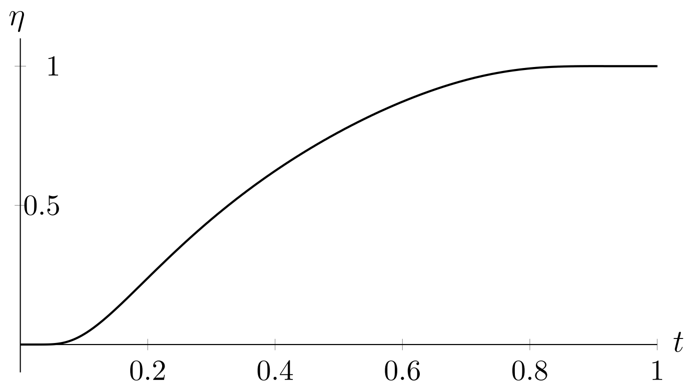

Figure 2.1: The function $\eta(t)$.

and $t_2 < n$, so we obtain the estimate

$$\begin{aligned}
&\sup_{t_1,t_2>0} \frac{1}{g(t_1)^2 g(t_2)^2} \|\psi^{\text{free}} - \psi_n\|^2_{L^2(\mathbb{R}^6)} \\
&= \sup_{t_1,t_2>0} \frac{1}{g(t_1)^2 g(t_2)^2} \left|1 - e^{-\eta(t_1-n)(t_1-n)} e^{-\eta(t_2-n)(t_2-n)}\right| && (2.86) \\
&\leqslant \frac{1}{g(n)^2}. && (2.87)
\end{aligned}$$

For the other terms we use that ψ^{free} solves the free Dirac equation in each variable and that $\sup_{t>0} \partial_t e^{-\eta(t)t} =: \alpha < \infty$ is realized for some positive value of t. So we find for $i \in \{0, 1\}$:

$$\begin{aligned}
&\sup_{t_1,t_2>0} \frac{1}{g(t_1)^2 g(t_2)^2} \|D_i(\psi^{\text{free}} - \psi_n)\|^2_{L^2(\mathbb{R}^6)}(t_1, t_2) \\
&= \sup_{t_1,t_2>0} \frac{1}{g(t_1)^2 g(t_2)^2} && (2.88) \\
&\quad \times \|\gamma_i^0 \psi^{\text{free}}(t_1, \cdot, t_2, \cdot) e^{-\eta(t_{3-i}-n)(t_{3-i}-n)} \partial_{t_i} e^{-\eta(t_i-n)(t_i-n)}\|^2_{L^2(\mathbb{R}^6)} && (2.89)
\end{aligned}$$

$$\leqslant \frac{\alpha}{g(n)^2}. \tag{2.90}$$

For the first inequality it has been used that the factor with a derivative vanishes for $t_i < n$.
An analogous estimate repeated for the $D_1 D_2$-term yields

$$\sup_{t_1,t_2>0} \frac{1}{g(t_1)^2 g(t_2)^2} \|D_1 D_2(\psi^{\text{free}} - \psi_n)\|^2_{L^2(\mathbb{R}^6)}(t_1,t_2) \leqslant \frac{\alpha^2}{g(n)^4} \leqslant \frac{\alpha^2}{g(n)^2}.$$

All in all, adding the estimates and taking the square root we find $\|\psi^{\text{free}} - \psi_n\|_g \leqslant \frac{1+\alpha}{g(n)}$, which together with the asymptotic behavior of g implies convergence. It follows that the free solution ψ^{free} can be approximated by Cauchy sequences in $\mathcal{D}$ and hence is contained in $\mathcal{B}_g$ which, we recall, has been defined as the completion of $\mathcal{D}$ with respect to $\|\cdot\|_g$. □

2.2.4.2 Proof of Lemma 7

Throughout the following subsections, let $\psi \in \mathcal{D}$ and $K : \mathbb{R}^8 \to \mathbb{C}$ be a smooth function. Furthermore define $\delta := 1 - \|K\|^2 > 0, \mu = \max\{m_1, m_2\}$ and let g be as in the statement of Thm. 8.
We begin with some lemmas which are useful for estimating $[A\psi]^2(x_1^0, x_2^0)$.

Lemma 12. *Let the following operators be defined on $C([0,\infty))$:*

$$\begin{aligned}
\big(\mathcal{A}^{(1)}(m)f\big)(t) &= t\int_0^t d\rho\ (t-\rho)^2 f(\rho),\\
\big(\mathcal{A}^{(2)}(m)f\big)(t) &= \frac{m^4 t^4}{2^4\, 3^2}\int_0^t d\rho\ (t-\rho)^3 f(\rho),\\
\big(\mathcal{A}^{(3)}(m)f\big)(t) &= t^2 f(0),\\
\big(\mathcal{A}^{(4)}(m)f\big)(t) &= \frac{m^4 t^6}{2^2\, 3^2} f(0).
\end{aligned} \tag{2.91}$$

Then, for $j = 1, 2$ and $k = 1, 2, 3, 4$, we define the operator $\mathcal{A}_j^{(k)}(m)$ acting on functions $\phi \in C([0, \infty)^2)$ by letting $\mathcal{A}^{(k)}(m)$ act on the j-th variable of $\phi(t_1, t_2)$. Then we have for all $\psi \in \mathcal{D}$, all $m_1, m_2 \geqslant 0$ and all $k, l = 1, 2, 3, 4$:

$$\left\| A_1^{(k)}(m_1) A_2^{(l)}(m_2) \psi(t_1, \cdot, t_2, \cdot) \right\|_{L^2}^2 \leqslant \mathcal{A}_j^{(k)}(m_1) \mathcal{A}_j^{(l)}(m_2) \left\| \psi(t_1, \cdot, t_2, \cdot) \right\|_{L^2}^2. \tag{2.92}$$

Here, it is understood that the operators $\mathcal{A}_j^{(k)}$ are applied to the functions defined by the norms which follow them, e.g.
$\mathcal{A}_1^{(4)}(m_1) \left\| \psi(t_1, \cdot, t_2, \cdot) \right\|_{L^2}^2 = \frac{m_1^4 t_1^6}{2^2\, 3^2} \left\| \psi(0, \cdot, t_2, \cdot) \right\|_{L^2}^2$.

Proof. We prove (2.92) for $k = 1$, $l = 2$ and $k = 3$, $l = 4$. The remaining cases can be treated in the same way. We begin with $k = 1$, $l = 2$, using $|J_1(x)/x| \leqslant \frac{1}{2}$:

$$\begin{aligned}
\| A_1^{(1)}(m_1) A_2^{(2)}(m_2)\, \psi(x_1^0, \cdot, x_2^0, \cdot) \|_{L^2}^2 \; &= \frac{m_2^2}{(4\pi)^4} \int_{\mathbb{R}^3 \times \mathbb{R}^3} d^3\mathbf{x}_1\, d^3\mathbf{x}_2 \\
\times \left| \int_{B_{x_1^0}(0)} d^3\mathbf{y}_1 \int_{-x_2^0}^0 dy_2^0 \int_{B_{|y_2^0|}(0)} d^3\mathbf{y}_2 \; \frac{1}{|\mathbf{y}_1|} \frac{J_1(m_2 \sqrt{y_2^2})}{\sqrt{y_2^2}} \psi(x_1 + y_1, x_2 + y_2)|_{y_1^0 = -|\mathbf{y}_1|} \right|^2 \\
&\leqslant \frac{m_2^2}{(4\pi)^4} \int_{\mathbb{R}^3 \times \mathbb{R}^3} d^3\mathbf{x}_1\, d^3\mathbf{x}_2 \\
&\times \left(\int_{B_{x_1^0}(0)} d^3\mathbf{y}_1 \int_{-x_2^0}^0 dy_2^0 \int_{B_{|y_2^0|}(0)} d^3\mathbf{y}_2 \; \frac{1}{|\mathbf{y}_1|^2} \left| \frac{J_1(m_2 \sqrt{y_2^2})}{\sqrt{y_2^2}} \right|^2 \right) \\
&\times \left(\int_{B_{x_1^0}(0)} d^3\mathbf{y}_1 \int_{-x_2^0}^0 dy_2^0 \int_{B_{|y_2^0|}(0)} d^3\mathbf{y}_2 \; |\psi|^2(x_1 + y_1, x_2 + y_2)|_{y_1^0 = -|\mathbf{y}_1|} \right) \\
&\leqslant \frac{m_2^2}{(4\pi)^4} \int_{\mathbb{R}^3 \times \mathbb{R}^3} d^3\mathbf{x}_1\, d^3\mathbf{x}_2 \; 4\pi x_1^0 \left(\frac{\pi m_2^2 (x_2^0)^4}{12} \right)
\end{aligned}$$

$$
\begin{aligned}
&\times \left(\int_{B_{x_1^0}(0)} d^3\mathbf{y}_1 \int_{-x_2^0}^{0} dy_2^0 \int_{B_{|y_2^0|}(0)} d^3\mathbf{y}_2 \; |\psi|^2(x_1+y_1, x_2+y_2)|_{y_1^0=-|\mathbf{y}_1|} \right) \\
&\leqslant \frac{m_2^4\, x_1^0\, (x_2^0)^4}{3\pi^2\, 2^8} \int_{\mathbb{R}^3\times\mathbb{R}^3} d^3\mathbf{x}_1\, d^3\mathbf{x}_2 \int_{B_{x_1^0}(0)} d^3\mathbf{y}_1 \int_{-x_2^0}^{0} dy_2^0 \int_{B_{|y_2^0|}(0)} d^3\mathbf{y}_2 \\
&\times |\psi|^2(x_1^0 - |\mathbf{y}_1|, \mathbf{x}_1+\mathbf{y}_1, x_2^0+y_2^0, \mathbf{x}_2+\mathbf{y}_2).
\end{aligned}
$$

Exchanging the x and y integrals yields:

$$
\begin{aligned}
(2.93) &\leqslant \frac{m_2^4\, x_1^0\, (x_2^0)^4}{3\pi^2\, 2^8} \int_{B_{x_1^0}(0)} d^3\mathbf{y}_1 \int_{-x_2^0}^{0} dy_2^0 \int_{B_{|y_2^0|}(0)} d^3\mathbf{y}_2 \\
&\qquad\qquad \times \|\psi(x_1^0 - |\mathbf{y}_1|, \cdot, x_2^0 + y_2^0, \cdot)\|_{L^2} \\
&\leqslant \frac{m_2^4\, x_1^0\, (x_2^0)^4}{3\pi^2\, 2^8}\, 4\pi \int_0^{x_1^0} dr_1\; r_1^2 \int_{-x_2^0}^{0} dy_2^0\; \frac{4\pi}{3} |y_2^0|^3 \\
&\qquad\qquad \times \|\psi(x_1^0 - |\mathbf{y}_1|, \cdot, x_2^0 + y_2^0, \cdot)\|_{L^2} \\
&\leqslant \frac{m_2^4\, x_1^0\, (x_2^0)^4}{2^4\, 3^2} \int_0^{x_1^0} d\rho_1\; (x_1^0 - \rho_1)^2 \int_0^{x_2^0} d\rho_2\; (x_2^0 - \rho_2)^3\, \|\psi(\rho_1, \cdot, \rho_2, \cdot)\|_{L^2} \\
&= \mathcal{A}_1^{(1)}(m_1)\mathcal{A}_2^{(2)}(m_2)\, \|\psi(x_1^0, \cdot, x_2^0, \cdot)\|_{L^2}^2. \qquad (2.93)
\end{aligned}
$$

Next, we turn to the case $k = 3, l = 4$. Using that the modulus of the largest eigenvalue of γ^0 is 1, we obtain:

$$
\begin{aligned}
&\|A_1^{(3)}(m_1)A_2^{(4)}(m_2)\, \psi(x_1^0, \cdot, x_2^0, \cdot)\|_{L^2}^2 \;\leqslant\; \frac{m_2^2}{(4\pi)^4 (x_1^0)^2} \int_{\mathbb{R}^3\times\mathbb{R}^3} d^3\mathbf{x}_1\, d^3\mathbf{x}_1 \\
&\times \left| \int_{\partial B_{x_1^0}(0)} d\sigma(\mathbf{y}_1) \int_{B_{x_2^0}(0)} d^3\mathbf{y}_2\; \frac{J_1\left(m_2\sqrt{(x_2^0)^2 - \mathbf{y}_2^2}\right)}{\sqrt{(x_2^0)^2 - \mathbf{y}_2^2}} |\psi|(0, \mathbf{x}_1+\mathbf{y}_2, 0, \mathbf{x}_2+\mathbf{y}_2) \right|^2 \\
&\leqslant \frac{m_2^4}{(4\pi)^4 (x_1^0)^2} \int_{\mathbb{R}^3\times\mathbb{R}^3} d^3\mathbf{x}_1\, d^3\mathbf{x}_1
\end{aligned}
$$

$$
\begin{aligned}
&\times\left(\int_{\partial B_{x_1^0}(0)} d\sigma(\mathbf{y}_1)\int_{B_{x_2^0}(0)} d^3\mathbf{y}_2\left|\frac{J_1\left(m_2\sqrt{(x_2^0)^2-\mathbf{y}_2^2}\right)}{m_2\sqrt{(x_2^0)^2-\mathbf{y}_2^2}}\right|^2\right)\\
&\times\left(\int_{\partial B_{x_1^0}(0)} d\sigma(\mathbf{y}_1)\int_{\partial B_{x_2^0}(0)} d\sigma(\mathbf{y}_2)\ |\psi|^2(0,\mathbf{x}_1+\mathbf{y}_2,0,\mathbf{x}_2+\mathbf{y}_2)\right)\\
&= \frac{m_2^4}{(4\pi)^4(x_1^0)^2}\,4\pi(x_1^0)^2\,\frac{\pi(x_2^0)^3}{3}\int_{\mathbb{R}^3\times\mathbb{R}^3} d^3\mathbf{x}_1\,d^3\mathbf{x}_1\int_{\partial B_{x_1^0}(0)} d\sigma(\mathbf{y}_1)\\
&\qquad\times\int_{B_{x_2^0}(0)} d^3\mathbf{y}_2\ |\psi|^2(0,\mathbf{x}_1+\mathbf{y}_2,0,\mathbf{x}_2+\mathbf{y}_2). \qquad (2.94)
\end{aligned}
$$

Exchanging the order of the x and y integrals yields:

$$
\begin{aligned}
(2.94) &= \frac{m_2^4}{(4\pi)^4}\pi(x_2^0)^2\int_{\partial B_{x_1^0}(0)} d\sigma(\mathbf{y}_1)\int_{B_{x_2^0}(0)} d^3\mathbf{y}_2\ \|\psi(0,\cdot,0,\cdot)\|_{L^2}^2\\
&= \frac{m_2^4\,(x_1^0)^2\,(x_2^0)^6}{2^2\,3^2}\,\|\psi(0,\cdot,0,\cdot)\|_{L^2}^2\\
&= \mathcal{A}_1^{(3)}(m_1)\mathcal{A}_2^{(4)}(m_2)\,\|\psi(x_1^0,\cdot,x_2^0,\cdot)\|_{L^2}^2. \qquad (2.95)
\end{aligned}
$$

□

Lemma 13. *For $j=1,2$ let $\mathcal{A}_j(m)=\sum_{k=1}^4\mathcal{A}_j^{(k)}(m)$. Then the following estimates hold:*

$$
\begin{aligned}
\|(A\psi)(x_1^0,\cdot,x_2^0,\cdot)\|_{L^2}^2 &\leqslant 64\,\|K\|^2\mathcal{A}_1(m_1)\mathcal{A}_2(m_2)\,[\psi]^2(x_1^0,x_2^0), && (2.96)\\
\|(D_1(A\psi))(x_1^0,\cdot,x_2^0,\cdot)\|_{L^2}^2 &\leqslant 8\,\|K\|^2\,\mathcal{A}_2(m_2)\,[\psi](x_1^0,x_2^0), && (2.97)\\
\|(D_2(A\psi))(x_1^0,\cdot,x_2^0,\cdot)\|_{L^2}^2 &\leqslant 8\,\|K\|^2\,\mathcal{A}_1(m_1)\,[\psi](x_1^0,x_2^0), && (2.98)\\
\|(D_1D_2(A\psi))(x_1^0,\cdot,x_2^0,\cdot)\|_{L^2}^2 &\leqslant \|K\|^2\,[\psi]^2(x_1^0,x_2^0), && (2.99)
\end{aligned}
$$

where $[\psi]^2(x_1^0,x_2^0)$ is regarded as a function of x_1^0,x_2^0 to which the operators in front of it are applied.

Proof. We start with (2.96). Recalling (2.46), the expression $A\psi$ contains terms such as $D_1 D_2(K\psi)$ and $D_i(K\psi)$, $i = 1, 2$. Recalling also the definition of $\mathcal{D}_k$ (Eq. (2.58)), we have:

$$D_1 D_2(K\psi) = \sum_{k=0}^{3} (\nabla_{3-k} K)(\mathcal{D}_k \psi) \tag{2.100}$$

with

$$\nabla_k := \begin{cases} 1, & k = 0 \\ i\partial\!\!\!/_1, & k = 1 \\ i\partial\!\!\!/_2, & k = 2 \\ -\partial\!\!\!/_1 \partial\!\!\!/_2, & k = 3. \end{cases} \tag{2.101}$$

Hence, noting (2.60):

$$|D_1 D_2 \psi| \leqslant \|K\| \sum_{k=0}^{3} |\mathcal{D}_k \psi|. \tag{2.102}$$

Similarly, we find:

$$D_i(K\psi) \leqslant \|K\| \sum_{k=0}^{3} |\mathcal{D}_k \psi|, \quad i = 1, 2. \tag{2.103}$$

Considering the definition of $A_j^{(k)}(m)$, $j = 1, 2$, $k = 1, 2, 3, 4$ it follows that

$$\begin{aligned} |A\psi| \;\leqslant\; \|K\| \sum_{k=0}^{3} \prod_{j=1,2} \big[A_j(m_j)^{(1)} + A_j^{(2)}(m_j) + A_j^{(3)}(m_j) \\ + A_j^{(4)}(m_j) \big] |\mathcal{D}_k \psi|. \end{aligned} \tag{2.104}$$

In slight abuse of notation, we here use the same symbols for the operators $A_j^{(k)}(m)$ acting on functions with and without spin components.

The idea now is to make use of lemma 12. In order to be able to apply the lemma, we first note that by Young's inequality for $a_1, ..., a_N \in \mathbb{R}$, we have $\left(\sum_{i=1}^N a_i\right)^2 \leqslant N \sum_{i=1}^N a_i^2$ and thus:

$$|A\psi(x_1, x_2)|^2 \;\leqslant\; 64\,\|K\|^2 \sum_{i,j=1}^{4} \sum_{k=0}^{3} \left|A_1^{(i)}(m_1) A_2^{(j)}(m_2)\,|\mathcal{D}_k \psi|\right|^2. \quad (2.105)$$

Integrating over this expression and using lemma 12, we obtain:

$$\begin{aligned} \|(A\psi)(x_1^0, \cdot, x_2^0, \cdot)\|_{L^2}^2 \;\leqslant\; 64\,\|K\|^2 \sum_{i,j=1}^{4} \sum_{k=0}^{3} \mathcal{A}_1^{(i)}(m_1) \mathcal{A}_2^{(j)}(m_2) \\ \times \|(\mathcal{D}_k \psi)(x_1^0, \cdot, x_2^0, \cdot)\|_{L^2}^2. \end{aligned} \quad (2.106)$$

Recalling the definition of $[\psi]^2(x_1^0, x_2^0)$, Eq. (2.57) yields (2.96).
Next, we turn to (2.97). We start from the initial form of the integral equation (2.129) and use that as a distributional identity on test functions $\psi \in \mathcal{D}_T$, we have $D_1 S^{\mathrm{ret}}(x_1 - x_1') = \delta^{(4)}(x_1 - x_1')$. Thus, we obtain:

$$(D_1 A\psi)(x_1, x_2) = \int_{\frac{1}{2}\mathbb{M}} d^4x_2' \; S_2^{\mathrm{ret}}(x_2 - x_2')(K\psi)(x_1, x_2'). \quad (2.107)$$

Proceeding similarly as for (2.46) we rewrite this as:

$$D_1(A\psi) = \Big(A_2^{(1)}(m_2)\, D_2 + A_2^{(2)}(m_2) D_2 + A_2^{(3)}(m_2) + A_2^{(4)}(m_2)\Big)(K\psi). \quad (2.108)$$

Considering the form of $A_j^{(k)}(m_j)$ this implies:

$$|D_1(A\psi)| \;\leqslant\; \|K\| \sum_{i=1}^{4} \sum_{k \in \{0,2\}} A_2^{(i)}(m_2)\, |\mathcal{D}_k \psi|. \quad (2.109)$$

We now square and use Young's inequality, finding:

$$|D_1(A\psi)|^2 \;\leqslant\; 8\,\|K\|^2 \sum_{i=1}^{4} \sum_{k\in\{0,2\}} A_2^{(i)}(m_2)\,|\mathcal{D}_k\psi|^2. \tag{2.110}$$

Integrating and using lemma 12 yields:

$$\begin{aligned}\|D_1(A\psi)(x_1^0,\cdot,x_2^0,\cdot)\|_{L^2}^2 \;\leqslant\; 8\,\|K\|^2 \sum_{i=1}^{4} \sum_{k\in\{0,2\}} \mathcal{A}_2^{(i)}(m_2)&\\ \|(\mathcal{D}_k\psi)(x_1^0,\cdot,x_2^0,\cdot)\|_{L^2}^2.&\end{aligned} \tag{2.111}$$

Adding the terms with $k=1,3$ and using the definition of $[\psi]^2(x_1^0,x_2^0)$ gives us (2.97).
The estimate (2.98) follows in an analogous way.
Finally, for (2.99) we also start from the initial integral equation (2.129) and use $D_i S_i^{\mathrm{ret}}(x_i - x_i') = \delta^{(4)}(x_i - x_i')$. This results in:

$$D_1 D_2(A\psi) = K\psi. \tag{2.112}$$

Squaring and integrating gives us:

$$\begin{aligned}\|D_1D_2(A\psi)(x_1^0,\cdot,x_2^0,\cdot)\|^2 \;&\leqslant\; \|K\|^2\,\|\psi(x_1^0,\cdot,x_2^0,\cdot)\|_{L^2}^2\\ &\leqslant\; \|K\|^2\,[\psi]^2(x_1^0,x_2^0),\end{aligned} \tag{2.113}$$

which yields (2.99). □

These estimates are the core of:

Proof of Lemma 7: We use lemma 13 together with the definition of $[\psi]^2(x_1^0,x_2^0)$ to obtain:

$$[A\psi]^2(x_1^0,x_2^0) \;\leqslant\; (2.96) + (2.97) + (2.98) + (2.99). \tag{2.114}$$

Summarizing the operators into a product yields (2.61).

□

2.2.4.3 Proof of Theorem 8

In order to prove Thm. 8, we combine the previous estimates to show that $\|A\| < 1$, first on test functions $\psi \in \mathcal{D}$ and by linear extension also on the whole of $\mathcal{B}_g$. We start with Eq. (2.61) of Thm. 2.2.3.1 using the definition of $\mathcal{A}_j$ for $j = 1, 2$, as well as the following estimate, valid for all $\psi \in \mathcal{D}, t_1, t_2 > 0$:

$$[\psi](t_1,t_2) \;=\; [\psi](t_1,t_2)\,\frac{g(t_1)g(t_2)}{g(t_1)g(t_2)} \;\leqslant\; \|\psi\|_g\, g(t_1)g(t_2). \tag{2.115}$$

Using this in (2.61) yields:

$$\|A\psi\|_g^2 \;\leqslant\; \sup_{x_1^0,x_2^0>0} \frac{1}{(g(x_1^0)g(x_2^0))^2}\,\|K\|^2 \prod_{j=1,2} (\mathbb{1} + 8\mathcal{A}_j(m_j))\,[\psi]^2(x_1^0,x_2^0), \tag{2.116}$$

$$\leqslant\; \sup_{x_1^0,x_2^0>0} \frac{\|\psi\|^2}{(g(x_1^0)g(x_2^0))^2}\,\|K\|^2 \times \prod_{j=1,2} (\mathbb{1} + 8\mathcal{A}_j(m_j))\;(g^2\otimes g^2)(x_1^0,x_2^0), \tag{2.117}$$

$$\leqslant\; \|K\|^2\,\|\psi\|_g^2 \left(\sup_{t>0} \frac{1}{g(t)^2}\,(\mathbb{1} + 8\mathcal{A}(\mu))\,g^2(t)\right)^2, \tag{2.118}$$

where $\mu = \max\{m_1, m_2\}$ and $\mathcal{A}(\mu) = \sum_{k=1}^4 \mathcal{A}^{(k)}(\mu)$ with $\mathcal{A}^{(k)}(\mu)$ as in (2.91).

Next, we shall estimate the term in the big round bracket. To this end, we first note some special properties of g^2, which motivated choosing g as in (2.62).

Lemma 14. *For all $t > 0$, we have*

$$\int_0^t d\tau\, g^2(\tau) = \frac{t}{1+bt^8}\, g^2(t). \tag{2.119}$$

Proof: Differentiating the right side of the equation and using the concrete function g^2 as in (2.62) shows that it is, indeed, the anti-derivative of g^2. Since this function vanishes at $t = 0$, the claim follows. □ □

Lemma 15. *For $c < 8$ we have*

$$\sup_{t>0} \frac{t^c}{1+bt^8} = \frac{c}{8}\, b^{-c/8} \left(\frac{8}{c} - 1\right)^{-c/8}, \tag{2.120}$$

and furthermore for $c = 8$:

$$\sup_{t>0} \frac{t^8}{1+bt^8} = \frac{1}{b}. \tag{2.121}$$

Proof. To prove (2.120), considering the shape of the function $h(t) = t^c/(1+bt^8)$ we find that the supremum is in fact a maximum which is located at $t = b^{-1/8}\,(8/c-1)^{-1/8}$. Inserting this back into the function $h(t)$ yields (2.120). Equation (2.121) follows from $\frac{t^8}{1+bt^8} = \frac{1}{b}\frac{1}{1/(bt^8)+1} \leqslant \frac{1}{b}$. □ □

2.2.4.3.1 Proof of Thm. 8: Applying Lemma 14 to $\mathcal{A}(\mu)\, g^2$ yields:

$$\begin{aligned}
\big(\mathcal{A}^{(1)}(\mu)\, g^2\big)(t) &= t\int_0^t d\rho\,(t-\rho)^2\, g^2(\rho) \leqslant t^3 \int_0^t d\rho\, g^2(\rho)\\
&= \frac{t^4}{1+bt^8}\, g^2(t),\\
\big(\mathcal{A}^{(2)}(\mu)\, g^2\big)(t) &= \frac{\mu^4\, t^4}{2^4\, 3^2}\int_0^t d\rho\,(t-\rho)^3\, g^2(\rho) \leqslant \frac{\mu^4\, t^8}{2^4\, 3^2}\frac{g^2(t)}{1+bt^8},\\
\big(\mathcal{A}^{(3)}(\mu)\, g^2\big)(t) &= t^2,\\
\big(\mathcal{A}^{(4)}(\mu)\, g^2\big)(t) &= \frac{\mu^4\, t^6}{2^2\, 3^2}.
\end{aligned} \tag{2.122}$$

Multiplying with $1/g^2(t)$ and using Lemma 15 as well as $1/g(t)^2 \leqslant (1+bt^8)^{-1}$, we find:

$$\begin{aligned}
g^{-2}(t)\,\big(\mathcal{A}^{(1)}(\mu)\,g^2\big)(t) &\leqslant \frac{1}{2}\,b^{-\frac{1}{2}}, \\
g^{-2}(t)\,\big(\mathcal{A}^{(2)}(\mu)\,g^2\big)(t) &\leqslant \frac{\mu^4}{2^4\,3^2\,b}, \\
g^{-2}(t)\,\big(\mathcal{A}^{(3)}(\mu)\,g^2\big)(t) &\leqslant \frac{1}{2^2\,(3b)^{1/4}}, \\
g^{-2}(t)\,\big(\mathcal{A}^{(4)}(\mu)\,g^2\big)(t) &\leqslant \frac{\mu^4}{2^4\,3^{1/4}}\,b^{-3/4}. \qquad (2.123)
\end{aligned}$$

Using (2.118), we can employ these inequalities (whose right hand sides are inversely proportional to powers of b) to estimate the norm of A. According to (2.118), we have, first on $\mathcal{D}$ and by linear extension also on the whole of $\mathcal{B}_g$:

$$\|A\| \leqslant \|K\| \sup_{t>0} g^{-2}(t)\big((\mathbb{1}+8\mathcal{A}(\mu))\,g^2\big)(t). \qquad (2.124)$$

Now we use (2.123) for the various contributions $A^{(k)}(\mu)$ to $\mathcal{A}(\mu) = \sum_{k=1}^{4} A^{(k)}(\mu)$, finding:

$$\begin{aligned}
\|A\| &\leqslant \|K\| + \frac{4\|K\|}{b^{1/2}} + \frac{\mu^4\|K\|}{18b} + \frac{2\|K\|}{(3b)^{1/4}} + \frac{\mu^4\|K\|}{2(3b^3)^{1/4}} \\
&\overset{b\geqslant 1}{\leqslant} \|K\| + \frac{\|K\|}{b^{1/4}}\big(4+\mu^4/18+2/3^{1/4}+\mu^4/(2\cdot 3^{1/4})\big) \qquad (2.125) \\
&< \|K\| + \frac{\|K\|}{b^{1/4}}(6+\mu^4). \qquad (2.126)
\end{aligned}$$

Recalling that $b = \frac{\|K\|^4}{(1-\|K\|)^4}(6+\mu^4)^4$ (see (2.63)), we finally obtain that:

$$\|A\| < \|K\| + \frac{\|K\|}{b^{1/4}}(6+\mu^4) = \|K\| + 1 - \|K\| = 1. \qquad (2.127)$$

We have thus shown that A defines (by linear extension) a contraction on $\mathcal{B}_g$. Thus, the Neumann series $\psi = \sum_{k=0}^{\infty} A^k \psi^{\text{free}}$ yields the unique (global-in-time) solution of the equation $\psi = \psi^{\text{free}} + A\psi$. □

2.2.5 Conclusion and outlook

Extending previous work for Klein-Gordon particles [37, 35] to the Dirac case, we have established the existence of dynamics for a class of integral equations which express direct interactions with time delay at the quantum level. To obtain this result, we have assumed a cutoff of the spacetime before $t = 0$. It has been demonstrated that the Big Bang singularity can naturally provide such a cutoff. Remarkably, this yields a class of rigorous interacting models in 1+3 spacetime dimensions.

Compared to the previous works [37, **?**], our techniques have been modified and improved. Instead of demonstrating explicitly the convergence of the Neumann series by iterating the estimate (2.61) – which is lengthy – we have here succeeded in directly showing that A is a contraction on the weighted space $\mathcal{B}_g$ for a suitable g. This also has the advantage that no arbitrary final time T as in [37, **?**] had to be introduced which could only later be taken to infinity by an additional argument (involving a change of Banach space).

The main challenge in our work has been the non-Markovian nature of the dynamics. This has made it necessary to directly prove global existence in time instead of concatenating short-time solutions on small time intervals (which would have been easier to obtain). Apart from this, the distributional derivatives in the Green's functions of the Dirac equation have made the analysis substantially more difficult than in the Klein-Gordon case. Compared to the latter, we have also treated the massive case (which was not considered in [37] for 1+3 dimensions).

Our results are furthermore characterized as follows. We have shown that the wave function is determined by Cauchy data at the initial time (corresponding to the Big Bang singularity); however, no Cauchy problem is available at different times. The main requirement of our theorems is a smallness condition on the interaction kernel K, demanding that both K and certain first and second order derivatives of K must be bounded and not too large. This still admits a wide class of interaction kernels, and we emphasize that in no way the interaction needs to be small compared to the size of the domain of the wave function. The latter is a common requirement for Fredholm integral equations but it would make the result worthless for infinite spatio-temporal domains.
Besides, we have assumed that K is complex-valued while it could be matrix-valued in the most general case. The reason for this assumption is that our proof requires the integral operator A to be a map from a certain Sobolev space onto itself in which weak derivatives with respect to the Dirac operators of the two particles can be taken. If K were matrix-valued, it would not commute with these Dirac operators in general. Then $A\psi$ would contain new types of weak derivatives which cannot be taken in the initial Sobolev space. As illustrated in Sec. 2.2.2.2, this creates a situation where more and more derivatives have to be controlled, possibly up to infinite order where the success of an iteration scheme seems unlikely. At present, we do not know how to deal with this issue. Improving on this point, however, defines an important task for future research, as e.g. electromagnetic interactions involve interaction kernels proportional to $\gamma_1^\mu \gamma_{2\mu}$ (see [29]).
In addition, it would be desirable to generalize our work in the following regards.

- *N particle integral equations.* Our hope is that our work could contribute to the formulation of a rigorous relativistic many-body theory that can be applied for finite times, not only for scattering processes. An important step in this direction is to treat an arbitrary fixed number $N \in \mathbb{N}$ of particles (setting aside particle creation and annihilation). A class of possible N-particle integral equations has been suggested in [29]. It has the schematic form

$$\psi(x_1, ..., x_N) = \psi^{\text{free}} + \sum_{i<j} \int d^4x_i'\, d^4x_j' S_i(x_i - x_i')\, S_j(x_j - x_j') \\ \times K_{ij}(x_i', x_j')\, \psi(...x_i', ..., x_j', ...). \quad (2.128)$$

It might well be possible to prove the existence and uniqueness of solutions for that equation using the methods developed in the present paper.

- *Singular interaction kernels.* The physically most natural interaction kernel is given by a delta function along the light cone, $K(x_1, x_2) \propto \delta((x_1 - x_2)_\mu (x_1 - x_2)^\mu)$. Getting closer to this case is one of our central goals. Apart from approaching the problem head-on by suitably interpreting the distributional expressions and trying to prove the existence of solutions of the resulting singular integral equation, which seems difficult, one could also try to make smaller steps first. For example, one could decompose $\delta((x_1 - x_2)_\mu (x_1 - x_2)^\mu))$ into $\frac{1}{2|\mathbf{x}_1 - \mathbf{x}_2|}[\delta(x_1^0 - x_2^0 - |\mathbf{x}_1 - \mathbf{x}_2|) + \delta(x_1^0 - x_2^0 + |\mathbf{x}_1 - \mathbf{x}_2|)]$ and only then replace the delta functions with a peaked but smooth function, keeping the singular factor $1/|\mathbf{x}_1 - \mathbf{x}_2|$. This has been done in [37] for the Klein-Gordon case. In the Dirac case, the distributional derivatives make a generalization of that result difficult, and we have not attempted it here. However, it is conceivable that a suitable modification of our techniques could make it possible to treat this case.
  Another interesting question is whether the smallness condition on K can be alleviated such that arbitrarily peaked functions are

admitted. This could allow taking a limit where K approaches the delta function along the light cone.

2.3 Singular light cone interactions of scalar particles in 1+3 dimensions

Here we consider an integral equation describing a fixed number of scalar particles which interact not through boson exchange but directly along light cones, similarly as in bound state equations such as the Bethe-Salpeter equation. The equation involves a multi-time wave function $\psi(x_1, ..., x_N)$ with $x_i = (t_i, \mathbf{x}_i) \in \mathbb{R}^4$ as a crucial concept. Assuming a cutoff in time, we prove that it has a unique solution for all data at the initial time. The cutoff is justified by considering the integral equation for a particular curved spacetime with a Big Bang singularity where an initial time occurs naturally without violating any spacetime symmetries. The main feature of our work is that we treat the highly singular case that interactions occur exactly at zero Minkowski distance, reflected by a delta distribution along the light cone. We also extend the existence and uniqueness result to an arbitrary number $N \geqslant 2$ of particles. Overall, we provide a rigorous example for a certain type of interacting relativistic quantum dynamics in 1+3 spacetime dimensions.

2.3.1 Introduction

2.3.1.1 Motivation

The goal of this paper is to prove the existence and uniqueness of solutions of the equation

$$\psi(x,y) = \psi^{\text{free}}(x,y)+\int d^4x'\, d^4y'\, G_1^{\text{ret}}(x-x')G_2^{\text{ret}}(y-y')K(x',y')\psi(x',y') \tag{2.129}$$

and its N-particle generalization for the singular case of light cone interactions, i.e., for

$$K(x,y) \ = \ \frac{\lambda}{4\pi}\delta((x-y)^2). \tag{2.130}$$

Here, $\lambda > 0$ is a coupling constant and $(x-y)^2 = (x^0-y^0)^2 - |\mathbf{x}-\mathbf{y}|^2$ stands for the Minkowski distance of the spacetime points $x = (x^0, \mathbf{x})$ and $y = (y^0, \mathbf{y})$. Moreover, ψ^{free} is a solution of the free Klein-Gordon equation in each variable, i.e., $(\Box_k + m_k^2)\psi^{free}(x_1, x_2) = 0$, $k = 1, 2$. We shall later see that ψ^{free} plays the role of initial data for Eq. (2.129). G_k^{ret} stands for the retarded Green's function of the respective Klein-Gordon equation. ψ is a *multi-time wave function*, i.e., for $N = 2$ particle, a map

$$\psi : \text{spacetime} \times \text{spacetime} \to \mathbb{C}, \quad (x,y) \mapsto \psi(x,y). \tag{2.131}$$

The crucial point about Eq. (2.129) is that it describes a fixed number of (here $N = 2$) interacting particles in a manifestly Lorentz invariant way. Interactions happen directly along the light cones instead of through particle exchange as in quantum field theory. Such a relativistically invariant interacting dynamics for a fixed number of particles in 1+3 spacetime dimensions is difficult to achieve; in fact, for Hamiltonian theories this is generally believed to be impossible, and there have long been no-go theorems in that direction (see e.g. [4, 25]). It is therefore not surprising that Eq. (2.129) has a distinctly non-Hamiltonian character. This is evident from the fact that the interaction term involves values of ψ in the past, not only on a Cauchy surface which defines the present. In fact, the time delay of the interaction is an important resource of the kind of dynamics defined by Eq. (2.129), and it is only made possible by the more general notion of wave function, the multi-time wave function ψ.

The concept of multi-time wave functions enjoys a long history, going back to well-known physicists such as Eddington [16]; Dirac [13]; Dirac, Fock, Podolsky [12]; Bloch [2]; Tomonaga [54] and Schwinger [51]. While it was intermittently picked up during the years (see e.g. [20, 39, 50, 14, 49, 55], it has recently received renewed attention and undergone significant developments [45, 46, 44, 28, 30, 27, 9, 34, 36, 26, 41, 10, 38]; an overview from 2016 can be found in [33]. The idea is straightforward: to seek a Lorentz covariant generalization of the usual Schrödinger picture wave function (here for $N = 2$)

$$\varphi : \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{C}, \qquad (t, \mathbf{x}, \mathbf{y}) \mapsto \varphi(t, \mathbf{x}, \mathbf{y}). \tag{2.132}$$

In fact, the relation of ψ to φ is just given by evaluation of ψ at equal times in a given Lorentz frame:

$$\varphi(t, \mathbf{x}, \mathbf{y}) \;=\; \psi(t, \mathbf{x}, t, \mathbf{y}). \tag{2.133}$$

For the present purposes, the point of interest is that the concept of a multi-time wave function allows us to see the integral equation (2.129) as the natural generalization of the Schrödinger equation $(i\partial_t - H_1^{\text{free}} - H_2^{\text{free}} - V(t, \mathbf{x}, \mathbf{y}))\varphi(t, \mathbf{x}, \mathbf{y}) = 0$ when formulated as an integral equation (see [29] for a more detailed discussion). The latter can namely be written as

$$\begin{aligned}\varphi(t,\mathbf{x},\mathbf{y}) =& \varphi^{\text{free}}(t,\mathbf{x}_1,\mathbf{x}_2) + \int_0^\infty dt' \int d^3\mathbf{x}'\, d^3\mathbf{y}' G_1^{\text{ret}}(t-t',\mathbf{x}-\mathbf{x}') \\ & \times G_2^{\text{ret}}(t-t',\mathbf{y}-\mathbf{y}')V(t',\mathbf{x}',\mathbf{y}')\varphi(t',\mathbf{x}',\mathbf{y}'),\end{aligned} \tag{2.134}$$

where φ^{free} is a solution of $(i\partial_t - H_1^{\text{free}} - H_2^{\text{free}})\varphi(t,\mathbf{x},\mathbf{y}) = 0$ and G_k^{ret} stands for the retarded Green's function of the operator $(i\partial_t - H_k^{\text{free}})$. Now, our previous integral equation (2.129) reduces to a very similar equation when we neglect the time delay $|\mathbf{x}-\mathbf{y}|$ (we here work with units where $\hbar = 1 = c$) by replacing

$$\delta((x'-y')^2) = \frac{\delta(x'^0 - y'^0 - |\mathbf{x}'-\mathbf{y}'|) + \delta(x'^0 - y'^0 + |\mathbf{x}'-\mathbf{y}'|)}{2|\mathbf{x}'-\mathbf{y}'|} \tag{2.135}$$

with $\delta(x'^0 - y'^0)/|\mathbf{x}'-\mathbf{y}'|$. After performing the time integration over y'^0, renaming the remaining time variable x'^0 as t' and considering on the left hand side of (2.129) at equal times $x^0 = t = y^0$ in the given Lorentz frame, we arrive at (2.134) with Klein-Gordon Green's functions and $V(t,\mathbf{x},\mathbf{y}) \propto 1/|\mathbf{x}-\mathbf{y}|$, the Coulomb potential.

This train of thought suggests that Eq. (2.129) constitutes a natural generalization of the Schrödinger equation (2.134) for relativistic quantum phenomena (here for two scalar particles with electromagnetic interactions) – at least for processes where particle creation and annihilation are not relevant, such as relativistic bound states. Interestingly, this is also the domain where the well-known Bethe-Salpeter equation [48, 40] of quantum field theory is usually applied. The equation also involves a multi-time wave function (2.131) and has a similar form as (2.129). The main differences, however, are that for the Bethe-Salpeter equation (a) the interaction kernel K is not given by a clear-cut expression such as $\delta((x-y)^2)$ but by a (potentially divergent) infinite series of Feynman diagrams, and (b) that it involves Feynman Green's functions instead of retarded Green's functions. Contrary to

retarded Green's functions, Feynman Green's functions have support not only along and inside backward light cones. Nevertheless, the similarity of (2.129) with the Bethe-Salpeter equation constitutes further motivation for its study.
In this context, it is interesting to note that previous works about the existence and solutions of simplified models for the Bethe-Salpeter equation (as in the so-called Wick-Cutkosky model, see [58, 5, 19, 3, 53, 42]) have (to the best of our knowledge) not answered the question of the existence and uniqueness of solutions. Rather, they omit the free solution ψ^{free} from the equation (which, as we will see, leads only to the trivial solution in our case), perform a Fourier transform in all eight variables, assume a plane wave in the center-of-momentum coordinate, use a Wick rotation and then study the eigenvalue problem of a resulting equation of the qualitative form $\widetilde{\psi} = \lambda \widehat{K}(E)\widetilde{\psi}$ in λ (instead of the energy E in the center-of-momentum frame). A transformation back to the original problem is not attempted (and may not always be possible). While these results are nevertheless interesting as they reveal features of possible stationary states of the actual problem (i.e., for the physical value of λ), they are far-removed from the physical problem of time evolution of the quantum-mechanical wave function which we attempt to address here.

2.3.1.2 Previous Works

The physical ideas underlying the multi-time integral equation (2.129) were first introduced in [29]. That paper includes a more detailed derivation of (2.129) as a relativistic generalization of the integral version of the Schrödinger equation, the treatment of the non-retarded limit as well as a comparison with differential multi-time equations. Furthermore, it discusses the parallels of (2.129) with classical action-at-a-distance electrodynamics (where interactions also occur directly along the light cone). In addition, different N-particle generalizations

of (2.129) are compared and analyzed.
The first rigorous results about the existence and uniqueness of solutions of multi-time integral equations of the form (2.129) were obtained in [37]. This article also focuses on the (simpler) case of scalar particles; in addition, it makes two important assumptions: (i) Only bounded or weakly singular interactions kernels K instead of $\delta((x-y)^2)$ are considered which makes the problem much easier to treat. (ii) A cutoff in time is assumed, meaning that the domain of integration is only $(\frac{1}{2}\mathbb{M}) \times (\frac{1}{2}\mathbb{M})$ where $\frac{1}{2}\mathbb{M} = [0,\infty) \times \mathbb{R}^3$ denotes a Minkowski half-space. This assumption together with the fact that the retarded Green's functions are only supported on and inside of the backward light cone has the important effect of rendering the domain of integration in (2.129) finite. In addition, one obtains a Volterra-structure in the time variables, meaning that the time integrations in x'^0 and y'^0 only run from 0 to x^0 and from 0 to y^0, respectively. This, in turn, made it possible to utilize an efficient iteration scheme for the proof of existence and uniqueness of solutions. The result was that for every free solution ψ^{free} in a suitable Banach space, the integral equation posseses a unique solution ψ in that space which, furthermore, agrees with ψ^{free} at the initial time, i.e., for $x^0 = 0 = y^0$.
The assumption of a cutoff in time was made in [37] with reference to a potential Big Bang singularity without, however, considering (2.129) on curved spacetimes. To carry out this task for a class of spacetimes where the Green's functions of the (conformal) wave equation are explicitly known was the topic of [35]. There, it was shown that Eq. (2.129) has a straightforward generalization to curved spacetimes. A number of explicit examples (flat, open and closed Friedman-Lemaître-Robertson-Walker (FLRW) spacetimes) was formulated, and it was shown that for most of these cases, conformal transformations could be used to reduce the proof of existence and uniqueness of solutions to the one on $(\frac{1}{2}\mathbb{M})^2$. Thereby, the point was made that a cutoff in time can arise naturally in a cosmological context, without violating

any spacetime symmetries.

The most recent work about multi-time integral equation is [31]. It is concerned with extending the previous results to the case of Dirac particles. This has been achieved for a class of sufficiently regular interaction kernels $K(x, y)$. The main difficulty in the Dirac case compared to the Klein-Gordon case is that the Green's functions involve distributional derivatives which complicates the analysis. In particular, it becomes necessary to achieve a delicate balance of the regularity of solutions with the form of the integral equation. Apart from this, the work [31] also led to some technical developments where the method of proof was refined by directly using a contraction argument on a weighted Sobolev space instead of the Volterra iteration scheme of [37, 35].

2.3.1.3 Overview of the paper

The goal of the present paper is to extend the previous results for scalar particles to the physically most interesting case $K(x, y) \propto \delta((x - y)^2)$. This is, at the same time, a highly singular and therefore challenging case. It becomes necessary to define the particular combination of the three distributions $G_1^{\text{ret}}(x - x')$, $G_2^{\text{ret}}(y - y')$ and $\delta((x - y)^2)$ which occurs in (2.129) and then prove the existence and uniqueness of the resulting singular integral equation. This equation significantly differs from that considered in [37] where through admitting only less singular interaction kernels K only two singular distributions acting on different variables needed to be considered.

The paper is structured as follows. In Sec. 2.3.2 it is shown how to define the integral equation in a rigorous way (by using the delta distributions to eliminate certain integration variables). To this end, we again consider the equation on the Minkowski half-space (assuming a cutoff in time). Section 2.3.3 contains our main results: Thm. 16 contains explicit bounds for the integral operator in terms of a general

weight function of a weighted L^∞ space. Thm. 17 shows that in the case of massless particles already an exponential weight function leads to the existence and uniqueness of solutions of the integral equation. Our main result is Thm. 18, an existence and uniqueness theorem for the full (massive) case. In that case, a different weight function growing like the exponential of a polynomial is used.
Section 2.3.3.2 deals with generalizing this existence and uniqueness theorem to N scalar particles; the corresponding theorem, Thm. 19, is a direct consequence of Thm. 18. To the best of our knowledge, this is the first rigorous result about a multi-time integral equation for N-particles.
In Section 2.3.3.3 we show by considering a specific example (an open FLRW spacetime) that the cutoff in time can be achieved naturally for a cosmological spacetime with a Big Bang singularity, without breaking any spacetime symmetries. That is, we show the equivalent result of [35] for singular light cone interactions. The respective existence and uniqueness theorem is Thm. 21.
Section 2.3.4 contains the proofs. In Sec. 2.3.5, we conclude.

2.3.2 Precise formulation of the problem

In the following, we show how to precisely define the integral equation (2.129) for the case of two scalar particles with masses m_1 and m_2 on the Minkowski half space $\frac{1}{2}\mathbb{M} = [0,\infty) \times \mathbb{R}^3$. Strictly speaking, to introduce a cutoff in time in this way breaks the Poincaré invariance of (2.129); however, we will give an argument for its use in Sec. 2.3.3.3. It is necessary to take special care of the definition of the integral equation as it contains certain combinations (convolutions and products) of distributions (the Green's functions). Our strategy is to consider the integral operator acting on test functions first where its action can be defined straightforwardly. Later it will be shown that it is bounded on test functions with respect to a suitably chosen weighted norm.

This will make it possible to linearly extend the integral operator to the completion of test functions with respect to that norm.
The retarded Green's function of the Klein-Gordon equation is given by:

$$G^{\mathrm{ret}}(x) \;=\; \frac{1}{4\pi|\mathbf{x}|}\delta(x^0-|\mathbf{x}|) - \frac{m}{4\pi\sqrt{x^2}}H(x^0-|\mathbf{x}|)\frac{J_1(m\sqrt{x^2})}{\sqrt{x^2}} \tag{2.136}$$

where H denotes the Heaviside function and J_1 stands for a Bessel function of the first kind. Then, with $K(x,y) = \frac{\lambda}{4\pi}\delta((x-y)^2)$, our integral equation (2.129) on $(\frac{1}{2}\mathbb{M})^2$ becomes:

$$\psi \;=\; \psi^{\mathrm{free}} + A\psi \tag{2.137}$$

where $A = A_0 + A_1 + A_2 + A_{12}$ and

$$\begin{aligned}
(A_0\psi)(x,y) \;=& \frac{\lambda}{(4\pi)^3}\int_0^{x^0} dx'^0 \int_{\mathbb{R}^3} d^3\mathbf{x}' \int_0^{y^0} dy'^0 \int_{\mathbb{R}^3} \\
&\times \frac{\delta(x^0-x'^0-|\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|}\frac{\delta(y^0-y'^0-|\mathbf{y}-\mathbf{y}'|)}{|\mathbf{y}-\mathbf{y}'|} \\
&\times \delta((x'-y')^2)\psi(x',y'), \qquad (2.138)\\
(A_1\psi)(x,y) \;=& -\frac{\lambda\, m_1}{(4\pi)^3}\int_0^{\infty} dx'^0 \int d^3\mathbf{x}' \int_0^{\infty} dy'^0 \int d^3\mathbf{y}' \\
&\times H(x^0-x'^0-|\mathbf{x}-\mathbf{x}'|)\frac{J_1(m_1\sqrt{(x-x')^2})}{\sqrt{(x-x')^2}} \\
&\times \frac{\delta(y^0-y'^0-|\mathbf{y}-\mathbf{y}'|)}{|\mathbf{y}-\mathbf{y}'|}\delta((x'-y')^2)\psi(x',y') \qquad (2.139)\\
(A_2\psi)(x,y) \;=& -\frac{\lambda\, m_2}{(4\pi)^3}\int_0^{\infty} dx'^0 \int d^3\mathbf{x}' \int_0^{\infty} dy'^0 \int d^3\mathbf{y}' \\
&\times \frac{\delta(x^0-x'^0-|\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|}H(y^0-y'^0-|\mathbf{y}-\mathbf{y}'|)
\end{aligned}$$

$$\times \frac{J_1(m_2\sqrt{(y-y')^2})}{\sqrt{(y-y')^2}}\delta((x'-y')^2)\psi(x',y') \qquad (2.140)$$

$$\begin{aligned}(A_{12}\psi)(x,y) =& \frac{\lambda\, m_1 m_2}{(4\pi)^3}\int_0^\infty dx'^0 \int d^3\mathbf{x}' \int_0^\infty dy'^0 \int d^3\mathbf{y}' \\ &\times H(x^0-x'^0-|\mathbf{x}-\mathbf{x}'|)\frac{J_1(m_1\sqrt{(x-x')^2})}{\sqrt{(x-x')^2}} \\ &\times H(y^0-y'^0-|\mathbf{y}-\mathbf{y}'|)\frac{J_1(m_2\sqrt{(y-y')^2})}{\sqrt{(y-y')^2}} \\ &\times \delta((x'-y')^2)\psi(x',y').\end{aligned} \qquad (2.141)$$

We now formally manipulate these informal expressions in such a way that the end results can be given a precise meaning on test functions. Let $\mathcal{S} = \mathcal{S}((\frac{1}{2}\mathbb{M})^2)$ denote the space of Schwartz functions on $(\frac{1}{2}\mathbb{M})^2$, and let $\psi \in \mathcal{S}$.

2.3.2.1 Definition of A_0.

We consider the massless term A_0 first which is also the most singular term. Using the δ-functions to eliminate the integration over x'^0 and y'^0 results in:

$$\begin{aligned}(A_0\psi)(x,y) =& \frac{\lambda}{(4\pi)^3}\int_{B_{x^0}(\mathbf{x})} d^3\mathbf{x}' \int_{B_{y^0}(\mathbf{y})} d^3\mathbf{y}' \\ &\times \frac{\delta((x^0-y^0-|\mathbf{x}'|+|\mathbf{y}'|)^2-|\mathbf{x}-\mathbf{y}+\mathbf{x}'-\mathbf{y}'|^2)}{|\mathbf{x}'||\mathbf{y}'|} \\ &\times \psi(x+x',y+y')|_{x'^0=-|\mathbf{x}'|,\, y'^0=-|\mathbf{y}'|},\end{aligned} \qquad (2.142)$$

Note that the domain of integration has been reduced to a compact region whose size depends on x^0 and y^0. There is still one more δ-distribution left. We choose to use it to eliminate $|\mathbf{x}'| =: r$. It is convenient to introduce the vector

$$b = x - y - (-|\mathbf{y}'|, \mathbf{y}'). \tag{2.143}$$

Then the argument of the delta function can be written as:

$$(b^0 - |\mathbf{x}'|)^2 - |\mathbf{b} + \mathbf{x}'|^2. \tag{2.144}$$

This expression has a root in r for

$$r = r^* := \frac{1}{2}\frac{b^2}{b^0 + |\mathbf{b}|\cos\vartheta} \tag{2.145}$$

where ϑ is the angle between $\mathbf{b}$ and $\mathbf{x}'$. Of course, r^* inherits the restrictions of the range of r, thus is only a valid root for

$$0 < r^* < x^0. \tag{2.146}$$

The requirement $0 < r^*$ can be satisfied in two cases, either $b^2 > 0$ and $b^0 > 0$, or $b^2 < 0$ and $\cos\vartheta < -\frac{b^0}{|\mathbf{b}|}$. Using these restrictions, the condition $r^* < x^0$ can be converted into a restriction of the domain of integration in ϑ:

$$\begin{aligned}
& \frac{1}{2}\frac{b^2}{b^0 + |\mathbf{b}|\cos\vartheta} &<& \; x^0 \\
\Longleftrightarrow \quad & \operatorname{sgn}(b^2)b^2 &<& \; 2x^0\operatorname{sgn}(b^2)(b^0 + |\mathbf{b}|\cos\vartheta) \\
\Longleftrightarrow \quad & \frac{|b^2|}{2x^0|\mathbf{b}|} - \frac{\operatorname{sgn}(b^2)b^0}{|\mathbf{b}|} &<& \; \operatorname{sgn}(b^2)\cos\vartheta \\
\Longleftrightarrow \quad & \begin{cases} \cos\vartheta > \frac{b^2}{2x^0|\mathbf{b}|} - \frac{b^0}{|\mathbf{b}|}, & \text{for } b^2 > 0 \\ \cos\vartheta < \frac{b^2}{2x^0|\mathbf{b}|} - \frac{b^0}{|\mathbf{b}|}, & \text{for } b^2 < 0. \end{cases}
\end{aligned} \tag{2.147}$$

In case of $b^2 < 0$, the new restriction on $\cos\vartheta$ is stricter than $\cos\vartheta < -\frac{b^0}{|\mathbf{b}|}$; we thus use it to replace the latter. We evaluate the δ-function using spherical coordinates in $\mathbf{y}'$ and the usual rule

$$\delta(f(z)) = \sum_{z^*: f(z^*)=0} \frac{\delta(z-z^*)}{|f'(z^*)|}, \tag{2.148}$$

where $f(r) = (b^0 - r)^2 - (\mathbf{b} + x')^2 = -(r - r^*)2(b^0 + |\mathbf{b}|\cos\vartheta)$. The result is an expression for $A_0\psi$ which does not contain distributions anymore:

$$(A_0\psi)(x,y) = \frac{\lambda}{(4\pi)^3} \int_{B_{y^0}(\mathbf{y})} d^3\mathbf{y}' \int_0^{2\pi} d\varphi \int_{-1}^{1} d\cos\vartheta \; \frac{|b^2|}{4(b^0 + |\mathbf{b}|\cos\vartheta)^2 |\mathbf{y}'|}$$
$$\left(1_{b^2>0} 1_{b^0>0} 1_{\cos\vartheta > \frac{b^2}{2x^0|\mathbf{b}|} - \frac{b^0}{|\mathbf{b}|}} + 1_{b^2<0} 1_{\cos\vartheta < \frac{b^2}{2x^0|\mathbf{b}|} - \frac{b^0}{|\mathbf{b}|}} \right) \psi(x+x', y+y'), \tag{2.149}$$

still subject to $x'^0 = -r^* = -|\mathbf{x}'|, y'^0 = -|\mathbf{y}'|$. The different cases for b have been implemented through the various indicator functions. Eq. (2.149) will serve as our *definition* of A_0 on test functions $\psi \in \mathcal{S}$.

2.3.2.2 Definition of A_1.

Next, we turn to the definition of A_1, starting from the informal expression (2.139). We first split up the δ-function of the interaction kernel according to (2.135). Then we use $\delta(y^0 - y'^0 - |\mathbf{y} - \mathbf{y}'|)$ to eliminate y'^0 $(= y^0 - |\mathbf{y} - \mathbf{y}'|)$. Note that the order of these two steps does not matter. This yields:

$$(A_1\psi)(x,y) = -\frac{\lambda m_1}{2(4\pi)^3} \int_0^\infty dx'^0 \int d^3\mathbf{x}' \int d^3\mathbf{y}' \; H(x^0 - x'^0 - |\mathbf{x} - \mathbf{x}'|)$$
$$\times \frac{J_1(m_1\sqrt{(x-x')^2})}{\sqrt{(x-x')^2}} \frac{H(y^0 - |\mathbf{y} - \mathbf{y}'|)}{|\mathbf{y} - \mathbf{y}'|} \frac{1}{|\mathbf{x}' - \mathbf{y}'|}$$
$$\left[\delta(x'^0 - y^0 + |\mathbf{y} - \mathbf{y}'| - |\mathbf{x}' - \mathbf{y}'|) + \delta(x'^0 - y^0 + |\mathbf{y} - \mathbf{y}'| + |\mathbf{x}' - \mathbf{y}'|) \right]$$

$$\times\, \psi(x', y^0 - |\mathbf{y} - \mathbf{y}'|, \mathbf{y}'). \tag{2.150}$$

Finally, we use the remaining δ-functions to eliminate x'^0. We obtain:

$$\begin{aligned}(A_1\psi)(x,y) = -\frac{\lambda\, m_1}{2(4\pi)^3} \int d^3\mathbf{x}' \int d^3\mathbf{y}'\ &\frac{H(y^0 - |\mathbf{y}-\mathbf{y}'|)}{|\mathbf{y}-\mathbf{y}'|} \frac{1}{|\mathbf{x}'-\mathbf{y}'|}\\ \Bigg[H(x'^0) H(x^0 - x'^0 - |\mathbf{x}-\mathbf{x}'|) &\\ \times \frac{J_1(m_1\sqrt{(x-x')^2})}{\sqrt{(x-x')^2}} \psi(x',y')\Bigg|_{\substack{y'^0 = y^0 - |\mathbf{y}-\mathbf{y}'|,\\ x'^0 = y^0 - |\mathbf{y}-\mathbf{y}'| + |\mathbf{x}'-\mathbf{y}'|}} &\\ +\ H(x'^0) H(x^0 - x'^0 - |\mathbf{x}-\mathbf{x}'|) &\\ \times \frac{J_1(m_1\sqrt{(x-x')^2})}{\sqrt{(x-x')^2}} \psi(x',y')\Bigg|_{\substack{y'^0 = y^0 - |\mathbf{y}-\mathbf{y}'|,\\ x'^0 = y^0 - |\mathbf{y}-\mathbf{y}'| - |\mathbf{x}'-\mathbf{y}'|}} \Bigg]. &\end{aligned} \tag{2.151}$$

This expression is free of distributions, so it will serve as our definition of A_1 on test functions $\psi \in \mathcal{S}$. Note that the domain of integration is effectively finite due to the Heaviside functions.

2.3.2.3 Definition of A_2.

Starting from (2.140), the analogous steps as for A_1 yield:

$$\begin{aligned}(A_2\psi)(x,y) = -\frac{\lambda m_2}{2(4\pi)^3} \int d^3\mathbf{x}' \int d^3\mathbf{y}'\ &\frac{H(x^0 - |\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|} \frac{1}{|\mathbf{x}'-\mathbf{y}'|}\\ \Bigg[H(y'^0) H(y^0 - y'^0 - |\mathbf{y}-\mathbf{y}'|) &\\ \times \frac{J_1(m_2\sqrt{(y-y')^2})}{\sqrt{(y-y')^2}} \psi(x',y')\Bigg|_{\substack{x'^0 = x^0 - |\mathbf{x}-\mathbf{x}'|,\\ y'^0 = x^0 - |\mathbf{x}-\mathbf{x}'| + |\mathbf{x}'-\mathbf{y}'|}} &\end{aligned}$$

$$+ \; H(y'^0)H(y^0 - y'^0 - |\mathbf{y} - \mathbf{y}'|) \\ \times \; \frac{J_1(m_2\sqrt{(y-y')^2})}{\sqrt{(y-y')^2}}\psi(x',y')\Big|_{\substack{x'^0 = x^0 - |\mathbf{x}-\mathbf{x}'|, \\ y'^0 = x^0 - |\mathbf{x}-\mathbf{x}'| - |\mathbf{x}'-\mathbf{y}'|}} \Bigg]. \quad (2.152)$$

This serves as our definition of A_2 on test functions $\psi \in \mathcal{S}$.

2.3.2.4 Definition of A_{12}.

Here, we start with (2.141). We change variables $(\mathbf{x}', \mathbf{y}') \to (\mathbf{x}', \mathbf{z} = \mathbf{x}' - \mathbf{y}')$ (Jacobi determinant = 1), with the goal of using the remaining δ-function to eliminate $|\mathbf{z}| = |\mathbf{x}' - \mathbf{y}'|$ in mind. We find:

$$(A_{12}\psi)(x,y) = \frac{\lambda m_1 m_2}{(4\pi)^3}\int_0^\infty dx'^0 \int d^3\mathbf{x}' \int_0^\infty dy'^0 \int d^3\mathbf{z} H(x^0 - x'^0 - |\mathbf{x}-\mathbf{x}'|) \\ \times \; \frac{J_1(m_1\sqrt{(x-x')^2})}{\sqrt{(x-x')^2}} H(y^0 - y'^0 - |\mathbf{y} - \mathbf{x}' + \mathbf{z}|) \\ \times \; \frac{J_1(m_2\sqrt{(y-y')^2})}{\sqrt{(y-y')^2}} \delta((x'^0 - y'^0)^2 - |\mathbf{z}|^2)\psi(x',y')\Big|_{\mathbf{y}'=\mathbf{x}'-\mathbf{z}}. \quad (2.153)$$

Now we use spherical coordinates for $\mathbf{z}$ and eliminate $|\mathbf{z}|$ through the δ-function, using

$$\delta((x'^0 - y'^0)^2 - |\mathbf{z}|^2) = \frac{1}{2|\mathbf{z}|}\delta(|x^{0\prime} - y^{0\prime}| - |\mathbf{z}|). \quad (2.154)$$

This yields:

$$(A_{12}\psi)(x,y) = \frac{\lambda\, m_1 m_2}{2(4\pi)^3}\int_0^\infty dx'^0 \int d^3\mathbf{x}' \int_0^\infty dy'^0 \int_0^{2\pi} d\varphi \int_0^\pi d\vartheta \\ \times \cos(\vartheta)|x'^0 - y'^0| H(x^0 - x'^0 - |\mathbf{x}-\mathbf{x}'|)\frac{J_1(m_1\sqrt{(x-x')^2})}{\sqrt{(x-x')^2}}$$

$$\times H(y^0 - y'^0 - |\mathbf{y} - \mathbf{x}' + \mathbf{z}|) \frac{J_1(m_2\sqrt{(y-y')^2})}{\sqrt{(y-y')^2}} \psi(x', y')\Big|_{\mathbf{y}'=\mathbf{x}'-\mathbf{z},\, |\mathbf{z}|=|x^{0\prime}-y^{0\prime}|}. \tag{2.155}$$

The resulting expression does not contain distributions anymore and will serve as our definition of A_{12} on test functions $\psi \in \mathcal{S}$. Note that the domain of integration is again effectively finite.

2.3.2.5 Lifting of the integral operator from test functions to a suitable Banach space.

In order to prove the existence and uniqueness of solutions of the integral equation $\psi = \psi^{\text{free}} + A\psi$, we need to define the operator A not only on test functions but on a suitable Banach space which includes (at least) sufficiently many solutions ψ^{free} of the free multi-time Klein-Gordon equations, $(\Box_k + m_k^2)\psi^{\text{free}}(x_1, x_2) = 0, \; k = 1, 2$. We shall define this Banach space as the completion of $\mathcal{S} = \mathcal{S}((\frac{1}{2}\mathbb{M})^2)$ with respect to a suitable norm. A good choice which works well for the upcoming existence and uniqueness proofs is the class of weighted L^∞-norms

$$\|\psi\|_g := \operatorname*{ess\,sup}_{x,y \in \frac{1}{2}\mathbb{M}} \frac{|\psi(x,y)|}{g(x^0)g(y^0)}, \tag{2.156}$$

where $g : \mathbb{R}_0^+ \to \mathbb{R}^+$ is assumed to be a monotonically increasing function such that $1/g$ is bounded. Then our Banach space is given by the completion

$$\mathcal{B}_g = \overline{\mathcal{S}}^{\|\cdot\|_g}. \tag{2.157}$$

Our next goal is to find a weight function g such that the operator A is not only bounded but even defines a contraction on $\mathcal{B}_g$. By linear extension, it is sufficient to estimate $\|A\psi\|_g$ on test functions $\psi \in \mathcal{S}$.

Remarks: 1. We have attempted to use an $L_t^\infty L_{\mathbf{x}}^2$-based norm ($L^\infty$ in the times and L^2 in the space variables). However, we did not succeed with obtaining suitable estimates for that case. This might not be a problem in principle, but its treatment would require further technical innovation. More precisely, one would need to understand integral operators such as (2.149) whose kernel is in L^1 but not in L^2.

2. Nevertheless, our definition of $\mathcal{B}_g$ contains a large class of free solutions of the Klein-Gordon equation. As the Klein-Gordon equation preserves boundedness, all bounded initial data for ψ^{free} lead to a free solution $\psi^{\text{free}} \in \mathcal{B}_g$ which can be used as an input to our integral equation.

2.3.3 Results

This section is structured as follows. Sec. 2.3.3.1 (which is about the two-particle case) contains the main results: the estimates of the integral operators as well as the theorems about existence and uniqueness of solutions. Sec. 2.3.3.2 we extend these results to the N-particle case and in Sec. 2.3.3.3 we show that a curved spacetime with a Big Bang singularity can provide a natural reason for a cutoff in time.

2.3.3.1 The two-particle case

For $t \geqslant 0$, we define the functions:

$$\begin{aligned} & g_0(t) = g(t), \\ \text{and for } n \in \mathbb{N}: \quad & g_n(t) = \int_0^t dt'\, g_{n-1}(t'). \end{aligned} \tag{2.158}$$

Note that due to the properties of g, the functions g_n are monotonically increasing for all $n \in \mathbb{N}$; furthermore, by definition, they satisfy $g_n(0) = 0$.

Our first theorem gives explicit bounds for the operators A_0, A_1, A_2, A_{12} in terms of the functions g_n. The proof can be found in Sec. 2.3.4.1.

Theorem 16 (Bounds of the integral operators on $\mathcal{S}$.)**.** *For all $\psi \in \mathcal{S}((\frac{1}{2}\mathbb{M})^2)$, the integral operators A_0, A_1, A_2, A_{12} satisfy the following bounds:*

$$\sup_{\psi\in\mathcal{S}((\frac{1}{2}\mathbb{M})^2)} \frac{\|A_0\psi\|_g}{\|\psi\|_g} \leqslant \frac{\lambda}{8\pi}\left(\sup_{t\geqslant 0}\frac{g_1(t)}{g(t)}\right)^2, \tag{2.159}$$

$$\begin{aligned}\sup_{\psi\in\mathcal{S}((\frac{1}{2}\mathbb{M})^2)} \frac{\|A_1\psi\|_g}{\|\psi\|_g} \leqslant \frac{\lambda\, m_1^2}{16\pi}\Bigg[&3\left(\sup_{t\geqslant 0}\frac{tg_1(t)}{g(t)}\right)\left(\sup_{t\geqslant 0}\frac{g_2(t)}{g(t)}\right)\\ &+3\left(\sup_{t\geqslant 0}\frac{g_1(t)}{g(t)}\right)\left(\sup_{t\geqslant 0}\frac{tg_2(t)}{g(t)}\right)\\ &+2\left(\sup_{t\geqslant 0}\frac{g_1(t)}{g(t)}\right)\left(\sup_{t\geqslant 0}\frac{g_3(t)}{g(t)}\right)\Bigg],\end{aligned} \tag{2.160}$$

$$\begin{aligned}\sup_{\psi\in\mathcal{S}((\frac{1}{2}\mathbb{M})^2)} \frac{\|A_2\psi\|_g}{\|\psi\|_g} \leqslant \frac{\lambda\, m_2^2}{16\pi}\Bigg[&3\left(\sup_{t\geqslant 0}\frac{tg_1(t)}{g(t)}\right)\left(\sup_{t\geqslant 0}\frac{g_2(t)}{g(t)}\right)\\ &+3\left(\sup_{t\geqslant 0}\frac{g_1(t)}{g(t)}\right)\left(\sup_{t\geqslant 0}\frac{tg_2(t)}{g(t)}\right)\\ &+2\left(\sup_{t\geqslant 0}\frac{g_1(t)}{g(t)}\right)\left(\sup_{t\geqslant 0}\frac{g_3(t)}{g(t)}\right)\Bigg],\end{aligned} \tag{2.161}$$

$$\begin{aligned}\sup_{\psi\in\mathcal{S}((\frac{1}{2}\mathbb{M})^2)} \frac{\|A_{12}\psi\|_g}{\|\psi\|_g} \leqslant \frac{\lambda\, m_1^2 m_2^2}{96\pi}\Bigg[&\left(\sup_{t\geqslant 0}\frac{t^2g_2(t)}{g(t)}\right)\left(\sup_{t\geqslant 0}\frac{tg_1(t)}{g(t)}\right)\\ &+\frac{1}{2}\left(\sup_{t\geqslant 0}\frac{t^2g_3(t)}{g(t)}\right)\left(\sup_{t\geqslant 0}\frac{g_1(t)}{g(t)}\right)\Bigg].\end{aligned} \tag{2.162}$$

In case these expressions are finite, A_0, A_1, A_2, A_{12} extend to linear operators on $\mathcal{B}_g$ with the same norms. Our next task is to find suitable weight functions g such that this is actually the case. We begin with the massless case where already an exponential weight function leads to an estimate which remains finite after taking the supremum. The massive case is treated subsequently; it is a little more difficult as all the estimates for the operators A_0, A_1, A_2, A_{12} have to be finite at the same time. This requires a different choice of weight function (see Thm. 18).

Theorem 17 (Bounds for A_0 and $g(t) = e^{\gamma t}$; existence of massless dynamics.)**.**
For any $\gamma > 0$, let $g(t) = e^{\gamma t}$. Then A_0 can be linearly extended to a bounded operator on $\mathcal{B}_g$ with norm

$$\|A_0\| \leqslant \frac{\lambda}{8\pi\gamma^2}. \tag{2.163}$$

Consequently, for all $\gamma > \sqrt{\frac{\lambda}{8\pi}}$, the integral equation $\psi = \psi^{\text{free}} + A_0\psi$ has a unique solution $\psi \in \mathcal{B}_g$ for every $\psi^{\text{free}} \in \mathcal{B}_g$.

Now we come to our main result.

Theorem 18 (Existence of dynamics in the massive case.)**.**
For any $\alpha > 0$, let

$$g(t) = (1 + \alpha t^2)e^{\alpha t^2/2}. \tag{2.164}$$

Then A_0, A_1, A_2 and A_{12} can be linearly extended to bounded operators *on $\mathcal{B}_g$ with norms*

$$\|A_0\| \leqslant \frac{\lambda}{32\pi}\frac{1}{\alpha}, \tag{2.165}$$

$$\|A_1\| \leqslant \frac{5\lambda\, m_1^2}{16\pi}\frac{1}{\alpha^2}, \tag{2.166}$$

$$\|A_2\| \leqslant \frac{5\lambda\, m_2^2}{16\pi}\frac{1}{\alpha^2}, \tag{2.167}$$

$$\|A_{12}\| \leqslant \frac{\lambda\, m_1^2 m_2^2}{80\pi}\frac{1}{\alpha^3}. \tag{2.168}$$

Consequently, for all $\alpha > 0$ with

$$\frac{\lambda}{8\pi\alpha}\left(\frac{1}{4} + \frac{5(m_1^2+m_2^2)}{2}\frac{1}{\alpha} + \frac{m_1^2\, m_2^2}{10}\frac{1}{\alpha^2}\right) \;<\; 1, \tag{2.169}$$

the integral equation $\psi = \psi^{\text{free}} + A\psi$ has a unique solution $\psi \in \mathcal{B}_g$ for every $\psi^{\text{free}} \in \mathcal{B}_g$.

The proof can be found in Sec. 2.3.4.3.

Remarks: 1. *Comparison of Thms. 17 and 18 in the massless case.* On the first glance, the result of Thm. 17 looks stronger in the sense that for $g(t) = e^{\gamma t}$, the estimate of $\|A_0\|$ goes with γ^{-2} while for $g(t) = (1+\alpha t^2)e^{\alpha t^2/2}$, the estimate of $\|A_0\|$ goes with α^{-1}. However, one should note that γ is the constant in front of t while α occurs in combination with t^2. Thus, if one wants to draw a comparison between these different cases at all, then it should be between γ and $\sqrt{\alpha}$. Of course, the main difference between the two theorems is the admitted growth rate of the solutions. In this regard, Thm. 17 contains the stronger statement.

2. A *physically realistic value of* λ is $\frac{1}{137}$, the value of the fine structure constant. In that case, α need not even be particularly large in order for condition (2.169) to be satisfied.

3. *Initial value problem.* By the integral equation (2.129), we obtain that the solution ψ satisfies $\psi(0, \mathbf{x}, 0, \mathbf{y}) = \psi^{\text{free}}(0, \mathbf{x}, 0, \mathbf{y})$. If ψ^{free} is a solution of the free multi-time Klein-Gordon equations, then it is itself determined by initial data at $x_1^0, x_2^0 = 0$. (As the Klein-Gordon equation is of second order in time, these initial data include data for $\partial_{x^0}\psi$, $\partial_{y^0}\psi$ and $\partial_{x^0}\partial_{y^0}\psi$, see [41, chap. 5].) Thus, we find that ψ is determined by these data at $x_1^0, x_2^0 = 0$ as well. Note that for later times, ψ and ψ^{free} do not, in general, coincide and consequently a similar statement does not hold.

4. *Finite propagation speed.* The theorem implies that $\psi = \sum_{k=0}^{\infty} A^k \psi^{\text{free}}$. As $(A\psi^{\text{free}})(x, y)$ involves only values of ψ^{free} in $\text{past}(x) \times \text{past}(y)$ where $\text{past}(x)$ denotes the causal past of $x \in \frac{1}{2}\mathbb{M}$ (see Eqs. (2.149), (2.151), (2.152), (2.155)), so do $A^k \psi^{\text{free}}$ for all $k \in \mathbb{N}$ and ψ. Therefore, we obtain: if the initial data for ψ^{free} at $x^0 = 0 = y^0$ are compactly supported in a region $R \subset (\{0\} \times \mathbb{R}^3)^2$, then for all Cauchy surfaces $\Sigma \subset \frac{1}{2}\mathbb{M}$, $\psi|_{\Sigma \times \Sigma}$ is supported in the causally grown set $\text{Gr}(R, \Sigma) = \left(\bigcup_{(x,y) \in R} \text{future}(x) \times \text{future}(y) \right) \cap (\Sigma \times \Sigma)$ where $\text{future}(x)$ stands for the causal future of $x \in \frac{1}{2}\mathbb{M}$.

5. *Square integrable solutions.* As a consequence of the previous item, compactly supported and bounded initial data for ψ^{free} lead to a compactly supported and bounded solution ψ. In particular, this implies that $\psi(x^0, \cdot, y^0)$ lies in $L^2(\mathbb{R}^6)$ for all times $x^0, y^0 \geqslant 0$.

2.3.3.2 The N-particle case

Here we extend Thm. 18 from two to $N \geqslant 3$ scalar particles. While there are different possibilities to generalize the two-particle integral equation (2.129), we focus on the one advocated in [29] as the most promising. For

$$\psi : \left(\tfrac{1}{2}\mathbb{M}\right)^N \to \mathbb{C}, \qquad (x_1, ..., x_N) \mapsto \psi(x_1, ..., x_N) \tag{2.170}$$

we consider the integral equation

$$\begin{aligned}\psi(x_1, ..., x_N) &= \psi^{\text{free}}(x_1, ..., x_N) + \frac{\lambda}{4\pi} \sum_{i,j=1,...,N;\, i<j} \int_{\frac{1}{2}\mathbb{M}} d^4x_i \int_{\frac{1}{2}\mathbb{M}} d^4x_j\; G_i^{\text{ret}}(x_i - x_i') \\ &\times\; G^{\text{ret}}(x_j - x_j')\delta((x_i' - x_j')^2)\psi(x_1, ..., x_i, ..., x_j, ..., x_N).\end{aligned} \tag{2.171}$$

Here, ψ^{free} is again a solution of the free Klein-Gordon equations $(\Box_k + m_k^2)\phi(x_k)$ in each spacetime variable and G_k^{ret} stands for the retarded Green's function of the operator $(\Box_k + m_k^2)$, $k = 1, 2, ..., N$.
Eq. (2.171) is written down in an informal way. To define a rigorous version, let $\psi \in \mathcal{S}\big((\tfrac{1}{2}\mathbb{M})^N\big)$ be a test function. Moreover, let $A^{(ij)}$ be the integral operator of the two-particle problem acting on the variables x_i and x_j instead of $x = x_1$ and $y = x_2$. We define

$$^{(N)}A \;=\; \sum_{i,j=1,...,N;\, i<j} A^{(ij)}. \tag{2.172}$$

As will be shown below, $^{(N)}A$ can be linearly extended to a bounded operator on the Banach space $^{(N)}\mathcal{B}_g$. That space is defined as the completion of $\mathcal{S}\big((\tfrac{1}{2}\mathbb{M})^N\big)$ with respect to the norm

$$\|\psi\|_g \;=\; \operatorname*{ess\,sup}_{x_1,...,x_N \in \frac{1}{2}\mathbb{M}} \frac{|\psi|(x_1, ..., x_N)}{g(x_1^0)\cdots g(x_N^0)}, \tag{2.173}$$

where the function g is defined as before.
Then we take the equation

$$^{(N)}A \;=\; \psi^{\text{free}} + {}^{(N)}A\psi. \tag{2.174}$$

to be the rigorous version of (2.171) on ${}^{(N)}\mathcal{B}_g$.
With these preparations, we are ready to formulate the N-particle existence and uniqueness theorem.

Theorem 19 (Existence of dynamics for N particles.)**.**
For any $\alpha > 0$, let $g(t) = (1+\alpha t^2)e^{\alpha t^2/2}$. Then the operator ${}^{(N)}A$ can be linearly extended to a bounded operator on ${}^{(N)}\mathcal{B}_g$ with norm

$$\|{}^{(N)}A\| \;\leqslant\; \frac{\lambda}{8\pi\alpha} \sum_{i,j=1,\ldots,N;\, i<j} \left(\frac{1}{4} + \frac{5(m_i^2+m_j^2)}{2}\frac{1}{\alpha} + \frac{m_i^2\, m_j^2}{10}\frac{1}{\alpha^2} \right). \tag{2.175}$$

If $\alpha > 0$ is such that this expression is strictly smaller than one, the integral equation (2.174) has a unique solution $\psi \in {}^{(N)}\mathcal{B}_g$ for every $\psi^{\text{free}} \in {}^{(N)}\mathcal{B}_g$.

The proof follows straightforwardly from that of Thm. 18 using

$$\|{}^{(N)}A\| \leqslant \sum_{i,j=1,\ldots,N;\, i<j} \|A^{(ij)}\|_g. \tag{2.176}$$

For the norms of the operators $A^{(ij)}$, one can use the previous expressions as these operators act only as the identity on variables x_k with $k \notin \{i,j\}$.

Remark 20. *To the best of our knowledge, Thm. 19 is the first result about the existence and uniqueness of solutions of multi-time integral equations for N particles. While for the present contraction argument the generalization to N particles has been straightforward, this is not the case for other works. For example, the Volterra iterations used* *in [37] become increasingly complicated with increasing particle number N. For Dirac particles, a similar technique as ours was used in [31]. However, as the Dirac Green's functions contain distributional derivatives, one has to control weak derivatives of the solutions, and the number of such derivatives depends on N. That situation also does not allow for such a straightforward generalization to N particles as has been possible here.*

2.3.3.3 On the possible origin of a cutoff in time

So far, we have assumed a cutoff in time. In the way this has been treated so far, this cutoff breaks the manifest Poincaré invariance of our integral equation. In this section, we demonstrate at a particular (simple and tractable) example that such a cutoff can arise naturally if the considered spacetime has a Big Bang singularity. Then the Big Bang defines the initial time. To consider a simple example is necessary as otherwise the Green's functions may not be known in detail, and in that case it would not be possible to explicitly define the integral operator, let alone to carry out an analysis of that operator comparable to the one before.
Our example consists of two massless scalar particles which, in absence of interactions, obey the conformally invariant wave equation on a curved spacetime $\mathcal{M}$ with metric g,

$$(\Box_g - \xi R)\,\chi = 0, \tag{2.177}$$

where R denotes the Ricci scalar and in 1+3 dimensions $\xi = \frac{1}{6}$.
We consider these particles on a flat Friedman-Lemaître-Robertson-Walker (FLRW) spacetime which is described by the metric

$$ds^2 = a^2(\eta)\left(d\eta^2 - dr^2 - r^2 d\Omega^2\right), \tag{2.178}$$

where η denotes conformal time, $d\Omega$ denotes the surface measure on $\mathbb{S}^2$ and $a(\eta)$ is the so-called *scale function*, a continuous function with

$a(0) = 0$ and $a(\eta) > 0$ for $\eta > 0$. This form makes it obvious that the spacetime is conformally equivalent to a Minkowski half space $\frac{1}{2}\mathbb{M}$, with conformal factor $a(\eta)$.

In this case, it is well-known that the Green's functions of (2.177) on the flat FLRW spacetime $\mathcal{M}$ can be obtained from those of the usual wave equation on $\frac{1}{2}\mathbb{M}$ as follows (using coordinates $x = (\eta, \mathbf{x})$ and $x' = (\eta', \mathbf{x}')$ with $\eta, \eta' \in [0, \infty)$ and $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^3$; see [35] for a more detailed explanation):

$$G_{\mathcal{M}}(x, x') = \frac{1}{a(\eta)} \frac{1}{a(\eta')} G_{\frac{1}{2}\mathbb{M}}(x, x'). \tag{2.179}$$

Inserting the well-known expression for the retarded and symmetric Green's functions on $\frac{1}{2}\mathbb{M}$ (see (2.136)) yields:

$$\begin{aligned} G^{\text{ret}}_{\mathcal{M}}(x, x') &= \frac{1}{4\pi} \frac{1}{a(\eta)a(\eta')} \frac{\delta(\eta - \eta' - |\mathbf{x} - \mathbf{x}'|)}{|\mathbf{x} - \mathbf{x}'|} \\ G^{sym}_{\mathcal{M}}(x, x') &= \frac{1}{4\pi} \frac{1}{a(\eta)a(\eta')} \delta((\eta - \eta')^2 - |\mathbf{x} - \mathbf{x}'|^2). \end{aligned} \tag{2.180}$$

With this information, we are ready to write down the integral equation on $\mathcal{M}$. The generalization of (2.129) to curved spacetimes is straightforward: ψ becomes a scalar function on $\mathcal{M} \times \mathcal{M}$, one exchanges the Minkowski spacetime volume elements with the invariant 4-volume elements on $\mathcal{M}$, and the Green's functions on $\frac{1}{2}\mathbb{M}$ get replaced with those on $\mathcal{M}$ as well. As in the Minkowski case, the interaction kernel is given by the symmetric Green's function. With this, the relevant integral equation becomes:

$$\psi(x, y) = \psi^{\text{free}}(x, y) + \lambda \int_{\mathcal{M} \times \mathcal{M}} dV(x)\, dV(y)\, G^{\text{ret}}_1(x, x') G^{\text{ret}}_2(y, y') G^{sym}(x', y') \psi(x', y'), \tag{2.181}$$

For regular and only weakly singular interaction kernels $K(x, y)$ instead of $G^{sym}(x', y')$, the problem of existence and uniqueness of solutions of this equation has been treated in [35] for flat, open and closed

FLRW universes; the case of Dirac particles and smooth interaction kernels has been addressed in [31]. For flat FLRW universes and scalar particles, we here extend [35] to the physically most interesting and mathematically challenging case $K(x,y) = G^{sym}(x,y)$.
We now formulate (2.181) explicitly. The spacetime volume element is given by:

$$dV(x) = a^4(\eta)\, d\eta\, d^3\mathbf{x}. \tag{2.182}$$

With this information, (2.181) becomes:

$$\begin{aligned}
\psi(\eta_1, \mathbf{x}_1, \eta_2, \mathbf{x}_2) = \;& \psi^{\text{free}}(\eta_1, \mathbf{x}_1, \eta_2, \mathbf{x}_2) + \frac{\lambda}{(4\pi)^3} \frac{1}{a(\eta_1)a(\eta_2)} \int_0^{\eta_1} d\eta_1' \int d^3\mathbf{x}_1' \int_0^{\eta_2} d\eta_2' \int \\
\times\;& a^2(\eta_1')a^2(\eta_2') \frac{\delta(\eta_1 - \eta_1' - |\mathbf{x}_1 - \mathbf{x}_1'|)}{|\mathbf{x}_1 - \mathbf{x}_1'|} \frac{\delta(\eta_2 - \eta_2' - |\mathbf{x}_2 - \mathbf{x}_2'|)}{|\mathbf{x}_2 - \mathbf{x}_2'|} \\
\times\;& \delta((\eta_1' - \eta_2')^2 - |\mathbf{x}_1' - \mathbf{x}_2'|^2)\psi(\eta_1', \mathbf{x}_1', \eta_2', \mathbf{x}_2').
\end{aligned} \tag{2.183}$$

Now let

$$\chi(\eta_1, \mathbf{x}_1, \eta_2) = a(\eta_1)a(\eta_2)\psi(\eta_1, \mathbf{x}_1, \eta_2). \tag{2.184}$$

and $\chi^{free}(\eta_1, \mathbf{x}_1, \eta_2) = a(\eta_1)a(\eta_2)\psi^{free}(\eta_1, \mathbf{x}_1, \eta_2)$. Then (2.183) is equivalent to:

$$\begin{aligned}
\chi(\eta_1, \mathbf{x}_1, \eta_2, \mathbf{x}_2) = \;& \chi^{\text{free}}(\eta_1, \mathbf{x}_1, \eta_2, \mathbf{x}_2) + \frac{\lambda}{(4\pi)^3} \int_0^{\eta_1} d\eta_1' \int d^3\mathbf{x}_1' \int_0^{\eta_2} d\eta_2' \int d^3\mathbf{x}_2' \\
\times\;& a(\eta_1')a(\eta_2') \frac{\delta(\eta_1 - \eta_1' - |\mathbf{x}_1 - \mathbf{x}_1'|)}{|\mathbf{x}_1 - \mathbf{x}_1'|} \frac{\delta(\eta_2 - \eta_2' - |\mathbf{x}_2 - \mathbf{x}_2'|)}{|\mathbf{x}_2 - \mathbf{x}_2'|} \\
\times\;& \delta((\eta_1' - \eta_2')^2 - |\mathbf{x}_1' - \mathbf{x}_2'|^2)\chi(\eta_1', \mathbf{x}_1', \eta_2', \mathbf{x}_2').
\end{aligned} \tag{2.185}$$

We can see that this equation has almost exactly the same form as the massless version of (2.129) on $\frac{1}{2}\mathbb{M}$ (see (2.138)). The only difference is the additional appearance of the factor $a(\eta_1')a(\eta_2')$ inside the integrals.

Going through the same steps as for (2.149) before, (2.185) can be defined on test functions $\chi \in \mathcal{S}$ by

$$\chi \;=\; \chi^{free} + \widetilde{A}_0\chi, \tag{2.186}$$

where $\widetilde{A}_0$ is defined by (using coordinates $x = (\eta_1, \mathbf{x})$, $y = (\eta_2, \mathbf{y})$):

$$(\widetilde{A}_0\psi)(x,y) = \frac{\lambda}{(4\pi)^3}\int_{B_{y^0}(\mathbf{y})} d^3\mathbf{y}' \int_0^{2\pi} d\varphi \int_{-1}^{1} d\cos\vartheta \; \frac{|b^2|}{4(b^0+|\mathbf{b}|\cos\vartheta)^2|\mathbf{y}'|} \; \times$$
$$a(\eta_1+\eta_1')a(\eta_2+\eta_2')\psi(x+x',y+y')\left(1_{b^2>0}1_{b^0>0}1_{\cos\vartheta>\frac{b^2}{2x^0|\mathbf{b}|}-\frac{b^0}{|\mathbf{b}|}} + 1_{b^2<0}1_{\cos\vartheta<\frac{b^2}{2x^0|\mathbf{b}|}-\frac{b^0}{|\mathbf{b}|}}\right), \tag{2.187}$$

with $\eta_1' = -r^* = -|\mathbf{x}'|, \eta_2' = -|\mathbf{y}'|$. (Here, b and r^* are defined as in (2.143) and (2.145), respectively).

Knowing precisely how our integral equation on the flat FLRW spacetime is to be understood, we can formulate the respective existence and uniqueness theorem:

Theorem 21 (Existence of dynamics for an open FLRW universe.)**.** *Let $a : [0,\infty) \to [0,\infty)$ be a continuous function with $a(0) = 0$ and $a(\eta) > 0$ for $\eta > 0$. Moreover, let*

$$g(t) = \exp\left(\gamma\int_0^t d\tau\, a(\tau)\right). \tag{2.188}$$

Then, the operator $\widetilde{A}_0$ satisfies the following estimate:

$$\sup_{\chi\in\mathcal{S}\left(([0,\infty)\times\mathbb{R}^3)^2\right)} \frac{\|\widetilde{A}_0\chi\|_g}{\|\chi\|_g} \;\leqslant\; \frac{\lambda}{8\pi\gamma^2}. \tag{2.189}$$

$\widetilde{A}_0$ can be extended to a linear operator on $\mathcal{B}_g$ which satisfies the same bound. Moreover, for $\gamma < \sqrt{\frac{\lambda}{8\pi}}$, the equation $\chi = \chi^{free} + \widetilde{A}_0\chi$ has a unique solution $\chi \in \mathcal{B}_g$ for every $\psi^{free} \in \mathcal{B}_g$.

The proof can be found in Sec. 2.3.4.4.

Remarks: 1. *Manifest covariance.* The theorem shows the existence and uniqueness of solutions of the manifestly covariant integral equation (2.181). Our example of a particular FLRW spacetime thus achieves its goal of demonstrating that a cutoff in time can arise naturally in a cosmological context.

2. *Initial value problem.* As in the case of $\frac{1}{2}\mathbb{M}$, the solution χ satisfies $\chi(0, \mathbf{x}, 0, \mathbf{y}) = \chi^{\text{free}}(0, \mathbf{x}, 0, \mathbf{y})$ where χ^{free} is determined by the solution ψ^{free} of the free conformal wave equation (2.177) in both spacetime variables. Since ψ^{free} is determined by initial data at $\eta_1 = 0 = \eta_2$, so are χ^{free} and χ.

3. *Behavior of ψ towards the Big Bang singularity.* While the transformed wave function χ remains bounded for $\eta_1, \eta_2 \to 0$, the physical wave function $\psi(\eta_1, \mathbf{x}, \eta_2, \mathbf{y}) = \frac{1}{a(\eta_1)a(\eta_2)}\chi(\eta_1, \mathbf{x}, \eta_2, \mathbf{y})$ diverges like $\frac{1}{a(\eta_1)a(\eta_2)}$. This is to be expected, as the Klein-Gordon equation has a preserved "energy" (given by a certain spatial integral) and as the volume in $\mathbf{x}, \mathbf{y}$ contracts to zero towards the Big Bang.

4. *N-particle generalization.* As shown in Sec. 2.3.3.2 for the Minkowski half-space, it would also be easy to extend Thm. 21 to N particles. To avoid duplication, we do not carry this out explicitly for the curved spacetime example here.

2.3.4 Proofs

2.3.4.1 Proof of Theorem 16

The proof is divided into the proofs of the estimates (2.159), (2.160), (2.161) and (2.162), respectively. Here, (2.159) is the most singular and difficult term which deserves the greatest attention.

Throughout this subsection, let $\psi \in \mathcal{S}((\frac{1}{2}\mathbb{M})^2)$.

2.3.4.1.1 Estimate of the massless term (2.159)**.** We start with Eq. (2.149) and take the absolute value. Using, in addition, that

$$|\psi(x,y)| \;\leqslant\; \|\psi\|_g\, g(x^0)g(y^0) \tag{2.190}$$

leads us to:

$$|A_0\psi|(x,y) \leqslant \frac{\lambda\|\psi\|_g}{4(4\pi)^3}\int_{B_{y^0}(\mathbf{y})} d^3\mathbf{y}' \int_0^{2\pi} d\varphi \int_{-1}^{1} d\cos\vartheta\, \frac{|b^2|}{(b^0+|\mathbf{b}|\cos\vartheta)^2|\mathbf{y}'|} g(y^0-|\mathbf{y}'|)$$
$$\times\, g\left(x^0 - \frac{1}{2}\frac{b^2}{b^2+|\mathbf{b}|\cos\vartheta}\right)\left(1_{b^2>0}1_{b^0>0}1_{\cos\vartheta > \frac{b^2}{2x^0|\mathbf{b}|}-\frac{b^0}{|\mathbf{b}|}} + 1_{b^2<0}1_{\cos\vartheta<\frac{b^2}{2x^0|\mathbf{b}|}-\frac{b^0}{|\mathbf{b}|}}\right). \tag{2.191}$$

Next, we observe that the fraction $\frac{|b^2|}{(b^0+|\mathbf{b}|\cos\vartheta)^2|\mathbf{y}'|}$ is the derivative of the fraction which occurs in the argument of the second g-function. Introducing $u = \cos\vartheta$ allows us to rewrite (2.191) as

$$(2.191) = \frac{\lambda\|\psi\|_g}{8(4\pi)^2}\int_{B_{y^0}(\mathbf{y})} d^3\mathbf{y}' \int_{-1}^{1} du\; 2\,\mathrm{sgn}(b^2)\,\partial_u g_1\left(x^0 - \frac{1}{2}\frac{b^2}{b^0+|\mathbf{b}|u}\right) g(y^0-|\mathbf{y}'|) \tag{2.192}$$
$$\times\left(1_{b^2>0}1_{b^0>0}1_{u>\frac{b^2}{2x^0|\mathbf{b}|}-\frac{b^0}{|\mathbf{b}|}} + 1_{b^2<0}1_{u<\frac{b^2}{2x^0|\mathbf{b}|}-\frac{b^0}{|\mathbf{b}|}}\right)\frac{1}{|\mathbf{b}||\mathbf{y}'|}$$
$$= \frac{\lambda\|\psi\|_g}{4(4\pi)^2}\int_{B_{y^0}(\mathbf{y})} d^3\mathbf{y}' \int_{-1}^{1} du\; \partial_u g_1\left(x^0 - \frac{1}{2}\frac{b^2}{b^0+|\mathbf{b}|u}\right) g(y^0-|\mathbf{y}'|) \tag{2.193}$$
$$\times\left(1_{b^2>0}1_{b^0>0}1_{u>\frac{b^2}{2x^0|\mathbf{b}|}-\frac{b^0}{|\mathbf{b}|}} - 1_{b^2<0}1_{u<\frac{b^2}{2x^0|\mathbf{b}|}-\frac{b^0}{|\mathbf{b}|}}\right)\frac{1}{|\mathbf{b}||\mathbf{y}'|}.$$

This form allows for a direct integration with respect to u. Before we integrate, we check whether the conditions implicit in the characteristic functions can always be satisfied. (Otherwise, the respective term would not contribute any further and we could drop it.) Recall that $b = x - y - (-|\mathbf{y}'|, \mathbf{y}')$. First we check whether in the case $b^2 > 0, b^0 > 0$ it is true that $1 > \frac{b^2}{2x^0|\mathbf{b}|} - \frac{b^0}{|\mathbf{b}|}$ holds. (The comparison with 1 is due to the upper range for u.) We compute

$$\begin{aligned} 1 > \frac{b^2}{2x^0|\mathbf{b}|} - \frac{b^0}{|\mathbf{b}|} &\iff 2x^0|\mathbf{b}| + 2x^0 b^0 > b^2 \\ &\iff 2x^0(b^0 + |\mathbf{b}|) > (b^0 + |\mathbf{b}|)(b^0 - |\mathbf{b}|) \\ &\overset{b^2>0, b^0>0}{\iff} 2x^0 > b^0 - |\mathbf{b}| \\ &\iff x^0 + y^0 - |\mathbf{y}'| > -|\mathbf{b}|. \end{aligned} \tag{2.194}$$

Now because of $|\mathbf{y}'| < y^0$ we see that this inequality always holds true. Hence the respective term in (2.194) contributes without further restrictions.

Next, we turn to the case $b^2 < 0$. Here we check whether (or when) $-1 < \frac{b^2}{2x^0|\mathbf{b}|} - \frac{b^0}{|\mathbf{b}|}$ holds. (The comparison with -1 is due to the lower bound for u.) A similar calculation yields

$$-1 < \frac{b^2}{2x^0|\mathbf{b}|} - \frac{b^0}{|\mathbf{b}|} \iff -2x^0|\mathbf{b}| + 2x^0|\mathbf{b}| < b^2 \tag{2.195}$$

$$\iff 2x^0(b^0 - |\mathbf{b}|) < (b^0 - |\mathbf{b}|)(b^0 + |\mathbf{b}|) \tag{2.196}$$

$$\overset{b^2<0}{\iff} 2x^0 > b^0 + |\mathbf{b}|. \tag{2.197}$$

This inequality need not always hold, as we can increase $|\mathbf{b}|$ with respect to b^0 as much as we like, e.g., by picking $|\mathbf{x} - \mathbf{y}|$ large. Therefore, in this case, the respective term is only sometimes nonzero. We make this clear by including the characteristic function $1_{2x^0 > b^0 + |\mathbf{b}|}$.

Taking these considerations into account, we now carry out the u-

integration in (2.194):

$$|A_0\psi|(x,y) \leqslant \frac{\lambda\|\psi\|_g}{4(4\pi)^2}\int_{B_{y^0}(\mathbf{y})} d^3\mathbf{y}'\ \frac{g(y^0-|\mathbf{y}'|)}{|\mathbf{b}||\mathbf{y}'|} \tag{2.198}$$

$$\times\left(1_{b^2>0,b^0>0}\left[g_1\left(x^0-\frac{1}{2}\frac{b^2}{b^0+|\mathbf{b}|}\right)-g_1\left(x^0-\frac{1}{2}\frac{b^2}{b^0+|\mathbf{b}|\max(-1,\frac{b^2}{2x^0|\mathbf{b}|}-\frac{b^0}{|\mathbf{b}|})}\right)\right]\right. \tag{2.199}$$

$$\left.-1_{b^2<0}1_{2x^0>b^0+|\mathbf{b}|}\left[g_1\left(x^0-\frac{1}{2}\frac{b^2}{b^0+|\mathbf{b}|\min(1,\frac{b^2}{2x^0|\mathbf{b}|}-\frac{b^0}{|\mathbf{b}|})}\right)-g_1\left(x^0-\frac{1}{2}\frac{b^2}{b^0-|\mathbf{b}|}\right)\right]\right). \tag{2.200}$$

The minima and maxima in this expression result from the indicator functions $1_{u>\frac{b^2}{2x^0|\mathbf{b}|}-\frac{b^0}{|\mathbf{b}|}}$ and $1_{u<\frac{b^2}{2x^0|\mathbf{b}|}-\frac{b^0}{|\mathbf{b}|}}$, respectively.
Our next step is to simplify the complicated fractions in (2.200) involving min and max. For the first one we use that $1/\max(a,b) = \min(1/a,1/b)$ whenever $a,b>0$ or $a,b<0$ holds. Therefore, we have:

$$\begin{aligned}\frac{1}{2}\frac{b^2}{b^0+|\mathbf{b}|\max\left(-1,\frac{b^2}{2x^0|\mathbf{b}|}-\frac{b^0}{|\mathbf{b}|}\right)} &= \frac{1}{2}\frac{b^2}{\max\left(b^0-|\mathbf{b}|,\frac{b^2}{2x^0}\right)}\\ &= \frac{1}{2}\min\left(\frac{b^2}{b^0-|\mathbf{b}|},2x^0\right)\\ &= \min\left(\frac{b^0+|\mathbf{b}|}{2},x^0\right).\end{aligned} \tag{2.201}$$

The fraction in (2.200) which contains a minimum can be simplified by observing that

$$b^0+|\mathbf{b}|\min\left(1,\frac{b^2}{2x^0|\mathbf{b}|}-\frac{b^0}{|\mathbf{b}|}\right)=\min\left(b^0+|\mathbf{b}|,\frac{b^2}{2x^0}\right)=\frac{b^2}{2x^0} \tag{2.202}$$

as the term contributes only for $b^2 < 0$ and $2x^0 > b^0 + |\mathbf{b}|$ (note that then $\frac{b^2}{2x^0} < \frac{(b^0)^2 - |\mathbf{b}|^2}{b^0 + |\mathbf{b}|} = b^0 - |\mathbf{b}| < b^0 + |\mathbf{b}|$). Thus,

$$\frac{1}{2} \frac{b^2}{b^0 + |\mathbf{b}| \min(1, \frac{b^2}{2x^0|\mathbf{b}|} - \frac{b^0}{|\mathbf{b}|})} = x^0. \tag{2.203}$$

With these simplifications, we obtain (using $g_1(0) = 0$):

$$\begin{aligned}
|A_0\psi|(x,y) \leqslant\ & \frac{\lambda\|\psi\|_g}{4(4\pi)^2} \int_{B_{y^0}(\mathbf{y})} d^3\mathbf{y}'\ \frac{g(y^0 - |\mathbf{y}'|)}{|\mathbf{b}||\mathbf{y}'|} \\
& \times \left(1_{b^2>0, b^0>0} \left[g_1\left(x^0 - \frac{b^0 - |\mathbf{b}|}{2}\right) - g_1\left(x^0 - \min\left(\frac{b^0 + |\mathbf{b}|}{2}, x^0\right)\right)\right] \right. \\
& \left. - 1_{b^2<0}\ 1_{2x^0 > b^0 + |\mathbf{b}|} \left[g_1\left(x^0 - x^0\right) - g_1\left(x^0 - \frac{b^0 + |\mathbf{b}|}{2}\right)\right]\right) \\
= & \frac{\lambda\|\psi\|_g}{4(4\pi)^2} \int_{B_{y^0}(\mathbf{y})} d^3\mathbf{y}' \frac{g(y^0 - |\mathbf{y}'|)}{|\mathbf{b}||\mathbf{y}'|} 1_{b^2>0, b^0>0}\, g_1\left(\frac{x^0 + y^0 - |\mathbf{y}'| + |\mathbf{b}|}{2}\right) \qquad (2.204) \\
& - \frac{\lambda\|\psi\|_g}{4(4\pi)^2} \int_{B_{y^0}(\mathbf{y})} d^3\mathbf{y}'\ \frac{g(y^0 - |\mathbf{y}'|)}{|\mathbf{b}||\mathbf{y}'|} 1_{b^2>0, b^0>0}\, g_1\left(\max\left(\frac{x^0 + y^0 - |\mathbf{y}'| - |\mathbf{b}|}{2}, 0\right)\right) \qquad (2.205) \\
& + \frac{\lambda\|\psi\|_g}{4(4\pi)^2} \int_{B_{y^0}(\mathbf{y})} d^3\mathbf{y}'\ \frac{g(y^0 - |\mathbf{y}'|)}{|\mathbf{b}||\mathbf{y}'|} 1_{b^2<0}\ 1_{x^0 + y^0 - |\mathbf{y}'| > |\mathbf{b}|}\ g_1\left(\frac{x^0 + y^0 - |\mathbf{y}'| - |\mathbf{b}|}{2}\right). \qquad (2.206)
\end{aligned}$$

We now want to carry out as many of the remaining $\mathbf{y}'$-integrations as possible. In order to do so, we orient the coordinates such that $\mathbf{x} - \mathbf{y}$ is parallel to the $(\mathbf{y}')_3$ axis. Then the integrands in (2.204)-(2.206) are independent of the azimuthal angle φ of the respective spherical coordinate system (ρ, θ, φ) with standard conventions.

In order to perform the remaining angular and then the radial integral, we need to find out which boundaries for θ and r result from the characteristic functions. First we analyze for which arguments the maximum in (2.205) is greater than zero and therefore contributes to the integral (as $g_1(0) = 0$). We have:

$$\frac{x^0 + y^0 - |\mathbf{y}'| - |\mathbf{b}|}{2} > 0 \iff (x^0 + y^0 - |\mathbf{y}'|)^2 > |\mathbf{x} - \mathbf{y}|^2 + |\mathbf{y}'|^2 + 2|\mathbf{y}'||\mathbf{x} - \mathbf{y}| \cos\theta$$
$$\iff \cos\theta < \frac{(x^0 + y^0)^2}{2|\mathbf{y}'||\mathbf{x} - \mathbf{y}|} - \frac{|\mathbf{x} - \mathbf{y}|}{2|\mathbf{y}'|} - \frac{x^0 + y^0}{|\mathbf{x} - \mathbf{y}|} =: P_{x,y}(|\mathbf{y}'|). \tag{2.207}$$

This calculation also helps to reformulate the second indicator function $1_{b^2<0}\ 1_{x^0+y^0-|\mathbf{y}'|>|\mathbf{b}|}$ in (2.206) (for which we have $b^2 < 0$). The condition $b^0 > 0$ in (2.204) and (2.205) is readily seen to be equivalent to

$$|\mathbf{y}'| > y^0 - x^0. \tag{2.208}$$

In order to perform the θ-integral we have to translate $b^2 \gtrless 0$ into conditions on θ. We have:

$$b^2 > 0 \iff (x^0 - y^0 + |\mathbf{y}'|)^2 > |\mathbf{x} - \mathbf{y}|^2 + |\mathbf{y}'|^2 + 2|\mathbf{y}'||\mathbf{x} - \mathbf{y}| \cos\theta$$
$$\iff \cos\theta < \frac{(x - y)^2}{2|\mathbf{y}'||\mathbf{x} - \mathbf{y}|} + \frac{x^0 - y^0}{|\mathbf{x} - \mathbf{y}|} := K_{x-y}(|\mathbf{y}'|). \tag{2.209}$$

With these considerations, we have extracted relatively simple conditions on the boundaries of the integrals in spherical coordinates. However, if different restrictions of the boundaries conflict with each other, it may happen that for some parameter values the domain of integration is the empty set. We check whether this is so term by term, focusing on the θ-integration first. For term (2.204), θ needs to satisfy $-1 < \cos\theta < \min(1, K_{x-y}(|\mathbf{y}'|))$, so we need to check whether

$-1 < K_{x-y}(|\mathbf{y}'|)$ holds. We have:

$$
\begin{aligned}
-1 < K_{x-y}(|\mathbf{y}'|) &\iff -2|\mathbf{y}'||\mathbf{x}-\mathbf{y}| < (x-y)^2 + 2|\mathbf{y}'|(x^0 - y^0)\\
&\iff 0 \; < \; (x-y)^2 + 2|\mathbf{y}'|(x^0 - y^0 + |\mathbf{x}-\mathbf{y}|)\\
&\iff \begin{cases} \frac{y^0-x^0+|\mathbf{x}-\mathbf{y}|}{2} < |\mathbf{y}'| & \text{for } |\mathbf{x}-\mathbf{y}| > y^0 - x^0\\ \frac{y^0-x^0+|\mathbf{x}-\mathbf{y}|}{2} > |\mathbf{y}'| & \text{for } |\mathbf{x}-\mathbf{y}| < y^0 - x^0. \end{cases}
\end{aligned}
\tag{2.210}
$$

Together with (2.208), we obtain the condition $y^0 - x^0 < |\mathbf{y}'| < \frac{y^0-x^0+|\mathbf{x}-\mathbf{y}|}{2} < y^0 - x^0$ in the second case which means that there is no contribution to the integral. So we focus on the first case,

$$
\frac{y^0 - x^0 + |\mathbf{x}-\mathbf{y}|}{2} < |\mathbf{y}'| \quad \text{and } |\mathbf{x}-\mathbf{y}| > y^0 - x^0, \tag{2.211}
$$

by including the characteristic function $1_{|\mathbf{x}-\mathbf{y}|>y^0-x^0}$ in the integral. Next, we turn to the radial integral. By comparing its upper limit $|\mathbf{y}'| < y^0$ and lower limit $(y^0 - x^0 + |\mathbf{x}-\mathbf{y}|)/2$, we find that the integral can only be nonzero for

$$
y^0 + x^0 > |\mathbf{x}-\mathbf{y}|. \tag{2.212}
$$

We make this clear by including the respective characteristic function.

2.3.4.1.1.1 Simplification of term (2.204). These considerations allow us to continue computing (2.204):

$$
\begin{aligned}
(2.204) = \frac{\lambda\|\psi\|_g}{4(4\pi)^2} 1_{y^0+x^0>|\mathbf{x}-\mathbf{y}|} \int_{\max(0,y^0-x^0)}^{y^0} d\rho \int_0^{2\pi} d\varphi \; 1_{\frac{y^0-x^0+|\mathbf{x}-\mathbf{y}|}{2}<\rho} 1_{|\mathbf{x}-\mathbf{y}|>y^0-x^0}\\
\times \int_{-1}^{\min(1,K_{x-y}(\rho))} d\cos\theta \; \frac{\rho\, g(y^0-\rho)}{\sqrt{|\mathbf{x}-\mathbf{y}|^2 + \rho^2 + 2|\mathbf{x}-\mathbf{y}|\rho\cos\theta}}\\
\times g_1\left(\frac{x^0 + y^0 - \rho + \sqrt{|\mathbf{x}-\mathbf{y}|^2 + \rho^2 + 2\rho|\mathbf{x}-\mathbf{y}|\cos\theta}}{2}\right).
\end{aligned}
\tag{2.213}
$$

Now we carry out the φ-integration and use the same trick for the θ-integral as for the ϑ-integral in the $\mathbf{x}'$-integration earlier. Moreover, we absorb some of the restrictions of ρ into the limits of the integrals. This yields:

$$
\begin{aligned}
(2.204) = {} & \frac{\lambda \|\psi\|_g}{8(4\pi)} 1_{y^0+x^0>|\mathbf{x}-\mathbf{y}|>y^0-x^0} \int_{\max\left(0,y^0-x^0,\frac{y^0-x^0+|\mathbf{x}-\mathbf{y}|}{2}\right)}^{y^0} d\rho \int_{-1}^{\min(1,K_{x-y}(\rho))} dw \, \frac{2g(y^0-\rho)}{|\mathbf{x}-\mathbf{y}|} \\
& \times \partial_w g_2 \left(\frac{x^0+y^0-\rho+\sqrt{|\mathbf{x}-\mathbf{y}|^2+\rho^2+2\rho|\mathbf{x}-\mathbf{y}|w}}{2} \right) \\
= {} & \frac{\lambda \|\psi\|_g}{4(4\pi)} 1_{x^0+y^0>|\mathbf{x}-\mathbf{y}|>y^0-x^0} \int_{\max\left(0,y^0-x^0,\frac{y^0-x^0+|\mathbf{x}-\mathbf{y}|}{2}\right)}^{y^0} d\rho \, \frac{g(y^0-\rho)}{|\mathbf{x}-\mathbf{y}|} \\
& \times \Bigg[g_2 \left(\frac{x^0+y^0-\rho+\sqrt{|\mathbf{x}-\mathbf{y}|^2+\rho^2+2\rho|\mathbf{x}-\mathbf{y}|\min(1,K_{x-y}(\rho))}}{2} \right) \\
& -g_2 \left(\frac{x^0+y^0-\rho+||\mathbf{x}-\mathbf{y}|-\rho|}{2} \right) \Bigg]
\end{aligned}
\tag{2.214}
$$

The square root can be simplified using the following identity:

$$
\sqrt{|\mathbf{x}-\mathbf{y}|^2+\rho^2+2\rho|\mathbf{x}-\mathbf{y}|K_{x-y}(\rho)} = \sqrt{\rho^2+(x^0-y^0)^2+2\rho(x^0-y^0)} = |x^0-y^0+\rho|.
\tag{2.215}
$$

Using this, we can effectively pull the minimum out of the square root. We obtain:

$$
\begin{aligned}
(2.204) = {} & \frac{\lambda \|\psi\|_g}{16\pi} 1_{x^0+y^0>|\mathbf{x}-\mathbf{y}|>y^0-x^0} \int_{\max\left(0,y^0-x^0,\frac{y^0-x^0+|\mathbf{x}-\mathbf{y}|}{2}\right)}^{y^0} d\rho \, \frac{g(y^0-\rho)}{|\mathbf{x}-\mathbf{y}|} \\
\times \Bigg[& g_2 \left(\frac{x^0+y^0-\rho+\min(|\mathbf{x}-\mathbf{y}|+\rho,|x^0-y^0+\rho|)}{2} \right) - g_2 \left(\frac{x^0+y^0-\rho+||\mathbf{x}-\mathbf{y}|-
\end{aligned}
\tag{2.216}
$$

Next, we subdivide the conditions in the first indicator function into two cases, (a) $(x-y)^2 > 0$ and (b) $(x-y)^2 < 0$. In case (a), the condition $|\mathbf{x}-\mathbf{y}| > y^0 - x^0$ implies $x^0 > y^0$. This, in turn, yields $\max\left(0, y^0 - x^0, \frac{y^0-x^0+|\mathbf{x}-\mathbf{y}|}{2}\right) = 0$. Moreover, the condition $x^0 + y^0 > |\mathbf{x}-\mathbf{y}|$ is automatically satisfied (note that $x^0, y^0 > 0$). In case (b), the condition $|\mathbf{x}-\mathbf{y}| > 0$ is automatically satisfied. We find:

$$
\begin{aligned}
(2.204) = {} & \frac{\lambda\|\psi\|_g}{16\pi} 1_{(x-y)^2>0, x^0>y^0} \int_0^{y^0} d\rho \, \frac{g(y^0-\rho)}{|\mathbf{x}-\mathbf{y}|} \\
& \times \left[g_2\left(\frac{x^0+y^0+|\mathbf{x}-\mathbf{y}|}{2}\right) - g_2\left(\frac{x^0+y^0-\rho+||\mathbf{x}-\mathbf{y}|-\rho|}{2}\right)\right] \\
& + \frac{\lambda\|\psi\|_g}{16\pi} 1_{(x-y)^2<0} \; 1_{x^0+y^0>|\mathbf{x}-\mathbf{y}|} \int_{\frac{y^0-x^0+|\mathbf{x}-\mathbf{y}|}{2}}^{y^0} d\rho \, \frac{g(y^0-\rho)}{|\mathbf{x}-\mathbf{y}|} \\
& \times \left[g_2\left(\frac{x^0+y^0-\rho+|x^0-y^0+\rho|}{2}\right) - g_2\left(\frac{x^0+y^0-\rho+||\mathbf{x}-\mathbf{y}|-\rho|}{2}\right)\right] \\
= {} & \frac{\lambda\|\psi\|_g}{16\pi} 1_{(x-y)^2>0, x^0>y^0} \int_0^{y^0} d\rho \, \frac{g(y^0-\rho)}{|\mathbf{x}-\mathbf{y}|} \\
& \times \Bigg[g_2\left(\frac{x^0+y^0+|\mathbf{x}-\mathbf{y}|}{2}\right) - g_2 \max\Bigg(\frac{x^0+y^0-|\mathbf{x}-\mathbf{y}|}{2}, \frac{x^0+y^0+|\mathbf{x}-\mathbf{y}|}{2} - \rho \\
& + \frac{\lambda\|\psi\|_g}{16\pi} 1_{(x-y)^2<0} \; 1_{x^0+y^0>|\mathbf{x}-\mathbf{y}|} \int_{\frac{y^0-x^0+|\mathbf{x}-\mathbf{y}|}{2}}^{y^0} d\rho \, \frac{g(y^0-\rho)}{|\mathbf{x}-\mathbf{y}|} \\
& \times \left[g_2 \max\left(x^0, y^0-\rho\right) - g_2 \max\left(\frac{x^0+y^0-|\mathbf{x}-\mathbf{y}|}{2}, \frac{x^0+y^0+|\mathbf{x}-\mathbf{y}|}{2} - \rho\right)\right].
\end{aligned}
\tag{2.217}
$$

Here and in the following we abbreviate $g_2(\max(\cdots))$ as $g_2 \max(\cdots)$, and similarly for the minimum. This ends the calculation of (2.204): we have arrived at an expression where no more exact calculations can be done and further estimates are needed.

2.3.4.1.1.2 Simplification of term (2.205). Next, we proceed with (2.205) in a similar fashion. In case the reader is not interested in the details of the calculation, the result can be found in (2.224).

The restrictions of the integration variables for (2.205) are the same as for (2.204), namely:

$$\cos\theta < K_{x-y}(|\mathbf{y}'|) \qquad \text{from (2.209)}, \tag{2.218}$$

$$\frac{y^0 - x^0 + |\mathbf{x}-\mathbf{y}|}{2} < |\mathbf{y}'| \qquad \text{from (2.211)} \tag{2.219}$$

$$y^0 - x^0 < |\mathbf{x}-\mathbf{y}| < y^0 + x^0 \qquad \text{from (2.211) and from (2.212)}. \tag{2.220}$$

The only difference is that from the maximum in (2.205), we obtain the additional restriction (2.207), i.e.

$$\cos\theta < P_{x,y}(|\mathbf{y}'|). \tag{2.221}$$

We need to check if there are new restrictions imposed by $P_{x,y}(|\mathbf{y}'|) > -1$. We compute

$$\begin{aligned} P_{x,y}(|\mathbf{y}'|) &> -1 \iff \\ \frac{(x^0+y^0)^2}{2|\mathbf{y}'||\mathbf{x}-\mathbf{y}|} - \frac{|\mathbf{x}-\mathbf{y}|}{2|\mathbf{y}'|} - \frac{x^0+y^0}{|\mathbf{x}-\mathbf{y}|} &> -1 \iff \\ |\mathbf{y}'| &< \frac{x^0+y^0+|\mathbf{x}-\mathbf{y}|}{2}; \end{aligned} \tag{2.222}$$

however, the last inequality is already ensured by (2.219) and $x^0 > 0$. In order to be able to evaluate (2.205) further, we next plug the condition $\cos\theta < P_{x,y}(|\mathbf{y}'|)$ into the expression for $|\mathbf{b}|$. This yields (recall that we use spherical variables for $|\mathbf{y}'|$):

$$\begin{aligned} |\mathbf{b}| &= \sqrt{|\mathbf{x}-\mathbf{y}|^2 + \rho^2 + 2\rho|\mathbf{x}-\mathbf{y}|\cos\theta} < \sqrt{|\mathbf{x}-\mathbf{y}|^2 + \rho^2 + 2\rho|\mathbf{x}-\mathbf{y}|P_{x,y}(\rho)} \\ &= \sqrt{\rho^2 - 2\rho(x^0+y^0) + (x^0+y^0)^2} = x^0 + y^0 - \rho. \end{aligned} \tag{2.223}$$

With this, we perform for (2.205) the analogous calculation to (2.213)–(2.216). This yields:

$$
\begin{aligned}
(2.205) = {} & \frac{\lambda \|\psi\|_g}{16\pi} 1_{y^0 - x^0 < |\mathbf{x}-\mathbf{y}| < x^0 + y^0} \int_{\max\left(0, y^0 - x^0, \frac{y^0 - x^0 + |\mathbf{x}-\mathbf{y}|}{2}\right)}^{y^0} d\rho \; \frac{g(y^0-\rho)}{|\mathbf{x}-\mathbf{y}|} \\
& \times \left[g_2 \left(\frac{x^0 + y^0 - \rho - \min(|\mathbf{x}-\mathbf{y}| + \rho, |x^0 - y^0 + \rho|, x^0 + y^0 - \rho)}{2} \right) \right. \\
& \left. - g_2 \left(\frac{x^0 + y^0 - \rho - ||\mathbf{x}-\mathbf{y}| - \rho|}{2} \right) \right] \\
= {} & \frac{\lambda \|\psi\|_g}{16\pi} 1_{(x-y)^2 > 0, x^0 > y^0} \int_0^{y^0} d\rho \; \frac{g(y^0-\rho)}{|\mathbf{x}-\mathbf{y}|} \\
& \times \left[g_2 \left(\frac{x^0 + y^0 - |\mathbf{x}-\mathbf{y}|}{2} - \rho \right) - g_2 \left(\frac{x^0 + y^0 - \rho - ||\mathbf{x}-\mathbf{y}| - \rho|}{2} \right) \right] \\
& + \frac{\lambda \|\psi\|_g}{16\pi} 1_{(x-y)^2 < 0} \; 1_{x^0 + y^0 > |\mathbf{x}-\mathbf{y}|} \int_{\frac{y^0 - x^0 + |\mathbf{x}-\mathbf{y}|}{2}}^{y^0} d\rho \; \frac{g(y^0-\rho)}{|\mathbf{x}-\mathbf{y}|} \\
& \times \left[g_2 \left(\frac{x^0 + y^0 - \rho - |x^0 - y^0 + \rho|}{2} \right) - g_2 \left(\frac{x^0 + y^0 - \rho - ||\mathbf{x}-\mathbf{y}| - \rho|}{2} \right) \right. \\
= {} & \frac{\lambda \|\psi\|_g}{16\pi} 1_{(x-y)^2 > 0, x^0 > y^0} \int_0^{y^0} d\rho \; \frac{g(y^0-\rho)}{|\mathbf{x}-\mathbf{y}|} \\
& \times \left[g_2 \left(\frac{x^0 + y^0 - |\mathbf{x}-\mathbf{y}|}{2} - \rho \right) - g_2 \min \left(\frac{x^0 + y^0 - |\mathbf{x}-\mathbf{y}|}{2}, \frac{x^0 + y^0 + |}{2} \right. \right. \\
& + \frac{\lambda \|\psi\|_g}{16\pi} 1_{(x-y)^2 < 0} \; 1_{x^0 + y^0 > |\mathbf{x}-\mathbf{y}|} \int_{\frac{y^0 - x^0 + |\mathbf{x}-\mathbf{y}|}{2}}^{y^0} d\rho \; \frac{g(y^0-\rho)}{|\mathbf{x}-\mathbf{y}|} \\
& \times \left[g_2 \min\left(x^0, y^0 - \rho\right) - g_2 \min \left(\frac{x^0 + y^0 - |\mathbf{x}-\mathbf{y}|}{2}, \frac{x^0 + y^0 + |\mathbf{x}-\mathbf{y}|}{2} - \right. \right.
\end{aligned}
\tag{2.224}
$$

This ends the calculation of (2.205).

2.3.4.1.1.3 Simplification of term (2.206). We next turn to (2.206). In case the reader is not interested in the details of the computation, the result can be found in (2.237). First we note that the restriction imposed by the first indicator function here is $\cos\theta > K_{x-y}(|\mathbf{y}'|)$ and the condition of the second indicator function is $\cos\theta < P_{x,y}(|\mathbf{y}'|)$. In order to to satisfy these conditions (and the restrictions of the regular range of integration) it is required that

$$\max(-1, K_{x-y}(|\mathbf{y}'|) < \cos\theta < \min(1, P_{x,y}(|\mathbf{y}'|)). \tag{2.225}$$

This leads us to ask which restrictions on $|\mathbf{y}'|$ are imposed by the conditions

$$\begin{aligned} K_{x-y}(|\mathbf{y}'|) &< 1, & (2.226)\\ P_{x,y}(|\mathbf{y}'|) &> -1, & (2.227)\\ K_{x-y}(|\mathbf{y}'|) &< P_{x,y}(|\mathbf{y}'|). & (2.228) \end{aligned}$$

These restrictions shall be computed next. With $|\mathbf{y}'| = \rho$, we find:

$$\begin{aligned} K_{x-y}(|\mathbf{y}'|) < 1 \quad &\Longleftrightarrow \quad \frac{(x-y)^2}{2\rho|\mathbf{x}-\mathbf{y}|} + \frac{x^0-y^0}{|\mathbf{x}-\mathbf{y}|} < 1 \\ &\Longleftrightarrow \quad (x-y)^2 < 2\rho(y^0-x^0+|\mathbf{x}-\mathbf{y}|) \\ &\Longleftrightarrow \quad \begin{cases} \rho > \frac{y^0-x^0-|\mathbf{x}-\mathbf{y}|}{2} & \text{for } |\mathbf{x}-\mathbf{y}| > x^0-y^0, \\ \rho < \frac{y^0-x^0-|\mathbf{x}-\mathbf{y}|}{2} & \text{for } |\mathbf{x}-\mathbf{y}| < x^0-y^0. \end{cases} \end{aligned} \tag{2.229}$$

The second case in the last line is in conflict with $\rho > 0$, so we have to impose the first condition on (2.206). We continue with $P_{x,y}(\rho) > -1$.

$$\begin{aligned} P_{x,y}(\rho) > -1 \quad &\Longleftrightarrow \quad \frac{(x^0+y^0)^2}{2\rho|\mathbf{x}-\mathbf{y}|} - \frac{|\mathbf{x}-\mathbf{y}|}{2\rho} - \frac{x^0+y^0}{|\mathbf{x}-\mathbf{y}|} > -1 \\ &\Longleftrightarrow \quad (x^0+y^0)^2 - |\mathbf{x}-\mathbf{y}|^2 > 2\rho(x^0+y^0-|\mathbf{x}-\mathbf{y}|) \end{aligned}$$

$$\iff \quad \begin{cases} \rho < \frac{x^0+y^0+|\mathbf{x}-\mathbf{y}|}{2} & \text{for } x^0+y^0 > |\mathbf{x}-\mathbf{y}|, \\ \rho > \frac{x^0+y^0+|\mathbf{x}-\mathbf{y}|}{2} & \text{for } x^0+y^0 < |\mathbf{x}-\mathbf{y}|. \end{cases} \tag{2.230}$$

The second case is in conflict with $\rho < y^0$, so we implement indicator functions corresponding only to the first case in (2.206). The third condition $K_{x-y}(\rho) < P_{x,y}(\rho)$ in fact does not impose any additional conditions. This can be seen as follows:

$$\begin{aligned} K_{x-y}(\rho) \; < \; P_{x,y}(\rho) \quad & \iff \quad \frac{(x-y)^2}{2\rho|\mathbf{x}-\mathbf{y}|} + \frac{x^0-y^0}{|\mathbf{x}-\mathbf{y}|} \; < \; \frac{(x^0+y^0)^2}{2\rho|\mathbf{x}-\mathbf{y}|} - \frac{|\mathbf{x}-\mathbf{y}|}{2\rho} - \frac{x^0}{|\mathbf{x}} \\ & \iff \quad -2x^0y^0 + 4\rho x^0 \; < \; 2x^0y^0 \\ & \iff \quad \rho < y^0, \end{aligned} \tag{2.231}$$

which always holds true.

Taking into account the computed restrictions, we arrive at:

$$\begin{aligned} (2.206) \overset{\cos\theta = w}{=} & \frac{\lambda\|\psi\|_g}{4(4\pi)^2} \int_0^{2\pi} d\varphi \int_0^{y^0} d\rho \int_{-1}^{1} dw \, 1_{K_{x-y}(\rho)<w<P_{x,y}(\rho)} 1_{\frac{y^0-x^0-|\mathbf{x}-\mathbf{y}|}{2}<\rho<\frac{x^0+y^0+|\mathbf{x}-}{2}} \\ & \times 1_{x^0-y^0<|\mathbf{x}-\mathbf{y}|<x^0+y^0} \frac{g(y^0-\rho)\rho}{\sqrt{\rho^2+|\mathbf{x}-\mathbf{y}|^2+2\rho|\mathbf{x}-\mathbf{y}|w}} \\ & \times g_1\left(\frac{x^0+y^0-\sqrt{\rho^2+|\mathbf{x}-\mathbf{y}|^2+2\rho|\mathbf{x}-\mathbf{y}|w}}{2}\right) \\ = & \frac{\lambda\|\psi\|_g 2\pi}{4(4\pi)^2} 1_{x^0-y^0<|\mathbf{x}-\mathbf{y}|<x^0+y^0} \int_{\max\left(0,\frac{y^0-x^0-|\mathbf{x}-\mathbf{y}|}{2}\right)}^{\min\left(y^0,\frac{x^0+y^0+|\mathbf{x}-\mathbf{y}|}{2}\right)} d\rho \int_{\max(-1,K_{x-y}(\rho))}^{\min(1,P_{x,y}(\rho))} dw \, \frac{-2g(y^0-\rho}{|\mathbf{x}-\mathbf{y}|} \\ & \times \partial_w g_2\left(\frac{x^0+y^0-\sqrt{\rho^2+|\mathbf{x}-\mathbf{y}|^2+2\rho|\mathbf{x}-\mathbf{y}|w}}{2}\right) \\ = & \frac{\lambda\|\psi\|_g}{16\pi} 1_{x^0-y^0<|\mathbf{x}-\mathbf{y}|<x^0+y^0} \int_{\max\left(0,\frac{y^0-x^0-|\mathbf{x}-\mathbf{y}|}{2}\right)}^{\min\left(y^0,\frac{x^0+y^0+|\mathbf{x}-\mathbf{y}|}{2}\right)} d\rho \, \frac{g(y^0-\rho)}{|\mathbf{x}-\mathbf{y}|} \end{aligned}$$

$$\times \left[g_2 \left(\frac{x^0 + y^0 - \rho - \sqrt{\rho^2 + |\mathbf{x}-\mathbf{y}|^2 + 2\rho|\mathbf{x}-\mathbf{y}| \max(-1, K_{x-y}(\rho))}}{2} \right) \right.$$
$$\left. -g_2 \left(\frac{x^0 + y^0 - \rho - \sqrt{\rho^2 + |\mathbf{x}-\mathbf{y}|^2 + 2\rho|\mathbf{x}-\mathbf{y}| \min(1, P_{x,y}(\rho))}}{2} \right) \right]. \tag{2.232}$$

At this point, the expressions look quite formidable. We can, however, achieve significant simplifications by inserting the functional form of $K_{x,y}(\rho)$ and $P_{x,y}(\rho)$ as in (2.223) and (2.215). This yields:

$$(2.206) = \frac{\lambda \|\psi\|_g}{16\pi} 1_{x^0-y^0<|\mathbf{x}-\mathbf{y}|<x^0+y^0} \int_{\max\left(0, \frac{y^0-x^0-|\mathbf{x}-\mathbf{y}|}{2}\right)}^{\min\left(y^0, \frac{x^0+y^0+|\mathbf{x}-\mathbf{y}|}{2}\right)} d\rho \, \frac{g(y^0-\rho)}{|\mathbf{x}-\mathbf{y}|}$$
$$\times \left[g_2 \left(\frac{x^0+y^0-\rho-\max(||\mathbf{x}-\mathbf{y}|-\rho|, |x^0-y^0+\rho|)}{2} \right) \right.$$
$$\left. -g_2 \left(\frac{x^0+y^0-\rho-\min(|\mathbf{x}-\mathbf{y}|+\rho, x^0+y^0-\rho)}{2} \right) \right] \tag{2.233}$$

Now we simplify the arguments of the g_2-functions. For the first one, we have:

$$\begin{aligned} &x^0+y^0-\rho-\max(||\mathbf{x}-\mathbf{y}|-\rho|, |x^0-y^0+\rho|) \\ &\quad = \; x^0+y^0-\rho-\max(|\mathbf{x}-\mathbf{y}|-\rho, \rho-|\mathbf{x}-\mathbf{y}|, x^0-y^0+\rho, y^0-\rho-x^0) \\ &\quad = \min\left(x^0+y^0-\rho-|\mathbf{x}-\mathbf{y}|, x^0+y^0+|\mathbf{x}-\mathbf{y}|-2\rho, 2(y^0-\rho), 2x^0\right). \end{aligned} \tag{2.234}$$

And for the second one:

$$x^0+y^0-\min(|\mathbf{x}-\mathbf{y}|+\rho, x^0+y^0-\rho) \;=\; \max(x^0+y^0-|\mathbf{x}-\mathbf{y}|-2\rho, 0). \tag{2.235}$$

Using this in (2.233), we find:

$$
\begin{aligned}
(2.206) = \frac{\lambda\|\psi\|_g}{16\pi} 1_{x^0-y^0<|\mathbf{x}-\mathbf{y}|<x^0+y^0} & \int_{\max\left(0,\frac{y^0-x^0-|\mathbf{x}-\mathbf{y}|}{2}\right)}^{\min\left(y^0,\frac{x^0+y^0+|\mathbf{x}-\mathbf{y}|}{2}\right)} d\rho\, \frac{g(y^0-\rho)}{|\mathbf{x}-\mathbf{y}|} \\
& \times \left[g_2 \min\left(\frac{x^0+y^0-|\mathbf{x}-\mathbf{y}|}{2}, \frac{x^0+y^0+|\mathbf{x}-\mathbf{y}|}{2} - \rho, y^0-\rho, x^0 \right) \right. \\
& \left. - g_2 \max\left(\frac{x^0+y^0-|\mathbf{x}-\mathbf{y}|}{2} - \rho, 0 \right) \right]. \qquad (2.236)
\end{aligned}
$$

As in the consideration below (2.216), we split the expression into separate terms with $(x-y)^2 \gtrless 0$. Using $y^0 \gtrless x^0 + |\mathbf{x}-\mathbf{y}|$, we can simplify the expressions involving the minimum. This results in:

$$
\begin{aligned}
(2.206) = \frac{\lambda\|\psi\|_g}{16\pi} 1_{(x-y)^2>0,y^0>x^0} & \int_{\frac{y^0-x^0-|\mathbf{x}-\mathbf{y}|}{2}}^{\frac{x^0+y^0+|\mathbf{x}-\mathbf{y}|}{2}} d\rho\, \frac{g(y^0-\rho)}{|\mathbf{x}-\mathbf{y}|} \\
\times & \left[g_2 \min\left(\frac{x^0+y^0+|\mathbf{x}-\mathbf{y}|}{2} - \rho, x^0 \right) - g_2 \max\left(\frac{x^0+y^0-|\mathbf{x}-\mathbf{y}|}{2} - \rho, 0 \right) \right. \\
+ & \frac{\lambda\|\psi\|_g}{16\pi} 1_{(x-y)^2<0,|\mathbf{x}-\mathbf{y}|<x^0+y^0} \int_0^{y^0} d\rho\, \frac{g(y^0-\rho)}{|\mathbf{x}-\mathbf{y}|} \\
\times & \left[g_2 \min\left(\frac{x^0+y^0-|\mathbf{x}-\mathbf{y}|}{2}, y^0-\rho \right) - g_2 \max\left(\frac{x^0+y^0-|\mathbf{x}-\mathbf{y}|}{2} - \rho, 0 \right. \right. \\
& \qquad (2.237)
\end{aligned}
$$

This concludes the calculation of (2.206).

2.3.4.1.1.4 Summary of the first estimate. We have obtained the following bound for $|A_0\psi|(x,y)$:

$$
\frac{16\pi}{\lambda\|\psi\|_g} |A_0\psi|(x,y) \leqslant 1_{(x-y)^2>0,x^0>y^0} \int_0^{y^0} d\rho\, \frac{g(y^0-\rho)}{|\mathbf{x}-\mathbf{y}|}
$$

$$
\begin{aligned}
&\times \left[g_2 \left(\frac{x^0 + y^0 + |\mathbf{x} - \mathbf{y}|}{2} \right) - g_2 \max \left(\frac{x^0 + y^0 - |\mathbf{x} - \mathbf{y}|}{2}, \frac{x^0 + y^0 + |\mathbf{x} - \mathbf{y}|}{2} - \rho \right) \right] \\
&+ 1_{(x-y)^2<0} \; 1_{x^0+y^0>|\mathbf{x}-\mathbf{y}|} \int_{\frac{y^0 - x^0 + |\mathbf{x}-\mathbf{y}|}{2}}^{y^0} d\rho \, \frac{g(y^0 - \rho)}{|\mathbf{x} - \mathbf{y}|} \\
&\times \left[g_2 \max \left(x^0, y^0 - \rho \right) - g_2 \max \left(\frac{x^0 + y^0 - |\mathbf{x} - \mathbf{y}|}{2}, \frac{x^0 + y^0 + |\mathbf{x} - \mathbf{y}|}{2} - \rho \right) \right] \\
&+ 1_{(x-y)^2>0, x^0>y^0} \int_0^{y^0} d\rho \, \frac{g(y^0 - \rho)}{|\mathbf{x} - \mathbf{y}|} \\
&\times \left[g_2 \left(\frac{x^0 + y^0 - |\mathbf{x} - \mathbf{y}|}{2} - \rho \right) - g_2 \min \left(\frac{x^0 + y^0 - |\mathbf{x} - \mathbf{y}|}{2}, \frac{x^0 + y^0 + |\mathbf{x} - \mathbf{y}|}{2} - \rho \right) \right] \\
&+ 1_{(x-y)^2<0} \; 1_{x^0+y^0>|\mathbf{x}-\mathbf{y}|} \int_{\frac{y^0 - x^0 + |\mathbf{x}-\mathbf{y}|}{2}}^{y^0} d\rho \, \frac{g(y^0 - \rho)}{|\mathbf{x} - \mathbf{y}|} \\
&\times \left[g_2 \min \left(x^0, y^0 - \rho \right) - g_2 \min \left(\frac{x^0 + y^0 - |\mathbf{x} - \mathbf{y}|}{2}, \frac{x^0 + y^0 + |\mathbf{x} - \mathbf{y}|}{2} - \rho \right) \right] \\
&+ 1_{(x-y)^2>0, y^0>x^0} \int_{\frac{y^0 - x^0 - |\mathbf{x}-\mathbf{y}|}{2}}^{\frac{x^0 + y^0 + |\mathbf{x}-\mathbf{y}|}{2}} d\rho \, \frac{g(y^0 - \rho)}{|\mathbf{x} - \mathbf{y}|} \\
&\times \left[g_2 \min \left(\frac{x^0 + y^0 + |\mathbf{x} - \mathbf{y}|}{2} - \rho, x^0 \right) - g_2 \max \left(\frac{x^0 + y^0 - |\mathbf{x} - \mathbf{y}|}{2} - \rho, 0 \right) \right] \\
&+ 1_{(x-y)^2<0, |\mathbf{x}-\mathbf{y}|<x^0+y^0} \int_0^{y^0} d\rho \, \frac{g(y^0 - \rho)}{|\mathbf{x} - \mathbf{y}|} \\
&\times \left[g_2 \min \left(\frac{x^0 + y^0 - |\mathbf{x} - \mathbf{y}|}{2}, y^0 - \rho \right) - g_2 \max \left(\frac{x^0 + y^0 - |\mathbf{x} - \mathbf{y}|}{2} - \rho, 0 \right) \right].
\end{aligned}
\tag{2.238}
$$

In order to simplify the result, we introduce the variables

$$
\xi^+ := \frac{x^0 + y^0 + |\mathbf{x} - \mathbf{y}|}{2}, \tag{2.239}
$$

$$\xi^- := \frac{x^0 + y^0 - |\mathbf{x} - \mathbf{y}|}{2}. \tag{2.240}$$

Moreover, we collect terms with the same indicator functions. This results in:

$$\frac{16\pi}{\lambda\|\psi\|_g}|A_0\psi|(x,y) \leqslant 1_{(x-y)^2<0,\xi^->0} \int_0^{y^0} d\rho\, \frac{g(y^0-\rho)}{|\mathbf{x}-\mathbf{y}|}\Big[g_2 \min(\xi^-, y^0-\rho) - g_2 \max$$

$$+ 1_{\frac{y^0-x^0+|\mathbf{x}-\mathbf{y}|}{2}<\rho}\Big(g_2(x^0) + g_2(y^0-\rho) - g_2(\xi^-) - g_2(\xi^+-\rho)\Big)\Big] \tag{2.241}$$

$$+ 1_{(x-y)^2>0,x^0>y^0} \int_0^{y^0} d\rho\, \frac{g(y^0-\rho)}{|\mathbf{x}-\mathbf{y}|}\big[g_2(\xi^+) + g_2(\xi^- - \rho) - g_2(\xi^-) - g_2(\xi^+-\rho)\big] \tag{2.242}$$

$$+ 1_{(x-y)^2>0,y^0>x^0} \int_{\frac{y^0-x^0-|\mathbf{x}-\mathbf{y}|}{2}}^{\xi^+} d\rho\, \frac{g(y^0-\rho)}{|\mathbf{x}-\mathbf{y}|}\big[g_2 \min(\xi^+-\rho, x^0) - g_2 \max(\xi^- - \rho, 0)\big] \tag{2.243}$$

This estimate is an important stepping stone in the proof. Except for special weight functions, the resulting expressions are too complicated to be computed explicitly. We therefore continue with further estimates. The main difficulty in these estimates is that the $1/|\mathbf{x}-\mathbf{y}|$ singularity in the expressions needs to be compensated by the integrand and that this cancellation needs to be preserved by the respective estimate. Fortunately, the mean value theorem turns out suitable to provide such estimates.

2.3.4.1.1.5 Simplification of (2.241)-(2.243). First, we note that since g, g_1 and g_2 are monotonously increasing and since $\xi^- \leqslant \xi^+$, we have in (2.242):

$$g_2(\xi^- - \rho) - g_2(\xi^+ - \rho) \leqslant 0. \tag{2.244}$$

As the remaining terms in (2.242) still vanish in the limit $|\mathbf{x}-\mathbf{y}| \to 0$, we may replace this difference by zero to obtain a suitable estimate.

Similarly, a brief calculations shows that we have $\xi^+ > y^0$ for $(x-y)^2 < 0$. It follows that:

$$g_2(y^0 - \rho) - g_2(\xi^+ - \rho) < 0. \tag{2.245}$$

We shall use this in (2.241).

Further simplifications can be obtained using the mean value theorem. We begin with the expression in the square brackets in (2.243). The mean value theorem then implies that there is a $\chi \in [\max(\xi^- - \rho, 0), \min(\xi^+ - \rho, x^0)]$ such that

$$g_2 \min(\xi^+ - \rho, x^0) - g_2 \max(\xi^- - \rho, 0) = \big[\min(\xi^+ - \rho, x^0) - \max(\xi^- - \rho, 0)\big] g_1(\chi). \tag{2.246}$$

Therefore, we have:

$$\begin{aligned} & g_2 \min(\xi^+ - \rho, x^0) - g_2 \max(\xi^- - \rho, 0) \\ & \quad \leqslant \ \min(\xi^+ - \xi^-, \xi^+ - \rho, x^0 - \xi^- + \rho, x^0)\, g_1 \min(\xi^+ - \rho, x^0) \\ & \quad \leqslant \ |\mathbf{x} - \mathbf{y}|\, g_1 \min(\xi^+ - \rho, x^0) \ \leqslant \ |\mathbf{x} - \mathbf{y}|\, g_1(x^0). \end{aligned} \tag{2.247}$$

Note that the factor $|\mathbf{x} - \mathbf{y}|$ exactly compensates the $1/|\mathbf{x} - \mathbf{y}|$ singularity. This is the main reason the mean value theorem is so useful here.

Analogously we find for the expression in the square bracket in the first line of (2.241):

$$\begin{aligned} & g_2 \min(\xi^-, y^0 - \rho) - g_2 \max(\xi^- - \rho, 0) \\ & \quad \leqslant \ \big[\min(\xi^-, y^0 - \rho) - \max(\xi^- - \rho, 0)\big] g_1 \min(\xi^-, y^0 - \rho) \\ & \quad = \ \min(\rho, \xi^-, y^0 - \xi^-, y^0 - \rho)\, g_1 \min(\xi^-, y^0 - \rho) \\ & \quad \leqslant \ (y^0 - \xi^-)\, g_1 \min(\xi^-, y^0 - \rho) \\ & \quad \leqslant \ |\mathbf{x} - \mathbf{y}|\, g_1 \min(\xi^-, y^0 - \rho), \end{aligned} \tag{2.248}$$

where we have used that the further restriction of that term, $(x-y)^2 < 0$, implies $|\mathbf{x} - \mathbf{y}| > |x^0 - y^0| \geqslant y^0 - x^0$.

With these considerations, we obtain a rougher but simpler estimate than (2.241)-(2.243):

$$\frac{16\pi}{\lambda\|\psi\|_g}|A_0\psi|(x,y) \leqslant 1_{(x-y)^2<0,\xi^->0}\int_0^{y^0} d\rho\; g(y^0-\rho)\Big[g_1\min(\xi^-,y^0-\rho) \tag{2.249}$$

$$+\,1_{\frac{y^0-x^0+|\mathbf{x}-\mathbf{y}|}{2}<\rho}\frac{g_2(x^0)-g_2(\xi^-)}{|\mathbf{x}-\mathbf{y}|}\Big] \tag{2.250}$$

$$+\,1_{(x-y)^2>0,x^0>y^0}\frac{g_2(\xi^+)-g_2(\xi^-)}{|\mathbf{x}-\mathbf{y}|}\int_0^{y^0} d\rho\; g(y^0-\rho) \tag{2.251}$$

$$+\,1_{(x-y)^2>0,y^0>x^0}\,g_1(x^0)\int_{\frac{y^0-x^0-|\mathbf{x}-\mathbf{y}|}{2}}^{\xi^+} d\rho\; g(y^0-\rho). \tag{2.252}$$

Next, we continue estimating these terms separately so that only expressions without integrals remain.

2.3.4.1.1.6 Further estimate of (2.249). Using the monotonicity of g_1 as well as $\min(\xi^-,y^0-\rho)\leqslant\xi^-$, we find:

$$(2.249) \leqslant 1_{(x-y)^2<0,\xi^->0}\,g_1(\xi^-)\int_0^{y^0} ds\; g(s) = 1_{(x-y)^2<0,\xi^->0}\,g_1(\xi^-)g_1(y^0). \tag{2.253}$$

For the constraints given by the indicator function, we have $\xi^-<x^0$. Thus:

$$(2.249) \leqslant 1_{(x-y)^2<0,\xi^->0}\,g_1(x^0)g_1(y^0). \tag{2.254}$$

2.3.4.1.1.7 Further estimate of (2.250). We have:

$$\begin{aligned}
(2.250) &= 1_{(x-y)^2<0,\xi^->0}\,\frac{g_2(x^0)-g_2(\xi^-)}{|\mathbf{x}-\mathbf{y}|}\int_{\frac{y^0-x^0+|\mathbf{x}-\mathbf{y}|}{2}}^{y_0} d\rho\, g(y^0-\rho)\\
&= 1_{(x-y)^2<0,\xi^->0}\,\frac{g_2(x^0)-g_2(\xi^-)}{|\mathbf{x}-\mathbf{y}|}\int_0^{\xi^-} ds\, g(s)\\
&= 1_{(x-y)^2<0,\xi^->0}\,\frac{g_2(x^0)-g_2(\xi^-)}{|\mathbf{x}-\mathbf{y}|}\big[g_1(\xi^-)-\underbrace{g_1(0)}_{=0}\big]. \qquad (2.255)
\end{aligned}$$

Applying the mean value theorem to g_2 in the interval $[\xi^-,x^0]$ (note that here $\xi^-<x^0$), we obtain that:

$$(2.250) \leqslant 1_{(x-y)^2<0,\xi^->0}\,\frac{x^0-\xi^-}{|\mathbf{x}-\mathbf{y}|}\,g_1(x^0)g_1(\xi^-). \qquad (2.256)$$

Next, we use that $\frac{x^0-\xi^-}{|\mathbf{x}-\mathbf{y}|}=\frac{x^0-y^0+|\mathbf{x}-\mathbf{y}|}{2|\mathbf{x}-\mathbf{y}|}\leqslant 1$ as $|x^0-y^0|<|\mathbf{x}-\mathbf{y}|$. Thus:

$$(2.250) \leqslant 1_{(x-y)^2<0,\xi^->0}\,g_1(x^0)g_1(\xi^-). \qquad (2.257)$$

Using also that for the given constrains $\xi^-<y^0$, we finally obtain:

$$(2.250) \leqslant 1_{(x-y)^2<0,\xi^->0}\,g_1(x^0)g_1(y^0). \qquad (2.258)$$

2.3.4.1.1.8 Further estimate of (2.251). Here, we can directly carry out the remaining integral using the definition of g_1 as the integral of g:

$$(2.251) = 1_{(x-y)^2>0,x^0>y^0}\,\frac{g_2(\xi^+)-g_2(\xi^-)}{|\mathbf{x}-\mathbf{y}|}g_1(y^0). \qquad (2.259)$$

Next, we apply the mean value theorem to g_2 in the interval $[\xi^-,\xi^+]$ noting that $\xi^+-\xi^-=|\mathbf{x}-\mathbf{y}|$. This implies:

$$(2.251) \leqslant 1_{(x-y)^2>0,x^0>y^0}\,g_1(\xi^+)g_1(y^0). \qquad (2.260)$$

Next, we note that $(x-y)^2 > 0 \Leftrightarrow |x^0 - y^0| > |\mathbf{x} - \mathbf{y}|$. Together with $x^0 > y^0$, we obtain $x^0 > y^0 + |\mathbf{x} - \mathbf{y}|$ and therefore:

$$\xi^+ = \frac{x^0 + y^0 + |\mathbf{x} - \mathbf{y}|}{2} \leqslant x^0. \tag{2.261}$$

Thus, we obtain:

$$(2.251) \leqslant 1_{(x-y)^2>0, x^0>y^0}\, g_1(x^0) g_1(y^0). \tag{2.262}$$

2.3.4.1.1.9 Further estimate of (2.252). Here, we carry out the remaining integral as well.

$$\begin{aligned}(2.243) \;&\leqslant\; 1_{(x-y)^2>0, y^0>x^0}\, g_1(x^0)[g_1(\xi^+) - g_1((y^0 - x^0 - |\mathbf{x} - \mathbf{y}|)/2)] \\ &\leqslant\; 1_{(x-y)^2>0, y^0>x^0}\, g_1(x^0) g_1(y^0). \end{aligned} \tag{2.263}$$

as $\xi^+ \leqslant y^0$.

2.3.4.1.1.10 Summary of the result. Gathering the terms (2.254), (2.258), (2.262) and (2.263) yields:

$$\frac{16\pi}{\lambda \|\psi\|_g} |A_0 \psi|(x,y) \leqslant g_1(x^0) g_1(y^0) \big(2 \times 1_{(x-y)^2<0, \xi^->0} + 1_{(x-y)^2>0, x^0>y^0} + 1_{(x-y)^2>0, y^0} \tag{2.264}$$

Considering that the conditions in different indicator functions are mutually exclusive, we finally obtain:

$$\frac{16\pi}{\lambda \|\psi\|_g} |A_0 \psi|(x,y) \leqslant 2 g_1(x^0) g_1(y^0). \tag{2.265}$$

Dividing by $g(x^0)g(y^0)$, taking the supremum over $x, y \in \frac{1}{2}\mathbb{M}$ and factorizing into one-dimensional suprema finally yields the claim (2.159).

2.3.4.1.2 Estimate of the mixed terms (2.160) and (2.161). We focus on A_2 first, starting from its definition (2.152). We take the absolute value and make use of $|\psi(x,y)| \leqslant g(x^0)g(y^0)\,\|\psi\|_g$. Moreover, we use:

$$|J_1(t)/t| \leqslant \frac{1}{2}. \tag{2.266}$$

This yields:

$$\begin{aligned}
|A_2\psi|(x,y) \leqslant \frac{\lambda\, m_2^2\,\|\psi\|_g}{4(4\pi)^3} \int d^3\mathbf{x}' \int d^3\mathbf{y}'\ & \frac{H(x^0-|\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|}\frac{g(x^0-|\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}'-\mathbf{y}'|}\\
& \times \big[H(x^0-|\mathbf{x}-\mathbf{x}'|+|\mathbf{x}'-\mathbf{y}'|)H(y^0-x^0+|\mathbf{x}-\mathbf{x}'|-|\mathbf{x}'-\mathbf{y}'|-|\mathbf{y}-\mathbf{y}'|)\\
& \times g(x^0-|\mathbf{x}-\mathbf{x}'|+|\mathbf{x}'-\mathbf{y}'|)\\
& + H(x^0-|\mathbf{x}-\mathbf{x}'|-|\mathbf{x}'-\mathbf{y}'|)H(y^0-x^0+|\mathbf{x}-\mathbf{x}'|+|\mathbf{x}'-\mathbf{y}'|-|\mathbf{y}-\mathbf{y}'|)\\
& \times g(x^0-|\mathbf{x}-\mathbf{x}'|-|\mathbf{x}'-\mathbf{y}'|)\big]\,.
\end{aligned} \tag{2.267}$$

As the remaining singularities are independent of each other for a suitable choice of integration variables (see below), we are left with an integrable function on a finite domain.
The next task is to bring the expressions into a simpler form. One possibility to do this is to use

$$H(y^0-x^0+|\mathbf{x}-\mathbf{x}'|+|\mathbf{x}'-\mathbf{y}'|-|\mathbf{y}-\mathbf{y}'|) \leqslant H(y^0-x^0+|\mathbf{x}-\mathbf{x}'|+|\mathbf{x}'-\mathbf{y}'|) \tag{2.268}$$

for the second Heaviside function in the second summand. The first Heaviside function in the first summand equals 1 anyway, as $|\mathbf{x}-\mathbf{x}'| < x^0$. We furthermore use

$$H(y^0-x^0+|\mathbf{x}-\mathbf{x}'|-|\mathbf{x}'-\mathbf{y}'|-|\mathbf{y}-\mathbf{y}'|) \leqslant H(y^0-x^0+|\mathbf{x}-\mathbf{x}'|-|\mathbf{x}'-\mathbf{y}'|), \tag{2.269}$$

as it simplifies the domain of integration. Overall, the domain of integration remains bounded. Introducing $\mathbf{z}_1 = \mathbf{x}-\mathbf{x}'$, $\mathbf{z}_2 = \mathbf{x}'-\mathbf{y}'$ (with

Jacobi determinant of modulus 1) and using spherical coordinates for $\mathbf{z}_2$, this leads to:

$$
\begin{aligned}
&|A_2\psi|(x,y)\,\frac{4(4\pi)^3}{\lambda\, m_2^2\,\|\psi\|_g}\\
&\leqslant \int_{B_{x^0}(0)} d^3\mathbf{z}_1 4\pi \int_0^{\max(0,y^0-x^0+|\mathbf{z}_1|)} d^3\mathbf{z}_2\ |\mathbf{z}_2|^2\,\frac{1}{|\mathbf{z}_1|}\frac{1}{|\mathbf{z}_2|} g(x^0-|\mathbf{z}_1|)g(x^0-|\mathbf{z}_1|+|\mathbf{z}_2|)\\
&+ \int_{B_{x^0}(0)} d^3\mathbf{z}_1\ 4\pi \int_{\max(0,x^0-y^0-|\mathbf{z}_1|)}^{x^0-|\mathbf{z}_1|} d|\mathbf{z}_2|\ |\mathbf{z}_2|^2\,\frac{1}{|\mathbf{z}_1|}\frac{1}{|\mathbf{z}_2|} g(x^0-|\mathbf{z}_1|)g(x^0-|\mathbf{z}_1|-|\mathbf{z}_2|).
\end{aligned}
\tag{2.270}
$$

Using spherical coordinates also for $\mathbf{z}_1$, this can be further simplified to:

$$
|A_2\psi|(x,y)\,\frac{16\pi}{\lambda\, m_2^2\,\|\psi\|_g} \leqslant \int_0^{x^0} dr_1 \int_0^{\max(0,y^0-x^0+r_1)} dr_2\ r_1 r_2\, g(x^0-r_1)g(x^0-r_1+r_2)
\tag{2.271}
$$

$$
+ \int_0^{x^0} dr_1 \int_{\max(0,x^0-r_1-y^0)}^{t_1-r_1} dr_2\ r_1 r_2\ g(x^0-r_1)g(x^0-r_1-r_2).
\tag{2.272}
$$

Our next task is to simplify the remaining integrals. We begin with making the change of variables $\rho = x^0 - r_1$:

$$
\begin{aligned}
|A_2\psi|(x,y)\,\frac{16\pi}{\lambda\, m_2^2\,\|\psi\|_g} \ \leqslant\ & \int_0^{x^0} d\rho\ (x^0-\rho)g(\rho) \int_0^{\max(0,y^0-\rho)} dr_2\ r_2\, g(\rho+r_2)\\
&+ \int_0^{x^0} d\rho\ (x^0-\rho)g(\rho) \int_{\max(0,\rho-y^0)}^{\rho} dr_2\ r_2\, g(\rho-r_2).
\end{aligned}
\tag{2.273}
$$

Now we consider the r_2-integral in both terms and integrate by parts. This yields:

$$\int_0^{\max(0,y^0-\rho)} dr_2\; r_2\, g(\rho+r_2) = \max(0,y^0-\rho) g_1(y^0) - g_2(\max(\rho,y^0)) + g_2(\rho), \tag{2.274}$$

$$\int_{\max(0,\rho-y^0)}^{\rho} dr_2\; r_2\, g(\rho-r_2) \;=\; \max(0,\rho-y^0) g_1(y^0) + g_2(\min(\rho,y^0)). \tag{2.275}$$

We now use $-g_2(\max(\rho,y^0))+g_2(\rho) \leqslant 0$ in the first term and then re-insert the resulting estimate into (2.273). Considering also $\max(0,y^0-\rho)+\max(0,\rho-y^0) = |y^0-\rho|$, this yields:

$$|A_2\psi|(x,y)\,\frac{16\pi}{\lambda\, m_2^2\, \|\psi\|_g} \;\leqslant\; \int_0^{x^0} d\rho\; (x^0-\rho) g(\rho) \left[|y^0-\rho| g_1(y^0) + g_2(\min(\rho,y^0))\right]. \tag{2.276}$$

The first summand of (2.276) can be treated as follows. First we focus on whether $x^0 > y^0$ or $x^0 \leqslant y^0$. In the first case, we then differentiate between the cases $\rho < y^0$ and $\rho \geqslant y^0$ and split up the integrals accordingly. This yields:

$$\int_0^{x^0} d\rho\; (x^0-\rho) g(\rho) |y^0-\rho| g_1(y^0)$$

$$= \; g_1(y^0)\, H(x^0-y^0) \int_0^{y^0} d\rho\; (x^0-\rho)(y^0-\rho) g(\rho) \tag{2.277}$$

$$- \, g_1(y^0)\, H(x^0-y^0) \int_{y^0}^{x^0} d\rho\; (x^0-\rho)(y^0-\rho) g(\rho) \tag{2.278}$$

$$+ \, g_1(y^0)\, H(y^0-x^0) \int_0^{x^0} d\rho\; (x^0-\rho)(y^0-\rho) g(\rho). \tag{2.279}$$

We now calculate these terms separately using integration by parts. The first term yields:

$$(2.277) \;=\; g_1(y^0)H(x^0-y^0)\left[(x^0-y^0)g_2(y^0)+2g_3(y^0)\right]. \tag{2.280}$$

We turn to (2.278):

$$(2.278) \;=\; -\,g_1(y^0)H(x^0-y^0)\left[(y^0-x^0)(g_2(x^0)+g_2(y^0))+2g_3(x^0)-2g_3(y^0)\right]. \tag{2.281}$$

The result of (2.279) is:

$$(2.279) \;=\; g_1(y^0)H(y^0-x^0)\left[(y^0-x^0)g_2(x^0)+2g_3(x^0)\right]. \tag{2.282}$$

Gathering the terms (2.280), (2.281) and (2.282) yields:

$$\begin{aligned}
|A_2\psi|(x,y)\,\frac{16\pi}{\lambda\, m_2^2\,\|\psi\|_g} \;&\leqslant\; g_1(y^0)H(x^0-y^0)\left[2(x^0-y^0)g_2(y^0)+4g_3(y^0)-2g_3(x^0)\right] \\
&\quad + g_1(y^0)|x^0-y^0|g_2(x^0)+2g_1(y^0)H(y^0-x^0)g_3(x^0) \\
&\leqslant\; 2g_1(y^0)|x^0-y^0|g_2(x^0)+2g_1(y^0)g_3(x^0)H(x^0-y^0) \\
&\quad + g_1(y^0)|x^0-y^0|g_2(x^0)+2g_1(y^0)g_3(x^0)H(y^0-x^0) \\
&=\; 3g_1(y^0)|x^0-y^0|g_2(x^0)+2g_1(y^0)g_3(x^0) \\
&\leqslant\; 3(x^0+y^0)g_1(y^0)g_2(x^0)+2g_1(y^0)g_3(x^0).
\end{aligned} \tag{2.283}$$

In order to obtain $\|A_2\psi\|_g$, we divide by $g(x^0)g(y^0)$ and take the supremum over $x,y\in\frac{1}{2}\mathbb{M}$. This results in:

$$\sup_{\psi\in\mathcal{S}((\frac{1}{2}\mathbb{M})^2)}\frac{\|A_2\psi\|_g}{\|\psi\|_g} \;\leqslant\; \frac{\lambda\, m_2^2}{16\pi}\Bigg(3\sup_{x^0,y^0\geqslant 0}\frac{(x^0+y^0)g_2(x^0)\,g_1(y^0)}{g(x^0)g(y^0)}+2\sup_{x^0,y^0\geqslant 0}\frac{g_3(x^0)g_1(y^0}{g(x^0)g(y^0)} \tag{2.284}$$

After factorizing the two-dimensional suprema into one-dimensional ones, this exactly yields the claim, (2.161).

For the operator A_1, we find analogously:

$$\sup_{\psi \in \mathcal{S}((\frac{1}{2}\mathbb{M})^2)} \frac{\|A_1\psi\|_g}{\|\psi\|_g} \leqslant \frac{\lambda\, m_1^2}{16\pi}\left(3 \sup_{x^0,y^0 \geqslant 0} \frac{(x^0+y^0)g_1(x^0)\, g_2(y^0)}{g(x^0)g(y^0)} + 2 \sup_{x^0,y^0\geqslant 0} \frac{g_1(x^0)g_3(y^0)}{g(x^0)g(y^0)}\right). \tag{2.285}$$

which, after factorization into one-dimensional suprema, yields the claim (2.160).

2.3.4.1.3 Estimate of the mass-mass term (2.162)**.** We begin with (2.155). Taking the absolute value and using $|\psi(x,y)| \leqslant \|\psi\|_g\, g(x^0)g(y^0)$ as well as $|J_1(t)/t| \leqslant \frac{1}{2}$ yields:

$$|A_{12}\psi|(x,y) \leqslant \frac{\lambda\, m_1 m_2\, \|\psi\|_g}{4(4\pi)^3} \int_0^\infty dx'^0 \int d^3\mathbf{x}' \int_0^\infty dy'^0 \int_0^{2\pi} d\varphi \int_0^\pi d\vartheta\, \cos(\vartheta)|x'^0 - y'^0|$$
$$\times\, H(x^0 - x'^0 - |\mathbf{x}-\mathbf{x}'|)H(y^0 - y'^0 - |\mathbf{y} - \mathbf{x}' + \mathbf{z}|)g(x'^0)g(y'^0)\Big|_{|\mathbf{z}| = |x^{0\prime} - y^{0\prime}|}, \tag{2.286}$$

where, we recall, $\mathbf{z}$ is the variable for which the spherical coordinates are used.

Next, we consider the ranges of integration which the Heaviside functions imply. $H(x^0 - x'^0 - |\mathbf{x}-\mathbf{x}'|)$ restricts the range of integration of $\mathbf{x}'$ to the ball $B_{x^0 - x'^0}(\mathbf{x})$ and the range of the x'^0-integration to $(0, x^0)$. The range implied by the second Heaviside function is more complicated. We therefore use the estimate

$$H(y^0 - y'^0 - |\mathbf{y} - \mathbf{x}' + \mathbf{z}|) \leqslant H(y^0 - y'^0). \tag{2.287}$$

Then $y'^0 \in (0, y^0)$ and there is no further restriction for the angular variables. We obtain:

$$|A_{12}\psi|(x,y) \leqslant \frac{\lambda\, m_1 m_2\, \|\psi\|_g}{8(4\pi)^3} \int_0^{x^0} d'^0 \int_{B_{x^0-x'^0}(\mathbf{x})} d^3\mathbf{x}' \int_0^{y^0} dy'^0 \int_0^{2\pi} d\varphi \int_0^\pi d\vartheta$$
$$\times \cos(\vartheta)|x'^0 - y'^0|\, g(x'^0)g(y'^0). \tag{2.288}$$

Performing the $\mathbf{x}'$-integration, as well as the angular integrals yields:

$$|A_{12}\psi|(x,y) \leqslant \frac{\lambda\, m_1 m_2\, \|\psi\|_g}{96\pi} \int_0^{x^0} dx'^0 |x^0 - x'^0|^3 g(x'^0) \int_0^{y^0} dy'^0\, |x'^0 - y'^0|\, g(y'^0). \tag{2.289}$$

Our next task is to estimate the term explicitly in terms of the functions g_n only. To do so, we use

$$|x'^0 - y'^0| \leqslant x'^0 + y'^0. \tag{2.290}$$

This yields:

$$|A_{12}\psi|(x,y) \leqslant \frac{\lambda\, m_1 m_2\, \|\psi\|_g}{48\pi} \int_0^{x^0} dx'^0 |x^0 - x'^0|^3 g(x'^0) \int_0^{y^0} dy'^0\, (x'^0 + y'^0) g(y'^0). \tag{2.291}$$

Let

$$I(x^0, y^0) = \int_0^{x^0} dx'^0 |x^0 - x'^0|^3 g(x'^0) \int_0^{y^0} dy'^0\, (x'^0 + y'^0) g(y'^0) \tag{2.292}$$

and

$$L(x'^0, y^0) = \int_0^{y^0} dy'^0\, (x'^0 + y'^0) g(y'^0). \tag{2.293}$$

Integration by parts yields:

$$L(x'^0, y^0) = x'^0 g_1(y^0) + y^0 g_1(y^0) - g_2(y^0) \leqslant x'^0 g_1(y^0) + y^0 g_1(y^0). \tag{2.294}$$

Next, let

$$\begin{aligned} I_a(x^0) &= \int_0^{x^0} dx'^0\, |x^0 - x'^0|^3 g(x'^0), \\ I_b(x^0) &= \int_0^{x^0} dx'^0\, x'^0 |x^0 - x'^0|^3 g(x'^0). \end{aligned} \tag{2.295}$$

Then:

$$I(x^0,y^0) \leqslant I_a(x^0)\, y^0 g_1(y^0) + I_b(x^0)\, g_1(y^0). \tag{2.296}$$

We consider I_a first, using $(x^0 - x'^0)^2 \leqslant (x^0)^2$ and integrating by parts:

$$\begin{aligned} I_a(x^0) &\leqslant (x^0)^2 \int_0^{x^0} dx'^0 \ (x^0 - x'^0) g(x'^0) \\ &= (x^0)^2 \left(\underbrace{(x^0 - x'^0) g_1(x'^0)|_{x'^0=0}^{x^0}}_{=0} + g_2(x^0) \right) = (x^0)^2 g_2(x^0). \end{aligned} \tag{2.297}$$

We turn to I_b, using $x'^0(x^0 - x'^0) \leqslant \frac{1}{4}(x^0)^2$ and integrating by parts twice. This results in:

$$I_b(x^0) \ \leqslant \ \frac{(x^0)^2}{4} \int_0^{x^0} dx'^0 \ (x^0 - x'^0)^2 g(x'^0) \ = \ \frac{(x^0)^2}{2}\, g_3(x^0). \tag{2.298}$$

Considering (2.296), we therefore obtain:

$$I(x^0,y^0) \ \leqslant \ (x^0)^2 g_2(x^0)\, y^0 g_1(y^0) + \frac{(x^0)^2}{2}\, g_3(x^0)\, g_1(y^0). \tag{2.299}$$

Returning to (2.291), we divide by $g(x^0)g(y^0)$ and take the supremum, with the result:

$$\begin{aligned} \sup_{\psi \in \mathcal{S}((\frac{1}{2}\mathbb{M})^2)} \frac{\|A_{12}\psi\|_g}{\|\psi\|_g} \ \leqslant \ \frac{\lambda\, m_1 m_2\, \|\psi\|_g}{96\pi} \Bigg[& \sup_{x^0,y^0 \geqslant 0} \frac{(x^0)^2 g_2(x^0)\, y^0 g_1(y^0)}{g(x^0)g(y^0)} \\ & + \frac{1}{2} \sup_{x^0,y^0 \geqslant 0} \frac{(x^0)^2 g_3(x^0)\, g_1(y^0)}{g(x^0)g(y^0)} \Bigg]. \end{aligned} \tag{2.300}$$

Factorizing the two-dimensional suprema into one-dimensional ones yields the claim, (2.162).

2.3.4.2 Proof of Theorem 17

Let $\psi \in \mathcal{S}$. It only remains to calculate the supremum in (2.159) for $g(t) = e^{\gamma t}$. We have:

$$g_1(t) = \frac{1}{\gamma}\left(e^{\gamma t} - 1\right) \tag{2.301}$$

and hence

$$\sup_{\psi \in \mathcal{S}((\frac{1}{2}\mathbb{M})^2)} \frac{\|A_0\psi\|_g}{\|\psi\|_g} \leqslant \frac{\lambda}{8\pi}\left(\sup_{t\geqslant 0}\frac{g_1(t)}{g(t)}\right)^2 = \frac{\lambda}{4\pi}\left(\sup_{t\geqslant 0}\frac{1}{\gamma}(1-e^{-\gamma t})\right)^2 = \frac{\lambda}{8\pi\gamma^2}. \tag{2.302}$$

This shows that A_0 can be linearly extended to a bounded operator on $\mathcal{B}_g$ which satisfies the same estimate, (2.163). Moreover, for $\gamma > \sqrt{\frac{\lambda}{4\pi}}$, A_0 is a contraction and Banach's fixed point theorem implies the existence of a unique solution $\psi \in \mathcal{B}_g$ of the equation $\psi = \psi^{\text{free}} + A_0\psi$ for every $\psi^{\text{free}} \in \mathcal{B}_g$.

2.3.4.3 Proof of Theorem 18

Let again $\psi \in \mathcal{S}$. We need to calculate the suprema in (2.159)-(2.162) for $g(t) = (1+\alpha t^2)e^{\alpha t^2/2}$. We first note:

$$\begin{aligned}
g_1(t) &= t e^{\alpha t^2/2},\\
g_2(t) &= \frac{1}{\alpha}\left(e^{\alpha t^2/2} - 1\right),\\
g_3(t) &= \frac{1}{\alpha}\left[\sqrt{\frac{\pi}{2\alpha}}\,\mathrm{erfi}(\sqrt{\alpha/2}\,t) - t\right].
\end{aligned} \tag{2.303}$$

We can see that with each successive integration, the functions g_n grow slower as $t \to \infty$. Furthermore, the leading terms in g_n are inversely proportional to increasing powers of α. These two properties (and of course the fact that g_1, g_2, g_3 can be written down in terms

of elementary functions) make this particular function $g(t)$ a suitable choice for the proof.
As we need to estimate the behavior of quotients like $g_3(t)/g(t)$ for $t \to \infty$, we look for a simpler estimate of g_3 in terms of exponential functions. We note:

$$\begin{aligned} g_3(t) &= \int_0^t dt' \, \frac{1}{\alpha} \left(e^{\alpha t'^2/2} - 1 \right) \\ &\leqslant \frac{e^{\alpha t^2/2}}{\alpha} e^{-\alpha t^2/2} \sqrt{2/\alpha} \int_0^{\sqrt{\alpha/2} t} d\tau \, e^{\tau^2} \\ &= \frac{\sqrt{2}}{\alpha^{3/2}} \, e^{\alpha t^2/2} \, D(\sqrt{\alpha/2}\, t), \end{aligned} \tag{2.304}$$

where $D(t) = e^{-t^2} \int_0^t d\tau \, e^{\tau^2}$ denotes the Dawson function. Using the property $|tD(t)| < \frac{2}{3}$, we obtain:

$$t g_3(t) \leqslant \frac{4}{3} \frac{e^{\alpha t^2/2}}{\alpha^2}. \tag{2.305}$$

We are now well-equipped to calculate the suprema occurring in (2.159)-(2.162). Using

$$\sup_{t \geqslant 0} \frac{t^\beta}{1+t^2} = \begin{cases} 1 & \text{for } \beta = 0 \\ \frac{1}{2} & \text{for } \beta = 1 \\ 1 & \text{for } \beta = 2 \end{cases} \tag{2.306}$$

we obtain:

$$\sup_{t \geqslant 0} \frac{g_1(t)}{g(t)} = \sup_{t \geqslant 0} \frac{t}{1+\alpha t^2} = \frac{1}{2} \frac{1}{\sqrt{\alpha}}, \tag{2.307}$$

$$\sup_{t \geqslant 0} \frac{t g_1(t)}{g(t)} = \sup_{t \geqslant 0} \frac{t^2}{1+\alpha t^2} = \frac{1}{\alpha}, \tag{2.308}$$

$$\sup_{t \geqslant 0} \frac{g_2(t)}{g(t)} \leqslant \sup_{t \geqslant 0} \frac{1}{\alpha} \frac{1}{1+\alpha t^2} = \frac{1}{\alpha}, \tag{2.309}$$

$$\sup_{t\geqslant 0}\frac{tg_2(t)}{g(t)} \leqslant \sup_{t\geqslant 0}\frac{1}{\alpha}\frac{t}{1+\alpha t^2} = \frac{1}{2}\frac{1}{\alpha^{3/2}}, \tag{2.310}$$

$$\sup_{t\geqslant 0}\frac{t^2g_2(t)}{g(t)} \leqslant \sup_{t\geqslant 0}\frac{1}{\alpha}\frac{t^2}{1+\alpha t^2} = \frac{1}{\alpha^2}. \tag{2.311}$$

Using, in addition, the property $|D(t)| < \frac{3}{5}$, we find:

$$\sup_{t\geqslant 0}\frac{g_3(t)}{g(t)} \leqslant \sup_{t\geqslant 0}\frac{\sqrt{2}}{\alpha^{3/2}}\frac{D(\sqrt{\alpha/2}t)}{1+\alpha t^2} = \frac{3\sqrt{2}}{5}\frac{1}{\alpha^{3/2}} < \frac{1}{\alpha^{3/2}}, \tag{2.312}$$

$$\sup_{t\geqslant 0}\frac{t^2g_3(t)}{g(t)} \leqslant \sup_{t\geqslant 0}\frac{4}{3}\frac{1}{\alpha^2}\frac{t}{1+\alpha t^2} = \frac{2}{3}\frac{1}{\alpha^{5/2}}. \tag{2.313}$$

In the last line, we have made use of (2.305).
With these results, we find for A_0:

$$(2.159) \leqslant \frac{\lambda}{8\pi}\left(\frac{1}{2}\frac{1}{\sqrt{\alpha}}\right)^2 = \frac{\lambda}{32\pi}\frac{1}{\alpha}. \tag{2.314}$$

This yields (2.165).
We continue with A_1.

$$(2.160) \leqslant \frac{\lambda\, m_1^2}{16\pi}\left[3\,\frac{1}{\alpha}\,\frac{1}{\alpha} + 3\,\frac{1}{2}\frac{1}{\sqrt{\alpha}}\,\frac{1}{2}\frac{1}{\alpha^{3/2}} + 2\,\frac{1}{2}\frac{1}{\sqrt{\alpha}}\,\frac{1}{\alpha^{3/2}}\right] = \frac{\lambda\, m_1^2}{16\pi}\,\frac{19}{4}\,\frac{1}{\alpha^2} < \frac{\lambda\, m_1^2}{16\pi} \tag{2.315}$$

This yields (2.166). Analogously, we obtain the estimate (2.167) for A_2.
Finally, for A_{12}, we have

$$(2.162) \leqslant \frac{\lambda\, m_1^2\, m_2^2}{96\pi}\left[\frac{1}{\alpha^2}\,\frac{1}{\alpha} + \frac{1}{2}\,\frac{2}{3}\,\frac{1}{\alpha^{5/2}}\,\frac{1}{2}\,\frac{1}{\sqrt{\alpha}}\right] = \frac{\lambda\, m_1^2\, m_2^2}{96\pi}\,\frac{7}{6}\,\frac{1}{\alpha^3} < \frac{\lambda\, m_1^2\, m_2^2}{80\pi}\,\frac{1}{\alpha^3}, \tag{2.316}$$

which yields (2.168).

Now, the estimates (2.165)-(2.168) show that the operators A_0, A_1, A_2 and A_{12} are bounded on test functions. Thus, they can be linearly extended to bounded operators on $\mathcal{B}_g$ with the same bounds.
The operator $A = A_0 + A_1 + A_2 + A_{12}$ then also defines a bounded linear operator on $\mathcal{B}_g$ with norm

$$\|A\| \leqslant \|A_0\| + \|A_1\| + \|A_2\| + \|A_{12}\|. \tag{2.317}$$

Using the previous results (2.165)-(2.168), we obtain:

$$\|A\| \leqslant \frac{\lambda}{8\pi\alpha}\left(\frac{1}{4} + \frac{5(m_1^2+m_2^2)}{2}\frac{1}{\alpha} + \frac{m_1^2 m_2^2}{10}\frac{1}{\alpha^2}\right). \tag{2.318}$$

If α is chosen such that this expression is strictly smaller than unity, A becomes a contraction and the existence and uniqueness of solutions of the equation $\psi = \psi^{free} + A\psi$ follows. This yields condition (2.169) and ends the proof.

2.3.4.4 Proof of Theorem 21

The proof can be reduced to the one for $\frac{1}{2}\mathbb{M}$. To do so, we take the absolute value of (2.187) and use $|\psi|(\eta_1, \mathbf{x}, \eta_2, \mathbf{y}) \leqslant g(\eta_1)g(\eta_2)\|\psi\|_g$. With

$$G(\eta) = a(\eta)\exp\left(\gamma\int_0^\eta d\eta'\; a(\eta')\right) \tag{2.319}$$

$$G_1(\eta) = \int_0^\eta d\eta'\; G(\eta) \tag{2.320}$$

we obtain the estimate

$$|\widetilde{A}_0\psi|(x,y) \leqslant \frac{\lambda\|\psi\|_g}{4(4\pi)^3}\int_{B_{\eta_2}(\mathbf{y})} d^3\mathbf{y}' \int_0^{2\pi} d\varphi \int_{-1}^1 d\cos\vartheta\, \frac{|b^2|}{(b^0+|\mathbf{b}|\cos\vartheta)^2|\mathbf{y}'|} G(\eta_2 - |\mathbf{y}'|)$$
$$\times\, G\left(\eta_1 - \frac{1}{2}\frac{b^2}{b^2+|\mathbf{b}|\cos\vartheta}\right)\left(1_{b^2>0}1_{b^0>0}1_{\cos\vartheta > \frac{b^2}{2\eta_1^0|\mathbf{b}|} - \frac{b^0}{|\mathbf{b}|}} + 1_{b^2<0}1_{\cos\vartheta < \frac{b^2}{2\eta_1|\mathbf{b}|} - \frac{b^0}{|\mathbf{b}|}}\right). \tag{2.321}$$

■

This estimate is identical to (2.191) with the only difference that the function g is exchanged with G in the integral (but not in $\|\cdot\|_g$). Thus, going through the same steps as in Secs. 2.3.4.1, 2.3.4.3, we obtain:

$$\sup_{\psi \in \mathcal{S}(([0,\infty)\times\mathbb{R}^3)^2)} \frac{\|\widetilde{A}_0\psi\|_g}{\|\psi\|_g} \leqslant \frac{\lambda}{8\pi}\left(\sup_{t\geqslant 0}\frac{G_1(t)}{g(t)}\right)^2. \tag{2.322}$$

Now, recalling $g(t) = \exp\left(\gamma \int_0^t d\tau\, a(\tau)\right)$ we have

$$G_1(t) = \frac{1}{\gamma} g(t) \tag{2.323}$$

and it follows that

$$\sup_{\psi \in \mathcal{S}(([0,\infty)\times\mathbb{R}^3)^2)} \frac{\|\widetilde{A}_0\psi\|_g}{\|\psi\|_g} \leqslant \frac{\lambda}{8\pi\gamma^2}, \tag{2.324}$$

which yields (2.189). The rest of the claim follows as before.

2.3.5 Conclusions

In this paper we have given what we think of as a satisfactory answer to the problem posed: to prove the existence and uniqueness of solution of the integral equation (2.129) and its N-particle generalization (2.171). Following previous works, we have assumed a cutoff in time. By considering an example for our integral equation on a cosmological spacetime with a Big Bang singularity, we have shown that such a cutoff can arise naturally and without violating any spacetime symmetries.

Our work provides a rigorous proof of the existence of interacting relativistic quantum dynamics in 1+3 spacetime dimensions; in particular, our model does not suffer from ultraviolet divergences which are typically encountered in quantum field theoretic models. Of course, our model does not describe particle creation and annihilation and is therefore a toy model rather than an alternative to QFT. Nevertheless, we find the fact that direct interactions, even singular ones along the light cone, can be made mathematically rigorous, remarkable. We wonder whether in the long run the mechanism of interaction through multi-time integral equations and direct interactions could contribute to a rigorous formulation of quantum field theory.

In the more immediate future, it would first of all be desirable to extend our results to Dirac particles (meaning that the Green's functions in (2.129) are replaced with Green's functions of the Dirac equation). As the Dirac Green's functions involve distributional derivatives, it would then be more difficult than in the Klein-Gordon case to define the combination of the three distributions G_1^{ret}, G_2^{ret} and $\delta((x-y)^2)$ which occur in (2.129). Moreover, as the previous work [31] on Dirac particles but regular interaction kernels K suggests, the occurrence of the distributional derivatives in the Green's functions alone leads to technical complications, as one then needs to prove a higher regularity of the solutions. In the N-particle case, this regularity would have to be greater than in the two-particle case so that one cannot simply add up estimates for the norm of the two-particle integral operator to obtain an estimate for the N-particle integral operator anymore. There would be further terms to consider.

This is the set of questions which a work on the Dirac case of Eq. (2.129) would have to answer.

Besides the Dirac case, there is also a range of more detailed technical questions for the Klein-Gordon case which would be interesting to address. While we have here worked with a weighted L^∞ norm both for time and space variables, one could also try to use a weighted $L^\infty L^2$ norm instead (L^∞ for the time variables and L^2 for the space variables). It would then be a challenging task to find the right inequalities to obtain similar estimates as we did. Moreover, one could also try to prove higher regularity not only in the sense of integrability but also differentiability. An interesting question, for example, is whether one can apply the Klein-Gordon operators $(\Box_k + m_k^2)$ to the solutions of (2.129) in a weak sense. For the Dirac case, an analogous property was, in fact, established in [31].

Chapter 3

Quantum Field Theoretic Approach to Interactions

3.1 Introduction

thorough introduction to Franz and Dirk stuff for handling the geometry part as independent chapter, orientiere dich an perspective of external field qed paper

3.2 The Relationship Between Hadamard States and Admissible Polarisation Classes

insert QFT communications paper

3.3 Analyticity of the One Particle Scattering Operator

In this section we analyse the construction of the one particle scattering operator S_A carried out in [6] and answer the question whether operators like

$$P^+ \partial_B S_A^* S_{A+B} P^- \tag{3.1}$$

are Hilbert-Schmidt operators. This will turn out to be important for the geometric construction carried out in chapter 3.4.
Since this section is heavily inspired from [6], we need to introduce some notation from this paper.

Definition 22. *We define the set $\mathcal{V}$ of four potentials*

$$\mathcal{V} := C_c^\infty(\mathbb{R}^4, \mathbb{R}^4). \tag{3.2}$$

Let $A \in \mathcal{V}$, we define the the integral operator $Q^A : \mathcal{H} \circlearrowleft$ by giving its integral kernel, which is also denoted by Q^A:

$$\mathbb{R}^3 \times \mathbb{R}^3 \ni (p,q) \mapsto Q^A(p,q) := \frac{Z^A_{+-}(p,q) - Z^A_{-+}(p,q)}{i(E(p)+E(q))} \tag{3.3}$$

$$\text{with } Z^A_{\pm\mp}(p,q) := P_\pm(p) Z^A(p-q) P_\mp(q), \tag{3.4}$$

$$Z^A = -ie\gamma^0\gamma^\alpha \hat{A}_\alpha, \tag{3.5}$$

$$\hat{A}_\mu := \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} A_\mu(x) e^{-ipx} d^3x, \tag{3.6}$$

$$\text{and } E(p) := \sqrt{m^2 + |p|^2}. \tag{3.7}$$

think of some more intuitive notation for the projector.

introduce the standard polarisation of $\mathcal{H}$ for the free Dirac equation

$$P^- := 1_{\mathrm{spec}(H^0)<0}, \quad P^+ = 1 - P^-. \tag{3.8}$$

Lastly we introduce the partial derivative in the direction of any four-potential F of an operator valued function $F : \mathcal{V} \to \mathcal{B}(\mathcal{F})$ by

$$\partial_F T(F) := \partial_\varepsilon T(\varepsilon F)|_{\varepsilon=0}, \tag{3.9}$$

where the limit is taken with respect to the operator norm topology.

Lemma 23. *For general $A, F \in \mathcal{V}$ and $t_0, t_1 \in \mathbb{R}$ we have the well known equations for the one-particle time evolution operators*

$$U^A(t_1,t_0) = U^0(t_1,t_0) + \int_{t_0}^{t_1} dt\ U^0(t_1,t)Z^A(t)U^A(t,t_0) \tag{3.10}$$

$$U^{A+F}(t_1,t_0) = U^A(t_1,t_0) + \int_{t_0}^{t_1} dt\ U^A(t_1,t)Z^F(t)U^{A+F}(t,t_0). \tag{3.11}$$

Proof. ... □

Definition 24. *For any $A \in \mathcal{V}$, we introduce the integral operator $Q'^A : \mathcal{H} \circlearrowleft$ by it's kernel*

$$\mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \ni (t,p,q) \mapsto Q'^A(t,p,q) = \partial_t Q^A(t,p,q), \tag{3.12}$$

where the time dependence is due to the time dependence of the four-potential A. The following notion of even and odd part of an arbitrary bounded linear operator $T : \mathcal{F} \circlearrowleft$ on Fock space will come in handy:

$$T_{\text{odd}} := P^+TP^- + P^-TP^+ \tag{3.13}$$

$$T_{\text{ev}} := P^+TP^+ + P^-TP^-. \tag{3.14}$$

Additionally, we define the norm

$$T : \mathcal{H} \circlearrowleft \|T\|_{\text{op}+I_2} = \|T\| + \|T_{\text{odd}}\|_{I_2}, \tag{3.15}$$

where $\|\cdot\|$ is the operator norm and $\|\cdot\|_{I_2}$ is the Hilbert-Schmidt norm and the space

$$I_2^{\text{odd}} := \{T : \mathcal{F} \to \mathcal{F} \mid \|T\| < \infty, \|T_{\text{odd}}\|_{I_2} < \infty\}. \tag{3.16}$$

Lemma 25. *The space* I_2^{odd} *equipped with the norm* $\|\cdot\|_{\text{op}+I_2}$ *is a Banach space.*

Proof. Let $(T_n)_{n\in\mathbb{R}} \subset I_2^{\text{odd}}$ be a Cauchy sequence with respect to $\|\cdot\|_{\text{op}+I_2}$. Then it follows directly that $(T_n)_{n\in\mathbb{N}}$ is also a Cauchy sequence with respect to $\|\cdot\|$ and $(T_{n,\text{odd}})_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to $\|\cdot\|_{I_2}$. Since the space of bounded operators equipped with $\|\cdot\|$ and the space of Hilbert-Schmidt oeprators equipped with $\|\cdot\|_{I_2}$ both are complete we have

$$T_n \xrightarrow[\|\cdot\|]{n\to\infty} T^1 \tag{3.17}$$

$$T_{n,\text{odd}} \xrightarrow[\|\cdot\|_{I_2}]{n\to\infty} T^2 \tag{3.18}$$

for some bounded operator T^1 and some Hilbert-Schmidt operator T^2. Now because the Hilbert-Schmidt norm fulfills

$$\|T\| \leqslant \|T\|_{I_2}, \tag{3.19}$$

we obtain directly

$$T_{n,\text{odd}} \xrightarrow[\|\cdot\|]{n\to\infty} T^2, \tag{3.20}$$

hence $T^1_{\text{odd}} = T^2$. Therefore, $T^1 \in I_2^{\text{odd}}$ holds. Finally, since $\|\cdot\|_{\text{op}+I_2} = \|\cdot\| + \|\cdot_{\text{odd}}\|_{I_2}$ is true, we find

$$T_n \xrightarrow[\|\cdot\|_{\text{op}+I_2}]{n\to\infty} T^1, \tag{3.21}$$

proving completeness. $\square$

Theorem 26 (Smoothness of S)**.** *Let* $n \in \mathbb{N}$, $A, H_k \in \mathcal{V}$ *for* $k \leqslant n$, *pick* t_1 *after* $\operatorname{supp} A \cup \bigcup_{k\leqslant n} \operatorname{supp} H_k$ *and* t_0 *before* $\operatorname{supp} A \cup \bigcup_{k\leqslant n} \operatorname{supp} H_k$ *then the derivative*

$$\partial_{H_1} \dots \partial_{H_k} U^{A+\sum_{b=1}^{k} H_b}(t_1, t_0) \tag{3.22}$$

exists with respect to the topology induced by the norm $\| \cdot \|_{\mathrm{op}+I_2}$.

Proof. Following [6] throughout this proof, we make use of the following shorthand notation. For operator valued maps $T_1, T_2 : \mathbb{R}^2 \to \mathcal{B}(\mathcal{H})$ we define for $t_1, t_0 \in \mathbb{R}$

$$T_1 T_2 = \int_{t_0}^{t_1} dt \; T_1(t_1, t) T_2(t, t_0), \tag{3.23}$$

as a map of the same type as T_1 and T_2 whenever this is well defined. Furthermore, for operator valued functions $W_1, W_2 : \mathbb{R} \to \mathcal{B}(\mathcal{H})$ we define

$$T_1 W_1 \; (t', t) = T_1(t', t) C_1(t) \tag{3.24}$$

$$W_1 T_1 \; (t', t) = W(t_1) T_1(t', t) \tag{3.25}$$

$$W_1 W_2 \; (t) = W_1(t) W_2(t), \tag{3.26}$$

as maps of the same type as T_1, T_1 and C_1 respectively.
Pick $k \in \mathbb{N}$, $A, H_b \in \mathcal{V}$ for $b \leqslant k$ and t_1, t_0 as in the theorem. Whenever the shorthand (3.24) and (3.24) is used without specific arguments, by convention $t' = t_1, t = t_0$. We abbreviate

$$H := \sum_{b=1}^{k} H_b, \quad B := A + H. \tag{3.27}$$

We introduce

$$R^B(t', t) = (1 - Q^B) U^B (1 + Q^B)(t', t), \tag{3.28}$$

for general $t', t \in \mathbb{R}$. Because of the choice of t_1, t_0 we have

$$R^B(t_1, t_0) = (1 - Q^B) U^B (1 + Q^B) = U^B(t_1, t_0), \tag{3.29}$$

because $B = 0$ both at t_1 and t_0.

So it suffices to study the family of operators R^B. As shown in the proof of [6, lemma 3.5] R^B for $B \in \mathcal{V}$ is the limit in the sense of the operator norm of the sequence

$$R_0^B := 0, \quad R_{n+1}^B := U^0 F^B R_n^B + U^0 + G^B, \tag{3.30}$$

where F and G are given By

$$F^B := (-Q'^B + Z_{\mathrm{ev}}^B - Q^B Z^B)(1 + Q^B), \tag{3.31}$$

$$G^B := -U^0 Q^B Q^B \tag{3.32}$$

$$+U^0(-Q'^B + Z_{\mathrm{ev}} - Q^B Z^B) Q^B Q^B U^B (1 + Q^B). \tag{3.33}$$

First we introduce the auxiliary norms for operators T and W depending on one and two scalar variables respectively.

$$\|T\|_{\mathrm{op}+I_2,\gamma} := \sup_{t \in [t_1,t_0]} e^{-\gamma(t-t_0)} \left(\|T(t)\| + \|T_{\mathrm{odd}}(t)\|_{I_2}\right) \tag{3.34}$$

$$\|T\|_\gamma := \sup_{t \in [t_1,t_0]} e^{-\gamma(t-t_0)} \|T(t)\| \tag{3.35}$$

$$\|W\|_0 := \sup_{t,t' \in [t_1,t_0]} \|W(t,t')\| \tag{3.36}$$

$$\|T\|_{I_2,\gamma} := \sup_{t \in [t_1,t_0]} e^{-\gamma(t-t_0)} \|T(t)\|_{I_2}, \tag{3.37}$$

$$\|W\|_{I_2,0} := \sup_{t,t' \in [t_1,t_0]} \|W(t,t')\|_{I_2}, \tag{3.38}$$

for $\gamma \geqslant 0$. In the proof of the same lemma in [6, equation (3.42)] we also have the recursive equation

$$R_n^B = U^0 F_{\mathrm{ev}}^B R_{n-1}^B + U^0 F_{\mathrm{odd}}^B U^0 F^B R_{n-2}^B \tag{3.39}$$

$$+U^0 F_{\mathrm{odd}}^B + U^0 F_{\mathrm{odd}}^B U^0 + U^0 + G^B \tag{3.40}$$

is fulfilled by the same sequence of operators for $n \geqslant 2$. Furthermore, we introduce the notation

$$[k] = \{l \in \mathbb{N} \mid l \leqslant k\} \tag{3.41}$$

$$\forall u \subseteq [k] : \partial_u = \prod_{k \in u} \partial_{H_k}, \tag{3.42}$$

$$\Delta^n = R^B_{n+1} - R^B_n, \tag{3.43}$$

where the product of derivatives is to be understood as the mixed derivative with respect to all the factors.
Hence we have for such n:

$$\Delta_n = U^0 F^B_{\mathrm{ev}} \Delta_{n-1} + U^0 F^B_{\mathrm{odd}} U^0 F^B \Delta_{n-2}.$$

Abbreviating $U^0 F^B_{\mathrm{ev}} =: a, \quad U^0 F^B_{\mathrm{odd}} U^0 F^B := b$, we obtain

$$\Delta_n = a\Delta_{n-1} + b\Delta_{n-2}. \tag{3.44}$$

Using the derivative defined in (3.9) and the recursion (3.44), we estimate for any set $u \subset [k]$:

$$\sup_{p \subseteq u} \|\partial_p \Delta^n_{\mathrm{odd}}\|_{I_2,\gamma} \tag{3.45}$$

$$\leqslant \sup_{p \subseteq u} \sup_{t \in [t_0,t_1]} e^{-\gamma(t-t_0)} \left\| \sum_{w \subseteq u} (\partial_{u \setminus w} a \; \partial_w \Delta^{n-1}_{\mathrm{odd}})(t,t_0) \right\|_{I_2} \tag{3.46}$$

$$+ \sup_{p \subseteq u} \sup_{t \in [t_0,t_1]} e^{-\gamma(t-t_0)} \left\| P^+ \sum_{w \subseteq u} (\partial_{u \setminus w} b \; \partial_w \Delta^{n-2})(t,t_0) P^- \right\|_{I_2} \tag{3.47}$$

$$+ \sup_{p \subseteq u} \sup_{t \in [t_0,t_1]} e^{-\gamma(t-t_0)} \left\| P^- \sum_{w \subseteq u} (\partial_{u \setminus w} b \; \partial_w \Delta^{n-2})(t,t_0) P^+ \right\|_{I_2} \tag{3.48}$$

$$\leqslant \sup_{p \subseteq u} \sup_{t \in [t_0,t_1]} e^{-\gamma(t-t_0)} \sum_{w \subseteq u} \|\partial_{u \setminus w} (a \; \partial_w \Delta^{n-1}_{\mathrm{odd}})(t,t_0)\|_{I_2} \tag{3.49}$$

$$+ 2 \sup_{p\subseteq u} \sup_{t\in[t_0,t_1]} e^{-\gamma(t-t_0)} \sum_{w\subseteq u} \|(\partial_{u\setminus w} b \; \partial_w \Delta^{n-2})(t,t_0)\|_{I_2} \tag{3.50}$$

$$\leqslant \sup_{p\subseteq u} \sup_{t\in[t_0,t_1]} e^{-\gamma(t-t_0)} \sum_{w\subseteq u} \int_{t_0}^{t} dt' \|\partial_{u\setminus w} a(t,t') \; \partial_w \Delta^{n-1}_{\text{odd}}(t',t_0)\|_{I_2} \tag{3.51}$$

$$+ 2 \sup_{p\subseteq u} \sup_{t\in[t_0,t_1]} e^{-\gamma(t-t_0)} \sum_{w\subseteq u} \int_{t_0}^{t} dt' \|\partial_{u\setminus w} b(t,t') \; \partial_w \Delta^{n-2}(t',t_0)\|_{I_2} \tag{3.52}$$

$$\leqslant \sup_{p\subseteq u} \sup_{t\in[t_0,t_1]} e^{-\gamma(t-t_0)} \sum_{w\subseteq u} \int_{t_0}^{t} dt' \|\partial_{u\setminus w} a\|_0 \|\partial_w \Delta^{n-1}_{\text{odd}}(t',t_0)\|_{I_2} \tag{3.53}$$

$$+ 2 \sup_{p\subseteq u} \sup_{t\in[t_0,t_1]} e^{-\gamma(t-t_0)} \sum_{w\subseteq u} \int_{t_0}^{t} dt' \|\partial_{u\setminus w} b(t,t')\|_{I_2} \|\partial_w \Delta^{n-2}(t',t_0)\| \tag{3.54}$$

$$\leqslant \sup_{p\subseteq u} \sup_{t\in[t_0,t_1]} e^{-\gamma t} \sum_{w\subseteq u} \int_{t_0}^{t} dt' e^{\gamma t'} \|\partial_{u\setminus w} a\|_0 \|\partial_w \Delta^{n-1}_{\text{odd}}(\cdot,t_0)\|_{I_2,\gamma} \tag{3.55}$$

$$+ 2 \sup_{p\subseteq u} \sup_{t\in[t_0,t_1]} e^{-\gamma t} \sum_{w\subseteq u} \int_{t_0}^{t} dt' e^{\gamma t'} \|\partial_{u\setminus w} b\|_{I_2,0} \|\partial_w \Delta^{n-2}(\cdot,t_0)\|_{\gamma} \tag{3.56}$$

$$\leqslant \frac{1}{\gamma} \sup_{p\subseteq u} \sum_{w\subseteq u} \|\partial_{u\setminus w} a\|_0 \|\partial_w \Delta^{n-1}_{\text{odd}}(\cdot,t_0)\|_{I_2,\gamma} \tag{3.57}$$

$$+ \frac{2}{\gamma} \sup_{p\subseteq u} \sum_{w\subseteq u} \|\partial_{u\setminus w} b\|_{I_2,0} \|\partial_w \Delta^{n-2}(\cdot,t_0)\|_{\gamma} \tag{3.58}$$

$$\leqslant \frac{2^{|u|}}{\gamma} \sup_{u'\subseteq u} \|\partial_{u'} a\|_0 \sup_{p\subseteq u} \|\partial_p \Delta^{n-1}_{\text{odd}}(\cdot,t_0)\|_{I_2,\gamma} \tag{3.59}$$

$$+ \frac{2^{|u|+1}}{\gamma} \sup_{u'\subseteq u} \|\partial_{u'} b\|_{I_2,0} \sup_{p\subseteq u} \|\partial_p \Delta^{n-2}(\cdot,t_0)\|_{\gamma} \tag{3.60}$$

Similarly we compute the operator norm:

$$\sup_{p\subseteq u} \|\partial_p \Delta^n\|_{\gamma} \tag{3.61}$$

$$\leqslant \sup_{t\in[t_0,t_1]} e^{-\gamma(t-t_0)} \sup_{p\subseteq u} \sum_{w\subseteq p} \|\partial_{p\backslash w}(a\ \partial_w \Delta^{n-1})(t,t_0)\| \tag{3.62}$$

$$+ \sup_{t\in[t_0,t_1]} e^{-\gamma(t-t_0)} \sup_{p\subseteq u} \sum_{w\subseteq p} \|(\partial_{p\backslash w} b\ \partial_p \Delta^{n-2})(t,t_0)\| \tag{3.63}$$

$$\leqslant \sup_{t\in[t_0,t_1]} e^{-\gamma(t-t_0)} \sup_{p\subseteq u} \sum_{w\subseteq p} \int_{t_0}^{t} dt' \|\partial_{p\backslash w} a(t,t')\ \partial_p \Delta^{n-1}(t',t_0)\| \tag{3.64}$$

$$+ \sup_{t\in[t_0,t_1]} e^{-\gamma(t-t_0)} \sup_{p\subseteq u} \sum_{w\subseteq u} \int_{t_0}^{t} dt' \|\partial_{p\backslash w} b(t,t')\ \partial_w \Delta^{n-2}(t',t_0)\| \tag{3.65}$$

$$\leqslant \sup_{t\in[t_0,t_1]} e^{-\gamma(t-t_0)} \sup_{p\subseteq u} \sum_{w\subseteq p} \int_{t_0}^{t} dt' \|\partial_{p\backslash w} a(t,t')\| \|\partial_w \Delta^{n-1}(t',t_0)\| \tag{3.66}$$

$$+ \sup_{t\in[t_0,t_1]} e^{-\gamma(t-t_0)} \sup_{p\subseteq u} \sum_{w\subseteq p} \int_{t_0}^{t} dt' \|\partial_{p\backslash w} b(t,t')\| \|\partial_w \Delta^{n-2}(t',t_0)\| \tag{3.67}$$

$$\leqslant \sup_{t\in[t_0,t_1]} e^{-\gamma t} \sup_{p\subseteq u} \sum_{w\subseteq p} \int_{t_0}^{t} dt' e^{\gamma t'} \|\partial_{p\backslash w} a\|_0 \|\partial_w \Delta^{n-1}(\cdot,t_0)\|_\gamma \tag{3.68}$$

$$+ \sup_{t\in[t_0,t_1]} e^{-\gamma t} \sup_{p\subseteq u} \sum_{w\subseteq p} \int_{t_0}^{t} dt' e^{\gamma t} \|\partial_{u\backslash p} b\|_0 \|\partial_w \Delta^{n-2}(\cdot,t_0)\|_\gamma \tag{3.69}$$

$$\leqslant \frac{1}{\gamma} \sup_{p\subseteq u} \sum_{w\subseteq p} \|\partial_{p\backslash w} a\|_0 \|\partial_w \Delta^{n-1}(\cdot,t_0)\|_\gamma \tag{3.70}$$

$$+ \frac{1}{\gamma} \sup_{p\subseteq u} \sum_{w\subseteq p} \|\partial_{p\backslash w} b\|_0 \|\partial_w \Delta^{n-2}(\cdot,t_0)\|_\gamma \tag{3.71}$$

$$\leqslant \frac{2^{|u|}}{\gamma} \sup_{u'\subseteq u} \|\partial_{u'} a\|_0 \sup_{w\subseteq u} \|\partial_w \Delta^{n-1}(\cdot,t_0)\|_\gamma \tag{3.72}$$

$$+ \frac{2^{|u|}}{\gamma} \sup_{u'\subseteq u} \|\partial_{u'} b\|_0 \sup_{w\subseteq u} \|\partial_w \Delta^{n-2}(\cdot,t_0)\|_\gamma \tag{3.73}$$

We can summarise the last calculations more briefly using the abbreviation

$$\alpha = \frac{2^{|u|+1}}{\gamma} \sup_{u' \subseteq u} \left\{ \|\partial_{u'} a\|_0, \|\partial_{u'} b\|_{I_2,\infty}, \|\partial_{u'} b\|_0 \right\}. \tag{3.74}$$

Here α is finite. This can be seen as follows: firstly, $\partial_u b = U^0 \partial_u F^B_{\mathrm{odd}} U^0$ vanishes if $|u| \geqslant 6$ because the factors Q'^B, Q^B and Z^B are all linear in B and the longest product of such operators appearing in b has three factors, analogously all derivatives $\partial_u a = 0$ for $|b| \geqslant 3$. Secondly, each of the operators Q'^C, Q^C and Z^C are bounded for every $C \in \mathcal{V}$, hence the polynomials a and b of these operators are also bounded. This shows finiteness of the two operator norms appearing in the expression for α. For the Hilbert-Schmidt norm we see that $\partial_u b$ is always a sum of terms where each term has a factor $U^0 \partial_p F^B_{\mathrm{odd}} U^0$ with $p \subseteq u$. This factor has finite Hilbert-Schmidt norm due to the I_2 estimate lemma 27.
We can thus summarise the last two calculations

$$\begin{pmatrix} \sup_{p\subseteq u} \|\partial_p \Delta^n\|_{\mathrm{op}+I_2,\gamma} \\ \sup_{p\subseteq u} \|\partial_p \Delta^{n-1}\|_{\mathrm{op}+I_2,\gamma} \end{pmatrix} \tag{3.75}$$

$$= \begin{pmatrix} \alpha & \alpha \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \sup_{p\subseteq u} \|\partial_p \Delta^{n-1}\|_{\mathrm{op}+I_2,\gamma} \\ \sup_{p\subseteq u} \|\partial_p \Delta^{n-2}\|_{\mathrm{op}+I_2,\gamma} \end{pmatrix} \tag{3.76}$$

$$= \begin{pmatrix} \alpha & \alpha \\ 1 & 0 \end{pmatrix}^{n-1} \begin{pmatrix} \sup_{p\subseteq u} \|\partial_p \Delta^{1}\|_{\mathrm{op}+I_2,\gamma} \\ \sup_{p\subseteq u} \|\partial_p \Delta^{0}\|_{\mathrm{op}+I_2,\gamma} \end{pmatrix}. \tag{3.77}$$

This matrix can be diagonalised, it's eigenvalues are

$$\lambda_\pm = \frac{\alpha}{2}\left(1 \pm \sqrt{1 + \frac{4}{\alpha}}\right). \tag{3.78}$$

The larger eigenvalue λ_+ is less than 1 if and only if $0 < \alpha < 0.5$ holds true, as can be seen from a quick calculation:

$$\frac{\alpha}{2}\left(1+\sqrt{1+\frac{4}{\alpha}}\right) < 1 \tag{3.79}$$

$$\iff \sqrt{1+\frac{4}{\alpha}} < \frac{2}{\alpha} - 1. \tag{3.80}$$

If $\alpha \geqslant \frac{1}{2}$ or $\alpha < 0$ this inequality is not satisfied, otherwise we may square both sides to find

$$1+\frac{4}{\alpha} < (2/\alpha - 1)^2 = 4/\alpha^2 - 4/\alpha + 1 \tag{3.81}$$

$$\iff \alpha < \frac{1}{2}. \tag{3.82}$$

So we conclude that for γ large enough the right hand side of (3.77) tends to zero as $c\lambda_+^n$ for $n \to \infty$, with

$$c = \sqrt{\sup_{p\subseteq u} \|\partial_p \Delta^1\|^2_{\mathrm{op}+I_2,\gamma} + \sup_{p\subseteq u} \|\partial_p \Delta^0\|^2_{\mathrm{op}+I_2,\gamma}}. \tag{3.83}$$

The two summands in the last equation are finite due to lemma 27. That is, we have

$$\sup_{p\subseteq u} \|\partial_p \Delta^n\|_{\mathrm{op}+I_2,\gamma} \leqslant \lambda_+^n c \xrightarrow{n\to\infty} 0. \tag{3.84}$$

For $2 \leqslant m \leqslant n$ we obtain

$$\sup_{p\subseteq u} \|\partial_p R_n^B - \partial_p R_m^B\|_{\mathrm{op}+I_2,\gamma} \leqslant \sum_{k=m}^{n-1} \sup_{p\subseteq u} \|\partial_p \Delta^k\|_{\mathrm{op}+I_2,\gamma} \tag{3.85}$$

$$\leqslant \sum_{k=m}^{\infty} \lambda_+^k c = \frac{\lambda_+^m}{1-\lambda_+} c \xrightarrow{m\to\infty} 0 \tag{3.86}$$

since the the norms $\|\cdot\|_{\mathrm{op}+I_2}$ and $\|\cdot\|_{\mathrm{op}+I_2,\gamma}$ are equivalent, we have just proven that $\partial_{[k]}R_m^B$ is a Cauchy sequence with respect to the norm $\|\cdot\|_{\mathrm{op}+I_2}$ and hence convergence by lemma 25.

□

The following lemma is a necessary ingredient for theorem 26. Morally, it has already been proven in [6, Lemma 3.7]; however, as that paper was not concerned with multiple four-potentials the lemma was not formulated general enough for our needs here. So we restate it and show how to modify the original proof.

Lemma 27 (I_2 estimates)**.** *Let $k \in \mathbb{N}$ and $A, H_b \in \mathcal{V}$ for $b \leqslant k$. Using the abbreviations introduced in (3.41) and (3.42) we have for any $u \subset [k]$ the following bounds:*

$$\|\partial_u U^0 F_{\mathrm{odd}}^{A+\sum_{b=1}^k H_b} U^0\|_{I_2,0} < \infty \tag{3.87}$$

$$\|\partial_u G^{A+\sum_{b=1}^k H_b}\|_{I_2,0} < \infty. \tag{3.88}$$

Proof. For $B \in \mathcal{V}$ recall

$$F_{\mathrm{odd}}^B := ((-Q'^B + Z_{\mathrm{ev}}^B - Q^B Z^B)(1+Q^B))_{\mathrm{odd}} \tag{3.89}$$

$$= -Q'^B + Z_{\mathrm{ev}}^B Q^B - Q^B Z_{\mathrm{ev}}^B - Q^B Z_{\mathrm{odd}}^B Q^B, \tag{3.90}$$

$$G^B := -U^0 Q^B Q^B \tag{3.91}$$

$$+U^0(-Q'^B + Z_{\mathrm{ev}} - Q^B Z^B) Q^B Q^B U^B (1+Q^B). \tag{3.92}$$

Pick $k \in \mathbb{N}$ and $A, B, H_b \in \mathcal{V}$ for $b \leqslant k$.
According to [6, lemma 3.7] the operators $U^0 Z_{\mathrm{ev}}^B Q^B U^0$, $U^0 Q^B Z_{\mathrm{ev}}^B U^0$, $U^0 Q'^B U^0$, $Q^B Q^B, Q'^B Q^B$ and $Q^B Z^B Q^B$ are Hilbert-Schmidt operators. Additionally, their Hilbert-Schmidt norm is uniformly bounded in time and F and G fulfil the following norm bound:

$$\|U^0 F^B_{\mathrm{odd}} U^0\|_{I_2,0} < \infty \text{ and } \|G^B\|_{I_2,0} < \infty. \tag{3.93}$$

In fact, the proof given in [6] also workes for non identical four-potentials $A, B, C \in \mathcal{V}$ proving

$$\|U^0 Z^A_{\mathrm{ev}} Q^B U^0\|_{I_2,0} < \infty, \tag{3.94}$$
$$\|U^0 Q^A Z^B_{\mathrm{ev}} U^0\|_{I_2,0} < \infty, \tag{3.95}$$
$$\|U^0 {Q'}^B U^0\|_{I_2,0} < \infty, \tag{3.96}$$
$$\|Q^B Q^A\|_{I_2,0} < \infty, \tag{3.97}$$
$$\|{Q'}^B Q^A\|_{I_2,0} < \infty, \tag{3.98}$$
$$\|Q^A Z^B Q^C\|_{I_2,0} < \infty \tag{3.99}$$

and therefore also

$$\|\partial_u U^0 F^{A+\sum_{b=0}^k H_b}_{\mathrm{odd}} U^0\|_{I_2,0} < \infty \text{ and } \|\partial_u G^{A+\sum_{b=0}^k H_b}\|_{I_2,0} < \infty \tag{3.100}$$

for any $u \subseteq [k]$.
For the benefit of the reader, we will reproduce the proof of the estimate

$$\|\partial_u G^{A+\sum_{b=0}^k H_b}\|_{I_2,0} < \infty, \tag{3.101}$$

to make clear the structure of the entire proof.
The operator $G^{A+\sum_{b=0}^k H_b}$ consists of two summands. Each summand is a product of operators with operator norm uniformly bounded in time and containing a factor of $Q^{A+\sum_{b=0}^k H_b} Q^{A+\sum_{b=0}^k H_b}$. All the other factors contributing to G stay bounded when differentiated and the map $Q : \mathcal{V} \to I_2$ is linear, so the bound

$$\|Q^B Q^D\|_{I_2,0} < \infty \tag{3.102}$$

for general $B, D \in \mathcal{V}$ will suffice to prove (3.101).

Pick $B, D \in \mathcal{V}$, we estimate

$$\sup_{t\in\mathbb{R}} \|Q^B(t)Q^D(t)\|_{I_2} \tag{3.103}$$

$$\leqslant \sum_{\mu,\nu=0}^{3} \Big(\sup_{p,k,q\in\mathbb{R}^3} |P_+(p)\gamma^0\gamma^\mu P_-(k)\gamma^0\gamma^\nu P_+(q)| \tag{3.104}$$

$$+ \sup_{p,k,q\in\mathbb{R}^3} |P_-(p)\gamma^0\gamma^\mu P_+(k)\gamma^0\gamma^\nu P_-(q)|\Big) \tag{3.105}$$

$$\sup_{t\in\mathbb{R}} \left\| \int_{\mathbb{R}^3} dk \frac{\hat{B}_\mu(t,p-k)\hat{D}_\nu(t,k-q)|}{(E(p)+E(k))(E(k)+E(q))} \right\|_{I_2,(p,q)}, \tag{3.106}$$

where the index in the norm indicates with respect to which variables the integral of the norm is to be performed. The prefactor is finite since $P_\pm(p) : \mathbb{C}^4 \to \mathbb{C}^4$ is a projector for any $p \in \mathbb{R}^3$. Abbreviating

$$\tilde{c} := \sum_{\mu,\nu=0}^{3} (\sup_{p,k,q\in\mathbb{R}^3} |P_+(p)\gamma^0\gamma^\mu P_-(k)\gamma^0\gamma^\nu P_+(q)| \tag{3.107}$$

$$+ \sup_{p,k,q\in\mathbb{R}^3} |P_-(p)\gamma^0\gamma^\mu P_+(k)\gamma^0\gamma^\nu P_-(q)|), \tag{3.108}$$

and using the integral estimate lemma [6, lemma 3.8 (iii)] we find

$$(3.103) \leqslant \tilde{c}\; C_{8,\text{of [6]}} \sum_{\mu,\nu=0}^{3} \sup_{t\in\mathbb{R}} \|\hat{B}_\mu(t)\|_{I_1} \|\hat{D}_\nu(t)\|_{I_2}. \tag{3.109}$$

Because of $B, D \in C_c^\infty(\mathbb{R}^4)$, we have $\hat{B}(t), \hat{D}(t)$ are analytic functions decaying faster than any negative power at anfinity for any t, so (3.109) is finite.

Also in order to proof the first estimate in (3.87) the proof of [6, lemma 3.7] can be followed almost verbatim. First one dissects F_{odd} into the sum $U^0(3.90)U^0$, next the summands have to be bounded individually.

This is achieved by repeating the proof of the partial integration lemma [6, lemma 3.6], estimates of the form of (3.103)-(3.106) and making use of the integral estimate lemma [6, lemma 3.8].

□

Theorem 28 (Properties of Derivatives of S). *Let $A, H \in \mathcal{V}$, pick t_1 after* $\operatorname{supp} A \cup \operatorname{supp} H$ *and t_0 before* $\operatorname{supp} A \cup \operatorname{supp} H$*, let $T_1 \in I_2(\mathcal{F})$ then the following equalities are satisfied:*

$$\begin{aligned}&\partial_H \operatorname{tr}(T_1 P^+ U^A(t_0,t_1) U^{A+H}(t_1,t_0) P^-)\\&= \operatorname{tr}(T_1 P^\pm U^A(t_0,t_1)\partial_H U^{A+H}(t_1,t_0) P^\mp)\end{aligned} \tag{3.110}$$

Proof. Let $A, H \in \mathcal{V}$ and $t_0, t_1 \in \mathbb{R}$ and T_1 be as in the theorem. The proof of the two equalities is analogous, so we only explicitly prove the first one. The trace is linear, so we have

$$\begin{aligned}&\Bigg|\operatorname{tr}\left(T_1 P^+ U^A(t_0,t_1)\frac{1}{\varepsilon}(U^{A+\varepsilon H}(t_1,t_0) - U^A(t_1,t_0))P^-\right)\\&\quad - \operatorname{tr}(T_1 P^+ U^A(t_0,t_1)\partial_H U^{A+H}(t_1,t_0)P^-)\Bigg|\end{aligned} \tag{3.111}$$

$$\begin{aligned}&\leqslant \|T_1\|_{I_2}\Bigg\| P^+ U^A(t_0,t_1)\frac{1}{\varepsilon}\big(U^{A+\varepsilon H}(t_1,t_0) - U^A(t_1,t_0)\big)P^-\\&\quad - P^+ U^A(t_0,t_1)\partial_H U^{A+H}(t_1,t_0)P^-\Bigg\|_{I_2}\end{aligned} \tag{3.112}$$

For the first summand we insert the identity in the form $P^+ + P^-$ and obtain

$$P^+ U^A(t_0,t_1)\frac{1}{\varepsilon}\left(U^{A+\varepsilon H}(t_1,t_0) - U^A(t_1,t_0)\right)P^- \tag{3.113}$$

$$= P^+ U^A(t_0,t_1) P^+ \frac{1}{\varepsilon}\left(U^{A+\varepsilon H}(t_1,t_0) - U^A(t_1,t_0)\right)P^- \tag{3.114}$$

$$+ P^+ U^A(t_0,t_1) P^- \frac{1}{\varepsilon}\left(U^{A+\varepsilon H}(t_1,t_0) - U^A(t_1,t_0)\right)P^-. \tag{3.115}$$

Analogously for the second summand. Now because of the Smoothness of S theorem 26 we know that

$$P^{-}\frac{1}{\varepsilon}\left(U^{A+\varepsilon H}(t_1,t_0)-U^{A}(t_1,t_0)\right)P^{-}\xrightarrow[\|\cdot\|]{\varepsilon\to 0}P^{-}\partial_H U^{A+H}(t_1,t_0)P^{-} \tag{3.116}$$

$$P^{+}\frac{1}{\varepsilon}\left(U^{A+\varepsilon H}(t_1,t_0)-U^{A}(t_1,t_0)\right)P^{-}\xrightarrow[\|\cdot\|_{I_2}]{\varepsilon\to 0}P^{+}\partial_H U^{A+H}(t_1,t_0)P^{-} \tag{3.117}$$

holds true. Hence we find in total

$$\frac{(3.112)}{\|T_1\|_{I_2}} \tag{3.118}$$

$$\leqslant \left\|P^{+}U^{A}(t_0,t_1)\right\|\left\|P^{+}\frac{1}{\varepsilon}\left(U^{A+\varepsilon H}(t_1,t_0)-U^{A}(t_1,t_0)\right)P^{-}\right.$$

$$\left.-\,P^{+}\partial_H U^{A+H}(t_1,t_0)P^{-}\right\|_{I_2} \tag{3.119}$$

$$+\left\|P^{+}U^{A}(t_0,t_1)P^{-}\right\|\left\|\frac{1}{\varepsilon}\left(U^{A+\varepsilon H}(t_1,t_0)-U^{A}(t_1,t_0)\right)P^{-}\right.$$

$$\left.-\,\partial_H U^{A+H}(t_1,t_0)P^{-}\right\|_{I_2}\xrightarrow{\varepsilon\to 0}0. \tag{3.120}$$

□

3.4 Geometric Construction of the Phase

Next we introduce the set of four potentials we work with, as well as the argument of a complex number and an invertible bounded operator. For complex numbers the convention we chose here differs slightly from the standard in the literature, which is why we also use a slightly non standard name for this function.

Definition 29 (polar decomposition and spectral projections). *We denote by $\mathcal{H} = L^2(\mathbb{R}^3, \mathbb{C})$. For $X : \mathcal{H} \to \mathcal{H}$ bounded*

$$\mathrm{AG}(X) := X|X|^{-1}. \tag{3.121}$$

Furthermore, we define for any complex number $z \in \mathbb{C}\backslash\{0\}$

$$\mathrm{ag}(z) := \frac{z}{|z|}. \tag{3.122}$$

In abuse of notation we will define the expression

$$\partial_t \ln f(t) := \frac{\partial_t f(t)}{f(t)}, \tag{3.123}$$

for any differentiable $f : \mathbb{R} \to \mathbb{C}\backslash\{0\}$, even if the expression $\ln f(t)$ cannot be interpreted as the principle branch of the logarithm.
We also introduce $S^1 := \{z \in \mathbb{C} \mid |z| = 1\}$.

Definition 30 (scattering operator and phases). *We define for all $A, B \in \mathcal{V}$*

$$S_{A,B} := U^A_{\Sigma_{\mathrm{in}},\Sigma_{\mathrm{out}}} U^B_{\Sigma_{\mathrm{out}},\Sigma_{\mathrm{in}}}, \tag{3.124}$$

where Σ_{out} and Σ_{int} are Cauchy-surfaces of Minkowski spacetime such that

$$\forall (x,y) \in \operatorname{supp} A \cup \operatorname{supp} B \times \Sigma_{\mathrm{in}} : (x-y)^2 \geqslant 0 \Rightarrow x^0 > y^0, \tag{3.125}$$
$$\forall (x,y) \in \operatorname{supp} A \cup \operatorname{supp} B \times \Sigma_{\mathrm{out}} : (x-y)^2 \geqslant 0 \Rightarrow x^0 < y^0 \tag{3.126}$$

holds. Let

$$\mathrm{dm} := \{(A,B) \in \mathcal{V}^2 \mid P^- S_{A,B} P^- \textit{ and} \tag{3.127}$$
$$P^- S_{B,A} P^- : \mathcal{H}^- \circlearrowleft \textit{ are invertible}\}, \tag{3.128}$$

we define

$$\operatorname{dom}\overline{S} := \{(A,B) \in \mathrm{dm} \mid \overline{A\ B} \times \overline{A\ B} \subseteq \mathrm{dm}\}, \tag{3.129}$$

where $\overline{A\ B}$ *is the line segment connecting* A *and* B *in* $\mathcal{V}$*. Furthermore, we choose for all* $A, B \in \operatorname{dom}\overline{S}$ *the lift*

$$\overline{S}_{A,B} = \mathcal{R}_{\mathrm{AG}((P^- S_{A,B} P^-)^{-1})} \mathcal{L}_{S_{A,B}}. \tag{3.130}$$

For $(A,B),(B,C),(C,A) \in \operatorname{dom}\overline{S}$*, we define the complex numbers*

$$\gamma_{A,B,C} := \mathrm{ag}(\det_{\mathcal{H}^-}(P^- S_{A,B} P^- S_{B,C} P^- S_{C,A} P^-)), \tag{3.131}$$

$$\Gamma_{A,B,C} := \mathrm{ag}(\gamma_{A,B,C}). \tag{3.132}$$

We will see in lemma 35 that $\gamma_{A,B,C} \neq 0$ *and* $P^- S_{A,B} P^- S_{B,C} P^- S_{C,A} - 1$ *is traceclass, so that* $\Gamma_{A,B,C}$ *is well-defined. Lastly we introduce for* $A, B, C \in \mathcal{V}$ *the function*

$$c_A(F,G) := -i\partial_F \partial_G \Im \operatorname{tr}[P^- S_{A,A+F} P^+ S_{A,A+G} P^-]. \tag{3.133}$$

Lemma 31 (properties of $\operatorname{dom}\overline{S}$)**.** *The set* $\operatorname{dom}\overline{S}$ *has the following properties:*

1. *contains the diagonal:* $\{(A,A) \mid A \in \mathcal{V}\} \subseteq \operatorname{dom}\overline{S}$.
2. *openness:* $\forall n \in \mathbb{N} : \left\{s \in \mathbb{R}^{2n} \mid \left(\sum_{k=1}^{n} s_k A_k, \sum_{k=n+1}^{2n} s_k A_k\right) \in \operatorname{dom}\overline{S}\right\}$ *is an open subset of* $\mathbb{R}^{2n}$ *for all* $A_1, \ldots A_{2n} \in \mathcal{V}$.
3. *symmetry:* $(A,A') \in \operatorname{dom}\overline{S} \iff (A',A) \in \operatorname{dom}\overline{S}$
4. *star-shaped:* $(A,tA) \in \operatorname{dom}\overline{S} \Rightarrow \forall s \in \overline{1\ t} : (A,sA) \in \operatorname{dom}\overline{S}$
5. *well defined-ness of* $\overline{S}$*:* $\operatorname{dom}\overline{S} \subseteq \{A, B \in \mathcal{V} \mid P^- S_{A,B} P^- : \mathcal{H}^- \circlearrowleft$ *is invertible*$\}$.

We will only prove openness, as the other properties follow directly from the definition (3.129)*. So pick* $n \in \mathbb{N}$*,* $A_i \in \mathcal{V}$ *for* $i \in \mathbb{N}, i \leqslant 2n$ *and* $s \in \mathbb{R}^{2n}$ *such that* $\left(\sum_{k=1}^{n} s_k A_k, \sum_{k=n+1}^{2n} s_k A_k\right) \in \operatorname{dom} \overline{S}$*. We have to find a neighbourhood* $U \subseteq \mathbb{R}^{2n}$ *of* s *such that* $\{\left(\sum_{k=1}^{n} s'_k A_k, \sum_{k=n+1}^{2n} s'_k A_k\right) \mid s' \in U\} \subseteq \operatorname{dom} \overline{S}$ *holds. In doing so we have to ensure that the square*

$$\overline{\sum_{k=1}^{n} s_k A_k \qquad \sum_{k=n+1}^{2n} s_k A_k}^{\,2} \tag{3.134}$$

stays a subsets of dm *for all* $s' \in U$*. Now pick a metric* d *on* $\mathbb{R}^n$ *and define*

$$r := \inf \left\{ d(s, s') \mid \overline{\sum_{k=1}^{n} s'_k A_k \qquad \sum_{k=n+1}^{2n} s'_k A_k}^{\,2} \quad \cap \, \mathrm{dm}^c \neq \varnothing \right\}.$$

It cannot be the case that $r = 0$*, because the metric is continuous, the square compact in* $\mathbb{R}^{2n}$ *and the set of invertible bounded operators (defining* dm*) is open in the topology generated by the operator norm. If* $r = \infty$ *then* $U = \mathbb{R}^{2n}$ *will suffice. If* $r \in \mathbb{R}^+$ *then* $U = B_r(s, t)$ *the open ball of radius* r *around* s *works.*

3.4.1 Main Result of Construction

Definition 32 (causal splitting)**.** *We define a causal splitting as a function*

$$c^+ : \mathcal{V}^3 \to \mathbb{C}, \tag{3.135}$$

$$(A, F, G) \mapsto c^+_A(F, G), \tag{3.136}$$

such that c^+ *restricted to any finite dimensional subspace is smooth in the first argument and linear in the second and third argument.*

Furthermore c^+ *should satisfy*

$$c_A(F,G) = c_A^+(F,G) - c_A^+(G,F), \tag{3.137}$$

$$\partial_H c_{A+H}^+(F,G) = \partial_G c_{A+G}^+(F,H), \tag{3.138}$$

$$\forall F \prec G : c_A^+(F,G) = 0. \tag{3.139}$$

Definition 33 (current)**.** *Given a lift* $\hat{S}_{A,B}$ *of the one-particle scattering operator* $S_{A,B}$ *for which the derivative in the following expression exists, we define the associated current by Bogolyubov's formula:*

$$j_A^{\hat{S}}(F) := i\partial_F \left\langle \Omega, \hat{S}_{A,A+F}\Omega \right\rangle. \tag{3.140}$$

Theorem 34 (existence of causal lift)**.** *Given a causal splitting* c^+*, there is a second quantised scattering operator* $\tilde{S}$*, lift of the one-particle scattering operator* S *with the following properties*

$$\forall A, B, C \in \mathcal{V} : \tilde{S}_{A,B}\tilde{S}_{B,C} = \tilde{S}_{A,C} \tag{3.141}$$

$$\forall F \prec G : \tilde{S}_{A,A+F} = \tilde{S}_{A+G,A+F+G} \tag{3.142}$$

and the associated current satisfies

$$\partial_G j_{A+G}^{\tilde{S}}(F) = \begin{cases} -2ic_A(F,G) & \textit{for } G \prec F \\ 0 & \textit{otherwise.} \end{cases} \tag{3.143}$$

3.4.2 Proofs

Since the phase of a lift relative to any other lift is fixed by a single matrix element, we may use the vacuum expectation values to characterise the phase of a lift. The function c captures the dependence of this object on variation of the external fields, the connection between vacuum expectation values and c becomes clearer with the next lemma.

Lemma 35 (properties of Γ)**.** *The function Γ has the following properties for all $A, B, C, D \in \mathcal{V}$ such that the expressions occurring in each equation are well defined:*

$$\gamma_{A,B,C} \neq 0 \tag{3.144}$$

$$\Gamma_{A,B,C} = \det_{\mathcal{H}^-}(P^- - P^- S_{A,C} P^+ S_{C,A} P^- \\ - P^- S_{A,B} P^+ S_{B,C} P^- S_{C,A} P^-) \tag{3.145}$$

$$\Gamma^{-1}_{A,B,C} = \mathrm{ag}(\langle \Omega, \overline{S}_{A,B} \overline{S}_{B,C} \overline{S}_{C,A} \Omega \rangle) \tag{3.146}$$

$$\Gamma_{A,B,C} = \Gamma_{B,C,A} = \frac{1}{\Gamma_{B,A,C}} \tag{3.147}$$

$$\Gamma_{A,A,B} = 1 \tag{3.148}$$

$$\Gamma_{A,B,C}\Gamma_{B,A,D}\Gamma_{A,C,D}\Gamma_{C,B,D} = 1 \tag{3.149}$$

$$\Gamma_{A,B,C} = \Gamma_{D,B,C}\Gamma_{A,D,C}\Gamma_{A,B,D} \tag{3.150}$$

$$\overline{S}_{A,C} = \Gamma_{A,B,C} \overline{S}_{A,B} \overline{S}_{B,C} \tag{3.151}$$

$$c_A(B, C) = \partial_B \partial_C \ln \Gamma_{A,A+B,A+C}. \tag{3.152}$$

Proof. Pick $A, B, C \in \mathcal{V}$ such that $\|1 - S_{X,Y}\| < 1$ for $X, Y \in \{A, B, C\}$. By definition γ is

$$\gamma_{A,B,C} = \det_{\mathcal{H}^-}(P^- S_{A,B} P^- S_{B,C} P^- S_{C,A} P^-). \tag{3.153}$$

The operator whose determinant we take in the last line is a product

$$P^- S_{A,B} P^- S_{B,C} P^- S_{C,A} P^- = P^- S_{A,B} P^- \quad P^- S_{B,C} P^- \quad P^- S_{C,A} P^-. \tag{3.154}$$

The three factors appearing in this product are all invertible, hence the product is also invertible as operators of type $\mathcal{H}^- \to \mathcal{H}^-$ because of the conditions of $\{A, B, C\}$ imply that $\|P^- - P^- S_{X,Y} P^-\| < 1$ which means that the Von Neumann series of the inverse converges, therefore if the determinant exists we have $\gamma_{A,B,C} \neq 0$. To see that it does exist, we reformulate

$$\begin{aligned}\gamma_{A,B,C} &= \det_{\mathcal{H}^-}(P^- S_{A,B} P^- S_{B,C} P^- S_{C,A} P^-) &(3.155)\\ &= \det_{\mathcal{H}^-}(P^- S_{A,C} P^- S_{C,A} P^- - P^- S_{A,B} P^+ S_{B,C} P^- S_{C,A} P^-) &(3.156)\\ &= \det_{\mathcal{H}^-}(P^- - P^- S_{A,C} P^+ S_{C,A} P^- - P^- S_{A,B} P^+ S_{B,C} P^- S_{C,A} P^-), &(3.157)\end{aligned}$$

now we know by a classic result of Ruisnaars [?] that $P^+ S_{X,Y} P^-$ is a Hilbert-Schmidt operator for our setting, hence γ and also Γ are well defined.

Equation (3.157) also proves (3.145). Next we show (3.146). Borrowing notation from [6, section 2] to identify $\Omega = \bigwedge \Phi$ with the injection $\Phi : \mathcal{H}^- \hookrightarrow \mathcal{H}$ and $\bigwedge$ is used to construct the infinite wedge spaces that are the perspective of Fock space introduced in [6]. We begin by reformulating the right hand side of (3.146)

$$\begin{aligned}&\langle \Omega, \overline{S}_{A,B} \overline{S}_{B,C} \overline{S}_{C,A} \Omega \rangle &(3.158)\\ &= \Big\langle \bigwedge \Phi, \bigwedge \big(S_{A,B} S_{B,C} S_{C,A} \Phi \,\mathrm{AG}(P^- S_{C,A} P^-)^{-1} \\ &\qquad \mathrm{AG}(P^- S_{B,C} P^-)^{-1} \,\mathrm{AG}(P^- S_{A,B} P^-)^{-1} \big) \Big\rangle \\ &= \Big\langle \bigwedge \Phi, \bigwedge \big(\Phi \,\mathrm{AG}(P^- S_{C,A} P^-)^{-1} &(3.159)\\ &\qquad \times \mathrm{AG}(P^- S_{B,C} P^-)^{-1} \,\mathrm{AG}(P^- S_{A,B} P^-)^{-1} \big) \Big\rangle \\ &= \det_{\mathcal{H}^-} \big((\Phi)^* \big[\Phi \,\mathrm{AG}(P^- S_{C,A} P^-)^{-1} \,\mathrm{AG}(P^- S_{B,C} P^-)^{-1} &(3.160)\\ &\qquad \times \mathrm{AG}(P^- S_{A,B} P^-)^{-1} \big] \big) \\ &= \det_{\mathcal{H}^-} \big(\big) \,\mathrm{AG}(P^- S_{C,A} P^-)^{-1} \,\mathrm{AG}(P^- S_{B,C} P^-)^{-1} &(3.161)\\ &\qquad \times \mathrm{AG}(P^- S_{A,B} P^-)^{-1} \big) \\ &= \frac{1}{\det_{\mathcal{H}^-} \mathrm{AG}(P^- S_{A,B} P^-) \,\mathrm{AG}(P^- S_{B,C} P^-) \,\mathrm{AG}(P^- S_{C,A} P^-)}. &(3.162)\end{aligned}$$

We first note that $\det_{\mathcal{H}^-} |P^- S_{X,Y} P^-| \in \mathbb{R}^+$ for $X, Y \in \{A, B, C\}$. This is well defined because

$$\langle \Omega, \overline{S}_{X,Y}\Omega \rangle = \langle \bigwedge \Phi, \bigwedge (S_{X,Y}\Phi \, \mathrm{AG}(P^- S_{X,Y} P^-)^{-1}) \rangle \tag{3.163}$$

$$= \det_{\mathcal{H}^-} \big(\Phi^* S_{X,Y} \Phi \, \mathrm{AG}(P^- S_{X,Y} P^-)^{-1} \big) \tag{3.164}$$

$$= \det_{\mathcal{H}^-} \big(P^- S_{X,Y} P^- \, \mathrm{AG}(P^- S_{X,Y} P^-)^{-1} \big) \tag{3.165}$$

$$= \det_{\mathcal{H}^-} \big(\mathrm{AG}(P^- S_{X,Y} P^-)^{-1} P^- S_{X,Y} P^- \big) \tag{3.166}$$

$$= \det_{\mathcal{H}^-} \big(\mathrm{AG}(P^- S_{X,Y} P^-)^{-1} \, \mathrm{AG}(P^- S_{X,Y} P^-) |P^- S_{X,Y} P^-| \big) \tag{3.167}$$

$$= \det_{\mathcal{H}^-} |P^- S_{X,Y} P^-| \tag{3.168}$$

holds. Moreover this determinant does not vanish, since the $P^- S_{X,Y} P^-$ is invertible. Also clearly the eigenvalues are positive since $|P^- S_{X,Y} P^-|$ is an absolute value. We continue with the result of (3.162). Thus, we find

$$\langle \Omega, \overline{S}_{A,B}\overline{S}_{B,C}\overline{S}_{C,A}\Omega \rangle^{-1} \tag{3.169}$$

$$= \det_{\mathcal{H}^-} \big(\mathrm{AG}(P^- S_{A,B} P^-) \, \mathrm{AG}(P^- S_{B,C} P^-) \, \mathrm{AG}(P^- S_{C,A} P^-) \big) \tag{3.170}$$

$$= \det_{\mathcal{H}^-} \big(\mathrm{AG}(P^- S_{A,B} P^-) \, \mathrm{AG}(P^- S_{B,C} P^-) P^- S_{C,A} P^- \times |P^- S_{C,A} P^-|^{-1} \big) \tag{3.171}$$

$$= \det_{\mathcal{H}^-} \big(\mathrm{AG}(P^- S_{A,B} P^-) \, \mathrm{AG}(P^- S_{B,C} P^-) P^- S_{C,A} P^- \big) \times \det_{\mathcal{H}^-} |P^- S_{C,A} P^-|^{-1} \tag{3.172}$$

$$= \det_{\mathcal{H}^-} \big(P^- S_{C,A} P^- \, \mathrm{AG}(P^- S_{A,B} P^-) \, \mathrm{AG}(P^- S_{B,C} P^-) \big) \times \det_{\mathcal{H}^-} |P^- S_{C,A} P^-|^{-1} \tag{3.173}$$

$$= \frac{\det_{\mathcal{H}^-} (P^- S_{A,B} P^- P^- S_{B,C} P^- P^- S_{C,A} P^-)}{\det_{\mathcal{H}^-} |P^- S_{A,B} P^-| \cdot \det_{\mathcal{H}^-} |P^- S_{B,C} P^-|}$$

$$\times \frac{1}{\det_{\mathcal{H}^-} |P^- S_{C,A} P^-|}. \tag{3.174}$$

Now since the denominator of this fraction is real we can use (3.145) to identity

$$\mathrm{ag}(\langle \Omega, \overline{S}_{A,B} \overline{S}_{B,C} \overline{S}_{C,A} \Omega \rangle) = \Gamma^{-1}_{A,B,C}, \tag{3.175}$$

which proves (3.146).
For the first equality in (3.147) we use $\det X(1+Y)X^{-1} = \det(1+Y)$ for any Y trace-class and X bounded and invertible. So we can cyclicly permute the factors $P^- S_{X,Y} P^-$ in the determinant and find

$$\begin{aligned}\Gamma_{A,B,C} &= \mathrm{ag}(\det_{\mathcal{H}^-} P^- S_{A,B} P^- S_{B,C} P^- S_{C,A} P^-) \\ &= \mathrm{ag}(\det_{\mathcal{H}^-} P^- S_{C,A} P^- S_{A,B} P^- S_{B,C} P^-) = \Gamma_{C,A,B}.\end{aligned}$$

For the second equality of (3.147) we use (3.145) to represent both $\Gamma_{A,B,C}$ and $\Gamma_{B,A,C}$. Using this and the manipulations of the determinant we already employed, we arrive at

$$\begin{aligned}
&\Gamma_{A,B,C}\Gamma_{B,A,C} && (3.176)\\
&= \mathrm{ag}(\det_{\mathcal{H}^-}(P^- S_{A,B} P^- S_{B,C} P^- S_{C,A} P^-)) && (3.177)\\
&\quad \times \mathrm{ag}(\det_{\mathcal{H}^-}(P^- S_{B,A} P^- S_{A,C} P^- S_{C,B} P^-)) && (3.178)\\
&= \mathrm{ag}(\det_{\mathcal{H}^-}(P^- S_{A,B} P^- S_{B,C} P^- S_{C,A} P^-)) && (3.179)\\
&\quad \times (\mathrm{ag}(\det_{\mathcal{H}^-}(P^- S_{B,C} P^- S_{C,A} P^- S_{A,B} P^-)))^* && (3.180)\\
&= \mathrm{ag}(\det_{\mathcal{H}^-}(P^- S_{A,B} P^- S_{B,C} P^- S_{C,A} P^-)) && (3.181)\\
&\quad \times (\mathrm{ag}(\det_{\mathcal{H}^-}(P^- S_{A,B} P^- S_{B,C} P^- S_{C,A} P^-)))^* && (3.182)\\
&= |\,\mathrm{ag}(\det_{\mathcal{H}^-}(P^- S_{A,B} P^- S_{B,C} P^- S_{C,A} P^-))|^2 = 1, && (3.183)
\end{aligned}$$

which proves (3.147).
Next, using (3.131) inserting twice the same argument yields

$$\gamma_{A,A,C} = \det_{\mathcal{H}^-} P^- S_{A,C} P^- S_{C,A} P^- = \det_{\mathcal{H}^-} (P^- S_{C,A} P^-)^* P^- S_{C,A} P^- \in \mathbb{R}^+, \tag{3.184}$$

hence (3.148) follows.
For proving (3.149) we will use the definition of Γ directly and repeatedly use that we can cyclicly permute operator groups of the form $P^- S_{X,Y} P^-$ for $X, Y \in \{A, B, C, D\}$ in the determinant, i.e.

$$\det P^- S_{X,Y} P^- O = \det O P^- S_{X,Y} P^-. \tag{$\circlearrowleft$}$$

This is possible, because $P^- S_{X,Y} P^-$ is bounded and invertible. Furthermore we will use that

$$\det O_1 O_2 = \det O_1 \det O_2 \tag{$\leftrightarrow$}$$

holds whenever both O_1 and O_2 have a determinant. Moreover for any $(P^- S_{X,Y} P^-)^* P^- S_{X,Y} P^-$ is the modulus squared of an invertible operator and hence its determinant is positive which means that

$$\operatorname{ag} \det (P^- S_{X,Y} P^-)^* P^- S_{X,Y} P^- = 1. \tag{ag$|\ |$}$$

These three rules will be repeatedly used. We calculate

$$\begin{aligned}
&\Gamma_{A,B,C}\Gamma_{B,A,D}\Gamma_{A,C,D}\Gamma_{C,B,D} && (3.185)\\
&= \operatorname{ag} \det_{\mathcal{H}^-} P^- S_{A,B} P^- S_{B,C} P^- S_{C,A} P^- \\
&\qquad \times \operatorname{ag} \det_{\mathcal{H}^-} P^- S_{B,A} P^- S_{A,D} P^- S_{D,B} P^- \quad \Gamma_{A,C,D}\Gamma_{C,B,D} && (3.186)\\
&\overset{(\circlearrowleft)}{=} \operatorname{ag} \det_{\mathcal{H}^-} P^- S_{A,D} P^- S_{D,B} P^- S_{B,A} P^- \\
&\qquad \times \operatorname{ag} \det_{\mathcal{H}^-} P^- S_{A,B} P^- S_{B,C} P^- S_{C,A} P^- \quad \Gamma_{A,C,D}\Gamma_{C,B,D} && (3.187)
\end{aligned}$$

$$\overset{(\leftrightarrow)}{=} \operatorname{ag}\det_{\mathcal{H}^-} \big(P^- S_{A,D} P^- S_{D,B} \big[P^- S_{B,A} P^- S_{A,B} P^-\big] \times S_{B,C} P^- S_{C,A} P^-\big)\ \Gamma_{A,C,D}\Gamma_{C,B,D} \tag{3.188}$$

$$\overset{(\circlearrowleft)}{=} \operatorname{ag}\det_{\mathcal{H}^-} P^- S_{B,C} P^- S_{C,A} P^- S_{A,D} P^- S_{D,B} \big[P^- S_{B,A} P^- S_{A,B} P^-\big] \times \Gamma_{A,C,D}\Gamma_{C,B,D} \tag{3.189}$$

$$\overset{(\leftrightarrow)}{=} \operatorname{ag}\det_{\mathcal{H}^-} P^- S_{B,C} P^- S_{C,A} P^- S_{A,D} P^- S_{D,B} P^- \times \operatorname{ag}\det_{\mathcal{H}^-} P^- S_{B,A} P^- S_{A,B} P^-\ \ \Gamma_{A,C,D}\Gamma_{C,B,D} \tag{3.190}$$

$$\overset{(\operatorname{ag}|\,|)}{=} \operatorname{ag}\det_{\mathcal{H}^-} P^- S_{B,C} P^- S_{C,A} P^- S_{A,D} P^- S_{D,B} P^-\ \ \Gamma_{A,C,D}\Gamma_{C,B,D} \tag{3.191}$$

$$\overset{(\circlearrowleft)}{=} \operatorname{ag}\det_{\mathcal{H}^-} P^- S_{A,D} P^- S_{D,B} P^- S_{B,C} P^- S_{C,A} P^-\ \ \Gamma_{A,C,D}\Gamma_{C,B,D} \tag{3.192}$$

$$= \operatorname{ag}\det_{\mathcal{H}^-} P^- S_{A,C} P^- S_{C,D} P^- S_{D,A} P^- \times \operatorname{ag}\det_{\mathcal{H}^-} P^- S_{A,D} P^- S_{D,B} P^- S_{B,C} P^- S_{C,A} P^-\ \ \Gamma_{C,B,D} \tag{3.193}$$

$$\overset{(\leftrightarrow)}{=} \operatorname{ag}\det_{\mathcal{H}^-} \big(P^- S_{A,C} P^- S_{C,D} P^- \big[P^- S_{D,A} P^- P^- S_{A,D} P^-\big] \times S_{D,B} P^- S_{B,C} P^- S_{C,A} P^-\big)\ \ \Gamma_{C,B,D} \tag{3.194}$$

$$\overset{(\circlearrowleft)}{=} \operatorname{ag}\det_{\mathcal{H}^-} \big(P^- S_{D,B} P^- S_{B,C} P^- S_{C,A} P^- S_{A,C} P^- S_{C,D} P^- \times \big[P^- S_{D,A} P^- P^- S_{A,D} P^-\big]\big)\ \ \Gamma_{C,B,D} \tag{3.195}$$

$$\overset{(\leftrightarrow)}{=} \operatorname{ag}\det_{\mathcal{H}^-} P^- S_{D,B} P^- S_{B,C} P^- S_{C,A} P^- S_{A,C} P^- S_{C,D} P^- \times \operatorname{ag}\det_{\mathcal{H}^-} P^- S_{D,A} P^- P^- S_{A,D} P^-\ \ \Gamma_{C,B,D} \tag{3.196}$$

$$\begin{aligned}
&\overset{(\mathrm{ag}|\,|)}{=} \operatorname{ag}\det_{\mathcal{H}^-}\big(P^-S_{D,B}P^-S_{B,C}P^-\big[P^-S_{C,A}P^-S_{A,C}P^-\big] \\
&\qquad\qquad \times P^-S_{C,D}P^-\big)\Gamma_{C,B,D} && (3.197)\\
&\overset{(\circlearrowleft)}{=} \operatorname{ag}\det_{\mathcal{H}^-} P^-S_{C,D}P^-S_{D,B}P^-S_{B,C}P^-\big[P^-S_{C,A}P^-S_{A,C}P^-\big] \\
&\qquad\qquad \times \Gamma_{C,B,D} && (3.198)\\
&\overset{(\leftrightarrow)}{=} \operatorname{ag}\det_{\mathcal{H}^-} P^-S_{C,D}P^-S_{D,B}P^-S_{B,C}P^- \\
&\qquad\qquad \times \operatorname{ag}\det P^-S_{C,A}P^-S_{A,C}P^- \quad \Gamma_{C,B,D} && (3.199)\\
&\overset{(\mathrm{ag}|\,|)}{=} \operatorname{ag}\det_{\mathcal{H}^-} P^-S_{C,D}P^-S_{D,B}P^-S_{B,C}P^- \quad \Gamma_{C,B,D} && (3.200)\\
&= \operatorname{ag}\det_{\mathcal{H}^-} P^-S_{C,D}P^-S_{D,B}P^-S_{B,C}P^- \\
&\qquad\qquad \times \operatorname{ag}\det_{\mathcal{H}^-} P^-S_{C,B}P^-S_{B,D}P^-S_{D,C}P^- && (3.201)\\
&= |\operatorname{ag}\det_{\mathcal{H}^-} P^-S_{C,D}P^-S_{D,B}P^-S_{B,C}P^-|^2 = 1. && (3.202)
\end{aligned}$$

Equation (3.150) is a direct consequence of (3.149) and (3.147).
For (3.151) we realise that according to [6] that two lifts can only differ by a phase, that is

$$\overline{S}_{A,C} = \alpha\overline{S}_{A,B}\overline{S}_{B,C} \tag{3.203}$$

for some $\alpha \in \mathbb{C}, |\alpha| = 1$.
In order to identify α we recognise that $\overline{S}_{X,Y} = \overline{S}^{-1}_{Y,X}$ for four potentials X, Y and find

$$1\alpha^{-1} = \overline{S}_{A,B}\overline{S}_{B,C}\overline{S}_{C,A}. \tag{3.204}$$

Now we take the vacuum expectation value on both sides of this equation and use (3.146) to find

$$\alpha^{-1} = \langle\Omega, \overline{S}_{A,B}\overline{S}_{B,C}\overline{S}_{C,A}\Omega\rangle = \Gamma^{-1}_{A,B,C}. \tag{3.205}$$

Finally we prove (3.152). We start from the right hand side of this equation and work our way towards the left hand side of it. In the following calculation we will repeatedly make use of the fact that $(P^- S_{A,A+B} P^- S_{A+B,A} P^-)$ is the absolute value squared of an invertible operator and has a determinant, which is therefore positive. For the marked equality we will use that for a differentiable function $z : \mathbb{R} \to \mathbb{C}$ at points t where $z(t) \in \mathbb{R}^+$ holds, we have

$$(z/|z|)'(t) = \frac{z'}{|z|}(t) + \frac{-z}{|z|^2}\frac{z'z^* + z^{*\prime}z}{2|z|}(t) = \frac{z'}{2|z|}(t) - \frac{z^2 z^{*\prime}}{2|z|^3}(t)$$
$$= i(\Im(z'))/z(t). \quad (3.206)$$

Furthermore, we will use the following expressions for the derivative of the determinant which holds for all functions $M : \mathbb{R} \to (\mathcal{H} \to \mathcal{H})$ such that $M(t) - 1$ is traceclass and M is invertible for all $t \in \mathbb{R}$

$$\partial_\varepsilon \det M(\varepsilon)|_{\varepsilon=0} = \det M(0)\, \mathrm{tr}(M^{-1}(0)\partial_\varepsilon M(\varepsilon)|_\varepsilon), \quad (3.207)$$

likewise we need the following expression for the derivative of M^{-1} for $M : \mathbb{R} \to (\mathcal{H} \to \mathcal{H})$ such that $M(t)$ is invertible and bounded for every $t \in \mathbb{R}$

$$\partial_\varepsilon M^{-1}(\varepsilon)|_{\varepsilon=0} = -M^{-1}(0)\partial_\varepsilon M(\varepsilon)|_{\varepsilon=0} M^{-1}(0). \quad (3.208)$$

We compute

$$\partial_B \partial_C \ln \Gamma_{A,A+B,A+C} \quad (3.209)$$
$$\overset{(3.145)}{=} \partial_B \partial_C \ln \mathrm{ag}(\det_{\mathcal{H}^-}(P^- S_{A,A+B} P^- S_{A+B,A+C} P^- S_{A+C,A} P^-)) \quad (3.210)$$
$$= \partial_B \frac{\partial_C\, \mathrm{ag}(\det_{\mathcal{H}^-}(P^- S_{A,A+B} P^- S_{A+B,A+C} P^- S_{A+C,A} P^-))}{\mathrm{ag}(\det_{\mathcal{H}^-}(P^- S_{A,A+B} P^- S_{A+B,A} P^-))} \quad (3.211)$$
$$= \partial_B \partial_C\, \mathrm{ag}(\det_{\mathcal{H}^-}(P^- S_{A,A+B} P^- S_{A+B,A+C} P^- S_{A+C,A} P^-)) \quad (3.212)$$
$$\overset{*}{=} i\partial_B \frac{\Im \partial_C \det_{\mathcal{H}^-}(P^- S_{A,A+B} P^- S_{A+B,A+C} P^- S_{A+C,A} P^-)}{\det_{\mathcal{H}^-}(P^- S_{A,A+B} P^- S_{A+B,A} P^-)} \quad (3.213)$$

$$
\begin{aligned}
&= i\partial_B\Big[\frac{\det_{\mathcal{H}^-}(P^-S_{A,A+B}P^-S_{A+B,A}P^-)}{\det_{\mathcal{H}^-}(P^-S_{A,A+B}P^-S_{A+B,A}P^-)} \\
&\qquad\times \Im\,\mathrm{tr}((P^-S_{A,A+B}P^-S_{A+B,A}P^-)^{-1} \\
&\qquad\times \partial_C P^-S_{A,A+B}P^-S_{A+B,A+C}P^-S_{A+C,A}P^-)\Big] \qquad (3.214)
\end{aligned}
$$

The fraction in front of the trace equals 1. As a next step we replace the second but last projector $P^- = 1-P^+$, the resulting first summand vanishes, because the dependence on C cancels. This results in

$$
\begin{aligned}
(3.214) = &-i\partial_B\Im\,\mathrm{tr}((P^-S_{A,A+B}P^-S_{A+B,A}P^-)^{-1} \\
&\times \partial_C P^-S_{A,A+B}P^-S_{A+B,A+C}P^+S_{A+C,A}P^-). \qquad (3.215)
\end{aligned}
$$

Now, because $P^+P^- = 0$ only one summand of the product rule survives:

$$
\begin{aligned}
(3.215) = &-i\partial_B\Im\,\mathrm{tr}((P^-S_{A,A+B}P^-S_{A+B,A}P^-)^{-1} \\
&\times \partial_C P^-S_{A,A+B}P^-S_{A+B,A}P^+S_{A+C,A}P^-). \qquad (3.216)
\end{aligned}
$$

Next we use $(MN)^{-1} = N^{-1}M^{-1}$ for invertible operators M and N for the first factor in the trace and cancel as much as possible of the second factor:

$$
\begin{aligned}
(3.216) &= -i\partial_B\Im\,\mathrm{tr}((P^-S_{A+B,A}P^-)^{-1}P^-S_{A+B,A} \\
&\qquad \times P^+\partial_C S_{A+C,A}P^-) && (3.217)\\
&= -i\Im\,\mathrm{tr}(\partial_B[(P^-S_{A+B,A}P^-)^{-1}P^-S_{A+B,A} \\
&\qquad \times P^+\partial_C S_{A+C,A}P^-]) && (3.218)\\
&= -i\Im\,\mathrm{tr}(\partial_B P^-S_{A+B,A}P^+\partial_C S_{A+C,A}P^-) && (3.219)\\
&= -i\Im\,\mathrm{tr}(\partial_B P^-S_{A,A+B}P^+\partial_C S_{A,A+C}P^-) && (3.220)\\
&= -i\partial_B\partial_C\Im\,\mathrm{tr}(P^-S_{A,A+B}P^+S_{A,A+C}P^-) && (3.221)
\end{aligned}
$$

which proves the claim.

□

In order to construct the lift announced in theorem 34, we first construct a reference lift $\hat{S}$, that is well defined on all of $\mathcal{V}$. Afterwards we will study the dependence of the relative phase between this global lift $\hat{S}_{0,A}$ and a local lift given by $\hat{S}_{0,B}\overline{S}_{B,A}$ for $B-A$ small. By exploiting properties of this phase and the causal splitting c^+ we will construct a global lift that has the desired properties.
Since $\mathcal{V}$ is star shaped, we may reach any four-potential A from 0 through the straight line $\{tA \mid t \in [0,1]\}$.

Definition 36 (ratio of lifts)**.** *For any $A, B \in \mathcal{V}$ and any two lifts $S'_{A,B}, S''_{A,B}$ of the one particle scattering operator $S_{A,B}$ we define the ratio*

$$\frac{S'_{A,B}}{S''_{A,B}} \in S^1 \tag{3.222}$$

to be the unique complex number $z \in S^1$ such that

$$z\ S''_{A,B} = S'_{A,B} \tag{3.223}$$

holds.

Theorem 37 (existence of global lift)**.** *There is a unique map $\hat{S}_{0,\cdot} : \mathcal{V} \to U(\mathcal{F})$ which maps $A \in \mathcal{V}$ to a lift of $S_{0,A}$ and solves the parallel transport differential equation*

$$A, B \in \mathcal{V} \textit{ linearly dependent} \Rightarrow \partial_B \frac{\hat{S}_{0,A+B}}{\hat{S}_{0,A}\overline{S}_{A,A+B}} = 0, \tag{3.224}$$

subject to the initial condition $\hat{S}_{0,0} = 1$.

The proof of theorem 37 is divided into two lemmas due to its length. We will introduce the integral flow ϕ_A associated with the differential equation (3.224) for some $A \in \mathcal{V}$. We will then study the properties of ϕ_A in the two lemmas and finally construct $\hat{S}_{0,A} = 1\phi_A(0,1)$. In the first lemma we will establish the existence of a local solution. The solution will be constructed along the line $\overline{0\ \ A}$. In the second lemma we patch local solutions together to a global one.

Lemma 38 (ϕ local existence and uniqueness)**.** *There is a unique* $\phi_A : \{(t,s) \in \mathbb{R}^2 \mid (tA, sA) \in \operatorname{dom} \overline{S}\} \to U(\mathcal{F})$ *for every* $A \in \mathcal{V}$ *satisfying*

$$\forall (t,s) \in \operatorname{dom} \phi_A : \phi_A(t,s) \textit{ is a lift of } S_{tA,sA} \tag{3.225}$$

$$\forall (t,s),(s,l),(l,t) \in \operatorname{dom} \phi_A : \phi_A(t,s)\phi_A(s,l) = \phi_A(t,l) \tag{3.226}$$

$$\forall t \in \mathbb{R} : \phi_A(t,t) = 1 \tag{3.227}$$

$$\forall s \in \mathbb{R} : \partial_t \left. \frac{\phi_A(s,t)}{\overline{S}_{sA,tA}} \right|_{t=s} = 0. \tag{3.228}$$

Proof. We first define the phase

$$z : \{(A,B) \in \operatorname{dom} \overline{S} \mid A, B \text{ linearly dependent}\} \to S^1 \tag{3.229}$$

by the differential equation

$$\frac{d}{dx} \ln z(tA, xA) = - \left. \left(\frac{d}{dy} \ln \Gamma_{tA,xA,yA} \right) \right|_{y=x} \tag{3.230}$$

and the initial condition

$$z(A,A) = 1 \tag{3.231}$$

for any $A \in \mathcal{V}$. The phase z takes the form

$$z(tA, xA) = \exp \left(- \int_t^x dx' \left. \left(\frac{d}{dx'} \ln \Gamma_{tA,yA,x'A} \right) \right|_{y=x'} \right). \tag{3.232}$$

Please note that both differential equation and initial condition are invariant under rescaling of the potential A, so z is well defined. We will now construct a local solution to (3.224) and define ϕ_A using this solution. Pick $A \in \mathcal{V}$ the expression

$$\hat{S}_{0,sA} = \hat{S}_{0,A}\overline{S}_{A,sA}z(A, sA) \tag{3.233}$$

solves (3.224) locally. Local here means that s is close enough to 1 such that $(A, sA) \in \operatorname{dom}\overline{S}$. Calculating the argument of the derivative of (3.224) we find:

$$0 = \frac{\hat{S}_{0,(s+\varepsilon)A}}{\hat{S}_{0,sA}\overline{S}_{sA,(s+\varepsilon)A}} = \frac{\hat{S}_{0,A}\overline{S}_{A,(s+\varepsilon)A}z(A,(s+\varepsilon)A)}{\hat{S}_{0,A}\overline{S}_{A,sA}\overline{S}_{sA,(s+\varepsilon)}z(A,sA)} \tag{3.234}$$

$$\overset{(3.150)}{=} \frac{\hat{S}_{0,A}\overline{S}_{A,sA}\overline{S}_{sA,(s+\varepsilon)}\Gamma_{A,sA,(s+\varepsilon)A}z(A,(s+\varepsilon)A)}{\hat{S}_{0,A}\overline{S}_{tA,sA}\overline{S}_{sA,(s+\varepsilon)}z(A,sA)} \tag{3.235}$$

$$= \frac{\Gamma_{tA,sA,(s+\varepsilon)A}z(tA,(s+\varepsilon)A)}{z(A,sA)} \tag{3.236}$$

Now we take the derivative with respect to ε at $\varepsilon = 0$, cancel the factor that does not depend on ε and relabel $s = x$ to obtain

$$0 = \left(\frac{d}{dy}(\Gamma_{A,xA,yA}\ z(A, yA))\right)\bigg|_{y=x} \tag{3.237}$$

$$\iff \frac{d}{dx}\ln z(tA, xA) = \left(-\frac{d}{dy}\ln\Gamma_{tA,xA,yA}\right)\bigg|_{y=x}, \tag{3.238}$$

which is exactly the defining differential equation of z. The initial condition of z equation (3.231) is necessary to match the initial condition in (3.233) for $s = 1$. The connection to ϕ from the statement of the lemma can now be made. We define

$$\phi_A(t, s) := z(tA, sA)\overline{S}_{tA,sA}, \tag{3.239}$$

for $(tA, sA) \in \operatorname{dom}\overline{S}$. Since $\overline{S}$ is a lift of S, we see that (3.225) holds. Equation (3.227) follows from (3.231) and $\overline{S}_{tA,tA} = 1$ for general

$t \in \mathbb{R}$. Equation (3.228) follows by plugging in (3.239) and using the differential equation for z (3.231):

$$\partial_s \left.\frac{\phi_A(t,s)}{\overline{S}_{tA,sA}}\right|_{s=t} = \partial_s \left.\frac{z(tA,sA)\overline{S}_{tA,sA}}{\overline{S}_{tA,sA}}\right|_{s=t} \tag{3.240}$$

$$= \partial_t \left. z(tA,sA)\right|_{t=s} = 0. \tag{3.241}$$

It remains to see that (3.226), i.e. that

$$\phi_A(t,s)\phi_A(s,l) = \phi_A(t,l) \tag{3.242}$$

holds for $(tA,sA),(sA,lA),(tA,lA) \in \operatorname{dom}\overline{S}$. In order to do so we plug in the definition (3.239) of ϕ_A and obtain

$$\phi_A(t,s)\phi_A(s,l) = \phi_A(t,l) \tag{3.243}$$

$$\Longleftrightarrow\; z(tA,sA)z(sA,lA)\overline{S}_{tA,sA}\overline{S}_{sA,lA} = z(tA,lA)\overline{S}_{tA,lA} \tag{3.244}$$

$$\Longleftrightarrow\; z(tA,sA)z(sA,lA)\overline{S}_{tA,sA}\overline{S}_{sA,lA} \tag{3.245}$$

$$= z(tA,lA)\overline{S}_{tA,sA}\overline{S}_{sA,lA}\Gamma_{tA,sA,lA} \tag{3.246}$$

$$\Longleftrightarrow\; z(tA,sA)z(sA,lA)z(tA,lA)^{-1} = \Gamma_{tA,sA,lA}. \tag{3.247}$$

In order to check the validity of the last equality we plug in the ntegral formula (3.232) for z, we also abbreviate $\frac{d}{dx} = \partial_x$

$$z(tA,sA)z(sA,lA)z(tA,lA)^{-1} \tag{3.248}$$

$$= e^{-\int_t^s dx'\,(\partial_{x'}\ln\Gamma_{tA,yA,x'A})\big|_{y=x'} - \int_s^l dx'\,(\partial_{x'}\ln\Gamma_{sA,yA,x'A})\big|_{y=x'}} \tag{3.249}$$

$$\times e^{+\int_t^l dx'\,(\partial_{x'}\ln\Gamma_{tA,yA,x'A})\big|_{y=x'}} \tag{3.250}$$

$$= e^{-\int_l^s dx'\,(\partial_{x'}\ln\Gamma_{tA,yA,x'A})\big|_{y=x'} - \int_s^l dx'\,(\partial_{x'}\ln\Gamma_{sA,yA,x'A})\big|_{y=x'}} \tag{3.251}$$

$$\overset{(3.150)}{=} e^{-\int_l^s dx'\,(\partial_{x'}\ln\Gamma_{sA,yA,x'A})\big|_{y=x'} - \int_l^s dx'\,(\partial_{x'}\ln\Gamma_{tA,sA,x'A})\big|_{y=x'}} \tag{3.252}$$

$$\times e^{-\int_l^s dx' \, (\partial_{x'} \ln \Gamma_{tA,yA,sA})\big|_{y=x'} - \int_s^l dx' \, (\partial_{x'} \ln \Gamma_{sA,yA,x'A})\big|_{y=x'}} \tag{3.253}$$

$$= e^{-\int_l^s dx' \, (\partial_{x'} \ln \Gamma_{tA,sA,x'A})\big|_{y=x'}} \tag{3.254}$$

$$= e^{-\int_l^s dx' \partial_{x'} \ln \Gamma_{tA,sA,x'A}} \tag{3.255}$$

$$\overset{(3.148)}{=} \Gamma_{tA,sA,lA}, \tag{3.256}$$

which proves the validity of the consistency relation (3.242).
In order to prove uniqueness we pick $A \in \mathcal{V}$ and assume there is ϕ' also defined on $\operatorname{dom} \phi_A$ and satisfies (3.225)-(3.228). Then we may use (3.225) to conclude that for any $(t, s) \in \operatorname{dom} \phi_A$ there is $\gamma(t, s) \in S^1$ such that

$$\phi_A(t, s) = \phi'(t, s)\gamma(t, s) \tag{3.257}$$

holds true. Picking l such that $(t, s), (s, l), (t, l) \in \operatorname{dom} \phi_A$ and using (3.226) we find

$$\phi'(t, s)\gamma(t, s) = \phi_A(t, s) = \phi_A(t, l)\phi_A(l, s) \tag{3.258}$$

$$= \gamma(t, l)\phi'(t, l)\gamma(l, s)\phi'(l, s) = \gamma(t, l)\gamma(l, s)\phi'(t, s), \tag{3.259}$$

hence we have

$$\gamma(t, s) = \gamma(t, l)\gamma(l, s). \tag{3.260}$$

From property (3.227) we find

$$\gamma(t, t) = 1, \tag{3.261}$$

for any t. Using equation (3.228) we conclude that

$$0 = \partial_t \left. \frac{\phi'(s, t)}{\overline{S}_{sA,tA}} \right|_{t=s} = \partial_t \left. \frac{\phi_A(s, t)\gamma(s, t)}{\overline{S}_{sA,tA}} \right|_{t=s} \tag{3.262}$$

$$= \partial_t \, \gamma(s, t) \left. \frac{\phi_A(s, t)}{\overline{S}_{sA,tA}} \right|_{t=s} = \partial_t \, \gamma(s, t)|_{t=s} + \partial_t \left. \frac{\phi_A(s, t)}{\overline{S}_{sA,tA}} \right|_{t=s} \tag{3.263}$$

$$= \partial_t \, \gamma(s, t)|_{t=s}. \tag{3.264}$$

Finally we find for general $(s,t) \in \operatorname{dom} \phi_A$:

$$\partial_x \gamma(s,x)|_{x=t} = \partial_x(\gamma(s,t)\gamma(t,x))|_{x=t} = \gamma(s,t)\partial_x\gamma(t,x)|_{x=t} = 0. \tag{3.265}$$

So $\gamma(t,s) = 1$ everywhere. We conclude $\phi_A = \phi'$.

□

Lemma 39 (ϕ global existence and uniqueness)**.** *For any $A \in \mathcal{V}$ the map ϕ_A constructed in lemma 38 can be uniquely extended to all of $\mathbb{R}^2$ keeping its defining properties*

$$\forall (t,s) \in \mathbb{R}^2 : \phi_A(t,s) \text{ is a lift of } S_{tA,sA} \tag{3.266}$$

$$\forall (t,s),(s,l),(l,t) \in \mathbb{R}^2 : \phi_A(t,s)\phi_A(s,l) = \phi_A(t,l) \tag{3.267}$$

$$\forall t \in \mathbb{R} : \phi_A(t,t) = 1 \tag{3.268}$$

$$\forall s \in \mathbb{R} : \partial_t \left. \frac{\phi_A(s,t)}{\overline{S}_{sA,tA}} \right|_{t=s} = 0. \tag{3.269}$$

Proof. Pick $A \in \mathcal{V}$. For $x \in \mathbb{R}$ we define the set

$$U_x := \{y \in \mathbb{R} \mid (xA, yA) \in \operatorname{dom} \overline{S}\}, \tag{3.270}$$

which according to properties 2 and 4 of lemma 31 is an open interval and fulfills that $\bigcup_{x \in \mathbb{R}} U_x \times U_x$ is an open neighbourhood of the diagonal $\{(x,x) \mid x \in \mathbb{R}\}$. Therefore ϕ_A is defined for arguments that are close enough to each other. Since properties (3.269) and (3.268) only concern the behavior of ϕ_A at the diagonal any extension fulfils them. We pick a sequence $(x_k)_{k\in\mathbb{N}} \subset \mathbb{R}$ such that

$$\bigcup_{k \in \mathbb{N}_0} U_{x_k} = \mathbb{R} \tag{3.271}$$

holds and

$$\forall n \in \mathbb{N}_0 : \bigcup_{k=0}^{n} U_{x_k} =: \operatorname{dom}_n \tag{3.272}$$

is an open interval. Please note that such a sequence always exists. We are going to prove that for any $n \in \mathbb{R}_0$ There is a function $\psi_n : \mathrm{dom}_n \times \mathrm{dom}_n \to U(\mathcal{F})$, which satisfies the conditions

$$\forall (t,s) \in \mathrm{dom}_n \times \mathrm{dom}_n : \psi_n(t,s) \text{ is a lift of } S_{tA,sA} \tag{3.273}$$

$$\forall s,k,l \in \mathrm{dom}_n : \psi_n(k,s)\psi_n(s,l) = \psi_n(k,l) \tag{3.274}$$

$$\forall x,y \in \mathrm{dom}_n : (xA, yA) \in \mathrm{dom}\,\overline{S} \Rightarrow \psi_n(x,y) = \phi_A(x,y) \tag{3.275}$$

and is the unique function to do so, i.e. any other function $\tilde{\psi}_n$ fulfilling properties (3.273)-(3.275) possibly being defined on a larger domain coincides with ψ_n on $\mathrm{dom}_n \times \mathrm{dom}_n$.
We start with $\psi_0 = \phi_A$ restricted to $U_{x_0} \times U_{x_0}$. This function is a restriction of ϕ_A and because of lemma 38 it fulfils all of the required properties directly.
For the induction step we define ψ_{n+1} on the domain $\mathrm{dom}_{n+1} \times \mathrm{dom}_{n+1}$ by

$$\psi_{n+1}(x,y) := \begin{cases} \psi_n(x,y) & \text{for } x,y \in \mathrm{dom}_n \\ \phi_A(x,y) & \text{for } x,y \in U_{x_{n+1}} \\ \psi_n(x,t)\phi_A(t,y) & \text{for } x \in \mathrm{dom}_n, y \in U_{x_{n+1}} \\ \phi_A(x,t)\psi_n(t,y) & \text{for } y \in \mathrm{dom}_n, x \in U_{x_{n+1}}. \end{cases} \tag{3.276}$$

In order to complete the induction step we have to show that ψ_{n+1} is well defined and fulfils properties (3.273)-(3.275) with n replaced by $n+1$ and is the unique function to do so.
To see that ψ_{n+1} is well defined we have to check that the cases in the definition agree when they overlap.

1. If we have $x,y \in \mathrm{dom}_n \cap U_{x_{n+1}}$ all four cases overlap; however, the alternative definitions all equal $\phi_A(x,y)$:

$$\begin{aligned}\psi_n(x,y) &\overset{(3.275)}{=} \phi_A(x,y) \overset{(3.242)}{=} \phi_A(x,t)\phi_A(t,y) \\ &\overset{(3.275)}{=} \begin{cases} \psi_A(x,t)\phi_n(t,y) \\ \phi_A(x,t)\psi_n(t,y). \end{cases}\end{aligned} \tag{3.277}$$

2. Furthermore, if we have $x \in \mathrm{dom}_n$, $y \in \mathrm{dom}_n \cap U_{x_{n+1}}$ case one and three overlap. Here both calternatives are equal to $\psi_n(x,y)$, since $x, y \in \mathrm{dom}_n$ and we obtain:

$$\psi_n(x,y) \overset{(3.274)}{=} \psi_n(x,t)\psi_n(t,y) \overset{(3.275)}{=} \psi_n(x,t)\phi_A(t,y). \quad (3.278)$$

3. Additionally, if $y \in \mathrm{dom}_n$, $x \in \mathrm{dom}_n \cap U_{x_{n+1}}$ case one and four overlap. Here they are equal to $\psi_n(x,y)$, since $x, y \in \mathrm{dom}_n$ a quick calculation yields:

$$\psi_n(x,y) \overset{(3.274)}{=} \psi_n(x,t)\psi_n(t,y) \overset{(3.275)}{=} \psi_A(x,t)\psi_n(t,y). \quad (3.279)$$

4. Moreover, if we have $y \in U_{x_{n+1}}$, $x \in \mathrm{dom}_n \cap U_z$ case two and three overlap. Here both candidate definitions are equal to $\phi_A(x,y)$, since $x, t \in U_z$ we arrive at:

$$\phi_A(x,y) \overset{(3.242)}{=} \phi_A(x,t)\phi_A(t,y) \overset{(3.275)}{=} \psi_n(x,t)\phi_A(t,y). \quad (3.280)$$

5. Also, if we have $x \in U_{x_{n+1}}$, $y \in \mathrm{dom}_n \cap U_{x_{n+1}}$ case two and four overlap. In this case both alternatives are equal to $\phi_A(x,y)$, since $y, t \in U_{x_{n+1}}$ we get:

$$\phi_A(x,y) \overset{(3.242)}{=} \phi_A(x,t)\phi_A(t,y) \overset{(3.275)}{=} \phi_A(x,t)\psi_n(t,y). \quad (3.281)$$

We proceed to show the induction claim, starting with $(3.273)_{n+1}$. By the induction hypothesis we know that $\psi_n(x,y)$ as well as $\phi_A(x,y)$ are lifts of $S_{xA,yA}$ for any (x,y) in their domain of definition. Therefore we have for $x, y \in \mathrm{dom}_n \cup U_{x_{n+1}}$

$$\psi_{n+1}(x,y) = \begin{cases} \psi_n(x,y) & \text{for } x, y \in \mathrm{dom}_n, \\ \phi_A(x,y) & \text{for } x, y \in U_{x_{n+1}}, \\ \psi_n(x,t)\phi_A(t,y) & \text{for } x \in \mathrm{dom}_n, y \in U_{x_{n+1}}, \\ \phi_A(x,t)\psi_n(t,y) & \text{for } y \in \mathrm{dom}_n, x \in U_{x_{n+1}}, \end{cases} \quad (3.276)$$

where each of the lines is a lift of $S_{xA,yA}$ whenever the expression is defined.
Equation $(3.274)_{n+1}$ we will again show in a case by case manner depending on the s, k and l:

1. $s, k, l \in \mathrm{dom}_n$: $(3.274)_{n+1}$ follows directly from the induction hypothesis;

2. $s, k \in \mathrm{dom}_n$ and $l \in U_{x_{n+1}}$:
$$\begin{aligned}\psi_{n+1}(s,k)\psi_{n+1}(k,l) &= \psi_n(s,k)\psi_n(k,t)\phi_A(t,l)\\ &\overset{(3.274)}{=} \psi_n(s,t)\phi_A(t,l) = \psi_{n+1}(s,l),\end{aligned} \tag{3.282}$$

3. $s, l \in \mathrm{dom}_n$ and $k \in U_{x_{n+1}}$:
$$\begin{aligned}\psi_{n+1}(s,k)\psi_{n+1}(k,l) &= \psi_n(s,t)\phi_A(t,k)\phi_A(t,k)\psi_n(t,l)\\ &\overset{(3.227),(3.226)}{=} \psi_n(s,t)\psi_n(t,l) \overset{(3.274)}{=} \psi_n(s,l) = \psi_{n+1}(s,l),\end{aligned}$$

4. $s \in \mathrm{dom}_n$ and $k, l \in U_{x_{n+1}}$:
$$\begin{aligned}\psi_{n+1}(s,k)\psi_{n+1}(k,l) &= \psi_n(s,t)\phi_A(t,k)\phi_A(k,l)\\ &\overset{(3.226)}{=} \psi_n(s,t)\phi_A(t,l) = \psi_{n+1}(s,l),\end{aligned}$$

5. $k, l \in \mathrm{dom}_n$ and $s \in U_{x_{n+1}}$:
$$\begin{aligned}\psi_{n+1}(s,k)\psi_{n+1}(k,l) &= \phi_A(s,t)\psi_n(t,k)\psi_n(k,l)\\ &\overset{(3.274)}{=} \phi_A(s,t)\psi_n(t,l) = \psi_{n+1}(s,l),\end{aligned}$$

6. $k \in \mathrm{dom}_n$ and $s, l \in U_{x_{n+1}}$:
$$\begin{aligned}\psi_{n+1}(s,k)\psi_{n+1}(k,l) &= \phi_A(s,t)\psi_n(t,k)\psi_n(k,t)\phi_A(t,l)\\ &\overset{(3.274)}{=} \phi_A(s,t)\psi(t,t)\phi_A(t,l) \overset{(3.275),(3.227)}{=} \phi_A(s,t)\phi_A(t,l)\\ &\overset{(3.226)}{=} \phi_A(s,l) = \psi_{n+1}(s,l),\end{aligned}$$

7. $l \in \mathrm{dom}_n$ and $s, k \in U_{x_{n+1}}$:

$$\begin{aligned}\psi_{n+1}(s,k)\psi_{n+1}(k,l) &= \phi_A(s,k)\phi_A(k,t)\psi_n(t,l)\\ &\overset{(3.226)}{=} \phi_A(s,t)\psi_n(t,l) = \psi_{n+1}(s,l),\end{aligned}$$

8. and if $s, k, l \in U_z$:

$$\begin{aligned}\psi_{n+1}(s,k)\psi_{n+1}(k,l) &= \phi_A(s,k)\phi_A(k,l)\\ &\overset{(3.226)}{=} \phi_A(s,l) = \psi_{n+1}(s,l).\end{aligned}$$

To see $(3.275)_{n+1}$, i.e. that ψ_{n+1} is coincides with ϕ_A where both functions are defined pick $x, y \in \mathrm{dom}_{n+1}$ such that $(xA, yA) \in \mathrm{dom}\,\overline{S})$. Recall the definition of ψ_{n+1}

$$\psi_{n+1}(x,y) = \begin{cases} \psi_n(x,y) & \text{for } x, y \in \mathrm{dom}_n, \\ \phi_A(x,y) & \text{for } x, y \in U_{x_{n+1}}, \\ \psi_n(x,t)\phi_A(t,y) & \text{for } x \in \mathrm{dom}_n, y \in U_{x_{n+1}}, \\ \phi_A(x,t)\psi_n(t,y) & \text{for } y \in \mathrm{dom}_n, x \in U_{x_{n+1}}. \end{cases} \tag{3.276}$$

Therefore if $x, y \in \mathrm{dom}_{n+1}$ we may use the induction hypothesis directly and if $x, y \in U_{x_{n+1}}$ we also arrived a the claim we want to prove. Excluding these cases, we are left with rows number three and four of this definition with the restriction

3. $x \in \mathrm{dom}_n \setminus U_{x_{n+1}}, y \in U_{x_{n+1}} \setminus \mathrm{dom}_n$ or

4. $y \in \mathrm{dom}_n \setminus U_{x_{n+1}}, x \in U_{x_{n+1}} \setminus \mathrm{dom}_n$,

respectively. Because t satisfies $t \in \mathrm{dom}_n \cap U_{x_{n+1}}$, we have in both cases $t \in \overline{x\,y}$. By using property 4 of lemma 31 we infer from $(xA, yA) \in \mathrm{dom}\,\overline{S}$ that in both cases $(xA, tA), (tA, yA) \in \mathrm{dom}\,\overline{S}$ also holds. Hence we may apply the induction hypothesis $(3.275)_n$.

It remains to show uniqueness. So let $\tilde{\psi}_{n+1}$ be defined on $\mathrm{dom}_{n+1} \times \mathrm{dom}_{n+1}$ fulfil

$$\forall (t,s) \in \mathrm{dom}_{n+1} \times \mathrm{dom}_{n+1} : \tilde{\psi}(t,s) \text{ is a lift of } S_{tA,sA}, \quad (3.273_{\tilde{\psi}})$$

$$\forall s,k,l \in \mathbb{R} : \tilde{\psi}(k,s)\tilde{\psi}(s,l) = \tilde{\psi}(k,l), \quad (3.274_{\tilde{\psi}})$$

$$\forall (x,y) \in \mathrm{dom}_{n+1} : (xA, yA) \in \mathrm{dom}\,\overline{S} \Rightarrow \tilde{\psi}(x,y) = \phi_A(x,y). \quad (3.275_{\tilde{\psi}})$$

Now pick $x, y \in (\mathrm{dom}_n \cup U_{x_{n+1}})$. We proceed in a case by case manner

1. If $x, y \in \mathrm{dom}_n$ holds, then $\psi_{n+1}(x,y) = \tilde{\psi}_{n+1}(x,y)$ follows directly from the induction hypothesis.

2. Similarly if $x, y \in U_{x_{n+1}}$ holds, we have

$$\psi_{n+1}(x,y) = \phi_A(x,y) = \tilde{\psi}_{n+1}(x,y). \quad (3.283)$$

3. Additionally, if $x \in \mathrm{dom}_n, y \in U_{x_{n+1}}$ holds, then

$$\psi_{n+1}(x,y) \overset{(3.274)}{=} \psi_{n+1}(x,t)\psi_{n+1}(t,y) \quad (3.284)$$

$$\overset{t \in \mathrm{dom}_n \cap U_{x_{n+1}}}{=} \tilde{\psi}_{n+1}(x,t)\tilde{\psi}_{n+1}(t,y) \overset{(3.274_{\tilde{\psi}})}{=} \tilde{\psi}_{n+1}(x,y) \quad (3.285)$$

   is satisfied.

4. Conversely, if $y \in \mathrm{dom}_n, x \in U_{x_{n+1}}$ holds, we may use the same calculation to obtain

$$\psi_{n+1}(x,y) \overset{(3.274)}{=} \psi_{n+1}(x,t)\psi_{n+1}(t,y) \quad (3.286)$$

$$\overset{t \in \mathrm{dom}_n \cap U_{x_{n+1}}}{=} \tilde{\psi}_{n+1}(x,t)\tilde{\psi}_{n+1}(t,y) \overset{(3.274_{\tilde{\psi}})}{=} \tilde{\psi}_{n+1}(x,y). \quad (3.287)$$

Now we have established an extension ψ_n of ϕ_A fulfilling properties (3.273)-(3.275).

We know that for each $n \in \mathbb{N}$ the function $\psi_{n+1} : \mathrm{dom}^2_{n+1} \to U(\mathcal{F})$ is an extention of $\psi_{n+1} : \mathrm{dom}^2_n \to U(\mathcal{F})$. Furhtermore, the sets dom_n cover $\mathbb{R}$ according to equation (3.271). Consequently there is a unique common extension, by small abuse of notation again called $\phi_A : \mathbb{R}^2 \to U(\mathcal{F})$, of all ψ_n. This function fulfills the claim (3.266)-(3.269), because any $t, l, s \in \mathbb{R}$ are contained in some dom_n.

□

Lemma 39 enables us to define a global lift.

Definition 40 (global lift)**.** *For any $A \in \mathcal{V}$ we define*

$$\hat{S}_{0,A} := \phi_A(1, 0). \tag{3.288}$$

Using lemma 39 we are now in a position to prove theorem 37.

proof of theorem 37. The operator $\hat{S}$ fulfils the claimed differential equation (3.224) due to the global multiplication property (3.267) and the differential equation (3.269). Its uniqueness is inherited from the uniqueness of ϕ_A for $A \in \mathcal{V}$ from lemma 39. □

Definition 41 (relative phase)**.** *Let $(A, B) \in \mathrm{dom}\,\overline{S}$, we define $z(A, B) \in S^1$ by*

$$z(A, B) := \frac{\hat{S}_{0,B}}{\hat{S}_{0,A}\overline{S}_{A,B}}. \tag{3.289}$$

Please note that for such A, B the lift $\bar{S}_{A,B}$ is well defined. This means that the product in the denominator is a lift of $S_{0,B}$ and according to definition 40 the ratio is well defined.

Remark 42. *The global function z defined here is an extension of the funtion z appearing locally in the the proof of lemma 38, cf. formula (3.229).*
Please note that z is smooth when restricted to $\mathcal{W}^2 \cap \mathrm{dom}\,\overline{S}$ for any finite dimensional subspace $\mathcal{W} \subseteq \mathcal{V}$.

Lemma 43 (properties of the relative phase). *For all* $(A, F), (F, G), (G, A) \in \operatorname{dom} \overline{S}$, *as well as or all* $H, K \in \mathcal{V}$, *we have*

$$z(A, F) = z(F, A)^{-1} \tag{3.290}$$

$$z(F, A)z(A, G)z(G, F) = \Gamma_{F,A,G} \tag{3.291}$$

$$\partial_H \partial_K \ln z(A + H, A + K) = c_A(H, K). \tag{3.292}$$

Proof. Pick $A, F, G \in \mathcal{V}$ as in the lemma. We start off by analysing

$$\hat{S}_{0,F}\overline{S}_{F,G} \overset{(3.289)}{=} z(A, F)\hat{S}_{0,A}\overline{S}_{A,F}\overline{S}_{F,G} \tag{3.293}$$

$$\overset{(3.150)}{=} z(A, F)\Gamma^{-1}_{A,F,G}\hat{S}_{0,A}\overline{S}_{A,G}. \tag{3.294}$$

Exchanging A and F in this equation yields

$$\hat{S}_{0,A}\overline{S}_{A,G} = z(F, A)\Gamma^{-1}_{F,A,G}\hat{S}_{0,F}\overline{S}_{F,G}. \tag{3.295}$$

This is equivalent to

$$\hat{S}_{0,F}\overline{S}_{F,G} = z(F, A)^{-1}\Gamma_{F,A,G}\hat{S}_{0,A}\overline{S}_{A,G} \ . \tag{3.296}$$

Comparing the last equation with formula (3.294) and taking the permutation properties (3.147) of Γ into account this implies that

$$z(A, F) = z(F, A)^{-1} \tag{3.297}$$

holds true. Equation (3.294) solved for $\hat{S}_{0,A}\overline{S}_{A,G}$ also gives us

$$\hat{S}_{0,G} \overset{(3.289)}{=} z(A, G)\hat{S}_{0,A}\overline{S}_{A,G} \tag{3.298}$$

$$\overset{(3.294)}{=} z(A, G)z(A, F)^{-1}\Gamma_{A,F,G}\hat{S}_{0,F}\overline{S}_{F,G}. \tag{3.299}$$

The latter equation compared with

$$\hat{S}_{0,G} \overset{(3.289)}{=} z(F, G)\hat{S}_{0,F}\overline{S}_{F,G}, \tag{3.300}$$

yields a direct connection between Γ and z:

$$\frac{z(A,G)}{z(A,F)}\Gamma_{A,F,G} = z(F,G), \tag{3.301}$$

which we rewrite using the antisymmetry (3.290) of z as

$$\Gamma_{A,F,G} = z(F,G)z(A,F)z(G,A). \tag{3.302}$$

Finally, in this equation, we substitute $F = A + \varepsilon_1 H$ as well as $G = A + \varepsilon_2 K$, where $\varepsilon_1, \varepsilon_2$ is small enough so that z and Γ are still well defined. Then we take the second logarithmic derivative to find

$$\begin{aligned}\partial_{\varepsilon_1}\partial_{\varepsilon_2} \ln z(A+\varepsilon_1 H, A+\varepsilon_2 K) &= \partial_{\varepsilon_1}\partial_{\varepsilon_2} \ln \Gamma_{A,A+\varepsilon_1 H, A+\varepsilon_2 K} \\ &\overset{(3.152)}{=} c_A(H,K). \end{aligned}\tag{3.303}$$

□

So we find that c_A is the second mixed logarithmic derivative of z. In the following we will characterise z more thoroughly by c and c^+.

Definition 44 (p-forms of four potentials, phase integral)**.** *For $p \in \mathbb{N}$, we introduce the set Ω^p of p-forms to consist of all maps $\omega : \mathcal{V}\times\mathcal{V}^p \to \mathbb{C}$ such that ω is linear and antisymmetric in its p last arguments and smooth in its first argument when restricted to any finite dimensional subspace of $\mathcal{V}$.*
Additionally, we define the 1-form $\chi \in \Omega^1(\mathcal{V})$ by

$$\chi_A(B) := \partial_B \ln z(A, A+B) \tag{3.304}$$

for all $A, B \in \mathcal{V}$. Furthermore, for $p \in \mathbb{N}$ and any differential form $\omega \in \Omega^p(\mathcal{V})$, we define its exterior derivative, $d\omega \in \Omega^{p+1}(\mathcal{V})$ by

$$(d\omega)_A(B_1,\ldots,B_{p+1}) := \sum_{k=1}^{p+1}(-1)^{k+1}\partial_{B_k}\omega_{A+B_k}(B_1,\ldots,\widehat{B_k},\ldots,B_{p+1}), \tag{3.305}$$

for $A, B_1, \ldots, B_{p+1} \in \mathcal{V}$*, where the notation* $\widehat{B_k}$ *denotes that* B_k *is dropped as an argument.*

Lemma 45 (connection between c and the relative phase)**.** *The differential form* χ *fulfils*

$$(d\chi)_A(F,G) = 2c_A(F,G) \tag{3.306}$$

for all $A, F, G \in \mathcal{V}$*.*

Proof. Pick $A, F, G \in \mathcal{V}$, we calculate

$$\begin{aligned}(d\chi)_A(F,G) &= \partial_F \partial_G \ln z(A+F, A+F+G) \\ &\quad - \partial_F \partial_G \ln z(A+G, A+F+G) &(3.307)\\ &= \partial_F \partial_G (\ln z(A, A+F+G) + \ln z(A+F, A+G)) &(3.308)\\ &\quad - \partial_F \partial_G (\ln z(A, A+F+G) + \ln z(A+G, A+F)) &(3.309)\\ &\overset{(3.290)}{=} 2\partial_F \partial_G \ln z(A+F, A+G) \overset{(3.292)}{=} 2c_A(F,G). &(3.310)\end{aligned}$$

□

Now since $dc = 0$, we might use Poincaré's lemma as a method independent of z to construct a differential form ω such that $d\omega = c$. In order to execute this plan, we first need to prove Poincaré's lemma for our setting:

Lemma 46 (Poincaré)**.** *Let* $\omega \in \Omega^p(\mathcal{V})$ *for* $p \in \mathbb{N}$ *be closed, i.e.* $d\omega = 0$*. Then* ω *is also exact, more precisely we have*

$$\omega = d\int_0^1 \iota_t^* i_X f^* \omega dt, \tag{3.311}$$

where X, ι_t for $t \in \mathbb{R}$ and f are given by

$$X : \mathbb{R} \times \mathcal{V} \to \mathbb{R} \times \mathcal{V}, \tag{3.312}$$
$$(t, B) \mapsto (1, 0) \tag{3.313}$$
$$\forall t \in \mathbb{R} : \iota_t : \mathcal{V} \to \mathbb{R} \times \mathcal{V}, \tag{3.314}$$
$$B \mapsto (t, B) \tag{3.315}$$
$$f : \mathbb{R} \times \mathcal{V} \mapsto \mathcal{V}, \tag{3.316}$$
$$(t, B) \mapsto tB \tag{3.317}$$
$$i_X : \Omega^p(\mathcal{V}) \to \Omega^{p-1}(\mathcal{V}), \tag{3.318}$$
$$\omega \mapsto ((A; Y_1, \ldots, Y_{p-1}) \mapsto \omega_A(X, Y_1, \ldots, Y_{p-1})) \tag{3.319}$$

Proof. Pick some $\omega \in \Omega^p(\mathcal{V})$. We will first show the more general formula

put proof into appendix

$$f_b^*\omega - f_a^*\omega = d\int_a^b \iota_t^* i_X f^*\omega \; dt + \int_a^b \iota_t^* i_X f^* d\omega dt, \tag{3.320}$$

where f_t is defined as

$$\forall t \in \mathbb{R} : f_t := f(t, \cdot). \tag{3.321}$$

The lemma follows then by $b = 1$, $a = 0$, $f_1^*\omega = \omega$, $f_0^*\omega = 0$ and $d\omega = 0$ for a closed ω. We begin by rewriting the right hand side of (3.320):

$$d\int_a^b \iota_t^* i_X f^*\omega \; dt + \int_a^b \iota_t^* i_X f^* d\omega dt$$
$$= \int_a^b (d\iota_t^* i_X f^*\omega + \iota_t^* i_X f^* d\omega) dt. \tag{3.322}$$

Next we look at both of these terms separately. Let therefore $p \in \mathbb{N}$, $t, s_k \in \mathbb{R}$ and $A, B_k \in \mathcal{V}$ for each $p + 1 \geqslant k \in \mathbb{N}$. First, we calculate

$d\iota_t^* i_X f^*\omega$:

$$(f^*\omega)_{(t,A)}((s_1, B_1), \ldots, (s_p, B_p)) = \omega_{tA}(s_1 A + tB_1, \ldots, s_p A + tB_p) \tag{3.323}$$

$$\Rightarrow (i_X f^*\omega)_{(t,A)}((s_1, B_1), \ldots, (s_{p-1}, B_{p-1})) = \omega_{tA}(A, s_1 A + tB_1, \ldots, s_{p-1} A + tB_{p-1}) \tag{3.324}$$

$$\Rightarrow (\iota_t^* i_X f^*\omega)_A(B_1, \ldots, B_{p-1}) = t^{p-1}\omega_{tA}(A, B_1, \ldots, B_{p-1}) \tag{3.325}$$

$$\Rightarrow (d\iota_t^* i_X f^*\omega)_A(B_1, \ldots, B_p)$$

$$= \partial_\varepsilon|_{\varepsilon=0} \sum_{k=1}^{p} (-1)^{k+1} t^{p-1} \omega_{tA+\varepsilon t B_k}(A, B_1, \ldots, \widehat{B_k}, \ldots, B_p) \tag{3.326}$$

$$+ \partial_\varepsilon|_{\varepsilon=0} \sum_{k=1}^{p} (-1)^{k+1} t^{p-1} \omega_{tA}(A + \varepsilon B_k, B_1, \ldots, \widehat{B_k}, \ldots, B_p) \tag{3.327}$$

$$= \partial_\varepsilon|_{\varepsilon=0} \sum_{k=1}^{p} t^p (-1)^{k+1} \omega_{tA+\varepsilon B_k}(A, B_1, \ldots, \widehat{B_k}, \ldots, B_p) + p t^{p-1}\omega_{tA}(B_1, \ldots, B_p). \tag{3.328}$$

Now, we calculate $\iota_t^* i_X f^* d\omega$:

$$(d\omega)_A(B_1, \cdots, B_{p+1}) = \partial_\varepsilon|_{\varepsilon=0} \sum_{k=1}^{p+1} (-1)^{k+1} \omega_{A+\varepsilon B_k}(B_1, \ldots, \widehat{B_k}, \ldots, B_{p+1}) \tag{3.329}$$

$$(f^* d\omega)(t, A)((s_1, B_1), \ldots, (s_{p+1}, B_{p+1})) \tag{3.330}$$

$$= (d\omega)_{tA}(s_1 A + tB_1, \ldots, s_{p+1} A + tB_{p+1})$$

$$= \partial_\varepsilon|_{\varepsilon=0} \sum_{k=1}^{p+1} (-1)^{k+1} \tag{3.331}$$

$$\times\, \omega_{tA+\varepsilon(s_kA+tB_k)}(s_1A+tB_1,\dots,s_k\widehat{A+t}B_k,\dots,s_pA+tB_p)$$

$$\begin{aligned}
&(i_Xf^*d\omega)_{(t,A)}((s_1,B_1),\dots,(s_p,B_p)) && (3.332)\\
&\quad = \partial_\varepsilon|_{\varepsilon=0}\omega_{(t+\varepsilon)A}(s_1A+tB_1,\dots,s_pA+tB_p)\\
&+ \partial_\varepsilon|_{\varepsilon=0}\sum_{k=1}^{p}(-1)^k\\
&\times\, \omega_{tA+\varepsilon(s_kA+tB_k)}(A,s_1A+tB_1,\dots,s_k\widehat{A+t}B_k,\dots,s_pA+tB_p)\\
&= t^p\partial_\varepsilon|_{\varepsilon=0}\omega_{(t+\varepsilon)A}(B_1,\dots,,\widehat{B_k},\dots,B_p)) && (3.333)\\
&\qquad + \sum_{k=1}^{p} s_kt^{p-1}(-1)^{k+1}\partial_\varepsilon|_{\varepsilon=0}\omega_{(t+\varepsilon)A}(A,B_1,\dots,B_p) && (3.334)\\
&+ \partial_\varepsilon|_{\varepsilon=0}\sum_{k=1}^{p}(-1)^kt^{p-1}(\omega_{(t+s_k\varepsilon)A}(A,B_1,\dots,\widehat{B_k},\dots,B_p) && (3.335)\\
&\qquad + \omega_{tA+\varepsilon tB_k}(A,B_1,\dots,\widehat{B_k},\dots,B_p)) && (3.336)\\
&= t^p\partial_\varepsilon|_{\varepsilon=0}\Big(\omega_{(t+\varepsilon)A}(B_1,\dots,B_p) && (3.337)\\
&\qquad + \sum_{k=1}^{p}(-1)^k\omega_{tA+\varepsilon B_k}(A,B_1,\dots,\widehat{B_k},\dots,B_p)\Big)\\
&(\iota_t^*i_Xf^*d\omega)_A(B_1,\dots,B_p) = t^p\partial_\varepsilon|_{\varepsilon=0}\Big(\omega_{(t+\varepsilon)A}(B_1,\dots,B_p) && (3.338)\\
&\qquad + \sum_{k=1}^{p}(-1)^k\omega_{tA+\varepsilon B_k}(A,B_1,\dots,\widehat{B_k},\dots,B_p)\Big)
\end{aligned}$$

Adding (3.328) and (3.338) we find for (3.322):

$$\int_a^b(d\iota_t^*i_Xf^*\omega+\iota_t^*i_Xf^*d\omega)dt = \qquad (3.339)$$

$$\int_a^b\Big(t^p\partial_\varepsilon|_{\varepsilon=0}\omega_{(t+\varepsilon)A}(B_1,\dots,B_p)+pt^{p-1}\omega_{tA}(B_1,\dots,B_p)\Big)dt \qquad (3.340)$$

$$= \int_a^b \frac{d}{dt}(t^p \omega_{tA}(B_1, \ldots, B_p))dt = \int_a^b \frac{d}{dt}(f_t^* \omega)_A(B_1, \ldots, B_p)dt \quad (3.341)$$

$$= (f_b^* \omega)_A(B_1, \ldots, B_p) - (f_a^* \omega)_A(B_1, \ldots, B_p). \quad (3.342)$$

□

Definition 47 (antiderivative of a closed p form)**.** *For a closed exterior form $\omega \in \Omega^p(\mathcal{V})$ we define the form $\Pi[\omega]$*

$$\Pi[\omega] := \int_0^1 \iota_t^* i_X f^* \omega dt. \quad (3.343)$$

For $A, B_1, \ldots, B_{p-1} \in \mathcal{V}$ it takes the form

$$\Pi[\omega]_A(B_1, \ldots, B_{p-1}) = \int_0^1 t^{p-1} \omega_{tA}(A, B_1, \ldots, B_{p-1})dt. \quad (3.344)$$

By lemma 46 we know $d\Pi[\omega] = \omega$ if $d\omega = 0$.

Now we found two one forms each produces c when the exterior derivative is taken. The next lemma informs us about their relationship.

Lemma 48 (inversion of lemma 45)**.** *The following equality holds*

$$\chi = 2\Pi[c]. \quad (3.345)$$

Proof. By definition 47 of Π and lemma 45 we have $d(\chi - 2\Pi[c]) = 0$. Hence, by the Poincaré lemma 46, we know that there is $v : \mathcal{V} \to \mathbb{R}$ such that

$$dv = \chi - 2\Pi[c] \quad (3.346)$$

holds. Using the definition 41 of z, the parallel transport equation (3.224) translates into the following ODE for z:

$$\partial_B \ln z(0, B) = 0, \quad \partial_\varepsilon \ln z(A, (1 + \varepsilon)A)|_{\varepsilon=0} = 0 \quad (3.347)$$

for all $A, B \in \mathcal{V}$. Therefore have

$$\chi_0(B) = 0 = \Pi[c]_0(B), \quad \chi_A(A) = 0 = \Pi[c]_A(A), \tag{3.348}$$

which implies

$$\partial_\varepsilon v_{\varepsilon A}|_{\varepsilon=0} = 0, \quad \partial_\varepsilon v_{A+\varepsilon A}|_{\varepsilon=0} = 0. \tag{3.349}$$

In conclusion, v is constant. □

From this point on we will assume the existence of a function c^+ fulfilling (3.137),(3.138) and (3.139). Recall property (3.138):

$$\forall A, F, G, H : \partial_H c^+_{A+H}(F, G) = \partial_G c^+_{A+G}(F, H). \tag{3.350}$$

For a fixed $F \in \mathcal{V}$, this condition can be read as $d(c^+_\cdot(F, \cdot)) = 0$. As a consequence we can apply lemma 46 to define a one form.

Definition 49 (integral of the causal splitting)**.** *For any $A, F \in \mathcal{V}$, we define*

$$\beta_A(F) := 2\Pi[c^+_\cdot(F, \cdot)]_A. \tag{3.351}$$

Lemma 50 (relation between the integral of the causal splitting and the phase integral)**.** *The following two equations hold:*

$$d\beta = -2c, \tag{3.352}$$

$$d(\beta + \chi) = 0. \tag{3.353}$$

Proof. We start with the exterior derivative of β. Pick $A, F, G \in \mathcal{V}$:

$$d\beta_A(F, G) = \partial_F \beta_{A+F}(G) - \partial_G \beta_{A+G}(F) \tag{3.354}$$

$$= d\Big(\Pi[c^+_\cdot(G, \cdot)]\Big)_A(F) - d\Big(\Pi[c^+_\cdot(F, \cdot)]\Big)_A(G) \tag{3.355}$$

$$= 2c^+_A(G, F) - 2c^+_A(F, G) \overset{(3.137)}{=} -2c_A(F, G). \tag{3.356}$$

This proves the first equality. The second equality follows directly by $d\chi = 2c$. □

Definition 51 (corrected lift)**.** *Since $\beta + \chi$ is closed, we may use lemma 46 again to define the phase*

$$\alpha := \Pi[\beta + \chi]. \tag{3.357}$$

Furthermore, for all $A, B \in \mathcal{V}$ we define the corrected second quantised scattering operator

$$\tilde{S}_{0,A} := e^{-\alpha_A} \hat{S}_{0,A}, \tag{3.358}$$

$$\tilde{S}_{A,B} := \tilde{S}_{0,A}^{-1} \tilde{S}_{0,B}. \tag{3.359}$$

Using this definition one immediately gets:

Corollary 52 (group structure of the corrected lift)**.** *We have* $\tilde{S}_{A,B}\tilde{S}_{B,C} = \tilde{S}_{A,C}$ *for all $A, B, C \in \mathcal{V}$.*

Theorem 53 (causality of the corrected lift)**.** *The corrected second quantised scattering operator fulfils the following causality condition for all $A, F, G \in \mathcal{V}$ such that $F \prec G$:*

$$\tilde{S}_{A,A+F} = \tilde{S}_{A+G,A+G+F}. \tag{3.360}$$

Proof. Let $A, F, G \in \mathcal{V}$ such that $F \prec G$. For the first quantised scattering operator we have

$$S_{A+G,A+G+F} = S_{A,A+F}, \tag{3.361}$$

so that by definition of $\overline{S}$ we obtain

$$\overline{S}_{A+G,A+G+F} = \overline{S}_{A,A+F}. \tag{3.362}$$

So for any lift this equality is true up to a phase, meaning that

$$f(A, F, G) := \frac{\tilde{S}_{A+G,A+G+F}}{\tilde{S}_{A,A+F}} \tag{3.363}$$

is well defined. We see immediately

$$f(A,0,G) = 1 = f(A,F,0). \tag{3.364}$$

Pick $F_1, F_2 \prec G_1, G_2$. We abbreviate $F = F_1 + F_2, G = G_1 + G_2$ and we calculate

$$
\begin{aligned}
f(A,F,G) &= \frac{\tilde{S}_{A+G,A+F+G}}{\tilde{S}_{A,A+F}} && (3.365)\\
&= \frac{\tilde{S}_{A+G,A+F+G}}{\tilde{S}_{A+G_1,A+G_1+F}} \frac{\tilde{S}_{A+G_1,A+G_1+F}}{\tilde{S}_{A,A+F}} && (3.366)\\
&= \frac{\tilde{S}_{A+G,A+G+F_1}\tilde{S}_{A+G+F_1,A+F+G}}{\tilde{S}_{A+G_1,A+F_1+G_1}\tilde{S}_{A+G_1+F_1,A+G_1+F}} \frac{\tilde{S}_{A+G_1,A+G_1+F}}{\tilde{S}_{A,A+F}} && (3.367)\\
&= \frac{\tilde{S}_{A+G,A+G+F_1}}{\tilde{S}_{A+G_1,A+F_1+G_1}} \frac{\tilde{S}_{A+G+F_1,A+F+G}}{\tilde{S}_{A+G_1+F_1,A+G_1+F}} f(A,G_1,F_1+F_2) && (3.368)\\
&= f(A+G_1,F_1,G_2)f(A+G_1+F_1,G_2,F_2)f(A,G_1,F_1+F_2). && (3.369)
\end{aligned}
$$

Taking the logarithmic derivative we find:

$$\partial_{F_2}\partial_{G_2} \ln f(A,F_1+F_2,G_1+G_2) = \partial_{F_2}\partial_{G_2} \ln f(A+F_1+G_1,F_2,G_2). \tag{3.370}$$

Next we pick $F_2 = \alpha_1 F_1$ and $G_2 = \alpha_2 G_1$ for $\alpha_1, \alpha_2 \in \mathbb{R}^+$ small enough so that

$$
\begin{aligned}
\|1 - S_{A+F_1+F_2+G_1+G_2,A+F_1+G_1}\| &< 1 && (3.371)\\
\|1 - S_{A+F_1+F_2+G_1+G_2,A+F_1+G_1+G_2}\| &< 1 && (3.372)\\
\|1 - S_{A+F_1+F_2+G_1+G_2,A+F_1+F_2+G_1}\| &< 1 && (3.373)
\end{aligned}
$$

hold. We abbreviate $A' = A + G_1 + F_1$, use (3.289) and compute

$$\begin{aligned}
&f(A', F_2, G_2)\\
&\overset{(3.358)}{=} \exp(-\alpha_{A'+F_2+G_2} + \alpha_{A'+G_2} + \alpha_{A'+F_2} - \alpha_{A'})\\
&\qquad\qquad \times \frac{\hat{S}^{-1}_{0,A'+G_2}\hat{S}_{0,A'+G_2+F_2}}{\hat{S}^{-1}_{0,A'}\hat{S}_{0,A'+F_2}} &(3.374)\\
&\overset{(3.289)}{=} \exp(-\alpha_{A'+F_2+G_2} + \alpha_{A'+G_2} + \alpha_{A'+F_2} - \alpha_{A'})\\
&\qquad\qquad \times \frac{z(A'+G_2, A'+G_2+F_2)}{z(A', A'+F_2)} \frac{\overline{S}_{A'+G_2,A'+G_2+F_2}}{\overline{S}_{A',A'+F_2}} &(3.375)\\
&\overset{F_2 \precsim G_2}{=} \exp(-\alpha_{A'+F_2+G_2} + \alpha_{A'+G_2} + \alpha_{A'+F_2} - \alpha_{A'})\\
&\qquad\qquad \times \frac{z(A'+G_2, A'+G_2+F_2)}{z(A', A'+F_2)} &(3.376)
\end{aligned}$$

Most of the factors do not depend on F_2 and G_2, so taking the mixed logarithmic derivative things simplify:

$$\begin{aligned}
&\partial_{G_2}\partial_{F_2} \ln f(A', F_2, G_2) =\\
&\partial_{G_2}\partial_{F_2}(-\alpha_{A'+F_2+G_2} + \ln z(A'+G_2, A'+G_2+F_2)) &(3.377)\\
&\overset{(3.357),(3.304)}{=} \partial_{G_2}(-\beta_{A'+G_2}(F_2) - \chi_{A'+G_2}(F_2) + \chi_{A'+G_2}(F_2)) &(3.378)\\
&\overset{(3.352)}{=} 2c^{+}_{A'}(F_2, G_2) \overset{F_2 \prec G_2,(3.139)}{=} 0. &(3.379)
\end{aligned}$$

So by (3.370) we also have

$$\begin{aligned}
\partial_{F_2}\partial_{G_2} \ln f(A, F_1 + F_2, G_1 + G_2) = 0 \qquad &(3.380)\\
= \partial_{\alpha_1}\partial_{\alpha_2} \ln f(A, F_1(1+\alpha_1), G_1(1+\alpha_2)). \qquad &(3.381)
\end{aligned}$$

Using this then we can integrate and obtain

$$0 = \int_{-1}^{0} d\alpha_1 \int_{-1}^{0} d\alpha_2 \partial_{\alpha_1}\partial_{\alpha_2} \ln f(A, F_1(1+\alpha_1), G_1(1+\alpha_2)) \tag{3.382}$$

$$= \ln f(A, F_1, G_1) - \ln f(A, 0, G_1) - \ln f(A, F_1, 0) + \ln f(A, 0, 0) \tag{3.383}$$

$$\overset{(3.364)}{=} \ln f(A, F_1, G_1). \tag{3.384}$$

remembering equation (3.363), the definition of f, this ends our proof.
□

Using $\tilde{S}$ we introduce the current associated to it.

Theorem 54 (evaluation of the current of the corrected lift)**.** *For general $A, F \in \mathcal{V}$ we have*

$$j_A(F) = -i\beta_A(F). \tag{3.385}$$

So in particular for $G \in \mathcal{V}$

$$\partial_G j_{A+G}(F) = -2ic_A^+(F, G). \tag{3.386}$$

holds.

Proof. Pick $A, F \in \mathcal{V}$ as in the theorem. We calculate

$$i\partial_F \ln\left\langle \Omega, \tilde{S}_{A,A+F}\Omega \right\rangle \tag{3.387}$$

$$\overset{(3.358)}{=} i\partial_F \left(-\alpha_{A+F} - \alpha_A + \ln\left\langle \Omega, \hat{S}_{0,A}^{-1}\hat{S}_{0,A+F}\Omega \right\rangle\right) \tag{3.388}$$

$$\overset{(3.289)}{=} i\partial_F \left(-\alpha_{A+F} + \ln z(A, A+F) + \ln\langle \Omega, \overline{S}_{A,A+F}\Omega\rangle\right) \tag{3.389}$$

The last summand vanishes, as can be seen by the following calculation

$$\partial_F \ln \left\langle \Omega, \overline{S}_{A,A+F}\Omega \right\rangle \tag{3.390}$$
$$= i\partial_F \ln \det_{\mathcal{H}^-}(P^- S_{A,A+F} P^- \operatorname{AG}(P^- S_{A,A+F} P^-)^{-1}) \tag{3.391}$$
$$\overset{(3.121)}{=} i\partial_F \ln \det_{\mathcal{H}^-} |P^- S_{A,A+F} P^-| \tag{3.392}$$
$$= \frac{i}{2}\partial_F \ln \det_{\mathcal{H}^-}((P^- S_{A,A+F} P^-)^* P^- S_{A,A+F} P^-) \tag{3.393}$$
$$= \frac{i}{2}\partial_F \det_{\mathcal{H}^-}(P^- S_{A+F,A} P^- S_{A,A+F} P^-) \tag{3.394}$$
$$= \frac{i}{2} \operatorname{tr}(\partial_F P^- S_{A+F,A} P^- S_{A,A+F} P^-) \tag{3.395}$$
$$= \frac{i}{2} \operatorname{tr}(\partial_F P^- S_{A,A+F} P^- + \partial_F P^- S_{A+F,A} P^-) = 0 \tag{3.396}$$

where we made use of (3.207). Where after the substitution $P^- = 1 - P^+$ we may use theorem 26 to see that the derivatives are well-defined. So we are left with

$$j_A(F) = i\partial_F(-\alpha_{A+F} + \ln z(A, A+F)) \tag{3.397}$$
$$= i(-\beta_A(F) - \chi_A(F) + \chi_A(F)) = -i\beta_A(F). \tag{3.398}$$

Finally by taking the derivative with respect to $G \in \mathcal{V}$ and using the definition of β we find

$$\partial_G j_{A+G}(F) = -2ic_A^+(F, G). \tag{3.399}$$

□

3.5 Simple Formula for the Scattering Operator

3.5.1 Defining One-Particle Scattering-Matrix

In order to introduce the one-particle dynamics I introduce Diracs equation (3.400) and reformulate it in integral form in equation (**??**). By iterating this equation we will naturally be led to the informal series expansion of the scattering operator equation (**??**), whose convergence is discussed in the next section.
Throughout this thesis I will consider four-potentials A, F or G to be smooth functions in $C_c^\infty(\mathbb{R}^4) \otimes \mathbb{C}^4$, where the index c denotes that the elements have compact support. The Dirac equation for a wave function $\phi \in L^2(\mathbb{R}^3) \otimes \mathbb{C}^4$ is

$$0 = (i\partial\!\!\!/ - eA\!\!\!/ - m\mathbb{1})\phi, \tag{3.400}$$

where m is the mass of the electron, $\mathbb{1} : \mathbb{C}^4 \to \mathbb{C}^4$ is the identity and crossed out letters mean that their four-index is contracted with Dirac matrices

$$A\!\!\!/ := A_\alpha \gamma^\alpha, \tag{3.401}$$

where Einstein's summation convention is used. These matrices fulfil the anti-commutation relation

$$\forall \alpha, \beta \in \{0, 1, 2, 3\} : \{\gamma^\alpha, \gamma^\beta\} := \gamma^\alpha\gamma^\beta + \gamma^\beta\gamma^\alpha = g^{\alpha\beta}, \tag{3.402}$$

where g is the Minkowski metric. I work with the $+ - --$ metric signature and the Dirac representation of this algebra. Squared four dimensional objects always refer to the Minkowski square, meaning for all $a \in \mathbb{C}^4$, $a^2 := a^\alpha a_\alpha$.
In order to define Lorentz invariant measures for four dimensional integrals I employ the same notation as in [8]. The standard volume

form over $\mathbb{R}^4$ is denoted by $\mathrm{d}^4x = \mathrm{d}x^0\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3$, the product of forms is understood as the wedge product. The symbol d^3x means the 3-form $\mathrm{d}^3x = \mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3$ on $\mathbb{R}^4$. Contraction of a form ω with a vector v is denoted by $\mathfrak{i}_v(\omega)$. The notation $\mathfrak{i}_v(\omega)$ is also used for the spinor matrix valued vector $\gamma = (\gamma^0, \gamma^1, \gamma^2, \gamma^3) = \gamma^\alpha e_\alpha$:

$$\mathfrak{i}_\gamma(\mathrm{d}^4x) := \gamma^\alpha \mathfrak{i}_{e_\alpha}(\mathrm{d}^4x), \tag{3.403}$$

with $(e_\alpha)_\alpha$ being the canonical basis of $\mathbb{C}^4$. Let $\mathcal{C}_A$ be the space of solutions to (3.400) which have compact support on any spacelike hyperplane Σ. Let ϕ, ψ be in $\mathcal{C}_\mathcal{A}$, the scalar product $\langle \cdot, \cdot \rangle$ of elements of is defined as

Todo: delete derivation; simply state series representation and convergence result citing something

3. .2 Construction of the Second Quantised Scattering-Matrix

In he following I outline how the construction of the second quantised scattering operator is to be carried out, we will naturally be led to an informal power series representation for the scattering operator S.
First I fix some more notation in agreement with [7]. Using a general Hilbertspace $\mathcal{H}$ as a one-particle Hilbertspace. A closed subspace $\mathcal{H}^+$ of $\mathcal{H}$ is called polarisation if both $\mathcal{H}^+$ and $\mathcal{H}^- := (\mathcal{H}^+)^\perp$ are infinite dimensional, where by $\perp$ I denote the orthogonal complement. With a polarisation $\mathcal{H}^+$ comes also the orthogonal projection operator P^+ onto the subspace $\mathcal{H}^+$ and its complement $P^- = 1 - P^+$. For one particle operators C we introduce the notation $C_{\#\ddagger} := P^\# C P^\ddagger$, where $\#, \ddagger \in \{+, -\}$. One constructs the Fock space associated with $\mathcal{H}$ and a polarisation $\mathcal{H}^+$ of $\mathcal{H}$ in the following way. We define $\overline{\mathcal{H}^-}$ identical with $\mathcal{H}^-$ as a set, but scalar multiplication as $\mathbb{C} \times \overline{\mathcal{H}} \ni (a, \psi) \mapsto \overline{a}\psi$ where the bar denotes complex conjugation of complex numbers. A wedge $\wedge$ in the exponent denotes that only elements which are antisymmetric with respect to permutations are allowed. This antisymmetric product

as well as the tensor product are to be understood in the Hilbert space sense. The Factor $(\mathcal{H}^{\pm})^0$ is understood as $\mathbb{C}$. We now define Fock space as

$$\mathcal{F} := \bigoplus_{m,p=0}^{\infty} \left(\mathcal{H}^+\right)^{\wedge m} \otimes \left(\overline{\mathcal{H}^-}\right)^{\wedge p}. \tag{3.404}$$

I will denote the sectors of Fock space of fixed particle numbers by $\mathcal{F}_{m,p}$. The element of $\mathcal{F}_{0,0}$ of norm 1 will be denoted by Ω. The simplest and yet interesting example of this construction is the Fock space constructed on a hyperplane prior to the support of an external field, in this case $\mathcal{H} = L^2(\mathbb{R}^3, \mathbb{C}^4)$ and $\mathcal{H}^+$ consists of the wavefunctions that can be constructed from the generalised eigenfunctions of positive energy with respect to the free Dirac Hamiltonian.
The annihilation operator a acts on an arbitrary sector of Fock space $\mathcal{F}_{m,p}$, for any $m, p \in \mathbb{N}_0$ with either of the operator types

$$a : \overline{\mathcal{H}} \otimes \mathcal{F}_{m,p} \to \mathcal{F}_{m-1,p} \oplus \mathcal{F}_{m,p+1} \tag{3.405}$$

$$a : \overline{\mathcal{H}} \times \mathcal{F}_{m,p} \to \mathcal{F}_{m-1,p} \oplus \mathcal{F}_{m,p+1} \tag{3.406}$$

regardless of the exact type of the annihilation operator I will denote it by a. Also here the tensor product is understood in the algebraic sense. I start out by defining a on elements of $\{\bigwedge_{l=1}^{m} \varphi_l \otimes \bigwedge_{c=1}^{p} \phi_c \mid \forall c : \varphi_c \in \mathcal{H}^+, \phi_c \in \mathcal{H}^-\}$ which spans a dense subset of $\mathcal{F}_{m,p}$, then one continues this operator uniquely by linearity and finally by the bounded linear extension theorem to all of $\mathcal{F}_{m,p}$ and then again by linearity to all of $\overline{\mathcal{H}} \otimes \mathcal{F}_{m,p}$.

$$a\left(\phi \otimes \bigwedge_{l=1}^{m} \varphi_l \otimes \bigwedge_{c=1}^{p} \phi_c\right) = a\left(\phi, \bigwedge_{l=1}^{m} \varphi_l \otimes \bigwedge_{c=1}^{p} \phi_c\right) \tag{3.407}$$

$$= \sum_{k=1}^{m}(-1)^{1+k}\langle P^{+}\phi, \varphi_k\rangle \bigwedge_{\substack{l=1\\ l\neq k}}^{m}\varphi_l \otimes \bigwedge_{c=1}^{p}\phi_c + \bigwedge_{l=1}^{m}\varphi_l \otimes P^{-}\phi \wedge \bigwedge_{c=1}^{p}\phi_c \tag{3.408}$$

where $\langle , \rangle$ denotes that the scalar product of $\mathcal{H}$. The first summand on the right hand side is taken to vanish for $m = 0$. For $\varphi \in \mathcal{H}$ I will also use the abbreviation $a(\varphi) := a(\varphi, \cdot)$.
Now we turn to the construction of the S-matrix, the second quantised analogue of U^A. This construction is carried out axiomatically. The first axiom makes sure that the following diagram, and the analogue for the adjoint of the annihilation operator commute.

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{S^A} & \mathcal{F} \\ \uparrow{\scriptstyle a} & & \uparrow{\scriptstyle a} \\ \overline{\mathcal{H}} \otimes \mathcal{F} & \xrightarrow{U^A \otimes S^A} & \overline{\mathcal{H}} \otimes \mathcal{F} \end{array} \tag{3.409}$$

Axiom 55. *The S operator fulfils the "lift condition".*

$$\forall \phi \in \mathcal{H} : \quad S^A \circ a(\phi) = a\left(U^A\phi\right) \circ S^A, \qquad \text{(lift condition)}$$
$$\forall \phi \in \mathcal{H} : \quad S^A \circ a^*(\phi) = a^*\left(U^A\phi\right) \circ S^A, \qquad \text{(adjoint lift condition)}$$

where a^* is the adjoint of the annihilation operator, the creation operator.

There is a convergent power series of the one-particle scattering operator U^A:

$$U^A = \sum_{k=0}^{\infty} \frac{1}{k!} Z_k(A), \tag{3.410}$$

where $Z_k(A)$ are bounded operators on $\mathcal{H}$, which are homogeneous of degree k in A. We try an analogous formal power series ansatz for the second quantised scattering operator S^A

$$S^A = \sum_{k=0}^{\infty} \frac{1}{k!} T_k(A). \tag{3.411}$$

Here T_k are assumed to be homogeneous of degree k in A; however, they will only turn out to be bounded on fixed particle number subspaces $\mathcal{F}_{m,p}$ of Fock space. It is the goal of the following sections to show that this ansatz indeed works. That is, we can identify operators T_k such that (3.411) holds up to a global phase and furthermore the question of convergence can be settled if one assumes that the phase is analytic in the external field A. In order to fully characterise S^A it is enough to characterise all of the T_k operators. Using the (lift condition) one can derive commutation relations for the operators T_k by plugging in (3.410) and (3.411) into (lift condition) and (adjoint lift condition) and collecting all terms with the same degree of homogeneity. They are given by

$$\left[T_m(A), a^{\#}(\phi)\right] = \sum_{j=1}^{m} \binom{m}{j} a^{\#}\left(Z_j(A)\phi\right) T_{m-j}(A), \tag{3.412}$$

where $a^{\#}$ is either a or a^*. Together T_k and $\langle T_k \rangle$ characterise the operator T_k on the whole algebraic direct sum, it can then be further extended to all of Fock space.
Before we go on to construct a concrete form of the scattering operator, we will first define a certain kind of unitary operator on Fock space.

3.5.3 Differential second quantisation

Let $B : \mathcal{H} \to \mathcal{H}$ be a bounded operator on $\mathcal{H}$, such that iB is self adjoint and B_{+-} is a Hilbert-Schmidt operator. We would like to construct a version $\mathrm{d}\Gamma(B)$ of B that acts on Fock space and also is skew adjoint. The strategy of this section is to construct an operator in two steps that is essentially self adjoint of the Fock space of finitely many particles, a dense subset of Fock space. It is denoted by

Definition 56.
$$\mathcal{F}' := \bigoplus_{m,p=0}^{\infty} \mathcal{F}_{m,p}, \tag{3.413}$$

where $\bigoplus$ *refers to the algebraic direct sum.*

Because $B_{-+} : \mathcal{H}^+ \to \mathcal{H}^-$ is compact, there is an ONB $(\varphi_n)_{n\in\mathbb{N}}$ of $\mathcal{H}^+$ and likewise an ONB $(\varphi_{-n})_{n\in\mathbb{N}}$ of $\mathcal{H}^-$ such that it takes the canonical form of compact operators

$$B_{-+} = \sum_{n\in\mathbb{N}} \lambda_n |\varphi_{-n}\rangle\langle\varphi_n|, \quad \lambda_n \geqslant 0. \tag{3.414}$$

Here the numbers λ_n fulfil $\sum_{k=1}^{\infty} \lambda_k^2 = \|B_{-+}\|_{\mathrm{HS}} < \infty$. As a consequence we have

$$B_{+-} = -\sum_{n\in\mathbb{N}} \lambda_n |\varphi_n\rangle\langle\varphi_{-n}|. \tag{3.415}$$

With respect to this basis we define the set of finite linear combinations of product states of finitely many particles

Definition 57. *We define*

$$\mathcal{F}^0 := \mathrm{span}\left\{ \prod_{k=1}^{m} a^*(\varphi_{L_k}) \prod_{c=1}^{p} a(\varphi_{-C_c})\Omega \mid m, p\in\mathbb{N}, (L_k)_k, (C_c)_c \subset \mathbb{N} \right\}, \tag{3.416}$$

we will refer to a subset of this set for fixed values of m and p by $\mathcal{F}^0_{m,p}$.

In order to do so, the following splitting turns out to be advantageous.

Definition 58. *We define the following operators of type $\mathcal{F}^0 \to \mathcal{F}$*

$$\mathrm{d}\Gamma(B_{++}) := \sum_{n\in\mathbb{N}} a^*(B_{++}\varphi_n)a(\varphi_n) \tag{3.417}$$

$$\mathrm{d}\Gamma(B_{--}) := -\sum_{n\in\mathbb{N}} a(\varphi_{-n})a^*(B_{--}\varphi_{-n}) \tag{3.418}$$

$$\mathrm{d}\Gamma(B_{-+}) := \sum_{n\in\mathbb{N}} a^*(B_{-+}\varphi_n)a(\varphi_n) \tag{3.419}$$

where the sum converges in the strong operator topology and $(\varphi_n)_n, (\varphi_{-n})_n$ are arbitrary ONBs of $\mathcal{H}^+$ and $\mathcal{H}^-$.

Lemma 59. *The operators $\mathrm{d}\Gamma(B_{++}), \mathrm{d}\Gamma(B_{--})$ and $\mathrm{d}\Gamma(B_{-+})$ restricted to $|_{\mathcal{F}^0_{m,p}}$ they have the following type*

$$\mathrm{d}\Gamma(B_{++})|_{\mathcal{F}^0_{m,p}} : \quad \mathcal{F}^0_{m,p} \to \mathcal{F}_{m,p} \tag{3.420}$$

$$\mathrm{d}\Gamma(B_{--})|_{\mathcal{F}^0_{m,p}} : \quad \mathcal{F}^0_{m,p} \to \mathcal{F}_{m,p} \tag{3.421}$$

$$\mathrm{d}\Gamma(B_{-+})|_{\mathcal{F}^0_{m,p}} : \quad \mathcal{F}^0_{m,p} \to \mathcal{F}_{m-1,p-1} \tag{3.422}$$

and fulfil the following bounds for all m, p

$$\|\mathrm{d}\Gamma(B_{++})|_{\mathcal{F}^0_{m,p}}\| \leqslant (m+1)\|B_{++}\| \tag{3.423}$$

$$\|\mathrm{d}\Gamma(B_{--})|_{\mathcal{F}^0_{m,p}}\| \leqslant (p+1)\|B_{--}\| \tag{3.424}$$

$$\|\mathrm{d}\Gamma(B_{-+})|_{\mathcal{F}^0_{m,p}}\| \leqslant \|B_{-+}\|_{HS}. \tag{3.425}$$

Proof. Pick $\alpha \in \mathcal{F}^0_{m,p}$ for $m, p \in \mathbb{N}_0$, α can be expressed in terms of a general ONB $(\tilde{\varphi}_k)_{k\in\mathbb{N}}$ of $\mathcal{H}^+$ and $(\tilde{\varphi}_{-k})_{k\in\mathbb{N}}$ of $\mathcal{H}^-$

$$\alpha = \sum_{\substack{L,C\subset\mathbb{N}\\ |L|=m,|C|=p}} \alpha_{L,C} \prod_{l=1}^{m} a^*(\tilde{\varphi}_{L_l}) \prod_{c=1}^{p} a(\tilde{\varphi}_{-C_c})\Omega. \tag{3.426}$$

In this expansion only finitely many coefficients $\alpha_{\cdot,\cdot}$ are nonzero. Our operators all map the vacuum onto the zero vector, so commuting them through the products of of creation and annihilation operators in the expansion of α we can make the action of them more explicit:

$$\mathrm{d}\Gamma(B_{++})\alpha = \sum_{\substack{L,C\subset\mathbb{N}\\|L|=m,|C|=p}} \alpha_{L,C} \sum_{b=1}^{m} \prod_{l=1}^{b-1} a^*(\tilde{\varphi}_{L_l}) \sum_{n\in\mathbb{N}} a^*(B_{++}\varphi_n)\langle\varphi_n, \tilde{\varphi}_{L_b}\rangle \prod_{l=b+1}^{m} a^*(\tilde{\varphi}_l) \prod_{c=1}^{p} a(\tilde{\varphi}_{-C_c})\Omega \tag{3.427}$$

$$= \sum_{\substack{L,C\subset\mathbb{N}\\|L|=m,|C|=p}} \alpha_{L,C} \sum_{b=1}^{m} \prod_{l=1}^{b-1} a^*(\tilde{\varphi}_{L_l}) \quad a^*(B_{++}\tilde{\varphi}_{L_b}) \prod_{l=b+1}^{m} a^*(\tilde{\varphi}_l) \prod_{c=1}^{p} a(\tilde{\varphi}_{-C_c})\Omega. \tag{3.428}$$

We notice, that $\mathrm{d}\Gamma(B_{++})\alpha \in \mathcal{F}_{m,p}$ holds. What is left to show for the first operator is therefore its norm. For estimating this we see that B_{++} in the last line can be replaced by

$$B^L_{L_b} := \left(1 - \sum_{\substack{l=1\\l\neq b}}^{m} |\tilde{\varphi}_{L_l}\rangle\langle\tilde{\varphi}_{L_l}|\right) B_{++}, \tag{3.429}$$

due to the antisymmetry of fermions. Expanding

$$\|\mathrm{d}\Gamma(B_{++})\alpha\|^2 = \langle \mathrm{d}\Gamma(B_{++})\alpha, \mathrm{d}\Gamma(B_{++})\alpha\rangle$$
$$= \sum_{\substack{L,C,L',C'\subset\mathbb{N}\\|L'|=|L|=m,|C'|=|C|=p}} \overline{\alpha_{L,C}}\alpha_{L',C'} \sum_{b,b'=1}^{m} \left\langle \prod_{l=1}^{b-1} a^*(\tilde{\varphi}_{L_l}) \quad a^*(B^L_{L_b}\tilde{\varphi}_{L_b}) \right.$$

$$\prod_{l=b+1}^{m} a^*(\tilde\varphi_{L_l}) \prod_{c=1}^{p} a(\tilde\varphi_{-C_c})\Omega, \prod_{l=1}^{b'-1} a^*(\tilde\varphi_{L'_l}) \;\; a^*(B^{L'}_{L'_{b'}} \tilde\varphi_{L'_b})$$
$$\prod_{l=b'+1}^{m} a^*(\tilde\varphi_{L'_l}) \prod_{c=1}^{p} a(\tilde\varphi_{-C'_c})\Omega \Big\rangle \tag{3.430}$$

we see that in fact C and C' need to agree, because we can just commute the corresponding annihilation operators from one end of the scalar product to the other. Furthermore only a single wavefunction on each side of the scalar product is modified, this implies that in order for the scalar product not to vanish $|L \cap L'| \geqslant m-2$ has to hold. If $L \neq L'$ the double sum over n, n' has only the contribution where $b = L_l \notin L'$ and $b' = L'_{l'} \notin L$ are selected. Otherwise the full sum contributes, yielding

$$\begin{aligned}
&\|\mathrm{d}\Gamma(B_{++})\alpha\|^2 = \\
&= \sum_{\substack{L,C\subset\mathbb{N}\\|C|=p\\|L|=m-1}} \sum_{n\neq n'\in\mathbb{N}\setminus L} \overline{\alpha_{L\cup\{n\},C}}\alpha_{L\cup\{n'\},C} \langle B_n^{L\cup\{n\}}\tilde\varphi_n, B_{n'}^{L\cup\{n'\}}\tilde\varphi_{n'}\rangle (-1)^{g(L,n)+g(L,n')} \\
&+ \sum_{\substack{L,C\subset\mathbb{N}\\|L|=m,|C|=p}} |\alpha_{L,C}|^2 \sum_{b,b'=1}^{m} \Big\langle \prod_{l=1}^{b-1} a^*(\tilde\varphi_{L_l}) \;\; a^*(B^L_{L_b}\tilde\varphi_{L_b}) \qquad (3.431)\\
&\prod_{l=b+1}^{m} a^*(\tilde\varphi_{L_l})\Omega, \prod_{l=1}^{b'-1} a^*(\tilde\varphi_{L_l}) \;\; a^*(B^L_{L_{b'}}\tilde\varphi_{L_b}) \prod_{l=b'+1}^{m} a^*(\tilde\varphi_{L_l}))\Omega \Big\rangle,
\end{aligned}$$

where $g(L,n) := |\{l \in L \mid l < n\}|$ keeps track of the number of anti commutations. In the first sum we add and subtract the terms where $n = n'$. The enlarged sum can then be reformulated

$$\sum_{\substack{L,C\subset\mathbb{N}\\|L|=m-1\\|C|=p}}\sum_{n,n'\in\mathbb{N}\setminus L}\overline{\alpha_{L\cup\{n\},C}}\alpha_{L\cup\{n'\},C}\langle B_n^{L\cup\{n\}}\tilde{\varphi}_n, B_{n'}^{L\cup\{n'\}}\tilde{\varphi}_{n'}\rangle(-1)^{g(L,n)+g(L,n')}$$

$$= \sum_{\substack{L,C\subset\mathbb{N}\\|L|=m-1,|C|=p}}\left\|\sum_{n\in\mathbb{N}\setminus L}\alpha_{L\cup\{n\},C}B_n^{L\cup\{n\}}\tilde{\varphi}_n(-1)^{g(L,n)}\right\|^2$$

$$= \sum_{\substack{L,C\subset\mathbb{N}\\|L|=m-1\\|C|=p}}\left\|\left(1-\sum_{l\in L}|\tilde{\varphi}_l\rangle\langle\tilde{\varphi}_l|\right)B_{++}\sum_{n\in\mathbb{N}\setminus L}\alpha_{L\cup\{n\},C}\tilde{\varphi}_n(-1)^{g(L,n)}\right\|^2 \quad (3.432)$$

Now the operator product inside the norm has operator norm $\|B_{++}\|$ and so we can estimate the whole object by

$$(3.432) \leqslant \|\alpha\|^2\|B_{++}\|^2. \quad (3.433)$$

Now for the first term in (3.431) we need to estimate the term we added to complete the norm square, this is done as follows

$$\sum_{\substack{L,C\subset\mathbb{N}\\|L|=m-1,|C|=p}}\sum_{n\in\mathbb{N}\setminus L}|\alpha_{L\cup\{n\},C}|^2\|B_n^{L\cup\{n\}}\tilde{\varphi}_n\|^2$$
$$\leqslant \sum_{\substack{L,C\subset\mathbb{N}\\|L|=m,|C|=p}}\|B_{++}\|^2|\alpha_{L,C}|^2 = \|\alpha\|^2\|B_{++}\|^2. \quad (3.434)$$

What remains is the second sum in (3.431), for this term there are two cases. If $b=b'$ then the scalar product is equal to $\langle B^L_{L_b}\tilde{\varphi}_b, B^L_{L_b}\tilde{\varphi}_b\rangle$. If $b\neq b'$ the scalar product is, up to a sign, equal to $\langle B^L_{L_b}\tilde{\varphi}_b, \tilde{\varphi}_b\rangle\langle\tilde{\varphi}_{b'}, B^L_{L_{b'}}\tilde{\varphi}_{b'}\rangle$. However both of these terms can be estimated by $\|B_{++}\|^2$. So all m^2

summands of this sum contribute $\|B_{++}\|^2$. Overall this estimate yields

$$\begin{aligned}&\|\mathrm{d}\Gamma(B_{++})\alpha\|^2 \leqslant (3.433) + (3.434) + \|\alpha\|^2 m^2 \|B_{++}\|^2\\ &= \|\alpha\|^2(2+m^2)\|B_{++}\|^2.\end{aligned}$$

For convenience of notation the estimate can be weakened to

$$\|\mathrm{d}\Gamma(B_{++})\alpha\| \leqslant (m+1)\|B_{++}\|, \tag{3.435}$$

because for all $m \neq 0$ this estimate is and upper bound on what we found, but for $m = 0$ the operator $\mathrm{d}\Gamma(B_{++})$ is actually the zero operator. A completely analogous argument works for $\mathrm{d}\Gamma(B_{--})$.
So lets move on to $\mathrm{d}\Gamma(B_{-+})$. Applying it to the same $\alpha \in \mathcal{F}^0_{m,p}$ again we permute all the operators to the right, where they annihilate the vacuum. The remaining terms are

$$\begin{aligned}&\sum_{n\in\mathbb{N}} a^*(B_{-+}\varphi_n)a(\varphi_n) \sum_{\substack{L,C\subset\mathbb{N}\\|L|=m,|C|=p}} \alpha_{L,C} \prod_{l=1}^m a^*(\tilde{\varphi}_{L_l}) \prod_{c=1}^p a(\tilde{\varphi}_{-C_c})\Omega\\ &= \sum_{\substack{L,C\subset\mathbb{N}\\|L|=m,|C|=p}} \alpha_{L,C} \sum_{b=1}^m \sum_{d=1}^p (-1)^{m-1+b+d} \langle B_{+-}\tilde{\varphi}_{-C_d}, \tilde{\varphi}_{L_b}\rangle\\ &\qquad \prod_{\substack{l=1\\l\neq b}}^m a^*(\tilde{\varphi}_{L_l}) \prod_{\substack{c=1\\c\neq d}}^p a(\tilde{\varphi}_{-C_c})\Omega.\end{aligned} \tag{3.436}$$

By counting the remaining creation and annihilation operators we immediately see that $\mathrm{d}\Gamma(B_{-+})\alpha \in \mathcal{F}_{m-1,p-1}$. For estimating the norm of this vector, we switch basis from $(\tilde{\varphi}_{\pm n})_n$ to $(\varphi'_{\pm n})_n$, the basis where B_{-+} takes its canonical form. Then the scalar product involving B_{-+} reduces to $\lambda_{L_b}\delta_{L_b,C_d}$. We estimate

$$\|\mathrm{d}\Gamma(B_{-+})\alpha\|^2 = \sum_{\substack{L,L',C,C'\subset\mathbb{N}\\ |L|=|L'|=m\\ |C|=|C'|=p}} \sum_{a,a'=1}^{m} \sum_{b,b'=1}^{p} \overline{\alpha}_{L,C}\alpha_{L',C'}(-1)^{b+d+b'+d'}\lambda_{L_b}\lambda_{L'_{b'}}$$
$$\delta_{L_b,C_d}\delta_{L'_{b'},C'_{d'}} \left\langle \prod_{\substack{l=1\\ l\neq b}}^{m} a^*(\varphi'_{L_l}) \prod_{\substack{c=1\\ c\neq d}}^{p} a(\varphi'_{-C_c})\Omega, \prod_{\substack{l=1\\ l\neq b'}}^{m} a^*(\varphi'_{L'_l}) \prod_{\substack{c=1\\ c\neq d'}}^{p} a(\varphi'_{-C'_c})\Omega \right\rangle. \tag{3.437}$$

The scalar product in the second line tells us that $L\backslash\{L_b\} = L'\backslash\{L'_{b'}\}$ and $C\backslash\{C_d\} = C'\backslash\{C'_{d'}\}$ have to hold in order for the term not to vanish. So this means that L and L' as well as C and C' can respectively differ at most by one element which then has to be in the intersection $L \cap C$. Because this sum is really just a finite sum, we can reorder it in the following way

$$\|\mathrm{d}\Gamma(B_{-+})\alpha\|^2 = \sum_{\substack{L,C\subset\mathbb{N}\\ |L|=m-1\\ |C|=p-1}} \sum_{b,b'\in\mathbb{N}\backslash(L\cup C)} \lambda_b\lambda_{b'}\overline{\alpha}_{L\cup\{b\},C\cup\{b\}}\alpha_{L\cup\{b'\},C\cup\{b'\}}$$
$$(-1)^{g(L,b)+g(C,b)+g(L,b')+g(C,b')}, \tag{3.438}$$

where $g(L,b) = |\{l \in L \mid l < b\}|$ as before. This expression can be rewritten in terms of a scalar product in $\ell^2(\mathbb{N})$

$$\begin{aligned}\|\mathrm{d}\Gamma(B_{-+})\alpha\|^2 &= \sum_{\substack{L,C\subset\mathbb{N}\\ |L|=m-1\\ |C|=p-1}} \left|\left\langle 1_{(L\cup C)^c}\alpha_{L\cup\{\cdot\},C\cup\{\cdot\}}(-1)^{g(L,\cdot)+g(C,\cdot)}, \lambda_\cdot\right\rangle_{\ell^2}\right|^2 \\ &\leqslant \sum_{\substack{L,C\subset\mathbb{N}\\ |L|=m-1\\ |C|=p-1}} \sum_{b\in\mathbb{N}} 1_{(L\cup C)^c}(b)|\alpha_{L\cup\{b\},C\cup\{b\}}|^2 \sum_{d\in\mathbb{N}} \lambda_d^2 \end{aligned} \tag{3.439}$$

$$\leqslant \|\alpha\|^2 \|B_{-+}\|_{\mathrm{HS}}^2. \tag{3.440}$$

□

Corollary 60. *The operators* $\mathrm{d}\Gamma(B_{--})$ *and* $\mathrm{d}\Gamma(B_{++})$ *can be extended by continuity on* $\mathcal{F}^0_{m,p}$ *to unbounded operators on all of* $\mathcal{F}'$*. The operator* $\mathrm{d}\Gamma(B_{-+})$ *can be continuously extended to all of* $\mathcal{F}$*.*

Lemma 61. *The operator* $(\mathrm{d}\Gamma(B_{-+}))^*$ *acts on elements of* $\mathcal{F}^0$ *as*

$$-\sum_{n\in\mathbb{N}} a^*(B_{+-}\varphi_{-n})a(\varphi_{-n}) =: -\mathrm{d}\Gamma(B_{+-}). \tag{3.441}$$

So also $\mathrm{d}\Gamma(B_{+-}) : \mathcal{F}^0 \to \mathcal{F}$ *can be extended continuously to all of* $\mathcal{F}$*. Moreover* $\mathrm{d}\Gamma(B_{-+}) + \mathrm{d}\Gamma(B_{+-})$ *is skew-adjoint.*

Proof. Pick $\beta, \alpha \in \mathcal{F}^0$. We expand those states with respect to the basis $(\varphi'_k)_{k\in\mathbb{Z}\setminus\{0\}}$. Consider

$$\begin{aligned}
\langle \beta, \mathrm{d}\Gamma(B_{-+})\alpha\rangle &= \left\langle \beta, \sum_{n\in\mathbb{N}} a^*(B_{-+}\varphi_n)a(\varphi_n)\alpha \right\rangle \\
&= \sum_{n\in\mathbb{N}} \langle \beta, a^*(B_{-+}\varphi_n)a(\varphi_n)\alpha\rangle = \sum_{n\in\mathbb{N}} \langle a^*(\varphi_n)a(B_{-+}\varphi_n)\beta, \alpha\rangle \\
&= \left\langle \sum_{n\in\mathbb{N}} a^*(\varphi_n)a(B_{-+}\varphi_n)\beta, \alpha \right\rangle = \left\langle \sum_{n\in\mathbb{N}} \lambda_n a^*(\varphi_n)a(\varphi_{-n})\beta, \alpha \right\rangle \\
&= \left\langle \sum_{n\in\mathbb{N}} a^*(-B_{+-}\varphi_{-n})a(\varphi_{-n})\beta, \alpha \right\rangle = -\langle \mathrm{d}\Gamma(B_{+-})\beta, \alpha\rangle,
\end{aligned} \tag{3.442}$$

So we see that $\mathrm{d}\Gamma(B_{+-})$ and $\mathrm{d}\Gamma(B_{-+})^*$ agree on $\mathcal{F}^0$ which is dense. So they are the same bounded and continuous operator on all of Fock space. □

Lemma 62. *The operator* $\mathrm{d}\Gamma(B) : \mathcal{F}' \to \mathcal{F}$,

$$\mathrm{d}\Gamma(B) := \mathrm{d}\Gamma(B_{++}) + \mathrm{d}\Gamma(B_{+-}) + \mathrm{d}\Gamma(B_{-+}) + \mathrm{d}\Gamma(B_{--}) \tag{3.443}$$

is skew symmetric.

Proof. Since the sum of skew symmetric operators is skew symmetric, it suffices to show skew symmetry of $\mathrm{d}\Gamma(B_{++})$ and $\mathrm{d}\Gamma(B_{--})$. Moreover since both of these operators are extended versions of operators of the same name of type $\mathcal{F}^0 \to \mathcal{F}$ it suffices to show skew symmetry on this domain. We will only do the calculation for $\mathrm{d}\Gamma(B_{++})$, the other calculation is analogous. First we notice that

$$\mathrm{d}\Gamma(B_{++}) = \sum_{n\in\mathbb{N}} a^*(B_{++}\varphi_n)a(\varphi_n) = \sum_{n\in\mathbb{N}}\sum_{m\in\mathbb{N}} \langle\varphi_m, B_{++}\varphi_n\rangle a^*(\varphi_m)a(\varphi_n) \tag{3.444}$$

holds. Pick $\alpha, \beta \in \mathcal{F}^0$. Consider

$$\begin{aligned}
\langle\beta, \mathrm{d}\Gamma(B_{++})\alpha\rangle = \sum_{L,L',C,C'\subset\mathbb{N}} \overline{\beta}_{L',C'}\alpha_{L,C} \Bigg\langle \prod_{l=1}^{|L'|} a^*(\varphi_{L'_l}) \prod_{c=1}^{|C'|} a(\varphi_{-C'_c})\Omega,\\
\sum_{n\in\mathbb{N}} a^*(B_{++}\varphi_n)a(\varphi_n) \prod_{l=1}^{|L|} a^*(\varphi_{L_l}) \prod_{c=1}^{|C|} a(\varphi_{-C_c})\Omega \Bigg\rangle\\
= \sum_{L,L',C,C'\subset\mathbb{N}} \overline{\beta}_{L',C'}\alpha_{L,C} \sum_{n\in\mathbb{N}} \Bigg\langle \prod_{l=1}^{|L'|} a^*(\varphi_{L'_l}) \prod_{c=1}^{|C'|} a(\varphi_{-C'_c})\Omega,\\
a^*(B_{++}\varphi_n)a(\varphi_n) \prod_{l=1}^{|L|} a^*(\varphi_{L_l}) \prod_{c=1}^{|C|} a(\varphi_{-C_c})\Omega \Bigg\rangle\\
= \sum_{L,L',C,C'\subset\mathbb{N}} \overline{\beta}_{L',C'}\alpha_{L,C} \sum_{n\in\mathbb{N}} \Bigg\langle a^*(\varphi_n)a(B_{++}\varphi_n) \prod_{l=1}^{|L'|} a^*(\varphi_{L'_l}) \prod_{c=1}^{|C'|} a(\varphi_{-C'_c})\Omega,\\
\prod_{l=1}^{|L|} a^*(\varphi_{L_l}) \prod_{c=1}^{|C|} a(\varphi_{-C_c})\Omega \Bigg\rangle
\end{aligned}$$

$$= \sum_{L,L',C,C' \subset \mathbb{N}} \overline{\beta}_{L',C'} \alpha_{L,C} \left\langle \sum_{n\in\mathbb{N}} \sum_{m\in\mathbb{N}} \langle B_{++}\varphi_n, \varphi_m \rangle a^*(\varphi_n) a(\varphi_m) \prod_{l=1}^{|L'|} a^*(\varphi_{L'_l}) \prod_{c=1}^{|C'|} a(\varphi_{-C'_c})\Omega, \prod_{l=1}^{|L|} a^*(\varphi_{L_l}) \prod_{c=1}^{|C|} a(\varphi_{-C_c})\Omega \right\rangle .$$

Now because $B_{++}^* = -B_{++}$ we see that

$$\langle \beta, \mathrm{d}\Gamma(B_{++})\alpha \rangle = -\langle \mathrm{d}\Gamma(B_{++})\beta, \alpha \rangle \tag{3.445}$$

holds. □

Now we would like to define $e^{\mathrm{d}\Gamma(B)}$, in order to do so, we will show that $\mathrm{d}\Gamma(B)$ is essentially skew-adjoint. One way of doing so is by Nelson's analytic vector theorem.

Theorem 63 (Nelson's analytic vector theorem)**.** *Let C be a symmetric operator on a Hilbert space $\mathcal{H}$. If $D(C)$ contains a total set $S \subset \bigcap_{n=1}^{\infty} D(C^n)$ of analytic vectors, then C is essentially self adjoint. A vector $\phi \in \bigcap_{n=1}^{\infty} D(C^n)$ is called analytic if there is $t > 0$ such that $\sum_{k=0}^{\infty} \frac{\|C^n\phi\|}{n!} t^n < \infty$ holds. A set S is said to be total if $\overline{span(S)} = \mathcal{H}$*

For a proof see e.g. [47].

Lemma 64. *For any $\alpha \in \mathcal{F}', t > 0$ the operator $\mathrm{d}\Gamma(B) : \mathcal{F}' \to \mathcal{F}$ satisfies*

$$\sum_{k=0}^{\infty} \frac{\|\mathrm{d}\Gamma(B)^k \alpha\|}{k!} t^k < \infty. \tag{3.446}$$

Proof. By definition of $\mathcal{F}'$ there are $m, p \in \mathbb{N}$ such that $\alpha \in \bigoplus_{l=0}^{m} \bigoplus_{c=0}^{p} \mathcal{F}_{l,p}$. Fix $t > 0$. We dissect α into its parts of fixed particle numbers:

$$\sum_{k=0}^{\infty} \frac{\|\mathrm{d}\Gamma(B)^k \alpha\|}{k!} t^k \leqslant \sum_{l=0}^{m} \sum_{c=0}^{p} \sum_{k=0}^{\infty} \frac{\|\mathrm{d}\Gamma(B)^k \alpha_{l,c}\|}{k!} t^k. \tag{3.447}$$

Using the following abbreviations

$$\Gamma_{-1} := \mathrm{d}\Gamma(B)_{-+} \tag{3.448}$$
$$\Gamma_0 := \mathrm{d}\Gamma(B)_{++} + \mathrm{d}\Gamma(B)_{--} \tag{3.449}$$
$$\Gamma_{+1} := \mathrm{d}\Gamma(B)_{+-} \tag{3.450}$$
$$\beta := \max\{\|B_{++}\| + \|B_{--}\|, \|B_{-+}, \|_{\mathrm{HS}}\} \tag{3.451}$$

we estimate

$$\|\mathrm{d}\Gamma(B)^k \alpha_{l,c}\| \leqslant \sum_{x\in\{-1,0,+1\}^k} \left\| \prod_{b=1}^{k} \Gamma_{x_b} \alpha_{l,c} \right\|$$
$$\leqslant \sum_{x\in\{-1,0,+1\}^k} \prod_{b=1}^{k} \left\| \Gamma_{x_b}|_{\mathcal{F}_{l+\sum_{d=1}^{b-1} x_d, c+\sum_{d=1}^{b-1} x_d}} \right\| \|\alpha_{l,c}\| \tag{3.452}$$
$$\leqslant 3^k \|\alpha\| \max_{x\in\{-1,0,+1\}^k} \prod_{b=1}^{k} \left\| \Gamma_{x_b}|_{\mathcal{F}_{l+\sum_{d=1}^{b-1} x_d, c+\sum_{d=1}^{b-1} x_d}} \right\|. \tag{3.453}$$

At this point the factors only depend on the number of particles the Fock space vector attains as we act on it with the operators $\Gamma_\cdot$. As these bounds increase with the particle number we can restrict the set $\{-1, 0, +1\}$ in the last line to $\{0, +1\}$. We notice that the bound in (3.452) will only increase if we exchange each pair $x_i = 1, x_h = 0$ by the pair $x_{\max\{i,h\}} = 1, x_{\min\{i,h\}} = 0$ so that the norm of the operator that acts like a particle number operator is taken after the particle number is increased. The maximum therefore has the form $(c + l + 2 + 2d)^{k-d}$, which we bound by $2^k(c/2 + d/2 + 1 + d)^{k-d}$. For maximising this object we treat d as a continuous variable take the derivative and set it to zero. From the form of the function to be maximised it is clear that it is equal to 1 for $d = k$ and at $d = -c/2 - l/2$, but for k large it will be bigger in between. We abbreviate $y = c/2 + l/2 + 1$.

$$0 = (y+d)^{k-d}(-\ln(y+d) + \frac{k-d}{y+d}) \tag{3.454}$$
$$\iff \frac{k-d}{y+d} = \ln(y+d) \tag{3.455}$$
$$\iff \frac{k+y}{y+d} - 1 = -1 + \ln(e(y+d)) \tag{3.456}$$
$$\iff e(k+y) = e(y+d)\ln(e(y+d)) \tag{3.457}$$
$$\iff e(k+y) = \ln(e(y+d))e^{\ln(e(y+d))} \tag{3.458}$$
$$\iff W_0(e(k+y)) = \ln(e(y+d)) \tag{3.459}$$
$$\iff e^{W_0(e(k+y))-1} - y = d, \tag{3.460}$$

where we made use of the Lambert W function, which is the inverse function of $x \mapsto xe^x$ and has multiple branches; however as $e(y+d) > 0$ W_0 is the only real branch which is applicable here, it corresponds to the inverse of $x \mapsto xe^x$ for $x > -1$. From the form of the maximising value we see, that it is always bigger than $-y$. Plugging this back onto our function we find its maximum

$$\begin{aligned}&\max_{d\in]-y,\infty[} (y+d)^{k-d} = e^{(W_0(e(k+y))-1)(k+y)-(W_0(e(k+y))-1)e^{W_0(e(k+y))-1}}\\ &= e^{-(k+y)+(k+y)W_0(e(k+y))+e^{W_0(e(k+y))-1}-((k+y)e)/e}\\ &= e^{-2(k+y)+(k+y)W_0((k+y)e)+\frac{e(k+y)}{eW_0((k+y)e)}}\\ &= e^{(k+y)(-2+W_0((k+y)e)+W_0((k+y)e)^{-1})},\end{aligned} \tag{3.461}$$

where we repeatedly used $W_0(x)e^{W_0(x)} = x$. Putting things together we find

$$\|\Gamma(B)^k\alpha_{l,c}\| \leqslant (6\beta)^k\|\alpha\|e^{(k+y)(-2+W_0((k+y)e)+W_0((k+y)e)^{-1})}. \tag{3.462}$$

Dividing this by $k!$ and using the lower bound given by Sterling's formula we would like to prove that

$$\sum_{k=1}^{\infty}(6\beta t)^k e^{k(1-\ln(k))-\frac{1}{2}\ln(k)+(k+y)(-2+W_0((k+y)e)+W_0((k+y)e)^{-1})} < \infty \tag{3.463}$$

holds, where we neglected constant factors and the summand $k = 0$ which do not matter for the task at hand. Next we are going to use an inequality about the growth of W_0 proven in [21]. For any $x \geqslant e$

$$W_0(x) \leqslant \ln(x) - \ln(\ln(x)) + \frac{e}{e-1}\frac{\ln(\ln(x))}{\ln(x)} \tag{3.464}$$

holds true. Plugging this into our sum the exponent is bounded from above by

$$\begin{aligned}
&k(1-\ln(k)) - \frac{1}{2}\ln(k) + (k+y)\Big[-1+\ln(k+y)-\ln(1+\ln(k+y)) \\
&+\frac{e}{e-1}\frac{\ln(1+\ln(k+y))}{1+\ln(k+y)} + W_0((k+y)e)^{-1}\Big] \\
&= -y + k\ln\left(1+\frac{y}{k}\right) + y\ln(k+y) - \frac{1}{2}\ln(k) + \\
&(k+y)\left[\ln(1+\ln(k+y))\frac{1-(e-1)\ln(k+y)}{(e-1)(1+\ln(k+y))} + W_0((k+y)e)^{-1}\right] \\
&\leqslant y\ln(k+y) - \frac{1}{2}\ln(k) + (k+y)W_0((k+y)e)^{-1} + \\
&(k+y)\ln(1+\ln(k+y))\frac{1-(e-1)\ln(k+y)}{(e-1)(1+\ln(k+y))}.
\end{aligned} \tag{3.465}$$

Now it is important to notice that the only remaining term that grows faster than linearly in magnitude is the last summand. This term;

however, is negative for large k, as the fraction converges to $-(e-1)$ for large k, while the double logarithm in front grows without bounds. So there is a k^* big enough such that for all $k > k^*$ (3.465) is smaller than $-k(\ln(6\beta t)+1)$, proving that (3.463) in fact holds.

□

Theorem 65. *The operator* $\mathrm{d}\Gamma(B) : \mathcal{F}' \to \mathcal{F}$ *is essentially skew adjoint and hence by Stones theorem generates a strongly continuous unitary group* $\left(e^{t\ \widehat{\mathrm{d}\Gamma(B)}}\right)_t$*, where* $\widehat{\mathrm{d}\Gamma(B)}$ *is the closure of* $\mathrm{d}\Gamma(B)$*.*

Proof. In order to apply Nelson's analytic vector theorem we pick $C = \mathcal{F}'$. Pick $\alpha \in \mathcal{F}'$. We need to show that there is $t > 0$ such that

$$\sum_{k=0}^{\infty} \frac{\|\mathrm{d}\Gamma(B)^k \alpha\|}{k!} t^k < \infty \tag{3.466}$$

holds. This is guaranteed by the last lemma. □

Lastly in this chapter, we will investigate the commutation properties of $\mathrm{d}\Gamma(B)$ with general creation and annihilation operators. These properties are the reason we are interested in this operator, they will prove to be very useful in the next chapter.

Theorem 66. *For* $\psi \in \mathcal{H}$ *we have*

$$[\mathrm{d}\Gamma(B), a^{\#}(\psi)] = a^{\#}(B\psi), \tag{3.467}$$

where $a^{\#}$ *can be either* a *or* a^**.*

Proof. Because $\mathrm{d}\Gamma(B)$ is defined as the extension of an operator on $\mathcal{F}^0$ it suffices to show the desired identity on this space. In order to do so we first restrict $\psi \in \mathrm{span}\{\varphi_n | n \in \mathbb{Z}\backslash\{0\}\}$. We will first cover the case $a(\psi)$. As a first step we decompose $\mathrm{d}\Gamma(B)$ into its four parts

$$[\mathrm{d}\Gamma(B), a(\psi)] = [\mathrm{d}\Gamma(B_{++}) + \mathrm{d}\Gamma(B_{-+}) + \mathrm{d}\Gamma(B_{-+}) + \mathrm{d}\Gamma(B_{--}), a(\psi)]\,, \tag{3.468}$$

each of those parts is evaluated directly. We begin with the B_{++} part, this can be expressed as

$$[\mathrm{d}\Gamma(B_{++}), a(\psi)] \tag{3.469}$$
$$= \sum_{n\in\mathbb{N}} a^*(B_{++}\varphi_n)a(\varphi_n)a(\psi) - \sum_{n\in\mathbb{N}} a(\psi)a^*(B_{++}\varphi_n)a(\varphi_n) \tag{3.470}$$
$$= \sum_{n\in\mathbb{N}} [-\langle\psi, B_{++}\varphi_n\rangle a(\varphi_n) + a(\psi)a^*(B_{++}\varphi_n)a(\varphi_n)] \tag{3.471}$$
$$- \sum_{n\in\mathbb{N}} a(\psi)a^*(B_{++}\varphi_n)a(\varphi_n). \tag{3.472}$$

Let $\alpha \in \mathcal{F}^0$. Now applying the expression in the last two lines to α online finitely many elements of the sum in fact contribute. So we may split the firs sum into two and observe the cancellation between the last two terms. Continuing we find

$$[\mathrm{d}\Gamma(B_{++}), a(\psi)]\,\alpha = -\sum_{n\in\mathbb{N}}\langle\psi, B_{++}\varphi_n\rangle a(\varphi_n)\alpha \tag{3.473}$$
$$= -a\left(\sum_{n\in\mathbb{N}}\langle B_{++}\varphi_n, \psi\rangle\varphi_n\right)\alpha = a\left(\sum_{n\in\mathbb{N}}\langle\varphi_n, B_{++}\psi\rangle\varphi_n\right)\alpha \tag{3.474}$$
$$= a(B_{++}\psi)\alpha, \tag{3.475}$$

where we used $B^* = -B^*$. Now for a general $\psi \in \mathcal{H}$ we pick a sequence $(\psi_k)_{k\in\mathbb{N}} \subset \operatorname{span}\{\varphi_n | n \in \mathbb{Z}\backslash\{0\}\}$ such that $\lim_{k\to\infty}\psi_k = \psi$. Now because of the calculation we have the equality

$$[\mathrm{d}\Gamma(B_{++}), a(\psi_k)]\, \alpha = a(B_{++}\psi_k)\alpha \tag{3.476}$$

for each $k \in \mathbb{N}$ and hence if a limit exists it also holds in the limit. Now on the right hand side, because a is a bounded operator on all of $\mathcal{F}$ clearly the limit exists and is equal to $a(B_{++}\psi)\alpha$. On the left hand side we know $\mathrm{d}\Gamma(B)$ to be bounded and hence continuous on every $\mathcal{F}_{m,p}$ for every $m, p \in \mathbb{N}$. Furhtermore since $\alpha \in \mathcal{F}^0$ there is are $m', p' \in \mathbb{N}$ such that $\alpha \in \bigoplus_{m=0}^{m'} \bigoplus_{p=0}^{p'} \mathcal{F}_{m,p}$ holds and hence we can exchange the limit also with $\mathrm{d}\Gamma(B)$ and find

$$[\mathrm{d}\Gamma(B_{++}), a(\psi)]\, \alpha = a(B_{++}\psi)\alpha \tag{3.477}$$

for general $\psi \in \mathcal{H}$. The final extension of this equation to all $\alpha \in \mathcal{F}'$ happens via the continuous linear extension theorem on $\mathcal{F}_{m,p}$ for each $m, p \in \mathbb{N}$. The proof in all seven other cases are completely analogous. Putting thins together again we obtain

$$\begin{aligned} [\mathrm{d}\Gamma(B_{++}), a(\psi)] + [\mathrm{d}\Gamma(B_{-+}), a(\psi)] \quad &(3.478)\\ + [\mathrm{d}\Gamma(B_{+-}), a(\psi)] + [\mathrm{d}\Gamma(B_{--}), a(\psi)] = \quad &(3.479)\\ a(B_{++}\psi) + a(B_{+-}\psi) + a(B_{-+}\psi) + a(B_{--}\psi) \iff \quad &(3.480)\\ [\mathrm{d}\Gamma(B), a(\psi)] = a(B\psi) \quad &(3.481) \end{aligned}$$

on all of $\mathcal{F}'$.

□

3.5.4 Presentation and Proof of the Formula

In this chapter we verify the formula for the S-matrix directly. For a heuristic derivation in the appendix in section3.5.5

Theorem 67. *For A such that*

$$\|\mathbb{1} - U^A\| < 1. \tag{3.482}$$

The second quantized scattering operator fulfils

$$S^A = e^{i\varphi^A} e^{\mathrm{d}\Gamma(\ln(U^A))} \tag{3.483}$$

for some phase $\varphi^A \in \mathbb{R}$, that may depend on the external field A.

Proof. In order to establish this theorem we need to verify that the expression given in (67) for the scattering operator is a well defined object and fulfils (lift condition) and (adjoint lift condition). Because these conditions uniquely fix the Operator S^A up to a phase this suffices as a proof.
Well definedness is established, by theorem 65, because for unitary U^A with $\|1 - U^A\| < 1$ the power series of the logarithm converges and fulfils

$$\|\ln(U^A)\| = \|\ln(1 - (1 - U^A))\| = \left\| - \sum_{k=1}^{\infty} \frac{(1 - U^A)^k}{k} \right\| \tag{3.484}$$

$$\leqslant \sum_{k=1}^{\infty} \frac{\|1 - U^A\|^k}{k} = -\ln(1 - \|1 - U^A\|) \tag{3.485}$$

implying that the power series of the logarithm around the identity is a well defined map from the one particle operators of norm less than one to the bounded one particle operators. Moreover this operator fulfils $[\ln(U^A)]^* = \ln(U^A)^* = \ln(U^A)^{-1} = -\ln(U^A)$, so $\mathrm{d}\Gamma(\ln U^A)$ is a well defined unbounded operator that is essentially self adjoint on the finite particle sector of Fockspace $\mathcal{F}'$.

Let $\varphi \in \mathcal{H}$, for any $k \in \mathbb{N}_0$ we see applying the commutation relation of $\mathrm{d}\Gamma$:

$$
\begin{aligned}
\mathrm{d}\Gamma(\ln U) \sum_{l=0}^{k} \binom{k}{l} a^{\#}\left((\ln U)^{l}\varphi\right)(\mathrm{d}\Gamma(\ln U))^{k-l} = \\
\sum_{l=0}^{k} \binom{k}{l} a^{\#}\left((\ln U)^{l+1}\varphi\right)(\mathrm{d}\Gamma(\ln U))^{k-l} \\
+ \sum_{l=0}^{k} \binom{k}{l} a^{\#}\left((\ln U)^{l}\varphi\right)(\mathrm{d}\Gamma(\ln U))^{k-l+1} \\
= \sum_{b=0}^{k+1} \left(\binom{k}{b-1} + \binom{k}{b}\right) a^{\#}\left((\ln U)^{b}\varphi\right)(\mathrm{d}\Gamma(\ln U))^{k+1-b} \\
= \sum_{b=0}^{k+1} \binom{k+1}{b} a^{\#}\left((\ln U)^{b}\varphi\right)(\mathrm{d}\Gamma(\ln U))^{k+1-b},
\end{aligned}
$$

so we see that for $k \in \mathbb{N}_0$

$$
(\mathrm{d}\Gamma(\ln U))^{k} a^{\#}(\varphi) = \sum_{b=0}^{k} \binom{k}{b} a^{\#}\left((\ln U)^{b}\varphi\right)(\mathrm{d}\Gamma(\ln U))^{k-b} \tag{3.486}
$$

holds. Let $\alpha \in \mathcal{F}'$. Using what we just obtained, we conclude

$$
\begin{aligned}
e^{\mathrm{d}\Gamma(\ln U)} a^{\#}(\varphi) &= \sum_{k=0}^{\infty} \frac{1}{k!} (\mathrm{d}\Gamma(\ln U))^{k} a^{\#}(\varphi)\alpha \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{b=0}^{k} \binom{k}{b} a^{\#}\left((\ln U)^{b}\varphi\right)(\mathrm{d}\Gamma(\ln U))^{k-b} \alpha \\
&\overset{*}{=} \sum_{c=0}^{\infty} \sum_{l=0}^{\infty} \frac{1}{c!l!} a^{\#}\left((\ln U)^{c}\varphi\right)(\mathrm{d}\Gamma(\ln U))^{l} \alpha \\
&= a^{\#}\left(e^{\ln U}\varphi\right) e^{\mathrm{d}\Gamma(\ln U)}\alpha = a^{\#}(U\varphi)\, e^{\mathrm{d}\Gamma(\ln U)}\alpha.
\end{aligned}
$$

For the marked equality changing oder of summation is justified, because by the bounds $\|a^{\#}((\ln U)^c\varphi)\| \leqslant \|\ln U\|^c$ and lemma 64 the sum obtained by changing the order of summands converges absolutely. Clearly multiplying the second quantised operator by an additional phase as in (67) does not influence this calculation. $\square$

Todo: decide whether to use or delete this part.

As a preparation for calculating the vacuum polarisation current we proof the following

Lemma 68. *Let $P_k, P_l \in Q$ then the following holds*

$$[G(P_k), G(P_l)] = \operatorname{tr}\left(P_{\substack{k\\-+}} P_{\substack{l\\+-}}\right) - \operatorname{tr}\left(P_{\substack{l\\-+}} P_{\substack{k\\+-}}\right) + G\left([P_k, P_l]\right). \tag{3.487}$$

For a proof of this lemma let $P_k, P_l \in Q$, we compute

$$\begin{aligned}
&[G(P_k), G(P_l)] \overset{(??)}{=} \\
&= \sum_{n,b\in\mathbb{N}} \left[a^*\left(P_k\varphi_n\right) a(\varphi_n), a^*\left(P_l\varphi_b\right) a(\varphi_n)\right] \\
&- \sum_{-b,n\in\mathbb{N}} \left[a^*\left(P_k\varphi_n\right) a(\varphi_n), a(\varphi_b) a^*\left(P_l\varphi_b\right)\right] \\
&- \sum_{-n,b\in\mathbb{N}} \left[a(\varphi_n) a^*\left(P_k\varphi_n\right), a^*\left(P_l\varphi_b\right) a(\varphi_b)\right] \\
&+ \sum_{n,b\in\mathbb{N}} \left[a(\varphi_n) a^*\left(P_k\varphi_n\right), a(\varphi_b) a^*\left(P_l\varphi_b\right)\right] \\
&= \sum_{n,b\in\mathbb{N}} \left(a^*\left(P_k\varphi_n\right)\langle\varphi_n, P_l\varphi_b\rangle a(\varphi_b) - a^*\left(P_l\varphi_b\right)\langle\varphi_b, P_k\varphi_n\rangle a(\varphi_n)\right) \\
&- \sum_{-b,n\in\mathbb{N}} \left(-\langle\varphi_b, P_k\varphi_n\rangle a(\varphi_n) a^*\left(P_l\varphi_b\right) + a(\varphi_b) a^*\left(P_k\varphi_n\right)\langle\varphi_n, P_l\varphi_b\rangle\right) \\
&- \sum_{-n,b\in\mathbb{N}} \left(-\langle\varphi_n, P_l\varphi_b\rangle a^*\left(P_k\varphi_n\right) a(\varphi_b) + a^*\left(P_l\varphi_b\right) a(\varphi_n)\langle\varphi_b, P_k\varphi_n\rangle\right)
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{n,b\in-\mathbb{N}} \left(a(\varphi_n)\left\langle \varphi_b, P_k\varphi_n\right\rangle a^*\left(P_l\varphi_b\right) - a(\varphi_b)\left\langle \varphi_n, P_l\varphi_b\right\rangle a^*\left(P_k\varphi_n\right)\right)\\
&= \sum_{b\in\mathbb{N}} a^*\left(P_kP_{\substack{l\\++}}\varphi_b\right)a(\varphi_b) - \sum_{n\in\mathbb{N}} a^*\left(P_lP_{\substack{k\\++}}\varphi_n\right)a(\varphi_n)\\
&+ \sum_{n\in\mathbb{N}} a(\varphi_n)a^*\left(P_lP_{\substack{k\\-+}}\varphi_n\right) - \sum_{-b\in\mathbb{N}} a(\varphi_b)a^*\left(P_kP_{\substack{l\\+-}}\varphi_b\right)\\
&+ \sum_{b\in\mathbb{N}} a^*\left(P_kP_{\substack{l\\-+}}\varphi_b\right)a(\varphi_b) - \sum_{-n\in\mathbb{N}} a^*\left(P_lP_{\substack{k\\+-}}\varphi_n\right)a(\varphi_n)\\
&+ \sum_{-n\in\mathbb{N}} a(\varphi_n)a^*\left(P_lP_{\substack{k\\--}}\varphi_n\right) - \sum_{-b\in\mathbb{N}} a(\varphi_b)a^*\left(P_kP_{\substack{l\\--}}\varphi_b\right)\\
={}& \sum_{n\in\mathbb{N}} a^*\left(P_kP_l\varphi_n\right)a(\varphi_n) - \sum_{n\in\mathbb{N}} a^*\left(P_lP_{\substack{k\\++}}\varphi_n\right)a(\varphi_n)\\
&+ \operatorname{tr}\left(P_{\substack{l\\+-}}P_{\substack{k\\-+}}\right) - \sum_{n\in\mathbb{N}} a^*\left(P_lP_{\substack{k\\-+}}\varphi_n\right)a(\varphi_n)\\
&- \operatorname{tr}\left(P_{\substack{l\\-+}}P_{\substack{k\\+-}}\right) + \sum_{-b\in\mathbb{N}} a(\varphi_b)a^*\left(P_lP_{\substack{k\\+-}}\varphi_b\right)\\
&+ \sum_{-b\in\mathbb{N}} a(\varphi_b)a^*\left(P_lP_{\substack{k\\--}}\varphi_b\right) - \sum_{-b\in\mathbb{N}} a(\varphi_b)a^*\left(P_kP_l\varphi_b\right)\\
&= \operatorname{tr}\left(P_{\substack{l\\+-}}P_{\substack{k\\-+}}\right) - \operatorname{tr}\left(P_{\substack{l\\-+}}P_{\substack{k\\+-}}\right)\\
&+ \sum_{n\in\mathbb{N}} a^*\left(\left[P_k, P_l\right]\varphi_n\right)a(\varphi_n) + \sum_{-b\in\mathbb{N}} a(\varphi_b)a^*\left(\left[P_l, P_k\right]\varphi_b\right)\\
&= \operatorname{tr}\left(P_{\substack{l\\+-}}P_{\substack{k\\-+}}\right) - \operatorname{tr}\left(P_{\substack{l\\-+}}P_{\substack{k\\+-}}\right) + G\left(\left[P_k, P_l\right]\right)
\end{aligned}
$$

$\square$

Definition 69. *For $k \in \mathbb{N}_0$, $X, Y \in \mathcal{B}(\mathcal{H})$ the nested commutator $[X,Y]_k$ is defined inductively as*

$$[X,Y]_0 := Y$$

$$[X,Y]_{k+1} := [X,[X,Y]_k] \quad \forall k \in \mathbb{N}_0.$$

Lemma 70. *For $m \in \mathbb{N}$ and $B, C \in Q$ the following holds*

$$[G(B),G(C)]_m = \operatorname{tr}\left(P_- B P_+ [B,C]_{m-1}\right) - \operatorname{tr}\left(P_+ B P_- [B,C]_{m-1}\right) \\ + G\left([B,C]_m\right). \quad (3.488)$$

Proof: Proof by Induction is the first thing that comes to mind, looking at the claim. Indeed, $m = 1$ is the consequence of the lemma 68. For m general we have

$$\begin{aligned}
[G(B),G(C)]_{m+1} &= [G(B),[G(B),G(C)]_m] \\
&\overset{\text{ind.hyp.}}{=} [G(B), \operatorname{tr}\left(P_- B P_+ [B,C]_{m-1}\right) - \operatorname{tr}\left(P_+ B P_- [B,C]_{m-1}\right) + G\left([B,C]_m\right)] \\
&= [G(B), G\left([B,C]_m\right)] \\
&\overset{\text{lemma 68}}{=} \operatorname{tr}\left(P_- B P_+ [B,C]_m\right) - \operatorname{tr}\left(P_+ B P_- [B,C]_m\right) + G\left([B,[B,C]_m]\right) \\
&= \operatorname{tr}\left(P_- B P_+ [B,C]_m\right) - \operatorname{tr}\left(P_+ B P_- [B,C]_m\right) + G\left([B,C]_{m+1}\right)
\end{aligned} \quad (3.489)$$

□

Lemma 71. *For external potentials A, X small enough the derivatives of the scattering operator can be computed to fulfil*

$$\partial_\varepsilon|_{\varepsilon=0} e^{G \ln U^{A+\varepsilon X}} = e^{G \ln U^A} j_A^0(X) + e^{G \ln U^A} G((U^A)^{-1} \partial_\varepsilon U^{A+\varepsilon X}) \quad (3.490)$$

$$\partial_\varepsilon|_{\varepsilon=0} e^{-G \ln U^{A+\varepsilon X}} = -e^{-G \ln U^A} j_A^0(X) + G(\partial_\varepsilon \left(U^{A+\varepsilon X}\right)^{-1} U^A) e^{-G \ln U^A}, \quad (3.491)$$

with

$$j_A^0(X) := \sum_{l \in \mathbb{N}_0} \frac{(-1)^{l+1}}{(l+2)!} \Big(\operatorname{tr} P_- \ln U^A P_+ \left[\ln U^A, \partial_\varepsilon \ln U^{A+\varepsilon X}\right]_l \\ - \operatorname{tr} P_+ \ln U^A P_- \left[\ln U^A, \partial_\varepsilon \ln U^{A+\varepsilon X}\right]_l \Big). \quad (3.492)$$

Proof: We start out by employing Duhamel's and Hadamard's formulas. These are

ref!! + restrictions, something better than this

$$\partial_\alpha \, e^{Y+\alpha X}\Big|_{\alpha=0} = \int_0^1 dt e^{(1-t)Y} X e^{tY} \qquad \text{(Duhamel's formula)}$$

and

$$e^X Y e^{-X} = \sum_{k=0}^{\infty} \frac{1}{k!}[X,Y]_k. \qquad \text{(Hadamard's formula)}$$

So one gets

$$\begin{aligned}\partial_\varepsilon|_{\varepsilon=0} e^{G \ln U^{A+\varepsilon X}} &= \int_0^1 dz e^{(1-z)G\ln U^A} \partial_\varepsilon|_{\varepsilon=0} G \ln U^{A+\varepsilon X} e^{zG\ln U^A} \qquad (3.493)\\ &= e^{G\ln U^A} \int_0^1 dz \sum_{l\in\mathbb{N}_0} \frac{1}{l!}\left[-zG\ln U^A, \partial_\varepsilon|_{\varepsilon=0} G \ln U^{A+\varepsilon X}\right]_l \\ &= e^{G\ln U^A} \int_0^1 dz \sum_{l\in\mathbb{N}_0} \frac{(-z)^l}{l!} \partial_\varepsilon|_{\varepsilon=0}\left[G\ln U^A, G\ln U^{A+\varepsilon X}\right]_l.\end{aligned}$$

At this point we see that for $l = 0$ the summand vanishes. For all other values of l we use lemma 70, yielding

$$\begin{aligned}\partial_\varepsilon|_{\varepsilon=0} e^{G\ln U^{A+\varepsilon X}} &= e^{G\ln U^A} \int_0^1 dz \sum_{l\in\mathbb{N}} \frac{(-z)^l}{l!} \partial_\varepsilon|_{\varepsilon=0} \left(G\left(\left[\ln U^A, \ln U^{A+\varepsilon X}\right]\right)\right.\\ &+ \operatorname{tr} P_- \ln U^A P_+ \left[\ln U^A, \partial_\varepsilon|_{\varepsilon=0} \ln U^{A+\varepsilon X}\right]_{l-1} \\ &\left. - \operatorname{tr} P_+ \ln U^A P_- \left[\ln U^A, \partial_\varepsilon|_{\varepsilon=0} \ln U^{A+\varepsilon X}\right]_{l-1}\right). \qquad (3.494)\end{aligned}$$

The last two terms together result in the first term of (3.490) after performing the integration and shifting the summation index. For the first term we will use linearity and continuity of G and use the same identities backwards to give

contiuity of G!!

$$
\begin{aligned}
&e^{G\ln U^A}\int_0^1 dz \sum_{l\in\mathbb{N}} \frac{(-z)^l}{l!}\partial_\varepsilon|_{\varepsilon=0} G\left(\left[\ln U^A, \ln U^{A+\varepsilon X}\right]\right)\\
&\qquad = e^{G\ln U^A} G\left(\left(\int_0^1 dz \sum_{l\in\mathbb{N}} \frac{1}{l!}\left[-z\ln U^A, \partial_\varepsilon|_{\varepsilon=0}\ln U^{A+\varepsilon X}\right]\right)\right)\\
&\qquad = e^{G\ln U^A} G\left(e^{-\ln U^A}\int_0^1 dz e^{\ln U^A} e^{-z\ln U^A}\partial_\varepsilon|_{\varepsilon=0}\ln U^{A+\varepsilon X} e^{z\ln U^A}\right)\\
&\qquad = e^{G\ln U^A} G\left(e^{\ln\left(U^A\right)^{-1}}\partial_\varepsilon|_{\varepsilon=0} e^{\ln U^{A+\varepsilon X}}\right)\\
&\qquad = e^{G\ln U^A} G\left(\left(U^A\right)^{-1}\partial_\varepsilon|_{\varepsilon=0} U^{A+\varepsilon X}\right). \qquad (3.495)
\end{aligned}
$$

Putting things together results in the first equality we wanted to prove. For the second one the computation is completely analogous, except for after applying Duhamel's formula as in (3.493) we substitute $u = 1-z$. The minus sign in front of the first term then arises by the chain rule, where as the second term does not share the sign change with the first, since we have to revert the use of the chain rule in the second half of the calculation when we apply (Duhamel's formula) backwards. □

Definition 72. *We use Bogoliubov's formula to define the vacuum*
ref!!
ectation value of the current

$$j_A(F) = i\partial_\varepsilon \left.\left\langle \Omega, S^{A^*} S^{A+\varepsilon F}\Omega\right\rangle\right|_{\varepsilon=0}. \qquad (3.496)$$

Theorem 73. *The vacuum expectation value of the current of the* *scattering operator takes the form*

$$j_A(F) = -\partial_\varepsilon\, \varphi(A+\varepsilon F)\big|_{\varepsilon=0}$$
$$-2\int_0^1 dz(1-z)\Im\,\mathrm{tr}\left(P_+ \ln U^A P_- e^{-z\ln U^A}\partial_\varepsilon \ln U^{A+\varepsilon F}\big|_{\varepsilon=0} e^{z\ln U^A}\right)$$
$$= -\partial_\varepsilon\, \varphi(A+\varepsilon F)\big|_{\varepsilon=0}$$
$$-2\Im\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+2)!}\,\mathrm{tr}\left(P_+\ln U^A P_-\left[\ln U^A,\, \partial_\varepsilon \ln U^{A+\varepsilon F}\big|_{\varepsilon=0}\right]_k\right)$$

Proof: By theorem 67 and abbreviating $\varphi(A) = \sum_{n\in\mathbb{N}}\frac{C_n(A)}{n!}$ we see that the current can be written in the form

$$\begin{aligned}
j_A(F) &= i\partial_\varepsilon \left\langle \Omega, S^{A^*}S^{A+\varepsilon F}\Omega\right\rangle\Big|_{\varepsilon=0}\\
&= i\partial_\varepsilon \left\langle \Omega, e^{-i\varphi(A)}e^{-G(\ln(U^A))}e^{i\varphi(A+\varepsilon F)}e^{G(\ln(U^{A+\varepsilon F}))}\Omega\right\rangle\Big|_{\varepsilon=0}\\
&= -\partial_\varepsilon\, \varphi(A+\varepsilon F)\big|_{\varepsilon=0} + i\langle\Omega, \partial_\varepsilon\, e^{-G(\ln(U^A))}e^{G(\ln(U^{A+\varepsilon F}))}\Omega\rangle\Big|_{\varepsilon=0},
\end{aligned} \tag{3.497}$$

so the first summand works out just as claimed. For the second summand we employ lemma 71 and note that the vacuum expectation value of G vanishes no matter its argument.

$$\begin{aligned}
&i\langle\Omega, \partial_\varepsilon\, e^{-G(\ln(U^A))}e^{G(\ln(U^{A+\varepsilon F}))}\Omega\rangle\Big|_{\varepsilon=0}\\
&\quad= -i\partial_\varepsilon \sum_{k=0}^{\infty}\frac{(-1)^k}{(k+2)!}\,\mathrm{tr}\left(P_-\ln U^A P_+[\ln U^A, \ln U^{A+\varepsilon F}]_k\right)\Big|_{\varepsilon=0}\\
&\quad\quad + i\partial_\varepsilon \sum_{k=0}^{\infty}\frac{(-1)^k}{(k+2)!}\,\mathrm{tr}\left(P_+\ln U^A P_-[\ln U^A, \ln U^{A+\varepsilon F}]_k\right)\Big|_{\varepsilon=0}
\end{aligned} \tag{3.498}$$

In order to apply Hadamard's formula once again in the opposite direction, we introduce two auxiliary integrals. The second term then becomes

$$
\begin{aligned}
& i\partial_\varepsilon \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+2)!} \operatorname{tr}\left(P_+ \ln U^A P_- [\ln U^A, \ln U^{A+\varepsilon F}]_k\right)\Big|_{\varepsilon=0} \\
&= i\partial_\varepsilon \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_0^1 dt \int_0^1 s^k t^{k+1} \operatorname{tr}\left(P_+ \ln U^A P_- [\ln U^A, \ln U^{A+\varepsilon F}]_k\right)\Big|_{\varepsilon=0} \\
&= i\partial_\varepsilon \sum_{k=0}^{\infty} \frac{1}{k!} \int_0^1 dt \int_0^1 ds\; t \operatorname{tr}\left(P_+ \ln U^A P_- [-ts \ln U^A, \ln U^{A+\varepsilon F}]_k\right)\Big|_{\varepsilon=0} \\
&= i\partial_\varepsilon \sum_{k=0}^{\infty} \frac{1}{k!} \int_0^1 dz \int_z^1 ds\; \operatorname{tr}\left(P_+ \ln U^A P_- [-z \ln U^A, \ln U^{A+\varepsilon F}]_k\right)\Big|_{\varepsilon=0} \\
&= i\partial_\varepsilon \sum_{k=0}^{\infty} \frac{1}{k!} \int_0^1 dz (1-z) \operatorname{tr}\left(P_+ \ln U^A P_- [-z \ln U^A, \ln U^{A+\varepsilon F}]_k\right)\Big|_{\varepsilon=0} \\
&= i\partial_\varepsilon \int_0^1 dz (1-z) \operatorname{tr}\left(P_+ \ln U^A P_- \sum_{k=0}^{\infty} \frac{1}{k!} [-z \ln U^A, \ln U^{A+\varepsilon F}]_k\right)\Big|_{\varepsilon=0} \\
&\overset{\text{(Hadamard's formula)}}{=} i\partial_\varepsilon \int_0^1 dz (1-z) \operatorname{tr}\left(P_+ \ln U^A P_- e^{-z \ln U^A} \ln U^{A+\varepsilon F} e^{z \ln U^A}\right)\Big|_{\varepsilon=0} \\
&= i \int_0^1 dz (1-z) \operatorname{tr}\left(P_+ \ln U^A P_- e^{-z \ln U^A} \partial_\varepsilon \ln U^{A+\varepsilon F}\big|_{\varepsilon=0} e^{z \ln U^A}\right).
\end{aligned}
\tag{3.499}
$$

The calculation for the first term of (3.498) is identical. At this point we notice that (3.499) and the term where the projectors are exchanged are complex conjugates of one another. So summarising we find

$$
\begin{aligned}
&j_A(F) = -\partial_\varepsilon \, \varphi(A+\varepsilon F)|_{\varepsilon=0} \\
&-2\int_0^1 dz(1-z)\Im \operatorname{tr}\left(P_+ \ln U^A P_- e^{-z\ln U^A} \partial_\varepsilon \ln U^{A+\varepsilon F}\big|_{\varepsilon=0} e^{z\ln U^A}\right).
\end{aligned}
$$

□

Theorem 74. *Independent of the phase that is used to correct the scattering operator the following formula holds true for any four potentials A, F, H, with A small enough so that the relevant series converge.*

$$
\begin{aligned}
&\partial_\varepsilon|_{\varepsilon=0}(j_{A+\varepsilon H}(F) - j_{A+\varepsilon F}(H)) = \\
&\quad 2\Im \operatorname{tr}\left(P_+ \left(U^A\right)^{-1} \partial_\varepsilon|_{\varepsilon=0} U^{A+\varepsilon F} P_- \left(U^A\right)^{-1} \partial_\delta|_{\delta=0} U^{A+\delta H}\right) \qquad (3.500)
\end{aligned}
$$

Proof: We compute $\partial_\varepsilon|_{\varepsilon=0} j_{A+\varepsilon F}(H)$.

$$
\begin{aligned}
&-i\partial_\varepsilon|_{\varepsilon=0} j_{A+\varepsilon H}(F) = \\
&\partial_\varepsilon|_{\varepsilon=0}\partial_\delta|_{\delta=0}\langle \Omega, e^{i\varphi(A+\varepsilon H+\delta F)-i\varphi(A+\varepsilon H)} e^{-G\ln U^{A+\varepsilon H}} e^{G\ln U^{A+\varepsilon H+\delta F}} \Omega\rangle
\end{aligned}
$$

We first act with the derivative with respect to H, fixing F.

$$
\begin{aligned}
&-i\partial_\varepsilon|_{\varepsilon=0} j_{A+\varepsilon H}(F) = \\
&\partial_\delta|_{\delta=0} i(\partial_\varepsilon|_{\varepsilon=0}\varphi(A+\varepsilon H+\delta F) - \partial_\varepsilon|_{\varepsilon=0}\varphi(A+\varepsilon H)) e^{i\varphi(A+\delta F)-i\varphi(A)} \\
&\langle \Omega, e^{-G\ln U^A} e^{G\ln U^{A+\delta F}} \Omega\rangle \\
&+\partial_\delta|_{\delta=0} e^{i\varphi(A+\delta F)-i\varphi(A)} \langle \Omega \partial_\varepsilon|_{\varepsilon=0} e^{-G\ln U^{A+\varepsilon H}} e^{G\ln U^{A+\delta F}} \Omega\rangle \\
&+\partial_\delta|_{\delta=0} e^{i\varphi(A+\delta F)-i\varphi(A)} \langle \Omega e^{-G\ln U^A} \partial_\varepsilon|_{\varepsilon=0} e^{G\ln U^{A+\varepsilon H+\delta F}} \Omega\rangle
\end{aligned}
$$

In computing further one can notice a few cancellations. For the first summand the first factor vanishes if δ is set to zero, so the only the first summand in the product rule will not vanish. For the second and third summand we will use lemma 71, giving

$$
\begin{aligned}
&- i\partial_\varepsilon|_{\varepsilon=0} j_{A+\varepsilon H}(F) = \\
&\partial_\delta|_{\delta=0} i\partial_\varepsilon|_{\varepsilon=0}\varphi(A+\varepsilon H+\delta F) \\
&- \partial_\delta|_{\delta=0} e^{i\varphi(A+\delta F)-i\varphi(A)} j^0_A(H)\langle\Omega, e^{-G\ln U^A} e^{G\ln U^{A+\delta F}}\Omega\rangle \\
&+ \partial_\delta|_{\delta=0} e^{i\varphi(A+\delta F)-i\varphi(A)}\langle\Omega, G\left(\partial_\varepsilon|_{\varepsilon=0}\left(U^{A+\varepsilon H}\right)^{-1} U^A\right) e^{-G\ln U^A} e^{G\ln U^{A+\delta F}}\Omega\rangle \\
&+ \partial_\delta|_{\delta=0} e^{i\varphi(A+\delta F)-i\varphi(A)} j^0_{A+\delta F}(H)\langle\Omega, e^{-G\ln U^A} e^{G\ln U^{A+\delta F}}\Omega\rangle \\
&+ \partial_\delta|_{\delta=0} e^{i\varphi(A+\delta F)-i\varphi(A)}\langle\Omega, e^{-G\ln U^A} e^{G\ln U^{A+\delta F}} G\left(\left(U^{A+\delta F}\right)^{-1}\partial_\varepsilon|_{\varepsilon=0} U^{A+\varepsilon H+\delta F}\right)\Omega\rangle.
\end{aligned}
$$

Now there are a few further simplifications to appreciate: since $\langle\Omega, G\Omega\rangle = 0$, in the third and last summand only the derivatives with respect to δ which produce by lemma 71 another factor of G will contribute to the sum. For the other summands except for the first we can spot the appearance of j^0. Respecting all this results in

$$
\begin{aligned}
&- i\partial_\varepsilon|_{\varepsilon=0} j_{A+\varepsilon H}(F) = i\partial_\delta|_{\delta=0}\partial_\varepsilon|_{\varepsilon=0}\varphi(A+\varepsilon H+\delta F) \\
&- i\partial_\delta|_{\delta=0}\varphi(A+\delta F) j^0_A(H) - j^0_A(H) j^0_A(F) \\
&+ \langle\Omega, G\left(\partial_\varepsilon|_{\varepsilon=0}\left(U^{A+\varepsilon H}\right)^{-1} U^A\right) G\left(\left(U^A\right)^{-1}\partial_\delta|_{\delta=0} U^{A+\delta F}\right)\Omega\rangle \\
&+ i\partial_\delta|_{\delta=0}\varphi(A+\delta F) j^0_A(H) + \partial_\delta|_{\delta=0} j^0_{A+\delta F}(H) + j^0_A(H) j^0_A(F) \\
&+ \langle\Omega, G\left(\left(U^A\right)^{-1}\partial_\delta|_{\delta=0} U^{A+\delta F}\right) G\left(\left(U^A\right)^{-1}\partial_\varepsilon|_{\varepsilon=0} U^{A+\varepsilon H}\right)\Omega\rangle.
\end{aligned}
$$

A few more terms cancel in the second and fourth line, also since $\partial_\varepsilon|_{\varepsilon=0}\left(U^{A+\varepsilon H}\right)^{-1} U^{A+\varepsilon H} = 0$ we can combine the two products of G into a commutator:

$$
\begin{aligned}
&-i\partial_\varepsilon|_{\varepsilon=0} j_{A+\varepsilon H}(F) = i\partial_\delta|_{\delta=0}\partial_\varepsilon|_{\varepsilon=0}\varphi(A+\varepsilon H+\delta F)\\
&+\partial_\delta|_{\delta=0} j^0_{A+\delta F}(H)\\
&+\langle\Omega, \left[G\left(\left(U^A\right)^{-1}\partial_\delta|_{\delta=0}U^{A+\delta F}\right), G\left(\left(U^A\right)^{-1}\partial_\varepsilon|_{\varepsilon=0}U^{A+\varepsilon H}\right)\right]\Omega\rangle.
\end{aligned}
$$

So we can once again apply lemma 68, which results in exactly right hand side of the equation we claimed to produce in the statement of this theorem. So all that is left is to recognise that one can combine the first two summands into $-i\partial_\varepsilon j_{A+\varepsilon H}(F)$, which is a direct consequence of theorem 73. □

3.5.5 Quantitative Estimates

Since we do not only want to give an expression for the time evolution operator, but also give bounds on the numerical errors which are due to truncate the occurring series we need to look at these series a little closer. The series involve powers of the second quantisation operator G, so we start by examining these in greater depth. In order to do so we define an object closely related to G.

Todo: probably this part cannot be made rigorous. Decide whether to keep it as heuristics

Definition 75.

$$L : \{M \subset B(\mathcal{H}) \mid |M| < \infty\} \times \{M \subset B(\mathcal{H}) \mid |M| < \infty\} \to B(\mathcal{F})$$

$$L(\{A_1,\dots,A_c\},\{B_1,\dots,B_m\}) := \prod_{l=1}^{m} a(\varphi_{-k_l})$$

$$\prod_{l=1}^{c} a^*(A_l\varphi_{n_l}) \prod_{l=1}^{m} a^*(B_l\varphi_{-k_l}) \prod_{l=1}^{c} a(\varphi_{n_l}), \quad (3.501)$$

where for notational reasons we chose to list the occurring one-particle operators in a specific order; however, the order does not matter, since *commutation of the relevant creation and annihilation operators always results an overall factor of one.*

Since this operator L occurs when computing powers of G we compute is product with some G with the following

Lemma 76. *For any $a, b, \in \mathbb{N}_0$ and apropriate one particle operators A_k, B_l, C for $1 \leqslant k \leqslant a$, $1 \leqslant l \leqslant b$ we have the following equality*

$$L\Big(\bigcup_{l=1}^{a}\{A_l\};\bigcup_{l=1}^{b}\{B_l\}\Big)G(C) = \tag{3.502}$$

$$(-1)^{a+b}L\Big(\bigcup_{l=1}^{a}\{A_l\}\cup\{C\};\bigcup_{l=1}^{b}\{B_l\}\Big) \tag{3.503}$$

$$+(-1)^{a+b+1}L\Big(\bigcup_{l=1}^{a}\{A_l\};\bigcup_{l=1}^{b}\{B_l\}\cup\{C\}\Big) \tag{3.504}$$

$$+\sum_{f=1}^{a}L\Big(\bigcup_{\substack{l=1\\ l\neq f}}^{a}\{A_l\}\cup\{A_fP_+C\};\bigcup_{l=1}^{b}\{B_l\}\Big) \tag{3.505}$$

$$+\sum_{f=1}^{a}L\Big(\bigcup_{\substack{l=1\\ f\neq l}}^{a}\{A_l\}\cup\{-CP_-A_f\};\bigcup_{l=1}^{b}\{B_l\}\Big) \tag{3.506}$$

$$-\sum_{f=1}^{a}L\Big(\bigcup_{\substack{l=1\\ f\neq l}}^{a}\{A_l\};\bigcup_{l=1}^{b}\{B_l\}\cup\{A_fP_+C\}\Big) \tag{3.507}$$

$$+\sum_{f=1}^{b}L\Big(\bigcup_{l=1}^{a}\{A_l\};\bigcup_{\substack{l=1\\ l\neq f}}^{b}\{B_l\}\cup\{-CP_-B_f\}\Big) \tag{3.508}$$

$$+(-1)^{a+b+1}\sum_{f=1}^{a}\operatorname{tr}\Big(P_+CP_-A_f\Big)L\Big(\bigcup_{\substack{l=1\\ l\neq f}}^{a}\{A_l\};\bigcup_{l=1}^{b}\{B_l\}\Big) \tag{3.509}$$

$$+(-1)^{a+b+1}\sum_{\substack{f_1,f_2=1\\ f_1\neq f_2}}^{a} L\Big(\bigcup_{\substack{l=1\\ l\neq f_1,f_2}}^{a}\{A_l\}\cup\{-A_{f_2}P_+CP_-A_{f_1}\};\bigcup_{l=1}^{b}\{B_l\}\Big) \tag{3.510}$$

$$+(-1)^{a+b+1}\sum_{f=1}^{b}\sum_{g=1}^{a} L\Big(\bigcup_{\substack{l=1\\ l\neq g}}^{a}\{A_l\};\bigcup_{\substack{l=1\\ l\neq f}}^{b}\{B_l\}\cup\{-A_gP_+CP_-B_f\}\Big). \tag{3.511}$$

Proof: The proof of this equality is a rather long calculation, where (3.501) is used repeatedly. We break up the calculation into several parts. Let us start with

$$\begin{aligned} &L\left(\bigcup_{l=1}^{a}\{A_l\};\bigcup_{l=1}^{b}\{B_l\}\right)L(C;)= \\ &\prod_{l=1}^{b}a(\varphi_{-k_l})\prod_{l=1}^{a}a^*(A_l\varphi_{n_l})\prod_{l=1}^{b}a^*(B_l\varphi_{-k_l})\prod_{l=1}^{a}a(\varphi_{n_l})a^*(C\varphi_m)a(\varphi_m). \end{aligned} \tag{3.512}$$

We (anti)commute the creation operator involving C to its place at the end of the second product, after that the term will be normally ordered and can be rephrased in terms of Ls. During the commutation the creation operator in question can be picked up by any of the annihilation operators in the rightmost product. For each term where that happens we can perform the sum over the basis of $\mathcal{H}^-$ related to the annihilation operator whose anticommutator triggered. After this sum the corresponding term is also normally ordered and can be rephrased in terms of an L after some reshuffling which may only produce signs. So performing these steps we get

$$
\begin{aligned}
L\left(\bigcup_{l=1}^{a}\{A_l\};\bigcup_{l=1}^{b}\{B_l\}\right) L(C;) = \\
\sum_{f=a}^{1}(-1)^{a-f}\prod_{l=1}^{b} a(\varphi_{-k_l})\prod_{l=1}^{f-1} a^*(A_l\varphi_{n_l})a^*(A_f P_+ C\varphi_m) \\
\prod_{l=f+1}^{a} a^*(A_l\varphi_{n_l})\prod_{l=1}^{b} a^*(B_l\varphi_{-k_l})\prod_{\substack{l=1\\ l\neq f}}^{a} a(\varphi_{n_l})a(\varphi_m) \\
+ L\left(\bigcup_{l=1}^{a}\{A_l\}\cup\{C\};\bigcup_{l=1}^{b}\{B_l\}\right) \\
= \sum_{f=1}^{a} L\left(\bigcup_{\substack{l=1\\ l\neq f}}^{a}\{A_l\}\cup\{A_f P_+ C\};\bigcup_{l=1}^{b}\{B_l\}\right) \\
+ L\left(\bigcup_{l=1}^{a}\{A_l\}\cup\{C\};\bigcup_{l=1}^{b}\{B_l\}\right). \quad (3.513)
\end{aligned}
$$

Now the remaining case is more laborious, that is why we will split off and treat some of the appearing terms separately. We start off analogous to before

$$
\begin{aligned}
L\left(\bigcup_{l=1}^{a}\{A_l\};\bigcup_{l=1}^{b}\{B_l\}\right) L(;C) = \\
\prod_{l=1}^{b} a(\varphi_{-k_l})\prod_{l=1}^{a} a^*(A_l\varphi_{n_l})\prod_{l=1}^{b} a^*(B_l\varphi_{-k_l})\prod_{l=1}^{a} a(\varphi_{n_l})a(\varphi_{-m})a^*(C\varphi_{-m}).
\end{aligned}
\tag{3.514}
$$

This time we need to (anti)commute the rightmost annihilation operator all the way to the end of the first product and the creation operator to the end of the second but last product. So there will be several qualitatively different terms. From the first step alone we get

$$L\left(\bigcup_{l=1}^{a}\{A_l\};\bigcup_{l=1}^{b}\{B_l\}\right)L(;C) =$$

$$(-1)^a\sum_{f=b}^{1}(-1)^{b-f}\prod_{l=1}^{b}a(\varphi_{-k_l})\prod_{l=1}^{a}a^*(A_l\varphi_{n_l})\prod_{\substack{l=1\\l\neq f}}^{b}a^*(B_l\varphi_{-k_l})$$
$$\prod_{l=1}^{a}a(\varphi_{n_l})a^*(CP_-B_f\varphi_{-k_f}) \tag{3.515}$$

$$+(-1)^{a+b}\sum_{f=a}^{1}(-1)^{b-f}\prod_{l=1}^{b}a(\varphi_{-k_l})\prod_{\substack{l=1\\l\neq f}}^{a}a^*(A_l\varphi_{n_l})$$
$$\prod_{l=1}^{b}a^*(B_l\varphi_{-k_l})\prod_{l=1}^{a}a(\varphi_{n_l})a^*(CP_-\varphi_{n_f}) \tag{3.516}$$

$$+(-1)^b\prod_{l=1}^{b}a(\varphi_{-k_l})a(\varphi_{-m})\prod_{l=1}^{a}a^*(A_l\varphi_{n_l})$$
$$\prod_{l=1}^{b}a^*(B_l\varphi_{-k_l})\prod_{l=1}^{a}a(\varphi_{n_l})a^*(C\varphi_{-m}). \tag{3.517}$$

We will discuss terms (3.515), (3.516) and (3.517) separately. In Term (3.515) we need to commute the last creation operator into its place in the third product, it can be picked up by one of the annihilation operators of the last product, but after performing the sum over the corresponding basis the resulting term can be rephrased in terms of an L operator by commuting only creation operators of the second and third product. Performing these steps yields the identity

$$
\begin{aligned}
(3.515) = &\sum_{f=1}^{b} L\left(\bigcup_{l=1}^{a}\{A_l\};\bigcup_{\substack{l=1\\l\neq f}}^{b}\{B_l\}\cup\{CP_-B_f\}\right)\\
&+(-1)^{a+b+1}\sum_{f=1}^{b}\sum_{g=1}^{a} L\left(\bigcup_{\substack{l=1\\l\neq g}}^{a}\{A_l\};\bigcup_{\substack{l=1\\l\neq f}}^{b}\{B_l\}\{A_gP_+CP_-B_f\}\right).
\end{aligned}
\tag{3.518}
$$

For (3.516) the last creation operator needs to be commuted to the end of the second product. It can be picked up by one of the annihilation operators of the last product, but here we have to distinguish between two cases. If the index of this annihilation operator equals f the resulting commutator will be $\operatorname{tr} P_+CP_-A_f$ otherwise one can again perform the sum over the corresponding index and express the whole Product in terms of an L operator. All this results in

$$
\begin{aligned}
(3.516) = &\sum_{f=1}^{a} L\left(\bigcup_{\substack{l=1\\l\neq f}}^{a}\{A_l\}\cup\{CP_-A_f\};\bigcup_{l=1}^{b}\{B_l\}\right)\\
&+(-1)^{a+b}\sum_{f=1}^{a} L\left(\bigcup_{\substack{l=1\\l\neq f}}^{a}\{A_l\};\bigcup_{l=1}^{b}\{B_l\}\right)\operatorname{tr}(P_+CP_-A_f)
\end{aligned}
$$

$$+ (-1)^{a+b+1} \sum_{\substack{f_1,f_2=1\\ f_1\neq f_2}}^{a} L\left(\bigcup_{\substack{l=1\\ l\neq f_1,f_2}}^{a} \{A_l\} \cup \{A_{f_2}P_+CP_-A_{f_1}\}; \bigcup_{l=1}^{b}\{B_l\} \right). \tag{3.519}$$

For (3.517) the procedure is basically the same as for (3.515), it results in

$$\begin{aligned}(3.517) = (-1)^{a+b} L\left(\bigcup_{l=1}^{a}\{A_l\}; \bigcup_{l=1}^{b}\{B_l\}\right)\\ + \sum_{f=1}^{a} L\left(\bigcup_{\substack{l=1\\ l\neq f}}^{a}\{A_l\} \cup \{CP_-A_f\}; \bigcup_{l=1}^{b}\{B_l\} \cup \{A_fP_+C\}\right).\end{aligned} \tag{3.520}$$

Putting the results of the calculation together results in the claimed equation, after pulling in some factors of -1 into L. □

We carry on with defining the important quantities for powers of G. First we introduce for each $k \in \mathbb{N}$ a linear bounded operator on $\mathcal{H}$, X_k which fulfils $\operatorname{tr} P_+X_kP_-X_k < \infty \wedge \operatorname{tr} P_-X_kP_+X_k < \infty$.

Definition 77. *Let*

$$Y := \{X_k \mid k \in \mathbb{N}\}.$$

Let for $n \in \mathbb{N}$

$$\langle n \rangle := \{X_l \mid l \in \mathbb{N}, l \leqslant n\}.$$

Definition 78. *Let for* $b \subset Y$*, such that* $|b| < \infty$

$$\begin{aligned} f_b : \{l \in \mathbb{N} \mid l \leqslant |b|\} \to b\\ \forall k < |b| : f_b(k) = X_l \wedge f_b(k+1) = X_m \to l < m \end{aligned} \tag{3.521}$$

Definition 79. *For any set b, we denote by $S(b)$ the symmetric group (group of permutations) over b.*

Definition 80. *Let for $b \subset Y$, such that $|b| < \infty$ and $\sigma_b \in S(b)$*

$$VZ^b_{\sigma_b} : \{k \in \mathbb{N} \mid k < |b|\} \to \{-1, 1\}$$
$$VZ^b_{\sigma_b}(k) := sgn\left[f_b^{-1}(\sigma_b(f_b(k+1))) - f_b^{-1}(\sigma_b(f_b(k)))\right]$$

In what is to follow the order of one particle operators will be changed in all possible ways, to keep track of this by use of a compact notation we introduce

Definition 81.

$$W : \{(b, \sigma_b) \mid b \subseteq Y \wedge |b| < \infty \wedge \sigma_b \in S(b)\} \to B(\mathcal{H})$$
$$W(b, \sigma_b) := \left(\prod_{k=1}^{|b|-1} \sigma_b(f_b(k)) P_{VZ^b_{\sigma_b}(k)} VZ^b_{\sigma_b}(k)\right) \sigma_b(f_b(|b|))$$

Definition 82. *Let l be any finite subset of Y. Denote by X^l_{max} the operator $X_k \in l$ such that for any $X_c \in l$ the relation $k \geqslant c$ is fulfilled. Furthermore define*

$$PT : \{T \subset \mathcal{P}(Y) \mid |T| < \infty, \forall b \in T : |b| < \infty\} \to \mathbb{C}$$
$$\textit{for: } T = \varnothing : PT(T) = 1, \ \textit{otherwise:}$$
$$PT(T) = \sum_{\substack{\forall l \in T: \\ \sigma_l \in S(l \backslash \{X^l_{max}\})}} \prod_{l \in T} \mathrm{tr}[P_+ X^l_{max} P_- W(l, \sigma_l)]$$

There is one more function left to define

Definition 83.

$$Op : \{R \in \mathcal{P}(Y) \mid |R| < \infty\} \times \{D \subset \mathcal{P}(Y) \mid |T| < \infty\} \to \mathcal{B}(\mathcal{F})$$
$$Op(R, D) = \sum_{\substack{\forall l \in D: \\ \sigma_l \in S(l)}} \sum_{a \subseteq R \cup \bigcup_{l \in D}\{W(l, \sigma_l)\}} L(a, a^c)(-1)^{|a| + \frac{(|R|+|D|)(|R|+|D|+1)}{2}}$$

Now we are able to state the main theorem which will help us do quantitative estimates.

Theorem 84. *Let $n \in \mathbb{N}$, $X_1, \ldots, X_n \in Y$ then the following equation holds*

$$\prod_{k=1}^{n} G(X_k) = \sum_{\substack{\langle n\rangle = \dot{\bigcup}_{l\in T} l \,\dot{\cup}\, \dot{\bigcup}_{l\in D} l \,\dot{\cup}\, R \\ \forall l \in T\cup D: |l|\geqslant 2}} PT(T)\,Op(R,D), \tag{3.522}$$

where the abbreviation $\langle n\rangle := \{X_k \mid k \leqslant n\}$ was used.

Proof: The proof will be by induction on n. Since the formula in the claim reduces to 1 for $n = 0$ we will not spend any more time on the start of the induction. The general strategy of the proof is to break up the right hand side of (3.522) for $n+1$ into small pieces and show for each piece that it corresponds to one of the contributions of lemma 76, while also each term in this lemma is represented by one of the terms obtained by breaking up (3.522).
As a first step we break the right hand side of (3.522) into three pieces separated by in which set X_{n+1} ends up in :

$$\sum_{\substack{\langle n+1\rangle = \dot{\bigcup}_{l\in T} l \,\dot{\cup}\, \dot{\bigcup}_{l\in D} l \,\dot{\cup}\, R \\ \forall l \in T\cup D: |l|\geqslant 2}} \mathrm{PT}(T)\mathrm{Op}(R,D) =$$

$$\sum_{\substack{\langle n+1\rangle = \dot{\bigcup}_{l\in T} l \,\dot{\cup}\, \dot{\bigcup}_{l\in D} l \,\dot{\cup}\, R \\ \exists l\in T: X_{n+1}\in l \\ \forall l \in T\cup D: |l|\geqslant 2}} \mathrm{PT}(T)\mathrm{Op}(R,D) \tag{3.523}$$

$$+ \sum_{\substack{\langle n+1\rangle = \dot{\bigcup}_{l\in T} l \,\dot{\cup}\, \dot{\bigcup}_{l\in D} l \,\dot{\cup}\, R \\ \exists l\in D: X_{n+1}\in l \\ \forall l \in T\cup D: |l|\geqslant 2}} \mathrm{PT}(T)\mathrm{Op}(R,D) \tag{3.524}$$

$$+ \sum_{\substack{\langle n+1\rangle = \biguplus_{l\in T} l \uplus \biguplus_{l\in D} l \uplus R \\ X_{n+1}\in R \\ \forall l\in T\cup D:|l|\geqslant 2}} \mathrm{PT}(T)\mathrm{Op}(R,D), \tag{3.525}$$

We will discuss each term separately. For term (3.523) the term containing X_{n+1} is in one of the elements l' of T, but each such element has to have more than one element. So if we were to sum over the partitions of $\langle n\rangle$ instead, the rest of $l'\backslash\{X_{n+1}\}$ is either an element of D or, if it contains only one element, of R. Picking D instead of T is at this stage an arbitrary choice, but this choice leads to the terms of lemma 76. Alls this means that one correct rewriting of term (3.523) is

$$\begin{aligned}(3.523) = &\sum_{\substack{\langle n\rangle = \biguplus_{l\in T} l \uplus \biguplus_{l\in D} l \uplus R \\ \forall l\in D\cup T:|l|>2}} \sum_{b\in D\cup\{\{r\}|r\in R\}} \\ &\qquad \mathrm{PT}(T\cup\{\{X_{n+1}\cup f\}\})\mathrm{Op}(R\backslash b, D\backslash\{b\}).\end{aligned} \tag{3.526}$$

Next we pull one factor and the corresponding sum out of PT and write out Op. Then we see that the sums over permutations can be merged into one. There we take the convention that for any set f such that $|f|=1$ holds, we define σ_f to be the identity on that set.
This results in

$$\begin{aligned}(3.526) = &\sum_{\substack{\langle n\rangle = \biguplus_{l\in T} l \uplus \biguplus_{l\in D} l \uplus R \\ \forall l\in D\cup T:|l|>2}} \sum_{b\in D\cup\{\{r\}|r\in R\}} \sum_{\sigma_b\in S(b)} \\ &\qquad \mathrm{tr}[P_+X_{n+1}P_-W(b,\sigma_b)]\mathrm{PT}(T) \sum_{\substack{\forall l\in D\backslash\{b\} \\ \sigma_l\in S(l)}} \\ &\sum_{a\subseteq R\backslash b\cup\bigcup_{l\in D\backslash\{b\}}\{W(l,\sigma_l)\}} L(a,a^c)(-1)^{|a|+\frac{(|R|+|D|-1)(|R|+|D|)}{2}}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\substack{\langle n\rangle = \dot{\bigcup}_{l\in T} l \,\dot\cup\, \dot{\bigcup}_{l\in D} l \,\dot\cup\, R \\ \forall l\in D\cup T: |l|>2}} \sum_{\substack{\forall l\in D \\ \sigma_l\in S(l)}} \sum_{b\in D\cup\{\{r\}|r\in R\}} \mathrm{PT}(T) \\
&\quad \sum_{a\subseteq R\backslash b\cup\bigcup_{l\in D\backslash\{b\}}\{W(l,\sigma_l)\}} L(a,a^c)(-1)^{|a|+\frac{(|R|+|D|-1)(|R|+|D|)}{2}} \\
&\quad \mathrm{tr}[P_+X_{n+1}P_-W(b,\sigma_b)] \\
&= \sum_{\substack{\langle n\rangle = \dot{\bigcup}_{l\in T} l \,\dot\cup\, \dot{\bigcup}_{l\in D} l \,\dot\cup\, R \\ \forall l\in D\cup T: |l|>2}} \sum_{\substack{\forall l\in D \\ \sigma_l\in S(l)}} \mathrm{PT}(T) \sum_{a\subseteq R\cup\bigcup_{l\in D}\{W(l,\sigma_l)\}} \\
&\quad \sum_{b\in D\cup\{\{r\}|r\in R\}} \mathbb{1}_{W(b,\sigma_b)\in a} L(a\backslash\{b\},a^c)(-1)^{|a|+1} \\
&\quad (-1)^{\frac{(|R|+|D|-1)(|R|+|D|)}{2}} \mathrm{tr}[P_+X_{n+1}P_-W(b,\sigma_b)] \\
&= \sum_{\substack{\langle n\rangle = \dot{\bigcup}_{l\in T} l \,\dot\cup\, \dot{\bigcup}_{l\in D} l \,\dot\cup\, R \\ \forall l\in D\cup T: |l|>2}} \mathrm{PT}(T) \sum_{\substack{\forall l\in D \\ \sigma_l\in S(l)}} \sum_{a\subseteq R\cup\bigcup_{l\in D}\{W(l,\sigma_l)\}} \\
&\quad \sum_{b\in a} L(a\backslash\{b\},a^c)(-1)^{|a|+\frac{(|R|+|D|+1)(|R|+|D|)}{2}} \\
&\quad (-1)^{1+|R|+|D|} \mathrm{tr}[P_+X_{n+1}P_-W(b,\sigma_b)] \\
&= \sum_{\substack{\langle n\rangle = \dot{\bigcup}_{l\in T} l \,\dot\cup\, \dot{\bigcup}_{l\in D} l \,\dot\cup\, R \\ \forall l\in D\cup T: |l|>2}} \mathrm{PT}(T) \sum_{\substack{\forall l\in D \\ \sigma_l\in S(l)}} \sum_{a\subseteq R\cup\bigcup_{l\in D}\{W(l,\sigma_l)\}} \\
&\quad (-1)^{|a|+\frac{(|R|+|D|+1)(|R|+|D|)}{2}} (3.509)_{L(a,a^c)G(X_{n+1})},
\end{aligned}
\tag{3.527}
$$

where the notation in the last line is to be taken as "apply Lemma 76 apply it to $L(a,a^c)G(X_{n+1})$ and pick only term (3.509)". We will use this abbreviating notation also for the next terms.

The next term is (3.524). Here we need a few more notational conventions. For any set $b\subseteq\langle n\rangle$ and corresponding permutation $\sigma_b\in S(b)$, we denote by the same symbol σ_b the continuation of σ_b to $b\cup\{X_{n+1}\}$, where for this continuation X_{n+1} is a fixed point. Furthermore we define for any set $b \subseteq \langle n \rangle$, σ_c^b by

$$\begin{aligned} \sigma_c^b &\in S(b \cup \{X_{n+1}\}), \\ \forall k \leqslant |b| : \sigma_c^b(f_{b\cup\{X_{n+1}\}}(k)) &= f_{b\cup\{X_{n+1}\}}(k+1) \\ \sigma_c^b(X_{n+1}) &= f_b(1). \end{aligned} \tag{3.528}$$

Finally we define for sets $b_1, b_2 \subseteq \langle n \rangle, b_1 \cap b_2 = \varnothing$ and corresponding permutations $\sigma_{b_1} \in S(b_1), \sigma_{b_2} \in S(b_2)$ the permutation σ_{b_1,b_2}^{n+1} by

$$\begin{aligned} M_{b_1,b_2}^{n+1} &:= b_1 \cup b_2 \cup \{X_{n+1}\} \\ \sigma_{b_1,b_2}^{n+1} &\in S(M_{b_1,b_2}^{n+1}) \\ \forall 1 \leqslant k \leqslant |b_1| : \sigma_{b_1,b_2}^{n+1}(f_{M_{b_1,b_2}^{n+1}}(k)) &= \sigma_{b_1}(f_{b_1}(k)) \\ \sigma_{b_1,b_2}^{n+1}(f_{M_{b_1,b_2}^{n+1}}(|b_1|+1)) &= X_{n+1} \\ \forall |b_1| + 2 \leqslant k \leqslant |b_1| + |b_2| + 1 : \\ \sigma_{b_1,b_2}^{n+1}(f_{M_{b_1,b_2}^{n+1}}(k)) &= \sigma_{b_2}(f_{b_2}(k - |b_1| - 1)) \end{aligned} \tag{3.529}$$

The beginning of the treatment of term (3.524) is analogous to (3.523), we rewrite the partition of $\langle n+1 \rangle$ into one of $\langle n \rangle$ with an additional sum over where the other operators packed to together with X_{n+1} come from. This splits into three parts, either X_{n+1} is put at the beginning of the compound operator, or its put at the end of the compound object, or to the left as well as to the right are operators with smaller index. Since the overall sign is decided by how often the operator index rises or falls, this separation into cases is helpful. The last case we then rewrite as picking two sets of operators, one of which will be in front of X_{n+1} and the other one behind this operator.
The calculation is as follows

$$
(3.524) = \sum_{\substack{\langle n+1\rangle = \bigparr_{l\in T} l \parr \bigparr_{l\in D} l \parr R \\ \exists l\in D: X_{n+1}\in l \\ \forall l\in T\cup D: |l|\geqslant 2}} \mathrm{PT}(T)\mathrm{Op}(R,D)
$$

$$
= \sum_{\substack{\langle n\rangle = \bigparr_{l\in T} l \parr \bigparr_{l\in D} l \parr R \\ \forall l\in D\cup T: |l|\geqslant 2}} \mathrm{PT}(T) \sum_{b\in D\cup\{\{r\}|r\in R\}} \mathrm{Op}(R\backslash b, D\cup\{b\cup\{X_{n+1}\}\backslash\{b\}\})
$$

$$
= \sum_{\substack{\langle n\rangle = \bigparr_{l\in T} l \parr \bigparr_{l\in D} l \parr R \\ \forall l\in D\cup T: |l|\geqslant 2}} \mathrm{PT}(T) \sum_{b\in D\cup\{\{r\}|r\in R\}} \sum_{\substack{\forall l\in D\cup\{b\cup\{X_{n+1}\}\} \\ \sigma_l\in S(l)}}
$$

$$
\sum_{a\subseteq R\backslash b\cup\bigcup_{l\in D\cup\{b\cup\{X_{n+1}\}\}\backslash\{b\}}\{W(l,\sigma_l)\}} L(a,a^c)(-1)^{|a|+\frac{(|R|+|D|)(|R|+|D|+1)}{2}}
$$

$$
= \sum_{\substack{\langle n\rangle = \bigparr_{l\in T} l \parr \bigparr_{l\in D} l \parr R \\ \forall l\in D\cup T: |l|\geqslant 2}} \mathrm{PT}(T) \sum_{b\in D\cup\{\{r\}|r\in R\}} \sum_{\substack{\forall l\in D \\ \sigma_l\in S(l)}} \Bigg[
$$

$$
\sum_{a\subseteq R\backslash b\cup\bigcup_{l\in D\backslash\{b\}}\{W(l,\sigma_l)\}\cup\{W(b\cup\{X_{n+1}\},\sigma_b)\}} L(a,a^c)(-1)^{|a|+\frac{(|R|+|D|)(|R|+|D|+1)}{2}} \tag{3.530}
$$

$$
+ \sum_{a\subseteq R\backslash b\cup\bigcup_{l\in D\backslash\{b\}}\{W(l,\sigma_l)\}\cup\{W(b\cup\{X_{n+1}\},\sigma_c^b\circ\sigma_b)\}} L(a,a^c)(-1)^{|a|+\frac{(|R|+|D|)(|R|+|D|+1)}{2}} \Bigg] \tag{3.531}
$$

$$
+ \sum_{\substack{\langle n\rangle = \bigparr_{l\in T} l \parr \bigparr_{l\in \overline{D}} l \parr \overline{R} \\ \forall l\in \overline{D}\cup T: |l|\geqslant 2}} \mathrm{PT}(T) \sum_{\substack{b_1,b_2\in\overline{D}\cup\{\{r\}|r\in\overline{R}\} \\ b_1\neq b_2}} \sum_{\substack{\forall l\in\overline{D} \\ \sigma_l\in S(l)}}
$$

$$
\sum_{a\subseteq\tilde{R}\cup\bigcup_{l\in\tilde{D}}\{W(l,\sigma_l)\}\cup\{W(b_1\cup\{X_{n+1}\}\cup b_2,\sigma_{b_1,b_2}^{n+1})\}} L(a,a^c)(-1)^{|a|+\frac{(|\overline{R}|+|\overline{D}|-1)(|\overline{R}|+|\overline{D}|)}{2}} \tag{3.532}
$$

$\blacksquare$

where $\tilde{R} = \overline{R}\backslash(b_1\cup b_2)$ and $\tilde{D} := \overline{D}\cup\{b_1\cup\{X_{n+1}\}\cup b_2\}\backslash\{b_1,b_2\}$. For the term (3.532) we had to reshuffle the outermost sum a bit. For each term in the original sum where X_{n+1} is neither the first nor the last factor in its product (we will call the set of factors in front of X_{n+1} α and the factors behind it β) there is a different splitting of $\langle n\rangle$ into $\overline{R}$ and $\overline{D}$ such that α and β are separate elements of $\overline{D} \cup \{\{r\} \mid r \in \overline{R}\}$. So we replace the original sum over D and R into one of $\overline{D}$ and $\overline{R}$. Since this is a one to one correspondence and the sum is finite this is always possible. The exponent of the sign also changes for this reason, since $|R|+|D| = |\overline{R}|+|\overline{D}|-1$ holds. Continuing with (3.530) the next steps are similar to the last steps in treating (3.523). They are

$$
\begin{aligned}
(3.530) = & \sum_{\substack{\langle n\rangle = \bigshuffle_{l\in T} l \shuffle \bigshuffle_{l\in D} l \shuffle R \\ \forall l\in D\cup T: |l|\geqslant 2}} \mathrm{PT}(T) \sum_{b\in D\cup\{\{r\}|r\in R\}} \sum_{\substack{\forall l\in D \\ \sigma_l\in S(l)}} \\
& \sum_{a\subseteq R\setminus b\cup\bigcup_{l\in D\setminus\{b\}}\{W(l,\sigma_l)\}\cup\{W(b\cup\{X_{n+1}\},\sigma_b)\}} L(a,a^c)(-1)^{|a|+\frac{(|R|+|D|)(|R|+|D|+1)}{2}} \\
= & \sum_{\substack{\langle n\rangle = \bigshuffle_{l\in T} l \shuffle \bigshuffle_{l\in D} l \shuffle R \\ \forall l\in D\cup T: |l|\geqslant 2}} \mathrm{PT}(T) \sum_{b\in D\cup\{\{r\}|r\in R\}} \sum_{\substack{\forall l\in D \\ \sigma_l\in S(l)}} \\
& \sum_{a\subseteq R\cup\bigcup_{l\in D}\{W(l,\sigma_l)\}} (-1)^{|a|+\frac{(|R|+|D|)(|R|+|D|+1)}{2}} \mathbb{1}_{W(b,\sigma_b)\in a} \\
& \big[L(a\setminus\{W(b,\sigma_b)\}\cup\{W(b\cup\{X_{n+1}\},\sigma_b)\},a^c) \\
& - L(a\setminus\{W(b,\sigma_b)\},a^c\cup\{W(b\cup\{X_{n+1}\},\sigma_b)\})\big] \\
= & \sum_{\substack{\langle n\rangle = \bigshuffle_{l\in T} l \shuffle \bigshuffle_{l\in D} l \shuffle R \\ \forall l\in D\cup T: |l|\geqslant 2}} \mathrm{PT}(T) \sum_{\substack{\forall l\in D \\ \sigma_l\in S(l)}} \\
& \sum_{a\subseteq R\cup\bigcup_{l\in D}\{W(l,\sigma_l)\}} (-1)^{|a|+\frac{(|R|+|D|)(|R|+|D|+1)}{2}} \sum_{W(b,\sigma_b)\in a} \\
& \big[L(a\setminus\{W(b,\sigma_b)\}\cup\{W(b\cup\{X_{n+1}\},\sigma_b)\},a^c)
\end{aligned}
$$

$$
\begin{aligned}
&- L(a\backslash\{W(b,\sigma_b)\}, a^c \cup \{W(b \cup \{X_{n+1}\}, \sigma_b)\})\Big] \\
&= \sum_{\substack{\langle n\rangle = \biguplus_{l\in T} l \uplus \biguplus_{l\in D} l \uplus R \\ \forall l\in D\cup T: |l|\geqslant 2}} \mathrm{PT}(T) \sum_{\substack{\forall l\in D \\ \sigma_l \in S(l)}} \sum_{a\subseteq R\cup\bigcup_{l\in D}\{W(l,\sigma_l)\}} \\
&(-1)^{|a|+\frac{(|R|+|D|)(|R|+|D|+1)}{2}} ((3.505)+(3.507))_{L(a,a^c)G(X_{n+1})}. \qquad (3.533)
\end{aligned}
$$

Almost the same procedure applies to (3.531). It yields

$$
\begin{aligned}
&(3.531) = \sum_{\substack{\langle n\rangle = \biguplus_{l\in T} l \uplus \biguplus_{l\in D} l \uplus R \\ \forall l\in D\cup T: |l|\geqslant 2}} \mathrm{PT}(T) \sum_{b\in D\cup\{\{r\}|r\in R\}} \sum_{\substack{\forall l\in D \\ \sigma_l \in S(l)}} \\
&\sum_{a\subseteq R\backslash b\cup\bigcup_{l\in D\backslash\{b\}}\{W(l,\sigma_l)\}\cup\{W(b\cup\{X_{n+1}\},\sigma_c^b\circ\sigma_b)\}} L(a,a^c)(-1)^{|a|+\frac{(|R|+|D|)(|R|+|D|+1)}{2}} \\
&= \sum_{\substack{\langle n\rangle = \biguplus_{l\in T} l \uplus \biguplus_{l\in D} l \uplus R \\ \forall l\in D\cup T: |l|\geqslant 2}} \mathrm{PT}(T) \sum_{b\in D\cup\{\{r\}|r\in R\}} \sum_{\substack{\forall l\in D \\ \sigma_l \in S(l)}} \\
&\sum_{a\subseteq R\cup\bigcup_{l\in D}\{W(l,\sigma_l)\}} (-1)^{|a|+\frac{(|R|+|D|)(|R|+|D|+1)}{2}} \\
&\Big[\mathbb{1}_{W(b,\sigma_c^b\circ\sigma_b)\in a} L(a\backslash\{W(b,\sigma_b)\}\cup\{W(b\cup\{X_{n+1}\},\sigma_c^b\circ\sigma_b)\},a^c) \\
&+ \mathbb{1}_{W(b,\sigma_c^b\circ\sigma_b)\in a^c} L(a\backslash\{W(b,\sigma_b)\},a^c\cup\{W(b\cup\{X_{n+1}\},\sigma_c^b\circ\sigma_b)\})\Big] \\
&= \sum_{\substack{\langle n\rangle = \biguplus_{l\in T} l \uplus \biguplus_{l\in D} l \uplus R \\ \forall l\in D\cup T: |l|\geqslant 2}} \mathrm{PT}(T) \sum_{\substack{\forall l\in D \\ \sigma_l \in S(l)}} \\
&\sum_{a\subseteq R\cup\bigcup_{l\in D}\{W(l,\sigma_l)\}} (-1)^{|a|+\frac{(|R|+|D|)(|R|+|D|+1)}{2}} \\
&\Big[\sum_{W(b,\sigma_b)\in a} L(a\backslash\{W(b,\sigma_b)\}\cup\{W(b\cup\{X_{n+1}\},\sigma_c^b\circ\sigma_b)\},a^c) \\
&+ \sum_{W(b,\sigma_b)\in a^c} L(a,a^c\backslash\{W(b,\sigma_b)\}\cup\{W(b\cup\{X_{n+1}\},\sigma_c^b\circ\sigma_b)\})\Big]
\end{aligned}
$$

$$= \sum_{\substack{\langle n\rangle = \biguplus_{l\in T} l \uplus \biguplus_{l\in D} l \uplus R \\ \forall l\in D\cup T: |l|\geqslant 2}} \mathrm{PT}(T) \sum_{\substack{\forall l\in D \\ \sigma_l\in S(l)}} \sum_{a\subseteq R\cup\bigcup_{l\in D}\{W(l,\sigma_l)\}}$$

$$(-1)^{|a|+\frac{(|R|+|D|)(|R|+|D|+1)}{2}} ((3.506) + (3.508))_{L(a,a^c)G(X_{n+1})}. \tag{3.534}$$

Also for (3.532) the procedure is almost the same. We bring the sums into a form such that one can read off the terms generated by the induction. We begin be renaming the sets which we had to change by resumming back to the names of the original sets.

$$(3.532) = \sum_{\substack{\langle n\rangle = \biguplus_{l\in T} l \uplus \biguplus_{l\in \overline{D}} l \uplus \overline{R} \\ \forall l\in \overline{D}\cup T: |l|\geqslant 2}} \mathrm{PT}(T) \sum_{\substack{b_1,b_2\in\overline{D}\cup\{\{r\}|r\in\overline{R}\} \\ b_1\neq b_2}} \sum_{\substack{\forall l\in\overline{D} \\ \sigma_l\in S(l)}}$$

$$\sum_{a\subseteq\tilde{R}\cup\bigcup_{l\in\tilde{D}}\{W(l,\sigma_l)\}\cup\{W(b_1\cup\{X_{n+1}\}\cup b_2,\sigma^{n+1}_{b_1,b_2})\}} L(a,a^c)(-1)^{|a|+\frac{(|\overline{R}|+|\overline{D}|-1)(|\overline{R}|+|\overline{D}|)}{2}}$$

$$= \sum_{\substack{\langle n\rangle = \biguplus_{l\in T} l \uplus \biguplus_{l\in D} l \uplus R \\ \forall l\in D\cup T: |l|\geqslant 2}} \mathrm{PT}(T) \sum_{\substack{b_1,b_2\in D\cup\{\{r\}|r\in R\} \\ b_1\neq b_2}} \sum_{\substack{\forall l\in D \\ \sigma_l\in S(l)}}$$

$$\sum_{a\subseteq R\cup\bigcup_{l\in D}\{W(l,\sigma_l)\}\cup\{W(b_1\cup\{X_{n+1}\}\cup b_2,\sigma^{n+1}_{b_1,b_2})\}} L(a,a^c)(-1)^{|a|+\frac{(|R|+|D|-1)(|R|+|D|)}{2}}$$

$$= \sum_{\substack{\langle n\rangle = \biguplus_{l\in T} l \uplus \biguplus_{l\in D} l \uplus R \\ \forall l\in D\cup T: |l|\geqslant 2}} \mathrm{PT}(T) \sum_{\substack{b_1,b_2\in D\cup\{\{r\}|r\in R\} \\ b_1\neq b_2}} \sum_{\substack{\forall l\in D \\ \sigma_l\in S(l)}} \sum_{a\subseteq R\cup\bigcup_{l\in D}\{W(l,\sigma_l)\}}$$

$$(-1)^{|R|+|D|+\frac{(|R|+|D|)(|R|+|D|+1)}{2}} \mathbb{1}_{W(b_1,\sigma_1)\in a}$$

$$\Big[+(-1)^{|a|+1}\mathbb{1}_{W(b_2,\sigma_2)\in a} L\Big(a\backslash\{W(b_1,\sigma_1),W(b_2,\sigma_2)\}\cup$$

$$\cup\,\{W(b_1\cup\{X_{n+1}\}\cup b_2,\sigma^{n+1}_{b_1,b_2})\},a^c\Big)$$

$$+\,(-1)^{|a|+1}\mathbb{1}_{W(b_2,\sigma_2)\in a^c} L\Big(a\backslash\{W(b_1,\sigma_1)\},a^c\backslash\{W(b_2,\sigma_2)\}\cup$$

$$\cup\{W(b_1 \cup \{X_{n+1}\} \cup f_2, \sigma^{n+1}_{b_1,b_2})\}\Big)\Big]$$
$$= \sum_{\substack{\langle n\rangle = \biguplus_{l\in T} l \uplus \biguplus_{l\in D} l \uplus R \\ \forall l\in D\cup T: |l|\geqslant 2}} \mathrm{PT}(T) \sum_{\substack{\forall l\in D \\ \sigma_l\in S(l)}} \sum_{a\subseteq R\cup\bigcup_{l\in D}\{W(l,\sigma_l)\}}$$
$$(-1)^{|R|+|D|+\frac{(|R|+|D|)(|R|+|D|+1)}{2}}$$
$$\Big[(-1)^{|a|+1} \sum_{\substack{b_1,b_2\in a \\ b_1\neq b_2}} L\Big(a\backslash\{W(b_1,\sigma_1), W(b_2,\sigma_2)\}\cup$$
$$\cup\, \{W(b_1 \cup \{X_{n+1}\} \cup b_2, \sigma^{n+1}_{b_1,b_2})\}, a^c\Big)$$
$$+\, (-1)^{|a|+1} \sum_{b_1\in a, b_2\in a^c} L\Big(a\backslash\{W(b_1,\sigma_1)\}, a^c\backslash\{W(b_2,\sigma_2)\}\cup$$
$$\cup\{W(b_1 \cup \{X_{n+1}\} \cup f_2, \sigma^{n+1}_{b_1,b_2})\}\Big)\Big]$$
$$= \sum_{\substack{\langle n\rangle = \biguplus_{l\in T} l \uplus \biguplus_{l\in D} l \uplus R \\ \forall l\in D\cup T: |l|\geqslant 2}} \mathrm{PT}(T) \sum_{\substack{\forall l\in D \\ \sigma_l\in S(l)}} \sum_{a\subseteq R\cup\bigcup_{l\in D}\{W(l,\sigma_l)\}}$$
$$(-1)^{|a|+\frac{(|R|+|D|)(|R|+|D|+1)}{2}} ((3.510) + (3.511))_{L(a,a^c)G(X_{n+1})}$$

Lastly we will discuss term (3.525); luckily, this term is less involved than the other two. The general procedure; however, stays the same. First we reformulate the partition of $\langle n+1\rangle$ into one of $\langle n\rangle$, where the terms acquire modifications. Secondly we massage these terms until the involved sums look exactly like the one in our induction hypothesis (3.522) and realise that the terms are produced by lemma 76. For term (3.525) this results in

$$(3.525) = \sum_{\substack{\langle n+1\rangle = \biguplus_{l\in T} l \uplus \biguplus_{l\in D} l \uplus R \\ X_{n+1}\in R \\ \forall l\in T\cup D: |l|\geqslant 2}} \mathrm{PT}(T)\mathrm{Op}(R,D)$$

$$
\begin{aligned}
&= \sum_{\substack{\langle n\rangle = \biguplus_{l\in T} l \uplus \biguplus_{l\in D} l \uplus R \\ \forall l\in T\cup D: |l|\geqslant 2}} \mathrm{PT}(T)\mathrm{Op}(R\cup\{X_{n+1}\}, D) \\
&= \sum_{\substack{\langle n\rangle = \biguplus_{l\in T} l \uplus \biguplus_{l\in D} l \uplus R \\ \forall l\in T\cup D: |l|\geqslant 2}} \mathrm{PT}(T) \sum_{\substack{\forall l\in D: \\ \sigma_l\in S(l)}} \sum_{a\subseteq R\cup\{X_{n+1}\}\cup\bigcup_{l\in D}\{W(l,\sigma_l)\}} \\
&L(a,a^c)(-1)^{|a|+\frac{(|R|+1+|D|)(|R|+|D|+2)}{2}} \\
&= \sum_{\substack{\langle n\rangle = \biguplus_{l\in T} l \uplus \biguplus_{l\in D} l \uplus R \\ \forall l\in T\cup D: |l|\geqslant 2}} \mathrm{PT}(T) \sum_{\substack{\forall l\in D: \\ \sigma_l\in S(l)}} \sum_{a\subseteq R\cup\bigcup_{l\in D}\{W(l,\sigma_l)\}} (-1)^{|a|+\frac{(|R|+1+|D|)(|R|+|D|)}{2}} \\
&\big(-L(a\cup\{X_{n+1}\},a^c) + L(a,a^c\cup\{X_{n+1}\})\big)(-1)^{|R|+|D|+1} \\
&= \sum_{\substack{\langle n\rangle = \biguplus_{l\in T} l \uplus \biguplus_{l\in D} l \uplus R \\ \forall l\in T\cup D: |l|\geqslant 2}} \mathrm{PT}(T) \sum_{\substack{\forall l\in D: \\ \sigma_l\in S(l)}} \sum_{a\subseteq R\cup\bigcup_{l\in D}\{W(l,\sigma_l)\}} (-1)^{|a|+\frac{(|R|+1+|D|)(|R|+|D|)}{2}} \\
&\big(L(a\cup\{X_{n+1}\},a^c)(-1)^{|R|+|D|} + L(a,a^c\cup\{X_{n+1}\})(-1)^{|R|+|D|+1}\big) \\
&= \sum_{\substack{\langle n\rangle = \biguplus_{l\in T} l \uplus \biguplus_{l\in D} l \uplus R \\ \forall l\in T\cup D: |l|\geqslant 2}} \mathrm{PT}(T) \sum_{\substack{\forall l\in D: \\ \sigma_l\in S(l)}} \sum_{a\subseteq R\cup\bigcup_{l\in D}\{W(l,\sigma_l)\}} (-1)^{|a|+\frac{(|R|+1+|D|)(|R|+|D|)}{2}} \\
&\big((3.503) + (3.504)\big)_{L(a,a^c)G(X_{n+1})}. \qquad \blacksquare
\end{aligned}
$$

Summarising we showed

$$
\begin{aligned}
&\sum_{\substack{\langle n+1\rangle = \biguplus_{l\in T} l \uplus \biguplus_{l\in D} l \uplus R \\ \forall l\in T\cup D: |l|\geqslant 2}} \mathrm{PT}(T)\mathrm{Op}(R, D) \\
&= \sum_{\substack{\langle n\rangle = \biguplus_{l\in T} l \uplus \biguplus_{l\in D} l \uplus R \\ \forall l\in D\cup T: |l|>2}} \mathrm{PT}(T) \sum_{\substack{\forall l\in D \\ \sigma_l\in S(l)}} \sum_{a\subseteq R\cup\bigcup_{l\in D}\{W(l,\sigma_l)\}} \\
&(-1)^{|a|+\frac{(|R|+|D|+1)(|R|+|D|)}{2}} \\
&\big((3.509) + (3.505) + (3.507) + (3.506) + (3.508) \\
&+ (3.510) + (3.511) + (3.503) + (3.504)\big)_{L(a,a^c)G(X_{n+1})}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{\substack{\langle n\rangle = \biguplus_{l\in T} l \uplus \biguplus_{l\in D} l \uplus R \\ \forall l\in D\cup T : |l|>2}} \mathrm{PT}(T) \sum_{\substack{\forall l\in D \\ \sigma_l\in S(l)}} \sum_{a\subseteq R\cup\bigcup_{l\in D}\{W(l,\sigma_l)\}} \\
&(-1)^{|a|+\frac{(|R|+|D|+1)(|R|+|D|)}{2}} L(a,a^c) G(X_{n+1}) \\
&= \sum_{\substack{\langle n\rangle = \biguplus_{l\in T} l \uplus \biguplus_{l\in D} l \uplus R \\ \forall l\in D\cup T : |l|>2}} \mathrm{PT}(T)\mathrm{Op}(R,D) G(X_{n+1}) \\
&= \prod_{l=1}^{n} G(X_l) \quad G(X_{n+1}),
\end{aligned}
$$

which ends our proof by induction. $\square$

Appendix

Heuristic Construction of S-Matrix expression

In the following I derive a recursive equation for the coefficients of the expansion of the second quantized scattering operator. The starting point of this derivation is the commutator of T_m, equation (3.412).

Guessing Equations

Why at this point one might suspect that such a representation exists is, because looking at equation (3.412) for a while, one comes to the conclusion that if one replaces T_m by

$$T_m - \frac{1}{2}\sum_{k=1}^{m-1}\binom{m}{k} T_k T_{m-k}, \tag{535}$$

no T_k with $k > m-2$ will occur on the right hand side of the resulting equation. So if one subtracts the right polynomial in T_k for suitable k one might achieve a commutator which contains only the creation respectively annihilation operator concatenated with some one particle operator. From our treatment of T_1 we know which operators have such commutation relations.

Todo: place proper reference to definition of G operator

So having this in Mind we start with the ansatz

$$\Gamma_m := \sum_{g=2}^{m} \sum_{\substack{b \in \mathbb{N}^g \\ |b|=m}} c_b \prod_{k=1}^{g} T_{b_k}. \tag{536}$$

Now in order to show that T_m and Γ_m agree up to operators which have a commutation relation of the form (**??**), we calculate $\left[T_m - \Gamma_m, a^{\#}(\varphi_n)\right]$ for arbitrary $n \in \mathbb{Z}$ and try to choose the coefficients c_b of (536) such that all contributions vanish which do not have the form $a^{\#}\left(\prod_k Z_{\alpha_k}\right)$ for any suitable $(\alpha_k)_k \subset \mathbb{N}$. If one does so, one is led to a system of equations of which I wrote down a few to give an overview of its structure. The objects α_k, β_l in the system of equations can be any natural Number for any $k, l \in \mathbb{N}$.

$$
\begin{aligned}
0 &= c_{\alpha_1,\beta_1} + c_{\beta_1,\alpha_1} + \binom{\alpha_1+\beta_1}{\alpha_1} \\
0 &= c_{\alpha_1,\alpha_2,\beta_1} + c_{\beta_1,\alpha_1,\alpha_2} + c_{\alpha_2,\alpha_1,\beta_1} + \binom{\alpha_2+\beta_1}{\alpha_2} c_{\alpha_1,\alpha_2+\beta_1} \\
&\qquad + \binom{\alpha_1+\beta_1}{\alpha_1} c_{\alpha_1+\beta_1,\alpha_2} \\
0 &= c_{\alpha_1,\alpha_2,\alpha_3,\beta_1} + c_{\alpha_1,\alpha_2,\beta_1,\alpha_3} + c_{\alpha_1,\beta_1,\alpha_2,\alpha_3} + c_{\beta_1,\alpha_1,\alpha_2,\alpha_3} \\
&+ \binom{\alpha_1+\beta_1}{\beta_1} c_{\alpha_1+\beta_1,\alpha_2,\alpha_3} + \binom{\alpha_2+\beta_1}{\beta_1} c_{\alpha_1,\alpha_2+\beta_1,\alpha_3} \\
&\qquad + \binom{\alpha_3+\beta_1}{\beta_1} c_{\alpha_1,\alpha_2,\alpha_3+\beta_1} \\
0 &= c_{\alpha_1,\alpha_2,\beta_1,\beta_2} + c_{\alpha_1,\beta_1,\alpha_2,\beta_2} + c_{\beta_1,\alpha_1,\alpha_2,\beta_2} + c_{\alpha_1,\beta_1,\beta_2,\alpha_2}
\end{aligned}
$$

$$
\begin{aligned}
&+ c_{\beta_1,\alpha_1,\beta_2,\alpha_2} + c_{\beta_1,\beta_2,\alpha_1,\alpha_2} + \binom{\alpha_1+\beta_1}{\alpha_1}\big(c_{\alpha_1+\beta_1,\alpha_2,\beta_2} \\
&+ c_{\alpha_1+\beta_1,\beta_2,\alpha_2}\big) + \binom{\alpha_1+\beta_2}{c}_{\beta_1,\alpha_1+\beta_2,\alpha_1} \\
&+ \binom{\alpha_2+\beta_1}{\alpha_2} c_{\alpha_1,\alpha_2+\beta_1,\beta_2} + \binom{\alpha_2+\beta_2}{\alpha_2}\big(c_{\alpha_1,\beta_1,\alpha_2+\beta_2} \\
&+ c_{\beta_1,\alpha_1,\alpha_2+\beta_2}\big) + \binom{\alpha_1+\beta_1}{\alpha_1}\binom{\alpha_2+\beta_2}{\alpha_2} c_{\alpha_1+\beta_1,\alpha_2+\beta_2}
\end{aligned}
$$

$$
\begin{aligned}
0 &= c_{\alpha_1,\beta_1,\beta_2,\beta_3,\beta_4} + c_{\beta_1,\alpha_1,\beta_2,\beta_3,\beta_4} + c_{\beta_1,\beta_2,\alpha_1,\beta_3,\beta_4} \\
&\qquad + c_{\beta_1,\beta_2,\beta_3,\alpha_1,\beta_4} + c_{\beta_1,\beta_2,\beta_3,\beta_4,\alpha_1} \\
&+ \binom{\alpha_1+\beta_1}{\alpha_1} c_{\alpha_1+\beta_1,\beta_2,\beta_3,\beta_4} + \binom{\alpha_1+\beta_2}{\alpha_1} c_{\beta_1,\alpha_1+\beta_2,\beta_3,\beta_4} \\
&+ \binom{\alpha_1+\beta_3}{\alpha_1} c_{\beta_1,\beta_2,\alpha_1+\beta_3,\beta_4} + \binom{\alpha_1+\beta_4}{\alpha_1} c_{\beta_1,\beta_2,\beta_3,\alpha_1+\beta_4}
\end{aligned}
$$

$$
\begin{aligned}
0 &= c_{\alpha_1,\alpha_2,\beta_1,\beta_2,\beta_3} + c_{\alpha_1,\beta_1,\alpha_2,\beta_2,\beta_3} + c_{\beta_1,\alpha_1,\alpha_2,\beta_2,\beta_3} \\
&+ c_{\alpha_1,\beta_1,\beta_2,\alpha_2,\beta_3} + c_{\beta_1,\alpha_1,\beta_2,\alpha_2,\beta_3} + c_{\beta_1,\beta_2,\alpha_1,\alpha_2,\beta_3} \\
&+ c_{\alpha_1,\beta_1,\beta_2,\beta_3,\alpha_2} + c_{\beta_1,\alpha_1,\beta_2,\beta_3,\alpha_2} + c_{\beta_1,\beta_2,\alpha_1,\beta_3,\alpha_2} \\
&+ c_{\beta_1,\beta_2,\beta_3,\alpha_1,\alpha_2} + \binom{\alpha_1+\beta_1}{\beta_1}\big(c_{\alpha_1+\beta_1,\alpha_2,\beta_2,\beta_3} \\
&+ c_{\alpha_1+\beta_1,\beta_2,\alpha_2,\beta_3} + c_{\alpha_1+\beta_1,\beta_2,\beta_3,\alpha_2}\big) \\
&+ \binom{\alpha_2+\beta_1}{\beta_1} c_{\alpha_1,\alpha_2+\beta_1,\beta_2,\beta_3} \\
&+ \binom{\alpha_2+\beta_2}{\beta_2}\big(c_{\beta_1,\alpha_1,\alpha_2+\beta_2,\beta_3} + c_{\alpha_1,\beta_1,\alpha_2+\beta_2,\beta_3}\big) \\
&+ \binom{\alpha_1+\beta_2}{\beta_2}\big(c_{\beta_1,\alpha_1+\beta_2,\alpha_2,\beta_3} + c_{\beta_1,\alpha_1+\beta_2,\beta_3,\alpha_2}\big) \\
&+ \binom{\alpha_2+\beta_3}{\beta_3}\big(c_{\alpha_1,\beta_1,\beta_2,\alpha_2+\beta_3} + c_{\beta_1,\alpha_1,\beta_2,\alpha_2+\beta_3}
\end{aligned}
$$

$$
\begin{aligned}
&+ c_{\beta_1,\beta_2,\alpha_1,\alpha_2+\beta_3}\big) + \binom{\alpha_1+\beta_3}{\beta_3} c_{\beta_1,\beta_2,\alpha_1+\beta_3,\alpha_2} \\
&+ \binom{\alpha_1+\beta_1}{\alpha_1}\binom{\alpha_2+\beta_2}{\alpha_2} c_{\alpha_1+\beta_1,\alpha_2+\beta_2,\beta_3} \\
&+ \binom{\alpha_1+\beta_2}{\alpha_1}\binom{\alpha_2+\beta_3}{\alpha_2} c_{\beta_1,\alpha_1+\beta_2,\alpha_2+\beta_3} \\
&+ \binom{\alpha_1+\beta_1}{\alpha_1}\binom{\alpha_2+\beta_3}{\alpha_2} c_{\alpha_1+\beta_1,\beta_2,\alpha_2+\beta_3} \\
&\qquad\qquad\vdots
\end{aligned}
$$

Solving the first few equations and plugging the solution into the consecutive equations one can see that at least the first equations are solved by

$$
c_{\alpha_1,\ldots,\alpha_k} = \frac{(-1)^k}{k} \binom{\sum_{l=1}^{k} \alpha_l}{\alpha_1\ \alpha_2\ \cdots\ \alpha_k}, \tag{537}
$$

where the last factor is the multinomial coefficient of the indices $\alpha_1, \ldots, \alpha_k \in \mathbb{N}$.

Recursive equation for Coefficients of the second quantised scattering operator

For the rest of this chapter, we are going to derive a concrete form of the second quantised scattering matrix. In order to turn the "conjectures" into "theorems" not only would one have to turn the rough sketches of the combinatorics into proofs, one also would have to show linearity (over real numbers) and continuity of $\mathrm{d}\Gamma(B)$ in B. However, since the final result can be verified to be well defined and to fulfil the lift conditions, this will not be necessary. We will nonetheless come across various combinatorial assertions that we are going to prove rigorously. These will be clearly marked: "lemma" and "proof".

We are going to use the following definition of binomial coefficients:

Definition 85. *For $a \in \mathbb{C}, b \in \mathbb{Z}$ we define*

$$\binom{a}{b} := \begin{cases} \prod_{l=0}^{b-1} \frac{a-l}{l+1} & \text{for } b \geqslant 0 \\ 0 & \text{otherwise.} \end{cases} \tag{538}$$

Defining the binomial coefficient for negative lower index to be zero has the merit, that one can extend the range of validity of many rules and sums involving binomial coefficients, also one does not have to worry about the range of summation in many cases.
The coefficients which we have already guessed more generally to be

Conjecture 86. *For any $n \in \mathbb{N}$ the n-th expansion coefficient of the second quantised scattering operator T_n is given by*

$$\begin{aligned} T_n = &\sum_{g=2}^{n} \sum_{\substack{\vec{b} \in \mathbb{N}^g \\ |\vec{b}|=n}} \frac{(-1)^g}{g} \binom{n}{\vec{b}} \prod_{l=1}^{g} T_{b_l} + C_n \mathbb{1}_{\mathcal{F}} \\ &+ d\Gamma \left(\sum_{g=1}^{n} \sum_{\substack{\vec{b} \in \mathbb{N}^g \\ |\vec{b}|=n}} \frac{(-1)^{g+1}}{g} \binom{n}{\vec{b}} \prod_{l=1}^{g} Z_{b_l} \right), \end{aligned} \tag{539}$$

for some $C_n \in \mathbb{C}$ which depends on the external field A. The last summand will henceforth be abbreviated by Γ_n.

Motivation: The way we will prove this is to compute the commutator of the difference between T_n and the first summand of (539) with the creation and annihilation operator of an element of the basis of $\mathcal{H}$. This will turn out to be exactly equal to the corresponding commutator of the second summand of (539), since two operators on Fock space only have the same commutator with general creation and annihilation operators if they agree up to multiples of the identity this will conclude the motivation of this conjecture.

In order to simplify the notation as much as possible, I will denote by $a^{\#}z$ either $a(z(\varphi_p))$ or $a^*(z(\varphi_p))$ for any one particle operator z and any element φ_p of the orthonormal basis $(\varphi_p)_{p\in\mathbb{Z}\setminus\{0\}}$ of $\mathcal{H}$. (We need not decide between creation and annihilation operator, since the expressions all agree)

In order to organize the bookkeeping of all the summands which arise from iteratively making use of the commutation rule (3.412) we organize them by the looking at a spanning set of the possible terms that arise my choice is

$$\left\{ a^{\#} \prod_{k=1}^{m_1} Z_{\alpha_k} \prod_{k=1}^{m_2} T_{\beta_k} \middle| m_1 \in \mathbb{N}, m_2 \in \mathbb{N}_0, \alpha \in \mathbb{N}^{m_1}, \beta \in \mathbb{N}^{m_2}, |\alpha| + |\beta| = n \right\}. \tag{540}$$

As a first step of computing the commutator in question we look at the summand corresponding to a fixed value of the summation index g of

$$-\sum_{g=1}^{n} \sum_{\substack{\vec{b}\in\mathbb{N}^g \\ |\vec{b}|=n}} \frac{(-1)^g}{g} \binom{n}{\vec{b}} \prod_{l=1}^{g} T_{b_l}. \tag{541}$$

We need to bring this object into the form of a sum of terms which are multiples of elements of the set (540). This we will commit ourselves to for the next few pages. First we apply the product rule for the commutator:

$$\left[\sum_{\substack{\vec{l}\in\mathbb{N}^g\\|\vec{l}|=n}}\frac{(-1)^g}{g}\binom{n}{\vec{l}}\prod_{k=1}^{g}T_{l_k},a^{\#}\right]$$
$$=\sum_{\substack{\vec{l}\in\mathbb{N}^g\\|\vec{l}|=n}}\frac{(-1)^g}{g}\binom{n}{\vec{l}}\sum_{\tilde{k}=1}^{g}\prod_{j=1}^{\tilde{k}-1}T_{l_j}\left[T_{l_{\tilde{k}}},a^{\#}\right]\prod_{j=\tilde{k}+1}^{g}T_{l_j}$$
$$=\sum_{\substack{\vec{l}\in\mathbb{N}^g\\|\vec{l}|=n}}\frac{(-1)^g}{g}\binom{n}{\vec{l}}\sum_{\tilde{k}=1}^{g}\prod_{j=1}^{\tilde{k}-1}T_{l_j}\sum_{\sigma_{\tilde{k}}=1}^{l_{\tilde{k}}}\binom{l_{\tilde{k}}}{\sigma_{\tilde{k}}}a^{\#}Z_fT_{l_{\tilde{k}}-\sigma_{\tilde{k}}}\prod_{j=\tilde{k}+1}^{g}T_{l_j},$$

in the second step we used (3.412). Now we commute all the T_ls to the left of $a^{\#}$ to its right:

$$=\sum_{\substack{\vec{l}\in\mathbb{N}^g\\|l|=n}}\frac{(-1)^g}{g}\binom{n}{\vec{l}}\sum_{\tilde{k}=1}^{g}\sum_{\substack{\forall 1\leqslant j<\tilde{k}\\0\leqslant\sigma_j\leqslant l_j}}\sum_{\sigma_{\tilde{k}}=1}^{l_{\tilde{k}}}\prod_{j=1}^{\tilde{k}}\binom{l_j}{\sigma_j}a^{\#}\prod_{j=1}^{\tilde{k}}Z_{\sigma_j}\prod_{j=1}^{\tilde{k}}T_{l_j-\sigma_j}\prod_{j=\tilde{k}+1}^{g}T_{l_j}. \tag{542}$$

At this point we notice that the multinomial coefficient can be combined with all the binomial coefficients to form a single multinomial coefficient of degree $g+\tilde{k}$. Incidentally this is also the amount of Z operators plus the amount of T operators in each product. Moreover the indices of the Multinomial index agree with the indices of the Z and T operators in the product. Because of this, we see that if we fix an element of the spanning set (540) $a^{\#}\prod_{k=1}^{m_1}Z_{\alpha_k}\prod_{k=1}^{m_2}T_{\beta_k}$, each summand of (542) which contributes to this element, has the prefactor

$$\frac{(-1)^g}{g}\binom{n}{\alpha_1\ \cdots\alpha_{m_1}\ \beta_1\cdots\beta_{m_2}} \tag{543}$$

no matter which summation index $l \in \mathbb{N}^g$ it corresponds to. In order to do the matching one may ignore the indices σ_j and $l_j - \sigma_j$ which vanish, because the corresponding operators Z_0 and T_0 are equal to the identity operator on $\mathcal{H}$ respectively Fock space.
Since we know that

$$\left[d\Gamma \left(\sum_{g=1}^{n} \sum_{\substack{\vec{b}\in\mathbb{N}^g \\ |\vec{b}|=n}} \frac{(-1)^g}{g} \binom{n}{\vec{b}} \prod_{l=1}^{g} Z_{b_l} \right), a^{\#} \right]$$
$$= a^{\#} \sum_{g=1}^{n} \sum_{\substack{\vec{b}\in\mathbb{N}^g \\ |\vec{b}|=n}} \frac{(-1)^g}{g} \binom{n}{\vec{b}} \prod_{l=1}^{g} Z_{b_l}$$

holds, all that is left to show is that

$$\left[-\sum_{g=1}^{n} \sum_{\substack{\vec{b}\in\mathbb{N}^g \\ |\vec{b}|=n}} \frac{(-1)^g}{g} \binom{n}{\vec{b}} \prod_{l=1}^{g} T_{b_l}, a^{\#} \right] \tag{544}$$
$$= a^{\#} \sum_{g=1}^{n} \sum_{\substack{\vec{b}\in\mathbb{N}^g \\ |\vec{b}|=n}} \frac{(-1)^{g+1}}{g} \binom{n}{\vec{b}} \prod_{l=1}^{g} Z_{b_l}$$

also holds. For which we need to count the summands which are multiples of each element of (540) corresponding to each g in (541). So let us fix some element $K(m_1, m_2)$ of (540) corresponding to some $m_1 \in \mathbb{N}, m_2 \in \mathbb{N}_0, \alpha \in \mathbb{N}^{m_1}$ and $\beta \in \mathbb{N}^{m_2}$. Rephrasing this problem, we can ask which products

$$\prod_{l=1}^{g} T_{\gamma_l} \tag{545}$$

for suitable g and $(\gamma_l)_l$ produce, when commuted with a creation or annihilation operator, multiples of $K(m_1, m_2)$? We will call this number of total contributions weighted with the factor $-\frac{(-1)^g}{g}$ borrowed from (541) $\#K(m_1, m_2)$. Looking at the commutation relations (3.412) we split the set of indices $\{\gamma_1 \ldots \gamma_g\}$ into three sets A, B and C, where the commutation relation has to be used in such a way, that

$$\begin{aligned} \forall k : \gamma_k \in A &\iff \exists j \leqslant m_1 : \gamma_k = \alpha_j, \\ \wedge \forall k : \gamma_k \in B &\iff \exists j \leqslant m_2 : \gamma_k = \beta_j \\ \wedge \forall k : \gamma_k \in C &\iff \exists j \leqslant m_1, l \leqslant m_2 : \gamma_k = \alpha_j + \beta_l \end{aligned}$$

holds. Unfortunately not every splitting corresponds to a contribution and not every order of multiplication of a legal splitting corresponds to a contribution either. However $\prod_j T_{\alpha_j} \prod_j T_{\beta_j}$ gives a contribution and it is in fact the longes product that does. We may apply the commutation relations backwards to obtain any shorter valid combination and hence all combinations. Transforming the commutation rule for T_k read from right to left into a game results in the following rules. Starting from the string

$$A_1 A_2 \ldots A_{m_1} B_1 B_2 \ldots B_{m_2}, \tag{546}$$

representing the longes product, where here and in the following A's represent operators T_k which will turn into Z_k by the commutation rule, B's represent operators T_k which will stay T_k after commutation and C's represent operators T_k which will produce both a Z_l in the creation/annihilation operator and a T_{k-l} behind that operator. The indices are merely there to keep track of which operator moved where. So the game consists in the answering how many strings can we produce by applying the following rules to the initial string?

1. You may replace any occurrence of $A_k B_j$ by $B_j A_k$ for any j and k.

2. You may replace any occurrence of $A_k B_j$ by $C_{k,j}$ for any j and k.

Where we have to count the number of times we applied the second rule, or equivalently the number $\#C$ of C's in the resulting string, because the summation index g in (541) corresponds to $m_1 + m_2 - \#C$. Fix $\#C \in \{0, \ldots, \min(m_1, m_2)\}$. A valid string has $m_1 + m_2 - \#C$ characters, because the number of its Cs is $\#C$, its number of As is $m_1 - \#C$ and its number of Bs is $m_2 - \#C$. Ignoring the labelling of the As, Bs and Cs there are $\binom{m_1+m_2-\#C}{\#C\ (m_1-\#C)\ (m_2-\#C)}$ such strings. Now if we consider one such string without labelling, e.g.

$$CAABACCBBACBBABBBB, \tag{547}$$

there is only one correct labelling to be restored, namely the one where each A and the first index of any C receive increasing labels from left to right and analogously for B and the second index of any C, resulting for our example in

$$C_{1,1}A_2A_3B_2A_4C_{5,3}C_{6,4}B_5B_6A_7C_{8,7}B_8B_9A_9B_{10}B_{11}B_{12}B_{13}. \tag{548}$$

So any unlabelled string corresponds to exactly one labelled string which in turn corresponds to exactly one choice of operator product $\prod T$. So returning to our Operators, we found the number $\#K(m_1, m_2)$ it is

$$\#K(m_1, m_2) = -\sum_{g=\max(m_1,m_2)}^{m_1+m_2} \frac{(-1)^g}{g} \binom{g}{(m_1 + m_2 - g)\ (g - m_1)\ (g - m_2)}, \tag{549}$$

where the total minus sign comes from the total minus sign in front of (544) with respect to (539).

Now since we introduced the slightly non-standard definition of binomial coefficients used in [18] we can make use of the rules for summing binomial coefficients derived there. As a first step to evaluate (549) we split the trinomial coefficient into binomial ones and make use of the absorption identity

$$\forall a \in \mathbb{C}\ \forall b \in \mathbb{Z} : b\binom{a}{b} = a\binom{a-1}{b-1} \qquad \text{(absorption)}$$

for $m_2, m_1 \neq 0$ as follows

$$\begin{aligned}
&\#K(m_1, m_2)\\
&= -\sum_{g=\max(m_1,m_2)}^{m_1+m_2} \frac{(-1)^g}{g}\binom{g}{(m_1+m_2-g)\ (g-m_1)\ (g-m_2)}\\
&= -\sum_{g=\max(m_1,m_2)}^{m_1+m_2} \frac{(-1)^g}{g}\binom{g}{m_2}\binom{m_2}{g-m_1}\\
&\overset{\text{(absorption)}}{=} -\sum_{g=\max(m_1,m_2)}^{m_1+m_2} \frac{(-1)^g}{m_2}\binom{g-1}{m_2-1}\binom{m_2}{g-m_1}\\
&= \frac{-1}{m_2}\sum_{g=\max(m_1,m_2)}^{m_1+m_2} (-1)^g\binom{g-1}{m_2-1}\binom{m_2}{g-m_1}\\
&\overset{m_1>0}{=} \frac{-1}{m_2}\sum_{g\in\mathbb{Z}}(-1)^g\binom{m_2}{g-m_1}\binom{g-1}{m_2-1}\\
&\overset{*}{=} \frac{-1}{m_2}(-1)^{m_2-m_1}\binom{m_1-1}{-1} = 0,
\end{aligned}$$

where for the second but last equality $m_1 > 0$ is needed for the $g = 0$ summand not to contribute and for the marked equality we used summation rule (5.24) of [18]. So all the coefficients vanish that fulfil

$m_1, m_2 \neq 0$. The sum for the remaining cases is readily computed, since there is just one summand. Summarising we find

$$\#K(m_1, m_2) = \delta_{m_2,0}\frac{(-1)^{1+m_1}}{m_1} + \delta_{m_1,0}\frac{(-1)^{1+m_2}}{m_2},$$

where the second summand can be ignored, since terms with $m_1 = 0$ are irrelevant for our considerations.
So the left hand side of (544) can be evaluated

$$\begin{aligned}&\left[-\sum_{g=1}^{n}\sum_{\substack{\vec{b}\in\mathbb{N}^g\\|\vec{b}|=n}}\frac{(-1)^g}{g}\binom{n}{\vec{b}}\prod_{l=1}^{g}T_{b_l}, a^{\#}\right]\\&=\sum_{g=1}^{n}\sum_{\substack{\vec{b}\in\mathbb{N}^g\\|b|=n}}\frac{(-1)^{g+1}}{g}\binom{n}{\vec{b}}a^{\#}\prod_{l=1}^{g}Z_{b_l},\end{aligned}$$

which is exactly equal to the right hand side of (544). This ends the motivation of the conjecture.

Solution to Recursive Equation

So we found a recursive equation for the T_ns, now we need to solve it. In order to do so we need the following lemma about combinatorial distributions

Lemma 87. *For any $g \in \mathbb{N}, k \in \mathbb{N}$*

$$\sum_{\substack{\vec{g}\in\mathbb{N}^g\\|\vec{g}|=k}}\binom{k}{\vec{g}} = \sum_{l=0}^{g}(-1)^l(g-l)^k\binom{g}{l} \tag{550}$$

holds. The reader interested in terminology may be eager to know, that the right hand side is equal to $g!$ *times the Stirling number of the second kind* $\left\{ {k \atop g} \right\}$.

Proof: We would like to apply the multinomial theorem but there are all the summands missing where at least one of the entries of $\vec{g}$ is zero, so we add an appropriate expression of zero. We also give the expression in question a name, since we will later on arrive at a recursive expression.

$$
\begin{aligned}
F(g,k) &:= \sum_{\substack{\vec{g}\in\mathbb{N}^g \\ |\vec{g}|=k}} \binom{k}{\vec{g}} = \sum_{\substack{\vec{g}\in\mathbb{N}_0^g \\ |\vec{g}|=k}} \binom{k}{\vec{g}} - \sum_{\substack{\vec{g}\in\mathbb{N}_0^g \\ |\vec{g}|=k \\ \exists l: g_l=0}} \binom{k}{\vec{g}} \\
&= g^k - \sum_{\substack{\vec{g}\in\mathbb{N}_0^g \\ |\vec{g}|=k \\ \exists l: g_l=0}} \binom{k}{\vec{g}} = g^k - \sum_{n=1}^{g-1} \sum_{\substack{\vec{g}\in\mathbb{N}_0^g \\ |\vec{g}|=k}} \binom{k}{\vec{g}} 1_{\exists! i_1 \ldots i_n: \forall i_l \neq i_k \wedge \forall l: g_{i_l}=0} \qquad (551)
\end{aligned}
$$

where in the last line the indicator function is to enforce there being exactly n different indices i_l for which $g_{i_l} = 0$ holds. Now since it does not matter which entries of the vector vanish because the multinomial coefficient is symmetric and its value is identical to the corresponding multinomial coefficient where the vanishing entries are omitted, we can further simplify the sum:

$$
F(g,k) = g^k - \sum_{n=1}^{g-1} \binom{g}{n} \sum_{\substack{\vec{g}\in\mathbb{N}^n \\ |\vec{g}|=k}} \binom{k}{\vec{g}}
$$

The inner sum turns out to be $F(g-n,k)$, so we found the recursive relation for F:

$$F(g,k) = g^k - \sum_{n=1}^{g-1} \binom{g}{n} F(n,k) = g^k - \sum_{n=1}^{g-1} \binom{g}{n} F(g-n,k), \quad (552)$$

where for the last equality we used the symmetry of binomial coefficients. By iteratively applying this equation, we find the following formula, which we will now prove by induction

$$\forall d \in \mathbb{N}_0 : F(g,k) = \sum_{l=0}^{d} (-1)^l (g-l)^k \binom{g}{l} + (-1)^{d+1} \sum_{n=1}^{g-d-1} \binom{n+d-1}{d} \binom{g}{n+d} F(g-d-n,k). \quad (553)$$

We already showed the start of the induction, so what's left is the induction step. Before we do so the following remark is in order: We are only interested in the case $d = g$ and the formula seems meaningless for $d > g$; however, the additional summands in the left sum vanish, where as the right sum is empty for these values of d since the upper bound of the summation index is lower than its lower bound.
For the induction step, pick $d \in \mathbb{N}_0$, use (553) and pull the first summand out of the second sum, on this summand we apply the recursive relation (552) resulting in

$$F(g,k) = \sum_{l=0}^{d} (-1)^l (g-l)^k \binom{g}{l} + (-1)^{d+1} \sum_{n=2}^{g-d-1} \binom{n+d-1}{d} \binom{g}{n+d} F(g-d-n,k)$$

$$
\begin{aligned}
&+(-1)^{d+1}\binom{d}{d}\binom{g}{d+1}F(g-d-1,k)\\
&\overset{(552)}{=}\sum_{l=0}^{d+1}(-1)^l(g-l)^k\binom{g}{l}\\
&+(-1)^{d+1}\sum_{n=2}^{g-d-1}\binom{n+d-1}{d}\binom{g}{n+d}F(g-d-n,k)\\
&-(-1)^{d+1}\binom{g}{d+1}\sum_{n=1}^{g-d-2}\binom{g-d-1}{n}F(g-d-1-n,k)\\
&=\sum_{l=0}^{d+1}(-1)^l(g-l)^k\binom{g}{l}\\
&+(-1)^{d+1}\sum_{n=1}^{g-d-2}\binom{n+d}{d}\binom{g}{n+d+1}F(g-d-1-n,k)\\
&-(-1)^{d+1}\binom{g}{d+1}\sum_{n=1}^{g-d-2}\binom{g-d-1}{n}F(g-d-1-n,k). \quad (554)
\end{aligned}
$$

After the index shift we can combine the last two sums.

$$
\begin{aligned}
F(g,k) &= \sum_{l=0}^{d+1}(-1)^l(g-l)^k\binom{g}{l}\\
&+\sum_{n=1}^{g-d-2}\left[\binom{g}{d+1}\binom{g-d-1}{n}-\binom{n+d}{d}\binom{g}{n+d+1}\right]\\
&\qquad (-1)^{d+2}F(g-d-1-n,k). \quad (555)
\end{aligned}
$$

In order to combine the two binomials we reassemble $\binom{g}{d+1}\binom{g-d-1}{n}$ into $\binom{g}{n+d+1}\binom{n+d+1}{d+1}$, which can be seen to be possible by representing everything in terms of factorials. This results in

$$F(g,k) = \sum_{l=0}^{d+1} (-1)^l (g-l)^k \binom{g}{l}$$
$$+(-1)^{d+2} \sum_{n=1}^{g-d-2} \left[\binom{n+d+1}{d+1} - \binom{n+d}{d} \right] \binom{g}{n+d+1} F(g-d-1-n,k)$$
$$= \sum_{l=0}^{d+1} (-1)^l (g-l)^k \binom{g}{l}$$
$$+ (-1)^{d+2} \sum_{n=1}^{g-d-2} \binom{n+d}{d+1} \binom{g}{n+d+1} F(g-d-1-n,k), \quad (556)$$

where we used the addition formula for binomials:

$$\forall n \in \mathbb{C} \forall k \in \mathbb{Z} : \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}. \quad (557)$$

This conclues the proof by induction. By setting $d = g$ in equation (553) we arrive at the desired result. □
Using the previous lemma, we are able to show the next

Lemma 88. *For any $k \in \mathbb{N} \backslash \{1\}$ the following equation holds*

$$\sum_{g=1}^{k} \frac{(-1)^g}{g} \sum_{\substack{\vec{g} \in \mathbb{N}^g \\ |\vec{g}|=k}} \binom{k}{\vec{g}} = 0. \quad (558)$$

Proof: Let $k \in \mathbb{N} \backslash \{1\}$, as a first step we apply lemma 87. We change the order of summation, use (absorption), extend the range of summation and shift summation index to arrive at

$$\begin{aligned}\sum_{g=1}^{k}\frac{(-1)^g}{g}\sum_{l=0}^{g}(-1)^l(g-l)^k\binom{g}{l} &= \sum_{g=1}^{k}\frac{1}{g}\sum_{l=0}^{g}(-1)^{g-l}(g-l)^k\binom{g}{g-l}\\ &= \sum_{g=1}^{k}\sum_{p=0}^{g}(-1)^p p^k\frac{1}{g}\binom{g}{p} = \sum_{g=1}^{k}\sum_{p=0}^{g}(-1)^p p^k\frac{1}{p}\binom{g-1}{p-1}\\ &= \sum_{g=1}^{k}\sum_{p\in\mathbb{Z}}(-1)^p p^{k-1}\binom{g-1}{p-1} = \sum_{p\in\mathbb{Z}}(-1)^p p^{k-1}\sum_{g=1}^{k}\binom{g-1}{p-1}\\ &= \sum_{p\in\mathbb{Z}}(-1)^p p^{k-1}\sum_{g=0}^{k-1}\binom{g}{p-1}. \end{aligned} \tag{559}$$

Now we use equation (5.10) of [18]:

$$\forall m,n\in\mathbb{N}_0:\sum_{k=0}^{n}\binom{k}{m}=\binom{n+1}{m+1}, \qquad \text{(upper summation)}$$

which can for example be proven by induction on n.
We furthermore rewrite the power of the summation index p in terms of the derivative of an exponential and change order summation and differentiation. This results in

$$\begin{aligned}\sum_{g=1}^{k}\frac{(-1)^g}{g}\sum_{l=0}^{g}(-1)^l(g-l)^k\binom{g}{l} &= \sum_{p\in\mathbb{Z}}(-1)^p p^{k-1}\binom{k}{p}\\ &= \sum_{p=0}^{k}(-1)^p\,\frac{\partial^{k-1}}{\partial\alpha^{k-1}}e^{\alpha p}\bigg|_{\alpha=0}\binom{k}{p} = \frac{\partial^{k-1}}{\partial\alpha^{k-1}}\sum_{p=0}^{k}(-1)^p e^{\alpha p}\binom{k}{p}\Bigg|_{\alpha=0}\\ &= \frac{\partial^{k-1}}{\partial\alpha^{k-1}}\left(1-e^{\alpha p}\right)^k\bigg|_{\alpha=0} = (-1)^k\,\frac{\partial^{k-1}}{\partial\alpha^{k-1}}\left(\sum_{l=1}^{\infty}\frac{(\alpha p)^l}{l!}\right)^k\Bigg|_{\alpha=0}\end{aligned}$$

$$= (-1)^k \frac{\partial^{k-1}}{\partial \alpha^{k-1}} ((\alpha p)^k + \mathcal{O}((\alpha p)^{k+1}))\Big|_{\alpha=0} = 0.$$

□

We are now in a position to state the solution to the recursive equation (539) and motivate that it is in fact a solution.

Conjecture 89. *For $n \in \mathbb{N}$ the solution of the recursive equation* (539) *solely in terms of G_a and C_a is given by*

$$T_n = \sum_{g=1}^{n} \sum_{\substack{\vec{b} \in \mathbb{N}^g \\ |\vec{b}|=n}} \sum_{\vec{d} \in \{0,1\}^g} \frac{1}{g!} \binom{n}{\vec{b}} \prod_{l=1}^{g} F_{b_l, d_l}, \tag{560}$$

where F is given by

$$F_{a,b} = \begin{cases} \Gamma_a & \text{for } b = 0 \\ C_a & \text{for } b = 1 \end{cases}. \tag{561}$$

For the readers convenience we remind her, that Γ_a and the constants C_n are defined in theorem 86.

Motivation: The structure of this proof will be induction over n. For $n = 1$ the whole expression on the right hand side collapses to $C_1 + \Gamma_1$, which we already know to be equal to T_1. For arbitrary $n + 1 \in \mathbb{N} \backslash \{1\}$ we apply the recursive equation (539) once and use the induction hypothesis for all $k \leqslant n$ and thereby arrive at the rather convoluted expression

$$
\begin{aligned}
T_{n+1} &\overset{(539)}{=} \Gamma_{n+1} + C_{n+1} + \sum_{g=2}^{n+1} \sum_{\substack{\vec{b}\in\mathbb{N}^g \\ |\vec{b}|=n+1}} \frac{(-1)^g}{g} \binom{n+1}{\vec{b}} \prod_{l=1}^{g} T_{b_l} \\
&\overset{\text{induction hyp}}{=} \Gamma_{n+1} + C_{n+1} + \sum_{g=2}^{n+1} \sum_{\substack{\vec{b}\in\mathbb{N}^g \\ |\vec{b}|=n+1}} \frac{(-1)^g}{g} \binom{n+1}{\vec{b}} \prod_{l=1}^{g} \\
&\qquad \sum_{g_l=1}^{b_l} \sum_{\substack{\vec{c}_l\in\mathbb{N}^{g_l} \\ |\vec{c}_l|=b_l}} \sum_{\vec{e}_l\in\{0,1\}^{g_l}} \frac{1}{g_l!} \binom{b_l}{\vec{c}_l} \prod_{k=1}^{g_l} F_{c_{l,k},e_{l,k}}.
\end{aligned} \tag{562}
$$

If we were to count the contributions of this sum to a specific product $\prod F_{c_j,e_j}$ for some choice of $(c_j)_j, (e_j)_j$ we would first recognize that all the multinomial factors in (562) combine to a single one whose indices are given by the first indices of all the F factors involved. Other than this factor each contribution adds $\frac{(-1)^g}{g}\prod_{l=1}^{g}\frac{1}{g_l!}$ to the sum. So we need to keep track of how many contributions there are and which distributions of g_l they belong to.

Fix some product $\prod F := \prod_{j=1}^{\tilde{g}} F_{\tilde{b}_j,\tilde{d}_j}$. In the sum (562) we pick some initial short product of length g and split each factor into g_l pieces to arrive at one of length $\tilde{g}$ if the product is to contribute to $\prod F$. So clearly $\sum_{l=1}^{g} g_l = \tilde{g}$ holds for any contribution to $\prod F$. The reverse is also true, for any g and $g_1,\dots,g_g \in \mathbb{N}$ such that $\sum_{l=1}^{g} g_l = \tilde{g}$ holds the corresponding expression in (562) contributes to $\prod F$. Furthermore $\prod F$ and g, $g_1,\dots g_g$ is enough to uniquely determine the summand of (562) the contribution belongs to. For an illustration of this splitting see

$$\underbrace{\underbrace{F^1_{3,1}F^2_{2,0}F^3_{7,1}}_{g_1=3}\;\underbrace{F^4_{5,0}}_{g_2=1}\;\underbrace{F^5_{4,1}F^6_{2,1}}_{g_3=2}\;\underbrace{F^7_{1,1}F^8_{3,0}F^9_{4,1}}_{g_4=3}\;\underbrace{F^{10}_{4,1}F^{11}_{1,0}}_{g_5=2}}_{g=5}$$

$$g_1+g_2+g_3+g_4+g_5=11=\tilde{g},$$

where I labelled the factors in the upper right index for the readers convenience. We recognize that the sum we are about to perform is by no means unique for each order of n but only depends on the number of appearing factors and the number of splittings performed on them. By the preceding argument we need

$$\sum_{g=2}^{\tilde{g}}\frac{(-1)^g}{g}\sum_{\substack{\vec{g}\in\mathbb{N}^g\\|\vec{g}|=\tilde{g}}}\prod_{l=1}^{g}\frac{1}{g_l!}=\frac{1}{\tilde{g}!} \tag{563}$$

to hold for $\tilde{g}>1$, in order to find agreement with the proposed solution (561). Now proving (563) is done by realizing, that one can include the right hand side into the sum as the $g=1$ summand, dividing the equation by $\tilde{g}!$ and using lemma 88 with $k=\tilde{g}$. The remaining case, $\tilde{g}=1$, can directly be read off of (562). This ends the motivation of this conjecture.

Conjecture 90. *For $n\in\mathbb{N}$, T_n can be written as*

$$\frac{1}{n!}T_n=\sum_{\substack{1\leqslant c+g\leqslant n\\c,g\in\mathbb{N}_0}}\;\sum_{\substack{\vec{g}\in\mathbb{N}^g\\\vec{c}\in\mathbb{N}^c\\|\vec{c}|+|\vec{g}|=n}}\frac{1}{c!g!}\prod_{l=1}^{c}\frac{1}{c_l!}C_{c_l}\prod_{l=1}^{g}\frac{1}{g_l!}\Gamma_{g_l}. \tag{564}$$

Please note that for ease of notation we defined $\mathbb{N}^0:=\{1\}$.

Motivation: By an argument completely analogous to the combinatorial argument in the motivation of conjecture (86) we see that we can disentangle the Fs in (560) into Γs and Cs if we multiply by a factor of $\binom{c+g}{c}$ where c is the number of Cs and g is the number of Γs giving

$$T_n = \sum_{\substack{1 \leqslant c+g \leqslant n \\ c,g \in \mathbb{N}_0}} \sum_{\substack{\vec{g} \in \mathbb{N}^g \\ \vec{c} \in \mathbb{N}^c \\ |\vec{c}|+|\vec{g}|=n}} \binom{c+g}{c} \frac{1}{(c+g)!} \binom{n}{\vec{g} \oplus \vec{c}} \prod_{l=1}^{c} C_{c_l} \prod_{l=1}^{g} \Gamma_{g_l}, \quad (565)$$

which directly reduces to the equation we wanted to prove, by plugging in the multinomials in terms of factorials.

Conjecture 91. *As a formal power series, the second quantized scattering operator can be written in the form*

$$S = e^{\sum_{l \in \mathbb{N}} \frac{C_l}{l!}} e^{\sum_{l \in \mathbb{N}} \frac{\Gamma_l}{l!}}. \quad (566)$$

Proof: We plug conjecture 90 into the defining Series for the T_ns giving

$$S = \sum_{n \in \mathbb{N}_0} \frac{1}{n!} T_n \quad (567)$$

$$= \mathbb{1}_{\mathcal{F}} + \sum_{n \in \mathbb{N}} \sum_{\substack{1 \leqslant c+g \leqslant n \\ c,g \in \mathbb{N}_0}} \sum_{\substack{\vec{g} \in \mathbb{N}^g \\ \vec{c} \in \mathbb{N}^c \\ |\vec{c}|+|\vec{g}|=n}} \frac{1}{c!g!} \prod_{l=1}^{c} \frac{1}{c_l!} C_{c_l} \prod_{l=1}^{g} \frac{1}{g_l!} \Gamma_{g_l} \quad (568)$$

$$= \mathbb{1}_{\mathcal{F}} + \sum_{\substack{1 \leqslant c+g \\ c,g \in \mathbb{N}_0}} \sum_{\substack{\vec{g} \in \mathbb{N}^g \\ \vec{c} \in \mathbb{N}^c}} \frac{1}{c!g!} \prod_{l=1}^{c} \frac{1}{c_l!} C_{c_l} \prod_{l=1}^{g} \frac{1}{g_l!} \Gamma_{g_l} \quad (569)$$

$$= \sum_{c,g \in \mathbb{N}_0} \sum_{\substack{\vec{g} \in \mathbb{N}^g \\ \vec{c} \in \mathbb{N}^c}} \frac{1}{c!g!} \prod_{l=1}^{c} \frac{1}{c_l!} C_{c_l} \prod_{l=1}^{g} \frac{1}{g_l!} \Gamma_{g_l} \quad (570)$$

$$= \sum_{c\in\mathbb{N}_0} \frac{1}{c!} \sum_{\vec{c}\in\mathbb{N}^c} \prod_{l=1}^{c} \frac{1}{c_l!} C_{c_l} \sum_{g\in\mathbb{N}_0} \frac{1}{g!} \sum_{\vec{g}\in\mathbb{N}^g} \prod_{l=1}^{g} \frac{1}{g_l!} \Gamma_{g_l} \tag{571}$$

$$= \sum_{c\in\mathbb{N}_0} \frac{1}{c!} \prod_{l=1}^{c} \sum_{k\in\mathbb{N}} \frac{1}{k!} C_k \sum_{g\in\mathbb{N}_0} \frac{1}{g!} \prod_{l=1}^{g} \sum_{b\in\mathbb{N}} \frac{1}{b!} \Gamma_b \tag{572}$$

$$= \sum_{c\in\mathbb{N}_0} \frac{1}{c!} \left(\sum_{k\in\mathbb{N}} \frac{1}{k!} C_k \right)^c \sum_{g\in\mathbb{N}_0} \frac{1}{g!} \left(\sum_{b\in\mathbb{N}} \frac{1}{b!} \Gamma_b \right)^g \tag{573}$$

$$= e^{\sum_{l\in\mathbb{N}} \frac{1}{l!} C_l} e^{\sum_{l\in\mathbb{N}} \frac{1}{l!} \Gamma_l}. \tag{574}$$

Conjecture 92. *For A such that*

$$\|\mathbb{1} - U^A\| < 1. \tag{575}$$

The second quantized scattering operator fulfils

$$S = e^{\sum_{n\in\mathbb{N}} \frac{C_n}{n!}} e^{\mathrm{d}\Gamma(\ln(U))} \tag{576}$$

where C_n must be imaginary for any $n \in \mathbb{N}$ in order to satisfy unitarity.

Motivation: First the remark about $C_n \in i\mathbb{R}$ for any n is a direct consequence of the second factor of (576) begin unitary. This in turn follows directly from $\mathrm{d}\Gamma^*(K) = -\mathrm{d}\Gamma(K)$ for any K in the domain of $\mathrm{d}\Gamma$. That $\ln U$ is in the domain of $\mathrm{d}\Gamma$ follows from $(\ln U)^* = \ln U^* = \ln U^{-1} = -\ln U$ and $\|U - \mathbb{1}\| < 1$.

We are going to change the sum in the second exponential of (566), so let's take a closer look at that: by exchanging summation we can step by step simplify

$$
\begin{aligned}
\sum_{l\in\mathbb{N}}\frac{\Gamma_l}{l!} &= \sum_{n\in\mathbb{N}}\frac{1}{n!}\mathrm{d}\Gamma\left(\sum_{g=1}^{n}\sum_{\substack{\vec{b}\in\mathbb{N}^g\\|\vec{b}|=n}}\frac{(-1)^{g+1}}{g}\binom{n}{\vec{b}}\prod_{l=1}^{g}Z_{b_l}\right)\\
&= \mathrm{d}\Gamma\left(\sum_{n\in\mathbb{N}}\frac{1}{n!}\sum_{g=1}^{n}\sum_{\substack{\vec{b}\in\mathbb{N}^g\\|\vec{b}|=n}}\frac{(-1)^{g+1}}{g}\binom{n}{\vec{b}}\prod_{l=1}^{g}Z_{b_l}\right)\\
&= \mathrm{d}\Gamma\left(\sum_{n\in\mathbb{N}}\sum_{g=1}^{n}\sum_{\substack{\vec{b}\in\mathbb{N}^g\\|\vec{b}|=n}}\frac{(-1)^{g+1}}{g}\prod_{l=1}^{g}\frac{Z_{b_l}}{b_l!}\right)\\
&= \mathrm{d}\Gamma\left(\sum_{g\in\mathbb{N}}\sum_{\vec{b}\in\mathbb{N}^g}\frac{(-1)^{g+1}}{g}\prod_{l=1}^{g}\frac{Z_{b_l}}{b_l!}\right)\\
&= \mathrm{d}\Gamma\left(\sum_{g\in\mathbb{N}}\frac{(-1)^{g+1}}{g}\prod_{l=1}^{g}\left(\sum_{b_l\in\mathbb{N}}\frac{Z_{b_l}}{b_l!}\right)\right)\\
&= \mathrm{d}\Gamma\left(\sum_{g\in\mathbb{N}}\frac{(-1)^{g+1}}{g}\left(\sum_{b\in\mathbb{N}}\frac{Z_b}{b!}\right)^g\right)\\
&= \mathrm{d}\Gamma\left(\sum_{g\in\mathbb{N}}\frac{(-1)^{g+1}}{g}\left(U-\mathbb{1}\right)^g\right) = \mathrm{d}\Gamma\left(-\sum_{g\in\mathbb{N}}\frac{1}{g}\left(\mathbb{1}-U\right)^g\right)\\
&= \mathrm{d}\Gamma\left(\ln\left(\mathbb{1}-\left(\mathbb{1}-U\right)\right)\right) = \mathrm{d}\Gamma\left(\ln\left(U\right)\right). \qquad (577)
\end{aligned}
$$

The last conjecture is proven directly in section 3.5.4

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