LaTex Project: Problem 4.3.11

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1 Monotone Convergence Theorem Proof

Claim A monotone sequence is convergent if and only if it is bounded.

Proof. Let S_n be a monotone sequence. Assume that S_n is convergent. By Theorem 4.1.13, "Every convergent sequence is bounded", S_n must also be bounded.

Since a monotone sequence can either be increasing or decreasing, there must exist two cases. One case in which the sequence S_n is increasing and one in which the sequence S_n is decreasing.

Case 1: Suppose for case 1 that S_n is an increasing sequence. We have that the sequence must be bounded and therefore there must exist M such that for all n in $\mathbb{N}$, $S_n \leq M$ and therefore, since S_n is bounded, then the set $\{S_n : n \in \mathbb{N}\}$ must also be bounded. By the **Completeness Axiom**, this set has a supremum in $\mathbb{R}$, define L as the supremum of S_n.

Now, we will let there exist some ϵ exist, such that $\epsilon > 0$. L is the supremum of (S_n) and therefore, it is the least upper bound of (S_n), since L is the least upper bound and ϵ is greater than zero, we can say that $L - \epsilon$ cannot be an upper bound to the set. Since (S_n) is an increasing set there must also exist some N such that $n \geq N$ and $S_N \leq S_n$. Therefore:

$$L - \epsilon < L \leq S_N \leq S_n < L + \epsilon$$

After simplifying, we have:

$$L - \epsilon < S_n < L + \epsilon$$

And furthermore:

$$|S_n - L| < \epsilon$$

Therefore, the limit of S_n is L and S_n converges to L.

Case 2: Suppose for case 2 that S_n is a decreasing sequence. We have that S_n is bounded and therefore there must exist some M in $\mathbb{R}$ such that for all n in $\mathbb{N}$, we have that $M \leq S_n$. Considering the set $\{S_n : n \in \mathbb{N}\}$, given that S_n is convergent, we can also say that the set is bounded. Define L as the infimum of S_n.

Now, we will let there exist some ϵ, such that $\epsilon > 0$. Given that L is the infimum, L must be the greatest lower bound of S_n and since we know ϵ is greater than zero, $L + \epsilon$ cannot be lower bound to the set. Furthermore, since S_n is a decreasing sequence, there must exist some N such that for all $n \geq N$ and $S_N \geq S_n$. Therefore:

$$L - \epsilon < L \leq S_n \leq S_N < L + \epsilon$$

And after simplifying:

$$L - \epsilon \leq S_n < L + \epsilon \ |S_n - L| < \epsilon$$

Therefore, the limit of S_n is L and S_n converges to L.

Since both the increasing and decreasing cases converge, we can conclude that a monotone sequence is convergent if and only if it is bounded.

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