

# Math 2311 — Assignment 1

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# 1 Question 1

1. Determine if each of the following sets is a vector space.

**1.1 Answer: No,  $V$  is not a vector space.**

(a)  $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid x \geq y \right\}$  with the usual scalar multiplication and vector addition from  $\mathbb{R}^2$

*Proof.* let  $(x_0, y_0), (x_1, y_1) \in V$ , and take the vector space operations on  $V$  to be the usual operations of *vector* addition and *scalar* multiplication; that is,

$$(x_0, y_0) + (x_1, y_1) = (x_0 + x_1, y_0 + y_1) \quad (1)$$

$$k(x_0, y_0) = (kx_0, ky_0) \quad (2)$$

$V$  is closed under scalar addition since  $x_0 + x_1 \geq y_0 + y_1$

However, by properties of inequalities if the constant,  $k$ , is negative, we must reverse the symbol to preserve the inequality relation.

Given that  $k$  is negative,  $x \geq y \rightarrow kx \leq ky$

□

## 1.2 Answer: Yes, $W$ is a vector space.

- (b) Consider the set  $W = \{f \in F(-\infty, \infty) \mid f(1) = 0\}$  with the usual scalar multiplication and vector addition from  $F(-\infty, \infty)$ . Is  $W$  a vector space?

*Proof.* Since we know that  $F(-\infty, \infty)$  (with the usual operations) is a vector space, and since  $W$  is a subset of  $F(-\infty, \infty)$  (with the same operations), it suffices to prove that  $W$  is a subspace of  $F(-\infty, \infty)$ . To this end we must show three things:

- (a) That  $W$  is non-empty.
- (b) That  $W$  is closed under addition.
- (c) That  $W$  is closed under scalar multiplication.

There exists a function  $\mathbf{0}$  in  $F(-\infty, \infty)$  defined by  $\mathbf{0}(x) = 0$  for all  $x$ . Clearly  $\mathbf{0}(1) = 0$  so  $W$  is non-empty.

Now suppose  $f$  and  $g$  are two functions in  $W$ . We must show that  $f + g$  is in  $W$ .

$$\begin{aligned}(f + g)(1) &= f(1) + g(1) && \text{(definition of addition of functions)} \\ &= 0 && (f \text{ and } g \text{ are in } W)\end{aligned}$$

Finally, to show that  $W$  is closed under scalar multiplication, suppose  $f$  is in  $W$  and  $k$  is a scalar, then

$$\begin{aligned}(kf)(1) &= kf(1) && \text{(definition of scalar multiplication on functions)} \\ &= 0 && (f \text{ is in } W)\end{aligned}$$

so  $(kf)$  is in  $W$  and  $W$  is closed under scalar multiplication.

Therefore  $W$  is a subspace of  $F(-\infty, \infty)$  and hence is a vector space. □

## 2 Question 2

2. Let  $V$  be a vector space.

### 2.1

(a) If  $k$  is any scalar, prove that  $k\vec{0} = \vec{0}$ .

*Proof.*

$$\begin{aligned}k(\vec{0} + \vec{0}) &= k\vec{0} + k\vec{0} && \text{(vector space axiom 7)} \\k\vec{0} &= k\vec{0} + k\vec{0} && \text{(vector space axiom 4)} \\k\vec{0} + (-k\vec{0}) &= (-k\vec{0}) + (k\vec{0} + k\vec{0}) && \text{(vector space axiom 5)} \\k\vec{0} + (-k\vec{0}) &= ((-k\vec{0}) + k\vec{0}) + k\vec{0} && \text{(vector space axiom 3)} \\\vec{0} &= \vec{0} + k\vec{0} && \text{(vector space axiom 5)} \\&= k\vec{0} && \text{(vector space axiom 4)}\end{aligned}$$

□

### 2.2

(b) Prove that the zero vector in  $V$  is unique.

*Proof.* We must show that there is only one vector,  $\vec{0}$ , with the property that  $\vec{0} + \vec{v} = \vec{v} + \vec{0} = \vec{v}$ .

Suppose  $\vec{0}_1$  and  $\vec{0}$  are zero vectors in  $V$ . Then

$$\begin{aligned}\vec{0}_1 &= \vec{0}_1 + \vec{0} && \text{(vector space axiom 4)} \\&= \vec{0} + \vec{0}_1 && \text{(vector space axiom 2)} \\&= \vec{0} && \text{(vector space axiom 4)}\end{aligned}$$

Therefore  $\vec{0}_1 = \vec{0}$ . So, the zero vector is unique.

□

### 3 Question 3

3. Determine if each of the following are subspaces of  $M_{nn}$

**3.1 Answer: No,  $A \in M_{nn}$  is not a subspace of  $M_{nn}$ .**

(a)  $\{A \in M_{nn} \mid \det(A) = 0\}$

*Proof.*

$$\det \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0, \det \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\det \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$$

□

**3.2 Answer: Yes,  $W$  is a subspace of  $M_{nn}$**

(b)  $\{A \in M_{nn} \mid \text{tr}(A) = 0\}$

*Proof.* Let  $[B = (b_{ii})] \in W$  be a square matrix of order  $n$  such that  $\text{tr}(B) = 0$  and let  $k$  be any scalar.

The set  $W$  is non empty because if we let  $a_{ii} = 0$  for all  $i$  then  $\text{tr}(A) = 0$  therefore  $W$  contains the  $\mathbf{0}$  matrices.

*addition :*

$$\begin{aligned} \text{tr}(A + B) &= \sum_{i=1}^n (a_{ii} + b_{ii}) \\ &= \sum_{i=1}^n a_{ii} + \sum_{i=1}^n b_{ii} \\ &= \text{tr}(A) + \text{tr}(B) = 0 + 0 = 0 \end{aligned}$$

*multiplication :*

$$\begin{aligned} \text{tr}(kA) &= \sum_{i=1}^n (k \cdot a_{ii}) \\ &= k \cdot \sum_{i=1}^n a_{ii} \\ &= k \cdot \text{tr}(A) = k \cdot 0 = 0 \end{aligned}$$

To be clear, if we take some  $C = A + B$  such that  $A$  and  $B$  are in  $W$ , then for all  $C = (c_{ii})$ , the sum will be 0, so  $C$  is also in  $W$ . Therefore  $W$  is a subspace of  $M_{nn}$ . □

### 3.3 Answer: Yes, $W$ is a subspace of $M_{nn}$

(c)  $\{A \in M_{nn} \mid A^T = A\}$

**Answer:** Yes,  $W = \{A \in M_{nn} \mid A^T = A\}$  is a subspace of  $M_{nn}$ .

*Proof.* Let  $[B = (b_{ij})] \in W$  be a square matrix of order  $n$  such that  $b_{ij} = b_{ji}$ , and let  $k$  be any scalar.

The set  $W$  is non empty because, if we let  $[A = (a_{ij})] = 0$  for all  $(i, j)$  then  $W$  contains the  $\mathbf{0}$  matrices. It remains to show that  $W$  is closed under addition and scalar multiplication.

*addition :*

$$\begin{aligned} C &= (A + B)^T \\ &= A^T + B^T \\ &= A + B \end{aligned}$$

*multiplication :*

$$\begin{aligned} (k \cdot A)^T &= k \cdot A^T \\ &= k \cdot A \end{aligned}$$

so  $W$  is a subspace of  $M_{nn}$ . □

## 4 Question 4

### 4.1 Answer: Yes, $(p_1, p_2, p_3) = (4, -5, 1)$

4. Consider the following vectors in  $P_2$ :  $p_1 = 2+x+4x^2$ ,  $p_2 = 1-x+3x^2$ ,  $p_3 = 3+2x+5x^2$ .

(a) Express the vector  $g = 6 + 11x + 6x^2$  as a linear combination of  $p_1, p_2, p_3$ .

*Proof.*

$$\begin{aligned}(6 + 11x + 6x^2) &= k_0(2 + x + 4x^2) + k_1(1 - x + 3x^2) + k_2(3 + 2x + 5x^2) \\ &= (k_0 \cdot 2 + k_1 + k_2 \cdot 3) + (k_0 \cdot x - k_1 \cdot x + k_2 \cdot 2x) + (k_0 \cdot 4x^2 + k_1 \cdot 3x^2 + k_2 \cdot 5x^2) \\ &= (k_0 \cdot 2 + k_1 + k_2 \cdot 3) + (k_0 - k_1 + k_2 \cdot 2)x + (k_0 \cdot 4 + k_1 \cdot 3 + k_2 \cdot 5)x^2\end{aligned}$$

$$\begin{aligned} & \begin{bmatrix} 2 & 1 & 3 & | & 6 \\ 1 & -1 & 2 & | & 11 \\ 4 & 3 & 5 & | & 6 \end{bmatrix} \\ [-2r_2 + r_1] \wedge [-4r_2 + r_1] & \begin{bmatrix} 0 & 3 & -1 & | & -16 \\ 1 & -1 & 2 & | & 11 \\ 0 & 7 & -3 & | & -38 \end{bmatrix} \\ r_2 \leftrightarrow r_1 & \begin{bmatrix} 1 & -1 & 2 & | & 11 \\ 0 & 3 & -1 & | & -16 \\ 0 & 7 & -3 & | & -38 \end{bmatrix} \\ \frac{1}{3}r_2 & \begin{bmatrix} 1 & -1 & 2 & | & 11 \\ 0 & 1 & \frac{-1}{3} & | & \frac{-16}{3} \\ 0 & 7 & -3 & | & -38 \end{bmatrix} \\ -7r_2 + r_3 & \begin{bmatrix} 1 & -1 & 2 & | & 11 \\ 0 & 1 & \frac{-1}{3} & | & \frac{-16}{3} \\ 0 & 0 & \frac{\frac{-2}{3}}{3} & | & \frac{\frac{-2}{3}}{3} \end{bmatrix} \\ -\frac{3}{2}r_3 & \begin{bmatrix} 1 & -1 & 2 & | & 11 \\ 0 & 1 & \frac{-1}{3} & | & \frac{-16}{3} \\ 0 & 0 & 1 & | & 1 \end{bmatrix} \\ \frac{1}{3}r_3 + r_2 & \begin{bmatrix} 1 & -1 & 2 & | & 11 \\ 0 & 1 & 0 & | & -5 \\ 0 & 0 & 1 & | & 1 \end{bmatrix} \\ r_1 + r_2 & \begin{bmatrix} 1 & 0 & 2 & | & 6 \\ 0 & 1 & 0 & | & -5 \\ 0 & 0 & 1 & | & 1 \end{bmatrix} \\ r_1 + r_3 & \begin{bmatrix} 1 & 0 & 0 & | & 4 \\ 0 & 1 & 0 & | & -5 \\ 0 & 0 & 1 & | & 1 \end{bmatrix} \end{aligned}$$

$$\therefore (k_1, k_2, k_3) = (4, -5, 1)$$

□

## 4.2 Answer: Yes, $\text{span}(\{\vec{p}_1, \vec{p}_2, \vec{p}_3\}) = P_2$

(b) Does  $\{p_1, p_2, p_3\}$  span  $P_2$ ?

*Proof.* An arbitrary vector in  $P_2$  is of the form  $\vec{p} = a + bx + cx^2$  and so becomes,

$$k_0(2 + x + 4x^2) + k_1(1 - x + 3x^2) + k_2(3 + 2x + 5x^2) = a + bx + cx^2$$

which we can rewrite as

$$(k_0 2 + k_1 + k_2 3) + (k_0 - k_1 + k_2 2)x + (k_0 4 + k_1 3 + k_2 5)x^2 = a + bx + cx^2$$

Equating corresponding coefficients yields a linear system whose augmented matrix is

$$A = \left[ \begin{array}{ccc|c} 2 & 1 & 3 & a \\ 1 & -1 & 2 & b \\ 4 & 3 & 5 & c \end{array} \right]$$

Our problem reduces to ascertaining whether this system is consistent for all values of  $a$ ,  $b$ , and  $c$ . This can be determined if its coefficient matrix has a nonzero determinant, from our theorem for equivalent statements. If  $A$  is an  $n \times n$  matrix such that  $\det(A) \neq 0$  then  $A\vec{x} = \vec{0}$ .

It follows from solution (a) that

$$\left[ \begin{array}{ccc|c} 2 & 1 & 3 & 0 \\ 1 & -1 & 2 & 0 \\ 4 & 3 & 5 & 0 \end{array} \right] = \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right]$$

so  $A$  is consistent for every choice  $a, b$ , and  $c$ . Thus, the vectors in  $\{\vec{p}_1, \vec{p}_2, \vec{p}_3\}$  span  $P_2$ .  $\square$

## 4.3 Answer: Yes, $k_1\vec{p}_1 + k_2\vec{p}_2 + k_3\vec{p}_3 = \vec{0}$

(c) Is  $\{p_1, p_2, p_3\}$  linearly independent?

*Proof.* From part (b) we have

$$\left[ \begin{array}{ccc|c} 2 & 1 & 3 & 0 \\ 1 & -1 & 2 & 0 \\ 4 & 3 & 5 & 0 \end{array} \right] = \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right]$$

Therefore  $(k_1, k_2, k_3) = (0, 0, 0)$  so  $\{\vec{p}_1, \vec{p}_2, \vec{p}_3\}$  is linearly independent  $\square$



## 5 Question 5

5. Consider the following planes in  $\mathbb{R}^3$ .  $P_1 : 2x + 3y - z = 0$  and  $P_2 : x + 2y - 2z = 0$

**5.1 Answer:**  $\{(-\frac{3}{2}, 1, 0), (\frac{1}{2}, 0, 1)\}$

(a) Find a set of vectors that spans  $P_1$ .

*Proof.* Solve the following system  $(2, 3, -1)^T = \vec{0}$

$$\begin{array}{r} [2 \quad 3 \quad -1 \quad | \quad 0] \\ \frac{1}{2}r1 [1 \quad \frac{3}{2} \quad \frac{-1}{2} \quad | \quad 0] \\ \therefore (z = r), (y = q), (x = -\frac{3}{2}q + \frac{1}{2}r) \end{array}$$

giving the following set of vectors.

$$\begin{aligned} (x, y, z) &= (-\frac{3}{2}q + \frac{1}{2}r, q, r) \\ &= q(-\frac{3}{2}, 1, 0) + r(\frac{1}{2}, 0, 1) \end{aligned}$$

a set of vectors that spans  $P_1$  are  $\{(-\frac{3}{2}, 1, 0), (\frac{1}{2}, 0, 1)\}$  □

**5.2 Answer: any 2 vectors from the set  $\{(0, 2, 2), (-2, 0, -1), (-2, 1, 0)\}$**

(b) Find a set of vectors that spans  $P_2$ .

*Proof.*

$$\text{Let } \vec{e}_1 = (1, 0, 0), \vec{e}_2 = (0, 1, 0), \vec{e}_3 = (0, 0, 1)$$

be unit vectors in  $\mathbb{R}^3$ , we will compute the cross products

$$\vec{e}_1 \times (1, 2, -2), \vec{e}_2 \times (1, 2, -2), \vec{e}_3 \times (1, 2, -2)$$

to find a set of vectors that spans  $P_2$

$$\begin{aligned} (1, 0, 0) \times (1, 2, -2) &= ((0)(-2) - (0)(2), (0)(1) - (1)(-2), (1)(2) - (0)(1)) \\ &= (0, 2, 2) \end{aligned}$$

$$\begin{aligned} (0, 1, 0) \times (1, 2, -2) &= ((1)(-2) - (0)(2), (0)(1) - (0)(-2), (0)(3) - (1)(1)) \\ &= (-2, 0, -1) \end{aligned}$$

$$\begin{aligned} (0, 0, 1) \times (1, 2, -2) &= ((0)(-2) - (1)(2), (1)(1) - (0)(-2), (0)(3) - (0)(1)) \\ &= (-2, 1, 0) \end{aligned}$$

any 2 vectors from the set  $\{(0, 2, 2), (-2, 0, -1), (-2, 1, 0)\}$  will span the plane  $P_2$ . □

### 5.3 Answer: $(-4, 3, 1)$ spans $P_1 \cap P_2$

(c) Find a set of vectors that spans the intersection of  $P_1$  and  $P_2$ .

*Proof.* We will find the span of the solution space by solving the following system of equations

$$P_1 : 2x + 3y - z = 0$$

$$P_2 : x + 2y - 2z = 0$$

$$\begin{aligned} & \begin{bmatrix} 2 & 3 & -1 & | & 0 \\ 1 & 2 & -2 & | & 0 \end{bmatrix} \\ [r1] & \leftrightarrow [r2] \begin{bmatrix} 2 & 3 & -1 & | & 0 \\ 1 & 2 & -2 & | & 0 \end{bmatrix} \\ (-2)r1 & + r2 \begin{bmatrix} 1 & 2 & -2 & | & 0 \\ 0 & -1 & 3 & | & 0 \end{bmatrix} \\ 2(r2) & + r1 \begin{bmatrix} 1 & 0 & 4 & | & 0 \\ 0 & -1 & 3 & | & 0 \end{bmatrix} \\ \therefore (z = t), & (y = 3t), (x = -4t) \end{aligned}$$

giving the following vector.

$$(x, y, z) = (-4t, 3t, t) = t(-4, 3, 1)$$

therefore the set  $\{(-4, 3, 1)\}$  spans  $P_1 \cap P_2$

□

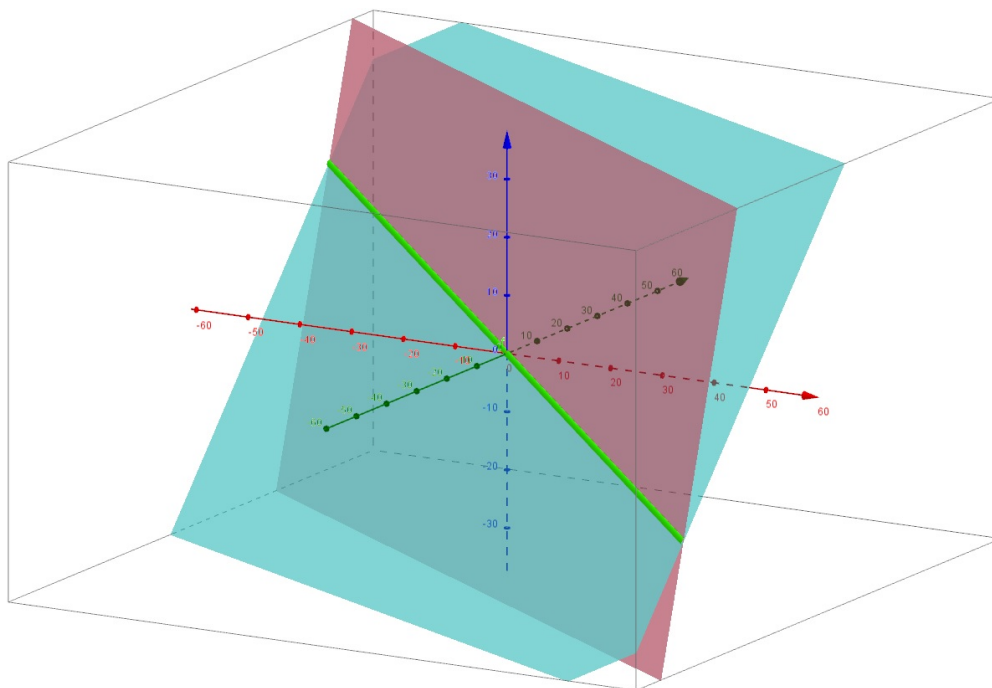


Figure 1:  $p_1 \cap p_2$  <https://www.geogebra.org/3d/kuawemzp>