Geometric Class Field Theory

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Contents

1 Introduction

Throughout this work we will be working over fields with characteristic $p \neq 0$ unless otherwise stated.

2 Class Field Theory In the language of Ideals

In this section we describe the main results of classical class field theory for global fields, following [Mil20]. We copy most of the content here from Milne.

2.1 Ideals, Moduli and Ray Class Groups

Let K be a global field of $\mathrm{char}(K) = p$. A modulus $\mathfrak{m}$ is a formal sum of places of K with non-negative integer coefficients. Let $S(K, \mathfrak{m}) = S(\mathfrak{m}) = \{v \in \mathfrak{m}\}$ be the set of places appearing in $\mathfrak{m}$ with non-zero coefficient.

Define $K_{\mathfrak{m},1} = \{x \in K^\times \mid v(x-1) \geq n_v \text{ for all } v \in S(\mathfrak{m})\}$ where n_v is the coefficient of v in $\mathfrak{m}$.

For every set of primes S we define

$$I_K^S = \{ \text{ fractional ideals of } K \text{ generated by primes not in } S\}$$

There is a natural map $i : K_{\mathfrak{m},1} \to I_K^{S(\mathfrak{m})}$ sending $x \mapsto (x)$

The qoutient

$$C_{\mathfrak{m}} = I_K^{S(\mathfrak{m})}/i(K_{\mathfrak{m},1})$$

is called the **(ray) class group** of K modulo $\mathfrak{m}$.

Let S be a finite set of primes of K. And G a finite abelian group. We shall say that a homomorphism $\psi : I^S \to G$ **admits a modulus** if there exists a modulus $\mathfrak{m}$ with $S(\mathfrak{m}) \supset S$ such that $\psi(i(K_{\mathfrak{m},1})) = 0$. Thus ψ admits a modulus if and only if it factors through $C_{\mathfrak{m}}$ for some $\mathfrak{m}$ with $S(\mathfrak{m}) \supset S$.

maybe we don't need this Milne states and prove a known theorem:

Theorem 1. *For every modulus $\mathfrak{m}$ of K there is an exact sequence:*

$$0 \to \mathcal{O}_K^\times/\mathcal{O}_K^\times \cap K_{\mathfrak{m},1} \to K_{\mathfrak{m}}/K_{\mathfrak{m},1} \to C_{\mathfrak{m}} \to C \to 0$$

Where

$$K_{\mathfrak{m}} = \{x \in K^\times \mid v(x) = 0 \text{ for all } v \in S(\mathfrak{m})\}$$

And C is the usual class group of K.

2.2 The Main Theorems

Theorem 2 (Artin Reciprocity Law)**.** *Let L be a finite abelian extension of a global field K. and let S be the set of primes of K ramifying in L. Then the Artin map* add here reference of the definition to milne *. $\psi : I^S \to Gal(L/K)$ admits a modulus $\mathfrak{m}$ with $S(\mathfrak{m}) = S$ and it defines an isomorphism:*

$$I^S / \Big(i(K_{\mathfrak{m},1}) \cdot N_{L/K}(I_L^{S(\mathfrak{m})}) \Big) \to Gal(L/K)$$

A modulus $\mathfrak{m}$ as in the statement of the theorem is called a definning modulus for L. Next, we write $I_K^{\mathfrak{m}}$ for the group of $S(\mathfrak{m})$-ideals in K, and $I_L^{\mathfrak{m}}$ for the group of $S(\mathfrak{m})'$-ideals in L where $S(\mathfrak{m})'$ is the set of primes of L lying above primes in $S(\mathfrak{m})$. Call a subgroup H of $I_K^{\mathfrak{m}}$ a **congruence subgroup** modulo $\mathfrak{m}$ if it contains $i(K_{\mathfrak{m},1})$.

Theorem 3. *[Existence Theorem of Class Field Theory] For every congruence subgroup H modulo $\mathfrak{m}$ there exists a unique finite abelian extension L/K, unramified at all primes not in $S(\mathfrak{m})$, such that the Artin map induces an isomorphism:*

$$I^{S(\mathfrak{m})}/H \to Gal(L/K)$$

.

More of the idealic class field theory in Milne.

Theorems 2 and 3 show that there is a canonical group isomorphism:

$$\varprojlim_{\mathfrak{m}} C_{\mathfrak{m}} \to \mathrm{Gal}(K^{\mathrm{ab}}/K). \tag{1}$$

Rather than studying $\varprojlim_{m} C_m$ directly, it turns out to be more natural to introduce another group that has it as a quotient - this is the idele class group. replace very where idele with ide'le

3 Class Field Theory In the language of Adeles and Ideles

Can we already say we are only considering function fields here? The modern formulation of Global Class Field Theory is given in terms of the adele and idele groups of a global field. In this chapter we will define these objects and state the main theorems of Class Field Theory in this language.

3.1 Adeles and Ideles

Let K be a global field. For each place v of K, we denote:

1. K_v = the completion of K at v
2. $\mathfrak{p}_v$ = the corresponding prime ideal in the ring of integers $\mathcal{O}_K$ of K
3. $\mathcal{O}_v$ = the ring of integers of K_v
4. $\hat{\mathfrak{p}}_v$ = the completion of $\mathfrak{p}_v$ = the maximal ideal of $\mathcal{O}_v$

We define the **adele ring** of K as the restricted direct product

$$\mathbb{A}_K = \prod_v{}' K_v$$

where the restriction is taken with respect to the rings of integers $\mathcal{O}_v$ of K_v for all non-archimedean (IS IT NECESSEARY TO STATE HERE? WE WORK OVER P ANYWAY) places v. In other words, an adele is a tuple $(x_v)_v$ with $x_v \in K_v$ such that $x_v \in \mathcal{O}_v$ for all but finitely many non-archimedean places v. The **idele group** of K is defined as the group of units of the adele ring:

$$\mathbb{I}_K = \mathbb{A}_K^\times = \prod_v{}' K_v^\times$$

where the restriction is taken with respect to the unit groups $\mathcal{O}_v^\times$ of the rings of integers $\mathcal{O}_v$ for all non-archimedean places v. An idele is thus a tuple $(x_v)_v$ with $x_v \in K_v^\times$ such that $x_v \in \mathcal{O}_v^\times$ for all but finitely many non-archimedean places v.

The field K embeds diagonally into $\mathbb{A}_K$, and thus $K^\times$ embeds diagonally into $\mathbb{I}_K$ as the subgroup of principal ideles. The **idele class group $\mathbf{C}_K$** is the quotient:

$$\mathbf{C}_K = \mathbb{I}_K / K^\times$$

There is a natural isomorphism between certain quotients of the idele group and the ideal group of K, which ultimately follows by understanding ideles as thickening of ideals: There is a canonical surjective homomophism id:

$$\begin{aligned} \mathrm{id} : \mathbb{I}_K &\to I_K \\ (x_v)_v &\mapsto \prod_v \mathfrak{p}_v^{v(x_v)} \end{aligned}$$

Thus, composing with $I_K \to C$ gives a surjective homomorphism $\mathbb{I}_K \to C$, noting that $K^\times \to \mathbb{I}_K \to C$ is 0, we realize $C = I_K/i(K^\times)$ as a quotient of $\mathbf{C}_K = \mathbb{I}_K/K^\times$.

The same thing is true for $C_\mathfrak{m}$: Let $\mathfrak{m} = \sum n_v \mathfrak{p}_v$ be a modulus of K, set:

$$W_\mathfrak{m}(v) = \begin{cases} \mathcal{O}_v^\times & v \notin \mathrm{Supp}(\mathfrak{m}) \\ 1 + \hat{\mathfrak{p}_v}^{n_v} & v \in \mathrm{Supp}(\mathfrak{m}) \end{cases}$$

And define

$$\mathbb{I}_\mathfrak{m} = \left(\prod_{v \notin \mathrm{Supp}(\mathfrak{m})} K_v^\times \times \prod_{v \in \mathrm{Supp}(\mathfrak{m})} W_\mathfrak{m}(v) \right) \bigcap \mathbb{I}_K$$

And

$$\mathbb{O}_\mathfrak{m}^\times = \prod_v W_\mathfrak{m}(v)$$

Note that:

$$K_{\mathfrak{m},1} = K^\times \cap \prod_{v \in \mathfrak{m}} W_\mathfrak{m}(v) \qquad \text{Intersection inside } \prod_{v \in \mathfrak{m}} K_v^\times$$

and that

$$K_{\mathfrak{m},1} = K^\times \cap \mathbb{I}_\mathfrak{m} \qquad \text{Intersection inside } \mathbb{I}_K$$

Milne shows the following proposition:

Proposition 4. *Let* $\mathfrak{m}$ *be a modulus of* K*.*

1. *The map* $\mathrm{id} : \mathbb{I}_\mathfrak{m} \to I_K^{S(\mathfrak{m})}$ *defines an isomorphism*

$$\mathbb{I}_\mathfrak{m}/K_{\mathfrak{m},1}\mathbb{O}_\mathfrak{m}^\times \xrightarrow{\sim} I_K^{S(\mathfrak{m})}/i(K_{\mathfrak{m},1}) = C_\mathfrak{m}$$

2. *The inclusion* $\mathbb{I}_\mathfrak{m} \hookrightarrow \mathbb{I}_K$ *defines an isomorphism*

$$\mathbb{I}_\mathfrak{m}/K_{\mathfrak{m},1} \xrightarrow{\sim} \mathbb{I}_K/K^\times$$

Taking the qoutient into character form?

3.1.1 Topology on Adeles and Ideles

We state quickly the topology on the adele ring and the idele group. More can be found in add referenceRecall that, for all v, K_v is locally compact more over, $\mathcal{O}_v$ is a compact neighborhood of 0. Similarly $K_v^\times$ is locally compact, in fact:

$$1 + \hat{\mathfrak{p}}_v \supset 1 + \hat{\mathfrak{p}}_v^2 \supset 1 + \hat{\mathfrak{p}}_v^3 \dots$$

is a fundamental system of neighborhoods of 1 consisting of compact open subgroups of $K_v^\times$.

For every finite set S of places of K, define:

$$\mathbb{I}_S = \prod_{v \in S} K_v^\times \times \prod_{v \notin S} \mathcal{O}_v^\times$$

with the product topology. $\mathbb{I}_S$ is locally compact and as sets we have:

$$\mathbb{I}_K = \bigcup_S \mathbb{I}_S$$

where the union is taken over all finite sets of places of K. We define a topology on I_K by giving a basis for the open sets $\prod_v V_v$ with $V_v \subseteq K_v^\times$ open for all v and and $V_v = \mathcal{O}_v^\times$ for all but finitely many v. This makes $\mathbb{I}_K$ a locally compact topological group, such that each $\mathbb{I}_S$ is open in $\mathbb{I}_K$, and inherits the product topology. The following sets form a fundamental system of neighborhoods of 1: for each finite set of primes S and $n > 0$, define

$$U_{S,n} = \left\{ (x_v)_v \in \mathbb{I}_K \mid v(x_v - 1) > n \text{ for all } v \in S, x_v \in \mathcal{O}_v^\times \text{ for } v \notin S \right\}$$

Note that the embedding $K^\times \to \mathbb{I}_K$ is discrete and thus the idele class group $\mathbf{C}_K = \mathbb{I}_K/K^\times$ is a locally compact topological group as well. Moreover the canonical injective homomorphism

$$K_v^\times \to \mathbb{I}_K \tag{2}$$
$$x \mapsto (1, \dots, 1, x, 1, \dots, 1) \qquad (x \text{ in the } v\text{-th position}) \tag{3}$$

is a topological embedding for each place v of K.

3.1.2 Characters of ideals and of ideles

in [Mil20], Milne proves:

Proposition 5. *Let G be a finite abelian group. If $\psi : I^S \to G$ admits a modulus, then there exists a unique homomorphism $\phi : \mathbb{I} \to G$ such that*

1. *ϕ is continuous (G with the discrete topology)*
2. *$\phi(K^\times) = 1$;*
3. *$\phi(\mathbf{a}) = \psi(id(\mathbf{a}))$, all $\mathbf{a} \in \mathbb{I}^S \overset{def}{=} \{\mathbf{a} \mid a_v = 1 \text{ all } v \in S\}$.*

Moreover, every continuous homomorphism $\phi : \mathbb{I} \to G$ satisfying (2) arises from a ψ. More over, ϕ and ψ fit in the following chain:

$$
\begin{tikzcd}
 & I^{\mathfrak{m}} \arrow[r] & C_{\mathfrak{m}} \arrow[r, "\psi"] & G \\
 & \mathbb{I}_{\mathfrak{m}}/K_{\mathfrak{m},1} \arrow[r] \arrow[d, "\cong"] & \mathbb{I}_{\mathfrak{m}}/K_{\mathfrak{m},1}\mathbb{O}_{\mathfrak{m}}^\times \arrow[u, "\cong"'] & \\
\mathbb{I} \arrow[r] \arrow[rrruu, bend right, "\phi"] & \mathbb{I}_K/K^\times & &
\end{tikzcd}
\tag{4}
$$

3.1.3 Norms of ideles

Let L be a finite extension of the number field K.

For an idèle $\mathbf{a} = (a_w) \in \mathbb{I}_L$, define $\mathrm{Nm}_{L/K}(\mathbf{a})$ to be the idèle $\mathbf{b} \in \mathbb{I}_K$ with $b_v = \prod_{w|v} \mathrm{Nm}_{L_w/K_v} a_w$. Then, one can show that the following diagram commutes:

$$
\begin{tikzcd}
L^\times \arrow[r] \arrow[d, "\mathrm{Nm}_{L/K}"] & \mathbb{I}_L \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{Nm}_{L/K}"] & I_L \arrow[d, "\mathrm{Nm}_{L/K}"] \\
K^\times \arrow[r] & \mathbb{I}_K \arrow[r, "\mathrm{id}"] & I_K.
\end{tikzcd}
$$

Thus getting a commutative diagram:

$$
\begin{tikzcd}
\mathbf{C}_L \arrow[r] \arrow[d, "\mathrm{Nm}_{L/K}"] & C_L \arrow[d, "\mathrm{Nm}_{L/K}"] \\
\mathbf{C}_K \arrow[r] & C_K
\end{tikzcd}
$$

(where C_L, C_K are the ideal class groups of L and K respectively).

3.2 The main theorems

The theory establishes a fundamental connection between the idele class group $\mathbf{C}_K$ and the Galois group of the maximal abelian extension of K, denoted K^{ab}.

Theorem 6 (Reciprocity Law)**.** *There exists a unique continious homomorphism* $\phi_K : \mathbb{I}_K \to Gal(K^{ab}/K)$ *called the* ***Artin map*** *with the following properties:*

1. $\phi_K(K^\times) = 1$*;*
2. *For every finite abelian extension* L/K*,* ϕ_K *defines an isomorphism:*

$$\phi_{L/K} : \mathbb{I}_K/(K^\times \cdot Nm_{L/K}(\mathbb{I}_L)) \xrightarrow{\sim} Gal(L/K)$$

   or, equivalently, an isomorphism:

$$\mathbf{C}_K/Nm_{L/K}(\mathbf{C}_L) \xrightarrow{\sim} Gal(L/K)$$

3. $\phi_{L/K}$ *arises from the global* $\mathbb{I}_K \to \mathrm{Gal}(L/K)$ *coming from the ideal-theoetic global artin map, as in proposition 5.*

Theorem 7 (The Existence Theorem)**.** *There is a one-to-one, inclusion-reversing correspondence between the set of finite abelian extensions of* K *and the set of open subgroups of finite index in the idele class group* $\mathbf{C}_K$*.*

$$\left\{\begin{array}{c}\textit{Finite abelian}\\ \textit{extensions } L/K\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\textit{Open subgroups } H \subseteq \mathbf{C}_K\\ \textit{of finite index}\end{array}\right\}$$

Under this correspondence, an extension L *corresponds to the subgroup* $H = N_{L/K}(\mathbf{C}_L)$*.*

Theorem 8. *Ideal-Theoetic and Idele-Theoetic formulations of CFT are equivilant through proposition 5.*

1. Above is finite adelic formulation of CFT, state somethign about the infinite extension CFT theorem
2. State something about the topology on the idele class group, Say how the fintie implies the infinte by taking inverse limits.
3. Find sources for the above. for exampel milne? maybe other?
4. The restriction is for the non-archimedean, are you sure?
5. What is the topology of the "continious" homomorphism?

4 Class Field Theory In the language of Characters

The character formulation of Class Field Theory provides a correspondence between characters of the idele class group and characters of the Galois group of the maximal abelian extension of a global field.

4.1 About Characters

We need to make precise what we mean by characters on both sides of the correspondence.

Definition 9. *Let G be an abelian group.*

1. *A character $\rho : G_K \to G$ is unramified at a place v if it is trivial on the inertia group $I_v \subseteq G_K$. ρ is called unramified if it is unramified at all places v of K.*
2. *A character $\rho : G_K \to G$ has ramification bounded by a modulus $\mathfrak{m} = \sum_v n_v v$ if for each place $v \in \mathfrak{m}$, the restriction of ρ to the higher ramification group $G_v^{n_v}$ is trivial.*

Note that since G is abelian, the value of ρ on the Frobenius element Fr_v is well-defined for unramified places v.

A useful theorem about characters is as follows:

Theorem 10. *Let G be an abelian group such that for every $n \in \mathbb{N}$, the n-torsion subgroup $G[n] = \{g \in G | ng = 0\}$ is cyclic of order n. Then for every finite abelian group A denote by $\hat{A} = \mathrm{Hom}(A, G)$ the group of characters from A to G.*

Then the functor $A \mapsto \hat{A}$ is a contravariant equivalence of categories between the category of finite abelian groups and itself. Moreover the natural map $A \to \hat{\hat{A}}$ is an isomorphism.

4.2 The Main Theorems

One formulation of Global Class Field Theory in terms of characters is as follows:

Theorem 11 (Character Formulation of Unramified Global Class Field Theory)**.**

1. *For each character $\xi : K^\times \backslash \mathbb{A}_K^\times / \mathbb{O}_K^\times \to \bar{\mathbb{Q}}_\ell^\times$ there exists a unique continuous unramified character $\rho : G_K \to \bar{\mathbb{Q}}_\ell^\times$ such that $\rho(Fr_v) = \xi(\pi_v)$ for all v.*
2. *For each continuous unramified character $\rho : G_K \to \bar{\mathbb{Q}}_\ell^\times$ there exists a unique character $\xi : K^\times \backslash \mathbb{A}_K^\times / \mathbb{O}_K^\times \to \bar{\mathbb{Q}}_\ell^\times$ such that $\rho(Fr_v) = \xi(\pi_v)$ for all v.*

Where $\mathbb{O}_K^\times = \mathbb{O}_0$

Theorem 12 (Character Formulation of Ramified class field theory)**.** *In the above notations:*

1. *For each character $\xi : K^\times \backslash \mathbb{A}_K^\times / \mathcal{O}_\mathfrak{m}^\times \to \bar{\mathbb{Q}}_\ell^\times$ there exists a unique continuous character $\rho : G_K \to \bar{\mathbb{Q}}_\ell^\times$ with $\mathrm{ram}(\rho) \subseteq \mathfrak{m}$ and $\rho(Fr_v) = \xi(\pi_v)$ for all primes $v \notin \mathrm{Supp}(\mathfrak{m})$.*
2. *For each continuous character $\rho : G_K \to \bar{\mathbb{Q}}_\ell^\times$ with $\mathrm{ram}(\rho) \subseteq \mathfrak{m}$ there exists a unique character $\xi : K^\times \backslash \mathbb{A}_K^\times / \mathcal{O}_\mathfrak{m}^\times \to \bar{\mathbb{Q}}_\ell^\times$ such that $\rho(Fr_v) = \xi(\pi_v)$ for all primes $v \notin \mathrm{Supp}(\mathfrak{m})$.*

In fact, for the above theorems, we can replace $\bar{\mathbb{Q}}_\ell^\times$ by any finite abelian group G with the discrete topology, and the theorems would still hold.

Theorem 13. *Assume 11, 12 are true for all finite abelian groups G with the discrete topology (as values of the characters). Then 11, 12 are true as stated.*

Proof. Indeed, assume such theorem would be true for such cases, then by varying G over $(\mathbb{Z}/l^n\mathbb{Z})^\times$ (for all $n \in \mathbb{N}$), we will get a compatible system of characters and a corresponding isomorphism of character groups with values in $\mathbb{Z}_l^\times$. And since $Tors(\mathbb{Z}_l^\times) \cong Tors(\mathbb{Q}_\ell^\times)$, this is equivilant to the theorems for characters with values in $\mathbb{Q}_\ell^\times$. Similiarly, every finite extension $\mathbb{Q}_l \subset F$ comes as inverse limit of its finite subgroups of units, so the same argument applies to characters with values in $F^\times$. We have compatability between those characters (from uniqueness) for all finite $\mathbb{Q}_l \subset F$ and by going to the colimit (and since all groups involved are fintely generated, hence in **Ab** $Hom(A, -)$ preserve colimtis) we get the result for characters with values in $\bar{\mathbb{Q}}_\ell^\times$ as well.

□

Theorem 14. *Let $\mathfrak{m}$ be a modulus of K. Assume 11 and 12 are true. Then the Artin map induces isomorphisms:*

$$\begin{aligned}
&Hom_{cont}(C_\mathfrak{m}, \bar{\mathbb{Q}}_\ell^\times) \cong Hom_{cont}(K^\times \backslash \mathbb{A}_K^\times / \mathbb{O}_\mathfrak{m}^\times, \bar{\mathbb{Q}}_\ell^\times) \\
&\cong Hom_{cont, ram \leq \mathfrak{m}}(G_K, \bar{\mathbb{Q}}_\ell^\times) \cong Hom_{(cont, ram \leq m)}(G_K^{ab}, \bar{\mathbb{Q}}_\ell^\times) \\
&\cong Hom_{cont}(Gal(L_\mathfrak{m}/K), \bar{\mathbb{Q}}_\ell^\times)
\end{aligned}$$

Where $L_\mathfrak{m}$ is the maximal abelian extension of K with ramification bounded by $\mathfrak{m}$. Hence by theorem 10 we get that the artin map induces isomorphism $C_\mathfrak{m} \cong Gal(L_\mathfrak{m}/K)$, which implies the statement of theorem 2 and theorem 3. (Details omitted, like how to go from $C_\mathfrak{m}$ to every congurgence subgroups, etc.)

1. Is this formulation *equivilant* to adeles language? is it dervied from it?
2. Give amichai reference for this formulation
3. Over what field are we working? what is l, what is p?
4. Fix the qoutient of adeles no match the subgroup
5. $\mathfrak{m}$ vs $\mathfrak{m}$ notation for divisors
6. maybe make theorem 13 more precise
7. maybe make theorem 14 more precise
8. replace l by ℓ everywhere

See milne, amichai, for more details.

5 Geometric Class Field Theory

Next, we state the main theorem of geometric class field theory. And see how it relates to the classical class field theory. This is copied from Daichi's and some adapated from Giroud's thesis: Let k be a perfect field, and let C be a projective smooth geometrically connected curve over k. The geometric class field theory gives a geometric description of abelian coverings of C by using generalized jacobian varieties. Let us recall its precise statement. Fix a modulus $\mathfrak{m}$, i.e. an effective Cartier divisor of C and let U be its complement in C.

The pairs $(\mathcal{L}, \alpha)$, where $\mathcal{L}$ is an invertible $\mathcal{O}_C$-module and α is a rigidification of $\mathcal{L}$ along $\mathfrak{m}$, are parametrized by a k-group scheme $\mathrm{Pic}_{C,\mathfrak{m}}$, called the rigidified Picard scheme. The Abel-Jacobi morphism

$$\Phi : U \to \mathrm{Pic}_{C,\mathfrak{m}}$$

is the morphism which sends a section x of U to the pair $(\mathcal{O}(x), 1)$. We prove the following version of geometric class field theory:

Theorem 15 (Geometric Class Field Theory)**.** *Let Λ be a finite ring of cardinality invertible in k, and let $\mathcal{F}$ be an étale sheaf of Λ-modules, locally free of rank 1 on U, with ramification bounded by $\mathfrak{m}$. Then, there exists a unique (up to isomorphism) multiplicative étale sheaf of Λ-modules $\mathcal{G}$ on $\mathrm{Pic}_{C,\mathfrak{m}}$, locally free of rank 1, such that the pullback of $\mathcal{G}$ by Φ is isomorphic to $\mathcal{F}$.*

Let d be a positive integer. We denote by $U^{(d)}$ the d-th symmetric power of U over k. For an étale sheaf $\mathcal{F}$ on U, we denote by $\mathcal{F}^{(d)}$ the d-th symmetric power of $\mathcal{F}$ on $U^{(d)}$. We have a natural morphism Check if this morphism only work for large enough d?

$$\Phi_d : U^{(d)} \to \mathrm{Pic}^d_{C,\mathfrak{m}}$$

is this actually the map? which sends a section $x_1 + \cdots + x_d$ of $U^{(d)}$ to the pair $(\mathcal{O}(x_1 + \cdots + x_d), 1)$.

Theorem 15 can be reduced to the following statement:

Theorem 16. *Let Λ be a finite ring of cardinality invertible in k, and let $\mathcal{F}$ be an étale sheaf of Λ-modules, locally free of rank 1 on U, with ramification bounded by $\mathfrak{m}$. Then, for sufficiently large integer d, there exists a unique (up to isomorphism) multiplicative étale sheaf of Λ-modules $\mathcal{G}_d$ on $\mathrm{Pic}^d_{C,\mathfrak{m}}$, locally free of rank 1, such that the pullback of $\mathcal{G}_d$ by Φ_d is isomorphic to $\mathcal{F}^{(d)}$.*

1. A word about the decompositon of the Picard scheme into connected components indexed by degree.
2. Definition of the abel jacobi-map, and its properties, its fibers, see how amichai does it.
3. Definition of multiplicative sheaves.
4. Say something about the ramification condition.
5. Put references inside the theorem like Giroud does

Proof of geometric CFT

6 Proof of theorem 16

Let $\mathfrak{m}$ be an effective Cartier divisor on C, and let d be a positive integer satisfying $d \geq max\{2g - 1 + \deg \mathfrak{m}, \deg \mathfrak{m}\}$ where g is the genus of C. We denote by $C^{(d)}$ the d-th symmetric power of C over k. By add reference, the fibers of the map for d large enough? under what conditions?

$$\Phi_d : U^{(d)} \to \mathrm{Pic}^d_{C,\mathfrak{m}}$$

over any point are isomorphic to

$$\begin{cases} \mathbb{A}_k^{d-\deg \mathfrak{m}-g+1} & \text{if } m > 0 \\ \mathbb{P}_k^{d-g} & \text{if } m = 0 \end{cases}$$

One can show this makes Φ_d into a fibration in affine spaces or projective spaces, depending on whether $\mathfrak{m}$ is non-zero or zero.

Using a fundamental theorem about etale fundamental groups:

Theorem 17 (Homotopy Exact Sequence [Stacks, Tag 0C0J])**.** *Let $f : X \to S$ be a flat proper morphism of finite presentation whose geometric fibres are connected and reduced. Assume S is connected and let $\overline{s}$ be a geometric point of S. Then there is an exact sequence*

$$\pi_1(X_{\overline{s}}) \to \pi_1(X) \to \pi_1(S) \to 1$$

of fundamental groups.

We get, in the case when $\mathfrak{m} = 0$ (Hence Φ_d is smooth and proper) that there is an isomorphism of fundamental groups:

$$\pi_1^{et}(C^{(d)}) \cong \pi_1^{et}(\mathrm{Pic}_C^d)$$

Implying theorem 16 explain exactly how, becuase here it is fundamental groups and there it is etale sheafves of Λ-modules. Also, we need it to be **multiplicative**!

In the case when $\mathfrak{m} > 0$, Φ_d is not proper anymore, so we cannot apply the homotopy exact sequence directly. In [Tak19] Takeuchi constructs a compactifiaction of $U^{(d)}$ by adding a boundary hyperplane, over which $\mathcal{F}^{(d)}$ is tamely ramified. More over, this compactification, denoted by $\tilde{C}_{\mathfrak{m}}^{(d)}$ is fibered over $\mathrm{Pic}_{C,\mathfrak{m}}^d$ with fibers isomorphic to projective spaces. Thus we get

$$\pi_1(\tilde{C}_{\mathfrak{m}}^{(d)}) \cong \pi_1(\mathrm{Pic}_{C,\mathfrak{m}}^d)$$

From here, one can go in two routes, **Route 1** - Showing $\ker\{\pi_1^{ab}(U^{(d)}) \to \pi_1^{ab}(\mathrm{Pic}_{C,\mathfrak{m}}^d)\}$ is pro-p group.

Route 2 - Showing

$$\pi_1^{t,ab}(U^{(d)}) \to \pi_1^{t,ab}(\mathrm{Pic}_{C,\mathfrak{m}}^d)$$

is isomorphism to its image. here one needs to be precise.

In both cases, one would then get theorem 16.

We start by defintions and basic proposition of everything we need.

7 Generalized Picard Scheme, Abel Jacobi Map, And The Blowup

7.1 Generalized Picard Scheme

We copy from [Gui19] [Tak19]. Let S be a scheme, C be a projective smooth S-scheme whose geometric fibers are connected and of dimension 1. Let $\mathfrak{m}$ be an effective Cartier divisor of C/S, i.e. a closed subscheme of C which is finite flat of finite presentation over S. We also call $\mathfrak{m}$ a modulus. Let us denote, for S-schemes T, the projections $C \times_S T \to T$ by the same symbol pr. In this section, we recall and study the notion of generalized jacobian varieties. Let d be an integer and $\mathfrak{m}$ be a modulus. Let T be an S-scheme. Consider a datum $(\mathcal{L}, \psi)$ such that

- $\mathcal{L}$ is an invertible sheaf of $\deg = d$ on C_T.
- ψ is an isomorphism $\mathcal{O}_{m_T} \to \mathcal{L}|_{m_T}$.

We say that two such data $(\mathcal{L}, \psi)$ and $(\mathcal{L}', \psi')$ are isomorphic if there exists an isomorphism of invertible sheaves $f : \mathcal{L} \to \mathcal{L}'$ making the following diagram commutes

$$\begin{tikzcd} & \mathcal{O}_{m_T} \arrow[dl, "\psi'"'] \arrow[dr, "\psi"] & \\ \mathcal{L}'|_{m_T} \arrow[rr, "f|_{m_T}"] & & \mathcal{L}|_{m_T} \end{tikzcd}$$

For an S-scheme T, define a set

$$\mathrm{Pic}^{d,\mathrm{pre}}_{C,\mathfrak{m}}(T) := \{\text{the isomorphism class of } (\mathcal{L}, \psi) \text{ defined as above}\}.$$

$\mathrm{Pic}^{d,\mathrm{pre}}_{C,\mathfrak{m}}$ extends in an obvious way to a presheaf on Sch/S, which we denote by $\mathrm{Pic}^{d,\mathrm{pre}}_{C,\mathfrak{m}}$ also. Define $\mathrm{Pic}^{d}_{C,\mathfrak{m}}$ as the étale sheafification of $\mathrm{Pic}^{d,\mathrm{pre}}_{C,\mathfrak{m}}$. Their fundamental properties which we use without proofs are:

- $\mathrm{Pic}^{d}_{C,\mathfrak{m}}$ are represented by S-schemes. When $\mathfrak{m}$ is faithfully flat over S, $\mathrm{Pic}^{d,\mathrm{pre}}_{C,\mathfrak{m}}$ are already étale sheaves.
- $\mathrm{Pic}^{0}_{C,\mathfrak{m}}$ is a smooth commutative group S-scheme with geometrically connected fibers.
-
- $\mathrm{Pic}^{d}_{C,\mathfrak{m}}$ are $\mathrm{Pic}^{0}_{C,\mathfrak{m}}$-torsors.

$\mathrm{Pic}^{0}_{C,\mathfrak{m}}$ is called the generalized jacobian variety of C with modulus $\mathfrak{m}$. When $\mathfrak{m} = 0$, this is the jacobian variety of C. In this case, we also denote Pic^{d}_{C} for $\mathrm{Pic}^{d}_{C,\mathfrak{m}}$.

maybe here, instead of saying what he says, just summarize, and refer to it/don't actually include it For a finite flat S-scheme of finite presentation D, define a presheaf $\mathcal{O}_D^\times$ on Sch/S by sending an S-scheme T to the multiplicative group $\Gamma(T, \mathcal{O}^\times_{D\times_S T})$, which is called the Weil restriction of $\mathbb{G}_{m,D}$ to S. This is an étale sheaf, and represented by a smooth group S-scheme. When $D = S$, this is $\mathbb{G}_{m,S}$. Define a map $\mathbb{G}_{m,S} \to \mathcal{O}_D^\times$ from the map of S-schemes $D \to S$. When $\deg D$ is strictly positive everywhere on S, this is an injection of étale sheaves.

Consider a map $\mathcal{O}_\mathfrak{m}^\times \to \mathrm{Pic}^{0}_{C,\mathfrak{m}}$ sending $s \in \mathcal{O}_\mathfrak{m}^\times$ to the pair $(\mathcal{O}_C, \mathcal{O}_\mathfrak{m} \to \mathcal{O}_\mathfrak{m})$. The image of this map coincides with the kernel of the map $\mathrm{Pic}^{0}_{C,\mathfrak{m}} \to \mathrm{Pic}^{0}_{C}$, and the kernel of the map $\mathcal{O}_\mathfrak{m}^\times \to \mathrm{Pic}^{0}_{C,\mathfrak{m}}$ is the image of $\mathbb{G}_{m,S} \to \mathcal{O}_\mathfrak{m}^\times$ induced by the morphism of S-schemes $\mathfrak{m} \to S$. In summary, if $\deg \mathfrak{m}$ is everywhere strictly positive, we have a short exact sequence:

$$0 \to \mathcal{O}_\mathfrak{m}^\times / \mathbb{G}_{m,S} \to \mathrm{Pic}^{0}_{C,\mathfrak{m}} \to \mathrm{Pic}^{0}_{C} \to 0.$$

In particular, when $C \to S$ has a section $P : S \to C$, $\mathrm{Pic}^{0}_{C,P}$ is isomorphic to Pic^{0}_{C}. In this case, Pic^{d}_{C} has an expression as a sheaf which does not depend on the choice of P. T be an S-scheme, and $\mathcal{L}_1$ and $\mathcal{L}_2$ are invertible sheaves of deg $= d$ on C_T. Define an equivalence relation on $\mathrm{Pic}_C^{d,\mathrm{pre}}$ such that $\mathcal{L}_1$ and $\mathcal{L}_2$ are equivalent if and only if there exists an invertible sheaf M on T such that $\mathcal{L}_1 \cong \mathcal{L}_2 \otimes \mathrm{pr}^* M$. If $C \to S$ has a section, the quotient presheaf of $\mathrm{Pic}_C^{d,\mathrm{pre}}$ by this equivalence relation is an étale sheaf and coincides with the étale sheafification of $\mathrm{Pic}_C^{d,\mathrm{pre}}$ via the natural surjection. In particular, the identity map $\mathrm{Pic}_C^d \to \mathrm{Pic}_C^d$ corresponds to an equivalence class of invertible sheaves on $C \times_S \mathrm{Pic}_C^d$. In this paper, we call this class the universal class of invertible sheaves of deg $= d$.

From now on we fix a modulus $\mathfrak{m}$ which is everywhere strictly positive. Then, $\mathrm{Pic}_{C,\mathfrak{m}}^d$ has an explicit expression as a sheaf, as explained before.

Denote the genus of C by g. This is a locally constant function on S. We consider a condition on an integer d as below:

$$d \geq \max\{2g - 1 + \deg \mathfrak{m}, \deg \mathfrak{m}\}. \tag{5}$$

When S is quasi-compact, such a d always exists.

Fix an integer d satisfying the condition above. Let T be an S-scheme and $\mathcal{L}$ be an invertible sheaf of deg $= d$ on C_T. One can show that $\mathrm{pr}_* \mathcal{L}(-\tilde{\mathfrak{m}})$ and $\mathrm{pr}_* \mathcal{L}$ are locally free sheaves and their formations commute with any base change, i.e. for any morphism of S-schemes $f : T' \to T$, the base change morphisms $f^* \mathrm{pr}_* \mathcal{L} \to \mathrm{pr}_* f^* \mathcal{L}$ and $f^* \mathrm{pr}_*(\mathcal{L}(-\tilde{\mathfrak{m}})) \to \mathrm{pr}_* f^*(\mathcal{L}(-\tilde{\mathfrak{m}}))$ are isomorphisms.

Moreover (from [Gui19]) one can show that if $\mathcal{L}$ is invertible $\mathcal{O}_C$-module with degree d on each fiber of f Then, the $\mathcal{O}_S$-module $pr_* \mathcal{L}$ is locally free of rank $d - g + 1$

1. It may be helpful to consult Milne's "Abelian Varieties" for further background and confidence in these constructions. https://www.jmilne.org/math/CourseNotes/AV.pdf

7.2 Abel-Jacobi Map

We copy from [Tót11] We assume d as in (5)

Definition 18 (Abel-Jacobi Map of degree d)**.**

The *Abel-Jacobi map of degree d*

$$\Phi_d : \mathrm{Div}_C^d \to \mathrm{Pic}_C^d$$

is defined for a scheme T over $\mathrm{Spec}(k)$ and for a relative effective Cartier divisor D of degree d on $(C \times_{\mathrm{Spec}(k)} T)/T$ by

$$\Phi_d(T)(D) := [\mathcal{O}(D)]$$

where $[\mathcal{O}(D)]$ is the class of the invertible sheaf $\mathcal{O}(D)$ on $(C \times_{\mathrm{Spec}(k)} T)/T$. Equivalently if D is represented by the pair $(\mathcal{G}, s)$ (1.2.6), then the Abel-Jacobi map is given by

$$\Phi_d(T)((\mathcal{G}, s)) := [\mathcal{G}].$$

A theorem in milne [Mil08] shows that over a field k, for any $d \geq 1$ we have that the functor Div_C^d is representable by the d-th symmetric power $C^{(d)}$.

7.2.1 Generalized Effective Cartier Divisors

We copy from [Gui19] Let $f : X \to S$ be a smooth morphism of schemes of relative dimension 1, with connected geometric fibers of genus g, which is Zariski-locally projective over S.

and let $i : Y \to X$ be a closed subscheme of X, which is finite locally free over S of degree $N \geq 1$, and let $U = X \setminus Y$ be its complement. A **Y-trivial effective Cartier divisor of degree d on X** is a pair $(\mathcal{L}, \sigma)$ such that $\mathcal{L}$ is a locally free $\mathcal{O}_X$-module of rank 1 and $\sigma : \mathcal{O}_X \to \mathcal{L}$ is an injective homomorphism such that $i^*\sigma$ is an isomorphism and such that the closed subscheme $V(\sigma)$ of X defined by the vanishing of the ideal $\sigma\mathcal{L}^{-1}$ of $\mathcal{O}_X$ is finite locally free of rank d over S. Two Y-trivial effective divisors $(\mathcal{L}, \sigma)$ and $(\mathcal{L}', \sigma')$ are **equivalent** if there is an isomorphism $\beta : \mathcal{L} \to \mathcal{L}'$ of $\mathcal{O}_X$-modules such that $\beta\sigma = \sigma'$. As in 4.7 - This is not "trivial" show/or skip but say somethinig about it, if such an isomorphism exists then it is unique. [Gui19] shows:

Proposition 19 (4.11 In Guignard)**.** *The map $(\mathcal{L}, \sigma) \mapsto (V(\sigma) \hookrightarrow X)$ is a bijection from the set of equivalence classes of Y-trivial effective Cartiers divisor of degree d on X onto the set of closed subschemes of U which are finite locally free of degree d over S.*

Proposition 20 (4.12 In Guignard)**.** *Let d be an integer and let $Div_S^{d,+}(X, Y)$ be the functor which to an S-scheme T associates the set of equivalence classes of Y_T-trivial effective Cartier divisors of degree d on X_T. Then $Div_S^{d,+}(X, Y)$ is representable by the S-scheme $Sym_S^d(U)$, the d-th symmetric power of $U = X \setminus Y$ over S. In particular $Div_S^{d,+}(X, Y)$ is smooth of relative dimension d over S.*

Proposition 21 (4.14 In Guignard)**.** *Let $d \geq N + 2g - 1$ be an integer, and let $Pic_S^d(X, Y)$ be the inverse image of $Pic_S^d(X)$ by the natural morphism $Pic_S(X, Y) \to Pic_S(X)$. Then the Abel-Jacobi morphism*

$$\Phi_d : Div_S^{d,+}(X, Y) \to Pic_S^d(X, Y)$$
$$(\mathcal{L}, \sigma) \mapsto (\mathcal{L}, i^*\sigma)$$

is surjective smooth of relative dimension $d - N - g + 1$ and it has geometrically connected fibers. N is the degree of Y so $\deg \mathfrak{m}$ - replace it

1. The notation here from giroud of f is as pr in takeuchi, so we need to make it precise in coheret
2. Some places they work over a family of curves, and some places over k. - make this uniform as well.
3. Here gioured use the term Y, we need to formulate it like $\mathfrak{m}$ and N by $N = \deg \mathfrak{m}$
4. Maybe add somewhere the definition of Sym_U like 2.22 in Guingard. ?

7.3 The Blowup

In this section we define the blowup of $C^{(d)}$ by Z_0 and give its basic properties. As outlined in [Tak19] Let $C^(d)$ be the d'th symmetric product of C, which parametrizes effective Cartier divisor of $deg = d$ on C. Let Z_0 be the closed subscheme of $C^{(d)}$ defined by the map $C^{(d-\deg \mathfrak{m})} \to C^{(d)}$ adding $\mathfrak{m}$. And let $X_\mathfrak{m}$ be the blowup of $C^{(d)}$ along Z_0. In [Tak19][Section 3] Takeuchi proves:

Theorem 22. *There exists a commutative diagram:*

$$\begin{tikzcd}
 & U^{(d)} \arrow[d, dashed, hook] \arrow[dd, dashed, bend right] \arrow[ddd, dashed, bend right] \arrow[dr, "\Phi_d"] & \\
Z_0 \times_{C^{(d)}} \tilde{C}^{(d)}_{\mathfrak{m}} \arrow[r] \arrow[d] & \tilde{C}^{(d)}_{\mathfrak{m}} \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\square"] & \mathrm{Pic}^d_{C,\mathfrak{m}} \arrow[d, hook] \\
Z_0 \times_{C^{(d)}} X_{\mathfrak{m}} \arrow[r] \arrow[d] & X_{\mathfrak{m}} \arrow[r] \arrow[d] & P^d_{\mathfrak{m}} \arrow[d] \\
Z_0 \arrow[r] & C^{(d)} & \mathrm{Pic}^d_C
\end{tikzcd} \tag{6}$$

Where

1. *[Tak19][Lemma 3.1.] $P^d_{\mathfrak{m}}$ is a sheaf in $Sch/\operatorname{Pic}^d_C$ defined by: Add sheaf definition and continue this theorem isomorphic to the projectivization $\mathbb{P}(pr_*\mathcal{L}'_{\mathfrak{m}})$. represented by a proper smooth S-algebraic space. Assume that $C \to S$ has a section. Let $\mathcal{L}'$ be a representative invertible sheaf of the universal class. Then, as sheaves on $Sch/Pic^{d_m}_C$, $\mathcal{P}^{d_m}_m$ is isomorphic to the projectivization $\mathbb{P}(pr_\star\mathcal{L}'_m)$ of $pr_\star\mathcal{L}'_m$.*

Where for a scheme X and a locally free sheaf of finite rank $\mathcal{F}$ on X. We use a contra-Grothendieck notation for a projective space. Thus the X-scheme $\mathbb{P}(\mathcal{F})$ parametrizes invertible subsheaves of $\mathcal{F}$.

7.4 Other

add reference or proof $\mathcal{F}^{(d)}$ can be extended to a sheaf on this compactification with tame ramification along the boundary. The compactification, denoted by $\tilde{C}^{(d)}_{\mathfrak{m}}$ is fibered over $\mathrm{Pic}^d_{C,\mathfrak{m}}$ with fibers isomorphic to projective spaces. Hence, show/add ref/add explanation

Here we will show The compactification has

blow-up $X_{\mathfrak{m}}$ of $C^{(d)}$ along a the closed subscheme $Z_0 \subset C^{(d)}$ defined by the map:

$$\phi\colon C^{(d-\deg\mathfrak{m})} \to C^{(d)}$$
$$E \mapsto E + \mathfrak{m}$$

He shows the existence of a commutative diagram:

$$\begin{tikzcd}
 & \tilde{C}^{(d_m)}_m \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{Pic}^{d_m}_{C,m} \arrow[d, "(3.7)"'] \\
X_m \arrow[r, "\cong"] \arrow[dr] & \mathbb{P}(\mathcal{E}_m) \arrow[r] \arrow[d, "(3.5)"'] & P^{d_m}_m \arrow[d, "(3.2)"'] \\
 & C^{(d_m)} & \mathrm{Pic}^{d_m}_C
\end{tikzcd} \tag{7}$$

1. Here, C is defined over S, a family of curves, usually?
2. Maybe add a section about Symmetric Powers of a curve? probably should.

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