

Math 2311 — Assignment 1

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1 Determine if each of the following sets is a vector space.

1.a Question: $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid x \geq y \right\}$ with the usual scalar multiplication and vector addition from \mathbb{R}^2

1.a Answer: No, V is not a vector space.

Proof. Counter example Axiom 5 fails.

$$\begin{aligned} (3, 2) &\in S : 3 \geq 2 \\ (-3, -2) &\notin S : -3 < -2 \end{aligned}$$

For V to be a vector space, all 10 of our vector space axioms must hold; this means it is enough to demonstrate one axiom fails; we have shown $\exists \vec{u} \in V : -\vec{u} \notin V$, therefore, V is not a vector space. \square

1.b Question: Consider the set $W = \{f \in F(-\infty, \infty) \mid f(1) = 0\}$ with the usual scalar multiplication and vector addition from $F(-\infty, \infty)$. Is W a vector space?

1.b Answer: Yes, W is a vector space.

Proof. Since we know that $F(-\infty, \infty)$ (with the usual operations) is a vector space, and since W is a subset of $F(-\infty, \infty)$ (with the same operations), it suffices to prove that W is a subspace of $F(-\infty, \infty)$. To this end we must show three things.

(1) Prove that W is non-empty.

Clearly the zero function $(\mathbf{0})(1) = f(1) = 0 \therefore W$ is non-empty.

(2) Prove that W is closed under addition.

Let $g \in W$. We must show that $f + g \in W$.

$$\begin{aligned} (f + g)(1) &= f(1) + g(1) && \text{(definition of addition of functions)} \\ &= 0 && (f \text{ and } g \text{ are in } W) \end{aligned}$$

(3) Prove that W is closed under scalar multiplication.

Let k be any scalar.

$$\begin{aligned} (kf)(1) &= kf(1) && \text{(definition of scalar multiplication on functions)} \\ &= 0 && (f \text{ is in } W) \end{aligned}$$

so W is closed under scalar multiplication.

Therefore W is a subspace of $F(-\infty, \infty)$ and hence is a vector space. \square

2 Let V be a vector space.

2.a Question: If k is any scalar, prove that $k\vec{0} = \vec{0}$.

2.a

Proof.

$$\begin{aligned}
 k\vec{0} &= k(\vec{0} + \vec{0}) && (\vec{0} = \vec{0} + \vec{0} \text{ by axiom 4}) \\
 &= k\vec{0} + k\vec{0} && (\text{by axiom 7}) \\
 k\vec{0} + (-k\vec{0}) &= [k\vec{0} + k\vec{0}] + (-k\vec{0}) && (\text{by axiom 5 } k\vec{0} \text{ has a negative}) \\
 k\vec{0} + (-k\vec{0}) &= k\vec{0} + [k\vec{0} + (-k\vec{0})] && (\text{by axiom 3}) \\
 \vec{0} &= \vec{0} + k\vec{0} && (\text{by axiom 5}) \\
 \vec{0} &= k\vec{0} && (\text{by axiom 4})
 \end{aligned}$$

□

2.b Question: Prove that the zero vector in V is unique.

2.b

Proof. We must show that there is only one vector, $\vec{0}$, with the property that $\vec{0} + \vec{v} = \vec{v} + \vec{0} = \vec{v}$.

Suppose $\vec{0}_1$ and $\vec{0}$ are zero vectors in V Then $\vec{v} + \vec{0} = \vec{v} \wedge \vec{v} + \vec{0}_1 = \vec{v}$

$$\begin{aligned}
 \vec{0}_1 &= \vec{0}_1 + \vec{0} && (\text{vector space axiom 4}) \\
 &= \vec{0} + \vec{0}_1 && (\text{vector space axiom 2}) \\
 &= \vec{0} && (\text{vector space axiom 4})
 \end{aligned}$$

Therefore $\vec{0}_1 = \vec{0}$. So, the zero vector is unique.

□

3 Determine if each of the following are subspaces of M_{nn}

3.a Question: $\{A \in M_{nn} \mid \det(A) = 0\}$

3.a Answer: No, W is not a subspace of M_{nn} .

Proof. Counter example Axiom 1 fails

$$\det \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0, \det \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0$$
$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$\det \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$$

A subset W of M_{nn} is a subspace of M_{nn} if and only if W satisfies the following three conditions W is nonempty, W is closed under addition, W is closed under scalar multiplication. We have shown W is not closed under addition; therefore, W is not a subspace of M_{nn} . \square

3.b Question: $\{A \in M_{nn} \mid \text{tr}(A) = 0\}$

3.b Answer: Yes, W is a subspace of M_{nn}

Proof. To prove that W is a subspace of M_{nn} we must show three things.

(1) Prove that W is non empty.

$$A = (a_{ij}) = 0 \forall ij \implies \text{tr}(A) = 0 \therefore \vec{0} \in W$$

(2) Prove that W is closed under addition.

Let $A = (a_{ij})$ and $B = (b_{ij}) \in W$ be square matrices of order n such that $\text{tr}(A) = \text{tr}(B) = 0$

$$\begin{aligned} \text{tr}(A + B) &= \sum_{i=1}^n (a_{ii} + b_{ii}) = \sum_{i=1}^n a_{ii} + \sum_{i=1}^n b_{ii} \\ &= \text{tr}(A) + \text{tr}(B) = 0 + 0 = 0 \\ \implies C \in W : C &= A + B \end{aligned}$$

(3) Prove that W is closed under multiplication.

Let k be any scalar.

$$\begin{aligned} \text{tr}(kA) &= \sum_{i=1}^n (k \cdot a_{ii}) = k \cdot \sum_{i=1}^n a_{ii} \\ &= k \cdot \text{tr}(A) = k \cdot 0 = 0 \\ \implies kA &\in W \end{aligned}$$

$\therefore W$ is a subspace of M_{nn} . \square

3.c Question: $\{A \in M_{nn} \mid A^T = A\}$

3.c Answer: Yes, W is a subspace of M_{nn}

Proof. To prove that W is a subspace of M_{nn} we must show three things.

(1) Prove that W is non empty.

$$A = (a_{ij}) = 0 \ \forall \ ij \implies A = A^T = 0 \therefore \vec{0} \in W$$

(2) Prove that W is closed under addition.

Let $A = (a_{ij}) \wedge B = (b_{ij}) \in W$ be square matrices of order n such that $a_{ij} = a_{ji} \wedge b_{ij} = b_{ji}$

$$\begin{aligned}(A + B)^T &= A^T + B^T \\ &= A + B \\ \implies C &\in W : C = A + B\end{aligned}$$

(3) Prove that W is closed under multiplication.

Let k be any scalar.

$$\begin{aligned}(k \cdot A)^T &= k \cdot A^T \\ &= k \cdot A \\ \implies kA &\in W\end{aligned}$$

$\therefore W$ is a subspace of M_{nn} .

□

4 Consider the following vectors in P_2 : $\vec{p}_1 = 2 + x + 4x^2$, $\vec{p}_2 = 1 - x + 3x^2$, $\vec{p}_3 = 3 + 2x + 5x^2$

4.a Question: Express the vector $\vec{g} = 6 + 11x + 6x^2$ as a linear combination of $\{\vec{p}_1, \vec{p}_2, \vec{p}_3\}$.

4.a Answer: $\vec{g} = 4(2 + x + 4x^2) - 5(1 - x + 3x^2) + 1(3 + 2x + 5x^2)$

Proof. We will show $\vec{g} = k_1\vec{p}_1 + k_2\vec{p}_2 + k_3\vec{p}_3$

$$\begin{aligned} (6 + 11x + 6x^2) &= k_1(2 + x + 4x^2) + k_2(1 - x + 3x^2) + k_3(3 + 2x + 5x^2) \\ &= (k_1 \cdot 2 + k_2 + k_3 \cdot 3) + (k_1 \cdot 1 - k_2 + k_3 \cdot 2)x + (k_1 \cdot 4 + k_2 \cdot 3 + k_3 \cdot 5)x^2 \\ &= (k_1 \cdot 2 + k_2 + k_3 \cdot 3) + (k_1 - k_2 + k_3 \cdot 2)x + (k_1 \cdot 4 + k_2 \cdot 3 + k_3 \cdot 5)x^2 \end{aligned}$$

$$\begin{aligned} &\begin{bmatrix} 2 & 1 & 3 & | & 6 \\ 1 & -1 & 2 & | & 11 \\ 4 & 3 & 5 & | & 6 \end{bmatrix} \\ [-2r_2 + r_1] \wedge [-4r_2 + r_1] &\begin{bmatrix} 0 & 3 & -1 & | & -16 \\ 1 & -1 & 2 & | & 11 \\ 0 & 7 & -3 & | & -38 \end{bmatrix} \\ r_2 \leftrightarrow r_1 &\begin{bmatrix} 1 & -1 & 2 & | & 11 \\ 0 & 3 & -1 & | & -16 \\ 0 & 7 & -3 & | & -38 \end{bmatrix} \\ \frac{1}{3}r_2 &\begin{bmatrix} 1 & -1 & 2 & | & 11 \\ 0 & 1 & \frac{-1}{3} & | & \frac{-16}{3} \\ 0 & 7 & -3 & | & -38 \end{bmatrix} \\ -7r_2 + r_3 &\begin{bmatrix} 1 & -1 & 2 & | & 11 \\ 0 & 1 & \frac{-1}{3} & | & \frac{-16}{3} \\ 0 & 0 & \frac{-2}{3} & | & \frac{-2}{3} \end{bmatrix} \\ -\frac{3}{2}r_3 &\begin{bmatrix} 1 & -1 & 2 & | & 11 \\ 0 & 1 & \frac{-1}{3} & | & \frac{-16}{3} \\ 0 & 0 & 1 & | & 1 \end{bmatrix} \\ \frac{1}{3}r_3 + r_2 &\begin{bmatrix} 1 & -1 & 2 & | & 11 \\ 0 & 1 & 0 & | & -5 \\ 0 & 0 & 1 & | & 1 \end{bmatrix} \\ r_1 + r_2 &\begin{bmatrix} 1 & 0 & 2 & | & 6 \\ 0 & 1 & 0 & | & -5 \\ 0 & 0 & 1 & | & 1 \end{bmatrix} \\ r_1 + r_3 &\begin{bmatrix} 1 & 0 & 0 & | & 4 \\ 0 & 1 & 0 & | & -5 \\ 0 & 0 & 1 & | & 1 \end{bmatrix} \end{aligned}$$

$$\implies (k_1, k_2, k_3) = (4, -5, 1)$$

$\therefore \vec{g}$ can be expressed as the following linear combination $\vec{g} = k_1\vec{p}_1 + k_2\vec{p}_2 + k_3\vec{p}_3$ □

4.b Question: Does $\{\vec{p}_1, \vec{p}_2, \vec{p}_3\}$ span P_2 ?

4.b Answer: Yes, $\text{span}(\{\vec{p}_1, \vec{p}_2, \vec{p}_3\}) = P_2$

Proof. An arbitrary vector in P_2 is of the form $\vec{p} = a + bx + cx^2$ and so becomes,

$$k_0(2 + x + 4x^2) + k_1(1 - x + 3x^2) + k_2(3 + 2x + 5x^2) = a + bx + cx^2$$

which we can rewrite as

$$(k_0 2 + k_1 + k_2 3) + (k_0 - k_1 + k_2 2)x + (k_0 4 + k_1 3 + k_2 5)x^2 = a + bx + cx^2$$

Equating corresponding coefficients yields a linear system whose augmented matrix is

$$A = \left[\begin{array}{ccc|c} 2 & 1 & 3 & a \\ 1 & -1 & 2 & b \\ 4 & 3 & 5 & c \end{array} \right]$$

Our problem reduces to ascertaining whether this system is consistent for all values of a , b , and c . This can be determined if its coefficient matrix has a nonzero determinant, from our theorem for equivalent statements. If A is an $n \times n$ matrix such that $\det(A) \neq 0$ then $A\vec{x} = \vec{0}$ has only the trivial solution.

It follows from 4.a that

$$A = \left[\begin{array}{ccc|c} 2 & 1 & 3 & 0 \\ 1 & -1 & 2 & 0 \\ 4 & 3 & 5 & 0 \end{array} \right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right]$$

We have shown A is consistent for every choice a, b , and c . Thus, the vectors in $\{\vec{p}_1, \vec{p}_2, \vec{p}_3\}$ span P_2 . \square

4.c Question: Is $\{\vec{p}_1, \vec{p}_2, \vec{p}_3\}$ linearly independent?

4.c Answer: Yes, $\{\vec{p}_1, \vec{p}_2, \vec{p}_3\}$ is linearly independent

From 4.b we already know this set is linearly independent.

Proof. The nonempty set $\{\vec{p}_1, \vec{p}_2, \vec{p}_3\}$ in a vector space V is linearly independent if and only if the coefficients satisfying

$$k_0\vec{p}_1 + k_1\vec{p}_2 + k_2\vec{p}_3 = \vec{0}$$

are $k_0 = 0, k_1 = 0, k_2 = 0$.

From our theorem for equivalent statements. If A is an $n \times n$ matrix such that $\det(A) \neq 0$ then $A\vec{x} = \vec{0}$ has only the trivial solution. We will show $\det(A) \neq 0$, to convince our selves this theorem holds.

$$\det \begin{vmatrix} 2 & 1 & 3 \\ 1 & -1 & 2 \\ 4 & 3 & 5 \end{vmatrix}$$

minor entry \wedge cofactor

$$a_{11}C_{11} = (-1)^{1+1}(2) \cdot [(-1)(5) - (2)(3)] = -22$$

$$a_{12}C_{12} = (-1)^{1+2}(1) \cdot [(1)(5) - (2)(4)] = 3$$

$$a_{13}C_{13} = (-1)^{1+3}(3) \cdot [(1)(3) - (-1)(4)] = 21$$

cofactor expansion

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = 2$$

$\therefore (k_0, k_1, k_2) = (0, 0, 0)$ so $\{\vec{p}_1, \vec{p}_2, \vec{p}_3\}$ is linearly independent

□

5 Consider the following planes in \mathbb{R}^3 . $P_1 : 2x + 3y - z = 0$ and $P_2 : x + 2y - 2z = 0$

5.a Question Find a set of vectors that spans $P_1 : 2x + 3y - z = 0$.

5.a Answer: $P_1 = \text{span}\{(-\frac{3}{2}, 1, 0), (\frac{1}{2}, 0, 1)\}$

Proof. Solve the following system $(2, 3, -1)^T = \vec{0}$ with $(2, 3, -1)^T$ a row vector.

$$\begin{array}{rcl} & [2 & 3 & -1 & | & 0] \\ & \frac{1}{2}r1 & [1 & \frac{3}{2} & -\frac{1}{2} & | & 0] \\ \therefore (z = r), & (y = q), & (x = -\frac{3}{2}q + \frac{1}{2}r) \end{array}$$

giving the following.

$$\begin{aligned} (x, y, z) &= (-\frac{3}{2}q + \frac{1}{2}r, q, r) \\ &= q(-\frac{3}{2}, 1, 0) + r(\frac{1}{2}, 0, 1) \end{aligned}$$

$$\therefore P_1 = \text{span}\{(-\frac{3}{2}, 1, 0), (\frac{1}{2}, 0, 1)\}$$

□

5.b Question Find a set of vectors that spans $P_2 : x + 2y - 2z = 0$.

5.b Answer: $P_2 = \text{span}\{(-2, 1, 0), (2, 0, 1)\}$

Proof. From $P_2 : x + 2y - 2z = 0$ we have the row vector

$$[1 \quad 2 \quad -2 \quad | \quad 0]$$

giving the following.

$$\begin{aligned} z &= r \\ y &= q \\ x &= -2q + 2r. \\ \implies (x, y, z) &= (-2q + 2r, q, r) \\ &= q(-2, 1, 0) + r(2, 0, 1) \end{aligned}$$

$$\therefore P_2 = \text{span}\{(-2, 1, 0), (2, 0, 1)\}$$

□

5.c Question: Find a set of vectors that spans $P_1 \cap P_2$.

5.c Answer: $P_1 \cap P_2 = \text{span}\{(-4, 3, 1)\}$

Proof. We will find the span of $P_1 \cap P_2$ by solving a system of equations

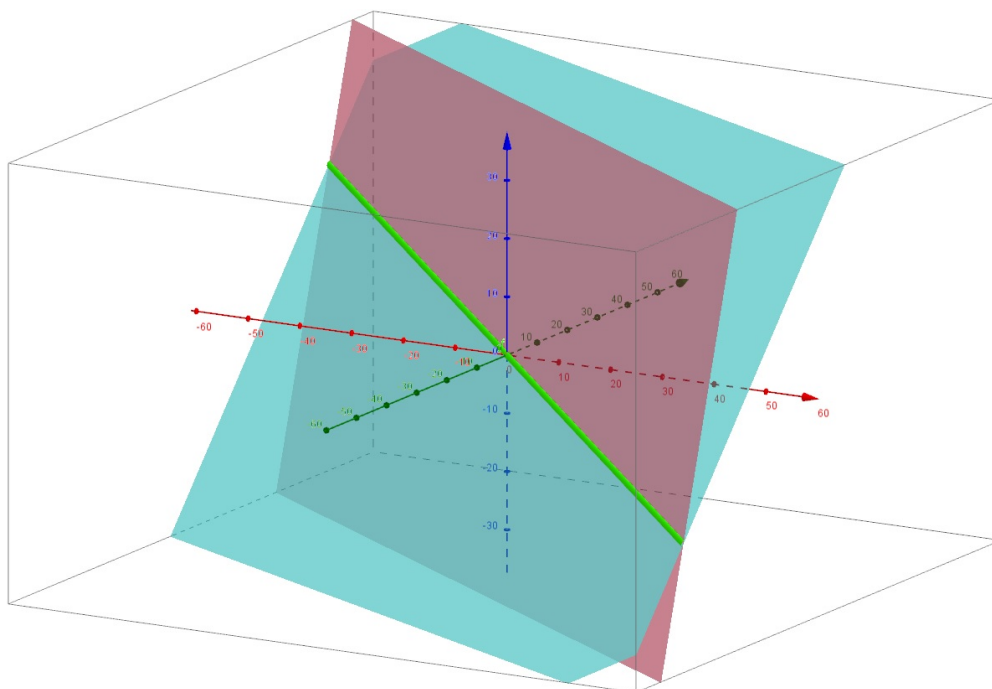
$$P_1 : 2x + 3y - z = 0$$

$$P_2 : x + 2y - 2z = 0$$

$$\begin{aligned} & \begin{bmatrix} 2 & 3 & -1 & | & 0 \\ 1 & 2 & -2 & | & 0 \end{bmatrix} \\ [r1] \leftrightarrow [r2] & \begin{bmatrix} 2 & 3 & -1 & | & 0 \\ 1 & 2 & -2 & | & 0 \end{bmatrix} \\ (-2)r1 + r2 & \begin{bmatrix} 1 & 2 & -2 & | & 0 \\ 0 & -1 & 3 & | & 0 \end{bmatrix} \\ 2(r2) + r1 & \begin{bmatrix} 1 & 0 & 4 & | & 0 \\ 0 & -1 & 3 & | & 0 \end{bmatrix} \\ \therefore (z = t), (y = 3t), (x = -4t) & \end{aligned}$$

giving the following.

$$(x, y, z) = (-4t, 3t, t) = t(-4, 3, 1)$$



$$\therefore P_1 \cap P_2 = \text{span} \{(-4, 3, 1)\}$$

□