

# Math 2311 — Assignment 1

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**1 Determine if each of the following sets is a vector space.**

1.a Question:  $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid x \geq y \right\}$  with the usual scalar multiplication and vector addition from  $\mathbb{R}^2$

**1.a Answer: No,  $V$  is not a vector space.**

*Proof.* Counter example Axiom 5 fails.

$$\begin{aligned} (3, 2) &\in S : 3 \geq 2 \\ (-3, -2) &\notin S : -3 < -2 \end{aligned}$$

For  $V$  to be a vector space, all 10 of our vector space axioms must hold; this means it is enough to demonstrate one axiom fails; we have shown  $\exists \vec{u} \in V : -\vec{u} \notin V$ , therefore,  $V$  is not a vector space.  $\square$

1.b Question: Consider the set  $W = \{f \in F(-\infty, \infty) \mid f(1) = 0\}$  with the usual scalar multiplication and vector addition from  $F(-\infty, \infty)$ . Is  $W$  a vector space?

**1.b Answer: Yes,  $W$  is a vector space.**

*Proof.* Since we know that  $F(-\infty, \infty)$  (with the usual operations) is a vector space, and since  $W$  is a subset of  $F(-\infty, \infty)$  (with the same operations), it suffices to prove that  $W$  is a subspace of  $F(-\infty, \infty)$ . To this end we must show three things.

(1) Prove that  $W$  is non-empty.

Clearly the zero function  $(\mathbf{0})(1) = f(1) = 0 \therefore W$  is non-empty.

(2) Prove that  $W$  is closed under addition.

Let  $g \in W$ . We must show that  $f + g \in W$ .

$$\begin{aligned} (f + g)(1) &= f(1) + g(1) && \text{(definition of addition of functions)} \\ &= 0 && (f \text{ and } g \text{ are in } W) \end{aligned}$$

(3) Prove that  $W$  is closed under scalar multiplication.

Let scalar  $k \in \mathbb{R}$ , then

$$\begin{aligned} (kf)(1) &= kf(1) && \text{(definition of scalar multiplication on functions)} \\ &= 0 && (f \text{ is in } W) \end{aligned}$$

so  $W$  is closed under scalar multiplication.

Therefore  $W$  is a subspace of  $F(-\infty, \infty)$  and hence is a vector space.  $\square$

**2 Let  $V$  be a vector space.**

2.a Question: If  $k$  is any scalar, prove that  $k\vec{0} = \vec{0}$ .

**2.a**

*Proof.*

$$\begin{aligned}
 k\vec{0} &= k(\vec{0} + \vec{0}) && (\vec{0} = \vec{0} + \vec{0} \text{ by axiom 4}) \\
 &= k\vec{0} + k\vec{0} && (\text{by axiom 7}) \\
 k\vec{0} + (-k\vec{0}) &= [k\vec{0} + k\vec{0}] + (-k\vec{0}) && (\text{by axiom 5 } k\vec{0} \text{ has a negative}) \\
 k\vec{0} + (-k\vec{0}) &= k\vec{0} + [k\vec{0} + (-k\vec{0})] && (\text{by axiom 3}) \\
 \vec{0} &= \vec{0} + k\vec{0} && (\text{by axiom 5}) \\
 \vec{0} &= k\vec{0} && (\text{by axiom 4})
 \end{aligned}$$

□

2.b Question: Prove that the zero vector in  $V$  is unique.

**2.b**

*Proof.* We must show that there is only one vector,  $\vec{0}$ , with the property that  $\vec{0} + \vec{v} = \vec{v} + \vec{0} = \vec{v}$ .

Suppose  $\vec{0}_1$  and  $\vec{0}$  are zero vectors in  $V$  Then  $\vec{v} + \vec{0} = \vec{v} \wedge \vec{v} + \vec{0}_1 = \vec{v}$

$$\begin{aligned}
 \vec{0}_1 &= \vec{0}_1 + \vec{0} && (\text{vector space axiom 4}) \\
 &= \vec{0} + \vec{0}_1 && (\text{vector space axiom 2}) \\
 &= \vec{0} && (\text{vector space axiom 4})
 \end{aligned}$$

Therefore  $\vec{0}_1 = \vec{0}$ . So, the zero vector is unique.

□

**3 Determine if each of the following are subspaces of  $M_{nn}$**

3.a Question:  $\{A \in M_{nn} \mid \det(A) = 0\}$

**3.a Answer: No,  $W$  is not a subspace of  $M_{nn}$ .**

*Proof.* Counter example Axiom 1 fails

$$\det \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0, \det \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0$$
$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$\det \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$$

A subset  $W$  of  $M_{nn}$  is a subspace of  $M_{nn}$  if and only if  $W$  satisfies the following three conditions  $W$  is nonempty,  $W$  is closed under addition,  $W$  is closed under scalar multiplication. We have shown  $W$  is not closed under addition; therefore,  $W$  is not a subspace of  $M_{nn}$ .  $\square$

3.b Question:  $\{A \in M_{nn} \mid \text{tr}(A) = 0\}$

**3.b Answer: Yes,  $W$  is a subspace of  $M_{nn}$**

*Proof.* To prove that  $W$  is a subspace of  $M_{nn}$  we must show three things.

(1) Prove that  $W$  is non empty.

$$A = (a_{ij}) = 0 \forall ij \implies \text{tr}(A) = 0 \therefore \vec{0} \in W$$

(2) Prove that  $W$  is closed under addition.

Let  $A = (a_{ij}) \wedge B = (b_{ij}) \in W$  be square matrices of order  $n$  such that  $\text{tr}(A) = \text{tr}(B) = 0$

$$\begin{aligned} \text{tr}(A + B) &= \sum_{i=1}^n (a_{ii} + b_{ii}) = \sum_{i=1}^n a_{ii} + \sum_{i=1}^n b_{ii} \\ &= \text{tr}(A) + \text{tr}(B) = 0 + 0 = 0 \\ \implies C \in W : C &= A + B \end{aligned}$$

(3) Prove that  $W$  is closed under multiplication.

Let  $k$  be any scalar.

$$\begin{aligned} \text{tr}(kA) &= \sum_{i=1}^n (k \cdot a_{ii}) = k \cdot \sum_{i=1}^n a_{ii} \\ &= k \cdot \text{tr}(A) = k \cdot 0 = 0 \\ \implies kA &\in W \end{aligned}$$

$\therefore W$  is a subspace of  $M_{nn}$ .  $\square$

3.c Question:  $\{A \in M_{nn} \mid A^T = A\}$

**3.c Answer: Yes,  $W$  is a subspace of  $M_{nn}$**

*Proof.* To prove that  $W$  is a subspace of  $M_{nn}$  we must show three things.

(1) Prove that  $W$  is non empty.

$$A = (a_{ij}) = 0 \ \forall \ ij \implies A = A^T = 0 \therefore \vec{0} \in W$$

(2) Prove that  $W$  is closed under addition.

Let  $A = (a_{ij})$  and  $B = (b_{ij}) \in W$  be square matrices of order  $n$  such that  $a_{ij} = a_{ji}$  and  $b_{ij} = b_{ji}$

$$\begin{aligned}(A + B)^T &= A^T + B^T \\ &= A + B \\ \implies C &\in W : C = A + B\end{aligned}$$

(3) Prove that  $W$  is closed under multiplication.

Let  $k$  be any scalar.

$$\begin{aligned}(k \cdot A)^T &= k \cdot A^T \\ &= k \cdot A \\ \implies kA &\in W\end{aligned}$$

$\therefore W$  is a subspace of  $M_{nn}$ .

□

4 Consider the following vectors in  $P_2$ :  $\vec{p}_1 = 2 + x + 4x^2$ ,  $\vec{p}_2 = 1 - x + 3x^2$ ,  $\vec{p}_3 = 3 + 2x + 5x^2$

4.a Question: Express the vector  $\vec{g} = 6 + 11x + 6x^2$  as a linear combination of  $\{\vec{p}_1, \vec{p}_2, \vec{p}_3\}$ .

4.a Answer:  $\vec{g} = 4(2 + x + 4x^2) - 5(1 - x + 3x^2) + 1(3 + 2x + 5x^2)$

*Proof.* We will show  $\vec{g} = k_1\vec{p}_1 + k_2\vec{p}_2 + k_3\vec{p}_3$

$$\begin{aligned} (6 + 11x + 6x^2) &= k_1(2 + x + 4x^2) + k_2(1 - x + 3x^2) + k_3(3 + 2x + 5x^2) \\ &= (k_1 2 + k_2 + k_3 3) + (k_1 x - k_2 x + k_3 2x) + (k_1 4x^2 + k_2 3x^2 + k_3 5x^2) \\ &= (k_1 2 + k_2 + k_3 3) + (k_1 - k_2 + k_3 2)x + (k_1 4 + k_2 3 + k_3 5)x^2 \end{aligned}$$

$$\begin{aligned} &\begin{bmatrix} 2 & 1 & 3 & | & 6 \\ 1 & -1 & 2 & | & 11 \\ 4 & 3 & 5 & | & 6 \end{bmatrix} \\ [-2r_2 + r_1] \wedge [-4r_2 + r_1] &\begin{bmatrix} 0 & 3 & -1 & | & -16 \\ 1 & -1 & 2 & | & 11 \\ 0 & 7 & -3 & | & -38 \end{bmatrix} \\ r_2 \leftrightarrow r_1 &\begin{bmatrix} 1 & -1 & 2 & | & 11 \\ 0 & 3 & -1 & | & -16 \\ 0 & 7 & -3 & | & -38 \end{bmatrix} \\ \frac{1}{3}r_2 &\begin{bmatrix} 1 & -1 & 2 & | & 11 \\ 0 & 1 & \frac{-1}{3} & | & \frac{-16}{3} \\ 0 & 7 & -3 & | & -38 \end{bmatrix} \\ -7r_2 + r_3 &\begin{bmatrix} 1 & -1 & 2 & | & 11 \\ 0 & 1 & \frac{-1}{3} & | & \frac{-16}{3} \\ 0 & 0 & \frac{-2}{3} & | & \frac{-2}{3} \end{bmatrix} \\ -\frac{3}{2}r_3 &\begin{bmatrix} 1 & -1 & 2 & | & 11 \\ 0 & 1 & \frac{-1}{3} & | & \frac{-16}{3} \\ 0 & 0 & 1 & | & 1 \end{bmatrix} \\ \frac{1}{3}r_3 + r_2 &\begin{bmatrix} 1 & -1 & 2 & | & 11 \\ 0 & 1 & 0 & | & -5 \\ 0 & 0 & 1 & | & 1 \end{bmatrix} \\ r_1 + r_2 &\begin{bmatrix} 1 & 0 & 2 & | & 6 \\ 0 & 1 & 0 & | & -5 \\ 0 & 0 & 1 & | & 1 \end{bmatrix} \\ r_1 + r_3 &\begin{bmatrix} 1 & 0 & 0 & | & 4 \\ 0 & 1 & 0 & | & -5 \\ 0 & 0 & 1 & | & 1 \end{bmatrix} \end{aligned}$$

$$\implies (k_1, k_2, k_3) = (4, -5, 1)$$

$\therefore \vec{g}$  can be expressed as the following linear combination  $\vec{g} = k_1\vec{p}_1 + k_2\vec{p}_2 + k_3\vec{p}_3$  □

4.b Question: Does  $\{\vec{p}_1, \vec{p}_2, \vec{p}_3\}$  span  $P_2$ ?

**4.b Answer: Yes,**  $\text{span}(\{\vec{p}_1, \vec{p}_2, \vec{p}_3\}) = P_2$

*Proof.* An arbitrary vector in  $P_2$  is of the form  $\vec{p} = a + bx + cx^2$  and so becomes,

$$k_0(2 + x + 4x^2) + k_1(1 - x + 3x^2) + k_2(3 + 2x + 5x^2) = a + bx + cx^2$$

which we can rewrite as

$$(k_0 2 + k_1 + k_2 3) + (k_0 - k_1 + k_2 2)x + (k_0 4 + k_1 3 + k_2 5)x^2 = a + bx + cx^2$$

Equating corresponding coefficients yields a linear system whose augmented matrix is

$$A = \left[ \begin{array}{ccc|c} 2 & 1 & 3 & a \\ 1 & -1 & 2 & b \\ 4 & 3 & 5 & c \end{array} \right]$$

Our problem reduces to ascertaining whether this system is consistent for all values of  $a$ ,  $b$ , and  $c$ . This can be determined if its coefficient matrix has a nonzero determinant, from our theorem for equivalent statements. If  $A$  is an  $n \times n$  matrix such that  $\det(A) \neq 0$  then  $A\vec{x} = \vec{0}$  has only the trivial solution.

It follows from 4.a that

$$A = \left[ \begin{array}{ccc|c} 2 & 1 & 3 & 0 \\ 1 & -1 & 2 & 0 \\ 4 & 3 & 5 & 0 \end{array} \right] = \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right]$$

We have shown  $A$  is consistent for every choice  $a, b$ , and  $c$ . Thus, the vectors in  $\{\vec{p}_1, \vec{p}_2, \vec{p}_3\}$  span  $P_2$ .  $\square$

4.c Question: Is  $\{\vec{p}_1, \vec{p}_2, \vec{p}_3\}$  linearly independent?

**4.c Answer: Yes,  $\{\vec{p}_1, \vec{p}_2, \vec{p}_3\}$  is linearly independent**

From 4.b we already know this set is linearly independent.

*Proof.* The nonempty set  $\{\vec{p}_1, \vec{p}_2, \vec{p}_3\}$  in a vector space  $V$  is linearly independent if and only if the coefficients satisfying

$$k_0\vec{p}_1 + k_1\vec{p}_2 + k_2\vec{p}_3 = \vec{0}$$

are  $k_0 = 0, k_1 = 0, k_2 = 0$ .

From our theorem for equivalent statements. If  $A$  is an  $n \times n$  matrix such that  $\det(A) \neq 0$  then  $A\vec{x} = \vec{0}$  has only the trivial solution. We will show  $\det(A) \neq 0$ , to convince our selves this theorem holds.

$$\det \begin{vmatrix} 2 & 1 & 3 \\ 1 & -1 & 2 \\ 4 & 3 & 5 \end{vmatrix}$$

minor entry  $\wedge$  cofactor

$$a_{11}C_{11} = (-1)^{1+1}(2) \cdot [(-1)(5) - (2)(3)] = -22$$

$$a_{12}C_{12} = (-1)^{1+2}(1) \cdot [(1)(5) - (2)(4)] = 3$$

$$a_{13}C_{13} = (-1)^{1+3}(3) \cdot [(1)(3) - (-1)(4)] = 21$$

cofactor expansion

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = 2$$

$\therefore (k_0, k_1, k_2) = (0, 0, 0)$  so  $\{\vec{p}_1, \vec{p}_2, \vec{p}_3\}$  is linearly independent

□



5 Consider the following planes in  $\mathbb{R}^3$ .  $P_1 : 2x + 3y - z = 0$  and  $P_2 : x + 2y - 2z = 0$

5.a Question Find a set of vectors that spans  $P_1 : 2x + 3y - z = 0$ .

5.a Answer:  $P_1 = \text{span}\{(-\frac{3}{2}, 1, 0), (\frac{1}{2}, 0, 1)\}$

*Proof.* Solve the following system  $(2, 3, -1)^T = \vec{0}$  with  $(2, 3, -1)^T$  a row vector.

$$\begin{array}{ccc|c} 2 & 3 & -1 & 0 \\ \frac{1}{2}r1 & [1 & \frac{3}{2} & -\frac{1}{2} & 0] \\ \therefore (z = r), (y = q), (x = -\frac{3}{2}q + \frac{1}{2}r) \end{array}$$

giving the following.

$$\begin{aligned} (x, y, z) &= (-\frac{3}{2}q + \frac{1}{2}r, q, r) \\ &= q(-\frac{3}{2}, 1, 0) + r(\frac{1}{2}, 0, 1) \end{aligned}$$

$$\therefore P_1 = \text{span}\{(-\frac{3}{2}, 1, 0), (\frac{1}{2}, 0, 1)\}$$

□

5.b Question Find a set of vectors that spans  $P_2 : x + 2y - 2z = 0$ .

5.b Answer:  $P_2 = \text{span}\{(-2, 1, 0), (2, 0, 1)\}$

*Proof.* From  $P_2 : x + 2y - 2z = 0$  we have the row vector

$$[1 \quad 2 \quad -2 \quad | \quad 0]$$

giving the following.

$$\begin{aligned} z &= r \\ y &= q \\ x &= -2q + 2r. \\ \implies (x, y, z) &= (-2q + 2r, q, r) \\ &= q(-2, 1, 0) + r(2, 0, 1) \end{aligned}$$

$$\therefore P_2 = \text{span}\{(-2, 1, 0), (2, 0, 1)\}$$

□

5.c Question: Find a set of vectors that spans  $P_1 \cap P_2$ .

**5.c Answer:**  $P_1 \cap P_2 = \text{span}\{(-4, 3, 1)\}$

*Proof.* We will find the span of  $P_1 \cap P_2$  by solving a system of equations

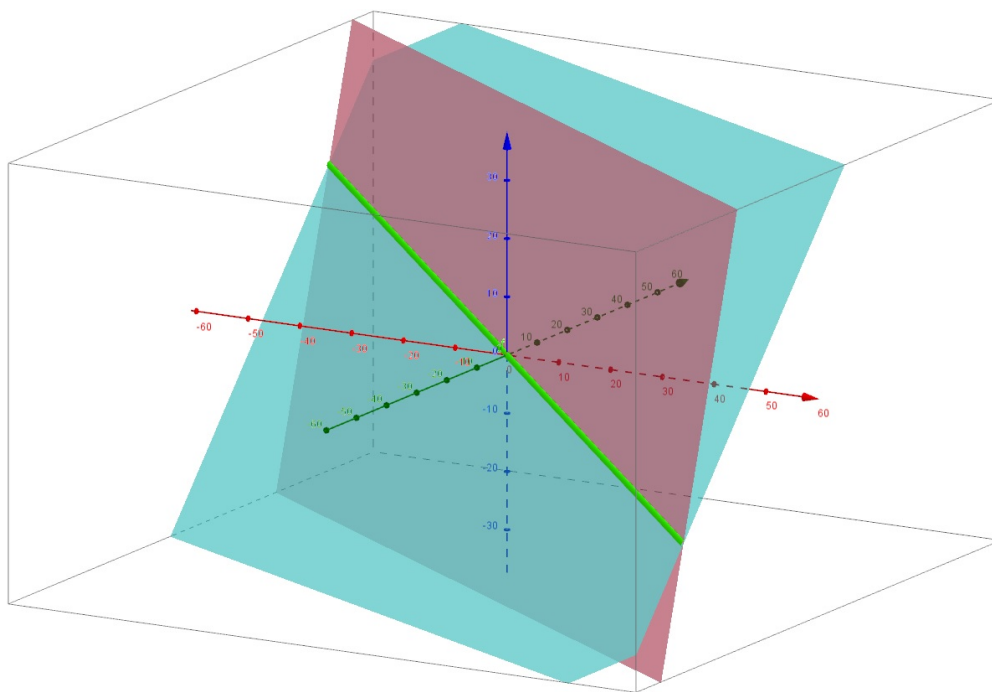
$$P_1 : 2x + 3y - z = 0$$

$$P_2 : x + 2y - 2z = 0$$

$$\begin{aligned} & \begin{bmatrix} 2 & 3 & -1 & | & 0 \\ 1 & 2 & -2 & | & 0 \end{bmatrix} \\ [r1] \leftrightarrow [r2] & \begin{bmatrix} 2 & 3 & -1 & | & 0 \\ 1 & 2 & -2 & | & 0 \end{bmatrix} \\ (-2)r1 + r2 & \begin{bmatrix} 1 & 2 & -2 & | & 0 \\ 0 & -1 & 3 & | & 0 \end{bmatrix} \\ 2(r2) + r1 & \begin{bmatrix} 1 & 0 & 4 & | & 0 \\ 0 & -1 & 3 & | & 0 \end{bmatrix} \\ \therefore (z = t), (y = 3t), (x = -4t) & \end{aligned}$$

giving the following.

$$(x, y, z) = (-4t, 3t, t) = t(-4, 3, 1)$$



$$\therefore P_1 \cap P_2 = \text{span} \{(-4, 3, 1)\}$$

□