Math 2311 – Assignment 1

Michael Walker

January 19, 2022

1. Determine if each of the following sets is a vector space.

   (a) $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid x \geq y \right\}$ with the usual scalar multiplication and vector addition from $\mathbb{R}^2$

Answer: No, V is not a vector space.

Proof. let $(x_0, y_0)^T, (x_1, y_1)^T \in V$, and take the vector space operations on V to be the usual operations of *vector* addition and *scalar* multiplication; that is,

$$(x_0, y_0)^T + (x_1, y_1)^T = (x_0 + x_1, y_0 + y_1)^T \tag{1}$$
$$k(x_0, y_0)^T = (kx_0, ky_0)^T \tag{2}$$

V is closed under scalar addition since $x_0 + x_1 \geq y_0 + y_1$

However, by properties of inequalities if the constant, k, is negative, we must reverse the symbol to preserve the inequality relation.

Given that k is negative, $x \geq y \rightarrow kx \leq ky$ □

   (b) Consider the set $W = \{f \in F(-\infty, \infty) \mid f(1) = 0\}$ with the usual scalar multiplication and vector addition from $F(-\infty, \infty)$. Is W a vector space?

Answer: Yes, W is a vector space.

Proof. Since we know that $F(-\infty, \infty)$ (with the usual operations) is a vector space, and since W is a subset of $F(-\infty, \infty)$ (with the same operations), it suffices to prove that W is a subspace of $F(-\infty, \infty)$. To this end we must show three things:

(a) That W is non-empty.

(b) That W is closed under addition.

(c) That W is closed under scalar multiplication.

There exists a function $\mathbf{0}$ in $F(-\infty,\infty)$ defined by $\mathbf{0}(x)=0$ for all x. Clearly $\mathbf{0}(1)=f(1)=0$ so W is non-empty.

Now suppose f and g are two functions in W. We must show that $f+g$ is in W.

$$\begin{aligned}(f+g)(1) &= f(1)+g(1) && \text{(definition of addition of functions)}\\ &= 0 && (f \text{ and } g \text{ are in } W)\end{aligned}$$

Finally, to show that W is closed under scalar multiplication, suppose f is in W and k is a scalar, then

$$\begin{aligned}(kf)(1) &= kf(1) && \text{(definition of scalar multiplication on functions)}\\ &= 0 && (f \text{ is in } W)\end{aligned}$$

so (kf) is in W and W is closed under scalar multiplication.

Therefore W is a subspace of $F(-\infty,\infty)$ and hence is a vector space. □

2. Let V be a vector space.

   (a) If k is any scalar, prove that $k\vec{0}=\vec{0}$.

Proof.

$$\begin{aligned}k(\vec{0}+\vec{0}) &= k\vec{0}+k\vec{0} && \text{(vector space axiom 7)}\\ k\vec{0} &= k\vec{0}+k\vec{0} && \text{(vector space axiom 4)}\\ k\vec{0}+(-k\vec{0}) &= (-k\vec{0})+(k\vec{0}+k\vec{0}) && \text{(vector space axiom 5)}\\ k\vec{0}+(-k\vec{0}) &= ((-k\vec{0})+k\vec{0})+k\vec{0} && \text{(vector space axiom 3)}\\ \vec{0} &= \vec{0}+k\vec{0} && \text{(vector space axiom 5)}\\ &= k\vec{0} && \text{(vector space axiom 4)}\end{aligned}$$

□

   (b) Prove that the zero vector in V is unique.

Proof. We must show that there is only one vector, $\vec{0}$, with the property that $\vec{0}+\vec{v}=\vec{v}+\vec{0}=\vec{v}$.

Suppose $\vec{0_1}$ and $\vec{0}$ are zero vectors in V Then

$$\begin{aligned}\vec{0_1} &= \vec{0_1}+\vec{0} && \text{(vector space axiom 4)}\\ &= \vec{0}+\vec{0_1} && \text{(vector space axiom 2)}\\ &= \vec{0} && \text{(vector space axiom 4)}\end{aligned}$$

Therefore $\vec{0_1}=\vec{0}$. So, the zero vector is unique. □

3. Determine if each of the following are subspaces of M_{nn}

   (a) $\{A\in M_{nn}\ |det(A)=0\}$

Answer: No, $\{A \in M_{nn} \mid \det(A) = 0\}$ is not a subspace of M_{nn}.

Proof.

$$\det \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0, \det \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\det \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$$

□

(b) $\{A \in M_{nn} \, | tr(A) = 0\}$

Answer: Yes, $W = \{A \in M_{nn} \, | tr(A) = 0\}$ is a subspace of M_{nn}.

Proof. Let $[B = (b_{ii})] \in W$ be a square matrix of order n such that $tr(B) = 0$ and let k be any scalar.

The set W is non empty because if we let $a_{ii} = 0$ for all i then $tr(A) = 0$ therefore W contains the $\mathbf{0}$ matrices. It remains to show that W is closed under addition and scalar multiplication.

$$\begin{aligned}
&addition: \\
tr(A+B) &= \sum_{i=1}^{n} (a_{ii} + b_{ii}) \\
&= \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii} \\
&= tr(A) + tr(B) \\
&= 0 + 0 \\
&= 0 \\
multiplication: & \\
tr(kA) &= \sum_{i=1}^{n} (k \cdot a_{ii}) \\
&= k \cdot \sum_{i=1}^{n} a_{ii} \\
&= k \cdot tr(A) \\
&= k \cdot 0 \\
&= 0
\end{aligned}$$

To be clear, if we take some $C = A + B$ such that A and B are in W, then for all $C = (c_{ii})$, the sum will be 0, so C is also in W. Therefore W is a subspace of M_{nn}. □

(c) $\left\{A \in M_{nn} \, | A^T = A\right\}$

Answer: Yes, $W = \{A \in M_{nn} \mid A^T = A\}$ is a subspace of M_{nn}.

Proof. Let $[B = (b_{ij})] \in W$ be a square matrix of order n such that $b_{ij} = b_{ji}$, and let k be any scalar.

The set W is non empty because, if we let $[A = (a_{ij})] = 0$ for all (i, j) then W contains the **0** matrices. It remains to show that W is closed under addition and scalar multiplication.

$$\begin{aligned} addition: & \\ C &= (A+B)^T \\ &= A^T + B^T \\ &= A + B \\ multiplication: & \\ (k \cdot A)^T &= k \cdot A^T \\ &= k \cdot A \end{aligned}$$

so W is a subspace of M_{nn}. □

4. Consider the following vectors in P_2: $p_1 = 2+x+4x^2$, $p_2 = 1-x+3x^2$, $p_3 = 3+2x+5x^2$.

   (a) Express the vector $g = 6 + 11x + 6x^2$ as a linear combination of p_1, p_2, p_3.

Answer: Yes, (p_1, p_2, p_3) = (4, -5, 1)

Proof.

$$\begin{aligned} (6 + 11x + 6x^2) &= k_0(2 + x + 4x^2) + k_1(1 - x + 3x^2) + k_2(3 + 2x + 5x^2) \\ &= (k_02 + k_1 + k_23) + (k_0x - k_1x + k_22x) + (k_04x^2 + k_13x^2 + k_25x^2) \\ &= (k_02 + k_1 + k_23) + (k_0 - k_1 + k_22)x + (k_04 + k_13 + k_25)x^2 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 6 \\ 1 & -1 & 2 & 11 \\ 4 & 3 & 5 & 6 \end{array}\right]$$

$$[-2r2 + r1] \wedge [-4r2 + r1] \left[\begin{array}{ccc|c} 0 & 3 & -1 & -16 \\ 1 & -1 & 2 & 11 \\ 0 & 7 & -3 & -38 \end{array}\right]$$

$$r2 \leftrightarrow r1 \left[\begin{array}{ccc|c} 1 & -1 & 2 & 11 \\ 0 & 3 & -1 & -16 \\ 0 & 7 & -3 & -38 \end{array}\right]$$

$$\frac{1}{3}r2 \left[\begin{array}{ccc|c} 1 & -1 & 2 & 11 \\ 0 & 1 & \frac{-1}{3} & \frac{-16}{3} \\ 0 & 7 & -3 & -38 \end{array}\right]$$

$$-7r2 + r3 \left[\begin{array}{ccc|c} 1 & -1 & 2 & 11 \\ 0 & 1 & \frac{-1}{3} & \frac{-16}{3} \\ 0 & 0 & \frac{-2}{3} & \frac{-2}{3} \end{array}\right]$$

$$-\frac{3}{2}r3 \begin{bmatrix} 1 & -1 & 2 & \bigm| & 11 \\ 0 & 1 & \frac{-1}{3} & \bigm| & \frac{-16}{3} \\ 0 & 0 & 1 & \bigm| & 1 \end{bmatrix}$$

$$\frac{1}{3}r3 + r2 \begin{bmatrix} 1 & -1 & 2 & \bigm| & 11 \\ 0 & 1 & 0 & \bigm| & -5 \\ 0 & 0 & 1 & \bigm| & 1 \end{bmatrix}$$

$$r1 + r2 \begin{bmatrix} 1 & 0 & 2 & \bigm| & 6 \\ 0 & 1 & 0 & \bigm| & -5 \\ 0 & 0 & 1 & \bigm| & 1 \end{bmatrix}$$

$$r1 + r3 \begin{bmatrix} 1 & 0 & 0 & \bigm| & 4 \\ 0 & 1 & 0 & \bigm| & -5 \\ 0 & 0 & 1 & \bigm| & 1 \end{bmatrix}$$

$$\therefore (k_1, k_2, k_3) = (4, -5, 1)$$

□

(b) Does $\{p_1, p_2, p_3\}$ span P_2?

Answer: Yes, $span(\{p_1, p_2, p_3\}) = P_2$

Proof. An arbitrary vector in P_2 is of the form $\vec{p} = a + bx + cx^2$ and so becomes,

$k_0(2 + x + 4x^2) + k_1(1 - x + 3x^2) + k_2(3 + 2x + 5x^2) = a + bx + cx^2$

which we can rewrite as

$(k_0 2 + k_1 + k_2 3) + (k_0 - k_1 + k_2 2)x + (k_0 4 + k_1 3 + k_2 5)x^2 = a + bx + cx^2$

Equating corresponding coefficients yields a linear system whose augmented matrix is

$$A = \begin{bmatrix} 2 & 1 & 3 & \bigm| & a \\ 1 & -1 & 2 & \bigm| & b \\ 4 & 3 & 5 & \bigm| & c \end{bmatrix}$$

Our problem reduces to ascertaining whether this system is consistent for all values of a, b, and c. This can be determined if its coefficient matrix has a nonzero determinant, from our theorem for equivalent statements. If A is an n x n matrix such that $\det(A) \neq 0$ then $A\vec{x} = \vec{0}$.

It follows from solution (a) that

$$\begin{bmatrix} 2 & 1 & 3 & \bigm| & 0 \\ 1 & -1 & 2 & \bigm| & 0 \\ 4 & 3 & 5 & \bigm| & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \bigm| & 0 \\ 0 & 1 & 0 & \bigm| & 0 \\ 0 & 0 & 1 & \bigm| & 0 \end{bmatrix}$$

so A is consistent for every choice $a, b,$ and c. Thus, the vectors in $\{p_1, p_2, p_3\}$ span P_2. □

(c) Is $\{p_1, p_2, p_3\}$ linearly independent?