

# Math 2311 – Assignment 1

Michael Walker

January 17, 2022

1. Determine if each of the following sets is a vector space.

(a)  $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid x \geq y \right\}$  with the usual scalar multiplication and vector addition from  $\mathbb{R}^2$

**Answer:** No,  $V$  is not a vector space.

*Proof.* let  $(x_0, y_0)^T, (x_1, y_1)^T \in V$ , and take the vector space operations on  $V$  to be the usual operations of *vector* addition and *scalar* multiplication; that is,

$$(x_0, y_0)^T + (x_1, y_1)^T = (x_0 + x_1, y_0 + y_1)^T \quad (1)$$

$$k(x_0, y_0)^T = (kx_0, ky_0)^T \quad (2)$$

$V$  is closed under scalar addition since  $x_0 + x_1 \geq y_0 + y_1$

However, by properties of inequalities if the constant,  $k$ , is negative, we must reverse the symbol to preserve the inequality relation.

Given that  $k$  is negative,  $x \geq y \rightarrow kx \leq ky$  □

- (b) Consider the set  $W = \{f \in F(-\infty, \infty) \mid f(1) = 0\}$  with the usual scalar multiplication and vector addition from  $F(-\infty, \infty)$ . Is  $W$  a vector space?

**Answer:** Yes,  $W$  is a vector space.

*Proof.* Since we know that  $F(-\infty, \infty)$  (with the usual operations) is a vector space, and since  $W$  is a subset of  $F(-\infty, \infty)$  (with the same operations), it suffices to prove that  $W$  is a subspace of  $F(-\infty, \infty)$ . To this end we must show three things:

- (a) That  $W$  is non-empty.
- (b) That  $W$  is closed under addition.
- (c) That  $W$  is closed under scalar multiplication.

There exists a function  $\mathbf{0}$  in  $F(-\infty, \infty)$  defined by  $\mathbf{0}(x) = 0$  for all  $x$ . Clearly  $\mathbf{0}(1) = f(1) = 0$  so  $W$  is non-empty.

Now suppose  $f$  and  $g$  are two functions in  $W$ . We must show that  $f + g$  is in  $W$ .

$$\begin{aligned}(f + g)(1) &= f(1) + g(1) && \text{(definition of addition of functions)} \\ &= 0 && (f \text{ and } g \text{ are in } W)\end{aligned}$$

Finally, to show that  $W$  is closed under scalar multiplication, suppose  $f$  is in  $W$  and  $k$  is a scalar, then

$$\begin{aligned}(kf)(1) &= kf(1) && \text{(definition of scalar multiplication on functions)} \\ &= 0 && (f \text{ is in } W)\end{aligned}$$

so  $(kf)$  is in  $W$  and  $W$  is closed under scalar multiplication.

Therefore  $W$  is a subspace of  $F(-\infty, \infty)$  and hence is a vector space.  $\square$

2. Let  $V$  be a vector space.

(a) If  $k$  is any scalar, prove that  $k\vec{0} = \vec{0}$ .

*Proof.*

$$\begin{aligned}k(\vec{0} + \vec{0}) &= k\vec{0} + k\vec{0} && \text{(vector space axiom 7)} \\ k\vec{0} &= k\vec{0} + k\vec{0} && \text{(vector space axiom 4)} \\ k\vec{0} + (-k\vec{0}) &= (-k\vec{0}) + (k\vec{0} + k\vec{0}) && \text{(vector space axiom 5)} \\ k\vec{0} + (-k\vec{0}) &= ((-k\vec{0}) + k\vec{0}) + k\vec{0} && \text{(vector space axiom 3)} \\ \vec{0} &= \vec{0} + k\vec{0} && \text{(vector space axiom 5)} \\ &= k\vec{0} && \text{(vector space axiom 4)}\end{aligned}$$

$\square$

(b) Prove that the zero vector in  $V$  is unique.

*Proof.* We must show that there is only one vector,  $\vec{0}$ , with the property that  $\vec{0} + \vec{v} = \vec{v} + \vec{0} = \vec{v}$ .

Suppose  $\vec{0}_1$  and  $\vec{0}$  are zero vectors in  $V$  Then

$$\begin{aligned}\vec{0}_1 &= \vec{0}_1 + \vec{0} && \text{(vector space axiom 4)} \\ &= \vec{0} + \vec{0}_1 && \text{(vector space axiom 2)} \\ &= \vec{0} && \text{(vector space axiom 4)}\end{aligned}$$

Therefore  $\vec{0}_1 = \vec{0}$ . So, the zero vector is unique.  $\square$

3. Determine if each of the following are subspaces of  $M_{nn}$

(a)  $\{A \in M_{nn} \mid \det(A) = 0\}$

**Answer:** No,  $\{A \in M_{nn} \mid \det(A) = 0\}$  is not a subspace of  $M_{nn}$ .

*Proof.*

$$\det \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0, \det \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0$$
$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$\det \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$$

□

(b)  $\{A \in M_{nn} \mid \text{tr}(A) = 0\}$

**Answer:** Yes,  $W = \{A \in M_{nn} \mid \text{tr}(A) = 0\}$  is a subspace of  $M_{nn}$ .

*Proof.* Let  $[B = (b_{ii})] \in W$  be a square matrix of order  $n$  such that  $\text{tr}(B) = 0$  and let  $k$  be any scalar.

The set  $W$  is non empty because if we let  $a_{ii} = 0$  for all  $i$  then  $\text{tr}(A) = 0$  therefore  $W$  contains the  $\mathbf{0}$  matrices. It remains to show that  $W$  is closed under addition and scalar multiplication.

*addition :*

$$\begin{aligned} \text{tr}(A + B) &= \sum_{i=1}^n (a_{ii} + b_{ii}) \\ &= \sum_{i=1}^n a_{ii} + \sum_{i=1}^n b_{ii} \\ &= \text{tr}(A) + \text{tr}(B) \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

*multiplication :*

$$\begin{aligned} \text{tr}(kA) &= \sum_{i=1}^n (k \cdot a_{ii}) \\ &= k \cdot \sum_{i=1}^n a_{ii} \\ &= k \cdot \text{tr}(A) \\ &= k \cdot 0 \\ &= 0 \end{aligned}$$

To be clear, if we take some  $C = A + B$  such that  $A$  and  $B$  are in  $W$ , then for all  $C = (c_{ii})$ , the sum will be 0, so  $C$  is also in  $W$ . Therefore  $W$  is a subspace of  $M_{nn}$ . □

(c)  $\{A \in M_{nn} \mid A^T = A\}$

**Answer:** Yes,  $W = \{A \in M_{nn} \mid A^T = A\}$  is a subspace of  $M_{nn}$ .

*Proof.* Let  $[B = (b_{ij})] \in W$  be a square matrix of order  $n$  such that  $b_{ij} = b_{ji}$ , and let  $k$  be any scalar.

The set  $W$  is non empty because, if we let  $[A = (a_{ij})] = 0$  for all  $(i, j)$  then  $W$  contains the  $\mathbf{0}$  matrices. It remains to show that  $W$  is closed under addition and scalar multiplication.

*addition :*

$$\begin{aligned} C &= (A + B)^T \\ &= A^T + B^T \\ &= A + B \end{aligned}$$

*multiplication :*

$$\begin{aligned} (k \cdot A)^T &= k \cdot A^T \\ &= k \cdot A \end{aligned}$$

so  $W$  is a subspace of  $M_{nn}$ .

□