Math 2311 – Assignment 1

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1. Determine if each of the following sets is a vector space.

   (a) $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid x \geq y \right\}$ with the usual scalar multiplication and vector addition from $\mathbb{R}^2$

Answer: No, V is not a vector space.

Proof. let $(x_0, y_0)^T, (x_1, y_1)^T \in V$, and take the vector space operations on V to be the usual operations of *vector* addition and *scalar* multiplication; that is,

$$(x_0, y_0)^T + (x_1, y_1)^T = (x_0 + x_1, y_0 + y_1)^T \quad (1)$$

$$k(x_0, y_0)^T = (kx_0, ky_0)^T \quad (2)$$

V is closed under scalar addition since $x_0 + x_1 \geq y_0 + y_1$

However, by properties of inequalities if the constant, k, is negative, we must reverse the symbol to preserve the inequality relation.

Given that k is negative, $x \geq y \rightarrow kx \leq ky$ □

   (b) Consider the set $W = \{f \in F(-\infty, \infty) \, | f(1) = 0\}$ with the usual scalar multiplication and vector addition from $F(-\infty, \infty)$. Is W a vector space?

Answer: Yes, W is a vector space.

Proof. Since we know that $F(-\infty, \infty)$ (with the usual operations) is a vector space, and since W is a subset of $F(-\infty, \infty)$ (with the same operations), it suffices to prove that W is a subspace of $F(-\infty, \infty)$. To this end we must show three things:

(a) That W is non-empty.

(b) That W is closed under addition.

(c) That W is closed under scalar multiplication.

There exists a function $\mathbf{0}$ in $F(-\infty, \infty)$ defined by $\mathbf{0}(x) = 0$ for all x. Clearly $\mathbf{0}(1) = f(1) = 0$ so W is non-empty.

Now suppose f and g are two functions in W. We must show that $f+g$ is in W.

$$
\begin{aligned}
(f+g)(1) &= f(1) + g(1) && \text{(definition of addition of functions)}\\
&= 0 && (f \text{ and } g \text{ are in } W)
\end{aligned}
$$

Finally, to show that W is closed under scalar multiplication, suppose f is in W and k is a scalar, then

$$
\begin{aligned}
(kf)(1) &= kf(1) && \text{(definition of scalar multiplication on functions)}\\
&= 0 && (f \text{ is in } W)
\end{aligned}
$$

so (kf) is in W and W is closed under scalar multiplication.

Therefore W is a subspace of $F(-\infty, \infty)$ and hence is a vector space. □

2. Let V be a vector space.

   (a) If k is any scalar, prove that $k\vec{0} = \vec{0}$.

Proof.

$$
\begin{aligned}
k(\vec{0} + \vec{u}) &= k\vec{0} + k\vec{u} && \text{(vector space axiom 7)}\\
k\vec{u} &= k\vec{0} + k\vec{u} && \text{(vector space axiom 4)}\\
k(\vec{u}) + (-k\vec{u}) &= (-k\vec{u}) + (k\vec{0} + k\vec{u}) && \text{(vector space axiom 5)}\\
\vec{\mathbf{0}} &= (-k\vec{u}) + (k\vec{0} + k\vec{u}) && \text{(vector space axiom 5)}\\
&= (k\vec{0} + k\vec{u}) + (-k\vec{u}) && \text{(vector space axiom 2)}\\
&= k\vec{0} + (k\vec{u} + (-k\vec{u})) && \text{(vector space axiom 3)}\\
&= k\vec{0} + \vec{0} && \text{(vector space axiom 5)}\\
&= \mathbf{k}\vec{\mathbf{0}} && \text{(vector space axiom 4)}
\end{aligned}
$$

□

   (b) Prove that the zero vector in V is unique.

Proof. We must show that there is only one vector, $\vec{0}$, with the property that $\vec{0} + \vec{v} = \vec{v} + \vec{0} = \vec{v}$.

Suppose $\vec{0_1}$ and $\vec{0}$ are zero vectors in V, and $\vec{v}$ is also in V. Then

$$
\begin{aligned}
\vec{v} &= \vec{0_1} + \vec{v} && \text{(vector space axiom 4)}\\
\vec{v} + (-\vec{v}) &= (-\vec{v}) + (\vec{0_1} + \vec{v}) && \text{(vector space axiom 5)}\\
\vec{\mathbf{0}} &= (-\vec{v}) + (\vec{0_1} + \vec{v}) && \text{(vector space axiom 5)}\\
&= (\vec{0_1} + \vec{v}) + (-\vec{v}) && \text{(vector space axiom 2)}\\
&= \vec{0_1} + (\vec{v} + (-\vec{v})) && \text{(vector space axiom 3)}\\
&= \vec{0_1} + \vec{0} && \text{(vector space axiom 5)}\\
&= \vec{\mathbf{0_1}} && \text{(vector space axiom 4)}
\end{aligned}
$$

Since we have shown that any two zero vectors must be equal to each other, we can conclude that $\vec{0}$ is unique. □

3. Determine if each of the following are subspaces of M_{nn}

   (a) $\{A \in M_{nn}\ |det(A) = 0\}$

   Answer: No, $\{A \in M_{nn} \mid \det(A) = 0\}$ is not a subspace of M_{nn}.

   Proof. $\det(A + B) \neq \det(A) + \det(B)$

$$\det \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0, \det \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\det \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$$

   □

   (b) $\{A \in M_{nn}\ |tr(A) = 0\}$

   Answer: Yes, $W = \{A \in M_{nn}\ |tr(A) = 0\}$ is a subspace of M_{nn}.

   Proof. Let $[B = (b_{ii})] \in W$ such that $tr(B) = 0$ and let k be any skalar

   The set W is non empty because, if we let $a_{ii} = 0$ for all i then $tr(A) = 0$ therefore W contains the **0** matrices. It remains to show that W is closed under addition and scalar multiplication

$$\begin{aligned} tr(A + B) &= \sum_{i=1}^{n} (a_{ii} + b_{ii}) \\ &= \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii} \\ &= tr(A) + tr(B) \\ &= 0 + 0 \\ &= 0. \end{aligned}$$

$$\begin{aligned} tr(kA) &= \sum_{i=1}^{n} (k \cdot a_{ii}) \\ &= k \cdot \sum_{i=1}^{n} a_{ii} \\ &= k \cdot tr(A) \\ &= k \cdot 0 \\ &= 0 \end{aligned}$$

   To be clear, if we take some $C = A + B$ such that A and B are in W, then for all $C = (c_{ii})$, the sum will be 0, so C is also in W. □

   (c) $\left\{A \in M_{nn}\ |A^T = A\right\}$

Answer: Yes, $W = \left\{A \in M_{nn} \mid A^T = A\right\}$ is a subspace of M_{nn}.

Proof. Let $[B = (b_{ij})] \in W$ be a square matrix of order n such that $b_{ij} = b_{ji}$, and let k be any skalar

The set W is non empty because, if we let $[A = (a_{ii})] = 0$ for all i then W contains the $\mathbf{0}$ matrices. It remains to show that W is closed under addition and scalar multiplication

$C = (A+B)^T = A^T + B^T = A + B$ is in W, and $(k \cdot A)T = k \cdot AT = k \cdot A$ is also in W. Therefore W is non-empty and closed under addition and scalar multiplication. $\square$