Math 2311 – Assignment 1

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1. Determine if each of the following sets is a vector space.

   (a) $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid x \geq y \right\}$ with the usual scalar multiplication and vector addition from $\mathbb{R}^2$

Answer: No, V is not a vector space.

Proof. let $(x_0, y_0), (x_1, y_1) \in V$, and take the vector space operations on V to be the usual operations of *vector* addition and *scalar* multiplication; that is,

$$(x_0, y_0) + (x_1, y_1) = (x_0 + x_1, y_0 + y_1) \tag{1}$$

$$k(x_0, y_0) = (kx_0, ky_0) \tag{2}$$

V is closed under scalar addition since $x_0 + x_1 \geq y_0 + y_1$

However, by properties of inequalities if the constant, k, is negative, we must reverse the symbol to preserve the inequality relation.

Given that k is negative, $x \geq y \rightarrow kx \leq ky$ $\square$

(b) Consider the set $W = \{f \in F(-\infty, \infty) \,|\, f(1) = 0\}$ with the usual scalar multiplication and vector addition from $F(-\infty, \infty)$. Is W a vector space?

Answer: Yes, W is a vector space.

Proof. Since we know that $F(-\infty, \infty)$ (with the usual operations) is a vector space, and since W is a subset of $F(-\infty, \infty)$ (with the same operations), it suffices to prove that W is a subspace of $F(-\infty, \infty)$. To this end we must show three things:

(a) That W is non-empty.

(b) That W is closed under addition.

(c) That W is closed under scalar multiplication.

There exists a function $\mathbf{0}$ in $F(-\infty, \infty)$ defined by $\mathbf{0}(x) = 0$ for all x. Clearly $\mathbf{0}(1) = f(1) = 0$ so W is non-empty.

Now suppose f and g are two functions in W. We must show that $f + g$ is in W.

$$\begin{aligned} (f+g)(1) &= f(1) + g(1) && \text{(definition of addition of functions)} \\ &= 0 && (f \text{ and } g \text{ are in } W) \end{aligned}$$

Finally, to show that W is closed under scalar multiplication, suppose f is in W and k is a scalar, then

$$\begin{aligned} (kf)(1) &= kf(1) && \text{(definition of scalar multiplication on functions)} \\ &= 0 && (f \text{ is in } W) \end{aligned}$$

so (kf) is in W and W is closed under scalar multiplication.

Therefore W is a subspace of $F(-\infty, \infty)$ and hence is a vector space. □

2. Let V be a vector space.

   (a) If k is any scalar, prove that $k\vec{0} = \vec{0}$.

Proof.

$$\begin{aligned}
k(\vec{0}+\vec{0}) &= k\vec{0}+k\vec{0} && \text{(vector space axiom 7)}\\
k\vec{0} &= k\vec{0}+k\vec{0} && \text{(vector space axiom 4)}\\
k\vec{0}+(-k\vec{0}) &= (-k\vec{0})+(k\vec{0}+k\vec{0}) && \text{(vector space axiom 5)}\\
k\vec{0}+(-k\vec{0}) &= ((-k\vec{0})+k\vec{0})+k\vec{0} && \text{(vector space axiom 3)}\\
\vec{0} &= \vec{0}+k\vec{0} && \text{(vector space axiom 5)}\\
&= k\vec{0} && \text{(vector space axiom 4)}
\end{aligned}$$

□

   (b) Prove that the zero vector in V is unique.

Proof. We must show that there is only one vector, $\vec{0}$, with the property that $\vec{0}+\vec{v} = \vec{v}+\vec{0} = \vec{v}$.

Suppose $\vec{0_1}$ and $\vec{0}$ are zero vectors in V Then

$$\begin{aligned}
\vec{0_1} &= \vec{0_1}+\vec{0} && \text{(vector space axiom 4)}\\
&= \vec{0}+\vec{0_1} && \text{(vector space axiom 2)}\\
&= \vec{0} && \text{(vector space axiom 4)}
\end{aligned}$$

Therefore $\vec{0_1} = \vec{0}$. So, the zero vector is unique. □

3. Determine if each of the following are subspaces of M_{nn}

   (a) $\{A \in M_{nn} \mid det(A) = 0\}$

Answer: No, $\{A \in M_{nn} \mid \det(A) = 0\}$ is not a subspace of M_{nn}.

Proof.

$$\det\begin{vmatrix}1 & 0\\ 0 & 0\end{vmatrix} = 0, \det\begin{vmatrix}0 & 0\\ 0 & 1\end{vmatrix} = 0$$

$$\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} + \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$\det\begin{vmatrix}1 & 0\\ 0 & 1\end{vmatrix} = 1 \neq 0$$

□

(b) $\{A \in M_{nn} \, |tr(A) = 0\}$

Answer: Yes, $W = \{A \in M_{nn} \, |tr(A) = 0\}$ is a subspace of M_{nn}.

Proof. Let $[B = (b_{ii})] \in W$ be a square matrix of order n such that $tr(B) = 0$ and let k be any scalar.

The set W is non empty because if we let $a_{ii} = 0$ for all i then $tr(A) = 0$ therefore W contains the **0** matrices. It remains to show that W is closed under addition and scalar multiplication.

$$
\begin{aligned}
&addition: \\
tr(A+B) &= \sum_{i=1}^{n}(a_{ii} + b_{ii}) \\
&= \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii} \\
&= tr(A) + tr(B) \\
&= 0 + 0 \\
&= 0 \\
multiplication: & \\
tr(kA) &= \sum_{i=1}^{n}(k \cdot a_{ii}) \\
&= k \cdot \sum_{i=1}^{n} a_{ii} \\
&= k \cdot tr(A) \\
&= k \cdot 0 \\
&= 0
\end{aligned}
$$

To be clear, if we take some $C = A + B$ such that A and B are in W, then for all $C = (c_{ii})$, the sum will be 0, so C is also in W. Therefore W is a subspace of M_{nn}. □

(c) $\left\{A \in M_{nn} \mid A^T = A\right\}$

Answer: Yes, $W = \left\{A \in M_{nn} \mid A^T = A\right\}$ is a subspace of M_{nn}.

Proof. Let $[B = (b_{ij})] \in W$ be a square matrix of order n such that $b_{ij} = b_{ji}$, and let k be any scalar.

The set W is non empty because, if we let $[A = (a_{ij})] = 0$ for all (i, j) then W contains the $\mathbf{0}$ matrices. It remains to show that W is closed under addition and scalar multiplication.

$$\begin{aligned}
\mathit{addition} :& \\
C &= (A + B)^T \\
&= A^T + B^T \\
&= A + B \\
\mathit{multiplication} :& \\
(k \cdot A)^T &= k \cdot A^T \\
&= k \cdot A
\end{aligned}$$

so W is a subspace of M_{nn}. □

4. Consider the following vectors in P_2: $p_1 = 2+x+4x^2$, $p_2 = 1-x+3x^2$, $p_3 = 3+2x+5x^2$.

(a) Express the vector $g = 6 + 11x + 6x^2$ as a linear combination of p_1, p_2, p_3.

Answer: Yes, (p_1, p_2, p_3) **= (4, -5, 1)**

Proof.

$$\begin{aligned}(6 + 11x + 6x^2) &= k_0(2 + x + 4x^2) + k_1(1 - x + 3x^2) + k_2(3 + 2x + 5x^2)\\ &= (k_02 + k_1 + k_23) + (k_0x - k_1x + k_22x) + (k_04x^2 + k_13x^2 + k_25x^2)\\ &= (k_02 + k_1 + k_23) + (k_0 - k_1 + k_22)x + (k_04 + k_13 + k_25)x^2\end{aligned}$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 6 \\ 1 & -1 & 2 & 11 \\ 4 & 3 & 5 & 6 \end{array}\right]$$

$$[-2r2 + r1] \wedge [-4r2 + r1] \left[\begin{array}{ccc|c} 0 & 3 & -1 & -16 \\ 1 & -1 & 2 & 11 \\ 0 & 7 & -3 & -38 \end{array}\right]$$

$$r2 \leftrightarrow r1 \left[\begin{array}{ccc|c} 1 & -1 & 2 & 11 \\ 0 & 3 & -1 & -16 \\ 0 & 7 & -3 & -38 \end{array}\right]$$

$$\frac{1}{3}r2 \left[\begin{array}{ccc|c} 1 & -1 & 2 & 11 \\ 0 & 1 & \frac{-1}{3} & \frac{-16}{3} \\ 0 & 7 & -3 & -38 \end{array}\right]$$

$$-7r2 + r3 \left[\begin{array}{ccc|c} 1 & -1 & 2 & 11 \\ 0 & 1 & \frac{-1}{3} & \frac{-16}{3} \\ 0 & 0 & \frac{-2}{3} & \frac{-2}{3} \end{array}\right]$$

$$-\frac{3}{2}r3 \left[\begin{array}{ccc|c} 1 & -1 & 2 & 11 \\ 0 & 1 & \frac{-1}{3} & \frac{-16}{3} \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\frac{1}{3}r3 + r2 \left[\begin{array}{ccc|c} 1 & -1 & 2 & 11 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$r1 + r2 \left[\begin{array}{ccc|c} 1 & 0 & 2 & 6 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$r1 + r3 \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\therefore (k_1, k_2, k_3) = (4, -5, 1)$$

$\square$

(b) Does $\{p_1, p_2, p_3\}$ span P_2?

Answer: Yes, $span(\{\vec{p_1}, \vec{p_2}, \vec{p_3}\}) = P_2$

Proof. An arbitrary vector in P_2 is of the form $\vec{p} = a + bx + cx^2$ and so becomes,

$k_0(2 + x + 4x^2) + k_1(1 - x + 3x^2) + k_2(3 + 2x + 5x^2) = a + bx + cx^2$

which we can rewrite as

$(k_0 2 + k_1 + k_2 3) + (k_0 - k_1 + k_2 2)x + (k_0 4 + k_1 3 + k_2 5)x^2 = a + bx + cx^2$

Equating corresponding coefficients yields a linear system whose augmented matrix is

$$A = \left[\begin{array}{ccc|c} 2 & 1 & 3 & a \\ 1 & -1 & 2 & b \\ 4 & 3 & 5 & c \end{array}\right]$$

Our problem reduces to ascertaining whether this system is consistent for all values of a, b, and c. This can be determined if its coefficient matrix has a nonzero determinant, from our theorem for equivalent statements. If A is an $n\ x\ n$ matrix such that $\det(A) \neq 0$ then $A\vec{x} = \vec{0}$.

It follows from solution (a) that

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 0 \\ 1 & -1 & 2 & 0 \\ 4 & 3 & 5 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

so A is consistent for every choice $a, b,$ and c. Thus, the vectors in $\{\vec{p_1}, \vec{p_2}, \vec{p_3}\}$ span P_2. $\square$

(c) Is $\{p_1, p_2, p_3\}$ linearly independent?

Answer: Yes, $k_1 p\vec{1} + k_2 p\vec{2} + k_3 p\vec{3} = \vec{0}$

Proof. From part (b) we have

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 0 \\ 1 & -1 & 2 & 0 \\ 4 & 3 & 5 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Therefore $(k_1, k_2, k_3) = (0, 0, 0)$ so $\{\vec{p_1}, \vec{p_2}, \vec{p_3}\}$ is linearly independent $\square$

5. Consider the following planes in $\mathbb{R}^3$. $P_1 : 2x + 3y - z = 0$ and $P_2 : x + 2y - 2z = 0$

(a) Find a set of vectors that spans P_1.

Answer: any 2 **vectors from the set** $\{(0,1,3),(-1,0,2),(-3,2,0)\}$

Proof.

$$\text{Let } \vec{e_1} = (1,0,0),\ \vec{e_2} = (0,1,0)\ \vec{e_3} = (0,0,1)$$

be unit vectors in R^3, we will compute the cross products

$$\vec{e_1} \times (2,3,-1),\ \vec{e_2} \times (2,3,-1),\ \vec{e_3} \times (2,3,-1)$$

to find a set of vectors that spans P_1

$$\begin{aligned}(1,0,0) \times (2,3,-1) &= ((0)(-1) - (0)(3), (0)(2) - (1)(-1), (1)(3) - (0)(2)) \\ &= (0,1,3)\end{aligned}$$

$$\begin{aligned}(0,1,0) \times (2,3,-1) &= ((1)(-1) - (0)(3), (0)(2) - (0)(-1), (0)(3) - (1)(2)) \\ &= (-1,0,-2)\end{aligned}$$

$$\begin{aligned}(0,0,1) \times (2,3,-1) &= ((0)(-1) - (1)(3), (1)(2) - (0)(-1), (0)(3) - (0)(2)) \\ &= (-3,2,0)\end{aligned}$$

any 2 vectors from the set $\{(0,1,3),(-1,0,2),(-3,2,0)\}$ will span the plane P_1. □

(b) Find a set of vectors that spans P_2.

Answer: any 2 **vectors from the set** $\{(0,2,2),(-2,0,-1),(-2,1,0)\}$

Proof.

$$\text{Let } \vec{e_1} = (1,0,0),\ \vec{e_2} = (0,1,0)\ \vec{e_3} = (0,0,1)$$

be unit vectors in R^3, we will compute the cross products

$$\vec{e_1} \times (1,2,-2),\ \vec{e_2} \times (1,2,-2),\ \vec{e_3} \times (1,2,-2)$$

to find a set of vectors that spans P_2

$$\begin{aligned}(1,0,0) \times (1,2,-2) &= ((0)(-2) - (0)(2), (0)(1) - (1)(-2), (1)(2) - (0)(1)) \\ &= (0,2,2)\end{aligned}$$

$$\begin{aligned}(0,1,0) \times (1,2,-2) &= ((1)(-2) - (0)(2), (0)(1) - (0)(-2), (0)(3) - (1)(1)) \\ &= (-2,0,-1)\end{aligned}$$

$$\begin{aligned}(0,0,1) \times (1,2,-2) &= ((0)(-2) - (1)(2), (1)(1) - (0)(-2), (0)(3) - (0)(1)) \\ &= (-2,1,0)\end{aligned}$$

any 2 vectors from the set $\{(0,2,2),(-2,0,-1),(-2,1,0)\}$ will span the plane P_2. □

(c) Find a set of vectors that spans the intersection of P_1 and P_2. (Recall that we showed the intersection of two subspaces is a subspace).

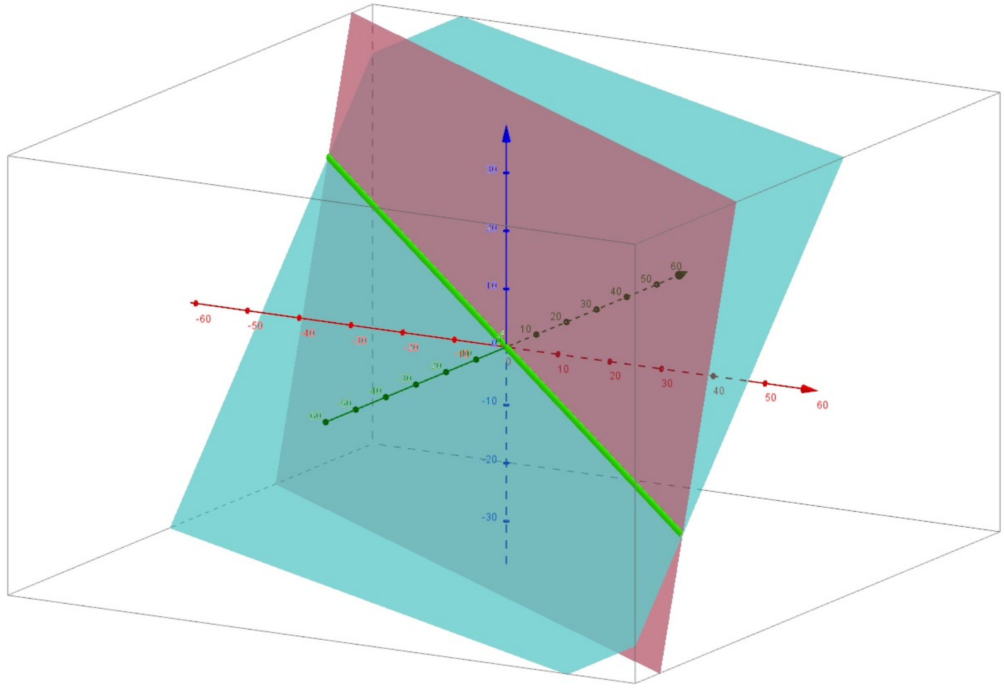

Figure 1: $p_1 \bigcap p_2$ `https://www.geogebra.org/3d/kuawemzp`