1 Let $\vec{x} = [1\ 2\ 3]^T$, $\mathcal{B} = \{[1\ 0\ 0]^T,\ [1\ 1\ 0]^T,\ [1\ 1\ 1]^T\}$, and $\mathcal{C} = \{[1\ 1\ 0]^T,\ [0\ 1\ 1]^T,\ [1\ 0\ 1]^T\}$.

1.a Find $[\vec{x}]_{\mathcal{B}}$

$\boxed{\text{Solution: } [\vec{x}]_{\mathcal{B}} = [-1\ \ -1\ 3]^T}$

Proof. By inspection: $\vec{x} = (-1)[1\ 0\ 0]^T + (-1)[1\ 1\ 0]^T + (3)[1\ 1\ 1]^T [\vec{x}]_{\mathcal{B}} = [-1\ \ -1\ 3]^T$ □

1.b Find $[\vec{x}]_{\mathcal{C}}$

$\boxed{\text{Solution: } [\vec{x}]_{\mathcal{C}} = [0\ 2\ 1]^T}$

Proof. By inspection: $\vec{x} = (0)[1\ 1\ 0]^T + (2)[0\ 1\ 1]^T + (1)[1\ 0\ 1]^T \implies [\vec{x}]_{\mathcal{C}} = [0\ 2\ 1]^T$ □

1.c Find $P_{\mathcal{C}\leftarrow\mathcal{B}}$ and compute $P_{\mathcal{C}\leftarrow\mathcal{B}}[\vec{x}]_{\mathcal{B}}$

$$\boxed{Solutions : P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 1/2 & 1 & 1/2 \\ -1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 \end{bmatrix}; [\vec{x}]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}[\vec{x}]_{\mathcal{B}} = [0\ 2\ 1]^T}$$

Proof.

$$\text{Partitioned matrix } [\ \mathcal{C} \mid \mathcal{B}\ ] = \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\text{Transition matrix } [\ I_3 \mid \mathcal{C} \leftarrow \mathcal{B}\ ] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 1 & 1/2 \\ 0 & 1 & 0 & -1/2 & 0 & 1/2 \\ 0 & 0 & 1 & 1/2 & 0 & 1/2 \end{array}\right]$$

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 1/2 & 1 & 1/2 \\ -1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

$$\begin{aligned} [\vec{x}]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}[\vec{x}]_{\mathcal{B}} &= \begin{bmatrix} 1/2 & 1 & 1/2 \\ -1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 \end{bmatrix} [-1\ \ -1\ 3]^T \\ &= [-1] \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \end{bmatrix} + [-1] \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + [3] \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} \\ &= [0\ 2\ 1]^T \end{aligned}$$

□

1.d Find $P_{\mathcal{B}\leftarrow\mathcal{C}}$ and compute $P_{\mathcal{B}\leftarrow\mathcal{C}}[\vec{x}]_{\mathcal{C}}$

$$\boxed{Solutions : P_{\mathcal{B}\leftarrow\mathcal{C}} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}; [\vec{x}]_{\mathcal{B}} = P_{\mathcal{B}\leftarrow\mathcal{C}}[\vec{x}]_{\mathcal{C}} = [-1\ \ -1\ 3]^T}$$

Proof.

$$\text{Partitioned matrix } [\ \mathcal{B} \mid \mathcal{C}\ ] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right]$$

$$\text{Transition matrix } [\ I_3 \mid \mathcal{B} \leftarrow \mathcal{C}\ ] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right]$$

$$P_{\mathcal{B}\leftarrow\mathcal{C}} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\begin{aligned} [\vec{x}]_{\mathcal{B}} = P_{\mathcal{B}\leftarrow\mathcal{C}}[\vec{x}]_{\mathcal{C}} &= \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} [0\ 2\ 1]^T \\ &= [0] \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + [2] \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + [1] \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \\ &= [-1\ \ -1\ 3]^T \end{aligned}$$

$\square$