## 1 Let $\vec{x} = [\ 1\ 2\ 3\ ]^T$, $\mathcal{B} = \{[\ 1\ 0\ 0\ ]^T,\ [\ 1\ 1\ 0\ ]^T,\ [\ 1\ 1\ 1\ ]^T\}$, and $\mathcal{C} = \{[\ 1\ 1\ 0\ ]^T,\ [\ 0\ 1\ 1\ ]^T,\ [\ 1\ 0\ 1\ ]^T\}$.

### 1.a Find $[\vec{x}]_{\mathcal{B}}$

$\boxed{\text{Solution: } [\vec{x}]_{\mathcal{B}} = [\ -1\ \ -1\ 3\ ]^T}$

*Proof.* By inspection: $\vec{x} = (-1)[\ 1\ 0\ 0\ ]^T + (-1)[\ 1\ 1\ 0\ ]^T + (3)[\ 1\ 1\ 1\ ]^T[\vec{x}]_{\mathcal{B}} = [\ -1\ \ -1\ 3\ ]^T$ □

### 1.b Find $[\vec{x}]_{\mathcal{C}}$

$\boxed{\text{Solution: } [\vec{x}]_{\mathcal{C}} = [\ 0\ 2\ 1\ ]^T}$

*Proof.* By inspection: $\vec{x} = (0)[\ 1\ 1\ 0\ ]^T + (2)[\ 0\ 1\ 1\ ]^T + (1)[\ 1\ 0\ 1\ ]^T \implies [\vec{x}]_{\mathcal{C}} = [\ 0\ 2\ 1\ ]^T$ □

### 1.c Find $P_{\mathcal{C}\leftarrow\mathcal{B}}$ and compute $P_{\mathcal{C}\leftarrow\mathcal{B}}[\vec{x}]_{\mathcal{B}}$

$$\boxed{Solutions: P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 1/2 & 1 & 1/2 \\ -1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 \end{bmatrix}; [\vec{x}]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}[\vec{x}]_{\mathcal{B}} = [\ 0\ 2\ 1\ ]^T}$$

*Proof.*

$$\begin{aligned}
\text{Partitioned matrix } [\ \mathcal{C} \mid \mathcal{B}\ ] &= \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \\
\text{Transition matrix } [\ I_3 \mid \mathcal{C} \leftarrow \mathcal{B}\ ] &= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 1 & 1/2 \\ 0 & 1 & 0 & -1/2 & 0 & 1/2 \\ 0 & 0 & 1 & 1/2 & 0 & 1/2 \end{array}\right] \\
P_{\mathcal{C}\leftarrow\mathcal{B}} &= \begin{bmatrix} 1/2 & 1 & 1/2 \\ -1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
[\vec{x}]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}[\vec{x}]_{\mathcal{B}} &= \begin{bmatrix} 1/2 & 1 & 1/2 \\ -1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 \end{bmatrix} [\ -1\ \ -1\ 3\ ]^T \\
&= [-1] \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \end{bmatrix} + [-1] \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + [3] \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} \\
&= [\ 0\ 2\ 1\ ]^T
\end{aligned}$$

□

### 1.d Find $P_{\mathcal{B}\leftarrow\mathcal{C}}$ and compute $P_{\mathcal{B}\leftarrow\mathcal{C}}[\vec{x}]_{\mathcal{C}}$

$$\boxed{Solutions: P_{\mathcal{B}\leftarrow\mathcal{C}} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}; [\vec{x}]_{\mathcal{B}} = P_{\mathcal{B}\leftarrow\mathcal{C}}[\vec{x}]_{\mathcal{C}} = [\ -1\ \ -1\ 3\ ]^T}$$

*Proof.*

$$\text{Partitioned matrix } [\ \mathcal{B} \mid \mathcal{C}\ ] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right]$$

$$\text{Transition matrix } [\ I_3 \mid \mathcal{B} \leftarrow \mathcal{C}\ ] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right]$$

$$P_{\mathcal{B}\leftarrow\mathcal{C}} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\begin{aligned} [\vec{x}]_{\mathcal{B}} = P_{\mathcal{B}\leftarrow\mathcal{C}}[\vec{x}]_{\mathcal{C}} &= \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} [\ 0\ 2\ 1\ ]^T \\ &= [0]\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + [2]\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + [1]\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \\ &= [\ -1\ \ -1\ 3\ ]^T \end{aligned}$$

□