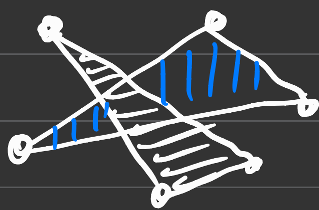
OCTOBER 9, 2019.

Rank Reduction for SDPs

THEOREM. [Radon's Theorem] Let S be a set of $d+2$ points in $\mathbb{R}^d$. Then $\exists\, P, Q$ such that

$$P \cap Q = \phi \; ; \; P \cup Q = S \; ; \; \mathrm{conv}(P) \cap \mathrm{conv}(Q) \neq \phi.$$



HOMOGENOUS

$\sum \alpha_i x_i = 0$ $\quad$ $d+1$ equations.

$\sum \alpha_i = 0$ $\quad$ $d+2$ variables.

$\Downarrow$

$\exists\, \alpha \neq 0$ satisfying this.

$P = \{i : \alpha_i > 0\}$, $Q = [d] - P$. $\quad \sum_P \alpha_i = Z$.

$$\frac{1}{Z} \sum_P \alpha_i x_i = \frac{1}{Z} \sum_Q \alpha_i x_i.$$

S_P, S_Q witness. ■

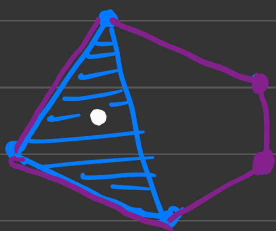
APPLICATION [VC Dimen] Take linear classifiers in d dimensions. $VC = \max_n \{n\text{-set } S \text{ is perfectly classifiable } \forall \text{ labellings}\}$.

Linear classifier can shatter $d+1$ points (simplex).

But given $d+2$, $\exists$ a labelling such that polytope distance between $P, N = 0$.

$$VC(\text{Linear Classifier}) = d+1.$$

THEOREM. [CARATHEODORY] If $x \in \mathrm{conv}(P)$, then $\exists Q \subseteq P$, $|Q| = d+1$, $x \in \mathrm{conv}(Q)$.



$x = \sum_{i=1}^{m} \alpha_i x_i$; $x_i \in P$. $\sum \alpha_i = 1$ $\alpha_i > 0$.

finitely supported.

minimal representation.

wlog $m \geq d+2$.

Claim that $\exists B \neq 0$ $\sum B_i x_i = 0$; $\sum B_i = 0$ (RADON-like)

($d+1$ equations ; $d+2$ variables).

Define $\gamma = \max\{\gamma \geq 0 : \alpha_i - \gamma B_i \geq 0 \ \forall i\}$.

$\Downarrow$ exists because $\exists i$, $B_i > 0$.

$\Delta_i = \alpha_i - \gamma B_i$. $x = \sum_{i=1}^{m} \Delta_i x_i$, $\sum \Delta_i = 1$, $\Delta_i \geq 0$.

Also it is $m-1$ supported ie. $\Delta_i = 0$ for $\exists i$. ∎

SPARSIFYING LPs

In general, sparsest solution to a linear system is NP-hard.

Let $x = \mathrm{SOLVE}(\min_x c^T x ; Ax = b ; x \geq 0)$. Posit $x^+ = x + \alpha\delta$, $\alpha > 0$.

x^+ is OPT. $c^T\delta = 0$. ← Subsume. O/w x is NOT OPT. why? Add $\pm$(small) δ.

x^+ is feasible. $A\delta = 0$

$x^+ \geq 0$ or $\delta_j = 0 \Leftrightarrow x_j = 0$.

* m equations, $|x|$ variables. ALA $m < |x|$, $\exists \delta \neq 0$, $|x^+| < |x|$.

∴ can reduce until $|x| \leq m$.

SDP EQUIVALENT

Let $X = \text{SOLVE}(\min_X \langle C, X\rangle ; A(X) = b ; X \succeq 0)$. Posit $X^+ = X + VSV^T\alpha$.

where $X = V\Sigma V^T$, S is symmetric.

X^+ is OPT. $\langle C, VSV^T\rangle = 0$ SUBSUMED.

X^+ is feasible. $A(VSV^T) = 0$ * m constraints, $r(r+1)/2$ variables.

CAN find $S \neq 0$ to construct $r(X^+) < r(X)$

whenever $m < r(r+1)/2$.

$\therefore$ can reduce $\underline{\underline{r(r+1)/2 \leq m.}}$ [PATAKI, BARVINOK].

$$\boxed{\text{RANK}(X) \leq \sqrt{2m + \frac{1}{4}} - \frac{1}{2}.}$$