UNIT - 1

RANK OF A MATRIX

Rank of a Matrix:-

If $A_{m \times n} \Rightarrow$ No. of linearly independent row or column vector.

$V_1 \sim V_2$

$\exists \alpha \neq 0 \in \mathbb{R}$, such that, $V_1 = \alpha V_2$.

Triangular matrix, $a_{ij} = 0$, if $i < j$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$

If the element under the diagonals are zero

Row Reduced - Form:-

To reduce upper triangular matrix

$\Rightarrow$ No. of linearly independent matrix

① $R_i \leftrightarrow R_j$ (Transverse)

② αR_i (or) R_i/α

Problem 1:-

Find the rank of the matrix $A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix} \quad R_2 \rightarrow R_2 - 4R_1$$

rank $(A) = 2$.

Problem 2:-

Find the rank of the matrix $A = \begin{bmatrix} 2 & 3 \\ 8 & 12 \end{bmatrix}$

$$\sim \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix} \quad R_2 \rightarrow R_2 - 4R_1$$

$r(A) = 1$

Problem 3:-

Find the rank of the matrix $A = \begin{bmatrix} 3 & 7 \\ 1 & 5 \end{bmatrix}$

$$\sim \begin{bmatrix} 3 & 7 \\ 1 & 5 \end{bmatrix} \quad R_2 \rightarrow 3R_2 - R_1$$

$$\begin{bmatrix} 3 & 7 \\ 0 & 8 \end{bmatrix}$$

Rank of A = 2.

Problem 4:-

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad R_2 \rightarrow R_2 - 4R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \end{bmatrix} \quad R_2 \rightarrow R_2 - 4R_1$$

Rank (A) = 2.

Property:-

If $A_{m \times n} \Rightarrow \text{rank}(A) \leq \min \langle n, m \rangle$

Problem 5:-

$$A = \begin{bmatrix} 1 & 4 & -1 \\ 2 & -2 & 1 \\ 4 & 6 & -3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 4 & -1 \\ 0 & -10 & 3 \\ 0 & -10 & 1 \end{bmatrix} \quad \begin{matrix} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - 4R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 4 & -1 \\ 0 & -10 & 3 \\ 0 & 0 & -2 \end{bmatrix} \quad R_3 \rightarrow R_3 - R_2$$

$\gamma(A) = 3.$

Problem 6:-

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 3 & -2 & 0 \\ 0 & -2 & -3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 3 \\ 0 & 4 & -9 \\ 0 & -2 & -3 \end{bmatrix} \quad R_2 \to R_2 - 3R_1$$

$$\sim \begin{bmatrix} 1 & -2 & 3 \\ 0 & 4 & -9 \\ 0 & 0 & -15 \end{bmatrix} \quad R_3 \to 2R_3 + R_2$$

$$\text{Rank}(A) = 3$$

Problem 7:-

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 1 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & -1 \\ 3 & 1 & 2 \end{bmatrix} \quad R_2 \to R_2 - 2R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & -1 \\ 0 & -5 & -7 \end{bmatrix} \quad R_3 \to R_3 - 3R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -7 \\ 0 & 0 & -1 \end{bmatrix} \quad R_2 \leftrightarrow R_3$$

$$\text{Rank}(A) = 3.$$

Problem 8:-

$$A = \begin{bmatrix} 1 & -1 & -1 & 3 \\ 2 & 1 & -2 & -1 \\ 7 & 2 & -7 & 4 \end{bmatrix}_{3\times 4}$$

$$\sim \begin{bmatrix} 1 & -1 & -1 & 3 \\ 0 & 3 & 0 & -7 \\ 0 & 9 & 0 & -17 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 7R_1 \end{matrix}$$

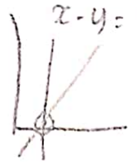
$$\sim \begin{bmatrix} 1 & -1 & -1 & 3 \\ 0 & 3 & 0 & -7 \\ 0 & 0 & 0 & 4 \end{bmatrix} \quad R_3 \to 3R_3 - 9R_2$$

$r(A) = 3$

System Of Linear Equation :-

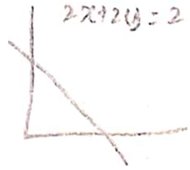
$ax + by = c \to$ Infinite number of solutions.

$x + y = 1$
$x - y = 1$



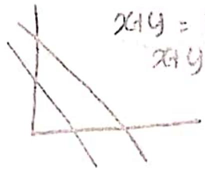
Unique solution

$x + y = 1$
$2x + 2y = 2$



No solution

$x + y = 1$
$x + y = 2$



Infinite Number of solution.

Theorem :-

Any system of linear equation $\underset{m\times n}{A}\underset{n\times 1}{x} = b$, then solutions are,

i) Unique solution
ii) Infinitely many solution
iii) No solution

$\underset{m\times n}{A} x = b$ $\left[\begin{array}{l} \because m \to \text{Number of equation} \\ n \to \text{number of unknowns} \end{array}\right]$

Consistant: $r(A) = r(A, b)$

- Unique: $r(A) = r(A, b) = n$ (No. of Unknowns)
- infinite: $r(A) = r(A, b) < n$ (No. of unknowns)

Inconsistant: $r(A) \neq r(A, b)$ → No solution.

① Test for a consistency of a system,

$x - y + z = -9$

$2x - 2y + 2z = -18$

$3x - 3y + 3z = -27$

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & -2 & 2 \\ 3 & -3 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -9 \\ -18 \\ -27 \end{bmatrix}$$

$$[A, b] = \left[\begin{array}{ccc|c} 1 & -1 & 1 & -9 \\ 2 & -2 & 2 & -18 \\ 3 & -3 & 3 & -27 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & -9 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$r(A, b) = 1$

$r(A) = 1$

$r(A) = r(A, b) < 3$

$\therefore$ Infinitely many solutions.

2. $4x - 2y + 6z = 8$

$x + y - 3z = -1$

$15x - 3y + 9z = 21$

$$\begin{bmatrix} 4 & -2 & 6 \\ 1 & 1 & -3 \\ 15 & -3 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -1 \\ 21 \end{bmatrix}$$

$$[A, b] = \left[\begin{array}{ccc|c} 4 & -2 & 6 & 8 \\ 1 & 1 & -3 & -1 \\ 15 & -3 & 9 & 21 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & 1 & -3 & -1 \\ 4 & -2 & 6 & 8 \\ 15 & -3 & 9 & 21 \end{array}\right] \quad R_1 \leftrightarrow R_2$$

$$= \left[\begin{array}{ccc|c} 1 & 1 & -3 & -1 \\ 0 & -6 & 18 & 12 \\ 0 & -18 & 54 & 36 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 4R_1 \\ R_3 \to R_3 - 15R_1 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 1 & -3 & -1 \\ 0 & -6 & 18 & 12 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_3 \to 3R_2 + R_3$$

$r(A, b) = 2$

$r(A) = 2$

$n = 3$

$r(A, b) = r(a) < 3$

$\therefore$ Infinitely many solutions.

3. $x + 2y - z = 3$

$3x - y + 2z = 1$

$x - 2y + 3z = 3$

$x - y + z = -1.$

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 1 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 3 \\ -1 \end{bmatrix}$$

$$[A, b] = \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \\ 1 & -2 & 3 & 3 \\ 1 & -1 & 1 & -1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -1 & 5 & -8 \\ 1 & -2 & 3 & 3 \\ 0 & -3 & -2 & -4 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_4 \to R_4 - R_3 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 1 & -2 & 3 & 3 \\ 0 & -1 & 5 & -8 \\ 0 & -3 & -2 & -4 \end{array}\right] \quad R_2 \leftrightarrow R_3$$

$$= \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -4 & 4 & 0 \\ 0 & -1 & 5 & -8 \\ 0 & -3 & -2 & -4 \end{array}\right] \quad R_2 \to R_2 - R_1$$

$$= \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -4 & 4 & 0 \\ 0 & -1 & 5 & -8 \\ 0 & 0 & 13 & -28 \end{array}\right] R_4 \to R_4 + 3R_3$$

Another method,

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 1 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 3 \\ -1 \end{bmatrix}$$

$$[A, b] = \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \\ 1 & -2 & 3 & 3 \\ 1 & -1 & 1 & -1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & -1 & 1 & -1 \\ 3 & -1 & 2 & 1 \\ 1 & -2 & 3 & 3 \\ 1 & 2 & -1 & 3 \end{array}\right] R_4 \leftrightarrow R_1$$

$$= \left[\begin{array}{ccc|c} 1 & -1 & 1 & -1 \\ 0 & 2 & -1 & 4 \\ 0 & -1 & 2 & 4 \\ 0 & 3 & -2 & 4 \end{array}\right] \begin{array}{l} R_4 \to R_4 - R_1 \\ R_3 \to R_3 - R_1 \\ R_2 \to R_2 - 3R_1 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & -1 & 1 & -1 \\ 0 & 2 & -1 & 4 \\ 0 & 0 & 3 & 12 \\ 0 & 0 & -1 & -4 \end{array}\right] \begin{array}{l} R_3 \to 2R_3 + R_2 \\ R_4 \to 2R_4 - 3R_2 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & -1 & 1 & -1 \\ 0 & 2 & -1 & 4 \\ 0 & 0 & 3 & 12 \\ 0 & 0 & 0 & 0 \end{array}\right] R_4 \to 3R_4 + R_3$$

$\text{rank}(A) = 3$

$\text{rank}(A, b) = 3$

$n = 3$

$r(A) = r(A, b) = 3$

$\therefore$ consistent Unique solution.

4. $x_1+2x_2+3x_3+4x_4=5$

$6x_1+7x_2+8x_3+9x_4=10$

$11x_1+12x_2+13x_3+14x_4=15$

$16x_1+17x_2+18x_3+19x_4=20$

$21x_1+22x_2+23x_3+24x_4=25$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 6 & 7 & 8 & 9 \\ 11 & 12 & 13 & 14 \\ 16 & 17 & 18 & 19 \\ 21 & 22 & 23 & 24 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 15 \\ 20 \\ 25 \end{bmatrix}$$

$$[A,b] = \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \\ 16 & 17 & 18 & 19 & 20 \\ 21 & 22 & 23 & 24 & 25 \end{array}\right]$$

$$= \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 5 \\ 5 & 5 & 5 & 5 & 5 \\ 10 & 10 & 10 & 10 & 10 \\ 15 & 15 & 15 & 15 & 15 \\ 20 & 20 & 20 & 20 & 20 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 - R_1 \\ R_5 \to R_5 - R_1 \end{array}$$

$$= \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 5 \\ 0 & -5 & -10 & -15 & -20 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 5R_1 \\ R_3 \to R_3 - 2R_2 \\ R_4 \to R_4 - 3R_2 \\ R_5 \to R_5 - 4R_2 \end{array}$$

$r(A,b) = 2 = r(A) < 4$

∴ Infinitely many solutions.

① ← $x_1+\cancel{2x_2}+3x_3+4x_4=5$

② ← $x_2+2x_3+3x_4=+4$

① − 2×② ⇒ $x_1 - x_3 - 2x_4 = +3$

$x_1 = k$, $x_3 = t$, $x_2 = 0$

$x_4 = \dfrac{k-t+3}{2}$

$k+2x_2+3(t)+4\left(\frac{k-t+3}{2}\right)=5$

$k+3t+2k-2t+6=5-2x_2$

$3k+t+6-5=-2x_2$

$3k+t+1=-2x_2$

$x_2=-\frac{3k+t+1}{2}$

$k-t-2x_4=3$

$k-t+3=2x_4$

5.

$$x_1 - 2x_2 - 3x_3 = 2$$
$$3x_1 - 2x_2 \qquad = -1$$
$$-2x_2 - 3x_3 = 2$$
$$x_2 + 2x_3 = 1$$

$$\begin{bmatrix} 1 & -2 & -3 \\ 3 & -2 & 0 \\ 0 & -2 & -3 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 2 \\ 1 \end{bmatrix}$$

$$[A, b] = \left[\begin{array}{ccc|c} 1 & -2 & -3 & 2 \\ 3 & -2 & 0 & -1 \\ 0 & -2 & -3 & 2 \\ 0 & 1 & 2 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & -2 & -3 & 2 \\ 0 & 4 & +9 & -7 \\ 0 & -2 & -3 & 2 \\ 0 & 1 & 2 & 1 \end{array}\right] R_2 \to R_2 - 3R_1$$

$$= \left[\begin{array}{ccc|c} 1 & -2 & -3 & 2 \\ 0 & 4 & 9 & -7 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 \end{array}\right] \quad R_3 \to R_3 - R_1$$

$$= \left[\begin{array}{ccc|c} 1 & -2 & -3 & 2 \\ 0 & 4 & 9 & -7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 11 \end{array}\right] R_4 \to 4R_4 - R_2$$

$$= \left[\begin{array}{ccc|c} 1 & -2 & -3 & 2 \\ 0 & 4 & 9 & -7 \\ 0 & 0 & -1 & 11 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_3 \leftrightarrow R_4$$

$$r(A, b) = 3 = r(A) = n$$

$$[A, b] = \left[\begin{array}{ccc|c} 1 & -2 & -3 & 2 \\ 3 & -2 & 0 & -1 \\ 0 & -2 & -3 & 2 \\ 0 & 1 & 2 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & -2 & -3 & 2 \\ 0 & 4 & 9 & -7 \\ 0 & 0 & 1 & 4 \\ 0 & 1 & 2 & 1 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 + 2R_4 \end{array}$$

$-2+2(1)$

0

$$= \left[\begin{array}{ccc|c} 1 & -2 & -3 & 2 \\ 0 & 4 & 9 & -7 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & -1/4 & 11/4 \end{array}\right] \quad R_4 \to R_4 - \tfrac{1}{4} R_2$$

$1+2(2)$ $1+9(2)$

$4 \neq 3$

$R(A, b) \neq R(A)$

$\therefore$ No solution

$R_4 \to R_4 + R_3$

$2 - 1/4(9)$

$2 - 9/4$

$8 - 9/4$

$= -1/4$

$-1/4 + 1/4$

$$\left[\begin{array}{ccc|c} 1 & -2 & -3 & 2 \\ 0 & 4 & 9 & -7 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 5/2 \end{array}\right]$$

$-1/4 + 1/4$

0

$\frac{-1}{4}+1$

6. $x-y+z+1=0$

$x-3y+4z+6=0$

$4x+3y-2z+3=0$

$7x-4y+7z+16=0$

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & -3 & 4 \\ 4 & 3 & -2 \\ 7 & -4 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ -6 \\ -3 \\ -16 \end{bmatrix}$$

$$[A,b] = \left[\begin{array}{ccc|c} 1 & -1 & 1 & -1 \\ 1 & -3 & 4 & -6 \\ 4 & 3 & -2 & -3 \\ 7 & -4 & 7 & -16 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & -1 & 1 & -1 \\ 1 & -3 & 4 & -6 \\ 4 & 3 & -2 & -3 \\ 7 & -4 & 7 & -16 \end{array}\right] \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 4R_1 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & -1 & 1 & -1 \\ 0 & -2 & 3 & -5 \\ 0 & 7 & -6 & 1 \\ 7 & -4 & 7 & -16 \end{array}\right] R_4 \to R_4 - 7R_1$$

$$= \left[\begin{array}{ccc|c} 1 & -1 & 1 & -1 \\ 0 & -2 & 3 & -5 \\ 0 & 7 & -6 & 1 \\ 0 & 3 & 0 & -9 \end{array}\right] \begin{array}{l} R_3 \to 2R_3 + 7R_2 \\ R_4 \to 2R_4 + 3R_2 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & -1 & 1 & -1 \\ 0 & -2 & 3 & -5 \\ 0 & 0 & 9 & -33 \\ 0 & 0 & 9 & -33 \end{array}\right] R_4 \to R_4 - R_3$$

$$= \left[\begin{array}{ccc|c} 1 & -1 & 1 & -1 \\ 0 & -2 & 3 & -5 \\ 0 & 0 & 9 & -33 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$r(A,b) = 3 = (r(A))$

$\therefore$ Unique solution.

$$x - y + z = -1$$
$$-2y + 3z = -5$$
$$9z = -33$$
$$z = -33/9$$
$$z = -11/3$$

$$2x - 2y + 2z = -2$$
$$\underline{\begin{matrix} -2y & +3z = -5 \\ (-)\ (+) & (-)\quad (+) \end{matrix}}$$
$$2x - z = 3$$

$y = 0,\ x = k,\ z = -11/3$

$$z = 2k - 3$$
$$-11/3 = 2k - 3$$
$$2k = -11/3 + 3$$
$$2k = -2/3$$
$$2k = -2/3$$
$$k = -1/3$$

$x = -1/3,\ y = 0,\ z = -11/3.$

$-1/3 - y - 11/3 = -1$

$-1/3 - y = -1 + 11/3$

$-y = -1 + 11/3 + 1/3$

$-y = \dfrac{-3 + 11 + 1}{3}$

$-y =$

$2x = 3 - 11/3$ $\quad 2x + \frac{11}{3} = 3$ $\quad 2x = 3 - \frac{11}{3}$ $\quad 2x = \frac{9-11}{3}$ $\quad 2x = \frac{-2}{3}$ $\quad \boxed{x = -\frac{1}{3}}$

$2x - \left(\frac{-11}{3}\right) = 3$ $\quad 2x + 11/3 = 3$ $\quad \frac{6x+11}{3} = 3$ $\quad 6x + 11 = 9$ $\quad 6x = 9 - 11$ $\quad 6x = -2$ $\quad x = -2/6$

$2x - \left(\frac{-11}{3}\right) = 3$ $\quad 2x - z = 3$ $\quad 2x - (-11/3) = 3$ $\quad 2x + \frac{11}{3} = 3$ $\quad \frac{6x+11}{3} = 3$ $\quad 6x + 11 = 9$ $\quad 6x = 9 - 11$ $\quad 6x = -2$

$-2/3 - y - 11/3 = -1$ $\quad \dfrac{-2 - 3y - 11}{3} = -1$ $\quad -2 - 3y - 11 = -3$ $\quad -3y = -3 + 13$ $\quad -3y = 10$ $\quad 3y = -10/3$

7. $3x + y + z = 8$

$-x + y - 2z = -5$

$x + y + z = 6$

$-2x + 2y - 3z = -7$

$$[A, b] = \left[\begin{array}{ccc|c} 3 & 1 & 1 & 8 \\ -1 & 1 & -2 & -5 \\ 1 & 1 & 1 & 6 \\ -2 & 2 & -3 & -7 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ -1 & 1 & -2 & -5 \\ 3 & 1 & 1 & 8 \\ -2 & 2 & -3 & -7 \end{array}\right] \quad R_1 \leftrightarrow R_3$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 2 & -1 & 1 \\ 0 & -2 & -2 & -10 \\ 0 & 4 & -1 & 5 \end{array}\right] \begin{array}{l} R_2 \to R_2 + R_1 \\ R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 + 2R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 2 & -1 & 1 \\ 0 & 0 & -3 & -9 \\ 0 & 0 & 1 & 3 \end{array}\right] \begin{array}{l} R_3 \to R_3 + R_2 \\ R_4 \to R_4 - 2R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 2 & -1 & 1 \\ 0 & 0 & -3 & -9 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_4 \to 3R_4 + R_3$$

$$\gamma(A) = \gamma(A, b) = 3$$

$\therefore$ Unique solution.

8. Find the value k, for

$x + y + z = 1$,

$x + 2y + 3z = k$,

$x + 5y + 9z = k^2$ have a solution. (0)

solution :-

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 5 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ k \\ k^2 \end{bmatrix}$$

$$[A, b] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & k \\ 1 & 5 & 9 & k^2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & k-1 \\ 0 & 4 & 8 & k^2 - 1 \end{array}\right] \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & k-1 \\ 0 & 0 & 0 & k^2-4k+3 \end{array}\right] R_3 \to R_3 - 4R_2$$

$k^2-1-4(k-1)$

$k^2-\cancel{1}-4k+\cancel{4}$

k^2-4k+3

$r(A, B) = r(A) = 2$, Then, Unique solution

$$k^2-4k+3 = 0$$

k^2-4k+3

3

$-3 \mid -1$

-4

$$k^2-3k-k+3 = 0$$

$$k(k-3) - 3(k-1) = 0$$

$$(k-3)(k-1) = 0$$

$$k = 3 \text{ or } k = 1$$

9. Find the condition satisfied a, b, c such that,

$$x+2y-3z = a$$

$$3x-y+2z = b$$

$$x-5y+8z = c$$ has solution exist.

$$[A, b] = \left[\begin{array}{ccc|c} 1 & 2 & -3 & a \\ 3 & -1 & 2 & b \\ 1 & -5 & 8 & c \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & 2 & -3 & a \\ 0 & -7 & 11 & b-3a \\ 0 & -7 & 11 & c-a \end{array}\right] \begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 2 & -3 & a \\ 0 & -7 & 11 & b-3a \\ 0 & 0 & 0 & 2a-b+c \end{array}\right] R_3 - R_2$$

$c-a-b+3a$

$2a-b+c$

$r(A, B) = r(A) = 2$, Then

$$2a-b+c = 0$$

Infinite solution.

10. Find the value of 'k' such that, i) Unique solution,
ii) Infinite solution,
iii) No solution.

$$kx+y+z=1$$
$$x+ky+z=1$$
$$x+y+kz=1$$

Note:

i) If $A_{n\times n}$, then $\text{rank}(A)=n$, iff $\det(A)\neq 0$

ii) If $A_{n\times n}$, then $\text{rank}(A)<n$, iff $\det(A)=0$.

$$|A| = \begin{bmatrix} k & 1 & 1 \\ 1 & k & 1 \\ 1 & 1 & k \end{bmatrix}$$ (Possibility)

$$|A| = k(k^2-1)-1(k-1)+1(1-k)$$
$$=(k^3-k)(-k+1)+(1-k)$$
$$=k^3-k-k+1+1-k$$
$$=k^3-3k+2.$$

$$|A| \quad k^3-3k+2=0$$

$$\begin{array}{c|cccc} & 1 & 0 & -3 & 2 \\ 1 & 0 & 1 & 1 & -2 \\ \hline & 1 & 1 & -2 & \boxed{0} \end{array}$$

$$\begin{array}{c|c} & -2 \\ \hline 2 & -1 \\ \hline & 1 \end{array}$$

$$(k-1)(k^2+1-2)=0$$
$$(k-1)(k-1)(k+2)=0$$

$$k=1,1,-2.$$

Case (i)

If $k\neq 1$ and $k\neq -2$

$$|A|\neq 0 \Rightarrow r(A,b)=r(A)=3=n$$

$\therefore$ Unique solution.

Case (ii) $k=1$

$$[A,b] = \begin{bmatrix} 1 & 1 & 1 & | & 1 \\ 1 & 1 & 1 & | & 1 \\ 1 & 1 & 1 & | & 1 \end{bmatrix}$$

$1<3$

$r(A) = r(A,b) = n < 3$

$\therefore$ Infinitely many solution.

Case (iii) $k=-2$

$$[A,b] = \begin{bmatrix} -2 & 1 & 1 & | & 1 \\ 1 & -2 & 1 & | & 1 \\ 1 & 1 & -2 & | & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 1 & 1 & | & 1 \\ 0 & -3 & 3 & | & 3 \\ 0 & 3 & -3 & | & 0 \end{bmatrix} \quad \begin{matrix} R_2 \to 2R_2 + R_1 \\ R_3 \to R_3 - R_2 \end{matrix}$$

$$= \begin{bmatrix} -2 & 1 & 1 & | & 1 \\ 0 & -3 & 2 & | & 3 \\ 0 & 0 & 0 & | & 3 \end{bmatrix} \quad R_3 \to R_3 + R_2$$

$r(A) = 2, \; r(A,b) = 3$

$r(A) \neq r(A,b)$

$\therefore$ No solution.

11. Investigate for what values of λ, μ the equation.

$x+y+z=6$

$x+2y+3z=10$

$x+2y+\lambda z=\mu$

have i) no solution

ii) unique solution

iii) Infinite solution

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right]$$

$$\simeq \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{array}\right] \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\simeq \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \underset{3-3}{\lambda-3} & \underset{10-10}{\mu-10} \end{array}\right] R_3 \to R_3 - R_2$$

when $\lambda = 3$ and $\mu = 10$

$r(A) = r(A, b) = 2$

$2 < 3$

$\therefore$ Infinite solution.

When $\lambda \neq 3$, $\mu = 10$,

$r(A) = 3 = r(A, b) = n$

Unique solution.

When $\lambda = 3$, $\mu \neq 10$

$r(A) = 2$

$r(A, b) = 3$

$r(A) \neq r(A, b)$

$\therefore$ No solution.

If $\underset{5\times5}{A}X = 0$ has non-trivial solution then, $|A| = 0$.

Problem 1 :-

Find non-trivial solution,

$$x+2y+3z=0$$
$$3x+4y+4z=0$$
$$7x+10y+11z=0$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 3 & 4 & 4 & 0 \\ 7 & 10 & 11 & 0 \end{array}\right] \quad R_2 \to R_2 - 3R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -2 & -5 & 0 \\ 0 & -4 & -10 & 0 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 7R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -2 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_3 \to R_3 - 2R_2$$

$$x+2y+3z=0$$
$$2y+5z=0 \Rightarrow$$
$$2y = -5z \qquad 2y = -5k, \quad y = -\frac{5}{2}k$$
$$x-5z+3z=0$$
$$x-2z=0$$

substitute, $z = k$

$$x = 2k$$
$$y = -5/2\, k$$
$$(2k, -5/2\, k, k) \quad k \in R.$$

Problem 2:-

Test for the consistency of the following system and solve it.

$x - 3y - 8z = 10$

$3x + y - 4z = 0$

$2x + 5y + 6z = 8$

$$\sim \left[\begin{array}{ccc|c} 1 & -3 & -8 & 10 \\ 3 & 1 & -4 & 0 \\ 2 & 5 & 6 & 8 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -3 & -8 & 10 \\ 0 & 10 & 20 & -30 \\ 0 & 11 & 22 & -12 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -3 & -8 & 10 \\ 0 & 10 & 20 & -30 \\ 0 & 0 & 0 & 21 \end{array}\right] \quad R_3 \to R_3 - \frac{11R_2}{10}$$

$11 - \frac{11(10)}{10}$

$r(A) \neq r(A, b)$

$\therefore$ No solution.

$\frac{110 - 110}{10} = 0/10$

Problem 3:

$x - 4y - 3z = -16$

$2x + 7y + 12z = 48$

$4x - y + 6z = 16$

$5x - 5y + 3z = 0$

$$\sim \left[\begin{array}{ccc|c} 1 & -4 & -3 & -16 \\ 2 & 7 & 12 & 48 \\ 4 & -1 & 6 & 16 \\ 5 & -5 & 3 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -4 & -3 & -16 \\ 0 & 15 & 18 & 80 \\ 0 & 15 & 18 & 80 \\ 0 & 15 & 18 & 80 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 4R_1 \\ R_4 \to R_4 - 5R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -4 & -3 & -16 \\ 0 & 15 & 18 & 80 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} R_3 \to R_3 - R_2 \\ R_4 \to R_4 - R_2 \end{array}$$

$r(A) = 2 = r(A, b) < 3$

$\therefore$ Infinitely many solutions.

$x - 4y - 3z = -16$

$+15y + 18z = 80$

$15y + 18z = +80$

Let $z = k$

$$y = \frac{80 - 18k}{15}$$

$$x - 4\left(\frac{80 - 18k}{15}\right) - 3z = -16$$

$$x - 4\left(\frac{80 - 18k}{15}\right) - 3k = -16$$

$$x = -16 + 4\left(\frac{80 - 18k}{15}\right) + 3k$$

$$x = -16 + \left(\frac{320 - 72k}{15}\right) + 3k$$

$$x = \frac{-240 + 320 - 72k + 45k}{15}$$

$$x = \frac{80 - 27k}{15}$$

Problem 4:

Let $A \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 2 & 3 & \alpha \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 3 \\ \beta \end{bmatrix}$

$Ax = b$ over $\mathbb{R}$

i) No solution whenever $\beta \neq 7$

ii) Infinite no. of solutions, $\alpha \neq 2$

iii) Infinite no. of solutions, If $\alpha = 2, \beta \neq 7$.

iv) Unique solution If $\alpha \neq 2$.

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 3 \\ 2 & 3 & \alpha & \beta \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 5 & \alpha-2 & \beta-2 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & \alpha+2 & \beta+7 \end{array}\right] \quad R_3 \to R_3 - \frac{5R_2}{2}$$

$r(A) = r(A,b) = 3$

iv) $\therefore$ Unique solution.

Inverse Method:

$\underset{n\times m}{A}\, X = b$

$|A| \neq 0$

$X = A^{-1} b$

$A^{-1} = \frac{1}{|A|}(\text{adj} A)$

Problem 1:

Solve: $x + 2y = 1$

$3x + 4y = 2$ by inverse method.

$$\begin{array}{ccccc} A & & X & = & b \\ \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} & & \begin{bmatrix} x \\ y \end{bmatrix} & = & \begin{bmatrix} 1 \\ 2 \end{bmatrix} \end{array}$$

$|A| = 4 - 6 = -2$

$$\text{adj} A = \begin{bmatrix} 4 & -2 \\ -3 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{-2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-2}\begin{bmatrix} 1 \\ 2 \end{bmatrix}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{-1}{2}\begin{bmatrix} 1 \\ 2 \end{bmatrix}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

$$= \cancel{\begin{bmatrix} \frac{-4}{2} & \frac{4}{2} \\ \frac{3}{2} & \frac{-2}{2} \end{bmatrix}}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} 4-4 \\ -3+2 \end{bmatrix}$$

$$= -\frac{1}{2}\begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}$$

$x = 0$

$y = \frac{1}{2}$

Problem 2:-

Solve $5x + 2y = 3$

$3x + 2y = 5$

$$\begin{bmatrix} 5 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

Product of Diagonal entrices

$|A| = 10 - 6 = 4$

$$\text{adj}A = \begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{adj}(A)$$

$$= \frac{1}{4}\begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 3 \\ 5 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 6-10 \\ -9+25 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} -4 \\ 16 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -4/4 \\ 16/4 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

$x = -1$

$y = 4.$

Problem 3 :-

$a_{12} = 7$

$M_{12} = (-1)^{i+j} \times a_{12}$

$= (-1)^3 \times 7$

$= -7$

$x - y + z = 4$

$x - 2y - 2z = 9$

$2x + y + 3z = 1$

$$\begin{array}{ccc} a_{11} & a_{12} & \\ \end{array}$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix}$$

$a_{11} = -6+2 = -4$ $\quad a_{21} = -3-1 = -4$

$a_{12} = 3+4 = 7$ $\quad a_{22} = 3-2 = 1$

$a_{13} = 1+4 = 5$ $\quad a_{23} = 1+2 = 3$

$$\begin{bmatrix} -4 & -7 & 5 \\ 4 & 1 & -3 \\ 4 & 3 & -1 \end{bmatrix}$$

$a_{31} = 2+2 = 4$

$a_{32} = -2-1 = -3$

$a_{33} = -2+1 = -1$

$$\begin{bmatrix} -4 & -7 & 5 \\ +4 & 1 & -3 \\ 4 & +3 & -1 \end{bmatrix} \Rightarrow \text{adj}A = \begin{bmatrix} -4 & +4 & +4 \\ -7 & +1 & +3 \\ +5 & -3 & +1 \end{bmatrix}$$

$M_{ij} = (-1)^{i+j} a_{ij}$

For A,

$$|A| = 1(-6+2) + 1(3+4) + 1(1+4)$$

$$= 1(-4) + (7) + 5$$

$$= -4 + 7 + 5$$

$$= 8.$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{8} \begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix} \begin{bmatrix} -4 & 4 & +4 \\ -7 & -1 & 3 \\ +5 & -3 & 1 \end{bmatrix}$$

$$= \frac{1}{8} \begin{bmatrix} -4 & +4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix}$$

$$= \frac{1}{8} \begin{bmatrix} -16+36+4 \\ -28+9+3 \\ 20-27-1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{8} \begin{bmatrix} +24 \\ -16 \\ 8 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

$x = 3$, $y = -2$, $z = 1$.

Problem 4:

Cramer's rule,

$|A| \neq 0$

$$x = \frac{|A_1|}{|A|}$$

$$y = \frac{|A_2|}{|A|}$$

$$z = \frac{|A_3|}{|A|}$$

solve, $x - y = 3$

$2x + 3y + 4z = 17$

$y + 2z = 7.$

$$\begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 17 \\ 7 \end{bmatrix}$$

$$|A| = 1(6-4) + 1(4-0) + 0(2-0)$$

$$= 2 + 4 = 6$$

$$|A| \neq 0.$$

$$A_1 = \begin{bmatrix} 3 & -1 & 0 \\ 17 & 3 & 4 \\ 7 & 1 & 2 \end{bmatrix}$$

$$|A_1| = 3(6-4) + 1(34-28) + 0$$

$$= 3(2) + 6$$

$$= 12$$

$$A_2 = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 17 & 4 \\ 0 & 7 & 2 \end{bmatrix}$$

$$|A_2| = 1(34-28) - 3(4-0) + 0$$

$$= 6 - 12$$

$$= -6$$

$$A_3 = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 3 & 17 \\ 0 & 1 & 7 \end{bmatrix}$$

$$|A_3| = 1(21-17) + 1(14-0) + 3(2-0)$$

$$= 4 + 14 + 6 = 24$$

$$x = \frac{|A_1|}{A}$$

$$= \frac{12}{6}$$

$$= 2$$

$$y = \frac{|A_2|}{A}$$

$$= \frac{-6}{6}$$

$$= -1$$

$$z = \frac{|A_3|}{A}$$

$$= \frac{24}{6}$$

$$= 4$$

$x = 2,\ y = -1,\ z = 4.$

Problem 5:

$$5x - 2y + 16 = 0$$

$$x + 3y - 7 = 0$$

$$5x - 2y = -16$$

$$x + 3y = 7$$

$$\begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -16 \\ 7 \end{bmatrix}$$

$$|A| = 15 + 2$$

$$|A| = 17$$

$$\text{adj}\, A = \begin{bmatrix} 3 & 2 \\ -1 & 5 \end{bmatrix}$$

$$A^{-1} = \frac{1}{17} \begin{bmatrix} 3 & 2 \\ -1 & 5 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{17}\begin{bmatrix} -16 \\ 7 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ -1 & 5 \end{bmatrix}$$

$$= \frac{1}{17}\begin{bmatrix} -48+14 \\ 16+35 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{17}\begin{bmatrix} -34 \\ 51 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

$x = -2$
$y = 3$

Problem 3:-

$3x + 3y - z = 11$
$2x - y + 2z = 9$
$4x + 3y + 2z = 25$.

$$A = \begin{bmatrix} 3 & 3 & -1 \\ 2 & -1 & 2 \\ 4 & 3 & 2 \end{bmatrix}$$

$|A| = 3(-2-6) - 3(4-8) - 1(6+4)$

$= 3(-8) - 3(-4) - 1(10)$

$= -24 + 12 - 10$

$= -22$

$|A| \neq 0$

$$A_1 = \begin{bmatrix} 11 & 3 & -1 \\ 9 & -1 & 2 \\ 25 & 3 & 2 \end{bmatrix}$$

$= 11(-2-6) - 3(18-50) - 1(27+25)$
$= 11(-8) - 3(-32) - 1(52)$
$= -88 + 96 - 52 \quad = -44$

$$|A_2| = \begin{bmatrix} 3 & 11 & -1 \\ 2 & 9 & 2 \\ 4 & 25 & 2 \end{bmatrix}$$

$$= 3(18-50) - 11(4-8) - 1(50-36)$$

$$= 3(-32) - 11(-4) - 1(14)$$

$$= -96 + 44 - 14$$

$$= -66$$

$$|A_3| = \begin{bmatrix} 3 & 3 & 11 \\ 2 & -1 & 9 \\ 4 & 3 & 25 \end{bmatrix}$$

$$= 3(-25-27) - 3(50-36) + 11(6+4)$$

$$= -3(52) - 3(14) + 11(10)$$

$$= 156 - 42 + 110$$

$$= -88$$

$$x = \frac{|A_1|}{A} = \frac{-44}{-22} = 2$$

$$y = \frac{|A_2|}{} = \frac{-66}{-22} = 3$$

$$z = \frac{|A_3|}{} = \frac{-88}{-22} = 4$$

Properties of Determinant:-

1. $\det(A) \neq 0 \Rightarrow |AB| = |A||B|$
2. $A_{n \times n} \Rightarrow |A| = |A^T|$
3. $A_{n \times n} = |kA| = k^n |A|$
4. $A_{n \times n}$, $A(\text{adj}A) = (\text{adj}A)A = |A| = I_n$

5. $A \to$ upper triangular matrix $\Rightarrow |A| = -|A|$

let $A \Rightarrow$ product of the diagonal entries.

Theorem 1:-

If $A \to$ invertible, then $A^{-1} = \frac{1}{|A|}(\text{adj } A)$

i.e $\text{adj} A = |A| A^{-1}$.

Proof:-

We know that,

$$A(\text{adj} A) = |A| I$$

$$(\text{adj } A) A = |A| I$$

Invertible,

$\therefore A$ is invertible $\to$ $AB = BA$

$I = I \to B = A^{-1}$

A^{-1} multiply on both side,

$$A^{-1} A (\text{adj} A) = |A| A^{-1}$$

$$\text{adj} A = |A| A^{-1}$$

Theorem 2:-

If A is non-singular then (i) $|A^{-1}| = \frac{1}{|A|}$

ii) $(A^{-1})^T = (A^T)^{-1}$

$A^{-1} A = I$

iii) $(\lambda A)^{-1} = \frac{1}{\lambda} A^{-1}, \lambda \neq 0$

i) Given A is non singular.

(i.e) $A . A^{-1} = I$

$$|A . A^{-1}| = |I| \quad \text{[Using } 1^{st} \text{ property]}$$

$$|A| |A^{-1}| = 1$$

$$|A^{-1}| = \frac{1}{|A|}$$

ii) A is non singular. A is invertile

$A.A^{-1} = I = A^{-1}A$ rank $A = n =$ rank A^T

$(A.A^{-1})^T = I^T = (A^{-1}A)^T$ $(A^{-1})^T = (A^{-1})^T A^T (A^T)^{-1}$

$A^T.(A^{-1})^T = I^T = (A^{-1}A)^T$ $(A^{-1})^T = (A^{-1}A)^T (A^T)^{-1}$

$(A^{-1})^T = (I^T) (A^T)$ $(A^{-1})^T = (I)^T (A^T)^{-1}$

$(A^T).(A^{-1})^T = I = (A^T)(A^{-1})^T$ $(A^{-1})^T = (A^T)^{-1}$

$(A^T).(A^{-1})^T = (A^T)(A^{-1})^T$

$(A^{-1})^T = (A^T)^{-1}$.

iii) A is non singular,

$(\lambda A)^{-1} = \frac{1}{\lambda} A^{-1}$

$\therefore |A| \neq 0 \Rightarrow (\lambda A) \neq 0$

$+ (\lambda A)(\lambda A)^{-1} = I$

$(\cancel{\lambda}A)\left(\frac{1}{\cancel{\lambda}}A^{-1}\right) = AA^{-1} = I.$

iv) A is non singular,

$A.A^{-1} = I$

$(A.A^{-1})^{-1} = I$

$(A^{-1})^{-1} A^{-1} = I$

$(A^{-1})^{-1} = \frac{I}{A^{-1}}$

$(A^{-1})^{-1} = A.$

3. If A is invertible of order n, then.

i) $(\operatorname{adj} A)^{-1} = \frac{1}{|A|} A.$

$$A^{-1} = \frac{1}{|A|} \operatorname{adj} A$$

Take inverse on both sides,

$$(A^{-1})^{-1} = \left(\frac{1}{|A|} \operatorname{adj} A\right)^{-1}$$

$$A = |A| \operatorname{adj} A^{-1}$$

$$\operatorname{adj} A^{-1} = \frac{1}{|A|} A.$$

ii) $|\operatorname{adj} A| = |A|^{n-1}$

$$\operatorname{adj} A = |A| . A^{-1}$$

Take determinant on both sides,

$$|\operatorname{adj} A| = \big| |A| . A^{-1} \big|$$

$$|\operatorname{adj} A| = |A|^n . |A|^{-1} \qquad [\because A_{n \times n} = |kA| = k^n |A|]$$

$$|\operatorname{adj} A| = |A|^{n-1}$$

iii) $\operatorname{adj}(\operatorname{adj} A) = |A|^{n-2} A$

We know that,

$$B (\operatorname{adj} B) = |B| I$$

$$B = \operatorname{adj} A$$

$$\operatorname{adj} A (\operatorname{adj}(\operatorname{adj} A)) = |\operatorname{adj} A| I \qquad [\because \operatorname{adj} A = |A| . A^{-1}$$

$$|A| A^{-1} (\operatorname{adj}(\operatorname{adj} A)) = |A|^{n-1} \qquad \therefore |\operatorname{adj} A| = |A|^{n-1}]$$

$$A^{-1} (\operatorname{adj}(\operatorname{adj} A)) = |A|^{n-2}$$

$$\operatorname{adj}(\operatorname{adj} A) = |A|^{n-2} . A.$$

iv) $|adj(adj A)| = |A|^{(n-1)^2}$.

We know that,

$$|adj\ A| = |A|^{n-1}$$

$$|adj(adj\ A)| = |adj A|^{n-1}$$

$$= \left(|A|^{n-1}\right)^{n-1}$$

$$= |A|^{(n-1)^2}.$$

UNIT - III

EIGEN VALUE

Eigen value and Eigen Vector:

If $A_{n\times n}$, $A_{n\times n} V_{n\times 1} = \lambda V_\lambda$ always square matrix

where, $V_{n\times 1} \neq 0 \rightarrow$ Eigen vector

$\lambda \rightarrow$ Eigen value

Theorem:

λ is an eigen value if A iff $|A-\lambda I| = 0$.

Proof:

λ is an eigen value of A

To prove $|A-\lambda I| = 0$

We know that,

$$Av = \lambda v$$

$$Av - \lambda v = 0$$

$AV = V\lambda$

$AV - V\lambda = 0$

$(A-\lambda I)V = 0$

$V \neq 0$

$|A - \lambda I| = 0$

$$(A-\lambda I)V = 0$$

$\therefore V \neq 0 \Rightarrow$ Infinitely many solutions.

$$|A - \lambda I| = 0.$$

Problem 1:

Find the Eigen value of $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$.

$$|A - \lambda I| = 0$$

Product of eigen value = Determinant.

$$\left| \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

Trace of A = Sum of all diagonal entries

$$\begin{vmatrix} 1-\lambda & 2 \\ 0 & 3-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(3-\lambda) - 0 = 0$$

$$3 - 3\lambda - \lambda + \lambda^2 = 0$$

$$\lambda^2 - 4\lambda + 3 = 0$$

$$(\lambda - 3)(\lambda - 1) = 0$$

3

$-3 \quad -1$

-4

$\lambda = 1, 3.$

Problem 2:-

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$\left| \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(4-\lambda) - 6 = 0$$

$$4 - \lambda - 4\lambda + \lambda^2 - 6 = 0$$

$\lambda^2 - 5\lambda - 2 = 0 \rightarrow$ characteristic polynomial

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$a = 1, \; b = -5, \; c = -2$$

$$\lambda = \frac{+5 \pm \sqrt{(-5)^2 - 4(1)(-2)}}{2(a)}$$

$$\lambda = \frac{5 \pm \sqrt{25 + 8}}{2}$$

$$\lambda = \frac{5 \pm \sqrt{33}}{2}$$

$$\lambda = \frac{5 + \sqrt{33}}{2} \quad \text{or} \quad \lambda = \frac{5 - \sqrt{33}}{2}$$

Problem 3:-

$$\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}.$$

$$|A - \lambda I| = 0$$

$$\left| \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

$\because$ C.P: $\lambda^2 - Tr(A)\lambda + \det(A) = 0$

$\Rightarrow \lambda^2 - 7\lambda + 6 = 0$

$$\begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$(5-\lambda)(2-\lambda) - 4 = 0$$

$$10 - 5\lambda - 2\lambda + \lambda^2 - 4 = 0$$

$$\lambda^2 - 7\lambda + 6 = 0$$

$$(\lambda - 6)(\lambda - 1) = 0$$

$$\lambda = 6, 1.$$

6; $-6 \mid -1$; -7

Problem 4:-

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$\left| \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\big((5-\lambda)(1-\lambda) - 1)\big) - 1\big((1-\lambda) - 3\big) + 3\big(1 - 3(5-\lambda)\big) = 0$$

$$(1-\lambda)\big((5 - 5\lambda - \lambda + \lambda^2 - 1)\big) - (1 - \lambda - 3) + 3(1 - 15 + 3\lambda) = 0$$

$$(1-\lambda)(\lambda^2 - 6\lambda + 4) - (1 - \lambda - 3) + 3 - 45 + 9\lambda = 0$$

$$\lambda^2 - \cancel{6\lambda} + 4 - \lambda^3 + 6\lambda^2 - \cancel{4\lambda} - 1 + \cancel{\lambda} + 3 + 3 - 45 + \cancel{9\lambda} = 0$$

$-\lambda^3 + 7\lambda^2 - 36 = 0$

$\lambda^3 - 7\lambda^2 + 36 = 0$

	1	$\cancel{7}$	0	36
6	0	1	-6	-36
	1	-1	-6	0

-6

-3 | 2

-1

$\lambda^2 - \lambda - 6 = 0$

$(\lambda - 3)(\lambda + 2) = 0$

$\lambda = 3, -2, 6.$

Another Method,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$C.P = \lambda^2 - Tr(A)\lambda + \det(A) = 0$$

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

$$C.P = \lambda^3 - Tr(A)\lambda^2 + \left[\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right]\lambda -$$

$\det A = 0.$

Problem continuation:-

Find the eigen value, $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$.

case (i): when $\lambda = -2$,

The eigen vector $X = (x, y, z)'$ is given by $(A - \lambda I)X = 0$

i.e. $(A + 2I)X = 0$

$3x + y + 3z = 0 \rightarrow$ ②

$x + 7y + z = 0 \rightarrow$ ③

$3x + y + 3z = 0 \rightarrow$ ④

Solving ② and ③, by cross multiplication,

$$\begin{array}{cccccccc} & x & & y & & z & & \\ 1 & & 3 & & 3 & & 1 & \\ 7 & & 1 & & 1 & & 7 & \end{array} \quad y = xy$$

$$\frac{x}{\begin{vmatrix} 1 & 3 \\ 7 & 1 \end{vmatrix}} = \frac{y}{\begin{vmatrix} 3 & 3 \\ 1 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 3 & 1 \\ 1 & 7 \end{vmatrix}}$$

$$\Rightarrow \frac{x}{-20} = \frac{y}{0} = \frac{z}{20}$$

$$\Rightarrow \frac{x}{1} = \frac{y}{0} = \frac{z}{-1}$$

$$\therefore X_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

When (ii), case $\lambda = 3.$,

The eigen vector $X = (x, y, z)'$ is given by

$(A - \lambda I)X = 0.$

i.e, $(A - 3I)X = 0$

$$-2x + 4y + 3z = 0 \rightarrow ⑤$$
$$x + 2y + z = 0 \rightarrow ⑥$$
$$3x + y - 2z = 0 \rightarrow ⑦.$$

Solve ⑤ and ⑥, by cross multiplication,

$$\begin{array}{cccccccc} & x & & y & & z & & \\ 4 & & 3 & & -2 & & 1 & \\ 2 & & 1 & & 1 & & 2 & \end{array} \quad y = xy$$

$$\frac{x}{\begin{vmatrix} 4 & 3 \\ 2 & 1 \end{vmatrix}} = \frac{y}{\begin{vmatrix} 3 & -2 \\ 1 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -2 & 1 \\ 1 & 2 \end{vmatrix}}$$

$$\Rightarrow \frac{x}{-5} = \frac{y}{5} = \frac{z}{-5}$$

$$\Rightarrow \frac{x}{1} = \frac{y}{-1} = \frac{z}{1}$$

$$\therefore X_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

case iii) : when $\lambda = 6$.

The eigen vector $X = (x, y, z)'$ is given by

$(A - \lambda I) X = 0$.

$$\begin{bmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{bmatrix}$$

$$-5x + y + 3z = 0 \rightarrow (8)$$

$$x - y + z = 0 \rightarrow (9)$$

$$3x + y - 5z = 0 \rightarrow (10).$$

solve (8) and (9), by cross multiplication.

$$\begin{array}{cccccc} x & & y & z & & \\ 1 & 3 & -5 & 1 & & y \; z \; x \; y \\ -1 & 1 & 1 & -1 & & \end{array}$$

$$\frac{x}{\begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix}} = \frac{y}{\begin{vmatrix} 3 & -5 \\ 1 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -5 & 1 \\ 1 & -1 \end{vmatrix}}$$

$$\frac{x}{4} = \frac{y}{8} = \frac{z}{4}$$

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{1}$$

$$X_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

Problem 5:-

Find the eigen values and eigen vectors of,

$$A = \begin{bmatrix} \overset{a_1}{2} & \overset{a_2}{-2} & \overset{a_3}{2} \\ \overset{b_1}{1} & \overset{b_2}{1} & \overset{b_3}{1} \\ \overset{c_1}{1} & \overset{c_2}{3} & \overset{c_3}{-1} \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$\left| \begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} 2-\lambda & -2 & 2 \\ 1 & 1-\lambda & 0 \\ 1 & 3 & -1-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)\big((1-\lambda)(-1-\lambda) - 3\big) + 2\big((-1-\lambda) - 1\big) + 2\big(3 - (1-\lambda)\big)$$

$$C.P \Rightarrow \lambda^3 - Tr(A)\lambda^2 + \left[\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right]\lambda - \det A = 0$$

$$\Rightarrow \lambda^3 - 2\lambda^2 + \left[\begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} + \begin{vmatrix} 2 & +2 \\ 1 & -1 \end{vmatrix} + \begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix} \right]\lambda - \big[2(-1-3) + 2(-1-1) + 2(3-1)\big] = 0$$

$$\Rightarrow \lambda^3 - 2\lambda^2 + \left[\cancel{-4} - 4 + \cancel{4}\right]\lambda - \left[2(-4) + 2(-2) + 2(2)\right] = 0$$

$$\Rightarrow \lambda^3 - 2\lambda^2 - 4\lambda - \left[-8 - \cancel{4} + \cancel{4}\right] = 0$$

$$\Rightarrow \lambda^3 - 2\lambda^2 - 4\lambda + 8 = 0$$

$$\begin{array}{c|cccc} & 1 & -2 & -4 & 8 \\ 2 & 0 & 2 & 0 & -8 \\ \hline & 1 & 0 & -4 & \boxed{0} \end{array}$$

$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

$\lambda^2 + 0\lambda - 4 = 0$

$\lambda^2 - 4 = 0$

$\lambda = 2, -2, 2.$

case (i), when $\lambda = 2$,

The eigen vector $X = (x, y, z)'$ is given by $(A - \lambda I)X = 0$

i.e, $\begin{bmatrix} 0 & -2 & 2 \\ 1 & -1 & 1 \\ 1 & 3 & -3 \end{bmatrix}$

$-2y + 2z = 0 \rightarrow (2)$

$x - y + z = 0 \rightarrow (3)$

$x + 3y - 3z = 0 \rightarrow (4)$

solving (2) and (3), cross multiplication,

$$\begin{array}{cccc} & x & y & z \\ -2 & 2 & 0 & -2 \\ -1 & -1 & 1 & -1 \end{array}$$

yzxy

$$\frac{x}{\begin{vmatrix} -2 & 2 \\ -1 & -1 \end{vmatrix}} = \frac{y}{\begin{vmatrix} 2 & 0 \\ -1 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 0 & -2 \\ 1 & -1 \end{vmatrix}}$$

$$\frac{x}{0} = \frac{y}{2} = \frac{z}{2}$$

$x_1 = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$

case (ii), when $\lambda = -2$.

The eigen vector $X = (x, y, z)'$ is given by

$(A - \lambda I) X = 0$.

i.e, $\begin{bmatrix} 4 & -2 & 2 \\ 1 & 3 & 1 \\ 1 & 3 & 1 \end{bmatrix}$

$4x - 2y + 2z = 0 \rightarrow (5)$

$x + 3y + z = 0 \rightarrow (6)$

$x + 3y + z = 0 \rightarrow (7)$.

	x	y	z	
-2	2	4	-2	
3	1	1	3	

$$\frac{x}{-2-6} = \frac{y}{2-4} = \frac{z}{12+2}$$

$$\frac{x}{-8} = \frac{y}{-2} = \frac{z}{14}$$

$$\frac{x}{-4} = \frac{y}{-1} = \frac{z}{7}$$

$$x_3 = \begin{bmatrix} -4 \\ -1 \\ 7 \end{bmatrix}$$

Problem 6 :-

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

$$|A - \lambda I| = 0.$$

$$\left|\begin{bmatrix}-2 & 2 & -3\\ 2 & 1 & -6\\ -1 & -2 & 0\end{bmatrix} - \lambda\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right| = 0$$

$$\begin{vmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & -\lambda\end{vmatrix} = 0$$

$$C.P \Rightarrow \lambda^3 - Tr(A)\lambda^2 + \left[\begin{vmatrix}b_2 & b_3\\ c_2 & c_3\end{vmatrix} + \begin{vmatrix}a_1 & a_3\\ c_1 & c_3\end{vmatrix} + \begin{vmatrix}a_1 & a_2\\ b_1 & b_2\end{vmatrix}\right]\lambda$$

$$-\det A = 0$$

$$\lambda^3 + 1\lambda^2 + \left[\begin{vmatrix}1 & -6\\ -2 & 0\end{vmatrix} + \begin{vmatrix}-2 & -3\\ -1 & 0\end{vmatrix} + \begin{vmatrix}-2 & 2\\ 2 & 1\end{vmatrix}\right]\lambda -$$

$$\left[-2(0-12) - 2(0-6) - 3(-4+1)\right] = 0$$

$$\lambda^3 + \lambda^2 + \left[-12 + (-3) + (-6)\right]\lambda - \left[-2(-12) - 2(-6) - 3(-3)\right] = 0$$

$$\lambda^3 + \lambda^2 + \left[-12 - 3 - 6\right]\lambda - \left[24 + 12 + 9\right] = 0$$

$$\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$$

$$\begin{array}{r|rrrr} & 1 & 1 & -21 & -45\\ 5 & 0 & 5 & 30 & 45\\ \hline & 1 & 6 & +9 & \boxed{0}\end{array}$$

$$\lambda^2 + 6\lambda + 9 = 0$$

9

3 | 3

6

$$(\lambda+3)(\lambda+3) = 0$$

$$\lambda = -3, -3$$

$$\lambda = 5, -3, -3.$$

case 1: $\lambda = 5$.

The eigen vector $X = (x, y, z)'$ is given by $(A - \lambda I)X = 0$.

i.e $\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}$

$-7x + 2y - 3z = 0 \rightarrow ②$

$2x - 4y - 6z = 0 \rightarrow ③$

$-x - 2y - 5z = 0 \rightarrow ④$

solving ② and ③, cross multiplication.

$$\begin{array}{cccccc} & x & y & z & \\ 2 & -3 & -7 & 2 \\ -4 & -6 & 2 & -4 \end{array}$$

$$\frac{x}{\begin{vmatrix} 2 & 3 \\ -4 & -6 \end{vmatrix}} = \frac{y}{\begin{vmatrix} -3 & -7 \\ -6 & 2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -7 & 2 \\ 2 & -4 \end{vmatrix}}$$

$$\frac{x}{-24} = \frac{y}{-48} = \frac{z}{24}$$

$$\frac{x}{-1} = \frac{y}{-2} = \frac{z}{1}$$

$$x_1 = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$$

case (ii) $\lambda = -3$,

The eigen vector $X = (x, y, z)'$ is given by $(A - \lambda I)X = 0$.

i.e $\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} = 0$

$x + 2y - 3z = 0$

$2x + 4y - 6z = 0$

$-x - 2y + 3z = 0$

	x	y	z	
2	-3	1	2	
4	-6	2	4	

$$\frac{x}{0} = \frac{y}{0} = \frac{z}{0}$$

$$X_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$X_1 = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix},\ X_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},\ X_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Problem 7:-

Find the eigen values and eigen vectors of

$$A = \begin{bmatrix} 6 & -6 & 5 \\ 14 & -13 & 10 \\ 7 & -6 & 4 \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$\left| \begin{bmatrix} 6 & -6 & 5 \\ 14 & -13 & 10 \\ 7 & -6 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right|$$

$$\begin{vmatrix} 6-\lambda & -6 & 5 \\ 14 & -13-\lambda & 10 \\ 7 & -6 & 4-\lambda \end{vmatrix} = 0$$

C.P $\Rightarrow$ $\lambda^3 - Tr(A)\lambda^2 + \left[\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right]$

$- \det A = 0$

6×7 / 42

13×7 / 91 ?

14×6 / 84 ?

$$\lambda^3 + 3\lambda^2 + \left[\begin{vmatrix} -13 & 10 \\ -6 & 4 \end{vmatrix} + \begin{vmatrix} 6 & 5 \\ 7 & 4 \end{vmatrix} + \begin{vmatrix} 6 & -6 \\ 14 & -13 \end{vmatrix} \right]\lambda -$$

$$6(-52+60) + 6(56-70) + 5(-84+91) = 0$$

$$\lambda^3+3\lambda^2+\left[(-52+60)+(24-35)+(-78+84)\right]\lambda-$$
$$6(8)+6(-14)+5(7)=0$$

$$\lambda^3+3\lambda^2+\left[8+(-11)+6\right]\lambda-(48-84+35)=0$$

$$\lambda^3+3\lambda^2+3\lambda-(-1)=0$$

$$\lambda^3+3\lambda^2+3\lambda+1=0$$

$$\begin{array}{r|rrrr} & 1 & 3 & 3 & 1 \\ -1 & 0 & -1 & -2 & -1 \\ \hline & 1 & 2 & 1 & \boxed{0} \end{array}$$

48
35
83

14×4
56

13×4
52

7×13
91

$$\lambda^2+2\lambda+1=0$$

$$(\lambda+1)(\lambda+1)=0$$

$$\lambda=-1,-1,-1.$$

case i) $\lambda=-1$

The eigen vector $X=(x,y,z)'$ is given by $(A-\lambda I)X=0$

i.e $\begin{bmatrix} 7 & -6 & 5 \\ 14 & -12 & 10 \\ 7 & -6 & 5 \end{bmatrix}$

$$7x-6y+5z=0 \rightarrow ②$$

$$14x-12y+10z=0 \rightarrow ③$$

$$7x-6y+5z=0 \rightarrow ④$$

Solving ② and ③, cross multiplication,

$$\begin{array}{cccc} x & y & z & \\ -6 & 5 & 7 & -6 \\ -12 & 10 & 14 & -12 \end{array}$$

$$\frac{x}{\begin{vmatrix} -6 & 5 \\ -12 & 10 \end{vmatrix}}=\frac{y}{\begin{vmatrix} 5 & 7 \\ 10 & 14 \end{vmatrix}}=\frac{z}{\begin{vmatrix} 7 & -6 \\ 14 & -12 \end{vmatrix}}$$

$$\frac{x}{-60+60} = \frac{y}{70-70} = \frac{z}{-84+84}$$

$$\frac{x}{0} = \frac{y}{0} = \frac{z}{0}.$$

Everything is dependent, so

$$7x-6y+5z=0$$

$$(0, 5, 6)$$

$$(5, 0, -7)$$

$$(6, 7, 0)$$

Problem 8:-

Find the eigen values and eigen vectors of

$$A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$\left| \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = 0$$

$$(-5-\lambda)(-2-\lambda) - 4 = 0$$

$$10+5\lambda+2\lambda+\lambda^2-4=0$$

$$\lambda^2+7\lambda+6=0$$

6

1 | 6

7.

$$(\lambda+6)(\lambda+1)=0$$

$$\lambda=-6, \quad \lambda=-1.$$

case (i) $\lambda = -1$.

$$-4x + 2y = 0$$
$$2x - y = 0$$

$\cancel{2x - 2y = 0}$
$\cancel{-4x + 2y = 0}$

$\cancel{-2y = 0}$
$\cancel{y = -2}$

$$2x - y = 0$$
$$2x = y$$
$$x = 1,$$
$$y = 2$$

$(1/2, 1), (1, 2), (3, 6)$

UNIT - 2

Properties of Eigen Vector:-

Symmetric Matrix:-

$A_{n \times n}$, s.t $A^T = A$

$$a_{ij} = a_{ji}$$

Skew - symmetric :-

$$A^T = -A$$
$$a_{ij} = -a_{ji}$$
$$i \neq j$$

Orthogonal Matrix:-

$AA^T = I \quad [\because AA^{-1} = I]$

$$A^T = A^{-1}$$

Eg:-

$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ $\quad \begin{matrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{matrix}$

$|AA^T| = |I|$ $\quad \cos^2\theta + \sin^2\theta$

$|A||A^T| = 1$ $\quad \sin^2\theta - \cos^2\theta$

$$|A|^2 = 1$$
$$|A| = \pm 1$$

Nil Potent Matrix :-

$A_{n\times n}$, $\exists\ K \leq n$

s.t $A^K = 0$

Eg :-

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

① $AV = \lambda V$

wher $\lambda \neq 0 \Rightarrow A^{-1}$ is exists

$$A^{-1}AV = A^{-1}(\lambda V)$$

$$IV = \lambda A^{-1}V$$

$$V = \lambda A^{-1}V$$

$$A^{-1}V = \frac{1}{\lambda}V$$

$\Rightarrow$ If $\lambda \neq 0$ is an eigen value of A^{-1}.

② $AV = \lambda V$

$A^2V = \lambda A V$

$A^2V = \lambda^2 V$

$\therefore$ Here $\lambda A = A^2$

③ $A \rightarrow \lambda$

$KA - K\lambda$

④ A and A^T have the same eigen values.

⑤ sum of the eigen values = trace (A)

⑥ product of the eigen values = $|A|$.

⑦ Eigen value of symmetric matrices are real.

⑧ Eigen values of Hermitian matrices are real.

⑨ Eigen values of skew-Hermitian matrices are either 0 or pure imaginary.

NOTE:-

Any odd order of $A = -A^T$, the $|A| = 0$,

(one of the element. $v\lambda$ must be 0).

⑩. If λ is an eigen value of A, then $\lambda - k$ is an eigen value of $A - KI$.

$$A \to \lambda$$

$$KA \to K\lambda$$

$$A + KI = \lambda + k.$$

$\lambda \to \lambda$ $KA \to \lambda\lambda$ $AKI = \lambda - k$ $\lambda \to -\lambda$ $\lambda - k\lambda$ $\lambda + k$

⑪. Eigen value of a Δ matrix are its diagonal elements.

⑫. If all the eigen value are distinct.

Example,

Find the eigen value for $\begin{bmatrix} 100 & 1 & 1 & \cdots & 1 \\ 1 & 100 & 1 & & \cdots \\ 1 & 1 & 100 & & \cdots \\ \cdots & & \cdots & & \cdots \end{bmatrix}_{100 \times 100}$

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \end{bmatrix}_{100 \times 100}$$

characteristic equation,

$$\lambda^{100} - 100\lambda^{99} = 0$$

$$\lambda^{99}(\lambda - 100) = 0$$

$$\lambda = 0, \lambda = 100.$$

Problem ①:

Find the sum of the eigen value and product of the eigen value.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

sum of eigen value $= 1 + 2 + 3 = 6$.

product of eigen value $= |A|$

$= 1(6-4) - 1(3-2) + 1(2-2)$

$= 2-1+0$

$= 1.$

2. If 2, 2, 8 are eigen value of $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & 1 \\ 2 & -1 & 3 \end{bmatrix}$, find eigen value of A^2, A^{-1} and $A-2I$.

Eigen value of $A^2 = 4, 4, 64$

Eigen value of $A^{-1} = \frac{1}{2}, \frac{1}{2}, \frac{1}{8}$

Eigen value of $A - 2I = 0, 0, 6.$

$A \to \lambda$	$A \to \lambda$	$A \to \lambda$
$A^2 \to \lambda^2$	$A^{-1} = \frac{1}{\lambda}$	$A+kI \to \lambda + k$

3. If one of the eigen value of, $A = \begin{bmatrix} 7 & 4 & -4 \\ 4 & -8 & -1 \\ 4 & -1 & -8 \end{bmatrix}$ is -9, Find the remaining two eigen value.

$\lambda_1 + \lambda_2 - 9 = 7 - 8 - 8$

$\lambda_1 + \lambda_2 = -9 + 9$

$\lambda_1 + \lambda_2 = 0 \to ①$

$\lambda_1 \times \lambda_2 \times (-9) = 7(64-1) - 4(-32+4) - 4(-4+32)$

$-9\lambda_1\lambda_2 = 441 + 4(\cancel{28}) - 4(\cancel{28})$

$-9\lambda_1\lambda_2 = +441 + 4(\cancel{28}) - 4(\cancel{28})$

$\lambda_1\lambda_2 = \frac{-\cancel{441}\ 49}{\cancel{9}\ 1}$

$\lambda_1\lambda_2 = -49 \to ②$

$① \Rightarrow \lambda_1 = -\lambda_2$

$② \Rightarrow -\lambda_2 \times \lambda_2 = -49$

$\cancel{-}\lambda_2^2 = \cancel{-}49$

$\lambda_2 = 7$

$\lambda_1 = -7.$

$\therefore$ The eigen values are $7, -7, -9$.

4. If 2 and 3 are two eigen values of $A = \begin{bmatrix} 3 & -4 & 4 \\ 1 & -2 & 4 \\ 1 & -1 & 3 \end{bmatrix}$

find the eigen value of A^3, A^{-1} and adj A.

sum of eigen value $\Rightarrow 2+3+\lambda_3 = 3-2+3$

$2+3+\lambda_3 = 3-2+3$

$5+\lambda_3 = 4$

$\lambda_3 = -1.$

eigen value of $A = (2, 3, -1)$

eigen value of $A^3 = (8, 27, -1)$

eigen value of $A^{-1} = (\frac{1}{2}, \frac{1}{3}, -1)$

$adj(A) = |A| A^{-1}.$

$|A| = 3(-6+4) + 4(3-4) + 4(-1+2)$

$= -6 - 4 + \cancel{4}$

$|A| = -6$

eigen value of $adjA = (-3, -2, 6).$

5. If $A_{2\times2}$ with elements $a_{11} = a_{12} = a_{21} = 1$, $a_{22} = -1$,

find eigen value of A^{19}

$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$\lambda_1 + \lambda_2 = 0$

$\lambda_1 \lambda_2 = -1-1 = -2$

$\lambda_1 \lambda_2 = -2$

$\lambda_1(-\lambda_1) = -2$

$\cancel{-}\lambda_1^2 = \cancel{-}2$

$\lambda_1 = \sqrt{2}$ $(\sqrt{2}, -\sqrt{2})$

$\lambda_2 = -\sqrt{2}$

$A^{19} \Rightarrow (\sqrt{2}^{19}, -\sqrt{2}^{19})$

$\sqrt{2} \times \sqrt{2}^{18}$

$A^{19} \Rightarrow (512\sqrt{2}, -512\sqrt{5})$

$\sqrt{2} \times 2^{9}$

$\sqrt{2} \times 16 \times 16 \times 2$

$512\sqrt{2}$

6.

11th property

$$A = \begin{bmatrix} 20 & 21 & 22 & 23 \\ 0 & 2021 & 4 & 5 \\ 0 & 0 & 2022 & 6 \\ 0 & 0 & 0 & 2000 \end{bmatrix}$$

Ans:-

Eigen value of A (20, 2021, 2022, 2000).

7. the product of two eigen values of a Matrix $A = \begin{bmatrix} 6 & -2 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ is 16. find the eigen value.

$|A| = 6(9-1)+2(-9+2)+2(3-6)$

$= 48-14-6$

$= 48-20$

$= 28$

$16 \times \lambda_3 = 28$

$\lambda_3 = \cancel{28}^{7} / \cancel{16}_{4}$

$\lambda_3 = 7/4$

$\lambda_1 + \lambda_2 + \lambda_3 = 12$

$\lambda_1 + \lambda_2 = 12 - 7/4$

$\lambda_1 + \lambda_2 = \frac{41}{4} \rightarrow ①$

Given

$\lambda_1 \times \lambda_2 = 16 \rightarrow ②$

$\lambda_1 (41/4 - \lambda_1) = 16$

$\frac{41\lambda_1}{4} - \lambda_1^2 = 16$

$41\lambda_1 - 4\lambda_1^2 = 64$

$4\lambda_1^2 - 41\lambda_1 + 64 = 0$

$$\lambda_1 = \frac{41 \pm \sqrt{1681 - 1024}}{8}$$

$$\lambda_1 = \frac{41 \pm \sqrt{657}}{8}$$

$$\lambda_1 = \frac{41 \pm 3\sqrt{73}}{8}$$

$$\lambda_1 = \frac{15.369}{8} = 1.9209$$

$$\lambda_1 = \frac{66.632}{8} = 8.329$$

when $\lambda_1 = 1.921$,

$$\lambda_2 = \frac{16}{1.921} = 8.329$$

when $\lambda_2 = 8.329$,

$$\lambda_1 = \frac{16}{8.329} = 1.921$$

$(1.921, 8.329)$

CAYLEY HAMILTON THEOREM:-

$A_{n \times n} \to \lambda^n - a_1\lambda^{n-1} + \dots + a_n = 0$

then $A^n - 0, A^{n-1} + \dots + a_n I = 0$

Problem 1 :-

If $A_{2\times 2}$ such that 2 and 3 are eigen value then $A^{-1} = \alpha A + \beta I$, find α and β = ?

$\lambda^2 - 5\lambda + 6 = 0 \to ①$

By cayley Hamilton Theorem,

$A^2 - 5A + 6I = 0$

$A^2 = 5A - 6I$

$A^2 = 5A - 6AA^{-1}$

Determinant is 6, so inverse exists

multiply by A^{-1},

$A = 5I - 6A^{-1}$

$-6A^{-1} = A - 5I$

$A^{-1} = -1/6 A + 5/6 I$

$\alpha = -1/6$, $\beta = 5/6$.

2. If $A_{3\times3}$, eigen value are 2, 3, 4. Then

$A^{-1} = \alpha A^2 + \beta A + \gamma I$

$\lambda^3 - 9\lambda^2 + \lambda - 24 = 0$

sum of minors $2(3) + 3(4) + 2(4)$

$\lambda^3 - 9\lambda^2 + 26\lambda - 24 = 0 \rightarrow ①$

By cayley Hamilton Theorem,

$A^3 - 9A^2 + 26A - 24I = 0$

Multiply by A^{-1},

$A^2 - 9A + 26I - 24A^{-1} = 0$

$24A^{-1} = A^2 - 9A + 26I$

$A^{-1} = 1/24 A^2 - 9/24 A + 26/24 I$.

$\alpha = 1/24$, $\beta = -9/24$, $\gamma = 26/24$.

3. Verify that the matrix,

$A = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}_{3\times3}$ satisfies the characteristic equation and find A^4.

$$C.P \Rightarrow \lambda^3 - 6\lambda^2 + (4-1+4-2+4-1)\lambda + [2(3)+1(-1)+2(-1)] = 0$$

$$\lambda^3 - 6\lambda^2 + 8\lambda - 3 = 0$$

By cayley Hamilton theorem,

$$A^3 - 6A^2 + 8A - 3I = 0 \rightarrow (1)$$

Multiply by A, $\quad A^3 = 6A^2 - 8A + 3I$

$$A^4 - 6A^3 + 8A^2 - 3A = 0$$

$$A^4 = 6A^3 - 8A^2 + 3A$$

$$\cancel{A^4 = 28A^2 - 51A + 3I \rightarrow (2)}$$

$$A^2 = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4+1+2 & -2-2-2 & 4+1+4 \\ -2-2-1 & 1+4+1 & -2-2-2 \\ 2+1+2 & -1-2-2 & 2+1+4 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 14+6+9 & -7-12-9 & 14+6+18 \\ -10-6-6 & 5+12+6 & -10-6-12 \\ 10+5+7 & -5-10-7 & 10+5+14 \end{bmatrix}$$

$$= \begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix}$$

Verify,

$$A^3 - 6A^2 + 8A - 3I = 0$$

L.H.S,

$$= \begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix} - \begin{bmatrix} 42 & -36 & 54 \\ -30 & 36 & -36 \\ 30 & -30 & 42 \end{bmatrix} + \begin{bmatrix} 16 & -8 & 16 \\ -8 & 16 & -8 \\ 8 & -8 & 16 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = R.H.S$$

$$\therefore A^3 - 6A^2 + 8A + 3I = 0.$$

To find A^4,

$$A^4 = 6A^3 - 8A^2 + 3A$$

$$= 6 \begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix} - \begin{bmatrix} 56 & -48 & 72 \\ -40 & 48 & -48 \\ 40 & -40 & 56 \end{bmatrix} + 3 \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 174 & -168 & 228 \\ -132 & 138 & -168 \\ 132 & -132 & 174 \end{bmatrix} - \begin{bmatrix} 56 & -48 & 72 \\ -40 & 48 & -48 \\ 40 & -40 & 56 \end{bmatrix} + \begin{bmatrix} 6 & -3 & 6 \\ -3 & 6 & -3 \\ 3 & -3 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 174-56+6 & -168+48-3 & 228-72+6 \\ -132+40-3 & 138-48+6 & -168+48-3 \\ 132-40+3 & -132+40-3 & 174-56+6 \end{bmatrix}$$

$$= \begin{bmatrix} 124 & -123 & 162 \\ -89 & 96 & -123 \\ 95 & -95 & 124 \end{bmatrix}$$

4. Verify Cayley - Hamilton theorem for a Matrix

$A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$ and also use it to find A^{-1}.

Given:

$$A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

$$\lambda^3 - a_1\lambda^2 + a_2\lambda - a_3 = 0$$

$$Tr(A) = 1+2+1$$
$$= 4.$$

$$|A| = 1(2-6) - 3(4-3) + 7(8-2)$$
$$= (-4) - 3(1) + 7(6)$$
$$= -4 - 3 + 42$$
$$|A| = 35.$$

$$\begin{bmatrix} 1-\lambda & 3 & 7 \\ 4 & 2-\lambda & 3 \\ 1 & 2 & 1-\lambda \end{bmatrix}$$

$$= (1-\lambda)\left((2-\lambda)(1-\lambda)-6\right) - 3\left(4(1-\lambda)-3\right) + 7\left(8-(2-\lambda)\right)$$

$$= (1-\lambda)\left(2-2\lambda-\lambda+\lambda^2-6\right) - 3(4-4\lambda-3) + 7(8-2+\lambda)$$

$$= 2-2\lambda-\lambda+\lambda^2-6-2\lambda+2\lambda^2+\lambda^2-\lambda^3+6\lambda-12+12\lambda+9+56-14+7\lambda$$

$$-\lambda^3+4\lambda^2+20\lambda+35=0$$

$$\lambda^3-4\lambda^2-20\lambda-35=0$$

By Cayley - Hamilton Theorem,

$$A^3-4A^2-20A-35I=0$$

$$A^2=\begin{bmatrix}1&8&7\\4&2&3\\1&2&1\end{bmatrix}\begin{bmatrix}1&3&7\\4&2&3\\1&2&1\end{bmatrix}$$

$$=\begin{bmatrix}1+12+7&3+6+14&7+9+7\\4+8+3&12+4+6&28+6+3\\1+8+1&3+4+2&7+6+1\end{bmatrix}$$

$$=\begin{bmatrix}20&23&23\\15&22&37\\10&9&14\end{bmatrix}$$

$$A^3=\begin{bmatrix}20&23&23\\15&22&37\\10&9&14\end{bmatrix}\begin{bmatrix}1&3&7\\4&2&3\\1&2&1\end{bmatrix}$$

$$=\begin{bmatrix}135&152&232\\140&163&208\\60&76&111\end{bmatrix}$$

$$\Rightarrow A^3-4A^2-20A-35I=$$

$$\Rightarrow\begin{bmatrix}135&152&232\\140&163&208\\60&76&111\end{bmatrix}-4\begin{bmatrix}20&23&23\\15&22&37\\10&9&14\end{bmatrix}-20\begin{bmatrix}1&3&7\\4&2&3\\1&2&1\end{bmatrix}-35\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

= R.H.S.

$(A^3 - 4A^2 - 20A - 35I) = 0$

Multiply by A^{-1}.

$$A^2 - 4A - 20I - 35A^{-1} = 0.$$

$$\begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix} - 4\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} - 20I\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 35A^{-1} = 0$$

$$\begin{bmatrix} -4 & 11 & -5 \\ -1 & -6 & 25 \\ 6 & 1 & -10 \end{bmatrix} = 35A^{-1}$$

$$A^{-1} = \frac{1}{35}\begin{bmatrix} -4 & 11 & -5 \\ -1 & -6 & 25 \\ 6 & 1 & -10 \end{bmatrix}$$

5. Use Cayley Hamilton theorem to find the value of the matrix given by $(A^8 - 5A^7 + 7A^6 - 3A^5 + 1A^4 - 5A^3 + 8A^2 - 2A + I)$ if the matrix, $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$.

Trace (A) = 5 $\quad \lambda^3 - 5\lambda^2 + 7\lambda - 3I = 0$

$|A| = 3$. By Caley Theorem,

$A^3 - 5A^2 + 7A - 3I = 0$

sum of minor, $2+4-1+2 = 7$

$$= A^8 - 5A^7 + 7A^6 - 3A^5 + 1A^4 - 5A^3 + 8A^2 - 2A + I$$

$$= A^5(A^3 - 5A^2 + 7A - 3) + A^4 - 5A^3 + 8A^2 - 2A + I$$

$$= A^5(0) + A^4 - 5A^3 + 8A^2 - 2A + I$$

$$= A^4 - 5A^3 + 7A^2 - 3A + A^2 + A + I$$

$$= A(A^3 - 5A^2 + 7A - 3) + A^2 + A + I$$

$$\Downarrow$$
$$0$$

$$= A^2 + A + I.$$

$$A^2 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4+0+1 & 2+1+1 & 2+0+2 \\ 0+0+0 & 0+1+0 & 0+0+0 \\ 2+0+2 & 1+1+2 & 1+0+4 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$$

$$A^2 + A + I$$

$$= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

Similar Matrix:

$$\underset{n\times n}{A} \sim \underset{n\times n}{B}$$

$\exists\ |P| \neq 0$ such that $PAP^{-1} = B$

(or)

$P^{-1}BP = A$

1. **Lemma:-**

Let $A \sim B$ and λ is an eigen value of A then B also eigen value λ.

Proof:-

let λ is an eigen value $\Rightarrow |A-\lambda I| = 0$

Given $A \sim B \Rightarrow PAP^{-1} = B$

Aim: $|B-\lambda I| = 0$

$$|A-\lambda I| = 0$$

$$|PBP^{-1} - \lambda I| = 0$$

$$|PBP^{-1} - \lambda PP^{-1}| = 0$$

$$|P(B-\lambda I)P^{-1}| = 0$$

$$|P|\,|B-\lambda I|\,|P^{-1}| = 0 \quad (\because |AB| = |A|\,|B|)$$

$$|B-\lambda I| = 0$$

$\therefore$ λ is an eigen value B.

$PAP^{-1} = B \quad P^{-1}BP = A$

2. If $A \sim B$ then rank(A) = rank(B).

rank(A) = rank($P^{-1}BP$) $\leq$ rank(PB) $\leq$ rank(B)

Diagonalizable :

A is diagonalizable if $A \sim D$,

$\exists\ |P| \neq 0$ such that $P^{-1}AP = D$

where,

Inverse of P $\rightarrow$ Each column vector is an eigen vector of A.

1. $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ diagonalize ?

λ^2

solution :-

$\lambda = 1, 3$

case 1: $\lambda = 1$

$(A-\lambda I)X = 0$

$$\begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$0x_1 + 2x_2 = 0$

$x_2 = 0$

$\lambda = 1 \Rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\lambda = 3 \Rightarrow \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$P^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

$P^{-1}AP = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$

2. $A = \begin{bmatrix} 2 & 2 & -7 \\ 2 & 1 & 2 \\ 0 & 1 & -3 \end{bmatrix}$, Diagonalize of A by similar transformation.

$Tr(A) = 3 - 3 = 0$

$|A| = 2(-3-2) - 2(-6+0) - 7(2-0)$

$= 2(-5) + 12 - 14$

$= -10 + 12 - 14$

$= -12$

$\begin{vmatrix} 1 & 2 \\ 1 & -3 \end{vmatrix} + \begin{vmatrix} 2 & -7 \\ 0 & -3 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 2 & 1 \end{vmatrix}$

$= (-3-2) + (-6) + (2-4)$

$= -5 - 6 - 2$

$= -13$

$\lambda^3 - 0\lambda^2 - 13\lambda + 12 = 0$

$$\begin{array}{c|cccc} & 1 & +0 & -13 & +12 \\ 1 & 0 & 1 & 1 & -12 \\ \hline & 1 & 1 & -12 & \boxed{0} \end{array}$$

$\lambda^2 + \lambda - 12 = 0$ $\quad$ -12, 4, -3, 1

$(\lambda + 4)(\lambda - 3) = 0$

$\lambda = -4, 3, 1$

case i) $\lambda = 1$.

The eigen vector $X = (x, y, z)'$ is given by $(A - \lambda I)X = 0$.

i.e $\begin{bmatrix} 1 & 2 & -7 \\ 2 & 0 & 2 \\ 0 & 1 & -4 \end{bmatrix}$

$x + 2y - 7 = 0 \rightarrow ②$

$2x + 0y + 2 = 0 \rightarrow ③$

$y - 4 = 0 \rightarrow ③$

solving ② and ③, cross multiplication,

$$\begin{array}{cccccc} & x & & y & & z \\ 2 & & -7 & & 1 & & 2 \\ 0 & & 2 & & 2 & & 0 \end{array}$$

$y = x\,y$

$$\frac{x}{\begin{vmatrix} 2 & -7 \\ 0 & 2 \end{vmatrix}} = \frac{y}{\begin{vmatrix} -7 & 1 \\ 2 & 2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix}}$$

$$\frac{x}{4} = \frac{y}{-12} = \frac{z}{-4}$$

$$\frac{x}{1} = \frac{y}{-3} = \frac{z}{-1}, \quad X_1 = \begin{bmatrix} 1 \\ -3 \\ -1 \end{bmatrix}$$

case (ii) $\lambda = -4$,

i.e $\begin{bmatrix} 6 & 2 & -7 \\ 2 & 5 & 2 \\ 0 & 1 & 1 \end{bmatrix}$

$$+6x + 2y - 7 = 0 \rightarrow (5)$$

$$2x + 5y + 2 = 0 \rightarrow (6)$$

$$y + 1 = 0 \rightarrow (7)$$

	x	y	z	
2	-7	6	2	
5	2	2	5	

$$\frac{x}{\begin{vmatrix} 2 & -7 \\ 5 & 2 \end{vmatrix}} = \frac{y}{\begin{vmatrix} -7 & 6 \\ 2 & 2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 6 & 2 \\ 2 & 5 \end{vmatrix}}$$

$$\frac{x}{4+35} = \frac{y}{-14-12} = \frac{z}{30-4}$$

$$\frac{x}{+39} = \frac{y}{-26} = \frac{z}{26}$$

$$\frac{x}{3} = \frac{y}{-2} = \frac{z}{2}$$

$$X_2 = \begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix}.$$

case iii) $\lambda = 3$.

i.e $\begin{bmatrix} -1 & 2 & -7 \\ 2 & -2 & 2 \\ 0 & 1 & -6 \end{bmatrix}$

$$-x+2y-7=0 \rightarrow (8)$$

$$2x-2y+2=0 \rightarrow (9)$$

$$y-6=0 \rightarrow (10)$$

$$\begin{array}{cccc} & x & y & z \\ 2 & -7 & -1 & 2 \\ -2 & 2 & 2 & -2 \end{array}$$

$$\frac{x}{\begin{vmatrix} 2 & -7 \\ -2 & 2 \end{vmatrix}} = \frac{y}{\begin{vmatrix} -7 & -1 \\ 2 & 2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -1 & 2 \\ 2 & -2 \end{vmatrix}}$$

$$\frac{x}{4-14} = \frac{y}{-14+2} = \frac{z}{2-4}$$

$$\frac{x}{-10} = \frac{y}{-12} = \frac{z}{-2}$$

$$\frac{x}{-5} = \frac{y}{-6} = \frac{z}{-1}$$

$$\frac{x}{5} = \frac{y}{6} = \frac{z}{1}$$

$$X_3 = \begin{bmatrix} 5 \\ 6 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 5 & 3 \\ -3 & 6 & -2 \\ -1 & 1 & 2 \end{bmatrix}$$

$$P^{-1} = \frac{1}{|P|}(\operatorname{adj} P)$$

$$|P| = 1(12+2) - 5(-6-2) + 3(-3+6)$$

$= 14 - 5(-8) + 3(3)$

$= 14 + 40 + 9$

$= 63.$

$a_{11} = 14 \quad a_{12} = -8 \quad a_{13} = 3$

$a_{21} = 7 \quad a_{22} = 5 \quad a_{23} = 6$

$a_{31} = -28 \quad a_{32} = 7 \quad a_{33} = 21$

$$\begin{bmatrix} 14 & 8 & 3 \\ -7 & 5 & -6 \\ -28 & -7 & 21 \end{bmatrix}$$

$$\text{adj}(P) = \begin{bmatrix} 14 & -7 & -28 \\ 8 & 5 & -7 \\ 3 & -6 & 21 \end{bmatrix}$$

$$P^{-1} = \frac{1}{63} \begin{bmatrix} 14 & -7 & -28 \\ 8 & 5 & -7 \\ 3 & -6 & 21 \end{bmatrix}$$

$$P^{-1}AP = \frac{1}{63}\begin{bmatrix} 14 & -7 & -28 \\ 8 & 5 & -7 \\ 3 & -6 & 21 \end{bmatrix} \begin{bmatrix} 2 & 2 & -7 \\ 2 & 1 & 2 \\ 0 & 1 & -3 \end{bmatrix} \begin{bmatrix} 1 & 5 & 3 \\ -3 & 6 & -2 \\ -1 & 1 & 2 \end{bmatrix}$$

$$= \frac{1}{63} \begin{bmatrix} 28-14 & 28-7-28 & -98-14+84 \\ 16+10 & 16+5-7 & -56+10+21 \\ 6-12 & 6-6+21 & -21-12-63 \end{bmatrix} \begin{bmatrix} 1 & 5 & 3 \\ -3 & 6 & -2 \\ -1 & 1 & 2 \end{bmatrix}$$

$$= \frac{1}{63} \begin{bmatrix} 14 & -7 & -28 \\ 26 & 14 & -25 \\ -6 & 21 & -96 \end{bmatrix} \begin{bmatrix} 1 & 5 & 3 \\ -3 & 6 & -2 \\ -1 & 1 & 2 \end{bmatrix}$$

$$= \frac{1}{63}\begin{bmatrix} 14+21+28 & 70-42-28 & 42+14-56 \\ 26-42+25 & 130+84-25 & 78-28-50 \\ -6-63+96 & -30+126-96 & -18-42-192 \end{bmatrix}$$

$$= \frac{1}{63}\begin{bmatrix} 63 & 0 & 0 \\ 9 & 189 & 0 \\ 27 & 0 & -252 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 9/63 & 3 & 0 \\ 27/63 & 0 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1/7 & 3 & 0 \\ 3/7 & 0 & -4 \end{bmatrix}$$

class Test

1. Find the non-trivial solution of the equations:

$$x+2y+z+2w=0$$

$$x+3y+2z+2w=0$$

$$2x+4y+3z+6w=0$$

$$3x+7y+4z+6w=0$$

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 3 & 2 & 2 \\ 2 & 4 & 3 & 6 \\ 3 & 7 & 4 & 6 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & \cancel{2} & 1 & 2 \\ 0 & 1 & 1 & 0 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \\ R_4 \to R_4 - 3R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_4 \to R_4 - R_2$$

$$r(A) = r(A,b) = 3 < 4$$

$\therefore$ Infinitely many solutions.

$$x+2y+z=2 \to (1)$$

$$y+z=0 \to (2)$$

$$z=2. \to (3)$$

substitute z in (2)

$$y+2=0$$

$$y=-2$$

substitute $y=-2$, $z=2$ in (1),

$$x+2(-2)+2=2$$

$$x-4+2=2$$

$$x-2=2$$

$$x=2+2$$

$$x=4.$$

2. Find the k such that $kx - 2y + z = 1$

$$x - 2ky + z = -2$$

$$x - 2y + kz = 1.$$

i) no solution

ii) unique solution

iii) Infinitely many solutions.

$$\sim \left[\begin{array}{ccc|c} k & -2 & 1 & 1 \\ 1 & -2k & 1 & -2 \\ 1 & -2 & k & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} k & -2 & 1 & 1 \\ 1 & -2k & 1 & -2 \\ 0 & -2+2k & k-1 & 3 \end{array}\right] \quad R_3 \to R_3 - R_2$$

$$\sim \left[\begin{array}{ccc|c} k & -2 & 1 & 1 \\ 1-k & -2k+2 & 0 & -3 \\ 0 & -2+2k & k-1 & 3 \end{array}\right] \quad R_2 \to R_2 - R_1$$

3. Find the eigen value and Eigen vector.

$$A = \begin{array}{c} \begin{array}{ccc} a_1 & a_2 & a_3 \end{array} \\ \begin{bmatrix} 8^{b_1} & -6^{b_2} & 2^{b_3} \\ -6 & 7 & -4 \\ 2^{c_1} & -4^{c_2} & 3^{c_3} \end{bmatrix} \end{array}$$

$$|A - I\lambda| = 0$$

$$\left| \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

characteristic equation,

$$\lambda^3 - \text{Trace}(A)\lambda^2 + \left[\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right]$$

$$\lambda - \text{Det } A = 0.$$

$$\lambda^3 - 18\lambda^2 + \left[\begin{vmatrix} +7 & -4 \\ -4 & +3 \end{vmatrix} + \begin{vmatrix} 8 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 8 & -6 \\ -6 & +7 \end{vmatrix} \right] \lambda -$$

$$8(21-16) + 6(-18+8) + 2(24-14) = 0$$

$$\lambda^3 - 18\lambda^2 + [(21-16) + (24-4) + (56-36)]\lambda -$$

$$8(5) + 6(-10) + 2(10) = 0$$

$$\lambda^3 - 18\lambda^2 + [5 + 20 + 20)\lambda - [40 - 60 + 20] = 0$$

$$\lambda^3 - 18\lambda^2 + [45]\lambda - 0 = 0$$

$\lambda^3-18\lambda^2+45\lambda-0=0$

$\lambda^3-18\lambda^2+45\lambda=0$

$$\begin{array}{r|rrr} & 1 & -18 & 45 \\ 15 & 0 & 15 & -45 \\ \hline & 1 & -3 & \boxed{0} \end{array}$$

$\lambda^2-3\lambda=0$

$\lambda(\lambda-3)=0$

$\lambda=0,\ \lambda=3,\ \lambda=15.$

Case 1 :-

$\lambda=0.$

i.e $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

$8x-6y+2z=0 \rightarrow (1)$

$-6x+7y-4z=0 \rightarrow (2)$

$2x-4y+3z=0 \rightarrow (3)$

	x	y	z
-6	2	8	-6
$+7$	-4	-6	7

$$\frac{x}{\begin{vmatrix} 6 & 2 \\ 7 & -4 \end{vmatrix}} = \frac{y}{\begin{vmatrix} 2 & 8 \\ -4 & -6 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 8 & 6 \\ -6 & 7 \end{vmatrix}}$$

$$\frac{x}{+24-14} = \frac{y}{-24+32} = \frac{z}{56-36}$$

$$\frac{x}{10} = \frac{y}{8} = \frac{z}{10} \Rightarrow \frac{x}{5} = \frac{y}{4} = \frac{z}{5}$$

$$x_1 = \begin{bmatrix} 5 \\ 4 \\ 5 \end{bmatrix}$$

(case ii) $\lambda = 15$,

i.e. $$A = \begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix}$$

$$-7x - 6y + 2 = 0 \rightarrow (5)$$

$$-6x - 8y - 4 = 0 \rightarrow (6)$$

$$2x - 4y - 12 = 0 \rightarrow (7)$$

$$\begin{array}{cccc} & x & y & z \\ -6 & 2 & -7 & -6 \\ -8 & 4 & -6 & -8 \end{array}$$

$$\frac{x}{\begin{vmatrix} -6 & 2 \\ -8 & 4 \end{vmatrix}} = \frac{y}{\begin{vmatrix} 2 & -7 \\ 4 & -6 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -7 & -6 \\ -6 & -8 \end{vmatrix}}$$

$$\frac{x}{-24+16} = \frac{y}{-12+28} = \frac{z}{56-36}$$

$$\frac{x}{-8} = \frac{y}{16} = \frac{z}{10}$$

$$X_2 = \begin{bmatrix} -8 \\ 16 \\ 10 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} -4 \\ 8 \\ 5 \end{bmatrix}$$

case iii) $\lambda = 3$

i.e $\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix}$

$$5x - 6y + 2 = 0 \rightarrow (8)$$

$$-6x + 4y - 4 = 0 \rightarrow (9)$$

$$2x - 4y + 0 = 0 \rightarrow (10)$$

$$\begin{array}{cccc} & x & y & z \\ -6 & 2 & 5 & -6 \\ 4 & -4 & -6 & 4 \end{array}$$

$$\frac{x}{\begin{vmatrix} -6 & 2 \\ 4 & -4 \end{vmatrix}} = \frac{y}{\begin{vmatrix} 2 & 5 \\ -4 & -6 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 5 & -6 \\ -6 & 4 \end{vmatrix}}$$

$$\frac{x}{24-8} = \frac{y}{-12+20} = \frac{z}{20-36}$$

$$\frac{x}{16} = \frac{y}{8} = \frac{z}{-16}$$

$$\frac{x}{2} = \frac{y}{1} = \frac{z}{-2}$$

$$X_3 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

$$X_1 = \begin{bmatrix} 5 \\ 4 \\ 5 \end{bmatrix}, \quad X_2 = \begin{bmatrix} -4 \\ 8 \\ 5 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

4. Verify that the matrix $A = \frac{1}{3}\begin{bmatrix} -1 & 2 & -2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ is orthogonal and eigen value are of unit modulus.

$$A = \frac{1}{3}\begin{bmatrix} -1 & 2 & -2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$|A| = \frac{1}{3}\left[-1(1-4) - 2(-2-4) - 2(-4-2)\right]$$

$$= \frac{1}{3}\left[-1(-3) - 2(-6) - 2(-6)\right]$$

$$= \frac{1}{3}\left[3 + 12 + 12\right]$$

$$= \frac{27}{3}.$$

$$= 9.$$

$$A^{-1} = \frac{1}{|A|} \text{ adj } A$$

adj A,

$a_{11} = -3$ $\quad a_{12} = -6$ $\quad a_{13} = -6$

$a_{21} = 6$ $\quad a_{22} = 3$ $\quad a_{23} = -6$

$a_{31} = 6$ $\quad a_{32} = -6$ $\quad a_{33} = 3$

$$\begin{bmatrix} -3 & 6 & -6 \\ -6 & 3 & 6 \\ 6 & 6 & 3 \end{bmatrix}$$

$$\text{adj } A = \begin{bmatrix} -3 & -6 & 6 \\ 6 & 3 & 6 \\ -6 & 6 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{9}\begin{bmatrix} -3 & -6 & 6 \\ 6 & 3 & 6 \\ -6 & 6 & 3 \end{bmatrix}$$

$$A^T = \frac{1}{3}\begin{bmatrix} -1 & -2 & 2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{bmatrix}$$

$$= \frac{1}{3}\left[-1(1-4)+2(2+4)+2(4+2)\right]$$

$$= \frac{1}{3}\left(-1(-3)+2(6)+2(6)\right)$$

$$= \frac{1}{3}(3+12+12)$$

$$= 9$$

$\therefore |A| = |A^T|$

$\therefore$ It is a arthogonal matrix.

1.

$x + 2y + z + 2w = 0$

$x + 3y + 2z + 2w = 0$

$2x + 4y + 3z + 6w = 0$

$3x + 7y + 4z + 6w = 0$

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 3 & 2 & 2 \\ 2 & 4 & 3 & 6 \\ 3 & 7 & 4 & 6 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{matrix} \\ R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \\ R_4 \to R_4 - 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_4 \to R_4 - R_2$$

$r(A) = r(A,b) = 3 < 4$

$\therefore$ Infinitely many solutions.

$x + 2y + z + 2w = 0$

$y + z = 0$

$z + 2w = 0$

Let $w = t$,

$z + 2t = 0$

$z = -2t$

$y + z = 0$

$y - 2t = 0$

$y = 2t$

$$x + 2y + z + 2w = 0$$

$$x + 4t - \cancel{2t} + \cancel{2t} = 0$$

$$x = -4t$$

solution is $(-4t, 2t, -2t, t)$.

2. $$\begin{bmatrix} k & -2 & 1 \\ 1 & -2k & 1 \\ 1 & -2 & k \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$A = \begin{pmatrix} k & -2 & 1 \\ 1 & -2k & 1 \\ 1 & -2 & k \end{pmatrix}$$

$$|A| = \begin{vmatrix} k & -2 & 1 \\ 1 & -2k & 1 \\ 1 & -2 & k \end{vmatrix}$$

$$= k(-2k^2 + 2) + 2(k-1) + 1(-2+2k)$$

$$= -2k^3 + 2k^2 + 2k - 2 + 2k - 2$$

$$|A| = -2k^3 + 2k^2 + 4k - 4 \rightarrow (1).$$

Let us consider $|A| = 0$,

$$-2k^3 + 2k^2 + 4k - 4 = 0$$

$$\div -2$$

$$k^3 - k^2 - 2k + 2 = 0$$

$$\begin{array}{c|cccc} 1 & 1 & -1 & -2 & 2 \\ & 0 & 1 & 0 & -2 \\ \hline & 1 & 0 & -2 & \boxed{0} \end{array}$$

$(k-1)(k^2-2)=0$

$(k-1)(k-\sqrt{2})(k+\sqrt{2})=0.$

when $k=1$ or $k=\sqrt{2}$ or $k=-\sqrt{2}$

$|A|=0$

$\text{rank}(A) < n$

$\therefore$ It has no solution or Infinite.

when $k\neq 1$ and $k\neq\sqrt{2}$ and $k\neq-\sqrt{2}$

$|A|\neq 0$

$\text{rank}(A)=n$

$\therefore$ unique solution.

1. Diagnalize the matrix $A=\begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix}$ by means of an orthogonal transformation.

$T(A)=4.$

$|A|=2(1-4)-1(1+2)-1(-2-1)$

$=2(-3)-1(3)-1(-3)$

$=-6-3+3$

$=-6.$

$$\begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}$$

$=(1-4)+(2-1)+(2-1)$

$=-3+1+1$

$=-1.$

$$\lambda^3 - 4\lambda^2 - \lambda + 4 = 0$$

$$\begin{array}{c|cccc} & 1 & -4 & -1 & 4 \\ 1 & 0 & 1 & -3 & -4 \\ \hline & 1 & -3 & -4 & \boxed{0} \end{array}$$

$$\lambda^2 - 3\lambda - 4 = 0. \qquad \begin{array}{c} -4 \\ -4 \;\; 1 \\ \hline -3 \end{array}$$

$$(\lambda - 4)(\lambda + 1) = 0$$

$$\lambda = 4, \; \lambda = -1.$$

$$\lambda = 4, 1, -1.$$

case 1 :-

$$\lambda = 1$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & -2 \\ -1 & -2 & 0 \end{bmatrix}$$

$$\begin{array}{cccc} & x & y & z \\ 1 & -1 & 1 & 1 \\ 0 & -2 & 1 & 0 \end{array}$$

$$\frac{x}{\begin{vmatrix} 1 & -1 \\ 0 & -2 \end{vmatrix}} = \frac{y}{\begin{vmatrix} -1 & 1 \\ -2 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}}$$

$$\frac{x}{-2} = \frac{y}{1} = \frac{z}{-1}$$

$$X_1 = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}$$

case 2 : $\lambda = 1$.

$$\begin{bmatrix} 3 & 1 & -1 \\ 1 & 2 & -2 \\ -1 & -2 & 2 \end{bmatrix}$$

$$\begin{array}{cccc} & x & y & z \\ 1 & -1 & 3 & 1 \\ 2 & -2 & 1 & 2 \end{array}$$

$$\frac{x}{(-2+2)} = \frac{y}{(-1+6)} = \frac{z}{(6-1)}$$

$$\frac{x}{0} = \frac{y}{5} = \frac{z}{5}$$

$$X_2 = \begin{bmatrix} 0 \\ 5 \\ 5 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

case 3 : $\lambda = 4$.

$$\begin{bmatrix} -2 & 1 & -1 \\ 1 & -3 & -2 \\ -1 & -2 & -3 \end{bmatrix}$$

$$\begin{array}{cccc} & x & y & z \\ 1 & -1 & -2 & 1 \\ -3 & -2 & 1 & -3 \end{array}$$

$$\frac{x}{(-2-3)} = \frac{y}{(-1-4)} = \frac{z}{6-1}$$

$$\frac{x}{-5} = \frac{-y}{-5} = \frac{z}{5}$$

$$X_3 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} -2 & 0 & -1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$$

$(-2,1,-1)(-2,1,-1)$

$4+1+1$

$= 6 \neq 1$

$$\boxed{\forall x, y \Rightarrow \quad x.x = 1 \quad x \cdot y = 0.}$$

$$P = \begin{bmatrix} -1/\sqrt{3} & -2/\sqrt{6} & 0 \\ -1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{6} & 1/\sqrt{2} \end{bmatrix}$$

$$x.x = \left(-1/\sqrt{3}, -1/\sqrt{3}, 1/\sqrt{3}\right).\left(-1/\sqrt{3}, -1/\sqrt{3}, 1/\sqrt{3}\right)$$

$= 1/3 + 1/3 + 1/3$

$= 3/3$

$= 1.$

$$x.x = \left(-2/\sqrt{6}, 1/\sqrt{6}, -1/\sqrt{6}\right) \cdot \left(-2/\sqrt{6}, 1/\sqrt{6}, -1/\sqrt{6}\right)$$

$= \frac{4}{6} + \frac{1}{6} + 1/6$

$= \frac{5+1}{6}$

$= 6/6$

$= 1$

$$x.x = \left(0, 1/\sqrt{2}, 1/\sqrt{2}\right) \cdot \left(0, 1/\sqrt{2}, 1/\sqrt{2}\right)$$

$= 0 + \frac{1}{2} + \frac{1}{2}$

$= 2/2$

$= 1.$

$\therefore$ It is orthogonal.

$P^{-1}AP = D$

As it is orthogonal,

$P^{T}AP = D$.

$$= \begin{bmatrix} -1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \\ -2/\sqrt{6} & 1/\sqrt{6} & -1/\sqrt{6} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix} \begin{bmatrix} -1/\sqrt{3} & -2/\sqrt{6} & 0 \\ -1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{6} & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

2. Diagonalize the matrix $A = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 6 & 0 \\ 4 & 0 & 2 \end{bmatrix}$ by means of an orthogonal transformation.

Trace $(A) = 10$

$|A| = 2(12) - 0 + 4(-24)$

$= 24 - 24(4)$

$= 24 - 96$

$= -72$

$$\begin{vmatrix} 6 & 0 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 4 \\ 4 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & 6 \end{vmatrix}$$

$= (12) + (4 - 16) + (12$

$= 12 - \cancel{12} + \cancel{12}$

$= 12.$

$$\lambda^3 - 10\lambda^2 + 12\lambda + 72 = 0$$

$$\begin{array}{r|rrrr} & 1 & -10 & 12 & 72 \\ -2 & 0 & -2 & 24 & -72 \\ \hline & 1 & -12 & 36 & \boxed{0} \end{array}$$

$$\lambda^2 - 12\lambda + 36 = 0 \qquad \begin{array}{c} 36 \\ -6 \quad -6 \\ -12 \end{array}$$

$$(\lambda-6)(\lambda-6) = 0$$

$$\lambda = 6, 6, -2.$$

Case 1:

$$\lambda = -2.$$

$$A = \begin{bmatrix} 4 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 4 \end{bmatrix}$$

$$\begin{array}{cccccc} & x & & y & & z \\ 0 & & 4 & & 4 & & 0 \\ 8 & & 0 & & 0 & & 8 \end{array}$$

$$\frac{x}{-32} = \frac{y}{0} = \frac{z}{32}$$

$$\frac{x}{-1} = \frac{y}{0} = \frac{z}{1}$$

$$X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Case 2:

$$\lambda = 6$$

$$\begin{bmatrix} -4 & 0 & 4 \\ 0 & 0 & 0 \\ 4 & 0 & -4 \end{bmatrix}$$

$$\begin{array}{cccc} & x & y & z \\ 0 & 4 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{array}$$

$$\frac{x}{0} = \frac{y}{0} = \frac{z}{0}$$

$$x_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-4x+4z=0$

$-4x=-4z$

$x=1, z=1, y=0$

$$x_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$x.x = \left(-1/\sqrt{2}, 0, 1/\sqrt{2}\right) \cdot \left(-1/\sqrt{2}, 0, 1/\sqrt{2}\right)$$

$$= \left(1/2 + 0 + 1/2\right)$$

$$= 2/2$$

$$= 1.$$

$$P^T A P = \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 & 0 & 4 \\ 0 & 6 & 0 \\ 4 & 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$\begin{bmatrix} -2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

1. Diagnolise $\begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$ by orthogonal transformation.

$$\lambda^3 - 8\lambda^2 + \left[\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & a_3 \\ a_1 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right] \lambda -$$

$\det A = 0$

$$\lambda^3 - 8\lambda^2 + \left[\begin{vmatrix} +2 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 0 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} 3 & -1 \\ -1 & 2 \end{vmatrix} \right] \lambda -$$

$$3(6-1) + 1(-3+0) = 0$$

$$\lambda^3 - 8\lambda^2 + [(6-1) + (9) + (6-1)]\lambda - [15 + 1(-3)] = 0$$

$$\lambda^3 - 8\lambda^2 + [5 + 9 + 5]\lambda - [12] = 0$$

$$\lambda^3 - 8\lambda^2 + 19\lambda - 12 = 0.$$

	1	-8	19	-12
1	0	1	-7	12
	1	-7	12	$\boxed{0}$

$$\lambda^2 - 7\lambda + 12 = 0 \rightarrow ①$$

$$(\lambda-3)(\lambda-4)=0$$

$$\begin{array}{c|c} \multicolumn{2}{c}{12} \\ -3 & -4 \\ \multicolumn{2}{c}{-7} \end{array}$$

$$\lambda = 3, 4, 1.$$

Case 1:

$$\lambda = 1.$$

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$2x - y = 0 \rightarrow ②$$

$$-x + y - z = 0 \rightarrow ③$$

$$-y + 2z = 0 \rightarrow ④$$

$$\begin{array}{cccc} & x & y & z \\ -1 & 0 & 2 & -1 \\ 1 & -1 & -1 & 1 \end{array}$$

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{1}$$

$$X_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Case 2:

$$\lambda = 3.$$

$$\begin{bmatrix} 0 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$\begin{array}{cccc} & x & y & z \\ -1 & 0 & 0 & -1 \\ -1 & -1 & -1 & -1 \end{array}$$

$$\frac{x}{1} = \frac{y}{0} = \frac{z}{-1}$$

$$x_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Case 3:-

$\lambda = 4.$

$$\begin{bmatrix} -1 & -1 & 0 \\ -1 & -2 & -1 \\ 0 & -1 & -1 \end{bmatrix}$$

$$\begin{array}{cccccc} & x & & y & & z \\ -1 & & 0 & & -1 & & -1 \\ -2 & & -1 & & -1 & & -2 \end{array}$$

$$\frac{x}{1} = \frac{y}{-1} = \frac{z}{2-1}$$

$$\frac{x}{1} = \frac{y}{-1} = \frac{z}{1}$$

$$x_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

$(1,2,1)\ (1,2,1)$

$1+4+1$

$= 6$

$$P = \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}$$

$$x \cdot x = \left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right) \cdot \left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$$

$$= \frac{1}{6} + \frac{4}{6} + \frac{1}{6}$$

$$= \frac{6}{6}$$

$$= 1.$$

$\therefore$ It is orthogonal.

$P^{-1}AP = D$

As it is orthogonal.

$P^TAP = D$

$$= \begin{bmatrix} 1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Quadratic Form:

$x^T A x_{n\times 1}$

$x_{n\times 1}$, $A_{n\times n} = A^T$

If $n = 2$,

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

$$x^TAx = [x_1 x_2] \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= [ax_1 + bx_2 \quad bx_1 + cx_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= ax_1^2 + bx_1x_2 + bx_1x_2 + cx_2^2$$

$$= ax_1^2 + 2bx_1x_2 + cx_2^2.$$

1. $f(x,y) = x^2 + 4xy + y^2$

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

2. $f(x,y) = 5x^2+7y^2+8xy.$

$$A = \begin{bmatrix} 5 & 4 \\ 4 & 7 \end{bmatrix}$$

3. $f(x,y,z) = x^2+y^2-z^2+4xy+8yz+2xz.$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 4 \\ 1 & 4 & -1 \end{bmatrix}$$

4. $2x^2+x_2^2+x_3^2+2x_1x_2-2x_1x_3-4x_2x_3$

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix}$$

5. $2x_1^2+6x_2^2+2x_3^2+8x_1x_3$

$$A = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 6 & 0 \\ 4 & 0 & 2 \end{bmatrix}$$

4. $$A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix}$$

$\text{Trace}(A) = 4$

$|A| = -4$

$\lambda^3-4\lambda^2-\lambda+4=0$

$\lambda = 4, 1, -1.$

5.

$$A = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 6 & 0 \\ 4 & 0 & 2 \end{bmatrix}$$

$$\text{Trace}(A) = 2 + 6 + 2$$

$$= 10.$$

$$|A| = 2(12) + 4(-24)$$

$$= 24 - 96$$

$$= -72$$

$$\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$= \begin{vmatrix} 6 & 0 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 4 \\ 4 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & 6 \end{vmatrix}$$

$$= 12 + (4 - 16) + 12$$

$$= \cancel{12} - \cancel{12} + 12$$

$$= 12$$

$$\lambda^3 - 10\lambda^2 + 12\lambda + 72 = 0.$$

$$\begin{array}{r|rrrr} & 1 & -10 & 12 & 72 \\ -2 & 0 & -2 & 24 & -72 \\ \hline & 1 & -12 & 36 & \boxed{0} \end{array}$$

$$\begin{array}{c} 36 \\ -6 \mid -6 \\ -12 \end{array}$$

$$\lambda^2 - 12\lambda + 36 = 0$$

$$(\lambda - 6)(\lambda - 6) = 0$$

$$\lambda = -2, 6, 6.$$

case 1 :- $\lambda = -2$.

$$\begin{bmatrix} 4 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 4 \end{bmatrix}$$

$$\frac{x}{\begin{vmatrix} 0 & 4 \\ 8 & 0 \end{vmatrix}} = \frac{y}{\begin{vmatrix} 4 & 4 \\ 0 & 0 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 4 & 0 \\ 0 & 8 \end{vmatrix}}$$

$$\frac{x}{-32} = \frac{y}{0} = \frac{z}{32}$$

$$X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

case (ii) $\lambda = 6$

$$\begin{bmatrix} -4 & 0 & 4 \\ 0 & 0 & 0 \\ 4 & 0 & -4 \end{bmatrix}$$

$$\frac{x}{\begin{vmatrix} 0 & 0 \\ 0 & -4 \end{vmatrix}} = \frac{y}{\begin{vmatrix} 0 & 0 \\ -4 & 4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 0 & 0 \\ -4 & 0 \end{vmatrix}}$$

$$\frac{x}{0} = \frac{y}{0} = \frac{z}{0}$$

$$X_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$x.x = \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) \cdot \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{2} + 0 + \frac{1}{2}$$

$$= \frac{2}{2}$$

$$= 1.$$

$$P^T A P = \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 & 0 & 4 \\ 0 & 6 & 0 \\ 4 & 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

Definite Matrix:

i) Positive Definite Matrix:

$$x^T A x > 0, \ \forall x \in R$$

(or)

$$\lambda_i > 0$$

ii) Negative Definite Matrix:

$$x^T A x < 0$$

(or)

$$\lambda_i < 0$$

iii) Positive semi Definite Matrix:

$$x^T A x \geq 0$$

or

$$\lambda_i \geq 0$$

iv) Negative semi Definite Matrix:

$$x^T A x \leq 0$$

(or)

$$\lambda_i \leq 0$$

v) Indefinite Matrix:

$$\lambda_i \leq 0, \lambda_i \geq 0.$$

If $A = A^T$

i) rank(A): No. of +ve eigen values + No. of -ve eigen values.

ii) Index : No. of positive eigen values.

iii) Sign : Difference between No. of +ve and -ve eigen values.

1. If $A = A^T$ is invertible $\Rightarrow$ A +ve definite

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$, Invertible $\det \neq 0$

$\therefore$ True.

A +ve definite,

$A = [-1]$

$\therefore$ False.

2. If $A = A^T$, $B = B^T$ invertible matrix $\Rightarrow$ $A+B$ +ve definite

For example,

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix} = 12$$

$\therefore$ Invertible

$\therefore$ $A+B$, +ve definite.

For example

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -3 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & 4 \end{bmatrix} = -8$$

$\therefore$ Invertible

$\therefore$ $A+B$, -ve definite

$\therefore$ False.

3. If $A \to$ +ve definite and $B \to$ +ve definite $\to$

$A+B$ +ve definite.

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} \Rightarrow A+B \begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix} = 12$$

A = positive definite

B = positive definite

A+B, +ve definite.

$x^T A x > 0$

$x^T B x > 0$

By Associate positive,

$$x^T (A+B) x = x^T (Ax + Bx)$$

$$\Rightarrow x^T A x + x^T B x > 0.$$

4. If $A \to +ve$ definite then AB +ve definite ?

$B \to +ve$ definite

$(AB)^T = B^T A^T \neq A^T B^T$

$x^T A x > 0$

$x^T B x > 0$

$x^T (AB) x = x^T A x \cdot x^T B x$

$= x^T A x \cdot x^T B x > 0$

Problem 1:

Reduce the quadratic form $2x_1^2 + 6x_2^2 + 2x_3^2 + 8x_1x_3$ to canonical form by orthogonal reduction. Find also nature of the quadratic form and rank and sign.

$$A = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 6 & 0 \\ 4 & 0 & 2 \end{bmatrix}$$

$2x_1^2 + 5x_2^2 + 3x_3^2 + 4x_1 x_2$.

$$A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Trace (A) = 10.

$|A| = 2(15) - 2(6)$

$= 30 - 12$

$= 18$

$$\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$= \begin{vmatrix} 5 & 0 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 2 & 5 \end{vmatrix}$$

$= (15) + 6 + (10 - 4)$

$= 15 + 6 + 6$

$= 15 + 12$

$= 27.$

$$\lambda^3 - 10\lambda^2 + 27\lambda - 18 = 0.$$

$$\begin{array}{c|cccc} & 1 & -10 & 27 & -18 \\ 1 & 0 & 1 & -9 & 18 \\ \hline & 1 & -9 & 18 & \boxed{0} \end{array}$$

$$\lambda^2 - 9\lambda + 18 = 0$$

$$\begin{array}{c|c} \multicolumn{2}{c}{-18} \\ -6 & -3 \\ \hline \multicolumn{2}{c}{-9} \end{array}$$

$$(\lambda - 6)(\lambda - 3) = 0$$

$$\lambda = 6, 3, 1$$

case 1 :-

$$\lambda = 1.$$

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\begin{array}{ccccccc} & x & & y & & z & \\ 2 & & 0 & & 1 & & 2 \\ 4 & & 0 & & 2 & & 4 \end{array}$$

$$\frac{x}{0} = \frac{y}{0} = \frac{z}{4-4}$$

$$\frac{x}{0} = \frac{y}{0} = \frac{z}{0}$$

$$x + 2y = 0$$

$$x = -2y$$

$$x = 2$$

$$y = -1$$

$$X_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

Case 2:-

$$\lambda = 6$$

$$A = \begin{bmatrix} -4 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

$$\begin{array}{cccc} x & y & z & \\ 2 & 0 & -4 & 2 \\ -1 & 0 & 2 & -1 \end{array}$$

$$\frac{x}{0} = \frac{y}{0} = \frac{z}{4-4}$$

$$\frac{x}{0} = \frac{y}{0} = \frac{z}{0}.$$

$$-4x = -2y$$

$$x = 1$$

$$y = 2.$$

$$x_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

case 3 :- $\lambda = 3$

$$\begin{bmatrix} -1 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{cccc} x & y & z & \\ 2 & 0 & -1 & 2 \\ 2 & 0 & 2 & 2 \end{array}$$

$$\frac{x}{0} = \frac{y}{0} = \frac{z}{-2-4}$$

$$x_3 = \begin{bmatrix} 0 \\ 0 \\ -6 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

$$P = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 0 & 2 \\ 0 & -1 & 0 \end{bmatrix}$$

$$P = \begin{bmatrix} 2/\sqrt{5} & 0 & 1/\sqrt{5} \\ -1/\sqrt{5} & 0 & 2/\sqrt{5} \\ 0 & -1 & 0 \end{bmatrix}$$

$$x.x = \left(2/\sqrt{5}, -1/\sqrt{5}, 0\right) . \left(2/\sqrt{5}, -1/\sqrt{5}, 0\right)$$

$$= \frac{4}{5} + \frac{1}{5}$$

$$= 5/5$$

$$= 1$$

$\therefore$ It is orthogonal.

$$P^{-1} A P = D.$$

As it is orthogonal.

$$P^T A P = D$$

$$= \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} & 0 \\ 0 & 0 & -1 \\ 1/\sqrt{5} & 2/\sqrt{5} & 0 \end{bmatrix} \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & 0 & 1/\sqrt{5} \\ -1/\sqrt{5} & 0 & 2/\sqrt{5} \\ 0 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4/\sqrt{5} - 2/\sqrt{5} + 0 & 4/\sqrt{5} - 5/\sqrt{5} & 0 \\ 0 & 0 & -3 \\ 2/\sqrt{5} + 4/\sqrt{5} + 0 & 2/\sqrt{5} + 10/\sqrt{5} + 0 & 0 \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & 0 & 1/\sqrt{5} \\ -1/\sqrt{5} & 0 & 2/\sqrt{5} \\ 0 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} & 0 \\ 0 & 0 & -3 \\ 6/\sqrt{5} & 12/\sqrt{5} & 0 \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & 0 & 1/\sqrt{5} \\ -1/\sqrt{5} & 0 & 2/\sqrt{5} \\ 0 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4/5 + 1/5 & 0 & 2/5 - 2/5 \\ 0 & 3 & 0 \\ 12/5 - 12/5 & 0 & 6/5 + 24/5 \end{bmatrix}$$

sign $= 3 - 0 = 3$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

rank $= 3 + 0 = 3$

Nature :- positive definite matrix.

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

Positive Definite:

$$|D_n| > 0, \quad D_1 = |a_1| = 0$$

$$D_2 = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} > 0$$

$$D_3 = |A| > 0$$

Positive semi-Definite:

$$|D_n| \geq 0, \ \exists n$$

Negative Definite:

$$D_1 < 0, \ D_2 > 0, \ D_3 < 0$$

$$\begin{cases} D_n < 0 & \text{if } n \text{ odd} \\ D_n > 0 & \text{if } n \text{ even} \end{cases}$$

Negative semi Definite:

$$\begin{cases} D_n \leq 0 & \text{if } n \text{ odd} \\ D_n \geq & \text{if } n \text{ even.} \end{cases}$$

1. Determine the Nature of Quadratic form.

i) $x_1^2 + 3x_2^2 + 6x_3^2 + 2x_1x_2 + 2x_2x_3 + 4x_1x_3.$

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 6 \end{bmatrix}$$

$$D_1 = |1| > 0$$

$$D_2 = \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix}$$

$$= 3-1$$

$$= 2 > 0.$$

$$D_3 = 1(18-1) - 1(6-2) + 2(1-6)$$
$$= 17 - 1(4) + 2(-5)$$
$$= 17 - 4 - 10$$
$$= 17 - 14$$
$$= 3 > 0.$$

$\therefore$ positive definite.

ii) $5x_1^2 + 5x_2^2 + 14x_3^2 + 2x_1x_2 - 16x_2x_3 - 8x_3x_1$

$$A = \begin{bmatrix} 5 & 1 & -4 \\ 1 & 5 & -8 \\ -4 & -8 & 14 \end{bmatrix}$$

$$D_1 = |5| > 0$$

$$D_2 = \begin{vmatrix} 5 & 1 \\ 1 & 5 \end{vmatrix} = 25 - 1$$
$$= 24 > 0$$

$$D_3 = 5(70 - 64) - 1(14 - 32) - 4(-8 + 20)$$
$$= 5(6) + (+18) - 4(12)$$
$$= 30 + 18 - 48.$$
$$= 0.$$

$D_3 = 0.$

$\therefore$ positive semi definite.

iii) $2x_1^2 + x_2^2 - 3x_3^2 + 12x_1x_2 - 8x_2x_3 - 4x_3x_1$

$$A = \begin{bmatrix} 2 & 6 & -2 \\ 6 & 1 & -4 \\ -2 & -4 & -3 \end{bmatrix}$$

$$D_1 = |2| > 0$$

$$D_2 = \begin{vmatrix} 2 & 6 \\ 6 & 1 \end{vmatrix} = 2 - 36 = -34 < 0$$

$D_3 = 2(-3-16)-6(-18-8)-2(24+2)$

$= 2(-19)-6(-26)-2(-22)$

$= -36+156+44$

$= 164 > 0.$

$\therefore$ Indefinite matrix.

UNIT - IV.

Singular Value Decomposition:-

$A_{m\times n}, A^T_{n\times m},$

$AA^T_{m\times m}.$

$(AA^T)^T = (A^T)^T A^T \quad [\because (AB)^T = B^T A^T]$

$= AA^T$

singular value:

λ eigen value $\to AA^T$

singular value $\sqrt{\lambda}$

1. Find the singular value of $A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}_{3\times 4}$

$AA^T = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$

$= \begin{bmatrix} 3 & 1 & 2 \\ 1 & 1 & 0 \\ 2 & 0 & 2 \end{bmatrix}$

Trace $(A) = 6$

$|A| = 3(2) - 1(2) + 2(-2)$

$= 6-2-4$

$= 0.$

$$\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 3 & 2 \\ 2 & 2 \end{vmatrix} + \begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix}$$

$= (2) + (6-4) + (3-1)$

$= 2+2+2$

$= 6.$

$\lambda^3 - 6\lambda^2 + 6\lambda = 0.$

$\lambda(\lambda^2 - 6\lambda + 6) = 0$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad a = 1,\ b = -6,\ c = 6$$

$$= \frac{6 \pm \sqrt{36 - 4(1)(6)}}{2(1)}$$

$$= \frac{6 \pm \sqrt{36-24}}{2}$$

$AA^T \neq A^TA$

$\downarrow$

But eigen values are same

$$= \frac{6 \pm \sqrt{12}}{2}$$

$$= \frac{6 \pm \cancel{4}\sqrt{3}}{2}$$

$\lambda = 3+\sqrt{3},\ 3-\sqrt{3},\ 0.$

singular value,

$\lambda = 0, \sqrt{3 \pm \sqrt{3}}.$

If $A_{m\times n}$ such that $rank(A) = r$

$\exists$ P, Q, D such that,

i) $P_{m\times r}$ with $P^TP = I_{r\times r}$ → Every column, vector is an eigen vector AA^T

ii) $Q_{n\times r}$ with $Q^TQ = I_{r\times r}$ → Every column, vector is an eigen vector A^TA

iii) D is an diagonal entries singular value then, $A = PDQ^T$

(or)

$D = P^T A Q$

Note:-

1. Non zero eigen value (AA^T) = eigen value (A^TA)

2. To find Q,

$$q_i = \frac{1}{\sqrt{\lambda}} A^T P_i, \ \lambda \neq 0.$$

1. If $A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ to reduce singular value decomposition.

Aim: $A = PDQ^T$

$$AA^T = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+1+0 & 0+0+0+0 \\ 0+0+0+0 & 0+1+0+1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$\lambda = 2, 2.$

S.V $= \lambda = \sqrt{2}, \sqrt{2}.$

To find P:

$$AA^T = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$(AA^T - 2I)x = 0$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$0.x + 0.y = 0$

$X_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; X_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad [\because P^T P = I]$

To find Q :-

$$A^T A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$P \sim Q$

$\lambda = 2, 2, 0, 0.$

If $\lambda = 2$ (twice),

$$q_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

If $\lambda = 2$,

$$q_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

If $\lambda = 0$ (twice),

$(A^T A - \lambda I) X = 0$

$x + z = 0$

$y + w = 0$

$X_3 = (1, 0, -1, 0)$

$X_4 = (0, 1, 0, -1)$

$$Q = \begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 & 0 & -1/\sqrt{2} \end{bmatrix}$$

$$Q = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Q. Reduce the singular Value decomposition $A = \begin{bmatrix} 0 & 1 & 1 \\ \sqrt{2} & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

$$A = PDQ^T$$

$$AA^T = \begin{bmatrix} 0 & 1 & 1 \\ \sqrt{2} & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & \sqrt{2} & 0 \\ 1 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0+1+1 & 0+2+0 & 0+1+1 \\ 0+2+0 & 2+4+0 & 0+2+0 \\ 0+1+1 & 0+2+0 & 0+1+1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 & 2 \\ 2 & 6 & 2 \\ 2 & 2 & 2 \end{bmatrix}$$

Trace of $A = 10$

$$|A| = 2(12-4) - 2(4-4) + 2(4-12)$$

$$= 2(8) - 0 + 2(-8)$$

$$= 16 - 16$$

$$= 0.$$

$$\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$= \begin{vmatrix} 6 & 2 \\ 2 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 2 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 2 & 6 \end{vmatrix}$$

$$= (12-4) + (4-4) + (12-4)$$

$$= 8+8$$

$$= 16.$$

$$\lambda^3 - 10\lambda^2 + 16\lambda = 0$$

$$\lambda(\lambda^2 - 10\lambda + 16) = 0$$

$$\lambda(\lambda-8)(\lambda-2) = 0$$

$$\lambda = 0, 8, 2.$$

$$\begin{array}{cc} & 16 \\ -8 & -2 \\ \hline & -10 \end{array}$$

case 1:-

$\lambda = 0.$

$$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 6 & 2 \\ 2 & 2 & 2 \end{bmatrix}$$

$$\begin{array}{ccccccc} & x & & y & & z & \\ 2 & & 2 & & 2 & & 2 \\ 6 & & 2 & & 2 & & 6 \end{array}$$

$$\frac{x}{4-12} = \frac{y}{4-4} = \frac{z}{12-4}$$

$$\frac{x}{-8} = \frac{y}{0} = \frac{z}{8}$$

$$x_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

case 2:-

$\lambda = 2$

$$\begin{bmatrix} 0 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 0 \end{bmatrix}$$

$$\begin{array}{ccccccc} & x & & y & & z & \\ 2 & & 2 & & 0 & & 2 \\ 4 & & 2 & & 2 & & 4 \end{array}$$

$$\frac{x}{4-8} = \frac{y}{4} = \frac{z}{-4}$$

$$\frac{x}{-4} = \frac{y}{4} = \frac{z}{-4}$$

$$x_2 = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}, x_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

case 3:-

$\lambda = 8$

$$\begin{bmatrix} -6 & 2 & 2 \\ 2 & -2 & 2 \\ 2 & 2 & -6 \end{bmatrix}$$

$$\begin{array}{ccccc} & x & & y & & z \\ 2 & & 2 & & -6 & & 2 \\ -2 & & 2 & & 2 & & -2 \end{array}$$

$$\frac{x}{4+4} = \frac{y}{4+12} = \frac{z}{12-4}$$

$$\frac{x}{8} = \frac{y}{16} = \frac{z}{8}$$

$$x_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$q = \frac{1}{\sqrt{\lambda}} A^T P_i$$

$$q = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & \sqrt{2} & 0 \\ 1 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 0-\sqrt{2}+0 \\ 1-2+1 \\ 1+0+1 \end{bmatrix}$$

$$= 1/\sqrt{2} \begin{bmatrix} -\sqrt{2} \\ 0 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} -\sqrt{2}/\sqrt{2} \\ 0/\sqrt{2} \\ 2/\sqrt{2} \end{bmatrix}$$

$$q_1 = \begin{bmatrix} -1 \\ 0 \\ 2/2 \end{bmatrix}$$

$$q_2 = \frac{1}{\sqrt{8}} \begin{bmatrix} 0 & \sqrt{2} & 0 \\ 1 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{8}} \begin{bmatrix} 0 + 2\sqrt{2} + 0 \\ 1 + 4 + 1 \\ 1 + 0 + 1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{8}} \begin{bmatrix} 2\sqrt{2} \\ 6 \\ 2 \end{bmatrix}$$

$$= \frac{1}{2\sqrt{2}} \begin{bmatrix} 2\sqrt{2} \\ 6 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 3/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$q_3 = \frac{1}{\sqrt{0}} \begin{bmatrix} -0 & \sqrt{2} & 0 \\ 1 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$(A^T A - \lambda I)x = 0 \Rightarrow \begin{bmatrix} 0 & \sqrt{2} & 0 \\ 1 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ \sqrt{2} & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\begin{bmatrix} 0+2+0 & 0+2\sqrt{2}+0 & 0+0+0 \\ 0+2\sqrt{2}+0 & 1+4+1 & 1+0+1 \\ 0+0+0 & 1+0+1 & 1+0+1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\begin{bmatrix} 2 & 2\sqrt{2} & 0 \\ 2\sqrt{2} & 6 & 2 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\begin{array}{cccc} x & y & z & \\ 2\sqrt{2} & 0 & 2 & 2\sqrt{2} \\ 6 & 2 & 2\sqrt{2} & 6 \end{array}$$

$$\frac{x}{4\sqrt{2}} = \frac{y}{0-4} = \frac{z}{12-8}$$

$$\frac{x}{4\sqrt{2}} = \frac{y}{-4} = \frac{z}{4}$$

$$x_3 = \begin{bmatrix} \sqrt{2} \\ -1 \\ 1 \end{bmatrix}$$

$$Q = \begin{bmatrix} -1 & 1 & \sqrt{2} \\ 0 & 3/\sqrt{2} & -1 \\ 2/\sqrt{2} & 1/\sqrt{2} & 1 \end{bmatrix}$$

$$Q^T \begin{bmatrix} -1 & 0 & 2/\sqrt{2} \\ 1 & 3/\sqrt{2} & 1/\sqrt{2} \\ \sqrt{2} & -1 & 1 \end{bmatrix}$$

$$A = PDQ^T$$

$$= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & -1/\sqrt{3} & 2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix} \begin{bmatrix} -1 & 0 & 2/\sqrt{2} \\ 1 & 3/\sqrt{2} & 1/\sqrt{2} \\ \sqrt{2} & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0+0+0 & 0+2/\sqrt{3}+0 & 0+0+8/\sqrt{6} \\ 0+0+0 & 0+(-2/\sqrt{3})+0 & 0+0+16/\sqrt{6} \\ 0+0+0 & 0+2/\sqrt{3}+0 & 0+0+8/\sqrt{6} \end{bmatrix} \begin{bmatrix} -1 & 0 & 2/\sqrt{2} \\ 1 & 3/\sqrt{2} & 1/\sqrt{2} \\ \sqrt{2} & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2/\sqrt{3} & 8/\sqrt{6} \\ 0 & -2/\sqrt{3} & 16/\sqrt{6} \\ 0 & 2/\sqrt{3} & 8/\sqrt{6} \end{bmatrix} \begin{bmatrix} -1 & 0 & 2/\sqrt{2} \\ 1 & 3/\sqrt{2} & 1/\sqrt{2} \\ \sqrt{2} & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0+2/\sqrt{3}+8\sqrt{2}/\sqrt{6} & 0+6/\sqrt{6}-8/\sqrt{6} & 0+2/\sqrt{6}+8/\sqrt{6} \\ 0-2/\sqrt{3}+\frac{16}{\sqrt{12}} & 0-6/\sqrt{6}-16/\sqrt{6} & 0-2/\sqrt{6}+16/\sqrt{6} \\ 0+2/\sqrt{3}+8/\sqrt{12} & 0+6/\sqrt{6}-8/\sqrt{6} & 0+2/\sqrt{6}+8/\sqrt{6} \end{bmatrix}$$

$$= \begin{bmatrix} \quad \end{bmatrix}$$

Vector space :-

1) $(R,+) \rightarrow$ is a group

2) $(Q,*) \rightarrow$ satisfies all the properties.

Abilien Group :-

Abilien group is a group which statisfies,

i) closure

ii) Associative

iii) Identity

iv) Inverse

v) commutative.

Group $(G,*)$

It should statisfy the following properties,

i) Associative

ii) closure

iii) Identity

iv) Inverse

1. $S = \{A \in M_n(\mathbb{R}) / tr(A) = 2022\}, +)$

i) $A, B \in S \Rightarrow tra(A) = 2022$

$tra(B) = 2022$

$A+B \in S$

$tr(A+B) = tr(A) + tr(B)$

$= 4044 \notin S.$

$\therefore$ It does not satisfy commutative property.

ii) $A * E = A \Rightarrow A+E = A$

$E = 0$

formate $\leftarrow$ $A + 0 = A$

$0 \in S.$

$Tr(0) = 0 \neq 2022$

$\therefore$ It does not satisfy identity property.

$\therefore$ It is not a group.

Field:

$(F, +, \cdot)$

(i) $(F, +)$ abelian group

(ii) $(F^*, \cdot)$ group (Non-zero element)

(iii) Distributive law

Ring:

$(R, +, \cdot)$

i) $(R, +)$ abelian subgroup

ii) $(R, \cdot)$ Semi sub group

iii) Distributive law

2. $S = \{A / A \in M_n(\mathbb{R})\}$ Field (or) not ?

i) closure
ii) Commutative
iii) Associative
iv) Identity

$\Big\}$ $\oplus$ Abelian group

① closure $A \in R$, $B \in R$, $A, B, \in R$

② Commutative $A \in R$, $B \in R$, $AB \neq BA$

③ $(A - (B - C)) \neq (A . B) . C$ [$\therefore$ Note group]

$\therefore$ It is not a field.

Vector space:

V is an vector space over F (Field).

if $+ : V * V \to V$

$\cdot : V * F \to V$

i) $(V, +)$ abelian

ii) $\alpha(u+v) = \alpha u + \alpha v$, $\alpha \in F$, u and $v \in V$

iii) $(\alpha\beta)u = \alpha(\beta u)$, $\alpha, \beta \in F$, $u \in V$

iv) $\underline{1} . u = u$, $u \in V$

v) $(\alpha + \beta) u = \alpha u + \beta u$, $\alpha, \beta \in F$, $u \in V$.

Example:

1) $\mathbb{R}$ is an Vector space over $\mathbb{R}$
2) $\mathbb{R}$ is an vector space over $\mathbb{Q}$
3) $\mathbb{Q}$ is not an Vector space over $\mathbb{R}$
4) $\mathbb{Q}$ is an Vector space over $\mathbb{Q}$
5) $\mathbb{Z}$ is not ~~is~~ an vector space over $\mathbb{Q}$.
6) $\mathbb{Z}$ is not an vector space over $\mathbb{Z}$

1. $V = \{ A \in M_n(\mathbb{R}) / |A| \neq 0 \}$ is not an vector space over $\mathbb{R}$.

$+ : V \times V \to V$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \notin V \quad |A+B| = 0$$

so, it is not an vector space over $\mathbb{R}$

2. $V = \{ A \in M_n(\mathbb{R}) / |A| = 0 \}$ not an vector space over $\mathbb{R}$?

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$|A| = 0$

$|B| = 0$

$|A+B| \neq 0$, Here $|A+B| = 1$

$\therefore$ This is not an vector space over $\mathbb{R}$

3. $V = \{ A \in M_n(\mathbb{R}) / Tr(A) = 0 \}$

$Trace(A+B) = Trace(A) + Trace(B)$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$$

$trace(A+B) = Trace\ A + Trace\ of\ B$

$0 = 0 + 0$

sub space :.

$W \subset V$ is a subspace of V if

i) $\forall\, w_1, w_2 \in W \Rightarrow w_1 + w_2 \in W$

ii) $\forall\, w \in W$ and $\alpha \in F \Rightarrow \alpha w \in W$

iii) $0 \in W$.

1. If $V = \mathbb{R}$, $F = \mathbb{Q}$ then $W = \mathbb{Q}$ is subspace of V.

$\alpha = \frac{1}{2}$, $W = 1$

$\alpha W = \frac{1}{2} \times 1$

$= \frac{1}{2} \quad \alpha W \in \mathbb{Q}$

$\therefore$ It is a subspace.

2. If $V = \mathbb{R}$, $F = \mathbb{R} \Rightarrow W = \mathbb{Q}$ is not subspace of V.

$\alpha w \notin W$

$\alpha = \sqrt{2}$, $w = 1$

$\alpha W = \sqrt{2} \times 1$

$= \sqrt{2}$.

$\sqrt{2}$ does not belongs to the subspace.

3. If $V = \mathbb{R}$, $\mathbb{F} = \mathbb{R}$ then $W = \{0\}$ is an subspace of V.

$\alpha = 0, \omega = 0$

$\alpha\omega = 0$

0 belongs to the subspace.

4. If $V = \mathbb{R}$, $\mathbb{F} = \mathbb{R}$ then $W = \{0, 1\}$ is not an subspace of V.

$\alpha = 1, \omega = 1$

$\alpha + \omega = 1 + 1$

$= 2$

2 not belongs to the subspace.

5. If $V = \mathbb{R}^2$ and $\mathbb{F} = \mathbb{R}$ then $W = \mathbb{R}$ is an subspace of V.

solution:

i) $\omega_1, \omega_2 \in \mathbb{R} \Rightarrow \omega_1 + \omega_2 \in \mathbb{R}$

ii) $\omega \in \mathbb{R}, \alpha \in \mathbb{R} \Rightarrow \alpha\omega \in \mathbb{R}$

iii) $0 \in \mathbb{R}$.

6. If $V = \mathbb{R}^2$ and $\mathbb{F} = \mathbb{R}$, then $W = \{(x, y) \in \mathbb{R}^2 / x = 0\}$ is an sub space of V.

solution:

i) $\forall\ \omega_1, \omega_2 \in W$, $\omega_1 = \{(x_1, y_1) / x_1 = 0\}$

$\omega_2 = \{(x_2, y_2) / x_2 = 0\}$

Aim:

$\omega_1 + \omega_2 \in W$

$\omega_1 + \omega_2 = \{(x_1 + x_2, y_1 + y_2)\}$,

$x_1 + x_2 = 0 + 0$

$= 0 \in W$

ii) $\omega \in W$ and $\alpha \in \mathbb{F}$.

Aim, $\alpha\omega \in W$

ie $W = \{(x, y) / x = 0\}$

$\alpha\omega = \{(\alpha x, \alpha y) / \alpha x = \alpha \cdot 0$

$= 0 \in W$.

iii) $\{(0,0)\} \cdot \{(0,0) / 0=0\}$

$\{0,0)\} \in W$.

1) $W = \{(x,y) \in R^2 / x+5y=0\}$ is a subspace of $\mathbb{R}$ over the field $\mathbb{R}$.

i) $(0,0) \in W$.

ii) $\forall\, w_1$ and $w_2 \in W$.

$w_1 = \{(x_1,y_1) / x_1+5y_1=0\}$

$w_2 = \{(x_2,y_2) / (x_2+5y_2)=0\}$

$w_1+w_2 = \{(x_1+x_2, y_1+y_2) / x_1+x_2+5(y_1+y_2)\}$

$= x_1+5y_1+x_2+5y_2 = 0+0=0.$

$\therefore w_1+w_2 \in W$.

iii) $\forall w \in W$, $\alpha \in F$.

$\alpha w = \{(\alpha x, \alpha y) / \alpha x+5\alpha y\} = \alpha(x+5y) = \alpha(0)=0\} \in W$.

2. $W = \{(x,y) \in \mathbb{R}^2 / x=y^2\}$ is subspace of $\mathbb{R}^2$ over field $\mathbb{R}$.

i) $(0,0) \in W$

ii) $\forall w_1, w_2 \in W$.

$w_1 = \{(x_1,y_1) / x_1 = y_1^2\}$

$w_2 = \{(x_2,y_2) / x_2 = y_2^2\}$

$w_1+w_2 = \{(x_1+x_2, y_1+y_2) / (x_1+x_2) = (y_1+y_2)^2$

$y_1^2+y_2^2 \neq (y_1+y_2)^2\}$

$w_1+w_2 \notin W$.

iii) $\forall w \in W$, $\alpha \in F$

$\alpha w = \{(\alpha x, \alpha y) / \alpha x = (\alpha y)^2\}$

$\alpha x = \alpha^2 y^2$

$x = \alpha y^2$

$y^2 \neq \alpha y^2$

$\alpha w \notin W$

$\therefore$ This is not a subspace.

iii) $\{(0,0)\} = \{(0,0) / 0 = 0\}$

$\{(0,0)\} \in W$.

1) $W = \{(x,y) \in R^2 / x+5y=0\}$ is a subspace of $\mathbb{R}$ over the field $\mathbb{R}$.

i) $(0,0) \in W$.

ii) $\forall$ w_1 and $w_2 \in W$.

$w_1 = \{(x_1, y_1) / x_1 + 5y_1 = 0\}$

$w_2 = \{(x_2, y_2) / (x_2 + 5y_2) = 0\}$

$w_1 + w_2 = \{(x_1+x_2, y_1+y_2) / x_1+x_2 + 5(y_1+y_2)\}$

$= x_1 + 5y_1 + x_2 + 5y_2 = 0+0 = 0.$

$\therefore w_1 + w_2 \in W$.

iii) $\forall w \in W$, $\alpha \in F$.

$\alpha w = \{(\alpha x, \alpha y) / \alpha x + 5\alpha y\} = \alpha(x+5y) = \alpha(0) = 0\} \in W$.

2. $W = \{(x,y) \in \mathbb{R}^2 / x = y^2\}$ is subspace of $\mathbb{R}^2$ over field $\mathbb{R}$.

i) $(0,0) \in W$

ii) $\forall w_1, w_2 \in W$.

$w_1 = \{(x_1, y_1) / x_1 = y_1^2\}$

$w_2 = \{(x_2, y_2) / x_2 = y_2^2\}$

$w_1 + w_2 = \{(x_1+x_2, y_1+y_2) / (x_1+x_2) = (y_1+y_2)^2$

$y_1^2 + y_2^2 \neq (y_1+y_2)^2\}$

$w_1 + w_2 \notin W$.

iii) $\forall w \in W$, $\alpha \in F$

$\alpha w = \{(\alpha x, \alpha y) / \alpha x = (\alpha y)^2\}$

$\alpha x = \alpha^2 y^2$.

$x = \alpha y^2$

$y^2 \neq \alpha y^2$

$\alpha w \notin W$

$\therefore$ This is not a subspace.

NOTE:- $\mathbb{R}^2$ is a subspace only for the straight line equation.

Theorem:-

In $\mathbb{R}^2$, subspace are only line passing through origin.

Eg:- Subspace | Not a subspace.

Subspace	Not a subspace.
$x+y=0$	$x+y=1$
$x+5y=0$	$x^3+1=1$
$\forall\ ax+by=0$	$xy=1$

3. $W=\{(x,y,z)\in\mathbb{R}^3/x=0\}$ is subspace of $\mathbb{R}^3$ over field $\mathbb{R}$.

i) $(0,0,0)\in W$

ii) $\forall\ w_1, w_2\in W$

$w_1+w_2=\{(x_1+x_2, y_1+y_2, z_1+z_2)/x_1+x_2=0\}$

Aim to prove:- $x_1+x_2=0$

L.H.S $= x_1+x_2$

$=0+0$

$=0.$

$\therefore w_1+w_2\in W$

iii) $\alpha\in F$ and $w\in W$

$\alpha w=((\alpha x, \alpha y, \alpha z)/\alpha x=0)$

Direct Sum:-

If V is an Vector space and W_1 and W_2 subspace of V, then $V = W_1 \oplus W_2$ if

i) $W_1 \oplus W_2 = V$

ii) $W_1 \cap W_2 = \{0\}$

1. In $R^2 = \{(x,y) / x \in R, y \in R\}$

$W_1 = \{(x,y) / (x=0)\}$

$W_2 = \{(x,y) / (y=0)\}$

i) $W_1 + W_2 = \mathbb{R}^2$

ii) $W_1 \cap W_2 = \{(0,0)\}$

$\mathbb{R}^2 = W_1 \oplus W_2$

2. In $M_n(\mathbb{R})$

$W_1 = \{A / A = A^T\}$

$W_2 = \{A / A = -A^T\}$

$$A = \underbrace{\left(\frac{A+A^T}{2}\right)}_{\text{symmetric Matrix}} + \underbrace{\left(\frac{A-A^T}{2}\right)}_{\text{skew symmetric matrix}}$$

[zero matrix is both symmetric and skew symmetric Matrix]

3. In $V = \{ f: \mathbb{R} \to \mathbb{R} / f$ is continuous fn$\}$

$W_1 = \{ f: R \to R / f(x) = f(-x)\} \to f(x) - f(-x) = 0$ Even fn

$W_2 = \{ f: R \to R / f(x) = -f(-x)\} \to f(x) + f(-x) = 0$ odd fn

$$f(x) = \underbrace{\frac{f(x) - f(-x)}{2}}_{\text{Even function}} + \underbrace{\frac{f(x) + f(-x)}{2}}_{\text{odd function}}$$

Basis:-

$\{v_1, v_2 \ldots v_n\}$ basis of

i) $\{v_1, v_2, \ldots v_n\}$ is a linearly Independent.

if $\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = 0$, $\alpha_i \in \mathbb{F}$

then $\alpha_1 = \alpha_2 = \alpha_3 = \cdot\ \alpha_n = 0$.

ii) span,

$\forall\ v \in V$ such that $v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$.

1. In $\mathbb{R}^2$, $\{(1,0), (0,1)\}$ standard Basis of $\mathbb{R}^2$

i) $\alpha_1 v_1 + \alpha_2 v_2 = 0$

$\alpha_1 (1,0) + \alpha_2 (0,1) = (0,0)$ $\quad$ [$\therefore (a,b) = \alpha(1,2) + \alpha_2(3,4)$ $\forall\ a,b \in \mathbb{R}$.]

$(\alpha_1, 0) + (0, \alpha_2) = (0,0)$

$\alpha_1 + 0 = 0 \Rightarrow \alpha_1 = 0$.

$0 + \alpha_2 = 0 \Rightarrow \alpha_2 = 0$.

$\alpha_1(1,0) + \alpha_2(0,1) = (2022, 2023)$

ii) $(2022, 2023) = 2022(1,0) + 2023(0,1)$.

$\alpha_1(1,0) = 2022$ $\quad$ $\alpha_2(0,1) = 2023$

$\alpha_1 = 2022$, $\quad$ $\alpha_2 = 2023$

2. In $\mathbb{R}^2$, $\{(1,2), (3,4)\}$ basis of $\mathbb{R}^2$

i) $\alpha_1(1,2) + \alpha_2(3,4) = (0,0)$

$\alpha_1 + 3\alpha_2 = 0$

$2\alpha_1 + 4\alpha_2 = 0$ $\quad$ [$\exists\ \alpha_k \neq 0$, Linearly Dependent]

$\alpha_2 = 0$, $\alpha = 0$.

ii) $(0,2) = \alpha_1(1,2) + \alpha_2(3,4)$

$\alpha_1(1,2) + \alpha_2(3,4) = (0,2)$

$\alpha_1(1,2) + \alpha_2(3,4) = (0,0)$

① × ② $\quad$ $\alpha_1 + 3\alpha_2 = 0$

$2\alpha_1 + 6\alpha_2 = 2 \rightarrow ①$

$\underline{\underset{(-)}{}\ 2\alpha_1 \underset{(-)}{+} 4\alpha_2 \underset{(-)}{=} 2 \rightarrow ②}$

$2\alpha_2 = 0$

$\alpha_2 = 0$

$\alpha_1 = 0$.

3. In R^3, $\{(1,0,0), (0,1,0), (0,0,1)\}$ basis of R^3.

i) $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$

$\alpha_1(1,0,0) + \alpha_2(0,1,0) + \alpha_3(0,0,1) = 0$

$\alpha_1 + 0 + 0 = 0$

$0 + \alpha_2 + 0 = 0$

$0 + 0 + \alpha_3 = 0$

$\alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0.$

ii) $\alpha_1(1,0,0) + \alpha_2(0,1,0) + \alpha_3(0,0,1) = (a,b,c)$

$\alpha_1 + 0 + 0 = a$

$0 + \alpha_2 + 0 = b$

$0 + 0 + \alpha_3 = c$

$\alpha_1 = a$

$\alpha_2 = b$

$\alpha_3 = c$

$a, b, c \in R.$

Dimension:-

$V \rightarrow$ Vector space

No. of basis.

1. Find the basis and dimension of the following set

$V = \{(x,y) \in R^2 / x = 0\}.$

$\{(0,1), (0,2), (0,3) \ldots\}$ $\quad V = \{0, y / y \in R\}$

$= \{y(0,1) / y \in R\}$

$B = \{(0,1)\}$ $\quad B = \{(0,1)\}$

Dimension $= 1$.

2. $V = \{(x,y) \in R^2 / x+y = 0\}.$

$V = \{(x,y) \in R^2 / x = -y\}$

$= \{(-y,y) / y \in R\}$

$= \{y(-1,1) / y \in R\}$

Basis $= \{(-1, 1)\}$

dimension $= 1$.

3. $V = \{(x, y) \in \mathbb{R}^2 / x + 5y = 0\}$

$= \{(x, y) \in \mathbb{R}^2 / x = -5y\}$

$= \{(-y, y) / y \in \mathbb{R}\}$

$= \{y(5, +1) / y \in \mathbb{R}\}$

Basis $= \{-5, +1\}$

dimension $= 2$.

4. $V = \{(x, y) \in \mathbb{R}^2 / x + y = 1)$

subspace and vector space is not possible.

so, basis and dimension are not possible. $(0, 0 \notin \mathbb{R})$

5. $V = \{(x, y, z)\} \in \mathbb{R}^3 / x = y = z\}$

$=$

7. $\{(x, y, z)\} \in \mathbb{R}^3 / x + y + z = 0\}$

$V = \{(x, y, z) / x = -y - z\}$

$= \{(-y, -z, y, z) / y, z \in \mathbb{R}\}$

$= \{y(-1, 1, 0) + z(-1, 0, 1) / y, z \in \mathbb{R}\}$

$B = \{(-1, 1, 0), (-1, 0, 1)\}$

dimension $= 2$.

6.

8. $V = \{(x,y,z) \in \mathbb{R}^3 \,/\, 5x+y+5z=0\}$

$V = \{(x,y,z) \,/\, y = -5x-5z$

1. $V = \{A \in M_2(\mathbb{R}) / A = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\}$

$= \{a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} / a \in \mathbb{R}\}$

$B = \{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\}$

Dimension : 1.

2. $V = \{A \in M_2(\mathbb{R}) / A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\}$

$= \{a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} / a, b \in \mathbb{R}\}$

$B = \{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\}$

Dimension = 2.

3. $V = \{A \in M_2(\mathbb{R}) / A \begin{bmatrix} a & b \\ c & d \end{bmatrix}\}$

$= \{a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} / a, b, c, d \in \mathbb{R}\}$

$B = \{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\}$

Dimension = 4.

Dimension:-

$A_{n \times n} \to n^2$

$A_{m \times n} \to m \times n$

Eg:- $A_{2022 \times 2022}$

$\dim V = (2022)^2$

4. $V = \{A \in M_2(\mathbb{R}) / A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}\}$

$= \{a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} / a \in \mathbb{R}\}$

$B = \{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\}$

Dimension = 1.

5. $V = \{A \in M_{2 \times 3}(\mathbb{R}) / A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}\}$

$= \{a_1\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + a_2\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + a_3\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + b_1\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$b_2\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + b_3\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} / a_1, a_2, \ldots, b_1, b_2, \ldots$

$$B = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

Dimension = 6.

Theorem:-

Theorem:

A list $(v_1, v_2 \ldots v_n)$ of vectors in V is a basis of V if and only if every $v \in V$ can be written uniquely in the form.

$$v = a_1 v_1 + \ldots + a_n v_n.$$

where $a_1 \ldots a_n \in F$.

Proof:

Given $\{v_1, v_2 \ldots v_n\}$ basis

Aim:

$v = a_1 v_1 + a_2 v_2 + \ldots a_n v_n \ \forall \ v \in V$ unique.

$\therefore V$ is basis

$\Rightarrow$ spaning list vector

$v = a_1 v_1 + \ldots + a_n v_n$

Unique:

$v = b_1 v_1 + b_2 v_2 + \ldots + b_n v_n$, for some $b_i \in F$

$a_1 v_1 + a_2 v_2 + \ldots + a_n v_n = b_1 v_1 + \ldots + b_n v_n$

$(a_1 - b_1) v_1 + (a_2 - b_2) v_2 + \ldots + (a_n - b_n) v_n = 0.$

$\therefore$ V is basis [Linearly Independent]

$a_1 - b_1 = 0, a_2 - b_2 = 0, \ldots \ a_n - b_n = 0$

$a_1 = b_1, a_2 = b_2, \ldots \ a_n = b_n.$

Converse Proof:

Given:

$v = a_1 v_1 + \ldots + a_n v_n$, Unique.

Aim:

$\{(v_1 \ldots v_n)\}$ basis

i) $\{(v_1, v_2 \ldots v_n)\}$ [Linearly Independent]

$a_1 v_1 + \ldots a_n v_n = 0$

$a_1 = 0, a_2 = 0 \ldots a_n = 0.$

NOTE 1:

Every spanning list in a vector space can be reduced to a basis of the Vector Space.

NOTE 2:

Every linearly independent list of vectors in a finite-dimensional vector space can be extended to a basis of the vector space.

THEOREM 2:

Suppose V is finite dimensional vector space and U is a subspace of V. Then there is a subspace W of V such that $V = U \oplus W$.

Proof:

Given $U \subset V$ and $\{u_1, u_2, \dots u_k\}$ basis of U.

By Note 2:

$\{u_1, u_2 \dots u_k, w_1, w_2 \dots w_n\}$ basis of V.

$\Rightarrow \{w_1, w_2 \dots w_n\}$ basis of W

To prove:

$V = U \oplus W$

i) $V = U + W$

ii) $U \cap W = \{0\}$

$\therefore \{u_1, u_2 \dots u_k, w_1, w_2 \dots w_n\}$ basis of V.

$\forall\, v \in V \Rightarrow v = a_1u_1 + a_2u_2 + \dots + a_nu_k + b_1w_1 + b_2w_2 + \dots b_nw_n$

$v = u + w$, $u \in U$, $w \in W$

$V = U + W$

ii) Aim:

$U \cap W = \{0\}$.

suppose $v \in U \cap W$

$$v \in U \Rightarrow \left\{ v = a_1u_1 + a_2u_2 + \dots + a_ku_k \right.$$
$$(*) \leftarrow \quad v \in W \Rightarrow \left\} v = b_1w_1 + b_2w_2 + \dots + b_nw_n \right.$$

$$a_1u_1 + a_2u_2 + \dots a_ku_k = b_1w_1 + \dots + b_nw_n$$

$$a_1u_1 + a_2u_2 + \dots + a_kv_k - b_1w_1 - b_2w_2 \dots b_nw_n = 0$$

$a_i = 0$; $b_i = 0$ ($\because$ V is basis)

In $(*)$ $v = 0$ $\quad 0 \in U \cap W$.

Theorem 3:-

If U_1 and U_2 are subspaces of a finite dimensional vector space, then

$$\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2)$$

Proof:-

Let $\{u_1, u_2 \dots u_k\}$ is basis of $U_1 \cap U_2$

$\Rightarrow$ $\{u_1, u_2 \dots u_k\}$ is Linearly Independent

$U_1 \Rightarrow$ $\{u_1, u_2 \dots u_k, v_1, v_2, \dots v_j\}$ extented basis V_1.

and $\Rightarrow$ $\{u_1, u_2 \dots u_k\}$ is Linearly Independent

$U_2 \Rightarrow$ $\{u_1, u_2, \dots u_k, w_1, w_2 \dots w_n\}$ extended basis V_2.

$\Rightarrow$ $\dim(U_1) = k+j$ and $\dim(U_2) = k+n$ and

$\dim(U_1 \cap U_2) = k$

Aim:-

$\{u_1, u_2 \dots u_k, v_1, v_2 \dots v_j, w_1, \dots w_n\}$ basis of $V_1 + U_2$.

Span:- $\forall$ $v \in U_1 + U_2$

$$v = (\alpha_1 + \alpha_1')u_1 + (\alpha_2 + \alpha_2')u_2 + \dots + (\alpha_k + \alpha_k')u_k +$$
$$\beta_1v_1 + \dots + \beta_jv_j + \gamma_1w_1 + \dots + \gamma_nw_n$$

$= \alpha_1 u_1 + \cdots + \alpha_k u_k + \beta_1 v_1 + \cdots + \beta_j v_j + \alpha'_1 u_1 \cdots + \alpha'_k u_k +$

$\gamma_1 w_1 + \cdots + \gamma_n w_n$

$\because U_1$ basis and U_2 basis.

$V = U_1 + U_2, \quad V \in U_1 + U_2$

Linearly Independent:-

$$\sum_{i=1}^{k} \alpha_i u_i + \sum_{i=1}^{j} \beta_i v_i + \sum_{i=1}^{n} \gamma_i w_i = 0$$

Aim:-

$\alpha_i = 0, \ \beta_i = 0, \ \gamma_i = 0$

$\because U_1 \cap U_2$ basis $\Rightarrow \{u_1, u_2 \ldots u_k\}$ Linearly Independent $\Rightarrow$

$\alpha_i = 0.$

$\because U_1$ basis $\{u_1, \ldots u_k, v_1, \ldots v_j\}$ Linearly Indepent $\Rightarrow$

$\alpha_i = 0$

$\because U_2$ basis $\{u_1, \ldots u_k, w_1, \ldots w_n\}$ Linearly Independent $\Rightarrow$

$\beta_i = 0$

$\alpha_i = 0$

$\gamma_i = 0$

$\alpha_i = \beta_i = \gamma_i = 0.$

Hence,

$\{u_1, u_2 \ldots u_k, v_1, v_2 \ldots v_j, w_1, \ldots w_n\}$ basis of $U_1 + U_2$

$\dim(U_1 + U_2) = k + j + n + k - k$

$\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2).$

Problems:-

1. $\dim W_1 = 1, \ \dim W_2 = 1, \ V = \mathbb{R}^2$

$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$

$= 1 + 1 - 0$

$= 1 + 1 - 1$

$\dim(W_1 + W_2) = 2$ (or) 1

2. $\dim(W_1) = 1, \dim(W_2) = 2, V = \mathbb{R}^3$.

$\dim(W_1 + W_2) = 1 + 2 - 0 = 3$

$= 1 + 2 - 1 = 2$

$\dim(W_1 + W_2) = 3$ (or) 2.

3. $\dim V = 7$.

$\dim W_1 = 4$ and $\dim W_2 = 5$.

then $\dim(W_1 \cap W_2) = ?$

[$|W_1| = m$, $\dim n$
$\min\{n, m\} \leq \dim(W_1 + W_2) \leq \dim V$]

$5 \leq \dim(W_1 + W_2) \leq 7$

$\dim(W_1 + W_2) = 5$ (or) 6 (or) 7.

Linear Map (Linear transformation):.

$T: V \to W$ is a linear map

i) $T(v_1 + v_2) = T(v_1) + T(v_2) \quad \forall\, v_1, v_2 \in V$

ii) $T(\alpha v) = \alpha T(v) \quad \alpha \in \mathbb{F}, v \in V$

iii) $T(0) = 0$.

$\int(f+g) = \int f + \int g$

i) $\frac{d}{dx}(e^{ax}) = ae^{ax}$

ii) $\frac{d}{dx}(x^n) = nx^{n-1}$

$d/dx\,(e^{ax} + x^n)$

$d/dx\,(e^{ax}) + \frac{d}{dx}(x^n)$

1. $f: \mathbb{R} \to \mathbb{R}$

$f(x) = x$

i) $f(x+y) = x + y$

$= f(x) + f(y), \forall\, x, y \in \mathbb{R}$

ii) $f(\alpha x) = \alpha f(x)$

iii) $f(0) = 0$.

$\therefore$ This is a linear map.

2. $F: \mathbb{R} \to \mathbb{R}$

$f(x) = x + 1$

$f(0) \neq 0$.

$\therefore$ This is not a linear map.

3. $f: M_n(\mathbb{R}) \to \mathbb{R}$

1) $f(A) = \det(A)$

i) $f(A+B) = \det(A+B) \neq \det A + \det B$.

$\therefore$ This is not a linear map.

i) $f(A) = trace(A)$

i) $f(A+B) = tr(A+B) = tr(A) + tr(B)$

ii) $tr(\alpha A) = \alpha \, tr(A)$

$\therefore$ This is a linear map.

4. $f: \mathbb{R}^2 \to \mathbb{R}$

i) $f(x,y) = x+y$

$\forall (x_1, y_1)$ and $(x_2, y_2) \in \mathbb{R}^2$

i) $f(x_1+x_2, y_1+y_2) = x_1+x_2+y_1+y_2$

$= x_1+y_1+x_2+y_2$

$= f_1(x_1, y_1) + f(x_2, y_2)$

ii) $f(\alpha x, \alpha y) = \alpha x_1 + \alpha y_1$

$= \alpha (x_1+y_1)$

$= \alpha f(x_1, y_1).$

iii) $f(0) = 0.$

$\therefore$ This is a linear map.

ii) $f(x,y) = xy$

i) $f(\alpha x_1, \alpha y_1) = \alpha x_1 \alpha y_1$

$= \alpha^2 (x_1, y_1)$

$\neq \alpha^2 f(x_1, y_1)$

$\therefore$ condition not satisfied.

$\therefore$ This is not a linear map.

5. $f: \mathbb{R}^2 \to \mathbb{R}^2$

i) $f(x,y) = (x+y, y)$

i) $f(x_1+x_2, y_1+y_2)$

$= (x_1+x_2+y_1+y_2, y_1+y_2)$

$= (x_1+y_1, y_1) + (x_2+y_2, y_2)$

$= f(x_1, y_1) + f(x_2, y_2).$

ii) $f(\alpha x, \alpha y) = (\alpha x + \alpha y, \alpha y)$

$= \alpha(x+y, y)$

$= \alpha f(x, y)$

iii) $f(0,0) = (0+0, 0) = (0,0)$.

$\therefore$ This is a linear map.

ii) $f(x, y) = (x^2, x+y)$.

$f(\alpha x, \alpha y) = (\alpha^2 x^2, \alpha x + \alpha y)$

$\neq \alpha(x^2, x+y)$

$\therefore$ This is not a linear map.

NULL SPACE OF T:-

$T : V \to W$ [Linear Transformation]

$Null(T) = \{ v \in V / T(v) = 0 \}$

$Null(A) = \{ x / Ax = 0 \}$

Lemma: 1

If $T: V \to W$ is linear Transformation. Then $Null(T)$ is a subspace of V.

Proof:-

To prove:- i) $0 \in Null(T)$

ii) $v+w \in Null(T)$, $v, w \in Null(T)$

iii) $\alpha v \in Null(T)$, $v \in Null(T)$, $\alpha \in T$.

i) T is a linear Transformation $\Rightarrow T(0) = 0$

$0 \in Null(T)$

ii) $v \in Null(T) = T(v) = 0$

$w \in Null(T) \quad T(w) = 0$

$\Rightarrow T(V+W) = T(V) + T(W)$

$= 0+0$

$T(V+W) = 0$

$V+W \in Null(T)$

iii) If $V \in Null(T) \Rightarrow T(V) = 0$

$T(\alpha V) = \alpha T(V)$

$= \alpha . 0 = 0$

$\alpha V \in Null(T)$.

$\therefore$ Hence proved.

Lemma:

If $T: V \to W$ is Linear Transformation. Then

$Null(T) = \{0\}$ iff T is 1-1

Proof:

To prove: T is 1-1

consider $Null(T) = \{0\}$

$T(V) = T(W)$

$T(V) - T(W) = 0$

$T(V-W) = 0$

$V - W = 0$ (Null = 0)

$V = W$.

To prove: $Null(T) = \{0\}$ — i) $\{0\} \leq Null(T)$; ii) $Null(T) \leq \{0\}$

$\because T$ is linear $\Rightarrow T(0) = 0$

$0 \in Null(T)$

$\{0\} \leq Null(T)$

To prove: $Null(T) \leq \{0\}$

Let $V \in Null(T) \Rightarrow T(V) = 0 = T(0)$

$T(V) = T(0)$

$V = 0$ ($\because T$ is 1-1)

$\therefore Null(T) = \{0\}$

1. Diagonolize the following Matrix by orthogonal transformation.

$$\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$Trace(A) = 6.$

$|A| = 2(4-1)+1(-2+1)+1(1-2)$

$= 2(3)+(-1)+(-1)$

$= 6-2$

$= 4.$

$$\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}$$

$= (4-1)+(4-1)+(4-1)$

$= 3+3+3$

$= 9.$

$\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0.$

$$\begin{array}{c|cccc} & 1 & -6 & 9 & -4 \\ 1 & 0 & 1 & -5 & 4 \\ \hline & 1 & -5 & 4 & \boxed{0} \end{array}$$

$\lambda^2 - 5\lambda + 4$

-5

$-4 \; \boxed{-1}$

4

$(\lambda-4)(\lambda-1)=0$

$\lambda = 1, 1, 4.$

Case 1:-

$\lambda = 1$

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x - y + z = 0$

$-x + y - z = 0$

$$\begin{array}{cccccccc} & x & & y & & z & \\ -1 & & 1 & & 1 & & -1 \\ 1 & & -1 & & -1 & & 1 \end{array}$$

$$\frac{x}{1-1} = \frac{y}{-1+1} = \frac{z}{1-1}$$

$$\frac{x}{0} = \frac{y}{0} = \frac{z}{0}$$

$$x_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

All are dependent.

when $x = 0$,

$$-y+z=0$$

$$\frac{y}{+1} = \frac{z}{1}$$

$$X_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

when $y = 0$,

$$x+z=0$$

$$X_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

case 2:-

$$\lambda = 4$$

$$\begin{bmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2x-y+z=0$$
$$-x-2y-z=0$$

$$\begin{array}{cccccccc} & x & & y & & z & \\ -1 & & 1 & & -2 & & -1 \\ -2 & & -1 & & -1 & & -2 \end{array}$$

$$\frac{x}{1+2} = \frac{y}{-1-2} = \frac{z}{4-1}$$

$$\frac{x}{3} = \frac{y}{-3} = \frac{z}{3}$$

$$x_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

$x . x = 1$

$= (0, 1, 1).(0, 1, 1)$

$= 0+1+1$

$= 2$

$\therefore$ It is not orthogonal.

$$P = \begin{bmatrix} 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$D = P^{-1} A P$$

As it is orthogonal,

$$D = P^T A P$$

$$= \begin{bmatrix} 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

2. i) $6x_1^2 + 3x_2^2 + 14x_3^2 + 4x_2x_3 + 18x_1x_3 + 4x_1x_2$

$$A = \begin{bmatrix} 6 & 2 & 9 \\ 2 & 3 & 2 \\ 9 & 2 & 14 \end{bmatrix}$$

Trace $(A) = 23$.

$|A| = 6(42-4) - 2(28-18) + 9(4-27)$

$= 6(38) - 2(10) + 9(-23)$

$= 228 - 20 - 207$

$= 1$.

$\begin{vmatrix} 3 & 2 \\ 2 & 14 \end{vmatrix} + \begin{vmatrix} 6 & 9 \\ 9 & 14 \end{vmatrix} + \begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix}$

$= [42-4] + [84-81] + [18-4]$

$= 38 + 3 + 14$

$= 55$.

$\lambda^3 - 23\lambda^2 + 55\lambda - 1 = 0$.

$1 \quad -23 \quad -55 \quad -1$

2. i) $6x_1^2 + 3x_2^2 + 14x_3^2 + 4x_2x_3 + 18x_1x_3 + 4x_1x_2.$

$$A = \begin{bmatrix} 6 & 2 & 9 \\ 2 & 3 & 2 \\ 9 & 2 & 4 \end{bmatrix}$$

$|D_1| = 6$

$|D_2| = 18 - 4$

$= 14.$

$|D_3| = 6(42-4) - 2(28-18) + 9(4-27)$

$= 6(38) - 2(10) + 9(-23)$

$= 228 - 20 - 207$

$= 1.$

$|D_1|, |D_2|, |D_3| > 0$

$\therefore$ positive definite. matrix

ii) $x_1^2 + 2x_2^2 + 3x_3^2 + 2x_1x_2 + 2x_2x_3 - 2x_3x_1$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

$|D_1| = 1$

$|D_2| = 6-1$
$= 5$

$|D_3| = 1(6-1) - 1(3+1) + 1(-1-2)$
$= 5-4-3$
$= 5-4$
$= -7.$

$\therefore$ Indifinite matrix.

3. $$A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}_{2\times3} \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix}_{3\times2}$$

$$= \begin{bmatrix} 9+4+4 & 6+6-4 \\ 6+6-4 & 4+9+4 \end{bmatrix}$$

$$= \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}$$

$|A| = 225$

Trace$(A) = 34$

$\lambda^2 - 34\lambda + 225 = 0.$

$(\lambda - 25)(\lambda - 9) = 0$

$\lambda = 25, 9.$

225
$-25 \quad -9$
-34

$\lambda = \sqrt{25}, \sqrt{9}$

case 1: $\lambda = 9$

$$\begin{bmatrix} 8 & 8 \\ 8 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$8x + 8y = 0$$

$$x_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

case 2: $\lambda = 25$

$$\begin{bmatrix} -8 & 8 \\ 8 & -8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-8x + 8y = 0$$

$$x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$q_1 = \frac{1}{\sqrt{\lambda}} A^T p_i$$

$$= \frac{1}{\sqrt{25}} \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix}_{3\times 2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{2\times 1}$$

$$= \frac{1}{5} \begin{bmatrix} 3+2 \\ 2+3 \\ 2-2 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} 5 \\ 5 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$q_2 = \frac{1}{\sqrt{\lambda}} A^T p_i$$

$$= \frac{1}{3}\begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 3-2 \\ 2-3 \\ 2+2 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 \\ -1/3 \\ 4/3 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 & 1/3 \\ 1 & -1/3 \\ 0 & 4/3 \end{bmatrix}$$

$$A = PDQ^T$$

$$= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 9 & 0 \\ 0 & 25 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 1/3 & -1/3 & 4/3 \end{bmatrix}$$

$$= \begin{bmatrix} 9+0 & 0+25 \\ -9+0 & +0+25 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 25 \\ -9 & 25 \end{bmatrix}_{2\times 2}\begin{bmatrix} 1 & 1 & 0 \\ 1/3 & -1/3 & 4/3 \end{bmatrix}_{2\times 3}$$

$$= \begin{bmatrix} 9+25/3 & \\ & \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1/3 & -1/3 & 4/3 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$$

Shinchan

1. $T: R^2 \to R^2$ Linear map

$T(x,y) = (2x+y, 3x+2y)$

write a matrix with respect to basis $\{(1,0),(0,1)\}$ to $\{(1,0),(0,1)\}$.

$T(1,0) = (2,3) = 2(1,0) + 3(0,1)$

$T(0,1) = (1,2) = 1(1,0) + 2(0,1)$

$$\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$$

2. $\{(1,0),(0,1)\}$ to $\{(1,2),(3,4)\}$

$T(1,0) = (2,3) = \alpha(1,2) + \beta(3,4)$

$T(0,1) = (1,2) = \alpha_1(1,2) + \beta_1(3,4)$

$T(1,0) = \quad \alpha + 3\beta = 2 \to ①$

$2\alpha + 4\beta = 3 \to ②$

$① \times 2 \Rightarrow \quad 2\alpha + 6\beta = 4$

$\begin{array}{r} 2\alpha + 4\beta = 3 \\ (-) \quad (-) \quad (-) \\ \hline 2\beta = 1 \end{array}$

$\beta = 1/2$

$2\alpha + \overset{3}{\cancel{6}}\left(\frac{1}{\cancel{2}_1}\right) = 4$

$2\alpha = 4 - 3$

$\alpha = 1/2.$

$T(0,1) = \quad \alpha_1 + 3\beta_1 = 1 \to ①$

$2\alpha_1 + 4\beta_1 = 2 \to ②$

$① \times 2 \Rightarrow \quad 2\alpha_1 + 6\beta_1 = 2$

$\begin{array}{r} 2\alpha_1 + 4\beta_1 = 2 \\ (-) \quad (-) \quad (-) \\ \hline 2\beta_1 = 0 \end{array}$

$\beta_1 = 0$

$2\alpha_1 + 0 = 2$

$\alpha_1 = 1$

$$\begin{bmatrix} 1/2 & 1/2 \\ 1 & 0 \end{bmatrix}$$

3. $T: \mathbb{R}^2 \to \mathbb{R}^3$

$T(x,y) = \{(x+2y), 0, 2x+3y)\}$ write matrix with respect to standard basis.

$T(1,0) = (1,0,2) = 1(1,0,0) + 0(0,1,0) + 2(0,0,1)$

$T(0,1) = (2,0,3) = 2(1,0,0) + 0(1,0,0) + 3(0,0,1)$

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 2 & 3 \end{bmatrix}$$

4. $D: V \to V$ linear map

$$D(f(t)) = \frac{d}{dt}(f(t))$$

Write a matrix with respect to basis $\{\sin t, \cos t, e^{3t}\}$.

$$D(\sin t) = \frac{d}{dt}(\sin t)$$

$$= \cos t = \underline{0}\sin t + \underline{1}\cos t + \underline{0}\, e^{3t}$$

$$D(\cos t) = \frac{d}{dt}(\cos t)$$

$$= -\sin t = \underline{-1}\sin t + \underline{0}\cos t + \underline{0}\, e^{3t}$$

$$D(e^{3t}) = \frac{d}{dt}(e^{3t})$$

$$= 3e^{3t} = \underline{0}\sin t + \underline{0}\cos t \; \underline{3}\, e^{3t}$$

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

5. $I: P_2(\mathbb{R}) \to P_3(\mathbb{R})$ Linear map.

$$I(f(t)) = \int f(t)\, dt$$

write matrix with respect to,

$\{1, t, t^2\} \to \{1, t, t^2, t^3\}$.

$I(1) = \int 1 \, dt$

$= t = 0 \cdot 1 + 1 \cdot t + 0 \cdot t^2 + 0 \cdot t^3$

$I(t) = \int t \, dt$

$= \frac{t^2}{2} = 0 \cdot 1 + 0 \cdot t + \frac{1}{2} t^2 + 0 \cdot t^3$

$I(t^2) = \int t^2 \, dt$

$= \frac{t^3}{3} = 0 \cdot 1 + 0 \cdot t + 0 \cdot t^2 + \frac{1}{3} t^3$

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$$

6. A and B invertible such that,

$AB = -BA$

then $\text{Trace}(A) = ?$

$\text{Trace}(B) = ?$

$A^{-1}AB = A^{-1}(-B)A$

$B = A^{-1}(-B)A$

$B \sim (-B)$

$\text{Trace}(B) = \text{Trace}(-B)$

$T(B) + T(B) = 0$

$2T(B) = 0$

$\text{Trace}(B) = 0$

$\left[\because A \sim B \quad \exists \, |P| \neq 0 \quad PAP^{-1} = D \right]$

$B^{-1}AB = B^{-1}(-B)A$

$A \sim -A$

$\text{Trace}(A) = \text{Trace}(-A)$

$\text{Trace}(A) + \text{Trace}(A) = 0$

$2T(A) = 0$

$T(A) = 0$

$\text{Trace}(A) = 0$

MATHS ASSIGNMENT

14. If W_1 and W_2 are subspaces of a finite dimensional vector space, then.

$$\dim(W_1+W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$$

Proof :-

Let $\{w_1, w_2 \dots w_k\}$ is basis of $W_1 \cap W_2$

$\Rightarrow \{w_1, w_2 \dots w_k\}$ is Linearly independent

$W_1 \Rightarrow \{w_1, w_2 \dots w_k, v_1, v_2 \dots v_j\}$ extended basis V_1 and

$\{w_1, w_2 \dots w_k\}$ is linearly Independent.

$W_1 \Rightarrow \{w_1, w_2 \dots w_k, u_1, u_2 \dots u_n\}$ extendend basis V_2.

$\Rightarrow \dim(W_1) = k+j$ and $\dim(W_2) = k+n$ and

$\dim(W_1 \cap W_2) = k$.

Aim:

$\{w_1, w_2 \dots w_k, v_1, v_2 \dots v_j, u_1, u_2 \dots u_n\}$ basis of W_1+W_2.

Span:-

$\forall\, v \in W_1 + W_2$

$$V = (\alpha_1+\alpha_1')w_1 + (\alpha_2+\alpha_2')w_2 + \dots + (\alpha_k+\alpha_k')w_k + \beta_1 v_1 + \dots + \beta_j v_j + \gamma_1 u_1 + \dots \gamma_n u_n$$

$$= \alpha_1 w_1 + \dots + \alpha_k w_k + \beta_1 v_1 + \dots \beta_j v_j + \alpha_1' w_1 + \alpha_k' w_k + \gamma_1 u_1 + \dots \gamma_n u_n.$$

$\therefore$ W_1 basis and W_2 basis.

$V = W_1 + W_2$, $V \in W_1 + W_2$.

Linearly Independent:-

$$\sum_{i=1}^{k} \alpha_i w_i + \sum_{i=1}^{j} \beta_i v_i + \sum_{i=1}^{n} \gamma_i u_i = 0.$$

Aim:

$\alpha_i = 0$, $\beta_i = 0$, $\gamma_i = 0$.

$\therefore$ $W_1 \cap W_2$ basis $\Rightarrow \{w_1, w_2 \dots w_k\}$ Linearly Independent $\Rightarrow$ $\alpha_i = 0$.

$\therefore$ W_1 basis $\{w_1, \dots w_k, v_1 \dots v_j\}$ Linearly Independent $\Rightarrow$ $\alpha_i = 0$, $\beta_i = 0$.

$\therefore$ W_2 basis $\{w_1, \dots w_k, u_1 \dots u_n\}$ Linearly Independent $\Rightarrow$ $\alpha_i = 0$, $\gamma_i = 0$

$\alpha_i = \beta_i = \gamma_i = 0$

$\{w_1, w_2 \dots w_k, v_1, v_2 \dots v_j, u_1 \dots u_n\}$ basis $W_1 + W_2$.

$$\dim(W_1+W_2) = k+j+n+k-k$$

$$\dim(W_1+W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$$

1. $x^TAx > 0, \; x^TBx > 0$

$$x^TAx + x^TBx = 0$$

$$x^T(A+B)x = 0.$$

$\therefore$ $A+B$ is positive definite.

9. $A \sim B$

$A = P^{-1}BP$, λ is an eigen value $\rightarrow |A-\lambda I| = 0$

Aim :- $|B-\lambda I| = 0$

$$|A-\lambda I| = 0$$

$$|PBP^{-1} - \lambda I| = 0$$

$$|PBP^{-1} - \lambda PP^{-1}| = 0$$

$$|P(B-\lambda I)P^{-1}| = 0$$

$$|P|\,|B-\lambda I|\,|P^{-1}| = 0$$

$$|B-\lambda I| = 0$$

$\therefore$ λ is an eigen value of B.

4. $W_1 = 5, W_2 = 8, \dim V = 10.$

$$\min\{n, m\} < \dim(W_1+W_2) \leq \dim V.$$

$$5 \leq \dim(W_1+W_2) \leq 10$$

$\dim(W_1+W_2) = 5$ or 10 or 6 or 7 or 8 or 9

$$\dim(W_1+W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$$

$$\dim(W_1 \cap W_2) = 5+8-5 = 8$$

$$= 5+8-6 = 7$$

$$= 5+8-7 = 6$$

$$= 5+8-8 = 5$$

$$= 5+8-9 = 4$$

$$= 5+8-10 = 3$$

Possible values of $\dim(W_1 \cap W_2)$ are $8, 7, 6, 5, 4, 3$.

5. $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}$

$AA^{-1} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & -1 \end{bmatrix}$

$= \begin{bmatrix} 1+4 & -2 \\ -2 & 1+1 \end{bmatrix}$

$= \begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix}$

$|AA^{-1}| = 10-4$

$= 6.$

$\begin{array}{c} 6 \\ -6 \;\; -1 \\ -7 \end{array}$

$\lambda^2 - 7\lambda + 6 = 0$

$(\lambda-6)(\lambda-1) = 0$

$\lambda = 6, \lambda = 1$

singular value $= \sqrt{\lambda}$

$= \sqrt{6}, 1.$

2. $A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

$D_1 = 1$

$D_2 = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

$D_3 = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$

$D_4 = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 1 \end{vmatrix} = -3$

$D_4 < 0$

$\therefore$ This is indefinite matrix.

3. a) This is a subspace over $\mathbb{R}$

b) This is not subspace over $\mathbb{R}$

c) This is not space over $\mathbb{R}$

d) This is not subspace over $\mathbb{R}$

e) This is a subspace over $\mathbb{R}$

f) This is a subspace ove $\mathbb{R}$

g) This is not subspace over $\mathbb{R}$.

6. a) Let symmetric matrix be $A = \begin{bmatrix} a & h & f \\ h & b & g \\ f & g & c \end{bmatrix}$

dim = no. of independent vectors = 6

= dim (A) = 6.

b) Let skew - symmetrix matrix be $\begin{bmatrix} a & h & f \\ -h & 0 & g \\ -f & -g & 0 \end{bmatrix}$

No. of independent vectors = 3

dim(A) = 3.

7. a) $V = \{(x_1, x_2, x_3, x_4\} \in \mathbb{R}^4 : x_1 + 2x_2 + 2x_3 + 4x_4 = 0\}$

$x_1 = -2x_2 - 3x_2 - 4x_4.$

$V = \{(-2x_2 - 3x_3 - 4x_4), x_2, x_3, x_4\}$, $x_2, x_3, x_4 \in \mathbb{R}$.

$V = \{x_2(-2, 1, 0, 0)\ x_3 [-3, 0, 1, 0]\ x_4 (-4, 0, 0, 1)\} x_2, x_3, x_4 = 8$

Basis : $[(-2,1,0,0), (-3,0,1,0), (-4,0,0,1)]$

b) $V = \{A \in M_3(\mathbb{R}) : Tr(A) = 0\}$

Let $A = \begin{bmatrix} a & b & c \\ e & -a & d \\ f & g & 0 \end{bmatrix}$

Basis :

$V = \left[a \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} b \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} c \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} d \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right.$

$\left. e \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} f \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} g \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \right] \{a, b, c, d \in \mathbb{R}\}$

8. a) False : It doesnot satisfy the condition of vectors space because $W_1 + W_2 \notin W$.

b) False : Because A is non-invertible.

10. Let $W_1 = \{A + A : A^T\}$

$W_2 = \{B\} - B = B^T\}$

$$A = \frac{A + A^T}{2} + \frac{A - A^T}{2}$$

$A = W_1 + W_2$

$\{0\}$ belongs to both W_1 and W_2.

Hence, $W_1 \oplus W_2$ is direct sum.

DOT PRODUCT :-

For $x, y \in \mathbb{R}^n$, the dot product of x and y, denoted $x \cdot y$ is denoted by

$$x \cdot y = x_1 y_1 + \ldots + x_n y_n$$

Where $x = (x_1 \ldots x_n)$ and $y = (y_1, \ldots, y_n)$

INNER PRODUCT SPACE :-

An inner product on V is a function that takes each ordered pair (u, v) of elements of V to a number $\langle u, v \rangle \in F$ and has the following properties :-

⇒ **Positivity :-**

$\langle v, v \rangle \geq 0$ for all $v \in V$;

⇒ **Definiteness :-**

$\langle v, v \rangle = 0$ if and only if $v = 0$;

⇒ **Additivity in first slot :-**

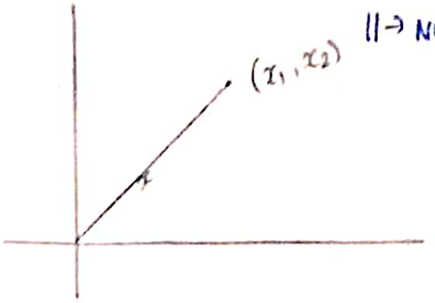
$\langle u+v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ for all $u, v, w \in V$;

⇒ **Homogeneity in first slot :-**

$\langle av, w \rangle = a \langle v, w \rangle$ for all $a \in F$ and all $v, w \in V$;

⇒ **Conjugate Symmetry :-**

$\langle v, w \rangle = \langle w, v \rangle$ for all $v, w \in V$.



$\|\to$ Norm.

$$\|x\| = \sqrt{(x_1-0)^2 + (x_1-0)^2}$$

$$\|x\| = \sqrt{x_1^2 + x_2^2}$$

$$\|x\|^2 = x_1^2 + x_2^2$$

$$\|x\|^2 = x \cdot x$$

$$\|x\|^2 = \langle x, x \rangle$$

Orthogonal :-

If $(u, v) = 0$, the order of vectors doesn't matter because $\langle v, v \rangle = 0$.

If $\langle v, u \rangle = 0$

Pythagorean Theorem :-

If u, v are orthogonal vectors in V then,

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2$$

Proof :

$$\|u + v\|^2 = \langle u+v, u+v \rangle$$

$$= \langle u, u \rangle, \langle v, v \rangle, \underbrace{\overset{0}{\langle u, v \rangle}, \overset{0}{\langle v, u \rangle}}_{\text{orthogonal}}$$

$$= \|u\|^2 + \|v\|^2$$

$\therefore$ Hence proved.

Triangle Inequality :-

If $u, v \in V$, then

$$\|u + v\| \leq \|u\| + \|v\|$$

Proof :-

$$\|u + v\|^2 = \langle u+v, u+v \rangle$$

Orthogonal Bases :-

A list of vectors is called orthonormal if the vectors in it are pair-wise orthogonal and each has norm.

$\langle a,b\rangle = 0$, where $a \neq b$.

$\langle a,a\rangle = 1$

Gram schmit ortho normalisation process :-

If $\{v_1, v_2 \ldots v_k\}$ is a basis,

then

$$u_j := v_j - \left(\frac{\langle v_j, u_1\rangle}{\langle u_1,u_1\rangle}\right)u_1 - \left(\frac{\langle v_j,u_2\rangle}{\langle u_2,u_2\rangle}\right)u_2 - \cdots - \left(\frac{\langle v_j,u_{j-1}\rangle}{\langle u_{j-1},u_{j-1}\rangle}\right)u_{j-1}$$

$\hookrightarrow$ orthogonal basis.

1. consider $\{(1,1,1), (-1,0,-1), (-1,2,3)\}$ basis
   Find orthonormal basis ?

$v_1 = (1,1,1)$, $v_2 = (-1,0,-1)$, $v_3 = (-1,2,3)$

W.K.T

$$u_j := v_j - \left(\frac{\langle v_j, u_1\rangle}{\langle u_1,u_1\rangle}\right)u_1 - \left(\frac{\langle v_j,u_2\rangle}{\langle u_2,u_2\rangle}\right)u_2 - \cdots - \left(\frac{\langle v_j,u_{j-1}\rangle}{\langle u_{j-1},u_{j-1}\rangle}\right)u_{j-1}$$

$$u_1 = v_1 \Rightarrow u_1 = (1,1,1)$$

$$u_2 = v_2 - \frac{\langle v_2 - u_1\rangle}{\langle u_1, u_1\rangle} u_1$$

$$= (-1,0,-1) - \frac{(-1+0-1)}{1+1+1}(1,1,1)$$

$$= (-1,0,-1) + 2/3\,(1,1,1)$$

$$u_2 = \left(\tfrac{-1}{3}, \tfrac{2}{3}, \tfrac{-1}{3}\right)$$

$$u_3 = v_3 - \frac{\langle v_3, u_1\rangle}{\langle u_1,u_1\rangle}u_1 - \frac{\langle v_3, u_2\rangle}{\langle u_2,u_2\rangle}\cdot u_2.$$

$$= (-1,2,3) - \frac{-1+2+3}{3}(1,1,1) - \frac{1/3 + 4/3 - 1}{1/9 + 4/9 + 1/9}\left(\tfrac{-1}{3}, \tfrac{2}{3}, \tfrac{-1}{3}\right)$$

$$= (-1,2,3) - 4/3\,(1,1,1) - \left(\tfrac{-1}{3}, \tfrac{2}{3}, \tfrac{-1}{3}\right)$$

$$= (-2,0,2)$$

2. $S = \{(1,0,1), (-1,4,1), (2,1,-2)\} \in \mathbb{R}^3$.

$\quad\quad\downarrow \quad\quad \downarrow \quad\quad \downarrow$

$\quad\quad V_1 \quad\quad V_2 \quad\quad V_3$

$$u_1 = v_1 = (1,0,1)$$

$$u_2 = v_2 - \frac{\langle v_2, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1$$

$$= (-1,4,1) - \left(\frac{-1+0+1}{1+0+1}\right)(1,0,1)$$

$$= (-1,4,1) - \frac{0}{2}(1,0,1)$$

$$= (-1,4,1) \Rightarrow \left(\frac{-1}{\sqrt{18}}, \frac{4}{\sqrt{18}}, \frac{1}{\sqrt{18}}\right)$$

$$u_3 = v_3 - \frac{\langle v_3, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 - \frac{\langle v_3, u_2 \rangle}{\langle u_2, u_2 \rangle} u_2$$

$$= (2,1,-2) - \left(\frac{-2+4-2}{1+16+1}\right)(-1,4,1) - \left(\frac{-1+1}{1+0+1}\right)(1,0,1)$$

$$u_3 = (2,1,-2)$$

$$u_1 = \left(-1/\sqrt{2}, 0, 1/\sqrt{2}\right)$$

$$u_2 = \left(-1/\sqrt{18}, 4/\sqrt{18}, 1/\sqrt{18}\right)$$

$$u_3 = \left(2/3, 1/3, -2/3\right).$$

Q.R. decomposition : 1

A $m \times n$ matrix A is said do have a Q.R. decomposition if there exists matrices Q and R with the following properties :-

i) Q is a $m \times n$ matrix whose coloumns forms an orthogonal basis for the column space for A.

ii) R is a $n \times n$ upper triangular invertile matrix with positive diagonal entrices.

iii) $A = QR$.

$A = QR$

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{31} \end{bmatrix}_{3\times3}, \quad Q = [q_1, q_2 \; q_3]$$

$$q_i = \frac{q_i'}{\|q_i'\|}, \; q_1' = a_1, \; q_2' = a_2 - r_{12} q_1, \; q_3' = a_3 - r_{13} q_1 - r_{23} q_2.$$

$$r_{ij} = \begin{cases} \|q_i'\| & \text{if } i = j \\ q_i^T \cdot a_j & \text{if } i \neq j \end{cases}$$

1. $A = \begin{bmatrix} 1 & -4 \\ 2 & 3 \\ 2 & 2 \end{bmatrix}$, Find QR decomposition using Gram Schmidt method.

$$Q = [q_1 \; q_2], \quad R = \begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{bmatrix}$$

$$q_1' = a_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$$q_1 = \frac{q_1'}{\|q_1'\|} = \frac{1}{\sqrt{9}} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$$

$$r_{11} = \|q_1'\| = 3 \qquad \therefore \left[r_{ij} = \begin{cases} \|q_i'\| & , \text{if } i = j \\ q_i^T \cdot a_j & , \text{if } i \neq j \end{cases} \right]$$

$$r_{12} = q_1^T \cdot a_2$$

$$= \begin{bmatrix} 1/3 & 2/3 & 2/3 \end{bmatrix} \begin{bmatrix} -4 \\ 3 \\ 2 \end{bmatrix} = -\frac{4}{3} + \frac{6}{3} + \frac{4}{3} = 2$$

$$q_2' = a_2 - r_{12} q_1$$

$$= \begin{bmatrix} -4 \\ 3 \\ 2 \end{bmatrix} - 2 \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$$

$$= \begin{bmatrix} -4 \\ 3 \\ 2 \end{bmatrix} - \begin{bmatrix} 2/3 \\ 4/3 \\ 4/3 \end{bmatrix} = \begin{bmatrix} -14/3 \\ 5/3 \\ 2/3 \end{bmatrix}$$

$$q_2 = \frac{q_2'}{\|q_2'\|} = \frac{1}{\sqrt{\frac{196}{9} + \frac{25}{9} + 4/9}} \begin{bmatrix} -14/3 \\ 5/3 \\ 2/3 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} -14/3 \\ 5/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} -14/15 \\ 5/15 \\ 2/15 \end{bmatrix}$$

$$r_{22} = \|q_2'\| = \sqrt{\frac{225}{9}} = 5$$

$$A = QR$$

$$= \begin{bmatrix} 1/3 & -14/15 \\ 2/3 & 5/15 \\ 2/3 & 2/15 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0 & 2/3 - 14/3 \\ 2+0 & 4/3 + 25/15 \\ 2+0 & 4/3 + 10/15 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -4 \\ 2 & 3 \\ 2 & 2 \end{bmatrix}$$

2. $A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$, Find QR decomposition using Gram schmidt method.

$$a_1 = q_1' = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}$$

$$q_1 = \frac{q_1'}{\|q_1'\|} = \frac{1}{\sqrt{29}} \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}.$$

$$r_{11} = \|q_1'\| = \sqrt{29}$$

$$r_{12} = q_1^T . a_2$$

$$= \begin{bmatrix} \frac{3}{\sqrt{29}} & \frac{2}{\sqrt{29}} & \frac{4}{\sqrt{29}} \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{6}{\sqrt{29}} + 0 + \frac{8}{\sqrt{29}} \end{bmatrix}$$

$$= \frac{14}{\sqrt{29}}.$$

$$q_2' = a_2 - r_{12} q_1$$

$$= \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} - \frac{14}{\sqrt{29}} \begin{bmatrix} 3/\sqrt{29} \\ 2/\sqrt{29} \\ 4/\sqrt{29} \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} - \begin{bmatrix} 42/29 \\ 28/29 \\ 56/29 \end{bmatrix}$$

$$= \begin{bmatrix} 16/29 \\ -28/29 \\ 2/29 \end{bmatrix} = \frac{1}{29} \begin{bmatrix} 16 \\ -28 \\ 2 \end{bmatrix}$$

$$\|q_2'\| = \sqrt{\frac{256}{841} + \frac{784}{841} + \frac{4}{841}}$$

$$= \sqrt{36/29} = 6/\sqrt{29}$$

$$q_2 = \frac{q_2'}{\|q_2'\|} = \frac{6}{29\sqrt{29}} \begin{bmatrix} 16 \\ -28 \\ 2 \end{bmatrix}$$

$$r_{22} = \|q_2'\| = 6/\sqrt{29}$$

$$r_{23} = q_2^T \cdot a_2$$

$$= \begin{bmatrix} \frac{16}{6\sqrt{29}} & \frac{-28}{6\sqrt{29}} & \frac{2}{6\sqrt{29}} \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}$$

$$r_{23} = \frac{7}{3\sqrt{29}}$$

$$r_{13} = \frac{28}{\sqrt{29}} = \begin{bmatrix} \frac{3}{\sqrt{29}} & \frac{2}{\sqrt{29}} & \frac{4}{\sqrt{29}} \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}$$

$$q_3' = a_3 - r_{13} q_1 - r_{23} q_2.$$

$$= \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix} - \frac{28}{29} \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} - \frac{7}{3} \times \frac{1}{6 \times 29} \begin{bmatrix} 16 \\ -28 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix} - \frac{28}{29} \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} - \frac{7}{522} \begin{bmatrix} 16 \\ -28 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 - 84/29 - \frac{112}{522} \\ 4 - 56/29 + 196/522 \\ 3 - \frac{112}{29} - 14/522 \end{bmatrix} = \begin{bmatrix} 8/9 \\ 22/9 \\ -8/9 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 8 \\ 22 \\ -8 \end{bmatrix}$$

$$\|q_3'\| = \sqrt{\frac{64 + 484 + 64}{87}}$$

$$\|q_3'\| = 2\sqrt{17}/3$$

$$q_3 = \frac{9}{6\sqrt{17}} \times \frac{1}{9} \begin{bmatrix} 8 \\ 22 \\ -8 \end{bmatrix} = \frac{1}{6\sqrt{17}} \begin{bmatrix} 8 \\ 22 \\ -8 \end{bmatrix}$$

$$r_{33} = \|q_3'\| = \frac{2\sqrt{17}}{3}$$

$$QR = X$$

$$= \begin{bmatrix} 3/\sqrt{29} & \frac{16}{6\sqrt{29}} & 8/6\sqrt{17} \\ 2/\sqrt{29} & \frac{-28}{6\sqrt{29}} & 22/6\sqrt{17} \\ 4/\sqrt{29} & 2/6\sqrt{29} & -8/6\sqrt{17} \end{bmatrix} \begin{bmatrix} \sqrt{29} & 14/\sqrt{29} & 28/\sqrt{29} \\ 0 & 6/\sqrt{29} & 7/3\sqrt{29} \\ 0 & 0 & 2\sqrt{17}/3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

3. i) $\begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{bmatrix}$

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{bmatrix}$$

$$a_1 = q_1' = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$q_1 = \frac{q_1'}{\|q_1'\|} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$r_{11} = \|q_1'\| = 2$$

$$r_{12} = q_1^T \cdot a_2$$

$$= \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \\ 4 \\ -1 \end{bmatrix} = \frac{-1}{2} + \frac{4}{2} + \frac{4}{2} - \frac{1}{2} = 3$$

$$q_2' = a_2 - r_{12} q_1$$

$$= \begin{bmatrix} -1 \\ 4 \\ 4 \\ -1 \end{bmatrix} - \frac{3}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -5/2 \\ 5/2 \\ 5/2 \\ -5/2 \end{bmatrix}$$

$$= \frac{5}{2}\begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\|q_2'\| = \sqrt{\frac{24}{4} \times 4}$$

$$= 5$$

$$q_2 = \frac{1}{\cancel{5}} \times \frac{\cancel{5}}{2} \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$$q_2 = \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$$r_{22} = \|q_2'\| = 5$$

$$r_{13} = q_1^T \times a_3$$

$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 4 \\ -2 \\ 2 \\ 0 \end{bmatrix}$$

$$= \frac{4}{2} - \cancel{\frac{2}{2}} + \cancel{\frac{2}{2}}$$

$$= 2$$

$$r_{23} = q_2^T \times a_3 = \begin{bmatrix} -\frac{5}{2} & \frac{5}{2} & \frac{5}{2} & -\frac{5}{2} \end{bmatrix} \begin{bmatrix} 4 \\ -2 \\ 2 \\ 0 \end{bmatrix}$$

$$\cancel{= -\frac{20}{2} - \frac{10}{2} + \frac{10}{2} + 0}$$

$$\cancel{= -\frac{20}{2}}$$

$$\cancel{= -10}$$

$$= -2 - 1 + 1$$

$$= -2$$

$$q_3' = a_3 - r_{13} q_1 - r_{23} q_2$$

$$= \begin{bmatrix} 4 \\ -2 \\ 2 \\ 0 \end{bmatrix} - \frac{2}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \frac{2}{2} \begin{bmatrix} -1 \\ +1 \\ +1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 4-1-5 \\ -2-1+5 \\ 2-1+5 \\ 0-1-5 \end{bmatrix}$$

$$= \begin{bmatrix} -2 \\ 2 \\ 2 \\ -2 \end{bmatrix}$$

$$\|q_3'\| = \sqrt{4+4+4+4} = \sqrt{16}$$

$$= 4.$$

$$q_3 = \frac{1}{4} \begin{bmatrix} -2 \\ 2 \\ 2 \\ -2 \end{bmatrix}$$

$$= \cancel{\frac{1}{2\sqrt{5}} \begin{bmatrix} -1 \\ \\ 3 \\ -3 \end{bmatrix}} = \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$r_{33} = 4$.

$$A = QR$$

$$\begin{bmatrix} 1/2 & -1/2 & -1/2 \\ 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 2 & 3 & 2 \\ 0 & 5 & -2 \\ 0 & 0 & 4\sqrt{5} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{bmatrix}.$$

ii) $\begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 3 & 6 \end{bmatrix}$

$$a_1 = q_1' = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\|q_1'\| = 1 = r_{11}$$

$$q_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$r_{12} = q_1^T \cdot a_2$$

$$= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$$

$$= 2 + 0 + 0$$

$$= 2$$

$$q_2' = a_2 - r_{12} q_1$$

$$= \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} - 2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$

$$\|q_2'\| = 3$$

$$q_2 = \frac{1}{3} \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$r_{22} = \| q_2' \| = 3$

$r_{13} = q_1^T \times a_3$

$= [1\ 0\ 0] \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$

$= 4$

$r_{23} = q_2^T \times a_3$

$= [0\ 0\ 1] \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$

$= 6.$

$q_3' = a_3 - r_{13} q_1 - r_{23} q_3$

$= \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} - \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix}$

$= \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}$

$\| q_3' \| = 5$

$q_3 = \frac{1}{5} \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$A = QR$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 3 & 6 \end{bmatrix}$$

PCA Algorithm: (principal component analysis):-

step 1:- Get data

step 2:- compute the mean vector (μ)

step 3: substract mean from the given data.

step 4:- calculate the covariance matrix.

step 5:- calculate the eigen vectors and given values of the covariance matrix.

Step 6:- choosing components and forming a feature vector.

Step 7:- Deriving the new data set.

Problem :- 01

Given data = {2,3,4,5,6,7; 1,5,3,6,7,8}

compute the principal component using PCA Algorithm.

(2,1), (3,5), (4,3), (5,6), (6,7), (7,8)

Step 1:- $\begin{bmatrix}2\\1\end{bmatrix} \begin{bmatrix}3\\5\end{bmatrix} \begin{bmatrix}4\\3\end{bmatrix} \begin{bmatrix}5\\6\end{bmatrix} \begin{bmatrix}6\\7\end{bmatrix} \begin{bmatrix}7\\8\end{bmatrix}$

$\mu_1 = \frac{\Sigma x_i}{n} = \frac{27}{6} = 4.5 \qquad \mu_2 = \frac{\Sigma y_i}{n} = \frac{30}{6} = 5$

Step 2:- $\mu = (4.5, 5)$

Step 3:- $\begin{bmatrix}-2.5\\-4\end{bmatrix} \begin{bmatrix}-1.5\\0\end{bmatrix} \begin{bmatrix}-0.5\\-2\end{bmatrix} \begin{bmatrix}0.5\\1\end{bmatrix} \begin{bmatrix}1.5\\2\end{bmatrix} \begin{bmatrix}2.3\\3\end{bmatrix}$

Step 4:- covariance:-

$(x_1 - \mu)(x_1 - \mu)^T = \begin{bmatrix}-2.5\\-4\end{bmatrix} \begin{bmatrix}-2.5 & -4\end{bmatrix} = \begin{bmatrix}6.25 & 10\\10 & 16\end{bmatrix}$

$(x_2 - \mu)(x_2 - \mu)^T = \begin{bmatrix}-1.5\\0\end{bmatrix} \begin{bmatrix}-1.5 & 0\end{bmatrix} = \begin{bmatrix}2.25 & 0\\0 & 0\end{bmatrix}$

$(x_3 - \mu)(x_3 - \mu)^T = \begin{bmatrix}-0.5\\-2\end{bmatrix} \begin{bmatrix}0.5 & -2\end{bmatrix} = \begin{bmatrix}0.25 & 1\\1 & 4\end{bmatrix}$

$(x_4 - \mu)(x_4 - \mu)^T = \begin{bmatrix}0.5\\1\end{bmatrix} \begin{bmatrix}0.5 & 1\end{bmatrix} = \begin{bmatrix}0.25 & 0.5\\0.5 & 1\end{bmatrix}$

$(x_5 - \mu)(x_5 - \mu)^T = \begin{bmatrix}1.5\\2\end{bmatrix} \begin{bmatrix}1.5 & 2\end{bmatrix} = \begin{bmatrix}2.25 & 3\\3 & 4\end{bmatrix}$

$(x_6 - \mu)(x_6 - \mu)^T = \begin{bmatrix}2.3\\3\end{bmatrix} \begin{bmatrix}2.3 & 3\end{bmatrix} = \begin{bmatrix}6.25 & 7.5\\7.5 & 9\end{bmatrix}$

$= \frac{\Sigma (x_i - \mu)(x_i - \mu)^T}{n}$

$= \frac{\begin{bmatrix}17.5 & 22\\22 & 34\end{bmatrix}}{6} \Rightarrow A = \begin{bmatrix}\frac{17.5}{6} & \frac{22}{6}\\ \frac{22}{6} & 34/6\end{bmatrix}$.

Trace of A = $\frac{51.5}{6}$

$$|A| = \frac{17.5}{6} \times \frac{34}{6} - \frac{22}{6} \times \frac{22}{6}$$

$$= \frac{595-487}{36} = \frac{37}{12}$$

$$\lambda^2 - \frac{51.5}{6}\lambda + \frac{37}{12} = 0$$

$$12\lambda^2 - 103\lambda + 37 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad a = 12, b = -103, c = 37$$

$$= \frac{103 \pm \sqrt{10609 - 1776}}{24}$$

$$= \frac{103 \pm 93.98}{24}$$

$$\lambda = 8.20, 0.375$$

$\lambda = 8.20$

$$\begin{bmatrix} -5.284 & 3.666 \\ 3.666 & -2.534 \end{bmatrix}$$

When, $\lambda = 8.215.$

$$(A - \lambda I)x = 0$$

$$\left(\begin{bmatrix} 2.92 & 3.67 \\ 3.67 & 5.67 \end{bmatrix} - \begin{bmatrix} 8.215 & 0 \\ 0 & 8.215 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -5.295 & 3.67 \\ 3.67 & -2.545 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-5.295x + 3.67y = 0 \rightarrow ①$$

$$3.67x - 2.545y = 0 \rightarrow ②$$

$$\Rightarrow \quad 5.295x = 3.67y$$

$$x = \frac{3.67}{5.295}y \Rightarrow x = 0.69y \rightarrow ③$$

$$\frac{x}{0.69} = \frac{y}{1}$$

$$x_1 = \begin{bmatrix} 0.69 \\ 1 \end{bmatrix}$$

$\lambda = 0.375$

$(A - \lambda I)x = 0$

$$\left[\begin{bmatrix} 2.92 & 3.67 \\ 3.67 & 5.67 \end{bmatrix} - \begin{bmatrix} 0.375 & 0 \\ 0 & 0.375 \end{bmatrix}\right]\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2.545 & 3.67 \\ 3.67 & 5.295 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$2.545x + 3.67y = 0$$

$$3.67x + 5.295y = 0$$

$$① \Rightarrow 2.545x = -3.67y$$

$$x = -1.44y$$

$$\frac{x}{-1.44} = \frac{y}{1} \Rightarrow X_2 = \begin{bmatrix} -1.44 \\ 1 \end{bmatrix}$$

2. Given data: $\{1, 2, 5, 7, 9;\ 3, 4, 6, 8, 9\}$

$(1,3), (2,4), (5,6), (7,8), (9,9)$

step 1: $\begin{bmatrix}1\\3\end{bmatrix} \begin{bmatrix}2\\4\end{bmatrix} \begin{bmatrix}5\\6\end{bmatrix} \begin{bmatrix}7\\8\end{bmatrix} \begin{bmatrix}9\\9\end{bmatrix}$

$$\mu_1 = \frac{\Sigma x_i}{n} = \frac{24}{5} = 4.8 \qquad \mu_2 = \frac{\Sigma y_i}{n} = \frac{30}{5} = 6$$

step 2: $\mu = (4.8, 6)$

step 3: $\begin{bmatrix}-3.8\\-3\end{bmatrix} \begin{bmatrix}-2.8\\-2\end{bmatrix} \begin{bmatrix}0.2\\0\end{bmatrix} \begin{bmatrix}2.2\\2\end{bmatrix} \begin{bmatrix}+4.2\\3\end{bmatrix}$

step 4: Covariance

$$(x_1-\mu)(x_1-\mu)^T = \begin{bmatrix}-3.8\\-3\end{bmatrix}\begin{bmatrix}-3.8 & -3\end{bmatrix} = \begin{bmatrix}14.44 & 11.4\\11.4 & 9\end{bmatrix}$$

$$(x_2-\mu)(x_2-\mu)^T = \begin{bmatrix}-2.8\\-2\end{bmatrix}\begin{bmatrix}-2.8 & -2\end{bmatrix} = \begin{bmatrix}7.84 & 5.6\\5.6 & 4\end{bmatrix}$$

$$(x_3-\mu)(x_3-\mu)^T = \begin{bmatrix}0.2\\0\end{bmatrix}\begin{bmatrix}0.2 & 0\end{bmatrix} = \begin{bmatrix}0.04 & 0\\0 & 0\end{bmatrix}$$

$$(x_4-\mu)(x_4-\mu)^T = \begin{bmatrix}2.2\\2\end{bmatrix}\begin{bmatrix}2.2 & 2\end{bmatrix} = \begin{bmatrix}4.84 & 4.4\\4.4 & 4\end{bmatrix}$$

$$(x_5-\mu)(x_5-\mu)^T = \begin{bmatrix}-4.2\\3\end{bmatrix}\begin{bmatrix}-4.2 & 3\end{bmatrix} = \begin{bmatrix}17.64 & 12.6\\12.6 & 9\end{bmatrix}$$

$$= \frac{\Sigma (x_i-\mu)(x_i-\mu)^T}{n}$$

$$= \frac{\begin{bmatrix}44.8 & 34\\34 & 26\end{bmatrix}}{5}$$

$$A = \begin{bmatrix}8.96 & 6.8\\6.8 & 5.2\end{bmatrix}$$

$$\text{Trace}(A) = \frac{70.8}{5}$$

$|A| = \frac{8.8}{25}$

$\lambda^2 - \frac{70.8}{5}\lambda + \frac{8.8}{25} = 0$

$\lambda^2 - \frac{354}{25}\lambda + \frac{8.8}{25} = 0.$

$25\lambda^2 - 354\lambda + 8.8 = 0$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $\quad a = 25,\ b = -354,\ c = 8.8$

$= \frac{354 \pm \sqrt{125316 - 880}}{50}$

$= \frac{354 \pm 18.78}{50}$

$\lambda = 14.91, \quad \lambda = 13.40$

Case 1 :- $\lambda = 14.91$

$$\begin{bmatrix} -5.95 & 6.8 \\ 6.8 & -9.71 \end{bmatrix}$$

$$\begin{bmatrix} -5.175 & -7.335 \\ -7.335 & -8.935 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-5.175x - 7.335y = 0$

$-7.335x - 8.935y = 0$

$-5.175x = 7.335y$

$x = \frac{-7.335}{5.175} y$

$x + 1.417y = 0.$