Periodic motion:

Any motion which is repeated after equal intervals of time is called ‘periodic motion’ or ‘harmonic motion’. If a particle having periodic motion moves back and forth over the same path, it is said be ‘oscillating’ or ‘vibrating’. The pendulum of a clock swings back and forth and is said to perform an oscillatory motion. Similarly, the motion of the prongs of a tuning fork, the internal motion of atoms within molecules, the motion of electric and magnetic fields in an electromagnetic wave are the examples of oscillatory motion.

Periodic time:

The ‘periodic time’ T of an oscillatory motion is defined as the time taken for one complete round trip of the motion *i.e.* for one oscillation (or one cycle).

Frequency:

The frequency of the motion, n, is defined as the number of oscillations in 1 second. Obviously, the frequency is the reciprocal of the periodic time.

$$n = \frac{1}{T} \text{ (cycles per sec.)}$$

Displacement:

There is a position for every oscillating particle at which no net force acts on the particle. This is called the ‘equilibrium position’ of the particle. The distance of the particle in any direction from its equilibrium position at any instant is called the ‘displacement’ of that particle in that direction at that instant. The displacement changes periodically in magnitude and direction.

Amplitude:

The ‘amplitude’ **a** of the oscillations is the distance between the equilibrium position and an extreme position. It is the maximum displacement.

Phase:

The ‘phase’ of an oscillating particle at any instant defines the stage of the particle as regards its position and direction of motion at that instant.

Mechanical energy:

The total mechanical energy E of an oscillating particle consists of kinetic energy K and potential energy U.

$$E=K+U$$

The particle has kinetic energy by virtue of its motion, and potential energy by virtue of its displacement from the equilibrium position.

If there are no non-conservative forces (like friction) acting on the particle, the total energy E remains constant.

Restoring force:

The oscillating particle can remain at rest in its equilibrium position at which no net force acts upon it. When it is displaced from this position, a periodic force acts upon it in such a direction as to return it to its equilibrium position. This is called the ‘restoring force’ F.

We know that the potential energy can also be defined as a function of position whose negative gradient gives the intrinsic force. Hence the (restoring) force acting on the particle at any position is derivable from the potential energy function:

$$F = -\frac{dU}{dx}$$

Simple harmonic motion:

The simple harmonic motion is the simplest possible periodic motion. *It is defined as the motion of the oscillating particle moving back and forth about an equilibrium position through a (restoring) force which is directly proportional to the displacement but opposite to it in it direction.* The particle is called a ‘simple harmonic oscillator’.

Examples: The backward and forward swing of a pendulum, the up and down motion of a weight hanging on a spring, and the twisting and untwisting motion of a body suspended by a wire are the examples of motions which are approximately simple harmonic.

Equation of mation of a Simple Harmonic Oscillator :-

Let us consider a particle P of mass 'm' executing S.H.M. about an equilibrium position 'O'. By definition, the force under which the particle is oscillating is proportional to the displacement of 'P' from 'O' and is directed toward 'O'. Thus if x be the displacement of 'P' from 'O' at any instant t. The instantaneous force F acting upon it is given by.

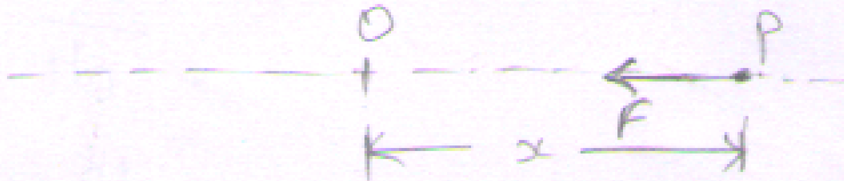


Fig (1)

$$F = -kx,$$

where k is the proportionality factor (force per unit displacement). The minus sign shows that the force F is opposite to the displacement x.

According to Newton's second law, the force acting on the particle is equal to the product of the mass and the acceleration. The instantaneous acceleration of the particle is d^2x/dt^2. Hence

$$-kx = m\frac{d^2x}{dt^2}$$

or

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x.$$

Let us put the constant, $k/m = \omega^2$. Then we have

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \quad \text{- - - - -} \quad (i)$$

This is the differential equation of a simple harmonic oscillator.

Let a solution to this equation be

$$x = C e^{\alpha t},$$

where C and α are arbitrary constants. On differentiating with respect to t, we get

$$\frac{dx}{dt} = C e^{\alpha t} \alpha$$

and

$$\frac{d^2x}{dt^2} = C e^{\alpha t} \alpha^2.$$

Substituting these values of $\frac{d^2x}{dt^2}$ and x in (i), we get

$$C e^{\alpha t} \alpha^2 + \omega^2 C e^{\alpha t} = 0$$

(or)

$$C e^{\alpha t} (\alpha^2 + \omega^2) = 0$$

(or)

$$\alpha^2 + \omega^2 = 0$$

(or)

$$\alpha = \pm \sqrt{(-\omega^2)} = \pm j\omega,$$

where $j = \sqrt{(-1)}$. Thus, there are two possible solutions for eq. (i), which are

$$x = C e^{+j\omega t} \quad \text{and} \quad x = C e^{-j\omega t}$$

Hence the general solution to eq (i) will be the sum of these two solutions, that is

$$x = C_1 e^{+j\omega t} + C_2 e^{-j\omega t}$$

where C_1 and C_2 are arbitrary constants. This gives

$$x = C_1 (\cos\omega t + j \sin\omega t) + C_2 (\cos\omega t - j \sin\omega t)$$

$$= (C_1 + C_2)\cos\omega t + j(C_1 - C_2)\sin\omega t.$$

Let us make a change in arbitrary constants by putting

$$C_1 + C_2 = a \sin\phi$$

and $j(C_1 - C_2) = a\cos\phi$,

where a and ϕ are new constants, this gives

$$x = a \sin\phi \cos\omega t + a\cos\phi \sin\omega t$$

(or) $$\boxed{x = a\sin(\omega t + \phi)} \quad \text{------ (ii)}$$

This is, in fact, a solution of the equation of S.H.M. The values of the constants 'a' and 'ϕ' depend upon how the motion is started,

<u>Amplitude and Phase</u>:- In the above equation the term $\sin(\omega t + \phi)$ takes on values from $+1$ to -1. Hence the maximum value of the displacement 'x' is 'a' which is the "amplitude" of the motion.

The quantity $(\omega t + \phi)$ is called the "phase" of the motion. The constant ϕ is called the "phase constant".

<u>Velocity</u>:- The velocity 'u' of the oscillating particle can be obtained by differentiating eq. (ii). Thus

$$u = \frac{dx}{dt} = \omega a \cos(\omega t + \phi) = \omega\sqrt{a^2 - x^2}$$

This shows that the velocity is maximum $(= \omega a)$

when the displacement x (and hence the force F) is zero and vice versa.

Periodic time:- If the time 't' in eq (ii) is increased by $2\pi/\omega$, the displacement becomes

$$x = a \sin[\omega(t + 2\pi/\omega) + \phi]$$
$$= a \sin(\omega t + 2\pi + \phi)$$
$$= a \sin(\omega t + \phi)$$

That is, the displacement repeats itself after a time $2\pi/\omega$. Therefore, $2\pi/\omega$ is the "period" of the motion, T. Thus

$$T = \frac{2\pi}{\omega}.$$

But $\omega^2 = \frac{k}{m}$. Therefore the periodic time is

$$T = 2\pi\sqrt{\frac{m}{k}} \quad \text{--- (iii)}$$

$$= 2\pi\sqrt{\left(\frac{\text{mass}}{\text{force per unit displacement}}\right)}$$

$$= 2\pi\sqrt{\left(\frac{1}{\text{acceleration per unit displacement}}\right)}$$

$$= 2\pi\sqrt{\left(\frac{\text{displacement}}{\text{acceleration}}\right)}$$

The eq (iii) shows that the period is independent of the amplitude of the motion.

<u>Frequency</u>: The frequency 'n' of the oscillator is

$$n = \frac{1}{T} = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\left(\frac{k}{m}\right)}$$

$\omega\,(=2\pi n)$ is called the angular frequency.

Relations between Displacement, Velocity and acceleration.

The displacement of the particle in S.H.M is given by eq (ii) i.e.

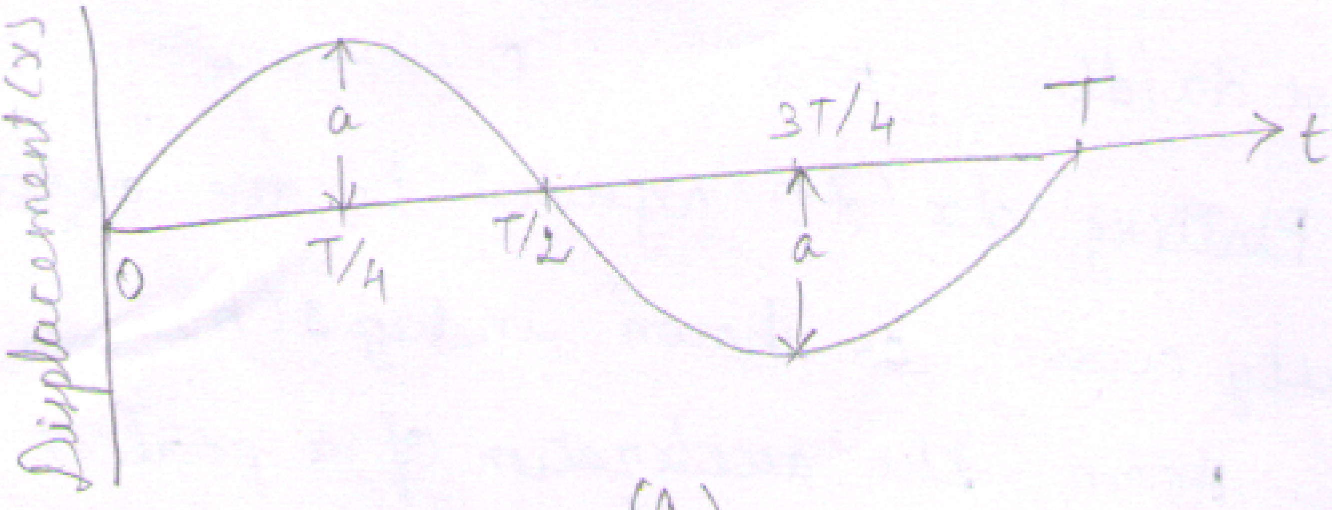
$$x = a\sin(\omega t + \phi)$$

Let us take, for simplicity $\phi = 0$ and put $\omega = \frac{2\pi}{T}$. Then

$$x = a\sin\frac{2\pi t}{T}$$

Thus at $t =$	0	$T/4$	$T/2$	$3T/4$	T
we have $x =$	0	a	0	$-a$	0

Plotting 'x' against 't', we obtain a "displacement curve" as shown in Fig. (2) (a).



(a)

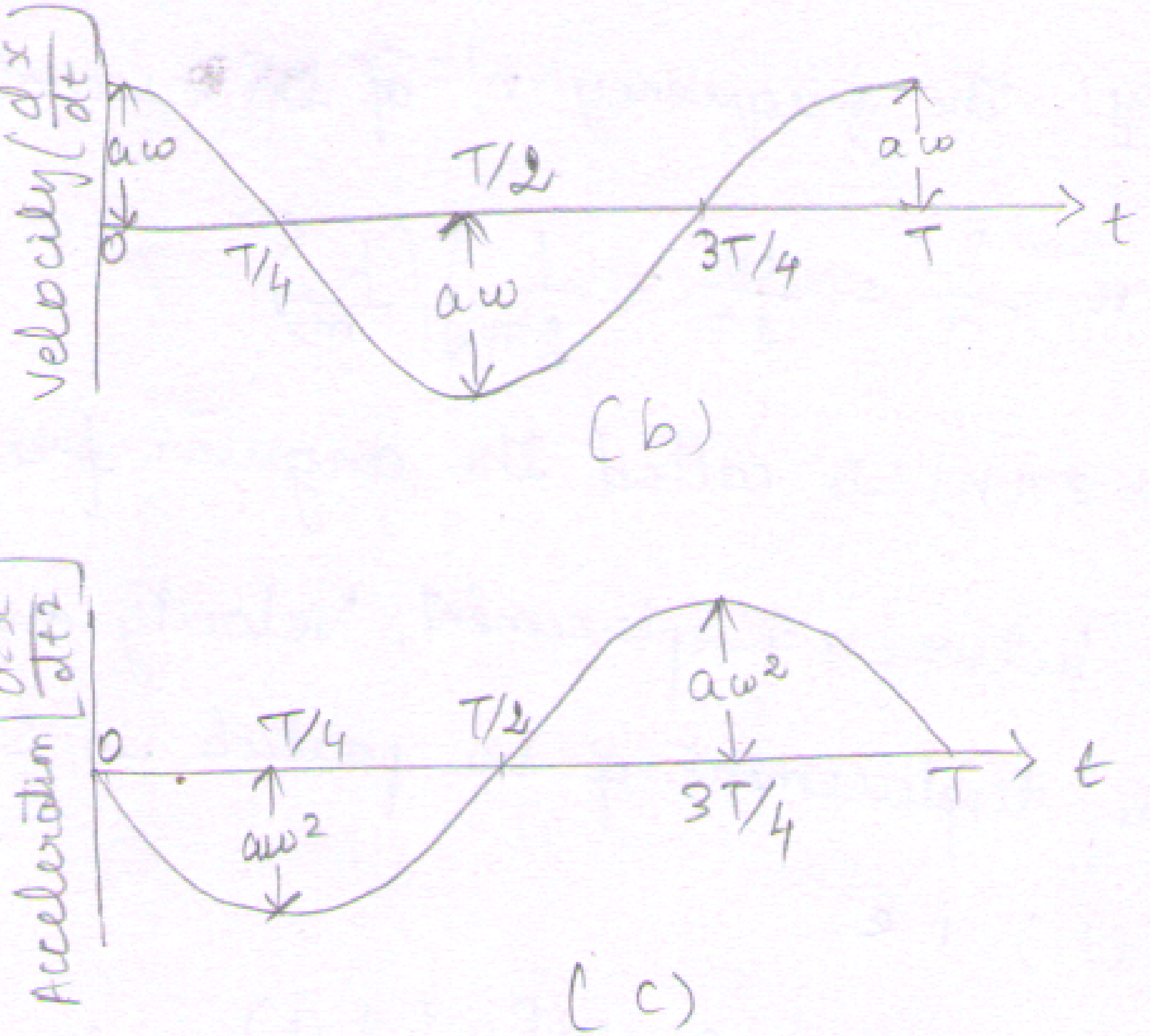


Fig. (2)

Now the velocity of a particle in S.H.M is given by

$$u = \frac{dx}{dt} = \omega a \cos(\omega t + \phi).$$

Again, taking $\phi = 0$ and putting $\omega = \frac{2\pi}{T}$, we get

$$u = \frac{dx}{dt} = \omega a \cos\frac{2\pi t}{T}$$

Thus at $t =$	0	$T/4$	$T/2$	$3T/4$	T
we have $dx/dt =$	ωa	0	$-\omega a$	0	ωa

Plotting dx/dt against t, we obtain a "velocity curve" as shown in Fig 2(b).

Again, the acceleration of a particle in S.H.M is obtained by differentiating eq (ii) twice, i.e.,

$$f = \frac{d^2x}{dt^2} = -\omega^2 a \sin(\omega t + \phi).$$

Thus taking $\phi = 0$ and putting $\omega = \frac{2\pi}{T}$, we get

$$f = \frac{d^2x}{dt^2} = -\omega^2 a \sin\frac{2\pi t}{T}$$

Thus at $t =$	0	$T/4$	$T/2$	$3T/4$	T
we have $d^2x/dt^2 =$	0	$-\omega^2 a$	0	$\omega^2 a$	0

Plotting d^2x/dt^2 against t, we obtain an "acceleration curve" as shown in Fig 2(c).

A look on these curves tells that the velocity is maximum when the displacement is zero, and is zero when the displacement is maximum. The acceleration curve is a mirror image of the displacement curve. This means that the acceleration is always proportional but opposite to the displacement.