1 Recall the standard basis of $\mathbb{R}^3$, $\vec{e_1} = [1\ 0\ 0]^T,\ \vec{e_2} = [0\ 1\ 0]^T,\ \vec{e_3} = [0\ 0\ 1]^T$

1.a Consider the matrix $A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 3 & 2 \\ 0 & 3 & 2 \end{bmatrix}$. Does the set of vectors $S_1 = \{A\vec{e_1}, A\vec{e_2}, A\vec{e_3}\}$ form a basis for $\mathbb{R}^3$?

$\boxed{\text{Solution: } S_1 \text{ is a basis for } \mathbb{R}^3}$

Proof. To see if the vectors in S_1 are a basis for $\mathbb{R}^3$, we must verify S_1 is linearly independent.

$$[0\ 0\ 0]^T = c_1 A\vec{e_1} + c_2 A\vec{e_2} + c_3 A\vec{e_1}$$

$$\begin{array}{ccccccc} 1c_1 & + & 0c_2 & + & 2c_3 & = & 0 \\ 1c_1 & + & 3c_2 & + & 2c_3 & = & 0 \\ 0c_1 & + & 3c_2 & + & 2c_3 & = & 0 \end{array}$$

The augmented matrix has the reduced row echelon form,

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] = I_3$$

showing that the homogenous system has only the trivial solution $\therefore S_1$ is a basis for $\mathbb{R}^3$ □

1.b Consider the matrix $B = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 3 & 0 \\ 0 & 3 & -2 \end{bmatrix}$. Does the set of vectors $S_2 = \{B\vec{e_1}, B\vec{e_2}, B\vec{e_3}\}$ form a basis for $\mathbb{R}^3$?

$\boxed{\text{Solution: } S_2 \text{ is not a basis for } \mathbb{R}^3}$

Proof. To see if the vectors in S_2 are a basis for $\mathbb{R}^3$, we must verify S_2 is linearly independent.

$$[0\ 0\ 0]^T = c_1 B\vec{e_1} + c_2 B\vec{e_2} + c_3 B\vec{e_1}$$

$$\begin{array}{ccccccc} 1c_1 & + & 0c_2 & + & 2c_3 & = & 0 \\ 1c_1 & + & 3c_2 & + & 0c_3 & = & 0 \\ 0c_1 & + & 3c_2 & + & -2c_3 & = & 0 \end{array}$$

The augmented matrix has the reduced row echelon form,

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \neq I_3$$

showing that the homogenous system has infinite solutions $\therefore S_2$ is not a basis for $\mathbb{R}^3$ □

1.c Make a conjecture of the form "$S = \{A\vec{e_1}, A\vec{e_2}, A\vec{e_3}\}$ forms a basis for $\mathbb{R}^3$ if and only if A *(insert appropriate property of A here)*".

$\boxed{\text{Conjecture } S = \{A\vec{e_1}, A\vec{e_2}, A\vec{e_3}\} \text{ forms a basis for } \mathbb{R}^3 \text{ if and only if } A \textit{ (is an invertible matrix).}}$

1.d Bonus: Prove your conjecture.

We will prove our conjecture $S = \{A\vec{e_1}, A\vec{e_2}, A\vec{e_3}\}$ forms a basis for $\mathbb{R}^3$ if and only if A *(is an invertible matrix)*, using our list of equivalent statements. Since we have already proven if A is invertible, then $A\vec{x} = \vec{0}$ has only the trivial solution, we will use this. Asserting if R is any row echelon form of a 3×3 matrix A, then either R has at least one row of zeros, or R is the identity matrix I_3.

We will prove the reverse direction first "if A is an invertible matrix, then $S = \{A\vec{e_1}, A\vec{e_2}, A\vec{e_3}\}$ forms a basis for $\mathbb{R}^3$."

Proof. Suppose R is the identity matrix I_3, then A has an inverse, and $A\vec{x}$ is a linear combination of the column vectors of A. Since $A\vec{x} = \vec{0}$ has only the trivial solution, the column vectors of A must be linearly independent. Since we know that the 3 column vectors of A are linearly independent in the 3-dimensional vector space $\mathbb{R}^3$, they must span $\mathbb{R}^3$, and form a basis for $\mathbb{R}^3$.

$$A\vec{e_1} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = (1) \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} + (0) \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} + (0) \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}$$

$$A\vec{e_2} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = (0) \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} + (1) \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} + (0) \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} = \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix}$$

$$A\vec{e_3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = (0) \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} + (0) \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} + (1) \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} = \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix}$$

$$A = [A\vec{e_1} \mid A\vec{e_2} \mid A\vec{e_3}]$$
$$\implies col(A) = S$$

Therefore $S = \{A\vec{e_1}, A\vec{e_2}, A\vec{e_3}\}$ is a basis for R^3 □

We will now prove the forward direction "if $S = \{A\vec{e_1}, A\vec{e_2}, A\vec{e_3}\}$ forms a basis for $\mathbb{R}^3$ then A is an invertible matrix." by proving its contrapositive "if A is not an invertible matrix, then $S = \{A\vec{e_1}, A\vec{e_2}, A\vec{e_3}\}$ does not form a basis for $\mathbb{R}^3$"

Proof. Suppose R has at least one row of zeros, then A has no inverse. We know from analysis of the positions of the $0's$ and $1's$ of R that elementary row operations don't change the dimension of the row space or the column space of our matrix, so it must be true that

dim(row space of A) $=$ dim(row space of R) and dim(column space of A) $=$ dim(column space of R).

Since these two numbers are the same, the row and column space have the same dimension $rank(A)$; the dimension of the null space of A is $nullity(A)$

$$0 < nullity(A) \leq dim(\mathbb{R}^3)$$
$$rank(A) + nullity(A) = dim(\mathbb{R}^3)$$
$$nullity(A) = dim(\mathbb{R}^3) - rank(A)$$
$$\implies 0 < [\ dim(\mathbb{R}^3) - rank(A)\ ] \leq dim(\mathbb{R}^3) \implies dim(\mathbb{R}^3) > rank(A) \geq 0$$
$$\therefore rank(A) < dim(\mathbb{R}^3)$$

This proves, if R has atleast one row of zeros then $rank(A) < dim(\mathbb{R}^3) \therefore S$ is not a basis for $\mathbb{R}^3$ □