

Geometric Class Field Theory

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1 Introduction

Throughout this work we will be working over fields with characteristic $p \neq 0$ unless otherwise stated.

2 Class Field Theory In the language of Ideals

In this section we describe the main results of classical class field theory for global fields, following [Mil20]. We copy most of the content here from Milne.

2.1 Ideals, Moduli and Ray Class Groups

Let K be a global field of $\text{char}(K) = p$. A modulus \mathfrak{m} is a formal sum of places of K with non-negative integer coefficients. Let $S(K, \mathfrak{m}) = S(\mathfrak{m}) = \{v \in \mathfrak{m}\}$ be the set of places appearing in \mathfrak{m} with non-zero coefficient.

Define $K_{\mathfrak{m},1} = \{x \in K^\times \mid v(x-1) \geq n_v \text{ for all } v \in S(\mathfrak{m})\}$ where n_v is the coefficient of v in \mathfrak{m} .

For every set of primes S we define

$$I_K^S = \{\text{fractional ideals of } K \text{ generated by primes not in } S\}$$

There is a natural map $i : K_{\mathfrak{m},1} \rightarrow I_K^{S(\mathfrak{m})}$ sending $x \mapsto (x)$

The quotient

$$C_{\mathfrak{m}} = I_K^{S(\mathfrak{m})} / i(K_{\mathfrak{m},1})$$

is called the **(ray) class group** of K modulo \mathfrak{m} .

Let S be a finite set of primes of K . And G a finite abelian group. We shall say that a homomorphism $\psi : I^S \rightarrow G$ **admits a modulus** if there exists a modulus \mathfrak{m} with $S(\mathfrak{m}) \supset S$ such that $\psi(i(K_{\mathfrak{m},1})) = 0$. Thus ψ admits a modulus if and only if it factors through $C_{\mathfrak{m}}$ for some \mathfrak{m} with $S(\mathfrak{m}) \supset S$.

maybe we don't need this Milne states and prove a known theorem:

Theorem 1. *For every modulus \mathfrak{m} of K there is an exact sequence:*

$$0 \rightarrow \mathcal{O}_K^\times / \mathcal{O}_K^\times \cap K_{\mathfrak{m},1} \rightarrow K_{\mathfrak{m}} / K_{\mathfrak{m},1} \rightarrow C_{\mathfrak{m}} \rightarrow C \rightarrow 0$$

Where

$$K_{\mathfrak{m}} = \{x \in K^\times \mid v(x) = 0 \text{ for all } v \in S(\mathfrak{m})\}$$

And C is the usual class group of K .

2.2 The Main Theorems

Theorem 2 (Artin Reciprocity Law). *Let L be a finite abelian extension of a global field K . and let S be the set of primes of K ramifying in L . Then the Artin map **add here reference of the definition to milne** . $\psi : I^S \rightarrow \text{Gal}(L/K)$ admits a modulus \mathfrak{m} with $S(\mathfrak{m}) = S$ and it defines an isomorphism:*

$$I^S / \left(i(K_{\mathfrak{m},1}) \cdot N_{L/K}(I_L^{S(\mathfrak{m})}) \right) \rightarrow \text{Gal}(L/K)$$

A modulus \mathfrak{m} as in the statement of the theorem is called a defining modulus for L . Next, we write $I_K^{\mathfrak{m}}$ for the group of $S(\mathfrak{m})$ -ideals in K , and $I_L^{\mathfrak{m}}$ for the group of $S(\mathfrak{m})'$ -ideals in L where $S(\mathfrak{m})'$ is the set of primes of L lying above primes in $S(\mathfrak{m})$. Call a subgroup H of $I_K^{\mathfrak{m}}$ a **congruence subgroup** modulo \mathfrak{m} if it contains $i(K_{\mathfrak{m},1})$.

Theorem 3. *[Existence Theorem of Class Field Theory] For every congruence subgroup H modulo \mathfrak{m} there exists a unique finite abelian extension L/K , unramified at all primes not in $S(\mathfrak{m})$, such that the Artin map induces an isomorphism:*

$$I^{S(\mathfrak{m})} / H \rightarrow \text{Gal}(L/K)$$

More of the idealic class field theory in Milne.

Theorems 2 and 3 show that there is a canonical group isomorphism:

$$\lim_{\leftarrow m} C_m \rightarrow \text{Gal}(K^{\text{ab}}/K). \quad (1)$$

Rather than studying $\lim_{\leftarrow m} C_m$ directly, it turns out to be more natural to introduce another group that has it as a quotient - this is the idele class group. replace very where idele with ide'le

3 Class Field Theory In the language of Adeles and Ideles

Can we already say we are only considering function fields here? The modern formulation of Global Class Field Theory is given in terms of the adele and idele groups of a global field. In this chapter we will define these objects and state the main theorems of Class Field Theory in this language.

3.1 Adeles and Ideles

Let K be a global field. For each place v of K , we denote:

1. K_v = the completion of K at v
2. \mathfrak{p}_v = the corresponding prime ideal in the ring of integers \mathcal{O}_K of K
3. \mathcal{O}_v = the ring of integers of K_v
4. $\hat{\mathfrak{p}}_v$ = the completion of \mathfrak{p}_v = the maximal ideal of \mathcal{O}_v

We define the **adele ring** of K as the restricted direct product

$$\mathbb{A}_K = \prod'_v K_v$$

where the restriction is taken with respect to the rings of integers \mathcal{O}_v of K_v for all non-archimedean (IS IT NECESSEARY TO STATE HERE? WE WORK OVER P ANYWAY) places v . In other words, an adele is a tuple $(x_v)_v$ with $x_v \in K_v$ such that $x_v \in \mathcal{O}_v$ for all but finitely many non-archimedean places v . The **idele group** of K is defined as the group of units of the adele ring:

$$\mathbb{I}_K = \mathbb{A}_K^\times = \prod'_v K_v^\times$$

where the restriction is taken with respect to the unit groups \mathcal{O}_v^\times of the rings of integers \mathcal{O}_v for all non-archimedean places v . An idele is thus a tuple $(x_v)_v$ with $x_v \in K_v^\times$ such that $x_v \in \mathcal{O}_v^\times$ for all but finitely many non-archimedean places v .

The field K embeds diagonally into \mathbb{A}_K , and thus K^\times embeds diagonally into \mathbb{I}_K as the subgroup of principal ideles. The **idele class group** \mathbf{C}_K is the quotient:

$$\mathbf{C}_K = \mathbb{I}_K / K^\times$$

There is a natural isomorphism between certain quotients of the idele group and the ideal group of K , which ultimately follows by understanding ideles as thickening of ideals: There is a canonical surjective homomorphism id :

$$\begin{aligned} \text{id} : \mathbb{I}_K &\rightarrow I_K \\ (x_v)_v &\mapsto \prod_v \mathfrak{p}_v^{v(x_v)} \end{aligned}$$

Thus, composing with $I_K \rightarrow C$ gives a surjective homomorphism $\mathbb{I}_K \rightarrow C$, noting that $K^\times \rightarrow \mathbb{I}_K \rightarrow C$ is 0, we realize $C = I_K/i(K^\times)$ as a quotient of $\mathbf{C}_K = \mathbb{I}_K/K^\times$.

The same thing is true for $C_{\mathfrak{m}}$: Let $\mathfrak{m} = \sum n_v \mathfrak{p}_v$ be a modulus of K , set:

$$W_{\mathfrak{m}}(v) = \begin{cases} \mathcal{O}_v^\times & v \notin \text{Supp}(\mathfrak{m}) \\ 1 + \hat{\mathfrak{p}}_v^{n_v} & v \in \text{Supp}(\mathfrak{m}) \end{cases}$$

And define

$$\mathbb{I}_{\mathfrak{m}} = \left(\prod_{v \notin \text{Supp}(\mathfrak{m})} K_v^\times \times \prod_{v \in \text{Supp}(\mathfrak{m})} W_{\mathfrak{m}}(v) \right) \cap \mathbb{I}_K$$

And

$$\mathbb{O}_{\mathfrak{m}}^\times = \prod_v W_{\mathfrak{m}}(v)$$

Note that:

$$K_{\mathfrak{m},1} = K^\times \cap \prod_{v \in \mathfrak{m}} W_{\mathfrak{m}}(v) \quad \text{Intersection inside } \prod_{v \in \mathfrak{m}} K_v^\times$$

and that

$$K_{\mathfrak{m},1} = K^\times \cap \mathbb{I}_{\mathfrak{m}} \quad \text{Intersection inside } \mathbb{I}_K$$

Milne **shows** the following proposition:

Proposition 4. *Let \mathfrak{m} be a modulus of K .*

1. *The map $\text{id} : \mathbb{I}_{\mathfrak{m}} \rightarrow I_K^{S(\mathfrak{m})}$ defines an isomorphism*

$$\mathbb{I}_{\mathfrak{m}}/K_{\mathfrak{m},1}\mathbb{O}_{\mathfrak{m}}^\times \xrightarrow{\sim} I_K^{S(\mathfrak{m})}/i(K_{\mathfrak{m},1}) = C_{\mathfrak{m}}$$

2. *The inclusion $\mathbb{I}_{\mathfrak{m}} \hookrightarrow \mathbb{I}_K$ defines an isomorphism*

$$\mathbb{I}_{\mathfrak{m}}/K_{\mathfrak{m},1} \xrightarrow{\sim} \mathbb{I}_K/K^\times$$

Taking the quotient into character form?

3.1.1 Topology on Adeles and Ideles

We state quickly the topology on the adèle ring and the idele group. More can be found in [add reference](#). Recall that, for all v , K_v is locally compact more over, \mathcal{O}_v is a compact neighborhood of 0. Similarly K_v^\times is locally compact, in fact:

$$1 + \hat{\mathfrak{p}}_v \supset 1 + \hat{\mathfrak{p}}_v^2 \supset 1 + \hat{\mathfrak{p}}_v^3 \dots$$

is a fundamental system of neighborhoods of 1 consisting of compact open subgroups of K_v^\times .

For every finite set S of places of K , define:

$$\mathbb{I}_S = \prod_{v \in S} K_v^\times \times \prod_{v \notin S} \mathcal{O}_v^\times$$

with the product topology. \mathbb{I}_S is locally compact and as sets we have:

$$\mathbb{I}_K = \bigcup_S \mathbb{I}_S$$

where the union is taken over all finite sets of places of K . We define a topology on \mathbb{I}_K by giving a basis for the open sets $\prod_v V_v$ with $V_v \subseteq K_v^\times$ open for all v and $V_v = \mathcal{O}_v^\times$ for all but finitely many v . This makes \mathbb{I}_K a locally compact topological group, such that each \mathbb{I}_S is open in \mathbb{I}_K , and inherits the product topology. The following sets form a fundamental system of neighborhoods of 1: for each finite set of primes S and $n > 0$, define

$$U_{S,n} = \left\{ (x_v)_v \in \mathbb{I}_K \mid v(x_v - 1) > n \text{ for all } v \in S, x_v \in \mathcal{O}_v^\times \text{ for } v \notin S \right\}$$

Note that the embedding $K^\times \rightarrow \mathbb{I}_K$ is discrete and thus the idele class group $\mathbf{C}_K = \mathbb{I}_K / K^\times$ is a locally compact topological group as well. Moreover the canonical injective homomorphism

$$K_v^\times \rightarrow \mathbb{I}_K \tag{2}$$

$$x \mapsto (1, \dots, 1, x, 1, \dots, 1) \quad (x \text{ in the } v\text{-th position}) \tag{3}$$

is a topological embedding for each place v of K .

3.1.2 Characters of ideals and of ideles

in [\[Mil20\]](#), Milne proves:

Proposition 5. *Let G be a finite abelian group. If $\psi : I^S \rightarrow G$ admits a modulus, then there exists a unique homomorphism $\phi : \mathbb{I} \rightarrow G$ such that*

1. ϕ is continuous (G with the discrete topology)
2. $\phi(K^\times) = 1$;
3. $\phi(\mathbf{a}) = \psi(\text{id}(\mathbf{a}))$, all $\mathbf{a} \in \mathbb{I}^S \stackrel{\text{def}}{=} \{\mathbf{a} \mid a_v = 1 \text{ all } v \in S\}$.

Moreover, every continuous homomorphism $\phi : \mathbb{I} \rightarrow G$ satisfying (2) arises from a ψ . Moreover, ϕ and ψ fit in the following chain:

$$\begin{array}{ccccc}
 I^{\mathfrak{m}} & \longrightarrow & C_{\mathfrak{m}} & \xrightarrow{\psi} & G \\
 & & \cong \uparrow & & \nearrow \\
 \mathbb{I}_{\mathfrak{m}}/K_{\mathfrak{m},1} & \longrightarrow & \mathbb{I}_{\mathfrak{m}}/K_{\mathfrak{m},1}\mathbb{O}_{\mathfrak{m}}^{\times} & & \\
 \downarrow \cong & & & & \\
 \mathbb{I} & \longrightarrow & \mathbb{I}_K/K^{\times} & \xrightarrow{\phi} & G
 \end{array} \tag{4}$$

3.1.3 Norms of ideles

Let L be a finite extension of the number field K .

For an idèle $\mathbf{a} = (a_w) \in \mathbb{I}_L$, define $\text{Nm}_{L/K}(\mathbf{a})$ to be the idèle $\mathbf{b} \in \mathbb{I}_K$ with $b_v = \prod_{w|v} \text{Nm}_{L_w/K_v} a_w$. Then, one can show that the following diagram commutes:

$$\begin{array}{ccccc}
 L^{\times} & \longrightarrow & \mathbb{I}_L & \xrightarrow{\text{id}} & I_L \\
 \downarrow \text{Nm}_{L/K} & & \downarrow \text{Nm}_{L/K} & & \downarrow \text{Nm}_{L/K} \\
 K^{\times} & \longrightarrow & \mathbb{I}_K & \xrightarrow{\text{id}} & I_K.
 \end{array}$$

Thus getting a commutative diagram:

$$\begin{array}{ccc}
 \mathbf{C}_L & \longrightarrow & C_L \\
 \downarrow \text{Nm}_{L/K} & & \downarrow \text{Nm}_{L/K} \\
 \mathbf{C}_K & \longrightarrow & C_K
 \end{array}$$

(where C_L, C_K are the ideal class groups of L and K respectively).

3.2 The main theorems

The theory establishes a fundamental connection between the idele class group \mathbf{C}_K and the Galois group of the maximal abelian extension of K , denoted K^{ab} .

Theorem 6 (Reciprocity Law). *There exists a unique continuous homomorphism $\phi_K : \mathbb{I}_K \rightarrow \text{Gal}(K^{ab}/K)$ called the **Artin map** with the following properties:*

1. $\phi_K(K^{\times}) = 1$;
2. For every finite abelian extension L/K , ϕ_K defines an isomorphism:

$$\phi_{L/K} : \mathbb{I}_K / (K^{\times} \cdot \text{Nm}_{L/K}(\mathbb{I}_L)) \xrightarrow{\sim} \text{Gal}(L/K)$$

or, equivalently, an isomorphism:

$$\mathbf{C}_K / \text{Nm}_{L/K}(\mathbf{C}_L) \xrightarrow{\sim} \text{Gal}(L/K)$$

3. $\phi_{L/K}$ arises from the global $\mathbb{I}_K \rightarrow \text{Gal}(L/K)$ coming from the ideal-theoretic global artin map, as in [proposition 5](#).

Theorem 7 (The Existence Theorem). *There is a one-to-one, inclusion-reversing correspondence between the set of finite abelian extensions of K and the set of open subgroups of finite index in the idele class group \mathbf{C}_K .*

$$\left\{ \begin{array}{c} \text{Finite abelian} \\ \text{extensions } L/K \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Open subgroups } H \subseteq \mathbf{C}_K \\ \text{of finite index} \end{array} \right\}$$

Under this correspondence, an extension L corresponds to the subgroup $H = N_{L/K}(\mathbf{C}_L)$.

Theorem 8. *Ideal-Theoretic and Idele-Theoretic formulations of CFT are equivalent through [proposition 5](#).*

1. Above is finite adelic formulation of CFT, state something about the infinite extension CFT theorem
2. State something about the topology on the idele class group, Say how the finite implies the infinite by taking inverse limits.
3. Find sources for the above. for example milne? maybe other?
4. The restriction is for the non-archimedean, are you sure?
5. What is the topology of the "continuous" homomorphism?

4 Class Field Theory In the language of Characters

The character formulation of Class Field Theory provides a correspondence between characters of the idele class group and characters of the Galois group of the maximal abelian extension of a global field.

4.1 About Characters

We need to make precise what we mean by characters on both sides of the correspondence.

Definition 9. *Let G be an abelian group.*

1. A character $\rho : G_K \rightarrow G$ is unramified at a place v if it is trivial on the inertia group $I_v \subseteq G_K$. ρ is called unramified if it is unramified at all places v of K .
2. A character $\rho : G_K \rightarrow G$ has ramification bounded by a modulus $\mathfrak{m} = \sum_v n_v v$ if for each place $v \in \mathfrak{m}$, the restriction of ρ to the higher ramification group $G_v^{n_v}$ is trivial.

Note that since G is abelian, the value of ρ on the Frobenius element Fr_v is well-defined for unramified places v .

A useful theorem about characters is as follows:

Theorem 10. *Let G be an abelian group such that for every $n \in \mathbb{N}$, the n -torsion subgroup $G[n] = \{g \in G \mid ng = 0\}$ is cyclic of order n . Then for every finite abelian group A denote by $\hat{A} = \text{Hom}(A, G)$ the group of characters from A to G .*

Then the functor $A \mapsto \hat{A}$ is a contravariant equivalence of categories between the category of finite abelian groups and itself. Moreover the natural map $A \rightarrow \hat{\hat{A}}$ is an isomorphism.

4.2 The Main Theorems

One formulation of Global Class Field Theory in terms of characters is as follows:

Theorem 11 (Character Formulation of Unramified Global Class Field Theory). *1. For each character $\xi : K^\times \backslash \mathbb{A}_K^\times / \mathbb{O}_K^\times \rightarrow \bar{\mathbb{Q}}_\ell^\times$ there exists a unique continuous unramified character $\rho : G_K \rightarrow \bar{\mathbb{Q}}_\ell^\times$ such that $\rho(\text{Fr}_v) = \xi(\pi_v)$ for all v .*

2. For each continuous unramified character $\rho : G_K \rightarrow \bar{\mathbb{Q}}_\ell^\times$ there exists a unique character $\xi : K^\times \backslash \mathbb{A}_K^\times / \mathbb{O}_K^\times \rightarrow \bar{\mathbb{Q}}_\ell^\times$ such that $\rho(\text{Fr}_v) = \xi(\pi_v)$ for all v .

Where $\mathbb{O}_K^\times = \mathbb{O}_0$

Theorem 12 (Character Formulation of Ramified class field theory). *In the above notations:*

1. For each character $\xi : K^\times \backslash \mathbb{A}_K^\times / \mathcal{O}_\mathfrak{m}^\times \rightarrow \bar{\mathbb{Q}}_\ell^\times$ there exists a unique continuous character $\rho : G_K \rightarrow \bar{\mathbb{Q}}_\ell^\times$ with $\text{ram}(\rho) \subseteq \mathfrak{m}$ and $\rho(\text{Fr}_v) = \xi(\pi_v)$ for all primes $v \notin \text{Supp}(\mathfrak{m})$.

2. For each continuous character $\rho : G_K \rightarrow \bar{\mathbb{Q}}_\ell^\times$ with $\text{ram}(\rho) \subseteq \mathfrak{m}$ there exists a unique character $\xi : K^\times \backslash \mathbb{A}_K^\times / \mathcal{O}_\mathfrak{m}^\times \rightarrow \bar{\mathbb{Q}}_\ell^\times$ such that $\rho(\text{Fr}_v) = \xi(\pi_v)$ for all primes $v \notin \text{Supp}(\mathfrak{m})$.

In fact, for the above theorems, we can replace $\bar{\mathbb{Q}}_\ell^\times$ by any finite abelian group G with the discrete topology, and the theorems would still hold.

Theorem 13. *Assume 11, 12 are true for all finite abelian groups G with the discrete topology (as values of the characters). Then 11, 12 are true as stated.*

Proof. Indeed, assume such theorem would be true for such cases, then by varying G over $(\mathbb{Z}/l^n\mathbb{Z})^\times$ (for all $n \in \mathbb{N}$), we will get a compatible system of characters and a corresponding isomorphism of character groups with values in \mathbb{Z}_l^\times . And since $\text{Tors}(\mathbb{Z}_l^\times) \cong \text{Tors}(\bar{\mathbb{Q}}_\ell^\times)$, this is equivalent to the theorems for characters with values in $\bar{\mathbb{Q}}_\ell^\times$. Similarly, every finite extension $\mathbb{Q}_l \subset F$ comes as inverse limit of its finite subgroups of units, so the same argument applies to characters with values in F^\times . We have compatibility between those characters (from uniqueness) for all finite $\mathbb{Q}_l \subset F$ and by going to the colimit (and since all groups involved are finitely generated, hence in **Ab** $\text{Hom}(A, -)$ preserve colimits) we get the result for characters with values in $\bar{\mathbb{Q}}_\ell^\times$ as well. □

Theorem 14. *Let \mathfrak{m} be a modulus of K . Assume 11 and 12 are true. Then the Artin map induces isomorphisms:*

$$\begin{aligned} \text{Hom}_{\text{cont}}(C_\mathfrak{m}, \bar{\mathbb{Q}}_\ell^\times) &\cong \text{Hom}_{\text{cont}}(K^\times \backslash \mathbb{A}_K^\times / \mathbb{O}_\mathfrak{m}^\times, \bar{\mathbb{Q}}_\ell^\times) \\ &\cong \text{Hom}_{\text{cont}, \text{ram} \leq \mathfrak{m}}(G_K, \bar{\mathbb{Q}}_\ell^\times) \cong \text{Hom}_{(\text{cont}, \text{ram} \leq \mathfrak{m})}(G_K^{\text{ab}}, \bar{\mathbb{Q}}_\ell^\times) \\ &\cong \text{Hom}_{\text{cont}}(\text{Gal}(L_\mathfrak{m}/K), \bar{\mathbb{Q}}_\ell^\times) \end{aligned}$$

Where $L_{\mathfrak{m}}$ is the maximal abelian extension of K with ramification bounded by \mathfrak{m} . Hence by [theorem 10](#) we get that the artin map induces isomorphism $C_{\mathfrak{m}} \cong \text{Gal}(L_{\mathfrak{m}}/K)$, which implies the statement of [theorem 2](#) and [theorem 3](#). (Details omitted, like how to go from $C_{\mathfrak{m}}$ to every congruence subgroups, etc.)

1. Is this formulation *equivilant* to adeles language? is it dervied from it?
2. Give amichai reference for this formulation
3. Over what field are we working? what is l , what is p ?
4. Fix the goutient of adeles no match the subgroup
5. \mathfrak{m} vs \mathfrak{m} notation for divisors
6. maybe make [theorem 13](#) more precise
7. maybe make [theorem 14](#) more precise
8. replace l by ℓ everywhere

See milne, amichai, for more details.

state in term of finite G , show it implies character formulation, etc... Proof of geometric CFT

References

[Mil20] J.S. Milne. *Class Field Theory (v4.03)*. Available at www.jmilne.org/math/. 2020.