Geometric Class Field Theory

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Abstract

We study the ramification of the symmetric product $\mathcal{F}^{(\deg \mathfrak{m})}$ of a local system $\mathcal{F}$ on a curve $C \setminus \mathfrak{m}$. Assuming the ramification of $\mathcal{F}$ is bounded by $\mathfrak{m}$, we prove that the symmetric product $\mathcal{F}^{(\deg \mathfrak{m})}$ is at most tamely ramified at the generic point of the exceptional divisor $E_{\mathfrak{m}}$ of the blowup of $C^{(\deg \mathfrak{m})}$ at $\mathfrak{m}$. As a primary application, we utilize this result to prove Geometric Class Field Theory. Our approach builds upon the geometric framework for the unramified case originally established by Deligne for the rank-one Langlands correspondence, following the subsequent extensions to the ramified case developed by Takeuchi and Guignard.

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1 Introduction

In this thesis, we give an elementary proof of a certain imporant geometric theorem occuring in Deligne's approach to geometric class field theory. We (usually) work over a perfect field k, C is a projective smooth geometrically connected curve over k, with genus g. One of the main geometric ingredients in the approach, is showing why a local system $\mathcal{F}$ with ramification bounded by a modulus $\mathfrak{m}$ on $U = C \setminus \mathfrak{m}$ descends via the Abel-Jacobi $\Phi : U \to \mathrm{Pic}_{C,\mathfrak{m}}$ to $\mathrm{Pic}_{C,\mathfrak{m}}$. The approach, innovated by Deligne, relies on analyzing the symmetric powers $\mathcal{F}^{(d)}$ of $\mathcal{F}$ on the symmetric powers $U^{(d)}$ of U, and showing that for sufficiently large d, $\mathcal{F}^{(d)}$ descends to $\mathrm{Pic}^d_{C,\mathfrak{m}}$ via the degree d Abel-Jacobi map $\Phi_d : U^{(d)} \to \mathrm{Pic}^d_{C,\mathfrak{m}}$. The geometric-fibers of Φ_d (for $d \geq \deg \mathfrak{m} + 2g - 1$) over any point are isomorphic to

$$\begin{cases} \mathbb{A}^{d-\deg \mathfrak{m}-g+1}_{k^{sep}} & \text{if } \mathfrak{m} > 0 \\ \mathbb{P}^{d-g}_{k^{sep}} & \text{if } \mathfrak{m} = 0 \end{cases}$$

Where g is the genus of the curve C. The unramified case ($\mathfrak{m} = 0$) is relatively simple, as the Abel-Jacobi map is proper, surjective with geometrically connected fibers, which follows from the fact that it is a fibration in projective spaces. Thus, by using the homotopy exact sequence for the étale fundamental group, one gets an isomorphism between the étale fundamental group of $U^{(d)} (= C^{(d)})$ and that of $\mathrm{Pic}^d_{C,\mathfrak{m}} (= \mathrm{Pic}^d_C)$.

The ramified case ($\mathfrak{m} > 0$) is more subtle, as the Abel-Jacobi map is not proper anymore, and one needs to analyze the ramification of $\mathcal{F}^{(d)}$ "along the boundary" of $U^{(d)}$ in $C^{(d)}$.

Previous work has generalized Deligne's approach to the ramified case, most notably by Guignard [Gui19] and Takeuchi [Tak19]. Their approaches differ. To descend, Guignard proves that the restriction of $\mathcal{F}^{(d)}$ to any line in the fiber of the degree d Abel-Jacobi map is a constant étale sheaf. He achieves this by demonstrating that the restriction is at most tamely ramified and invoking the triviality of the tame fundamental group of $\mathbb{A}^1_k$. His analysis relies on local geometric class field theory. It is also worth noting that Guignard's method generalizes to relative curves over arbitrary base schemes. Takeuchi, on the other hand, constructs a compactification of $U^{(d)}$ by blowing up $C^{(d)}$ along certain well-chosen centers. This compactification, denoted by $\tilde{C}^{(d)}_{\mathfrak{m}}$, has $U^{(d)}$ as an open subscheme with a codimension 1 closed subscheme H as complement. He then shows that the Abel-Jacobi map extends to a proper morphism from $\tilde{C}^{(d)}_{\mathfrak{m}}$ to $\mathrm{Pic}^d_{C,\mathfrak{m}}$, which is a fibration in projective spaces. Thus, by the homotopy exact sequence for the étale fundamental group, one gets an isomorphism between the étale fundamental group of $\tilde{C}^{(d)}_{\mathfrak{m}}$ and that of $\mathrm{Pic}^d_{C,\mathfrak{m}}$. To conclude the descent, Takeuchi analyzes the ramification of $\mathcal{F}^{(d)}$ along the boundary H of $\tilde{C}^{(d)}_{\mathfrak{m}}$, showing that it is tamely ramified there, which suffices. His methods relies on the theory of Witt vectors and refined Swan conductors.

For an account of these approaches, see [Gui19] and [Tak19]. For a full approach following Deligne's method in the unramified case, and the tamely ramified case see [Ten15], and [Tót11].

In this thesis, we calculate the ramification of $\mathcal{F}^{(d)}$ directly to show that it is at most tame at the generic point of H, avoiding the use of Swan conductors. To prove Geometric Class Field Theory, we utilize Takeuchi's construction of the compactification $\tilde{C}^{(d)}_{\mathfrak{m}}$ of $U^{(d)}$ via the blowing up of $C^{(d)}$.

In the rest of the introduction, we state the main theorem of geometric class field theory Theorem 1, and its reductions to Theorem 2 and Theorem 3, which we prove in this thesis.

Let k be a perfect field, and let C be a projective smooth geometrically connected curve over k, with genus g. Geometric class field theory gives a geometric description of abelian coverings of C by relating it to isogenies of the generalized Picard schemes.

Fix a modulus $\mathfrak{m}$, i.e. an effective Cartier divisor of C and let U be its complement in C. The pairs $(\mathcal{L}, \alpha)$, where $\mathcal{L}$ is an invertible $\mathcal{O}_C$-module and α is a rigidification of $\mathcal{L}$ along $\mathfrak{m}$, are parametrized by a k-group scheme $\mathrm{Pic}_{C,\mathfrak{m}}$, called the rigidified Picard scheme. The Abel-Jacobi morphism

$$\Phi : U \to \mathrm{Pic}_{C,\mathfrak{m}}$$

is the morphism which sends a section x of U to the pair $(\mathcal{O}(x), 1)$. The fundamental result of GCFT can be formulated as:

Theorem 1 (Geometric Class Field Theory). *Let Λ be a finite ring of cardinality invertible in k, and let $\mathcal{F}$ be an étale sheaf of Λ-modules, locally free of rank 1 on U, with ramification bounded by $\mathfrak{m}$. Then, there exists a unique (up to isomorphism) multiplicative[1] étale sheaf of Λ-modules $\mathcal{G}$ on $\mathrm{Pic}_{C,\mathfrak{m}}$, locally free of rank 1, such that the pullback of $\mathcal{G}$ by Φ is isomorphic to $\mathcal{F}$.*

Let d be a positive integer. We denote by $U^{(d)}$ the d-th symmetric power of U over k. For an étale sheaf $\mathcal{F}$ on $U_{\text{ét}}$, we denote by $\mathcal{F}^{(d)}$ the d-th symmetric power of $\mathcal{F}$ on $U^{(d)}$ The degree d Abel-Jacobi morphism is defined as the map

$$\Phi_d : U^{(d)} \to \mathrm{Pic}^d_{C,\mathfrak{m}}$$

which sends a section $x_1 + \cdots + x_d$ of $U^{(d)}$ to the pair $(\mathcal{O}(x_1 + \cdots + x_d), 1)$.

A method of descent shows that to prove Theorem 1, it suffices to prove the following reduced version[2]:

Theorem 2. *Let Λ be a finite ring of cardinality invertible in k, and let $\mathcal{F}$ be an étale sheaf of Λ-modules, locally free of rank 1 on $U_{ét}$, with ramification bounded by $\mathfrak{m}$. Then, for sufficiently large integer d, there exists a unique (up to isomorphism) étale sheaf of Λ-modules $\mathcal{G}_d$ on $\mathrm{Pic}^d_{C,\mathfrak{m}}$, locally free of rank 1, such that the pullback of $\mathcal{G}_d$ by Φ_d is isomorphic to $\mathcal{F}^{(d)}$.*

To prove Theorem 2 we follow the work of [Tak19] (and similiary done in [Tót11]), analyzing the ramification of $\mathcal{F}^{(d)}$ after blowing up $C^{(d)}$. We analyze this ramification using elementary methods. We prove the following Theorem:

Theorem 3. *Let Λ be a finite ring of cardinality invertible in k, and let $\mathcal{F}$ be an étale sheaf of Λ-modules, locally free of rank 1 on $U_{ét}$, with ramification bounded by $\mathfrak{m}$. Let $\mathfrak{n} = k_P P \subset \mathfrak{m}$ be a non trival sub modulus of $\mathfrak{m}$ such that $\mathcal{F}$ is bounded at P by k_P. Then $\mathcal{F}^{(\deg \mathfrak{n})}$ is tamely ramified at the generic point $\eta_\mathfrak{n}$ of the exceptional divisor $E_\mathfrak{n}$ of the blowup $X_\mathfrak{n}$ of $C^{(\deg \mathfrak{n})}$ at the closed point $\mathfrak{n}$.*

The thesis is organized as follows:

Chapter 1 - Preliminaries provides the necessary preliminaries and covers the foundational material upon which this work is based, generally without providing proofs.

Chapter 2 - Ramification After Blowup Is Tame is devoted to the proof of Theorem 3, along with several corollaries that will be instrumental in the proof of Theorem 2.

[1]The notion of a multiplicative locally free Λ-module of rank 1 is due to [Gui19] and corresponds to isogenies $G \to \mathrm{Pic}_{C,\mathfrak{m}}$ with constant kernel $\Lambda^\times$. This concept corresponds to multiplicative characters of $H^1(\mathrm{Pic}_{C,\mathfrak{m}}, \mathbb{Q}/\mathbb{Z})$ in the formulation of [Tak19], and generalizes Hecke eigensheaves in the context of [Ten15].

[2]See the last page of [Gui19], Section 8.3 of [Ten15], or the proof of Theorem 1.2 in [Tak19] for details on this reduction.

Chapter 3 - Geometric Class Field Theory presents the proof of Theorem 2.

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2 Preliminaries

This section establishes the foundational definitions and theorems necessary for the remainder of this work. We focus specifically on the theory of G-torsors, which are central to our study due to their correspondence with locally free sheaves of rank 1. Subsequently, we review the relevant background on ramification theory from the existing literature. Finally, we conclude with several algebraic geometric remarks and notational conventions that will be employed implicitly throughout this thesis.

2.1 Torsors

In what follows, we largely adhere to the treatment of torsors and group objects found in publised notes of Alex Youcis [You20]. Let $\mathscr{C} = (\mathcal{C}, J)$ be a site and let $\mathcal{E} = Sh(\mathscr{C})$ be the associated topos. Let $\mathcal{G}$ be a group object in $\mathcal{E}$. We denote by $\mathcal{G}\mathcal{E}$ the category of objects in $\mathcal{E}$ endowed with a left $\mathcal{G}$-action.

Definition 4. A **$\mathcal{G}$-torsor** in $\mathcal{E}$ is an object $\mathcal{P}$ of $\mathcal{G}\mathcal{E}$ satisfying the following conditions:

1. The structural morphism $\mathcal{P} \to 1$ is an epimorphism in $\mathcal{E}$ (i.e., $\mathcal{P}$ is locally non-empty).
2. The map $\mathcal{G} \times \mathcal{P} \to \mathcal{P} \times \mathcal{P}$ defined by $(g, p) \mapsto (g \cdot p, p)$ is an isomorphism in $\mathcal{E}$ (i.e., $\mathcal{G}$ acts simply transitively on $\mathcal{P}$).

Since $\mathcal{E}$ is the topos of sheaves on a site $\mathscr{C}$, the definition can be reformulated in terms of covers. A $\mathcal{G}$-sheaf $\mathcal{P}$ is a $\mathcal{G}$-torsor if:

1. For every object $X \in \mathcal{C}$, there exists a covering $\{U_i \to X\} \in J$ such that $\mathcal{P}(U_i) \neq \emptyset$ for all i.
2. For any $X \in \mathcal{C}$ where $\mathcal{P}(X)$ is non-empty, the action of $\mathcal{G}(X)$ on $\mathcal{P}(X)$ is simply transitively.

A fundamental property of torsors is their local triviality: a $\mathcal{G}$-sheaf $\mathcal{P}$ is a $\mathcal{G}$-torsor if and only if it is locally isomorphic to the trivial torsor. Specifically, for every $X \in \mathcal{C}$, there must exist a cover $\{U_i \to X\}$ such that the restriction $\mathcal{P}|_{U_i}$ is isomorphic, as a $\mathcal{G}|_{U_i}$-sheaf, to $\mathcal{G}|_{U_i}$ acting on itself by left multiplication.

A **morphism of $\mathcal{G}$-torsors** $f : \mathcal{P}_1 \to \mathcal{P}_2$ is a morphism of sheaves that is equivariant with respect to the $\mathcal{G}$-action.

It is a standard result that every morphism of $\mathcal{G}$-torsors is an isomorphism. Consequently, the category of $\mathcal{G}$-torsors in $\mathcal{E}$ forms a groupoid.

Definition 5. We denote the groupoid of $\mathcal{G}$-torsors in $\mathcal{E}$ by $\mathbf{Tors}(\mathcal{E},\mathcal{G})$. The set of isomorphism classes of $\mathcal{G}$-torsors is denoted by $\mathrm{Tors}(\mathcal{E},\mathcal{G})$.

Torsors exhibit functoriality with respect to the group object:

Definition 6. Let $\varphi : \mathcal{G}_1 \to \mathcal{G}_2$ be a morphism of group sheaves on $\mathscr{C}$, and let $\mathcal{P}$ be a $\mathcal{G}_1$-torsor. We define the **contracted product** $\mathcal{G}_2 \times^{\mathcal{G}_1} \mathcal{P}$ as the quotient sheaf $(\mathcal{G}_2 \times \mathcal{P})/\mathcal{G}_1$, where $\mathcal{G}_1$ acts on the product by:

$$g_1 \cdot (g_2, p) = (g_2\varphi(g_1)^{-1}, g_1 \cdot p)$$

The contracted product inherits a natural left $\mathcal{G}_2$-action given on local sections by $h \cdot [g_2, p] = [hg_2, p]$, which endows it with the structure of a $\mathcal{G}_2$-torsor.

This construction yields a functor:

$$\varphi_* : \mathbf{Tors}(\mathcal{E},\mathcal{G}_1) \to \mathbf{Tors}(\mathcal{E},\mathcal{G}_2), \quad \mathcal{P} \mapsto \mathcal{G}_2 \times^{\mathcal{G}_1} \mathcal{P}$$

On the level of isomorphism classes, φ_* induces a map of pointed sets $\mathrm{Tors}(\mathcal{E},\mathcal{G}_1) \to \mathrm{Tors}(\mathcal{E},\mathcal{G}_2)$, sending the class of the trivial $\mathcal{G}_1$-torsor to the class of the trivial $\mathcal{G}_2$-torsor.

When $\mathcal{G}$ is a **sheaf of abelian groups** (an *abelian sheaf*), the pointed set $\mathrm{Tors}(\mathcal{E},\mathcal{G})$ inherits the structure of an abelian group. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be objects of $\mathbf{Tors}(\mathcal{E},\mathcal{G})$. We define their sum $[\mathcal{P}_1] + [\mathcal{P}_2]$ to be the class $[\mathcal{P}_1 \otimes \mathcal{P}_2]$, where $\mathcal{P}_1 \otimes \mathcal{P}_2$ is defined as the quotient sheaf $(\mathcal{P}_1 \times \mathcal{P}_2)/\mathcal{G}$. In this construction, $\mathcal{G}$ acts on the product $\mathcal{P}_1 \times \mathcal{P}_2$ on T-points by:

$$g \cdot (f_1, f_2) := (gf_1, g^{-1}f_2)$$

The group object $\mathcal{G}$ then acts on the resulting quotient via its action on the presheaf quotient, which is given on classes by:

$$g \cdot [(f_1, f_2)] = [(gf_1, f_2)] = [(f_1, gf_2)]$$

where the square brackets denote the class in the quotient set. This structure turns $\mathrm{Tors}(\mathcal{G})$ into an abelian group, where the identity is the class of the trivial torsor and the inverse is obtained by the opposite action.[3]

Let Λ be a commutative ring object in the topos $\mathcal{E}$. We denote by $\mathbf{Pic}(\mathcal{E},\Lambda)$ the category of locally free Λ-modules of rank 1 in $\mathcal{E}$, also referred to as invertible modules. For the multiplicative group object $G = \Lambda^\times$, there exists a classical correspondence between such Λ-modules and G-torsors. Indeed, both categories are symmetric monoidal categories, $\mathbf{Pic}(\mathcal{E},\Lambda)$ under the tensor product $\otimes_\Lambda$, with unit object Λ. And $\mathbf{Tors}(\mathcal{E},G)$ Under contracted product $\times^G$, with unit object G acting on itself by translation.

Throughout this thesis, we shall primarily adopt the language of G-torsors, with the understanding that the associated Λ-module structure is implicitly inferred.

The following proposition formalizes this dictionary:

Proposition 7. *The associated module functor* $\Phi : \mathcal{P} \mapsto \mathcal{P} \times^G \Lambda$ *induces a canonical equivalence of monoidal categories:*

$$\Phi : (\mathbf{Tors}(\mathcal{E},G), \times^G, G) \xrightarrow{\sim} (\mathbf{Pic}(\mathcal{E},\Lambda), \otimes_\Lambda, \Lambda)$$

The inverse functor $\Psi : \mathcal{L} \mapsto \underline{\mathrm{Isom}}_\Lambda(\Lambda, \mathcal{L})$.

[3]Equivantly, the sum is obtained as the contracted product of the $\mathcal{G} \times \mathcal{G}$-torsor $\mathcal{P}_1 \times \mathcal{P}_2$ along the multiplication map $m : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$.

Lastly, for any object $\mathcal{X} \in \mathcal{E}$, there is a canonical identification between the slice category $(\mathcal{G}\mathcal{E})/\mathcal{X}$ and the category of group objects $\mathcal{G}(\mathcal{E}/\mathcal{X})$, where $\mathcal{X}$ is viewed as having the trivial $\mathcal{G}$-action.

We denote by $\mathbf{Tors}(\mathcal{X}, \mathcal{G})$ the category of G-torsors over $\mathcal{X}$ in $\mathcal{G}\mathcal{E}_{/\mathcal{X}}$.

2.2 Torsors over the Étale's Sites

Fix a scheme X, and let G be a smooth affine X-group scheme. Denote by $X_{\text{Ét}}$ and $X_{\text{ét}}$ the big and small étale sites on X, respectively. The functor of points of G, which we denote by $G(\cdot) = \mathrm{Hom}_X(-, G)$, defines a group sheaf on $X_{\text{ét}}$.

Definition 8. A *principal G-bundle* (or *principal homogeneous space*) for G is a scheme $f : Y \to X$ equipped with a left G-action $\rho : G \times_X Y \to Y$ satisfying:

1. The morphism $G \times_X Y \to Y \times_X Y$ sending $(g, y) \mapsto (gy, y)$ is an isomorphism of X-schemes.
2. There exists an étale covering $\{U_i \to X\}$ such that $Y_{U_i} \cong G_{U_i}$ as G-schemes, where G acts on itself by left translation.

Remark. Since $G \to X$ is smooth and Y is étale-locally isomorphic to G, the morphism $f : Y \to X$ is smooth and affine. Note that Y is generally not étale over X unless G is an étale group scheme (e.g., a finite constant group). Similarly, if G is finite then $Y \to X$ is finite.

Similarly to the case of G-torsors of a topos, we define a morphism of principal G-bundles to be a morphism of X-schemes commuting with the G-action. We now have:

Theorem 9. *The morphism sending $Y \mapsto \mathrm{Hom}_X(-, Y)$ is an equivalence of categories from the category of principal G-bundles to the category of G-torsors on $X_{\acute{E}t}$ Similarly, the morphism sending Y to $\mathrm{Hom}_X(-, Y)$ to the category of G-torsors on $X_{\acute{e}t}$ is an equivalence.*

Corollary 10. *There is a natural equivalence $\mathbf{Tors}(X_{\acute{e}t}, G) \cong \mathbf{Tors}(X_{\acute{E}t}, G)$ inducing a bijection of pointed sets $\mathrm{Tors}(X_{\acute{e}t}, G) \xrightarrow{\cong} \mathrm{Tors}(X_{\acute{E}t}, G)$ which is an isomorphism of abelian groups if G is abelian. And every G-torsor, in each of the above sites can be realized as a scheme, which is a prinicipal G-bundle.*

Constant Finite Group Torsors

Let G be a finite group, within this section, we denote the associated constant group scheme over X by $\underline{G}$ to maintain a clear distinction between the group as a set and the group as a scheme. In subsequent sections, we shall follow the standard practice of identifying G with its constant group scheme and omit the underline for brevity.

$\underline{G}$ is given by $\coprod_{g \in G} X$ with the action shuffeling the X's according to multiplication.

By following the definitions, one sees that if X be a connected scheme and $f : Y \to X$ is a finite Galois cover with Galois group G. Then, $f : Y \to X$ is a principal $\underline{G}$-bundle. (Recall that a *finite Galois cover* is a finite étale surjection $Y \to X$ with Y connected and such that $G = \mathrm{Aut}(Y/X)$ acts transitively on the geometric points of Y lying over any geometric point of X).

On the otherhand if $f : Y \to X$ is a principal $\underline{G}$-bundle with Y connected, then Y is a finite Galois cover with automorphism group G.

However, not all $\underline{G}$-torsors are connected. If $H \subset G$ is a proper subgroup then any connected finite étale cover $f : Y \to X$ with Galois group H gives rise to a non-connected $\underline{G}$-torsor by looking at the induced $\underline{G}$-torsor $\varphi_*(Y)$ under the inclusion $\varphi : \underline{H} \to \underline{G}$. On the otherhand, if we fix a geometric point $\overline{x} \to X$, then to give an homomorphism $\rho \in \mathrm{Hom}_{\mathrm{cont}}(\pi_1^{\mathrm{et}}(X,x), G)$ is equivilant to give a connected pointed Galois cover $(Y, \overline{y}) \to (X, \overline{x})$ with Galois group $H = \rho(\pi_1^{\mathrm{et}}(X,x)) \subset G$. Thus, pushing forward to $\underline{G}$ we get a prinicipal $\underline{G}$-bundle. The choice of a different geometric point $\overline{x'} \to X$ differ the homomorphism by an inner automorphism, thus we have:

Theorem 11. *Let X be a connected scheme and $\overline{x}$ a geometric point of X. Suppose in addition that G is a finite abstract group. Define a map*

$$\mathrm{Hom}_{\mathrm{cont}}(\pi_1^{\mathrm{et}}(X,\overline{x}), G)/\mathrm{Inn}(G) \to \mathrm{Tors}(X_{\acute{E}t}, \underline{G}) \tag{1}$$

by sending a homomorphism $\rho : \pi_1^{\mathrm{et}}(X,\overline{x}) \to G$ to the principal $\underline{G}$-bundle $\varphi_*(Y)$ where Y is the principal $\underline{\rho(\pi_1^{\mathrm{et}}(X,\overline{x}))}$-bundle obtained above and φ is the inclusion $\rho(\pi_1^{\mathrm{et}}(X,\overline{x})) \hookrightarrow G$. Then, the map is a bijection of pointed sets where the trivial homomorphisms (which is the only element of its $\mathrm{Inn}(G)$*-orbit) is the distinguished element of the left hand side.*

If G is abelian then $\mathrm{Inn}(G)$ is trivial, and we obtain:

Corollary 12. *Let G be a finite abelian group, X a connected scheme, and $\overline{x}$ a geometric point of X. Then, the map from Theorem 11 induces an isomorphism of abelian groups*

$$\mathrm{Hom}_{\mathrm{cont.}}(\pi_1^{\acute{e}t}(X,\overline{x}), G) \xrightarrow{\cong} \mathrm{Tors}(X_{\acute{E}t}, \underline{G}) \tag{2}$$

Decomposition of Torsors

In the case of finite groups, the functoriality of the contracted product has a concrete geometric interpretation as a tower of covers. Let G be a finite abelian group and $H \subseteq G$ a subgroup. Let $\pi : G \to G/H$ be the natural quotient map.

Proposition 13. *Let $\mathcal{P} \to X$ be a G-torsor in $X_{\acute{e}t}$. There exists a natural factorization of the morphism $\mathcal{P} \to X$:*

$$\mathcal{P} \xrightarrow{\phi} \mathcal{P}_{G/H} \xrightarrow{\psi} X$$

where:

1. *$\mathcal{P}_{G/H} := \mathcal{P} \times^G (G/H)$ is a G/H-torsor over X.*
2. *$\mathcal{P}$ is an H-torsor over the scheme $\mathcal{P}_{G/H}$.*

Proof. The first assertion follows directly from the definition of the contracted product functor $\pi_* :$ $\mathbf{Tors}(X,G) \to \mathbf{Tors}(X, G/H)$. To see the second assertion, we note that the map $\phi : \mathcal{P} \to \mathcal{P}_{G/H}$ is surjective. Locally on X, we may choose a cover $U \to X$ such that $\mathcal{P}|_U \cong G \times U$. Over this cover, the map ϕ is identified with the product of the quotient map and the identity:

$$\pi \times \mathrm{id}_U : G \times U \to (G/H) \times U$$

Since $G \to G/H$ is a trivial H-torsor (the fiber over any element $\bar{g} \in G/H$ is a coset gH, which is an H-set isomorphic to H acting on itself), $\mathcal{P}$ is locally an H-torsor over $\mathcal{P}_{G/H}$. By the descent of torsors, $\mathcal{P}$ is an H-torsor over $\mathcal{P}_{G/H}$ globally. ∎

2.3 Symmetric Powers of Schemes and Torsors

This section reviews the construction of quotients for schemes and torsors under finite group actions, specifically focusing on symmetric powers. To ensure these quotients exist as schemes, we utilize the framework of admissible actions from [SGA1]. Our treatment here closely follows the exposition in [Gui19] The definitions and results presented below are adapted from their work. This foundation provides the necessary criteria for admissibility and base change required to define the symmetric powers of a scheme X and a G-torsor $\mathcal{P}$ over X.

Let S be a scheme.

Definition 14 (([SGA1], V.1.7).)**.**

- Let T be an object of a category $\mathcal{C}$ endowed with a right action of a group Γ. We say that **the quotient** T/Γ **exists** in $\mathcal{C}$ if the covariant functor

$$\begin{aligned} \mathcal{C} &\to \text{Sets} \\ U &\mapsto \operatorname{Hom}_{\mathcal{C}}(T, U)^{\Gamma} \end{aligned}$$

  is representable by an object of $\mathcal{C}$.

- Let T be an S-scheme. An action of a finite group Γ on T is **admissible** if there exists an affine Γ-invariant morphism $f : T \to T'$ such that the canonical morphism $\mathcal{O}_{T'} \to f_*\mathcal{O}_T$ induces an isomorphism from $\mathcal{O}_{T'}$ to $(f_*\mathcal{O}_T)^{\Gamma}$.

Proposition 15. *The following holds:*

1. ***([SGA1] V.1.3).*** *Let T be an S-scheme endowed with an admissible right action of a finite group Γ. If $f : T \to T'$ is an affine Γ-invariant morphism such that the canonical morphism $\mathcal{O}_{T'} \to f_*\mathcal{O}_T$ induces an isomorphism from $\mathcal{O}_{T'}$ to $(f_*\mathcal{O}_T)^{\Gamma}$, then the quotient T/Γ exists and is isomorphic to T'.*

2. ***([SGA1], V.1.8).*** *Let T be an S-scheme endowed with a right action of a finite group Γ. Then, the action of Γ on T is admissible if and only if T is covered by Γ-invariant affine open subsets.*

3. ***([SGA1], V.1.9).*** *Let T be an S-scheme endowed with an admissible right action of a finite group Γ, and let S' be a flat S-scheme. Then, the action of Γ on the S'-scheme $T \times_S S'$ is admissible, and the canonical morphism*

$$(T \times_S S')/\Gamma \to (T/\Gamma) \times_S S'$$

   is an isomorphism.

Now let X be an S-scheme and let $d \geq 0$ be an integer. The group S_d of permutations of $[\![1, d]\!]$ acts on the right on the S-scheme $X^{\times_S d} = X \times_S \cdots \times_S X$ by the formula

$$(x_i)_{i \in [\![1,d]\!]} \cdot \sigma = (x_{\sigma(i)})_{i \in [\![1,d]\!]}.$$

Proposition 16 ([Gui19] Proposition 2.27)**.** *If X is a scheme, Zariski locally quasi-projective over S, then the right action of the symmetric group S_d on the d-fold fiber product $X^{\times_S d}$ is admissible. Consequently, the quotient $\operatorname{Sym}^d_S(X) = X^{\times_S d}/S_d$ exists as a scheme over S.*

Definition 17. Under the hypotheses of the proposition above, we define the **relative symmetric product** of X over S of degree d as the quotient

$$\mathrm{Sym}^d_S(X) \coloneqq X^{\times_S d}/S_d.$$

When the base scheme S is clear from the context, we shall denote this quotient by $X^{(d)}$.

Guingard shows that when $X = \mathrm{Spec}(B)$ and $S = \mathrm{Spec}(A)$ then $\mathrm{Sym}^d_S(X)$ is representable by an affine S-scheme (See [Gui19] Remark 2.28).

Proposition 18 ([Gui19] Proposition 2.28)**.** *If X is flat and Zariski-locally quasi-projective over S, then $Sym^d_S(X)$ is flat over S. Moreover, for any S-scheme S', the canonical morphism*

$$Sym^d_{S'}(X \times_S S') \to Sym^d_S(X) \times_S S'$$

is an isomorphism.

Now, let G be a finite abelian group, let $\mathcal{P}$ be a G-torsor over an S-scheme X in $S_{\text{Ét}}$.

Proposition 19 ([[SGA1]], IX.5.8)**.** *Assume that $\mathcal{P}$ and X are endowed with right actions from a finite group Γ such that the morphism $\mathcal{P} \to X$ is Γ-equivariant, and that the following properties hold:*

(a) The right Γ-action on $\mathcal{P}$ commutes with the left G-action.

(b) The right Γ-action on X is admissible, and the quotient morphism $X \to X/\Gamma$ is finite.

(c) For any geometric point $\bar{x}$ of X, the action of the stabilizer $\Gamma_{\bar{x}}$ of $\bar{x}$ in Γ on the fiber $\mathcal{P}_{\bar{x}}$ of $\mathcal{P}$ at $\bar{x}$ is trivial.

Then the action of Γ on $\mathcal{P}$ is admissible, and $\mathcal{P}/\Gamma$ is a G-torsor over X/Γ in $S_{\text{Ét}}$.

By Theorem 9, $\mathcal{P}$ is representable by a finite étale X-scheme. For each $i \in [\![1, d]\!]$ let $p_i : X^{\times_S d} \to X$ be the projection on i-th factor, and let us consider the G-torsor

$$p_1^{-1}\mathcal{P} \otimes \cdots \otimes p_d^{-1}\mathcal{P} = G_d \backslash \mathcal{P}^{\times_S d}$$

over $X^{\times_S d}$, where $G_d \subseteq G^d$ is the kernel of the multiplication morphism $G^d \to G$. The object $G_d \backslash \mathcal{P}^{\times_S d}$ of $S_{\text{Ét}}$ is too representable by an S-scheme which is finite étale over $X^{\times_S d}$. The group S_d acts on the right on $G_d \backslash \mathcal{P}^{\times_S d}$ by the formula

$$(p_i)_{i \in [\![1,d]\!]} \cdot \sigma = (p_{\sigma(i)})_{i \in [\![1,d]\!]}.$$

This action of S_d commutes with the left action of G on $G_d \backslash \mathcal{P}^{\times_S d}$.

Proposition 20 ([Gui19] Proposition 2.32.)**.** *If X is Zariski-locally quasi-projective on S, then the right action of S_d on $G_d \backslash \mathcal{P}^{\times_S d}$ is admissible, so that the quotient $\mathcal{P}^{(d)}$ of $G_d \backslash \mathcal{P}^{\times_S d}$ by S_d exists as a scheme over S. Moreover, the canonical morphism $\mathcal{P}^{(d)} \to \mathrm{Sym}^d_S(X)$ is a G-torsor, and the morphism*

$$p_1^{-1}\mathcal{P} \otimes \cdots \otimes p_d^{-1}\mathcal{P} \to r^{-1}\mathcal{P}^{(d)}$$

where $r : X^{\times_S d} \to \mathrm{Sym}^d_S(X)$ is the canonical projection, is an isomorphism of G-torsors over $X^{\times_S d}$.

The construction of symmetric powers is compatible with a natural addition law. For any integers $d_1, d_2 \geq 0$, there exists a canonical **addition morphism**

$$+_{d_1,d_2} : X^{(d_1)} \times_S X^{(d_2)} \to X^{(d_1+d_2)}$$

induced by the equivariant isomorphism $X^{\times_S d_1} \times_S X^{\times_S d_2} \cong X^{\times_S (d_1+d_2)}$ with respect to the inclusion of the product of symmetric groups $S_{d_1} \times S_{d_2} \subseteq S_{d_1+d_2}$.

Proposition 21. *Let $\mathcal{P}$ be a G-torsor over X. There is a canonical isomorphism of G-torsors over $X^{(d_1)} \times_S X^{(d_2)}$:*

$$+_{d_1,d_2}^{-1}\left(\mathcal{P}^{(d_1+d_2)}\right) \cong r_1^{-1}\mathcal{P}^{(d_1)} \otimes r_2^{-1}\mathcal{P}^{(d_2)}.$$

Proof. Let $\pi = r_{d_1} \times r_{d_2}$. By the commutativity of the following diagram

$$\begin{tikzcd} X^{\times_S d_1} \times_S X^{\times_S d_2} \arrow[r, "\sim"] \arrow[d, "\pi"'] & X^{\times_S (d_1+d_2)} \arrow[d, "r_{d_1+d_2}"] \\ X^{(d_1)} \times_S X^{(d_2)} \arrow[r, "+_{d_1,d_2}"] & X^{(d_1+d_2)} \end{tikzcd}$$

and applying Proposition 20, we obtain canonical isomorphisms:

$$\pi^{-1}\left(+_{d_1,d_2}^{-1}\mathcal{P}^{(d_1+d_2)}\right) \cong r_{d_1+d_2}^{-1}\mathcal{P}^{(d_1+d_2)} \cong \bigotimes_{i=1}^{d_1+d_2} p_i^{-1}\mathcal{P}$$

and

$$\pi^{-1}(r_1^{-1}\mathcal{P}^{(d_1)} \otimes r_2^{-1}\mathcal{P}^{(d_2)}) \cong \left(\bigotimes_{i=1}^{d_1} p_i^{-1}\mathcal{P}\right) \otimes \left(\bigotimes_{j=d_1+1}^{d_1+d_2} p_j^{-1}\mathcal{P}\right).$$

Since both pullbacks identify with $\bigotimes_{k=1}^{d_1+d_2} p_k^{-1}\mathcal{P}$ in an $S_{d_1} \times S_{d_2}$-equivariant manner, the isomorphism descends to the quotient $X^{(d_1)} \times_S X^{(d_2)}$ by Proposition 18. $\square$

2.4 Algebraic Preliminaries on Ramification

This section reviews the ramification of discrete valuations, beginning with the general theory of *discrete valuation rings* and their integral closures in finite separable extensions. We then specialize to complete rings in Galois extensions to describe the ramification filtration via lower and upper numbering, following the treatments in [Stacks, Tag 0EXQ] and [Ser79].

Definition 22 ([Stacks, Tag 09E4])**.** We say that $A \to B$ or $A \subset B$ is an extension of discrete valuation rings if A and B are discrete valuation rings and $A \to B$ is injective and local. In particular, if π_A and π_B are uniformizers of A and B, then $\pi_A = u\pi_B^e$ for some $e \geq 1$ and unit u of B. The integer e does not depend on the choice of the uniformizers as it is also the unique integer ≥ 1 such that

$$\mathfrak{m}_A B = \mathfrak{m}_B^e$$

The integer e is called the *ramification index* of B over A. We say that B is *weakly unramified* over A if $e = 1$. If the extension of residue fields $\kappa_A = A/\mathfrak{m}_A \subset \kappa_B = B/\mathfrak{m}_B$ is finite, then we set $f = [\kappa_B : \kappa_A]$ and we call it the *residual degree* or residue degree of the extension $A \subset B$.

Note that we do not require the extension of fraction fields to be finite.

Now let A be a discrete valuation ring with fraction field K, let L/K be a finite separable field extension and let $B \subset L$ be the integral closure of A in L. Then the ring extension $A \subset B$ is finite, hence B is Noetherian. The dimension of B is 1, hence B is a Dedekind domain. Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ be the maximal ideals of B (i.e., the primes lying over $\mathfrak{m}_A$). We obtain extensions of discrete valuation rings

$$A \subset B_{\mathfrak{m}_i}$$

and hence ramification indices e_i and residue degrees f_i. We have

$$[L : K] = \sum\nolimits_{i=1,\ldots,n} e_i f_i$$

Note that if A is henselian (e.g. A is complete) then $n = 1$.

Definition 23 ([Stacks, Tag 09E9])**.** Let A be a discrete valuation ring with fraction field K. Let L/K be a finite separable extension. With B and $\mathfrak{m}_i$, $i = 1, \ldots, n$ as above, we say the extension L/K is

1. unramified with respect to A if $e_i = 1$ and the extension $\kappa(\mathfrak{m}_i)/\kappa_A$ is separable for all i,
2. tamely ramified with respect to A if either the characteristic of κ_A is 0 or the characteristic of κ_A is $p > 0$, the field extensions $\kappa(\mathfrak{m}_i)/\kappa_A$ are separable, and the ramification indices e_i are prime to p, and
3. totally ramified with respect to A if $n = 1$ and the residue field extension $\kappa(\mathfrak{m}_1)/\kappa_A$ is trivial.

If the discrete valuation ring A is clear from context, then we sometimes say L/K is unramified, totally ramified, or tamely ramified for short.

Now and for the rest of the section assume A is a *complete discrete valuation ring* with uniformizer π and residue field κ, which we assume to be perfect. When A and κ are of the same characteristic $p > 0$, then A contains a coefficient field $k \cong \kappa$ and a well known structure theorem holds: $A = k[[\pi]] \cong k[[t]]$. Let K be the fraction field of A, then $K = k((\pi))$.

For the remainder of this section, assume A is a *complete discrete valuation ring* with uniformizer π and a perfect residue field κ. In the equal characteristic case, where $\mathrm{char}(A) = \mathrm{char}(\kappa) = p > 0$, the Cohen Structure Theorem implies that A contains a coefficient field $k \cong \kappa$, such that $A \cong k[[\pi]] \cong k[[t]]$. Consequently, the fraction field is given by $K = k((\pi))$. Let L/K be a finite Galois extension with Galois group G. The integral closure B of A in L is itself a complete discrete valuation ring. Since the residue field κ is perfect, the extension of residue fields is separable, and there exists a uniformizer $\Pi \in B$ such that $B = A[\Pi]$.

The *lower numbering ramification filtration* of G, denoted by $(G_i)_{i \geq -1}$, is defined as:

$$G_i = \{\sigma \in G \mid v_B(\sigma(x) - x) \geq i + 1 \text{ for all } x \in B\}$$

where v_B is the valuation on L associated with B. In this indexing, $G_{-1} = G$ and G_0 corresponds to the *inertia group* of the extension L/K. Note that L/K is unramified if and only if $G_0 = \{1\}$, and it is tamely ramified if and only if $G_1 = \{1\}$.

A standard result in the theory of local fields(complete local fields with a discrete valuation and perfect residue field) ensures that the condition in the definition of G_i need only be checked for the uniformizer Π of B. Specifically, if we define the function

$$i_K^L(\sigma) = v_B(\sigma(\Pi) - \Pi)$$

for $\sigma \in G$, then $G_i = \{\sigma \in G \mid i_K^L(\sigma) \geq i+1\}$. The subgroups G_i are normal in G and become trivial for sufficiently large i.

Consider a tower of fields $K \subset E \subset L$, and let $H = \mathrm{Gal}(L/E) \subset G$. The lower numbering is compatible with subgroups:

$$G_i \cap H = H_i \quad \text{for all } i \geq -1$$

which follows from the identity $i_E^L = i_K^L|_H$. However, the lower numbering does not behave as simply with respect to quotients. If H is normal in G, the image of G_i in G/H is generally not $(G/H)_i$. Instead, we have $G_iH/H = (G/H)_j$, where the index j is given by the formula:

$$j = \frac{1}{e_{L/E}} \sum_{\tau \in H} \min(i_K^L(\tau), i+1) - 1$$

In the literature, one reindexes the ramification groups by defining the Herbrand function $\phi_{L/K} : [-1, \infty) \to [-1, \infty)$:

$$\phi_{L/K}(i) = \frac{1}{e_{L/K}} \sum_{\sigma \in G} \min(i_K^L(\sigma), i+1) - 1 = \int_0^i \frac{1}{[G_0 : G_t]} dt$$

This function is continuous, strictly increasing, and piecewise linear, making it a bijection. It satisfies the composition law $\phi_{L/K} = \phi_{E/K} \circ \phi_{L/E}$ for a tower of extensions $K \subset E \subset L$. Using this function, we define the *upper numbering ramification groups* as:

$$G^i = G_{\phi_{L/K}^{-1}(i)}$$

The primary advantage of this reindexing is its compatibility with quotients: for any normal subgroup $H \subset G$, we have:

$$G^iH/H = (G/H)^i$$

for all $i \geq -1$.

Kummer and Artin-Schreier Theories

For cyclic extensions, Kummer and Artin-Schreier theories provide explicit ways to calculate ramification. These results show how the valuation of a defining element $a \in K$ determines the ramification index and the jumps in the upper numbering filtration. Throughout this section, let K be a discrete valuation field with a perfect residue field κ of characteristic $p > 0$.

Theorem 24 (Ramification in Kummer Extensions, [Koc97], Proposition 1.83)**.** *Assume K contains the n-th roots of unity μ_n. Let L/K be the extension given by the equation $X^n = a$ for some $a \in K^\times$ and denote by G its Galois group. Then we have:*

1. *If $v_K(a) \in n\mathbb{Z}$ and the image of $a\pi^{-v_K(a)}$ in the residue field κ is an n-th power, the extension L/K is trivial.*

2. *If $v_K(a) \in n\mathbb{Z}$ and the image of $a\pi^{-v_K(a)}$ in the residue field κ is not an n-th power, the extension L/K is cyclic and unramified.*

3. *If $v_K(a) \notin n\mathbb{Z}$, the extension L/K is cyclic and ramified. Specifically, if $\gcd(|v_K(a)|, n) = 1$, the extension is totally ramified of degree n. Otherwise it has ramification index $\frac{n}{|gcd(v_K(a)|,n)}$*

Conversly, Kummer theory ensures that every cyclic extension of degree n, prime to p of a field that contains n-th roots of unity, is of the above form. Moreover, in the above we can always take $a \in \mathcal{O}_K$

Note that in the case of total ramification the extesntion is tamely ramified.

Theorem 25 (Ramification in Artin-Schreier Extensions, [Tho05])**.** *Let $\wp(x) = x^p - x$ be the Artin-Schreier opeartor. Let L/K be the extension given by the equation $X^p - X = a$ for some $a \in K$ and denote by G its Galois group. Then we have:*

1. *If $v_K(a) > 0$ or if $v_K(a) = 0$ and $a \in \wp(K)$, the extension L/K is trivial.*

2. *If $v_K(a) = 0$ and if $a \notin \wp(K)$, the extension L/K is cyclic of degree p and unramified.*

3. *If $v_K(a) = -m < 0$ with $m \in \mathbb{Z}_{>0}$ and if m is prime to p, the extension L/K is cyclic of degree p again and totally ramified. Moreover, its ramification groups are given by:*

$$G = G^{(-1)} = \cdots = G^{(m)} \quad \text{and} \quad G^{(m+1)} = 1.$$

Conversely, Artin-Schreier theory ensures that every cyclic extension of degree p takes this form. Moreover, in the above and under the isomorphism $K \cong k((t))$, one can always take a of the form $ct^{-m} + a_{-m+1}t^{-m+1} + \cdots a_{-1}t^{-1} + a_0$. If k is algebraically closed then there is a change of variables such that $a = u^{-m}$.

2.5 Algebraic Geometry

In this section we group together some general theorems in algebraic geometry that we will be employing throughtout the text. All schemes are assumed to be locally of finite type.

Theorem 26. *Let $f : X \to Y$ be a finite flat map between integral schemes, of finite type over a field k. if $Z \subset X$ is a prime divisor with generic point η_Z, then $f(Z) \subset X$ is a prime divisor with generic point $\eta_{f(Z)}$ satisfying $f(\eta_Z) = \eta_{f(Z)}$*

Proof. f is finite hence proper hence closed so $f(Z)$ is closed subset of Y, it is irreducible as the image of an irreducible. Since $Z = \overline{\{\eta_Z\}}$ we get:

$$\{f(\eta_Z)\} \subset f(Z) = f(\overline{\{\eta_Z\}}) \subseteq \overline{f(\{\eta_Z\})} = \overline{\{f(\eta_Z)\}}$$

And since $f(Z)$ is closed we get $f(Z) = \overline{\{f(\eta_Z)\}}$.

For flat map of integral schemes we have for every $x \in X$, $y = f(x)$ the dimension formula:

$$\dim(\mathcal{O}_{X,x}) = \dim(\mathcal{O}_{Y,y}) + \dim(\mathcal{O}_{X_y,x})$$

And since $\dim(\mathcal{O}_{X_y,x}) = 0$ we get $\dim(\mathcal{O}_{X,x}) = \dim(\mathcal{O}_{Y,y})$ concluding that $f(Z)$ is a prime divisor as well. □

A known theorem states that:

Theorem 27. *Let X, Y be two integral schemes over a field k. If X is geometrically integral then $X \times_k Y$ is integral. If both X, Y are geometrically integral, then $X \times_k Y$ is geometrically integral.*

Theorem 28. *Let C be smooth projective curve geometrically connected over a field k. Then:*

1. *$C^{(d)}$ is smooth*
2. *For every d, $C^{(d)}$ is integral.*
3. *For every d, $C^{(d)}$ is geometrically integral.*
4. *The product of every finite number of $C^{(d)}$ is geometrically integral.*

Proof.

1. Let t_i be a local parameter for C at P_i. The local ring of the product C^d at the point $(P_1, \dots, P_d)$ is isomorphic $k[[t_1, t_2, \dots, t_d]]$ and the local ring of the qoutient at the divisor $D = \sum P_i$ is $k[[t_1, \dots, t_d]]^{S_d}$ which is isomorphic to $k[[t_1, \dots, t_d]]^{S_d} \cong k[[s_1, \dots, s_d]]$ where the s_i are the symmetric polynomials, hence this ring is regular local ring.
2. C is irreducible hence C^d is irreducible hence $C^{(d)}$ is irreducible. Since $C^{(d)}$ is smooth it is reduced.
3. By [Stacks, Tag 0366], C is geometrically integral, so it follows from the above.
4. Theorem 27

□

Blowups

Theorem 29 ([Stacks, Tag 0805])**.** *Let $X_1 \to X_2$ be a flat morphism of schemes. Let $Z_2 \subset X_2$ be a closed subscheme. Let Z_1 be the inverse image of Z_2 in X_1. Let X'_i be the blowup of Z_i in X_i. Then there exists a cartesian diagram*

$$\begin{tikzcd} X'_1 \arrow[r] \arrow[d] & X'_2 \arrow[d] \\ X_1 \arrow[r] & X_2 \end{tikzcd}$$

of schemes.

Theorem 30. *If X is integral then $Bl_Z(X)$ is integral.*

If X is a smooth curve, we have the following result regarding its symmetric power:

Proposition 31. *Let $f : X \to S$ be a smooth morphism of relative dimension 1. Suppose f is flat and Zariski-locally quasi-projective. Then the relative symmetric power $X_S^{(d)} = Sym_S^d(X)$ is smooth over S of relative dimension d.*

Proof. Since $f : X \to S$ is smooth, the d-fold product X_S^d is smooth over S. By Proposition 18, the quotient $X_S^{(d)}$ is flat over S. Smoothness is a fiberwise property for flat morphisms of finite presentation (cf. [Stacks, Tag 01V8]); thus, it suffices to verify the smoothness of the fibers. For every $s \in S$, the fiber of the symmetric power is given by:

$$(\mathrm{Sym}^d_S(X))_s \cong \mathrm{Sym}^d_{\kappa(s)}(X_s)$$

Consequently, we may reduce to the case where $S = \mathrm{Spec}(k)$ for a field k. Since smoothness is preserved under base change to the algebraic closure, we may further assume $k = \bar{k}$. Let $z = \sum_{i=1}^r d_i \cdot x_i \in X^{(d)}$ be a point represented by an effective cycle of degree d, where the points $x_i \in X(k)$ are distinct and $\sum d_i = d$. Choose pairwise disjoint Zariski-open subsets $U_i \subset X$ such that $x_i \in U_i$. Let $W \subset X^{(d)}$ be the open subset consisting of cycles whose support is contained in $\bigcup U_i$ and which meet each U_i with degree exactly d_i. There is a canonical isomorphism:

$$W \cong \prod_{i=1}^r U_i^{(d_i)}$$

Thus, it suffices to show that each $U_i^{(d_i)}$ is smooth of dimension d_i at the point $d_i \cdot x_i$.

We may therefore restrict our attention to the "worst" case: the diagonal point $z = d \cdot x$. Let $R = \mathcal{O}_{X,x}$ be the local ring of the curve at x. Since X is a smooth curve over k, R is a regular local ring of dimension 1, i.e., a discrete valuation ring. Its completion is $\hat{R} \cong k[[t]]$. The completed local ring of the product X^d at the point $(x, \dots, x)$ is:

$$\hat{\mathcal{O}}_{X^d,(x,\dots,x)} \cong \hat{R} \hat{\otimes}_k \dots \hat{\otimes}_k \hat{R} \cong k[[t_1, \dots, t_d]]$$

where t_i is the uniformizer for the i-th copy of X. The symmetric group S_d acts on this power series ring by permuting the variables t_i. By the *Fundamental Theorem of Symmetric Polynomials*, the ring of invariants is:

$$\left(k[[t_1, \dots, t_d]]\right)^{S^d} = k[[e_1, \dots, e_d]]$$

where e_j is the j-th elementary symmetric power series in the variables t_i.

The ring $k[[e_1, \dots, e_d]]$ is a formal power series ring in d variables. A fundamental theorem in commutative algebra states that a local ring is regular if and only if its completion is a formal power series ring over a field. Since $\hat{\mathcal{O}}_{X^{(d)},z} \cong k[[e_1, \dots, e_d]]$, the local ring $\mathcal{O}_{X^{(d)},z}$ is a regular local ring of dimension d. This implies that $X^{(d)}$ is smooth over k of dimension d, completing the proof. □

3 Ramification After Blowup Is Tame

3.1 Tame Ramification and Ramification of G-Torsors

Unramified scheme morphisms is not the same as unramified extensions of DVRs here, so be careful, and say something about that... Regarding Tame Ramification we follow [Stacks, Tag 0BSE]. It is worth mentioning [KS10] for the different notions of tameness in higher dimensions, and to what extent they agree.

Tame Ramification of etale covering in Codimension 1

Definition 32. Assume we are given:

1. a locally Noetherian scheme X,
2. a dense open $U \subset X$
3. a finite étale morphism $f : Y \to U$

such that for every prime divisor $Z \subset X$ with $Z \cap U = \emptyset$ the local ring $\mathcal{O}_{X,\xi}$ of X at the generic point ξ of Z is a discrete valuation ring. Setting K_ξ equal to the fraction field of $\mathcal{O}_{X,\xi}$ we obtain a cartesian square

$$\begin{tikzcd} \mathrm{Spec}(K_\xi) \arrow[r] \arrow[d] & U \arrow[d] \\ \mathrm{Spec}(\mathcal{O}_{X,\xi}) \arrow[r] & X \end{tikzcd}$$

of schemes. In particular, we see that $Y \times_U \mathrm{Spec}(K_\xi)$ is the spectrum of a finite separable algebra L_ξ/K_ξ. Then we say Y is unramified over X in codimension 1, resp. Y is tamely ramified over X in codimension 1 if L_ξ/K_ξ is unramified, resp. tamely ramified with respect to $\mathcal{O}_{X,\xi}$ for every (Z, ξ) as above, (Definition 23). More precisely, we decompose L_ξ into a product of finite separable field extensions of K_ξ and we require each of these to be unramified, resp. tamely ramified with respect to $\mathcal{O}_{X,\xi}$.

Ramification of G-Torsors over Curves

Let G be a finite abelian group. Let k be a perfect field and let C be a projective smooth geometrically connected curve over k, with genus g. Let $\mathfrak{m} = \sum_i n_i P_i$ be a modulus (i.e. an effective Cartier divisor) on C and let $U = C \setminus \mathfrak{m}$. Let $\mathcal{P}$ be a G-torsor in $U_{\text{ét}}$. By **??**, $\mathcal{P}$ is representable by a finite etale U-scheme.

Let $P \in \mathfrak{m} \subset C$ a closed point. Then $\mathcal{O}_{C,P}$ is a discrete valuation ring with fraction field K_P. After completion at the maximal ideal $\mathfrak{m}_P$ we obtain a complete discrete valuation ring $\widehat{\mathcal{O}_{C,P}}$ with fraction field $\widehat{K}_P$. Restricting the G-torsor $\mathcal{P}$ to $\mathrm{Spec}(K_P), \mathrm{Spec}(\widehat{K}_P)$ we obtain G-torsors in $\mathrm{Spec}(\widehat{K}_P)_{\text{Ét}}$, $\mathrm{Spec}(K_P)_{\text{Ét}}$ as in the diagram below:

$$\begin{tikzcd} \mathcal{P}|_{\mathrm{Spec}(\widehat{K}_P)} \arrow[d] & \mathcal{P}|_{\mathrm{Spec}(K_P)} \arrow[d] & \mathcal{P} \arrow[d] \\ \mathrm{Spec}(\widehat{K}_P) \arrow[r] \arrow[d] & \mathrm{Spec}(K_P) \arrow[r] \arrow[d] & U \arrow[d] \\ \mathrm{Spec}(\widehat{\mathcal{O}_{C,P}}) \arrow[r] & \mathrm{Spec}(\mathcal{O}_{C,P}) \arrow[r] & C \end{tikzcd}$$

The G-torsor $\mathcal{P}|_{\mathrm{Spec}(K_P)} \to \mathrm{Spec}(K_P)$ is an etale covering of $\mathrm{Spec}(K_P)$. Hence decompose into a disjoint union of spectra of finite separable field extensions of K_P.

$$\mathcal{P}|_{\mathrm{Spec}(K_P)} = \bigsqcup_i \mathrm{Spec}(M_i)$$

where each M_i/K_P is a finite separable field extension. Pulling back by $\mathrm{Spec}(\widehat{K_P})$ we get

$$\mathcal{P}|_{\mathrm{Spec}(\widehat{K}_P)} = \bigsqcup_i \mathrm{Spec}(M_i \otimes_{K_P} \widehat{K_P})$$

Each product $M_i \otimes_{K_P} \widehat{K_P}$ decomposes into a finite product of finite separable field extensions of $\widehat{K_P}$.

$$M_i \otimes_{K_P} \widehat{K_P} = \prod_Q \widehat{M}_{i,Q}$$

Where Q ranges over primes of M_i above P, and each $\widehat{M}_{i,Q}/\widehat{K_P}$ is a completion of M_i at that prime.

To summarize, we have decomposition:

$$\mathcal{P}|_{\mathrm{Spec}(\widehat{K}_P)} = \bigsqcup_i \mathrm{Spec}(F_i)$$

Where each $F_i/\widehat{K_P}$ is a finite separable field extension.

The fact that $\mathcal{P}$ is a G-torsor implies that:

1. The fields F_i are pairwise isomorphic
2. The fields F_i are Galois over $\widehat{K_P}$ with Galois group isomorphic to a subgroup of $H \subset G$.
3. The number of components F_i is equal to the index $[G : H]$.

We say the ramification of F_i over $\widehat{K_P}$ is bounded by r if the ramification group H^r (in upper numbering) is trivial.

We say that the G-torsor $\mathcal{P}|_{\mathrm{Spec}(\widehat{K}_P)}$ has ramification at P bounded by r if any of the $F_i/\widehat{K_P}$ has ramification bounded by r.

We say that the G-torsor $\mathcal{P}$ has ramification at P bounded by r if $\mathcal{P}|_{\mathrm{Spec}(\widehat{K}_P)}$ has ramification at P bounded by r.

Finally,

Definition 33. A G-torsor $\mathcal{P}$ on $U_{\text{ét}}$ has **ramification bounded by** $\mathfrak{m} = \sum n_i P_i$ **over** $\mathrm{Spec}(k)$ if for every i, the ramification of $\mathcal{P}|_{\mathrm{Spec}(\widehat{K}_{P_i})}$ at P_i is bounded by n_i.

Alternative Definition of Ramification of G-Torsors over Curves

Choose a geometric point $\bar{s} = \mathrm{Spec}(\bar{k}) \to \mathrm{Spec}(k)$. corresponding to a separable closure $k^{sep} = \bar{k}$ of k. By **??**, the higher ramification groups considered for $\mathcal{P}|_{\mathrm{Spec}(\widehat{K}_P)}$ and $\mathcal{P}_{\bar{k}}|_{\mathrm{Spec}(\widehat{K_P \otimes \bar{k}})}$ are isomorphic. Thus, we define:

Definition 34. A G-torsor $\mathcal{P}$ on $U_{\text{ét}}$ has **ramification bounded by** $\mathfrak{m}$ **over** $\mathrm{Spec}(k)$ if for every geometric point $\bar{x}$ of $\mathfrak{m}$, with image $\bar{s}$ in $\mathrm{Spec}(k)$, the restriction of $\mathcal{P}$ to

$$\mathrm{Spec}(\widehat{\mathcal{O}_{C_{\bar{k}},\bar{x}}}) \times_{C_{\bar{k}}} U_{\bar{k}}$$

has ramification bounded by the multiplicity of $\mathfrak{m}_{\bar{s}}$ at $\bar{x}$.

Explanation: The two definitions are equivalent. This is immediate as $\mathcal{O}_{C_{\bar{k}},\bar{x}}$ is the strict henselization of $\mathcal{O}_{C_{\bar{(k)}},P}$ So after completion it is $\widehat{\mathcal{O}_{C_{\bar{k}},\bar{x}}} \cong \widehat{\mathcal{O}_{C,P}} \otimes_k \bar{k}$. And taking the product with $Y_{\bar{k}}$ amounts to taking the fraction fields, i.e. we get $\mathrm{Spec}(\widehat{K_P \otimes \bar{k}}) = \mathrm{Spec}(\widehat{K_P} \otimes_k \bar{k})$.

Note that tame ramification and unramifiedness in terms of definition above coincide with the ones in Definition 32.

Ramification of G-torsors in terms of Characters

Since we are working over $X = \mathrm{Spec}(k)$, the group G is *etale* over $\mathrm{Spec}(k)$ Hence by Corollary 12 and **??** we get an isomosphism of groups: $\mathrm{Hom}_{\mathrm{cont.}}(\pi_1^{\text{ét}}(X, \overline{x}), G) \xrightarrow{\cong} \mathrm{Tors}(X_{\mathrm{et}}, G)$

When $X = \mathrm{Spec}(L)$ for a complete valued field L, $\pi_1^{\text{ét}}(X, \overline{x}) = G_L := Gal(L^{sep}/L)$ Where L^{sep} is a fixed seprable closure. And we conclude that $\mathcal{P}|_{\mathrm{Spec}(L)}$ correspond to a continious homomorphism $\rho : G_L \to G$ and one can check that it has ramificaiton bounded by r if and only if $\rho(G_L^r) = \{1\}$.

Basic Properties of Ramification of G-Torsors

In this section we prove some basic properties of the ramification of G-torsors.

Lemma 35. *Let G be a finite abelian group and X be a locally Noetherian scheme over a field k. Let $U \subset X$ be a dense open subset and let Z be a prime divisor in the complement $X \setminus U$, and let ξ denote its generic point.*

Assume $\mathcal{P}_1$ and $\mathcal{P}_2$ are two G-torsors on $U_{ét}$, Such that $\mathcal{P}_1$ has ramification bounded by r_1 at (Z, ξ), and $\mathcal{P}_2$ has ramification bounded by r_2 at (Z, ξ). Then their contracted product $\mathcal{P}_1 \wedge^G \mathcal{P}_2$ has ramification bounded by $max(r_1, r_2)$ at (Z, ξ).

Proof. Let $A = \widehat{\mathcal{O}_{X,\xi}}$ and let $K = Frac(A)$. Let $\rho_1, \rho_2 : G_K \to G$ be the associated continious homomorphisms corresponding to the G-torsors $\mathcal{P}_1|_{\mathrm{Spec}(K)}$, $\mathcal{P}_2|_{\mathrm{Spec}(K)}$. Then the associated character to $(\mathcal{P}_1 \wedge^G \mathcal{P}_2)|_{\mathrm{Spec}(K)}$ is $\rho = \rho_1 + \rho_2$. And the claim follows by Section 3.1 $\square$

Lemma 36. *Let $f : X \to Y$ be a morphism of locally Noetherian schemes. Let $U_X \subset X$ and $U_Y \subset Y$ be dense open subschemes such that $f^{-1}(U_Y) \subset U_X$.*

Let Z_X and Z_Y be prime divisors of X and Y with generic points η_X and η_Y, respectively, such that $Z_X \cap U_X = \emptyset$ and $Z_Y \cap U_Y = \emptyset$. Suppose $f(\eta_X) = \eta_Y$ and that f is étale at η_X.

Let $\mathcal{P}$ be a G-torsor over U_Y, and let $f^{-1}\mathcal{P}$ be its pullback to $f^{-1}(U_Y) \subset U_X$. Then, the ramification of $f^{-1}\mathcal{P}$ is bounded by r at η_X if and only if the ramification of $\mathcal{P}$ is bounded by r at η_Y.

Proof. The boundedness of ramification is determined by the behavior of the torsor over the completion of the local rings at the generic points.

Let $A = \mathcal{O}_{Y,\eta_Y}$ and $B = \mathcal{O}_{X,\eta_X}$ be the discrete valuation rings at the generic points, with fraction fields K and L respectively. Since f is étale at η_X, the map $A \to B$ is a flat, unramified local homomorphism. Consequently, the extension of completions $\widehat{L}/\widehat{K}$ is a finite unramified extension of complete discretely valued fields.

The upper numbering filtration on the absolute Galois group is compatible with unramified base change. Specifically, let $G_K = \mathrm{Gal}(K^{sep}/K)$ and $G_L = \mathrm{Gal}(L^{sep}/L)$. For an unramified extension, the Herbrand function is the identity, which implies that for any $r \geq 0$:

$$G_L^r = G_K^r \cap G_L$$

The ramification of the G-torsor $\mathcal{P}$ is bounded by r if and only if the corresponding Galois representation $\rho : G_K \to G$ satisfies $\rho(G_K^r) = \{1\}$.

By the filtration identity above, $\rho(G_L^r) = \{1\}$ if and only if $\rho(G_K^r) = \{1\}$. Thus, the pullback torsor $f^{-1}\mathcal{P}$ has ramification bounded by r at η_X if and only if $\mathcal{P}$ has ramification bounded by r at η_Y. □

3.2 Behavior of Ramification under Product of Blowups

Let X and Y be smooth schemes over a field k, and let $x \in X$ and $y \in Y$ be closed points. We denote the blowups of these schemes at the given points by $\pi_X : \mathrm{Bl}_x(X) \to X$ and $\pi_Y : \mathrm{Bl}_y(Y) \to Y$. Furthermore, let $\pi_{X\times Y} : \mathrm{Bl}_{(x,y)}(X \times_k Y) \to X \times_k Y$ be the blowup of the product scheme at the point (x, y). We denote by E_X, E_Y, and $E_{X\times Y}$ the respective exceptional divisors, and let η_X, η_Y, and $\eta_{X\times Y}$ be their generic points.

In this section, we establish the following result concerning the stability of ramification bounds under the external product of torsors.

Proposition 37. *Let G be a finite abelian group. Suppose $\mathcal{G}_X$ and $\mathcal{G}_Y$ are G-torsors defined on open subsets $U_X \subset \mathrm{Bl}_x(X)$ and $U_Y \subset \mathrm{Bl}_y(Y)$ that are disjoint from the exceptional divisors. If the ramification of $\mathcal{G}_X$ at η_X and $\mathcal{G}_Y$ at η_Y is bounded by r, then the external product torsors*

$$\mathcal{G}_{X\times Y} := pr_1^{-1}\mathcal{G}_X \otimes pr_2^{-1}\mathcal{G}_Y$$

has ramification bounded by r at the generic point $\eta_{X\times Y}$ of the exceptional divisor in the product blowup.

The proposition is purely local in nature, it suffices to consider the case where X and Y are affine. More precisely, by the smoothness of X and Y, we may restrict our attention to open neighborhoods of x and y that are isomorphic to affine spaces. The rest of this section treats that case.

The Affine Case

Let $X = \mathbb{A}_k^n$ be the affine n-space over a field k, and let $0 \in X$ be the origin. Let $\tilde{X} = \mathrm{Bl}_0(X)$ be the blowup of X at the origin. Recall that $\tilde{X} \subset X \times_k \mathbb{P}_k^{n-1}$ is defined by the equations $x_i u_j = x_j u_i$, where $[u_1 : \cdots : u_n]$ are the homogeneous coordinates of $\mathbb{P}_k^{n-1}$. The exceptional divisor $E \subset \tilde{X}$ is the fiber over the origin, $E = \{(0, [u_1 : \cdots : u_n])\}$, which is of codimension 1 in $\tilde{X}$. Let $\eta \in E$ be the generic point of E, and let $R = \mathcal{O}_{\tilde{X},\eta}$ be the associated local ring. This ring R is a discrete valuation ring (DVR) with fraction field $K = K(X) = k(x_1, \ldots, x_n)$.

On the affine chart U_1 where $u_1 \neq 0$, we have $x_i = \frac{u_i}{u_1}x_1$. The coordinate ring is:

$$\mathcal{O}_{\tilde{X}}(U_1) = k\left[x_1, \frac{u_2}{u_1}, \ldots, \frac{u_n}{u_1}\right]$$

In this chart, the generic point η corresponds to the prime ideal $\mathfrak{p}_1 = (x_1)$. Thus, the local ring is $R = k[x_1, \frac{u_2}{u_1}, \ldots, \frac{u_n}{u_1}]_{(x_1)}$. The residue field is $\kappa(\eta) = k(\frac{u_2}{u_1}, \ldots, \frac{u_n}{u_1})$, and the completion of R with respect to its maximal ideal is:

$$\hat{R} = \kappa(\eta)[[x_1]]$$

In this local ring, x_1 is a uniformizer. Note that any x_i (for $i > 1$) can also serve as a uniformizer, as $x_i = (\frac{u_i}{u_1})x_1$ and $\frac{u_i}{u_1}$ is a unit in R.

For any monomial $M = x_1^{a_1} \dots x_n^{a_n} \in K$, we can write:

$$M = x_1^{\sum a_i} \left(\frac{u_2}{u_1}\right)^{a_2} \dots \left(\frac{u_n}{u_1}\right)^{a_n}$$

Since the term in parentheses is a unit in R, the valuation ν_E associated with E satisfies:

$$\nu_E(M) = \sum a_i = \deg M$$

Consequently, for any polynomial $f = f_d + f_{d+1} + \cdots + f_l$, where f_i is the homogeneous part of degree i, we have $\nu_E(f) = d$ (the order of vanishing at the origin).

The Product Case

Now, let $X = \mathbb{A}^n$ and $Y = \mathbb{A}^m$ with origins $x = 0$ and $y = 0$. As before, $\mathrm{Bl}_0(X) \subset X \times \mathbb{P}^{n-1}$ and $\mathrm{Bl}_0(Y) \subset Y \times \mathbb{P}^{m-1}$ have exceptional divisors E_X and E_Y respectively. Consider the product $X \times_k Y \cong \mathbb{A}_k^{n+m}$. The blowup of the product at the origin $(0,0)$, denoted $\mathrm{Bl}_{(0,0)}(X \times_k Y)$, is a subscheme of $(X \times Y) \times \mathbb{P}^{n+m-1}$ defined by:

$$\begin{cases} x_i w_j = x_j w_i & 1 \le i, j \le n \\ y_k w_{n+l} = y_l w_{n+k} & 1 \le k, l \le m \\ x_i w_{n+j} = y_j w_i & 1 \le i \le n,\ 1 \le j \le m \end{cases}$$

where $[w_1 : \cdots : w_{n+m}]$ are the homogeneous coordinates of $\mathbb{P}^{n+m-1}$. The exceptional divisor $E_{X \times Y}$ is isomorphic to $\mathbb{P}^{n+m-1}$.

Comparison of Blowups

Both $\mathrm{Bl}_0(X) \times_k \mathrm{Bl}_0(Y)$ and $\mathrm{Bl}_{(0,0)}(X \times_k Y)$ are birational to $X \times_k Y$. They share a common dense open set $\tilde{U}$ defined by the condition that neither the X-coordinates nor the Y-coordinates vanish simultaneously in the projective space:

$$\tilde{U} = \{((x, y), [w_1 : \cdots : w_{n+m}]) \mid (w_1, \dots, w_n) \neq 0 \text{ and } (w_{n+1}, \dots, w_{n+m}) \neq 0\}$$

This yields a diagram of open immersions:

$$\begin{tikzcd} & \tilde{U} \arrow[dl, hook', "f_1"'] \arrow[dr, hook, "f_2"] & \\ \mathrm{Bl}_0(X) \times_k \mathrm{Bl}_0(Y) & & \mathrm{Bl}_{(0,0)}(X \times_k Y) \end{tikzcd}$$

where f_1 maps the coordinates to the respective projectivizations $[w_1 : \cdots : w_n]$ and $[w_{n+1} : \cdots : w_{n+m}]$. Also note that the generic point $\eta_{X \times Y}$ of $E_{X \times_k Y}$ is inside $\tilde{U}$

Extensions of DVRs

Let S be the local ring of the generic point $\eta_{X\times Y}$ of $E_{X\times Y}$ in $\tilde{U}$. On the chart where $w_1 \neq 0$ and $w_{n+1} \neq 0$, we have $x_1 = (\frac{w_1}{w_{n+1}})y_1$. Since $\frac{w_1}{w_{n+1}}$ is a unit in this chart, x_1 and y_1 are equivalent as uniformizers. We have:

$$S = k\left[x_1, \frac{w_2}{w_1} \cdots, \frac{w_n}{w_1}, \frac{w_{n+1}}{w_1} \cdots \frac{w_{n+m}}{w_1}\right]_{(x_1)} = k\left[\frac{w_1}{w_{n+1}}, \frac{w_2}{w_{n+1}} \cdots \frac{w_n}{w_{n+1}}, y_1 \cdots \frac{w_{n+m}}{w_{n+1}}\right]_{(y_1)}$$

$$k(\eta_{X\times Y}) = k\left(\frac{w_2}{w_1} \cdots, \frac{w_n}{w_1}, \frac{w_{n+1}}{w_1} \cdots \frac{w_{n+m}}{w_1}\right)$$

Let R_X be the local ring of the exceptional divisor E_X in $\mathrm{Bl}_0(X)$. The pullback of $E_X \times_k \mathrm{Bl}_0(Y)$ along f_1 induces an extension of DVRs $R_X \hookrightarrow S$. Which is:

1. Weakly Unramified: x_1 is a uniformizer in both R_X and S, so the ramification index is $e = 1$.
2. Residually Transcendental: The residue field extension $\kappa(\eta_X) \subset \kappa(\eta)$ is:

$$k\left(\frac{w_2}{w_1}, \dots, \frac{w_n}{w_1}\right) \subset k\left(\frac{w_2}{w_1}, \dots, \frac{w_n}{w_1}, \frac{w_{n+1}}{w_1}, \dots, \frac{w_{n+m}}{w_1}\right)$$

   Hence separable.

Since this extension is generated by transcendental elements, it is separable and formally smooth at the maximal ideal ([Stacks, Tag 09E7]).

Ramification of G-Torsors

Let G be a finite abelian group. Let $\mathcal{Q}$ be a G-torsor on an open $U \subset \mathrm{Bl}_0(X)$ disjoint from E_X. Let $V \subset \mathrm{Bl}_0(Y)$ be an open subscheme, and let $\pi_X : \mathrm{Bl}_0(X) \times_k V \to \mathrm{Bl}_0(X)$ be the projection onto the first factor. By restricting this projection to $U \times_k V$, we obtain the pullback G-torsor:

$$\pi_X^{-1}(\mathcal{Q}) \cong \mathcal{Q} \times_k V$$

which is defined on the open subset $U \times_k V \subset \mathrm{Bl}_0(X) \times_k \mathrm{Bl}_0(Y)$. The extension of local rings $R_X \to S$ is weakly unramified (the ramification index $e = 1$) and residually transcendental with seprable residue field exntesion. Under these conditions the ramification filtration is preserved. Therefore, the pullback $\pi_X^{-1}(\mathcal{Q})$ has ramification bounded by r at the generic point $\eta_{X\times Y}$ of the exceptional divisor $E_{X\times Y}$ if and only if the original torsor $\mathcal{Q}$ has ramification bounded by r at the generic point η_X of E_X

And we finish by Lemma 35.

4 Geometric Class Field Theory

4.1 Etale Fundamental Groups and Tame Fundamental Groups

We recall the definition and basic properties of the etale fundamental group, following stacks project [Stacks, Tag 0BQ6]

Proposition 38 ([Stacks, Tag 0C0J]). *Let $f : X \to S$ be a flat proper morphism of finite presentation whose geometric fibres are connected and reduced. Assume S is connected and let $\overline{s}$ be a geometric point of S. Then there is an exact sequence*

$$\pi_1(X_{\overline{s}}) \to \pi_1(X) \to \pi_1(S) \to 1$$

of fundamental groups.

Corollary 39. *Let $f : X \to S$ be a proper smooth morphism of finite presentation whose geometric fibres are connected. Assume S is connected and let $\overline{s}$ be a geometric point of S. Then there is an exact sequence*

$$\pi_1(X_{\overline{s}}) \to \pi_1(X) \to \pi_1(S) \to 1$$

of fundamental groups.

add about tameness?

4.2 Generalized Picard Scheme

In this section, we recall the notion of generalized Jacobian varieties and study their fundamental properties. The material presented here is primarily adapted from [Gui19] and [Tak19]. For further background on the general theory of abelian varieties and Jacobians, the reader may also consult [Mil08]. Let S be a scheme and let C be a projective smooth S-scheme whose geometric fibers are connected and of dimension 1. Let $\mathfrak{m}$ be a modulus on C, defined as an effective Cartier divisor of C/S (i.e., a closed subscheme of C which is finite flat of finite presentation over S). We denote the projection $C \times_S T \to T$ by pr for any S-scheme T.

The Functor of Points

Let d be an integer. For an S-scheme T, we consider the set of data $(\mathcal{L}, \psi)$ where:

- $\mathcal{L}$ is an invertible sheaf of degree d on C_T.
- $\psi : \mathcal{O}_{\mathfrak{m}_T} \xrightarrow{\sim} \mathcal{L}|_{\mathfrak{m}_T}$ is a trivialization of $\mathcal{L}$ along the modulus.

Two such pairs $(\mathcal{L}, \psi)$ and $(\mathcal{L}', \psi')$ are said to be isomorphic if there exists an isomorphism of invertible sheaves $f : \mathcal{L} \to \mathcal{L}'$ such that the following diagram commutes:

$$\begin{tikzcd} & \mathcal{O}_{\mathfrak{m}_T} \arrow[dl, "\psi'"'] \arrow[dr, "\psi"] & \\ \mathcal{L}'|_{\mathfrak{m}_T} \arrow[rr, "f|_{\mathfrak{m}_T}"] & & \mathcal{L}|_{\mathfrak{m}_T} \end{tikzcd}$$

We define the presheaf $\mathrm{Pic}^{d,\mathrm{pre}}_{C,\mathfrak{m}}$ on Sch$/S$ by assigning to T the set of isomorphism classes of such pairs. Let $\mathrm{Pic}^{d}_{C,\mathfrak{m}}$ denote the étale sheafification of this presheaf.

Representability and Structure

The fundamental properties of this functor are as follows:

1. $\mathrm{Pic}^d_{C,\mathfrak{m}}$ is represented by an S-scheme. (Note: If $\mathfrak{m}$ is faithfully flat over S, the presheaf is already a étale sheaf).
2. $\mathrm{Pic}^0_{C,\mathfrak{m}}$ is a smooth commutative group S-scheme with geometrically connected fibers, referred to as the *generalized Jacobian variety* of C with modulus $\mathfrak{m}$.
3. For any d, $\mathrm{Pic}^d_{C,\mathfrak{m}}$ is a $\mathrm{Pic}^0_{C,\mathfrak{m}}$-torsor.

In the case where $\mathfrak{m} = 0$, we recover the standard Jacobian variety, denoted simply as Pic^d_C.

Relation to the Standard Jacobian

We now examine the behavior of the generalized Picard scheme under the variation of the modulus. By viewing the structure along the modulus as an additional rigidification, we obtain natural transition maps corresponding to the inclusion of moduli.

Let $\mathfrak{m}_1$ and $\mathfrak{m}_2$ be moduli such that $\mathfrak{m}_1 \subset \mathfrak{m}_2$. There exists a natural map

$$\mathrm{Pic}^d_{C,\mathfrak{m}_2} \to \mathrm{Pic}^d_{C,\mathfrak{m}_1}$$

obtained by restricting the isomorphism ψ. Since $\mathfrak{m}_2$ is a finite S-scheme, this map is a surjection as a morphism of étale sheaves. In particular, for any modulus $\mathfrak{m}$, there is a natural surjective morphism of étale sheaves:

$$\mathrm{Pic}^d_{C,\mathfrak{m}} \to \mathrm{Pic}^d_C.$$

Local Freeness and Base Change

Let $\mathfrak{m}$ be a modulus which is everywhere strictly positive. Let g denote the genus of C, which is a locally constant function on S. We restrict our attention to degrees d satisfying the condition:

$$d \geq \max\{2g - 1 + \deg \mathfrak{m}, \deg \mathfrak{m}\}. \tag{3}$$

Assuming S is quasi-compact, such a d always exists.

Fix an integer d satisfying the condition above. Let T be an S-scheme and let $\mathcal{L}$ be an invertible sheaf of degree d on C_T. One can show that the pushforwards $\mathrm{pr}_*\mathcal{L}$ and $\mathrm{pr}_*\mathcal{L}(-\mathfrak{m})$ are locally free sheaves and their formations commute with any base change. Explicitly, for any morphism of S-schemes $f : T' \to T$, the base change morphisms are isomorphisms:

$$f^*\mathrm{pr}_*\mathcal{L} \xrightarrow{\sim} \mathrm{pr}_* f^*\mathcal{L}$$

and

$$f^*\mathrm{pr}_*(\mathcal{L}(-\mathfrak{m})) \xrightarrow{\sim} \mathrm{pr}_* f^*(\mathcal{L}(-\mathfrak{m})).$$

In particular, following [Gui19], if $\mathcal{L}$ is invertible $\mathcal{O}_C$-module with degree d satisfying 3 on each fiber of f then, $\mathrm{pr}_*\mathcal{L}$ is a locally free $\mathcal{O}_S$-module of rank $d - g + 1$.

For further background and verification of these constructions, we refer the reader to Milne's notes on abelian Varieties ([Mil08]).

4.3 The Abel-Jacobi Morphism and its Fibers

Let $U = C\backslash\mathfrak{m}$ be the complement of the modulus in C. The effective cartier divisors of degree d which are prime to $\mathfrak{m}$ are parameterized by the symmetric power $\mathrm{Sym}^d_S(U) = U^{(d)}$ over S (See [Gui19] Proposition 4.12, [Mil08] Theorem 3.13). For any such divisor $D \in U^{(d)}$, the associated line bundle $\mathcal{O}_C(D)$ admits a canonical trivialization along $\mathfrak{m}$. Specifically, the canonical section 1_D is regular and non-vanishing on $\mathfrak{m}$ because $\mathrm{supp}(D) \cap \mathrm{supp}(\mathfrak{m}) = \emptyset$. This section restricts to a nowhere-vanishing section on the subscheme $\mathfrak{m}$, thereby determining a trivialization $\psi_D^{-1} : \mathcal{O}_C(D)|_{\mathfrak{m}} \xrightarrow{\sim} \mathcal{O}_{\mathfrak{m}}$. This is done functorially in families, yielding a morphism from the symmetric power to the generalized Picard scheme (over S):

$$\Phi_d : U^{(d)} \to \mathrm{Pic}^d_{C,\mathfrak{m}}, \quad D \mapsto [(\mathcal{O}_C(D), \psi_D)], \tag{4}$$

When $\mathfrak{m} = 0$, $d \geq max\{2g-1, 0\}$ and C admits a section over S, $C^{(d)}$ is a projective space bundle over Pic^d_C, It is proper, surjective with geometrically connected fibers.

Guignard ([Gui19] Theorem 4.14) proves that for $\mathfrak{m} > 0$ and d satisfying (3), the Abel-Jacobi morphism Φ_d is surjective smooth of relative dimension $d - \deg\mathfrak{m} - g + 1$, with geometrically connected fibers.

When $S = \mathrm{spec}(k)$, the geometric-fibers of Φ_d are well understood:

Theorem 40. *Assuming $S = \mathrm{spec}(k)$ and $d \geq max\{2g-1+\deg\mathfrak{m}, \deg\mathfrak{m}\}$. Then, the geometric-fibers of the Abel-Jacobi morphism*

$$\Phi_d : U^{(d)} \to \mathrm{Pic}^d_{C,\mathfrak{m}}$$

over any point are isomorphic to

$$\begin{cases} \mathbb{A}^{d-\deg\mathfrak{m}-g+1}_{k^{sep}} & \textit{if } m > 0 \\ \mathbb{P}^{d-g}_{k^{sep}} & \textit{if } m = 0 \end{cases}$$

In both cases Φ_d is a fibration in affine spaces or projective spaces, depending on whether $\mathfrak{m}$ is non-zero or zero.

Proof. see [Ten15] Propositions 3.13-3.14, or [Tót11] Prop 2.1.4: □

4.4 Compactification of Blowup of Symmetric Powers of a Curve

We recall that our objective is to descend the local system $\mathcal{F}^{(d)}$ from $U^{(d)}$ to $\mathrm{Pic}^d_{C,\mathfrak{m}}$ along the Abel-Jacobi map Φ_d:

$$\begin{tikzcd} \mathcal{F}^{(d)} \arrow[d] & \\ U^{(d)} \arrow[r, "\Phi_d"] & \mathrm{Pic}^d_{C,\mathfrak{m}} \end{tikzcd}$$

(Here, the purple arrow emphasizes that the morphism is of sheaves on the étale site).

However, we encounter an obstruction: in the case we are considering ($\mathfrak{m} > 0$), the fibers of Φ_d are affine spaces (of the same degree) rather than the better-behaved projective spaces. This hint that a solution to this problem is to compactify the morphism to yield projective fibers.

This section describes the result of the compactification constructed by [Tak19] via the method of blowup.

Let $\mathfrak{m} = \sum_{i=1}^{n} k_P P$ with $\deg P = d_P$ be a modulus on C, and let d satisfy (3). Takeuchi ([Tak19]) defines $Z_0 = Z_0(\mathfrak{m}, d)$ as the closed subscheme of $C^{(d)}$ defined by the map $C^{(d-\deg \mathfrak{m})} \to C^{(d)}$ adding $\mathfrak{m}$. He also defines $X_{\mathfrak{m},d}$ as the blowup of $C^{(d)}$ along Z_0. Let $E_0 = E_{\mathfrak{m},d} = Z_0(\mathfrak{m}, d) \times_{C^{(d)}} X_{\mathfrak{m},d}$ be the exceptional divisor of the blowup. It is irreducible of codimension 1, and we let $\eta_0 = \eta_{\mathfrak{m},d}$ be its generic point.

Diagrammatically:

$$\begin{tikzcd}
\overline{\{\eta_0\}} = E_0 \arrow[r] \arrow[d] & X_{\mathfrak{m},d} \arrow[d, "\pi"] \\
Z_0 \arrow[r, "c.i"] & C^{(d)}
\end{tikzcd}$$

Incoporating $U^{(d)}$, the local system $\mathcal{F}^{(d)}$ and the Abel-Jacobi map, we have:

$$\begin{tikzcd}
 & & \mathcal{F}^{(d)} \arrow[d] & \\
\overline{\{\eta_0\}} = E_0 \arrow[r] \arrow[d] & X_{\mathfrak{m},d} \arrow[d, "\pi"] & U^{(d)} \arrow[l, hook'] \arrow[r, "\Phi_d"] \arrow[dl, hook'] & \mathrm{Pic}^d_{C,\mathfrak{m}} \\
Z_0 \arrow[r, "c.i"] & C^{(d)} & &
\end{tikzcd}$$

In Section 3 of [Tak19] Takeuchi constructs, for large enough d a compactification denoted by $\tilde{C}^{(d)}_{\mathfrak{m}}$ and proves the following: exactly determined the fate of that d

Theorem 41 (Takeuchi)**.** *The scheme $\tilde{C}^{(d)}_{\mathfrak{m}}$ is an open subscheme of $X_{\mathfrak{m},d}$ containing $U^{(d)}$. The morphism $\Phi_d : U^{(d)} \to \mathrm{Pic}^d_{C,\mathfrak{m}}$ extends to a morphism $\tilde{\Phi}_d : \tilde{C}^{(d)}_{\mathfrak{m}} \to \mathrm{Pic}^d_{C,\mathfrak{m}}$ which makes $\tilde{C}^{(d)}_{\mathfrak{m}}$ a projective space bundle over $\mathrm{Pic}^d_{C,\mathfrak{m}}$. Furthermore, the complement of $U^{(d)}$ in $\tilde{C}^{(d)}_{\mathfrak{m}}$ is isomorphic to the fiber product $E_0 \times_{C^{(d)}} \tilde{C}^{(d)}_{\mathfrak{m}}$.*

Proof. Add outline of construction and proofs □

Diagrammatically we have:

$$\begin{tikzcd}
 & & \mathcal{F}^{(d)} \arrow[d] \\
E_0 \times_{C^{(d)}} \tilde{C}_{\mathfrak{m}}^{(d)} \arrow[r] \arrow[d] & \tilde{C}_{\mathfrak{m}}^{(d)} \arrow[d] \arrow[dr, "\tilde{\Phi}_d"] & U^{(d)} \arrow[l, hook'] \arrow[d, "\Phi_d"] \\
\overline{\{\eta_0\}} = E_0 \arrow[r] \arrow[d] & X_{\mathfrak{m},d} \arrow[d, "\pi"] & \mathrm{Pic}^d_{C,\mathfrak{m}} \\
Z_0 \arrow[r, "c.i"] & C^{(d)} &
\end{tikzcd}$$

Also note that $E_0 \times_{C^{(d)}} \tilde{C}_{\mathfrak{m}}^{(d)} = Z_0 \times_{C^{(d)}} \tilde{C}_{\mathfrak{m}}^{(d)}$

5 Ramification of Sheaves after Blowup

do we assume here $S = k$? The main theorem of this section is

Theorem 42. *Let Λ be a finite ring of cardinality invertible in k, and let $\mathcal{F}$ be an étale sheaf of Λ-modules, locally free of rank 1 on U, with ramification bounded by $\mathfrak{m}$. Considering $U^{(d)}$ as an open subscheme of the blowup $\tilde{C}_{\mathfrak{m}}^{(d)}$ of $C^{(d)}$, we have that for sufficiently large integer d, $\mathcal{F}^{(d)}$ is tamely ramified on $H = \tilde{C}_{\mathfrak{m}}^{(d)} \setminus U^{(d)} = E_0 \times_{C^{(d)}} \tilde{C}_{\mathfrak{m}}^{(d)}$.*

Following the notation of Section 4.4, For any modulus $\mathfrak{n} \subset \mathfrak{m}$, we define $Z_{\mathfrak{n}}$ as the closed subscheme of $C^{(\deg \mathfrak{n})}$ defined by $\mathfrak{n}$ as a point of $C^{(\deg \mathfrak{n})}$.

We then define $X_{\mathfrak{n}}$ as the blowup of $C^{(\deg \mathfrak{n})}$ at $Z_{\mathfrak{n}}$, and we denote by $E_{\mathfrak{n}} = Z_{\mathfrak{n}} \times_{C^{(d)}} X_{\mathfrak{n}}$ the exceptional divisor of this blowup, it is irreducible of codimension 1. We denote by $\eta_{\mathfrak{n}}$ the generic point of $E_{\mathfrak{n}}$. Diagrammatically:

$$\begin{tikzcd}
\overline{\{\eta_{\mathfrak{n}}\}} = E_{\mathfrak{n}} \arrow[r, hook] \arrow[d] & X_{\mathfrak{n}} \arrow[d, "\pi_{\mathfrak{n}}"] \\
Z_{\mathfrak{n}} \arrow[r, hook] & C^{(\deg \mathfrak{n})}
\end{tikzcd} \tag{5}$$

Theorem 42 easily follows from:

/theorem:SymmetricPowerOfSheavesIsTamelyRamifiedReduction

Theorem 43. *:SymmetricPowerOfSheavesIsTamelyRamifiedReduction*

In the upcoming section, we perform the reduction and derive Theorem 42 from Theorem 3. We then prove Theorem 3 in the section that follows.

5.1 Reduction Lemmas

The following lemma is adapted from [Tak19] (Lemma 4.1)

Lemma 44. *Let C be a projective, smooth, and geometrically connected curve over a perfect field k. Let $\mathfrak{m} = \sum_{i=1}^{r} k_i P_i$ be an effective divisor where $P_1, \dots, P_r$ are distinct closed points. Let $U = C \setminus \mathfrak{m}$ and let $d \geq \deg \mathfrak{m}$.*

Suppose $\mathfrak{n}_1, \dots, \mathfrak{n}_l$ are pairwise coprime submoduli of $\mathfrak{m}$ such that $\mathfrak{m} = \sum_{j=1}^{l} \mathfrak{n}_j$. Consider the summation morphism:

$$\pi : C^{(\deg \mathfrak{n}_1)} \times_k \cdots \times_k C^{(\deg \mathfrak{n}_l)} \times_k C^{(d - \deg \mathfrak{m})} \longrightarrow C^{(d)}$$

defined by $(D_1, \dots, D_l, D_{extra}) \mapsto \sum_{j=1}^{l} D_j + D_{extra}$.

Then π is étale at the generic point of the closed subvariety

$$V = \{\mathfrak{n}_1\} \times_k \cdots \times_k \{\mathfrak{n}_l\} \times_k C^{(d - \deg \mathfrak{m})}$$

inside the domain $C^{(\deg \mathfrak{n}_1)} \times_k \cdots \times_k C^{(\deg \mathfrak{n}_l)} \times_k C^{(d - \deg \mathfrak{m})}$.

Proof. We may assume that k is algebraically closed (hence $\deg P_i = 1$ for all i). By miracle flatness π is flat, it is quasi-finite and projective as a map between projecive spaces. so we condlude π is finite and flat. It is enough to show that there exists a closed point Q of $\mathfrak{n}_1 + \dots \mathfrak{n}_l + C^{(d - \deg \mathfrak{m})} \subset C^{(d)}$ over which there are $\deg \pi$ points on $C^{(\mathfrak{n}_1)} \times_k \cdots \times_k C^{(\mathfrak{n}_l)} \times_k C^{(d - \deg \mathfrak{m})}$. (Because it will be unramified at this point and thus also at the generic point of V.) Choose Q as a point corresponding to a divisor $\mathfrak{n}_1 + \cdots + \mathfrak{n}_l + P_{r+1} + \cdots + P_{r+d-\deg \mathfrak{m}}$, where $P_1, \dots, P_{r+d-\deg \mathfrak{m}}$ are distinct points of $U(k)$. □

Corollary 45. *The morphism $C^{(\deg \mathfrak{m})} \times_k C^{(d - \deg \mathfrak{m})} \xrightarrow{\pi} C^{(d)}$ is finite flat everywhere, and étale at the generic point of the closed subvariety $Z_\mathfrak{m} \times_k C^{(d - \deg \mathfrak{m})} \subset C^{(\deg \mathfrak{m})} \times_k C^{(d - \deg \mathfrak{m})}$.*

Following from this, we look at the following diagram, coming from the flat base change $C^{(d - \deg \mathfrak{m})} \to \operatorname{Spec}(k)$ (Proposition 18) of (5): (Is this even smooth?)

$$\begin{tikzcd}
E_\mathfrak{m} \times_k C^{(d-\deg \mathfrak{m})} \arrow[r] \arrow[d] & X_\mathfrak{m} \times_k C^{(d-\deg \mathfrak{m})} \arrow[d] & \mathcal{F}^{(\deg \mathfrak{m})} \times_k C^{(d-\deg \mathfrak{m})} \arrow[d] \\
Z_\mathfrak{m} \times_k C^{(d-\deg \mathfrak{m})} \arrow[r] & C^{(\deg \mathfrak{m})} \times_k C^{(d-\deg \mathfrak{m})} & U^{(\deg \mathfrak{m})} \times_k C^{(d-\deg \mathfrak{m})} \arrow[ul]
\end{tikzcd}$$

Note that $U^{(\deg \mathfrak{m})} \times_k C^{(d-\deg \mathfrak{m})}$ is dense open subscheme of $X_\mathfrak{m} \times_k C^{(d-\deg \mathfrak{m})}$, And $E_\mathfrak{m} \times_k C^{(d-\deg \mathfrak{m})}$ is a prime divisor of $X_\mathfrak{m} \times_k C^{(d-\deg \mathfrak{m})}$. Hence it is well defined question according to Definition 32 to ask whether $\mathcal{F}^{(\deg \mathfrak{m})} \times_k C^{(d-\deg \mathfrak{m})}$ is tamely ramified at the generic point θ of $E_\mathfrak{m} \times_k C^{(d-\deg \mathfrak{m})}$.

Lemma 46. *If $\mathcal{F}^{(\deg \mathfrak{m})}$ is tamely ramified at $\eta_\mathfrak{m}$, then $\mathcal{F}^{(\deg \mathfrak{m})} \times_k C^{(d-\deg \mathfrak{m})}$ is tamely ramified at the generic point θ of $E_\mathfrak{m} \times_k C^{(d-\deg \mathfrak{m})}$.*

Proof. This follows from [Stacks, Tag 0EYD] make adjusments to definition and lemma, to only require that some prime divisors are with desired properties. □

Replacing $C^{(d-\deg \mathfrak{m})}$ with the dense open subscheme $U^{(d-\deg \mathfrak{m})} \subset C^{(d-\deg \mathfrak{m})}$, we get that the G-torsor $p_1^{-1}\mathcal{P}^{(\deg \mathfrak{m})}$ ($\mathcal{P}$ correspodns to $\mathcal{F}$ under Proposition 7) is tamely ramified at θ the generic point of $E_\mathfrak{m} \times_k U^{(d-\deg \mathfrak{m})} \subset U^{(\deg \mathfrak{m})} \times_k U^{(d-\deg \mathfrak{m})}$, where $p_1 : U^{(\deg \mathfrak{m})} \times_k U^{(d-\deg \mathfrak{m})} \to U^{(\deg \mathfrak{m})}$ is the projection to the first factor.

Looking at the second projection $p_2 : X_{\mathfrak{m}} \times_k U^{(d-\deg \mathfrak{m})} \to U^{(d-\deg \mathfrak{m})}$, and the fact that $\mathcal{P}^{(d-\deg \mathfrak{m})}$ is étale on $U^{(d-\deg \mathfrak{m})}$ we get that $p_2^{-1}\mathcal{P}^{(d-\deg \mathfrak{m})} = X_{\mathfrak{m}} \times_k \mathcal{P}^{(d-\deg \mathfrak{m})}$ is étale on $X_{\mathfrak{m}} \times_k U^{(d-\deg \mathfrak{m})}$. Hence, its restriction to $U^{(\deg \mathfrak{m})} \times_k U^{(d-\deg \mathfrak{m})}$ is unramified at θ.

Thus, by the following lemma, we conclude that $\mathcal{P}^{(\deg \mathfrak{m})} \boxtimes \mathcal{P}^{(d-\deg \mathfrak{m})} = p_1^{-1}\mathcal{P}^{(\deg \mathfrak{m})} \wedge^G p_2^{-1}\mathcal{P}^{(d-\deg \mathfrak{m})}$ is tamely ramified at θ the generic point of $E_{\mathfrak{m}} \times_k U^{(d-\deg \mathfrak{m})}$.

Lemma 47. *Let $X \to \operatorname{spec} k$ be a scheme over a field k, and let $\mathcal{P}_1, \mathcal{P}_2$ be two G-torsors on U_{et}. Let ξ be the generic point of a prime divisor $D \subset X$. If $\mathcal{P}_1$ is tamely ramified at ξ, and $\mathcal{P}_2$ is unramified at ξ, then the contracted product $\mathcal{P}_1 \wedge^G \mathcal{P}_2$ is tamely ramified at ξ.*

Proof. This follows from Lemma 35. $\square$

Combining this with Corollary 45 we get

Corollary 48. *Let $\mathfrak{m}$ be a modulus as above, and let $\eta_{\mathfrak{m}}$ be the generic point of $E_{\mathfrak{m}}$. Let $\mathcal{P}$ be a G-torsor on U_{et} with ramification bounded by $\mathfrak{m}$. Assume $\mathcal{P}^{(\deg \mathfrak{m})}$ is tamely ramified at $\eta_{\mathfrak{m}}$. Then $\mathcal{P}^{(\deg \mathfrak{m})} \boxtimes \mathcal{P}^{(d-\deg \mathfrak{m})}$ is tamely ramified at the generic point θ of $E_{\mathfrak{m}} \times_k C^{(d-\deg \mathfrak{m})} \subset C^{(\deg \mathfrak{m})} \times_k C^{(d-\deg \mathfrak{m})}$, and $\mathcal{P}^{(d)}$ is tamely ramified at the generic point $\eta_0 = \eta_{\mathfrak{m},d}$ of $E_0 = E_{\mathfrak{m},d}$*

Proof. The first assertion, that $\mathcal{P}^{(\deg \mathfrak{m})} \boxtimes \mathcal{P}^{(d-\deg \mathfrak{m})}$ is tamely ramified at θ, follows from the preceding discussion. Thus, it remains to show that $\mathcal{P}^{(d)}$ is tamely ramified at η_0.

Consider the blowup diagram defining $X_{\mathfrak{m},d}$:

$$\begin{tikzcd}
\overline{\{\eta_0\}} = E_0 \arrow[r] \arrow[d] & X_{\mathfrak{m},d} \arrow[d, "\pi"] \\
Z_0 \arrow[r, "c.i"] & C^{(d)}
\end{tikzcd}$$

By performing a base change along the flat addition map $+ : C^{(\deg \mathfrak{m})} \times_k U^{(d-\deg \mathfrak{m})} \to C^{(d)}$, we obtain the following commutative diagram:

$$\begin{tikzcd}
 & \left(C^{(\deg \mathfrak{m})} \times_k U^{(d-\deg \mathfrak{m})}\right) \times_{C^{(d)}} E_0 \arrow[r] \arrow[d] & \overline{\{\eta_0\}} = E_0 \arrow[d] \\
 & \left(C^{(\deg \mathfrak{m})} \times_k U^{(d-\deg \mathfrak{m})}\right) \times_{C^{(d)}} X_{\mathfrak{m},d} \arrow[r] \arrow[d] & X_{\mathfrak{m},d} \arrow[d, "\pi"] \\
\mathfrak{m} \times_k U^{(d-\deg \mathfrak{m})} \arrow[r, "c.i"] & C^{(\deg \mathfrak{m})} \times_k U^{(d-\deg \mathfrak{m})} \arrow[r, "+"] & C^{(d)}
\end{tikzcd}$$

Since blowups commute with flat base change, and the inverse image of the center Z_0 under the map $+$ is $\mathfrak{m} \times_k U^{(d-\deg \mathfrak{m})}$, the scheme $\left(C^{(\deg \mathfrak{m})} \times_k U^{(d-\deg \mathfrak{m})}\right) \times_{C^{(d)}} X_{\mathfrak{m},d}$ is the blowup of $C^{(\deg \mathfrak{m})} \times_k U^{(d-\deg \mathfrak{m})}$ along $\mathfrak{m} \times_k U^{(d-\deg \mathfrak{m})}$. This, in turn, is isomorphic to the base change of the blowup $X_{\mathfrak{m}}$ (of $C^{(\deg \mathfrak{m})}$ along $\mathfrak{m}$) via the (flat) projection $C^{(\deg \mathfrak{m})} \times_k U^{(d-\deg \mathfrak{m})} \to C^{(\deg \mathfrak{m})}$.

Assembling these facts, we obtain the following Cartesian square:

$$\begin{tikzcd}
 & E_{\mathfrak{m}} \times U^{(d-\deg \mathfrak{m})} \arrow[r, "\tilde{+}"] \arrow[d] & \overline{\{\eta_0\}} = E_0 \arrow[d] \\
 & X_{\mathfrak{m}} \times_k U^{(d-\deg \mathfrak{m})} \arrow[r, "\tilde{+}"] \arrow[d] & X_{\mathfrak{m},d} \arrow[d, "\pi"] \\
\mathfrak{m} \times_k U^{(d-\deg \mathfrak{m})} \arrow[r, "c.i"] & C^{(\deg \mathfrak{m})} \times_k U^{(d-\deg \mathfrak{m})} \arrow[r, "+"] & C^{(d)}
\end{tikzcd}$$

The point θ defined in the Corollary is the generic point of $E_{\mathfrak{m}} \times_k U^{(d-\deg \mathfrak{m})}$. Let η be the generic point of $\mathfrak{m} \times_k U^{(d-\deg \mathfrak{m})}$. By Corollary 45, the map $+$ is étale at η. Consequently, the lifted map $\tilde{+}$ is étale at θ. Given the isomorphism $(+^{-1})(\mathcal{P}^{(d)}) \cong \mathcal{P}^{(\deg \mathfrak{m})} \boxtimes \mathcal{P}^{(d-\deg \mathfrak{m})}$ from Proposition 21, the tame ramification of the box product at θ descends to the tame ramification of $\mathcal{P}^{(d)}$ at η_0 by applying Lemma 36.

□

Lemma 49. *Let $\mathfrak{n}_1, \mathfrak{n}_2 \subset \mathfrak{m}$ be two coprime sub moduli of $\mathfrak{m}$. Assume $\mathcal{P}^{(\deg \mathfrak{n}_1)}$, $\mathcal{P}^{(\deg \mathfrak{n}_2)}$ are at most tamely ramified at $\eta_{\mathfrak{n}_1}$, $\eta_{\mathfrak{n}_2}$ respectively. Then $\mathcal{P}^{(\deg \mathfrak{n}_1 + \deg \mathfrak{n}_2)}$ is at most tamely tamified at $\eta_{\mathfrak{n}_1+\mathfrak{n}_2}$.*

Proof. It follows from Proposition 37 and Proposition 21 □

5.2 Proof of Theorem 42

Proof of Theorem 42. Let $\mathcal{F}$ be as in Theorem 42, $\mathfrak{m} = \sum_{i=1}^n k_P P$ with $\deg P = d_P$ Then by Theorem 3 for every $\mathfrak{n} \subset \mathfrak{m}$ of the form $\mathfrak{n} = k_P P$, $\mathcal{F}^{(\deg \mathfrak{n})}$ is at most tamely ramified at $\eta_{\mathfrak{n}}$. By Lemma 49, $\mathcal{F}^{(\deg \mathfrak{m})}$ is then at most tamely ramified at $\eta_{\mathfrak{m}}$. And thus by Corollary 48 $\mathcal{F}^{(d)}$ is tamely ramified at the generic point η_0 of E_0 □

5.3 Proof of Theorem 3

Let G be a finite abelian group. Unless otherwise stated, assume that $C = \mathbb{P}^1_k$, $\mathfrak{m} = d \cdot 0$ and $\mathbb{G}_m = U \subset U' = C \setminus \mathfrak{m}$. Then $\deg \mathfrak{m} = d$ We also assume k is algebraicly closed. (We can etale base change, and this doesn't change ramification.) Our first result is:

Theorem 50. *Let $\mathcal{P} \to \mathbb{G}_m \subset \mathbb{P}^1_k$ be a G torsor which is either*

1. *tamely ramified at 0*
2. *wildly ramified at 0 with $G = \mathbb{Z}/p\mathbb{Z}$ and ramificaiton bounded by d.*

Then the ramification of the G-torsor $\mathcal{P}^{(d)} \to \mathbb{G}_m^{(d)}$ at $\eta_{\mathfrak{m}}$ the generic point of $E_{\mathfrak{m}} \subset X_{\mathfrak{m}}$ is

1. *tamely ramified if $\mathcal{P}$ was tamely ramified*
2. *unramified if $\mathcal{P}$ was wildly ramified with $G = \mathbb{Z}/p\mathbb{Z}$ and ramification bounded by d*

Proof. Recall that in Section 3.2, we saw that the local ring at the generic point of the excptional divisor of the blowup of the affine space at 0 point is $R = k[x_d, \frac{u_1}{u_d}, \dots, \frac{u_{d-1}}{u_d}]_{(x_d)}$. Where for every $i < d$, we have $x_i = \frac{u_i}{u_d} x_d$ are all uniformizers. The residue field was $\kappa(\eta) = k(\frac{u_1}{u_d}, \dots, \frac{u_{d-1}}{u_d})$ and the completion of R with respect to its maximal ideal is:

$$\hat{R} = \kappa(\eta)[[x_d]]$$

In our situation, when we take symmetric product of the affine space, the situation is similiar with different coordinates if we let $e_1, \dots, e_d$ be the symmetric polynomials in $x_1, \dots x_d$ then: The local ring is $R = k[e_d, \frac{u_2}{u_d}, \dots, \frac{u_{d-1}}{u_d}]_{(e_d)}$. $e_i = \frac{u_i}{u_d} e_d$ are all uniformizers. The residue field being: $\kappa(\eta) = k(\frac{u_1}{u_d}, \dots, \frac{u_{d-1}}{u_d})$ and the completion: $\hat{R} = \kappa(\eta)[[s_d]]$ Note that from $e_i = \frac{u_i}{u_d} e_d$ We get $\frac{e_i}{e_d} = \frac{u_i}{u_d}$ in the fracion field. hence

$$\hat{K} = k(\frac{e_1}{e_d}, \dots, \frac{e_{d-1}}{u_d})((e_d)) \tag{6}$$

We compute directly the extesntion of complete valued fields over the complete valued field at the generic point. Note that by Theorem 24 and Theorem 25 We can assume $\mathcal{P} = \operatorname{Spec} k[x, x^{-1}][X]/(X^n - a)$ for $a \in k[x, x^{-1}]$ or $\mathcal{P} = \operatorname{Spec} k[x, x^{-1}][X]/(X^p + X - f(x, x^{-1}))$ where $f(x, x^{-1}) = cx^{-m} + a_{-m+1}x^{-m+1} + \cdots a_{-1}x^{-1} + a_0 = cx^{-m} + f_{-m+1}(x^{-1})$ where $m < d$.

We deal with each case separately.

Artin-Schreier Extensions:

Set $R = k[x, x^{-1}]$, and $S = \operatorname{Spec} k[x, x^{-1}][X]/(X^p + X - f(x^{-1}))$ we have

$$\begin{aligned}
R^{\otimes_k d} &= k[x_1, x_1^{-1}, \dots, x_d, x_d^{-1}] \\
S^{\otimes_k d} &= k[x_1, x_1^{-1}, \dots, x_d, x_d^{-1}][X_1, \dots, X_d]/(X_1^p - X_1 - f(x_1), \dots, X_d^p - X_d - f(x_d)) \\
&= R^{\otimes_k d}[X_1, \dots, X_d]/(X_1^p - X_1 - f(x_1), \dots, X_d^p - X_d - f(x_d))
\end{aligned}$$

Next, we want to understand the ring corresponding to $p_1^{-1}(\mathcal{P}) \otimes \cdots \otimes p_d^{-1}(\mathcal{P})$ on C^d - the d'th-contracted product of the $G = \mathbb{Z}/p\mathbb{Z}$-torsors $p_1^{-1}(\mathcal{P}), \dots, p_d^{-1}(\mathcal{P})$ on U^d. It correspond to qoutient: $\Big(p_1^{-1}(\mathcal{P}) \times \cdots \times p_d^{-1}(\mathcal{P})\Big)/G^{d-1}$ where the action of G^{d-1} on the product is:

$$(g_1, \dots, g_{d-1}) \cdot (p_1, p_2, \dots, p_{d-1}, p_d) = (g_1(p_1), g_1^{-1}g_2(p_2), \dots, g_{d-2}^{-1}g_{d-1}(p_{d-1}), g_{d-1}^{-1}(p_d))$$

The affine ring corrsponding to the contracted product is $\left(S^{\otimes_k d}\right)^{G^{d-1}}$

Recall that the action of $g \in G = \mathbb{Z}/p\mathbb{Z}$ on X is $g(X) = X + g$ (g correspond to a number $0 \leq g \leq p-1$). So, the action of $(g_1, \dots, g_{d-1})$ on the generators $(X_1, X_2, \dots, X_{d-1}, X_d)$ Is $X_1 \mapsto X_1 + g_1$, $X_i \mapsto X_i - g_{i-1} + g_i$ for $1 < i < d$ and $X_d \mapsto X_d - g_{d-1}$. So we see that $Y = X_1 + \cdots + X_d$ is invariant. Moreover $Y^P - Y - \sum_{i=1}^d f(x_i) = 0$ is irreducible degree p equation for Y, Since we are quotienting a rank p^d extesntion by a group of order p^{d-1} the resulting invaraint subring must have rank p over $R^{\otimes_k d}$, So we conclude:

$$\left(S^{\otimes_k d}\right)^{G^{d-1}} \cong R^{\otimes_k d}[Y]/(Y^p - Y - \sum_{i=1}^d f(x_i))$$

The group S_d acts on $R^{\otimes_k d} = k[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_d^{\pm 1}]$ by permuting the variables $\{x_i\}_{i=1}^d$. And since $Y = \sum_i^d X_i$ it leaves Y invaraint. The invaraint subring $(R^{\otimes_k d})^{S_d}$ is simply $k[e_1, e_2, \dots, e_d, e_d^{-1}]$ where $\{e_i\}$ are the symmetric polynomials in $x_1, \dots, x_d$ e.g. $e_1 = x_1 + \dots + x_d$ give general defintion here... and $e_d = x_1 x_2 \dots x_d$.

To find $\left(R^{\otimes_k d}[Y]/(Y^p - Y - \sum_{i=1}^d f(x_i))\right)^{S_d}$ its enough to express $\sum_{i=1}^d f(x_i)$ in $e_1, \dots, e_d$, this can be done with the newton polynomials, moreover, we claim the following:

Lemma 51. *Let $f(x) = cx^{-m} + a_{-m+1}x^{-m+1} + \cdots a_{-1}x^{-1} + a_0$, define $\alpha(x_1, ..., x_d) = \sum_1^d f(x_i)$, and deonte by $e_1, ..., e_d$ the elementary symmetric polynomials in $x_1, ..., x_d$. If $m < d$ then $\alpha(x_1, ..., x_d) \in k(e_1/e_d, e_2/e_d, ..., e_{d-1}/e_d)$*

Proof. Changing variables $y_i = x_i^{-1}$ for each $i \in \{1, \dots, d\}$ We get

$$\alpha = \sum_{i=1}^d f(x_i) = \sum_{i=1}^d \left(cy_i^m + a_{-m+1}y_i^{m-1} + \cdots + a_{-1}y_i + a_0\right)$$

Rearranging the sums, we get

$$\alpha = c\sum_{i=1}^d y_i^m + a_{-m+1}\sum_{i=1}^d y_i^{m-1} + \cdots + a_{-1}\sum_{i=1}^d y_i + da_0$$

Let $p_k(y_1, \dots, y_d) = \sum_{i=1}^d y_i^k$ be the k-th power sum symmetric polynomial. The expression for α is a linear combination of these power sums:

$$\alpha = cp_m(y) + a_{-m+1}p_{m-1}(y) + \cdots + a_{-1}p_1(y) + da_0$$

According to the *Fundamental Theorem of Symmetric Polynomials*, any symmetric polynomial in $y_1, \dots, y_d$ can be expressed as a polynomial in the elementary symmetric polynomials $e_k(y_1, \dots, y_d)$. Since $m < d$, α is a polynomial in $e_1(y), e_2(y), \dots, e_m(y)$. $(y = (y_1, \dots, y_d))$ The elementary symmetric polynomials in $y_i = 1/x_i$ are related to the elementary symmetric polynomials in x_i as follows:

$$e_k(y_1, \dots, y_d) = \sum_{1 \le i_1 < \cdots < i_k \le d} \frac{1}{x_{i_1} \dots x_{i_k}} = \frac{\sum_{1 \le j_1 < \cdots < j_{d-k} \le d} x_{j_1} \dots x_{j_{d-k}}}{x_1 x_2 \dots x_d}$$

Thus,

$$e_k(y_1, \dots, y_d) = \frac{e_{d-k}(x_1, \dots, x_d)}{e_d(x_1, \dots, x_d)}$$

which concludes the proof. $\square$

Finally, restricting $\mathcal{P}^{(d)}$ to spec $\hat{K}$ we get by (6) and Theorem 25 the result. (that $\mathcal{P}$ is unramified at the generic point of the exceptional divisor of the blowup).

Kummer Extensions: Few things are different in that case,

Set $R = k[x, x^{-1}]$, and $S = R[X]/(X^n - f)$ where $f = f(x, 1/x) \in R$ In this case we have $char k = p$ and $gcd(p, n) = 1$.

We have

$$\begin{aligned} R^{\otimes_k d} &= k[x_1, x_1^{-1}, \dots, x_d, x_d^{-1}] \\ S^{\otimes_k d} &= k[x_1, x_1^{-1}, \dots, x_d, x_d^{-1}][X_1, \dots, X_d]/(X_1^n - f_1, \dots, X_d^n - f_d) \\ &= R^{\otimes_k d}[X_1, \dots, X_d]/(X_1^n - f_1, \dots, X_d^n - f_d) \end{aligned}$$

Where $f_i = f(x_i, x_i^{-1})$

Next, we want to figure out $\left(S^{\otimes_k d}\right)^{G^{d-1}}$

Recall that the action of $g \in G = \mathbb{Z}/n\mathbb{Z}$ on X is $g(X) = \zeta^g X$ (g correspond to a number $0 \le g \le n-1$). So, the action of $(g_1, \ldots, g_{d-1})$ on the generators $(X_1, X_2, \ldots, X_{d-1}, X_d)$ Is $X_1 \mapsto \zeta^{g_1} X_1$, $X_i \mapsto \zeta^{g_i - g_{i-1}} X_i$ for $1 < i < d$ and $X_d \mapsto \zeta^{-g_{d-1}} X_d$.

So we see that $Y = X_1 X_2 \ldots X_d$ is invariant. And $Y^n - \prod_{i=1}^d f_i$ is irreducible degree n equation for Y, So, like before, we conclude:

$$\left(S^{\otimes_k d}\right)^{G^{d-1}} \cong R^{\otimes_k d}[Y]/(Y^n - \prod_{i=1}^d f_i)$$

The group S_d acts on $R^{\otimes_k d}[Y]/(Y^n - \prod_{i=1}^d f_i)$ by permuting the indices. on the variables x_i, On $Y = \prod_1^d X_i$ it is invaraint. The invaraint subring $(R^{\otimes_k d})^{S_d}$ is simply $k[e_1, e_2, \ldots, e_d, e_d^{-1}]$ like before.

The polynomial $F = \prod_{i=1}^d f_i$ is symmetric in $\{x_i\}_1^d$ so it can be expressed as a polynomial $\tilde{F}(e_1, \ldots, e_d)$ in the elementary symmetric variables. Hence the qoutient ring is:

$$\left(\frac{k[x_1^{\pm 1}, \ldots, t_d^{\pm 1}][Y]}{(Y^n - \prod_{i=1}^d f_i)}\right)^{S_d} \cong \frac{k[e_1, \ldots, e_d, e_d^{-1}][Y]}{(Y^n - \tilde{F}(e_1, \ldots, e_d))}$$

So we see again, that restricting $\mathcal{P}^{(d)}$ to $\operatorname{spec} \hat{K}$ we get by (6) and Theorem 24, that $\mathcal{P}$ is tamely ramified at the generic point of the exceptional divisor of the blowup.

□

Now, G is fintie abelian. So by the Structure Theorem for Finite Abelian Groups we have a descending sequence of subgroups:

$$G = H_0 \supset H_1 supset H_2 \supset \cdots \supset H_l \supset 1$$

Where for all $i < l$ we have

6 Proof of Theorem 2

We work over $S = \operatorname{spec} k$, for k perfect.

By Proposition 7, its equivalent to prove:

Theorem 52. *Let $G = \Lambda^\times$ be a finite abelian group (Λ as before), and let $\mathcal{P}$ be a G-torsor on U, with ramification bounded by $\mathfrak{m}$. Then, for sufficiently large integer d, there exists a unique (up to isomorphism) G-torsor $\mathcal{Q}_d$ on $\mathrm{Pic}^d_{C,\mathfrak{m}}$, such that the pullback of $\mathcal{Q}_d$ by Φ_d is isomorphic to $\mathcal{P}^{(d)}$.*

Proof. We divide the proof into two cases, when $\mathfrak{m} = 0$ and when $\mathfrak{m} > 0$.

Case 1: $\mathfrak{m} = 0$. By Section 4.3, for d large enough, the Abel-Jacobi morphism $\Phi_d : C^{(d)} \to \mathrm{Pic}^d_C$ is proper surjective and smooth, with geometrically connected fibers, each isomorphic to $\mathbb{P}^{d-g}_{k^{sep}}$, Hence by Corollary 39 it induces an exact sequence of etale fundamental groups:

$$\pi_1^{et}(\mathbb{P}^{d-g}_{k^{sep}}) \to \pi_1^{et}(C^{(d)}) \to \pi_1^{et}(\mathrm{Pic}^d_C) \to 1$$

But $\mathbb{P}^{d-g}_{k^{sep}}$ is simply connected ([Ten15] Example 4.9, [Tót11] Example 1.4.12), hence its etale fundamental group is trivial, and we get an isomorphism of etale fundamental groups:

$$\pi_1^{et}(C^{(d)}) \cong \pi_1^{et}(\mathrm{Pic}^d_C)$$

Implying the theorem in this case.

Case 2: $\mathfrak{m} > 0$. In this case by Theorem 41, for d large enough, the Abel-Jacobi morphism $\Phi_d : U^{(d)} \to \mathrm{Pic}^d_{C,\mathfrak{m}}$ extends to a proper surjective and smooth map, with geometrically connected fibers isomorphic to projective spaces,

$$\tilde{\Phi}_d : \tilde{C}^{(d)}_{\mathfrak{m}} \to \mathrm{Pic}^d_{C,\mathfrak{m}}$$

Hence we get an isomorphism of etale fundamental groups:

$$\pi_1^{et}(\tilde{C}^{(d)}_{\mathfrak{m}}) \cong \pi_1^{et}(\mathrm{Pic}^d_{C,\mathfrak{m}})$$

By Theorem 42, $\mathcal{F}^{(d)}$ is tamely ramified on the boundary divisor $H = \tilde{C}^{(d)}_{\mathfrak{m}} \setminus U^{(d)}$.

Thus, by Lemma 53 below, we have: $\mathcal{F}^{(d)}$ extends to a locally constant sheaf $\tilde{\mathcal{F}}^{(d)}$ on $\tilde{C}^{(d)}_{\mathfrak{m}}$, which by the isomorphism of etale fundamental groups above, corresponds to a unique locally constant sheaf $\mathcal{G}_d$ on $\mathrm{Pic}^d_{C,\mathfrak{m}}$, such that $\tilde{\Phi}_d^*\mathcal{G}_d \cong \tilde{\mathcal{F}}^{(d)}$. Restricting back to $U^{(d)}$, we get $\Phi_d^*\mathcal{G}_d \cong \mathcal{F}^{(d)}$, as required. □

Lemma 53. *If $\mathcal{F}^{(d)}$ is a locally constant sheaf on $U^{(d)}$ which is tamely ramified along the boundary divisor $H = \tilde{C}^{(d)}_{\mathfrak{m}} \setminus U^{(d)}$, then $\mathcal{F}^{(d)}$ extends to a locally constant sheaf $\tilde{\mathcal{F}}^{(d)}$ on $\tilde{C}^{(d)}_{\mathfrak{m}}$.*

Proof. The lemma we are referencing above can be proved in two routes:
Route 1 - Showing $\ker\{\pi_1^{ab}(U^{(d)}) \to \pi_1^{ab}(\mathrm{Pic}^d_{C,\mathfrak{m}})\}$ is pro-p group.
Route 2 - Showing

$$\pi_1^{t,ab}(U^{(d)}) \to \pi_1^{t,ab}(\mathrm{Pic}^d_{C,\mathfrak{m}})$$

is isomorphism to its image. here one needs to be precise.

□

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