Geometric Class Field Theory

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Contents

1 Introduction

In this thesis, we give an elementary proof of a certain imporant geometric theorem occuring in Deligne's approach to geometric class field theory. We (usually) work over a perfect field k, C is a projective smooth geometrically connected curve over k, with genus g. One of the main geometric ingredients in the approach, is showing why a local system $\mathcal{F}$ with ramification bounded by a modulus $\mathfrak{m}$ on $U = C \setminus \mathfrak{m}$ descends via the Abel-Jacobi $\Phi : U \to \mathrm{Pic}_{C,\mathfrak{m}}$ to $\mathrm{Pic}_{C,\mathfrak{m}}$. The approach, innovated by Deligne, relies on analyzing the symmetric powers $\mathcal{F}^{(d)}$ of $\mathcal{F}$ on the symmetric powers $U^{(d)}$ of U, and showing that for sufficiently large d, $\mathcal{F}^{(d)}$ descends to $\mathrm{Pic}^d_{C,\mathfrak{m}}$ via the degree d Abel-Jacobi map $\Phi_d : U^{(d)} \to \mathrm{Pic}^d_{C,\mathfrak{m}}$. The geometric-fibers of Φ_d (for $d \geq \deg \mathfrak{m} + 2g - 1$) over any point are isomorphic to

$$\begin{cases} \mathbb{A}^{d-\deg \mathfrak{m}-g+1}_{k^{sep}} & \text{if } \mathfrak{m} > 0 \\ \mathbb{P}^{d-g}_{k^{sep}} & \text{if } \mathfrak{m} = 0 \end{cases}$$

Where g is the genus of the curve C. The unramified case ($\mathfrak{m} = 0$) is relatively simple, as the Abel-Jacobi map is proper, surjective with geometrically connected fibers, which follows from the fact that it is a fibration in projective spaces. Thus, by using the homotopy exact sequence for the etale fundamental group,

one gets an isomorphism between the etale fundamental group of $U^{(d)}(= C^{(d)})$ and that of $\mathrm{Pic}^d_{C,\mathfrak{m}}(= \mathrm{Pic}^d_C)$.

The ramified case ($\mathfrak{m} > 0$) is more subtle, as the Abel-Jacobi map is not proper anymore, and one needs to analyze the ramification of $\mathcal{F}^{(d)}$ "along the boundary" of $U^{(d)}$ in $C^{(d)}$.

Previous work has generalized Deligne's approach to the ramified case, most notably by Guignard [Gui19] and Takeuchi [Tak19]. Their approaches differ. To descend, Guignard proves that the restriction of $\mathcal{F}^{(d)}$ to any line in the fiber of the degree d Abel-Jacobi map is a constant étale sheaf. He achieves this by demonstrating that the restriction is at most tamely ramified and invoking the triviality of the tame fundamental group of $\mathbb{A}^1_k$. His analysis relies on local geometric class field theory. It is also worth noting that Guignard's method generalizes to relative curves over arbitrary base schemes. Takeuchi, on the other hand, constructs a compactification of $U^{(d)}$ by blowing up $C^{(d)}$ along certain well-chosen centers. This compactification, denoted by $\tilde{C}^{(d)}_{\mathfrak{m}}$, has $U^{(d)}$ as an open subscheme with a codimension 1 closed subscheme H as complement. He then shows that the Abel-Jacobi map extends to a proper morphism from $\tilde{C}^{(d)}_{\mathfrak{m}}$ to $\mathrm{Pic}^d_{C,\mathfrak{m}}$, which is a fibration in projective spaces. Thus, by the homotopy exact sequence for the etale fundamental group, one gets an isomorphism between the etale fundamental group of $\tilde{C}^{(d)}_{\mathfrak{m}}$ and that of $\mathrm{Pic}^d_{C,\mathfrak{m}}$. To conclude the descent, Takeuchi analyzes the ramification of $\mathcal{F}^{(d)}$ along the boundary H of $\tilde{C}^{(d)}_{\mathfrak{m}}$, showing that it is tamely ramified there, which suffices. His methods relies on the theory of Witt vectors and refined Swan conductors.

For an account of these approaches, see [Gui19] and [Tak19]. For a full approach following Deligne's method in the unramified case, and the tamely ramified case see [Ten15], and [Tót11].

In this thesis, we combine techniques and ideas from the approaches, and from [Ten15], to give an elementary proof of the ramified case of Deligne's approach to geometric class field theory. We follow Takeuchi's construction of the compactification $\tilde{C}^{(d)}_{\mathfrak{m}}$ of $U^{(d)}$ by blowing up $C^{(d)}$ and calculate the ramification of $\mathcal{F}^{(d)}$ along the boundary H of $\tilde{C}^{(d)}_{\mathfrak{m}}$ directly, avoiding the use of Swan conductors.

In the rest of the introduction, we state the main theorem of geometric class field theory Theorem 1, and its reduction to Theorem 2, which we prove in this thesis.

Let k be a perfect field, and let C be a projective smooth geometrically connected curve over k, with genus g. Geometric class field theory gives a geometric description of abelian coverings of C by relating it to isogenies of the generalized picard schemes.

Fix a modulus $\mathfrak{m}$, i.e. an effective Cartier divisor of C and let U be its complement in C. The pairs $(\mathcal{L}, \alpha)$, where $\mathcal{L}$ is an invertible $\mathcal{O}_C$-module and α is a rigidification of $\mathcal{L}$ along $\mathfrak{m}$, are parametrized by a k-group scheme $\mathrm{Pic}_{C,\mathfrak{m}}$, called the rigidified Picard scheme. The Abel-Jacobi morphism

$$\Phi : U \to \mathrm{Pic}_{C,\mathfrak{m}}$$

is the morphism which sends a section x of U to the pair $(\mathcal{O}(x), 1)$. The fundamental result of geometric class field theory can be formulated as:

Theorem 1 (Geometric Class Field Theory)**.** *Let Λ be a finite ring of cardinality invertible in k, and let $\mathcal{F}$ be an étale sheaf of Λ-modules, locally free of rank 1 on U, with ramification bounded by $\mathfrak{m}$. Then, there exists a unique (up to isomorphism) multiplicative étale sheaf of Λ-modules $\mathcal{G}$ on $\mathrm{Pic}_{C,\mathfrak{m}}$, locally free of rank 1, such that the pullback of $\mathcal{G}$ by Φ is isomorphic to $\mathcal{F}$.*

The notion of a multiplicative locally free Λ-module of rank 1 is due to [Gui19] and corresponds to isogenies $G \to \mathrm{Pic}_{C,\mathfrak{m}}$ with constant kernel $\Lambda^\times$. This concept corresponds to multiplicative characters of $H^1(\mathrm{Pic}_{C,\mathfrak{m}}, \mathbb{Q}/\mathbb{Z})$ in the formulation of [Tak19], and generalizes Hecke eigensheaves in the context of [Ten15].

Let d be a positive integer. We denote by $U^{(d)}$ the d-th symmetric power of U over k. For an étale sheaf $\mathcal{F}$ on U, we denote by $\mathcal{F}^{(d)}$ the d-th symmetric power of $\mathcal{F}$ on $U^{(d)}$. The degree d Abel-Jacobi morphism is defined as the map

$$\Phi_d : U^{(d)} \to \mathrm{Pic}^d_{C,\mathfrak{m}}$$

which sends a section $x_1 + \cdots + x_d$ of $U^{(d)}$ to the pair $(\mathcal{O}(x_1 + \cdots + x_d), 1)$.

The method of descent shows that to prove Theorem 1, it suffices to prove the following reduced version (see the last page of [Gui19], Section 8.3 of [Ten15], or the proof of Theorem 1.2 in [Tak19] for details on this reduction):

Theorem 2. *Let Λ be a finite ring of cardinality invertible in k, and let $\mathcal{F}$ be an étale sheaf of Λ-modules, locally free of rank 1 on U, with ramification bounded by $\mathfrak{m}$. Then, for sufficiently large integer d, there exists a unique (up to isomorphism) étale sheaf of Λ-modules $\mathcal{G}_d$ on $\mathrm{Pic}^d_{C,\mathfrak{m}}$, locally free of rank 1, such that the pullback of $\mathcal{G}_d$ by Φ_d is isomorphic to $\mathcal{F}^{(d)}$.*

Using the equivilance between G-torsors and locally free Λ-modules of rank 1 ($G = \Lambda^\times$, see Proposition 21), Theorem 2 can be reformulated in terms of G-torsors as follows:

Theorem 3. *Let $G = \Lambda^\times$ be a finite abelian group (Λ as before), and let $\mathcal{P}$ be a G-torsor on U, with ramification bounded by $\mathfrak{m}$. Then, for sufficiently large integer d, there exists a unique (up to isomorphism) G-torsor $\mathcal{Q}_d$ on $\mathrm{Pic}^d_{C,\mathfrak{m}}$, such that the pullback of $\mathcal{Q}_d$ by Φ_d is isomorphic to $\mathcal{P}^{(d)}$.*

To prove Theorem 3 we follow the work of [Tak19], there he analyzed the ramification of $\mathcal{P}^{(d)}$ after blowing up $C^{(d)}$, we analyze this ramification using elementary methods, drawing techniques and ideas from the works of [Gui19] and [Tak19], and [Ten15].

Notation and conventions.

- S is a base scheme.
- $C \to S$ is a relative curve. i.e. smooth morphism of schemes of relative dimension 1, with connected geometric fibers of genus g, which is Zariski-locally projective over S. Note that the genus g is a locally cosntant function on S.
- Most of the time we will assume that $S = \mathrm{Spec}\, k$, where k is a perfect field.

- A modulus $\mathfrak{m}$ on $C \to S$, is defined as an effective Cartier divisor of C over S (i.e., a closed subscheme of C which is finite flat of finite presentation (hence locally free) over S).

1. Say something about the ramification condition.
2. Add acknowledgemets to yakov, family, etc.

2 Preliminaries

Where is the first time you introduce $\mathcal{P}$ instead of P, mark it down, and explain In this section we recall the necessariy work, including work from [Gui19], [Ten15] and [Tak19].

2.1 Algebraic Geometry

In this section we group together some general theorems in algebraic geometry that we will be employing throughtout the text. All schemes are assumed to be locally of finite type.

Theorem 4. *Let $f : X \to Y$ be a finite flat map between integral schemes, of finite type over a field k. if $Z \subset X$ is a prime divisor with generic point η_Z, then $f(Z) \subset X$ is a prime divisor with generic point $\eta_{f(Z)}$ satisfying $f(\eta_Z) = \eta_{f(Z)}$*

Proof. f is finite hence proper hence closed so $f(Z)$ is closed subset of Y, it is irreducible as the image of an irreducible. Since $Z = \overline{\{\eta_Z\}}$ we get:

$$\{f(\eta_Z)\} \subset f(Z) = f(\overline{\{\eta_Z\}}) \subseteq \overline{f(\{\eta_Z\})} = \overline{\{f(\eta_Z)\}}$$

And since $f(Z)$ is closed we get $f(Z) = \overline{\{f(\eta_Z)\}}$.

For flat map of integral schemes we have for every $x \in X$, $y = f(x)$ the dimension formula:

$$\dim(\mathcal{O}_{X,x}) = \dim(\mathcal{O}_{Y,y}) + \dim(\mathcal{O}_{X_y,x})$$

And since $\dim(\mathcal{O}_{X_y,x}) = 0$ we get $\dim(\mathcal{O}_{X,x}) = \dim(\mathcal{O}_{Y,y})$ concluding that $f(Z)$ is a prime divisor as well. $\square$

A known theorem states that:

Theorem 5. *Let X, Y be two integral schemes over a field k. If X is geometrically integral then $X \times_k Y$ is integral. If both X, Y are geometrically integral, then $X \times_k Y$ is geometrically integral.*

Theorem 6. *Let C be smooth projective curve geometrically connected over a field k. Then:*

1. *$C^{(d)}$ is smooth*
2. *For every d, $C^{(d)}$ is integral.*
3. *For every d, $C^{(d)}$ is geometrically integral.*
4. *The product of every finite number of $C^{(d)}$ is geometrically integral.*

Proof. 1. Let t_i be a local parameter for C at P_i. The local ring of the product C^d at the point $(P_1, \ldots, P_d)$ is isomorphic $k[[t_1, t_2, \ldots, t_d]]$ and the local ring of the qoutient at the divisor $D = \sum P_i$ is $k[[t_1, \ldots, t_d]]^{S_d}$ which is isomorphic to $k[[t_1, \ldots, t_d]]^{S_d} \cong k[[s_1, \ldots, s_d]]$ where the s_i are the symmetric polynomials, hence this ring is regular local ring.

2. C is irreducible hence C^d is irreducible hence $C^{(d)}$ is irreducible. Since $C^{(d)}$ is smooth it is reduced.

3. By [Stacks, Tag 0366], C is geometrically integral, so it follows from the above.

4. Theorem 5

$\square$

Blowups

Theorem 7 ([Stacks, Tag 0805])**.** *Let $X_1 \to X_2$ be a flat morphism of schemes. Let $Z_2 \subset X_2$ be a closed subscheme. Let Z_1 be the inverse image of Z_2 in X_1. Let X_i' be the blowup of Z_i in X_i. Then there exists a cartesian diagram*

$$\begin{tikzcd} X_1' \arrow[r] \arrow[d] & X_2' \arrow[d] \\ X_1 \arrow[r] & X_2 \end{tikzcd}$$

of schemes.

Theorem 8. *If X is integral then $Bl_Z(X)$ is integral.*

2.2 Torsors

Torsors represent a fundamental bridge between two perspectives: they are simultaneously geometric objects amenable to structural manipulation and algebraic entities that allow for explicit computation. Their characterization as realizations of first cohomology classes, their correspondence with invertible modules over ring objects, and their deep-seated connection to Galois theory render them a remarkably flexible and potent tool in arithmetic geometry.

In this section, we provide an overview of the theory of torsors within a topos $\mathcal{E}$, with a particular focus on sheaves over the small and big étale sites. Let $\mathcal{G}$ be a group object in $\mathcal{E}$. We denote by $\mathcal{G}\mathcal{E}$ the category of objects in $\mathcal{E}$ endowed with a left $\mathcal{G}$-action. For any object $X \in \mathcal{E}$, there is a canonical identification between the slice category $(\mathcal{G}\mathcal{E})/X$ and the category of group objects $\mathcal{G}(\mathcal{E}/X)$, where X is viewed as having the trivial $\mathcal{G}$-action.

We follows notes by Alex Youcis Notes

Definitions and Basic Properties

Definition 9. A **$\mathcal{G}$-torsor** in $\mathcal{E}$ is an object $\mathcal{P}$ of $\mathcal{G}\mathcal{E}$ satisfying the following conditions:

1. The structural morphism $\mathcal{P} \to 1$ is an epimorphism in $\mathcal{E}$ (i.e., $\mathcal{P}$ is locally non-empty).

2. The map $\mathcal{G} \times \mathcal{P} \to \mathcal{P} \times \mathcal{P}$ defined by $(g, p) \mapsto (g \cdot p, p)$ is an isomorphism in $\mathcal{E}$ (i.e., $\mathcal{G}$ acts simply transitively on $\mathcal{P}$).

When $\mathcal{E}$ is the topos of sheaves on a site $\mathscr{C}$, the definition can be reformulated in terms of local sections. A $\mathcal{G}$-sheaf $\mathcal{P}$ is a $\mathcal{G}$-torsor if:

1. For every object $X \in \mathcal{C}$, there exists a covering $\{U_i \to X\}$ such that $\mathcal{P}(U_i) \neq \emptyset$ for all i.
2. For any $X \in \mathcal{C}$ where $\mathcal{P}(X)$ is non-empty, the action of $\mathcal{G}(X)$ on $\mathcal{P}(X)$ is simply transitively.

A fundamental property of torsors is their local triviality: a $\mathcal{G}$-sheaf $\mathcal{P}$ is a $\mathcal{G}$-torsor if and only if it is locally isomorphic to the trivial torsor. Specifically, for every $X \in \mathcal{C}$, there must exist a cover $\{U_i \to X\}$ such that the restriction $\mathcal{P}|_{U_i}$ is isomorphic, as a $\mathcal{G}|_{U_i}$-sheaf, to $\mathcal{G}|_{U_i}$ acting on itself by left multiplication.

The Category of Torsors

Definition 10. A **morphism of $\mathcal{G}$-torsors** $f : \mathcal{P}_1 \to \mathcal{P}_2$ is a morphism of sheaves that is equivariant with respect to the $\mathcal{G}$-action.

It is a standard result that every morphism of $\mathcal{G}$-torsors is an isomorphism. Consequently, the category of $\mathcal{G}$-torsors in $\mathcal{E}$ forms a groupoid.

Definition 11. We denote the groupoid of $\mathcal{G}$-torsors in $\mathcal{E}$ by $\mathbf{Tors}(\mathcal{E}, \mathcal{G})$. The set of isomorphism classes of $\mathcal{G}$-torsors is denoted by $\mathrm{Tors}(\mathcal{E}, \mathcal{G})$,.

Functoriality and the Contracted Product

Torsors exhibit functoriality with respect to the group object. Let $\varphi : \mathcal{G}_1 \to \mathcal{G}_2$ be a morphism of group sheaves on $\mathscr{C}$, and let $\mathcal{P}$ be a $\mathcal{G}_1$-torsor. We define the **contracted product** $\mathcal{G}_2 \times^{\mathcal{G}_1} \mathcal{P}$ as the quotient sheaf $(\mathcal{G}_2 \times \mathcal{P})/\mathcal{G}_1$, where $\mathcal{G}_1$ acts on the product by:

$$g_1 \cdot (g_2, p) = (g_2 \varphi(g_1)^{-1}, g_1 \cdot p)$$

The contracted product inherits a natural left $\mathcal{G}_2$-action given on local sections by $h \cdot [g_2, p] = [hg_2, p]$, which endows it with the structure of a $\mathcal{G}_2$-torsor. This construction yields a functor:

$$\varphi_* : \mathbf{Tors}(\mathcal{G}_1) \to \mathbf{Tors}(\mathcal{G}_2), \quad \mathcal{P} \mapsto \mathcal{G}_2 \times^{\mathcal{G}_1} \mathcal{P}$$

On the level of isomorphism classes, φ_* induces a map of pointed sets $\mathrm{Tors}(\mathcal{G}_1) \to \mathrm{Tors}(\mathcal{G}_2)$, sending the class of the trivial $\mathcal{G}_1$-torsor to the class of the trivial $\mathcal{G}_2$-torsor.

The Abelian Case

When $\mathcal{G}$ is a sheaf of abelian groups (an *abelian sheaf*), the pointed set $\mathrm{Tors}(\mathcal{G})$ inherits the structure of an abelian group.

Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be objects of $\mathbf{Tors}(\mathcal{G})$. We define their sum $[\mathcal{P}_1] + [\mathcal{P}_2]$ to be the class $[\mathcal{P}_3]$, where $\mathcal{P}_3$ is the quotient sheaf $(\mathcal{P}_1 \times \mathcal{P}_2)/\mathcal{G}$. In this construction, $\mathcal{G}$ acts on the product $\mathcal{P}_1 \times \mathcal{P}_2$ on T-points by:

$$g \cdot (f_1, f_2) := (gf_1, g^{-1} f_2)$$

The group object $\mathcal{G}$ then acts on the resulting quotient via its action on the presheaf quotient, which is given on classes by:

$$g \cdot [(f_1, f_2)] = [(gf_1, f_2)] = [(f_1, gf_2)]$$

where the square brackets denote the class in the quotient set. This structure turns $\mathrm{Tors}(\mathcal{G})$ into an abelian group, where the identity is the class of the trivial torsor and the inverse is obtained by the opposite action.

Equivantly, the sum is obtained as the contracted product of the $\mathcal{G} \times \mathcal{G}$-torsor $\mathcal{P}_1 \times \mathcal{P}_2$ along the multiplication map $m : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$.

Torsors as Flat Spaces

change Fl to be straight We now fix a scheme X. We start by focusing on the big fppf site. X_{Fl}. Let $\mathcal{G}$ be a group sheaf on X_{Fl}. A flat-torsor $\mathcal{G}$-torsor on X is just a $\mathcal{G}$-torsor on the site X_{Fl}.

To the end of this section assume G **is a flat affine-algebraic X-group.** (which by descent is associated also to a group sheaf)

Definition 12. Define a *principal G-bundle* (or *principal homogenous space* for G) to be a flat finite presentation X-scheme $f : Y \to X$ with an action of G satisfying the following equivalent properties:

1. The morphism $Y \times_X G \to Y \times_X Y$ defined on T-points by sending $(y, g) \in Y(T) \times G(T)$ to (y, gy) is an isomorphism of X-schemes.
2. There exists an open covering $\{U_i \to X\}$ in X_{fl} such that Y_{U_i} is isomorphic, as a G_{U_i}-space, to G_{U_i} with its left multiplication action.

Similarly to the case of G-torsors of a topos, we define a morphism of principal G-bundles to be a morphism of X-schemes commuting with the G-action. We now have:

Theorem 13. *The morphism sending $Y \mapsto \mathrm{Hom}_X(-, Y)$ is an equivalence of categories from the category of principal G-bundles to the category of G-torsors on X_{Fl}. Similarly, the morphism sending Y to $\mathrm{Hom}_X(-, Y)$ to the category of G-torsors on X_{fl} is an equivalence.*

Corollary 14. *There is a natural equivalence $\mathbf{Tors}(X_{\mathrm{fl}}, G) \cong \mathbf{Tors}(X_{\mathrm{Fl}}, G)$ inducing a bijection of pointed sets $\mathrm{Tors}(X_{\mathrm{fl}}, G) \xrightarrow{\cong} \mathrm{Tors}(X_{\mathrm{Fl}}, G)$ which is an isomorphism of abelian groups if G is abelian. And every G-torsor, in each of the above sites can be realized as a scheme, which is a prinicipal G-bundle.*

Torsors as Etale Spaces

Let X be a scheme, and let G be affine algebraic X-group. For any topology $\mathcal{T}$ on Sch/X coarser than the flat topology, we say that a flat torsor $\mathcal{P}$ for G is *locally trivial for the $\mathcal{T}$ topology* if, in fact, one can find a covering $\{U_i \to X\}$ in $\mathcal{T}$ such that $\mathcal{P}(U_i)$ or, equivalently, $\mathcal{P}_{U_i}$ is isomorphic to the trivial torsor for all i.

Then, it is immediate that $\mathrm{Tors}(X_{Et,G})$ is canonically isomorphic as pointed sets (abelian groups if G is abelian) to the subset of $\mathrm{Tors}(X_{Fl}, G)$ consisting of that flat torsors locally trivial for the etale topology.

For the small sites we have:

Theorem 15. *If G is smooth affine X group, then any G-torsor $\mathcal{P}$ on X_{fl} is locally trivial for the etale topology.*

Corollary 16. *If G is smooth affine X group, then there is a canonical bijection of pointed sets (abelian groups if G is abelian)* $\mathrm{Tors}(X_{et}, G) \cong \mathrm{Tors}(X_{fl}, G)$

And we also have:

Theorem 17. *If G is a smooth algebraic X-groupm there every flat G-torsor is locally trivial for the etale topology and every principal G-bundle $Y \to X$ is smooth. In other words the inclusion* $\mathbf{Tors}(X_{Et}, G) \to \mathbf{Tors}(X_{Fl}, G)$ *is actuall an isomorphism. ??(of what? of categories?)*

see which of the above theorems we leave intact, cuase it doesn't seem like we need all three

We now focus on the etale-topology which is coarser then the flat topology. Two theorems summarize what happends over the big and small sites:

Torsors and Cohomology

We qoute without proof:

Theorem 18. *There is a natural bijection of pointed sets Tors$(\mathcal{G}|_{\mathcal{T}}) \to \check{H}^1(T, \mathcal{G})$. Moreover, if $\mathcal{G}$ is an abelian group sheaf, it's an isomorphism of abelian groups.*

The idea is that for every $[\mathcal{P}] \in \mathrm{Tors}(T, \mathcal{G})$ We

1. Choose a covering $\{U_i \to T\}$ such that $\mathcal{P}(U_i) \neq \emptyset$ for all i
2. choose sections $\alpha_i \in \mathcal{F}(U_i)$

Then we note that for all (i, j) the elements $\alpha_i|_{U_i \times_X U_j}$ and $\alpha_j|_{U_i \times_X U_j}$ differ by a unique element of $\mathcal{G}(U_{ij})$ there exists a unique $s_{ij} \in \mathcal{G}(U_{ij})$ such that $\alpha_i|_{U_i \times_X U_j} = s_{ij}(\alpha_j|_{U_i \times_X U_j})$. One can then easily see that (s_{ij}) defines an element of $\check{H}^1(T, \mathcal{G})$ which is independent of the choice of repreantiative $\mathcal{P}$, choice of covering $\{U_i \to T\}$ and choice of sections.

Constant Finite Group Torsors

Let G be a finite group. We denote by $\underline{G}$ the constant group scheme $\underline{G}$ over X. Sometimes denoted by $\underline{G}_X$ and is given by $\coprod_{g \in G} X$ with the action shuffeling the X's according to multiplication.

By following the definitions, one sees that if X be a connected scheme and $f : Y \to X$ is a finite Galois cover with Galois group G. Then, $f : Y \to X$ is a principal $\underline{G}$-bundle. (Recall that a *finite Galois cover* is a finite étale surjection $Y \to X$ with Y connected and such that $G = \mathrm{Aut}(Y/X)$ acts transitively on the geometric points of Y lying over any geometric point of X.)

On the otherhand if $f : Y \to X$ is a principal $\underline{G}$-bundle with Y connected, then Y is a finite Galois cover with automorphism group G.

However, not all G-torsors are connected. If $H \subset G$ is a proper subgroup then any connected finite etale cover $f : Y \to X$ with Galois group H gives rise to a non-connected $\underline{G}$-torsor by looking at the induced G-torsor $\varphi_*(Y)$ under the inclusion $\varphi : \underline{H} \to \underline{G}$. On the otherhand, if we fix a geometric point $\overline{x} \to X$, then to give an homomorphism $\rho \in \mathrm{Hom}_{\mathrm{cont}}(\pi_1^{\mathrm{et}}(X, x), G)$ is equivilant to give a connected pointed Galois cover $(Y, \overline{y}) \to (X, \overline{x})$ with Galois group $H = \rho(\pi_1^{\mathrm{et}}(X, x)) \subset G$. Thus, pushing forward to G we get a prinicipal $\underline{G}$-bundle. The choice of a different geometric point $\overline{x'} \to X$ differ the homomorphism by an inner automorphism, thus we have:

Theorem 19. *Let X be a connected scheme and $\overline{x}$ a geometric point of X. Suppose in addition that G is a finite abstract group. Define a map*

$$\mathrm{Hom}_{\mathrm{cont}}(\pi_1^{\mathrm{et}}(X, \overline{x}), G)/\mathrm{Inn}(G) \to \mathrm{Tors}(X_{\mathrm{Fl}}, G) \tag{1}$$

by sending a homomorphism $\rho : \pi_1^{\mathrm{et}}(X, \overline{x}) \to G$ to the principal G-bundle $\varphi_*(Y)$ where Y is the principal $\underline{\rho(\pi_1^{\mathrm{et}}(X, \overline{x}))}$-bundle obtained above and φ is the inclusion $\rho(\pi_1^{\mathrm{et}}(X, \overline{x})) \hookrightarrow G$. Then, the map is a bijection of pointed sets where the trivial homomorphisms (which is the only element of its $\mathrm{Inn}(G)$*-orbit) is the distinguished element of the left hand side.*

If G is Abelian then $\mathrm{Inn}(G)$ is trivial, and we obtain:

Corollary 20. *Let G be a finite Abelian group, X a connected scheme, and $\overline{x}$ a geometric point of X. Then, the map from Theorem 19 induces an isomorphism of Abelian groups*

$$\mathrm{Hom}_{\mathrm{cont.}}(\pi_1^{\acute{e}t}(X, \overline{x}), G) \xrightarrow{\cong} \mathrm{Tors}(X_{\mathrm{Fl}}, G) \tag{2}$$

Let us give a final note that, evidently, $\mathrm{Aut}(G)$ acts on $\mathrm{Hom}_{\mathrm{cont.}}(\pi_1^{\mathrm{\acute{e}t}}(X, x), G)$ on the right, and if we consider the quotient $\mathrm{Hom}_{\mathrm{cont.}}(\pi_1^{\mathrm{\acute{e}t}}(X, x), G)/\mathrm{Aut}(G)$ we get the pointed set of all connected finite Galois covers of X with Galois group isomorphism to a subgroup of G.

Equivalence between Torsors and Invertible Modules

Edit this completly. Let $\mathcal{E}$ be a topos and let Λ be a commutative ring object in $\mathcal{E}$. Let $G = \Lambda^\times$ denote the internal group object of units of Λ. The following proposition establishes the fundamental dictionary between the geometric theory of principal homogeneous spaces and the algebraic theory of invertible modules. This equivalence allows us to transport the monoidal structure from the category of modules (with the tensor product over Λ) to the category of torsors (with the contracted product over G), strictly within the categorical framework.

Proposition 21. *There is a canonical equivalence of monoidal categories between the category of G-torsors in $\mathcal{E}$ and the category of locally free Λ-modules of rank 1 in $\mathcal{E}$:*

$$\Phi : \mathbf{Tors}(\mathcal{E}, \Lambda^\times) \xrightarrow{\sim} \mathbf{Pic}(\mathcal{E}, \Lambda)$$

The equivalence is defined by the associated module functor:

$$\mathcal{P} \longmapsto \mathcal{P} \times^{\Lambda^\times} \Lambda := \Lambda^\times \backslash (\Lambda \times \mathcal{P})$$

where the quotient is taken with respect to the diagonal action of $\Lambda^\times$ on $\Lambda \times \mathcal{P}$. The inverse functor associates to an invertible module $\mathcal{F}$ its sheaf of basis frames $\underline{\mathrm{Isom}}_\Lambda(\Lambda, \mathcal{F})$.

In light of this canonical equivalence, we will pass freely between the language of G-torsors and that of locally free Λ-modules throughout the text.

For a topos $\mathcal{E}$, a group object G in $\mathcal{E}$ and an object X in $\mathcal{E}$, there is a canonical identification between $(G\mathcal{E})_{/X}$ and $G(\mathcal{E}_{/X})$, given by endowing X with the trivial G-action.

We denote by $\mathbf{Tors}(X, G)$ the category of G-torsors over X in $G\mathcal{E}_{/X}$. Similarly, for a ring object Λ in $\mathcal{E}$, we denote by $\mathbf{Pic}(X, \Lambda)$ the category of locally free Λ-modules of rank 1 over X in $\mathcal{E}_{/X}$. The above equivilance of categories becomes

$$\Phi_X : \mathbf{Tors}(X, \Lambda^\times) \xrightarrow{\sim} \mathbf{Pic}(X, \Lambda)$$

.

For a morphism $f : Y \to X$ in $\mathcal{E}$, the equivalence is functorial with respect to:

- **Torsor Pullback:** $f^{-1}P = P \times_X Y$ (Fiber product).
- **Module Pullback:** $f^*\mathcal{L} = \Lambda_Y \otimes_{f^{-1}\Lambda_X} f^{-1}\mathcal{L}$ (Extension of scalars).

The following diagram commutes up to natural isomorphism:

$$\begin{tikzcd}
\mathbf{Tors}(X, G) \arrow[r, "\Phi_X", "\sim"'] \arrow[d, "f^{-1}"'] & \mathbf{Pic}(X, \Lambda) \arrow[d, "f^*"] \\
\mathbf{Tors}(Y, G) \arrow[r, "\Phi_Y", "\sim"'] & \mathbf{Pic}(Y, \Lambda)
\end{tikzcd}$$

More about G-torsors

We want to be more explicit about G-torsors, so let us recall the definition.

Other Theorems

Say something about the toposes of Etale and etale, maybe add them up in the notation. We recall some propositions about G-torsors that will be useful later.

Proposition 22 ([Gui19], Proposition 2.12)**.** *Let G be a finite abelian group, let S be a scheme, and let $\mathcal{P}$ be a G-torsor over an S-scheme X in S_{Et}. Then the etale sheaf $\mathcal{P}$ is representable by a finite etale X-scheme.*

Corollary 23 ([Gui19], Corollary 2.13)**.** *Let G be a finite abelian group, let S be a scheme, and let X be an S-scheme. Then the category of G-torsors over X in S_{Et} is equivalent to the category of G-torsors over X (the terminal object) in $X_{\acute{e}t}$.*

Next, we recall the definition of the contracted product of torsors, which endows the category of G-torsors with a monoidal structure.

2.3 Symmetric Powers of Schemes and Torsors

This section reviews the construction of quotients for schemes and torsors under finite group actions, specifically focusing on symmetric powers. To ensure these quotients exist as schemes, we utilize the framework of admissible actions from [SGA1]. Our treatment here closely follows the exposition in [Gui19] The definitions and results presented below are adapted from their work. This foundation provides the necessary criteria for admissibility and base change required to define the symmetric powers of a scheme X and a G-torsor $\mathcal{P}$ over X.

Let S be a scheme.

Definition 24 (([SGA1], V.1.7).)**.**

- Let T be an object of a category $\mathcal{C}$ endowed with a right action of a group Γ. We say that **the quotient** T/Γ **exists** in $\mathcal{C}$ if the covariant functor

$$\begin{aligned} \mathcal{C} &\to \mathrm{Sets} \\ U &\mapsto \mathrm{Hom}_{\mathcal{C}}(T,U)^{\Gamma} \end{aligned}$$

  is representable by an object of $\mathcal{C}$.

- Let T be an S-scheme. An action of a finite group Γ on T is **admissible** if there exists an affine Γ-invariant morphism $f : T \to T'$ such that the canonical morphism $\mathcal{O}_{T'} \to f_*\mathcal{O}_T$ induces an isomorphism from $\mathcal{O}_{T'}$ to $(f_*\mathcal{O}_T)^{\Gamma}$.

Proposition 25. *The following holds:*

1. ***([SGA1] V.1.3)****. Let T be an S-scheme endowed with an admissible right action of a finite group Γ. If $f : T \to T'$ is an affine Γ-invariant morphism such that the canonical morphism $\mathcal{O}_{T'} \to f_*\mathcal{O}_T$ induces an isomorphism from $\mathcal{O}_{T'}$ to $(f_*\mathcal{O}_T)^{\Gamma}$, then the quotient T/Γ exists and is isomorphic to T'.*

2. ***([SGA1], V.1.8)****. Let T be an S-scheme endowed with a right action of a finite group Γ. Then, the action of Γ on T is admissible if and only if T is covered by Γ-invariant affine open subsets.*

3. ***([SGA1], V.1.9)****. Let T be an S-scheme endowed with an admissible right action of a finite group Γ, and let S' be a flat S-scheme. Then, the action of Γ on the S'-scheme $T \times_S S'$ is admissible, and the canonical morphism*

$$(T \times_S S')/\Gamma \to (T/\Gamma) \times_S S'$$

   is an isomorphism.

Proposition 26 ([SGA1], IX.5.8)**.** *Let G be a finite abelian group, let $\mathcal{P}$ be a G-torsor over an S-scheme X in $S_{\acute{E}t}$. Assume that $\mathcal{P}$ and X are endowed with right actions from a finite group Γ such that the morphism $\mathcal{P} \to X$ is Γ-equivariant, and that the following properties hold:*

(a) The right Γ-action on $\mathcal{P}$ commutes with the left G-action.

(b) The right Γ-action on X is admissible, and the quotient morphism $X \to X/\Gamma$ is finite.

(c) For any geometric point $\bar{x}$ of X, the action of the stabilizer $\Gamma_{\bar{x}}$ of $\bar{x}$ in Γ on the fiber $\mathcal{P}_{\bar{x}}$ of $\mathcal{P}$ at $\bar{x}$ is trivial.

Then the action of Γ on $\mathcal{P}$ is admissible, and $\mathcal{P}/\Gamma$ is a G-torsor over X/Γ in $S_{\acute{E}t}$.

Symmetric Powers of Schemes

Let X be an S-scheme and let $d \geq 0$ be an integer. The group S_d of permutations of $[\![1, d]\!]$ acts on the right on the S-scheme $X^{\times_S d} = X \times_S \cdots \times_S X$ by the formula

$$(x_i)_{i \in [\![1,d]\!]} \cdot \sigma = (x_{\sigma(i)})_{i \in [\![1,d]\!]}.$$

Proposition 27 ([Gui19] Proposition 2.27)**.** *If X is a scheme, Zariski locally quasi-projective over S, then the right action of the symmetric group S_d on the d-fold fiber product $X^{\times_S d}$ is admissible. Consequently, the quotient $\mathrm{Sym}^d_S(X) = X^{\times_S d}/S_d$ exists as a scheme over S.*

Remark. When the base S is understood from context, this quotient is also denoted by $X^{(d)}$.

Guingard shows that when $X = \mathrm{Spec}(B)$ and $S = \mathrm{Spec}(A)$ then $\mathrm{Sym}^d_S(X)$ is representable by an affine S-scheme (See [Gui19] Remark 2.28).

Proposition 28 ([Gui19] Proposition 2.28)**.** *If X is flat and Zariski-locally quasi-projective over S, then $Sym^d_S(X)$ is flat over S. Moreover, for any S-scheme S', the canonical morphism*

$$Sym^d_{S'}(X \times_S S') \to Sym^d_S(X) \times_S S'$$

is an isomorphism.

Symmetric Powers of Torsors

change torsor tensor product to contracted product, and the analogy with sheaves to tensor product (which it is) below the exposition to be more accurate...

change below the exposition to be more accurate... Let S be a scheme, let X be an S-scheme and let $d \geq 1$ be an integer. Let G be a finite abelian group, and let $\mathcal{P} \to X$ be a G-torsor over X in $S_{\text{Ét}}$. It is easy to show that the sheaf $\mathcal{P}$ is representable by a finite étale X-scheme. (For example [Gui19] Proposition 2.12)

For each $i \in [\![1, d]\!]$ let $p_i : X^{\times_S d} \to X$ be the projection on i-th factor, and let us consider the G-torsor

$$p_1^{-1}\mathcal{P} \otimes \cdots \otimes p_d^{-1}\mathcal{P} = G_d \backslash \mathcal{P}^{\times_S d}$$

over $X^{\times_S d}$, where $G_d \subseteq G^d$ is the kernel of the multiplication morphism $G^d \to G$. The object $G_d \backslash \mathcal{P}^{\times_S d}$ of $S_{\text{Ét}}$ is too representable by an S-scheme which is finite étale over $X^{\times_S d}$. The group S_d acts on the right on $G_d \backslash \mathcal{P}^{\times_S d}$ by the formula

$$(p_i)_{i \in [\![1,d]\!]} \cdot \sigma = (p_{\sigma(i)})_{i \in [\![1,d]\!]}.$$

This action of S_d commutes with the left action of G on $G_d \backslash \mathcal{P}^{\times_S d}$.

Proposition 29 ([Gui19] Proposition 2.32.)**.** *If X is Zariski-locally quasi-projective on S, then the right action of S_d on $G_d \backslash \mathcal{P}^{\times_S d}$ is admissible, so that the quotient $\mathcal{P}^{(d)}$ of $G_d \backslash \mathcal{P}^{\times_S d}$ by S_d exists as a scheme over S. Moreover, the canonical morphism $\mathcal{P}^{(d)} \to \mathrm{Sym}^d_S(X)$ is a G-torsor, and the morphism*

$$p_1^{-1}\mathcal{P} \otimes \cdots \otimes p_d^{-1}\mathcal{P} \to r^{-1}\mathcal{P}^{(d)}$$

where $r : X^{\times_S d} \to \mathrm{Sym}^d_S(X)$ is the canonical projection, is an isomorphism of G-torsors over $X^{\times_S d}$.

consider replacing $\mathcal{P}$ with P because it is a scheme Add proposition about how it is being a scheme

2.4 Etale Fundamental Groups and Tame Fundamental Groups

We recall the definition and basic properties of the etale fundamental group, following stacks project [Stacks, Tag 0BQ6]

Proposition 30 ([Stacks, Tag 0C0J])**.** *Let $f : X \to S$ be a flat proper morphism of finite presentation whose geometric fibres are connected and reduced. Assume S is connected and let $\overline{s}$ be a geometric point of S. Then there is an exact sequence*

$$\pi_1(X_{\overline{s}}) \to \pi_1(X) \to \pi_1(S) \to 1$$

of fundamental groups.

Corollary 31. *Let $f : X \to S$ be a proper smooth morphism of finite presentation whose geometric fibres are connected. Assume S is connected and let $\overline{s}$ be a geometric point of S. Then there is an exact sequence*

$$\pi_1(X_{\overline{s}}) \to \pi_1(X) \to \pi_1(S) \to 1$$

of fundamental groups.

add about tameness?

2.5 Algebraic Preliminaries on Ramification

change? We recall the basic definitions and properties of the ramification of discrete valuations. We start with the general case of discrete valuation rings and their integral closures within finite separable field extensions. Then, we move to the specific setting of complete discrete valuation rings within Galois extensions, where we describe the ramification filtration of the Galois group via both lower and upper numbering. We follow [Stacks, Tag 0EXQ], and [Ser79].

Ramification of Discrete Valuation Rings

Let A be a discrete valuation ring with fraction field K. Let L/K be a finite separable field extension. Let $B \subset L$ be the integral closure of A in L. Picture:

$$\begin{tikzcd} B \arrow[r] & L \\ A \arrow[u] \arrow[r] & K \arrow[u] \end{tikzcd}$$

By [Stacks, Tag 032L] the ring extension $A \subset B$ is finite, hence B is Noetherian. By [Stacks, Tag 00OK] the dimension of B is 1, hence B is a Dedekind domain, see [Stacks, Tag 034X]. Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ be the maximal ideals of B (i.e., the primes lying over $\mathfrak{m}_A$). We obtain extensions of discrete valuation rings

$$A \subset B_{\mathfrak{m}_i}$$

and hence ramification indices e_i and residue degrees f_i. We have

$$[L:K] = \sum\nolimits_{i=1,\dots,n} e_i f_i$$

by [Stacks, Tag 02MJ] applied to a uniformizer in A. We observe that $n = 1$ if A is henselian (by [Stacks, Tag 04GH] and the fact that B is a domain), e.g. if A is complete.

Definition 32. Let A be a discrete valuation ring with fraction field K. Let L/K be a finite separable extension. With B and $\mathfrak{m}_i$, $i = 1, \dots, n$ as above, we say the extension L/K is

1. unramified with respect to A if $e_i = 1$ and the extension $\kappa(\mathfrak{m}_i)/\kappa_A$ is separable for all i,
2. tamely ramified with respect to A if either the characteristic of κ_A is 0 or the characteristic of κ_A is $p > 0$, the field extensions $\kappa(\mathfrak{m}_i)/\kappa_A$ are separable, and the ramification indices e_i are prime to p, and
3. totally ramified with respect to A if $n = 1$ and the residue field extension $\kappa(\mathfrak{m}_1)/\kappa_A$ is trivial.

If the discrete valuation ring A is clear from context, then we sometimes say L/K is unramified, totally ramified, or tamely ramified for short.

Structure Theorems and Some Lemmas

Let A be a complete discrete valuation ring over with uniformizer π and residue field κ, which we assume to be perfect. When A and κ are of the same characteristic $p > 0$, then A contains a coefficient field $k \cong \kappa$ and a well known structure theorem holds: $A = k[[\pi]] \cong k[[t]]$. Let K be the fraction field of A, then $K = k((\pi))$. By Kummer theory, unramified extensions of K correspond to separable extensions of k. The maximal unramified extension of K is $\overline{k}((\pi))$ where $\overline{k}$ is a separable closure of k. Maybe add something about the above facts.

Classical Ramification Filtration in the Galois Case

We now recall the classical ramification filtration in the Galois case. Assume A, B are complete DVRs. And that L/K is Galois with Galois group G. In that case there is uniformizer $\pi \in B$ such that $B = A[\pi]$

We have the ramification filtration of G by lower numbering $(G_i)_{i \geq -1}$, defined by

$$G_i = \{\sigma \in G \mid v_B(\sigma(x) - x) \geq i + 1 \text{ for all } x \in B\}$$

where v_B is the valuation on L associated to B. In particular, $G_{-1} = G$ and G_0 is the inertia group of the extension L/K. We have that L/K is unramified if and only if G_0 is trivial, and L/K is tamely ramified if and only if G_1 is trivial. It is easy exercise that in the definition of G_i it is enough to check the condition for the uniformizer π of B, if we define $i_K^L(\sigma) = v_B(\sigma(\pi) - \pi)$ for $\sigma \in G$, then we have $G_i = \{\sigma \in G \mid i_K^L(\sigma) \geq i + 1\}$. The groups G_i are normal in G and are trivial for large enough i. In a tower of fields $K \subset E \subset L$, where $H = \mathrm{Gal}(L/E)$ we have

$$G_i \cap H = H_i$$

for all $i \geq -1$, which corresponds to the fact that $i_E^L = i_K^L|_{\mathrm{Gal}(L/E)}$. Ramification groups also behave well with respect to quotients: $G_i H/H = (G/H)_j$. where

$$j = \frac{1}{e_{L/E}} \sum_{\tau \in H} \min(i_K^L(\tau), i+1) - 1$$

i.e. the quotient of a ramification group is itself a ramification group, but with a different index. In the literature, one reindexes the ramification groups by defining the Herbrand function $\phi_{L/K} : [-1, \infty) \to [-1, \infty)$:

$$\phi_{L/K}(i) = \frac{1}{e_{L/K}} \sum_{\sigma \in G} \min(i_K^L(\sigma), i+1) - 1 = \int_0^i \frac{1}{[G_0 : G_t]} dt$$

It is continuous, increasing, piecewise linear function, hence a bijection. It satisfies $\phi_{L/K} = \phi_{E/K} \circ \phi_{L/E}$ for $K \subset E \subset L$, and $G_i H/H = (G/H)_{\phi_{L/E}(i)}$. Thus, defining the ramification groups by upper numbering as $G^i = G_{\phi_{L/K}^{-1}(i)}$, we have:

$$G^i H/H = (G/H)^i$$

for all $i \geq -1$.

2.6 Tame Ramification and Ramification of G-Torsors

Unramified scheme morphisms is not the same as unramified extensions of DVRs here, so be careful, and say something about that... Regarding Tame Ramification we follow [Stacks, Tag 0BSE]. It is worth mentioning [KS10] for the different notions of tameness in higher dimensions, and to what extent they agree.

Tame Ramification of etale covering in Codimension 1

Definition 33. Assume we are given:

1. a locally Noetherian scheme X,
2. a dense open $U \subset X$
3. a finite étale morphism $f : Y \to U$

such that for every prime divisor $Z \subset X$ with $Z \cap U = \emptyset$ the local ring $\mathcal{O}_{X,\xi}$ of X at the generic point ξ of Z is a discrete valuation ring. Setting K_ξ equal to the fraction field of $\mathcal{O}_{X,\xi}$ we obtain a cartesian square

$$\begin{array}{ccc} \mathrm{Spec}(K_\xi) & \longrightarrow & U \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathcal{O}_{X,\xi}) & \longrightarrow & X \end{array}$$

of schemes. In particular, we see that $Y \times_U \mathrm{Spec}(K_\xi)$ is the spectrum of a finite separable algebra L_ξ/K_ξ. Then we say Y is unramified over X in codimension 1, resp. Y is tamely ramified over X in codimension 1 if L_ξ/K_ξ is unramified, resp. tamely ramified with respect to $\mathcal{O}_{X,\xi}$ for every (Z, ξ) as above, (Definition 32). More precisely, we decompose L_ξ into a product of finite separable field extensions of K_ξ and we require each of these to be unramified, resp. tamely ramified with respect to $\mathcal{O}_{X,\xi}$.

Ramification of G-Torsors over Curves

Let G be a finite abelian group. Let k be a perfect field and let C be a projective smooth geometrically connected curve over k, with genus g. Let $\mathfrak{m} = \sum_i n_i P_i$ be a modulus (i.e. an effective Cartier divisor) on C and let $U = C \setminus \mathfrak{m}$. Let $\mathcal{P}$ be a G-torsor in $U_{\text{ét}}$. By Proposition 22, $\mathcal{P}$ is representable by a finite etale U-scheme.

Let $P \in \mathfrak{m} \subset C$ a closed point. Then $\mathcal{O}_{C,P}$ is a discrete valuation ring with fraction field K_P. After completion at the maximal ideal $\mathfrak{m}_P$ we obtain a complete discrete valuation ring $\widehat{\mathcal{O}_{C,P}}$ with fraction field $\widehat{K}_P$. Restricting the G-torsor $\mathcal{P}$ to $\operatorname{Spec}(K_P), \operatorname{Spec}(\widehat{K}_P)$ we obtain G-torsors in $\operatorname{Spec}(\widehat{K}_P)_{\text{Ét}}$, $\operatorname{Spec}(K_P)_{\text{Ét}}$ as in the diagram below:

$$\begin{tikzcd}
\mathcal{P}|_{\operatorname{Spec}(\widehat{K}_P)} \arrow[d] & \mathcal{P}|_{\operatorname{Spec}(K_P)} \arrow[d] & \mathcal{P} \arrow[d] \\
\operatorname{Spec}(\widehat{K}_P) \arrow[r] \arrow[d] & \operatorname{Spec}(K_P) \arrow[r] \arrow[d] & U \arrow[d] \\
\operatorname{Spec}(\widehat{\mathcal{O}_{C,P}}) \arrow[r] & \operatorname{Spec}(\mathcal{O}_{C,P}) \arrow[r] & C
\end{tikzcd}$$

The G-torsor $\mathcal{P}|_{\operatorname{Spec}(K_P)} \to \operatorname{Spec}(K_P)$ is an etale covering of $\operatorname{Spec}(K_P)$. Hence decompose into a disjoint union of spectra of finite separable field extensions of K_P.

$$\mathcal{P}|_{\operatorname{Spec}(K_P)} = \bigsqcup_i \operatorname{Spec}(M_i)$$

where each M_i/K_P is a finite separable field extension. Pulling back by $\operatorname{Spec}(\widehat{K_P})$ we get

$$\mathcal{P}|_{\operatorname{Spec}(\widehat{K}_P)} = \bigsqcup_i \operatorname{Spec}(M_i \otimes_{K_P} \widehat{K_P})$$

Each product $M_i \otimes_{K_P} \widehat{K_P}$ decomposes into a finite product of finite separable field extensions of $\widehat{K_P}$.

$$M_i \otimes_{K_P} \widehat{K_P} = \prod_Q \widehat{M}_{i,Q}$$

Where Q ranges over primes of M_i above P, and each $\widehat{M}_{i,Q}/\widehat{K_P}$ is a completion of M_i at that prime.

To summarize, we have decomposition:

$$\mathcal{P}|_{\operatorname{Spec}(\widehat{K}_P)} = \bigsqcup_i \operatorname{Spec}(F_i)$$

Where each $F_i/\widehat{K_P}$ is a finite separable field extension.

The fact that $\mathcal{P}$ is a G-torsor implies that:

1. The fields F_i are pairwise isomorphic
2. The fields F_i are Galois over $\widehat{K_P}$ with Galois group isomorphic to a subgroup of $H \subset G$.

3. The number of components F_i is equal to the index $[G : H]$.

We say the ramification of F_i over $\widehat{K_P}$ is bounded by r if the ramification group H^r (in upper numbering) is trivial.

We say that the G-torsor $\mathcal{P}|_{\mathrm{Spec}(\widehat{K}_P)}$ has ramification at P bounded by r if any of the $F_i/\widehat{K_P}$ has ramification bounded by r.

We say that the G-torsor $\mathcal{P}$ has ramification at P bounded by r if $\mathcal{P}|_{\mathrm{Spec}(\widehat{K}_P)}$ has ramification at P bounded by r.

Finally,

Definition 34. A G-torsor $\mathcal{P}$ on $U_{\text{ét}}$ has **ramification bounded by** $\mathfrak{m} = \sum n_i P_i$ **over** $\mathrm{Spec}(k)$ if for every i, the ramification of $\mathcal{P}|_{\mathrm{Spec}(\widehat{K}_{P_i})}$ at P_i is bounded by n_i.

Alternative Definition of Ramification of G-Torsors over Curves

Choose a geometric point $\bar{s} = \mathrm{Spec}(\bar{k}) \to \mathrm{Spec}(k)$. corresponding to a separable closure $k^{sep} = \bar{k}$ of k. By Section 2.5, the higher ramification groups considered for $\mathcal{P}|_{\mathrm{Spec}(\widehat{K}_P)}$ and $\mathcal{P}_{\bar{k}}|_{\mathrm{Spec}(\widehat{K_P \otimes \bar{k}})}$ are isomorphic. Thus, we define:

Definition 35. A G-torsor $\mathcal{P}$ on $U_{\text{ét}}$ has **ramification bounded by** $\mathfrak{m}$ **over** $\mathrm{Spec}(k)$ if for every geometric point $\bar{x}$ of $\mathfrak{m}$, with image $\bar{s}$ in $\mathrm{Spec}(k)$, the restriction of $\mathcal{P}$ to

$$\mathrm{Spec}(\widehat{\mathcal{O}_{C_{\bar{k}},\bar{x}}}) \times_{C_{\bar{k}}} U_{\bar{k}}$$

has ramification bounded by the multiplicity of $\mathfrak{m}_{\bar{s}}$ at $\bar{x}$.

Explanation: The two definitions are equivalent. This is immediate as $\mathcal{O}_{C_{\bar{k}},\bar{x}}$ is the strict henselization of $\mathcal{O}_{C_{\bar{(k)}},P}$ So after completion it is $\widehat{\mathcal{O}_{C_{\bar{k}},\bar{x}}} \cong \widehat{\mathcal{O}_{C,P}} \otimes_k \bar{k}$. And taking the product with $Y_{\bar{k}}$ amounts to taking the fraction fields, i.e. we get $\mathrm{Spec}(\widehat{K_P \otimes \bar{k}}) = \mathrm{Spec}(\widehat{K_P} \otimes_k \bar{k})$.

Note that tame ramification and unramifiedness in terms of definition above coincide with the ones in Definition 33.

Ramification of G-torsors in terms of Characters

Since we are working over $X = \mathrm{Spec}(k)$, the group G is *etale* over $\mathrm{Spec}(k)$ Hence by Corollary 20 and Corollary 16 we get an isomosphism of groups: $\mathrm{Hom}_{\mathrm{cont.}}(\pi_1^{\text{ét}}(X, \overline{x}), G) \xrightarrow{\cong} \mathrm{Tors}(X_{\mathrm{et}}, G)$

When $X = \mathrm{Spec}(L)$ for a complete valued field L, $\pi_1^{\text{ét}}(X, \overline{x}) = G_L := Gal(L^{sep}/L)$ Where L^{sep} is a fixed seprable closure. And we conclude that $\mathcal{P}|_{\mathrm{Spec}(L)}$ correspond to a continious homomorphism $\rho : G_L \to G$ and one can check that it has ramificaiton bounded by r if and only if $\rho(G_L^r) = \{1\}$.

Basic Properties of Ramification of G-Torsors

In this section we prove some basic properties of the ramification of G-torsors.

Lemma 36. *Let G be a finite abelian group and X be a locally Noetherian scheme over a field k. Let $U \subset X$ be a dense open subset and let Z be a prime divisor in the complement $X \setminus U$, and let ξ denote its generic point.*

Assume $\mathcal{P}_1$ and $\mathcal{P}_2$ are two G-torsors on $U_{\acute{e}t}$, Such that $\mathcal{P}_1$ has ramification bounded by r_1 at (Z, ξ), and $\mathcal{P}_2$ has ramification bounded by r_2 at (Z, ξ). Then their contracted product $\mathcal{P}_1 \wedge^G \mathcal{P}_2$ has ramification bounded by $max(r_1, r_2)$ at (Z, ξ).

Proof. Let $A = \widehat{\mathcal{O}_{X,\xi}}$ and let $K = Frac(A)$. Let $\rho_1, \rho_2 : G_K \to G$ be the associated continious homomorphisms corresponding to the G-torsors $\mathcal{P}_1|_{\mathrm{Spec}(K)}$, $\mathcal{P}_2|_{\mathrm{Spec}(K)}$. Then the associated character to $(\mathcal{P}_1 \wedge^G \mathcal{P}_2)|_{\mathrm{Spec}(K)}$ is $\rho = \rho_1 + \rho_2$. And the claim follows by Section 2.6 □

Lemma 37. *Let $f : X \to Y$ be a morphism of locally Noetherian schemes. Let $U_X \subset X$ and $U_Y \subset Y$ be dense open subschemes such that $f^{-1}(U_Y) \subset U_X$.*

Let Z_X and Z_Y be prime divisors of X and Y with generic points η_X and η_Y, respectively, such that $Z_X \cap U_X = \emptyset$ and $Z_Y \cap U_Y = \emptyset$. Suppose $f(\eta_X) = \eta_Y$ and that f is étale at η_X.

Let $\mathcal{P}$ be a G-torsor over U_Y, and let $f^{-1}\mathcal{P}$ be its pullback to $f^{-1}(U_Y) \subset U_X$. Then, the ramification of $f^{-1}\mathcal{P}$ is bounded by r at η_X if and only if the ramification of $\mathcal{P}$ is bounded by r at η_Y.

Proof. The boundedness of ramification is determined by the behavior of the torsor over the completion of the local rings at the generic points.

Let $A = \mathcal{O}_{Y,\eta_Y}$ and $B = \mathcal{O}_{X,\eta_X}$ be the discrete valuation rings at the generic points, with fraction fields K and L respectively. Since f is étale at η_X, the map $A \to B$ is a flat, unramified local homomorphism. Consequently, the extension of completions $\widehat{L}/\widehat{K}$ is a finite unramified extension of complete discretely valued fields.

The upper numbering filtration on the absolute Galois group is compatible with unramified base change. Specifically, let $G_K = \mathrm{Gal}(K^{sep}/K)$ and $G_L = \mathrm{Gal}(L^{sep}/L)$. For an unramified extension, the Herbrand function is the identity, which implies that for any $r \geq 0$:

$$G_L^r = G_K^r \cap G_L$$

The ramification of the G-torsor $\mathcal{P}$ is bounded by r if and only if the corresponding Galois representation $\rho : G_K \to G$ satisfies $\rho(G_K^r) = \{1\}$.

By the filtration identity above, $\rho(G_L^r) = \{1\}$ if and only if $\rho(G_K^r) = \{1\}$. Thus, the pullback torsor $f^{-1}\mathcal{P}$ has ramification bounded by r at η_X if and only if $\mathcal{P}$ has ramification bounded by r at η_Y. □

The ramification in term of morphisms

maybe not needed.

EDIT AND REFINE THIS ENTIRE THING.

Definition 38 (Version 1: Pure Openness)**.** Let C be a curve over a field k. Let $P\ inC$ be closed point. $U = C - P$ be the complement. Let $\mathcal{F}$ be a local system on U. let $\eta \in P$ be the generic point of P (it is irreducible of codim=1 in C) Then $\mathcal{F}$ has ramification bounded by d on P if and only if $\mathcal{F}|_\eta$ is a field extension of ramification bounded by d.

We want to explain why $\mathcal{F}|_\eta$ is a finite field extension of $k(C)$. We are in the following situation:

$$\begin{array}{ccc} i^*\mathcal{F} & & \mathcal{F} \\ \downarrow & & \downarrow \\ P & \xrightarrow{i} & C \end{array}$$

So, we are asking, what are local system $\mathcal{G}$ on P. $P = \operatorname{spec}(k(P))$

Version 2: With etale coverings.

what is a local system on X, what is a local system on $P \to C$, what is a local system on $\operatorname{spec} \mathcal{O}_P \to C$

locally constant sheaves are given by: [Stacks, Tag 09Y8]

EDIT AN REFINE THIS ENTIRE THING The goal is to 1. define what ramificaiton of local system means in different settings 2. explore basic properties and ways to calculate.

Definition 39. Let G be a finite abelian group and let $d \geq 0$ be a rational number. A G-torsor over $\operatorname{Spec}(L)$ (in $\operatorname{Spec}(k)_{\text{Ét}}$), corresponding to a continuous homomorphism $\rho : G_L \to G$, is said to have ramification bounded by d if $\rho\left(G_L^d\right) = \{1\}$. A G-torsor over $\operatorname{Spec}(L)$ with ramification bounded by 0 (resp. 1) is said to be unramified (resp. tamely ramified).

Explain why a G torsor over $\operatorname{Spec}(L)$ **in** $\operatorname{Spec}(k)_{\textbf{Ét}}$ correspond to a continuous homomorphism $\rho : G_L \to G$

Explanation: Let L^{sep} be a separable closure of L, and let G_L be the Galois group of L^{sep} over L then

1. the small étale topos of $\operatorname{Spec}(L)$ is isomorphic to the topos of sets with continuous left G_L-action.
2. By 2.13, the category of G-torsors over $\operatorname{Spec}(L)$ in $\operatorname{Spec}(k)_{Et}$ is isomorphic to the category of G-torsors in the small étale topos $\operatorname{Spec}(L)_{et}$.
3. Correspondingly, for each finite abelian group G, the group of isomorphism classes of the category $Tors(\operatorname{Spec}(L), G)$ is isomorphic to the group of continuous homomorphisms from G_L to G.

We want to show 3 explicitly.

What is an isomorphism class of "G-sets with continuous left G_L action" before, we understand **What is an isomorphism class of "sets with continuous left G_L action" is a continuous G action**

We want to show How, from a G-Torsor P in the $\operatorname{Spec}(L)_{et}$ we get a set with ccontinious homomorphism from G_L

Example-Explain!!!

Remark 3.10. If $P \to \operatorname{Spec}(L)$ is a G-torsor in $\operatorname{Spec}(k)_{\text{Ét}}$, then we have a finite decomposition

$$P = \coprod_{i \in I} \operatorname{Spec}(L_i)$$

where each L_i is a finite separable extension of L, and are pairwise isomorphic. The G-torsor P has ramification bounded by d if and only if for each i (or, equivalently, for some i) the extension L_i/L has ramification bounded by d, in the sense G_L^d acts trivially on the finite set $\mathrm{Hom}_L(L_i, L^{\mathrm{sep}})$.

Definition 40. let X be good enough (irreducible, regular at codim=1? smooth?), U open subscheme and Z the complemenet which is irreducible closed subscheme of *codim* $= 1$. let η be the generic point of Z. and let $\mathcal{F}$ be a local system on U as in:

$$\begin{tikzcd} & & & \mathcal{F} \arrow[d] \\ \eta \arrow[r] & Z \arrow[r] & X & U \arrow[l, hook'] \end{tikzcd} \tag{3}$$

Example-Explain!!!

Then we say $\mathcal{F}$ is unramified on Z if and only if $\mathcal{F}|_\eta$ is unramified.

Note that

Theorem 41. *let $X, U, Z, \mathcal{F}$ be as in definition 40. If $\mathcal{F}$ is unramified on Z, then $\mathcal{F}$ is pulledback from a local system on X*

Next we have a general proposition.

Proposition 42. *Let X be a normal Noetherian scheme, and let $Z \subset X$ an irreducible closed subscheme of codimension 1 (a prime divisor) Let $U = X \setminus Z$ be the open complement, and η the generic point of Z. Let $f : Y \to X$ be a finite surjective morphism, where Y is also a normal Noetherian scheme. Suppose there exists an irreducible component $Z' \subset f^{-1}(Z)$ with generic point η' such that f is unramified at η'.*

Then:

Proof. For clarity, we draw the diagram:

$$\begin{tikzcd} Z \times_X Y \arrow[r] \arrow[d] & Y \arrow[d, "f"] & U \times_X Y \arrow[l] \arrow[d] \\ Z \arrow[r, "closed"] & X & U \arrow[l, "open"'] \end{tikzcd} \tag{1}$$

We have that $\eta' \to \eta$ is unramified. hence $\mathcal{F}$ unramified on η if and only if $f^*\mathcal{F}$ is unramified on η (from proposition in the beginning)

∎

2.7 Symmetric Powers of Local Systems on Curves

don't need modulus in this section, etc. Let k be a perfect field. Let C be a projective smooth geometrically connected curve over k, with genus g. Let $\mathfrak{m}$ be a modulus on C and let $U = C \setminus \mathfrak{m}$. Let G be a finite Abelian group and let $\mathcal{P}$ be a G-torsor on U with ramification bounded by $\mathfrak{m}$. Let $d \geq \deg m$.

We have the following diagram:

$$\begin{tikzcd} U^{(d_1)} \times_k U^{(d_2)} \arrow[r, "p_1"] \arrow[d, "p_2"] & U^{(d_1)} \\ U^{(d_2)} & \end{tikzcd}$$

pullbacking $\mathcal{P}^{(d_i)}$ along the projections we get a G-torsor

$$\mathcal{P}^{(d_1)} \boxtimes \mathcal{P}^{(d_2)} = p_1^{-1}\mathcal{P}^{(d_1)} \otimes p_2^{-1}\mathcal{P}^{(d_2)}$$

On $U^{(d_1)} \times_k U^{(d_2)}$

Note that the plus map $C^{(d_1)} \times_k C^{(d_2)} \xrightarrow{+} C^{(d_1+d_2)}$ is induced from

$$\begin{tikzcd} C^{d_1} \times_k C^{d_2} \arrow[r, "\cong"] \arrow[d, "r_1 \times r_2"] & C^{d_1+d_2} \arrow[d, "r"] \\ C^{(d_1)} \times_k C^{(d_2)} \arrow[r, "+"] & C^{(d_1+d_2)} \end{tikzcd}$$

Hence, by Proposition 29 (and replacing C with U above) we get canonical identification:

$$(+^{-1})(\mathcal{P}^{(d_1+d_2)}) \cong \mathcal{P}^{(d_1)} \boxtimes \mathcal{P}^{(d_2)}$$

2.8 Generalized Picard Scheme

In this section, we recall the notion of generalized Jacobian varieties and study their fundamental properties. The material presented here is primarily adapted from [Gui19] and [Tak19]. For further background on the general theory of abelian varieties and Jacobians, the reader may also consult [Mil08]. Let S be a scheme and let C be a projective smooth S-scheme whose geometric fibers are connected and of dimension 1. Let $\mathfrak{m}$ be a modulus on C, defined as an effective Cartier divisor of C/S (i.e., a closed subscheme of C which is finite flat of finite presentation over S). We denote the projection $C \times_S T \to T$ by pr for any S-scheme T.

The Functor of Points

Let d be an integer. For an S-scheme T, we consider the set of data $(\mathcal{L}, \psi)$ where:

- $\mathcal{L}$ is an invertible sheaf of degree d on C_T.
- $\psi : \mathcal{O}_{\mathfrak{m}_T} \xrightarrow{\sim} \mathcal{L}|_{\mathfrak{m}_T}$ is a trivialization of $\mathcal{L}$ along the modulus.

Two such pairs $(\mathcal{L}, \psi)$ and $(\mathcal{L}', \psi')$ are said to be isomorphic if there exists an isomorphism of invertible sheaves $f : \mathcal{L} \to \mathcal{L}'$ such that the following diagram commutes:

$$\begin{tikzcd} & \mathcal{O}_{\mathfrak{m}_T} \arrow[dl, "\psi'"'] \arrow[dr, "\psi"] & \\ \mathcal{L}'|_{\mathfrak{m}_T} \arrow[rr, "f|_{\mathfrak{m}_T}"] & & \mathcal{L}|_{\mathfrak{m}_T} \end{tikzcd}$$

We define the presheaf $\mathrm{Pic}^{d,\mathrm{pre}}_{C,\mathfrak{m}}$ on Sch/S by assigning to T the set of isomorphism classes of such pairs. Let $\mathrm{Pic}^d_{C,\mathfrak{m}}$ denote the étale sheafification of this presheaf.

Representability and Structure

The fundamental properties of this functor are as follows:

1. $\mathrm{Pic}^d_{C,\mathfrak{m}}$ is represented by an S-scheme. (Note: If $\mathfrak{m}$ is faithfully flat over S, the presheaf is already a étale sheaf).
2. $\mathrm{Pic}^0_{C,\mathfrak{m}}$ is a smooth commutative group S-scheme with geometrically connected fibers, referred to as the *generalized Jacobian variety* of C with modulus $\mathfrak{m}$.
3. For any d, $\mathrm{Pic}^d_{C,\mathfrak{m}}$ is a $\mathrm{Pic}^0_{C,\mathfrak{m}}$-torsor.

In the case where $\mathfrak{m} = 0$, we recover the standard Jacobian variety, denoted simply as Pic^d_C.

Relation to the Standard Jacobian

We now examine the behavior of the generalized Picard scheme under the variation of the modulus. By viewing the structure along the modulus as an additional rigidification, we obtain natural transition maps corresponding to the inclusion of moduli.

Let $\mathfrak{m}_1$ and $\mathfrak{m}_2$ be moduli such that $\mathfrak{m}_1 \subset \mathfrak{m}_2$. There exists a natural map

$$\mathrm{Pic}^d_{C,\mathfrak{m}_2} \to \mathrm{Pic}^d_{C,\mathfrak{m}_1}$$

obtained by restricting the isomorphism ψ. Since $\mathfrak{m}_2$ is a finite S-scheme, this map is a surjection as a morphism of étale sheaves. In particular, for any modulus $\mathfrak{m}$, there is a natural surjective morphism of étale sheaves:

$$\mathrm{Pic}^d_{C,\mathfrak{m}} \to \mathrm{Pic}^d_C.$$

Local Freeness and Base Change

Let $\mathfrak{m}$ be a modulus which is everywhere strictly positive. Let g denote the genus of C, which is a locally constant function on S. We restrict our attention to degrees d satisfying the condition:

$$d \geq \max\{2g - 1 + \deg \mathfrak{m}, \deg \mathfrak{m}\}. \tag{4}$$

Assuming S is quasi-compact, such a d always exists.

Fix an integer d satisfying the condition above. Let T be an S-scheme and let $\mathcal{L}$ be an invertible sheaf of degree d on C_T. One can show that the pushforwards $\mathrm{pr}_*\mathcal{L}$ and $\mathrm{pr}_*\mathcal{L}(-\mathfrak{m})$ are locally free sheaves and their formations commute with any base change. Explicitly, for any morphism of S-schemes $f : T' \to T$, the base change morphisms are isomorphisms:

$$f^*\mathrm{pr}_*\mathcal{L} \xrightarrow{\sim} \mathrm{pr}_*f^*\mathcal{L}$$

and

$$f^*\mathrm{pr}_*(\mathcal{L}(-\mathfrak{m})) \xrightarrow{\sim} \mathrm{pr}_*f^*(\mathcal{L}(-\mathfrak{m})).$$

In particular, following [Gui19], if $\mathcal{L}$ is invertible $\mathcal{O}_C$-module with degree d satisfying 4 on each fiber of f then, $\mathrm{pr}_*\mathcal{L}$ is a locally free $\mathcal{O}_S$-module of rank $d-g+1$.

For further background and verification of these constructions, we refer the reader to Milne's notes on Abelian Varieties ([Mil08]).

2.9 The Abel-Jacobi Morphism and its Fibers

Let $U = C\backslash\mathfrak{m}$ be the complement of the modulus in C. The effective cartier divisors of degree d which are prime to $\mathfrak{m}$ are parameterized by the symmetric power $\mathrm{Sym}^d_S(U) = U^{(d)}$ over S (See [Gui19] Proposition 4.12, [Mil08] Theorem 3.13). For any such divisor $D \in U^{(d)}$, the associated line bundle $\mathcal{O}_C(D)$ admits a canonical trivialization along $\mathfrak{m}$. Specifically, the canonical section 1_D is regular and non-vanishing on $\mathfrak{m}$ because $\mathrm{supp}(D) \cap \mathrm{supp}(\mathfrak{m}) = \emptyset$. This section restricts to a nowhere-vanishing section on the subscheme $\mathfrak{m}$, thereby determining a trivialization $\psi_D^{-1} : \mathcal{O}_C(D)|_{\mathfrak{m}} \xrightarrow{\sim} \mathcal{O}_{\mathfrak{m}}$. This is done functorially in families, yielding a morphism from the symmetric power to the generalized Picard scheme (over S):

$$\Phi_d : U^{(d)} \to \mathrm{Pic}^d_{C,\mathfrak{m}}, \quad D \mapsto [(\mathcal{O}_C(D), \psi_D)], \tag{5}$$

When $\mathfrak{m} = 0$, $d \geq max\{2g-1, 0\}$ and C admits a section over S, $C^{(d)}$ is a projective space bundle over Pic^d_C, It is proper, surjective with geometrically connected fibers.

Guignard ([Gui19] Theorem 4.14) proves that for $\mathfrak{m} > 0$ and d satisfying (4), the Abel-Jacobi morphism Φ_d is surjective smooth of relative dimension $d - \deg\mathfrak{m} - g + 1$, with geometrically connected fibers.

When $S = \mathrm{spec}(k)$, the geometric-fibers of Φ_d are well understood:

Theorem 43. *Assuming $S = \mathrm{spec}(k)$ and $d \geq max\{2g-1+\deg\mathfrak{m}, \deg\mathfrak{m}\}$. Then, the geometric-fibers of the Abel-Jacobi morphism*

$$\Phi_d : U^{(d)} \to \mathrm{Pic}^d_{C,\mathfrak{m}}$$

over any point are isomorphic to

$$\begin{cases} \mathbb{A}^{d-\deg\mathfrak{m}-g+1}_{k^{sep}} & \text{if } m > 0 \\ \mathbb{P}^{d-g}_{k^{sep}} & \text{if } m = 0 \end{cases}$$

In both cases Φ_d is a fibration in affine spaces or projective spaces, depending on whether $\mathfrak{m}$ is non-zero or zero.

Proof. see [Ten15] Propositions 3.13-3.14, or [Tót11] Prop 2.1.4:

□

2.10 Blowup of Smooth Schemes

complete this section In this seciton we prove some auxliary lemmas and propositions about blowups and local systems.

Proposition 44. *Let X, Y be smooth over k. $x \in X, y \in Y$ closed points. Let $\mathrm{Bl}_x(X), \mathrm{Bl}_y(Y)$, $Bl_{(x,y)}(X \times_k Y)$ be the respective blowups. Let $\eta_X, \eta_Y, \eta_{X\times Y}$ be the generic points of the exceptional* *divisor of the respective blowups. Then, there exists a scheme $\tilde{U}$ and maps f_1, f_2 making the diagram commute:*

$$\begin{tikzcd} & \tilde{U} \arrow[dl, "f_1"'] \arrow[dr, hook, "f_2"] & \\ \mathrm{Bl}_x(X) \times_k \mathrm{Bl}_y(Y) & & \mathrm{Bl}_{(x,y)}(X \times_k Y) \end{tikzcd} \tag{2}$$

such that:

1. *f_2 is open immersion*
2. *f_1 is open map (?open immersion?)*
3. *$\eta_{X\times Y} \in \tilde{U}$*
4. *$\eta_{X\times Y} \overset{f_1}{\mapsto} \eta_X \times \eta_Y$*

Let $P^{[d]} \to U^{(d)}$ be the corresponding G-torsor over $U^{(d)}$.

A conlusion maybe:

Lemma 45. *Let $p : C^{(d)} \times_k C^{(n-d)} \xrightarrow{+} C^{(n)}$ be the plus map, restricting p to $U^{(d)} \times_k U^{(n-d)}$ we get a map*

$$p : U^{(d)} \times_k U^{(n-d)} \xrightarrow{p} U^{(n)}$$

Then,

$$p^*(\mathcal{P}^{(n)}) \cong \mathcal{P}^{[d]} \boxtimes_k \mathcal{P}^{(n-d)}$$

define box product somewhere

Lemma 46. *rephrase this lemma Let $\mathfrak{n}_1, \mathfrak{n}_2 \subset \mathfrak{m}$ be two moduli of the form $\mathfrak{n}_1 = k_1P_1$, $\mathfrak{n}_2 = k_2P_2$ where P_1, P_2 are distinct points. Let $\mathcal{F}_1, \mathcal{F}_2$ be local systems on $U^{(\deg \mathfrak{n}_1)}$, $U^{(\deg \mathfrak{n}_2)}$, at most tamely ramified at $\eta_{\mathfrak{n}_1}$, $\eta_{\mathfrak{n}_2}$ respectively. Then the local system $\mathcal{F}_1 \boxtimes \mathcal{F}_2 := p_1^{-1}(\mathcal{F}_1) \otimes p_2^{-1}(\mathcal{F}_2)$ on is at most tamely ramified at the point $\eta_{\mathfrak{n}_1} \times \eta_{\mathfrak{n}_2}$ of*

2.11 Compactification of Blowup of Symmetric Powers of a Curve

We recall that our objective is to descend the local system $\mathcal{F}^{(d)}$ from $U^{(d)}$ to $\mathrm{Pic}^d_{C,\mathfrak{m}}$ along the Abel-Jacobi map Φ_d:

$$\begin{tikzcd} \mathcal{F}^{(d)} \arrow[d] & \\ U^{(d)} \arrow[r, "\Phi_d"] & \mathrm{Pic}^d_{C,\mathfrak{m}} \end{tikzcd}$$

(Here, the purple arrow emphasizes that the morphism is of sheaves on the étale site).

However, we encounter an obstruction: in the case we are considering ($\mathfrak{m} > 0$), the fibers of Φ_d are affine spaces (of the same degree) rather than the better-behaved projective spaces. This hint that a solution to this problem is to compactify the morphism to yield projective fibers.

This section describes the result of the compactification constructed by [Tak19] via the method of blowup.

Let $\mathfrak{m} = \sum_{i=1}^{n} k_P P$ with $\deg P = d_P$ be a modulus on C, and let d satisfy (4). Takeuchi ([Tak19]) defines $Z_0 = Z_0(\mathfrak{m}, d)$ as the closed subscheme of $C^{(d)}$ defined by the map $C^{(d-\deg \mathfrak{m})} \to C^{(d)}$ adding $\mathfrak{m}$. He also defines $X_{\mathfrak{m},d}$ as the blowup of $C^{(d)}$ along Z_0. Let $E_0 = E_{\mathfrak{m},d} = Z_0(\mathfrak{m}, d) \times_{C^{(d)}} X_{\mathfrak{m},d}$ be the exceptional divisor of the blowup. It is irreducible of codimension 1, and we let $\eta_0 = \eta_{\mathfrak{m},d}$ be its generic point.

Diagrammatically:

$$\begin{tikzcd}
\overline{\{\eta_0\}} = E_0 \arrow[r] \arrow[d] & X_{\mathfrak{m},d} \arrow[d, "\pi"] \\
Z_0 \arrow[r, "c.i"] & C^{(d)}
\end{tikzcd}$$

Incoporating $U^{(d)}$, the local system $\mathcal{F}^{(d)}$ and the Abel-Jacobi map, we have:

$$\begin{tikzcd}
 & & \mathcal{F}^{(d)} \arrow[d] & \\
\overline{\{\eta_0\}} = E_0 \arrow[r] \arrow[d] & X_{\mathfrak{m},d} \arrow[d, "\pi"] & U^{(d)} \arrow[l, hook'] \arrow[r, "\Phi_d"] \arrow[dl] & \mathrm{Pic}^d_{C,\mathfrak{m}} \\
Z_0 \arrow[r, "c.i"] & C^{(d)} & &
\end{tikzcd}$$

In Section 3 of [Tak19] Takeuchi constructs, for large enough d a compactification denoted by $\tilde{C}^{(d)}_{\mathfrak{m}}$ and proves the following: exactly determined the fate of that d

Theorem 47 (Takeuchi)**.** *The scheme $\tilde{C}^{(d)}_{\mathfrak{m}}$ is an open subscheme of $X_{\mathfrak{m},d}$ containing $U^{(d)}$. The morphism $\Phi_d : U^{(d)} \to \mathrm{Pic}^d_{C,\mathfrak{m}}$ extends to a morphism $\tilde{\Phi}_d : \tilde{C}^{(d)}_{\mathfrak{m}} \to \mathrm{Pic}^d_{C,\mathfrak{m}}$ which makes $\tilde{C}^{(d)}_{\mathfrak{m}}$ a projective space bundle over $\mathrm{Pic}^d_{C,\mathfrak{m}}$. Furthermore, the complement of $U^{(d)}$ in $\tilde{C}^{(d)}_{\mathfrak{m}}$ is isomorphic to the fiber product $E_0 \times_{C^{(d)}} \tilde{C}^{(d)}_{\mathfrak{m}}$.*

Proof. Add outline of construction and proofs $\square$

Diagrammatically we have:

$$\begin{tikzcd}
 & & \mathcal{F}^{(d)} \arrow[d] \\
E_0 \times_{C^{(d)}} \tilde{C}^{(d)}_{\mathfrak{m}} \arrow[r] \arrow[d] & \tilde{C}^{(d)}_{\mathfrak{m}} \arrow[d, hook] \arrow[dr, "\tilde{\Phi}_d"] & U^{(d)} \arrow[l, hook'] \arrow[d, "\Phi_d"] \\
\overline{\{\eta_0\}} = E_0 \arrow[r] \arrow[d] & X_{\mathfrak{m},d} \arrow[d, "\pi"] & \mathrm{Pic}^d_{C,\mathfrak{m}} \\
Z_0 \arrow[r, "c.i"] & C^{(d)} &
\end{tikzcd}$$

Also note that $E_0 \times_{C^{(d)}} \tilde{C}^{(d)}_{\mathfrak{m}} = Z_0 \times_{C^{(d)}} \tilde{C}^{(d)}_{\mathfrak{m}}$

3 Ramification of Sheaves after Blowup

do we assume here $S = k$? The main theorem of this section is

Theorem 48. *Let Λ be a finite ring of cardinality invertible in k, and let $\mathcal{F}$ be an étale sheaf of Λ-modules, locally free of rank 1 on U, with ramification bounded by $\mathfrak{m}$. Considering $U^{(d)}$ as an open subscheme of the blowup $\tilde{C}_{\mathfrak{m}}^{(d)}$ of $C^{(d)}$, we have that for sufficiently large integer d, $\mathcal{F}^{(d)}$ is tamely ramified on $H = \tilde{C}_{\mathfrak{m}}^{(d)} \setminus U^{(d)} = E_0 \times_{C^{(d)}} \tilde{C}_{\mathfrak{m}}^{(d)}$.*

Following the notation of Section 2.11, For any modulus $\mathfrak{n} \subset \mathfrak{m}$, we define $Z_{\mathfrak{n}}$ as the closed subscheme of $C^{(\deg \mathfrak{n})}$ defined by $\mathfrak{n}$ as a point of $C^{(\deg \mathfrak{n})}$.

We then define $X_{\mathfrak{n}}$ as the blowup of $C^{(\deg \mathfrak{n})}$ at $Z_{\mathfrak{n}}$, and we denote by $E_{\mathfrak{n}} = Z_{\mathfrak{n}} \times_{C^{(d)}} X_{\mathfrak{n}}$ the exceptional divisor of this blowup, it is irreducible of codimension 1. We denote by $\eta_{\mathfrak{n}}$ the generic point of $E_{\mathfrak{n}}$. Diagrammatically:

$$\begin{tikzcd}
\overline{\{\eta_{\mathfrak{n}}\}} = E_{\mathfrak{n}} \arrow[r, hook] \arrow[d] & X_{\mathfrak{n}} \arrow[d, "\pi_{\mathfrak{n}}"] \\
Z_{\mathfrak{n}} \arrow[r, hook] & C^{(\deg \mathfrak{n})}
\end{tikzcd} \tag{6}$$

Theorem 48 easily follows from:

Theorem 49. *Let $\mathcal{F}$ be a local system on U with ramification at P bounded by $\mathfrak{n} = k_P P \subset \mathfrak{m}$. Then $\mathcal{F}^{(\deg \mathfrak{n})}$ is tamely ramified at $\eta_{\mathfrak{n}}$ of $E_{\mathfrak{n}}$*

In the upcoming section, we perform the reduction and derive Theorem 48 from Theorem 49. We then prove Theorem 49 in the section that follows.

3.1 Reduction Lemmas

The following lemma is adapted from [Tak19] (Lemma 4.1)

Lemma 50. *Let C be a projective, smooth, and geometrically connected curve over a perfect field k. Let $\mathfrak{m} = \sum_{i=1}^{r} k_i P_i$ be an effective divisor where $P_1, \ldots, P_r$ are distinct closed points. Let $U = C \setminus \mathfrak{m}$ and let $d \geq \deg \mathfrak{m}$.*

Suppose $\mathfrak{n}_1, \ldots, \mathfrak{n}_l$ are pairwise coprime submoduli of $\mathfrak{m}$ such that $\mathfrak{m} = \sum_{j=1}^{l} \mathfrak{n}_j$. Consider the summation morphism:

$$\pi : C^{(\deg \mathfrak{n}_1)} \times_k \cdots \times_k C^{(\deg \mathfrak{n}_l)} \times_k C^{(d - \deg \mathfrak{m})} \longrightarrow C^{(d)}$$

defined by $(D_1, \ldots, D_l, D_{extra}) \mapsto \sum_{j=1}^{l} D_j + D_{extra}$.

Then π is étale at the generic point of the closed subvariety

$$V = \{\mathfrak{n}_1\} \times_k \cdots \times_k \{\mathfrak{n}_l\} \times_k C^{(d - \deg \mathfrak{m})}$$

inside the domain $C^{(\deg \mathfrak{n}_1)} \times_k \cdots \times_k C^{(\deg \mathfrak{n}_l)} \times_k C^{(d - \deg \mathfrak{m})}$.

Proof. We may assume that k is algebraically closed (hence $\deg P_i = 1$ for all i). By miracle flatness π is flat, it is quasi-finite and projective as a map between projecive spaces. so we condlude π is finite and flat. It is enough to show that there exists a closed point Q of $\mathfrak{n}_1 + \ldots \mathfrak{n}_l + C^{(d - \deg \mathfrak{m})} \subset C^{(d)}$ over which there are $\deg \pi$ points on $C^{(\mathfrak{n}_1)} \times_k \cdots \times_k C^{(\mathfrak{n}_l)} \times_k C^{(d-\deg \mathfrak{m})}$. (Because it will be unramified at this point and thus also at the generic point of V.) Choose Q as a point corresponding to a divisor $\mathfrak{n}_1 + \cdots + \mathfrak{n}_l + P_{r+1} + \cdots + P_{r+d-\deg \mathfrak{m}}$, where $P_1, \ldots, P_{r+d-\deg \mathfrak{m}}$ are distinct points of $U(k)$. ☐

Corollary 51. *The morphism $C^{(\deg \mathfrak{m})} \times_k C^{(d-\deg \mathfrak{m})} \xrightarrow{\pi} C^{(d)}$ is finite flat everywhere, and étale at the generic point of the closed subvariety $Z_{\mathfrak{m}} \times_k C^{(d-\deg \mathfrak{m})} \subset C^{(\deg \mathfrak{m})} \times_k C^{(d-\deg \mathfrak{m})}$.*

Following from this, we look at the following diagram, coming from the flat base change $C^{(d-\deg \mathfrak{m})} \to \mathrm{Spec}(k)$ (Proposition 28) of (6): (Is this even smooth?)

$$\begin{tikzcd}
E_{\mathfrak{m}} \times_k C^{(d-\deg \mathfrak{m})} \arrow[r] \arrow[d] & X_{\mathfrak{m}} \times_k C^{(d-\deg \mathfrak{m})} \arrow[d] & \mathcal{F}^{(\deg \mathfrak{m})} \times_k C^{(d-\deg \mathfrak{m})} \arrow[d] \\
Z_{\mathfrak{m}} \times_k C^{(d-\deg \mathfrak{m})} \arrow[r] & C^{(\deg \mathfrak{m})} \times_k C^{(d-\deg \mathfrak{m})} & U^{(\deg \mathfrak{m})} \times_k C^{(d-\deg \mathfrak{m})} \arrow[ul]
\end{tikzcd}$$

Note that $U^{(\deg \mathfrak{m})} \times_k C^{(d-\deg \mathfrak{m})}$ is dense open subscheme of $X_{\mathfrak{m}} \times_k C^{(d-\deg \mathfrak{m})}$, And $E_{\mathfrak{m}} \times_k C^{(d-\deg \mathfrak{m})}$ is a prime divisor of $X_{\mathfrak{m}} \times_k C^{(d-\deg \mathfrak{m})}$. Hence it is well defined question according to Definition 33 to ask whether $\mathcal{F}^{(\deg \mathfrak{m})} \times_k C^{(d-\deg \mathfrak{m})}$ is tamely ramified at the generic point θ of $E_{\mathfrak{m}} \times_k C^{(d-\deg \mathfrak{m})}$.

Lemma 52. *If $\mathcal{F}^{(\deg \mathfrak{m})}$ is tamely ramified at $\eta_{\mathfrak{m}}$, then $\mathcal{F}^{(\deg \mathfrak{m})} \times_k C^{(d-\deg \mathfrak{m})}$ is tamely ramified at the generic point θ of $E_{\mathfrak{m}} \times_k C^{(d-\deg \mathfrak{m})}$.*

Proof. This follows from [Stacks, Tag 0EYD] make adjusments to definition and lemma, to only require that some prime divisors are with desired properties. ☐

Replacing $C^{(d-\deg \mathfrak{m})}$ with the dense open subscheme $U^{(d-\deg \mathfrak{m})} \subset C^{(d-\deg \mathfrak{m})}$, we get that the G-torsor $p_1^{-1}\mathcal{P}^{(\deg \mathfrak{m})}$ ($\mathcal{P}$ correspodns to $\mathcal{F}$ under Proposition 21) is tamely ramified at θ the generic point of $E_{\mathfrak{m}} \times_k U^{(d-\deg \mathfrak{m})} \subset U^{(\deg \mathfrak{m})} \times_k U^{(d-\deg \mathfrak{m})}$, where $p_1 : U^{(\deg \mathfrak{m})} \times_k U^{(d-\deg \mathfrak{m})} \to U^{(\deg \mathfrak{m})}$ is the projection to the first factor.

Looking at the second projection $p_2 : X_{\mathfrak{m}} \times_k U^{(d-\deg \mathfrak{m})} \to U^{(d-\deg \mathfrak{m})}$, and the fact that $\mathcal{P}^{(d-\deg \mathfrak{m})}$ is étale on $U^{(d-\deg \mathfrak{m})}$ we get that $p_2^{-1}\mathcal{P}^{(d-\deg \mathfrak{m})} = X_{\mathfrak{m}} \times_k \mathcal{P}^{(d-\deg \mathfrak{m})}$ is étale on $X_{\mathfrak{m}} \times_k U^{(d-\deg \mathfrak{m})}$. Hence, its restriction to $U^{(\deg \mathfrak{m})} \times_k U^{(d-\deg \mathfrak{m})}$ is unramified at θ.

Thus, by the following lemma, we conclude that $\mathcal{P}^{(\deg \mathfrak{m})} \boxtimes \mathcal{P}^{(d-\deg \mathfrak{m})} = p_1^{-1}\mathcal{P}^{(\deg \mathfrak{m})} \wedge^G p_2^{-1}\mathcal{P}^{(d-\deg \mathfrak{m})}$ is tamely ramified at θ the generic point of $E_{\mathfrak{m}} \times_k U^{(d-\deg \mathfrak{m})}$.

Lemma 53. *Let $X \to \operatorname{spec} k$ be a scheme over a field k, and let $\mathcal{P}_1, \mathcal{P}_2$ be two G-torsors on U_{et}. Let ξ be the generic point of a prime divisor $D \subset X$. If $\mathcal{P}_1$ is tamely ramified at ξ, and $\mathcal{P}_2$ is unramified at ξ, then the contracted product $\mathcal{P}_1 \wedge^G \mathcal{P}_2$ is tamely ramified at ξ.*

Proof. This follows from Lemma 36. ☐

Combining this with Corollary 51 we get

Corollary 54. *Let $\mathfrak{m}$ be a modulus as above, and let $\eta_{\mathfrak{m}}$ be the generic point of $E_{\mathfrak{m}}$. Let $\mathcal{P}$ be a G-torsor on U_{et} with ramification bounded by $\mathfrak{m}$. Assume $\mathcal{P}^{(\deg \mathfrak{m})}$ is tamely ramified at $\eta_{\mathfrak{m}}$. Then $\mathcal{P}^{(\deg \mathfrak{m})} \boxtimes \mathcal{P}^{(d-\deg \mathfrak{m})}$ is tamely ramified at the generic point θ of $E_{\mathfrak{m}} \times_k C^{(d-\deg \mathfrak{m})} \subset C^{(\deg \mathfrak{m})} \times_k C^{(d-\deg \mathfrak{m})}$, and $\mathcal{P}^{(d)}$ is tamely ramified at the generic point $\eta_0 = \eta_{\mathfrak{m},d}$ of $E_0 = E_{\mathfrak{m},d}$*

Proof. The first assertion, that $\mathcal{P}^{(\deg \mathfrak{m})} \boxtimes \mathcal{P}^{(d-\deg \mathfrak{m})}$ is tamely ramified at θ, follows from the preceding discussion. Thus, it remains to show that $\mathcal{P}^{(d)}$ is tamely ramified at η_0.

Consider the blowup diagram defining $X_{\mathfrak{m},d}$:

$$\begin{tikzcd}
\overline{\{\eta_0\}} = E_0 \arrow[r] \arrow[d] & X_{\mathfrak{m},d} \arrow[d, "\pi"] \\
Z_0 \arrow[r, "c.i"] & C^{(d)}
\end{tikzcd}$$

By performing a base change along the flat addition map $+ : C^{(\deg \mathfrak{m})} \times_k U^{(d-\deg \mathfrak{m})} \to C^{(d)}$, we obtain the following commutative diagram:

$$\begin{tikzcd}
 & \left(C^{(\deg \mathfrak{m})} \times_k U^{(d-\deg \mathfrak{m})}\right) \times_{C^{(d)}} E_0 \arrow[r] \arrow[d] & \overline{\{\eta_0\}} = E_0 \arrow[d] \\
 & \left(C^{(\deg \mathfrak{m})} \times_k U^{(d-\deg \mathfrak{m})}\right) \times_{C^{(d)}} X_{\mathfrak{m},d} \arrow[r] \arrow[d] & X_{\mathfrak{m},d} \arrow[d, "\pi"] \\
\mathfrak{m} \times_k U^{(d-\deg \mathfrak{m})} \arrow[r, "c.i"] & C^{(\deg \mathfrak{m})} \times_k U^{(d-\deg \mathfrak{m})} \arrow[r, "+"] & C^{(d)}
\end{tikzcd}$$

Since blowups commute with flat base change, and the inverse image of the center Z_0 under the map $+$ is $\mathfrak{m} \times_k U^{(d-\deg \mathfrak{m})}$, the scheme $\left(C^{(\deg \mathfrak{m})} \times_k U^{(d-\deg \mathfrak{m})}\right) \times_{C^{(d)}} X_{\mathfrak{m},d}$ is the blowup of $C^{(\deg \mathfrak{m})} \times_k U^{(d-\deg \mathfrak{m})}$ along $\mathfrak{m} \times_k U^{(d-\deg \mathfrak{m})}$. This, in turn, is isomorphic to the base change of the blowup $X_{\mathfrak{m}}$ (of $C^{(\deg \mathfrak{m})}$ along $\mathfrak{m}$) via the (flat) projection $C^{(\deg \mathfrak{m})} \times_k U^{(d-\deg \mathfrak{m})} \to C^{(\deg \mathfrak{m})}$.

Assembling these facts, we obtain the following Cartesian square:

$$\begin{tikzcd}
 & E_{\mathfrak{m}} \times U^{(d-\deg \mathfrak{m})} \arrow[r, "\tilde{+}"] \arrow[d] & \overline{\{\eta_0\}} = E_0 \arrow[d] \\
 & X_{\mathfrak{m}} \times_k U^{(d-\deg \mathfrak{m})} \arrow[r, "\tilde{+}"] \arrow[d] & X_{\mathfrak{m},d} \arrow[d, "\pi"] \\
\mathfrak{m} \times_k U^{(d-\deg \mathfrak{m})} \arrow[r, "c.i"] & C^{(\deg \mathfrak{m})} \times_k U^{(d-\deg \mathfrak{m})} \arrow[r, "+"] & C^{(d)}
\end{tikzcd}$$

The point θ defined in the Corollary is the generic point of $E_{\mathfrak{m}} \times_k U^{(d-\deg \mathfrak{m})}$. Let η be the generic point of $\mathfrak{m} \times_k U^{(d-\deg \mathfrak{m})}$. By Corollary 51, the map $+$ is étale at η. Consequently, the lifted map $\tilde{+}$ is étale at θ. Given the isomorphism $(+^{-1})(\mathcal{P}^{(d)}) \cong \mathcal{P}^{(\deg \mathfrak{m})} \boxtimes \mathcal{P}^{(d-\deg \mathfrak{m})}$ from Section 2.7, the tame ramification of the box product at θ descends to the tame ramification of $\mathcal{P}^{(d)}$ at η_0 by applying Lemma 37.

$\square$

Lemma 55. *Let $\mathfrak{n}_1, \mathfrak{n}_2 \subset \mathfrak{m}$ be two moduli of the form $\mathfrak{n}_1 = k_1 P_1$, $\mathfrak{n}_2 = k_2 P_2$ where P_1, P_2 are distinct points. Assume $\mathcal{F}^{(\deg \mathfrak{n}_1)}$, $\mathcal{F}^{(\deg \mathfrak{n}_2)}$ are at most tamely ramified at $\eta_{\mathfrak{n}_1}$, $\eta_{\mathfrak{n}_2}$ respectively. Then $\mathcal{F}^{(\deg \mathfrak{n}_1 + \deg \mathfrak{n}_2)}$ is at most tamely tamified at $\eta_{\mathfrak{n}_1+\mathfrak{n}_2}$.*

Proof. Complete □

Lemma 56. *Let $\mathfrak{n}, \mathfrak{n}' \subset \mathfrak{m}$ be coprime sub moduli of $\mathfrak{m}$ where $\mathfrak{n}' = k_P P$. Assume $\mathcal{F}^{(\deg \mathfrak{n})}$, $\mathcal{F}^{(\deg \mathfrak{n}')}$ are at most tamely ramified at $\eta_{\mathfrak{n}}$, $\eta_{\mathfrak{n}'}$ respectively. Then $\mathcal{F}^{(\deg \mathfrak{n} + \deg \mathfrak{n}')}$ is at most tamely tamified at $\eta_{\mathfrak{n}+\mathfrak{n}'}$.*

Proof. Complete □

3.2 Proof of Theorem 48

Proof of Theorem 48. Let $\mathcal{F}$ be as in Theorem 48, $\mathfrak{m} = \sum_{i=1}^{n} k_P P$ with $\deg P = d_P$ Then by Theorem 49 for every $\mathfrak{n} \subset \mathfrak{m}$ of the form $\mathfrak{n} = k_P P$, $\mathcal{F}^{(\deg \mathfrak{n})}$ is at most tamely ramified at $\eta_{\mathfrak{n}}$. By Lemma 56, $\mathcal{F}^{(\deg \mathfrak{m})}$ is then at most tamely ramified at $\eta_{\mathfrak{m}}$. And thus by lemma Lemma 50 (maybe do another step here directly, in addition to the lemma?) $\mathcal{F}^{(d)}$ is tamely ramified at the generic point of H. □

3.3 Proof of Theorem 49

In this section we prove Theorem 49 complete - this is not finished

We will work this out along an example: Let $X = \mathbb{G}_m = \operatorname{spec} R[t, t^{-1}]$ and Let $\mathcal{P} = \mathbb{G}_m \xrightarrow{(\cdot)^n} G_m$ be the n'th power map. It is a $G = \mathbb{Z}/n\mathbb{Z}$ torsor. The ring map is $R[t, t^{-1}] \xrightarrow{t \mapsto t^n} R[t, t^{-1}]$ which corresponds to ring extension: $R[t, t^{-1}] \to R[t^{\frac{1}{n}}, t^{-\frac{1}{n}}]$. The field of fractions of $\mathbb{G}_m$ is $K(t)$ and the corresponding map between fields of $\mathcal{P} \to X$ is $K(t) \xrightarrow{t \to t^n} K(t)$. which corresponds to the field extension: $K(t) \hookrightarrow K(t^{\frac{1}{n}}) = K(t)[X]/(X^n - t)$. Where $K = FracR$

The points $0, \infty \in \mathbb{P}^1$ correspond to the local rings $\mathcal{O}_0 = K[t]_{(t)}$ and $\mathcal{O}_\infty = K[\frac{1}{t}]_{(\frac{1}{t})}$ of $K(t)$, which are DVRs. The corresponding valuations of $K(t)$ are given by:

$$v_0\left(\frac{f}{g}\right) = \text{maximal exponent } n \text{ s.t. } t^n \mid \frac{f}{g}, \quad v_\infty\left(\frac{f}{g}\right) = \deg g - \deg f$$

So the diagram of the G-torsor $\mathcal{P} = \mathbb{G}_m$ over $\mathbb{G}_m$ is:

$$\begin{tikzcd} & \mathcal{P} = \mathbb{G}_m \arrow[d, "(\cdot)^n"] \\ \mathbb{P}^1 & \mathbb{G}_m \arrow[l] \end{tikzcd} \tag{7}$$

The bounded ramification condition is given by:

$$\text{ram } P_\eta \leq k \tag{8}$$

We wish to understand

In the general case of a G torsor $P \to C$ we have similarly: $K(C) \cong K(t)$ the function ring of C for some variable t and a finite field extension $K/\mathbb{F}_p(t')$. If the basefield K contains n'th roots of unity, then the torsor is the same... and continue here: $\mathcal{O}_{P_1}$ the same.. and continue.

4 Proof of Theorem 2

We work over $S = \operatorname{spec} k$, for k perfect.

By Proposition 21, its equivalent to prove:

Theorem 3. *Let $G = \Lambda^\times$ be a finite abelian group (Λ as before), and let $\mathcal{P}$ be a G-torsor on U, with ramification bounded by $\mathfrak{m}$. Then, for sufficiently large integer d, there exists a unique (up to isomorphism) G-torsor $\mathcal{Q}_d$ on $\operatorname{Pic}^d_{C,\mathfrak{m}}$, such that the pullback of $\mathcal{Q}_d$ by Φ_d is isomorphic to $\mathcal{P}^{(d)}$.*

Proof. We divide the proof into two cases, when $\mathfrak{m} = 0$ and when $\mathfrak{m} > 0$.

Case 1: $\mathfrak{m} = 0$. By Section 2.9, for d large enough, the Abel-Jacobi morphism $\Phi_d : C^{(d)} \to \operatorname{Pic}^d_C$ is proper surjective and smooth, with geometrically connected fibers, each isomorphic to $\mathbb{P}^{d-g}_{k^{sep}}$, Hence by Corollary 31 it induces an exact sequence of etale fundamental groups:

$$\pi_1^{et}(\mathbb{P}^{d-g}_{k^{sep}}) \to \pi_1^{et}(C^{(d)}) \to \pi_1^{et}(\operatorname{Pic}^d_C) \to 1$$

But $\mathbb{P}^{d-g}_{k^{sep}}$ is simply connected ([Ten15] Example 4.9, [Tót11] Example 1.4.12), hence its etale fundamental group is trivial, and we get an isomorphism of etale fundamental groups:

$$\pi_1^{et}(C^{(d)}) \cong \pi_1^{et}(\operatorname{Pic}^d_C)$$

Implying the theorem in this case.

Case 2: $\mathfrak{m} > 0$. In this case by Theorem 47, for d large enough, the Abel-Jacobi morphism $\Phi_d : U^{(d)} \to \operatorname{Pic}^d_{C,\mathfrak{m}}$ extends to a proper surjective and smooth map, with geometrically connected fibers isomorphic to projective spaces,

$$\tilde{\Phi}_d : \tilde{C}^{(d)}_{\mathfrak{m}} \to \operatorname{Pic}^d_{C,\mathfrak{m}}$$

Hence we get an isomorphism of etale fundamental groups:

$$\pi_1^{et}(\tilde{C}^{(d)}_{\mathfrak{m}}) \cong \pi_1^{et}(\operatorname{Pic}^d_{C,\mathfrak{m}})$$

By Theorem 48, $\mathcal{F}^{(d)}$ is tamely ramified on the boundary divisor $H = \tilde{C}^{(d)}_{\mathfrak{m}} \setminus U^{(d)}$.

Thus, by Lemma 57 below, we have: $\mathcal{F}^{(d)}$ extends to a locally constant sheaf $\tilde{\mathcal{F}}^{(d)}$ on $\tilde{C}^{(d)}_{\mathfrak{m}}$, which by the isomorphism of etale fundamental groups above, corresponds to a unique locally constant sheaf $\mathcal{G}_d$ on $\operatorname{Pic}^d_{C,\mathfrak{m}}$, such that $\tilde{\Phi}_d^* \mathcal{G}_d \cong \tilde{\mathcal{F}}^{(d)}$. Restricting back to $U^{(d)}$, we get $\Phi_d^* \mathcal{G}_d \cong \mathcal{F}^{(d)}$, as required. $\square$

Lemma 57. *If $\mathcal{F}^{(d)}$ is a locally constant sheaf on $U^{(d)}$ which is tamely ramified along the boundary divisor $H = \tilde{C}^{(d)}_{\mathfrak{m}} \setminus U^{(d)}$, then $\mathcal{F}^{(d)}$ extends to a locally constant sheaf $\tilde{\mathcal{F}}^{(d)}$ on $\tilde{C}^{(d)}_{\mathfrak{m}}$.*

Proof. The lemma we are referencing above can be proved in two routes:
Route 1 - Showing $\ker\{\pi_1^{ab}(U^{(d)}) \to \pi_1^{ab}(\operatorname{Pic}^d_{C,\mathfrak{m}})\}$ is pro-p group.
Route 2 - Showing

$$\pi_1^{t,ab}(U^{(d)}) \to \pi_1^{t,ab}(\operatorname{Pic}^d_{C,\mathfrak{m}})$$

is isomorphism to its image. here one needs to be precise.

$\square$

References

[SGA1] Alexander Grothendieck. *Revêtements étales et groupe fondamental (SGA 1)*. Vol. 224. Lecture notes in mathematics. Springer-Verlag, 1971.

[Ser79] Jean-Pierre Serre. *Local Fields*. Springer, 1979.

[Mil08] James S. Milne. *Abelian Varieties (v2.00)*. Available at www.jmilne.org/math/. 2008.

[KS10] Moritz Kerz and Alexander Schmidt. "On different notions of tameness in arithmetic geometry". In: *Math. Ann.* 346.3 (2010), pp. 641–668. URL: https://dx.doi.org/10.1007/s00208-009-0409-6.

[Tót11] Péter Tóth. "Geometric Abelian Class Field Theory". In: (May 2011). Master of Science thesis. URL: https://math.bu.edu/people/rmagner/Seminar/GCFTthesis.pdf.

[Ten15] Avichai Tendler. *Geometric Class Field Theory*. 2015. arXiv: 1507.00104 [math.AG]. URL: https://arxiv.org/abs/1507.00104.

[Stacks] The Stacks Project Authors. *Stacks Project*. https://stacks.math.columbia.edu. 2018.

[Gui19] Quentin Guignard. "On the ramified class field theory of relative curves". In: *Algebra & Number Theory* 13 (May 2019). Revised version submitted on 29 May 2019, pp. 1299–1326. DOI: 10.2140/ant.2019.13.1299. arXiv: 1804.02243 [math.AG].

[Tak19] Daichi Takeuchi. "Blow-ups and the class field theory for curves". In: *Algebraic Number Theory* 13.6 (2019), pp. 1327–1351. DOI: 10.2140/ant.2019.13.1327. URL: https://doi.org/10.2140/ant.2019.13.1327.