Geometric Class Field Theory

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Contents

1 Introduction

Throughout this work we will be working over fields with characteristic $p \neq 0$ unless otherwise stated.

2 Class Field Theory In the language of Ideals

In this section we describe the main results of classical class field theory for global fields, following [Mil20]. We copy most of the content here from Milne.

2.1 Ideals, Moduli and Ray Class Groups

Let K be a global field of $\mathrm{char}(K) = p$. A modulus $\mathfrak{m}$ is a formal sum of places of K with non-negative integer coefficients. Let $S(K, \mathfrak{m}) = S(\mathfrak{m}) = \{v \in \mathfrak{m}\}$ be the set of places appearing in $\mathfrak{m}$ with non-zero coefficient.

Define $K_{\mathfrak{m},1} = \{x \in K^\times \mid v(x-1) \geq n_v \text{ for all } v \in S(\mathfrak{m})\}$ where n_v is the coefficient of v in $\mathfrak{m}$.

For every set of primes S we define

$$I_K^S = \{ \text{ fractional ideals of } K \text{ generated by primes not in } S\}$$

There is a natural map $i : K_{\mathfrak{m},1} \to I_K^{S(\mathfrak{m})}$ sending $x \mapsto (x)$

The qoutient

$$C_{\mathfrak{m}} = I_K^{S(\mathfrak{m})}/i(K_{\mathfrak{m},1})$$

is called the **(ray) class group** of K modulo $\mathfrak{m}$.

maybe we don't need this Milne states and prove a known theorem:

Theorem 1. *For every modulus $\mathfrak{m}$ of K there is an exact sequence:*

$$0 \to \mathcal{O}_K^\times/\mathcal{O}_K^\times \cap K_{\mathfrak{m},1} \to K_{\mathfrak{m}}/K_{\mathfrak{m},1} \to C_{\mathfrak{m}} \to C \to 0$$

Where

$$K_{\mathfrak{m}} = \{x \in K^\times \mid v(x) = 0 \textit{ for all } v \in S(\mathfrak{m})\}$$

And C is the usual class group of K.

2.2 The Main Theorems

Theorem 2 (Artin Reciprocity Law)**.** *Let L be a finite abelian extension of a global field K. and let S be the set of primes of K ramifying in L. Then the Artin map* add here reference of the definition to milne *. $\psi : I^S \to Gal(L/K)$ admits a modulus $\mathfrak{m}$ with $S(\mathfrak{m}) = S$ and it defines an isomorphism:*

$$I^S/\Big(i(K_{\mathfrak{m},1}) \cdot N_{L/K}(I_L^{S(\mathfrak{m})})\Big) \to Gal(L/K)$$

A modulus $\mathfrak{m}$ as in the statement of the theorem is called a definning modulus for L. Next, we write $I_K^{\mathfrak{m}}$ for the group of $S(\mathfrak{m})$-ideals in K, and $I_L^{\mathfrak{m}}$ for the group of $S(\mathfrak{m})'$-ideals in L where $S(\mathfrak{m})'$ is the set of primes of L lying above primes in $S(\mathfrak{m})$. Call a subgroup H of $I_K^{\mathfrak{m}}$ a **congruence subgroup** modulo $\mathfrak{m}$ if it contains $i(K_{\mathfrak{m},1})$.

Theorem 3. *[Existence Theorem of Class Field Theory] For every congruence subgroup H modulo $\mathfrak{m}$ there exists a unique finite abelian extension L/K, unramified at all primes not in $S(\mathfrak{m})$, such that the Artin map induces an isomorphism:*

$$I^{S(\mathfrak{m})}/H \to Gal(L/K)$$

.

More of the idealic class field theory in Milne.

Theorems 2 and 3 show that there is a canonical group isomorphism:

$$\varprojlim_{\mathfrak{m}} C_{\mathfrak{m}} \to \mathrm{Gal}(K^{\mathrm{ab}}/K).$$

Rather than studying $\varprojlim_{m} C_m$ directly, it turns out to be more natural to introduce another group that has it as a quotient - this is the idele class group. replace very where idele with ide'le

3 Class Field Theory In the language of Adeles and Ideles

Can we already say we are only considering function fields here? The modern formulation of Global Class Field Theory is given in terms of the adele and idele groups of a global field. In this chapter we will define these objects and state the main theorems of Class Field Theory in this language.

3.1 Adeles and Ideles

Let K be a global field. For each place v of K, we denote:

1. K_v = the completion of K at v
2. $\mathfrak{p}_v$ = the corresponding prime ideal in the ring of integers $\mathcal{O}_K$ of K
3. $\mathcal{O}_v$ = the ring of integers of K_v
4. $\hat{\mathfrak{p}}_v$ = the completion of $\mathfrak{p}_v$ = the maximal ideal of $\mathcal{O}_v$

We define the **adele ring** of K as the restricted direct product

$$\mathbb{A}_K = \prod_v{}' K_v$$

where the restriction is taken with respect to the rings of integers $\mathcal{O}_v$ of K_v for all non-archimedean (IS IT NECESSEARY TO STATE HERE? WE WORK OVER P ANYWAY) places v. In other words, an adele is a tuple $(x_v)_v$ with $x_v \in K_v$ such that $x_v \in \mathcal{O}_v$ for all but finitely many non-archimedean places v. The **idele group** of K is defined as the group of units of the adele ring:

$$\mathbb{I}_K = \mathbb{A}_K^\times = \prod_v{}' K_v^\times$$

where the restriction is taken with respect to the unit groups $\mathcal{O}_v^\times$ of the rings of integers $\mathcal{O}_v$ for all non-archimedean places v. An idele is thus a tuple $(x_v)_v$ with $x_v \in K_v^\times$ such that $x_v \in \mathcal{O}_v^\times$ for all but finitely many non-archimedean places v.

The field K embeds diagonally into $\mathbb{A}_K$, and thus $K^\times$ embeds diagonally into $\mathbb{I}_K$ as the subgroup of principal ideles. The **idele class group** C_K is the quotient:

$$C_K = \mathbb{I}_K/K^\times$$

3.1.1 Topology on Adeles and Ideles

We state quickly the topology on the adele ring and the idele group. More can be found in [Mil20]. Recall that, for all v, K_v is locally compact more over, $\mathcal{O}_v$ is a compact neighborhood of 0. Similarly $K_v^\times$ is locally compact, in fact:

$$1 + \hat{\mathfrak{p}}_v \supset 1 + \hat{\mathfrak{p}}_v^2 \supset 1 + \hat{\mathfrak{p}}_v^3 \dots$$

is a fundamental system of neighborhoods of 1 consisting of compact open subgroups of $K_v^\times$.

For every finite set S of places of K, define:

$$\mathbb{I}_S = \prod_{v \in S} K_v^\times \times \prod_{v \notin S} \mathcal{O}_v^\times$$

with the product topology. $\mathbb{I}_S$ is locally compact and as sets we have:

$$\mathbb{I}_K = \bigcup_S \mathbb{I}_S$$

where the union is taken over all finite sets of places of K. We define a topology on I_K by giving a basis for the open sets $\prod_v V_v$ with $V_v \subseteq K_v^\times$ open for all v and and $V_v = \mathcal{O}_v^\times$ for all but finitely many v. This makes $\mathbb{I}_K$ a locally compact topological group, such that each $\mathbb{I}_S$ is open in $\mathbb{I}_K$, and inherits the product topology. The following sets form a fundamental system of neighborhoods of 1: for each finite set of primes S and $n > 0$, define

$$U_{S,n} = \left\{ (x_v)_v \in \mathbb{I}_K \mid v(x_v - 1) > n \text{ for all } v \in S, x_v \in \mathcal{O}_v^\times \text{ for } v \notin S \right\}$$

Note that the embedding $K^\times \to \mathbb{I}_K$ is discrete and thus the idele class group $C_K = \mathbb{I}_K / K^\times$ is a locally compact topological group as well. Moreover the canonical injective homomorphism

$$K_v^\times \to \mathbb{I}_K \tag{1}$$
$$x \mapsto (1, \dots, 1, x, 1, \dots, 1) \qquad (x \text{ in the } v\text{-th position}) \tag{2}$$

is a topological embedding for each place v of K.

3.2 The Main Theorems

The theory establishes a fundamental connection between the idele class group C_K and the Galois group of the maximal abelian extension of K, denoted K^{ab}.

1. **The Global Reciprocity Map**: There exists a canonical continuous homomorphism, the **Artin map**, from the idele class group to the absolute abelian Galois group:

   $$\theta_K : C_K \to \mathrm{Gal}(K^{ab}/K)$$

   In the case of function fields this map is injective with dense image.

2. **Reciprocity Law**: For any finite abelian extension L/K, the Artin map induces a canonical isomorphism:

   $$C_K / N_{L/K}(C_L) \xrightarrow{\sim} \mathrm{Gal}(L/K)$$

   where $N_{L/K} : C_L \to C_K$ is the norm map on the idele class groups. The subgroup $N_{L/K}(C_L)$ is an open subgroup of finite index in C_K. The norm map on the idele class group is somewhat differet

3. **Existence Theorem**: There is a one-to-one, inclusion-reversing correspondence between the set of finite abelian extensions of K and the set of open subgroups of finite index in the idele class group C_K.

$$\begin{Bmatrix} \text{Finite abelian} \\ \text{extensions } L/K \end{Bmatrix} \longleftrightarrow \begin{Bmatrix} \text{Open subgroups } H \subseteq C_K \\ \text{of finite index} \end{Bmatrix}$$

Under this correspondence, an extension L corresponds to the subgroup $H = N_{L/K}(C_L)$.

1. Above is finite adelic formulation of CFT, state somethign about the infinite extension CFT theorem
2. State something about the topology on the idele class group, Say how the fintie implies the infinte by taking inverse limits.
3. Find sources for the above. for exampel milne? maybe other?
4. The restriction is for the non-archimedean, are you sure?
5. What is the topology of the "continious" homomorphism?

3.3 The connection between the idealic and adelic formulations

In this section we explain briefly how the idealic formulation of class field theory from the previous section is connected to the adelic formulation given here. We explain the natural isomorphism between certain quotients of the idele group and the ideal group of K, which ultimately follows by understanding ideles as thickening of ideals: There is a canonical surjective homomophism id:

$$\begin{aligned} \mathrm{id} : \mathbb{I}_K &\to I_K \\ (x_v)_v &\mapsto \prod_v \mathfrak{p}_v^{v(x_v)} \end{aligned}$$

Thus, composing with $I_K \to C$ gives a surjective homomorphism $\mathbb{I}_K \to C$, noting that $K^\times \to \mathbb{I}_K \to C$ is 0, we realize $C = I_K/i(K^\times)$ as a quotient of $C_K = \mathbb{I}_K/K^\times$.

The same thing is true for $C_\mathfrak{m}$: Let $\mathfrak{m} = \sum n_v \mathfrak{p}_v$ be a modulus of K, set:

$$W_\mathfrak{m}(v) = \begin{cases} \mathcal{O}_v^\times & v \notin \mathrm{Supp}(\mathfrak{m}) \\ 1 + \hat{\mathfrak{p}_v}^{n_v} & v \in \mathrm{Supp}(\mathfrak{m}) \end{cases}$$

And define

$$\mathbb{I}_\mathfrak{m} = \left(\prod_{v \notin \mathrm{Supp}(\mathfrak{m})} K_v^\times \times \prod_{v \in \mathrm{Supp}(\mathfrak{m})} W_\mathfrak{m}(v) \right) \bigcap \mathbb{I}_K$$

And

$$\mathbb{O}_{\mathfrak{m}}^{\times} = \prod_{v} W_{\mathfrak{m}}(v)$$

Note that:

$$K_{\mathfrak{m},1} = K^{\times} \cap \prod_{v \in \mathfrak{m}} W_{\mathfrak{m}}(v) \qquad \text{Intersection inside } \prod_{v \in \mathfrak{m}} K_v^{\times}$$

and that

$$K_{\mathfrak{m},1} = K^{\times} \cap \mathbb{I}_{\mathfrak{m}} \qquad \text{Intersection inside } \mathbb{I}_K$$

Milne shows the following proposition:

Proposition 4. *Let* $\mathfrak{m}$ *be a modulus of* K*.*

1. *The map* $\mathrm{id} : \mathbb{I}_{\mathfrak{m}} \to I_K^{S(\mathfrak{m})}$ *defines an isomorphism*

$$\mathbb{I}_{\mathfrak{m}} / K_{\mathfrak{m},1} \mathbb{O}_{\mathfrak{m}}^{\times} \xrightarrow{\sim} I_K^{S(\mathfrak{m})} / i(K_{\mathfrak{m},1}) = C_{\mathfrak{m}}$$

2. *The inclusion* $\mathbb{I}_{\mathfrak{m}} \hookrightarrow \mathbb{I}_K$ *defines an isomorphism*

$$\mathbb{I}_{\mathfrak{m}} / K_{\mathfrak{m},1} \xrightarrow{\sim} \mathbb{I}_K / K^{\times}$$

maybe add something about characters of idele class group AND/OR about norms of adeles/ideles

Show that acutal connection in this subsubsection.

4 Class Field Theory In the language of Characters

The character formulation of Class Field Theory provides a correspondence between characters of the idele class group and characters of the Galois group of the maximal abelian extension of a global field.

The Main Theorems

Theorem 5 (Character Formulation of Unramified Global Class Field Theory)**.**

1. For each character $\xi : K^{\times} \backslash \mathbb{A}_K^{\times} / \mathbb{O}_K^{\times} \to \bar{\mathbb{Q}}_{\ell}^{\times}$ there exists a unique continuous unramified character $\rho : G_K \to \bar{\mathbb{Q}}_{\ell}^{\times}$ such that $\rho(\mathrm{Fr}_v) = \xi(\pi_v)$ for all v.
2. For each continuous unramified character $\rho : G_K \to \bar{\mathbb{Q}}_{\ell}^{\times}$ there exists a unique character $\xi : K^{\times} \backslash \mathbb{A}_K^{\times} / \mathbb{O}_K^{\times} \to \bar{\mathbb{Q}}_{\ell}^{\times}$ such that $\rho(\mathrm{Fr}_v) = \xi(\pi_v)$ for all v.

Where $\mathbb{O}_K^{\times} = \mathbb{O}_0$

Theorem 6 (Character Formulation of Ramified class field theory)**.** *In the above notations:*

1. *For each character* $\xi : K^{\times} \backslash \mathbb{A}_K^{\times} / \mathcal{O}_{\mathfrak{m}}^{\times} \to \mathbb{Q}_{\ell}^{\times}$ *there exists a unique continuous character* $\rho : G_K \to \mathbb{Q}_{\ell}^{\times}$ *with* $\mathrm{ram}(\rho) \subseteq \mathfrak{m}$ *and* $\rho(Fr_v) = \xi(\pi_v)$ *for all primes* $v \notin \mathrm{Supp}(\mathfrak{m})$*.*

2. *For each continuous character* $\rho : G_K \to \mathbb{Q}_\ell^\times$ *with* $\mathrm{ram}(\rho) \subseteq \mathfrak{m}$ *there exists a unique character* $\xi : K^\times \backslash \mathbb{A}_K^\times / \mathcal{O}_\mathfrak{m}^\times \to \mathbb{Q}_\ell^\times$ *such that* $\rho(Fr_v) = \xi(\pi_v)$ *for all primes* $v \notin \mathrm{Supp}(\mathfrak{m})$.

Where The term unramified character, resp. character with ramification bounded by $\mathfrak{m}$, means that the character is trivial on the correspoding inertia group or higher ramification group of G_K corresponding to the relevant primes.

See milne, amichai, for more details.

We want to show how this formulation is equivalent to the adeles formulation given in the previous section.

1. Is this formulation *equivilant* to adeles language? is it dervied from it?
2. Give amichai reference for this formulation
3. Over what field are we working? what is l, what is p?
4. Fix the qoutient of adeles no match the subgroup
5. $\mathfrak{m}$ vs $\mathfrak{m}$ notation for divisors

Proof of geometric CFT

References

[Mil20] J.S. Milne. *Class Field Theory (v4.03)*. Available at www.jmilne.org/math/. 2020.