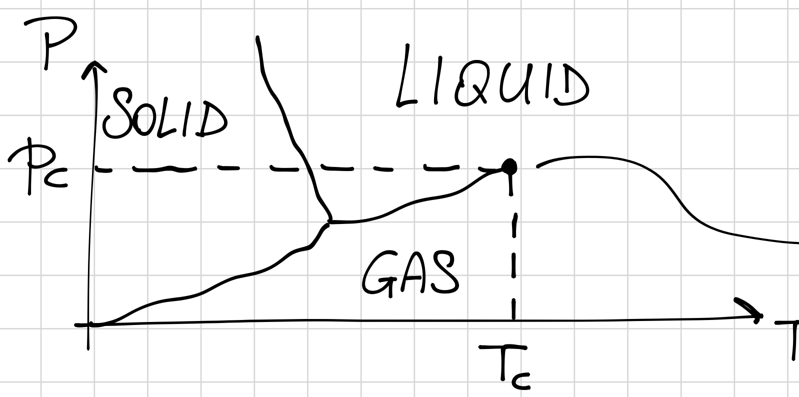
LACES 2018

CFT & BOOTSTRAP

[A. Vichi]

- D. Simmons Duffin, TASI Lectures, 1602.07982
- S. Rychkov, Lectures notes, 1601.05000
- D. Poland, S. Rychkov, A. Vichi, 1805.04405

e.g.: WATER



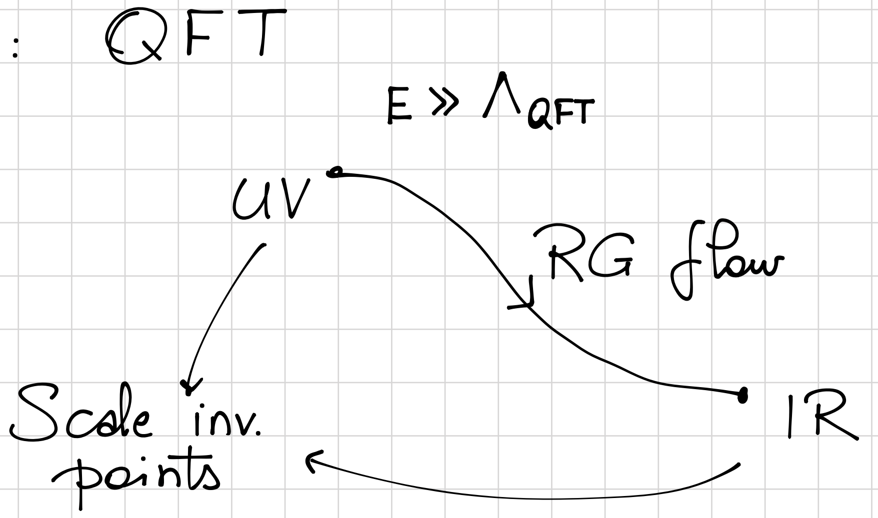
$\boxed{\xi \xrightarrow{T\to T_c} \infty} \Rightarrow$ NO LENGHT DEPENDENCE

$$\Rightarrow \quad \xi \sim (T-T_c)^{-\nu}$$

$$C \sim (T-T_c)^{2-3\nu}$$

"critical exponents"

e.g.: QFT

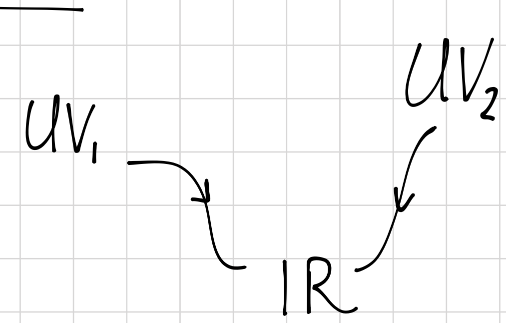


$E \ll \Lambda_{QFT}$

$\Rightarrow$ IR behaviour
- gapped (trivial)
- gapless dof (photons in QED, Goldstone)
- $\boxed{\text{fixed point} \Leftrightarrow \text{SCALE INV.}}$ $(\beta_i = 0)$

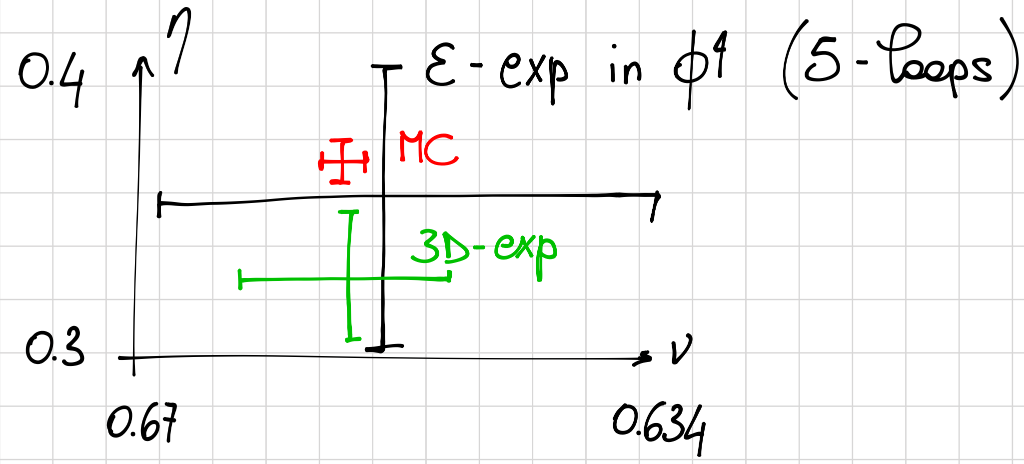
METHODS IN CFTs:

Universality:



"Universality classes"
(multiple UV descriptions can have the same IR)

e.g.:



$\Rightarrow$ BOOTSTRAP $\longrightarrow$ S-matrix: ('50 – '60)

Consistency conditions
- * Lorentz invariance
- * Unitarity
- * Crossing $\longrightarrow$

$\hookrightarrow$ CONFORMAL BOOTSTRAP

$\Rightarrow$ add CONFORMAL INVARIANCE!

(no more S matrices, though)

(no need for $\mathcal{L}$)

CONFORMAL INVARIANCE

$$x \longrightarrow x' = \lambda x \iff g_{\mu\nu} \longrightarrow \lambda^2 g_{\mu\nu}$$

$\hookrightarrow$ if invariant: $\delta S = \int d^d x \, T_{\mu\nu} \, \delta g^{\mu\nu} = 0$

$$\propto \int d^d x \, T_\mu{}^\mu = 0$$

$$\rightarrow T_\mu{}^\mu = \partial_\mu K^\mu \neq 0$$

(it is however very diff. to get:

$$[T_{\mu\nu}] = d \longrightarrow [K_\mu] = d-1$$)

$\Rightarrow$ In general there no example that contradict $T_\mu{}^\mu = 0$ (in 2D we can prove it)

$T_\mu^{\ \mu} = 0 \Rightarrow$ WEYL INVARIANCE (for infinitesimal Weyl transf)

$$g_{\mu\nu} \longrightarrow \lambda^2(x)\, g_{\mu\nu}$$

We still want flat background ($\Rightarrow \lambda^2 g_{\mu\nu}$ not curved).

$\hookrightarrow$ Conformal transf:

$$x \longrightarrow x' = x + \varepsilon(x)$$

s.t.

$$g_{\mu\nu}(x) \longrightarrow g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\, g_{\rho\sigma}(x) = \lambda^2 g_{\mu\nu}$$

Expand in ε and $\lambda(x) \sim 1 + O(\varepsilon^2)$:

$$\partial_\rho \varepsilon_\mu + \partial_\mu \varepsilon_\rho = \frac{2}{d}\partial^\sigma \varepsilon_\sigma\, g_{\mu\rho}$$

$\Rightarrow$ take one further derivative:

$$\partial_\mu \partial_\nu \varepsilon_\rho + \partial_\rho \partial_\nu \varepsilon_\mu = \partial_\nu \partial^\sigma \varepsilon_\sigma\, g_{\mu\rho}$$

$\Rightarrow$ Symm $\mu\nu\rho$: $\quad \partial_\mu\partial_\nu \varepsilon_\rho = \frac{1}{d}\left(\partial_\nu \partial^\sigma \varepsilon_\sigma\, g_{\mu\rho} - (\nu \leftrightarrow \mu) + (\mu \leftrightarrow \rho)\right)$

$$\Downarrow$$

$$\square \varepsilon_\mu = \frac{2-d}{d}\partial_\mu\left(\partial^\sigma \varepsilon_\sigma\right) \qquad (d=2 \text{ is particular})$$

Take $d \neq 2$:

$$\partial_\nu \square \varepsilon_\mu = \frac{2-d}{d}\partial_\nu\partial_\mu\left(\partial^\sigma\varepsilon_\sigma\right)$$

$\hookrightarrow$ Symm $\mu\nu \Rightarrow \quad \frac{2-d}{d}\partial_\mu\partial_\nu\left(\partial^\sigma\varepsilon_\sigma\right) = g_{\mu\nu}\,\square\left(\partial^\sigma\varepsilon_\sigma\right)$

$$\Downarrow$$

$$(d-1)\,\square f(x) = 0, \quad f(x) = \partial^\sigma \varepsilon_\sigma(x)$$

Then plug it into the previous eq: $\partial_\mu\partial_\nu f(x) = 0$

$$\Longrightarrow f(x) = a + b_\mu x^\mu.$$

$\Rightarrow$ $\varepsilon^\sigma(x)$ IS AT MOST QUADRATIC in the coord.!

$$\varepsilon_\mu(x) = c_\mu + \underbrace{a_{\mu\nu}}_{\downarrow} x^\nu + b_{\mu\nu\rho} x^\nu x^\rho$$

$$a_{\mu\nu} \longrightarrow \underset{\substack{\Downarrow \\ a g_{\mu\nu}}}{a_{(\mu\nu)}} + \underset{\substack{\Downarrow \\ \text{unconstrained } \omega_{\mu\nu}}}{a_{[\mu\nu]}}$$

Then plug it back to the original eq.:

$$b_{\mu\nu\rho} = \frac{1}{d}\left(b^\sigma{}_{\sigma\nu}\, g_{\mu\rho} + b^\sigma{}_{\sigma\rho}\, g_{\mu\nu} - b^\sigma{}_{\sigma\mu}\, g_{\nu\rho}\right)$$

$\hookrightarrow$ $\hat{b}_\nu$

Therefore:

$$\varepsilon_\mu(x) = \begin{cases} c_\mu : & \text{translations} \\ a\, x_\mu : & \text{dilatation} \\ \omega_{\mu\nu} x^\nu : & \text{Lorentz transf.} \\ 2(\hat{b}_\rho x^\rho) x_\mu - x^2 \hat{b}_\mu : & \text{Special conformal transf} \\ & (\text{"conformal boosts"}) \end{cases}$$

How many param? $d + \frac{d(d-1)}{2} + 1 = \frac{(d+1)(d+2)}{2}$ param

ALGEBRA of CFT

$$x \longrightarrow x + \varepsilon(x) = \zeta(x)$$

$$f(x) \longrightarrow f'(x') = f\left(\zeta^{-1}(x')\right) \longrightarrow f'(x) = e^{-G} f(x)$$

$\Rightarrow$ transl. $P_\mu = -i\partial_\mu$

Lorentz $M_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)$

dilat. $D = -i\, x_\mu \partial^\mu$

SCT $K_\mu = -i(2x_\mu x^\rho \partial_\rho - x^2 \partial_\mu)$

The non vanishing $[\,,\,]$ are (rescale $G \to -iG$):

$$[M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\nu\rho} M_{\mu\sigma} + \dots$$

$$[M_{\mu\nu}, P_\rho] = \eta_{\nu\rho} P_\mu - \eta_{\mu\rho} P_\nu$$

$$[M_{\mu\nu}, K_\rho] = \eta_{\nu\rho} K_\mu - \eta_{\mu\rho} K_\nu$$

$$[D, P_\mu] = P_\mu$$

$$[D, K_\mu] = -K_\mu$$

$$[K_\mu, K_\nu] = 2\eta_{\mu\nu} D - 2M_{\mu\nu}$$

Now let $A = 1, \dots, d+2$; $\mu = 1, \dots, d$.

Then

$$J_{d+1\,\mu} = (P_\mu - K_\mu)/2$$

$$J_{d+2\,\mu} = (P_\mu + K_\mu)/2$$

$$J_{\mu\nu} = M_{\mu\nu}$$

$$J_{d+1,\, d+2} = D$$

$$J_{AB} = -J_{BA}$$

$$\Rightarrow [J_{AB}, J_{CD}] = \eta_{BC} J_{AD} - \eta_{AC} J_{BD} + \eta_{BD} J_{AC} - \eta_{AD} J_{CB}$$

η_{AB} on $\mathbb{R}^{d+1,1}$

IRREPS OF CONFORMAL ALGEBRA

* Highest state irrep:

Lorentz quantum number(s) in d dim.

$$|\Delta\rangle : \quad D\underbrace{|\Delta, s\rangle}_{[M_{\mu\nu}, D] = 0} = \Delta |\Delta, s\rangle$$

• $\xi^\mu P_\mu |\Delta, s\rangle \longrightarrow |\Delta+1, s'\rangle$

$$\left[\ \xi^\mu D\left(P_\mu|\Delta,s\rangle\right) = \xi^\mu\left(P_\mu D|\Delta,s\rangle + P_\mu|\Delta,s\rangle\right) = \xi^\mu\,(\Delta+1)P_\mu|\Delta,s\rangle\ \right]$$

• $\xi^\mu K_\mu |\Delta, s\rangle \longrightarrow |\Delta-1, s'\rangle$

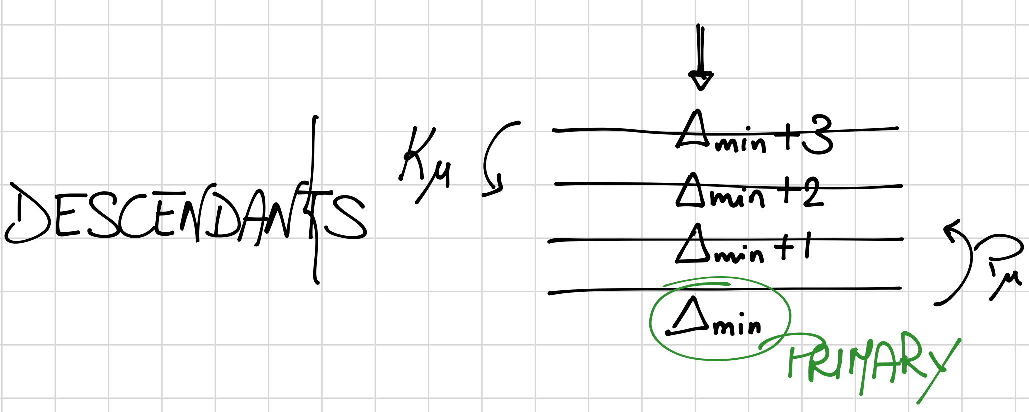
$\Rightarrow$ generically this is an ∞-dimensional rep $\rightarrow$ no bounds!

$\Rightarrow$ we impose a lower bound $\Rightarrow$ we don't want $\Delta \rightarrow -\infty$

$\hookrightarrow$ Restrict to irreps s.t. $\exists$ a state

$$\xi^\mu K_\mu \underbrace{|\Delta_{min}, s\rangle}_{\text{PRIMARY STATE}} = 0$$

(all the other can be obtained acting w/ P_μ's)



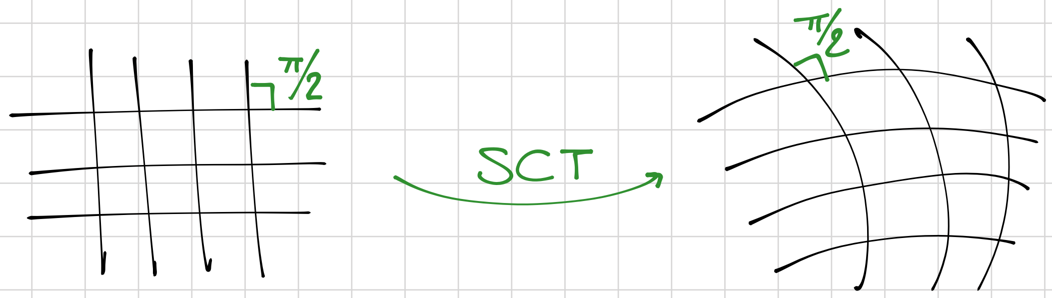
NB: be careful w/ Lorentz indices

in 3D $\rightarrow$ SO(3)

$$P_\mu|\Delta,s\rangle \begin{cases} |\Delta+1, s+1\rangle \\ |\Delta+1, s-1\rangle \end{cases}$$

CONFORMAL TRANSF.

* preserve the angles:



What are the invariants?

→ take $\underbrace{x_1, x_2, x_3}$

we can always use conf. inv. to set

$$x_1' = \vec{0}$$
$$x_2' = (0, \dots, 1)$$
$$x_3' = (0, \dots, \infty)$$

} gauge fixing of the conformal symm.

Is there residual symm?

$\Rightarrow$ STABILIZER of $x'_{1,2,3}$ is $SO(d-1)$

→ Now take $x_{1,\dots,4}$:

$$u = \frac{|x_1 - x_2|^2}{|x_1 - x_3|^2} \cdot \frac{|x_3 - x_4|^2}{|x_2 - x_4|^2}$$

$$v = \frac{|x_1 - x_4|^2}{|x_1 - x_3|^2} \cdot \frac{|x_2 - x_3|^2}{|x_2 - x_4|^2}$$

} CROSS RATIOS

$\Rightarrow$ $u = z\bar{z}$

$v = (1-z)(1-\bar{z})$

↳ in Euclidean $\bar{z} = z^*$

in Lorentzian $z, \bar{z} \in \mathbb{R}$ indep.

Generically, unless $s=0$ from the beginning, states have $\neq$ Spin!

$\Rightarrow$ FINITE CONFORMAL TRANSF:

$$x_\mu \longrightarrow \begin{array}{ll} x_\mu + a_\mu & \text{transl.} \\ \lambda x_\mu & \text{dilat.} \\ \Lambda_\mu{}^\nu x_\nu & \text{Lorentz. transf.} \\ \dfrac{x_\mu - b_\mu x^2}{1-2b\cdot x + b^2x^2} & \text{SCT} \end{array}$$

We shall consider also the "inversion" ($\notin \mathrm{Conf}(d)$):

$$I: \; x_\mu \mapsto \frac{x_\mu}{x^2} \Longrightarrow I\cdot P_\mu \cdot I = K_\mu \qquad (NB: \; I^2 = \mathbb{I})$$

Under finite conf. transf.:

$$g_{\mu\nu} \longrightarrow \Omega^2 g_{\mu\nu}$$

transl + Lorentz: $\Omega = 1$

dilat: $\Omega = \lambda$

SCT: $\Omega = 1 - 2b\cdot x + b^2 x^2$.

TRANSF. PROP. OF OPERATORS

$$\mathcal{O}(x) = e^{-P\cdot x}\,\mathcal{O}(0)\,e^{P\cdot x}$$

$$\Rightarrow [G, \mathcal{O}(x)] = e^{-P\cdot x}[\tilde{G}, \mathcal{O}(0)]e^{P\cdot x}$$

$$\Downarrow$$

$$\tilde{G} = e^{P\cdot x}\, G\, e^{-P\cdot x}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} x_{\mu_1}\dots x_{\mu_n} [P^{\mu_1}, [P^{\mu_2}, [\dots [P^{\mu_n}, G]\dots]]$$

$[D,P] = P$

$[M_{\mu\nu}, P_\rho] \sim P$

$[K_\mu, P_\nu] \sim 2(D g_{\mu\nu} - M_{\mu\nu})$

at most it stops at the 2nd order

$$\Rightarrow [M_{\mu\nu}, O_r^I(0)] = \underbrace{R(M_{\mu\nu})}_{\substack{\text{rep. of } M_{\mu\nu} \\ \text{acting on } O}}{}^I{}_J \, O_r^J(0)$$

$$[D, O_r^I(0)] = \Delta_O O_r^I(0)$$

$[K_\mu, O_r^I(0)] = 0$ if O_r^I is a PRIMARY OPERATORS

→ PRIMARY OP.

$$O_r^I(x) \longrightarrow \frac{1}{\Omega(x)^{\Delta_O}} R^I{}_J \, O_r^J(x)$$

CORRELATION FUNCTIONS OF PRIMARIES

Euclidean

$$\langle O_1(x_1) \dots O_n(x_n) \rangle = \int \mathcal{D}[O] \, e^{-S_{CFT}} O_1(x_1) \dots O_n(x_n)$$

e.g.: consider $\phi_1 : \Delta_1$
$\phi_2 : \Delta_2$

$\Downarrow$

$$\langle \phi_1(x_1) \phi_2(x_2) \rangle = \frac{1}{|x_1 - x_2|^{\Delta_1 + \Delta_2}}$$ for scale + Lor. + dil.

↳ SCT $\Rightarrow \Delta_1 = \Delta_2$

$$\Rightarrow \langle \phi_1(x_1) \phi_2(x_2) \rangle = \frac{1}{|x_1 - x_2|^{2\Delta}}$$

$$\rightarrow \langle \phi_1'(x_1') \phi_2'(x_2') \rangle = \frac{1}{|x_1' - x_2'|^{\Delta_1 + \Delta_2}}$$

$$= \frac{1}{\Omega_{SCT}^{\Delta_1}(x_1) \, \Omega_{SCT}^{\Delta_2}(x_2)} \langle \phi_1(x_1) \phi_2(x_2) \rangle$$

$$\Leftrightarrow \Delta_1 = \Delta_2$$

Now consider

$$\langle \phi_1(x_1)\,\phi_2(x_2)\,\phi_3(x_3)\rangle = \frac{C_{123}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3}\,x_{13}^{\Delta_1-\Delta_2+\Delta_3}\,x_{23}^{-\Delta_1+\Delta_2+\Delta_3}}$$

We could fix $x_1 = 0$, $x_1 = 1$, $x_3 = \infty$ on the LHS. On the RHS there can only be const. terms $\Rightarrow$ the 3 point f. is completely fixed

NB: if the th. is UNITARY:

$$\langle \phi_I(x_1)\,\phi_J(x_2)\rangle = \frac{\delta_{IJ}}{x_{12}^{2\Delta}}.$$

EMBEDDING FORMALISM

$\rightarrow$ Conf. group $\sim$ $SO(d+1,1)$

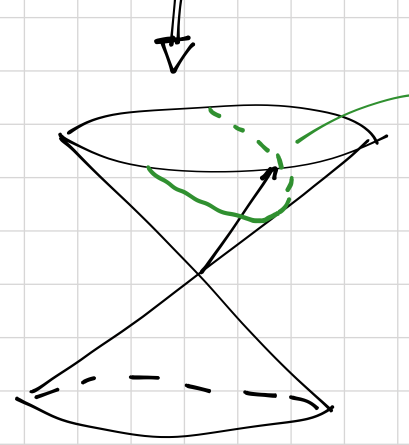
$$\begin{tikzcd} \text{Conf. group} \arrow[d, Rightarrow] & SO(d+1,1) \arrow[d, Rightarrow] \\ \mathbb{R}^d \arrow[r, bend right, "\text{EMBEDDING}"'] & \mathbb{R}^{d+1,1} \end{tikzcd}$$

$\Rightarrow$ $P^A \in \mathbb{R}^{d+1,1} \longrightarrow P^A P_A = P^A P^B \eta_{AB} = 0$ $\quad \eta_{AB} = \begin{pmatrix} + & & & \\ & + & & \\ & & + \ddots & \\ & & & - \end{pmatrix}$

$x^\mu \in \mathbb{R}^d$

use LC coords:

$$P^A = (P^+, P^-, P^\mu)$$
$$P^\pm = P^{d+2} \pm P^{d+1}$$



Poincaré section:
$$P^A = (1, x^2, x^\mu)$$

PROP:

$$P^A \longrightarrow \Lambda^A{}_B P^B$$

$\Lambda^A{}_B$: $SO(d+1,1)$ rot.

Hyperb. is invariant

$\Rightarrow$ the metric induced on the Poincaré section is flat.

$\Rightarrow$ The Poincaré section is not invariant

$$P_{PS}^A = (1, x^2, x^\mu) \xrightarrow{SO(d+1,1)} P^{A'} = (P^{+'}, P^{-'}, P^{'\mu}) = \Lambda^A{}_B P_{PS}^B$$

$$\hookrightarrow P_{PS}^{'A} = \frac{1}{P^{+'}} P^{'A} = (1, x'^2, x'^\mu)$$

$\Rightarrow$ x'^μ is related to x^μ by a conf. transf.

Now Consider

$$\mathcal{O}(P^A) \;\Big|\; \mathcal{O}(\lambda P^A) = \lambda^{-\Delta_\mathcal{O}} \mathcal{O}(P^A)$$

For scalars:

$$\underset{\text{in } \mathbb{R}^d}{\phi(x)} \longrightarrow \Phi(P_{PS}^A) \Rightarrow \Phi'(P'_{PS}) = \frac{1}{\underbrace{(P^{+'})}_{P^{+'} = \Omega(x)\ \checkmark}{}^{\Delta}} \Phi(P_{PS})$$

Take $\mathbb{R}^d$ $\qquad$ $\mathbb{R}^{d+1,1}$

$\Delta = [\phi(x)]$ $\qquad$ $\Phi(P^A)$ homogeneous of degree $-\Delta$ $\quad (P^2 = 0)$

$$\frac{1}{|x-y|^{2\Delta}} = \langle \phi(x)\phi(y)\rangle \qquad \langle \Phi_1(P_1)\,\Phi_2(P_2)\rangle = \frac{1}{(P_1\cdot P_2)^\alpha} \Rightarrow \Delta_1 = \Delta_2,\ \alpha = 2\Delta$$

$$\langle \Phi_1(P_1)\,\Phi_2(P_2)\,\Phi_3(P_3)\rangle = \frac{\tilde{C}_{123}}{(P_1\cdot P_2)^{\alpha_1}(P_2\cdot P_3)^{\alpha_2}(P_1\cdot P_3)^{\alpha_3}}$$

$$\hookrightarrow P_I \cdot P_J = -\frac{1}{2} X_{IJ}^2$$

TRACELESS SYMMETRIC OPERATORS

$$O_{\mu_1 \dots \mu_\ell}(x) \quad \text{in } \mathbb{R}^d$$

$$\Downarrow$$

$$\cancel{O_{A_1 \dots A_\ell}}(P) \Rightarrow \text{MUST PROJECT OUT SOME COMPONENTS}$$

$$\Rightarrow \quad O_{\mu_1 \dots \mu_\ell} = \overbrace{\frac{\partial P^{A_1}}{\partial X^{\mu_1}}}^{(0,\, 2x_\mu,\, \delta^\alpha_\mu)} \cdots \frac{\partial P^{A_\ell}}{\partial X^{\mu_\ell}} \, \hat{O}_{A_1 \dots A_\ell}(P)$$

$\Rightarrow$ if $\hat{O}$ is traceless symm., the same holds for O

$\Rightarrow$ any component of $\hat{O}$ $\propto$ to P is projected out : $P_A \dfrac{\partial P^A}{\partial X^\mu} = 0$

In conclusion:

Traceless Symmetric $O_{\mu_1 \dots \mu_\ell}$ $\longleftrightarrow$ Traceless Symmetric Transverse $\hat{O}_{A_1 \dots A_\ell}$

Therefore: e.g.

$$\langle \underbrace{V_1^{\mu_1}(x_1)\, V_2^{\mu_2}(x_2)}_{\text{VECTORS}} \rangle \Longrightarrow V_i^\mu(x_i) \to \hat{V}_i^A(P) : \begin{cases} \hat{V}_i^A(\lambda P) = \lambda^{-\Delta_i} \hat{V}_i^A(P) \\ \hat{V}_i^A(P)\, P_A = 0 \end{cases}$$

$$\Rightarrow \langle \hat{V}_1^A(P_1)\, \hat{V}_2^B(P_2) \rangle = G^{AB}(P_1, P_2)$$

$$G^{AB}(\lambda P_1, P_2) = \lambda^{-\Delta_1} G^{AB}(P_1, P_2) \qquad (\text{same for } P_2)$$

$$P_{1A}\, G^{AB} = P_{2B}\, G^{AB} = 0$$

$$\Rightarrow G^{AB}(P_1, P_2) = \frac{1}{(P_1 \cdot P_2)^\Delta} \left[C_1 \left(\eta^{AB} - \frac{P_2^A P_1^B}{P_1 \cdot P_2} \right) + \cancel{C_2 \left(\frac{P_1^A P_2^B}{P_1 \cdot P_2} \right)} \right]$$

PROJECTED OUT BY $\dfrac{\partial P_I}{\partial X}$

ex: prove that

$$\langle V^\mu(x_1)\, V^\nu(x_2)\rangle = \frac{1}{x_{12}^{2\Delta}} \underbrace{\left(\eta^{\mu\nu} - 2\, x_{12}^\mu x_{12}^\nu \frac{1}{x_{12}^2}\right)}_{I^{\mu\nu}}$$

In general:

$$\langle \mathcal{O}^{\mu_1\dots\mu_\ell}(x_1)\, \mathcal{O}^{\nu_1\dots\nu_\ell}(x_2)\rangle = \frac{1}{x_{12}^{2\Delta}}\left(I^{\mu_1\nu_1} I^{\mu_2\nu_2} \dots I^{\mu_\ell\nu_\ell} + \text{symm.}\right)$$

Now Consider a 3-point funct:

$$\text{SCAL - SCAL - SPIN } \ell \Rightarrow \langle \hat{\Phi}(P_1)\, \hat{\Phi}(P_2)\, \hat{\mathcal{O}}^{A_1\dots A_\ell}(P_3)\rangle$$

$\hookrightarrow -\Delta_\phi$ $\qquad\qquad$ $\hookrightarrow -\Delta_\mathcal{O}$

Encode a tensor in a polynomial:

$$f_{a_1\dots a_\ell} \longrightarrow f(z) = f_{a_1\dots a_\ell}\, z^{a_1} \dots z^{a_\ell}$$

$\hookrightarrow$ tracelessness: IMPOSE $z^2 = 0$

transversality: IMPOSE $z\cdot P = 0$

$$\Rightarrow \langle \hat{\Phi}(P_1)\, \hat{\Phi}(P_2)\, \mathcal{O}(P_3, z)\rangle = \mathcal{G}(P_1, P_2, P_3, z)$$

s.t. $P_1^2 = P_2^2 = P_3^2 = z^2 = 0$

$z\cdot P_3 = 0$

$\mathcal{G}(\dots)$ polyn. of order ℓ

$$\Rightarrow \mathcal{G}(P_1, P_2, P_3, z) = C_{123} \frac{\left((z\cdot P_1)(P_2\cdot P_3) - (z\cdot P_2)(P_1\cdot P_3)\right)^\ell}{(P_{12})^{\Delta_1+\Delta_2-\Delta_3+\ell}\, (P_{23})^{-\Delta_1+\Delta_2+\Delta_3+\ell}\, (P_{13})^{\Delta_1-\Delta_2+\Delta_3+\ell}}$$

Now we act w/ the proj:

$$\langle \phi(x_1)\phi(x_2)\,\mathcal{O}^{\mu_1\ldots\mu_\ell}(x_3)\rangle = \underset{\text{"OPE coeff"}}{C_{123}}\;\frac{\overbrace{Z_{123}^{\mu_1}\ldots Z_{123}^{\mu_\ell} - \text{traces}}^{\text{COMPLETELY FIXED by Conf. Sym.}}}{x_{12}^{\alpha_1}\,x_{23}^{\alpha_2}\,x_{13}^{\alpha_3}}$$

where $Z^{\mu}_{IJK} = \dfrac{x^{\mu}_{IK}}{|x_{IK}|} - \dfrac{x^{\mu}_{JK}}{|x_{JK}|}$.

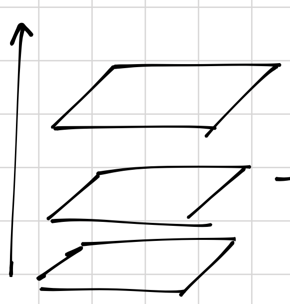
$$\alpha_1 = \Delta_1 + \Delta_2 - \Delta_3 + \ell$$

$$\alpha_2 = -\Delta_1 + \Delta_2 + \Delta_3 + \ell$$

$$\alpha_3 = \Delta_1 - \Delta_2 + \Delta_3 + \ell$$

RADIAL QUANTIZATION

on $\mathbb{R}^d$: t

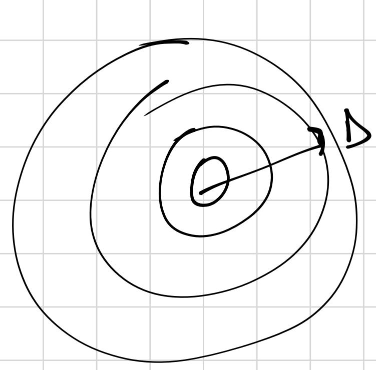


spacelike surfaces $\uparrow H = P^0$

However if I act with P_μ in CFT, I'll change the state.

D: use dilatation instead

$\Rightarrow$ Foliate using spheres:



$\Rightarrow$ Radial quant.

⇒ PATH INTEGRAL in QFT

$$\langle \phi(t_f) | \psi(t_i) \rangle = \int_{\substack{\theta(t_i)=\psi \\ \theta(t_f)=\phi}} \mathcal{D}[\theta]\, e^{-S[\theta]}$$

It also def. states: $\;U(t_f - t_i)|\psi(t_i)\rangle = \int_{\theta(t_i)=\psi} \mathcal{D}[\theta]\, e^{-S[\theta]}$

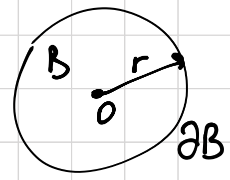
⇒ no fix endpoint

→ Initial state $t \to -\infty$ only $|0\rangle$ contr. to $e^{-S[\theta]}$:

$$U(t_f - t_i)|0,-\infty\rangle = |0, t_f\rangle_{un} = \lim_{t\to-\infty} \int_{\substack{\theta(t_i)=\phi \\ \theta(t_f)=?}} \mathcal{D}[\theta]\, e^{-S[\theta]}$$

⇒ with appropriate normal. it does not dep. on ϕ

Now consider



$$|0\rangle_r = \lim_{r\to 0} \int_{\substack{\theta(\partial B)=? \\ \theta(0)=\phi}} \mathcal{D}[\theta]$$

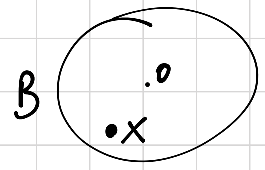
Operator ⟶ State

Then $\displaystyle\int_{B_r} \mathcal{D}[\theta]\, e^{-S[\theta]}\, \mathcal{O}(0) = |\mathcal{O}\rangle$ is a "primary state" def. on the

$= \mathcal{O}(0)|0\rangle$ boundary of B

primary

$$\Rightarrow \int_B \mathcal{D}[\theta]\, e^{-S[\theta]}\, \mathcal{O}(x) = \mathcal{O}(x)|0\rangle = e^{P\cdot x}\mathcal{O}(0)e^{-P\cdot x}|0\rangle =$$



$$= e^{P\cdot x}\mathcal{O}(0)|0\rangle =$$

$$= (1 + P\cdot x + \dots)\,\mathcal{O}(0)|0\rangle$$

∞ superpos. of primary and desc.

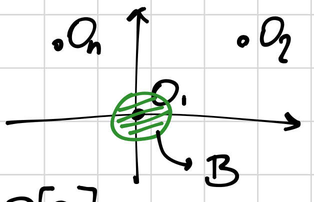
Given an op. we can therefore generate a state

$$|0\rangle \longleftrightarrow \mathbb{I}$$
$$|\mathcal{O}(0)\rangle \longleftrightarrow \mathcal{O}(0) \quad \text{primary}$$
$$|\mathcal{O}(x)\rangle \longleftrightarrow \mathcal{O}(x) \quad \text{prim. + desc.}$$

State $\to$ Operator

Suppose $\quad \langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle$

set $x_1 = 0 \longrightarrow$ Cut a hole around 0 :



$$\Rightarrow \langle \dots \rangle = \int_{\substack{\mathbb{R}^d \setminus B \\ \mathcal{O}(\partial B) = \text{fixed}}} \mathcal{D}[\mathcal{O}]\ \mathcal{O}_1(0)\mathcal{O}_2(x_2)\dots\mathcal{O}_n(x_n)\ e^{-S[\mathcal{O}]}$$

$$= \langle 0 | \mathcal{O}_2(x_2) \dots \mathcal{O}_n(x_n) \underbrace{|\mathcal{O}_1\rangle}_{\text{state}}$$

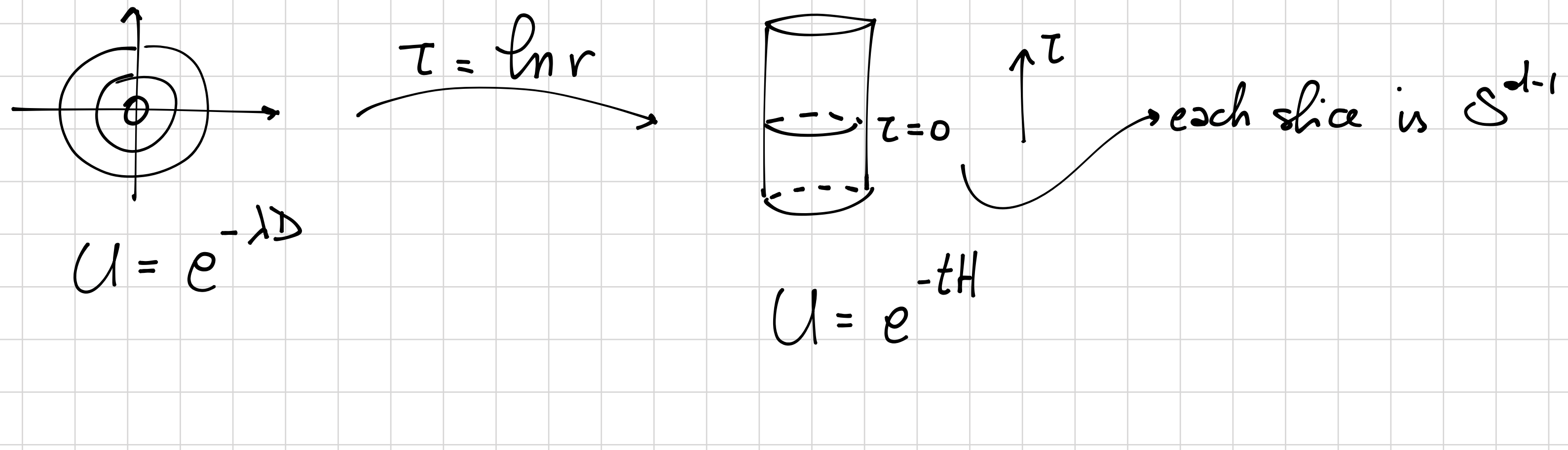
CFT $\Rightarrow$ operator $\longleftrightarrow$ state

$\rightarrow \quad x^\mu \in \mathbb{R}^d, \quad x^\mu = (r, \vec{n}) \qquad r \in \mathbb{R}^+ \quad \vec{n} \in S^{d-1}$

Def $\tau = \ln r.$

$\hookrightarrow \qquad r \to \lambda r \Rightarrow \tau \to \tau + \ln \lambda \sim$ "time translation"

$$\Rightarrow ds^2 = dr^2 + r^2 d\Omega_{d-1} = e^{2\tau} \underbrace{\left(d\tau^2 + d\Omega_{d-1} \right)}_{\text{CYLINDER } \mathbb{R} \times S^{d-1}}$$



Conseq. of Weyl inv.:

$$\langle O_1(x_1)\dots O_n(x_n)\rangle_{\underbrace{g_{\mu\nu}}_{\text{cyl.}}} = \langle O_1(x_1)\dots O_n(x_n)\rangle_{\underbrace{\Omega^2 g_{\mu\nu}}_{\text{flat}}} \prod_{i=1}^{n} \Omega^{\Delta_i}(x_i)$$

Then

$$ds^2_{flat} = e^{2\tau}\, ds^2_{cyl}$$

$$\Rightarrow O_{flat}(\tau,\vec{n}) \longrightarrow O_{cyl}(\tau,\vec{n}) = e^{\Delta_O \tau}\, O_{flat}(\tau,\vec{n})$$

↳ Consider a scalar on the cyl. ϕ:

$$\langle \phi(\tau_1,n_1)\,\phi(\tau_2,n_2)\rangle_{cyl} = \langle \underbrace{\phi_{cyl}(\tau_1,n_1)}_{= e^{\Delta_\phi \tau_1}\phi_{flat}}\,\phi_{cyl}(\tau_2,n_2)\rangle_{flat}$$

$$\Rightarrow \langle \phi_{cyl}\,\phi_{cyl}\rangle = \underbrace{e^{\Delta_\phi \tau_1}}_{= r_1^{\Delta_\phi}}\, \underbrace{e^{\Delta_\phi \tau_2}}_{= r_2^{\Delta_\phi}}\, \frac{1}{|x_1-x_2|^{2\Delta}} = \frac{(r_1/r_2)^{\Delta_\phi}}{\left|1-2\frac{r_1}{r_2}(\vec{n}_1\cdot\vec{n}_2)+\frac{r_1^2}{r_2^2}\right|^{\Delta_\phi}} =$$

$$= \frac{e^{\Delta_\phi \tau_{12}}}{\left|1-2e^{\tau_{12}}(\vec{n}_1\cdot\vec{n}_2)+e^{2\tau_{12}}\right|^{\Delta_\phi}} = f(\tau_{12},\vec{n}_1\cdot\vec{n}_2)$$

↳ $\langle \phi_{cyl} \,{}_{\wedge}\, \phi_{cyl}\rangle = \sum_n C_n\, e^{-(\Delta_\phi+n)\tau_{21}}$ $\qquad \tau_{21}>0$

$\sum_n |E_n\rangle\langle E_n|$ → eigenstate of en. have INCREASING ENERGY

$$\Rightarrow C_n \sim |\langle \phi_{cyl}|E_n\rangle|^2 > 0$$

Now the question is $|\psi\rangle \to \langle\psi| = ?$

CONJUGATION IN EUCLIDEAN QFT

* LORENTZIAN: $\mathcal{O}(t,\vec{x}) = e^{iHt - i\vec{x}\cdot\vec{p}}\,\mathcal{O}(0,\vec{0})\,e^{-iHt + i\vec{x}\cdot\vec{p}}$

↳ $\left(\mathcal{O}(t,\vec{x})\right)^\dagger = \mathcal{O}(t,\vec{x})$ if $\mathcal{O}(0,\vec{0})^\dagger = \mathcal{O}(0,\vec{0})$

* EUCLIDEAN: $t = -it_E$

$$\Rightarrow \mathcal{O}_E(t_E,\vec{x}) = \mathcal{O}(-it_E,\vec{x}) = e^{Ht_E - i\vec{x}\cdot\vec{P}}\,\mathcal{O}(0,\vec{0})\,e^{-Ht_E + i\vec{x}\cdot\vec{p}}$$

↳ $\left(\mathcal{O}_E(t_E,\vec{x})\right)^\dagger = \mathcal{O}_E(-t_E,\vec{x})$

TIME REFLECTION !

For spin-op.:

$\mathcal{O}^{\mu_1\ldots\mu_\ell}(t,\vec{x}) \xrightarrow{\text{WICK ROT.}}$ (mult. time comp. by $-i$)

↳ enforce $SO(d)$

$$\Rightarrow \left(\mathcal{O}_E^{\mu_1\ldots\mu_\ell}(t_E,\vec{x})\right)^\dagger = \Theta^{\mu_1}{}_{\nu_1}\cdots\Theta^{\mu_\ell}{}_{\nu_\ell}\,\mathcal{O}_E^{\nu_1\ldots\nu_\ell}(-t_E,\vec{x})$$

$$\Theta^{\mu}{}_{\nu} = \delta^{\mu}_{\nu} - 2\delta^{0}_{\nu}\delta^{\mu}_{0}$$

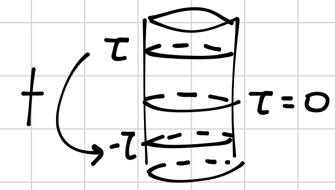
ex: in Lorentzian: $H = -\int d^{d-1}x\, T^{00}$ $\quad P^J = -\int d^{d-1}x\, T^{0J}$

↳ Wick rot. 0 ⇒ $H_E^\dagger = H_E$

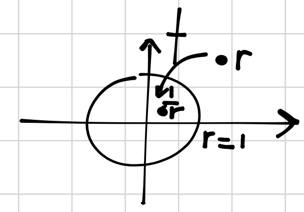
$P_E^{J\dagger} = -P_E^J$

On the cylinder:

$\left(\mathcal{O}(\tau,\vec{n})\right)^\dagger = \mathcal{O}(-\tau,\vec{n})$



In radial quant:



$$\tau = \ln r \Rightarrow \left(\mathcal{O}(r,\vec{n})\right)^\dagger = \mathcal{O}\left(\frac{1}{r},\vec{n}\right)$$

NB: $Q = \int_S dS_\mu J^\mu$

$$D^\dagger = D$$

$$(M_{\mu\nu})^\dagger = -M_{\mu\nu}$$

$$(P^\mu)^\dagger = I P^\mu I = K^\mu.$$

NB: $\langle \mathcal{O}_{\Delta,\ell}(x) \rangle \equiv 0.$

OPEs converge only for nearest fields (it conv. up to the nearest other field insertion).

$\Rightarrow$ We can then say

$$O_1 \underset{\text{"OPE"}}{\times} O_2 \supset O_3 \Leftrightarrow \langle O_1 O_2 O_3 \rangle \neq 0$$

$\Rightarrow \phi_\Delta(x) \times \phi_\Delta(0) \sim ?$

Conf. Symm.

1) $\langle \phi\phi\mathcal{O}_k \rangle \longrightarrow \langle \phi(x)\phi(-x)\mathcal{O}_k(0)\rangle = \dfrac{x^{\mu_1}\dots x^{\mu_\ell} - \text{traces}}{(\dots)(\dots)(\dots)}$

$\mathcal{O}_k$ can only transf. in traceless symm. irrep

$\hookrightarrow \phi(x) \times \phi(0) \sim x^{\mu_1}\dots x^{\mu_\ell}\,\mathcal{O}_{\mu_1\dots\mu_\ell}$

For $x \to -x$ invariance $\Rightarrow$ ℓ even $\to \langle \phi\phi\,(\text{even spin operator})\rangle$

CFT DATA

No $\mathcal{L}$, no S, we just need a set of op. $\left\{ \{ \Delta_i, s_i \} \right\} \longleftrightarrow \langle \mathcal{O}_{\Delta,s} \mathcal{O}_{\Delta,s} \rangle$

and the spectrum $\longleftrightarrow$ operator content, OPE coeff $\longleftrightarrow$ 3 pnt funct. (f_{ijk})

$\Rightarrow$ Use OPE to compute everything

$\rightarrow$ 4 pnt function (scalars):

$\hookrightarrow$ Conf. inv.: $u = \dfrac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$ $\quad v = \dfrac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$

$$\langle \mathcal{O}_1 \, \mathcal{O}_2 \, \mathcal{O}_3 \, \mathcal{O}_4 \rangle = (\cdots) \times \underbrace{g(u,v)}$$

$(\Delta_1 = \Delta_2 , \Delta_3 = \Delta_4)$

this DOES NOT spoil conf. inv. $\Rightarrow$ 4 pnt func. is NOT fixed!

$$\langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \mathcal{O}_3(x_3) \mathcal{O}_4(x_4) \rangle = $$

$$= \frac{g(u,v)}{x_{12}^{2\Delta_1} \; x_{34}^{2\Delta_3}} .$$

Now

$$\langle \underbrace{\mathcal{O}_1 \mathcal{O}_2} \underbrace{\mathcal{O}_3 \mathcal{O}_4} \rangle = \sum_k f_{12k} f_{34k} W_k(x_i) =$$

"partial wave"

$$= \sum_k C_{12k}(x_{12}, \partial_2) C_{34k}(x_{34}, \partial_4) \langle \mathcal{O}_k(x_2) \mathcal{O}_k(x_4) \rangle$$

behaves like 4 pnt. funct.

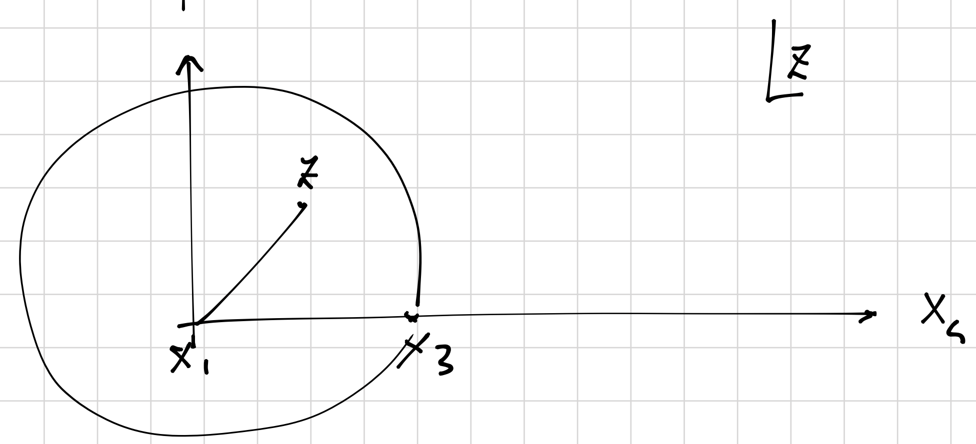
"kinematic factor"

$$\Rightarrow W_k(x_i) = K_4(x_i) \, \overbrace{g_{\Delta_k, \ell_k}(u,v)}$$

"CONFORMAL BLOCK" (contains info on all the ops)

$$\Delta_1 = \Delta_2 , \; \Delta_3 = \Delta_4 \Rightarrow K_4(x_i) = \frac{1}{x_{12}^{2\Delta_1} x_{34}^{2\Delta_3}}$$

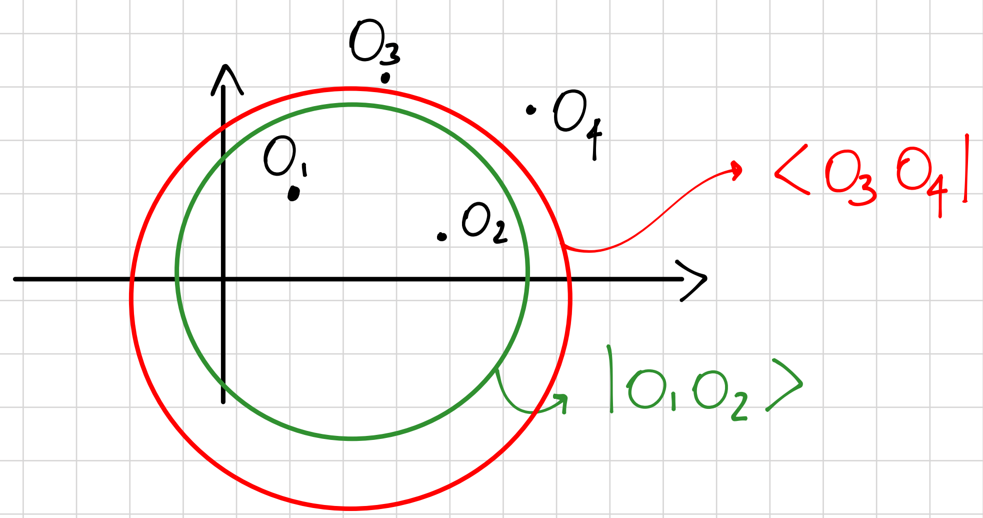
On the plane:



$u = z\bar z$

$v = (1-z)(1-\bar z)$

CASIMIR EQUATION FOR CONFORMAL BLOCKS



⇒ we are computing:

$$\langle O_3 O_4 \underset{\wedge}{|} O_1 O_2 \rangle = \sum_k \underbrace{f_{12k} f_{34k} W_k}$$

$\mathbb{I} = \sum_k P_k$

$$= \langle O_3 O_4 | P_k | O_1 O_2 \rangle$$

$$\Rightarrow P_k = \sum_{\alpha,\beta = |O_k\rangle, P_\mu|O_k\rangle, \dots} |\alpha\rangle\langle\beta| \, G^{\alpha\beta}$$

where $G^{\alpha\beta} = \langle \alpha | \beta \rangle$

$\Rightarrow \quad C_2 = J^{AB} J_{AB}$ where J_{AB} are gen. of $SO(d+1,1)$

↳ $[C_2, J_{AB}] = 0 \Rightarrow C_2 \propto \mathbb{I}$

on the left

$$\rightarrow \langle O_3 O_4 | P_k C_2 | O_1 O_2 \rangle = \underbrace{C_{\Delta,\ell}} \langle O_3 O_4 | P_k | O_1 O_2 \rangle$$

$$= \Delta(\Delta - d) + \ell(\ell + d - 2)$$

on the right

$$= D_{12} \langle O_3 O_4 | P_k | O_1 O_2 \rangle$$

↳ Diff op.

↳ $\underset{J_1^{AB} + J_2^{AB}}{\overset{}{J^{AB}}} O_1(x_1) O_2(x_2) |0\rangle = (D_1^{AB} + D_2^{AB}) O_1 O_2 |0\rangle$

Then :

$$\mathcal{D}_{12} \langle O_3 O_4 | P_k | O_1 O_2 \rangle = C_{\Delta,\ell} \langle O_3 O_4 | P_k | O_1 O_2 \rangle$$

$\Rightarrow$ Diff. eq. for $g_{\Delta,\ell}(u,v) = \tilde{g}_{\Delta,\ell}(z,\bar{z})$:

$$\mathcal{D}_{12}\, \tilde{g}_{\Delta,\ell}(z,\bar{z}) - C_{\Delta,\ell}\, \tilde{g}_{\Delta,\ell}(z,\bar{z})$$

$$\Rightarrow \mathcal{D}_{12} = \mathcal{D}_z + \mathcal{D}_{\bar{z}} + 2(d-2)\frac{z\bar{z}}{z-\bar{z}}\left[(1-z)\partial_z - (1-\bar{z})\partial_{\bar{z}}\right]$$

$$\mathcal{D}_z = 2z^2(1-z)\partial_z^2 - \left(2+\Delta_{34}-\Delta_{12}\right)z^2\partial_z + \frac{\Delta_{12}\Delta_{34}z}{2}$$

$$\mathcal{D}_{\bar{z}} = 2\bar{z}^2(1-\bar{z})\partial_{\bar{z}}^2 - \left(2+\Delta_{34}-\Delta_{12}\right)\bar{z}^2\partial_{\bar{z}} + \frac{\Delta_{12}\Delta_{34}\bar{z}}{2}$$

$$\Rightarrow g_{\Delta,\ell}(z,\bar{z}) \xrightarrow[z,\bar{z}\to 0]{} \underbrace{N_{d,\ell}}_{} \cdot (z\bar{z})^{\frac{\Delta}{2}} C_\ell^{\frac{d}{2}-1}\left(\frac{z+\bar{z}}{2\sqrt{z\bar{z}}}\right)$$

Normalization:

$$\frac{\ell!}{(-2)^\ell\left(\frac{d}{2}-1\right)_\ell}$$

When $d=2$: $\mathcal{D} = \mathcal{D}_z + \mathcal{D}_{\bar{z}}$

$$\Rightarrow \mathcal{D}_z K_\beta(z) = \frac{1}{2}\beta(\beta-2)K_\beta(z)$$

$$K_\beta(z) = z^{\beta/2}\, {}_2F_1\left(\frac{\beta}{2}, \frac{\beta}{2}; \beta; z\right)$$

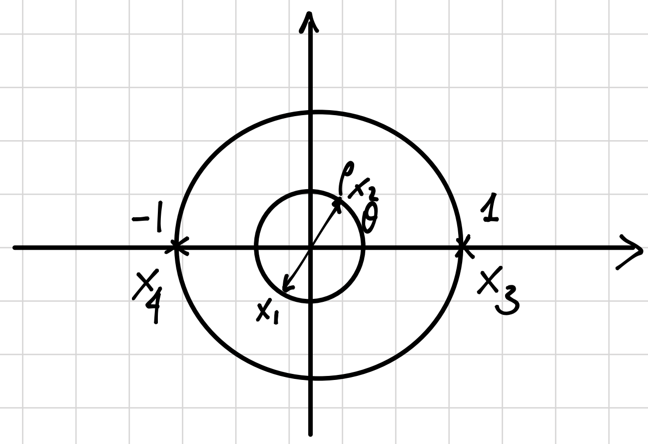
Then: $g_{\Delta,\ell}^{d=2}(z,\bar{z}) = K_{\beta_1}(z)K_{\beta_2}(\bar{z}) + K_{\beta_2}(z)K_{\beta_1}(\bar{z})$

$$\beta_1 = \Delta + \ell$$
$$\beta_2 = \Delta - \ell$$

In $d=4$: (2004: Dolan & Osborn)

$$g_{\Delta,\ell}(z,\bar z) = \frac{1}{(-2)^{\ell}}\,\frac{z\bar z}{z-\bar z}\Big(k_{\Delta+\ell}(z)\,k_{\Delta-\ell-2}(\bar z) - (z\leftrightarrow \bar z)\Big)$$

RADIAL EXPANSION



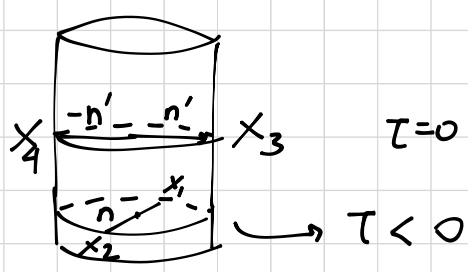
$\rho = r e^{i\theta}$

$\eta = \cos\theta$

$$z = \frac{4\rho}{(1+\rho)^2}$$

$\hookrightarrow \rho = \dfrac{z}{\left(1-\sqrt{1-z}\right)^2}$

$\Rightarrow \tau = \ln r$ to map to cylinder:



$$\hookrightarrow \langle \phi_3(\tau=0,n')\,\phi_4(\tau=0,-n')|\,\underbrace{\phi_1(\tau,-n)\,\phi_2(\tau,n)\rangle}_{}$$

$$= e^{\tau H}|\phi_1(0,-n)\,\phi_2(0,n)\rangle$$

$$\Longrightarrow \langle\phi_3\phi_4|\,P_k\, e^{\tau H}|\phi_1\phi_2\rangle = \sum_{n,J} r^{\Delta+n}\,\underbrace{\langle\phi_3\phi_4|\Delta+n,J\rangle}_{\propto\, n'_{\mu_1}\dots n'_{\mu_J}-\text{trace}}\,\underbrace{\langle\Delta+n,J|\phi_1\phi_2\rangle}_{\propto\, n^{\mu_1}\dots n^{\mu_J}-\text{traces}}$$

$$\sum_{\alpha,\beta}|\alpha\rangle\langle\beta| \to \sum_{n,J}|\Delta+n,J\rangle\langle\Delta+n,J|$$

$\max(0,J-n)\le J\le J+n$

$$= \sum_{n,J} r^{\Delta+n}\,\omega(n,J)\,C_J^{\frac{d}{2}-1}(\underbrace{n\cdot n'}_{\eta})$$

RECURSION RELATION FOR $\omega(n,J)$:

$$f_{n,J}(r,\eta) = r^n C_J^{\frac{d}{2}-1}(\eta) \qquad (\text{NB Gegenbauer polynomial are } \perp)$$

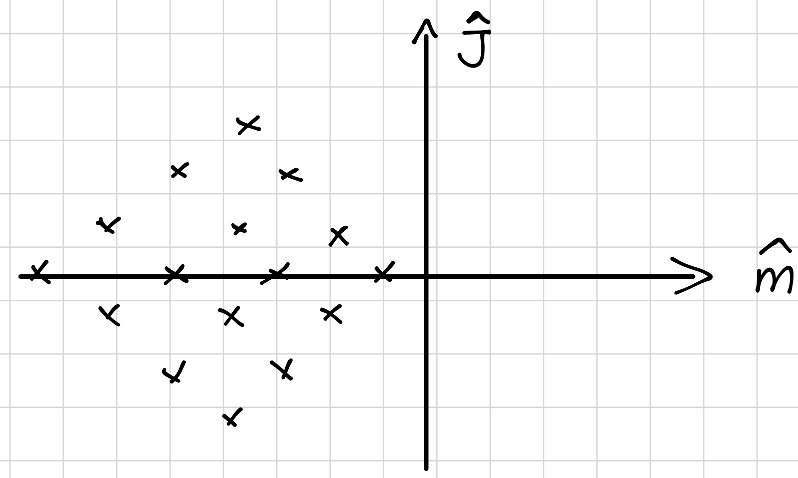
$$\{r, \eta, \partial_r, \partial_\eta\} f_{n,J}(r,\eta) \sim \sum_{n',J'} f_{n+n',J+J'}(r,\eta)$$

$$\hookrightarrow \mathcal{D}_{r,\eta}\, g_{\Delta,\ell}(r,\eta) - C_{\Delta\ell}\, g_{\Delta,\ell}(r,\eta) = 0$$

rational funct. of Δ, ℓ

$$\Rightarrow \sum_{\{\hat{m},\hat{J}\}\in S} C(\hat{m},\hat{J})\, \omega(m+\hat{m}, J+\hat{J}) = 0$$

$$S = \{\underbrace{(0,0), (-1,1), (-1,-1), \dots}_{30 \text{ points}}\}$$



We can then prove $\omega(0,J) = 0 \qquad (J \neq \ell)$

$\omega(0,\ell) = 1$

$\Rightarrow$ More recursion relations:

$$f_{12k} f_{34k} W_k = \langle \phi_3 \phi_4 | \sum_{\alpha,\beta} \frac{|\alpha\rangle\langle\beta|}{\underbrace{\langle\alpha|\beta\rangle}_{G^{\alpha\beta}}} |\phi_1 \phi_2\rangle$$

$\hookrightarrow \Delta \geq$ (unitarity bound) $\qquad G^{\alpha\beta} \geq 0$

for $\Delta \leq \Delta_{\text{unitarity}}$ $\exists$ more null states

Let Δ_A^* the value of Δ for which a descendant becomes null

$$\langle \mathcal{O}_A^{\text{null}} | \mathcal{O}_A^{\text{null}} \rangle \sim Q_A (\Delta - \Delta_A^*) \qquad \mathcal{O}_A^{\text{null}}$$

$\Rightarrow$ work in $\boxed{d : \text{odd}}$

$$\rightarrow \quad g_{\Delta,\ell}(r,\eta) \sim \frac{R_A}{\Delta - \Delta_A^*}\, g_{\Delta_A,\ell_A}(r,\eta)$$

$\Rightarrow$ Δ_A is the dim. of $\mathcal{O}_A^{null}$:

$$\Delta_A = \left[\mathcal{O}_A^{null}\right] = \Delta_A^* + n_A \qquad n_A \in \mathbb{N}$$

$$\ell_A = \mathrm{Spin}\left[\mathcal{O}_A^{null}\right]$$

We can show then:

$$K_\mu \,|\,\mathcal{O}_A^{null}\rangle = 0 \;\Rightarrow\; \mathcal{O}_A^{null} \text{ is a primary}$$

and that ALL DESCENDANTS OF $\mathcal{O}_A^{null}$ HAVE 0 NORM.

$\hookrightarrow \left|P_\mu\,|\mathcal{O}_A^{null}\rangle\right|^2 \sim (\Delta - \Delta_A^*)\, Q_A$ · (factor dictated by conf. symm.)

$$\Rightarrow \frac{R_A}{\Delta - \Delta_A^*} \sim \underbrace{\langle \phi_3\phi_4|\mathcal{O}_A^{null}\rangle}_{M_A^L}\underbrace{\langle \mathcal{O}_A^{null}|\phi_1\phi_2\rangle}_{M_A^R}\underbrace{\left(\langle \mathcal{O}_A^{null}|\mathcal{O}_A^{null}\rangle\right)^{-1}}_{\frac{Q_A^{-1}}{\Delta - \Delta_A^*}}$$

$$\Longleftrightarrow \quad R_A = M_A^L M_A^R Q_A^{-1}.$$

$\Rightarrow$ Where are the poles?

A		ℓ_A	Δ_A^*	n_A
I_n	$n \in \mathbb{N}$	$\ell + n$	$1 - \ell - n$	n
II_n	$1 \leq n \leq \ell$	$\ell - n$	$\ell + d - 1 - n$	n
III_n	$n \in \mathbb{N}$	ℓ	$\frac{d}{2} - n$	$2n$

It is then convenient:

$$g_{\Delta,\ell}(r,\eta) = (4r)^{\Delta}\, \underbrace{h_{\Delta,\ell}(r,\eta)}_{\sum_{n=0}^{\infty} r^n \ldots}$$

$\sum_{n=0}^{\infty} r^n \ldots \Rightarrow$ uniquely identified by the poles $\boxed{\text{in } \Delta}$

we miss: $h_{\infty,\ell}(r,\eta) = \lim_{\Delta\to\infty} h_{\Delta,\ell}(r,\eta)$

* Plug $g_\Delta = r^\Delta h_\Delta$ in the Casimir eq.
* Take leading terms as $\Delta \to \infty$:

$$h_{\infty,\ell}(r,\eta) = \frac{(1-r^2)^{1-\frac{d}{2}}\, N_{d,\ell}\, C_\ell^{\frac{d}{2}-1}(\eta)}{(1+r^2-2\eta r)^{1/2}(1+r^2+2\eta r)^{1/2}}$$

$$\Rightarrow h_{\Delta,\ell}(r,\eta) = h_{\infty,\ell}(r,\eta) + \sum_{A=\mathrm{I}_n,\mathrm{II}_n,\mathrm{III}_n} R_A \frac{1}{\Delta-\Delta_A^*} \underbrace{(4r)^{n_A}}\, h_{\Delta_A^*+n_A,\ell_A}(r,\eta)$$

whenever we have a pole $\exists$ suppression function r^{n_A}.

To have $O(r^N)$ in the LHS we just need $O(r^{N-n_A})$ in the RHS $\rightarrow$ recursion relation

$$\Rightarrow \partial_r^n \partial_\eta^m g_{\Delta,\ell}(r_*,\eta_*) = (4r_*)^\Delta \left(\frac{P_N^{mn}(\Delta)}{Q_N(\Delta)} + O(r_*^{N-n}) \right)$$

where N is the highest power of r in the h-exp

$$\Rightarrow Q_N(\Delta) = \prod_{\substack{A=\mathrm{I}_n,\mathrm{II}_n,\mathrm{III}_n \\ n\le N}} (\Delta-\Delta_A^*)$$

(>0 always because of unitarity bounds).

CFTs

CFT data: (Simmetry)
$$\begin{cases} \text{spectrum} \longleftrightarrow \langle O_i O_i \rangle \\ \text{OPE coeff} \longleftrightarrow \langle O_i O_j O_k \rangle \end{cases}$$

Unitarity:
$$\begin{cases} \Delta \geq \Delta_{unit}(\ell) \\ \langle \phi \phi O^{\mu_1 \dots \mu_n} \rangle \sim f_{\phi\phi O} \in \mathbb{R} \end{cases}$$

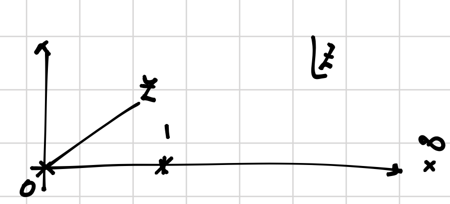
$\Rightarrow$ CROSSING SYMMETRY $\Leftrightarrow$ ASSOCIATIVITY of OP. ALG.

$$\langle \phi_1(x_1) \phi_2(x_2) \phi_3(x_3) \phi_4(x_4) \rangle = K_4(\Delta_i, x_i) \times g(u,v)$$

$$\Rightarrow \quad K_4(\Delta_i, x_i) = \frac{1}{|x_{12}|^{\Delta_1+\Delta_2} |x_{34}|^{\Delta_3+\Delta_4}} \left(\frac{|x_{24}|}{|x_{14}|} \right)^{\Delta_{12}} \left(\frac{|x_{14}|}{|x_{13}|} \right)^{\Delta_{34}}$$

$$\longrightarrow \sum_{\substack{O \in \phi_1 \times \phi_2 \\ O \in \phi_3 \times \phi_4}} f_{12O} f_{34O} \, g^{\Delta_{12} \Delta_{34}}_{\Delta_O, \ell_O}(z, \bar{z}) \frac{1}{(z\bar{z})^{\frac{\Delta_1+\Delta_2}{2}}}$$

$\hookrightarrow$ in conf. frame



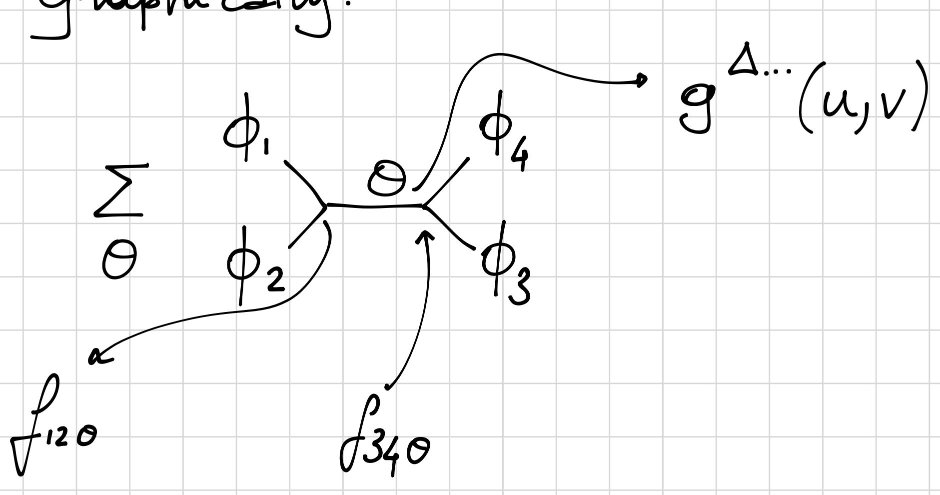
$x_1 = \vec{0}$

$x_3 = (0, \dots, 1)$

$x_4 = (0, \dots, \infty)$

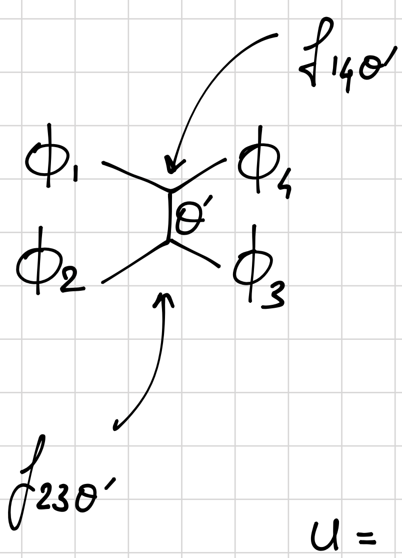
$x_2 = \left(0, \dots, \frac{z-\bar{z}}{2}, \frac{z+\bar{z}}{2}\right)$

Graphically:



Now take:

$$\langle \overbrace{\phi_1 \underbrace{\phi_2 \phi_3} \phi_4} \rangle \Longrightarrow \sum_{\mathcal{O}'}$$



$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \qquad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$$

$$\hookrightarrow \sum_{\substack{\mathcal{O}' \in \phi_1 \times \phi_4 \\ \mathcal{O}' \in \phi_2 \times \phi_3}} f_{14\mathcal{O}'} f_{23\mathcal{O}'} \, g^{\Delta_{14}\Delta_{23}}_{\Delta_{\mathcal{O}'}, \ell_{\mathcal{O}'}}(1-z, 1-\bar{z}) \frac{1}{\left((1-z)(1-\bar{z})\right)^{\frac{\Delta_1+\Delta_4}{2}}}$$

$$\Downarrow$$

$$x_2 \leftrightarrow x_4 \Rightarrow u \leftrightarrow v$$

$$\Downarrow$$

$$z \leftrightarrow 1-z$$
$$\bar{z} \leftrightarrow 1-\bar{z}$$

CROSS. SYM. $\Rightarrow$ the two OPEs MUST AGREE!

$\hookrightarrow$ for simplicity: $\phi_i = \phi \longrightarrow \Delta_i = \Delta_\phi$

$$\Rightarrow \sum_{\mathcal{O} \in \phi \times \phi} \Big(\quad - \quad \Big) = 0$$

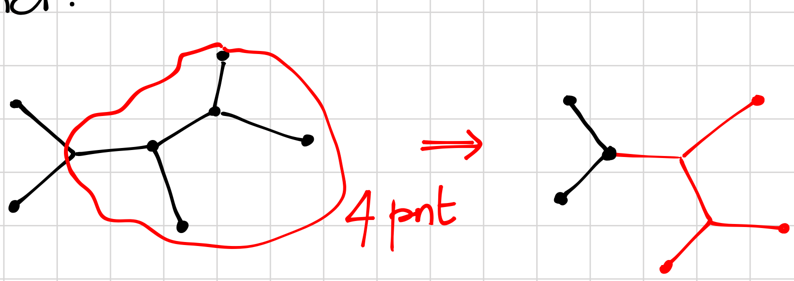
can only be traceless sym. op.

$$\hookrightarrow \sum_{\Delta, \ell} \overbrace{\lambda^2_{\Delta,\ell}}^{\lambda \geq 0} \Big(\underbrace{(z\bar{z})^{-\Delta_\phi} g_{\Delta,\ell}(z,\bar{z}) - \left((1-z)(1-\bar{z})\right)^{-\Delta_\phi} g_{\Delta,\ell}(1-z, 1-\bar{z})}_{\text{KNOWN! (and} \neq 0)\ \equiv F^{\Delta_\phi}_{\Delta,\ell}(z,\bar{z})} \Big) = 0$$

where $\lambda^2_{\Delta,\ell} = f^2_{\phi\phi\mathcal{O}_{\Delta,\ell}}$

$\Rightarrow$ to def. a CFT we have to check all the cross. symm. of <u>every</u> 4 pnt functions.

eg.: 5 pnt funct:



Let's expand around the crossing symmetric point:

$$z = \bar{z} = 1/2 \iff r = 3 - 2\sqrt{2}$$
$$\eta = 1$$

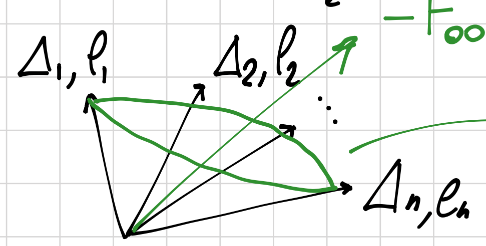
$$\hookrightarrow \sum_{\Delta,\ell} \lambda^2_{\Delta,\ell} \begin{pmatrix} F^{\Delta_\phi}_{\Delta,\ell}\left(\frac{1}{2},\frac{1}{2}\right) \\ \vdots \\ F^{\Delta_\phi\,(n,m)}_{\Delta,\ell}\left(\frac{1}{2},\frac{1}{2}\right) \end{pmatrix} = 0$$

$n + m \leq \Lambda$

(the full set is $\Lambda = \infty$ but we have to take a finite no. of terms)

$$\Rightarrow \phi \times \phi \sim \mathbb{I} + \dots$$

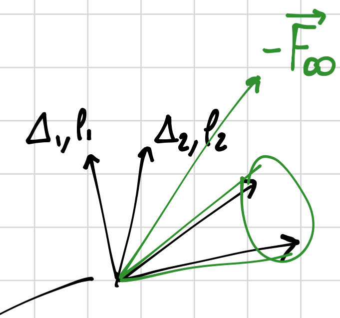
$$\hookrightarrow \sum_{\Delta,\ell}{}' \lambda^2_{\Delta,\ell} \left(\dots \right) = - \vec{F}^{\Delta_\phi}_\infty\left(\frac{1}{2},\frac{1}{2}\right)$$



$-\vec{F}^{\Delta_\phi}_\infty \Rightarrow$ IT MUST BE INSIDE THE CONE!

It's a cone!

Assume $\{\Delta_1, \ell_1\} \notin$ CFT
$\{\Delta_2, \ell_2\} \notin$ CFT $\Longrightarrow$



$\Rightarrow$ it's a narrower cone and $-\vec{F}_\infty$ is outside

$\Downarrow$

inconsistent w/ crossing symmetry

$\exists$ plane α containing $-\vec{F}^{\Delta_\phi}_\infty$ outside the cone

$\Downarrow$

$\exists$ linear functional:

$$\alpha : \quad \alpha\left[F^{\Delta_\phi}_{\Delta,\ell}(z,\bar{z})\right] \longrightarrow \mathbb{R}$$

s.t. $\quad \alpha\left[F^{\Delta_\phi}_{\Delta,\ell}\right] \geq 0 \quad \forall\, \Delta,\ell$ in the CFT

$$\alpha\left[-F^{\Delta_\phi}_\infty\right] < 0$$

$\hookrightarrow$ this is a contradiction

$$\text{e.g.: } \alpha = \sum_{n,m}^{n+m<\Lambda} \alpha_{nm}\, \partial^n_z \partial^m_{\bar{z}} \Big|_{\substack{z=1/2 \\ \bar{z}=1/2}}$$

$$\Rightarrow \sum_{\Delta,\ell}{}' \lambda^2_{\Delta,\ell} F^{\Delta\phi}_{\Delta,\ell}(z,\bar z) = - F^{\Delta\phi}_{00}(z,\bar z)$$

$$\alpha\left(\ldots = \ldots \right) \Rightarrow (\geq 0) = (<0) \rightarrow \text{imposs.!}$$

NUMERICAL BOOTSTRAP:

Spectrum (= set of assumptions) $\Longrightarrow$ $\boxed{\text{NUM. BOOTSTRAP}}$ $\nearrow$ no CFT, $\searrow$ maybe CFT

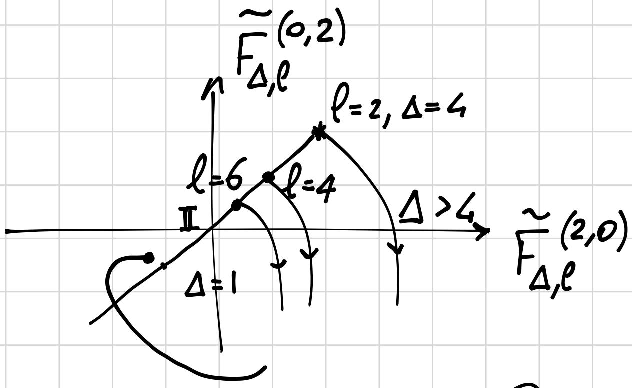
FREE THEORY in 4D

$\Delta\phi = 1$

$\phi \times \phi \sim \mathbb{I} + \phi^2 + \ldots$ "$\phi\, \partial_{\mu_1} \ldots \partial_{\mu_\ell}\, \phi$"

$\hookrightarrow$ study 4 pnt functions:

$\langle \phi\phi\phi\phi \rangle$ up to $\Lambda = 2$



$\rightarrow$ $\Delta_* = 2 \Rightarrow$ we need ϕ^2

$$\Rightarrow \sum_{\Delta,\ell} \lambda^2_{\Delta,\ell} \vec F_{\Delta,\ell} = 0$$

1) All $\ell = 0$ have $\Delta > \Delta_* \Rightarrow$ no CFT

1) Discretize Δ

$$\alpha\left[F^{\Delta\phi}_{\Delta,\ell}\right] \geqslant 0 \quad \text{for} \quad \Delta_1, \Delta_2, \dots, \Delta_{max}, \quad \ell = 0, 2, \dots, \ell_{max}$$

disc. + trunc. is the weak point

$$\left\{ \begin{array}{l} \sum_{n,m} \lambda_{nm} F^{(n,m)}(\Delta_i, \ell_i, \Delta_\phi) \geq 0 \\ \vdots \end{array} \right.$$

$\hookrightarrow$ one eq. $\forall\, \Delta_i, \ell_i$

$\Rightarrow$ Linear Programming Problem!

2) Use SDPB

$$\Rightarrow \sum_{n,m} \lambda_{nm} \underbrace{F^{(n,m)}(\Delta, \ell; \Delta_\phi)}_{\dfrac{P^{(n,m)}(\Delta,\ell,\Delta_\phi)}{Q(\Delta,\ell)}} > 0$$

$Q(\Delta,\ell)$: positive $\forall\, \Delta > \Delta_{unit}$

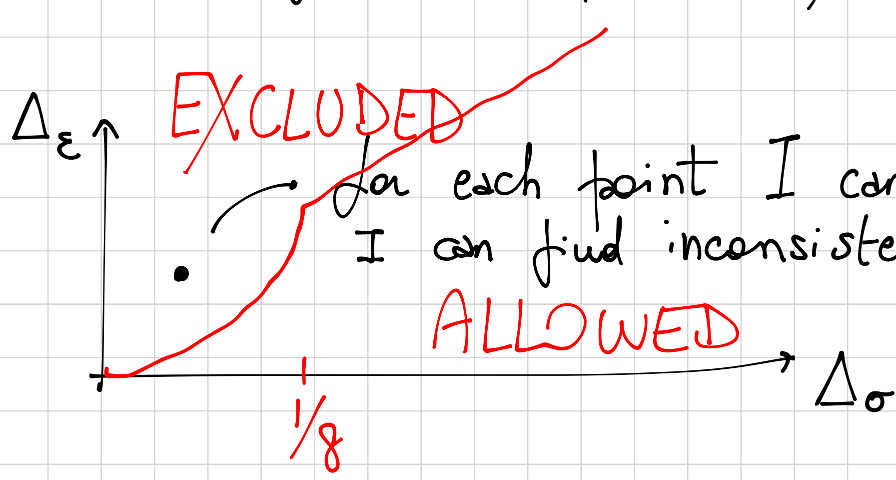
$$\Rightarrow \sum_{n,m} \lambda_{nm} P^{(n,m)}(\Delta, \ell, \Delta_\phi) > 0 \Rightarrow \text{SDPB problem!}$$

CFTs in $d=2$ (no Virasoro)

* consider a scalar $\sigma \to \Delta_\sigma$
  $\hookrightarrow \langle \sigma(x_1)\sigma(x_2)\sigma(x_3)\sigma(x_4)\rangle$

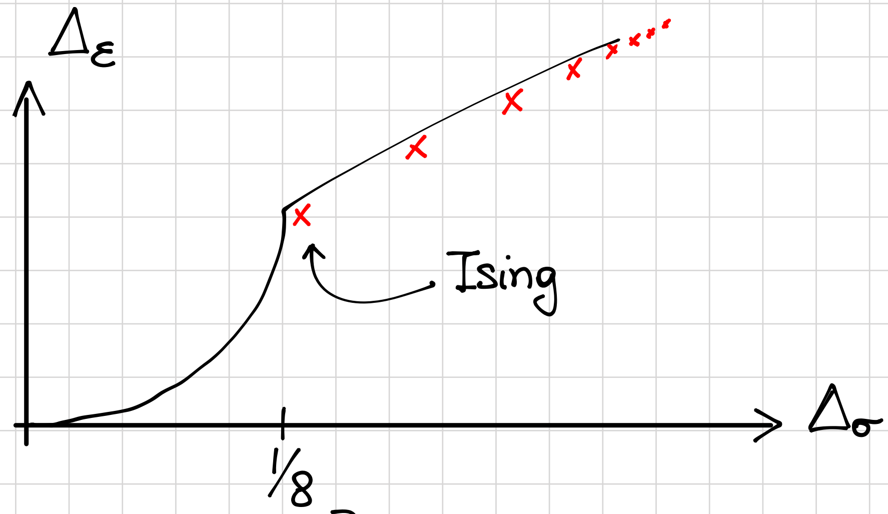
$$\sigma \times \sigma \sim \mathbb{I} + \varepsilon + (\text{higher dim. scalars}) + (\text{higher spins})$$

ε: the first scalar (dim. Δ_ε)



$\Rightarrow$ in 2D $\exists$ set of MINIMAL MODELS (exactly solvable):

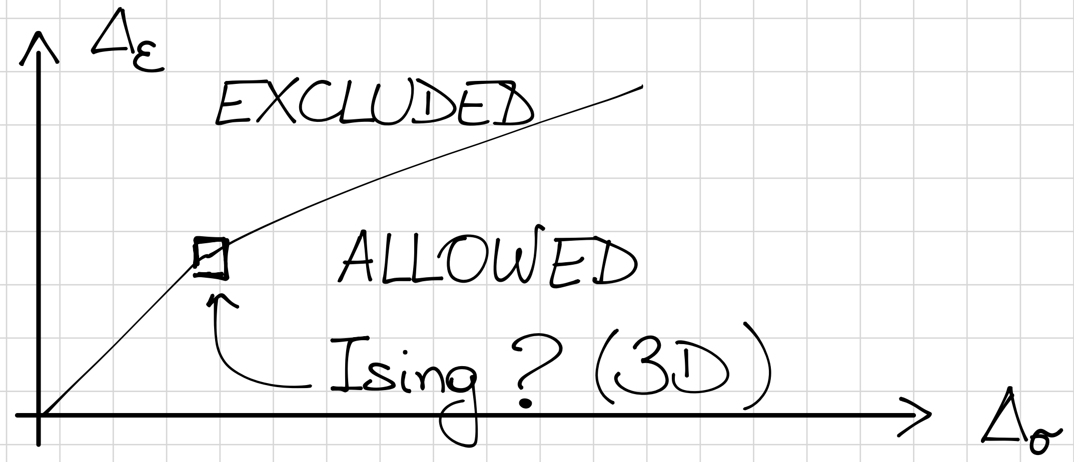
$\hookrightarrow \Delta_\theta$, f_{ijk} known



$\hookrightarrow$ $\Delta = h + \bar{h}$ $\quad \left(h = \frac{1}{16}\right)$

$\ell = h - \bar{h}$

We can also work in $d=3$:



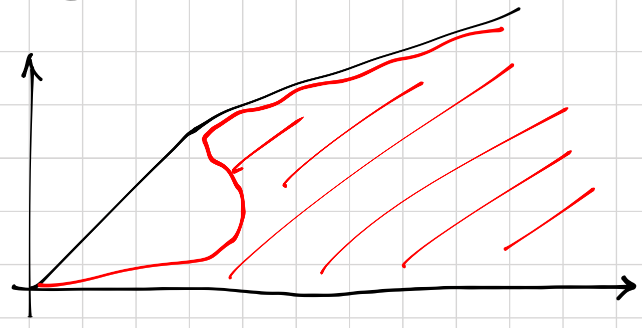
ISING MODEL

1 param. : $m^2 \leftrightarrow T - T_c$ to find criticality

$\hookrightarrow \underbrace{\sigma}_{\mathbb{Z}_2 \text{ odd}} \times \sigma \sim \mathbb{I} + \underbrace{\varepsilon}_{\Downarrow} + \dots$

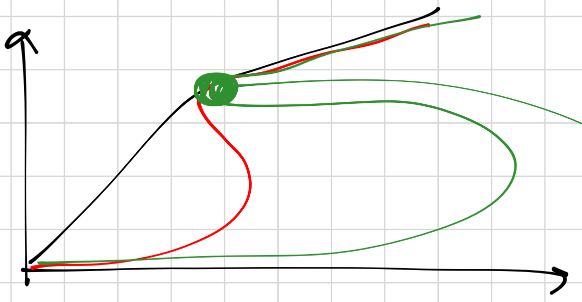
(add ext mag. field) the only unique relevant scalar ($\mathbb{Z}_2$ even)

Suppose $\sigma \times \sigma \sim \mathbb{I} + \varepsilon + \overbrace{\varepsilon'}^{\Delta_{scal} \geq 3} + \dots$



$\Rightarrow$ we can numerically compute $\Delta_{\varepsilon'} \simeq 3.84$

Set $\Delta_{scal} \geq 3.8 \Rightarrow$



So far we only used $\langle \sigma\sigma\sigma\sigma \rangle$

↳ include more operators:

the $\mathbb{Z}_2$ inv. $\begin{cases} \langle \sigma\sigma\sigma\sigma \rangle \\ \langle \varepsilon\varepsilon\varepsilon\varepsilon \rangle \\ \langle \sigma\varepsilon\sigma\varepsilon \rangle \sim \sum_{\mathcal{O}'_\varepsilon \in \sigma\times\varepsilon} f^2_{\sigma\varepsilon\mathcal{O}'} \end{cases}$

$\sim \sum_{\substack{\mathcal{O}\in\sigma\times\sigma \\ \epsilon\times\epsilon}} \underbrace{f_{\sigma\sigma\mathcal{O}} f_{\epsilon\epsilon\mathcal{O}}}_{\gtrless 0}$

$$\left(f_{\sigma\sigma\mathcal{O}} \; f_{\epsilon\epsilon\mathcal{O}} \right) \left(\text{METRIC} \right) \begin{pmatrix} f_{\sigma\sigma\mathcal{O}} \\ f_{\epsilon\epsilon\mathcal{O}} \end{pmatrix}$$

NB: $\sigma \times \epsilon \sim \sigma + \sigma' + \dots$

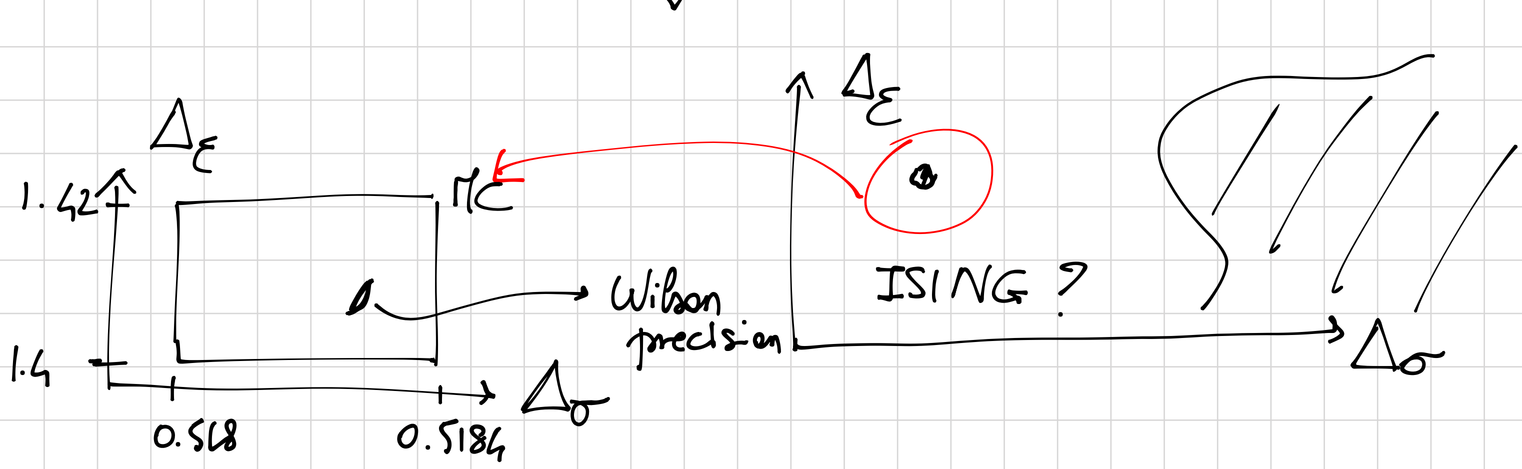
from a possible $\mathcal{L}$ descript. $\Rightarrow \mathcal{L} = \frac{1}{2}(\partial_\mu \sigma)^2 + \frac{\lambda}{4!}\sigma^4$

↳ $\Box\sigma \sim \underbrace{\sigma^3}_{\sigma'}$

$\Rightarrow$ also σ is the $\mathbb{Z}_2$ odd scal. (relevant)

$\Rightarrow \epsilon$ is the only relev. $\mathbb{Z}_2$ even scal

$\Downarrow$



SCFTs ($\mathcal{N}=1$, $d=4$)

$$ds^2 = \left(dx_\mu + i\theta\sigma^\mu d\bar\theta + i\bar\theta\sigma^\mu d\theta\right)^2$$

$$D, P_\mu, M_{\mu\nu}, K_\mu \; + \; \underbrace{Q_\alpha, \bar{Q}^{\dot\alpha}}_{\text{SUSY partners of } P_\mu}, \; \underbrace{S_{\mu\alpha}, \bar{S}_\mu^{\dot\alpha}}_{\text{SUSY part. of } K_\mu}, \; \underbrace{A}_{\text{susy part. of } D}$$

Diagonal simultaneously: $M_{\mu\nu}, D, A$

$\mathcal{O}^{(J,\bar J)}_{\Delta,r}$ $\quad q, \bar q, J, \bar J$

$\Delta = q + \bar q \qquad r = \frac{2}{3}(q - \bar q)$

SUPERPRIMARIES:

$$[K_\mu, \mathcal{O}(0)] = 0$$

$$[S, \mathcal{O}] = 0$$

$$[\bar S, \mathcal{O}] = 0$$

$\hookrightarrow \mathcal{O}^{(J,\bar J)}_{\Delta,r} \xrightarrow{Q} Q\mathcal{O}$, $\mathcal{O}^{(J,\bar J)}_{\Delta,r} \xrightarrow{\bar Q} \bar Q\mathcal{O}$ ... etc. (desc. + primaries)

$\hookrightarrow$ not superconf. though

* Ferrara-Zumino: $\supset T_{\mu\nu}$

S