Conformal Weight of a Vertex Operator

Riccardo Finotello

August 7, 2018

Consider:

$$
\begin{aligned}
T(z) &= \frac{1}{2}\eta_{\alpha\beta} : \partial_z X^\alpha(z)\, \partial_z X^\beta(z) : , \\
V^{\alpha\beta}(k,w) &= : \partial_w X^\alpha(w)\, \partial_w X^\beta(w) : e^{ik\cdot X(w)}.
\end{aligned}
$$

Then for $z \to w$:

$$
\begin{aligned}
T(z)\, V^{\mu\nu}(k,w) &= \frac{1}{2}\eta_{\alpha\beta} : \partial_z X^\alpha(z)\, \partial_z X^\beta(z) : : \partial_w X^\mu(w)\, \partial_w X^\nu(w)\, e^{ik\cdot X(w)} : = \\
&= \frac{1}{2}\eta_{\alpha\beta}\Big(\langle \partial_z X^\alpha(z)\, \partial_w X^\mu(w)\rangle\, \partial_z X^\beta(z)\, \partial_w X^\nu(w)\, e^{ik\cdot X(w)} + \\
&+ \langle \partial_z X^\alpha(z)\, \partial_w X^\nu(w)\rangle\, \partial_z X^\beta(z)\, \partial_w X^\mu(w)\, e^{ik\cdot X(w)} + \\
&+ \Big\langle \partial_z X^\alpha(z)\, e^{ik\cdot X(w)} \Big\rangle\, \partial_z X^\beta(z)\, \partial_w X^\mu(w)\, \partial_w X^\nu(w) \\
&+ \langle \partial_z X^\beta(z)\, \partial_w X^\mu(w)\rangle\, \partial_z X^\alpha(z)\, \partial_w X^\nu(w)\, e^{ik\cdot X(w)} + \\
&+ \langle \partial_z X^\beta(z)\, \partial_w X^\nu(w)\rangle\, \partial_z X^\alpha(z)\, \partial_w X^\mu(w)\, e^{ik\cdot X(w)} + \\
&+ \Big\langle \partial_z X^\beta(z)\, e^{ik\cdot X(w)} \Big\rangle\, \partial_z X^\alpha(z)\, \partial_w X^\mu(w)\, \partial_w X^\nu(w) + \\
&+ \langle \partial_z X^\alpha(z)\, \partial_w X^\mu(w)\rangle \langle \partial_z X^\beta(z)\, \partial_w X^\nu(w)\rangle\, e^{ik\cdot X(w)} \\
&+ \langle \partial_z X^\alpha(z)\, \partial_w X^\nu(w)\rangle \langle \partial_z X^\beta(z)\, \partial_w X^\mu(w)\rangle\, e^{ik\cdot X(w)} \\
&+ \langle \partial_z X^\alpha(z)\, \partial_w X^\mu(w)\rangle \Big\langle \partial_z X^\beta(z)\, e^{ik\cdot X(w)} \Big\rangle\, \partial_w X^\nu(w) + \\
&+ \langle \partial_z X^\alpha(z)\, \partial_w X^\nu(w)\rangle \Big\langle \partial_z X^\beta(z)\, e^{ik\cdot X(w)} \Big\rangle\, \partial_w X^\mu(w) + \\
&+ \Big\langle \partial_z X^\alpha(z)\, e^{ik\cdot X(w)} \Big\rangle \Big\langle \partial_z X^\beta(z)\, e^{ik\cdot X(w)} \Big\rangle\, \partial_w X^\mu(w)\, \partial_w X^\nu(w) \Big) = \\
&= \frac{1}{(z-w)^4}\eta^{\mu\nu} e^{ik\cdot X(w)} + \frac{i}{(z-w)^3} k^{(\mu}\, \partial_w X^{\nu)} e^{ik\cdot X(w)} + \\
&+ \frac{1}{(z-w)^2}\left(\frac{1}{2}k^2 + 1\right) \partial_w X^\mu(w)\, \partial_w X^\nu(w) + \\
&+ \frac{1}{z-w}\partial_w \left(\partial_w X^\mu(w)\, \partial_w X^\nu(w)\, e^{ik\cdot X(w)} \right) + \mathrm{O}(1)
\end{aligned}
$$

A similar treatment of $T(z)\, V^\mu(k,w)$, where $V^\mu(k,w) = \partial_w^2 X^\mu(w)\, e^{ik\cdot X(w)}$, leads to the conditions for

$$
\xi_{\mu\nu} V^{\mu\nu}(k,w) + \xi_\mu V^\mu(k,w)
$$

to be a physical vertex operator (e.g.: $k^2 = -2$).