

# Math 2311 — Assignment 5

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**1 Show that  $S = \{1 + 2x + x^2, 2 + 9x, 3 + 3x + 4x^2\}$  is a basis for  $P_2$  and write  $p = 2 + 17x - 3x^2$  as a linear combination of vectors in  $S$ . Finally, write  $[p]_S$ .**

**1.a Show that  $S = \{1 + 2x + x^2, 2 + 9x, 3 + 3x + 4x^2\}$  is a basis for  $P_2$**

Solution: vectors  $\{\vec{p}_1, \vec{p}_2, \vec{p}_3\}$  are a basis for  $P_3$

*Proof.* The set  $S = \{\vec{p}_1, \vec{p}_2, \vec{p}_3\}$  in a vector space  $P_2$ , is called a basis if.

- (1)  $S$  spans  $P_2$ .
- (2)  $S$  is linearly independent.

To prove that the vectors  $\text{span}\{S\} = P_2$  we must show that every vector  $\vec{p} = a_0 + a_1x + a_2x^2$  in  $P_2$  can be expressed as  $c_1\vec{p}_1 + c_2\vec{p}_2 + c_3\vec{p}_3 = \vec{p}$

$$c_1(1 + 2x + x^2) + c_2(2 + 9x) + c_3(3 + 3x + 4x^2) = \vec{v} \quad (1)$$

$$\begin{aligned} 1c_1 + 2c_2 + 3c_3 &= a_0 \\ 2c_1 + 9c_2 + 3c_3 &= a_1 \\ 1c_1 + 0c_2 + 4c_3 &= a_2 \end{aligned}$$

To prove linear independence we must show that  $c_1\vec{p}_1 + c_2\vec{p}_2 + c_3\vec{p}_3 = \vec{0}$  has only the trivial solution.

$$c_1(1 + 2x + x^2) + c_2(2 + 9x) + c_3(3 + 3x + 4x^2) = \vec{0} \quad (2)$$

$$\begin{aligned} 1c_1 + 2c_2 + 3c_3 &= 0 \\ 2c_1 + 9c_2 + 3c_3 &= 0 \\ 1c_1 + 0c_2 + 4c_3 &= 0 \end{aligned}$$

Thus, we have reduced the problem to showing that the homogenous system (2) has only the trivial solution, and that the nonhomogenous system (1) is consistent for all values  $c_1, c_2, c_3$ . The two systems have the same coefficient matrix.

$$A = \left[ \begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 9 & 3 & 0 \\ 1 & 0 & 4 & 0 \end{array} \right]$$

We will prove both results by showing that  $\det(A) \neq 0$

$$\begin{aligned} \det(A) &= (1) \begin{vmatrix} 2 & 3 \\ 9 & 3 \end{vmatrix} + (4) \begin{vmatrix} 1 & 2 \\ 2 & 9 \end{vmatrix} \\ &= 1[(2)(3) - (3)(9)] + 4[(1)(9) - (2)(2)] = -1. \end{aligned}$$

This proves that  $\{\vec{p}_1, \vec{p}_2, \vec{p}_3\}$  is a basis for  $P_2$ . □

**1.b** Write  $\vec{p} = 2 + 17x - 3x^2$  as a linear combination of vectors in  $S$

Solution:  $\vec{p} = (1)(1 + 2x + x^2) + (2)(2 + 9x) + (-1)(3 + 3x + 4x^2)$

*Proof.* The equation  $c_1\vec{p}_1 + c_2\vec{p}_2 + c_3\vec{p}_3 = \vec{p}$  which can be written as the linear system

$$\begin{array}{rrrrrcl} 1c_1 & + & 2c_2 & + & 3c_3 & = & 2 \\ 2c_1 & + & 9c_2 & + & 3c_3 & = & 17 \\ 1c_1 & + & 0c_2 & + & 4c_3 & = & -3 \end{array}$$

is an expression for a vector  $\vec{p}$  in terms of the basis  $S$ , with scalars  $c_1, c_2, c_3$  being the coordinates of  $\vec{p}$  relative to the basis  $S$ . Whose augmented matrix has the reduced row echelon form,

$$\left[ \begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array} \right]$$
$$c_1 = 1, c_2 = 2, c_3 = -1.$$

This gives  $\vec{p} = (1)(1 + 2x + x^2) + (2)(2 + 9x) + (-1)(3 + 3x + 4x^2)$

□

**1.c** Finally, write  $[p]_S$ .

Solution:  $[p]_S = [1 \ 2 \ -1]^T$

*Proof.* We use  $c_1, c_2, c_3$  from 1.b to construct the coordinate vector  $[1 \ 2 \ -1]^T$  of  $\vec{p}$  relative to  $S$ . □

**2 Recall the standard basis of  $\mathbb{R}^3$ ,  $\vec{e}_1 = [1 \ 0 \ 0]^T$ ,  $\vec{e}_2 = [0 \ 1 \ 0]^T$ ,  $\vec{e}_3 = [0 \ 0 \ 1]^T$**

**2.a Consider the matrix  $A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 3 & 2 \\ 0 & 3 & 2 \end{bmatrix}$ . Does the set of vectors  $S_1 = \{A\vec{e}_1, A\vec{e}_2, A\vec{e}_3\}$  form a basis for  $\mathbb{R}^3$ ?**

Solution:  $S_1$  is a basis for  $\mathbb{R}^3$

*Proof.* To see if the vectors in  $S_1$  are a basis for  $\mathbb{R}^3$ , we will verify  $A$  is invertable.

$$[0 \ 0 \ 0]^T = c_1 A\vec{e}_1 + c_2 A\vec{e}_2 + c_3 A\vec{e}_3$$

$$\begin{aligned} 1c_1 + 0c_2 + 2c_3 &= 0 \\ 1c_1 + 3c_2 + 2c_3 &= 0 \\ 0c_1 + 3c_2 + 2c_3 &= 0 \end{aligned}$$

The augmented matrix has the reduced row echelon form,

$$\left[ \begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right] = I_3$$

From inspecting the pivots of  $rref(A)$ , we can see that  $col(A) = \{[1 \ 1 \ 0]^T, [0 \ 3 \ 3]^T, [2 \ 2 \ 2]^T\}$ , is a basis for  $\mathbb{R}^3$ , but  $A = [A\vec{e}_1 \mid A\vec{e}_2 \mid A\vec{e}_3] \implies col(A) = S_1 \therefore S_1$  is a basis for  $\mathbb{R}^3$ .  $\square$

**2.b Consider the matrix  $B = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 3 & 0 \\ 0 & 3 & -2 \end{bmatrix}$ . Does the set of vectors  $S_2 = \{B\vec{e}_1, B\vec{e}_2, B\vec{e}_3\}$  form a basis for  $\mathbb{R}^3$ ?**

Solution:  $S_2$  is not a basis for  $\mathbb{R}^3$

*Proof.* To see if the vectors in  $S_2$  are a basis for  $\mathbb{R}^3$ , we will verify  $B$  is invertable.

$$[0 \ 0 \ 0]^T = c_1 B\vec{e}_1 + c_2 B\vec{e}_2 + c_3 B\vec{e}_3$$

$$\begin{aligned} 1c_1 + 0c_2 + 2c_3 &= 0 \\ 1c_1 + 3c_2 + 0c_3 &= 0 \\ 0c_1 + 3c_2 + -2c_3 &= 0 \end{aligned}$$

The augmented matrix has the reduced row echelon form,

$$\left[ \begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \neq I_3$$

$S_2$  is not a basis for  $\mathbb{R}^3$  because  $rank(B) = 2 \neq dim(\mathbb{R}^3)$   $\square$

**2.c Make a conjecture of the form “ $S = \{A\vec{e}_1, A\vec{e}_2, A\vec{e}_3\}$  forms a basis for  $\mathbb{R}^3$  if and only if  $A$  (insert appropriate property of  $A$  here)”.**

Conjecture  $S = \{A\vec{e}_1, A\vec{e}_2, A\vec{e}_3\}$  forms a basis for  $\mathbb{R}^3$  if and only if  $A$  (is an invertible matrix).

## 2.d Bonus: Prove your conjecture.

We will prove our conjecture  $S = \{A\vec{e}_1, A\vec{e}_2, A\vec{e}_3\}$  forms a basis for  $\mathbb{R}^3$  if and only if  $A$  (is an invertible matrix), using our list of equivalent statements. Since we have already proven if  $A$  is invertible, then  $A\vec{x} = \vec{0}$  has only the trivial solution, we will use this. Asserting if  $R$  is any row echelon form of a  $3 \times 3$  matrix  $A$ , then either  $R$  has at least one row of zeros, or  $R$  is the identity matrix  $I_3$ .

We will prove the reverse direction first "if  $A$  is an invertible matrix, then  $S = \{A\vec{e}_1, A\vec{e}_2, A\vec{e}_3\}$  forms a basis for  $\mathbb{R}^3$ ."

*Proof.* Suppose  $R$  is the identity matrix  $I_3$ , then  $A$  has an inverse, and  $A\vec{x}$  is a linear combination of the column vectors of  $A$ . Since  $A\vec{x} = \vec{0}$  has only the trivial solution, the column vectors of  $A$  must be linearly independent. Since we know that the 3 column vectors of  $A$  are linearly independent in the 3-dimensional vector space  $\mathbb{R}^3$ , they must span  $\mathbb{R}^3$ , and form a basis for  $\mathbb{R}^3$ .

$$\begin{aligned} A\vec{e}_1 &= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = (1) \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} + (0) \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} + (0) \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} \\ A\vec{e}_2 &= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = (0) \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} + (1) \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} + (0) \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} = \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} \\ A\vec{e}_3 &= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = (0) \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} + (0) \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} + (1) \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} = \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} \\ A &= [A\vec{e}_1 \mid A\vec{e}_2 \mid A\vec{e}_3] \\ &\implies \text{col}(A) = S \end{aligned}$$

Therefore  $S = \{A\vec{e}_1, A\vec{e}_2, A\vec{e}_3\}$  is a basis for  $\mathbb{R}^3$  □

We will now prove the forward direction "if  $S = \{A\vec{e}_1, A\vec{e}_2, A\vec{e}_3\}$  forms a basis for  $\mathbb{R}^3$  then  $A$  is an invertible matrix." by proving its contrapositive "if  $A$  is not an invertible matrix, then  $S = \{A\vec{e}_1, A\vec{e}_2, A\vec{e}_3\}$  does not form a basis for  $\mathbb{R}^3$ "

*Proof.* Suppose  $R$  has at least one row of zeros, then  $A$  has no inverse. We know from analysis of the positions of the 0's and 1's of  $R$  that elementary row operations don't change the dimension of the row space or the column space of our matrix, so it must be true that

$$\dim(\text{row space of } A) = \dim(\text{row space of } R) \text{ and } \dim(\text{column space of } A) = \dim(\text{column space of } R).$$

Since these two numbers are the same, the row and column space have the same dimension  $\text{rank}(A)$ ; the dimension of the null space of  $A$  is  $\text{nullity}(A)$

$$\begin{aligned} 0 &< \text{nullity}(A) \leq \dim(\mathbb{R}^3) \\ \text{rank}(A) + \text{nullity}(A) &= \dim(\mathbb{R}^3) \\ \text{nullity}(A) &= \dim(\mathbb{R}^3) - \text{rank}(A) \\ \implies 0 &< [\dim(\mathbb{R}^3) - \text{rank}(A)] \leq \dim(\mathbb{R}^3) \implies \dim(\mathbb{R}^3) > \text{rank}(A) \geq 0 \\ \therefore \text{rank}(A) &< \dim(\mathbb{R}^3) \end{aligned}$$

This proves, if  $R$  has atleast one row of zeros then  $\text{rank}(A) < \dim(\mathbb{R}^3) \therefore S$  is not a basis for  $\mathbb{R}^3$  □

**3 For each of the following subspaces of  $M_{33}$  find a basis and state the dimension.**

**3.a**  $W_1 = \{A \in M_{33} | A \text{ is a diagonal matrix}\}$

Solution:  $\dim(W_1) = 3$

*Proof.*

$$D = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}, A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore D = aA_1 + bA_2 + cA_3$$

The matrices  $A_1, A_2, A_3$  form a basis for  $W_1$  consequently, the dimension of  $W_1$  is 3.  $\square$

**3.b**  $W_2 = \{A \in M_{33} | A = A^T\}$  (the symmetric matrices)

Solution:  $\dim(W_2) = 6$

*Proof.*

$$S = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}, A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, A_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore S = aA_1 + bA_2 + cA_3 + dA_4 + eA_5 + fA_6$$

The matrices  $A_1, A_2, A_3, A_4, A_5, A_6$  form a basis for  $W_2$  consequently, the dimension of  $W_2$  is 6.  $\square$

**3.c**  $W_3 = \{A \in M_{33} | A = -A^T\}$  (the anti-symmetric matrices)

Solution:  $\dim(W_3) = 3$

*Proof.*

$$S = \begin{bmatrix} 0 & b & c \\ -b & 0 & d \\ -c & -d & 0 \end{bmatrix}, A_1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$\therefore S = aA_1 + bA_2 + cA_3$$

The matrices  $A_1, A_2, A_3$  form a basis for  $W_3$  consequently, the dimension of  $W_3$  is 3.  $\square$

**4 Find a basis for the subspace of  $P^3$  spanned by the following polynomials (vectors):**

$$p_1 = 1 + x + 3x^2 + 4x^3, \quad p_2 = 1 + 2x^2 + 3x^3, \quad p_3 = x + x^2 + 2x^3, \quad p_4 = 1 + x + 3x^2 + 5x^3$$

Solution: the vectors  $\vec{p}_1, \vec{p}_2, \vec{p}_3$  form a basis for  $\text{span}\{p_1, p_2, p_3, p_4\}$

*Proof.* The equation  $k_1\vec{p}_1 + k_2\vec{p}_2 + k_3\vec{p}_3 + k_4\vec{p}_4 = \vec{0}$  can be written as a linear system

$$\begin{aligned} 1k_1 + 1k_2 + 0k_3 + 1k_4 &= 0 \\ 1k_1 + 0k_2 + 1k_3 + 1k_4 &= 0 \\ 3k_1 + 2k_2 + 1k_3 + 3k_4 &= 0 \\ 4k_1 + 3k_2 + 2k_3 + 5k_4 &= 0 \end{aligned}$$

whose augmented matrix has the reduced row echelon form

$$\left[ \begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$
$$\implies k_1 = 0, \quad k_2 = -s, \quad k_3 = -s, \quad k_4 = s$$

removing vector  $\vec{p}_4$  gives  $\text{span}\{p_1, p_2, p_3\} = \text{span}\{p_1, p_2, p_3, p_4\}$ . Since the vector equation  $k_1\vec{p}_1 + k_2\vec{p}_2 + k_3\vec{p}_3 = \vec{0}$  has only the trivial solution. We conclude that the vectors  $\vec{p}_1, \vec{p}_2, \vec{p}_3$  form a basis for  $\text{span}\{p_1, p_2, p_3, p_4\}$ .  $\square$



**5**

**5.a**

*Solutions :*

*Proof.*

□

**5.b**

*Solutions :*

*Proof.*

□