Geometric Class Field Theory

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Contents

1 Introduction

Throughout this work we will be working over fields with characteristic $p \neq 0$ unless otherwise stated.

2 Preliminaries

In this section we understand some prelimenaries. Unless otherwise stated, we work within an arbitrary topos $\mathcal{E}$. The following proposition establishes the fundamental dictionary between the geometric theory of principal homogeneous spaces and the algebraic theory of invertible modules. This equivalence allows us to transport the monoidal structure from the category of modules (the tensor product) to the category of torsors (the contracted product), strictly within the categorical framework.

Proposition 1. *Let $\mathcal{E}$ be a topos and let Λ be a ring object in $\mathcal{E}$. Let $G = \Lambda^\times$ denote the internal group object of units of Λ.*

There is a canonical equivalence of monoidal categories between the category of G-torsors in $\mathcal{E}$ and the category of locally free Λ-modules of rank 1 in $\mathcal{E}$:

$$\Phi : \mathbf{Tors}(\mathcal{E}, \Lambda^\times) \xrightarrow{\sim} \mathbf{Pic}(\mathcal{E}, \Lambda)$$

The equivalence is defined by the associated module functor:

$$P \longmapsto P \times^{\Lambda^\times} \Lambda := \Lambda^\times \backslash (\Lambda \times P)$$

where the quotient is taken with respect to the diagonal action of $\Lambda^\times$ on $\Lambda \times P$. The inverse functor associates to an invertible module L its sheaf of basis frames $\underline{\mathrm{Isom}}_\Lambda(\Lambda, L)$.

In light of this canonical equivalence, we will pass freely between the language of G-torsors and that of locally free Λ-modules throughout the text.

For $P \to U$ a G-torsor. We denote by $P^{[d]} \to U^{(d)}$ the corresponding G-torsor over $U^{(d)}$. Let η^i be the generic point of the exceptional divisor E^i of the blowup of $C^{(d_i n_i)}$ by $n_i P_i$ (Where $degP_i = d_i$). Where $P_i \in C \setminus U$ We want to prove:

Proposition 2. *If $ramP_{\eta^1}^{[d_1 n_1]} \leq k_1$ and $ramP_{\eta^2}^{[d_2 n_2]} \leq k_2$ Then $ramP_{\eta}^{[d_1 n_1 + d_2 n_2]} \leq max(k_1, k_2)$ where η is the generic point of the exceptional divisor E of the blowup of $C^{(d_1 n_1 + d_2 n_2)}$ by $\mathfrak{m} = n_1 P_1 + n_2 P_2$*

We will work this out along an example: Let $X = G_m = \operatorname{spec} R[t, t^{-1}]$ and Let $P = G_m \xrightarrow{(\cdot)^n} G_m$ be the n'th power map. It is a $G = \mathbb{Z}/n\mathbb{Z}$ torsor. The ring map is $R[t, t^{-1}] \xrightarrow{t \mapsto t^n} R[t, t^{-1}]$ which corresponds to ring extension: $R[t, t^{-1}] \to R[t^{\frac{1}{n}}, t^{-\frac{1}{n}}]$. The field of fractions of G_m is $K(t)$ and the corresponding map between fields of $P \to X$ is $K(t) \xrightarrow{t \to t^n} K(t)$. which corresponds to the field extension: $K(t) \hookrightarrow K(t^{\frac{1}{n}}) = K(t)[X]/(X^n - t)$. Where $K = FracR$

The points $0, \infty \in \mathbb{P}^1$ corresponds to the local rings: $\mathcal{O}_0 = K[t]_{(t)}$ and $\mathcal{O}_\infty = K[\frac{1}{t}]_{(\frac{1}{t})}$ of $K(t)$ Which are DVRs. The corresponding valuations of $K(t)$ are given by: $v_0(\frac{f}{g}) =$ maximal exponent n s.t. $t^n \mid \frac{f}{g}$ and $v_\infty(\frac{f}{g}) = \deg g - \deg f$

In the general case of a G torsor $P \to C$ we have similarly: $K(C) \cong K(t)$ the function ring of C for some variable t and a finite field extension $K/\mathbb{F}_p(t')$. If the basefield K contains n'th roots of unity, then the torsor is the same... and continue here: $\mathcal{O}_{P_1}$ the same.. and continue.

This follows by the following: Start talking about local rings, and take everything locally and prove it locally.

Proposition 3. *Let*

3 Geometric Class Field Theory

Next, we state the main theorem of geometric class field theory. And see how it relates to the classical class field theory. This is copied from Daichi's and some adapated from Giroud's thesis: Let k be a perfect field, and let C be a projective smooth geometrically connected curve over k. The geometric class field theory gives a geometric description of abelian coverings of C by using generalized jacobian varieties. Let us recall its precise statement. Fix a modulus $\mathfrak{m}$, i.e. an effective Cartier divisor of C and let U be its complement in C.

The pairs $(\mathcal{L}, \alpha)$, where $\mathcal{L}$ is an invertible $\mathcal{O}_C$-module and α is a rigidification of $\mathcal{L}$ along $\mathfrak{m}$, are parametrized by a k-group scheme $\mathrm{Pic}_{C,\mathfrak{m}}$, called the rigidified Picard scheme. The Abel-Jacobi morphism

$$\Phi : U \to \mathrm{Pic}_{C,\mathfrak{m}}$$

is the morphism which sends a section x of U to the pair $(\mathcal{O}(x), 1)$. We prove the following version of geometric class field theory:

Theorem 4 (Geometric Class Field Theory)**.** *Let Λ be a finite ring of cardinality invertible in k, and let $\mathcal{F}$ be an étale sheaf of Λ-modules, locally free of rank 1 on U, with ramification bounded by $\mathfrak{m}$. Then, there exists a unique (up to isomorphism) multiplicative étale sheaf of Λ-modules $\mathcal{G}$ on $\mathrm{Pic}_{C,\mathfrak{m}}$, locally free of rank 1, such that the pullback of $\mathcal{G}$ by Φ is isomorphic to $\mathcal{F}$.*

Let d be a positive integer. We denote by $U^{(d)}$ the d-th symmetric power of U over k. For an étale sheaf $\mathcal{F}$ on U, we denote by $\mathcal{F}^{(d)}$ the d-th symmetric power of $\mathcal{F}$ on $U^{(d)}$. We have a natural morphism Check if this morphism only work for large enough d?

$$\Phi_d : U^{(d)} \to \mathrm{Pic}^d_{C,\mathfrak{m}}$$

is this actually the map? which sends a section $x_1 + \cdots + x_d$ of $U^{(d)}$ to the pair $(\mathcal{O}(x_1 + \cdots + x_d), 1)$.

Theorem 4 can be reduced to the following statement:

Theorem 5. *Let Λ be a finite ring of cardinality invertible in k, and let $\mathcal{F}$ be an étale sheaf of Λ-modules, locally free of rank 1 on U, with ramification bounded by $\mathfrak{m}$. Then, for sufficiently large integer d, there exists a unique (up to isomorphism) multiplicative étale sheaf of Λ-modules $\mathcal{G}_d$ on $\mathrm{Pic}^d_{C,\mathfrak{m}}$, locally free of rank 1, such that the pullback of $\mathcal{G}_d$ by Φ_d is isomorphic to $\mathcal{F}^{(d)}$.*

1. A word about the decompositon of the Picard scheme into connected components indexed by degree.
2. Definition of the abel jacobi-map, and its properties, its fibers, see how amichai does it.
3. Definition of multiplicative sheaves.
4. Say something about the ramification condition.
5. Put references inside the theorem like Giroud does

Proof of geometric CFT

4 Proof of theorem 5

Let $\mathfrak{m}$ be an effective Cartier divisor on C, and let d be a positive integer satisfying $d \geq max\{2g - 1 + \deg \mathfrak{m}, \deg \mathfrak{m}\}$ where g is the genus of C. We denote by $C^{(d)}$ the d-th symmetric power of C over k. By add reference, the fibers of the map for d large enough? under what conditions?

$$\Phi_d : U^{(d)} \to \mathrm{Pic}^d_{C,\mathfrak{m}}$$

over any point are isomorphic to

$$\begin{cases} \mathbb{A}_k^{d-\deg \mathfrak{m}-g+1} & \text{if } m > 0 \\ \mathbb{P}_k^{d-g} & \text{if } m = 0 \end{cases}$$

One can show this makes Φ_d into a fibration in affine spaces or projective spaces, depending on whether $\mathfrak{m}$ is non-zero or zero.

Using a fundamental theorem about etale fundamental groups:

Theorem 6 (Homotopy Exact Sequence [Stacks, Tag 0C0J])**.** *Let $f : X \to S$ be a flat proper morphism of finite presentation whose geometric fibres are connected and reduced. Assume S is connected and let $\overline{s}$ be a geometric point of S. Then there is an exact sequence*

$$\pi_1(X_{\overline{s}}) \to \pi_1(X) \to \pi_1(S) \to 1$$

of fundamental groups.

We get, in the case when $\mathfrak{m} = 0$ (Hence Φ_d is smooth and proper) that there is an isomorphism of fundamental groups:

$$\pi_1^{et}(C^{(d)}) \cong \pi_1^{et}(\mathrm{Pic}^d_C)$$

Implying theorem 5 explain exactly how, becuase here it is fundamental groups and there it is etale sheafves of Λ-modules. Also, we need it to be **multiplicative**!

In the case when $\mathfrak{m} > 0$, Φ_d is not proper anymore, so we cannot apply the homotopy exact sequence directly. In [Tak19] Takeuchi constructs a compactifiaction of $U^{(d)}$ by adding a boundary hyperplane, H, over which $\mathcal{F}^{(d)}$ is tamely ramified. This compactification is denoted by $\tilde{C}^{(d)}_{\mathfrak{m}}$, it has $U^{(d)}$ as an open subscheme with complement H.

This is a theorem we are going to prove in a different way:

Theorem 7. *Let Λ be a finite ring of cardinality invertible in k, and let $\mathcal{F}$ be an étale sheaf of Λ-modules, locally free of rank 1 on U, with ramification bounded by $\mathfrak{m}$. Then, for sufficiently large integer d, $\mathcal{F}^{(d)}$ is tamely ramified on H.*

Note that the above terminology of tamely ramified on H is defined exactly when H is the complement of $U^{(d)}$, so I want to say something about it, or make it clear, or somehow define it someplace else and refer to it

More over, this compactification, denoted by $\tilde{C}^{(d)}_{\mathfrak{m}}$ is fibered over $\mathrm{Pic}^d_{C,\mathfrak{m}}$ with fibers isomorphic to projective spaces. Thus we get

$$\pi_1(\tilde{C}^{(d)}_{\mathfrak{m}}) \cong \pi_1(\mathrm{Pic}^d_{C,\mathfrak{m}})$$

From here, one can go in two routes, **Route 1** - Showing $\ker\{\pi_1^{ab}(U^{(d)}) \to \pi_1^{ab}(\mathrm{Pic}^d_{C,\mathfrak{m}})\}$ is pro-p group.

Route 2 - Showing

$$\pi_1^{t,ab}(U^{(d)}) \to \pi_1^{t,ab}(\mathrm{Pic}^d_{C,\mathfrak{m}})$$

is isomorphism to its image. here one needs to be precise.

In both cases, one would then get theorem 5.

We start by defintions and basic proposition of everything we need.

5 Generalized Picard Scheme, Abel Jacobi Map, And The Blowup

5.1 Generalized Picard Scheme

We copy from [Gui19] [Tak19]. Let S be a scheme, C be a projective smooth S-scheme whose geometric fibers are connected and of dimension 1. Let $\mathfrak{m}$ be an effective Cartier divisor of C/S, i.e. a closed subscheme of C which is finite flat of finite presentation over S. We also call $\mathfrak{m}$ a modulus. Let us denote, for S-schemes T, the projections $C \times_S T \to T$ by the same symbol pr. In this section, we recall and study the notion of generalized jacobian varieties. Let d be an integer and $\mathfrak{m}$ be a modulus. Let T be an S-scheme. Consider a datum $(\mathcal{L}, \psi)$ such that

- $\mathcal{L}$ is an invertible sheaf of deg $= d$ on C_T.
- ψ is an isomorphism $\mathcal{O}_{m_T} \to \mathcal{L}|_{m_T}$.

We say that two such data $(\mathcal{L}, \psi)$ and $(\mathcal{L}', \psi')$ are isomorphic if there exists an isomorphism of invertible sheaves $f : \mathcal{L} \to \mathcal{L}'$ making the following diagram commutes

$$\begin{tikzcd} & \mathcal{O}_{m_T} \arrow[dl, "\psi'"'] \arrow[dr, "\psi"] & \\ \mathcal{L}'|_{m_T} \arrow[rr, "f|_{m_T}"] & & \mathcal{L}|_{m_T} \end{tikzcd}$$

For an S-scheme T, define a set

$$\mathrm{Pic}^{d,\mathrm{pre}}_{C,\mathfrak{m}}(T) := \{\text{the isomorphism class of } (\mathcal{L}, \psi) \text{ defined as above}\}.$$

$\mathrm{Pic}^{d,\mathrm{pre}}_{C,\mathfrak{m}}$ extends in an obvious way to a presheaf on Sch/S, which we denote by $\mathrm{Pic}^{d,\mathrm{pre}}_{C,\mathfrak{m}}$ also. Define $\mathrm{Pic}^{d}_{C,\mathfrak{m}}$ as the étale sheafification of $\mathrm{Pic}^{d,\mathrm{pre}}_{C,\mathfrak{m}}$. Their fundamental properties which we use without proofs are:

- $\mathrm{Pic}^{d}_{C,\mathfrak{m}}$ are represented by S-schemes. When $\mathfrak{m}$ is faithfully flat over S, $\mathrm{Pic}^{d,\mathrm{pre}}_{C,\mathfrak{m}}$ are already étale sheaves.
- $\mathrm{Pic}^{0}_{C,\mathfrak{m}}$ is a smooth commutative group S-scheme with geometrically connected fibers.
-
- $\mathrm{Pic}^{d}_{C,\mathfrak{m}}$ are $\mathrm{Pic}^{0}_{C,\mathfrak{m}}$-torsors.

$\mathrm{Pic}^0_{C,\mathfrak{m}}$ is called the generalized jacobian variety of C with modulus $\mathfrak{m}$. When $\mathfrak{m} = 0$, this is the jacobian variety of C. In this case, we also denote Pic^d_C for $\mathrm{Pic}^d_{C,\mathfrak{m}}$.

maybe here, instead of saying what he says, just summarize, and refer to it/don't actually include it For a finite flat S-scheme of finite presentation D, define a presheaf $\mathcal{O}_D^\times$ on Sch/S by sending an S-scheme T to the multiplicative group $\Gamma(T, \mathcal{O}^\times_{D\times_S T})$, which is called the Weil restriction of $\mathbb{G}_{m,D}$ to S. This is an étale sheaf, and represented by a smooth group S-scheme. When $D = S$, this is $\mathbb{G}_{m,S}$. Define a map $\mathbb{G}_{m,S} \to \mathcal{O}_D^\times$ from the map of S-schemes $D \to S$. When $\deg D$ is strictly positive everywhere on S, this is an injection of étale sheaves.

Consider a map $\mathcal{O}_\mathfrak{m}^\times \to \mathrm{Pic}^0_{C,\mathfrak{m}}$ sending $s \in \mathcal{O}_\mathfrak{m}^\times$ to the pair $(\mathcal{O}_C, \mathcal{O}_\mathfrak{m} \to \mathcal{O}_\mathfrak{m})$. The image of this map coincides with the kernel of the map $\mathrm{Pic}^0_{C,\mathfrak{m}} \to \mathrm{Pic}^0_C$, and the kernel of the map $\mathcal{O}_\mathfrak{m}^\times \to \mathrm{Pic}^0_{C,\mathfrak{m}}$ is the image of $\mathbb{G}_{m,S} \to \mathcal{O}_\mathfrak{m}^\times$ induced by the morphism of S-schemes $\mathfrak{m} \to S$. In summary, if $\deg \mathfrak{m}$ is everywhere strictly positive, we have a short exact sequence:

$$0 \to \mathcal{O}_\mathfrak{m}^\times/\mathbb{G}_{m,S} \to \mathrm{Pic}^0_{C,\mathfrak{m}} \to \mathrm{Pic}^0_C \to 0.$$

In particular, when $C \to S$ has a section $P : S \to C$, $\mathrm{Pic}^0_{C,P}$ is isomorphic to Pic^0_C. In this case, Pic^d_C has an expression as a sheaf which does not depend on the choice of P. T be an S-scheme, and $\mathcal{L}_1$ and $\mathcal{L}_2$ are invertible sheaves of $\deg = d$ on C_T. Define an equivalence relation on $\mathrm{Pic}^{d,\mathrm{pre}}_C$ such that $\mathcal{L}_1$ and $\mathcal{L}_2$ are equivalent if and only if there exists an invertible sheaf M on T such that $\mathcal{L}_1 \cong \mathcal{L}_2 \otimes \mathrm{pr}^* M$. If $C \to S$ has a section, the quotient presheaf of $\mathrm{Pic}^{d,\mathrm{pre}}_C$ by this equivalence relation is an étale sheaf and coincides with the étale sheafification of $\mathrm{Pic}^{d,\mathrm{pre}}_C$ via the natural surjection. In particular, the identity map $\mathrm{Pic}^d_C \to \mathrm{Pic}^d_C$ corresponds to an equivalence class of invertible sheaves on $C \times_S \mathrm{Pic}^d_C$. In this paper, we call this class the universal class of invertible sheaves of $\deg = d$.

From now on we fix a modulus $\mathfrak{m}$ which is everywhere strictly positive. Then, $\mathrm{Pic}^d_{C,\mathfrak{m}}$ has an explicit expression as a sheaf, as explained before.

Denote the genus of C by g. This is a locally constant function on S. We consider a condition on an integer d as below:

$$d \geq \max\{2g - 1 + \deg \mathfrak{m}, \deg \mathfrak{m}\}. \tag{1}$$

When S is quasi-compact, such a d always exists.

Fix an integer d satisfying the condition above. Let T be an S-scheme and $\mathcal{L}$ be an invertible sheaf of $\deg = d$ on C_T. One can show that $\mathrm{pr}_*\mathcal{L}(-\tilde{\mathfrak{m}})$ and $\mathrm{pr}_*\mathcal{L}$ are locally free sheaves and their formations commute with any base change, i.e. for any morphism of S-schemes $f : T' \to T$, the base change morphisms $f^*\mathrm{pr}_*\mathcal{L} \to \mathrm{pr}_* f^*\mathcal{L}$ and $f^*\mathrm{pr}_*(\mathcal{L}(-\tilde{\mathfrak{m}})) \to \mathrm{pr}_* f^*(\mathcal{L}(-\tilde{\mathfrak{m}}))$ are isomorphisms.

Moreover (from [Gui19]) one can show that if $\mathcal{L}$ is invertible $\mathcal{O}_C$-module with degree d on each fiber of f Then, the $\mathcal{O}_S$-module $pr_*\mathcal{L}$ is locally free of rank $d - g + 1$

1. It may be helpful to consult Milne's "Abelian Varieties" for further background and confidence in these constructions. https://www.jmilne.org/math/CourseNotes/AV.pdf

5.2 Abel-Jacobi Map

We copy from [Tót11] We assume d as in (1)

Definition 8 (Abel-Jacobi Map of degree d)**.**

The *Abel-Jacobi map of degree d*

$$\Phi_d : \mathrm{Div}_C^d \to \mathrm{Pic}_C^d$$

is defined for a scheme T over $\mathrm{Spec}(k)$ and for a relative effective Cartier divisor D of degree d on $(C \times_{\mathrm{Spec}(k)} T)/T$ by

$$\Phi_d(T)(D) := [\mathcal{O}(D)]$$

where $[\mathcal{O}(D)]$ is the class of the invertible sheaf $\mathcal{O}(D)$ on $(C \times_{\mathrm{Spec}(k)} T)/T$. Equivalently if D is represented by the pair $(\mathcal{G}, s)$ (1.2.6), then the Abel-Jacobi map is given by

$$\Phi_d(T)((\mathcal{G}, s)) := [\mathcal{G}].$$

A theorem in milne [Mil08] shows that over a field k, for any $d \geq 1$ we have that the functor Div_C^d is representable by the d-th symmetric power $C^{(d)}$.

5.2.1 Generalized Effective Cartier Divisors

We copy from [Gui19] Let $f : X \to S$ be a smooth morphism of schemes of relative dimension 1, with connected geometric fibers of genus g, which is Zariski-locally projective over S.

and let $i : Y \to X$ be a closed subscheme of X, which is finite locally free over S of degree $N \geq 1$, and let $U = X \setminus Y$ be its complement. A **Y-trivial effective Cartier divisor of degree d on X** is a pair $(\mathcal{L}, \sigma)$ such that $\mathcal{L}$ is a locally free $\mathcal{O}_X$-module of rank 1 and $\sigma : \mathcal{O}_X \to \mathcal{L}$ is an injective homomorphism such that $i^*\sigma$ is an isomorphism and such that the closed subscheme $V(\sigma)$ of X defined by the vanishing of the ideal $\sigma\mathcal{L}^{-1}$ of $\mathcal{O}_X$ is finite locally free of rank d over S. Two Y-trivial effective divisors $(\mathcal{L}, \sigma)$ and $(\mathcal{L}', \sigma')$ are **equivalent** if there is an isomorphism $\beta : \mathcal{L} \to \mathcal{L}'$ of $\mathcal{O}_X$-modules such that $\beta\sigma = \sigma'$. As in 4.7 - This is not "trivial" show/or skip but say somethinig about it, if such an isomorphism exists then it is unique. [Gui19] shows:

Proposition 9 (4.11 In Guignard)**.** *The map $(\mathcal{L}, \sigma) \mapsto (V(\sigma) \hookrightarrow X)$ is a bijection from the set of equivalence classes of Y-trivial effective Cartiers divisor of degree d on X onto the set of closed subschemes of U which are finite locally free of degree d over S.*

Proposition 10 (4.12 In Guignard)**.** *Let d be an integer and let $Div_S^{d,+}(X, Y)$ be the functor which to an S-scheme T associates the set of equivalence classes of Y_T-trivial effective Cartier divisors of degree d on X_T. Then $Div_S^{d,+}(X, Y)$ is representable by the S-scheme $Sym_S^d(U)$, the d-th symmetric power of $U = X \setminus Y$ over S. In particular $Div_S^{d,+}(X, Y)$ is smooth of relative dimension d over S.*

Proposition 11 (4.14 In Guignard)**.** *Let $d \geq N + 2g - 1$ be an integer, and let $Pic_S^d(X, Y)$ be the inverse image of $Pic_S^d(X)$ by the natural morphism $Pic_S(X, Y) \to Pic_S(X)$. Then the Abel-Jacobi morphism*

$$\Phi_d : Div_S^{d,+}(X, Y) \to Pic_S^d(X, Y)$$

$$(\mathcal{L}, \sigma) \mapsto (\mathcal{L}, i^*\sigma)$$

is surjective smooth of relative dimension $d - N - g + 1$ and it has geometrically connected fibers. *N is the degree of Y so* $\deg \mathfrak{m}$ *- replace it*

1. The notation here from giroud of f is as pr in takeuchi, so we need to make it precise in coheret
2. Some places they work over a family of curves, and some places over k. - make this uniform as well.
3. Here gioured use the term Y, we need to formulate it like $\mathfrak{m}$ and N by $N = \deg \mathfrak{m}$
4. Maybe add somewhere the definition of Sym_U like 2.22 in Guingard. ?

5.3 The Blowup

In this section we define the blowup of $C^{(d)}$ by Z_0 and give its basic properties. As outlined in [Tak19] Let $C^{(d)}$ be the d'th symmetric product of C, which parametrizes effective Cartier divisor of $deg = d$ on C. Let Z_0 be the closed subscheme of $C^{(d)}$ defined by the map $C^{(d-\deg \mathfrak{m})} \to C^{(d)}$ adding $\mathfrak{m}$. And let $X_{\mathfrak{m}}$ be the blowup of $C^{(d)}$ along Z_0. In [Tak19][Section 3] Takeuchi proves:

Theorem 12. *There exists a commutative diagram:*

$$\begin{tikzcd}
 & U^{(d)} \arrow[d, dashed, hook] \arrow[dd, dashed, bend right] \arrow[ddd, dashed, bend right=40] \arrow[dr, "\Phi_d"] & \\
Z_0 \times_{C^{(d)}} \tilde{C}^{(d)}_{\mathfrak{m}} \arrow[r] \arrow[d] & \tilde{C}^{(d)}_{\mathfrak{m}} \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\square"] & \mathrm{Pic}^d_{C,\mathfrak{m}} \arrow[d, hook] \\
Z_0 \times_{C^{(d)}} X_{\mathfrak{m}} \arrow[r] \arrow[d] & X_{\mathfrak{m}} \arrow[r] \arrow[d] & P^d_{\mathfrak{m}} \arrow[d] \\
Z_0 \arrow[r] & C^{(d)} & \mathrm{Pic}^d_C
\end{tikzcd} \tag{2}$$

Where

1. *[Tak19][Lemma 3.1.] $P^d_{\mathfrak{m}}$ is an etale sheaf on Sch/S defined by:*

$$P^d_{\mathfrak{m}}(T) = \left\{ (\mathcal{L}, \phi) \;\middle|\; \mathcal{L} \in \mathrm{Pic}^d(C_T),\ \phi : \mathcal{O}_T \hookrightarrow pr_*(\mathcal{L}/\mathcal{L}(-\mathfrak{m})) \text{ s.t. } \mathrm{coker}(\phi) \text{ is loc. free} \right\} \Big/ \cong$$

(Isomorphism between two $(\mathcal{L}, \phi), (\mathcal{L}', \phi')$ is an isomorphism $f : \mathcal{L} \xrightarrow{\cong} \mathcal{L}'$ such that $pr_*(f) \circ \phi = \phi'$) Moreover, it is represented by a proper smooth S-algebraic space. Assuming $C \to S$ has a section (This seems rather imporant, it is also used to show Pic^d_C as explicit expression as a sheaf, how do I refer to it), And letting $\mathcal{L}'$ be a representative invertible sheaf of the universal class, [Tak19] shows that as sheaves on $Sch/\mathrm{Pic}^d_{C,\mathfrak{m}}$, $P^d_{\mathfrak{m}}$ is isomorphic to the projectivization $\mathbb{P}(pr_*(\mathcal{L}'/\mathcal{L}'(-\mathfrak{m})))$.

2. *$\mathrm{Pic}^d_{C,\mathfrak{m}} \to P^d_{\mathfrak{m}}$ is defined by $(\mathcal{L}, \psi) \mapsto (\mathcal{L}, \phi)$. Where ϕ is defined as the composition*

$$\mathcal{O}_T \to pr_*\mathcal{O}_{\mathfrak{m}_T} \xrightarrow{pr_*\psi} pr_*(\mathcal{L}/\mathcal{L}(-\mathfrak{m}))$$

. [Tak19][Lemma 3.5] Shows this map is an open immersion.

3. *The map* $P^d_{\mathfrak{m}} \to \mathrm{Pic}^d_{C,\mathfrak{m}}$ *is by forgetting* ϕ

4. *The definition of* $X_{\mathfrak{m}} \to P_{\mathfrak{m}}$ *is more technically involved, so we refer the reader to [Tak19]. But an imporant feature it is that* $X_{\mathfrak{m}}$ *is a projective space bundle over* $P^d_{\mathfrak{m}}$ *via that map.*

5. $\tilde{C}^{(d)}_{\mathfrak{m}}$ *is an* S*-scheme (Is it obvious that it is an S-scheme, when the base is algebraic space?) defined as the fibered product*
$$\begin{tikzcd} \tilde{C}^{(d)}_{\mathfrak{m}} \arrow[r] \arrow[d, hook] & \mathrm{Pic}^d_{C,\mathfrak{m}} \arrow[d, hook] \\ X_{\mathfrak{m}} \arrow[r] & P^d_{\mathfrak{m}}, \end{tikzcd}$$
Hence, The S*-Scheme* $\tilde{C}^{(d)}_{\mathfrak{m}}$ *is a projective space bundle on* $\mathrm{Pic}^d_{C,\mathfrak{m}}$ *and an open subscheme of* $X_{\mathfrak{m}}$*.*

Where for a scheme X *and a locally free sheaf of finite rank* $\mathcal{F}$ *on* X*. We use a contra-Grothendieck notation for a projective space. Thus the* X*-scheme* $\mathbb{P}(\mathcal{F})$ *parametrizes invertible subsheaves of* $\mathcal{F}$*.*

Now focusing on the left side of (2), [Tak19] shows:

6. $U^{(d)} \to \tilde{C}^{(d)}_{\mathfrak{m}}$ *is an open immersion, and as an open subscheme,* $U^{(d)}$ *is the complement of* $Z_0 \times_{C^{(d)}} \tilde{C}^{(d)}_{\mathfrak{m}}$

.

5.4 Other Completions

add reference or proof $\mathcal{F}^{(d)}$ can be extended to a sheaf on this compactification with tame ramification along the boundary. The compactification, denoted by $\tilde{C}^{(d)}_{\mathfrak{m}}$ is fibered over $\mathrm{Pic}^d_{C,\mathfrak{m}}$ with fibers isomorphic to projective spaces. Hence, show/add ref/add explanation

Here we will show The compactification has

blow-up $X_{\mathfrak{m}}$ of $C^{(d)}$ along a the closed subscheme $Z_0 \subset C^{(d)}$ defined by the map:

1. Here, C is defined over S, a family of curves, usually?

2. Maybe add a section about Symmetric Powers of a curve? probably should.

6 Proof of theorem 7

In this section we prove theorem 7 Where here, we already know that $H = Z_0 \times_{C^{(d)}} \tilde{C}^{(d)}_{\mathfrak{m}}$ The method for the proof is by two steps, first, we are going to prove it for $\mathfrak{m} = dP$, and second we reduce the general case to this.

6.1 Assuming $\mathfrak{m} = dP$

In this subsection we are going to prove theorem 7 in the case $\mathfrak{m} = dP$

6.2 The general case

We do a reduction theorem 7 to the case $\mathfrak{m} = dP$.

Throughout this section Let $C \to S$ be a smooth morphism of schemes of relative dimension 1, with connected geometric fibers of genus g, which is OR

Let C be a projective smooth geometrically connected curve over a perfect field k. Let $\mathfrak{m}$ be a modulus on C and write $\mathfrak{m} = n_1P_1 + \cdots + n_rP_r$, where $P_1, \ldots, P_r$ are distinct closed points of $\mathfrak{m}$. Denote the complement of $\mathfrak{m}$ in C by U. Let $d_i := \deg P_i$. Take a positive integer d so that $d \geq \deg \mathfrak{m}$.

Zariski-locally projective over S.

The reduction is followed by 3 lemmas:

Proposition 13. *The morphism $\pi : C^{(n_1d_1)} \times_k \cdots \times_k C^{(n_rd_r)} \times_k C^{(d-\deg \mathfrak{m})} \to C^{(d)}$, taking the sum, is étale at the generic point of the closed subvariety $\{n_1P_1\} \times \cdots \times \{n_rP_r\} \times C^{(d-\deg \mathfrak{m})}$ of $C^{(n_1d_1)} \times_k \cdots \times_k C^{(n_rd_r)} \times_k C^{(d-\deg \mathfrak{m})}$.*

Proof. We may assume that k is algebraically closed (hence $d_i = 1$ for all i). Since the map $\pi : C^{(n_1)} \times_k \cdots \times_k C^{(n_r)} \times_k C^{(d-\deg \mathfrak{m})} \to C^{(d)}$ is finite flat, it is enough to show that there exists a closed point Q of $n_1P_1 + \ldots n_rP_r + C^{(d-\deg \mathfrak{m})}$ over which there are $\deg \pi$ points on $C^{(n_1)} \times_k \cdots \times_k C^{(n_r)} \times_k C^{(d-\deg \mathfrak{m})}$. Choose Q as a point corresponding to a divisor $n_1P_1 + \ldots n_rP_r + P_{r+1} + \cdots + P_{r+d-\deg \mathfrak{m}}$, where $P_1, \ldots, P_{r+d-\deg \mathfrak{m}}$ are distinct points of $U(k)$. $\square$

Proposition 14. *Denote $\mathfrak{m}_1 = n_1P_1$ and $\mathfrak{m}_2 = n_2P_2 + \ldots + n_rP_r$. Let $X_{\mathfrak{m}_1}, X_{\mathfrak{m}_2}$ be the blowups of $C^{(\deg \mathfrak{m}_1)}, C^{(\deg \mathfrak{m}_2)}$ by $\mathfrak{m}_1$, $\mathfrak{m}_2$ respectively. Let E_1, E_2 be the respective exceptional divisors (which are irreducibele of codim 1), and let η_1, η_2 be their generic points respectively. Assume $\mathcal{F}^{(\deg \mathfrak{m}_1)}$, $\mathcal{F}^{(\deg \mathfrak{m}_2)}$ are tamely ramified (is bounded ramification here good enough?) on η_1, η_2 respectively. Then $\mathcal{F}^{(\deg \mathfrak{m})}$ is tamely ramified (or any bounded ramification?) at η - the generic point of the exceiptional divisor of the blowup $X_\mathfrak{m}$ of $C^{(\deg \mathfrak{m})}$ by $\mathfrak{m}$*

Proposition 15. *In the notations of the previous proposition, suppose $\mathcal{F}^{(\deg \mathfrak{m})}$ is tamely ramified at η. Then $\mathcal{F}^{(\deg \mathfrak{m})} \boxtimes \mathcal{F}^{(n-\deg \mathfrak{m})}$ is tamely ramified at the generic point θ of $E \times_k C^{(n-\deg \mathfrak{m})}$*

Combining the above we get:

Proposition 16. *If for every $1 \leq i \leq r$, $\mathcal{F}^{(d_in_i)}$ is tamely ramified at η_i then $\mathcal{F}^{(n)}$ is tamely ramified at θ Make that precise and everything. notation wise etc...*

1. The first proposition copied from takeuchi, should we explain its proof? give another proof? exclude its proof and refer?
2. Which of the above definition of C are we going to use? (over k or s)
3. In the second proposition, add/explain why are the exceptioanl divisors are irreudcible of codmin 1
4. Say what is E - the exceiptional divisor of the blowup.

6.2.1 Proof of proposition 14

1.

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