Math 2311 - Assignment 4

Michael Walker

Problem 1 Find the least squares quadratic fit

$$y = a_0 + a_1 x + a_2 x^2$$

to the data points, and show that the result is reasonable by graphing the fitted curve and plotting the data in the same coordinate system.

$A(1, -2),\ B(0,\ -1),\ C(1,\ 0),\ D(2,\ 4)$

Solution

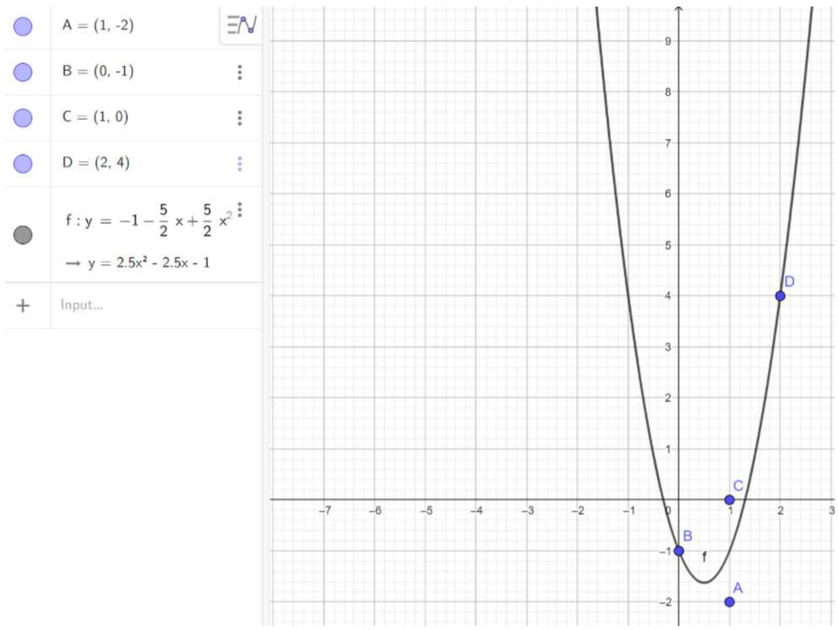

Figure 1: $y = -1 - \dfrac{5}{2}x + \dfrac{5}{2}x^2$ $\square$

Proof We will solve this sytem using cramers rule

$$a_0 = \frac{det(A_0)}{det(A)},\ a_1 = \frac{det(A_1)}{det(A)},\ a_2 = \frac{det(A_1)}{det(A)}$$

$$M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix};\ M^T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 \\ 1 & 0 & 1 & 4 \end{bmatrix};\ y = \begin{bmatrix} -2 \\ -1 \\ 0 \\ 4 \end{bmatrix}$$

$$A = M^T M = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 \\ 1 & 0 & 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 6 & 4 \\ 4 & 6 & 10 \\ 6 & 10 & 18 \end{bmatrix}$$

$$\vec{b} = M^T y = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 \\ 1 & 0 & 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ -1 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 14 \end{bmatrix}$$

$$A_0 = \begin{bmatrix} \mathbf{1} & 6 & 4 \\ \mathbf{6} & 6 & 10 \\ \mathbf{14} & 10 & 18 \end{bmatrix};\ A_1 = \begin{bmatrix} 4 & \mathbf{1} & 4 \\ 4 & \mathbf{6} & 10 \\ 6 & \mathbf{14} & 18 \end{bmatrix};\ A_2 = \begin{bmatrix} 4 & 6 & \mathbf{1} \\ 4 & 6 & \mathbf{6} \\ 6 & 10 & \mathbf{14} \end{bmatrix}$$

Therefore,

$$det(A) = 8$$
$$det(A_0) = -8$$
$$det(A_1) = -20$$
$$det(A_2) = 20$$

$$a_0 = \frac{det(A_0)}{det(A)} = \frac{-8}{8} = -1,\ a_1 = \frac{det(A_1)}{det(A)} = -\frac{20}{8} = -\frac{5}{2},\ a_2 = \frac{det(A_2)}{det(A)} = -\frac{20}{8} = \frac{5}{2}$$

Thus, the desired curve is $y = -1 - \frac{5}{2}x + \frac{5}{2}x^2$ $\square$

Problem 2 Let $\vec{u}$ be a unit vector in $\mathbb{R}^n$, and let $\vec{v}$ be any other vector in $\mathbb{R}^n$ (both $\vec{u}$ and $\vec{v}$ written as column vectors). Show that $\vec{u}\vec{u}^T\vec{v} = proj_{\vec{u}}(\overrightarrow{v})$ (Note: this shows the matrix $\vec{u}\vec{u}^T$ is the standard matrix for the linear transformation that projects onto the line spanned by $\vec{u}$).

Proof

let $\vec{u} \in \mathbb{R}^n : \vec{u} = \begin{bmatrix} u_1 & u_2 & \dots & u_n \end{bmatrix}^T \wedge \|\vec{u}\| = 1,\ u^T = \begin{bmatrix} u_1 & u_2 & \dots & u_n \end{bmatrix}$ and $\vec{v} \in \mathbb{R}^n : \vec{v} = \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix}^T$

$$\vec{u}\vec{u}^T\vec{v} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \begin{bmatrix} u_1 & u_2 & \dots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1u_1 & u_1u_2 & \dots & u_1u_n \\ u_2u_1 & u_2u_2 & \dots & u_2u_n \\ \vdots & \vdots & \vdots & \vdots \\ u_nu_1 & u_nu_2 & \dots & u_nu_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = v_1 \begin{bmatrix} u_1u_1 \\ u_2u_1 \\ \vdots \\ u_nu_1 \end{bmatrix} + v_2 \begin{bmatrix} u_1u_2 \\ u_2u_2 \\ \vdots \\ u_nu_2 \end{bmatrix} + \ \dots \ + v_n \begin{bmatrix} u_1u_n \\ u_2u_n \\ \vdots \\ u_nu_n \end{bmatrix}$$

$$= \begin{bmatrix} v_1(u_1u_1) + v_2(u_1u_2) + \ \dots \ + v_n(u_1u_n) \\ v_1(u_2u_1) + v_2(u_2u_2) + \ \dots \ + v_n(u_2u_n) \\ \vdots \\ v_1(u_nu_1) + v_2(u_nu_2) + \ \dots \ + v_n(u_nu_n) \end{bmatrix} = \begin{bmatrix} v_1(u_1u_1) + v_2(u_2u_1) + \ \dots \ + v_n(u_nu_1) \\ v_1(u_1u_2) + v_2(u_2u_2) + \ \dots \ + v_n(u_nu_2) \\ \vdots \\ v_1(u_1u_n) + v_2(u_2u_n) + \ \dots \ + v_n(u_nu_n) \end{bmatrix}$$

$$= \begin{bmatrix} (v_1u_1)u_1 + (v_2u_2)u_1 + \ \dots \ + (v_nu_n)u_1 \\ (v_1u_1)u_2 + (v_2u_2)u_2 + \ \dots \ + (v_nu_n)u_2 \\ \vdots \\ (v_1u_1)u_n + (v_2u_2)u_n + \ \dots \ + (v_nu_n)u_n \end{bmatrix} = \begin{bmatrix} (v_1u_1)u_1 + (v_2u_2)u_1 + \ \dots \ + (v_nu_n)u_1 \\ (v_1u_1)u_2 + (v_2u_2)u_2 + \ \dots \ + (v_nu_n)u_1 \\ \vdots \\ (v_1u_1)u_n + (v_2u_2)u_n + \ \dots \ + (v_nu_n)u_n \end{bmatrix}$$

$$= (\ (v_1u_1) + (v_2u_2) + \ \dots \ + (v_nu_n)\ ) \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \left(\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \right) \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = (\vec{v} \cdot \vec{u})\vec{u} = proj_{\vec{u}}(\vec{v})$$

Thus,

$$\vec{u}\vec{u}^T\vec{v} = proj_{\vec{u}}(\vec{v}) \qquad \square$$

Problem 3 Let W be a subspace of $\mathbb{R}^n$ with the Euclidean inner product. Let W have orthonormal basis $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}$. Let P be the $n \times k$ matrix with i^{th} column $\vec{w}_i$.

(a) Prove that $PP^T\vec{x} = proj_w(\vec{x})\ \forall \vec{x} \in \mathbb{R}^n$. (Hint, this can be done directly by thinking of this as $P\left(P^T\vec{x}\right)$, or it can be done by showing that $PP^T = \vec{w}_1\vec{w}_1^T + \vec{w}_2\vec{w}_2^T + \ldots + \vec{w}_k\vec{w}_k^T$ and using the results of the previous question)

(b) Find a basis for $col\left(PP^T\right)$. Hint: recall that $col(A) = \{\vec{b} \mid A\vec{x} = \vec{b} \text{ has a solution}\}$

(c) Find $rank\left(PP^T\right)$

(d) Find $det\left(PP^T\right)$ (consider separately the case $n < k$ and the case $n = k$)

(e) **Bonus** What are the eigenvalues of PP^T? What is a basis for each eigenspace?

proof (a)

$$PP^T = \begin{bmatrix} w_{11} & w_{12} & \ldots & w_{1k} \\ w_{21} & w_{22} & \ldots & w_{2k} \\ \vdots & \vdots & \vdots & \vdots \\ w_{n1} & w_{n2} & \ldots & w_{nk} \end{bmatrix} \cdot \begin{bmatrix} w_{11} & w_{21} & \ldots & w_{n1} \\ w_{12} & w_{22} & \ldots & w_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ w_{1k} & w_{2k} & \ldots & w_{nk} \end{bmatrix}$$

$$= \begin{bmatrix} (w_{11})^2 + (w_{12})^2 + \ldots + (w_{1k})^2 & w_{11}w_{21} + w_{12}w_{22} + \ldots + w_{1k}w_{2k} & \ldots & w_{11}w_{n1} + w_{12}w_{n2} + \ldots + w_{1k}w_{nk} \\ w_{21}w_{11} + w_{22}w_{12} + \ldots + w_{2k}w_{1k} & (w_{21})^2 + (w_{22})^2 + \ldots + (w_{2k})^2 & \ldots & w_{21}w_{n1} + w_{22}w_{n2} + \ldots + w_{2k}w_{nk} \\ \vdots & \vdots & \vdots & \vdots \\ w_{n1}w_{11} + w_{n2}w_{12} + \ldots + w_{nk}w_{1k} & w_{n1}w_{21} + w_{n2}w_{22} + \ldots + w_{nk}w_{2k} & \ldots & (w_{n1})^2 + (w_{n2})^2 + \ldots + (w_{nk})^2 \end{bmatrix}$$

$$= \begin{bmatrix} (w_{11})^2 & w_{11}w_{21} & \ldots & w_{11}w_{n1} \\ w_{21}w_{11} & (w_{21})^2 & \ldots & w_{21}w_{n1} \\ \vdots & \vdots & \vdots & \vdots \\ w_{n1}w_{11} & w_{n1}w_{21} & \ldots & (w_{n1})^2 \end{bmatrix} + \begin{bmatrix} (w_{12})^2 & w_{12}w_{22} & \ldots & w_{12}w_{n2} \\ w_{22}w_{12} & (w_{22})^2 & \ldots & w_{22}w_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ w_{n2}w_{12} & w_{n2}w_{22} & \ldots & (w_{n2})^2 \end{bmatrix} + \ldots + \begin{bmatrix} (w_{1k})^2 & w_{1k}w_{2k} & \ldots & w_{1k}w_{nk} \\ w_{2k}w_{1k} & (w_{2k})^2 & \ldots & w_{2k}w_{nk} \\ \vdots & \vdots & \vdots & \vdots \\ w_{nk}w_{1k} & w_{nk}w_{2k} & \ldots & (w_{nk})^2 \end{bmatrix}$$

Here we have

$$= \vec{w}_1\vec{w}_1^T + \vec{w}_2\vec{w}_2^T + \ldots + \vec{w}_k\vec{w}_k^T$$

From this we conclude that,

$$PP^T\vec{x} = \sum_{i=1}^{k} \vec{w}_i\vec{w}_i^T\vec{x}$$

we proved $\vec{u}\vec{u}^T\vec{v} = proj_{\vec{u}}(\vec{v})$ in problem 2, so it must be true that

$$\sum_{i=1}^{k} \vec{w}_i\vec{w}_i^T\vec{x} = \sum_{i=1}^{k} \langle \vec{x}, \vec{w}_i \rangle \vec{w}_i$$

Therefore,

$$PP^T\vec{x} = proj_w(\vec{x})\ \forall \vec{x} \in \mathbb{R}^n \qquad \square$$

proof (b) It follows from **(a)**, that

$$\left(PP^T\right)=\begin{bmatrix} \langle \vec{w}_1, \vec{w}_1^T \rangle & \langle \vec{w}_1, \vec{w}_2^T \rangle & \dots & \langle \vec{w}_1, \vec{w}_k^T \rangle \\ \langle \vec{w}_2, \vec{w}_1^T \rangle & \langle \vec{w}_2, \vec{w}_2^T \rangle & \dots & \langle \vec{w}_2, \vec{w}_k^T \rangle \\ \vdots & \vdots & \vdots & \vdots \\ \langle \vec{w}_k, \vec{w}_1^T \rangle & \langle \vec{w}_k, \vec{w}_2^T \rangle & \dots & \langle \vec{w}_k, \vec{w}_k^T \rangle \end{bmatrix}$$

but, our vectors $\vec{w}_i$ are orthogonal, so $\langle \vec{w}_i, \vec{w}_j^T \rangle = \begin{cases} 0 & for\ i \neq j \\ 1 & for\ i = j \end{cases}$. Therefore,

$$\left(PP^T\right)=\begin{bmatrix} \langle \vec{w}_1, \vec{w}_1^T \rangle & 0 & \dots & 0 \\ 0 & \langle \vec{w}_2, \vec{w}_2^T \rangle & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \langle \vec{w}_k, \vec{w}_k^T \rangle \end{bmatrix} = I_n$$

So, a basis for $col\left(PP^T\right)$ is $\{\vec{w}_1,\ \vec{w}_2,\ \dots\ , \vec{w}_k\}$ $\square$

proof (c) P is an $n \times k$ matrix and P^T is an $k \times n$ matrix, therefore PP^T is an $n \times n$ matrix. Because PP^T is $n \times n$ and in part **(b)** we showed $PP^T = I_n$ we conclude $rank\left(PP^T\right) = n$ $\square$.

proof (d)
case 1: unsure.
case 2:

$$det\left(PP^T\right)=\prod_{i=1}^{k} \langle \vec{w}_i, \vec{w}_i^T \rangle = 1 \qquad \square$$

solution (e)

$$\boxed{\begin{aligned} &\lambda = 1 \\ &B=\left\{\begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}^T + \begin{bmatrix} 0 & 1 & \dots & 0 \end{bmatrix}^T + \dots + \begin{bmatrix} 0 & 0 & \dots & 1 \end{bmatrix}^T\right\} \end{aligned}}$$

proof (e)

The eigen values of PP^T are going to be the values of $det\left(PP^T - \lambda I\right) = 0$,

$$det\left(PP^T - \lambda I\right) = \begin{vmatrix} 1-\lambda & 0 & \dots & 0 \\ 0 & 1-\lambda & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1-\lambda \end{vmatrix} = (1-\lambda)^n = 0$$

$$\therefore \lambda = 1.$$

The eigenspace of PP^T corresponding to $\lambda = 1$ is,

$$\begin{bmatrix} 1-1 & 0 & \dots & 0 \\ 0 & 1-1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1-1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}$$

$$\begin{aligned} \rightarrow \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}^T &= \begin{bmatrix} n_1 & n_2 & \dots & n_n \end{bmatrix}^T \\ &= n_1\begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}^T + n_2\begin{bmatrix} 0 & 1 & \dots & 0 \end{bmatrix}^T + \dots + n_n\begin{bmatrix} 0 & 0 & \dots & 1 \end{bmatrix}^T \end{aligned}$$

Thus,

$$B=\left\{\begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}^T + \begin{bmatrix} 0 & 1 & \dots & 0 \end{bmatrix}^T + \dots + \begin{bmatrix} 0 & 0 & \dots & 1 \end{bmatrix}^T\right\} \qquad \square$$

problem 4 Let A be an orthogonal $n \times n$ matrix.

a) Prove that A^{-1} is orthogonal. (Note, if you want to use the fact that A^T is orthogonal as a part of your proof, you must prove that A^T is orthogonal).

b) Prove that $(A\vec{u}) \cdot (A\vec{v}) = \vec{u} \cdot \vec{v}$ for all vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$.

proof (a)

A square matrix is said to be orthogonal if its transpose is the same as its inverse, that is if

$$A^{-1} = A^T$$

thus,

$$\left(A^{-1}\right)^{-1} = A = \left(A^T\right)^T = \left(A^{-1}\right)^T$$

So by definition, A^{-1} is orthogonal.

proof (b)

$$A\vec{u}A\vec{v} = \vec{u}A^T A\vec{v} = \vec{u}I\vec{v} = \vec{u}\vec{v} \quad \square$$

problem 5 Let $A = \begin{bmatrix} -3 & 1 & 2 \\ 1 & -3 & 2 \\ 2 & 2 & 0 \end{bmatrix}$

a) Find the orthogonal matrix P and diagonal matrix D such that $D = P^T AP$

b) Find the spectral decomposition of A

solution (a)

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

$$P = \begin{bmatrix} -\frac{\sqrt{6}}{6} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \\ -\frac{\sqrt{6}}{6} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \\ \frac{\sqrt{6}}{3} & 0 & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$\square$

proof (a) We must find an orthogonal matrix P that diagonalizes

$$A = \begin{bmatrix} -3 & 1 & 2 \\ 1 & -3 & 2 \\ 2 & 2 & 0 \end{bmatrix}$$

$$\begin{aligned} det(A - \lambda I) &= \begin{vmatrix} -3-\lambda & 1 & 2 \\ 1 & -3-\lambda & 2 \\ 2 & 2 & -\lambda \end{vmatrix} \\ &= 2\begin{vmatrix} 1 & 2 \\ -3-\lambda & 2 \end{vmatrix} + (-2)\begin{vmatrix} -3-\lambda & 2 \\ 1 & 2 \end{vmatrix} + (-\lambda)\begin{vmatrix} -3-\lambda & 1 \\ 1 & -3-\lambda \end{vmatrix} \\ &= 2[(1)(2) - (2)(-3-\lambda)] + (-2)[(-3-\lambda)(2) - (2)(1)] + (-\lambda)\left[(-3-\lambda)^2 - (1)\right] \\ &= 2[8+2\lambda] + (-2)[-8-2\lambda] + (-\lambda)\left[\lambda^2 + 6\lambda + 8\right] = (16+4\lambda) + (16+4\lambda) - \lambda^3 - 6\lambda^2 - 8\lambda \\ &= -\lambda^3 - 6\lambda^2 + 32 \end{aligned}$$

$$\begin{array}{r@{}l} & \;\; -1\lambda^2 - 8\lambda - 16 \\ \lambda - 2 \big) & \;\; -\lambda^3 - 6\lambda^2 + 0\lambda + 32 \\ & \underline{-\left(-\lambda^3 + 2\lambda^2\right)} \\ & \;\; -8\lambda^2 \\ & \underline{-\left(-8\lambda^2 + 16\lambda\right)} \\ & \;\; -16\lambda \\ & \underline{-(-16\lambda + 32)} \\ & \;\; 0 \end{array}$$

So, the characteristic equation of A is

$$det(A - \lambda I) = \begin{vmatrix} -3-\lambda & 1 & 2 \\ 1 & -3-\lambda & 2 \\ 2 & 2 & -\lambda \end{vmatrix} = -(\lambda - 2)(\lambda + 4)^2 = 0$$

Thus, the distinct eigenvalues of A are $\lambda = 2$ and $\lambda = -4$. A basis for eigenspace corresponding to $\lambda = 2$,

$$rref\left(\begin{bmatrix} -5 & 1 & 2 \\ 1 & -5 & 2 \\ 2 & 2 & -2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$
$$\rightarrow \vec{v}_1 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}^T$$

normalizing $\vec{v}_1$ gives the following,

$$\begin{aligned} \vec{q}_1 &= \frac{1}{\|\vec{v}_1\|} \cdot \vec{v}_1 \\ &= \frac{1}{\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + (1)^2}} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}^T \\ &= \begin{bmatrix} \frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{3} \end{bmatrix}^T \end{aligned}$$

A bases for the eigenspace corresponding $\lambda = -4$,

$$rref\left(\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$\rightarrow \vec{v}_2 = \begin{bmatrix} -1 & 1 & 0 \end{bmatrix}^T$$
$$\rightarrow \vec{v}_3 = \begin{bmatrix} -2 & 0 & 1 \end{bmatrix}^T$$

we will use gram-schmidt to make vectors $\vec{v}_2$ and $\vec{v}_1$ orthogonal,

$$\begin{aligned} \vec{u}_1 &= \begin{bmatrix} -1 & 1 & 0 \end{bmatrix}^T \\ \vec{u}_2 &= \vec{v}_3 - \frac{\langle \vec{v}_3, \vec{u}_1 \rangle}{\|\vec{u}_1\|^2} \vec{u}_1 \\ \|\vec{u}_1\|^2 &= \left(\sqrt{2}\right)^2 \\ \langle \vec{v}_3, \vec{u}_1 \rangle &= (-2)(-1) + (1)(0) + (1)(0) \\ \therefore \vec{u}_2 &= \vec{v}_3 - \frac{2}{2} \vec{u}_1 \\ &= \begin{bmatrix} -2 & 0 & 1 \end{bmatrix}^T - \begin{bmatrix} -1 & 1 & 0 \end{bmatrix}^T = \begin{bmatrix} -1 & -1 & 1 \end{bmatrix}^T \end{aligned}$$

normalizing $\vec{u}_1$ and $\vec{u}_2$ gives the following,

$$\vec{q}_2 = \frac{1}{\|\vec{u}_1\|} \cdot \vec{u}_1 = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}^T$$
$$\vec{q}_3 = \frac{1}{\|\vec{u}_2\|} \cdot \vec{u}_2 = \begin{bmatrix} -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}^T$$

Finally, using our matrix vectors $\vec{q}_1$, $\vec{q}_2$, $\vec{q}_3$ as column vectors, we obtain

$$P = \begin{bmatrix} \frac{\sqrt{6}}{6} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \\ \frac{\sqrt{6}}{6} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \\ \frac{\sqrt{6}}{3} & 0 & \frac{1}{\sqrt{3}} \end{bmatrix}$$

which orthogonally diagonalizes A. Our matrix D is a diagonal matrix with the eigenvalues in the diagonal positions

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{bmatrix} \qquad \square$$

solution (b)

$$\boxed{\begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{4}{3} \end{bmatrix} + \begin{bmatrix} -2 & 2 & 0 \\ 2 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \frac{-4}{3} & \frac{-4}{3} & \frac{4}{3} \\ \frac{-4}{3} & \frac{-4}{3} & \frac{4}{3} \\ \frac{4}{3} & \frac{4}{3} & \frac{-4}{3} \end{bmatrix} = \begin{bmatrix} -3 & 1 & 2 \\ 1 & -3 & 2 \\ 2 & 2 & 0 \end{bmatrix}}$$

proof (b) A spectral decomposition of $A = \lambda_1 \vec{q}_1 \vec{q}_1^{\,T} + \lambda_2 \vec{q}_2 \vec{q}_2^{\,T} + \lambda_3 \vec{q}_3 \vec{q}_3^{\,T}$,

$$\to \lambda_1 \vec{q}_1 \vec{q}_1^{\,T} = 2 \begin{bmatrix} \frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{3} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{3} \end{bmatrix} = \begin{bmatrix} \frac{2}{6} & \frac{2}{6} & \frac{2}{3} \\ \frac{2}{6} & \frac{2}{6} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{4}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{4}{3} \end{bmatrix}$$

$$\to \lambda_2 \vec{q}_2 \vec{q}_2^{\,T} = -4 \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} = \begin{bmatrix} \frac{-4}{2} & \frac{4}{2} & 0 \\ \frac{4}{2} & \frac{-4}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 2 & 0 \\ 2 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\to \lambda_3 \vec{q}_3 \vec{q}_3^{\,T} = -4 \begin{bmatrix} -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} \frac{-4}{3} & \frac{-4}{3} & \frac{4}{3} \\ \frac{-4}{3} & \frac{-4}{3} & \frac{4}{3} \\ \frac{4}{3} & \frac{4}{3} & \frac{-4}{3} \end{bmatrix}$$

Therefore,

$$\begin{aligned} A &= \lambda_1 \vec{q}_1 \vec{q}_1^{\,T} + \lambda_2 \vec{q}_2 \vec{q}_2^{\,T} + \lambda_3 \vec{q}_3 \vec{q}_3^{\,T} \\ &= \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{4}{3} \end{bmatrix} + \begin{bmatrix} -2 & 2 & 0 \\ 2 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \frac{-4}{3} & \frac{-4}{3} & \frac{4}{3} \\ \frac{-4}{3} & \frac{-4}{3} & \frac{4}{3} \\ \frac{4}{3} & \frac{4}{3} & \frac{-4}{3} \end{bmatrix} = \begin{bmatrix} -3 & 1 & 2 \\ 1 & -3 & 2 \\ 2 & 2 & 0 \end{bmatrix} \end{aligned} \qquad \square$$