

# Geometric Class Field Theory

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## 1 Introduction

In this thesis, we give an elementary proof of a certain important geometric theorem occurring in Deligne's approach to geometric class field theory. One of the main geometric ingredients in the approach, is showing why a local system  $\mathcal{F}$  with ramification bounded by a modulus  $\mathfrak{m}$  on  $U = C \setminus \mathfrak{m}$  descends via the Abel-Jacobi  $\Phi : U \rightarrow \mathrm{Pic}_{C,\mathfrak{m}}$  to  $\mathrm{Pic}_{C,\mathfrak{m}}$ . The approach, innovated by Deligne, relies on analyzing the symmetric powers  $\mathcal{F}^{(d)}$  of  $\mathcal{F}$  on the symmetric powers  $U^{(d)}$  of  $U$ , and showing that for sufficiently large  $d$ ,  $\mathcal{F}^{(d)}$  descends to  $\mathrm{Pic}_{C,\mathfrak{m}}^d$  via the degree  $d$  Abel-Jacobi map  $\Phi_d : U^{(d)} \rightarrow \mathrm{Pic}_{C,\mathfrak{m}}^d$ . The fibers of  $\Phi_d$  (for  $d \geq \deg \mathfrak{m} + 2g - 1$ ) over any point are isomorphic to

$$\begin{cases} \mathbb{A}_k^{d-\deg \mathfrak{m}-g+1} & \text{if } \mathfrak{m} > 0 \\ \mathbb{P}_k^{d-g} & \text{if } \mathfrak{m} = 0 \end{cases}$$

Where  $g$  is the genus of the curve  $C$ . The unramified case ( $\mathfrak{m} = 0$ ) is relatively simple, as the Abel-Jacobi map is proper, surjective with geometrically connected fibers, which follows from the fact that it is a fibration in projective spaces. Thus, by using the homotopy exact sequence for the etale fundamental group,

one gets an isomorphism between the etale fundamental group of  $U^{(d)}$  ( $= C^{(d)}$ ) and that of  $\mathrm{Pic}_{C,\mathfrak{m}}^d$  ( $= \mathrm{Pic}_C^d$ ).

The ramified case ( $\mathfrak{m} > 0$ ) is more subtle, as the Abel-Jacobi map is not proper anymore, and one needs to analyze the ramification of  $\mathcal{F}^{(d)}$  "along the boundary" of  $U^{(d)}$  in  $C^{(d)}$ .

Previous work has generalized Deligne's approach to the ramified case, most notably by Guignard [Gui19] and Takeuchi [Tak19]. Their approaches differ. To descend, Guignard proves that the restriction of  $\mathcal{F}^{(d)}$  to any line in the fiber of the degree  $d$  Abel-Jacobi map is a constant etale sheaf. He achieves this by demonstrating that the restriction is at most tamely ramified and invoking the triviality of the tame fundamental group of  $\mathbb{A}_k^1$ . His analysis relies on local geometric class field theory. It is also worth noting that Guignard's method generalizes to relative curves over arbitrary base schemes. Takeuchi, on the other hand, constructs a compactification of  $U^{(d)}$  by blowing up  $C^{(d)}$  along certain well-chosen centers. This compactification, denoted by  $\tilde{C}_{\mathfrak{m}}^{(d)}$ , has  $U^{(d)}$  as an open subscheme with a codimension 1 closed subscheme  $H$  as complement. He then shows that the Abel-Jacobi map extends to a proper morphism from  $\tilde{C}_{\mathfrak{m}}^{(d)}$  to  $\mathrm{Pic}_{C,\mathfrak{m}}^d$ , which is a fibration in projective spaces. Thus, by the homotopy exact sequence for the etale fundamental group, one gets an isomorphism between the etale fundamental group of  $\tilde{C}_{\mathfrak{m}}^{(d)}$  and that of  $\mathrm{Pic}_{C,\mathfrak{m}}^d$ . To conclude the descent, Takeuchi analyzes the ramification of  $\mathcal{F}^{(d)}$  along the boundary  $H$  of  $\tilde{C}_{\mathfrak{m}}^{(d)}$ , showing that it is tamely ramified there, which suffices. His methods relies on the theory of Witt vectors and refined Swan conductors.

For an account of these approaches, see [Gui19] and [Tak19]. For a full approach following Deligne's method in the unramified case, and the tamely ramified case see [Ten15], and [Tót11].

In this thesis, we combine techniques and ideas from the approaches, and from [Ten15], to give an elementary proof of the ramified case of Deligne's approach to geometric class field theory. We follow Takeuchi's construction of the compactification  $\tilde{C}_{\mathfrak{m}}^{(d)}$  of  $U^{(d)}$  by blowing up  $C^{(d)}$  and calculate the ramification of  $\mathcal{F}^{(d)}$  along the boundary  $H$  of  $\tilde{C}_{\mathfrak{m}}^{(d)}$  directly, avoiding the use of Swan conductors.

In the rest of the introduction, we state the main theorem of geometric class field theory [Theorem 1](#), and its reduction to [Theorem 2](#), which we prove in this thesis.

Let  $k$  be a perfect field, and let  $C$  be a projective smooth geometrically connected curve over  $k$ , with genus  $g$ . Geometric class field theory gives a geometric description of abelian coverings of  $C$  by relating it to isogenies of the generalized picard schemes.

Fix a modulus  $\mathfrak{m}$ , i.e. an effective Cartier divisor of  $C$  and let  $U$  be its complement in  $C$ . The pairs  $(\mathcal{L}, \alpha)$ , where  $\mathcal{L}$  is an invertible  $\mathcal{O}_C$ -module and  $\alpha$  is a rigidification of  $\mathcal{L}$  along  $\mathfrak{m}$ , are parametrized by a  $k$ -group scheme  $\text{Pic}_{C,\mathfrak{m}}$ , called the rigidified Picard scheme. The Abel-Jacobi morphism

$$\Phi : U \rightarrow \text{Pic}_{C,\mathfrak{m}}$$

is the morphism which sends a section  $x$  of  $U$  to the pair  $(\mathcal{O}(x), 1)$ . The fundamental result of geometric class field theory can be formulated as:

**Theorem 1** (Geometric Class Field Theory). *Let  $\Lambda$  be a finite ring of cardinality invertible in  $k$ , and let  $\mathcal{F}$  be an étale sheaf of  $\Lambda$ -modules, locally free of rank 1 on  $U$ , with ramification bounded by  $\mathfrak{m}$ . Then, there exists a unique (up to isomorphism) **multiplicative** étale sheaf of  $\Lambda$ -modules  $\mathcal{G}$  on  $\text{Pic}_{C,\mathfrak{m}}$ , locally free of rank 1, such that the pullback of  $\mathcal{G}$  by  $\Phi$  is isomorphic to  $\mathcal{F}$ .*

The notion of a multiplicative locally free  $\Lambda$ -module of rank 1 is due to [Gui19] and corresponds to isogenies  $G \rightarrow \text{Pic}_{C,\mathfrak{m}}$  with constant kernel  $\Lambda^\times$ . This concept corresponds to multiplicative characters of  $H^1(\text{Pic}_{C,\mathfrak{m}}, \mathbb{Q}/\mathbb{Z})$  in the formulation of [Tak19], and generalizes Hecke eigensheaves in the context of [Ten15].

Let  $d$  be a positive integer. We denote by  $U^{(d)}$  the  $d$ -th symmetric power of  $U$  over  $k$ . For an étale sheaf  $\mathcal{F}$  on  $U$ , we denote by  $\mathcal{F}^{(d)}$  the  $d$ -th symmetric power of  $\mathcal{F}$  on  $U^{(d)}$ . The degree  $d$  Abel-Jacobi morphism is defined as the map

$$\Phi_d : U^{(d)} \rightarrow \text{Pic}_{C,\mathfrak{m}}^d$$

which sends a section  $x_1 + \cdots + x_d$  of  $U^{(d)}$  to the pair  $(\mathcal{O}(x_1 + \cdots + x_d), 1)$ .

The method of descent shows that to prove [Theorem 1](#), it suffices to prove the following reduced version (see the last page of [Gui19], Section 8.3 of [Ten15], or the proof of Theorem 1.2 in [Tak19] for details on this reduction):

**Theorem 2.** *Let  $\Lambda$  be a finite ring of cardinality invertible in  $k$ , and let  $\mathcal{F}$  be an étale sheaf of  $\Lambda$ -modules, locally free of rank 1 on  $U$ , with ramification bounded by  $\mathfrak{m}$ . Then, for sufficiently large integer  $d$ , there exists a unique (up to isomorphism) étale sheaf of  $\Lambda$ -modules  $\mathcal{G}_d$  on  $\text{Pic}_{C,\mathfrak{m}}^d$ , locally free of rank 1, such that the pullback of  $\mathcal{G}_d$  by  $\Phi_d$  is isomorphic to  $\mathcal{F}^{(d)}$ .*

Using the equivalence between  $G$ -torsors and locally free  $\Lambda$ -modules of rank 1 ( $G = \Lambda^\times$ , see [Proposition 4](#)), Theorem [Theorem 2](#) can be reformulated in terms of  $G$ -torsors as follows:

**Theorem 3.** *Let  $G = \Lambda^\times$  be a finite abelian group ( $\Lambda$  as before), and let  $\mathcal{P}$  be a  $G$ -torsor on  $U$ , with ramification bounded by  $\mathfrak{m}$ . Then, for sufficiently large integer  $d$ , there exists a unique (up to isomorphism)  $G$ -torsor  $\mathcal{Q}_d$  on  $\text{Pic}_{C,\mathfrak{m}}^d$ , such that the pullback of  $\mathcal{Q}_d$  by  $\Phi_d$  is isomorphic to  $\mathcal{P}^{(d)}$ .*

To prove Theorem [Theorem 3](#) we follow the work of [Tak19], there he analyzed the ramification

of  $\mathcal{P}^{(d)}$  after blowing up  $C^{(d)}$ , we analyze this ramification using elementary methods, drawing techniques and ideas from the works of [Gui19] and [Tak19], and [Ten15].

### Notation and conventions.

- $S$  is a base scheme.
- $C \rightarrow S$  is a relative curve. i.e. smooth morphism of schemes of relative dimension 1, with connected geometric fibers of genus  $g$ , which is Zariski-locally projective over  $S$ . Note that the genus  $g$  is a locally constant function on  $S$ .
- Most of the time we will assume that  $S = \text{Spec } k$ , where  $k$  is a perfect field.
- A modulus  $\mathfrak{m}$  on  $C \rightarrow S$ , is defined as an effective Cartier divisor of  $C$  over  $S$  (i.e., a closed subscheme of  $C$  which is finite flat of finite presentation (hence locally free) over  $S$ ).

1. Say something about the ramification condition.

## 2 Preliminaries

In this section we recall the necessary work, including work from [Gui19], [Ten15] and [Tak19].

### 2.1 Generalities

#### 2.1.1 Equivalence between Torsors and Invertible Modules

The following proposition establishes the fundamental dictionary between the geometric theory of principal homogeneous spaces and the algebraic theory of invertible modules. This equivalence allows us to transport the monoidal structure from the category of modules (the tensor product) to the category of torsors (the contracted product), strictly within the categorical framework.

**Proposition 4.** *Let  $\mathcal{E}$  be a topos and let  $\Lambda$  be a ring object in  $\mathcal{E}$ . Let  $G = \Lambda^\times$  denote the internal group object of units of  $\Lambda$ .*

*There is a canonical equivalence of monoidal categories between the category of  $G$ -torsors in  $\mathcal{E}$  and the category of locally free  $\Lambda$ -modules of rank 1 in  $\mathcal{E}$ :*

$$\Phi : \mathbf{Tors}(\mathcal{E}, \Lambda^\times) \xrightarrow{\sim} \mathbf{Pic}(\mathcal{E}, \Lambda)$$

*The equivalence is defined by the associated module functor:*

$$P \longmapsto P \times^{\Lambda^\times} \Lambda := \Lambda^\times \backslash (\Lambda \times P)$$

*where the quotient is taken with respect to the diagonal action of  $\Lambda^\times$  on  $\Lambda \times P$ . The inverse functor associates to an invertible module  $L$  its sheaf of basis frames  $\underline{\text{Isom}}_\Lambda(\Lambda, L)$ .*

In light of this canonical equivalence, we will pass freely between the language of  $G$ -torsors and that of locally free  $\Lambda$ -modules throughout the text.

For a topos  $\mathcal{E}$ , a group object  $G$  in  $\mathcal{E}$  and an object  $X$  in  $\mathcal{E}$ , there is a canonical identification between  $(G\mathcal{E})_X$  and  $G(\mathcal{E}_X)$ , given by endowing  $X$  with the trivial  $G$ -action.

We denote by **Tors**( $X, G$ ) the category of  $G$ -torsors over  $X$  in  $G\mathcal{E}_X$ . Similarly, for a ring object  $\Lambda$  in  $\mathcal{E}$ , we denote by **Pic**( $X, \Lambda$ ) the category of locally free  $\Lambda$ -modules of rank 1 over  $X$  in  $\mathcal{E}_X$ . The above equivalence of categories becomes

$$\Phi_X : \mathbf{Tors}(X, \Lambda^\times) \xrightarrow{\sim} \mathbf{Pic}(X, \Lambda)$$

### 2.1.2 Symmetric Powers of Schemes and Torsors

There are several important theorems that repeat themselves throughout Deligne's approach to class field theory, one of them, is that the symmetric power of  $C$  over  $k$ , denoted by  $C^{(d)}$ , is a scheme over  $k$ , Guigard proves this in the most general setting so we quote his result here:

**Proposition 5.** *If  $X$  is Zariski locally quasi-projective over a scheme  $S$ , the right action of  $S_d$  on  $X^{\times sd}$  is admissible, and the quotient  $X^{(d)} = X^{\times sd}/S_d$  exists as a scheme over  $S$ .*

### 2.1.3 Etale Fundamental Groups and Tame Fundamental Groups

## 2.2 Generalized Picard Scheme

In this section, we recall the notion of generalized Jacobian varieties and study their fundamental properties. The material presented here is primarily adapted from [Gui19] and [Tak19]. For further background on the general theory of abelian varieties and Jacobians, the reader may also consult [Mil08]. Let  $S$  be a scheme and let  $C$  be a projective smooth  $S$ -scheme whose geometric fibers are connected and of dimension 1. Let  $\mathfrak{m}$  be a modulus on  $C$ , defined as an effective Cartier divisor of  $C/S$  (i.e., a closed subscheme of  $C$  which is finite flat of finite presentation over  $S$ ). We denote the projection  $C \times_S T \rightarrow T$  by  $\text{pr}$  for any  $S$ -scheme  $T$ .

### The Functor of Points

Let  $d$  be an integer. For an  $S$ -scheme  $T$ , we consider the set of data  $(\mathcal{L}, \psi)$  where:

- $\mathcal{L}$  is an invertible sheaf of degree  $d$  on  $C_T$ .
- $\psi : \mathcal{O}_{\mathfrak{m}_T} \xrightarrow{\sim} \mathcal{L}|_{\mathfrak{m}_T}$  is a trivialization of  $\mathcal{L}$  along the modulus.

Two such pairs  $(\mathcal{L}, \psi)$  and  $(\mathcal{L}', \psi')$  are said to be isomorphic if there exists an isomorphism of invertible sheaves  $f : \mathcal{L} \rightarrow \mathcal{L}'$  such that the following diagram commutes:

$$\begin{array}{ccc} & \mathcal{O}_{\mathfrak{m}_T} & \\ \psi' \swarrow & & \searrow \psi \\ \mathcal{L}'|_{\mathfrak{m}_T} & \xrightarrow{f|_{\mathfrak{m}_T}} & \mathcal{L}|_{\mathfrak{m}_T} \end{array}$$

We define the presheaf  $\text{Pic}_{C,\mathfrak{m}}^{d,\text{pre}}$  on  $\text{Sch}/S$  by assigning to  $T$  the set of isomorphism classes of such pairs. Let  $\text{Pic}_{C,\mathfrak{m}}^d$  denote the étale sheafification of this presheaf.

## Representability and Structure

The fundamental properties of this functor are as follows:

1.  $\text{Pic}_{C,\mathfrak{m}}^d$  is represented by an  $S$ -scheme. (Note: If  $\mathfrak{m}$  is faithfully flat over  $S$ , the presheaf is already an étale sheaf).
2.  $\text{Pic}_{C,\mathfrak{m}}^0$  is a smooth commutative group  $S$ -scheme with geometrically connected fibers, referred to as the *generalized Jacobian variety* of  $C$  with modulus  $\mathfrak{m}$ .
3. For any  $d$ ,  $\text{Pic}_{C,\mathfrak{m}}^d$  is a  $\text{Pic}_{C,\mathfrak{m}}^0$ -torsor.

In the case where  $\mathfrak{m} = 0$ , we recover the standard Jacobian variety, denoted simply as  $\text{Pic}_C^d$ .

## Relation to the Standard Jacobian

We now examine the behavior of the generalized Picard scheme under the variation of the modulus. By viewing the structure along the modulus as an additional rigidification, we obtain natural transition maps corresponding to the inclusion of moduli.

Let  $\mathfrak{m}_1$  and  $\mathfrak{m}_2$  be moduli such that  $\mathfrak{m}_1 \subset \mathfrak{m}_2$ . There exists a natural map

$$\text{Pic}_{C,\mathfrak{m}_2}^d \rightarrow \text{Pic}_{C,\mathfrak{m}_1}^d$$

obtained by restricting the isomorphism  $\psi$ . Since  $\mathfrak{m}_2$  is a finite  $S$ -scheme, this map is a surjection as a morphism of étale sheaves. In particular, for any modulus  $\mathfrak{m}$ , there is a natural surjective morphism of étale sheaves:

$$\text{Pic}_{C,\mathfrak{m}}^d \rightarrow \text{Pic}_C^d.$$

## Local Freeness and Base Change

From this point forward, we fix a modulus  $\mathfrak{m}$  which is everywhere strictly positive. Let  $g$  denote the genus of  $C$ , which is a locally constant function on  $S$ . We restrict our attention to degrees  $d$  satisfying the condition:

$$d \geq \max\{2g - 1 + \deg \mathfrak{m}, \deg \mathfrak{m}\}. \quad (1)$$

Assuming  $S$  is quasi-compact, such a  $d$  always exists.

Fix an integer  $d$  satisfying the condition above. Let  $T$  be an  $S$ -scheme and let  $\mathcal{L}$  be an invertible sheaf of degree  $d$  on  $C_T$ . One can show that the pushforwards  $\text{pr}_*\mathcal{L}$  and  $\text{pr}_*(\mathcal{L}(-\tilde{\mathfrak{m}}))$  are locally free sheaves and their formations commute with any base change. Explicitly, for any morphism of  $S$ -schemes  $f : T' \rightarrow T$ , the base change morphisms are isomorphisms:

$$f^*\text{pr}_*\mathcal{L} \xrightarrow{\sim} \text{pr}_*f^*\mathcal{L}$$

and

$$f^*\text{pr}_*(\mathcal{L}(-\tilde{\mathfrak{m}})) \xrightarrow{\sim} \text{pr}_*f^*(\mathcal{L}(-\tilde{\mathfrak{m}})).$$

In particular, following [Gui19], if  $\mathcal{L}$  is invertible  $\mathcal{O}_C$ -module with degree  $d$  on each fiber of  $f$  then,  $\text{pr}_*\mathcal{L}$  is a locally free  $\mathcal{O}_S$ -module of rank  $d - g + 1$ .

For further background and verification of these constructions, we refer the reader to Milne's notes on Abelian Varieties ([Mil08]).

### 2.3 The Abel-Jacobi Morphism and its Fibers

Let  $U = C \setminus \mathfrak{m}$  be the complement of the modulus in  $C$ . The effective Cartier divisors of degree  $d$  which are prime to  $\mathfrak{m}$  are parameterized by the symmetric power  $\text{Sym}_S^d(U) = U^{(d)}$  over  $S$  (See [Gui19] Proposition 4.12, [Mil08] Theorem 3.13). For any such divisor  $D \in U^{(d)}$ , the associated line bundle  $\mathcal{O}_C(D)$  admits a canonical trivialization along  $\mathfrak{m}$ . Specifically, the canonical section  $1_D$  is regular and non-vanishing on  $\mathfrak{m}$  because  $\text{supp}(D) \cap \text{supp}(\mathfrak{m}) = \emptyset$ . This section restricts to a nowhere-vanishing section on the subscheme  $\mathfrak{m}$ , thereby determining a trivialization  $\psi_D^{-1} : \mathcal{O}_C(D)|_{\mathfrak{m}} \xrightarrow{\sim} \mathcal{O}_{\mathfrak{m}}$ . This is done functorially in families, yielding a morphism from the symmetric power to the generalized Picard scheme (over  $S$ ):

$$\Phi_d : U^{(d)} \rightarrow \text{Pic}_{C,\mathfrak{m}}^d, \quad D \mapsto [(\mathcal{O}_C(D), \psi_D)], \quad (2)$$

When  $\mathfrak{m} = 0$ ,  $d \geq \max\{2g - 1, 0\}$  and  $C$  admits a section over  $S$ ,  $C^{(d)}$  is a projective space bundle over  $\text{Pic}_C^d$ , Guignard ([Gui19] Theorem 4.14) proves that for  $\mathfrak{m} > 0$  and  $d$  satisfying (1), the Abel-Jacobi morphism  $\Phi_d$  is surjective smooth of relative dimension  $d - \deg \mathfrak{m} - g + 1$ , with geometrically connected fibers.

When  $S = \text{spec}(k^{sep})$ , the fibers of  $\Phi_d$  are well understood:

**Theorem 6.** *Assuming  $S = \text{spec}(k^{sep})$ ,  $d \geq \max\{2g - 1 + \deg \mathfrak{m}, \deg \mathfrak{m}\}$ . Then, the fibers of the Abel-Jacobi morphism*

$$\Phi_d : U^{(d)} \rightarrow \text{Pic}_{C,\mathfrak{m}}^d$$

over any point are isomorphic to

$$\begin{cases} \mathbb{A}_k^{d-\deg \mathfrak{m}-g+1} & \text{if } \mathfrak{m} > 0 \\ \mathbb{P}_k^{d-g} & \text{if } \mathfrak{m} = 0 \end{cases}$$

In both cases  $\Phi_d$  is a fibration in affine spaces or projective spaces, depending on whether  $\mathfrak{m}$  is non-zero or zero.

*Proof.* see [Ten15] Propositions 3.13-3.14, or [Tóth11] Prop 2.1.4:

□

### 2.4 Symmetric Powers

### 2.5 Kummer Theory

A Kummer extension is a field extension  $L/K$  where for some given  $n \in \mathbb{N}$  we have:

1.  $K$  contains all  $n$ 'th roots of unity
2.  $L/K$  has abelian Galois group of exponent  $n$ .

A group  $G$  has exponent  $n$  if  $g^n = 1$  for all  $g \in G$ .

**Example 7.** Quadratic extensions are kummer extensions. Multi quadratic Extensions, etc...

**Example 8.** When  $K$  contains  $n$  distinct  $n$ 'th roots of unity (hence  $\text{char}(K) \nmid n$ ) then the extension  $L = K(a^{\frac{1}{n}})$  is a Kummer extension of degree  $m \mid n$ , for any element  $a \in K$ . The galois group  $G$  is cyclic of order  $m$ , and acts as multiplication by root of unity of order  $m$ .

Kummer theory gives us the converse of the above example: Let  $K$  be a field containing  $n$  distinct  $n$ 'th roots of unity, then we have a bijection:

$$\{\text{Kummer extensions } L/K \text{ of exponent dividing } n\} \longleftrightarrow \{\text{Subgroups } H \text{ of } K^\times/(K^\times)^n\}$$

This bijection is given by the maps:

$$\begin{aligned} L &\longmapsto (K^\times \cap (L^\times)^n)/(K^\times)^n \\ K(\sqrt[n]{H}) &\longleftarrow H \end{aligned}$$

Where  $(K^\times)^n$  is the group of  $n$ 'th powers in  $K^\times$ . And,  $K(\sqrt[n]{H}) := \{\sqrt[n]{a} \mid a \in K^\times, a \cdot (K^\times)^n \in H\}$

In the latter case we have:

$$\begin{aligned} H &\cong \text{Hom}_c(\text{Gal}(L/K), \mu_n) \\ a &\longmapsto \left( \sigma \mapsto \frac{\sigma(\alpha)}{\alpha} \right) \quad \text{where } \alpha \text{ is any } n\text{'th root of } a \text{ in } L \end{aligned}$$

Also note that if  $K$  contains all roots of unity then every finite abelian extension of  $K$  is a kummer extension. In this paper we will mainly be interested in kummer extensions of degree  $l^m - 1$  with galois group  $G = (\mathbb{Z}/l^m\mathbb{Z})^\times$  where  $l$  is a prime number and  $\text{char}K \neq l$ .

### 2.5.1 Ramification In Kummer extensions

Next, we turn to a brief discussion of ramification in kummer extensions.

We begin by the following useful lemma proved in [Lan05][9.1] Check if the lang bib entry is correct(i generated it using ai)

**Lemma 9.** *Let  $K$  be a field, and let  $2 \leq n \in \mathbb{N}$  be a natural number. Let  $0 \neq a \in K$  be an element of  $K$ . Assume that for every prime  $p$  dividing  $n$  we have  $a \notin K^p$ , and that if  $4 \mid n$  then  $a \not\in n - 4K^4$ . Then  $X^n - a$  is irreducible in  $K[X]$ .*

+

Using this lemma we can show the following proposition: Something about the proof here doesn't work, complete it

**Proposition 10.** *Let  $n \in \mathbb{N}$  be a natural number and let  $K$  be a field such that  $\gcd(n, \text{char}K) = 1$ . For  $b \in K^\times$ ,  $X^n - b$  is irreducible in  $K[X]$  if and only if  $\text{ord}(\bar{b}) = n$  in  $K^\times/(K^\times)^n$ .*

*Proof.* In one direction, note that  $\text{ord}(\bar{b}) = n$ , if and only if  $b^k \notin (K^\times)^n$  for every  $k \mid n, k < n$ , if and only if,  $b^{p^r} \notin (K^\times)^n$  for every prime  $p \mid n$  (What to do in the case  $n = p$ ?) and every  $r \leq \text{ord}_p(n)$ . Hence we can use the lemma above:

1. For every  $p \mid n$ ,  $b^p \notin K^n$  (otherwise  $b^{\frac{n}{p}} \in (K^\times)^n$ ).
2. If  $4 \mid n$  then  $i \in K$  ( $i^2 = -1$ ), hence  $b \notin -4K^4$  (because  $-4 = (2i)^2$ ).

Hence  $X^n - b$  is irreducible in  $K[X]$ . In the other direction, assume  $X^n - b$  is irreducible in  $K[X]$ . Then if  $p \mid n$  we can not have  $b \in K^p$  (By factoring  $X^n - b$ )

□

From now on we will focus on cyclic Kummer extensions. Those are of the form  $L = K(\sqrt[n]{a})$  where  $a \in K$  and  $n \in \mathbb{N}$  is a natural number. Their Galois group is cyclic of order  $n$ , and acts as multiplication by root of unity of order  $n$  on  $a^{\frac{1}{n}}$ .

$$\text{Gal}(L/K) \cong \mathbb{Z}/n\mathbb{Z} \quad \sigma \mapsto \sigma(a^{\frac{1}{n}}) = \zeta_n \sigma(a^{\frac{1}{n}})$$

1. Switch  $a^{\frac{1}{n}}$  to  $\sqrt[n]{a}$  or  $\alpha$  in the Galois group description

## 2.6 The equivalence between $G$ -torsors and locally free $\Lambda$ -modules

The following proposition establishes the fundamental dictionary between the geometric theory of principal homogeneous spaces and the algebraic theory of invertible modules. This equivalence allows us to transport the monoidal structure from the category of modules (the tensor product) to the category of torsors (the contracted product), strictly within the categorical framework.

**Proposition 11** ( ). Let  $\mathcal{E}$  be a topos and let  $\Lambda$  be a ring object in  $\mathcal{E}$ . Let  $G = \Lambda^\times$  denote the internal group object of units of  $\Lambda$ .

There is a canonical equivalence of monoidal categories between the category of  $G$ -torsors in  $\mathcal{E}$  and the category of locally free  $\Lambda$ -modules of rank 1 in  $\mathcal{E}$ :

$$\Phi : \mathbf{Tors}(\mathcal{E}, \Lambda^\times) \xrightarrow{\sim} \mathbf{Pic}(\mathcal{E}, \Lambda)$$

The equivalence is defined by the associated module functor:

$$P \longmapsto P \times^{\Lambda^\times} \Lambda := \Lambda^\times \backslash (\Lambda \times P)$$

where the quotient is taken with respect to the diagonal action of  $\Lambda^\times$  on  $\Lambda \times P$ . The inverse functor associates to an invertible module  $L$  its sheaf of basis frames  $\underline{\text{Isom}}_\Lambda(\Lambda, L)$ .

In light of this canonical equivalence, we will pass freely between the language of  $G$ -torsors and that of locally free  $\Lambda$ -modules throughout the text.

## 2.7 Ramification after Blowup

For  $P \rightarrow U$  a  $G$ -torsor. We denote by  $P^{[d]} \rightarrow U^{(d)}$  the corresponding  $G$ -torsor over  $U^{(d)}$ . Let  $\eta^i$  be the generic point of the exceptional divisor  $E^i$  of the blowup of  $C^{(d_i n_i)}$  by  $n_i P_i$  (Where  $\deg P_i = d_i$ ). Where  $P_i \in C \setminus U$  We want to prove:

**Proposition 12.** If  $\text{ram}P_{\eta_1}^{[d_1 n_1]} \leq k_1$  and  $\text{ram}P_{\eta_2}^{[d_2 n_2]} \leq k_2$  Then  $\text{ram}P_{\eta}^{[d_1 n_1 + d_2 n_2]} = \max(k_1, k_2)$  where  $\eta$  is the generic point of the exceptional divisor  $E$  of the blowup of  $C^{(d_1 n_1 + d_2 n_2)}$  by  $\mathfrak{m} = n_1 P_1 + n_2 P_2$

We will work this out along an example: Let  $X = \mathbb{G}_m = \text{spec } R[t, t^{-1}]$  and Let  $\mathcal{P} = \mathbb{G}_m \xrightarrow{(\cdot)^n} G_m$  be the  $n$ 'th power map. It is a  $G = \mathbb{Z}/n\mathbb{Z}$  torsor. The ring map is  $R[t, t^{-1}] \xrightarrow{t \mapsto t^n} R[t, t^{-1}]$  which corresponds to ring extension:  $R[t, t^{-1}] \rightarrow R[t^{\frac{1}{n}}, t^{-\frac{1}{n}}]$ . The field of fractions of  $\mathbb{G}_m$  is  $K(t)$  and the corresponding map between fields of  $\mathcal{P} \rightarrow X$  is  $K(t) \xrightarrow{t \mapsto t^n} K(t)$ . which corresponds to the field extension:  $K(t) \hookrightarrow K(t^{\frac{1}{n}}) = K(t)[X]/(X^n - t)$ . Where  $K = \text{Frac } R$

The points  $0, \infty \in \mathbb{P}^1$  correspond to the local rings  $\mathcal{O}_0 = K[t]_{(t)}$  and  $\mathcal{O}_{\infty} = K[\frac{1}{t}]_{(\frac{1}{t})}$  of  $K(t)$ , which are DVRs. The corresponding valuations of  $K(t)$  are given by:

$$v_0\left(\frac{f}{g}\right) = \text{maximal exponent } n \text{ s.t. } t^n \mid \frac{f}{g}, \quad v_{\infty}\left(\frac{f}{g}\right) = \deg g - \deg f$$

So the diagram of the  $G$ -torsor  $\mathcal{P} = \mathbb{G}_m$  over  $\mathbb{G}_m$  is:

$$\begin{array}{ccc} \mathcal{P} = \mathbb{G}_m & & \\ \downarrow (\cdot)^n & & \\ \mathbb{P}^1 & \longleftarrow & \mathbb{G}_m \end{array} \tag{3}$$

The bounded ramification condition is given by:

$$\text{ram } P_{\eta} \leq k \tag{4}$$

We wish to understand

In the general case of a  $G$  torsor  $P \rightarrow C$  we have similarly:  $K(C) \cong K(t)$  the function ring of  $C$  for some variable  $t$  and a finite field extension  $K/\mathbb{F}_p(t')$ . If the basefield  $K$  contains  $n$ 'th roots of unity, then the torsor is the same... and continue here:  $\mathcal{O}_{P_1}$  the same.. and continue.

This follows by the following: Start talking about local rings, and take everything locally and prove it locally.

**Proposition 13.** Let

sss

**Proposition 6 is proved in takeuchi's paper [Tak19].** We show here a simpler proof for a simpler case that we would need.

Let  $\mathcal{P}$  be a  $G$ -torsor over  $U$ . Let  $\mathfrak{m} = n_1 P_1 + n_2 P_2 = \mathfrak{m}_1 + \mathfrak{m}_2$  be a modulus on  $C$ . Let  $X_{\mathfrak{m}_1}, X_{\mathfrak{m}_2}$  be the blowups of  $C^{(\deg \mathfrak{m}_1)}, C^{(\deg \mathfrak{m}_2)}$  by  $\mathfrak{m}_1, \mathfrak{m}_2$  respectively. Let  $E_1, E_2$  be the respective exceptional divisors, and let  $\eta_1, \eta_2$  be their generic points respectively.

**Proposition 14.** Assume  $\mathcal{P}^{(\deg \mathfrak{m}_1)}, \mathcal{P}^{(\deg \mathfrak{m}_2)}$  are tamely ramified on  $\eta_1, \eta_2$  respectively. Then,  $\mathcal{P}^{(\deg \mathfrak{m}_1)} \boxtimes \mathcal{P}^{(\deg \mathfrak{m}_2)}$  is tamely ramified on  $\eta_1 \times \eta_2 \in C^{(\deg \mathfrak{m}_1)} \times C^{(\deg \mathfrak{m}_2)}$

*Proof.* Write proof here □

Next, we prove:

**Proposition 15.** Let  $X, Y$  be smooth over  $k$ .  $x \in X, y \in Y$  closed points. Let  $\text{Bl}_x(X), \text{Bl}_y(Y), \text{Bl}_{(x,y)}(X \times_k Y)$  be the respective blowups. Let  $\eta_X, \eta_Y, \eta_{X \times Y}$  be the generic points of the exceptional divisor of the respective blowups. Then, there exists a scheme  $\tilde{U}$  and maps  $f_1, f_2$  making the diagram commute:

$$\begin{array}{ccc} & \tilde{U} & \\ f_1 \swarrow & & \searrow f_2 \\ \text{Bl}_x(X) \times_k \text{Bl}_y(Y) & & \text{Bl}_{(x,y)}(X \times_k Y) \end{array} \quad (1)$$

such that:

1.  $f_2$  is open immersion
2.  $f_1$  is open map (?open immersion?)
3.  $\eta_{X \times Y} \in \tilde{U}$
4.  $\eta_{X \times Y} \xrightarrow{f_1} \eta_X \times \eta_Y$

Let  $P^{[d]} \rightarrow U^{(d)}$  be the corresponding  $G$ -torsor over  $U^{(d)}$ .

**Lemma 16.** Let  $p : C^{(d)} \times_k C^{(n-d)} \xrightarrow{\pm} C^{(n)}$  be the plus map, restricting  $p$  to  $U^{(d)} \times_k U^{(n-d)}$  we get a map

$$p : U^{(d)} \times_k U^{(n-d)} \xrightarrow{p} U^{(n)}$$

Then,

$$p^*(\mathcal{P}^{(n)}) \cong \mathcal{P}^{[d]} \boxtimes_k \mathcal{P}^{(n-d)}$$

**Proposition 17.** Combinining [Proposition 14](#), [Proposition 15](#) and [Lemma 16](#) we get that if  $P^{[d_1 n_1]}, P^{[d_2 n_2]}$  are tamely ramified on  $\eta_1, \eta_2$  respectively, then  $P^{[d_1 n_1 + d_2 n_2]}$  is tamely ramified on  $\eta$  the generic point of the exceptional divisor of the blowup of  $C^{(d_1 n_1 + d_2 n_2)}$  by  $\mathfrak{m} = n_1 P_1 + n_2 P_2$ .

### A few notes

1. takeuchi's paper [[Tak19](#)] proves a more general version of [Proposition 15](#) for arbitrary ramification, not necessarily smooth over a field. more generally, he proves:

**Proposition 18.** If  $\text{ram}P_{\eta_1}^{[d_1 n_1]} = k_1$  and  $\text{ram}P_{\eta_2}^{[d_2 n_2]} = k_2$  Then  $\text{ram}P_{\eta}^{[d_1 n_1 + d_2 n_2]} = \max(k_1, k_2)$  where  $\eta$  is the generic point of the exceptional divisor  $E$  of the blowup of  $C^{(d_1 n_1 + d_2 n_2)}$  by  $\mathfrak{m} = n_1 P_1 + n_2 P_2$

2. more generally, the above works for any finite number of points  $P_i$  with multiplicities  $n_i$ . the generalizaiton is easy.

ass somewhere the defintiion of box product

## 3 Proof of [Theorem 2](#)

Let  $\mathfrak{m}$  be an effective Cartier divisor on  $C$ , and let  $d$  be a positive integer satisfying  $d \geq \max\{2g - 1 + \deg \mathfrak{m}, \deg \mathfrak{m}\}$  where  $g$  is the genus of  $C$ . We denote by  $C^{(d)}$  the  $d$ -th symmetric power of  $C$  over

$k$ . By add reference, the fibers of the map for  $d$  large enough? under what conditions?

$$\Phi_d : U^{(d)} \rightarrow \mathrm{Pic}_{C,\mathfrak{m}}^d$$

over any point are isomorphic to

$$\begin{cases} \mathbb{A}_k^{d-\deg \mathfrak{m}-g+1} & \text{if } m > 0 \\ \mathbb{P}_k^{d-g} & \text{if } m = 0 \end{cases}$$

One can show this makes  $\Phi_d$  into a fibration in affine spaces or projective spaces, depending on whether  $\mathfrak{m}$  is non-zero or zero.

Using a fundamental theorem about etale fundamental groups:

**Theorem 19** (Homotopy Exact Sequence [Stacks, Tag 0C0J]). *Let  $f : X \rightarrow S$  be a flat proper morphism of finite presentation whose geometric fibres are connected and reduced. Assume  $S$  is connected and let  $\bar{s}$  be a geometric point of  $S$ . Then there is an exact sequence*

$$\pi_1(X_{\bar{s}}) \rightarrow \pi_1(X) \rightarrow \pi_1(S) \rightarrow 1$$

of fundamental groups.

We get, in the case when  $\mathfrak{m} = 0$  (Hence  $\Phi_d$  is smooth and proper) that there is an isomorphism of fundamental groups:

$$\pi_1^{et}(C^{(d)}) \cong \pi_1^{et}(\mathrm{Pic}_C^d)$$

Implying Theorem 2 explain exactly how, because here it is fundamental groups and there it is etale sheaves of  $\Lambda$ -modules. Also, we need it to be multiplicative!

In the case when  $\mathfrak{m} > 0$ ,  $\Phi_d$  is not proper anymore, so we cannot apply the homotopy exact sequence directly. In [Tak19] Takeuchi constructs a compactification of  $U^{(d)}$  by adding a boundary hyperplane,  $H$ , over which  $\mathcal{F}^{(d)}$  is tamely ramified. This compactification is denoted by  $\tilde{C}_{\mathfrak{m}}^{(d)}$ , it has  $U^{(d)}$  as an open subscheme with complement  $H$ .

This is a theorem we are going to prove in a different way:

**Theorem 20.** *Let  $\Lambda$  be a finite ring of cardinality invertible in  $k$ , and let  $\mathcal{F}$  be an étale sheaf of  $\Lambda$ -modules, locally free of rank 1 on  $U$ , with ramification bounded by  $\mathfrak{m}$ . Then, for sufficiently large integer  $d$ ,  $\mathcal{F}^{(d)}$  is tamely ramified on  $H$ .*

Note that the above terminology of tamely ramified on  $H$  is defined exactly when  $H$  is the complement of  $U^{(d)}$ , so I want to say something about it, or make it clear, or somehow define it someplace else and refer to it

More over, this compactification, denoted by  $\tilde{C}_{\mathfrak{m}}^{(d)}$  is fibered over  $\mathrm{Pic}_{C,\mathfrak{m}}^d$  with fibers isomorphic to projective spaces. Thus we get

$$\pi_1(\tilde{C}_{\mathfrak{m}}^{(d)}) \cong \pi_1(\mathrm{Pic}_{C,\mathfrak{m}}^d)$$

From here, one can go in two routes, **Route 1** - Showing  $\ker\{\pi_1^{ab}(U^{(d)}) \rightarrow \pi_1^{ab}(\mathrm{Pic}_{C,\mathfrak{m}}^d)\}$  is pro- $p$  group.

**Route 2** - Showing

$$\pi_1^{t,ab}(U^{(d)}) \rightarrow \pi_1^{t,ab}(\mathrm{Pic}_{C,\mathfrak{m}}^d)$$

is isomorphism to its image. here one needs to be precise.

In both cases, one would then get [Theorem 2](#).

We start by definitions and basic proposition of everything we need.

## 4 Generalized Picard Scheme, Abel Jacobi Map, And The Blowup

### 4.1 Abel-Jacobi Map

We copy from [\[Tót11\]](#) We assume  $d$  as in [\(1\)](#)

**Definition 21** (Abel-Jacobi Map of degree  $d$ ).

The *Abel-Jacobi map of degree  $d$*

$$\Phi_d : \text{Div}_C^d \rightarrow \text{Pic}_C^d$$

is defined for a scheme  $T$  over  $\text{Spec}(k)$  and for a relative effective Cartier divisor  $D$  of degree  $d$  on  $(C \times_{\text{Spec}(k)} T)/T$  by

$$\Phi_d(T)(D) := [\mathcal{O}(D)]$$

where  $[\mathcal{O}(D)]$  is the class of the invertible sheaf  $\mathcal{O}(D)$  on  $(C \times_{\text{Spec}(k)} T)/T$ . Equivalently if  $D$  is represented by the pair  $(\mathcal{G}, s)$  [\(1.2.6\)](#), then the Abel-Jacobi map is given by

$$\Phi_d(T)((\mathcal{G}, s)) := [\mathcal{G}].$$

A theorem in milne [\[Mil08\]](#) shows that over a field  $k$ , for any  $d \geq 1$  we have that the functor  $\text{Div}_C^d$  is representable by the  $d$ -th symmetric power  $C^{(d)}$ .

#### 4.1.1 Generalized Effective Cartier Divisors

We copy from [\[Gui19\]](#) Let  $f : X \rightarrow S$  be a smooth morphism of schemes of relative dimension 1, with connected geometric fibers of genus  $g$ , which is Zariski-locally projective over  $S$ .

and let  $i : Y \rightarrow X$  be a closed subscheme of  $X$ , which is finite locally free over  $S$  of degree  $N \geq 1$ , and let  $U = X \setminus Y$  be its complement. A **Y-trivial effective Cartier divisor of degree  $d$  on  $X$**  is a pair  $(\mathcal{L}, \sigma)$  such that  $\mathcal{L}$  is a locally free  $\mathcal{O}_X$ -module of rank 1 and  $\sigma : \mathcal{O}_X \rightarrow \mathcal{L}$  is an injective homomorphism such that  $i^*\sigma$  is an isomorphism and such that the closed subscheme  $V(\sigma)$  of  $X$  defined by the vanishing of the ideal  $\sigma\mathcal{L}^{-1}$  of  $\mathcal{O}_X$  is finite locally free of rank  $d$  over  $S$ . Two  $Y$ -trivial effective divisors  $(\mathcal{L}, \sigma)$  and  $(\mathcal{L}', \sigma')$  are **equivalent** if there is an isomorphism  $\beta : \mathcal{L} \rightarrow \mathcal{L}'$  of  $\mathcal{O}_X$ -modules such that  $\beta\sigma = \sigma'$ . As in [4.7 - This is not "trivial"](#) show/or skip but say somethinig about it, if such an isomorphism exists then it is unique. [\[Gui19\]](#) shows:

**Proposition 22** (4.11 In Guignard). *The map  $(\mathcal{L}, \sigma) \mapsto (V(\sigma) \hookrightarrow X)$  is a bijection from the set of equivalence classes of  $Y$ -trivial effective Cartiers divisor of degree  $d$  on  $X$  onto the set of closed subschemes of  $U$  which are finite locally free of degree  $d$  over  $S$ .*

**Proposition 23** (4.12 In Guignard). *Let  $d$  be an integer and let  $\text{Div}_S^{d,+}(X, Y)$  be the functor which to an  $S$ -scheme  $T$  associates the set of equivalence classes of  $Y_T$ -trivial effective Cartier divisors of degree  $d$  on  $X_T$ . Then  $\text{Div}_S^{d,+}(X, Y)$  is representable by the  $S$ -scheme  $\text{Sym}_S^d(U)$ , the  $d$ -th symmetric power of  $U = X \setminus Y$  over  $S$ . In particular  $\text{Div}_S^{d,+}(X, Y)$  is smooth of relative dimension  $d$  over  $S$ .*

**Proposition 24** (4.14 In Guignard). Let  $d \geq N + 2g - 1$  be an integer, and let  $\text{Pic}_S^d(X, Y)$  be the inverse image of  $\text{Pic}_S^d(X)$  by the natural morphism  $\text{Pic}_S(X, Y) \rightarrow \text{Pic}_S(X)$ . Then the Abel-Jacobi morphism

$$\begin{aligned}\Phi_d : \text{Div}_S^{d,+}(X, Y) &\rightarrow \text{Pic}_S^d(X, Y) \\ (\mathcal{L}, \sigma) &\mapsto (\mathcal{L}, i^*\sigma)\end{aligned}$$

is surjective smooth of relative dimension  $d - N - g + 1$  and it has geometrically connected fibers.  
N is the degree of Y so deg m - replace it

1. The notation here from giroud of  $f$  is as  $pr$  in takeuchi, so we need to make it precise in coherent
2. Some places they work over a family of curves, and some places over  $k$ . - make this uniform as well.
3. Here gioured use the term  $Y$ , we need to formulate it like  $m$  and  $N$  by  $N = \deg m$
4. Maybe add somewhere the definition of  $\text{Sym}_U$  like 2.22 in Guingard. ?

## 4.2 The Blowup

In this section we define the blowup of  $C^{(d)}$  by  $Z_0$  and give its basic properties. As outlined in [Tak19] Let  $C^{(d)}$  be the  $d$ 'th symmetric product of  $C$ , which parametrizes effective Cartier divisor of  $\deg = d$  on  $C$ . Let  $Z_0$  be the closed subscheme of  $C^{(d)}$  defined by the map  $C^{(d-\deg m)} \rightarrow C^{(d)}$  adding  $m$ . And let  $X_m$  be the blowup of  $C^{(d)}$  along  $Z_0$ . In [Tak19][Section 3] Takeuchi proves:

**Theorem 25.** There exists a commutative diagram:

$$\begin{array}{ccccc} & & U^{(d)} & & \\ & & \downarrow & & \\ Z_0 \times_{C^{(d)}} \tilde{C}_m^{(d)} & \xrightarrow{\quad} & \tilde{C}_m^{(d)} & \xrightarrow{\quad \Phi_d \quad} & \text{Pic}_{C,m}^d \\ \downarrow & & \downarrow & & \downarrow \\ Z_0 \times_{C^{(d)}} X_m & \xrightarrow{\quad} & X_m & \xrightarrow{\quad} & P_m^d \\ \downarrow & & \downarrow & & \downarrow \\ Z_0 & \xrightarrow{\quad} & C^{(d)} & & \text{Pic}_C^d \end{array} \quad (5)$$

Where

1. [Tak19]/Lemma 3.1.]  $P_m^d$  is an etale sheaf on  $\text{Sch}/S$  defined by:

$$P_m^d(T) = \left\{ (\mathcal{L}, \phi) \mid \mathcal{L} \in \text{Pic}^d(C_T), \phi : \mathcal{O}_T \hookrightarrow pr_*(\mathcal{L}/\mathcal{L}(-m)) \text{ s.t. } \text{coker}(\phi) \text{ is loc. free} \right\} \cong$$

(Isomorphism between two  $(\mathcal{L}, \phi), (\mathcal{L}', \phi')$  is an isomorphism  $f : \mathcal{L} \xrightarrow{\cong} \mathcal{L}'$  such that  $pr_*(f) \circ \phi = \phi'$ ) Moreover, it is represented by a proper smooth  $S$ -algebraic space. Assuming  $C \rightarrow S$  has a

section (*This seems rather important, it is also used to show  $\text{Pic}_C^d$  as explicit expression as a sheaf, how do I refer to it*), And letting  $\mathcal{L}'$  be a representative invertible sheaf of the universal class, [Tak19] shows that as sheaves on  $\text{Sch}/\text{Pic}_{C,\mathfrak{m}}^d$ ,  $P_{\mathfrak{m}}^d$  is isomorphic to the projectivization  $\mathbb{P}(\text{pr}_*(\mathcal{L}'/\mathcal{L}'(-\mathfrak{m})))$ .

2.  $\text{Pic}_{C,\mathfrak{m}}^d \rightarrow P_{\mathfrak{m}}^d$  is defined by  $(\mathcal{L}, \psi) \mapsto (\mathcal{L}, \phi)$ . Where  $\phi$  is defined as the composition

$$\mathcal{O}_T \rightarrow \text{pr}_*\mathcal{O}_{\mathfrak{m}_T} \xrightarrow{\text{pr}_*\psi} \text{pr}_*(\mathcal{L}/\mathcal{L}(-\mathfrak{m}))$$

- . [Tak19]/[Lemma 3.5] Shows this map is an open immersion.
3. The map  $P_{\mathfrak{m}}^d \rightarrow \text{Pic}_{C,\mathfrak{m}}^d$  is by forgetting  $\phi$
  4. The definition of  $X_{\mathfrak{m}} \rightarrow P_{\mathfrak{m}}$  is more technically involved, so we refer the reader to [Tak19]. But an important feature it is that  $X_{\mathfrak{m}}$  is a projective space bundle over  $P_{\mathfrak{m}}^d$  via that map.
  5.  $\tilde{C}_{\mathfrak{m}}^{(d)}$  is an  $S$ -scheme (*Is it obvious that it is an  $S$ -scheme, when the base is algebraic space?*)

$$\begin{array}{ccc} \tilde{C}_{\mathfrak{m}}^{(d)} & \longrightarrow & \text{Pic}_{C,\mathfrak{m}}^d \\ \text{defined as the fibered product} \quad \downarrow & & \downarrow \quad \text{Hence, The } S\text{-Scheme } \tilde{C}_{\mathfrak{m}}^{(d)} \text{ is a projective} \\ X_{\mathfrak{m}} & \longrightarrow & P_{\mathfrak{m}}^d, \end{array}$$

space bundle on  $\text{Pic}_{C,\mathfrak{m}}^d$  and an open subscheme of  $X_{\mathfrak{m}}$ .

Where for a scheme  $X$  and a locally free sheaf of finite rank  $\mathcal{F}$  on  $X$ . We use a contra-Grothendieck notation for a projective space. Thus the  $X$ -scheme  $\mathbb{P}(\mathcal{F})$  parametrizes invertible subsheaves of  $\mathcal{F}$ .

Now focusing on the left side of (5), [Tak19] shows:

6.  $U^{(d)} \rightarrow \tilde{C}_{\mathfrak{m}}^{(d)}$  is an open immersion, and as an open subscheme,  $U^{(d)}$  is the complement of  $Z_0 \times_{C^{(d)}} \tilde{C}_{\mathfrak{m}}^{(d)}$

### 4.3 Other Completions

**add reference or proof**  $\mathcal{F}^{(d)}$  can be extended to a sheaf on this compactification with tame ramification along the boundary. The compactification, denoted by  $\tilde{C}_{\mathfrak{m}}^{(d)}$  is fibered over  $\text{Pic}_{C,\mathfrak{m}}^d$  with fibers isomorphic to projective spaces. Hence, **show/add ref/add explanation**

Here we will show The compactification has

blow-up  $X_{\mathfrak{m}}$  of  $C^{(d)}$  along a the closed subscheme  $Z_0 \subset C^{(d)}$  defined by the map:

1. Here,  $C$  is defined over  $S$ , a family of curves, usually?
2. Maybe add a section about Symmetric Powers of a curve? probably should.

## 5 Proof of Theorem 20

In this section we prove [Theorem 20](#). Where here, we already know that  $H = Z_0 \times_{C^{(d)}} \tilde{C}_{\mathfrak{m}}^{(d)}$ . The method for the proof is by two steps, first, we are going to prove it for  $\mathfrak{m} = dP$ , and second we reduce the general case to this.

### 5.1 Assuming $\mathfrak{m} = dP$

In this subsection we are going to prove [Theorem 20](#) in the case  $\mathfrak{m} = dP$ .

### 5.2 The general case

We do a reduction theorem [Theorem 20](#) to the case  $\mathfrak{m} = dP$ .

Throughout this section Let  $C \rightarrow S$  be a smooth morphism of schemes of relative dimension 1, with connected geometric fibers of genus  $g$ , which is [OR](#)

Let  $C$  be a projective smooth geometrically connected curve over a perfect field  $k$ . Let  $\mathfrak{m}$  be a modulus on  $C$  and write  $\mathfrak{m} = n_1 P_1 + \dots + n_r P_r$ , where  $P_1, \dots, P_r$  are distinct closed points of  $\mathfrak{m}$ . Denote the complement of  $\mathfrak{m}$  in  $C$  by  $U$ . Let  $d_i := \deg P_i$ . Take a positive integer  $d$  so that  $d \geq \deg \mathfrak{m}$ .

Zariski-locally projective over  $S$ .

The reduction is followed by 3 lemmas:

**Proposition 26.** *The morphism  $\pi : C^{(n_1 d_1)} \times_k \dots \times_k C^{(n_r d_r)} \times_k C^{(d-\deg \mathfrak{m})} \rightarrow C^{(d)}$ , taking the sum, is étale at the generic point of the closed subvariety  $\{n_1 P_1\} \times \dots \times \{n_r P_r\} \times C^{(d-\deg \mathfrak{m})}$  of  $C^{(n_1 d_1)} \times_k \dots \times_k C^{(n_r d_r)} \times_k C^{(d-\deg \mathfrak{m})}$ .*

*Proof.* We may assume that  $k$  is algebraically closed (hence  $d_i = 1$  for all  $i$ ). Since the map  $\pi : C^{(n_1)} \times_k \dots \times_k C^{(n_r)} \times_k C^{(d-\deg \mathfrak{m})} \rightarrow C^{(d)}$  is finite flat, it is enough to show that there exists a closed point  $Q$  of  $n_1 P_1 + \dots + n_r P_r + C^{(d-\deg \mathfrak{m})}$  over which there are  $\deg \pi$  points on  $C^{(n_1)} \times_k \dots \times_k C^{(n_r)} \times_k C^{(d-\deg \mathfrak{m})}$ . Choose  $Q$  as a point corresponding to a divisor  $n_1 P_1 + \dots + n_r P_r + P_{r+1} + \dots + P_{r+d-\deg \mathfrak{m}}$ , where  $P_1, \dots, P_{r+d-\deg \mathfrak{m}}$  are distinct points of  $U(k)$ .  $\square$

**Proposition 27.** *Denote  $\mathfrak{m}_1 = n_1 P_1$  and  $\mathfrak{m}_2 = n_2 P_2 + \dots + n_r P_r$ . Let  $X_{\mathfrak{m}_1}, X_{\mathfrak{m}_2}$  be the blowups of  $C^{(\deg \mathfrak{m}_1)}, C^{(\deg \mathfrak{m}_2)}$  by  $\mathfrak{m}_1, \mathfrak{m}_2$  respectively. Let  $E_1, E_2$  be the respective exceptional divisors ([which are irreducible of codim 1](#)), and let  $\eta_1, \eta_2$  be their generic points respectively. Assume  $\mathcal{F}^{(\deg \mathfrak{m}_1)}, \mathcal{F}^{(\deg \mathfrak{m}_2)}$  are tamely ramified ([is bounded ramification here good enough?](#)) on  $\eta_1, \eta_2$  respectively. Then  $\mathcal{F}^{(\deg \mathfrak{m})}$  is tamely ramified ([or any bounded ramification?](#)) at  $\eta$  - the generic point of the exceptional divisor of the blowup  $X_{\mathfrak{m}}$  of  $C^{(\deg \mathfrak{m})}$  by  $\mathfrak{m}$*

**Proposition 28.** *In the notations of the previous proposition, suppose  $\mathcal{F}^{(\deg \mathfrak{m})}$  is tamely ramified at  $\eta$ . Then  $\mathcal{F}^{(\deg \mathfrak{m})} \boxtimes \mathcal{F}^{(n-\deg \mathfrak{m})}$  is tamely ramified at the generic point  $\theta$  of  $E \times_k C^{(n-\deg \mathfrak{m})}$*

Combining the above we get:

**Proposition 29.** *If for every  $1 \leq i \leq r$ ,  $\mathcal{F}^{(d_i n_i)}$  is tamely ramified at  $\eta_i$  then  $\mathcal{F}^{(n)}$  is tamely ramified at  $\theta$  [Make that precise and everything. notation wise etc...](#)*

1. The first proposition copied from takeuchi, should we explain its proof? give another proof? exclude its proof and refer?
2. Which of the above definition of  $C$  are we going to use? (over  $k$  or  $s$ )
3. In the second proposition, add/explain why are the exceptional divisors are irreducible of codmin 1
4. Say what is  $E$  - the exceptional divisor of the blowup.

### 5.2.1 Proof of Proposition 27

1. s

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