1 Let $A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$

1.a Find a basis for each of null(A), row(A), col(A), and state the dimension of each of these subspaces.

$$\boxed{\begin{array}{ll} & Bases: \\ & null(A) = \{[-1\ \ -1\ 1\ 0]^T,\ [\frac{2}{7}\ \ -\frac{4}{7}\ 0\ 1]^T\} \\ Solutions: & row(A) = \{[1\ 0\ 1\ \ -\frac{2}{7}],\ [0\ 1\ 1\ \frac{4}{7}]\} \\ & col(A) = \{[1\ 2\ \ -1]^T, [4\ 1\ 3]^T\} \\ & Dimensions: \\ & rank(A) = nullity(A) = 2 \end{array}}$$

Proof.

$$rref(A) = \begin{bmatrix} 1 & 0 & 1 & -\frac{2}{7} \\ 0 & 1 & 1 & \frac{4}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\therefore [x_1\ x_2\ x_3\ x_4]^T = [-\mathcal{S} + \frac{2}{7}\mathcal{T},\ -\mathcal{S} - \frac{4}{7}\mathcal{T},\ \mathcal{S},\ \mathcal{T}]^T$$

$$= \mathcal{S}[-1\ \ -1\ 10]^T + \mathcal{T}[\frac{2}{7}\ \ -\frac{4}{7}\ 0\ 1]^T$$

$$\implies null(A) = \{[-1\ \ -1\ 1\ 0]^T,\ [\frac{2}{7}\ \ -\frac{4}{7}\ 0\ 1]^T\}$$

$$\implies row(A) = \{[1\ 0\ 1\ \ -\frac{2}{7}],\ [0\ 1\ 1\ \frac{4}{7}]\}$$

$$\implies col(A) = \{[1\ 2\ \ -1]^T,\ [4\ 1\ 3]^T\}$$

$$\implies rank(A) = 2$$

$$\implies nullity(A) = 2$$

$row(A)$ and $null(A)$ are 2 dimensional subspaces of R^4, $col(A)$ is a 2 dimensional subspace of R^3. □

1.b Is the vector $\vec{b} = [\,4\;6\;-2\,]^T$ in the column space of A? If so, write $\vec{b}$ as a linear combination of the columns of A.

$\boxed{\text{Solution: Yes } \vec{b} \in col(A).\ [\,4\;6\;-2\,]^T = \tfrac{20}{7}[1\;2\;-1]^T + \tfrac{2}{7}[4\;1\;3]^T}$

Proof. If $\vec{b}$ is in the column space of A then the following system will be consistent.

$$[\,4\;6\;-2\,]^T = c_1[1\;2\;-1]^T + c_2[4\;1\;3]^T$$

which can be expressed as,

$$\begin{array}{rcrcr} 1c_1 & + & 4c_2 & = & 4 \\ 2c_1 & + & 1c_2 & = & 6 \\ -1c_1 & + & 3c_2 & = & -2 \end{array}$$

whose augmented matrix has the reduced row echelon form

$$\left[\begin{array}{cc|c} 1 & 0 & \frac{20}{7} \\ 0 & 1 & \frac{2}{7} \\ 0 & 0 & 0 \end{array}\right]$$

$$\implies c_1 = \frac{20}{7}, c_2 = \frac{2}{7}$$

$\therefore \vec{b}$ is in the column space of A. $[\,4\;6\;-2\,]^T = \frac{20}{7}[1\;2\;-1]^T + \frac{2}{7}[4\;1\;3]^T$ $\square$