# Amplitude-ology

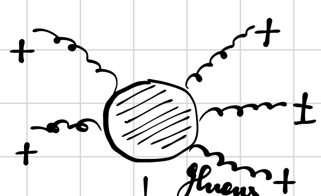
Motivations:

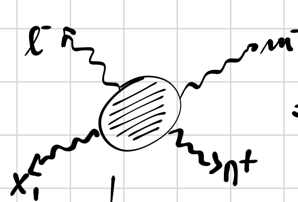
1) Practical (QCD @LHC) $\longrightarrow$ better technical tools

$2\to1$ } very good
$2\to2$ }
$2\to3$ ???

$\downarrow$ expansion in $\alpha_s$: LO, NLO, NNLO, ...

$\frac{\delta\sigma}{\sigma}$ can reach 100% at LO
10-30% at NLO
1-8% at NNLO
few % at $N^3LO$

2) Formal structure: 

$= 0$ 

$= \frac{\langle \ell m\rangle^4}{\langle 12\rangle\langle 23\rangle\ldots\langle n1\rangle}$

sum of all tree diagrams

Parke-Taylor amplitude (1986)

on shell ext. legs

$\Rightarrow$ amplitudes in different theories related: $\longrightarrow$ S-matrix analiticy (causality) $\oplus$ ⟨ factorize, unitarity

• Gravity $\sim (YM)^2$

perturbative scaffold

$\Rightarrow$ can we formulate them geometrically (i.e. w/o causality - locality)

↳ amplitude hedron

3) Other applications:

a) Quantum SUGRA: $\mathcal{N}=8 \longrightarrow$ UV finite through $\geq 4$ loops

b) (planar) $\mathcal{N}=4$ SYM [AdS/CFT]

↳ $SU(N_c\to\infty) \Rightarrow$ integrability ('t Hooft coupl: $\lambda = g_{YM}^2 N_c$ fixed)

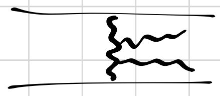
↳ exactly solvable

4) Mathematics

a) combinatorics

b) geometry

c) iterated integrals $\longleftrightarrow$ multiple zeta values $\left(\zeta_n = \sum_{k=1}^{\infty}\frac{1}{k^n} \to \zeta_{n_1,n_2,\ldots,n_m}\right)$

5) Compute LIGO inspiral waveforms: 

# LHC physics

$*$ ~ 3000 bunches of p ⟶ $10^{11}$ part per bunch ⟶ 6.5 TeV/part ⇒ ~300 MJ of energy

(kinetic en of an aircraft carrier or to melt 1 ton of Cu)

$pp \to i$

$\Rightarrow N^{(i)}_{events} = \sigma^{(i)} \int \mathcal{L}\, dt$ ⟶ integrated luminosity

2016 : ~ 35 $fb^{-1}$

2017 : ~ 50 $fb^{-1}$

$(\sigma^{tot}_{pp} \sim 100\ mb)$ ⟶ classically p $\sim 10^{-13}$ cm. → $\sigma \sim 10$ mb

e.g.: $\Rightarrow \sigma(pp \to Z + X) = 50$ nb ⟶ 5 billion Z's @ LHC

"standard candle"

$Z \to q\bar{q},\ e^+e^-,\ \mu^+\mu^-,\ \tau^+\tau^-$

$q\bar{q}$ ⟶ too much QCD background to be seen

(not everything is detected)

HIGGS BOSON ⟶ $\sigma(pp \to H + X) = 63$ pb ⟶ 6 million Higgses at LHC

(but not all of them can be detected)

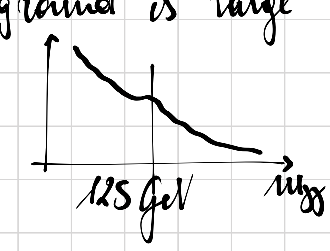
$H \to b\bar{b}$ (main, but very diff.)

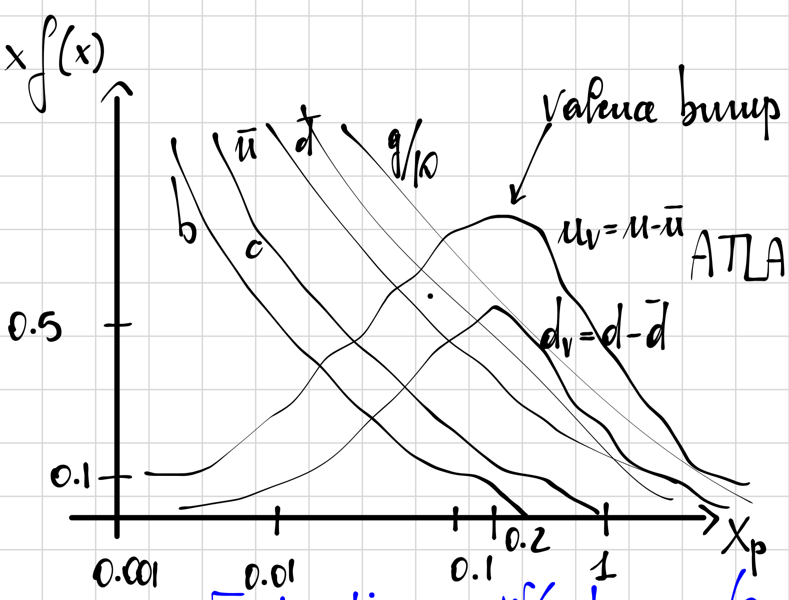
$\to ZZ^* \to \ell^+\ell^-,\ \ell\ell^-$

ATLAS @ 36 $fb^{-1}$ : 100 ± 10 events

$\frac{\delta\sigma}{\sigma} \sim 10\%$

$\to \gamma\gamma$ (~ 2000 events but $q\bar{q} \to 2\gamma$ background is large)




ex: $\sqrt{s}$ to discover W/Z bosons $(s x_1 x_2 = M^2_{W/Z})$

## QCD - improved parton model

Inclusive Z production: $\sigma(pp \to Z + X;\ s) = \sum_{a,b} \int_0^1 \int_0^1 dx_1 dx_2\, f_{a/p}(x_1, \mu_F)\, f_{b/p}(x_2, \mu_F)\, \hat{\sigma}(ab \to Z + X;\ s x_1 x_2;\ \mu_F, \mu_R)$

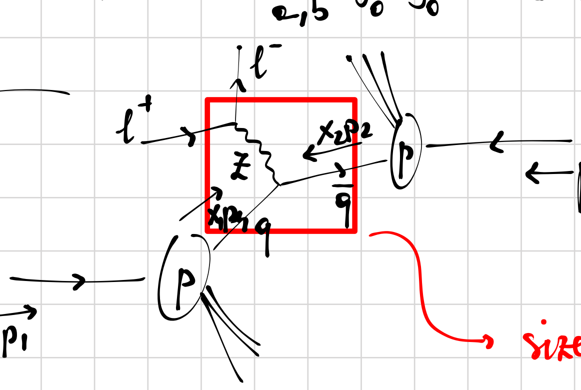
PDF's

short distance cross-section

factorization scale

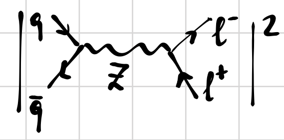
renormal scale $\alpha_s(\mu_R)$

Drell-Yan process




size of the box $\frac{1}{\mu_F}$ (independ. when consid. all orders)

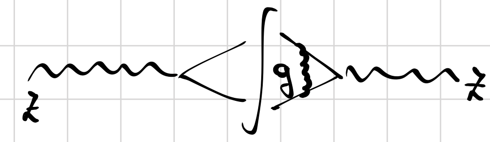
- gluons inside ⟶ $\hat{\sigma}$
- gluons outside ⟶ proton. structure



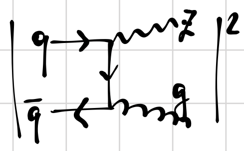

at LO:
$$\hat{\sigma} = \overline{\left| M^{(0)}_{\underset{q}{a}+\underset{\bar q}{b} \to Z} \right|^2} \quad \text{where} \quad \overline{|M|^2} = \frac{1}{\underset{2}{N_{S_1}}\,\underset{2}{N_{S_2}}\,\underset{3}{N_{C_1}}\,\underset{3}{N_{C_2}}} \sum_{S_1,S_2,C_1,C_2} |M|^2$$

at NLO:

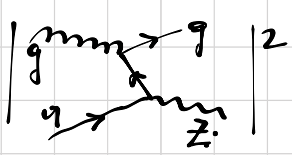
- 1-loop virtual: $M = M^{(0)} + \frac{g_S^2}{(4\pi)^2} M^{(1)} + \left(\frac{g_S^2}{(4\pi)^2}\right)^2 M^{(2)} + \ldots$
- real emission: $|M|^2 \geq |M^{(0)}|^2 + \frac{\alpha_s}{2\pi} \mathrm{Re}\left(M^{(0)\dagger} M^{(1)}\right)$




a) gluon emission:




b) quark emission:

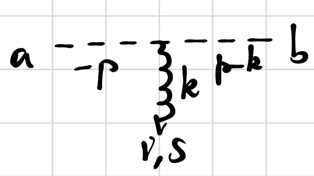



$\Rightarrow$ VIRTUAL EMISSION

- IR = soft + collinear

$*$ soft emission gluon $\longrightarrow$ classical (no spin, only color is relev.)

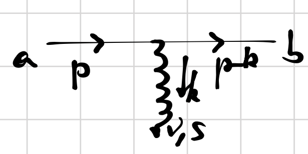
SCALAR:




$\sim f^{abs}(-2p + \cancel{k})^\nu$

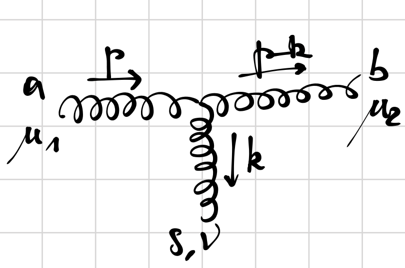
same!

FERM:



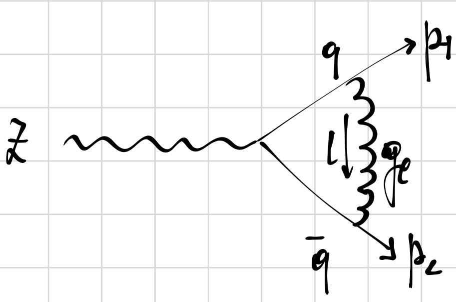

$\sim f^{abs}\, \not{p}\, \gamma^\nu u(p - \cancel{k}) \sim f^{abs}(2p^\nu)$

ex: same analysis for gluons




$\sim \eta^{\mu_1\mu_2} f^{abs}\, 2p^\nu$

Use:




$$\simeq p_1 \cdot p_2 \int \frac{d^{4-2\varepsilon}\ell}{(2\pi)^{4-2\varepsilon}}\, \mu_R^{2\varepsilon} \frac{1}{\ell^2 (\ell + p_1)^2 (\ell - p_2)^2} = *$$

$\Downarrow$

Divergences:

- SOFT: $\ell \to 0$
- COLLINEAR: $\ell \parallel p_1$

$$* \sim \frac{1}{4\pi} \cdot \frac{\Gamma(1+\varepsilon)\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)} \left(-\frac{1}{\varepsilon^2}\right) \left(\frac{\mu^2}{s_{12}}\right)^{\varepsilon} \longrightarrow \text{double poles in } \varepsilon!$$

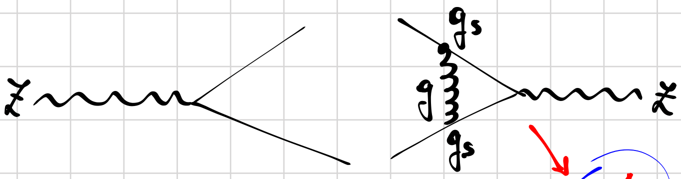
→ REAL EMISSION: $q\bar{q} \longrightarrow Zg \Rightarrow$ soft + collinear sing $\longrightarrow \frac{1}{\varepsilon^2}$

$qg \longrightarrow Zq \Rightarrow$ collinear sing. $\longrightarrow \frac{1}{\varepsilon}$

$\Rightarrow$ VIRTUAL + REAL EMISSION $\longrightarrow \frac{1}{\varepsilon}$ leftover $\Rightarrow$ renormalize pdf's

## Anatomy of NLO

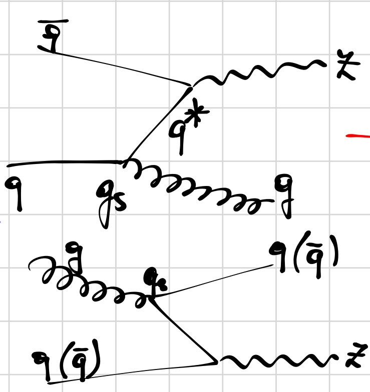
- 1-loop virtual: $Z$




$\left(\alpha_s = \frac{g_s^2}{4\pi}\right)$

$-\frac{1}{\varepsilon^2}$ divergence.

- real:




$\longrightarrow + \frac{1}{\varepsilon^2} + \frac{1}{\varepsilon}$

↳ cancellation

we need to renormalize pdf's

collinear limit:

$$M_{q\bar{q}\to Z+g} \overset{q\|g}{\sim} \mathrm{Split}_{q\to q^*g}(z)\cdot M_{q\bar{q}\to Z^0} \qquad \left(p_*^\mu = z P_q^\mu\,,\ p_q^\mu = (1-z)p_q^\mu\right)$$

↳ universality: $M_{n+1}(q, q, \dots) \sim \mathrm{Split}_{q\to qg}(z)\, M_n(q^*, \dots)$

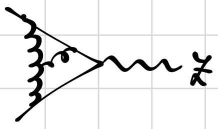
$\Rightarrow |\mathrm{Split}_{q\to qg}(z)|^2 \sim P_{qq}(z) \longrightarrow$ Altarelli-Parisi's splitting prob.

$$\longrightarrow \mu_F^2 \frac{\partial}{\partial \mu_F^2} f_{ap}(x) - \frac{\alpha_s}{2\pi}\sum_b P_{ab}\otimes f_b$$

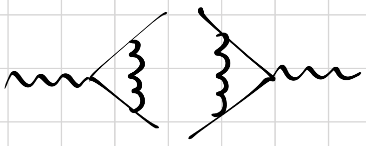
$$(P_{ab}\times f_b)(x) = \int_x^1 \frac{dy}{y} P_{ab}\left(\frac{x}{y}\right) f_b(y) \qquad x/y = z$$

## NNLO:

Virtual:

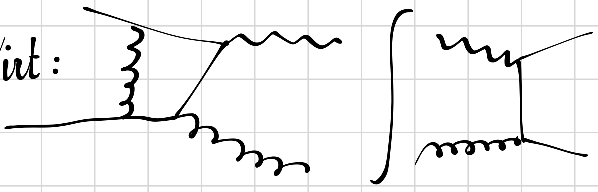



$\longrightarrow -\frac{1}{\varepsilon^2}$



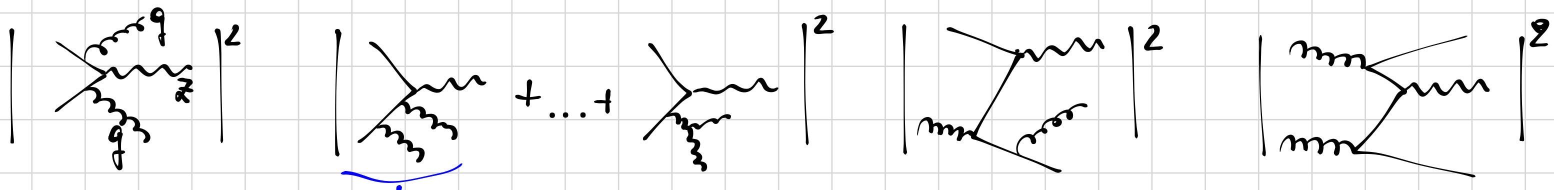
$\longrightarrow |1\text{-loop}|^2$

Mixed Real/Virt:



$\sim \mathrm{Re}\left(M_{n+1}^{(1)\dagger} M_{n+1}^{(0)}\right)$

Double real:




↳ also triple collinear!

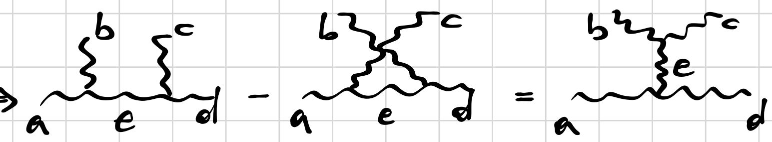
$\Rightarrow$ divergences cancel summing all contributions and using NLO Altarelli-Parisi

## Colour decomposition [illustrate for pure glue → all adj rep (pure $\mathcal{N}=1$ SYM, $\mathcal{N}=4$ SYM)]

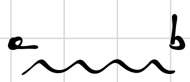
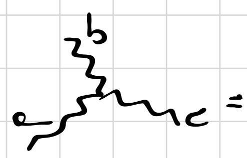
$$\underbrace{\mathcal{N}=1 \text{ SYM}}_{g,\,\tilde{g}} \qquad \underbrace{\mathcal{N}=4 \text{ SYM}}_{g,\,\tilde{g},\,\phi^{48}}$$

$$f^{abc} = (T^a_{adj})^{bc} \quad \text{(totally antisymm.)} \longrightarrow T^a \in SU(N_c)$$

- $[T^a, T^b] = f^{abc} T^c \Rightarrow$



$$f^{abe} f^{ecd} - f^{ace} f^{ebd} = f^{aed} f^{ebc} \qquad \text{(Jacobi identity)}$$

-  $= \delta_{ab}$
-  $= f^{abc}$

$$(a, b, c, \ldots = 1, \ldots, N_c^2 - 1)$$

Resolve this:

1) Introduce $T^a = T^a_{(\text{fundam})} \longrightarrow$ "Trace basis"

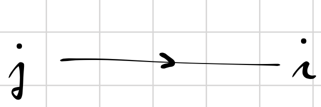
OR

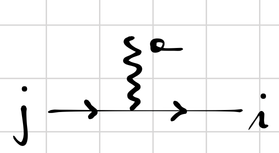
2) Solve the Jacobi's in some systematic way

"Birdtracks"
(P. Cvitanovic)

① Introduce $T^a$:

$$i \to N_c \qquad \bar{j} \to \bar{N}_c$$

 $= \delta_i^{\bar{j}}$

 $= (T^a)_i^{\bar{j}}$

$$\longrightarrow \mathrm{Tr}(T^a, T^b) = \delta^{ab} \quad (\text{instead of } \tfrac{1}{2}\delta^{ab})$$

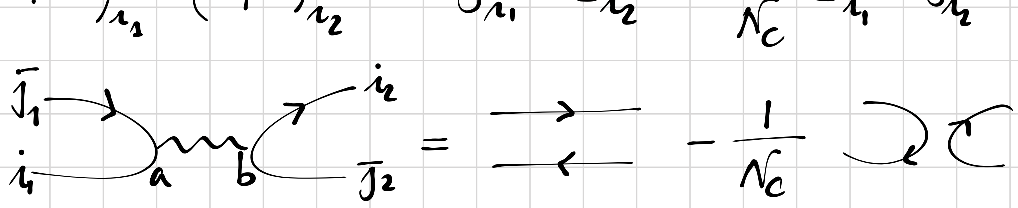
$$[T^a, T^b] = i\sqrt{2}\, f^{abc} T^c = \tilde{f}^{abc} T^c$$

$$\tilde{f}^{abc} = \mathrm{Tr}([T^a, T^b] T^c)$$

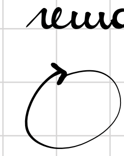
$\Rightarrow$ 

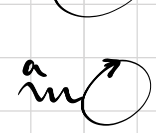
$\Rightarrow SU(N_c)$ Fierz identities:

$$\sum_a (T^a)_{i_1}^{\bar{j}_1} (T^a)_{i_2}^{\bar{j}_2} = \delta_{i_1}^{\bar{j}_2} \delta_{i_2}^{\bar{j}_1} - \frac{1}{N_c} \delta_{i_1}^{\bar{j}_1} \delta_{i_2}^{\bar{j}_2}$$

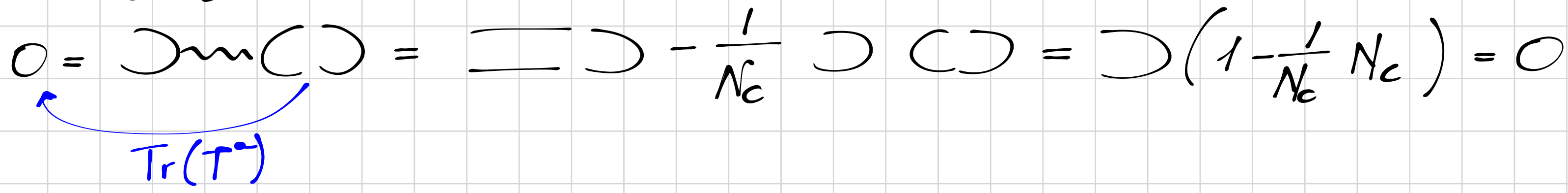


→ "remove the trace"

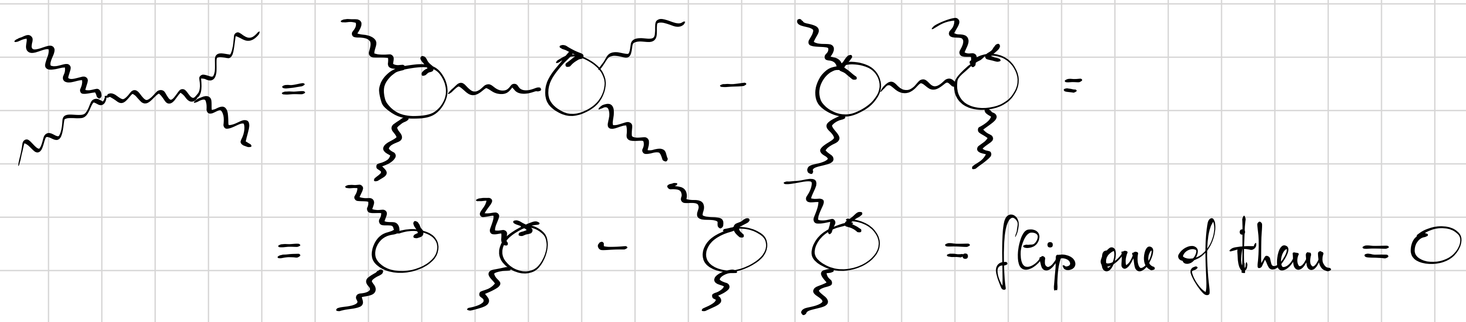
 $= \mathrm{Tr}_{(\text{fund})}(\mathbb{I}) = N_c$

 $= \mathrm{Tr}(T^a) = 0$

NB: Why $\frac{1}{N_c}$ when removing the trace?




If everything is adj (no ferm. lines) we don't need $-\frac{1}{N_c}$...

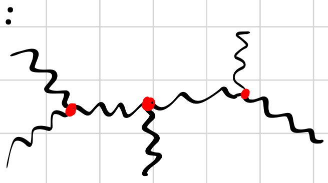



(no need to remove the trace)

$$\Rightarrow U(N_c) = SU(N_c) \otimes \cancel{U(1)} \longrightarrow f^{abx} = 0$$

$x$: $U(1)$

Consider:

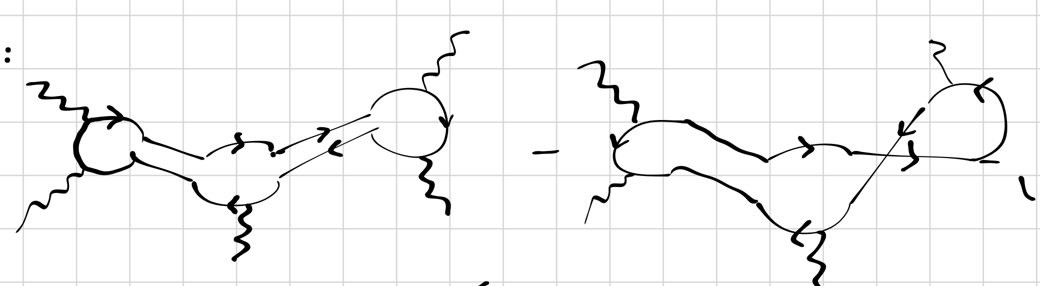


5-point $gg \longrightarrow ggg$

$\Downarrow$

I can insert  in 3 points $\rightarrow 2^3 = 8$ contributions

e.g.:



$\downarrow$

it's a single fermion trace!

full amplitude

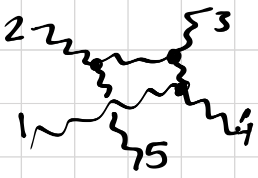
"colour stripped" amp.

$\rightarrow$ trees single traces: (n gluon amp.) $\rightarrow A_n^{tree}(\{k_i, h_i, a_i\}) =$

$$= g^{n-2} \sum_{\sigma \in S_n/\mathbb{Z}_n} \mathrm{Tr}\left(T^{a_{\sigma(1)}} T^{a_{\sigma(2)}} \dots T^{a_n}\right) A_n^{tree}\left(\sigma(1^{h_1}), \sigma(n-1)^{h_{n-1}}, \dots n^{h_n}\right)$$

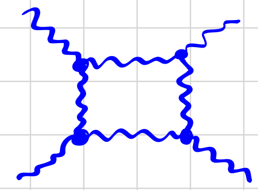
$\hookrightarrow$ non cyclic perm.

it's a decomp. into simpler pieces

eg:  $\longrightarrow \mathrm{Tr}(\dots T^{a_1} T^{a_4} \dots) \Rightarrow$ doesn't contribute to $\mathrm{Tr}(T^{a_1} T^{a_2} T^{a_4} T^{a_5})$

$\hookrightarrow$ $a_1$ and $a_4$ are not next to each other

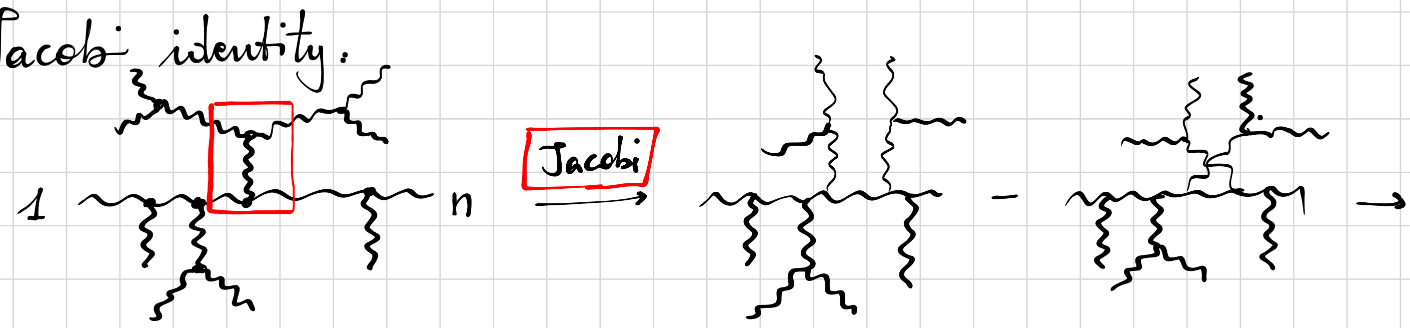
$\Rightarrow$ common to non planar diagrams ex: consider 1-loop case and compute color factors using this method



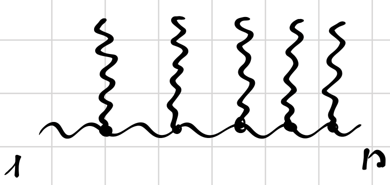
$\mathcal{A}_n^{tree}(1,2,\dots,n)$ only for planar diagrams

$\rightarrow$ only singularities are in $(k_i + k_{i+1} + \dots + k_{i+j})^2$

② Solve Jacobi identity.

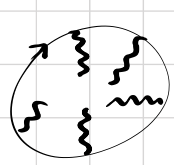


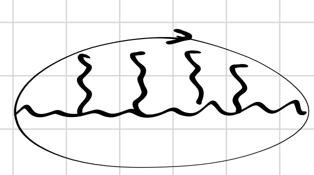

$\longrightarrow$ move all the vertices on the "main" line $\rightarrow$

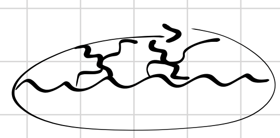
$\longrightarrow$ 


$$\Rightarrow \mathcal{A}_n^{tree} = g^{n-2} \sum_{\sigma\in S_{n-2}} \left(\tilde{f}^{\,a_{\sigma(2)}} \cdots \tilde{f}^{\,\sigma(n-1)}\right)_{a_1 a_n} \mathcal{A}_n^{tree\,\prime}(1,\sigma(2),\dots,\sigma(n-1),n) \rightarrow \text{"}f\text{-basis decomposition"}$$

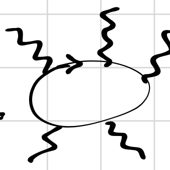
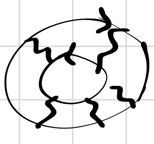
$\hookrightarrow$ in fact we'll see $\mathcal{A}_n = \mathcal{A}_n'$

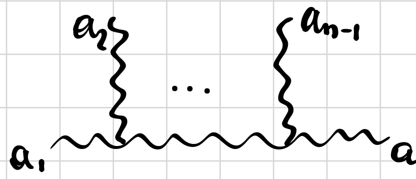
$\rightarrow$ contract with $\mathrm{Tr}(T^{a_1}\dots T^{a_n}) \rightarrow$ 

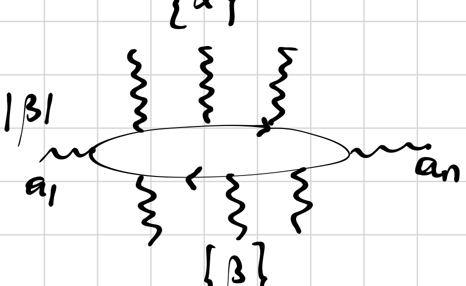
$\hookrightarrow$  $= N_c^{\,n} + \mathcal{O}(N_c^{\,n-2})$

 $= \mathcal{O}(N_c^{\,n-1})$

$\Rightarrow N_c \rightarrow \infty \rightarrow$ pick up one term

On the other side: $\mathcal{A}_n = \sum$  $\mathcal{A}_n^{tree} \xrightarrow{N_c\to\infty}$  $\longrightarrow$ the same happens

Consider. 

$= (-1)^{|\beta|}$ 


$$\Rightarrow \{a_2,\dots,a_{n-1}\} \in \{\alpha\} \sqcup\!\sqcup \{\beta\}^T$$

$\{\beta\}$ reversed

$\simeq$ shuffling a deck of cards but $\{\alpha\}$ and $\{\beta\}$ stay in order.

shuffle [preserves $\{\alpha\}$ and $\{\beta\}$ orderings]

The result is:

$$\mathcal{A}_n^{tree}(1,\{\alpha\},\{\beta\},n) = (-1)^{|\beta|} \sum_{\sigma\in\{\alpha\}\sqcup\{\beta\}^T} \mathcal{A}_n^{tree}(1,\sigma(2),\dots,\sigma(n-1),n)$$

$\Rightarrow$ KLEISS-KUIFF relations: $(n-1)! \longrightarrow (n-2)!$

# KINEMATICS

$\rightarrow$ 2 component spinors $\longrightarrow$ Pauli matrices $(\sigma^0 = \mathbb{I}_2)$

$\hookrightarrow \sigma^\mu_{\alpha\dot\alpha}$

RH $\nearrow$ $\nwarrow$ LH

Def: $k_{\alpha\dot\alpha} = k_\mu \sigma^\mu_{\alpha\dot\alpha} \rightarrow \det(k_{\alpha\dot\alpha}) = k_0^2 - |\vec{k}|^2 = k^2 = 0$

Since $\det(k)=0$, we can decompose on a prod of row and col. vect.:

$$k_{\alpha\dot\alpha} = \lambda_\alpha \tilde\lambda_{\dot\alpha} \longrightarrow \begin{array}{ll} \lambda_\alpha \sim \frac{1}{2}(1+\gamma_5)u(k) & [\text{RH spinor}] \\ \lambda_{\dot\alpha} \sim \frac{1}{2}(1-\gamma_5)u(k) & [\text{LH spinor}] \end{array}$$

$$\Rightarrow k_i^2 = 0 \rightarrow \sum_i k_i^\mu = 0 \longrightarrow \sum_i \lambda_\alpha^i \tilde\lambda_{\dot\alpha}^i = 0$$

$\rightarrow k^\mu$ real $\Rightarrow k_{\alpha\dot\alpha}$ = Hermitian $\quad \tilde\lambda_{\dot\alpha} = \pm \lambda_\alpha^*$

LORENTZ INV. Spinor prod $\longrightarrow$

$$\varepsilon^{\alpha\beta}\lambda_\alpha^i\lambda_\beta^j \equiv \langle ij\rangle \qquad (\langle ii\rangle = 0)$$

$$\varepsilon^{\dot\alpha\dot\beta}\tilde\lambda_{\dot\alpha}^i\tilde\lambda_{\dot\beta}^j \equiv [ij] \qquad ([ij]=0)$$

$$\Rightarrow \langle ij\rangle[ji] = \varepsilon^{\alpha\beta}\varepsilon^{\dot\alpha\dot\beta}\lambda_\alpha^i\lambda_\beta^j\tilde\lambda_{\dot\beta}^j\tilde\lambda_{\dot\alpha}^i - \frac{1}{2}\mathrm{Tr}(k_i k_j) = 2k_i k_j = s_{ij}$$

$$\Rightarrow \langle ij\rangle = \sqrt{s_{ij}}\, e^{i\phi_{ij}}$$

$$[ij] = \sqrt{s_{ij}}\, e^{-i\phi_{ij}}$$

$$\left(\tilde\lambda_{\dot\alpha} = \pm\lambda_\alpha^* \longrightarrow [ij] = \pm\langle ij\rangle^*\right)$$

eg: $k_1 + k_2 + k_3 = 0$ [3-point ampl]

↳ All $s_{ij} = 0 = 2k_ik_j = 2E_iE_j(1-\cos\theta_{ij})$ → $E=0 \Rightarrow$ sick! ; → exactly parallel → } kinematics is too singular!

$$\Rightarrow -k_1 = (2,0,0,2) = k_2 + k_3 = (1,1,i,1) + (1,-1,-i,1)$$

DEFINE

→ $\boxed{k_i \text{ complex}} \Rightarrow$ 2 non singular solutions

⇓

1) $\tilde\lambda_1 \propto \tilde\lambda_2 \propto \tilde\lambda_3 \longrightarrow$ all $[ij] \equiv 0$ but $\langle ij\rangle \neq 0$

2) $\lambda_1 \propto \lambda_2 \propto \lambda_3 \longrightarrow$ all $\langle ij\rangle \equiv 0$ but $[ij] \neq 0$

⇒ External states:

• $\frac{\text{Spinors}}{\text{fermions}}$ $h = +\frac{1}{2} \longrightarrow$ use $\tilde\lambda^i_{\dot\alpha} \sim 1/\lambda$ ; $h = -\frac{1}{2} \longrightarrow$ use $\lambda^i_{\alpha} \sim 1/\tilde\lambda$

"little group" scaling: $\lambda_i \to e^{i\phi}\lambda_i$
$\tilde\lambda_i \to e^{-i\phi}\tilde\lambda_i$

massless vectors: $\varepsilon^\mu(\sigma_\mu)_{\alpha\dot\alpha} = \varepsilon_{\alpha\dot\alpha}$ — two of these together are ok for gravi ten

↳ $h = +1 \rightarrow \varepsilon_{\alpha\dot\alpha}(q) = \dfrac{q_\alpha \tilde\lambda^i_{\dot\alpha}}{\varepsilon^{\beta\gamma} q_\beta \lambda^i_\gamma}$ — reference spinor

ex: $\varepsilon^i_\mu k^\mu_i = 0$
$\varepsilon^i_\mu q^\mu = 0$

$h = -1 \rightarrow \varepsilon_{\alpha\dot\alpha} = \dfrac{q_{\dot\alpha} \lambda^i_{\alpha}}{\varepsilon^{\dot\beta\dot\gamma} q_{\dot\beta} \tilde\lambda^i_{\dot\gamma}}$

$$\Rightarrow \mathcal{A}_n(i) \sim \lambda^{-2h_i}$$

⇒ 3-points amplitude

$$\mathcal{A}_3(1^{h_1}, 2^{h_2}, 3^{h_3}) = \# \langle 12\rangle^{h_3-h_1-h_2} \langle 23\rangle^{h_1-h_2-h_3} \langle 31\rangle^{h_2-h_3-h_1} \sim$$

$$\sim \lambda_1^{(h_3-h_1-h_2)+(h_2-h_3-h_1)} = \lambda_1^{-2h_1}$$

We found:

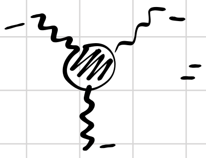
$$A_3(1^{h_1},2^{h_2},3^{h_3}) = \# \langle 12\rangle^{h_3-h_2-h_1}\langle 23\rangle^{h_1-h_3-h_2}\langle 31\rangle^{h_2-h_3-h_1} \quad \to \text{if choose all } [ij]\equiv 0$$

$$= \#[12]^{-h_3+h_2+h_1}[23]^{-h_1+h_3+h_2}[31]^{-h_2+h_3+h_1} \quad \to \text{if choose all } \langle ij\rangle\equiv 0$$

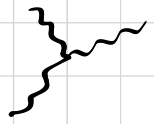
→ mass dimension of $\langle ij\rangle$ ?

$$\langle ij\rangle[ji] = s_{ij} \to [k^2] \longrightarrow [\langle ij\rangle] = [[ij]] = K$$

$$\Rightarrow A_3 \sim K^{-(h_1+h_2+h_3)} \sim K^{+(\dots)}$$

$\Rightarrow A_3(1^-,2^-,3^-) =$  $= \#\langle12\rangle\langle23\rangle\langle31\rangle \sim K^3$

$$= \frac{\#}{[12][23][31]} \sim K^{-3}$$

for YM  we expect $[A_3] \sim gK^{+1}$

$$\Rightarrow A_3 = \frac{\#}{\Lambda^2}\langle12\rangle\langle23\rangle\langle31\rangle \leftarrow \mathcal{L} \supseteq \frac{f^{abc}}{\Lambda^2} G^{a\nu}_{\mu} G^{b\rho}_{\nu} G^{c\mu}_{\rho}$$

↳ NOT YM

↳ the other one is TOO singular!

⇒ exchange particles $1 \longleftrightarrow 2$ → we need antisymm. factor for colour ⇒ $\boxed{f^{abc}}$ in front of $A_3$

Now take:

$$A_3(1^-,2^-,3^+) = (g)\frac{\langle12\rangle^3}{\langle23\rangle\langle31\rangle} = i\frac{\langle12\rangle^4}{\langle12\rangle\langle23\rangle\langle31\rangle}$$ "Parker-Taylor 3-points ampl"

Parity: $- \longleftrightarrow + \Rightarrow \lambda_i \longleftrightarrow \tilde{\lambda}_i$

$$A_3(1^+,2^+,3^-) = -i\frac{[12]^4}{[12][23][31]}$$

Gravity: [gravity ~ $(YM)^2$]

$$M_{tree}(1^{--},2^{--},3^{++}) = \left[A_3^{tree}(1^-,2^-,3^+)\right]^2$$

$$M_{tree}(1^{++},2^{++},3^{--}) = \left[A_3^{tree}(1^+,2^+,3^-)\right]^2$$

## Identities:

$$\langle i\, i \rangle = [i\, i] = 0$$
$$\langle ij \rangle = -\langle j i \rangle$$
$$[ij] = -[ji]$$
$$\langle ij \rangle [ji] = s_{ij}$$

Momentum conservation: $\displaystyle\sum_{i=1}^{n} \lambda_\alpha^i \tilde{\lambda}_{\dot\alpha}^i = 0$

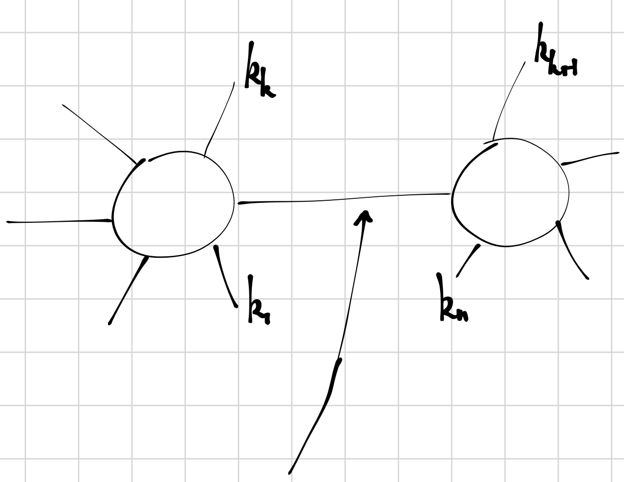
$$\sum_{i=1}^{n} \langle j i \rangle [i \ell] = 0$$

Schouten identities: $\langle ij \rangle\langle k\ell \rangle - \langle ik \rangle\langle j\ell \rangle = \langle i\ell \rangle\langle kj \rangle$

ex: show this. $\rightarrow$ $\lambda_k^\alpha = c_1 \lambda_i^\alpha + c_2 \lambda_j^\alpha$ for some #s $c_{1,2}$. $(\alpha = 1,2)$ Compute $c_{1,2}$.

## N point amplitude

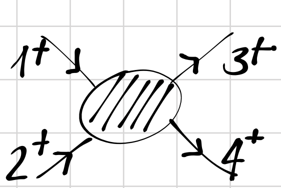
Suppose



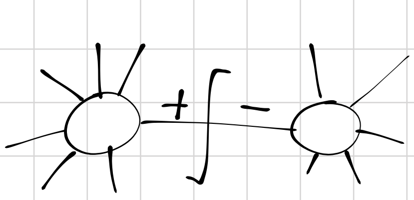

$K^2 = (k_1 + \ldots + k_k)^2 \rightarrow 0 \Rightarrow$ mass-shell $\rightarrow$ we can split the n-point

$$A_n \overset{K\to 0}{\sim} \frac{A_{k+1}\, A_{n-k+1}}{K^2} + O(1)$$

$\Rightarrow$ All outgoing helicity labelling:

$A_4(1^-, 2^-, 3^+, 4^+)$ is really 

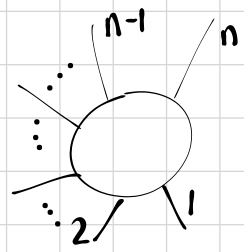

$\rightarrow$ when we cross the cut we have to switch $- \longleftrightarrow +$ :



BCFW "on-shell" recursion:

$$A_n^{\text{tree}} \longrightarrow A_n^{\text{tree}}(z)$$

$z$ ⟶ $\mathbb{C}$ parameter



$[n,1\rangle$ shift: $\tilde\lambda_n \to \tilde\lambda_n - z\tilde\lambda_1 \equiv \hat{\tilde\lambda}_n$ $\qquad \lambda_1 \to \lambda_1 + z\lambda_n \equiv \hat\lambda_1$

$\lambda_n \to \lambda_n \qquad \tilde\lambda_1 \to \tilde\lambda_1$

* is mom. still conserved?

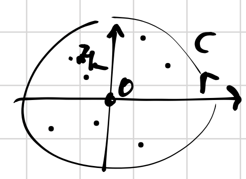
$$\lambda_1\tilde\lambda_1 + \lambda_n\tilde\lambda_n = 0 \rightsquigarrow z\left(\lambda_n\tilde\lambda_1 - \lambda_n\tilde\lambda_1\right) = 0$$

* $\hat{k}_1^2(z) = \hat{k}_n^2(z) = 0$

* shift vector: $V^2 = V\cdot k_1 = V\cdot k_n = 0$

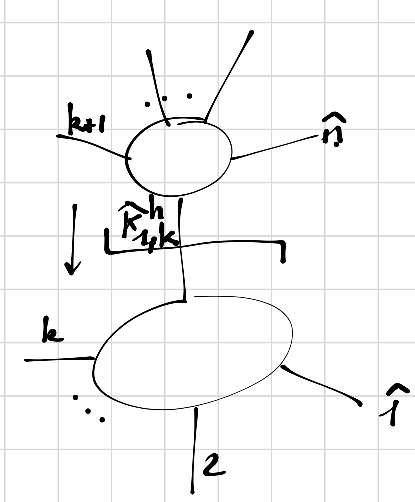
$\left( V_{\alpha\dot\alpha} = \lambda_{n,\alpha}\tilde\lambda_{1,\dot\alpha} \right)$ ⟶ complex vector!

Consider: what about $z\to\infty$



$$0 \overset{?}{=} \frac{1}{2\pi i}\oint_C dz\,\frac{A_n(z)}{z} = A_n(0) + \sum_{k=2}^{n-2} \operatorname{Res}\left[\frac{A_n(z)}{z}\right]_{z=z_k}$$

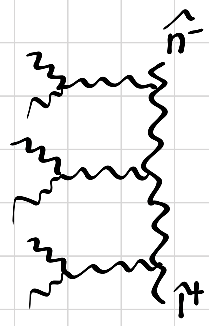
large contour!




$z\to\infty$: $n^-, 1^+$:

$$\varepsilon_n^- \sim \frac{\lambda_n}{\tilde\lambda_n} \longmapsto \frac{\lambda_n}{\tilde\lambda_n - z\tilde\lambda_1} \overset{z\to\infty}{\sim} \frac{1}{z}$$

$$\varepsilon_1^+ \sim \frac{\tilde\lambda_1}{\lambda_1} \sim \frac{\tilde\lambda_1}{\lambda_1 + z\lambda_n} \overset{z\to\infty}{\sim} \frac{1}{z}$$




$\Longrightarrow$ $A_n \sim \frac{1}{z}$

no contrib from $\infty$

↳ $\sim k \sim z \longrightarrow \frac{1}{P^2} \sim \frac{1}{(x+2v)^2} \sim \frac{1}{z(x\cdot v)}$

Then $\oint \frac{dz}{2\pi i}\cdot\frac{A_n(z)}{z} = A_n(0) + \sum_k \operatorname{Res}\left[\frac{A_n(z)}{z}\right]_{z=z_k} = 0$

What is $z_k$?

$$0 = \left(\hat{k}_1(z_k) + k_2 + k_3 + \dots + k_k\right)^2 = \left(z_k \lambda_n \tilde{\lambda}_1 + K_{1,k}\right)^2$$

$\to$ suppose $V = \sum_i k_i \longrightarrow \langle a|V|b] = \sum_i \langle a\, i\rangle [i\, b]$

$\hookrightarrow \quad 0 = z_k^2 \cdot 0 + z_k \langle n|K_{1,k}|1] + K_{1,k}^2 \longrightarrow z_k = \dfrac{-K_{1,k}^2}{\langle n|K_{1,k}|1]}$

We also need:

$$z\, K_{1,k}^2(z) \simeq \underbrace{z_k \langle n|K_{1,k}|1]}_{-K_{1,k}^2}(z - z_k)$$

ex: prove this

insert $z_k$ at the right point

$$\to A_n = A_n(0) = \sum_{h=\pm} \sum_{k=2}^{n-2} A_{k+1}\left(\hat{1}, 2, \dots, k, -\hat{K}_{1,k}^{-h}\right) \frac{i}{K_{1,k}^2} A_{n-k+1}\left(\hat{K}_{1,k}^{h}, k+1, \dots, n-1, \hat{n}\right)$$

BCFW RECURSION RELATION

NB: $A_n^{tree}(1^+, 2^+, \dots, n^+) = 0$

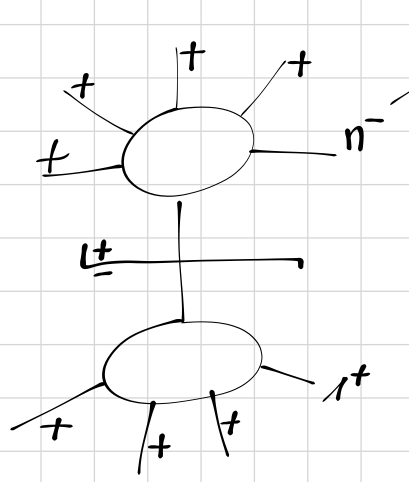
ex: show this. $\to$ 1) Show any tree amp. has $\geq 1$ $\varepsilon_i \cdot \varepsilon_j$

2) All + case choose $q_i \equiv q$ (all same)

$\hookrightarrow$ show $\varepsilon_i^+(q) \cdot \varepsilon_j^+(q) = 0$

Now show $A_{n>3}^{tree}(1^+, 2^+, \dots, (n-1)^+, n^-) = 0$:

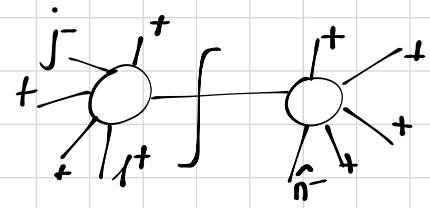
it has to stay here or on the other side would become $n^+ \to A_n = 0$ from the ex.




$= 0$ by induction

$\to$ Non vanishing case: Parke-Taylor amplitude [MHV amp $\to$ "Maximally Helicity Violating Amp"]

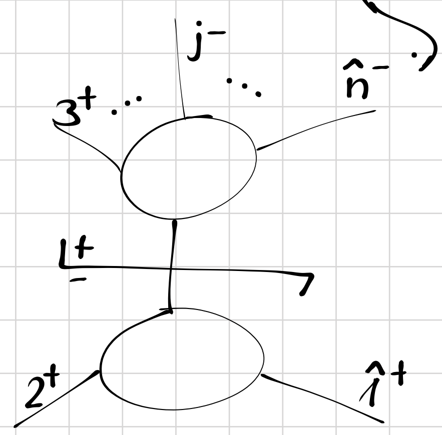
$$A_{jn}^{tree} = A_n^{tree}\left(1^+, \dots, j^-, \dots, (n-1)^+, n^-\right) = i \frac{\langle jn\rangle^4}{\langle 12\rangle\langle 23\rangle \dots \langle n1\rangle}$$




$= 0$

$$i\frac{\langle jn\rangle^4}{\langle \hat K 3\rangle\langle 34\rangle\dots\langle n-1,n\rangle\langle n,\hat K\rangle}$$

But $\underline{k=2}$:




$\tilde\lambda_1 + \tilde\lambda_2$ - NOT SHIFTED!

$$\mathcal{A}_3(1^+, 2^+, -\hat K^-_{1,2}) = -i\frac{[\hat 1 2]^{\cancel{4}\,3}}{\cancel{[\hat 1 2]}[2(-\hat K)][(-\hat K)\hat 1]} =$$

$$= i\frac{[12]^3}{[2\hat K][\hat K 1]}$$

$\Rightarrow$ BCFW:

$$A^{free}_{jn} = -i\frac{\langle jn\rangle^4}{\langle 34\rangle \dots \langle n-1,n\rangle}\ \frac{1}{\underset{\cancel{[12]}\langle 21\rangle}{s_{12}}}\ \frac{\cancel{[12]}^3}{\underbrace{\langle n\hat K\rangle[\hat K 2]}_{\cancel{[12]}}\underbrace{\langle 3\hat K\rangle[\hat K,1]}_{-\cancel{[12]}}}$$

$$\hat K = k_1 + k_2 + z_2\lambda_n\tilde\lambda_1$$

$$= i\frac{\langle jn\rangle^4}{\langle 12\rangle\langle 23\rangle\dots\langle n1\rangle}$$

---

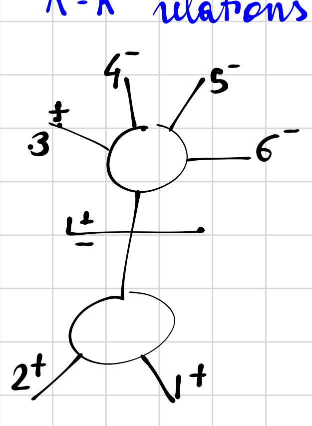
## NMHV 6-points ampl

$$(+++---),\ (++-+--),\ (+-+-+-)$$

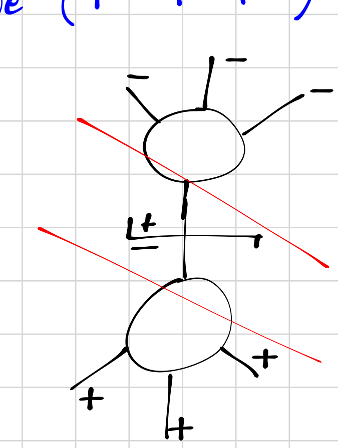
the rest is related by reflection + cyclic + parity

ex: use K-K relations to eliminate $(+-+-+-)$

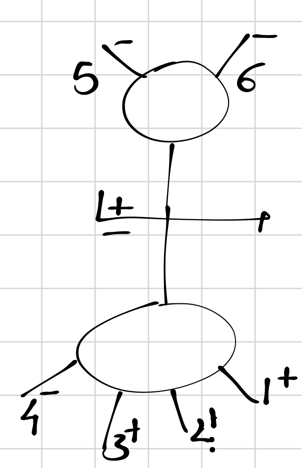
$\underline{k=2}$




$\underline{k=3}$




$\underline{k=4}$




$$\langle 1| \longleftrightarrow |6]$$
$$\vdots \qquad\qquad \vdots$$
$$\langle 6| \longleftrightarrow |1]$$

NB: $A^{\text{MHV}}_{jn}(1^+, \dots, j^-, \dots, (n-1)^+, n^-) = i \dfrac{\langle jn\rangle^4}{\langle 12\rangle\langle 23\rangle \dots \langle n1\rangle}$

$A^{\overline{\text{MHV}}}_{jn}(1^-, \dots, j^+, \dots, (n-1)^-, n^+) = -i \dfrac{[jn]^4}{[12][23]\dots[n1]}$ } PARITY

~~~

Go back to NMHV:

$$A_3(\hat{1}^+, 2^+, -\hat{K}_{1,2}) \frac{i}{s_{12}} A_5(\hat{K}^+_{1,2}, 3^+, 4^-, 5^-, \hat{6}^-)$$

$$\hookrightarrow \frac{[12]^3}{[2\hat{K}][\hat{K}1]\langle\hat{K}6\rangle\langle 6\hat{K}\rangle} \frac{i}{s_{12}} \frac{[\hat{K}3]^3 \langle 6\hat{K}\rangle^3}{[34][45][5\hat{6}][\hat{6}\hat{K}]\langle\hat{K}6\rangle}$$

$$= i \frac{\langle 6|(1+2)|3]^3}{\langle 61\rangle\langle 12\rangle[34][45] s_{612} \langle 2|(6+1)|5]}$$

$s_{612} = s_{61} + s_{62} + s_{12}$

$z_2 = -\dfrac{K^2_{12}}{\langle 6|K_{12}|1]} = -\dfrac{\langle 12\rangle[21]}{\langle 62\rangle[21]}$

$\hookrightarrow |\hat{1}] = |1]$

$|\hat{6}] = |6] + \dfrac{\langle 12\rangle}{\langle 62\rangle}|1]$

$\hat{K}_{1,2} = K_1 + K_2 - \dfrac{\langle 12\rangle}{\langle 62\rangle}|6\rangle[1|$

$\langle 6|\hat{K}|a] = \langle 6|(1+2)|a]$

$$\rightarrow A_6 = (k=2) + (k=4) = i\left[\frac{\langle 6|(1+2)|3]^3}{\langle 61\rangle\langle 12\rangle[34][45] s_{612}\langle 2|(6+1)|5]} + \frac{\langle 4|(5+6)|1]^3}{\langle 23\rangle\langle 34\rangle[56][61] s_{561}\langle 2|(6+1)|5]}\right]$$

$\hookrightarrow$ (difficult to show) THEY CANCEL!

$\Rightarrow \langle 2|(6+1)|5]$ : Spurious pole $\rightarrow K_6 + K_1 = c_1 K_1 + c_2 K_2$

$\langle 2| c_1 K_1 + c_2 K_2 |5] = 0$ if $(K_1 + K_6)_\perp = 0$

Now we want to compute stuff in real kinematics (almost):
~~~

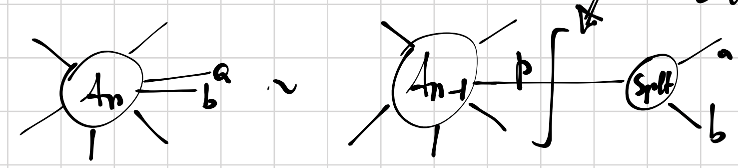
## Collinear behaviour (real momenta) and Altarelli-Parisi

Pick $a, b$ s.t.: $\longrightarrow$ i.e.: $a$ and $b$ COLLINEAR!

$k_a \simeq z k_p$

$k_b \simeq (1-z) k_p$

$k_p = k_a + k_b$

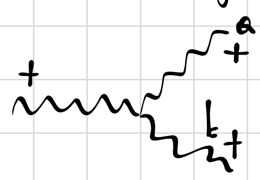



it's on shell since both a and b already were!

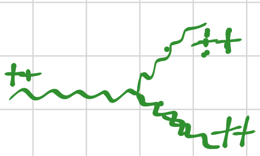
$$A_5(1^-, 2^-, 3^+, 4^+, 5^+) = i \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle} \sim \underbrace{i\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 3p\rangle\langle p1\rangle}}_{A_4(1^-, 2^-, 3^+, p^+)} \underbrace{\frac{1}{\sqrt{z(1-z)}\,\langle 45\rangle}}_{\text{Split}_-(4^+, 5^+)}$$

$\hookrightarrow$ $\text{split}_-(a^+, b^+) = \dfrac{1}{\sqrt{z(1-z)}\,\langle ab\rangle}$

$\langle ab\rangle \hookrightarrow \frac{1}{\sqrt{s_{ab}}}$ instead of $1/s_{ab}$ for angular mom. cons




$\Rightarrow$ Violates by an. unit $\rightarrow \dfrac{\sqrt{s_{ab}}}{s_{ab}}$

NB graviton: 

$\longrightarrow \dfrac{(\sqrt{s_{ab}})^2}{s_{ab}} \sim 1 \Rightarrow$ no singularities

$\rightarrow$ square of Non alg

NB
$$\text{Split}_-(a^+, b^+) = \frac{1}{\sqrt{z(1-z)}\,\langle ab\rangle}$$

$$\text{Split}_+(a^-, b^+) = \frac{z^2}{\sqrt{z(1-z)}\,\langle ab\rangle}$$

$$\text{Split}_+(a^+, b^-) = \frac{(1-z)^2}{\sqrt{z(1-z)}\,\langle ab\rangle}$$

$$\text{Split}_+(a^+, b^+) = 0$$

$$\text{Split}_+(a^-, b^-) = \frac{\pm 1}{\sqrt{z(1-z)}\,[ab]}$$

ex: check $3 \parallel 4$ limit of NMHV $A_6(+++---)$

$\rightarrow$ Altarelli-Parisi splitting funct.

DEF: $$P_{gg}(z) \propto s_{ab} \sum_{h_p, h_a, h_b} \left|\text{Split}_{-h_p}(a^{h_a}, b^{h_b})\right|^2 = \# C_A \frac{1 + z^4 + (1-z)^4}{z(1-z)}$$

$C_A$ $\rightarrow$ adj Casimir

We've seen:

$$A_n^{\text{tree}}(1,\dots,n)$$

$$(n-1)!$$
$$\downarrow \text{ Kleiss–Kuijf}$$
$$(n-2)! \qquad\qquad \rightarrow \text{"How many ampl to compute?"}$$
$$\downarrow \text{ BCJ}$$
$$(n-3)!$$

$\Rightarrow$ $st\, A_4^{\text{tree}}(1234)$ is totally symmetric
$$\| $$
$$\langle ij\rangle^4 X_{1234} \qquad X_{1234} = \frac{s_{12}s_{23}}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} = \dots = \frac{[12][34]}{\langle 12\rangle\langle 34\rangle} = X_{2134} = X_{1243} = X_{2143}$$

$$\frac{X_{1324}}{X_{1234}} = \mathbb{I}$$ ex: check this

$\Rightarrow$ $st\, A_4^{\text{tree}}(1234) = us\, A_4^{\text{tree}}(1342) = tu\, A_4^{\text{tree}}(1423)$

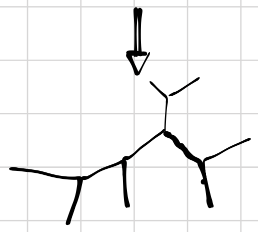
---

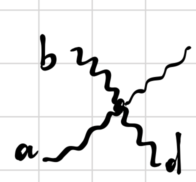
Write [BCJ]

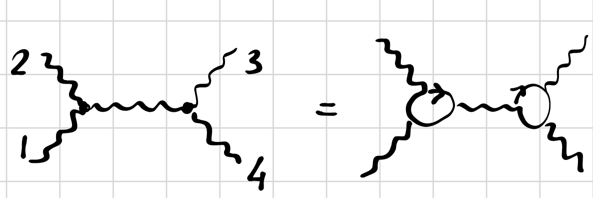
$$A_n^{\text{tree}} = \sum_{g \in \Gamma_{\text{cubic}}} \frac{n_g c_g}{\prod_{i \in \text{int lines of } g} p_i^2}$$

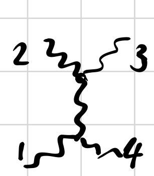
$n_g$: kinematics; $c_g$: colour factors $(f^{abc})$

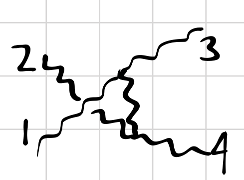
$\Downarrow$



 $= f^{abe} f^{cde} =$  with additional $p^2$.

$\rightarrow$ $c_s =$  $= \dots = \mathrm{tr}(1234) - \mathrm{tr}(1342) +$ reflections

$c_t =$  $= \mathrm{tr}(1234) - \mathrm{tr}(1423) +$ reflections

$c_u =$  $= -\mathrm{tr}(1342) + \mathrm{tr}(1423) +$ refl.

$$A_4^{\text{tree}}(1234) = \frac{n_s}{s} + \frac{n_t}{t}$$

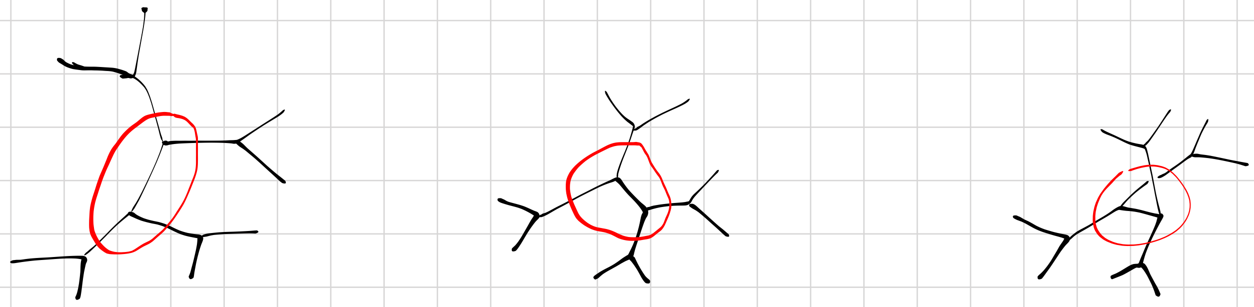
$$A_4^{\text{tree}}(1342) = -\frac{n_u}{u} - \frac{n_s}{s} \qquad \longrightarrow \quad st\left(\frac{n_s}{s}+\frac{n_t}{t}\right) = us\left(-\frac{n_u}{u}-\frac{n_s}{s}\right)$$

$$A_4^{\text{tree}}(1423) = -\frac{n_t}{t} + \frac{n_u}{u} \qquad \Longleftrightarrow \quad n_u = n_s - n_t$$

$\longrightarrow$ Jacobi relations for colour factors: $c_u = c_s - c_t$

$*$ BCJ :

consider



$\alpha$ $\qquad$ $\beta$ $\qquad$ $\gamma$

$\Rightarrow$ their triplets (differ for $u,t,s$ factors): whenever $c_\alpha - c_\beta + c_\gamma = 0$ then

$$n_\alpha - n_\beta + n_\gamma = 0.$$

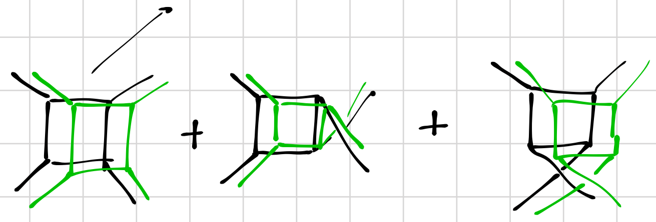
$\rightarrow$ BCJ showed:

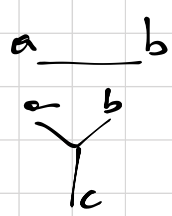
$$A_n^{\text{tree}}(1,2,\{\alpha\},3,\{\beta\}) = \sum_{\sigma\in \text{POP}\{\alpha\}\{\beta\}} A_n^{\text{tree}}(1,2,3,\sigma)\prod_{k=4}^{m}\frac{\overbrace{\mathcal{F}(\{\sigma,1/K)}^{\text{linear in } s_{ij}}}{s_{2,4,5,\dots,K}}$$

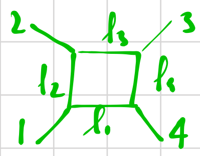
e.g.: $\mathcal{N}=4$ SYM

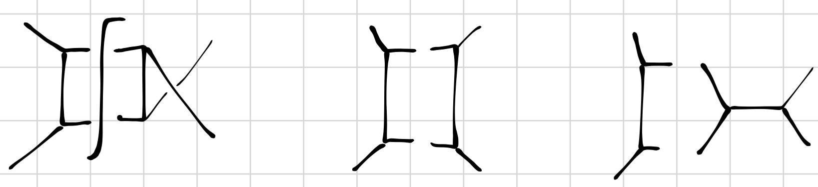
— : colour graph

— : kinematics

$$A_4^{1L} = st\, A_4^{\text{tree}}\Big[$$  $$\Big]$$

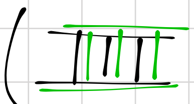
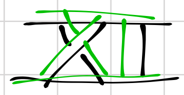
 $= \delta^{ab}$, $\; = f^{abc}$

 $$= \int \frac{d^4\ell}{(\dots)}\frac{1}{\ell_1^2\ell_2^2\ell_3^2\ell_4^2}$$



$n_u = 1$ $\qquad$ $n_s = 1$ $\qquad$ $n_t = 0$

$1 = 1 - 0$ ✓

$\mathcal{A}_4^{2L} = st\,\mathcal{A}_4^{\text{tree}}\Big[ s\Big($  $+$  $\Big) + \text{perm.}\Big]$ $\longrightarrow$ same checks...

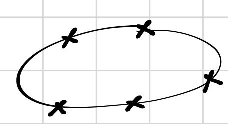
$\Rightarrow$ $\exists$ BCJ satisfying rep's for $L = 1, 2, 3, 4$ $\longrightarrow$ rep's for many theories

$\rightarrow$ "Gravity = YM$^2$" (Kawai, Lewellen, Tye 1986)

$\hookrightarrow$ KLT relations

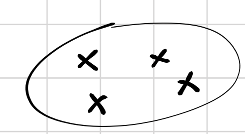
STRING THEORY

OPEN



gluons

CLOSED



gravitons

"products of open str. limit"

$$\underbrace{\mathcal{M}_4^{\text{tree}}(1234)}_{4\ \text{gravitons}} = -i s_{12}\, A_4^{\text{tree}}(1234)\, \tilde{A}_4^{\text{tree}}(1243)$$

$$\Rightarrow \mathcal{M}_4^{\text{tree}}(\pm,+,+,+) = 0$$

$$\Rightarrow \mathcal{M}_5^{\text{tree}}(12345) = i s_{12} s_{34}\, A_5^{\text{tree}}(12345)\, \tilde{A}_5^{\text{tree}}(21435) + (2 \leftrightarrow 3)$$

$\vdots$

If $\displaystyle A_n^{\text{tree}} = \sum_{g \in \Gamma_3} \frac{n_g c_g}{\prod_{i \in \text{int. lines}} p_i^2}$ then $\displaystyle \mathcal{M}_n^{\text{tree}} = \sum_{g \in \Gamma_3} \frac{n_g \tilde{n}_g}{\prod_{i \in \text{int. lines}} p_i^2}$ $\rightarrow \sim n_g^2$

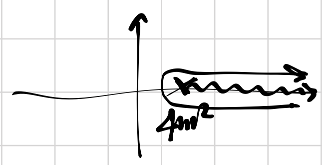
$\searrow$ at loop level may be $\neq$

$\mathcal{N}=8$ SUGRA $= \mathcal{N}=4$ SYM $\otimes$ $\mathcal{N}=4$ SYM

$\mathcal{N}=4$ SYM $\otimes$ pure gluons.

## LOOPS

$\rightarrow \begin{cases} S^\dagger S = \mathbb{I} \\ S = \mathbb{I} + iA \end{cases} \longrightarrow \underbrace{2\,\mathrm{Im}\,A}_{\mathrm{Disc}\,A} = A^\dagger A \longrightarrow$ perturbative expansion




(1) AMP

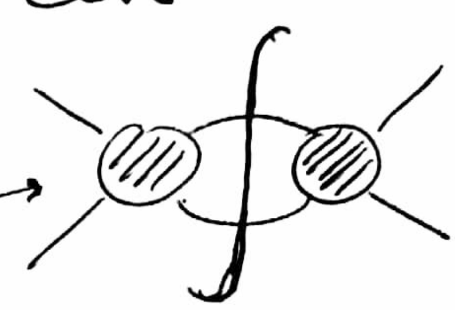
4 point ampl.

$$A_4 = g^2 A_4^{(0)} + g^4 A_4^{(1)} + \dots$$

( $\mathrm{Disc}\,A = A^\dagger A$ holds for real momenta including cuts )

$\Rightarrow$ • $\mathrm{Disc}\,A_4^{(0)} = 0$

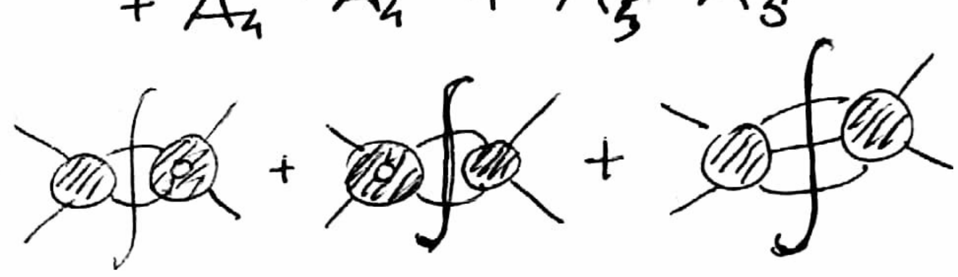
• $\mathrm{Disc}\,A_4^{(1)} = A_4^{(0)\dagger} A_4^{(0)}$



Disc 

$\Delta$ more generic than OPTICAL THEOREM

• $\mathrm{Disc}\,A_4^{(2)} = A_4^{(0)\dagger} A_4^{(1)} + A_4^{(1)\dagger} A_4^{(0)} + A_5^{(0)\dagger} A_5^{(0)}$

$=$ 

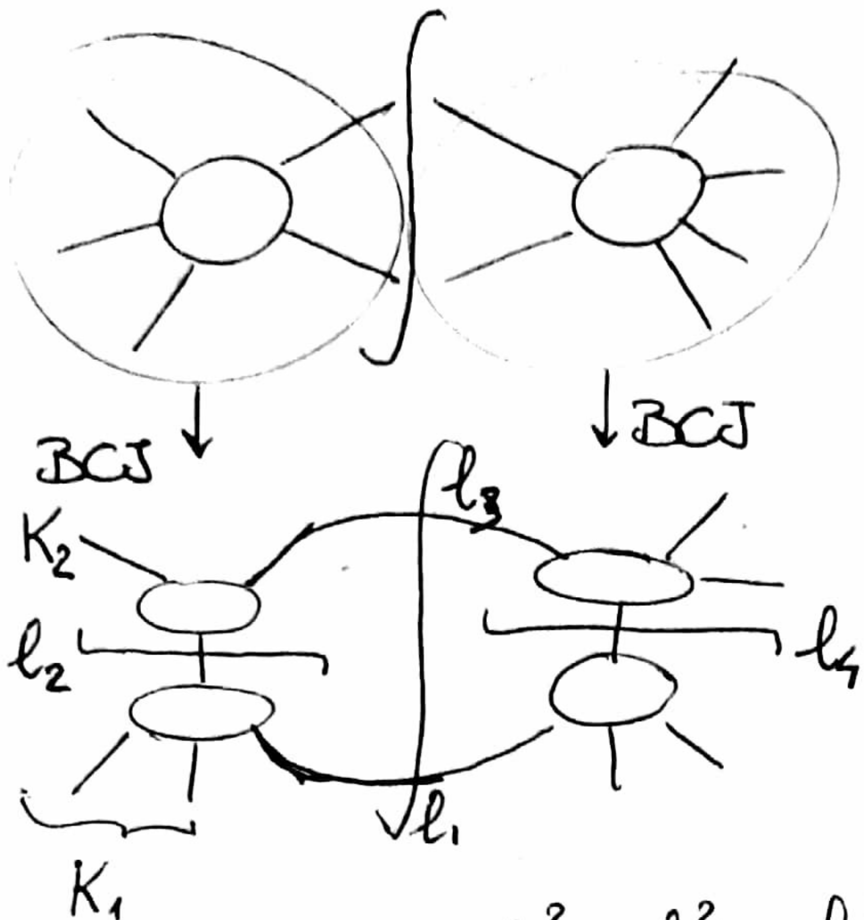
etc.

Therefore

1) Match these relations to a set of local integrands $\longrightarrow$ DETERMINE FULL AMPLITUDES cuts in $D = 4 - 2\varepsilon$

2) Let cut momenta be complex

e.g.:




($\cancel{4}$ 4 cuts !)

$\Rightarrow \ell_1^2 = \ell_2^2 = \ell_3^2 = \ell_4^2 = 0$ (mass shell)

↳ 4 equations

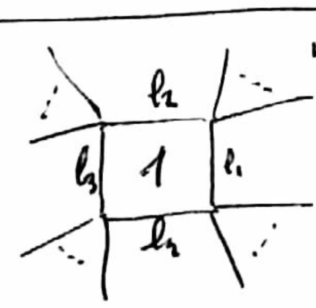
↳ 4 unknowns $\ell_1^{\mu=0,1,2,3}$ ("simple" kinematics with external momenta)

More equations
⇓
CANNOT SOLVE
(i.e.: no more than 4 cuts! in $D=4$)

→ I need $D=4-2\varepsilon$ which at 1-loop becomes $\sim D=5$

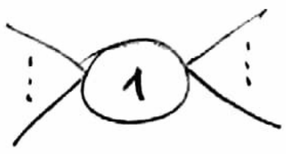
→ MATCH CUT INFO TO:

scalar box int $\int \frac{d^4 \ell_1}{(2\pi)^4} \frac{1}{\ell_1^2 \ell_2^2 \ell_3^2 \ell_4^2}$ (PASSARINO VELTMAN)

$A_n^{1\text{-loop}} = \sum_i d_i$  $+ \sum_i c_i$  $+$



no penthagon, exagon, ... integrals because they can be reduced

$+ \sum_i b_i$  $+ R_n + O(\varepsilon)$

- $d_i, c_i, b_i, R_n$ are RATIONAL FUNCTIONS (no cuts)
- $i \in \{$partition of $n$ legs into 4, 3, 2 sets$\}$
  - 4 → box, 3 → triangle, 2 → bubble

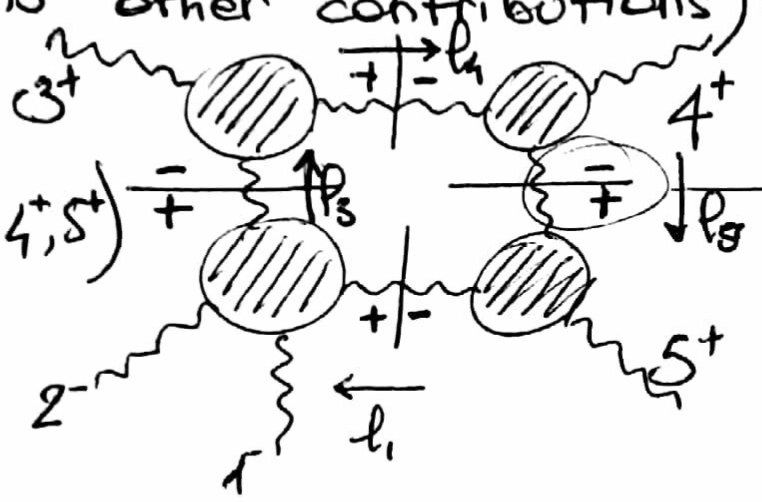
NB: suppose

$$\frac{\ell_1\cdot K_1}{\ell_1^2\ell_2^2\cdots} = \frac{-\frac{1}{2}\left[\overbrace{(\ell_1-K_1)^2}^{\ell_2^2}-\ell_1^2-K_1^2\right]}{\cdots} = -\frac{1}{2}\left\{\frac{1}{\ell_1^2\ell_3^2\cdots}-\frac{1}{\ell_2^2\ell_3^2\cdots}-\frac{K_1^2}{\ell_1^2\cdots}\right\}$$

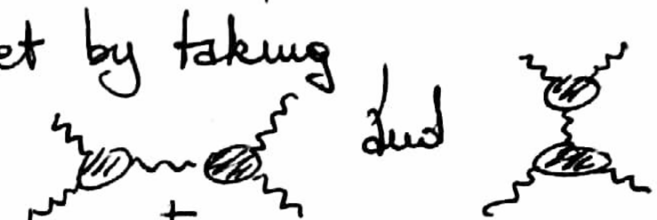
$\longrightarrow$ reduction to previous form!

Now consider e.g. a <u>quadrupole-cut</u> (2 cuts leads to contamination from [box] and [triangle] to ~~in~~ [bubble], while ~~[box]~~ 4 cuts has no other contributions):

$A_5^{1\,loop}(1^-,2^-,3^+,4^+,5^+)$




$\longrightarrow$ set by taking



$\Rightarrow A_4^{tree}$ has only $(--++)$ contrib $\neq 0$

$$\rightarrow \boxed{\ell_1^2=\ell_3^2=\ell_4^2=\ell_5^2=0}$$

~~2 sol.~~

$\ell_1^2=\ell_3^2=$
$=2(K_1+K_2)\cdot\ell_1-(k_1+k_2)^2$
$\Rightarrow \boxed{\text{linear}}$ $\longrightarrow$ only 1 equation is quad $\longrightarrow$ $\boxed{\text{2 SOLUTIONS}}$ (only one is NON VANISHING)

Then

$$A_5^{1\,loop}(1^-,2^-,3^+,4^+,5^+) = A_4^{tree}(-\ell_1^+,1^-,2^-,\ell_3^+)A_3^{tree}(-\ell_3^-,3^+,\ell_4^+)\cdot$$
$$\cdot A_3^{tree}(-\ell_4^+,4^+,\ell_5^-)A_3^{tree}(-\ell_5^+,5^+,\ell_1^-) =$$
$$= \frac{\langle 12\rangle^3}{\langle 2\ell_3\rangle\langle\ell_3\ell_1\rangle\langle\ell_1 1\rangle}\cdot\underbrace{\frac{[3\ell_4]^3}{[\ell_4\ell_3][\ell_3 3]}}_{\downarrow}\cdot\underbrace{\frac{\langle\ell_5\ell_4\rangle^3}{\langle 4\ell_5\rangle\langle\ell_4 4\rangle}}_{\downarrow}\cdot\frac{[\ell_5 5]^3}{[5\ell_5][\ell_1\ell_5]}$$

$\langle 3\ell_4\rangle=0$ $\quad$ $[4\ell_4]=0$

$$(\ell_4)_{\alpha\dot\alpha} = c(\lambda_3)_\alpha(\tilde\lambda_4)_{\dot\alpha}$$

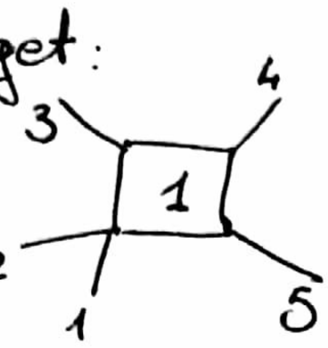
Therefore

$$\ell_1^2 = (\ell_4 - K_4 - K_5)^2 = 0$$
$$= S_{45} - 2\ell_4 \cdot (k_4 + k_5)$$

$$\left[ \text{ex: show} \quad c = \frac{\langle 45\rangle}{\langle 35\rangle} \right]$$



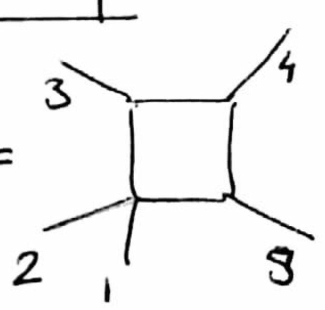
In the end we get:

• coeff of 


$$\Rightarrow d_{(12)} = -\frac{1}{2}\frac{\langle 12\rangle^3 S_{34}S_{45}}{\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle} =$$
$$= \frac{i}{2} S_{34} S_{45} A_5^{\text{tree}}(1^-,2^-,3^+,4^+,5^+)$$

~~Therefore:~~

$d_{(12)}$ multiplies:

$I_4^{(12)} =$ 

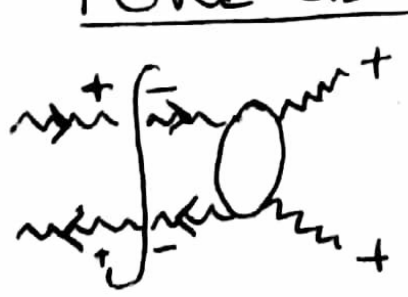

$$= -\frac{2i\,C_\Gamma}{S_{34}S_{45}}\left\{ -\frac{1}{\varepsilon^2}\left[\left(\frac{\mu^2}{-S_{34}}\right)^{\varepsilon} + \left(\frac{\mu^2}{-S_{45}}\right)^{\varepsilon} - \left(\frac{\mu^2}{-S_{12}}\right)^{\varepsilon}\right] + Li_2\left(1-\frac{S_{12}}{S_{34}}\right) + Li_2\left(1-\frac{S_{12}}{S_{45}}\right) + \frac{1}{2}\left(\ln\left(\frac{-S_{34}}{-S_{45}}\right)\right)^2 + \frac{\pi^2}{6}\right\}$$

where $Li_2(x) = -\int_0^x \frac{dt}{t}\ln(1-t)$

$\longrightarrow$ pure glue theory $\Longleftrightarrow$ $\boxed{\mathcal{N}=4 \text{ SYM}}$

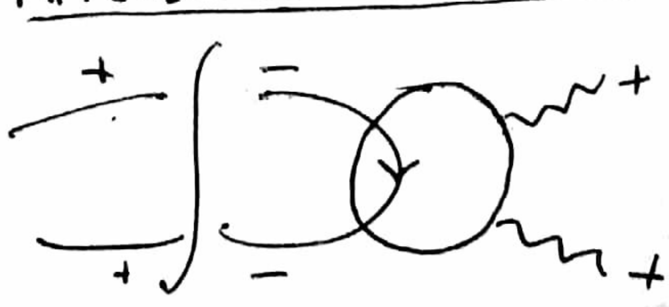
why?

PURE GLUE

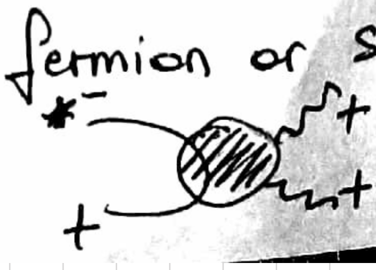



$\xleftrightarrow[\text{STRUCT}]{\text{SAME}}$

MASSLESS QUARKS




↳ no fermion or ~~scalar~~ loops!



$= 0$

- $\mathcal{N}=4$ SYM $\longrightarrow$ amplitudes are <u>ONLY BOXES</u>



$$A_5^{1\text{loop},\, \mathcal{N}=4\text{ SYM}} = A_5^{\text{tree}}\left[\{\dots\} + \text{cyclic}\right]$$

- QCD $\longrightarrow$ boxes as in $\mathcal{N}=4$ SYM
  
  \+ triangles
  
  \+ bubbles $\Big\}$ calculated separately

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## GENERALISED POLYLOGS

- General ampl $\Rightarrow$ general functions!

~~General~~

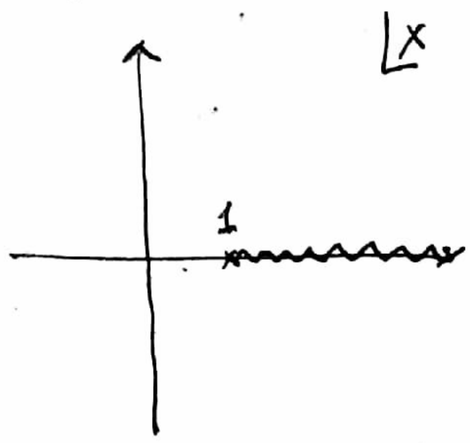
→ Classical polylog:

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \longrightarrow \int_0^x \frac{dt}{1-t} = -\ln(1-x) = \sum_{k=1}^{\infty} \frac{x^k}{k}$$

$$-\int_0^x \frac{dt}{t} \ln(1-t) = Li_2(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^2}$$

$$\Rightarrow \quad Li_n(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n} = \int_0^x \frac{dt}{t} Li_{n-1}(t)$$

$$Li_n(1) = \sum_{k=1}^{\infty} \frac{1}{k^n} = \zeta(n) = \zeta_n$$

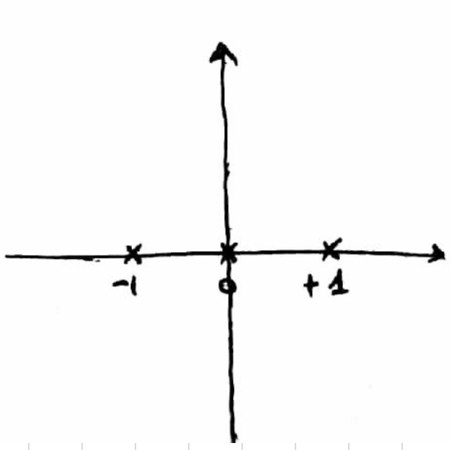



→ Harmonic polylog (HPL):

$$H_{\vec{w}}(x) \Rightarrow H_{0,\vec{w}}(x) = \int_0^x \frac{dt}{t} H_{\vec{w}}(t)$$

$$w_i \in \{-1,0,1\} \qquad H_{1,\vec{w}}(x) = \int_0^x \frac{dt}{1-t} H_{\vec{w}}(t)$$

$$H_{-1,\vec{w}}(x) = \int_0^x \frac{dt}{1+t} H_{\vec{w}}(t)$$



$$H_{\vec{0}_n}(x) = \frac{1}{n!}(\ln x)^n$$

Then HPL's $w_i \in \{0,1\} \rightarrow$ multiple zeta values (MZVs)

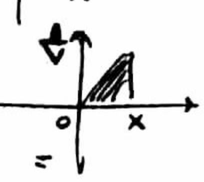
$$\zeta_{m_1,\dots,m_k} = \sum_{n_1>n_2>\dots>n_k>0} \frac{1}{n_1^{m_1} n_2^{m_2}\cdots n_k^{m_k}}, \; m_1 > 1$$

→ Goncharov polylog

$$G(a_1,\dots,a_n;x) = \int_0^x \frac{dt}{t-a_1}\, G(a_2,\dots,a_n;t)$$

$$a_i \in \mathbb{C} \quad \forall i = 1,\dots,n$$

IDENTITIES: e.g.: $G(a;x)G(b;x) = \int_0^x \int_0^x \frac{dt_1}{t_1-a}\,\frac{dt_2}{t_2-b} =$



$$= G(a,b;x) + G(b,a;x)$$

$$= \sum_{\sigma \in a ш b} G(\sigma;x)$$

⇓

SHUFFLE IDENTITIES

→ "Symbol" computations

• SYMBOL ⟶ Iterated derivatives

$$\left\{ \begin{aligned} & Li_2(1-x) = \frac{\pi^2}{6} - \ln x \cdot \ln(1-x) - Li_2(x) \\ & \frac{dLi_2(x)}{dx} = -\frac{\ln(1-x)}{x} = Li_1(x) \end{aligned} \right.$$

$$\hookrightarrow \cancel{\ln(1-x)}\; \cancel{\frac{\ln x}{1-x}} = -\frac{\ln(\cancel{1-x})}{\cancel{x}} + \cancel{\frac{\ln x}{1-x}} + \frac{\ln(\cancel{1-x})}{x} \quad \square$$

$$\Rightarrow Li_2(1) = \frac{\pi^2}{6} = \zeta(2)$$

function over K-dimens manifold (no. of integr. we have to do)

• $F$ WEIGHT $K \in \{1, 2, 3, \dots\}$ → "letters in symbol"

Variables $x_a$, $S_i(x_a) \in S$

$$\xrightarrow{\text{assume}} dF = \sum_{S_i \in S} \underbrace{F^{S_i}}_{\text{WEIGHT } K-1} d\ln S_i(x_a)$$

$$\Downarrow$$

$$\frac{\partial F}{\partial x_a} = \sum_{S_i \in S} F^{S_i} \frac{\partial \ln S_i(x_a)}{\partial x_a}$$

$$\rightarrow S = \{\overset{-1}{1+x}, \overset{0}{x}, \overset{+1}{1-x}\}$$

$$\rightarrow dF^{S_i} = \sum_{S_j \in S} F^{S_j, S_i} d\ln S_j(x_a)$$

$$d^2F = 0 = \sum_{S_i, S_j \in S} \underbrace{F^{S_j, S_i} d\ln S_i \wedge d\ln S_j}_{\boxed{\text{INTEGRABILITY RELATIONS}}}$$

$$\rightarrow \boxed{S(F) = \sum_{S_{i_1}, \dots, S_{i_n}} F^{S_{i_1}, \dots, S_{i_n}} d\ln S_{i_1} \dots d\ln S_{i_n} = \sum_{S_{i_1}, \dots, S_{i_n}} F^{S_{i_1}, \dots, S_{i_n}} S_{i_1} \otimes \dots \otimes S_{i_n}}$$

"SYMBOL OF F"

$\Rightarrow$ $\cancel{S(Li_2(x)) = -\ln}$

$$dLi_2(x) = -\ln(1-x)\, d\ln x \rightarrow S(Li_2(x)) = -(1-x) \otimes x$$

$$S(Li_2(1-x)) = -x \otimes (1-x)$$

Therefore

$$d(FG) = G\,dF + F\,dG = \sum_{S_i} (GF^{S_i} + FG^{S_i})\, d\ln S_i \ni F^{S_j S_i} + G^{S_j} F^{S_i} + \dots$$

$$S[FG] = S[F] ⧢ S[G]$$

$$\downarrow$$

e.g.:

$$S[\ln x \cdot \ln y] = x \otimes y + y \otimes x$$

$$S[(\ln x)^2 \cdot \ln y] = 2(x \otimes x \otimes y + x \otimes y \otimes x + y \otimes x \otimes x$$

⑧ AMP

$$\Rightarrow \mathcal{S}\left[Li_2(1-x)\right] = \mathcal{S}\left[\frac{\pi^2}{6} - \ln x \cdot \ln(1-x) - Li_2(x)\right]$$

$$\hookrightarrow \cancel{-x \otimes (1-x)} = 0 \cancel{- x \otimes (1-x)} \cancel{- (1-x) \otimes x} + \cancel{(1-x) \otimes x} \quad \square$$