

Assignment 2 MATH 4199

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February 2022

Question 1

Consider the convolution filter

$$g(n) = \frac{1}{4}f(n-2) + \frac{1}{2}f(n) + \frac{1}{4}f(n+1)$$

find its transfer function $H(\omega)$

Solution

$$H(\omega) = \frac{1}{4} [e^{i\omega} + e^{-2i\omega} + 2]$$

Proof

$$\begin{aligned} e^{i\omega n} &\implies \frac{1}{4}e^{i\omega(n-2)} + \frac{1}{2}e^{i\omega n} + \frac{1}{4}e^{i\omega(n+1)} \\ &= e^{i\omega n} \left[\frac{1}{4}e^{-i\omega 2} + \frac{1}{2} + \frac{1}{4}e^{i\omega} \right] \\ \text{Complex} &= e^{i\omega n} \left[\frac{1}{4}(\cos(2\omega) - i\sin(2\omega)) + \frac{1}{4}(\cos(\omega) + i\sin(\omega)) + \frac{1}{2} \right] \\ \therefore H(\omega) &= \frac{1}{4} [e^{i\omega} + e^{-2i\omega} + 2] \quad \square \end{aligned}$$

Question 2

Consider the convolution filter

$$z(n) = \frac{1}{8}f(n-3) + \frac{1}{2}f(n) + \frac{3}{8}f(n+1)$$

find its transfer function $H(\omega)$

Solution

$$H(\omega) = \frac{1}{8} [3e^{i\omega} + e^{-3i\omega} + 4]$$

Proof

$$\begin{aligned} e^{i\omega n} &\implies \frac{1}{8}e^{i\omega(n-3)} + \frac{1}{2}e^{i\omega n} + \frac{3}{8}e^{i\omega(n+1)} \\ &= e^{i\omega n} \left(\frac{1}{8}e^{-i\omega 3} + \frac{1}{2} + \frac{3}{8}e^{i\omega} \right) \\ \text{Complex} &= e^{i\omega n} \left(\frac{1}{8}(\cos(\omega 3) - i\sin(\omega 3)) + \frac{1}{2} + \frac{3}{8}(\cos(\omega) + i\sin(\omega)) \right) \\ \therefore H(\omega) &= \frac{1}{8} [3e^{i\omega} + e^{-3i\omega} + 4] \quad \square \end{aligned}$$

As our filters are not symmetric, we have complex values, meaning there are shifts in the phases so both a phase shift and an amplitude stretch occur. The following graphs capture our stretches (power spectrum) and angle between points (phase spectrum).

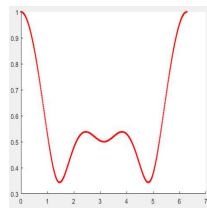


Figure 1: Question 1 power spectrum

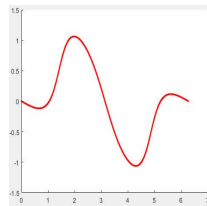


Figure 2: Question 1 phase spectrum

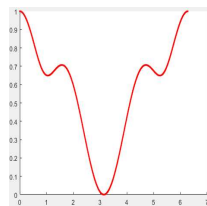


Figure 3: Question 2 power spectrum

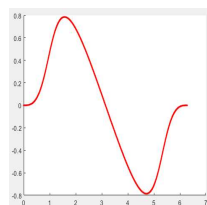


Figure 4: Question 2 phase spectrum

Question 3 Fix $k \in \{1, \dots, N-1\}$ and define a row vector of size $1 \times N$

$$\mathbf{c}_k(n) = \sqrt{\frac{2}{N}} \cos\left(\omega_k \left(n + \frac{1}{2}\right)\right) \text{ for } n \in \{0, 1, \dots, N-1\}$$

where

$$\omega_k = \frac{2\pi}{2N}k$$

For $k = 0$ set

$$\mathbf{c}_0 = \sqrt{\frac{1}{N}}(1, 1, \dots, 1)$$

(a) Show the set of vectors

$$\{\mathbf{c}_k\}_{k=0}^{N-1}$$

is an orthonormal basis for \mathbb{R}^N

(b) Convert each row vector \mathbf{c}_k into a column vector and show the $N \times N$ matrix C_N with its columns $\{\mathbf{c}_0, \mathbf{c}_1, \dots, \mathbf{c}_{N-1}\}$ is a unitary matrix.

Proof Since we are defining each \mathbf{c}_k as row a vector.

$$\begin{aligned} \mathbf{c}_k(n) &= \sqrt{\frac{2}{N}} \cos\left(\frac{(2n+1)k\pi}{2N}\right) \\ \rightarrow \mathbf{c}_0 &= \sqrt{\frac{1}{N}}(1, 1, \dots, 1) \\ \mathbf{c}_1 &= \sqrt{\frac{2}{N}} \left(\cos\left(\frac{\pi}{2N}\right), \cos\left(\frac{3\pi}{2N}\right), \dots, \cos\left(\frac{(2N-1)\pi}{2N}\right) \right) \\ \mathbf{c}_2 &= \sqrt{\frac{2}{N}} \left(\cos\left(\frac{2\pi}{2N}\right), \cos\left(\frac{6\pi}{2N}\right), \dots, \cos\left(\frac{2(2N-1)\pi}{2N}\right) \right) \\ &\vdots \\ \mathbf{c}_{N-1} &= \sqrt{\frac{2}{N}} \left(\cos\left(\frac{(N-1)\pi}{2N}\right), \cos\left(\frac{3(N-1)\pi}{2N}\right), \dots, \cos\left(\frac{(N-1)(2N-1)\pi}{2N}\right) \right) \end{aligned}$$

We must show the bases vectors $\{\mathbf{c}_0, \mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_{N-1}\}$ are orthonormal. We will do this by showing $CC^T = I_N$

$$\begin{aligned} C &= \sqrt{\frac{2}{N}} \sum_{n=0}^{N-1} \cos\left(\omega_k \left(n + \frac{1}{2}\right)\right) = \sqrt{\frac{2}{N}} \sum_{n=0}^{N-1} \cos\left(\frac{2\pi}{2N}k \left(n + \frac{1}{2}\right)\right) = \sqrt{\frac{2}{N}} \sum_{n=0}^{N-1} \cos\left(\frac{2\pi}{2N}kn + \frac{2\pi}{2N}k \cdot \frac{1}{2}\right) \\ &= \sqrt{\frac{2}{N}} \sum_{n=0}^{N-1} \cos\left(\frac{2\pi}{2N}kn + \frac{\pi}{2N}k\right) = \sqrt{\frac{2}{N}} \sum_{n=0}^{N-1} \cos\left(\frac{2\pi kn + \pi k}{2N}\right) = \sqrt{\frac{2}{N}} \sum_{n=0}^{N-1} \cos\left(\frac{(2n+1)k\pi}{2N}\right) \\ \therefore C^T &= \sqrt{\frac{2}{N}} \sum_{n=0}^{N-1} \cos\left(\frac{(2n+1)l\pi}{2N}\right) \end{aligned}$$

Consider the k, l th, element of CC^T for $k, l \neq 0$

$$\begin{aligned} [CC^T]_{k,l} &= \sqrt{\frac{2}{N}} \sum_{n=0}^{N-1} \cos\left(\frac{(2n+1)k\pi}{2N}\right) \cdot \sqrt{\frac{2}{N}} \sum_{n=0}^{N-1} \cos\left(\frac{(2n+1)l\pi}{2N}\right) \\ &= \frac{2}{N} \sum_{n=0}^{N-1} \cos\left(\frac{(2n+1)k\pi}{2N}\right) \cdot \cos\left(\frac{(2n+1)l\pi}{2N}\right) \end{aligned}$$

let $A = \frac{(2n+1)k\pi}{2N}$, and let $B = \frac{(2n+1)l\pi}{2N}$. We will simplify $\frac{2}{N} \sum_{n=0}^{N-1} \cos A \cdot \cos B$ with the product to sum identity.

$$\begin{aligned}
 \left. \begin{aligned}
 \cos(A) \cdot \cos(B) &= \frac{1}{2} \cos(A-B) + \frac{1}{2} \cos(A+B) \\
 &= \frac{1}{4} (e^{-iA} + e^{iA}) (e^{-iB} + e^{iB}) = \frac{1}{4} [(e^{-iA}) (e^{-iB}) + (e^{-iA}) (e^{iB}) + (e^{iA}) (e^{-iB}) + (e^{iA}) (e^{iB})] \\
 &= \frac{1}{4} [e^{-iA-iB} + e^{iA-iB} + e^{-iA+iB} + e^{iA+iB}] = \frac{1}{4} [e^{-i(A+B)} + e^{i(A+B)} + e^{-i(A-B)} + e^{i(A-B)}] \\
 &= \frac{1}{4} \left[e^{-i \left(\frac{(2n+1)k\pi}{2N} + \frac{(2n+1)l\pi}{2N} \right)} + e^{i \left(\frac{(2n+1)k\pi}{2N} + \frac{(2n+1)l\pi}{2N} \right)} + e^{-i \left(\frac{(2n+1)k\pi}{2N} - \frac{(2n+1)l\pi}{2N} \right)} + e^{i \left(\frac{(2n+1)k\pi}{2N} - \frac{(2n+1)l\pi}{2N} \right)} \right] \\
 &= \frac{1}{4} \left[e^{\frac{-i(2n+1)(k+l)\pi}{2N}} + e^{\frac{i(2n+1)(k+l)\pi}{2N}} + e^{\frac{i(2n+1)(k-l)\pi}{2N}} + e^{\frac{-i(2n+1)(k-l)\pi}{2N}} \right]
 \end{aligned} \right\} \text{Aside}
 \end{aligned}$$

$$\begin{aligned}
 \therefore [CC^T]_{k,l} &= \frac{2}{N} \cdot \frac{1}{4} \sum_{n=0}^{N-1} \left\{ e^{\frac{-i(2n+1)(k+l)\pi}{2N}} + e^{\frac{i(2n+1)(k+l)\pi}{2N}} + e^{\frac{i(2n+1)(k-l)\pi}{2N}} + e^{\frac{-i(2n+1)(k-l)\pi}{2N}} \right\} \\
 &= \frac{1}{2N} \sum_{n=0}^{N-1} \left\{ e^{\frac{-i(2n+1)(k+l)\pi}{2N}} + e^{\frac{i(2n+1)(k+l)\pi}{2N}} + e^{\frac{i(2n+1)(k-l)\pi}{2N}} + e^{\frac{-i(2n+1)(k-l)\pi}{2N}} \right\} \\
 &= \frac{1}{2N} \sum_{n=0}^{N-1} \cos \left(\frac{(2n+1)(k+l)\pi}{2N} \right) + \frac{1}{2N} \sum_{n=0}^{N-1} \cos \left(\frac{(2n+1)(k-l)\pi}{2N} \right)
 \end{aligned}$$

This simplifies to,

$$[CC^T]_{k,l} = \cos \left[\frac{(k+l)\pi}{2N} \right] \delta(k+l-N) + \cos \left[\frac{(k-l)\pi}{2N} \right] \delta(k-l)$$

where the first term is zero, and the second term is zero except when $k = l$. Therefore,

$$[CC^T]_{k,l} = \begin{cases} 0 & \text{for } k \neq l \\ 1 & \text{for } k = l \end{cases}$$

For $k, l = 0$, $[CC^T]_{0,0} = \sum_{n=0}^{N-1} \frac{1}{N} = 1 \therefore CC^T = I_N$. So, C is unitary, and orthogonal. As C is an orthogonal matrix, C must also be orthonormal. □

resources used

https://en.wikipedia.org/wiki/Kronecker_delta

https://www.engr.colostate.edu/ECE513/SP09/lectures/lectures11_12.pdf

resources read

<https://klein.mit.edu/~gs/papers/dct.pdf>

<https://math.mit.edu/~gs/highdegree/TplusH.pdf>

<https://www.jstor-org.libproxy.mtroyal.ca/stable/pdf/24248477.pdf>

https://www.ldeo.columbia.edu/~richards/webpage_rev_Jan06/Ch3_FourTrans%26DeltaFns.pdf

(c) Set $N = 8$ and verify the matrix

$$C_8 = \begin{pmatrix} 0.3536 & 0.4904 & 0.4619 & 0.4157 & 0.3536 & 0.2778 & 0.1913 & 0.0975 \\ 0.3536 & 0.4157 & 0.1913 & -0.0975 & -0.3536 & -0.4904 & -0.4619 & -0.2778 \\ 0.3536 & 0.2778 & -0.1913 & -0.4904 & -0.3536 & 0.0975 & 0.4619 & 0.4157 \\ 0.3536 & 0.0975 & -0.4619 & -0.2778 & 0.3536 & 0.4157 & -0.1913 & -0.4904 \\ 0.3536 & -0.0975 & -0.4619 & 0.2778 & 0.3536 & -0.4157 & -0.1913 & 0.4904 \\ 0.3536 & -0.2778 & -0.1913 & 0.4904 & -0.3536 & -0.0975 & 0.4619 & -0.4157 \\ 0.3536 & -0.4157 & 0.1913 & 0.0975 & -0.3536 & 0.4904 & -0.4619 & 0.2778 \\ 0.3536 & -0.4904 & 0.4619 & -0.4157 & 0.3536 & -0.2778 & 0.1913 & -0.0975 \end{pmatrix}$$

is a unitary matrix.

Proof

```

1: function [F] = m_dct(N)
2:
3: % Code for Assignment 2 MATH 4199
4:
5: constant0 = sqrt(1/N);
6: constant1 = sqrt(2/N);
7: W = (2*pi)/(2*N);
8:
9: F = zeros(N:N);
10:
11: for a = 1:N
12:     c0(a) = 1;
13: end
14: c0 = c0*constant0;
15: F = c0';
16:
17: for k = 1:N-1
18:     for n = 1:N
19:         angle = W*k;
20:         angle = angle*((n-1) + 1/2);
21:         ck(n) = cos(angle);
22:     end
23:     F = [F (ck * constant1)'];
24: end
25: end

```

```

>> F = m_dct(8)
F =
    0.3536    0.4904    0.4619    0.4157    0.3536    0.2778    0.1913    0.0975
    0.3536    0.4157    0.1913   -0.0975   -0.3536   -0.4904   -0.4619   -0.2778
    0.3536    0.2778   -0.1913   -0.4904   -0.3536    0.0975    0.4619    0.4157
    0.3536    0.0975   -0.4619   -0.2778    0.3536    0.4157   -0.1913   -0.4904
    0.3536   -0.0975   -0.4619    0.2778    0.3536   -0.4157   -0.1913    0.4904
    0.3536   -0.2778   -0.1913    0.4904   -0.3536   -0.0975    0.4619   -0.4157
    0.3536   -0.4157    0.1913    0.0975   -0.3536    0.4904   -0.4619    0.2778
    0.3536   -0.4904    0.4619   -0.4157    0.3536   -0.2778    0.1913   -0.0975

>> F*F'
ans =
    1.0000    0.0000    0.0000    0.0000   -0.0000    0.0000    0.0000   -0.0000
    0.0000    1.0000    0.0000    0.0000    0.0000   -0.0000   -0.0000    0.0000
    0.0000    0.0000    1.0000   -0.0000   -0.0000   -0.0000   -0.0000    0.0000
    0.0000    0.0000   -0.0000    1.0000    0.0000    0.0000    0.0000    0.0000
   -0.0000    0.0000   -0.0000    0.0000    1.0000         0    0.0000   -0.0000
    0.0000   -0.0000   -0.0000    0.0000         0    1.0000    0.0000    0.0000
    0.0000   -0.0000   -0.0000    0.0000    0.0000    0.0000    1.0000    0.0000
   -0.0000    0.0000    0.0000    0.0000   -0.0000    0.0000    0.0000    1.0000

```

Figure 5: $AA^* = A^*A = I$

□

(d) Let \mathbf{x} be any real row data vector of size N . The discrete cosine transform of \mathbf{x} is given by

$$DCT(\mathbf{x}) = 2 \sum_{n=0}^{N-1} \mathbf{x}(n) \cos\left(\frac{\pi k}{2N}(2n+1)\right) = 2 \sum_{n=0}^{N-1} \mathbf{x}(n) \cos\left(\omega_k \left(n + \frac{1}{2}\right)\right)$$

for $k \in \{0, 1, \dots, N-1\}$

Proof

(e) View the vector \mathbf{x} as a column vector and show that

$$DCT(\mathbf{x}) = C_N^T \mathbf{x}$$

proof Given $\mathbf{y} = (y_0, y_1, \dots, y_{N-1})^T$ the (DCT) of \mathbf{y} is the vector $\mathbf{x} = (x_0, x_1, \dots, x_{N-1})^T$ where $\mathbf{x} = C_N \mathbf{y}$

$$\begin{aligned}
 \mathbf{x} &= C_N \mathbf{y} \\
 C_N^T \mathbf{x} &= C_N^T C_N \mathbf{y} \\
 C_N^T \mathbf{x} &= I \mathbf{y} \\
 \therefore C_N^T \mathbf{x} &= \mathbf{y} = DCT(\mathbf{x})
 \end{aligned}$$

□

(f) The inverse discrete cosine transform, IDCT, is given by

$$\mathbf{x}(n) = \sqrt{\frac{2}{N}} \sum_{k=0}^{N-1} DCT(\mathbf{x})(k) \lambda_k \cos \left(\omega_k \left(n + \frac{1}{2} \right) \right)$$

for $n \in \{0, 1, \dots, N-1\}$ where $\lambda_0 = \frac{1}{\sqrt{2}}$ and $\lambda_k = 1$ for $k \in \{1, 2, \dots, N-1\}$

(g) Consider a column vector \mathbf{y} and show that

$$IDCT(\mathbf{y}) = C_N \mathbf{y}$$

Proof Given $\mathbf{x} = (x_0, x_2, \dots, x_{N-1})^T$ the (IDCT) of \mathbf{x} is the vector $\mathbf{y} = (y_0, y_2, \dots, y_{N-1})^T$ where $\mathbf{y} = C_N^T \mathbf{x}$.

$$\begin{aligned} \mathbf{y} &= C_N^T \mathbf{x} \\ C_N \mathbf{y} &= C_N C_N^T \mathbf{x} \\ C_N \mathbf{y} &= \mathbf{I} \mathbf{x} \\ \therefore C_N \mathbf{y} &= \mathbf{x} \quad \square \end{aligned}$$

(h) Consider a column vector \mathbf{x} show that

$$C_N C_N^T \mathbf{x} = \mathbf{x}$$

Proof

$$\begin{aligned} \mathbf{x} &= \mathbf{x} \\ C_N \mathbf{x} &= C_N \mathbf{x} \\ C_N C_N^T \mathbf{x} &= C_N C_N^T \mathbf{x} \\ C_N C_N^T \mathbf{x} &= \mathbf{I} \mathbf{x} \\ C_N C_N^T \mathbf{x} &= \mathbf{x} \quad \square \end{aligned}$$

(i) Consider a column vector $\mathbf{x} = (1, 2, -2, 2, 3, 0, 1, 3)^T$. Let P be a 8×7 matrix whose columns are the first 7 columns of the matrix C . We find

$$\mathbf{y} = PP^T \mathbf{x} = (1.0794, 1.17739, -1.6616, 1.6008, 3.3992, -0.3384, 1.2261, 2.9206)^T$$

Give an interpretation to the column vector \mathbf{y} .

Solution

$\mathbf{y} = PP^T \mathbf{x} = \text{proj}_C(\mathbf{x})$ is the orthogonal projection of the vector \mathbf{x} onto the columns of C , which is an approximation of the data points with a minimum error. So, we can zero the coefficients corresponding to significant frequencies in the last terms and still have a function that is a close approximation as the terms are arranged in order of importance, which removes the highest data frequency and keeps the lowest. □