

1 Find a basis for the subspace of P^3 spanned by the following polynomials (vectors):

$$p_1 = 1 + x + 3x^2 + 4x^3, \quad p_2 = 1 + 2x^2 + 3x^3, \quad p_3 = x + x^2 + 2x^3, \quad p_4 = 1 + x + 3x^2 + 5x^3$$

Solution: the vectors $\vec{p}_1, \vec{p}_2, \vec{p}_3$ form a basis for $\text{span}\{p_1, p_2, p_3, p_4\}$

Proof. The equation $k_1\vec{p}_1 + k_2\vec{p}_2 + k_3\vec{p}_3 + k_4\vec{p}_4 = \vec{0}$ can be written as a linear system

$$\begin{aligned} 1k_1 + 1k_2 + 0k_3 + 1k_4 &= 0 \\ 1k_1 + 0k_2 + 1k_3 + 1k_4 &= 0 \\ 3k_1 + 2k_2 + 1k_3 + 3k_4 &= 0 \\ 4k_1 + 3k_2 + 2k_3 + 5k_4 &= 0 \end{aligned}$$

whos augmented matrix has the reduced row echelon form

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$
$$\implies k_1 = 0, k_2 = -s, k_3 = -s, k_4 = s$$

removing vector \vec{p}_4 gives $\text{span}\{p_1, p_2, p_3\} = \text{span}\{p_1, p_2, p_3, p_4\}$. Since the vector equation $k_1\vec{p}_1 + k_2\vec{p}_2 + k_3\vec{p}_3 = \vec{0}$ has only the trivial solution. We conclude that the vectors $\vec{p}_1, \vec{p}_2, \vec{p}_3$ form a basis for $\text{span}\{p_1, p_2, p_3, p_4\}$. \square