

Math 2311 – Assignment 1

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January 17, 2022

1. Determine if each of the following sets is a vector space.

(a) $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid x \geq y \right\}$ with the usual scalar multiplication and vector addition from \mathbb{R}^2

Answer: No, V is not a vector space.

Proof. let $(x_0, y_0)^T, (x_1, y_1)^T \in V$, and take the vector space operations on V to be the usual operations of *vector* addition and *scalar* multiplication; that is,

$$(x_0, y_0)^T + (x_1, y_1)^T = (x_0 + x_1, y_0 + y_1)^T \quad (1)$$

$$k(x_0, y_0)^T = (kx_0, ky_0)^T \quad (2)$$

V is closed under scalar addition since $x_0 + x_1 \geq y_0 + y_1$

However, by properties of inequalities if the constant, k , is negative, we must reverse the symbol to preserve the inequality relation.

Given that k is negative, $x \geq y \rightarrow kx \leq ky$ □

- (b) Consider the set $W = \{f \in F(-\infty, \infty) \mid f(1) = 0\}$ with the usual scalar multiplication and vector addition from $F(-\infty, \infty)$. Is W a vector space?

Answer: Yes, W is a vector space.

Proof. Since we know that $F(-\infty, \infty)$ (with the usual operations) is a vector space, and since W is a subset of $F(-\infty, \infty)$ (with the same operations), it suffices to prove that W is a subspace of $F(-\infty, \infty)$. To this end we must show three things:

- (a) That W is non-empty.
- (b) That W is closed under addition.
- (c) That W is closed under scalar multiplication.

There exists a function $\mathbf{0}$ in $F(-\infty, \infty)$ defined by $\mathbf{0}(x) = 0$ for all x . Clearly $\mathbf{0}(1) = f(1) = 0$ so W is non-empty.

Now suppose f and g are two functions in W . We must show that $f + g$ is in W .

$$\begin{aligned}(f + g)(1) &= f(1) + g(1) && \text{(definition of addition of functions)} \\ &= 0 && (f \text{ and } g \text{ are in } W)\end{aligned}$$

Finally, to show that W is closed under scalar multiplication, suppose f is in W and k is a scalar, then

$$\begin{aligned}(kf)(1) &= kf(1) && \text{(definition of scalar multiplication on functions)} \\ &= 0 && (f \text{ is in } W)\end{aligned}$$

so (kf) is in W and W is closed under scalar multiplication.

Therefore W is a subspace of $F(-\infty, \infty)$ and hence is a vector space. \square

2. Let V be a vector space.

(a) If k is any scalar, prove that $k\vec{0} = \vec{0}$.

Proof.

$$\begin{aligned}k(\vec{0} + \vec{u}) &= k\vec{0} + k\vec{u} && \text{(vector space axiom 7)} \\ k\vec{u} &= k\vec{0} + k\vec{u} && \text{(vector space axiom 4)} \\ k(\vec{u}) + (-k\vec{u}) &= (-k\vec{u}) + (k\vec{0} + k\vec{u}) && \text{(vector space axiom 5)} \\ \vec{0} &= (-k\vec{u}) + (k\vec{0} + k\vec{u}) && \text{(vector space axiom 5)} \\ &= (k\vec{0} + k\vec{u}) + (-k\vec{u}) && \text{(vector space axiom 2)} \\ &= k\vec{0} + (k\vec{u} + (-k\vec{u})) && \text{(vector space axiom 3)} \\ &= k\vec{0} + \vec{0} && \text{(vector space axiom 5)} \\ &= k\vec{0} && \text{(vector space axiom 4)}\end{aligned}$$

\square

(b) Prove that the zero vector in V is unique.

Proof. We must show that there is only one vector, $\vec{0}$, with the property that $\vec{0} + \vec{v} = \vec{v} + \vec{0} = \vec{v}$.

Suppose $\vec{0}_1$ and $\vec{0}$ are zero vectors in V , and \vec{v} is also in V . Then

$$\begin{aligned}\vec{v} &= \vec{0}_1 + \vec{v} && \text{(vector space axiom 4)} \\ \vec{v} + (-\vec{v}) &= (-\vec{v}) + (\vec{0}_1 + \vec{v}) && \text{(vector space axiom 5)} \\ \vec{0} &= (-\vec{v}) + (\vec{0}_1 + \vec{v}) && \text{(vector space axiom 5)} \\ &= (\vec{0}_1 + \vec{v}) + (-\vec{v}) && \text{(vector space axiom 2)} \\ &= \vec{0}_1 + (\vec{v} + (-\vec{v})) && \text{(vector space axiom 3)} \\ &= \vec{0}_1 + \vec{0} && \text{(vector space axiom 5)} \\ &= \vec{0}_1 && \text{(vector space axiom 4)}\end{aligned}$$

Since we have shown that any two zero vectors must be equal to each other, we can conclude that $\vec{0}$ is unique. \square

3. Determine if each of the following are subspaces of M_{nn}

(a) $\{A \in M_{nn} \mid \det(A) = 0\}$

Answer: No, $\{A \in M_{nn} \mid \det(A) = 0\}$ is not a subspace of M_{nn} .

Proof.

$$\det \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0, \det \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0$$
$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$\det \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$$

□

(b) $\{A \in M_{nn} \mid \text{tr}(A) = 0\}$

Answer: Yes, $W = \{A \in M_{nn} \mid \text{tr}(A) = 0\}$ is a subspace of M_{nn} .

Proof. Let $[B = (b_{ii})] \in W$ be a square matrix of order n such that $\text{tr}(B) = 0$ and let k be any scalar.

The set W is non empty because if we let $a_{ii} = 0$ for all i then $\text{tr}(A) = 0$ therefore W contains the $\mathbf{0}$ matrices. It remains to show that W is closed under addition and scalar multiplication.

addition :

$$\begin{aligned} \text{tr}(A + B) &= \sum_{i=1}^n (a_{ii} + b_{ii}) \\ &= \sum_{i=1}^n a_{ii} + \sum_{i=1}^n b_{ii} \\ &= \text{tr}(A) + \text{tr}(B) \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

multiplication :

$$\begin{aligned} \text{tr}(kA) &= \sum_{i=1}^n (k \cdot a_{ii}) \\ &= k \cdot \sum_{i=1}^n a_{ii} \\ &= k \cdot \text{tr}(A) \\ &= k \cdot 0 \\ &= 0 \end{aligned}$$

To be clear, if we take some $C = A + B$ such that A and B are in W , then for all $C = (c_{ii})$, the sum will be 0, so C is also in W . Therefore W is a subspace of M_{nn} . □

(c) $\{A \in M_{nn} \mid A^T = A\}$

Answer: Yes, $W = \{A \in M_{nn} \mid A^T = A\}$ is a subspace of M_{nn} .

Proof. Let $[B = (b_{ij})] \in W$ be a square matrix of order n such that $b_{ij} = b_{ji}$, and let k be any scalar.

The set W is non empty because, if we let $[A = (a_{ij})] = 0$ for all (i, j) then W contains the $\mathbf{0}$ matrices. It remains to show that W is closed under addition and scalar multiplication.

addition :

$$\begin{aligned} C &= (A + B)^T \\ &= A^T + B^T \\ &= A + B \end{aligned}$$

multiplication :

$$\begin{aligned} (k \cdot A)^T &= k \cdot A^T \\ &= k \cdot A \end{aligned}$$

so W is a subspace of M_{nn} . □