Math 2311 – Assignment 2

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1 Show that $S = \{1 + 2x + x^2, 2 + 9x, 3 + 3x + 4x^2\}$ is a basis for P_2 and write $p = 2 + 17x - 3x^2$ as a linear combination of vectors in S. Finally, write $[p]_S$.

1.a Show that $S = \{1 + 2x + x^2, 2 + 9x, 3 + 3x + 4x^2\}$ is a basis for P_2

$\boxed{\text{Solution: vectors } \{\vec{p_1}, \vec{p_2}, \vec{p_3}\} \text{ are a basis for } P_3}$

Proof. The set $S = \{\vec{p_1}, \vec{p_2}, \vec{p_3}\}$ in a vector space P_2, is called a basis if.

(1) S spans P_2.
(2) S is linearly independent.

To prove that the vectors span$\{S\} = P_2$ we must show that every vector $\vec{p} = a_0 + a_1x + a_2x^2$ in P_2 can be expressed as $c_1\vec{p_1} + c_2\vec{p_2} + c_3\vec{p_3} = \vec{p}$

$$c_1(1 + 2x + x^2) + c_2(2 + 9x) + c_3(3 + 3x + 4x^2) = \vec{v} \tag{1}$$

$$\begin{array}{ccccccl} 1c_1 & + & 2c_2 & + & 3c_3 & = a_0 \\ 2c_1 & + & 9c_2 & + & 3c_3 & = a_1 \\ 1c_1 & + & 0c_2 & + & 4c_3 & = a_2 \end{array}$$

To prove linear independence we must show that $c_1\vec{p_1} + c_2\vec{p_2} + c_3\vec{p_3} = \vec{0}$ has only the trivial solution.

$$c_1(1 + 2x + x^2) + c_2(2 + 9x) + c_3(3 + 3x + 4x^2) = \vec{0} \tag{2}$$

$$\begin{array}{ccccccl} 1c_1 & + & 2c_2 & + & 3c_3 & = 0 \\ 2c_1 & + & 9c_2 & + & 3c_3 & = 0 \\ 1c_1 & + & 0c_2 & + & 4c_3 & = 0 \end{array}$$

Thus, we have reduced the problem to showing that the homogenous system (2) has only the trivial solution, and that the nonhomogenous system (1) is consistent for all values c1, c2, c3. The two systems have the same coefficient matrix.

$$A = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 9 & 3 & 0 \\ 1 & 0 & 4 & 0 \end{array}\right]$$

We will prove both results by showing that $\det(A) \neq 0$

$$\begin{aligned} det(A) &= (1)\begin{vmatrix} 2 & 3 \\ 9 & 3 \end{vmatrix} + (4)\begin{vmatrix} 1 & 2 \\ 2 & 9 \end{vmatrix} \\ &= 1[(2)(3) - (3)(9)] + 4[(1)(9) - (2)(2)] = -1. \end{aligned}$$

This proves that $\{\vec{p_1}, \vec{p_2}, \vec{p_3}\}$ is a basis for P_2. □

1.b Write $\vec{p} = 2 + 17x - 3x^2$ as a linear combination of vectors in S

$\boxed{\text{Solution: } \vec{p} = (1)(1 + 2x + x^2) + (2)(2 + 9x) + (-1)(3 + 3x + 4x^2)}$

Proof. The equation $c_1\vec{p_1} + c_2\vec{p_2} + c_3\vec{p_3} = \vec{p}$ which can be written as the linear system

$$\begin{array}{ccccccl} 1c_1 & + & 2c_2 & + & 3c_3 & = 2 \\ 2c_1 & + & 9c_2 & + & 3c_3 & = 17 \\ 1c_1 & + & 0c_2 & + & 4c_3 & = -3 \end{array}$$

is an expression for a vector $\vec{p}$ in terms of the basis S, with scalars $c1, c2, c3$ being the coordinates of $\vec{p}$ relative to the basis S. Whose augmented matrix has the reduced row echelon form,

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right]$$
$$c_1 = 1, c_2 = 2, c_3 = -1.$$

This gives $\vec{p} = (1)(1 + 2x + x^2) + (2)(2 + 9x) + (-1)(3 + 3x + 4x^2)$ □

1.c Finally, write $[p]_S$.

$\boxed{\text{Solution: } [p]_s = [1\ 2\ {-1}]^T}$

Proof. We use c_1, c_2, c_3 from 1.b to construct the coordinate vector $[1\ 2\ {-1}]^T$ of $\vec{p}$ relative to S. □

2 Recall the standard basis of $\mathbb{R}^3$, $\vec{e_1} = [1\ 0\ 0]^T$, $\vec{e_2} = [0\ 1\ 0]^T$, $\vec{e_3} = [0\ 0\ 1]^T$

2.a Consider the matrix $A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 3 & 2 \\ 0 & 3 & 2 \end{bmatrix}$. Does the set of vectors $S_1 = \{A\vec{e_1}, A\vec{e_2}, A\vec{e_3}\}$ form a basis for $\mathbb{R}^3$?

$\boxed{S_1 \text{ is a basis for } \mathbb{R}^3}$

Proof. To see if the vectors in S_1 are a basis for $\mathbb{R}^3$, we must verify two things

(1) $span\{S_1\} = \mathbb{R}^3$,

$$[a\ b\ c]^T = c_1 A\vec{e_1} + c_2 A\vec{e_2} + c_3 A\vec{e_1}$$

$$\begin{array}{ccccccc} 1c_1 & + & 0c_2 & + & 2c_3 & = & a \\ 1c_1 & + & 3c_2 & + & 2c_3 & = & b \\ 0c_1 & + & 3c_2 & + & 2c_3 & = & c \end{array}$$

(2) S_1 is linearly independent.

$$[0\ 0\ 0]^T = c_1 A\vec{e_1} + c_2 A\vec{e_2} + c_3 A\vec{e_1}$$

$$\begin{array}{ccccccc} 1c_1 & + & 0c_2 & + & 2c_3 & = & 0 \\ 1c_1 & + & 3c_2 & + & 2c_3 & = & 0 \\ 0c_1 & + & 3c_2 & + & 2c_3 & = & 0 \end{array}$$

Thus, we have reduced the problem to showing that the homogenous system (2) has only the trivial solution, and that the nonhomogenous system (1) is consistent for all values c1, c2, c3. The augmented matrix has the reduced row echelon form,

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] = I_3$$

$\therefore S_1$ is a basis for $\mathbb{R}^3$ □

2.b Consider the matrix $B = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 3 & 0 \\ 0 & 3 & -2 \end{bmatrix}$. Does the set of vectors $S_2 = \{B\vec{e_1}, B\vec{e_2}, B\vec{e_3}\}$ form a basis for $\mathbb{R}^3$?

$\boxed{S_2 \text{ is not a basis for } \mathbb{R}^3}$

Proof. To see if the vectors in S_2 are a basis for $\mathbb{R}^3$, we must verify two things

(1) $span\{S_2\} = \mathbb{R}^3$,

$$[a\ b\ c]^T = c_1 B\vec{e_1} + c_2 B\vec{e_2} + c_3 B\vec{e_1}$$

$$\begin{array}{ccccccc} 1c_1 & + & 0c_2 & + & 2c_3 & = & a \\ 1c_1 & + & 3c_2 & + & 0c_3 & = & b \\ 0c_1 & + & 3c_2 & + & -2c_3 & = & c \end{array}$$

(2) S_2 is linearly independent.

$$[0\ 0\ 0]^T = c_1 B\vec{e_1} + c_2 B\vec{e_2} + c_3 B\vec{e_1}$$

$$\begin{array}{ccccccc} 1c_1 & + & 0c_2 & + & 2c_3 & = & 0 \\ 1c_1 & + & 3c_2 & + & 0c_3 & = & 0 \\ 0c_1 & + & 3c_2 & + & -2c_3 & = & 0 \end{array}$$

Thus, we have reduced the problem to showing that the homogenous system (2) has only the trivial solution, and that the nonhomogenous system (1) is consistent for all values c1, c2, c3. The augmented matrix has the reduced row echelon form,

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \neq I_3$$

$\therefore S_2$ is not a basis for $\mathbb{R}^3$ □

2.c Make a conjecture of the form "$S = \{A\vec{e_1}, A\vec{e_2}, A\vec{e_3}\}$ forms a basis for $\mathbb{R}^3$ if and only if A *(insert appropriate property of A here)*".

$\boxed{\text{Conjecture } S = \{A\vec{e_1}, A\vec{e_2}, A\vec{e_3}\} \text{ forms a basis for } \mathbb{R}^3 \text{ if and only if } A \textit{ (is an invertible matrix).}}$

2.d Bonus: Prove your conjecture.

Proof.

$$\begin{aligned} A &= AI_3 && (3) \\ &= A[\vec{e_1}\ \vec{e_2}\ \vec{e_3}] && (4) \\ &= [A\vec{e_1}\ A\vec{e_2}\ A\vec{e_3}] && (5) \\ A^{-1}A &= A^{-1}[A\vec{e_1}\ A\vec{e_2}\ A\vec{e_3}] && (6) \\ &= [A^{-1}A\vec{e_1}\ A^{-1}A\vec{e_2}\ A^{-1}A\vec{e_3}] && (7) \\ &= [I_3\vec{e_1}\ I_3\vec{e_2}\ I_3\vec{e_3}] && (8) \\ &= [\vec{e_1}\ \vec{e_2}\ \vec{e_3}] && (9) \\ &= I_3 && (10) \end{aligned}$$

□

3 For each of the following subspaces of M_{33} find a basis and state the dimension.

3.a $W_1 = \{A \in M_{33} | A$ is a diagonal matrix$\}$

$\boxed{\dim(W_1) = 3}$

Proof.

$$D = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}, A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore D = aA_1 + bA_2 + cA_3$$

The matricies A_1, A_2, A_3 form a basis for W_1 consequently, the dimension of W_1 is 3. □

3.b $W_2 = \{A \in M_{33} | A = A^T\}$ (the symmetric matrices)

$\boxed{dim(W_2) = 6}$

Proof.

$$S = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}, A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, A_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore S = aA_1 + bA_2 + cA_3 + dA_4 + eA_5 + fA_6$$

The matricies $A_1, A_2, A_3, A_4, A_5, A_6$ form a basis for W_2 consequently, the dimension of W_2 is 6. □

3.c $W_1 = \{A \in M_{33} | A = -A^T\}$ (the anti-symmetric matrices)

$\boxed{dim(W_3) = 3}$

Proof.

$$S = \begin{bmatrix} 0 & b & c \\ -b & 0 & d \\ -c & -d & 0 \end{bmatrix}, A_1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$\therefore S = aA_1 + bA_2 + cA_3$$

The matricies A_1, A_2, A_3 form a basis for W_3 consequently, the dimension of W_2 is 3. □

4 Find a basis for the subspace of P3 spanned by the following polynomials (vectors):

$p_1 = 1 + x + 3x^2 + 4x^3,\ p_2 = 1 + 2x^2 + 3x^3,\ p_3 = x + x^2 + 2x^3,\ p_4 = 1 + x + 3x^2 + 5x^3$

$\boxed{\text{Solution: the vectors } \vec{p_1}, \vec{p_2}, \vec{p_3} \text{ form a basis for span}\{p_1, p_2, p_3, p_4\}}$

Proof. The equation $k_1\vec{p_1} + k_2\vec{p_2} + k_3\vec{p_3} + k_4\vec{p_4} = \vec{0}$ can be written as a linear system

$$\begin{aligned} 1k_1 + 1k_2 + 0k_3 + 1k_4 &= 0 \\ 1k_1 + 0k_2 + 1k_3 + 1k_4 &= 0 \\ 3k_1 + 2k_2 + 1k_3 + 3k_4 &= 0 \\ 4k_1 + 3k_2 + 2k_3 + 5k_4 &= 0 \end{aligned}$$

whos augmented matrix has the reduced row echelon form

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\implies k1 = 0,\ k2 = -s,\ k3 = -s,\ k4 = s$$

removing vector $\vec{p_4}$ gives span$\{p_1, p_2, p_3\}$ = span$\{p_1, p_2, p_3, p_4\}$. Since the vector equation $k_1\vec{p_1} + k_2\vec{p_2} + k_3\vec{p_3} = \vec{0}$ has only the trivial solution. We conclude that the vectors $\vec{p_1}, \vec{p_2}, \vec{p_3}$ form a basis for span$\{p_1, p_2, p_3, p_4\}$. □

5 Let $\vec{x} = [1\ 2\ 3]^T$, $\mathcal{B} = \{[1\ 0\ 0]^T,\ [1\ 1\ 0]^T,\ [1\ 1\ 1]^T\}$, and $\mathcal{C} = \{[1\ 1\ 0]^T,\ [0\ 1\ 1]^T,\ [1\ 0\ 1]^T\}$.

5.a Find $[\vec{x}]_\mathcal{B}$

$\boxed{\text{Solution: } [\vec{x}]_\mathcal{B} = [-1\ \ -1\ 3]^T}$

Proof.

$$\text{By inspection: } \vec{x} = (-1)[1\ 0\ 0]^T + (-1)[1\ 1\ 0]^T + (3)[1\ 1\ 1]^T$$

$\therefore [\vec{x}]_\mathcal{B} = [-1\ \ -1\ 3]^T$ □

5.b Find $[\vec{x}]_\mathcal{C}$

$\boxed{\text{Solution: } [\vec{x}]_\mathcal{C} = [0\ 2\ 1]^T}$

Proof.

$$\text{By inspection: } \vec{x} = (0)[1\ 1\ 0]^T + (2)[0\ 1\ 1]^T + (1)[1\ 0\ 1]^T$$

$\therefore [\vec{x}]_\mathcal{C} = [0\ 2\ 1]^T$ □

5.c Find $P_{\mathcal{C}\leftarrow\mathcal{B}}$ and compute $P_{\mathcal{C}\leftarrow\mathcal{B}}[\vec{x}]_{\mathcal{B}}$

$$\boxed{Solutions: P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 1/2 & 1 & 1/2 \\ -1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 \end{bmatrix}; [\vec{x}]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}[\vec{x}]_{\mathcal{B}} = [0\ 2\ 1]^T}$$

Proof.

$$\text{Partitioned matrix } [\ \mathcal{C} \mid \mathcal{B}\ ] = \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\text{Transition matrix } [\ I_3 \mid \mathcal{C} \leftarrow \mathcal{B}\ ] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 1 & 1/2 \\ 0 & 1 & 0 & -1/2 & 0 & 1/2 \\ 0 & 0 & 1 & 1/2 & 0 & 1/2 \end{array}\right]$$

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 1/2 & 1 & 1/2 \\ -1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

$$\begin{aligned} [\vec{x}]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}[\vec{x}]_{\mathcal{B}} &= \begin{bmatrix} 1/2 & 1 & 1/2 \\ -1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 \end{bmatrix} [-1\ -1\ 3]^T \\ &= [-1] \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \end{bmatrix} + [-1] \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + [3] \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} \\ &= [0\ 2\ 1]^T \end{aligned}$$

$\square$

5.d Find $P_{\mathcal{B}\leftarrow\mathcal{C}}$ and compute $P_{\mathcal{B}\leftarrow\mathcal{C}}[\vec{x}]_{\mathcal{C}}$

6 Let $A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$

6.a Find a basis for each of null(A), row(A), col(A), and state the dimension of each of these subspaces.

6.b Is the vector $\vec{b} = \begin{bmatrix} 4 \\ 6 \\ -2 \end{bmatrix}$ in the column space of A? If so, write $\vec{b}$ as a linear combination of the columns of A.