Assignment 3 MATH 2200

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Question 1: Page 1 of 2.

Consider the function $f(x,y) = xy - x^3 - y^2$. Find all points (x_o, y_o) where both of the partial derivatives are equal to zero. i.e $f_x(x_o, y_o) = f_y(x_o, y_0) = 0$

solution:

$$\left(\frac{1}{6}, \frac{1}{12}\right) \wedge (0,0)$$

$$\begin{aligned}
&\textbf{find partial derivatie:} \\
f(x,y) &= xy - x^3 - y^2 \\
\partial f / \partial x &= y - 3x^2 \\
\partial f / \partial y &= x - 2y \\
&\textbf{find zeros for x:} \\
x - 2y &= 0 \\
\rightarrow y &= \frac{x}{2} \\
\rightarrow \frac{x}{2} - 3x^2 &= x\left(\frac{1}{2} - 3x\right) \\
\rightarrow x &= 0 \\
\rightarrow \frac{1}{2} - 3x &= 0 \rightarrow x = \frac{1}{6} \\
&\textbf{find zeros for y:} \\
x &= -2y \\
\rightarrow y - 3(2y)^2 &= y - 12y^2 \\
\rightarrow 0 &= y(1 - 12y) \\
\rightarrow y &= 0 \\
\rightarrow y &= \frac{1}{12}
\end{aligned}$$

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test zeros:

$$\partial f / \partial x|_{(x,y)=1/6,1/12} = \frac{1}{12} - 3\left(\frac{1}{6}\right)^2 = \frac{1}{12} - \frac{3}{36} = \frac{1}{12} - \frac{1}{12} = 0$$

$$\partial f / \partial x|_{(x,y)=0,0} 0 - 3(0)^2 = 0$$

$$\partial f / \partial y|_{(x,y)=1/6,1/12} = \frac{1}{6} - 2\left(\frac{1}{12}\right) = \frac{1}{6} - \frac{2}{12} = \frac{1}{6} - \frac{1}{6} = 0$$

$$\partial f / \partial y|_{(x,y)=0,0} = 0 - 2(0) = 0$$

$$\therefore \left(\frac{1}{6}, \frac{1}{12}\right) \wedge (0, 0) \blacksquare$$

Question 2: Page 1 of 2.

Find the volume of the solid in the first octant which is bounded by the surface above by $z = 9 - x^2$, below by $z = 0$ and laterally by $y^2 = 3x$

Solution:

$$\int_0^3 \int_0^{\sqrt{3x}} \left(9 - x^2\right) \cdot dy \cdot dx = \frac{216}{7}$$

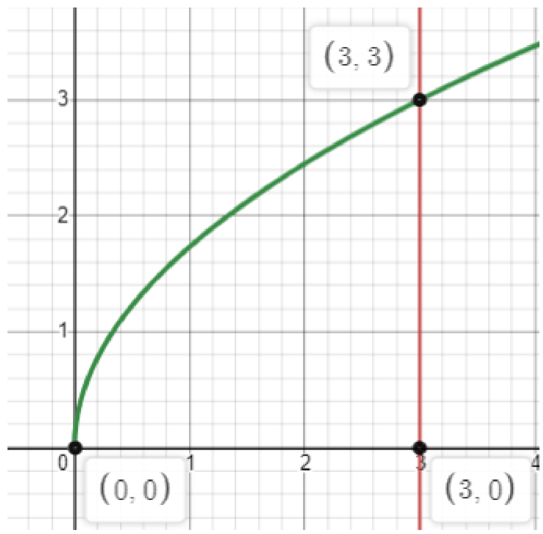

Figure 1: Bounds

type 1 region bounds :

$$y^2 = 3x \rightarrow y = \sqrt{3x}$$
$$\therefore 0 \leqslant y \leqslant \sqrt{3x}$$
$$9 - x^2 = 0 \rightarrow x = \pm 3$$
$$\therefore 0 \leqslant x \leqslant 3$$

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$$V = \iint_R \left(9-x^2\right)\cdot dA = \int_0^3 \int_0^{\sqrt{3x}} \left(9-x^2\right)\cdot dy \cdot dx = \int_0^3 \left[\left(9-x^2\right)\cdot y\right]_0^{\sqrt{3x}} \cdot dx$$

$$= \int_0^3 [\left(9-x^2\right)\cdot \sqrt{3x}\cdot dx = \int_0^3 [\left(9-x^2\right)\cdot (3)^{\frac{1}{2}}\cdot x^{\frac{1}{2}}\cdot dx$$

$$= \int_0^3 \left(3^{\frac{5}{2}}x^{\frac{1}{2}} - 3^{\frac{1}{2}}x^{\frac{5}{2}}\right)\cdot dx$$

$$= \left\{3^{\frac{5}{2}}\left[\frac{2}{3}\cdot x^{\frac{3}{2}}\right] - 3^{\frac{1}{2}}\left[\frac{2}{7}\cdot x^{\frac{7}{2}}\right]\right\}_{x=0}^{3}$$

$$= 3^{\frac{5}{2}}\left[\frac{2}{3}\cdot 3^{\frac{3}{2}}\right] - 3^{\frac{1}{2}}\left[\frac{2}{7}\cdot 3^{\frac{7}{2}}\right]$$

$$= 9\cdot 3^{\frac{1}{2}}\left[\frac{2}{3}\cdot 3\cdot 3^{\frac{1}{2}}\right] - 3^{\frac{1}{2}}\left[\frac{2}{7}\cdot 27\cdot 3^{\frac{1}{2}}\right]$$

$$= 9\cdot 3\left[\frac{2}{3}\cdot 3\right] - 3\left[\frac{2}{7}\cdot 27\right] = 27\left[\frac{6}{3}\right] - 3\left[\frac{54}{7}\right] = 54 - \frac{162}{7}$$

$$= \frac{378-162}{7} = \frac{216}{7}$$

$$\therefore \int_0^3 \int_0^{\sqrt{3x}} \left(9-x^2\right)\cdot dy\cdot dx = \frac{216}{7} \blacksquare$$

Question 3

In statistics, the standard normal distribution $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ plays an important role. The area under this curve between $x = a$ and $x = b$ gives the probability that a normally-distributed random variable has a z-score between a and b. As this is a probability density function, the total area under the curve must be equal to 1. In this question we will verify that this is the case. i.e we will compute $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \cdot dx$

Question 3a Page 1 of 2.

Can you find an antiderivative for $f(x)$? Either give the antiderivative or explain in a sentence or two why you are unable to find one.

part 1 solution (finding an antiderivative):

According to Liouville's Theorem. "Elementary anti derivatives, if they exist, must be in the same differential field as the function, plus possibly a finite number of logarithms".
https://en.wikipedia.org/wiki/Liouville%27s_theorem_(differential_algebra)#Basic_theorem

(Chapter 7.6 Page 451) our textbook asserts, the error function $erf(t) = \int \left[\frac{2}{\sqrt{\pi}} \cdot e^{-t^2} \right] \cdot dt$

is not an elementary function. We will use $erf(t)$ to show **F(x)** not elementary.

Note: (although the antiderivative is found here, it's not understood and may be incorrect).

show F(x) not elementary

$$\textbf{seprable} = \int \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{x^2}{2}} \cdot dx$$

$$let\ u = \frac{x}{\sqrt{2}} \to du = \frac{1}{\sqrt{2}} \cdot dx$$

$$\to \sqrt{2} \cdot du = dx$$

$$\therefore \int \frac{\sqrt{2}}{\sqrt{2\pi}} \cdot e^{-u^2} \cdot du = \frac{\sqrt{2}}{\sqrt{2}} \cdot \int \frac{\sqrt{2}}{\sqrt{2\pi}} \cdot e^{-u^2} \cdot du$$

$$= \frac{1}{2} \cdot \int \left[\frac{2}{\sqrt{\pi}} \cdot e^{-u^2} \right] \cdot du$$

$$\therefore F(x) = \frac{1}{2} \cdot erf(u) + C \blacksquare$$

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part 2 solution (not finding an antiderivative):

Tabular method of integration from (Chapter 7.2 Page 418)

$$let\ F(x) = e^{\frac{-x^2}{2}}$$
$$G(x) = 1$$
$$\rightarrow \frac{1}{\sqrt{2\pi}} \cdot \int e^{-\frac{x^2}{2}} \cdot dx = \frac{1}{\sqrt{2\pi}} \cdot \int F(x)G(x) \cdot dx$$

$$\begin{array}{rl} \textbf{derivative column} & \textbf{anti-derivative column} \\ +F & G \\ -F^1 & G^{-1} \\ +F^2 & G^{-2} \\ -F^3 & G^{-3} \\ . & . \\ . & . \\ . & . \\ (-1)^n F^n & G^{-n} \\ (-1)^{n+1} F^{(n+1)} & G^{-n-1} \end{array}$$

$$\therefore \int F(x)G(x) \cdot dx = F \cdot G^{-1} - F^1 \cdot G^{-2} + F^2 G^{-3} - \ldots + (-1)^n F^n G^{-n-1}$$
$$+(-1)^{n+1} \int F^{n+1}(x) \cdot G^{n-1}(x) \cdot x$$
$$= \sum_{k=0}^{n} (-1)^k F^k \cdot G^{-k-1} + (-1)^{n+1} \int F^{n+1}(x) \cdot G^{-n-1}(x) \cdot dx$$

$let\ a_k = F^k \cdot G^{-k-1} \rightarrow \sum_{k=0}^{\infty} (-1)^k a_k$ is divergent because $a_1 \leqslant a_2 \leqslant a_3 \leqslant \ldots$

This shows, no matter how many times we integrate, there are no combinations $(-1)^{n+1} \int F^{n+1}(x) \cdot G^{-n-1}(x) \cdot dx$ that result in "simple" functions or a finite number of logarithms of "simple" functions. So $F(x) = \int \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ is not an elementary function we can find. ■

Question 3(b,c,d,e,f) page 1 of 2

(b) Consider $g(y) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{y^2}{2}}$ how are $\int_{-\infty}^{\infty} f(x) \cdot dx \wedge \int_{-\infty}^{\infty} g(y) \cdot dy$ related

(c) $I = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \cdot dx \wedge h(x,y) = f(x) \cdot g(y)\ show \int\int_R h(x,y) \cdot dA = I^2$

(d) Change to polar coordinates and write $h(r,\theta)$

(e) Compute $\int\int_R h(x,y) \cdot dA = \int\int_R h(r,\theta) \cdot dA$

(f) Show that $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \cdot dx = 1$

$$\int_{-\infty}^{\infty} f(x) \cdot dx = \int_{-\infty}^{\infty} g(y) \cdot dy\ Becaue\ \forall a \in \Re, f(a) = g(a)\ \blacksquare \qquad \text{(b)}$$

seprable using b:

$$\int\int_R h(x,y) \cdot dA = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \cdot dxdy$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \left[\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \cdot dx \right] \cdot dy$$

$$I = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \cdot dx \rightarrow\ = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \cdot I = I \cdot I = I^2\ \blacksquare \qquad \text{(c)}$$

polar coordinates:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2-y^2}{2}} \cdot dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi} e^{\frac{-(x^2+y^2)}{2}} \cdot dxdy$$

$$(0 \leqslant r \leqslant \infty) \wedge (0 \leqslant \theta \leqslant 2\pi)$$

$$\left(\frac{x}{r} = cos \cdot \theta, \frac{y}{r} = sin \cdot \theta \right)$$

$$\therefore x^2 + y^2 = r^2 \left(cos^2\theta\right) + r^2 \left(sin^2\theta\right)$$

$$= r^2 \left(cos^2\theta + sin^2\theta\right) = r^2 \cdot 1 = r^2$$

$$\rightarrow h(r,\theta) = \frac{1}{2\pi} e^{\frac{-(r^2)}{2}}\ \blacksquare \qquad \text{(d)}$$

$$\rightarrow \int\int_R h(x,y)\cdot dA = \int\int_R h(r,\theta)\cdot dA = \int_0^\infty \int_0^{2\pi} \frac{1}{2\pi} e^{\frac{-(r^2)}{2}} \cdot rd\theta dr \;\blacksquare \qquad \text{(e)}$$

$$= \int_0^\infty \frac{2\pi}{2\pi} e^{\frac{-(r^2)}{2}} \cdot rdr$$

$$= \int_0^\infty e^{\frac{-(r^2)}{2}} \cdot rdr$$

$$let\ u = \frac{r^2}{2} \rightarrow \frac{du}{dr} = r \therefore du = r\cdot dr$$

$$\rightarrow \lim_{b\to\infty} \int_0^b e^{-u}\cdot du = \lim_{b\to\infty} \left[-e^{-u}\cdot\right]_{u=0}^{b}$$

$$= \lim_{b\to\infty} \left[-e^{-b} -- e^{-0}\right]$$

$$= (0+1)$$

$$= 1$$

$$So \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \cdot dxdy = I^2 = 1$$

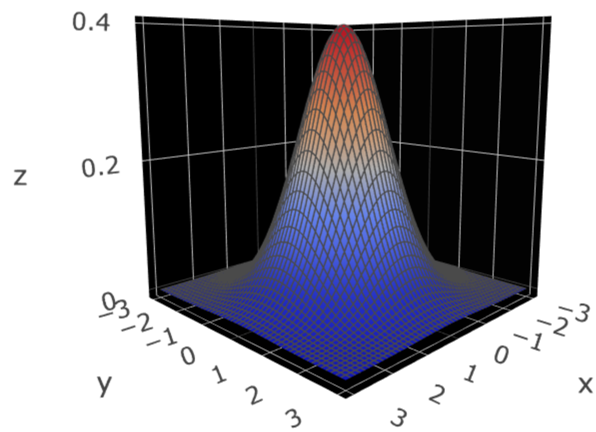
$$\sqrt{I^2} = \sqrt{1}$$

$$\rightarrow I = 1$$

$$\therefore \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \cdot dx = 1 \;\blacksquare \qquad \text{(f)}$$

Question 3g

(g) Explain why this technique cannot be used to evaluate $\int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \cdot dx$



The reason the technique used in 3f could not be used to evaluate $\int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \cdot dx$

is because we are calculating the volume under the 3d function $h(x, y)$ over all reals. When we convert this to $h(r, \theta)$ we can see from our r that this graph is symetrical no matter the orientation with respect to the z axis. i.e we can rotate it however we want. This means $h(r, \theta)$ is equvalent to $\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ rotated around the $y - axis$. In Chapter 6 we calculated this using the shell method. Using the idea of shells to find our volume accuratly, we would need an infinite number of shells, with an infinitly small thickness $r + dr$ and the radius r spanning from 0 out to infinity ■

Question 4

Consider the solid whose volume you computed in question 2. Suppose the density of this solid varies throughout the solid such that the density is $\delta(x,y,z) = y$.

Note: The mass of such a solid can be computed as $\int\int\int_V \delta(x,y,z)\cdot dV$ where V is the solid $3-dimensional$ region over which we integrate the density function and dV is the volume element $dx\cdot dy\cdot dz$ (or in what ever order makes the integration most convenient).

Solution:

$$\int_0^3\int_0^{\sqrt{3x}}\int_0^{(9-x^2)} y\cdot dz\cdot dy\cdot dx = \frac{243}{8}$$

$$\int\int\int_V \delta(x,y,z)\cdot dV = \int_0^3\int_0^{\sqrt{3x}}\int_0^{(9-x^2)} y\cdot dz\cdot dy\cdot dx$$

$$\textbf{see question 2 for work:} \quad = \int_0^3\int_0^{\sqrt{3x}} [yz]_{z=0}^{(9-x^2)}\cdot dy\cdot dx$$

$$= \int_0^3\int_0^{\sqrt{3x}} y(9-x^2)\cdot dy\cdot dx$$

$$= \int_0^3 \frac{1}{2}y^2(9-x^2)]_{y=0}^{\sqrt{3x}}\cdot dx$$

$$= \int_0^3 \frac{1}{2}\sqrt{3x}^2(9-x^2)\cdot dx$$

$$= \int_0^3 \frac{3}{2}x(9-x^2)\cdot dx$$

$$= \frac{3}{2}\int_0^3 (9x-x^3)\cdot dx$$

$$= \frac{3}{2}\left(\frac{9}{2}x^2-\frac{1}{4}x^4\right)_{x=0}^3 = \frac{3}{2}\left\{\left(\frac{9}{2}3^2-\frac{1}{4}3^4\right) - \left(\frac{9}{2}0^2-\frac{1}{4}0^4\right)\right\}$$

$$= \frac{3}{2}\left(\frac{81}{2}-\frac{81}{4}\right) = \frac{3}{2}\cdot\frac{81}{4} = \frac{243}{8}$$

$$\therefore \int\int\int_V \delta(x,y,z)\cdot dV = \frac{243}{8} \blacksquare$$