Math 2311 – Assignment 1

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1 Question 1

1. Determine if each of the following sets is a vector space.

1.1 Answer: No, V is not a vector space.

(a) $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid x \geq y \right\}$ with the usual scalar multiplication and vector addition from $\mathbb{R}^2$

Proof. let $(x_0, y_0), (x_1, y_1) \in V$, and take the vector space operations on V to be the usual operations of *vector* addition and *scalar* multiplication; that is,

$$(x_0, y_0) + (x_1, y_1) = (x_0 + x_1, y_0 + y_1) \tag{1}$$
$$k(x_0, y_0) = (kx_0, ky_0) \tag{2}$$

V is closed under scalar addition since $x_0 + x_1 \geq y_0 + y_1$

However, by properties of inequalities if the constant, k, is negative, we must reverse the symbol to preserve the inequality relation.

Given that k is negative, $x \geq y \rightarrow kx \leq ky$ □

1.2 Answer: Yes, W is a vector space.

(b) Consider the set $W = \{f \in F(-\infty, \infty) \mid f(1) = 0\}$ with the usual scalar multiplication and vector addition from $F(-\infty, \infty)$. Is W a vector space?

Proof. Since we know that $F(-\infty, \infty)$ (with the usual operations) is a vector space, and since W is a subset of $F(-\infty, \infty)$ (with the same operations), it suffices to prove that W is a subspace of $F(-\infty, \infty)$. To this end we must show three things:

(a) That W is non-empty.

(b) That W is closed under addition.

(c) That W is closed under scalar multiplication.

There exists a function $\mathbf{0}$ in $F(-\infty, \infty)$ defined by $\mathbf{0}(x) = 0$ for all x. Clearly $\mathbf{0}(1) = f(1) = 0$ so W is non-empty.

Now suppose f and g are two functions in W. We must show that $f + g$ is in W.

$$\begin{aligned} (f+g)(1) &= f(1) + g(1) && \text{(definition of addition of functions)} \\ &= 0 && (f \text{ and } g \text{ are in } W) \end{aligned}$$

Finally, to show that W is closed under scalar multiplication, suppose f is in W and k is a scalar, then

$$\begin{aligned} (kf)(1) &= kf(1) && \text{(definition of scalar multiplication on functions)} \\ &= 0 && (f \text{ is in } W) \end{aligned}$$

so (kf) is in W and W is closed under scalar multiplication.

Therefore W is a subspace of $F(-\infty, \infty)$ and hence is a vector space. □

2 Question 2

2. Let V be a vector space.

2.1

(a) If k is any scalar, prove that $k\vec{0} = \vec{0}$.

Proof.

$$
\begin{aligned}
k(\vec{0}+\vec{0}) &= k\vec{0}+k\vec{0} && \text{(vector space axiom 7)}\\
k\vec{0} &= k\vec{0}+k\vec{0} && \text{(vector space axiom 4)}\\
k\vec{0}+(-k\vec{0}) &= (-k\vec{0})+(k\vec{0}+k\vec{0}) && \text{(vector space axiom 5)}\\
k\vec{0}+(-k\vec{0}) &= ((-k\vec{0})+k\vec{0})+k\vec{0} && \text{(vector space axiom 3)}\\
\vec{0} &= \vec{0}+k\vec{0} && \text{(vector space axiom 5)}\\
&= k\vec{0} && \text{(vector space axiom 4)}
\end{aligned}
$$

□

2.2

(b) Prove that the zero vector in V is unique.

Proof. We must show that there is only one vector, $\vec{0}$, with the property that $\vec{0}+\vec{v} = \vec{v}+\vec{0} = \vec{v}$.

Suppose $\vec{0_1}$ and $\vec{0}$ are zero vectors in V Then

$$
\begin{aligned}
\vec{0_1} &= \vec{0_1}+\vec{0} && \text{(vector space axiom 4)}\\
&= \vec{0}+\vec{0_1} && \text{(vector space axiom 2)}\\
&= \vec{0} && \text{(vector space axiom 4)}
\end{aligned}
$$

Therefore $\vec{0_1} = \vec{0}$. So, the zero vector is unique. □

3 Question 3

3. Determine if each of the following are subspaces of M_{nn}

3.1 Answer: No, $A \in M_{nn}$ is not a subspace of M_{nn}.

(a) $\{A \in M_{nn} \mid det(A) = 0\}$

Proof.

$$\det \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0, \det \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\det \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$$

□

3.2 Answer: Yes, W is a subspace of M_{nn}

(b) $\{A \in M_{nn} \mid tr(A) = 0\}$

Proof. Let $[B = (b_{ii})] \in W$ be a square matrix of order n such that $tr(B) = 0$ and let k be any scalar.

The set W is non empty because if we let $a_{ii} = 0$ for all i then $tr(A) = 0$ therefore W contains the $\mathbf{0}$ matrices.

$$\begin{aligned} &addition: \\ tr(A+B) &= \sum_{i=1}^{n}(a_{ii} + b_{ii}) \\ &= \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii} \\ &= tr(A) + tr(B) = 0 + 0 = 0 \\ multiplication: & \\ tr(kA) &= \sum_{i=1}^{n}(k \cdot a_{ii}) \\ &= k \cdot \sum_{i=1}^{n} a_{ii} \\ &= k \cdot tr(A) = k \cdot 0 = 0 \end{aligned}$$

To be clear, if we take some $C = A + B$ such that A and B are in W, then for all $C = (c_{ii})$, the sum will be 0, so C is also in W. Therefore W is a subspace of M_{nn}. □

3.3 Answer: Yes, W is a subspace of M_{nn}

(c) $\{A \in M_{nn} \mid A^T = A\}$

Answer: Yes, $W = \{A \in M_{nn} \mid A^T = A\}$ is a subspace of M_{nn}.

Proof. Let $[B = (b_{ij})] \in W$ be a square matrix of order n such that $b_{ij} = b_{ji}$, and let k be any scalar.

The set W is non empty because, if we let $[A = (a_{ij})] = 0$ for all (i, j) then W contains the $\mathbf{0}$ matrices. It remains to show that W is closed under addition and scalar multiplication.

$$\begin{aligned} &\textit{addition}: \\ C &= (A+B)^T \\ &= A^T + B^T \\ &= A + B \\ &\textit{multiplication}: \\ (k \cdot A)^T &= k \cdot A^T \\ &= k \cdot A \end{aligned}$$

so W is a subspace of M_{nn}. $\square$

4 Question 4

4. Consider the following vectors in P_2: $p_1 = 2+x+4x^2$, $p_2 = 1-x+3x^2$, $p_3 = 3+2x+5x^2$.

4.1 Answer: Yes, $g = 4(2+x+4x^2) + -5(1-x+3x^2) + 1(3+2x+5x^2)$

(a) Express the vector $g = 6+11x+6x^2$ as a linear combination of p_1, p_2, p_3.

Proof.

$$\begin{aligned}(6+11x+6x^2) &= k_0(2+x+4x^2) + k_1(1-x+3x^2) + k_2(3+2x+5x^2)\\ &= (k_0 2 + k_1 + k_2 3) + (k_0 x - k_1 x + k_2 2x) + (k_0 4x^2 + k_1 3x^2 + k_2 5x^2)\\ &= (k_0 2 + k_1 + k_2 3) + (k_0 - k_1 + k_2 2)x + (k_0 4 + k_1 3 + k_2 5)x^2\end{aligned}$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 6\\ 1 & -1 & 2 & 11\\ 4 & 3 & 5 & 6\end{array}\right]$$

$$[-2r2+r1] \wedge [-4r2+r1] \left[\begin{array}{ccc|c} 0 & 3 & -1 & -16\\ 1 & -1 & 2 & 11\\ 0 & 7 & -3 & -38\end{array}\right]$$

$$r2 \leftrightarrow r1 \left[\begin{array}{ccc|c} 1 & -1 & 2 & 11\\ 0 & 3 & -1 & -16\\ 0 & 7 & -3 & -38\end{array}\right]$$

$$\frac{1}{3}r2 \left[\begin{array}{ccc|c} 1 & -1 & 2 & 11\\ 0 & 1 & \frac{-1}{3} & \frac{-16}{3}\\ 0 & 7 & -3 & -38\end{array}\right]$$

$$-7r2+r3 \left[\begin{array}{ccc|c} 1 & -1 & 2 & 11\\ 0 & 1 & \frac{-1}{3} & \frac{-16}{3}\\ 0 & 0 & \frac{-2}{3} & \frac{-2}{3}\end{array}\right]$$

$$-\frac{3}{2}r3 \left[\begin{array}{ccc|c} 1 & -1 & 2 & 11\\ 0 & 1 & \frac{-1}{3} & \frac{-16}{3}\\ 0 & 0 & 1 & 1\end{array}\right]$$

$$\frac{1}{3}r3+r2 \left[\begin{array}{ccc|c} 1 & -1 & 2 & 11\\ 0 & 1 & 0 & -5\\ 0 & 0 & 1 & 1\end{array}\right]$$

$$r1+r2 \left[\begin{array}{ccc|c} 1 & 0 & 2 & 6\\ 0 & 1 & 0 & -5\\ 0 & 0 & 1 & 1\end{array}\right]$$

$$r1+r3 \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4\\ 0 & 1 & 0 & -5\\ 0 & 0 & 1 & 1\end{array}\right]$$

$$\therefore (k_0, k_1, k_2) = (4, -5, 1)$$

$\square$

4.2 Answer: Yes, $span(\{\vec{p_1}, \vec{p_2}, \vec{p_3}\}) = P_2$

(b) Does $\{p_1, p_2, p_3\}$ span P_2?

Proof. An arbitrary vector in P_2 is of the form $\vec{p} = a + bx + cx^2$ and so becomes,

$k_0(2 + x + 4x^2) + k_1(1 - x + 3x^2) + k_2(3 + 2x + 5x^2) = a + bx + cx^2$

which we can rewrite as

$(k_0 2 + k_1 + k_2 3) + (k_0 - k_1 + k_2 2)x + (k_0 4 + k_1 3 + k_2 5)x^2 = a + bx + cx^2$

Equating corresponding coefficients yields a linear system whose augmented matrix is

$$A = \left[\begin{array}{ccc|c} 2 & 1 & 3 & a \\ 1 & -1 & 2 & b \\ 4 & 3 & 5 & c \end{array}\right]$$

Our problem reduces to ascertaining whether this system is consistent for all values of a, b, and c. This can be determined if its coefficient matrix has a nonzero determinant, from our theorem for equivalent statements. If A is an $n\ x\ n$ matrix such that $\det(A) \neq 0$ then $A\vec{x} = \vec{0}$ has only the trivial solution.

It follows from solution (a) that

$$A = \left[\begin{array}{ccc|c} 2 & 1 & 3 & 0 \\ 1 & -1 & 2 & 0 \\ 4 & 3 & 5 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

so A is consistent for every choice $a, b,$ and c. Thus, the vectors in $\{\vec{p_1}, \vec{p_2}, \vec{p_3}\}$ span P_2. $\square$

4.3 Answer: Yes, $\{p_1, p_2, p_3\}$ is linearly independent

(c) Is $\{p_1, p_2, p_3\}$ linearly independent? From part (b) we know this set is linearly independent

Proof. The nonempty set $\{\vec{p_1}, \vec{p_2}, \vec{p_3}\}$ in a vector space V is linearly independent if and only if the coefficients satisfying

$$k_0\vec{p_1} + k_1\vec{p_2} + k_2\vec{p_3} = \vec{0}$$

are $k_0 = 0,\ k_1 = 0,\ k_2 = 0.$

From our theorem for equivalent statements. If A is an $n\ x\ n$ matrix such that $\det(A) \neq 0$ then $A\vec{x} = \vec{0}$ has only the trivial solution. We will show $\det(A) \neq 0$

$$\det \begin{vmatrix} 2 & 1 & 3 \\ 1 & -1 & 2 \\ 4 & 3 & 5 \end{vmatrix}$$

minor entry $\wedge$ cofactor

$$a_{11}C_{11} = (-1)^{1+1}(2) \cdot [(-1)(5) - (2)(3)] = -22$$
$$a_{12}C_{12} = (-1)^{1+2}(1) \cdot [(1)(5) - (2)(4)] = 3$$
$$a_{13}C_{13} = (-1)^{1+3}(3) \cdot [(1)(3) - (-1)(4)] = 21$$

cofactor expansion

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = 2$$

Therefore $(k_0, k_1, k_2) = (0, 0, 0)$ so $\{\vec{p_1}, \vec{p_2}, \vec{p_3}\}$ is linearly independent $\square$

5 Question 5

5. Consider the following planes in $\mathbb{R}^3$. $P_1 : 2x + 3y - z = 0$ and $P_2 : x + 2y - 2z = 0$

5.1 Answer: $\{(-\frac{3}{2}, 1, 0), (\frac{1}{2}, 0, 1)\}$ spans the plane P_1

(a) Find a set of vectors that spans P_1.

Proof. Solve the following system $(2, 3, -1)^T = \vec{0}$

$$\begin{aligned} &\begin{bmatrix} 2 & 3 & -1 & | & 0 \end{bmatrix} \\ \frac{1}{2}r1 &\begin{bmatrix} 1 & \frac{3}{2} & \frac{-1}{2} & | & 0 \end{bmatrix} \\ \therefore (z = r),\ (y = q),\ (x &= -\frac{3}{2}q + \frac{1}{2}r) \end{aligned}$$

giving the following set of vectors.

$$\begin{aligned} (x, y, z) &= (-\frac{3}{2}q + \frac{1}{2}r, q, r) \\ &= q(-\frac{3}{2}, 1, 0) + r(\frac{1}{2}, 0, 1) \end{aligned}$$

a set of vectors that spans P_1 are $\{(-\frac{3}{2}, 1, 0), (\frac{1}{2}, 0, 1)\}$ □

5.2 Answer: $\{(0, 2, 2), (-2, 0, -1)\}$ spans the plane P_2

(b) Find a set of vectors that spans P_2.

Proof.

$$\text{Let } \vec{e_1} = (1, 0, 0),\ \vec{e_2} = (0, 1, 0)\ \vec{e_3} = (0, 0, 1)$$

be unit vectors in R^3, we will compute the cross products

$$\vec{e_1} \times (1, 2, -2),\ \vec{e_2} \times (1, 2, -2),\ \vec{e_3} \times (1, 2, -2)$$

to find a set of vectors that spans P_2

$$\begin{aligned} (1, 0, 0) \times (1, 2, -2) &= ((0)(-2) - (0)(2), (0)(1) - (1)(-2), (1)(2) - (0)(1)) \\ &= (0, 2, 2) \end{aligned}$$

$$\begin{aligned} (0, 1, 0) \times (1, 2, -2) &= ((1)(-2) - (0)(2), (0)(1) - (0)(-2), (0)(3) - (1)(1)) \\ &= (-2, 0, -1) \end{aligned}$$

$$\begin{aligned} (0, 0, 1) \times (1, 2, -2) &= ((0)(-2) - (1)(2), (1)(1) - (0)(-2), (0)(3) - (0)(1)) \\ &= (-2, 1, 0) \end{aligned}$$

any 2 vectors from the set $\{(0, 2, 2), (-2, 0, -1), (-2, 1, 0)\}$ will span the plane P_2. □

5.3 Answer: $\{(-4,3,1)\}$ spans $P_1 \bigcap P_2$

(c) Find a set of vectors that spans the intersection of P_1 and P_2.

Proof. We will find the span of the solution space by solving a system of equations

$$\begin{aligned} P_1 &: 2x + 3y - z = 0 \\ P_2 &: x + 2y - 2z = 0 \end{aligned}$$

$$\begin{aligned}
&\left[\begin{array}{ccc|c} 2 & 3 & -1 & 0 \\ 1 & 2 & -2 & 0 \end{array}\right] \\
[r1] \leftrightarrow [r2] &\left[\begin{array}{ccc|c} 2 & 3 & -1 & 0 \\ 1 & 2 & -2 & 0 \end{array}\right] \\
(-2)r1 + r2 &\left[\begin{array}{ccc|c} 1 & 2 & -2 & 0 \\ 0 & -1 & 3 & 0 \end{array}\right] \\
2(r2) + r1 &\left[\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & -1 & 3 & 0 \end{array}\right] \\
\therefore (z = t),\ (y = 3t),\ (x = -4t)&
\end{aligned}$$

giving the following vector.

$$(x, y, z) = (-4t, 3t, t) = t(-4, 3, 1)$$

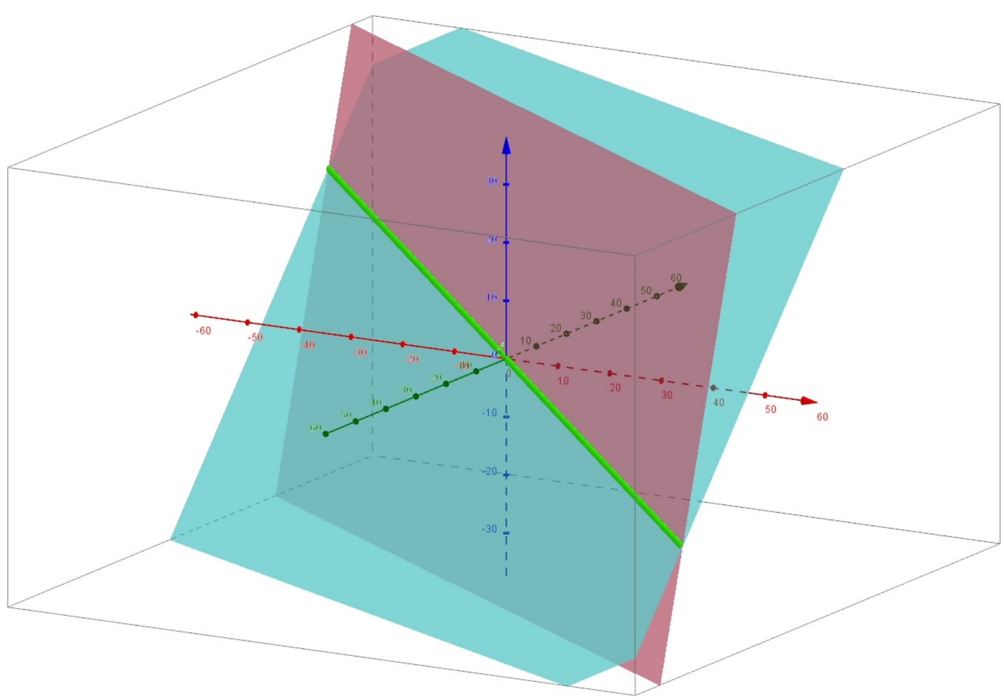

Figure 1: the set $\{(-4,3,1)\}$ spans $P_1 \bigcap P_2$

$\square$