Normal Ordering of a Free Boson

Riccardo Finotello

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Consider the boson field:

$$X^\mu(z) = x^\mu + \alpha' p^\mu \ln(z) + i\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{\alpha_n^\mu}{n} z^{-n},$$

and the commutation relations:

$$\begin{aligned}
[x^\mu, p^\nu] &= i\eta^{\mu\nu}, \\
[\alpha_n^\mu, \alpha_m^\nu] &= n\eta^{\mu\nu}\delta_{n+m,0},
\end{aligned}$$

where $\delta_{i,j}$ is the Kronecker delta.

Then consider the radially ordered product between $X^\mu(z)$ and $X^\nu(w)$ ($|z| > |w|$):

$$\begin{aligned}
&\mathrm{R}\left(X^\mu(z)\, X^\nu(w)\right) = \\
&= \left(x^\mu + \alpha' p^\mu \ln(z) + i\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{\alpha_n^\mu}{n} z^{-n} \right) \left(y^\nu + \alpha' q^\nu \ln(w) + i\sqrt{\frac{\alpha'}{2}} \sum_{m\in\mathbb{Z}\setminus\{0\}} \frac{\alpha_m^\nu}{m} w^{-m} \right) = \\
&= x^\mu y^\nu + \alpha' x^\mu q^\nu \ln(w) + i\sqrt{\frac{\alpha'}{2}} x^\mu \sum_{m\in\mathbb{Z}\setminus\{0\}} \frac{\alpha_m^\nu}{m} w^{-m} + \\
&+ \alpha' p^\mu y^\nu \ln(z) + \alpha'^2 p^\mu q^\nu \ln(z)\ln(w) + i\alpha'\sqrt{\frac{\alpha'}{2}} p^\mu \ln(z) \sum_{m\in\mathbb{Z}\setminus\{0\}} \frac{\alpha_m^\nu}{m} w^{-m} + \\
&+ i\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{\alpha_n^\mu}{n} y^\nu z^{-n} + i\alpha'\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{\alpha_n^\mu}{n} q^\nu \ln(w)\, z^{-n} - \frac{\alpha'}{2} \sum_{n\in\mathbb{Z}\setminus\{0\}} \sum_{m\in\mathbb{Z}\setminus\{0\}} \frac{\alpha_n^\mu}{n} \frac{\alpha_m^\nu}{m} w^{-m} = \\
&=: X^\mu(z)\, X^\nu(w) : \; - \frac{\alpha'}{2} \sum_{n=1}^{+\infty} \sum_{m=1}^{+\infty} \frac{1}{nm} \left[\alpha_n^\mu, \alpha_{-m}^\nu\right] z^{-n} w^m + [p^\mu, y^\nu] \ln(z) = \\
&=: X^\mu(z)\, X^\nu(w) : \; - \eta^{\mu\nu} \frac{\alpha'}{2} \sum_{n=1}^{+\infty} \frac{1}{n} \left(\frac{w}{z}\right)^n - i\eta^{\mu\nu} \ln(z) = \\
&=: X^\mu(z)\, X^\nu(w) : \; - \eta^{\mu\nu} \frac{\alpha'}{2} \left(\ln\left(1 - \frac{w}{z}\right) - \ln(z) \right) = \\
&=: X^\mu(z)\, X^\nu(w) : \; - \eta^{\mu\nu} \frac{\alpha'}{2} \ln(z - w).
\end{aligned}$$

Then, since:

$$\mathrm{R}\left(X^\mu(z)\, X^\nu(w)\right) =: X^\mu(z)\, X^\nu(w) : \; + \left\langle X^\mu(z)\, X^\nu(w) \right\rangle,$$

we find:

$$\left\langle X^\mu(z)\, X^\nu(w) \right\rangle = -\eta^{\mu\nu} \frac{\alpha'}{2} \ln(z - w).$$