Math 2311 – Assignment 1

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Contents

1 Determine if each of the following sets is a vector space.

1.a Question: $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid x \geq y \right\}$ with the usual scalar multiplication and vector addition from $\mathbb{R}^2$

1.a Answer: No, V is not a vector space.

Proof. Counter example Axiom 5 fails.

$$(3,2) \in S : 3 \geq 2$$
$$(-3,-2) \notin S : -3 < -2$$

For V to be a vector space, all 10 of our vector space axioms must hold; this means it is enough to demonstrate one axiom fails; we have shown $\exists\ \vec{u} \in V : -\vec{u} \notin V$, therefore, V is not a vector space. □

1.b Question: Consider the set $W = \{f \in F(-\infty,\infty)\ |f(1) = 0\}$ with the usual scalar multiplication and vector addition from $F(-\infty,\infty)$. Is W a vector space?

1.b Answer: Yes, W is a vector space.

Proof. Since we know that $F(-\infty,\infty)$ (with the usual operations) is a vector space, and since W is a subset of $F(-\infty,\infty)$ (with the same operations), it suffices to prove that W is a subspace of $F(-\infty,\infty)$. To this end we must show three things.

(1) Prove that W is non-empty.
Clearly the zero function $(\mathbf{0})(1) = f(1) = 0 \therefore W$ is non-empty.

(2) Prove that W is closed under addition.
Let $g \in W$. We must show that $f + g \in W$.

$$\begin{aligned} (f+g)(1) &= f(1) + g(1) && \text{(definition of addition of functions)} \\ &= 0 && (f \text{ and } g \text{ are in } W) \end{aligned}$$

(3) Prove that W is closed under scalar multiplication.
Let k be any scalar.

$$\begin{aligned} (kf)(1) &= kf(1) && \text{(definition of scalar multiplication on functions)} \\ &= 0 && (f \text{ is in } W) \end{aligned}$$

so W is closed under scalar multiplication.

Therefore W is a subspace of $F(-\infty,\infty)$ and hence is a vector space. □

2 Let V be a vector space.

2.a Question: If k is any scalar, prove that $k\vec{0} = \vec{0}$.

2.a

Proof.

$$\begin{aligned}
k\vec{0} &= k(\vec{0} + \vec{0}) && (\vec{0} = \vec{0} + \vec{0} \text{ by axiom 4}) \\
&= k\vec{0} + k\vec{0} && (\text{by axiom 7}) \\
k\vec{0} + (-k\vec{0}) &= [k\vec{0} + k\vec{0}] + (-k\vec{0}) && (\text{by axiom 5 } k\vec{0} \text{ has a negative}) \\
k\vec{0} + (-k\vec{0}) &= k\vec{0} + [k\vec{0} + (-k\vec{0})] && (\text{by axiom 3}) \\
\vec{0} &= \vec{0} + k\vec{0} && (\text{by axiom 5}) \\
\vec{0} &= k\vec{0} && (\text{by axiom 4})
\end{aligned}$$

□

2.b Question: Prove that the zero vector in V is unique.

2.b

Proof. We must show that there is only one vector, $\vec{0}$, with the property that $\vec{0} + \vec{v} = \vec{v} + \vec{0} = \vec{v}$.

Suppose $\vec{0_1}$ and $\vec{0}$ are zero vectors in V Then $\vec{v} + \vec{0} = \vec{v} \ \wedge \vec{v} + \vec{0_1} = \vec{v}$

$$\begin{aligned}
\vec{0_1} &= \vec{0_1} + \vec{0} && (\text{vector space axiom 4}) \\
&= \vec{0} + \vec{0_1} && (\text{vector space axiom 2}) \\
&= \vec{0} && (\text{vector space axiom 4})
\end{aligned}$$

Therefore $\vec{0_1} = \vec{0}$. So, the zero vector is unique. □

3 Determine if each of the following are subspaces of M_{nn}

3.a Question: $\{A \in M_{nn} \mid det(A) = 0\}$

3.a Answer: No, W is not a subspace of M_{nn}.

Proof. Counter example Axiom 1 fails

$$\det \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0, \det \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0$$
$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$\det \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$$

A subset W of M_{nn} is a subspace of M_{nn} if and only if W satisfies the following three conditions W is nonempty, W is closed under addition, W is closed under scalar multiplication. We have shown W is not closed under addition; therefore, W is not a subspace of M_{nn}. □

3.b Question: $\{A \in M_{nn} \mid tr(A) = 0\}$

3.b Answer: Yes, W is a subspace of M_{nn}

Proof. To prove that W is a subspace of M_{nn} we must show three things.

(1) Prove that W is non empty.

$$A = (a_{ij}) = 0 \ \forall \ ij \implies tr(A) = 0 \therefore \vec{0} \in W$$

(2) Prove that W is closed under addition.

Let $A = (a_{ij}) \wedge B = (b_{ij}) \in W$ be square matricies of order n such that $tr(A) = tr(B) = 0$

$$\begin{aligned} tr(A+B) &= \sum_{i=1}^{n} (a_{ii} + b_{ii}) = \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii} \\ &= tr(A) + tr(B) = 0 + 0 = 0 \\ \implies C \in W &: C = A + B \end{aligned}$$

(3) Prove that W is closed under multiplication.

Let k be any scalar.

$$\begin{aligned} tr(kA) &= \sum_{i=1}^{n} (k \cdot a_{ii}) = k \cdot \sum_{i=1}^{n} a_{ii} \\ &= k \cdot tr(A) = k \cdot 0 = 0 \\ \implies kA &\in W \end{aligned}$$

$\therefore W$ is a subspace of M_{nn}. □

3.c Question: $\left\{A \in M_{nn} \mid A^T = A\right\}$

3.c Answer: Yes, W is a subspace of M_{nn}

Proof. To prove that W is a subspace of M_{nn} we must show three things.

(1) Prove that W is non empty.

$$A = (a_{ij}) = 0 \; \forall \; ij \implies A = A^T = 0 \therefore \vec{0} \in W$$

(2) Prove that W is closed under addition.

Let $A = (a_{ij}) \wedge B = (b_{ij}) \in W$ be square matricies of order n such that $a_{ij} = aji \wedge b_{ij} = b_{ji}$

$$\begin{aligned} (A+B)^T &= A^T + B^T \\ &= A + B \\ \implies C \in W &: C = A + B \end{aligned}$$

(3) Prove that W is closed under multiplication.

Let k be any scalar.

$$\begin{aligned} (k \cdot A)^T &= k \cdot A^T \\ &= k \cdot A \\ \implies kA &\in W \end{aligned}$$

$\therefore W$ is a subspace of M_{nn}. □

4 Consider the following vectors in P_2: $\vec{p_1} = 2 + x + 4x^2$, $\vec{p_2} = 1 - x + 3x^2$, $\vec{p_3} = 3 + 2x + 5x^2$

4.a Question: Express the vector $\vec{g} = 6 + 11x + 6x^2$ as a linear combination of $\{\vec{p_1}, \vec{p_2}, \vec{p_3}\}$.

4.a Answer: $\vec{g} = 4(2 + x + 4x^2) + -5(1 - x + 3x^2) + 1(3 + 2x + 5x^2)$

Proof. We will show $\vec{g} = k_1\vec{p_1} + k_2\vec{p_2} + k_3\vec{p_3}$

$$\begin{aligned}(6 + 11x + 6x^2) &= k_1(2 + x + 4x^2) + k_2(1 - x + 3x^2) + k_3(3 + 2x + 5x^2) \\ &= (k_1 2 + k_2 + k_3 3) + (k_1 x - k_2 x + k_3 2x) + (k_1 4x^2 + k_2 3x^2 + k_3 5x^2) \\ &= (k_1 2 + k_2 + k_3 3) + (k_1 - k_2 + k_3 2)x + (k_1 4 + k_2 3 + k_3 5)x^2\end{aligned}$$

$$\begin{aligned}
&\left[\begin{array}{ccc|c} 2 & 1 & 3 & 6 \\ 1 & -1 & 2 & 11 \\ 4 & 3 & 5 & 6 \end{array}\right] \\
[-2r2 + r1] \wedge [-4r2 + r1] &\left[\begin{array}{ccc|c} 0 & 3 & -1 & -16 \\ 1 & -1 & 2 & 11 \\ 0 & 7 & -3 & -38 \end{array}\right] \\
r2 \leftrightarrow r1 &\left[\begin{array}{ccc|c} 1 & -1 & 2 & 11 \\ 0 & 3 & -1 & -16 \\ 0 & 7 & -3 & -38 \end{array}\right] \\
\frac{1}{3}r2 &\left[\begin{array}{ccc|c} 1 & -1 & 2 & 11 \\ 0 & 1 & \frac{-1}{3} & \frac{-16}{3} \\ 0 & 7 & -3 & -38 \end{array}\right] \\
-7r2 + r3 &\left[\begin{array}{ccc|c} 1 & -1 & 2 & 11 \\ 0 & 1 & \frac{-1}{3} & \frac{-16}{3} \\ 0 & 0 & \frac{-2}{3} & \frac{-2}{3} \end{array}\right] \\
-\frac{3}{2}r3 &\left[\begin{array}{ccc|c} 1 & -1 & 2 & 11 \\ 0 & 1 & \frac{-1}{3} & \frac{-16}{3} \\ 0 & 0 & 1 & 1 \end{array}\right] \\
\frac{1}{3}r3 + r2 &\left[\begin{array}{ccc|c} 1 & -1 & 2 & 11 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 1 \end{array}\right] \\
r1 + r2 &\left[\begin{array}{ccc|c} 1 & 0 & 2 & 6 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 1 \end{array}\right] \\
r1 + r3 &\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 1 \end{array}\right] \\
\implies &(k_1, k_2, k_3) = (4, -5, 1)
\end{aligned}$$

$\therefore \vec{g}$ can be expressed as the following linear combination $\vec{g} = k_1\vec{p_1} + k_2\vec{p_2} + k_3\vec{p_3}$ $\square$

4.b Question: Does $\{\vec{p_1}, \vec{p_2}, \vec{p_3}\}$ span P_2?

4.b Answer: Yes, $span(\{\vec{p_1}, \vec{p_2}, \vec{p_3}\}) = P_2$

Proof. An arbitrary vector in P_2 is of the form $\vec{p} = a + bx + cx^2$ and so becomes,

$k_0(2 + x + 4x^2) + k_1(1 - x + 3x^2) + k_2(3 + 2x + 5x^2) = a + bx + cx^2$

which we can rewrite as

$(k_0 2 + k_1 + k_2 3) + (k_0 - k_1 + k_2 2)x + (k_0 4 + k_1 3 + k_2 5)x^2 = a + bx + cx^2$

Equating corresponding coefficients yields a linear system whose augmented matrix is

$$A = \left[\begin{array}{ccc|c} 2 & 1 & 3 & a \\ 1 & -1 & 2 & b \\ 4 & 3 & 5 & c \end{array}\right]$$

Our problem reduces to ascertaining whether this system is consistent for all values of a, b, and c. This can be determined if its coefficient matrix has a nonzero determinant, from our theorem for equivalent statements. If A is an $n\ x\ n$ matrix such that $\det(A) \neq 0$ then $A\vec{x} = \vec{0}$ has only the trivial solution.

It follows from 4.a that

$$A = \left[\begin{array}{ccc|c} 2 & 1 & 3 & 0 \\ 1 & -1 & 2 & 0 \\ 4 & 3 & 5 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

We have shown A is consistent for every choice a, b, and c. Thus, the vectors in $\{\vec{p_1}, \vec{p_2}, \vec{p_3}\}$ span P_2. $\square$

4.c Question: Is $\{\vec{p_1}, \vec{p_2}, \vec{p_3}\}$ linearly independent?

4.c Answer: Yes, $\{\vec{p_1}, \vec{p_2}, \vec{p_3}\}$ is linearly independent

From 4.b we already know this set is linearly independent.

Proof. The nonempty set $\{\vec{p_1}, \vec{p_2}, \vec{p_3}\}$ in a vector space V is linearly independent if and only if the coefficients satisfying

$$k_0\vec{p_1} + k_1\vec{p_2} + k_2\vec{p_3} = \vec{0}$$

are $k_0 = 0,\ k_1 = 0,\ k_2 = 0$.

From our theorem for equivalent statements. If A is an $n\ x\ n$ matrix such that $\det(A) \neq 0$ then $A\vec{x} = \vec{0}$ has only the trivial solution. We will show $\det(A) \neq 0$, to convince our selves this theorem holds.

$$\det \begin{vmatrix} 2 & 1 & 3 \\ 1 & -1 & 2 \\ 4 & 3 & 5 \end{vmatrix}$$

minor entry $\wedge$ cofactor

$$a_{11}C_{11} = (-1)^{1+1}(2) \cdot [(-1)(5) - (2)(3)] = -22$$
$$a_{12}C_{12} = (-1)^{1+2}(1) \cdot [(1)(5) - (2)(4)] = 3$$
$$a_{13}C_{13} = (-1)^{1+3}(3) \cdot [(1)(3) - (-1)(4)] = 21$$

cofactor expansion

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = 2$$

$\therefore (k_0, k_1, k_2) = (0, 0, 0)$ so $\{\vec{p_1}, \vec{p_2}, \vec{p_3}\}$ is linearly independent $\square$

5 Consider the following planes in $\mathbb{R}^3$. $P_1 : 2x + 3y - z = 0$ and $P_2 : x + 2y - 2z = 0$

5.a Question Find a set of vectors that spans $P_1 : 2x + 3y - z = 0$.

5.a Answer: $P_1 = span\{(-\frac{3}{2}, 1, 0), (\frac{1}{2}, 0, 1)\}$

Proof. Solve the following system $(2, 3, -1)^T = \vec{0}$ with $(2, 3, -1)^T$ a row vector.

$$\begin{aligned}
&\begin{bmatrix} 2 & 3 & -1 & \mid & 0 \end{bmatrix} \\
\frac{1}{2}r1 &\begin{bmatrix} 1 & \frac{3}{2} & \frac{-1}{2} & \mid & 0 \end{bmatrix} \\
\therefore (z = r),\ (y = q),\ (x = -\frac{3}{2}q + \frac{1}{2}r)&
\end{aligned}$$

giving the following.

$$\begin{aligned}
(x, y, z) &= (-\frac{3}{2}q + \frac{1}{2}r, q, r) \\
&= q(-\frac{3}{2}, 1, 0) + r(\frac{1}{2}, 0, 1)
\end{aligned}$$

$\therefore P_1 = \text{span}\{(-\frac{3}{2}, 1, 0), (\frac{1}{2}, 0, 1)\}$ □

5.b Question Find a set of vectors that spans $P_2 : x + 2y - 2z = 0$.

5.b Answer: $P_2 =$ **span**$\{(-2, 1, 0), (2, 0, 1)\}$

Proof. From $P_2 : x + 2y - 2z = 0$ we have the row vector

$$\begin{bmatrix} 1 & 2 & -2 & \mid & 0 \end{bmatrix}$$

giving the following.

$$\begin{aligned}
z &= r \\
y &= q \\
x &= -2q + 2r. \\
\implies (x, y, z) &= (-2q + 2r, q, r) \\
&= q(-2, 1, 0) + r(2, 0, 1)
\end{aligned}$$

$\therefore P_2 = \text{span}\{(-2, 1, 0), (2, 0, 1)\}$ □

5.c Question: Find a set of vectors that spans $P_1 \bigcap P_2$.

5.c Answer: $P_1 \bigcap P_2 = \textbf{span}\{(-4,3,1)\}$

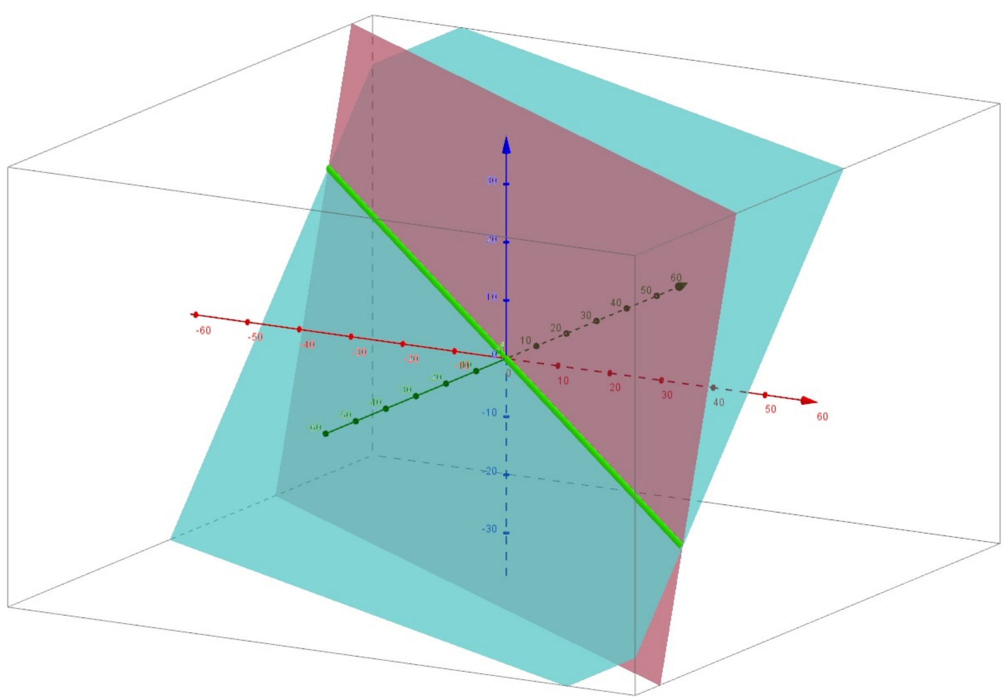
Proof. We will find the span of $P_1 \bigcap P_2$ by solving a system of equations

$$P_1 : 2x + 3y - z = 0$$
$$P_2 : x + 2y - 2z = 0$$

$$\begin{aligned}
&\left[\begin{array}{ccc|c} 2 & 3 & -1 & 0 \\ 1 & 2 & -2 & 0 \end{array}\right] \\
[r1] \leftrightarrow [r2] &\left[\begin{array}{ccc|c} 2 & 3 & -1 & 0 \\ 1 & 2 & -2 & 0 \end{array}\right] \\
(-2)r1 + r2 &\left[\begin{array}{ccc|c} 1 & 2 & -2 & 0 \\ 0 & -1 & 3 & 0 \end{array}\right] \\
2(r2) + r1 &\left[\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & -1 & 3 & 0 \end{array}\right] \\
\therefore (z = t),\ &(y = 3t),\ (x = -4t)
\end{aligned}$$

giving the following.

$$(x, y, z) = (-4t, 3t, t) = t(-4, 3, 1)$$



$\therefore P_1 \bigcap P_2 = \text{span } \{(-4, 3, 1)\}$ $\square$