1 Find a basis for the subspace of P^3 spanned by the following polynomials (vectors):

$p_1 = 1 + x + 3x^2 + 4x^3,\ p_2 = 1 + 2x^2 + 3x^3,\ p_3 = x + x^2 + 2x^3,\ p_4 = 1 + x + 3x^2 + 5x^3$

$\boxed{\text{Solution: the vectors } \vec{p_1}, \vec{p_2}, \vec{p_3} \text{ form a basis for } \operatorname{span}\{p_1, p_2, p_3, p_4\}}$

Proof. The equation $k_1\vec{p_1} + k_2\vec{p_2} + k_3\vec{p_3} + k_4\vec{p_4} = \vec{0}$ can be written as a linear system

$$\begin{aligned}
1k_1 + 1k_2 + 0k_3 + 1k_4 &= 0\\
1k_1 + 0k_2 + 1k_3 + 1k_4 &= 0\\
3k_1 + 2k_2 + 1k_3 + 3k_4 &= 0\\
4k_1 + 3k_2 + 2k_3 + 5k_4 &= 0
\end{aligned}$$

whose augmented matrix has the reduced row echelon form

$$\left[\begin{array}{cccc|c}
1 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 1 & 0\\
0 & 0 & 1 & 1 & 0\\
0 & 0 & 0 & 0 & 0
\end{array}\right]$$

$$\implies k1 = 0,\ k2 = -s,\ k3 = -s,\ k4 = s$$

removing vector $\vec{p_4}$ gives $\operatorname{span}\{p_1, p_2, p_3\} = \operatorname{span}\{p_1, p_2, p_3, p_4\}$. Since the vector equation $k_1\vec{p_1} + k_2\vec{p_2} + k_3\vec{p_3} = \vec{0}$ has only the trivial solution. We conclude that the vectors $\vec{p_1}, \vec{p_2}, \vec{p_3}$ form a basis for $\operatorname{span}\{p_1, p_2, p_3, p_4\}$. $\square$