

Math 2311 — Assignment 1

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1 Question 1

1. Determine if each of the following sets is a vector space.

1.1 Answer: No, V is not a vector space.

- (a) $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid x \geq y \right\}$ with the usual scalar multiplication and vector addition from \mathbb{R}^2

Proof. let $(x_0, y_0), (x_1, y_1) \in V$, and take the vector space operations on V to be the usual operations of *vector* addition and *scalar* multiplication; that is,

$$(x_0, y_0) + (x_1, y_1) = (x_0 + x_1, y_0 + y_1) \quad (1)$$

$$k(x_0, y_0) = (kx_0, ky_0) \quad (2)$$

V is closed under scalar addition since $x_0 + x_1 \geq y_0 + y_1$

However, by properties of inequalities if the constant, k , is negative, we must reverse the symbol to preserve the inequality relation.

Given that k is negative, $x \geq y \rightarrow kx \leq ky$ □

1.2 Answer: Yes, W is a vector space.

- (b) Consider the set $W = \{f \in F(-\infty, \infty) \mid f(1) = 0\}$ with the usual scalar multiplication and vector addition from $F(-\infty, \infty)$. Is W a vector space?

Proof. Since we know that $F(-\infty, \infty)$ (with the usual operations) is a vector space, and since W is a subset of $F(-\infty, \infty)$ (with the same operations), it suffices to prove that W is a subspace of $F(-\infty, \infty)$. To this end we must show three things:

- (a) That W is non-empty.
- (b) That W is closed under addition.
- (c) That W is closed under scalar multiplication.

There exists a function $\mathbf{0}$ in $F(-\infty, \infty)$ defined by $\mathbf{0}(x) = 0$ for all x . Clearly $\mathbf{0}(1) = f(1) = 0$ so W is non-empty.

Now suppose f and g are two functions in W . We must show that $f + g$ is in W .

$$\begin{aligned} (f + g)(1) &= f(1) + g(1) && \text{(definition of addition of functions)} \\ &= 0 && (f \text{ and } g \text{ are in } W) \end{aligned}$$

Finally, to show that W is closed under scalar multiplication, suppose f is in W and k is a scalar, then

$$\begin{aligned} (kf)(1) &= kf(1) && \text{(definition of scalar multiplication on functions)} \\ &= 0 && (f \text{ is in } W) \end{aligned}$$

so (kf) is in W and W is closed under scalar multiplication.

Therefore W is a subspace of $F(-\infty, \infty)$ and hence is a vector space. □

2 Question 2

2. Let V be a vector space.

2.1

(a) If k is any scalar, prove that $k\vec{0} = \vec{0}$.

Proof.

$$\begin{aligned}k(\vec{0} + \vec{0}) &= k\vec{0} + k\vec{0} && \text{(vector space axiom 7)} \\k\vec{0} &= k\vec{0} + k\vec{0} && \text{(vector space axiom 4)} \\k\vec{0} + (-k\vec{0}) &= (-k\vec{0}) + (k\vec{0} + k\vec{0}) && \text{(vector space axiom 5)} \\k\vec{0} + (-k\vec{0}) &= ((-k\vec{0}) + k\vec{0}) + k\vec{0} && \text{(vector space axiom 3)} \\\vec{0} &= \vec{0} + k\vec{0} && \text{(vector space axiom 5)} \\&= k\vec{0} && \text{(vector space axiom 4)}\end{aligned}$$

□

2.2

(b) Prove that the zero vector in V is unique.

Proof. We must show that there is only one vector, $\vec{0}$, with the property that $\vec{0} + \vec{v} = \vec{v} + \vec{0} = \vec{v}$.

Suppose $\vec{0}_1$ and $\vec{0}$ are zero vectors in V . Then

$$\begin{aligned}\vec{0}_1 &= \vec{0}_1 + \vec{0} && \text{(vector space axiom 4)} \\&= \vec{0} + \vec{0}_1 && \text{(vector space axiom 2)} \\&= \vec{0} && \text{(vector space axiom 4)}\end{aligned}$$

Therefore $\vec{0}_1 = \vec{0}$. So, the zero vector is unique.

□

3 Question 3

3. Determine if each of the following are subspaces of M_{nn}

3.1 Answer: No, $A \in M_{nn}$ is not a subspace of M_{nn} .

(a) $\{A \in M_{nn} \mid \det(A) = 0\}$

Proof.

$$\det \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0, \det \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\det \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$$

□

3.2 Answer: Yes, W is a subspace of M_{nn}

(b) $\{A \in M_{nn} \mid \text{tr}(A) = 0\}$

Proof. Let $[B = (b_{ii})] \in W$ be a square matrix of order n such that $\text{tr}(B) = 0$ and let k be any scalar.

The set W is non empty because if we let $a_{ii} = 0$ for all i then $\text{tr}(A) = 0$ therefore W contains the $\mathbf{0}$ matrices.

addition :

$$\begin{aligned} \text{tr}(A + B) &= \sum_{i=1}^n (a_{ii} + b_{ii}) \\ &= \sum_{i=1}^n a_{ii} + \sum_{i=1}^n b_{ii} \\ &= \text{tr}(A) + \text{tr}(B) = 0 + 0 = 0 \end{aligned}$$

multiplication :

$$\begin{aligned} \text{tr}(kA) &= \sum_{i=1}^n (k \cdot a_{ii}) \\ &= k \cdot \sum_{i=1}^n a_{ii} \\ &= k \cdot \text{tr}(A) = k \cdot 0 = 0 \end{aligned}$$

To be clear, if we take some $C = A + B$ such that A and B are in W , then for all $C = (c_{ii})$, the sum will be 0, so C is also in W . Therefore W is a subspace of M_{nn} . □

3.3 Answer: Yes, W is a subspace of M_{nn}

(c) $\{A \in M_{nn} \mid A^T = A\}$

Answer: Yes, $W = \{A \in M_{nn} \mid A^T = A\}$ is a subspace of M_{nn} .

Proof. Let $[B = (b_{ij})] \in W$ be a square matrix of order n such that $b_{ij} = b_{ji}$, and let k be any scalar.

The set W is non empty because, if we let $[A = (a_{ij})] = 0$ for all (i, j) then W contains the $\mathbf{0}$ matrices. It remains to show that W is closed under addition and scalar multiplication.

addition :

$$\begin{aligned} C &= (A + B)^T \\ &= A^T + B^T \\ &= A + B \end{aligned}$$

multiplication :

$$\begin{aligned} (k \cdot A)^T &= k \cdot A^T \\ &= k \cdot A \end{aligned}$$

so W is a subspace of M_{nn} . □

4 Question 4

4. Consider the following vectors in P_2 : $p_1 = 2+x+4x^2$, $p_2 = 1-x+3x^2$, $p_3 = 3+2x+5x^2$.

4.1 Answer: Yes, $g = 4(2+x+4x^2) + -5(1-x+3x^2) + 1(3+2x+5x^2)$

- (a) Express the vector $g = 6 + 11x + 6x^2$ as a linear combination of p_1, p_2, p_3 .

Proof.

$$\begin{aligned} (6 + 11x + 6x^2) &= k_0(2 + x + 4x^2) + k_1(1 - x + 3x^2) + k_2(3 + 2x + 5x^2) \\ &= (k_0 \cdot 2 + k_1 + k_2 \cdot 3) + (k_0 \cdot x - k_1 \cdot x + k_2 \cdot 2x) + (k_0 \cdot 4x^2 + k_1 \cdot 3x^2 + k_2 \cdot 5x^2) \\ &= (k_0 \cdot 2 + k_1 + k_2 \cdot 3) + (k_0 - k_1 + k_2 \cdot 2)x + (k_0 \cdot 4 + k_1 \cdot 3 + k_2 \cdot 5)x^2 \end{aligned}$$

$$\begin{aligned} &\begin{bmatrix} 2 & 1 & 3 & | & 6 \\ 1 & -1 & 2 & | & 11 \\ 4 & 3 & 5 & | & 6 \end{bmatrix} \\ [-2r_2 + r_1] \wedge [-4r_2 + r_1] &\begin{bmatrix} 0 & 3 & -1 & | & -16 \\ 1 & -1 & 2 & | & 11 \\ 0 & 7 & -3 & | & -38 \end{bmatrix} \\ r_2 \leftrightarrow r_1 &\begin{bmatrix} 1 & -1 & 2 & | & 11 \\ 0 & 3 & -1 & | & -16 \\ 0 & 7 & -3 & | & -38 \end{bmatrix} \\ \frac{1}{3}r_2 &\begin{bmatrix} 1 & -1 & 2 & | & 11 \\ 0 & 1 & \frac{-1}{3} & | & \frac{-16}{3} \\ 0 & 7 & -3 & | & -38 \end{bmatrix} \\ -7r_2 + r_3 &\begin{bmatrix} 1 & -1 & 2 & | & 11 \\ 0 & 1 & \frac{-1}{3} & | & \frac{-16}{3} \\ 0 & 0 & \frac{-2}{3} & | & \frac{-2}{3} \end{bmatrix} \\ -\frac{3}{2}r_3 &\begin{bmatrix} 1 & -1 & 2 & | & 11 \\ 0 & 1 & \frac{-1}{3} & | & \frac{-16}{3} \\ 0 & 0 & 1 & | & 1 \end{bmatrix} \\ \frac{1}{3}r_3 + r_2 &\begin{bmatrix} 1 & -1 & 2 & | & 11 \\ 0 & 1 & 0 & | & -5 \\ 0 & 0 & 1 & | & 1 \end{bmatrix} \\ r_1 + r_2 &\begin{bmatrix} 1 & 0 & 2 & | & 6 \\ 0 & 1 & 0 & | & -5 \\ 0 & 0 & 1 & | & 1 \end{bmatrix} \\ r_1 + r_3 &\begin{bmatrix} 1 & 0 & 0 & | & 4 \\ 0 & 1 & 0 & | & -5 \\ 0 & 0 & 1 & | & 1 \end{bmatrix} \end{aligned}$$

$$\therefore (k_0, k_1, k_2) = (4, -5, 1)$$

□

4.2 Answer: Yes, $\text{span}(\{\vec{p}_1, \vec{p}_2, \vec{p}_3\}) = P_2$

(b) Does $\{p_1, p_2, p_3\}$ span P_2 ?

Proof. An arbitrary vector in P_2 is of the form $\vec{p} = a + bx + cx^2$ and so becomes,

$$k_0(2 + x + 4x^2) + k_1(1 - x + 3x^2) + k_2(3 + 2x + 5x^2) = a + bx + cx^2$$

which we can rewrite as

$$(k_0 2 + k_1 + k_2 3) + (k_0 - k_1 + k_2 2)x + (k_0 4 + k_1 3 + k_2 5)x^2 = a + bx + cx^2$$

Equating corresponding coefficients yields a linear system whose augmented matrix is

$$A = \left[\begin{array}{ccc|c} 2 & 1 & 3 & a \\ 1 & -1 & 2 & b \\ 4 & 3 & 5 & c \end{array} \right]$$

Our problem reduces to ascertaining whether this system is consistent for all values of a , b , and c . This can be determined if its coefficient matrix has a nonzero determinant, from our theorem for equivalent statements. If A is an $n \times n$ matrix such that $\det(A) \neq 0$ then $A\vec{x} = \vec{0}$.

It follows from solution (a) that

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 0 \\ 1 & -1 & 2 & 0 \\ 4 & 3 & 5 & 0 \end{array} \right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right]$$

so A is consistent for every choice a, b , and c . Thus, the vectors in $\{\vec{p}_1, \vec{p}_2, \vec{p}_3\}$ span P_2 . \square

4.3 Answer: Yes, $k_1\vec{p}_1 + k_2\vec{p}_2 + k_3\vec{p}_3 = \vec{0}$

(c) Is $\{p_1, p_2, p_3\}$ linearly independent?

Proof. From part (b) we have

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 0 \\ 1 & -1 & 2 & 0 \\ 4 & 3 & 5 & 0 \end{array} \right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right]$$

Therefore $(k_1, k_2, k_3) = (0, 0, 0)$ so $\{\vec{p}_1, \vec{p}_2, \vec{p}_3\}$ is linearly independent \square

5 Question 5

5. Consider the following planes in \mathbb{R}^3 . $P_1 : 2x + 3y - z = 0$ and $P_2 : x + 2y - 2z = 0$

5.1 Answer: $\{(-\frac{3}{2}, 1, 0), (\frac{1}{2}, 0, 1)\}$ spans the plane P_1

(a) Find a set of vectors that spans P_1 .

Proof. Solve the following system $(2, 3, -1)^T = \vec{0}$

$$\begin{array}{rcl} & [2 & 3 & -1 & | & 0] \\ & \frac{1}{2}r1 & [1 & \frac{3}{2} & -\frac{1}{2} & | & 0] \\ \therefore & (z = r), & (y = q), & (x = -\frac{3}{2}q + \frac{1}{2}r) \end{array}$$

giving the following set of vectors.

$$\begin{aligned} (x, y, z) &= (-\frac{3}{2}q + \frac{1}{2}r, q, r) \\ &= q(-\frac{3}{2}, 1, 0) + r(\frac{1}{2}, 0, 1) \end{aligned}$$

a set of vectors that spans P_1 are $\{(-\frac{3}{2}, 1, 0), (\frac{1}{2}, 0, 1)\}$ □

5.2 Answer: $\{(0, 2, 2), (-2, 0, -1)\}$ spans the plane P_2

(b) Find a set of vectors that spans P_2 .

Proof.

$$\text{Let } \vec{e}_1 = (1, 0, 0), \vec{e}_2 = (0, 1, 0), \vec{e}_3 = (0, 0, 1)$$

be unit vectors in \mathbb{R}^3 , we will compute the cross products

$$\vec{e}_1 \times (1, 2, -2), \vec{e}_2 \times (1, 2, -2), \vec{e}_3 \times (1, 2, -2)$$

to find a set of vectors that spans P_2

$$\begin{aligned} (1, 0, 0) \times (1, 2, -2) &= ((0)(-2) - (0)(2), (0)(1) - (1)(-2), (1)(2) - (0)(1)) \\ &= (0, 2, 2) \end{aligned}$$

$$\begin{aligned} (0, 1, 0) \times (1, 2, -2) &= ((1)(-2) - (0)(2), (0)(1) - (0)(-2), (0)(3) - (1)(1)) \\ &= (-2, 0, -1) \end{aligned}$$

$$\begin{aligned} (0, 0, 1) \times (1, 2, -2) &= ((0)(-2) - (1)(2), (1)(1) - (0)(-2), (0)(3) - (0)(1)) \\ &= (-2, 1, 0) \end{aligned}$$

any 2 vectors from the set $\{(0, 2, 2), (-2, 0, -1), (-2, 1, 0)\}$ will span the plane P_2 . □

5.3 Answer: $\{(-4, 3, 1)\}$ spans $P_1 \cap P_2$

(c) Find a set of vectors that spans the intersection of P_1 and P_2 .

Proof. We will find the span of the solution space by solving the following system of equations

$$P_1 : 2x + 3y - z = 0$$

$$P_2 : x + 2y - 2z = 0$$

$$\begin{aligned} & \begin{bmatrix} 2 & 3 & -1 & | & 0 \\ 1 & 2 & -2 & | & 0 \end{bmatrix} \\ [r1] & \leftrightarrow [r2] \begin{bmatrix} 2 & 3 & -1 & | & 0 \\ 1 & 2 & -2 & | & 0 \end{bmatrix} \\ (-2)r1 & + r2 \begin{bmatrix} 1 & 2 & -2 & | & 0 \\ 0 & -1 & 3 & | & 0 \end{bmatrix} \\ 2(r2) & + r1 \begin{bmatrix} 1 & 0 & 4 & | & 0 \\ 0 & -1 & 3 & | & 0 \end{bmatrix} \\ \therefore (z = t), & (y = 3t), (x = -4t) \end{aligned}$$

giving the following vector.

$$(x, y, z) = (-4t, 3t, t) = t(-4, 3, 1)$$

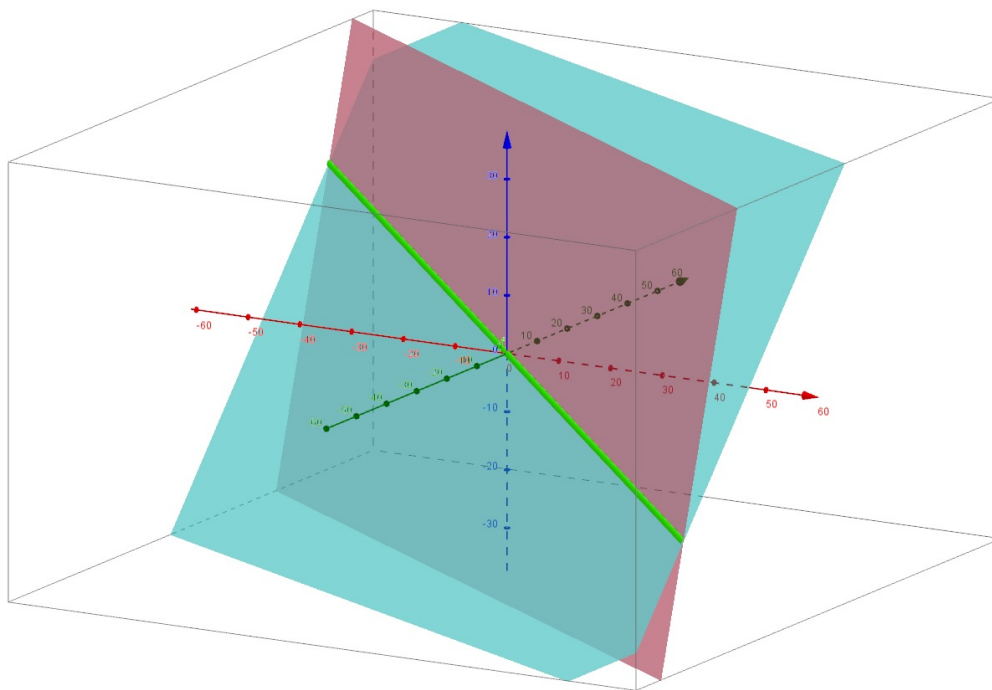


Figure 1: the set $\{(-4, 3, 1)\}$ spans $P_1 \cap P_2$

□