

Math 2311 – Assignment 1

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1. Determine if each of the following sets is a vector space.

(a) $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid x \geq y \right\}$ with the usual scalar multiplication and vector addition from \mathbb{R}^2

Answer: No, V is not a vector space.

Proof. let $(x_0, y_0)^T, (x_1, y_1)^T \in V$, and take the vector space operations on V to be the usual operations of *vector* addition and *scalar* multiplication; that is,

$$(x_0, y_0)^T + (x_1, y_1)^T = (x_0 + x_1, y_0 + y_1)^T \quad (1)$$

$$k(x_0, y_0)^T = (kx_0, ky_0)^T \quad (2)$$

V is closed under scalar addition since $x_0 + x_1 \geq y_0 + y_1$

However, by properties of inequalities if the constant, k , is negative, we must reverse the symbol to preserve the inequality relation.

Given that k is negative, $x \geq y \rightarrow kx \leq ky$

□

- (b) Consider the set $W = \{f \in F(-\infty, \infty) \mid f(1) = 0\}$ with the usual scalar multiplication and vector addition from $F(-\infty, \infty)$. Is W a vector space?

Answer: Yes, W is a vector space.

Proof. Since we know that $F(-\infty, \infty)$ (with the usual operations) is a vector space, and since W is a subset of $F(-\infty, \infty)$ (with the same operations), it suffices to prove that W is a subspace of $F(-\infty, \infty)$. To this end we must show three things:

- (a) That W is non-empty.
- (b) That W is closed under addition.
- (c) That W is closed under scalar multiplication.

There exists a function $\mathbf{0}$ in $F(-\infty, \infty)$ defined by $\mathbf{0}(x) = 0$ for all x . Clearly $\mathbf{0}(1) = f(1) = 0$ so W is non-empty.

Now suppose f and g are two functions in W . We must show that $f + g$ is in W .

$$\begin{aligned}(f + g)(1) &= f(1) + g(1) && \text{(definition of addition of functions)} \\ &= 0 && \text{(} f \text{ and } g \text{ are in } W\text{)}\end{aligned}$$

Finally, to show that W is closed under scalar multiplication, suppose f is in W and k is a scalar, then

$$\begin{aligned}(kf)(1) &= kf(1) && \text{(definition of scalar multiplication on functions)} \\ &= 0 && \text{(} f \text{ is in } W\text{)}\end{aligned}$$

so (kf) is in W and W is closed under scalar multiplication.

Therefore W is a subspace of $F(-\infty, \infty)$ and hence is a vector space. \square

2. Let V be a vector space.

(a) If k is any scalar, prove that $k\vec{0} = \vec{0}$.

Proof.

$$\begin{aligned}k(\vec{0} + \vec{0}) &= k\vec{0} + k\vec{0} && \text{(vector space axiom 7)} \\ k\vec{0} &= k\vec{0} + k\vec{0} && \text{(vector space axiom 4)} \\ k\vec{0} + (-k\vec{0}) &= (-k\vec{0}) + (k\vec{0} + k\vec{0}) && \text{(vector space axiom 5)} \\ k\vec{0} + (-k\vec{0}) &= ((-k\vec{0}) + k\vec{0}) + k\vec{0} && \text{(vector space axiom 3)} \\ \vec{0} &= \vec{0} + k\vec{0} && \text{(vector space axiom 5)} \\ &= k\vec{0} && \text{(vector space axiom 4)}\end{aligned}$$

\square

(b) Prove that the zero vector in V is unique.

Proof. We must show that there is only one vector, $\vec{0}$, with the property that $\vec{0} + \vec{v} = \vec{v} + \vec{0} = \vec{v}$.

Suppose $\vec{0}_1$ and $\vec{0}$ are zero vectors in V Then

$$\begin{aligned}\vec{0}_1 &= \vec{0}_1 + \vec{0} && \text{(vector space axiom 4)} \\ &= \vec{0} + \vec{0}_1 && \text{(vector space axiom 2)} \\ &= \vec{0} && \text{(vector space axiom 4)}\end{aligned}$$

Therefore $\vec{0}_1 = \vec{0}$. So, the zero vector is unique. \square

3. Determine if each of the following are subspaces of M_{nn}

(a) $\{A \in M_{nn} \mid \det(A) = 0\}$

Answer: No, $\{A \in M_{nn} \mid \det(A) = 0\}$ is not a subspace of M_{nn} .

Proof.

$$\det \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0, \det \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\det \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$$

□

(b) $\{A \in M_{nn} \mid \text{tr}(A) = 0\}$

Answer: Yes, $W = \{A \in M_{nn} \mid \text{tr}(A) = 0\}$ is a subspace of M_{nn} .

Proof. Let $[B = (b_{ii})] \in W$ be a square matrix of order n such that $\text{tr}(B) = 0$ and let k be any scalar.

The set W is non empty because if we let $a_{ii} = 0$ for all i then $\text{tr}(A) = 0$ therefore W contains the $\mathbf{0}$ matrices. It remains to show that W is closed under addition and scalar multiplication.

addition :

$$\begin{aligned} \text{tr}(A + B) &= \sum_{i=1}^n (a_{ii} + b_{ii}) \\ &= \sum_{i=1}^n a_{ii} + \sum_{i=1}^n b_{ii} \\ &= \text{tr}(A) + \text{tr}(B) \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

multiplication :

$$\begin{aligned} \text{tr}(kA) &= \sum_{i=1}^n (k \cdot a_{ii}) \\ &= k \cdot \sum_{i=1}^n a_{ii} \\ &= k \cdot \text{tr}(A) \\ &= k \cdot 0 \\ &= 0 \end{aligned}$$

To be clear, if we take some $C = A + B$ such that A and B are in W , then for all $C = (c_{ii})$, the sum will be 0, so C is also in W . Therefore W is a subspace of M_{nn} . □

(c) $\{A \in M_{nn} \mid A^T = A\}$

Answer: Yes, $W = \{A \in M_{nn} \mid A^T = A\}$ is a subspace of M_{nn} .

Proof. Let $[B = (b_{ij})] \in W$ be a square matrix of order n such that $b_{ij} = b_{ji}$, and let k be any scalar.

The set W is non empty because, if we let $[A = (a_{ij})] = 0$ for all (i, j) then W contains the $\mathbf{0}$ matrices. It remains to show that W is closed under addition and scalar multiplication.

addition :

$$\begin{aligned} C &= (A + B)^T \\ &= A^T + B^T \\ &= A + B \end{aligned}$$

multiplication :

$$\begin{aligned} (k \cdot A)^T &= k \cdot A^T \\ &= k \cdot A \end{aligned}$$

so W is a subspace of M_{nn} . □

4. Consider the following vectors in P_2 : $p_1 = 2 + x + 4x^2$, $p_2 = 1 - x + 3x^2$, $p_3 = 3 + 2x + 5x^2$.

(a) Express the vector $g = 6 + 11x + 6x^2$ as a linear combination of p_1, p_2, p_3 .

Answer: $(p_1, p_2, p_3) = (4, -5, 1)$

Proof.

$$\begin{aligned} (6 + 11x + 6x^2) &= k_0(2 + x + 4x^2) + k_1(1 - x + 3x^2) + k_2(3 + 2x + 5x^2) \\ &= (k_0 \cdot 2 + k_1 + k_2 \cdot 3) + (k_0 x - k_1 x + k_2 \cdot 2x) + (k_0 \cdot 4x^2 + k_1 \cdot 3x^2 + k_2 \cdot 5x^2) \\ &= (k_0 \cdot 2 + k_1 + k_2 \cdot 3) + (k_0 - k_1 + k_2 \cdot 2)x + (k_0 \cdot 4 + k_1 \cdot 3 + k_2 \cdot 5)x^2 \end{aligned}$$

$$\begin{aligned} \left[\begin{array}{ccc|c} 2 & 1 & 3 & 6 \\ 1 & -1 & 2 & 11 \\ 4 & 3 & 5 & 6 \end{array} \right] & \left[\begin{array}{ccc|c} 0 & 3 & -1 & -16 \\ 1 & -1 & 2 & 11 \\ 0 & 7 & -3 & -38 \end{array} \right] & \left[\begin{array}{ccc|c} 1 & -1 & 2 & 11 \\ 0 & 3 & -1 & -16 \\ 0 & 7 & -3 & -38 \end{array} \right] \\ \left[\begin{array}{ccc|c} 1 & -1 & 2 & 11 \\ 0 & 1 & \frac{-1}{3} & \frac{-16}{3} \\ 0 & 7 & -3 & -38 \end{array} \right] & \left[\begin{array}{ccc|c} 1 & -1 & 2 & 11 \\ 0 & 1 & \frac{-1}{3} & \frac{-16}{3} \\ 0 & 0 & \frac{3}{3} & \frac{-2}{3} \end{array} \right] & \left[\begin{array}{ccc|c} 1 & -1 & 2 & 11 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 1 \end{array} \right] \\ & \left[\begin{array}{ccc|c} 1 & 0 & 2 & 6 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 1 \end{array} \right] & \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 1 \end{array} \right] \end{aligned}$$

$$\therefore (k_1, k_2, k_3) = (4, -5, 1)$$

□

(b) Does $\{p_1, p_2, p_3\}$ span P_2 ?

(c) Is $\{p_1, p_2, p_3\}$ linearly independent?

Answer: Yes

Proof. Let $g = 0 + 0x + 0x^2$ then $[p_1 | p_2 | p_3] \vec{x} = \vec{0}$ gives the trivial solution, therefore $\{p_1, p_2, p_3\}$ spans p_2 and $\{p_1, p_2, p_3\}$ is linearly independent. □