

Assignment 2 MATH 4199

*Michael Walker,
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Question 1

Consider the convolution filter

$$g(n) = \frac{1}{4}f(n-2) + \frac{1}{2}f(n) + \frac{1}{4}f(n+1)$$

find its transfer function $H(\omega)$

Solution

$$H(\omega) = \frac{1}{4}[e^{iw} + e^{-2iw} + 2]$$

Proof

$$\begin{aligned} e^{i\omega n} &\implies \frac{1}{4}e^{i\omega(n-2)} + \frac{1}{2}e^{i\omega n} + \frac{1}{4}e^{i\omega(n+1)} \\ &= e^{i\omega n} \left[\frac{1}{4}e^{-i\omega 2} + \frac{1}{2} + \frac{1}{4}e^{i\omega} \right] \\ \text{Complex } &= e^{i\omega n} \left[\frac{1}{4}(\cos(2\omega) - i\sin(2\omega)) + \frac{1}{4}(\cos(\omega) + i\sin(\omega)) + \frac{1}{2} \right] \\ &\therefore H(\omega) = \frac{1}{4}[e^{iw} + e^{-2iw} + 2] \end{aligned}$$

□

Question 2

Consider the convolution filter

$$z(n) = \frac{1}{8}f(n-3) + \frac{1}{2}f(n) + \frac{3}{8}f(n+1)$$

find its transfer function $H(\omega)$

Solution

$$H(\omega) = \frac{1}{8}[3e^{i\omega} + e^{-3i\omega} + 4]$$

Proof

$$\begin{aligned} e^{i\omega n} &\implies \frac{1}{8}e^{i\omega(n-3)} + \frac{1}{2}e^{i\omega n} + \frac{3}{8}e^{i\omega(n+1)} \\ &= e^{i\omega n} \left[\frac{1}{8}e^{-i\omega 3} + \frac{1}{2} + \frac{3}{8}e^{i\omega} \right] \\ \text{Complex } &= e^{i\omega n} \left[\frac{1}{8}(\cos(\omega 3) - i\sin(\omega 3)) + \frac{1}{2} + \frac{3}{8}(\cos(\omega) + i\sin(\omega)) \right] \\ &\therefore H(\omega) = \frac{1}{8}[3e^{i\omega} + e^{-3i\omega} + 4] \end{aligned}$$

□

As our filters are not symmetric, we have complex values, meaning there are shifts in the phases so both a phase shift and an amplitude stretch occur. The following graphs capture our stretches (power spectrum) and angle between points (phase spectrum).

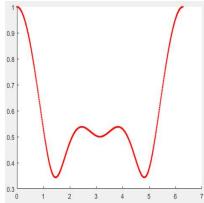


Figure 1: Question 1 power spectrum

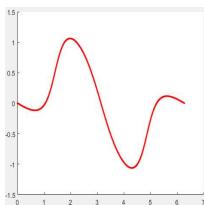


Figure 2: Question 1 phase spectrum

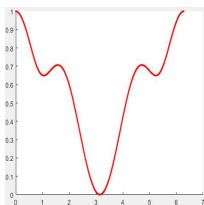


Figure 3: Question 2 power spectrum

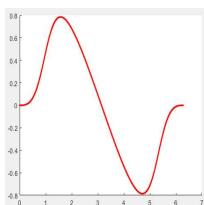


Figure 4: Question 2 phase spectrum

Question 3 Fix $k \in \{1, \dots, N-1\}$ and define a row vector of size $1 \times N$

$$\mathbf{c}_k(n) = \sqrt{\frac{2}{N}} \cos\left(\omega_k \left(n + \frac{1}{2}\right)\right) \text{ for } n \in \{0, 1, \dots, N-1\}$$

where

$$\omega_k = \frac{2\pi}{2N}k$$

For $k=0$ set

$$\mathbf{c}_0 = \sqrt{\frac{1}{N}}(1, 1, \dots, 1)$$

(a) Show the set of vectors

$$\{\mathbf{c}_k\}_{k=0}^{N-1}$$

is an orthonormal basis for \mathbb{R}^N

(b) Convert each row vector \mathbf{c}_k into a column vector and show the $N \times N$ matrix C_N with its columns $\{\mathbf{c}_0, \mathbf{c}_1, \dots, \mathbf{c}_{N-1}\}$ is a unitary matrix.

Proof Since we are defining each \mathbf{c}_k as row a vector.

$$\begin{aligned} \mathbf{c}_k(n) &= \sqrt{\frac{2}{N}} \cos\left(\frac{(2n+1)k\pi}{2N}\right) \\ \rightarrow \mathbf{c}_0 &= \sqrt{\frac{1}{N}}(1, 1, \dots, 1) \\ \mathbf{c}_1 &= \sqrt{\frac{2}{N}}\left(\cos\left(\frac{\pi}{2N}\right), \cos\left(\frac{3\pi}{2N}\right), \dots, \cos\left(\frac{(2N-1)\pi}{2N}\right)\right) \\ \mathbf{c}_2 &= \sqrt{\frac{2}{N}}\left(\cos\left(\frac{2\pi}{2N}\right), \cos\left(\frac{6\pi}{2N}\right), \dots, \cos\left(\frac{2(2N-1)\pi}{2N}\right)\right) \\ \vdots &= \vdots \\ \mathbf{c}_{N-1} &= \sqrt{\frac{2}{N}}\left(\cos\left(\frac{(N-1)\pi}{2N}\right), \cos\left(\frac{3(N-1)\pi}{2N}\right), \dots, \cos\left(\frac{(N-1)(2N-1)\pi}{2N}\right)\right) \end{aligned}$$

We must show the bases vectors $\{\mathbf{c}_0, \mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_{N-1}\}$ are orthonormal. We will do this by showing $CC^T = I_N$

$$\begin{aligned} C &= \sqrt{\frac{2}{N}} \sum_{n=0}^{N-1} \cos\left(\omega_k \left(n + \frac{1}{2}\right)\right) = \sqrt{\frac{2}{N}} \sum_{n=0}^{N-1} \cos\left(\frac{2\pi}{2N}k \left(n + \frac{1}{2}\right)\right) = \sqrt{\frac{2}{N}} \sum_{n=0}^{N-1} \cos\left(\frac{2\pi}{2N}kn + \frac{2\pi}{2N}k \cdot \frac{1}{2}\right) \\ &= \sqrt{\frac{2}{N}} \sum_{n=0}^{N-1} \cos\left(\frac{2\pi}{2N}kn + \frac{\pi}{2N}k\right) = \sqrt{\frac{2}{N}} \sum_{n=0}^{N-1} \cos\left(\frac{2\pi kn + \pi k}{2N}\right) = \sqrt{\frac{2}{N}} \sum_{n=0}^{N-1} \cos\left(\frac{(2n+1)k\pi}{2N}\right) \\ \therefore C^T &= \sqrt{\frac{2}{N}} \sum_{n=0}^{N-1} \cos\left(\frac{(2n+1)l\pi}{2N}\right) \end{aligned}$$

Consider the k, l th, element of CC^T for $k, l \neq 0$

$$\begin{aligned} [CC^T]_{k,l} &= \sqrt{\frac{2}{N}} \sum_{n=0}^{N-1} \cos\left(\frac{(2n+1)k\pi}{2N}\right) \cdot \sqrt{\frac{2}{N}} \sum_{n=0}^{N-1} \cos\left(\frac{(2n+1)l\pi}{2N}\right) \\ &= \frac{2}{N} \sum_{n=0}^{N-1} \cos\left(\frac{(2n+1)k\pi}{2N}\right) \cdot \cos\left(\frac{(2n+1)l\pi}{2N}\right) \end{aligned}$$

let $A = \frac{(2n+1)k\pi}{2N}$, and let $B = \frac{(2n+1)l\pi}{2N}$. We will simplify $\frac{2}{N} \sum_{n=0}^{N-1} \cos A \cdot \cos B$ with the product to sum identity.

$$\begin{aligned}
 \text{Aside} & \left\{ \begin{aligned}
 \cos(A) \cdot \cos(B) &= \frac{1}{2} \cos(A-B) + \frac{1}{2} \cos(A+B) \\
 &= \frac{1}{4} (e^{-iA} + e^{iA}) (e^{-iB} + e^{iB}) = \frac{1}{4} [(e^{-iA})(e^{-iB}) + (e^{-iA})(e^{iB}) + (e^{iA})(e^{-iB}) + (e^{iA})(e^{iB})] \\
 &= \frac{1}{4} [e^{-iA-iB} + e^{iA-iB} + e^{-iA+iB} + e^{iA+iB}] = \frac{1}{4} [e^{-i(A+B)} + e^{i(A+B)} + e^{-i(A-B)} + e^{i(A-B)}] \\
 &= \frac{1}{4} \left[e^{-i\left(\frac{(2n+1)k\pi}{2N} + \frac{(2n+1)l\pi}{2N}\right)} + e^{i\left(\frac{(2n+1)k\pi}{2N} + \frac{(2n+1)l\pi}{2N}\right)} + e^{-i\left(\frac{(2n+1)k\pi}{2N} - \frac{(2n+1)l\pi}{2N}\right)} + e^{i\left(\frac{(2n+1)k\pi}{2N} - \frac{(2n+1)l\pi}{2N}\right)} \right] \\
 &= \frac{1}{4} \left[e^{\frac{-i(2n+1)(k+l)\pi}{2N}} + e^{\frac{i(2n+1)(k+l)\pi}{2N}} + e^{\frac{i(2n+1)(k-l)\pi}{2N}} + e^{\frac{-i(2n+1)(k-l)\pi}{2N}} \right]
 \end{aligned} \right. \\
 & \therefore [CC^T]_{k,l} = \frac{2}{N} \cdot \frac{1}{4} \sum_{n=0}^{N-1} \left\{ e^{\frac{-i(2n+1)(k+l)\pi}{2N}} + e^{\frac{i(2n+1)(k+l)\pi}{2N}} + e^{\frac{i(2n+1)(k-l)\pi}{2N}} + e^{\frac{-i(2n+1)(k-l)\pi}{2N}} \right\} \\
 &= \frac{1}{2N} \sum_{n=0}^{N-1} \left\{ e^{\frac{-i(2n+1)(k+l)\pi}{2N}} + e^{\frac{i(2n+1)(k+l)\pi}{2N}} + e^{\frac{i(2n+1)(k-l)\pi}{2N}} + e^{\frac{-i(2n+1)(k-l)\pi}{2N}} \right\} \\
 &= \frac{1}{2N} \sum_{n=0}^{N-1} \cos\left(\frac{(2n+1)(k+l)\pi}{2N}\right) + \frac{1}{2N} \sum_{n=0}^{N-1} \cos\left(\frac{(2n+1)(k-l)\pi}{2N}\right)
 \end{aligned}$$

This simplifies to,

$$[CC^T]_{k,l} = \cos\left(\frac{(k+l)\pi}{2N}\right) \delta(k+l-N) + \cos\left(\frac{(k-l)\pi}{2N}\right) \delta(k-l)$$

where the first term is zero, and the second term is zero except when $k = l$. Therefore,

$$[CC^T]_{k,l} = \begin{cases} 0 & \text{for } k \neq l \\ 1 & \text{for } k = l \end{cases}$$

For $k, l = 0$, $[CC^T]_{0,0} = \sum_{n=0}^{N-1} \frac{1}{N} = 1 \therefore CC^T = I_N$. So, C is unitary, and orthogonal. As C is an orthogonal matrix, C must also be orthonormal. \square

resources used

https://en.wikipedia.org/wiki/Kronecker_delta

https://www.engr.colostate.edu/ECE513/SP09/lectures/lectures11_12.pdf

resources read

<https://klein.mit.edu/~gs/papers/dct.pdf>

<https://math.mit.edu/~gs/highdegree/TplusH.pdf>

<https://www-jstor-org.libproxy.mtroyal.ca/stable/pdf/24248477.pdf>

https://www.ideo.columbia.edu/~richards/webpage_rev_Jan06/Ch3_FourTrans%26DeltaFns.pdf

(c) Set $N = 8$ and verify the matrix

$$C_8 = \begin{pmatrix} 0.3536 & 0.4904 & 0.4619 & 0.4157 & 0.3536 & 0.2778 & 0.1913 & 0.0975 \\ 0.3536 & 0.4157 & 0.1913 & -0.0975 & -0.3536 & -0.4904 & -0.4619 & -0.2778 \\ 0.3536 & 0.2778 & -0.1913 & -0.4904 & -0.3536 & 0.0975 & 0.4619 & 0.4157 \\ 0.3536 & 0.0975 & -0.4619 & -0.2778 & 0.3536 & 0.4157 & -0.1913 & -0.4904 \\ 0.3536 & -0.0975 & -0.4619 & 0.2778 & 0.3536 & -0.4157 & -0.1913 & 0.4904 \\ 0.3536 & -0.2778 & -0.1913 & 0.4904 & -0.3536 & -0.0975 & 0.4619 & -0.4157 \\ 0.3536 & -0.4157 & 0.1913 & 0.0975 & -0.3536 & 0.4904 & -0.4619 & 0.2778 \\ 0.3536 & -0.4904 & 0.4619 & -0.4157 & 0.3536 & -0.2778 & 0.1913 & -0.0975 \end{pmatrix}$$

is a unitary matrix.

Proof

The figure shows a screenshot of the MATLAB environment. On the left, the code file `m_dct.m` is displayed, containing the implementation of the DCT matrix C_N . The code uses nested loops to calculate the elements of the matrix based on the formula provided in the text. On the right, the Command Window shows the execution of the code for $N=8$. It first displays the matrix F , then the result of F^*F' , which is the identity matrix I , confirming that F is unitary.

```

m_dct.m
function [F] = m_dct(N)
% Code for Assignment 2 MATH 4199
constant0 = sqrt(1/N);
constant1 = sqrt(2/N);
W = (2*pi)/(2*N);
F = zeros(N:N);
for a = 1:N
    c0(a) = 1;
end
c0 = c0*constant0;
F = c0';
for k = 1:N-1
    for n = 1:N
        angle = W*k;
        angle = angle*((n-1) + 1/2);
        ck(n) = cos(angle);
    end
    F = [F (ck * constant1)'];
end
end

```

```

>> F = m_dct(8)
F =
0.3536 0.4904 0.4619 0.4157 0.3536 0.2778 0.1913 0.0975
0.3536 0.4157 0.1913 -0.0975 -0.3536 -0.4904 -0.4619 -0.2778
0.3536 0.2778 -0.1913 -0.4904 -0.3536 0.0975 0.4619 0.4157
0.3536 0.0975 -0.4619 -0.2778 0.3536 0.4157 -0.1913 -0.4904
0.3536 -0.0975 -0.4619 0.2778 0.3536 -0.4157 -0.1913 0.4904
0.3536 -0.2778 -0.1913 0.4904 -0.3536 -0.0975 0.4619 -0.4157
0.3536 -0.4157 0.1913 0.0975 -0.3536 0.4904 -0.4619 0.2778
0.3536 -0.4904 0.4619 -0.4157 0.3536 -0.2778 0.1913 -0.0975
>> F'*F'
ans =
1.0000 0.0000 0.0000 0.0000 -0.0000 0.0000 0.0000 -0.0000
0.0000 1.0000 0.0000 0.0000 0.0000 -0.0000 -0.0000 0.0000
0.0000 0.0000 1.0000 -0.0000 -0.0000 -0.0000 -0.0000 0.0000
0.0000 0.0000 -0.0000 1.0000 0.0000 0.0000 0.0000 0.0000
-0.0000 0.0000 -0.0000 0.0000 1.0000 0 0.0000 -0.0000
0.0000 -0.0000 -0.0000 0.0000 0 1.0000 0.0000 0.0000
0.0000 -0.0000 -0.0000 0.0000 0.0000 0.0000 1.0000 0.0000
-0.0000 0.0000 0.0000 0.0000 -0.0000 0.0000 0.0000 1.0000
fx>>

```

Figure 5: $AA^*=A^*A=I$

□

(d) Let \mathbf{x} be any real row data vector of size N . The discrete cosine transform of \mathbf{x} is given by

$$DCT(\mathbf{x}) = 2 \sum_{n=0}^{N-1} \mathbf{x}(n) \cos\left(\frac{\pi k}{2N}(2n+1)\right) = 2 \sum_{n=0}^{N-1} \mathbf{x}(n) \cos\left(\omega_k \left(n + \frac{1}{2}\right)\right)$$

for $k \in \{0, 1, \dots, N-1\}$

Proof

(e) View the vector \mathbf{x} as a column vector and show that

$$DCT(\mathbf{x}) = C_N^T \mathbf{x}$$

proof Given $\mathbf{y} = (y_0, y_1, \dots, y_{N-1})^T$ the (DCT) of \mathbf{y} is the vector $\mathbf{x} = (x_0, x_1, \dots, x_{N-1})^T$ where $\mathbf{x} = C_N \mathbf{y}$

$$\begin{aligned} \mathbf{x} &= C_N \mathbf{y} \\ C_N^T \mathbf{x} &= C_N^T C_N \mathbf{y} \\ C_N^T \mathbf{x} &= I \mathbf{y} \\ \therefore C_N^T \mathbf{x} &= \mathbf{y} = DCT(\mathbf{x}) \end{aligned}$$

□

(f) The inverse discrete cosine transform, IDCT, is given by

$$\mathbf{x}(n) = \sqrt{\frac{2}{N}} \sum_{k=0}^{N-1} DCT(\mathbf{x})(k) \lambda_k \cos\left(\omega_k \left(n + \frac{1}{2}\right)\right)$$

for $n \in \{0, 1, \dots, N-1\}$ where $\lambda_0 = \frac{1}{\sqrt{2}}$ and $\lambda_k = 1$ for $k \in \{1, 2, \dots, N-1\}$

(g) Consider a column vector \mathbf{y} and show that

$$IDCT(\mathbf{y}) = C_N \mathbf{y}$$

Proof Given $\mathbf{x} = (x_0, x_1, \dots, x_{N-1})^T$ the (IDCT) of \mathbf{x} is the vector $\mathbf{y} = (y_0, y_1, \dots, y_{N-1})^T$ where $\mathbf{y} = C_N^T \mathbf{x}$.

$$\begin{aligned} \mathbf{y} &= C_N^T \mathbf{x} \\ C_N \mathbf{y} &= C_N C_N^T \mathbf{x} \\ C_N \mathbf{y} &= I \mathbf{x} \\ \therefore C_N \mathbf{y} &= \mathbf{x} \quad \square \end{aligned}$$

(h) Consider a column vector \mathbf{x} show that

$$C_N C_N^T \mathbf{x} = \mathbf{x}$$

Proof

$$\begin{aligned} \mathbf{x} &= \mathbf{x} \\ C_N \mathbf{x} &= C_N \mathbf{x} \\ C_N C_N^T \mathbf{x} &= C_N C_N^T \mathbf{x} \\ C_N C_N^T \mathbf{x} &= I \mathbf{x} \\ C_N C_N^T \mathbf{x} &= \mathbf{x} \quad \square \end{aligned}$$

(i) Consider a column vector $\mathbf{x} = (1, 2, -2, 2, 3, 0, 1, 3)^T$. Let P be a 8×7 matrix whose columns are the first 7 columns of the matrix C . We find

$$\mathbf{y} = P P^T \mathbf{x} = (1.0794, 1.17739, -1.6616, 1.6008, 3.3992, -0.3384, 1.2261, 2.9206)^T$$

Give an interpretation to the column vector \mathbf{y} .

Solution

$\mathbf{y} = P P^T \mathbf{x} = proj_C(\mathbf{x})$ is the orthogonal projection of the vector \mathbf{x} onto the columns of C , which is an approximation of the data points with a minimum error. So, we can zero the coefficients corresponding to significant frequencies in the last terms and still have a function that is a close approximation as the terms are arranged in order of importance, which removes the highest data frequency and keeps the lowest. \square