1 Let $A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$

1.a Find a basis for each of null(A), row(A), col(A), and state the dimension of each of these subspaces.

$$\boxed{\begin{array}{ll} & Bases: \\ & null(A) = \{[\ -1\ \ -1\ 1\ 0\ ]^T,\ [\ \frac{2}{7}\ \ -\frac{4}{7}\ 0\ 1\ ]^T\} \\ Solutions: & row(A) = \{[\ 1\ 0\ 1\ \ -\frac{2}{7}\ ],\ [\ 0\ 1\ 1\ \frac{4}{7}\ ]\} \\ & col(A) = \{[\ 1\ 2\ \ -1\ ]^T, [\ 4\ 1\ 3\ ]^T\} \\ & Dimensions: \\ & rank(A) = nullity(A) = 2 \end{array}}$$

Proof.

$$rref(A) = \begin{bmatrix} 1 & 0 & 1 & -\frac{2}{7} \\ 0 & 1 & 1 & \frac{4}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\therefore [\ x_1\ x_2\ x_3\ x_4\ ]^T = [\ -\mathcal{S} + \frac{2}{7}\mathcal{T},\ -\mathcal{S} - \frac{4}{7}\mathcal{T},\ \mathcal{S},\ \mathcal{T}\ ]^T$$

$$= \mathcal{S}[\ -1\ \ -1\ 10\ ]^T + \mathcal{T}[\ \frac{2}{7}\ \ -\frac{4}{7}\ 0\ 1\ ]^T$$

$$\implies null(A) = \{[\ -1\ \ -1\ 1\ 0\ ]^T,\ [\ \frac{2}{7}\ \ -\frac{4}{7}\ 0\ 1\ ]^T\}$$

$$\implies row(A) = \{[\ 1\ 0\ 1\ \ -\frac{2}{7}\ ],\ [\ \ 0\ 1\ 1\ \frac{4}{7}\ ]\}$$

$$\implies col(A) = \{[\ 1\ 2\ \ -1\ ]^T,\ [\ 4\ 1\ 3\ ]^T\}$$

$$\implies rank(A) = 2$$

$$\implies nullity(A) = 2$$

$row(A)$ and $null(A)$ are 2 dimensional subspaces of R^4, $col(A)$ is a 2 dimensional subspace of R^3. □

1.b Is the vector $\vec{b} = [\,4\ 6\ -2\,]^T$ in the column space of A? If so, write $\vec{b}$ as a linear combination of the columns of A.

$\boxed{\text{Solution: Yes } \vec{b} \in col(A) \wedge \vec{b} = \frac{20}{7}[\,1\ 2\ \ -1\,]^T + \frac{2}{7}[\,4\ 1\ 3\,]^T + (0)[\,5\ 3\ 2\,]^T + (0)[\,2\ 0\ 2\,]^T}$

let $\vec{c1} = [\,1\ 2\ \ -1\,]^T$, $\vec{c2} = [\,4\ 1\ 3\,]^T$, and let $C = [\,\vec{c1} \mid \vec{c2}\,]$ we will show $\vec{b} \in col(\,A\,) \implies rank(\,C\,) = rank(\,C \mid \vec{b}\,)$

Proof.

$$[\,4\ 6\ \ -2\,]^T = k_1[\,1\ 2\ \ -1\,]^T + k_2[\,4\ 1\ 3\,]^T$$

which can be expressed as,

$$\begin{array}{rcrcr} 1k_1 & + & 4k_2 & = & 4 \\ 2k_1 & + & 1k_2 & = & 6 \\ -1k_1 & + & 3k_2 & = & -2 \end{array}$$

whose augmented matrix has the reduced row echelon form

$$\left[\begin{array}{cc|c} 1 & 0 & \frac{20}{7} \\ 0 & 1 & \frac{2}{7} \\ 0 & 0 & 0 \end{array}\right]$$

because $rank(\,C\,) = 2$ and $rank(\,C \mid \vec{b}\,) = 2$ the system is consistent, so $\vec{b}$ is in the column space of A. $\therefore$ $\vec{b}$ as a linear combination of the columns of A can expressed by the following,
$\vec{b} = \frac{20}{7}[\,1\ 2\ \ -1\,]^T + \frac{2}{7}[\,4\ 1\ 3\,]^T + (0)[\,5\ 3\ 2\,]^T + (0)[\,2\ 0\ 2\,]^T$ $\square$