

1 Let $\vec{x} = [1 \ 2 \ 3]^T$, $\mathcal{B} = \{[1 \ 0 \ 0]^T, [1 \ 1 \ 0]^T, [1 \ 1 \ 1]^T\}$, and $\mathcal{C} = \{[1 \ 1 \ 0]^T, [0 \ 1 \ 1]^T, [1 \ 0 \ 1]^T\}$.

1.a Find $[\vec{x}]_{\mathcal{B}}$

Solution: $[\vec{x}]_{\mathcal{B}} = [-1 \ -1 \ 3]^T$

Proof.

$$\text{By inspection: } \vec{x} = (-1)[1 \ 0 \ 0]^T + (-1)[1 \ 1 \ 0]^T + (3)[1 \ 1 \ 1]^T$$

$$\therefore [\vec{x}]_{\mathcal{B}} = [-1 \ -1 \ 3]^T \quad \square$$

1.b Find $[\vec{x}]_{\mathcal{C}}$

Solution: $[\vec{x}]_{\mathcal{C}} = [0 \ 2 \ 1]^T$

Proof.

$$\text{By inspection: } \vec{x} = (0)[1 \ 1 \ 0]^T + (2)[0 \ 1 \ 1]^T + (1)[1 \ 0 \ 1]^T$$

$$\therefore [\vec{x}]_{\mathcal{C}} = [0 \ 2 \ 1]^T \quad \square$$

1.c Find $P_{\mathcal{C} \leftarrow \mathcal{B}}$ and compute $P_{\mathcal{C} \leftarrow \mathcal{B}}[\vec{x}]_{\mathcal{B}}$

$$\text{Solutions : } P_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{bmatrix} 1/2 & 1 & 1/2 \\ -1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 \end{bmatrix}; [\vec{x}]_{\mathcal{C}} = P_{\mathcal{C} \leftarrow \mathcal{B}}[\vec{x}]_{\mathcal{B}} = [0 \ 2 \ 1]^T$$

Proof.

$$\begin{aligned} \text{Partitioned matrix } [\mathcal{C} \mid \mathcal{B}] &= \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array} \right] \\ \text{Transition matrix } [I_3 \mid \mathcal{C} \leftarrow \mathcal{B}] &= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 1 & 1/2 \\ 0 & 1 & 0 & -1/2 & 0 & 1/2 \\ 0 & 0 & 1 & 1/2 & 0 & 1/2 \end{array} \right] \\ P_{\mathcal{C} \leftarrow \mathcal{B}} &= \begin{bmatrix} 1/2 & 1 & 1/2 \\ -1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} [\vec{x}]_{\mathcal{C}} &= P_{\mathcal{C} \leftarrow \mathcal{B}}[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 1/2 & 1 & 1/2 \\ -1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 \end{bmatrix} [-1 \ -1 \ 3]^T \\ &= [-1] \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \end{bmatrix} + [-1] \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + [3] \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} \\ &= [0 \ 2 \ 1]^T \quad \square \end{aligned}$$

1.d Find $P_{\mathcal{B} \leftarrow \mathcal{C}}$ and compute $P_{\mathcal{B} \leftarrow \mathcal{C}}[\vec{x}]_{\mathcal{C}}$

$$\text{Solutions : } P_{\mathcal{B} \leftarrow \mathcal{C}} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} ; [\vec{x}]_{\mathcal{B}} = P_{\mathcal{B} \leftarrow \mathcal{C}}[\vec{x}]_{\mathcal{C}} = [-1 \ -1 \ 3]^T$$

Proof.

$$\begin{aligned} \text{Partitioned matrix } [\mathcal{B} \mid \mathcal{C}] &= \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array} \right] \\ \text{Transition matrix } [I_3 \mid \mathcal{B} \leftarrow \mathcal{C}] &= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array} \right] \\ P_{\mathcal{B} \leftarrow \mathcal{C}} &= \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} [\vec{x}]_{\mathcal{B}} = P_{\mathcal{B} \leftarrow \mathcal{C}}[\vec{x}]_{\mathcal{C}} &= \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} [0 \ 2 \ 1]^T \\ &= [0] \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + [2] \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + [1] \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \\ &= [-1 \ -1 \ 3]^T \end{aligned}$$

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