Math 2311 − Assignment 2

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1 Show that $S = \{1 + 2x + x^2, 2 + 9x, 3 + 3x + 4x^2\}$ is a basis for P_2 and write $p = 2 + 17x - 3x^2$ as a linear combination of vectors in S. Finally, write $[p]_S$.

1.a Show that $S = \{1 + 2x + x^2, 2 + 9x, 3 + 3x + 4x^2\}$ is a basis for P_2

$\boxed{\text{Solution: vectors } \{\vec{p_1}, \vec{p_2}, \vec{p_3}\} \text{ are a basis for } P_3}$

Proof. The set $S = \{\vec{p_1}, \vec{p_2}, \vec{p_3}\}$ in a vector space P_2, is called a basis if.

(1) S spans P_2.
(2) S is linearly independent.

To prove that the vectors span$\{S\} = P_2$ we must show that every vector $\vec{p} = a_0 + a_1x + a_2x^2$ in P_2 can be expressed as $c_1\vec{p_1} + c_2\vec{p_2} + c_3\vec{p_3} = \vec{p}$

$$c_1(1 + 2x + x^2) + c_2(2 + 9x) + c_3(3 + 3x + 4x^2) = \vec{v} \tag{1}$$

$$\begin{array}{ccccccl} 1c_1 & + & 2c_2 & + & 3c_3 & = a_0 \\ 2c_1 & + & 9c_2 & + & 3c_3 & = a_1 \\ 1c_1 & + & 0c_2 & + & 4c_3 & = a_2 \end{array}$$

To prove linear independence we must show that $c_1\vec{p_1} + c_2\vec{p_2} + c_3\vec{p_3} = \vec{0}$ has only the trivial solution.

$$c_1(1 + 2x + x^2) + c_2(2 + 9x) + c_3(3 + 3x + 4x^2) = \vec{0} \tag{2}$$

$$\begin{array}{ccccccl} 1c_1 & + & 2c_2 & + & 3c_3 & = 0 \\ 2c_1 & + & 9c_2 & + & 3c_3 & = 0 \\ 1c_1 & + & 0c_2 & + & 4c_3 & = 0 \end{array}$$

Thus, we have reduced the problem to showing that the homogenous system (2) has only the trivial solution, and that the nonhomogenous system (1) is consistent for all values c1, c2, c3. The two systems have the same coefficient matrix.

$$A = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 9 & 3 & 0 \\ 1 & 0 & 4 & 0 \end{array}\right]$$

We will prove both results by showing that $\det(A) \neq 0$

$$\begin{aligned} det(A) &= (1)\begin{vmatrix} 2 & 3 \\ 9 & 3 \end{vmatrix} + (4)\begin{vmatrix} 1 & 2 \\ 2 & 9 \end{vmatrix} \\ &= 1[(2)(3) - (3)(9)] + 4[(1)(9) - (2)(2)] = -1. \end{aligned}$$

This proves that $\{\vec{p_1}, \vec{p_2}, \vec{p_3}\}$ is a basis for P_2. □

1.b Write $\vec{p} = 2 + 17x - 3x^2$ as a linear combination of vectors in S

$\boxed{\text{Solution: } \vec{p} = (1)(1 + 2x + x^2) + (2)(2 + 9x) + (-1)(3 + 3x + 4x^2)}$

Proof. The equation $c_1\vec{p_1} + c_2\vec{p_2} + c_3\vec{p_3} = \vec{p}$ which can be written as the linear system

$$\begin{array}{ccccccc} 1c_1 & + & 2c_2 & + & 3c_3 & = 2 \\ 2c_1 & + & 9c_2 & + & 3c_3 & = 17 \\ 1c_1 & + & 0c_2 & + & 4c_3 & = -3 \end{array}$$

is an expression for a vector $\vec{p}$ in terms of the basis S, with scalars $c1, c2, c3$ being the coordinates of $\vec{p}$ relative to the basis S. Whose augmented matrix has the reduced row echelon form,

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right]$$
$$c_1 = 1, c_2 = 2, c_3 = -1.$$

This gives $\vec{p} = (1)(1 + 2x + x^2) + (2)(2 + 9x) + (-1)(3 + 3x + 4x^2)$ □

1.c Finally, write $[p]_S$.

$\boxed{\text{Solution: } [p]_s = [1\ 2\ {-1}]^T}$

Proof. We use c_1, c_2, c_3 from 1.b to construct the coordinate vector $[1\ 2\ {-1}]^T$ of $\vec{p}$ relative to S. □

2 Recall the standard basis of $\mathbb{R}^3$, $\vec{e_1} = [1\ 0\ 0]^T,\ \vec{e_2} = [0\ 1\ 0]^T,\ \vec{e_3} = [0\ 0\ 1]^T$

2.a Consider the matrix $A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 3 & 2 \\ 0 & 3 & 2 \end{bmatrix}$**. Does the set of vectors** $S_1 = \{A\vec{e_1}, A\vec{e_2}, A\vec{e_3}\}$ **form a basis for** $\mathbb{R}^3$**?**

$\boxed{\text{Solution: } S_1 \text{ is a basis for } \mathbb{R}^3}$

Proof. To see if the vectors in S_1 are a basis for $\mathbb{R}^3$, we will verify A is invertable.

$$[0\ 0\ 0]^T = c_1 A\vec{e_1} + c_2 A\vec{e_2} + c_3 A\vec{e_1}$$

$$\begin{array}{ccccccc} 1c_1 & + & 0c_2 & + & 2c_3 & = & 0 \\ 1c_1 & + & 3c_2 & + & 2c_3 & = & 0 \\ 0c_1 & + & 3c_2 & + & 2c_3 & = & 0 \end{array}$$

The augmented matrix has the reduced row echelon form,

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] = I_3$$

From inspecting the pivots of $rref(A)$, we can see that $col(A) = \{[\ 1\ 1\ 0\ ]^T, [\ 0\ 3\ 3\ ]^T, [\ 2\ 2\ 2\ ]^T\}$, is a basis for $\mathbb{R}^3$, but $A = [A\vec{e_1} \mid A\vec{e_2} \mid A\vec{e_3}] \implies col(A) = S_1 \therefore S_1$ is a basis for $\mathbb{R}^3$. $\square$

2.b Consider the matrix $B = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 3 & 0 \\ 0 & 3 & -2 \end{bmatrix}$**. Does the set of vectors** $S_2 = \{B\vec{e_1}, B\vec{e_2}, B\vec{e_3}\}$ **form a basis for** $\mathbb{R}^3$**?**

$\boxed{\text{Solution: } S_2 \text{ is not a basis for } \mathbb{R}^3}$

Proof. To see if the vectors in S_2 are a basis for $\mathbb{R}^3$, we will verify B is invertable.

$$[0\ 0\ 0]^T = c_1 B\vec{e_1} + c_2 B\vec{e_2} + c_3 B\vec{e_1}$$

$$\begin{array}{ccccccc} 1c_1 & + & 0c_2 & + & 2c_3 & = & 0 \\ 1c_1 & + & 3c_2 & + & 0c_3 & = & 0 \\ 0c_1 & + & 3c_2 & + & -2c_3 & = & 0 \end{array}$$

The augmented matrix has the reduced row echelon form,

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \neq I_3$$

S_2 is not a basis for $\mathbb{R}^3$ because $rank(B) = 2 \neq dim(\mathbb{R}^3)$ $\square$

2.c Make a conjecture of the form "$S = \{A\vec{e_1}, A\vec{e_2}, A\vec{e_3}\}$ **forms a basis for** $\mathbb{R}^3$ **if and only if** A ***(insert appropriate property of A here)*".**

$\boxed{\text{Conjecture } S = \{A\vec{e_1}, A\vec{e_2}, A\vec{e_3}\} \text{ forms a basis for } \mathbb{R}^3 \text{ if and only if } A \textit{ (is an invertible matrix).}}$

2.d Bonus: Prove your conjecture.

We will prove our conjecture $S = \{A\vec{e_1}, A\vec{e_2}, A\vec{e_3}\}$ forms a basis for $\mathbb{R}^3$ if and only if A *(is an invertible matrix)*, using our list of equivalent statements. Since we have already proven if A is invertible, then $A\vec{x} = \vec{0}$ has only the trivial solution, we will use this. Asserting if R is any row echelon form of a 3×3 matrix A, then either R has at least one row of zeros, or R is the identity matrix I_3.

We will prove the reverse direction first "if A is an invertible matrix, then $S = \{A\vec{e_1}, A\vec{e_2}, A\vec{e_3}\}$ forms a basis for $\mathbb{R}^3$."

Proof. Suppose R is the identity matrix I_3, then A has an inverse, and $A\vec{x}$ is a linear combination of the column vectors of A. Since $A\vec{x} = \vec{0}$ has only the trivial solution, the column vectors of A must be linearly independent. Since we know that the 3 column vectors of A are linearly independent in the 3-dimensional vector space $\mathbb{R}^3$, they must span $\mathbb{R}^3$, and form a basis for $\mathbb{R}^3$.

$$A\vec{e_1} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = (1) \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} + (0) \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} + (0) \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}$$

$$A\vec{e_2} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = (0) \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} + (1) \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} + (0) \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} = \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix}$$

$$A\vec{e_3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = (0) \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} + (0) \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} + (1) \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} = \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix}$$

$$A = [A\vec{e_1} \mid A\vec{e_2} \mid A\vec{e_3}]$$
$$\implies col(A) = S$$

Therefore $S = \{A\vec{e_1}, A\vec{e_2}, A\vec{e_3}\}$ is a basis for R^3 □

We will now prove the forward direction "if $S = \{A\vec{e_1}, A\vec{e_2}, A\vec{e_3}\}$ forms a basis for $\mathbb{R}^3$ then A is an invertible matrix." by proving its contrapositive "if A is not an invertible matrix, then $S = \{A\vec{e_1}, A\vec{e_2}, A\vec{e_3}\}$ does not form a basis for $\mathbb{R}^3$"

Proof. Suppose R has at least one row of zeros, then A has no inverse. We know from analysis of the positions of the $0's$ and $1's$ of R that elementary row operations don't change the dimension of the row space or the column space of our matrix, so it must be true that

dim(row space of A) $= dim$(row space of R) and dim(column space of A) $=$ dim(column space of R).

Since these two numbers are the same, the row and column space have the same dimension $rank(A)$; the dimension of the null space of A is $nullity(A)$

$$\begin{aligned} 0 < nullity(A) &\leq dim(\mathbb{R}^3) \\ rank(A) + nullity(A) &= dim(\mathbb{R}^3) \\ nullity(A) &= dim(\mathbb{R}^3) - rank(A) \\ \implies 0 < [\ dim(\mathbb{R}^3) - rank(A)\ ] &\leq dim(\mathbb{R}^3) \implies dim(\mathbb{R}^3) > rank(A) \geq 0 \\ \therefore rank(A) &< dim(\mathbb{R}^3) \end{aligned}$$

This proves, if R has atleast one row of zeros then $rank(A) < dim(\mathbb{R}^3) \therefore S$ is not a basis for $\mathbb{R}^3$ □

3 For each of the following subspaces of M_{33} find a basis and state the dimension.

3.a $W_1 = \{A \in M_{33} | A$ is a diagonal matrix$\}$

$\boxed{\text{Solution: } \dim(W_1) = 3}$

Proof.

$$D = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}, A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore D = aA_1 + bA_2 + cA_3$$

The matricies A_1, A_2, A_3 form a basis for W_1 consequently, the dimension of W_1 is 3. □

3.b $W_2 = \{A \in M_{33} | A = A^T\}$ (the symmetric matrices)

$\boxed{\text{Solution: } dim(W_2) = 6}$

Proof.

$$S = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}, A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, A_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore S = aA_1 + bA_2 + cA_3 + dA_4 + eA_5 + fA_6$$

The matricies $A_1, A_2, A_3, A_4, A_5, A_6$ form a basis for W_2 consequently, the dimension of W_2 is 6. □

3.c $W_1 = \{A \in M_{33} | A = -A^T\}$ (the anti-symmetric matrices)

$\boxed{\text{Solution: } dim(W_3) = 3}$

Proof.

$$S = \begin{bmatrix} 0 & b & c \\ -b & 0 & d \\ -c & -d & 0 \end{bmatrix}, A_1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$\therefore S = aA_1 + bA_2 + cA_3$$

The matricies A_1, A_2, A_3 form a basis for W_3 consequently, the dimension of W_3 is 3. □

4 Find a basis for the subspace of P^3 spanned by the following polynomials (vectors):
$p_1 = 1 + x + 3x^2 + 4x^3,\ p_2 = 1 + 2x^2 + 3x^3,\ p_3 = x + x^2 + 2x^3,\ p_4 = 1 + x + 3x^2 + 5x^3$

$\boxed{\text{Solution: the vectors } \vec{p_1}, \vec{p_2}, \vec{p_3} \text{ form a basis for span}\{p_1, p_2, p_3, p_4\}}$

Proof. The equation $k_1\vec{p_1} + k_2\vec{p_2} + k_3\vec{p_3} + k_4\vec{p_4} = \vec{0}$ can be written as a linear system

$$\begin{aligned} 1k_1 + 1k_2 + 0k_3 + 1k_4 &= 0 \\ 1k_1 + 0k_2 + 1k_3 + 1k_4 &= 0 \\ 3k_1 + 2k_2 + 1k_3 + 3k_4 &= 0 \\ 4k_1 + 3k_2 + 2k_3 + 5k_4 &= 0 \end{aligned}$$

whose augmented matrix has the reduced row echelon form

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$
$$\implies k1 = 0,\ k2 = -s,\ k3 = -s,\ k4 = s$$

removing vector $\vec{p_4}$ gives span$\{p_1, p_2, p_3\}$ = span$\{p_1, p_2, p_3, p_4\}$. Since the vector equation $k_1\vec{p_1} + k_2\vec{p_2} + k_3\vec{p_3} = \vec{0}$ has only the trivial solution. We conclude that the vectors $\vec{p_1}, \vec{p_2}, \vec{p_3}$ form a basis for span$\{p_1, p_2, p_3, p_4\}$. □

5 Let $\vec{x} = [\ 1\ 2\ 3\ ]^T$, $\mathcal{B} = \{[\ 1\ 0\ 0\ ]^T,\ [\ 1\ 1\ 0\ ]^T,\ [\ 1\ 1\ 1\ ]^T\}$, and $\mathcal{C} = \{[\ 1\ 1\ 0\ ]^T,\ [\ 0\ 1\ 1\ ]^T,\ [\ 1\ 0\ 1\ ]^T\}$.

5.a Find $[\vec{x}]_{\mathcal{B}}$

$\boxed{\text{Solution: } [\vec{x}]_{\mathcal{B}} = [\ -1\ \ -1\ 3\ ]^T}$

Proof. By inspection: $\vec{x} = (-1)[\ 1\ 0\ 0\ ]^T + (-1)[\ 1\ 1\ 0\ ]^T + (3)[\ 1\ 1\ 1\ ]^T [\vec{x}]_{\mathcal{B}} = [\ -1\ \ -1\ 3\ ]^T$ □

5.b Find $[\vec{x}]_{\mathcal{C}}$

$\boxed{\text{Solution: } [\vec{x}]_{\mathcal{C}} = [\ 0\ 2\ 1\ ]^T}$

Proof. By inspection: $\vec{x} = (0)[\ 1\ 1\ 0\ ]^T + (2)[\ 0\ 1\ 1\ ]^T + (1)[\ 1\ 0\ 1\ ]^T \implies [\vec{x}]_{\mathcal{C}} = [\ 0\ 2\ 1\ ]^T$ □

5.c Find $P_{\mathcal{C}\leftarrow\mathcal{B}}$ and compute $P_{\mathcal{C}\leftarrow\mathcal{B}}[\vec{x}]_{\mathcal{B}}$

$$\boxed{Solutions: P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 1/2 & 1 & 1/2 \\ -1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 \end{bmatrix}; [\vec{x}]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}[\vec{x}]_{\mathcal{B}} = [\ 0\ 2\ 1\ ]^T}$$

Proof.

$$\text{Partitioned matrix } [\ \mathcal{C} \mid \mathcal{B}\ ] = \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\text{Transition matrix } [\ I_3 \mid \mathcal{C} \leftarrow \mathcal{B}\ ] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 1 & 1/2 \\ 0 & 1 & 0 & -1/2 & 0 & 1/2 \\ 0 & 0 & 1 & 1/2 & 0 & 1/2 \end{array}\right]$$

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 1/2 & 1 & 1/2 \\ -1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

$$\begin{aligned} [\vec{x}]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}[\vec{x}]_{\mathcal{B}} &= \begin{bmatrix} 1/2 & 1 & 1/2 \\ -1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 \end{bmatrix} [\ -1\ \ -1\ 3\ ]^T \\ &= [-1] \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \end{bmatrix} + [-1] \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + [3] \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} \\ &= [\ 0\ 2\ 1\ ]^T \end{aligned}$$

□

5.d Find $P_{\mathcal{B}\leftarrow\mathcal{C}}$ and compute $P_{\mathcal{B}\leftarrow\mathcal{C}}[\vec{x}]_{\mathcal{C}}$

$$\boxed{Solutions: P_{\mathcal{B}\leftarrow\mathcal{C}} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}; [\vec{x}]_{\mathcal{B}} = P_{\mathcal{B}\leftarrow\mathcal{C}}[\vec{x}]_{\mathcal{C}} = [\ -1\ \ -1\ 3\ ]^T}$$

Proof.

$$\text{Partitioned matrix } [\ \mathcal{B} \mid \mathcal{C}\ ] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right]$$

$$\text{Transition matrix } [\ I_3 \mid \mathcal{B} \leftarrow \mathcal{C}\ ] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right]$$

$$P_{\mathcal{B}\leftarrow\mathcal{C}} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\begin{aligned} [\vec{x}]_{\mathcal{B}} = P_{\mathcal{B}\leftarrow\mathcal{C}}[\vec{x}]_{\mathcal{C}} &= \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} [\ 0\ 2\ 1\ ]^T \\ &= [0]\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + [2]\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + [1]\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \\ &= [\ -1\ \ -1\ 3\ ]^T \end{aligned}$$

□

6 Let $A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$

6.a Find a basis for each of null(A), row(A), col(A), and state the dimension of each of these subspaces.

$$\boxed{\begin{array}{ll} & Bases: \\ & null(A) = \{[\ -1\ \ -1\ 1\ 0\ ]^T,\ [\ \frac{2}{7}\ \ -\frac{4}{7}\ 0\ 1\ ]^T\} \\ Solutions: & row(A) = \{[\ 1\ 0\ 1\ \ -\frac{2}{7}\ ],\ [\ 0\ 1\ 1\ \frac{4}{7}\ ]\} \\ & col(A) = \{[\ 1\ 2\ \ -1\ ]^T, [\ 4\ 1\ 3\ ]^T\} \\ & Dimensions: \\ & rank(A) = nullity(A) = 2 \end{array}}$$

Proof.

$$rref(A) = \begin{bmatrix} 1 & 0 & 1 & -\frac{2}{7} \\ 0 & 1 & 1 & \frac{4}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\therefore [\ x_1\ x_2\ x_3\ x_4\ ]^T = [\ -\mathcal{S} + \frac{2}{7}\mathcal{T},\ -\mathcal{S} - \frac{4}{7}\mathcal{T},\ \mathcal{S},\ \mathcal{T}\ ]^T$$

$$= \mathcal{S}[\ -1\ \ -1\ 10\ ]^T + \mathcal{T}[\ \frac{2}{7}\ \ -\frac{4}{7}\ 0\ 1\ ]^T$$

$$\implies null(A) = \{[\ -1\ \ -1\ 1\ 0\ ]^T,\ [\ \frac{2}{7}\ \ -\frac{4}{7}\ 0\ 1\ ]^T\}$$

$$\implies row(A) = \{[\ 1\ 0\ 1\ \ -\frac{2}{7}\ ],\ [\ \ 0\ 1\ 1\ \frac{4}{7}\ ]\}$$

$$\implies col(A) = \{[\ 1\ 2\ \ -1\ ]^T,\ [\ 4\ 1\ 3\ ]^T\}$$

$$\implies rank(A) = 2$$

$$\implies nullity(A) = 2$$

$row(A)$ and $null(A)$ are 2 dimensional subspaces of R^4, $col(A)$ is a 2 dimensional subspace of R^3. □

6.b **Is the vector $\vec{b} = [\ 4\ 6\ -2\ ]^T$ in the column space of A? If so, write $\vec{b}$ as a linear combination of the columns of A.**

$$\boxed{\text{Solution: Yes } \vec{b} \in col(A) \wedge \vec{b} = \tfrac{20}{7}[\ 1\ 2\ \ -1\ ]^T + \tfrac{2}{7}[\ 4\ 1\ 3\ ]^T + (0)[\ 5\ 3\ 2\ ]^T + (0)[\ 2\ 0\ 2\ ]^T}$$

let $\vec{c1} = [\ 1\ 2\ \ -1\ ]^T$, $\vec{c2} = [\ 4\ 1\ 3\ ]^T$, and let $C = [\ \vec{c1} \mid \vec{c2}\ ]$ we will show $\vec{b} \in col(\ A\ ) \implies rank(\ C\ ) = rank(\ C \mid \vec{b}\ )$

Proof.

$$[\ 4\ 6\ \ -2\ ]^T = k_1[\ 1\ 2\ \ -1\ ]^T + k_2[\ 4\ 1\ 3\ ]^T$$

which can be expressed as,

$$\begin{array}{rcrcr} 1k_1 & + & 4k_2 & = & 4 \\ 2k_1 & + & 1k_2 & = & 6 \\ -1k_1 & + & 3k_2 & = & -2 \end{array}$$

whose augmented matrix has the reduced row echelon form

$$\left[\begin{array}{cc|c} 1 & 0 & \frac{20}{7} \\ 0 & 1 & \frac{2}{7} \\ 0 & 0 & 0 \end{array}\right]$$

because $rank(\ C\ ) = 2$ and $rank(\ C \mid \vec{b}\ ) = 2$ the system is consistent, so $\vec{b}$ is in the column space of A. $\therefore$ $\vec{b}$ as a linear combination of the columns of A can expressed by the following,
$\vec{b} = \frac{20}{7}[\ 1\ 2\ \ -1\ ]^T + \frac{2}{7}[\ 4\ 1\ 3\ ]^T + (0)[\ 5\ 3\ 2\ ]^T + (0)[\ 2\ 0\ 2\ ]^T$ $\square$