Math 2311 - Assignment 6

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Question 1

Let $T_1 : \mathcal{P}_1 \to \mathcal{P}_2$ be the linear transformation defined by

$$T_1(p(x)) = xp(x)$$

and let $T_2 : \mathcal{P}_2 \to \mathcal{P}_2$ be the linear operator defined by

$$T_2(p(x)) = p(2x+1)$$

Let $B = \{1, x\}$ and $B' = \left\{1,\ x,\ x^2\right\}$ be the standard bases for P_1 and P_2.

a. Find $[T_2 \circ T_1]_{B',B}$ *and* $[T_2]_{B'}$ *and* $[T_1]_{B',B}$

b. State a formula relating the matricies in part (a)

c. Verify that the matricies in part (a) satisfy the formula you stated in part (b)

Proof

Find $[T_2 \circ T_1]_{B',B}$

$$\begin{aligned}
P_1 &= a_0 + a_1 x \\
\to T_2 \circ T_1 &= T_2(\ T_1(P_1)\ ) = T_2\left(a_0 x + a_1 x^2\right) \\
&= a_0(2x+1) + a_1(2x+1)^2 = 2a_0 x + a_0 + a_1\left(4x^2 + 4x + 1\right) \\
&= 2a_0 x + a_0 + a_1\left(4x^2 + 4x + 1\right) = 2a_0 x + a_0 + 4a_1 x^2 + 4a_1 x + a_1 \\
&= (a_0 + a_1) + (2a_0 + 4a_1)x + (4a_1)x^2 \\
\therefore [T_2 \circ T_1]_{B',B} &= \begin{bmatrix} 1 & 1 \\ 2 & 4 \\ 0 & 4 \end{bmatrix}
\end{aligned}$$

Find $[T_2]_{B'}$

$$\begin{aligned}
T_2(\ p(x)\ ) &= p(2x+1) \\
\therefore T_2(1) &= 1 \\
T_2(x) &= 2x+1 \\
T_2\left(x^2\right) &= 4x^2 + 4x + 1 \\
[T_2(1)]_{B'} &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\
[T_2(x)]_{B'} &= \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \\
\left[T_2\left(x^2\right)\right]_{B'} &= \begin{bmatrix} 4 \\ 4 \\ 1 \end{bmatrix} \\
\to [T_2]_{B'} &= \begin{bmatrix} 1 & 1 & 4 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

Find $[T_1]_{B',B}$

$$\begin{aligned}
T_1(\ p(x)\ ) &= xp(x) \\
\therefore T_1(1) &= x
\end{aligned}$$

$$
\begin{aligned}
T_2(x) &= x^2 \\
[T_1(1)]_{B'} &= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\
[T_1(1)]_{B'} &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
\rightarrow [T_1]_{B',B} &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}
\end{aligned}
$$

So, a formula that relates these matricies is

$$
\begin{aligned}
[T_1]_{B',B}[T_2]_{B'} &= [T_2 \circ T_1]_{B',B} \\
&= \begin{bmatrix} 1 & 1 & 4 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 4 \\ 0 & 4 \end{bmatrix} \quad \square
\end{aligned}
$$

Question 2
$T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is defined by

$$T(x_1, x_2, x_3) = (x_1 + 2x_2 - x_3,\ -2x_2, x_1 + 7x_3)$$

B is the standard basis, and $B' = \{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3\}$, where

$$\mathbf{v}_1 = (1,\ 0,\ 0),\ \mathbf{v}_2 = (1,\ 1,\ 0),\ \mathbf{v}_3 = (1,\ 1,\ 1)$$

Find the matrix for T relative to the basis B, and use **Theorem 8.5.2** to compute the matrix for T relative to the basis B'

Theorem 8.5.2

Let $T : V \rightarrow V$ be a linear operator on a finite-dimensional vector space V, and let B and B' be bases for V. Then

$$[T]_{B'} = P^{-1}[T]_B P$$

where $P = P_{B \leftarrow B'}$ and $P^{-1} = P_{B' \leftarrow B}$

Proof
By observation because our basis B is the standard basis,

$$[T]_B = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -2 & 0 \\ -1 & 0 & 7 \end{bmatrix}$$

Find $P = P_{B \leftarrow B'}$

$$
\begin{aligned}
P_{B \leftarrow B'} &= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \\
&= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

Find $P^{-1} = P_{B' \leftarrow B}$

$$(P_{B \leftarrow B'})^{-1} = \begin{bmatrix} 1 & 1 & 1 & | & 1 & 0 & 0 \\ 0 & 1 & 1 & | & 0 & 1 & 0 \\ 0 & 0 & 1 & | & 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Find $[T]_{B'} = P^{-1}[T]_B P$

$$P^{-1}[T]_B P = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 2 & -2 & 0 \\ -1 & 0 & 7 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 3 & -1 \\ -1 & -1 & -7 \\ 1 & 0 & 7 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 4 & 3 \\ -1 & 2 & -9 \\ 1 & 1 & 8 \end{bmatrix} \quad \square$$

Question 3

Consider the linear operator $T: \mathcal{P}_1 \to \mathcal{P}_1$ given by $T(a_0 + a_1 x) = (a_0 - a_1)(1 + x)$.

a. Find a basis for $\mathrm{k}er(T)$

b. Find a basis for $\mathrm{R}ange(T)$

c. Find $[T]_B$ where $B = \{1, x\}$ is the standard basis for $\mathcal{P}_1$

d. Find the eigenvalues and eigenvectors of $[T]_B$ (eigenvectors expressed as column vectors)

e. Find the eigenvalues and eigenvectors of T (eigenvectors expressed as polynomials).

f. Explain why there is no basis, B'' in which $[T]_{B''}$ is diagonal.

g. Let $B' = \{1 + x, 1 - x\}$ and find the two change of basis matrices $P_{B \leftarrow B'}$ and $P_{B' \leftarrow B}$.

h. Find $[T]_{B'}$.

i. What is the linear operator $T \circ T$? (you can compute this directly, or you can probably figure out the answer via matrix multiplication of some sort)

proof

(**a**). Find a basis for $ker(T)$

$$(a_0 - a_1)(1 + x) = a_0 + a_0 x - a_1 - a_1 x = (a_0 - a_1)x + (a_0 - a_1)$$
$$\to \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & -1 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \to t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$\therefore ker(T) = span\{1 + x\} \quad \square$$

(**b**). To find a basis for $Range(T)$, notice the pivot

$$\left[\begin{array}{cc|c} \mathbf{1} & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\therefore range(T) = span\{1 + x\} \quad \square$$

(**c**). Find $[T]_B$ where $B = \{1, x\}$

$$[T]_B = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \quad \square$$

(**d**). Find the eigenvalues and eigenvectors of $[T]_B$

$$\begin{aligned} det(\lambda I - [T]_B) &= \begin{vmatrix} \lambda - 1 & 1 \\ -1 & \lambda + 1 \end{vmatrix} \\ &= (\lambda-1)(\lambda+1) - (1)(-1) \\ &= \lambda^2 - 1 + 1 = \lambda^2 \\ \rightarrow \lambda &= 0 \end{aligned}$$

$$rref\left(\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

$$\rightarrow t\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \square$$

(**e**) So, the eigenvalues of $[T]_B$ are $\lambda = 0$, the eigenvectors are $(1,\ 1)^T$. The eigenvalue as a polynomial is 0 and the eigenvector as a polynomial is $1+x$ $\square$

(**f**) There is no basis, B'' in which $[T]_{B''}$ is diagonal because the algebraic multiplicity of $[T]_B$ is 2 and the geometric multiplicity is only 1. $\square$

(**g, h**) Let $B' = \{1+x, 1-x\}$ and find the two change of basis matrices $P_{B \leftarrow B'}$ and $P_{B' \leftarrow B}$

$$T(a_0 + a_1 x) = (a_0 - a_1)(1+x)$$

$$\rightarrow (1-1)(1+x) = 0 \qquad\qquad 1+x$$

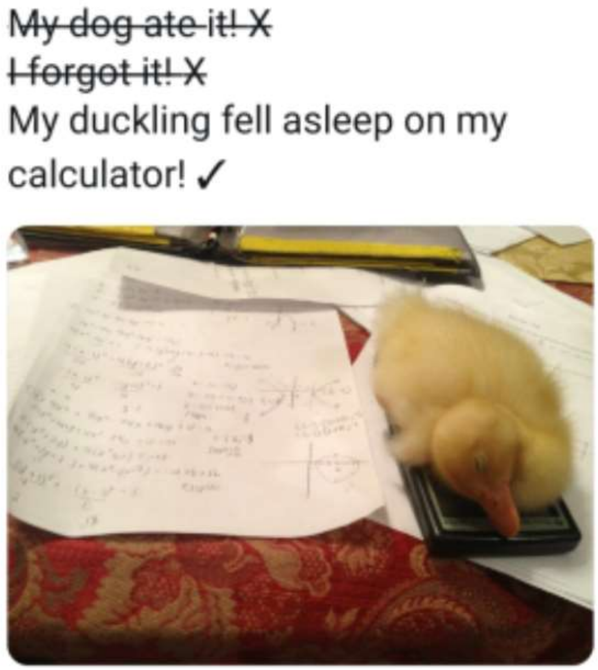
$$\rightarrow (1-(-1))(1+x) = 2(1+x) \qquad\qquad 1-x$$

$$\rightarrow [T]_{B'} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \quad \square$$

(**i**) Find $T \circ T$

$$\begin{aligned} T((a_0 - a_1) + (a_0 - a_1)x) &= [(a_0 - a_1) - (a_0 - a_1)](1+x) \\ &= 0 \quad \square \end{aligned}$$

Bonus Prove that for a linear operator $T : V \to V$, $T \circ T = 0$ (the zero transformation) if and only if $Range(T) \subseteq \ker(T)$



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