1 Recall the standard basis of $\mathbb{R}^3$, $\vec{e_1} = [1\ 0\ 0]^T,\ \vec{e_2} = [0\ 1\ 0]^T,\ \vec{e_3} = [0\ 0\ 1]^T$

1.a Consider the matrix $A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 3 & 2 \\ 0 & 3 & 2 \end{bmatrix}$. Does the set of vectors $S_1 = \{A\vec{e_1}, A\vec{e_2}, A\vec{e_3}\}$ form a basis for $\mathbb{R}^3$?

$\boxed{S_1 \text{ is a basis for } \mathbb{R}^3}$

Proof. To see if the vectors in S_1 are a basis for $\mathbb{R}^3$, we must verify two things

(1) $span\{S_1\} = \mathbb{R}^3$,

$$[a\ b\ c]^T = c_1 A\vec{e_1} + c_2 A\vec{e_2} + c_3 A\vec{e_1}$$

$$\begin{array}{ccccccc} 1c_1 & + & 0c_2 & + & 2c_3 & = & a \\ 1c_1 & + & 3c_2 & + & 2c_3 & = & b \\ 0c_1 & + & 3c_2 & + & 2c_3 & = & c \end{array}$$

(2) S_1 is linearly independent.

$$[0\ 0\ 0]^T = c_1 A\vec{e_1} + c_2 A\vec{e_2} + c_3 A\vec{e_1}$$

$$\begin{array}{ccccccc} 1c_1 & + & 0c_2 & + & 2c_3 & = & 0 \\ 1c_1 & + & 3c_2 & + & 2c_3 & = & 0 \\ 0c_1 & + & 3c_2 & + & 2c_3 & = & 0 \end{array}$$

Thus, we have reduced the problem to showing that the homogenous system (2) has only the trivial solution, and that the nonhomogenous system (1) is consistent for all values c1, c2, c3. The augmented matrix has the reduced row echelon form,

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] = I_3$$

$\therefore S_1$ is a basis for $\mathbb{R}^3$ □

1.b Consider the matrix $B = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 3 & 0 \\ 0 & 3 & -2 \end{bmatrix}$. Does the set of vectors $S_2 = \{B\vec{e_1}, B\vec{e_2}, B\vec{e_3}\}$ form a basis for $\mathbb{R}^3$?

$\boxed{S_2 \text{ is not a basis for } \mathbb{R}^3}$

Proof. To see if the vectors in S_2 are a basis for $\mathbb{R}^3$, we must verify two things

(1) $span\{S_2\} = \mathbb{R}^3$,

$$[a\ b\ c]^T = c_1 B\vec{e_1} + c_2 B\vec{e_2} + c_3 B\vec{e_1}$$

$$\begin{array}{ccccccc} 1c_1 & + & 0c_2 & + & 2c_3 & = & a \\ 1c_1 & + & 3c_2 & + & 0c_3 & = & b \\ 0c_1 & + & 3c_2 & + & -2c_3 & = & c \end{array}$$

(2) S_2 is linearly independent.

$$[0\ 0\ 0]^T = c_1 B\vec{e_1} + c_2 B\vec{e_2} + c_3 B\vec{e_1}$$

$$\begin{array}{rcrcrcl} 1c_1 & + & 0c_2 & + & 2c_3 & = & 0 \\ 1c_1 & + & 3c_2 & + & 0c_3 & = & 0 \\ 0c_1 & + & 3c_2 & + & -2c_3 & = & 0 \end{array}$$

Thus, we have reduced the problem to showing that the homogenous system (2) has only the trivial solution, and that the nonhomogenous system (1) is consistent for all values c1, c2, c3. The augmented matrix has the reduced row echelon form,

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \neq I_3$$

$\therefore S_2$ is not a basis for $\mathbb{R}^3$ □

1.c Make a conjecture of the form "$S = \{A\vec{e_1}, A\vec{e_2}, A\vec{e_3}\}$ forms a basis for $\mathbb{R}^3$ if and only if A *(insert appropriate property of A here)*".

$$\boxed{\text{Conjecture } S = \{A\vec{e_1}, A\vec{e_2}, A\vec{e_3}\} \text{ forms a basis for } \mathbb{R}^3 \text{ if and only if } A \textit{ (is an invertible matrix).}}$$

1.d Bonus: Prove your conjecture.

Proof.

$$\begin{aligned} A &= AI_3 && (1) \\ &= A[\vec{e_1}\ \vec{e_2}\ \vec{e_3}] && (2) \\ &= [A\vec{e_1}\ A\vec{e_2}\ A\vec{e_3}] && (3) \\ A^{-1}A &= A^{-1}[A\vec{e_1}\ A\vec{e_2}\ A\vec{e_3}] && (4) \\ &= [A^{-1}A\vec{e_1}\ A^{-1}A\vec{e_2}\ A^{-1}A\vec{e_3}] && (5) \\ &= [I_3\vec{e_1}\ I_3\vec{e_2}\ I_3\vec{e_3}] && (6) \\ &= [\vec{e_1}\ \vec{e_2}\ \vec{e_3}] && (7) \\ &= I_3 && (8) \end{aligned}$$

□