Math 2311 - Assignment 5
Due: Friday, April 1, 2022

1. For each of the following linear transformations $T : V \to W$ determine $\text{rank}(T)$ and $\text{nullity}(T)$.

   (a) $\dim(V) = 5$, $\dim(W) = 7$, and T is one-to-one.

   (b) $\dim(W) = 4$, and T is both one-to-one and onto.

   (c) $\dim(V) = 5$, $\dim(W) = 3$, and T is onto.

   (d) $T_A : \mathbb{R}^4 \to \mathbb{R}^3$ is given by $T_A(\vec{x}) = A\vec{x}$ where $A = \begin{bmatrix} 2 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & -1 \end{bmatrix}$.

2. Section 8.2, question 34 from the 12th edition (question 26 from 11th edition).

3. Let $T_1 : U \to V$ and $T_2 : V \to W$ be linear transformations.

   (a) Suppose $T_2 \circ T_1 : U \to W$ is one-to-one. Prove T_1 is one-to-one.

   (b) Suppose $T_2 \circ T_1 : U \to W$ is onto. Prove T_2 is onto.

4. Consider the linear transformation $D : \mathcal{P}_3 \to \mathcal{P}_2$ given by $D(p(x)) = p'(x)$ (i.e. differentiation). Consider also the linear transformation $S : \mathcal{P}_2 \to \mathcal{P}_3$ given by $S(p(x)) = \displaystyle\int_0^x p(t)dt$

   (a) For an arbitrary $p(x) \in \mathcal{P}_2$, what is $(D \circ S)(p(x))$? What is another name for the transformation $D \circ S$?

   (b) For an arbitrary $p(x) \in \mathcal{P}_3$, what is $(S \circ D)(p(x))$? What can you say about the transformation $S \circ D$? Compare to part (a).

   (c) Are S and D inverses? Explain.

   (d) Find a basis for $\text{Range}(S)$.

   (e) Find a basis for $\ker(D)$, and for $\ker(D)^\perp$ (with respect to the standard inner product - see example 7 from section 6.1). Compare to part (d).

   (f) Show that S is one-to-one. (and therefore $S : \mathcal{P}_2 \to \text{Range}(S)$ is an isomorphism).

   (g) Verify that $D : \text{Range}(S) \to \mathcal{P}_2$ and $S : \mathcal{P}_2 \to \text{Range}(S)$ are inverses.

   (h) If we use the standard inner products on $\mathcal{P}_3$ and $\mathcal{P}_2$, is D an inner product space isomorphism? Justify your answer.

Bonus: Prove that for any linear transformation $T : V \to W$, if $\tilde{T} : \ker(T)^\perp \to \text{Range}(T)$ is the restriction of T, then $\tilde{T}$ is an isomorphism.