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LOOP CORRECTIONS TO SUPERSTRING EQUATIONS OF MOTION

C.G. CALLAN[1], C. LOVELACE[2], C.R. NAPPI[3] and S.A. YOST[4]

Joseph Henry Laboratories, Princeton University, Princeton NJ 08544, USA

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We extend to the O(32) superstring our program of renormalizing string theory by cancelling BRST anomalies between different genus worldsheets. We calculate all the anomalies arising to lowest nontrivial loop order and in general background fields. At tree level, string consistency requires the background fields to satisfy equations of motion which ensure the conformal invariance of the underlying sigma model. We implement loop anomaly cancellation by adding to these equations appropriate loop-order source terms which necessarily break conformal invariance. The procedure leads to consistent loop-corrected equations of motion which can be derived from a spacetime effective action. The loop-order terms in this action incorporate a number of crucial aspects of spacetime physics, including the interaction of the photon with the graviton and the interactions of the antisymmetric tensor field with the gauge field which are responsible for spacetime gauge anomaly cancellation. We take these results as compelling evidence that loop-corrected string theory is associated with sigma models which are not conformally invariant in the usual sense. An extended version of conformal invariance, whose deep structure has yet to be uncovered, presumably incorporates string loop physics.

1. Introduction

It is well-known that the physical content of tree-level (classical) string theory can be extracted from conformally-invariant two-dimensional nonlinear sigma models. One way of revealing this connection is to interpret the coupling constants of the general sigma model as spacetime fields and the conformal invariance conditions on the coupling constants as spacetime equations of motion. In all cases that have been examined, the spacetime physics extracted in this way from conformal sigma models agrees with that extracted directly from classical string theory. Since algebraic and analytic methods for obtaining exact, nonperturbative results on two-dimensional conformal field theories exist, the deep reason for being interested in conformal

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[4] Now at University of Florida, Gainsville. Supported in part by DOE grant DE-FG05-87ER-40272.

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invariance is that it may give us a way of obtaining nonperturbative information about spacetime physics.

It is fairly obvious that string loop corrections must destroy the simple connection with conformal invariance which exists at the classical level. The reason is that, while string loop perturbation theory is given by a sum over sigma model amplitudes on two-dimensional worldsheets of different topologies, the beta function, which governs conformal invariance, knows only about short-distance physics and is indifferent to topology. If conformal invariance were the whole story, the equations of motion governing spacetime fields would receive no string quantum corrections, a very unlikely state of affairs. A plausible exit from this impasse has been suggested by several people [1–5]: string loop perturbation theory is a sum over sigma model amplitudes on worldsheets of all topologies. The contribution at a given order is an integral of a sigma model amplitude over the finite-dimensional moduli space of conformally inequivalent worldsheets of a given genus. This integration typically has endpoint divergences quite separate from the field theory divergences which give rise to the above-mentioned sigma model beta functions. The proposal is to cancel these divergences and anomalies against divergences and anomalies induced in amplitudes on lower-genus worldsheets by moving the sigma model away from its conformal fixed point in a suitable way. Thus, loop-corrected string theory would be associated with conformally non-invariant sigma models and the equations of motion for spacetime fields would differ from their "classical" expressions. The main question is whether such a scheme can consistently be carried out, i.e. whether *all* anomalies can be eliminated in this way and whether the resulting loop-corrected beta functions are mutually consistent.

The approach outlined above was first applied to the closed bosonic string in a flat spacetime background [2] and it was found that the one-loop worldsheet (torus) generates a cosmological constant contribution to the tree-level equations of motion for the graviton. For interacting closed and open strings, the lowest loop correction arises from adding a boundary to the worldsheet and one can easily consider the effect of an open string background gauge field coupled to the boundary. (The analogous computation for the closed string would involve the study of an interacting sigma model, corresponding to non-flat background metric, on the torus. This is technically a much harder problem.) The loop divergences and anomalies then have very nontrivial spacetime structure and very demanding consistency conditions can be studied. In a previous paper [3] we showed, for the bosonic open and closed strings, that the beta functions were all consistent and that essential physical effects, such as the gauge field energy-momentum tensor, arose directly from loop-induced conformal invariance violation. Others [6–8] have experienced difficulty in generating a consistent set of loop-corrected equations for both the graviton and the dilaton. We use a ghost treatment of the dilation, while the authors just cited use a worldsheet curvature approach. Since we have no difficulty in generating consistent equations we are inclined to suspect the existence of a subtlety in expressing the ghost contribution to the loop divergence in terms of the worldsheet curvature. A solution to this problem has recently been reported by Polchinski [9].

With this encouragement for the bosonic string, attention has turned to the technically more difficult, but phenomenologically more relevant, case of the superstring. Several preliminary steps have been taken [4, 10], but a complete analysis of all anomaly cancellations and consistency conditions has not been done. That is the subject of this paper, in which we study the O(32) unoriented open superstring in non-vanishing backgrounds of all the massless spacetime bosons of the theory including the O(32) gauge field itself. We focus on the anomalies in BRST invariance (the symmetry which decouples unphysical states) generated by loop corrections due to the insertion of a hole in the worldsheet. Background gauge fields are accounted for by including in the path integral an appropriate generalization of the usual Wilson line integral about the boundary:

$$W = \operatorname{tr} P \exp\left\{ -\frac{i}{4\pi} \int \mathrm{d}s \left[A_\mu(X) \frac{\partial}{\partial s} X^\mu - \tfrac{1}{2} i F_{\mu\nu}(X) \psi^\mu \psi^\nu \right] \right\}$$

(this expression is just the exponentiation of the supersymmetric photon vertex). This amounts to coupling an interacting one-dimensional field theory to the two-dimensional sigma model of the closed string. For general A_μ, the effect is very complicated, but for constant gauge field strength the whole problem is exactly soluble, because the boundary theory becomes a free theory. Adding a hole is equivalent to inserting into the worldsheet a gauge-field dependent *closed* string state. Demanding that it be invariant under boundary reparametrizations gives restrictions on the background field, which may be interpreted as equations of motion for A_μ. BRST anomalies in the decoupling of unphysical states come from the on-shell components of this "boundary state". In our approach, these anomalies are cancelled by interpreting them as loop corrections to the tree-level beta functions (conformal invariance conditions). It must be shown that appropriate beta functions exist to cancel all of the anomalies, and that the resulting loop-corrected equations of motion are consistent and incorporate the expected physics.

Our main achievement is the development of a method for computing the boundary state and reading off all its zero-mass pieces in the presence of the complications of fermions, NS versus R sectors, superghosts and the like. The picture ambiguity for states and vertex operators of the superstring is a major annoyance because the boundary state naturally appears as sum over infinitely many pictures. Although some problems remain, we have resolved this issue well enough to deal with the questions raised in this paper. Our results amount to a realization of the operator approach to string loop corrections [11] in the context of an *interacting* worldsheet field theory. An important consequence of these interactions is that the normalization of the boundary state depends in a nonpolynomial fashion on the background gauge field $F_{\mu\nu}$. This dependence must be correctly determined in order to verify consistency. Fortunately, our methods are powerful enough to do this.

The physics of the NS-NS sector turns out to be very similar to the physics of the bosonic string. The spacetime bosons of that sector are the graviton and dilaton. (The antisymmetric tensor of the bosonic string is not present in a nonorientable string theory.) The loop corrections to their equations of motion are nonpolynomial functions of the background gauge field strength which turn out to be, at least in the constant field case, identical to the results previously found for the bosonic string [3]. They are derivable from a lagrangian and guaranteed to be consistent. The R-R sector is where most of the intersecting physics lies. The only massless spacetime boson in this sector is a 3-form field strength whose vertex operator is a product of two spin fields or spacetime fermion vertices [12]. The sigma model corresponding to a background of this somewhat unconventional antisymmetric tensor field has not, to our knowledge, been studied previously in any detail. The classical equations of motion (beta function) for this field strength consist of one 4-form and one 8-form first-order equation. The BRST anomalies arising from the insertion of a boundary consist of 0-form, 4-form and 8-form functionals of the background gauge fields. By a remarkable Bose-Fermi cancellation, the 0-form is actually independent of the gauge field and the other two forms are simple polynomials in the field strength. As was first pointed out by Polchinski and Cai [10], there is no equation of motion for which the 0-form can be thought of as a loop-generated source term and it must be cancelled by some other loop-order effect. They showed that, in the absence of any background gauge field and if the gauge group is precisely O(32), the BRST anomaly of the crosscap cancels that of the boundary. Since the crosscap is not a boundary, there is no way to couple it to the gauge field, and its contribution to the anomaly must be field-independent. The field-independence of the 0-form part of the boundary anomaly thus guarantees that the total 0-form contribution of hole and crosscap vanishes for any O(32) background gauge field. The 4-form and 8-form pieces then serve as source terms for the antisymmetric tensor equation of motion. It is easy to verify, at least for constant background gauge field, that they are consistent and follow from the sort of effective action previously inferred from arguments about spacetime gauge anomaly cancellation [13]. In the R-R sector, there are many opportunities for the BRST anomaly-cancelling scheme to go wrong, but, as we shall demonstrate in some detail, the theory leaps gracefully over each potential pitfall.

We regard all this as convincing evidence that string loop corrections have the effect of driving the underlying sigma model away from the conformal fixed point that describes classical string theory. The real question is whether this "quantum" sigma model retains useful vestiges of the infinite-parameter conformal group which might eventually permit us to extract nonperturbative information about quantum string theory. That will have to be the subject of a future paper, but we will advance some speculations at the end of this one.

The paper is organized as follows. In sect. 2, we construct, in the fermionic language, the state of the closed superstring created by a worldsheet boundary in the presence of open string backgrounds. We find that this state can be expressed as the path integral for a certain renormalizable one-dimensional field theory. In sect. 3, we exploit the techniques of bosonization to obtain the production of the boundary state on all zero-mass levels in the presence of spacetime constant background gauge fields. This is what is needed to obtain a complete accounting of string-loop-generated BRST anomalies. We invent some tricks to circumvent technical difficulties associated with the superghost and the picture ambiguity. In sect. 4, we use the machinery developed in the previous sections to evaluate the BRST loop anomalies and to show that they can all be eliminated by modifying the tree-level equations of motion of the closed string background fields. Breakdown of conformal invariance of the sigma model is shown to be intimately related to many important features of the low-energy spacetime effective field theory, including gauge anomaly cancellation. In sect. 5, we present conclusions, speculations and suggestions for future developments of these ideas. Various technical issues are dealt with in appendices.

2. Boundary states

2.1. GENERAL CONSIDERATIONS

To carry out the program just outlined, we need a method for computing open string corrections to closed string tree amplitudes in the presence of nonvanishing spacetime backgrounds of open string fields. The lowest-order such loop correction arises from adding a boundary to the tree worldsheet. To account correctly for the moduli, the new boundary has to be glued into the old worldsheet via a cylinder of variable length. The boundary itself represents the creation out of the vacuum, via open string physics, of some closed string state, $|B\rangle$, while the cylinder is just the closed string propagator which connects that state to the subsequent tree-level process. Inserting the closed-string state $|B\rangle$ and its associated propagator into a worldsheet then reproduces the effect of adding a boundary. Our problem is to construct $|B\rangle$. In the case of constant backgrounds, this state will be completely determined, up to normalization, by boundary conditions on the worldsheet fields [4]. More generally, the state can be constructed by performing a functional integral over internal degrees of freedom living on the boundary. These methods are useful even in a constant background since the normalization of $|B\rangle$ will be a function of the gauge field.

In the absence of open string backgrounds, the closed string matter and ghost fields satisfy the standard free boundary conditions

$$\begin{aligned} \partial_+ X^\mu &= -\partial_- X^\mu , & \psi_+^\mu &= \pm i\psi_-^\mu , \\ c^+ &= -c^- , & b_{++} &= b_{--} , \\ \gamma^+ &= \mp i\gamma^- , & \beta_{++} &= \mp i\beta_{--} . \end{aligned} \tag{2.1}$$

The boundary is taken to lie at $\tau = 0$ and to be parametrized by σ with $0 \leqslant \sigma \leqslant 2\pi$, while $+$ and $-$ correspond respectively to left- and right-moving vector or spinor components. The fermion and superghost sign ambiguities correspond to the two spin structures which are eventually combined by the GSO projection. The factors of i in the boundary conditions look strange, but are correct: we will see that they ensure that a complete anticommuting (commuting) set of fermionic (superghost) fields are set to zero at the boundary. An equivalent set of boundary conditions, expressed in terms of mode creation and annihilation operators, is

$$\begin{aligned} \alpha^{\mu}_{m} &= -\tilde{\alpha}^{\mu}_{-m}, & \psi^{\mu}_{m} &= \pm i\tilde{\psi}^{\mu}_{-m}, \\ c_{m} &= -\tilde{c}_{-m}, & b_{m} &= \tilde{b}_{-m}, \\ \gamma_{m} &= \mp i\tilde{\gamma}_{-m}, & \beta_{m} &= \mp i\tilde{\beta}_{-m}. \end{aligned} \tag{2.2}$$

The boundary state, $|B\rangle$, is simply the state which is annihilated by the boundary conditions as operators.

The eigenstate of (2.2) is easily obtained, up to normalization, by a Bogoliubov transformation. The result is [4, 10]

$$\begin{aligned} |B\rangle &= |B\rangle_{\alpha}|B\rangle_{\psi}|B\rangle_{\text{gh}}, \\ |B\rangle_{\alpha} &= \exp\left(-\frac{1}{m}\sum_{m=1}^{\infty}\alpha^{\mu}_{-m}\tilde{\alpha}^{\mu}_{-m}\right)|0\rangle, \\ |B\rangle_{\psi} &= \exp\left(\pm i\sum_{m>0}\psi^{\mu}_{-m}\tilde{\psi}^{\mu}_{-m}\right)|0\rangle, \\ |B\rangle_{\text{gh}} &= \exp\left(\sum_{m=1}^{\infty}\left[c_{-m}\tilde{b}_{-m} + \tilde{c}_{-m}b_{-m}\right]\right. \\ &\qquad \left. \pm i\sum_{m>0}\left[\gamma_{-m}\tilde{\beta}_{-m} - \tilde{\gamma}_{-m}\beta_{-m}\right]\right)\tfrac{1}{2}(c_0 + \tilde{c}_0)|\downarrow\downarrow\rangle. \end{aligned} \tag{2.3}$$

Here, a tilde denotes right-moving modes, and $|\downarrow\downarrow\rangle$ is annihilated by all positive-frequency ghost and superghost oscillators, and the antighost zero-modes [14]. The fermion and superghost mode numbers are half-integers in the NS-NS sector, and integers in the R-R sector. The R-R zero-mode parts of $|B\rangle_{\psi}$ and $|B\rangle_{\text{gh}}$ have been temporarily omitted. In the presence of general open string backgrounds, the problem is more complicated: $|B\rangle$ will not satisfy simple linear boundary conditions and will not be a simple gaussian in the closed string creation operators. For constant background field strength, it retains its gaussian form, but acquires a field-dependent normalization which cannot be calculated from the boundary condition alone. In this section we will present a simple and general one-dimensional path integral algorithm which determines $|B\rangle$ in all cases.

Before proceeding with this program, we want to point out some features of the zero-field boundary state which have useful generalizations. First, note that the boundary conditions (2.1) on the fields all have the form $O_d(\sigma)=(-1)^d\tilde{O}_d(\sigma)$ where O and $\tilde{O}$ are left- and right-moving operators of conformal weight d. The peculiar factor of i in the fermion and superghost boundary conditions, as well as the sign ambiguity, thus arises because the corresponding fields have half-integral conformal weight. After GSO projection, the fermions will be accompanied by ghosts in combinations with integer total conformal weight, and the factors of i will disappear. How this happens will be explained in detail in the next section, where we study the bosonized formalism. A further important point is that the linear boundary conditions on the mode creation and annihilation operators imply similar conditions on composite operators such as the Virasoro generators, BRST charge and the spacetime Lorentz and supersymmetry generators. Specifically, the Virasoro generators and the BRST charges satisfy [4]

$$\begin{aligned}\left(L_n-\tilde{L}_{-n}\right)|B\rangle &= 0\,,\\ \left(Q+\tilde{Q}\right)|B\rangle &= 0,\end{aligned}\tag{2.4}$$

which are just the conditions that $|B\rangle$ be invariant to $\mathrm{Diff}(S_1)$ reparametrizations of the boundary coordinate σ [15,16]. This is essentially the condition that $|B\rangle$ be a physical state of the closed string and is clearly more general than any specific linear boundary condition on the worldsheet fields. In fact, we believe that reparametrization invariance is the general condition which picks out acceptable open string backgrounds.

The effect of a general background gauge field $A_\mu(X)$ is accounted for by including, for each boundary in the Polyakov path integral, a Wilson line factor

$$\operatorname{tr}\mathrm{P}\exp(-S_A)\,,\tag{2.5}$$

where S_A represents a condensate of photon vertices:

$$S_A=\frac{i}{4\pi}\int_0^{2\pi}\mathrm{d}\sigma\left[A_\mu(X)\,\partial_\sigma X^\mu-\tfrac{1}{2}iF_{\mu\nu}(X)\theta^\mu\theta^\nu\right]_{\tau=0}.\tag{2.6}$$

In this expression, $A_\mu=A^a_\mu\lambda_a$ and $F_{\mu\nu}=F^a_{\mu\nu}\lambda_a$ where λ_a are the generators of the gauge group, the P notation means path ordering on the boundary, and the fermionic term would be absent in a bosonic theory. The trace, represented by the notation tr, is over the indices of the group-matrices. The precise identity of the fermion $\theta^\mu(\sigma)$ is a subtlety which will be considered later. For now, we simply note that this interaction is both gauge-invariant and supersymmetric. The one-dimensional supersymmetry is

$$\delta X^{\mu} = \varepsilon\theta^{\mu},$$
$$\delta\theta^{\mu} = -i\varepsilon\,\partial_{\sigma}X^{\mu},$$
$$\delta\lambda_{a} = i\varepsilon\theta^{\mu}\left[A_{\mu}(X), \lambda_{a}\right], \tag{2.7}$$

where the rather unconventional supersymmetry variation of the generators is made necessary by the nonabelian nature of the gauge group but of course preserves their algebra.

Thus, instead of integrating freely over the boundary fields, a procedure which led to the boundary conditions and boundary states cited above, the worldsheet path integral is in general weighted by a non-trivial functional (2.5) of the boundary values of the worldsheet fields. The effect of this on $|B\rangle$ is best appreciated by working one's way up from a similar treatment of the one-dimensional path integral for a simple harmonic oscillator.

2.2. ONE-DIMENSIONAL PATH INTEGRAL APPROACH

Consider the following euclidean path integral for a simple harmonic oscillator:

$$\langle S|\mathrm{e}^{-Ht}|S\rangle = \int\mathscr{D}q(t)\exp\left(-\tfrac{1}{2}\int_0^T \mathrm{d}t\,(\dot{q}^2+q^2)\right)\mathrm{e}^{-S(q(T))}\mathrm{e}^{-S(q(0))}. \tag{2.8}$$

The insertion of the boundary action $S(q)$ serves to create the oscillator at $t=0$ (and annihilate it at $t=T$) in the state $|\Psi\rangle$ with coordinate representation

$$\langle x|\Psi\rangle = \mathrm{e}^{-S(x)}. \tag{2.9}$$

For an oscillator (of frequency $\frac{1}{2}$) the normalized coordinate eigenstates can be written in terms of the creation operators $a^{\dagger}$ as follows:

$$|x\rangle = (2\pi)^{-1/4}\mathrm{e}^{-x^2/4}e^{-1/2(a^{\dagger})^2+xa^{\dagger}}|0\rangle. \tag{2.10}$$

These states satisfy the eigenvalue condition

$$(a^{\dagger}+a-x)|x\rangle = 0 \tag{2.11}$$

and the completeness relation

$$\int_{-\infty}^{\infty}\mathrm{d}x\,|x\rangle\langle x| = 1, \tag{2.12}$$

so that the state (2.9) created by the boundary action $S(q)$ can be rewritten as

$$|\Psi\rangle = \int \mathrm{d}x\, \mathrm{e}^{-S(x)}|x\rangle$$
$$= \int \mathrm{d}x\, (2\pi)^{-1/4} \mathrm{e}^{-x^2/4 - S(x)} \mathrm{e}^{-(a^\dagger)/2 + xa^\dagger}|0\rangle . \tag{2.13}$$

This expresses the general harmonic oscillator boundary state in terms of creation operators and permits it to be explicitly decomposed in terms of energy eigenstates. If $S(x) = 0$, the integral can be done explicitly, yielding

$$|\Psi\rangle = (8\pi)^{1/4} \mathrm{e}^{(a^\dagger)^2/2}|0\rangle , \tag{2.14}$$

which satisfies the oscillator equivalent of free boundary conditions,

$$i\dot{q}|\Psi\rangle = (a - a^\dagger)|\Psi\rangle = 0 . \tag{2.15}$$

The integral representation (2.13) is a simple one-component realization of the structure we want to develop for the string.

To generalize this argument to the string, we need to identify a complete commuting set of coordinates on which the boundary action depends and we need to find the normalized eigenstates of these coordinates in terms of the closed string oscillators (the normalization is essential for the completeness integral). If the string is created at $\tau = 0$, its bosonic coordinates can be written in terms of modes as

$$X^\mu(\sigma, \tau) = q^\mu + \sum_{m=-\infty}^{\infty} |m|^{-1/2}\left[a_m^\mu \mathrm{e}^{-im\sigma} + \tilde{a}_m^\mu \mathrm{e}^{im\sigma}\right] . \tag{2.16}$$

We have defined rescaled mode operators

$$\alpha_m^\mu \equiv -i\sqrt{m}\, a_m^\mu , \qquad \alpha_{-m}^\mu \equiv i\sqrt{m}\, a_{-m}^\mu \tag{2.17}$$

to get standard harmonic oscillator commutators. Since the boundary involves only parallel derivatives $\partial/\partial\sigma$, the bosonic modes will appear only in the combinations

$$\begin{aligned} \bar{x}_m^\mu &= a_m^{\mu\dagger} + \tilde{a}_m^\mu , \\ x_m^\mu &= a_m^\mu + \tilde{a}_m^{\mu\dagger} , \end{aligned} \qquad (m > 0) . \tag{2.18}$$

Together with q^μ, this makes up a complete *commuting* set of bosonic coordinates for the string. We now interpret (2.18) as eigenvalue equations, with x_m^μ, $\bar{x}_m^\mu$ as

c-number eigenvalues and construct states $|x, \bar{x}\rangle$ satisfying

$$\left[a_m^{\mu\dagger} + \tilde{a}_m^{\mu} - \bar{x}_m^{\mu}\right]|x, \bar{x}\rangle = 0,$$

$$\left[a_m^{\mu} + \tilde{a}_m^{\mu\dagger} - x_m^{\mu}\right]|x, \bar{x}\rangle = 0, \tag{2.19}$$

as well as the completeness relation

$$\int \mathscr{D}\bar{x}\mathscr{D}x|x, \bar{x}\rangle\langle x, \bar{x}| = 1. \tag{2.20}$$

The explicit construction of these states in terms of mode creation operators is given in appendix A, where we show that

$$|x, \bar{x}\rangle = \exp\left\{-\tfrac{1}{2}(\bar{x}|x) - \left(a^{\dagger}|\tilde{a}^{\dagger}\right) + \left(a^{\dagger}|x\right) + \left(\bar{x}|\tilde{a}^{\dagger}\right)\right\}|0\rangle, \tag{2.21}$$

with the round brackets in the exponent denoting sums over spacetime and mode indices:

$$(\bar{x}|x) \equiv \sum_{\mu=1}^{D}\sum_{m=1}^{\infty}\bar{x}_m^{\mu}x_m^{\mu}. \tag{2.22}$$

That $|x, \bar{x}\rangle$ is an eigenvector of (2.19) is fairly obvious, while showing that it is normalized requires some work.

To study the superstring, we will need an analogous construction for the fermionic coordinates. The worldsheet fermions are two-component Majorana spinors whose left- and right-moving components have mode expansions at $\tau = 0$ given by

$$\psi_+^{\mu}(\sigma, 0) = \sum_m \psi_m^{\mu}e^{-im\sigma},$$

$$\psi_-^{\mu}(\sigma, 0) = \sum_m \tilde{\psi}_m^{\mu}e^{im\sigma}, \tag{2.23}$$

where m takes integer values in the R-R sector and half-integer values in the NS-NS sector. The correct "free" boundary conditions are found to be [4, 10]

$$\psi_+^{\mu} \mp i\psi_-^{\mu} = 0, \tag{2.24}$$

where the sign ambiguity corresponds to the two spin structures summed by the GSO projection. The boundary action (2.6) represents a sum of photon vertices, and should therefore depend upon only the combination of ψ_+^{μ} and ψ_-^{μ} not set to zero by the boundary conditions, namely

$$\psi_+^{\mu} \pm i\psi_-^{\mu} \equiv \theta^{\mu} = \sum_{m=-\infty}^{\infty}\theta_m^{\mu}e^{-im\sigma}. \tag{2.25}$$

This is the one-component fermion appearing in the Wilson line integral (2.6). The Majorana condition may be imposed by taking ψ_+ to be real. Then ψ_- is imaginary, θ is real, and the modes satisfy

$$\psi^{\mu}_{-m} = \psi^{\mu\dagger}_{m}, \qquad \tilde{\psi}^{\mu}_{-m} = -\tilde{\psi}^{\mu\dagger}_{m},$$
$$\theta^{\mu}_{-m} = \theta^{\mu\dagger}_{m} \equiv \bar{\theta}^{\mu}_{m}. \tag{2.26}$$

There is a zero mode in the R-R sector which will be considered separately below. The fermion anticommutation relations

$$\{\psi^{\mu}_{m}, \psi^{\nu}_{n}\} = \{\tilde{\psi}^{\mu}_{m}, \tilde{\psi}^{\nu}_{n}\} = \delta_{m+n,0}\delta^{\mu\nu} \tag{2.27}$$

imply that for $m>0$, $\psi^{\mu\dagger}_{m}$ and $-\tilde{\psi}^{\mu\dagger}_{m}$ are the creation operators conjugate to the annihilation operators ψ^{μ}_{m}, $\tilde{\psi}^{\mu}_{m}$. The factor of i in the boundary conditions plays a crucial role in guaranteeing that these fermionic boundary coordinates anticommute among themselves:

$$\{\theta^{\mu}_{m}, \theta^{\nu}_{n}\} = 0. \tag{2.28}$$

Note that the fermionic boundary coordinates for one spin structure become the boundary conditions (fermionic momenta) for the other and vice-versa.

We now interpret θ^{μ}_{m} and $\bar{\theta}^{\mu}_{m} \equiv \theta^{\mu}_{-m}$ as anticommuting c-numbers and look for fermionic position eigenstates satisfying

$$\begin{aligned} &\left(\bar{\theta}^{\mu}_{m} - \psi^{\mu\dagger}_{m} \mp i\tilde{\psi}^{\mu}_{m}\right)|\theta, \bar{\theta}; \pm\rangle = 0, \\ &\left(\theta^{\mu}_{m} - \psi^{\mu}_{m} \pm i\tilde{\psi}^{\mu\dagger}_{m}\right)|\theta, \bar{\theta}; \pm\rangle = 0, \end{aligned} \qquad (m>0). \tag{2.29}$$

In appendix A, we show that the states

$$|\theta, \bar{\theta}; \pm\rangle = \exp\left\{-\tfrac{1}{2}(\bar{\theta}|\theta) \pm i(\psi^{\dagger}|\tilde{\psi}^{\dagger}) + (\psi^{\dagger}|\theta) \mp i(\bar{\theta}|\tilde{\psi}^{\dagger})\right\}|0\rangle \tag{2.30}$$

satisfy the above eigenvalue conditions and the completeness relation

$$\int \mathcal{D}\bar{\theta}\mathcal{D}\theta |\theta, \bar{\theta}; \pm\rangle\langle\theta, \bar{\theta}; \pm| = 1. \tag{2.31}$$

The functional integral is grassmannian, and the round bracket notation is the same as defined in (2.22), summing over indices $m>0$.

The similarity between the forms of (2.21) and (2.30) reflects the worldsheet supersymmetry of both the boundary conditions and the boundary state. The 2D supersymmetry currents are

$$J_+ = \psi^{\mu}_{+}\partial_+ X^{\mu}, \qquad J_- = \psi^{\mu}_{-}\partial_- X^{\mu}. \tag{2.32}$$

The combination $J_+ \pm iJ_-$ vanishes on the boundary, and the orthogonal combination

$$J_+ \mp iJ_- = -\tfrac{1}{2} i\theta_\mu \partial_\sigma X^\mu \qquad (\text{at } \tau = 0) \tag{2.33}$$

generates a one-dimensional supersymmetry between X^μ and θ^μ in the boundary action. Thus, given the bosonic boundary state, the fermion boundary state could have been deduced from supersymmetry alone.

The zero modes in the bosonic sector and the R-R fermion sector require special treatment. The dependence of the zero-field state on the zero modes is so simple that it is convenient to use it as a base. The bosonic zero-mode coordinate q^μ is conjugate to the total momentum P^μ of the string. In the absence of an external field, the zero-mode boundary-state is the translation-invariant state $|0\rangle$ annihilated by P^μ. If the boundary action $S(q,\dots)$ depends on q^μ through a space-dependent background field, this zero-mode state is changed by the action of the zero-mode coordinate, to

$$\mathrm{e}^{-S(q,\dots)}|0\rangle . \tag{2.34}$$

Similar considerations apply to the R-R-sector fermionic zero modes. The boundary coordinates associated with the zero modes of the + spin structure are

$$\theta_0^\mu = \psi_0^\mu + i\tilde{\psi}_0^\mu , \tag{2.35}$$

and the corresponding conjugate momenta are

$$\pi_0^\mu = \tfrac{1}{2}\left(\psi_0^\mu - i\tilde{\psi}_0^\mu\right). \tag{2.36}$$

These coordinates and momenta satisfy the anticommutation relations

$$\{\theta_0^\mu, \theta_0^\nu\} = 0 , \qquad \{\pi_0^\mu, \pi_0^\nu\} = 0 , \qquad \{\theta_0^\mu, \pi_0^\nu\} = \delta^{\mu\nu} . \tag{2.37}$$

In the absence of an external field, the boundary state must, according to (2.2), be annihilated by the fermionic momenta:

$$\pi_0^\mu |0;+\rangle = \tfrac{1}{2}\left(\psi_0^\mu - i\tilde{\psi}_0^\mu\right)|0;+\rangle = 0 . \tag{2.38}$$

The fermionic zero modes, θ_0^μ, then act as fermionic creation operators [10] on the vacuum $|0;+\rangle$, generating a 2^D-dimensional Hilbert space spanned by the n-forms (with $n = 0, 1, \dots, D$) built out of the independent products of the θ_0^μ taken n at a time acting on the base state $|0;+\rangle$. In the presence of a nonvanishing background field, the zero-mode piece of the boundary action will generate some linear combination of these n-forms by its action on the vacuum:

$$e^{-S(\theta_0^\mu,\dots)}|0;+\rangle=\{\text{polynomial in }\theta_0^\mu\}|0;+\rangle. \tag{2.39}$$

Passing to the $-$ spin structure interchanges the roles of θ_0^μ and π_0^μ so that the creation operators of one spin structure are the annihilation operators of the other, and the vacuum of one is the filled Fermi sea of the other. Thus the analog of (2.39) for the $-$ spin structure is:

$$e^{-S(\pi_0^\mu,\dots)}|0;-\rangle=\{\text{polynomial in }\pi_0^\mu\}|0;-\rangle$$

$$|0;-\rangle=\prod_{\mu=1}^{D}\theta_0^\mu|0;+\rangle. \tag{2.40}$$

The n-forms built out of the θ_0^μ correspond to the D-dimensional Clifford algebra so that the states in the zero-mode Hilbert space can conveniently be represented by antisymmetrized products of D-dimensional gamma matrices:

$$\theta_0^{\mu_1}\dots\theta_0^{\mu_n}|0;+\rangle\sim\gamma^{\mu_1\cdots\mu_n}|0;+\rangle,$$

where we have used the notation

$$\gamma^{\mu_1\cdots\mu_n}=\frac{1}{n!}\sum_{\text{perms}}\varepsilon(p)\gamma^{\mu_{p(1)}}\cdots\gamma^{\mu_{p(n)}}. \tag{2.41}$$

The commutation relations of (2.37) induce a duality relation between the zero-mode forms appearing in (2.39) and (2.40),

$$\prod_{i=1}^{n}\pi_0^{\mu_i}|0;-\rangle=\frac{1}{(D-n)!}\varepsilon^{\mu_1\cdots\mu_D}\prod_{i=n+1}^{D}\theta_0^{\mu_i}|0;+\rangle, \tag{2.42}$$

which is precisely the transformation on antisymmetrized products of gamma matrices induced by multiplying them by γ_{D+1}. If we adopt the gamma matrix representation of the zero-mode forms, (2.39) and (2.40) take the form

$$e^{-S(\theta_0^\mu,\dots)}|0;+\rangle\equiv\{\text{sum of }\gamma^{\mu_1\cdots\mu_n}|0;+\rangle\},$$

$$e^{-S(\pi_0^\mu,\dots)}|0;-\rangle\equiv\{\text{sum of }\gamma^{\mu_1\cdots\mu_n}\}\gamma_{11}|0;+\rangle.$$

The gamma matrices of course only summarize the fermionic zero-mode structure and, in expressions like the above, we must imagine that they are tensored with states reflecting the other modes of the system. Since the boundary states of the two spin structures are related by γ_{11}, GSO projection amounts to multiplying the result for a given spin structure by $(1-\gamma_{11})$. The unification of the two spin structures provided by the gamma matrix notation will prove very convenient in what follows.

Now we turn to the construction of the boundary state generated by an arbitrary Wilson line action $S_A(X,\theta)$. For a single harmonic oscillator, we were able to write the boundary state as an integral over the normalized position eigenstates, weighted by the boundary action (2.13). Since the string is just a collection of oscillators, the obvious generalization of that construction to the string is

$$|A_\mu\rangle = \int \mathscr{D}\bar{x}\mathscr{D}x\mathscr{D}\bar{\theta}\mathscr{D}\theta \,\operatorname{tr} P\exp\left[-S_A\left(x,\bar{x},q;\theta,\bar{\theta},\theta_0\right)\right]|x,\bar{x}\rangle|\theta,\bar{\theta};\pm\rangle . \quad (2.43)$$

In this expression, the position eigenstates are those constructed above and the boundary action is the Wilson line appropriate to the background field. As in (2.34) and (2.39), the zero modes q^μ and $\psi_0^\mu \pm i\tilde{\psi}_0^\mu$ are allowed to act directly on the zero field vacuum. Using our explicit representations (2.21) and (2.30) for the position eigenstates, (2.43) can be expanded as

$$\begin{aligned}|A_\mu\rangle = {} & \exp\left\{-\left(a^\dagger|\tilde{a}^\dagger\right) \pm i\left(\psi^\dagger|\tilde{\psi}^\dagger\right)\right\}\\ & \times \int \mathscr{D}\bar{x}\mathscr{D}x \exp\left\{-\tfrac{1}{2}(\bar{x}|x) + \left(a^\dagger|x\right) + \left(\bar{x}|\tilde{a}^\dagger\right)\right\}\\ & \times \int \mathscr{D}\bar{\theta}\mathscr{D}\theta \exp\left\{-\tfrac{1}{2}(\bar{\theta}|\theta) + \left(\psi^\dagger|\theta\right) \mp i\left(\bar{\theta}|\tilde{\psi}^\dagger\right)\right\}\\ & \times \operatorname{tr} P\exp\left[-S_A\left(x,\bar{x},q;\theta,\bar{\theta},\theta_0\right)\right]|0;\pm\rangle . \end{aligned} \quad (2.44)$$

In principle, this construction solves the problem of calculating the effect of open string condensates on the closed string. It represents the boundary state as a path integral for a one-dimensional field theory with an action consisting of a free lagrangian quadratic in x and θ and an interaction term generated by S_A, plus linear source terms coupling these fields to the closed string creation operators. On closer examination, it turns out that the free action is linear in derivatives, so that the theory is renormalizable rather than finite. In general, this path integral has to be evaluated perturbatively, expanding the action and developing Feynman rules and a regularization and renormalization procedure. Because of renormalization anomalies, the reparametrization invariance conditions (2.4) will not be met for a general background. Thus, the equations of motion for open string backgrounds can be thought of as arising from the condition that these anomalies vanish. We will not pursue the general problem here, but will instead evaluate the path integral for the special case of constant abelian background field strength.

2.3. CONSTANT BACKGROUND FIELD STRENGTH

The importance of the constant-background case is that it is simple enough that (2.44) may be solved exactly and complex enough to teach us quite a lot about the mechanism of loop anomaly cancellation. By abelian we mean that the background is $F^a_{\mu\nu}\lambda_a$ with the λ_a in an abelian subgroup of O(32) so that the P-ordering instruction in (2.44) may be ignored. Nonabelian backgrounds are considered briefly in appendix B.

For the constant abelian gauge field case, the boundary action reduces to the simple quadratic

$$S_F = \frac{i}{8\pi} F_{\mu\nu} \int_0^{2\pi} \mathrm{d}\sigma \left[x^\mu \partial_\sigma x^\nu - i\theta^\mu\theta^\nu \right], \tag{2.45}$$

where $X^\mu(\sigma) \equiv q^\mu + x^\mu(\sigma)$. For a nonconstant field, (2.45) would be the beginning of an expansion of derivatives of F at the point q. Inserting the mode expansions (2.16), (2.18), and (2.25) for the fields and using the round bracket notation of (2.22) gives

$$S_F = \tfrac{1}{2}\left\{ (\bar{x}|F|x) + (\bar{\theta}|F|\theta) \right\}, \tag{2.46}$$

where we have defined the matrix

$$F_{\mu m, \nu n} = F_{\mu\nu}\delta_{mn}. \tag{2.47}$$

The functional integrals in (2.44) are then gaussian, and can be done explicitly to give the bosons and fermion factors of the boundary state:

$$|F\rangle_a = \mathrm{tr}[\mathrm{Det}(1+F)]^{-1} \exp\left(a^\dagger \left| \frac{1-F}{1+F} \right| \tilde{a}^\dagger \right) |0\rangle,$$

$$|F\rangle_\psi = \mathrm{tr}[\mathrm{Det}(1+F)]^{+1} \exp\left\{ \mp i \left(\psi^\dagger \left| \frac{1-F}{1+F} \right| \tilde{\psi}^\dagger \right) \right\} |0; \pm\rangle. \tag{2.48}$$

In the R-R sector, there is an extra fermion zero-mode factor of the form (2.39), which will be discussed shortly. The trace is over the group indices in an abelian subgroup of O(32). The determinant factors in front of these states are symbolic at this stage and will be explicitly evaluated in the next paragraph. Because $F_{\mu\nu}$ is independent of position, the boundary state has vanishing total momentum. Note that $\left(\frac{1-F}{1+F}\right)_{\mu\nu}$ is an orthogonal matrix. This means that, apart from the determinants, the effect of the gauge field, relative to the zero-field case, is a simple Lorentz rotation of the right-moving closed-string modes relative to the left-moving ones. Thus Lorentz-scalar functions of the left or right oscillators alone, such as the

Virasoro, Ramond or BRST charges will be unaffected by $F_{\mu\nu}$ so that the reparametrization invariance, $L_n - \tilde{L}_{-n} = 0$, and other worldsheet symmetry conditions will still hold. Spacetime symmetry generators will, in general, not annihilate the $F_{\mu\nu} \neq 0$ state, as is to be expected.

The determinants are really products of an infinite number of identical factors coming from the different modes. We use zeta-function regulation to evaluate these otherwise divergent quantities. Let Det denote the infinite determinant coming from the functional integral and let det denote the determinant of a 10×10 matrix:

$$\mathrm{Det}(1+F) = \prod_{m>0} [\det(1+F)],$$

$$\det(1+F) = \det(\delta_{\mu\nu} + F_{\mu\nu}). \tag{2.49}$$

For the bosons and the R-R sector fermions, the product is over integer modes and zeta-function techniques give

$$\sum_{m=1}^{\infty} 1 = \lim_{s\to 0} \sum_{m=1}^{\infty} m^{-s} = \zeta(0) = -\tfrac{1}{2}. \tag{2.50}$$

Consequently,

$$[\mathrm{Det}(1+F)]^{\mp 1} = [\det(1+F)]^{\pm 1/2} \tag{2.51}$$

and in the R-R sector the boson and fermion determinants cancel [17]. In the NS-NS sector, the situation is quite different. Zeta-function regulation gives

$$\begin{aligned}\sum_{m=1/2,3/2,\ldots} 1 &= \lim_{s\to 0} \sum_{n \text{ odd}} 2^s n^{-s} \\ &= \lim_{s\to 0} 2^s \left\{ \sum_{n=1}^{\infty} n^{-s} - \sum_{n=1}^{\infty} (2n)^{-s} \right\} \\ &= \lim_{s\to 0} (2^s - 1)\zeta(s) = 0. \end{aligned} \tag{2.52}$$

Therefore, in the NS-NS sector, the fermion determinant is unity, and the bosonic determinant

$$[\det(1+F)]^{1/2} \tag{2.53}$$

is not cancelled. This distinction between the two fermionic sectors will be crucial for the consistency of the theory. We will see later that we can also derive this result in a very simple way in the bosonized formalism, without using any zeta-function tricks.

Translating to standard oscillator notations using (2.17) and (2.26), we get the following NS-NS sector result:

$$|F\rangle_{\mathrm{NS}} = \operatorname{tr}\left[\det(1+F)\right]^{1/2}$$

$$\times \exp\left\{\left(\frac{1-F}{1+F}\right)_{\mu\nu}\left[-\sum_{m=1}^{\infty}\frac{1}{m}\alpha^{\mu}_{-m}\tilde{\alpha}^{\nu}_{-m} \pm i\sum_{m=1/2}^{\infty}\psi^{\mu}_{-m}\tilde{\psi}^{\mu}_{-m}\right]\right\}|0\rangle . \quad (2.54)$$

The ghost contribution is as in (2.3). This state reduces to (2.3) for $F_{\mu\nu}=0$, and to the bosonic string result [3] if the fermions are omitted. It is an eigenstate of the modified boundary conditions [18]

$$\partial_\tau X^\mu = iF_{\mu\nu}\,\partial_\sigma X^\nu ,$$

$$(\psi^\mu_+ \mp i\psi^\mu_-) = -F_{\mu\nu}(\psi^\nu_+ \pm i\psi^\nu_-) . \quad (2.55)$$

The exponent of (2.54) could have been deduced from (2.55) alone, but this functional integral was needed to derive the determinant factor.

In the R-R sector the determinant is absent and the ψ sum is over integer m. The action (2.45) also has a zero mode piece

$$S_0 = \tfrac{1}{4}F_{\mu\nu}\theta^\mu_0\theta^\nu_0 , \quad (2.56)$$

which, by (2.35) and (2.39), changes the R-R ground state to

$$|F;\pm\rangle_{\mathrm{R,\,ground\ state}} = \operatorname{tr}\exp\left\{-\tfrac{1}{4}F_{\mu\nu}(\psi^\mu_0 \pm i\tilde{\psi}^\mu_0)(\psi^\nu_0 \pm i\tilde{\psi}^\nu_0)\right\}|0,\pm\rangle . \quad (2.57)$$

Because $(\psi^\mu_0 \pm i\tilde{\psi}^\mu_0)$ anticommute, only the first six terms in the expansion of the exponential will survive, giving a sum of wedge products of $F_{\mu\nu}$ up to fifth order in F. Odd powers of F will be cancelled by the O(32) trace, since the generators are antisymmetric. According to the discussion after (2.42) we may represent the products of fermion zero modes as antisymmetrized products of gamma matrices. Thus we may write (2.57) in terms of an exponential of $-\frac{1}{2}F_{\mu\nu}\gamma^\mu\gamma^\nu$, provided we antisymmetrize the γ-matrices in each term. It is convenient to introduce a notation $Æ(-\frac{1}{2}\not{F})$ for this antisymmetrized exponential. All terms in the expansion with repeated Lorentz indices are to be dropped, leaving a polynomial in $F_{\mu\nu}$ of order $\frac{1}{2}D$. The γ-matrix representation introduced in the previous section allows us to write the zero-mode structure of the GSO-projected R-R ground state in the very simple form

$$\operatorname{tr} Æ(-\tfrac{1}{2}\not{F})(1-\gamma_{11})|0;+\rangle . \quad (2.58)$$

The full boundary state is obtained by acting on this ground state with the appropriate positive frequency creation operators but we will again refrain from writing it explicitly as we will, in the end, only need the projection of the boundary state onto its lowest-mass or ground state component. In the next section, we will rederive (2.58) in a bosonized language. In particular, we will obtain an independent verification of the zeta-function results for the fermion determinant in the two sectors.

There is also a superghost piece of the boundary state, a proper account of which is essential if we are to match (2.58) to the sphere. The argument from the superghost path integral is fairly subtle, but the end result is a fairly simple recipe (the justification may be found in appendix C): as far as the superghost zero modes are concerned, the boundary state is equivalent to a BRST-invariant super-Teichmüller insertion [19,10]

$$\left(G_0 \pm i\tilde{G}_0\right)\delta\left(\beta_0 \pm i\tilde{\beta}_0\right). \tag{2.59}$$

The need for the delta function is reasonably intuitive, and the super-Virasoro generator factor serves to render the whole insertion BRST-invariant. Acting on a massless state carrying momentum k, the super-Virasoro generators behave more or less like an inverse fermion propagator:

$$G_0 = \oint \frac{\mathrm{d}z}{2\pi i} z^{-1+3/2} \psi_\mu \partial X^\mu \cong -2^{-1/2} i \gamma_\mu k^\mu .$$

Although the boundary state in a constant background field carries zero momentum, so that the inverse propagator factor, strictly speaking, vanishes, we will find it useful to carry along a nonvanishing value of the momentum as a regulator. Compensating factors of k^{-1} will eventually appear so that nonvanishing results will be obtained in the $k \to 0$ limit. Using the boundary conditions to eliminate right-moving in favor of left-moving modes, the superghost insertions turn (2.58) into

$$2^{-3/2} i \not{k} \operatorname{tr} \not{\mathcal{E}}\left(-\tfrac{1}{2}\not{F}\right)\delta(\beta_0)(1-\gamma_{11})|0;+\rangle . \tag{2.60}$$

This result is heuristic at best and is presented here for comparison with the much more convincing bosonized treatment presented in the next section.

In our previous paper on this subject [4], field equations in the NS-NS sector followed easily once the boundary state had been constructed. This is not so easy for the R-R sector, because amplitudes on different genus worldsheets are constructed on different vacua for the ghosts and superghosts. We cannot even properly define the sigma model for the R-R sector without the spacetime fermion vertex, which can be defined only after bosonization [14]. Although R-R equations of motion for zero background field have been constructed by Polchinski and Cai [10] in the fermionic formalism, and the correct result was obtained, the derivation was heuristic and neglected problems associated with the ghost vacua at tree-level. In the next section, we will bosonize the fermions and superghosts, and then will be able to transform easily and consistently between worldsheets of different genus.

3. Bosonization

3.1. GENERAL REMARKS

In the last section, we constructed the fermionic functional integral and found it to be gaussian, and therefore calculable, at least for constant background gauge fields. Two-dimensional fermions can of course be rewritten as bosons and it is notorious that bosonization is essential to deal with the superghosts and spacetime fermions of the superstring. In particular, the vertex operators for spacetime fermions can *only* be written in terms of bosonized superghosts and the superghost charge "picture" structure of the theory can only be made evident (at least using current technology) via bosonization. It would therefore be useful to convert the results of the previous section to bosonized form. Rather than redoing the path integral, extending it to superghosts, we will show that we can determine the zero-mass piece of the bosonized boundary state from general symmetry arguments, using the results of sect. 2 only to fix some normalizations.

We therefore bosonize the matter fermions, ghosts and superghosts using the familiar results [14, 20, 21]

$$\psi^{\pm j} = \mathrm{e}^{\pm \rho_j}, \qquad j = 1, \dots, 5; \tag{3.1}$$

$$\begin{aligned} c &= \mathrm{e}^{\sigma}, \qquad b = \mathrm{e}^{-\sigma}, \\ \gamma &= \mathrm{e}^{\phi}\eta = \mathrm{e}^{\phi - \chi}, \\ \beta &= \mathrm{e}^{-\phi}\partial\xi = \partial\chi \mathrm{e}^{\chi - \phi}, \end{aligned} \tag{3.2}$$

and similar formulae with a tilde for the right-moving fields. Note that ψ^{μ} in (3.1) is in a Cartan-Weyl basis of O(10). The spin fields used in the fermion vertex are

$$S^{A} = \mathrm{e}^{(A \cdot \rho)} \tag{3.3}$$

where A is a spinor weight of O(10). Each of its five components is $\pm\frac{1}{2}$, and an odd/even number of negative signs determines the chirality (dot/no dot). Chirality is, in turn, coupled to the superghost number by the GSO projection, such that only combinations like

$$S^{A}\mathrm{e}^{-\phi/2}, \qquad S^{\dot{B}}\mathrm{e}^{-3\phi/2}, \text{ etc.} \tag{3.4}$$

are physical. To simplify notation, we will usually not show the chirality explicitly, but understand half the components of every spin field to be zero. The cocycles required to complete the bosonization can be arranged to generate the spinor metric [21]

$$C_{A\dot{B}} = -C_{\dot{B}A}, \qquad C^{A\dot{E}}C_{\dot{E}B} = \delta^A{}_B, \tag{3.5}$$

which is then used to raise and lower spin indices, and satisfies

$$\begin{aligned}(\gamma^\mu C)_{AB} &= (\gamma^\mu C)_{BA}, \\ (C\gamma^\mu)^{AB} &= (C\gamma^\mu)^{BA}, \\ (C\gamma^\mu C)^A{}_B &= -(\gamma^\mu)_B{}^A.\end{aligned} \tag{3.6}$$

Vertices for R-R states contain left and right spin fields joined by a matrix from the O(10) γ-matrix algebra

$$S^A \Gamma_{AB} \tilde{S}^B. \tag{3.7}$$

In expressions such as this we will raise and lower spinor indices according to the conventions

$$S^A = S_B C^{BA}, \qquad \tilde{S}^A = C^{AB}\tilde{S}_B.$$

This slightly unusual convention, carefully used in conjunction with (3.6), will eliminate unaesthetic signs and transposes in our final formulae. Products of an odd/even number of γ-matrices connect spinors of equal/opposite chirality. In particular, the antisymmetric tensor of the nonorientable superstring will have this form. We are now in a position to construct its vertex. The GSO projection requires that the two fermion vertices be of like chirality and that their left and right superghost numbers be equal modulo 2 (see (3.4)). The product of two like-chirality O(10) spinors decomposes into odd-rank antisymmetric tensors according to

$$\begin{aligned}S^A \times \tilde{S}^B &= \Psi_\mu + \Psi_{[\mu\nu\lambda]} + \Psi_{[\mu\nu\lambda\rho\sigma]}, \\ 16 \times 16 &= (10) + [120] + (126).\end{aligned} \tag{3.8}$$

(The fifth rank tensor is self-dual.) In fact, only the third rank tensor appears, for the following reasons:

The $O(N)$ open superstring has unoriented "quark" lines running around the boundary of the worldsheet. The closed superstring states that couple to it will therefore be symmetric under left-right interchange. The left and right fermion vertices anticommute (after inclusion of cocycles), so the wave-function that multiplies them must be antisymmetric in the two *spinor* indices. By (3.6),

$$\left(C\gamma^{\mu_1\cdots\mu_n}\right)^{AB}=(-1)^{n+1}\left(C\gamma^{\mu_n\cdots\mu_1}\right)^{BA}. \tag{3.9}$$

Now $123\to321$ is an odd permutation, but $1\to1$ and $12345\to54321$ are even permutations, so twist-invariance selects the third-rank tensor from (3.8) and rejects the others. This implies that the physical R-R vertex will have the form (3.7) with

$$\Gamma_{AB}\sim \not{H}_{AB}\equiv\left[\gamma^{\mu\nu\lambda}CH_{\mu\nu\lambda}(x)\right]_{AB}, \tag{3.10}$$

where $H_{\mu\nu\lambda}$ is a third-rank antisymmetric tensor. This differs from the case of the bosonic string, which coupled to a second-rank antisymmetric tensor. However, when we obtain the equations of motion in sect. 4, we will see that they imply that H is the field-strength of a second-rank tensor $B_{\mu\nu}$.

Each of the boson fields ρ_j, σ, ϕ, χ can be decomposed into oscillators of frequency $m=0,1,2,\dots$. The zero mode is a "momentum" with discrete eigenvalues. This is especially important for the ϕ superghosts, where $\mathrm{e}^{s\phi}$ has dimension

$$d_s\equiv d(\mathrm{e}^{s\phi})=-\tfrac{1}{2}s(s+2). \tag{3.11}$$

Equivalent vertices for emitting a boson can be constructed with any integer s, and for a fermion with any half-integer s. This ambiguity is called picture-choosing [14,22]. Pictures near $s=-1$ are easier to calculate [23]. Unfortunately the left minus right superghost number is not conserved at a boundary. To satisfy the boundary conditions on operators that change s, the boundary state must contain contributions from all pictures. Such a bosonized boundary state, in zero background field, has recently been constructed by Kostelecky et al. [24]. Their state is the direct product of independent states for the various worldsheet fields and contains a factor

$$\sum_s \mathrm{e}^{-i\pi d_s}\mathrm{e}^{s\phi}\mathrm{e}^{-(2+s)\tilde{\phi}}|0\rangle, \tag{3.12}$$

due to the zero modes of the superghost fields. Because of (3.11), this factor introduces tachyons of arbitrarily large negative mass-squared which should certainly not be present. Probably there is some superghost subtlety that invalidates the simple direct product construction of the full boundary state.

Fortunately, the zero-mass part of the bosonized boundary state is all we need to determine the anomaly and the problem alluded to just now does not have to be faced in its full generality. We have found a purely algebraic method of determining the zero-mass part, up to an overall normalization which has to be taken from the functional integral in sect. 2. In the pictures $-2\leqslant s\leqslant 0$, which include all the picture needed for our considerations, the results of ref. [24] are free of tachyons and agree with ours.

3.2. ALGEBRAIC STRATEGY

We first illustrate our algebraic strategy on the bosonic string in zero background field. Inspired by recent generalizations of the BRST formalism [25], we note that we can define an $\mathrm{OSp}(D|2)$ superalgebra which transforms states at a given mass level into each other. In zero background, the string has the usual invariance under $\mathrm{O}(D)$ rotations of the spacetime coordinates with generators

$$J^{\mu\nu} = \sum_{m=1}^{\infty} m^{-1}\left[\alpha^{\mu}_{-m}\alpha^{\nu}_{m} - \alpha^{\nu}_{-m}\alpha^{\mu}_{m}\right]. \tag{3.13}$$

The $\mathrm{OSp}(D|2)$ subalgebra also contains generators acting on the ghosts:

$$\begin{aligned}
J^{\mu c} &\equiv \sum_{m=1}^{\infty} m^{-1/2}\left[\alpha^{\mu}_{-m}c_{m} - c_{-m}\alpha^{\mu}_{m}\right],\\
J^{\mu b} &\equiv \sum_{m=1}^{\infty} m^{-1/2}\left[\alpha^{\mu}_{-m}b_{m} + b_{-m}\alpha^{\mu}_{m}\right],\\
J^{cb} &\equiv \sum_{m=1}^{\infty} \left[c_{-m}b_{m} - b_{-m}c_{m}\right],\\
J^{cc} &\equiv \sum_{m=1}^{\infty} \left[c_{-m}c_{m}\right],\\
J^{bb} &\equiv \sum_{m=1}^{\infty} \left[b_{-m}b_{m}\right].
\end{aligned} \tag{3.14}$$

What is of interest to us is that by virtue of our boundary conditions [4]

$$\alpha^{\mu}_{m} = -\tilde{\alpha}^{\mu}_{-m}, \qquad c_{m} = -\tilde{c}_{-m}, \qquad b_{m} = +\tilde{b}_{-m}, \tag{3.15}$$

the generators J^{MN} of the superalgebra all annihilate the zero field boundary state:

$$\left(J^{MN} + \tilde{J}^{MN}\right)|B\rangle = 0. \tag{3.16}$$

In other words, the boundary state is a scalar of the superalgebra. Since the bosonic zero mass piece must be constructed as a tensor product of one left $\mathrm{OSp}(D|2)$ vector

$$\left(\alpha^{\mu}_{-1}, c_{-1}, b_{-1}\right), \tag{3.17}$$

and one right vector, that piece can only be their $\mathrm{OSp}(D|2)$ scalar product

$$|B_0\rangle = \kappa\left[\left(\alpha_{-1}\cdot\tilde{\alpha}_{-1}\right) + b_{-1}\tilde{c}_{-1} - c_{-1}\tilde{b}_{-1}\right]|Z\rangle, \tag{3.18}$$

where κ is an undetermined normalization factor which we interpret as the open string coupling constant squared. This is precisely the result found in [4] and we recognize the Sp(2) ghost piece as the Siegel dilaton. The boundary conditions on b_0, c_0 determine the vacuum ket to be

$$|Z\rangle = \tfrac{1}{2}(c_0 + \tilde{c}_0) c_1 \tilde{c}_1 |\Omega\rangle , \tag{3.19}$$

where $|\Omega\rangle$ is the SL_2 invariant vacuum [14]. Since

$$i\partial x^\mu(z) = \sum_m \alpha^\mu_{-m} z^{m-1} , \tag{3.20}$$

$$c(z) = \sum_m c_{-m} z^{m+1} , \tag{3.21}$$

(3.18) can also be written as a field at $z=0$ acting on $|\Omega\rangle$:

$$|B_0\rangle = \kappa\left\{ -(\partial x \cdot \bar{\partial}\tilde{x}) c\tilde{c} + \tfrac{1}{2}\tilde{c}\bar{\partial}^2\tilde{c} - \tfrac{1}{2} c\partial^2 c\right\} \tfrac{1}{2}(\partial c + \bar{\partial}\tilde{c})(0)|\Omega\rangle . \tag{3.22}$$

Now we are ready for the superstring in zero gauge field. The bosonized BRST operator is [22]

$$Q_{\mathrm{BRST}} = \oint \frac{\mathrm{d}z}{2\pi i}\left\{ \mathrm{e}^\sigma T + \tfrac{1}{2} i\eta\psi \cdot \partial x\, \mathrm{e}^\phi - \tfrac{1}{4}\mathrm{e}^{-\sigma}\eta\, \partial\eta\, \mathrm{e}^{2\phi}\right\} , \tag{3.23}$$

where the fields are defined in (3.1), (3.2) and T is the total stress tensor (including ghosts). The picture changing operator [14, 22] is

$$\begin{aligned} \mathscr{X} &= 2\{ Q_{\mathrm{BRST}}, \xi(0)\} \\ &= \left\{ i\mathrm{e}^\phi \psi \cdot \partial x + \mathrm{e}^{2\phi}\partial\eta\, b + 2c\,\partial\xi \right\}(0) . \end{aligned} \tag{3.24}$$

The closed string has separate operators $Q, \tilde{Q}$ and $\mathscr{X}, \tilde{\mathscr{X}}$ for the left and right halves. Vertices in different pictures are defined by

$$\mathscr{X} V_s = V_{s+1} , \tag{3.25}$$

where s is the superghost number as in (3.11). Using short distance expansions, it is possible to compute the effect of picture-changing on any given operator. We give below the superghost charge $s = -2, -1, 0$ versions of three useful operators:

$$V_s^\mu :\ i\partial x^\mu \mathrm{e}^{-2\phi} c \to \psi^\mu \mathrm{e}^{-\phi} c \to i\partial x^\mu c + \cdots ; \tag{3.26a}$$

$$V_s^b :\ -\mathrm{e}^{-2\phi} \to 2\partial\xi \mathrm{e}^{-2\phi} c \to -1 ; \tag{3.26b}$$

$$V_s^c :\ \tfrac{1}{2} c\partial^2 c \mathrm{e}^{-2\phi} \to \tfrac{1}{2}\eta c \to \tfrac{1}{2} c\partial^2 c + \cdots . \tag{3.26c}$$

(In the notation V_s^μ, V_s^b and V_s^c, the superscript denotes the operator type and the subscript identifies the picture. The dots indicate some "inessential" extra ghost terms.) Eq. (3.26a) gives the vertex V_s^μ for emitting a zero-momentum photon [22] in pictures $s=-2,-1,0$. The graviton wave function is the outer product of left and right photon vertices. Comparing (3.26) with (3.22), we see that in the $s=0$ picture the graviton and dilaton vertex operators of the superstring are essentially identical to those of the bosonic string. In the $s=-2$ picture they differ [26] only by a factor $e^{-2\phi}$. As in (3.12), we want the left and right superghost numbers to add to $s_L+s_R=-2$. The $(-2,0)$ and $(0,-2)$ vertices for the superstring are thus given by inserting $e^{-2\phi}$ or $e^{-2\tilde{\phi}}$ into the bosonic formula (3.22). Now the picture-changing operator (3.24) has dimension 0, so its boundary condition should be

$$\mathscr{X}\,|B_0\rangle_{NS}=\tilde{\mathscr{X}}\,|B_0\rangle_{NS}\,. \tag{3.27}$$

This determines $|B_0\rangle_{NS}$ uniquely once a single picture is given. Thus

$$|B_0\rangle_{NS}=\kappa\sum_{s=-\infty}^{\infty}\left[V_s^\mu\tilde{V}^\mu_{-2-s}-V_s^b\tilde{V}^c_{-2-s}+V_s^c\tilde{V}^b_{-2-s}\right]\tfrac{1}{2}(c_0+\tilde{c}_0)|\Omega\rangle\,. \tag{3.28}$$

As a further check, note that in the picture $s_L=s_R=-1$, corresponding to the middle terms of (3.26), the graviton and dilaton wave functions are products of

$$V^\mu_{-1}=\psi^\mu e^{-\phi}c\,,$$

$$V^b_{-1}=2\partial\xi\, e^{-2\phi}c=2\beta e^{-\phi}c\,,$$

$$V^c_{-1}=\tfrac{1}{2}\eta c=\tfrac{1}{2}\gamma e^{-\phi}c\,. \tag{3.29}$$

(Here we have used (3.2) to write the dilaton wave-function in β_γ form.) Then (3.28) plus (3.29) manifestly reproduce the fermionic boundary state found in (2.3). The ghost vacuum $|\downarrow\downarrow\rangle$ is related to the SL_2-invariant ghost vacuum $|\Omega\rangle$ by [14]

$$|\downarrow\downarrow\rangle=e^{-\phi}ce^{-\tilde{\phi}}\tilde{c}|\Omega\rangle\,.$$

We note that the state (3.28) is invariant under left-right interchange, under which the tree-level GSO-projected states are also invariant. A left-right twist interchanges V and $\tilde{V}$, and changes the sign of $|\Omega\rangle$. It is important here to note that the inclusion of c-ghosts in the vertices (3.26) reverses their statistics from naive expectations.

Now we turn to the RR sector. In zero background field, we expect the zero-mass piece of the RR boundary state to have the general form (in the picture $s_L+s_R=-2$)

$$|B_0\rangle_R=\sum_s\left[V_s^A L^s_{AB}\tilde{V}^B_{-2-s}\right]\tfrac{1}{2}(c_0+\tilde{c}_0)|\Omega\rangle\,, \tag{3.30}$$

where the sum is over half-integer s, V^A and $\tilde{V}^A$ are the left- and right-moving zero-momentum fermion vertex operators and the matrix L_{AB} is to be determined. The explicit form of the fermion vector operators in the simple pictures $s = -\frac{1}{2}, -\frac{3}{2}$ is

$$V^A_{-1/2} = S^A e^{-\phi/2} c ,$$

$$V^A_{-3/2} = 2 S^A e^{-5\phi/2} \partial\xi c \, \partial c , \tag{3.31}$$

and the vertex operators in other pictures are defined as usual by $V^A_{s+1} = [Q, 2\xi V^A_s] \approx \mathscr{X} V^A_s$.

We will determine the matrix L_{AB} in terms of the known NS sector boundary with the help of spacetime supersymmetry. The spacetime supersymmetry generators are built out of the zero-momentum fermion vertex and, like it, can be constructed in different pictures. Explicitly

$$\Lambda^A_{-1/2} = \oint \frac{\mathrm{d}z}{2\pi i} S^A e^{-\phi/2}(z) ,$$

$$\Lambda^A_{1/2} = \mathscr{X} \Lambda^A_{-1/2} = \oint \frac{\mathrm{d}z}{2\pi i} \left\{ \tfrac{1}{2} b\eta S^A e^{3\phi/2} - i\sqrt{\tfrac{1}{2}}\, \partial X^\nu S^B e^{\phi/2} \gamma^{\nu A}_B \right\}(z) , \tag{3.32}$$

and the generators in other pictures are constructed by picture-changing in the usual way. These operators all have dimension 1 and, in the absence of background fields, satisfy the boundary condition $\Lambda^A_r = -\tilde{\Lambda}^A_r$ or

$$\left(\Lambda^A_r + \tilde{\Lambda}^A_r\right)\left(|B_0\rangle_{\mathrm{NS}} + |B_0\rangle_{\mathrm{R}}\right) = 0 . \tag{3.33}$$

This is just the condition that the spacetime supersymmetry annihilates the boundary state. As we shall see, it suffices to determine the RR part from the known NS part. In the next section, we see what to do when a background gauge field is present and supersymmetry is broken.

Although the physical boundary state carries zero momentum, in order to regulate $0/0$ ambiguities in the calculation of the anomaly, it is essential to attribute a momentum k to the vertex operators in the boundary state, taking the limit $k \to 0$ only at the very end. Because of the basic coordinate boundary condition $\partial_+ X^\mu = -\partial_- X^\mu$, it is essential to attribute opposite momenta to the left- and right-moving vertices:

$$\tilde{k}^\mu_{\mathrm{R}} = -k^\mu_{\mathrm{L}} . \tag{3.34}$$

Failure to do this amounts to letting the regulator change the physical nature of the boundary condition with disastrous effects on consistency. The regulated boundary states on which we act with the supercharges are therefore

$$|B_0\rangle_{\rm NS}=\kappa\sum_s\left[V^{\mu}_{s,k_{\rm L}}\tilde{V}^{\mu}_{-2-s,\tilde{k}_{\rm R}}-V^{b}_{s,k_{\rm L}}\tilde{V}^{c}_{-2-s,\tilde{k}_{\rm R}}+V^{c}_{s,k_{\rm L}}\tilde{V}^{b}_{-2-s,\tilde{k}_{\rm R}}\right]\tfrac{1}{2}(c_0+\tilde{c}_0)|\Omega\rangle,$$

$$|B_0\rangle_{\rm R}=\sum_s\left[V^{A}_{s,k_{\rm L}}L^{s}_{AB}\tilde{V}^{B}_{-2-s,\tilde{k}_{\rm R}}\right]\tfrac{1}{2}(c_0+\tilde{c}_0)|\Omega\rangle. \tag{3.35}$$

The momentum-dependent vertex operators (the momentum being denoted by a subscript) are defined in the obvious way: for a left-moving vertex append a factor of $e^{ik\cdot x_+}$ to the zero-momentum vertex in a base picture ($s=-1$ for bosons, $s=-\frac{1}{2}$ for fermions) and define the operator in other pictures by picture-changing.

To find the action of supersymmetry on the boundary state, we need to know the action of the supercharges on the individual vertices. This is in principle a straightforward matter of using the operator product expansions of the worldsheet fields to work out the products of the supercharges and the vertices. For future reference, we record here the precise expansions we use:

$$\psi^{\mu}(z)S^{A}(0)\sim-2^{-1/2}S^{B}(0)(\gamma^{\mu})_{B}{}^{A}z^{-1/2},$$

$$S^{A}(z)S^{B}(0)\sim C^{AB}z^{-5/4}+i2^{-1/2}\not{\psi}^{AB}(0)z^{-3/4},$$

$$e^{r\phi(z)}e^{s\phi(0)}\sim z^{-rs}e^{(r+s)\phi(0)}. \tag{3.36}$$

Our notation follows [21], except that we have converted the results of that paper to the more usual conventions that X^{μ} is real, and $\not{\psi}\equiv\gamma^{\mu}\psi_{\mu}$. An annoying subtlety is that one finds on carrying out this program that the actions of supersymmetry and picture-changing are not manifestly commutative. With a little effort, one can show that the failure to commute is due to terms which are themselves BRST transformations and make no contribution to our final result. This important point is explained in detail in appendix D. There we argue that, apart from ignorable BRS transforms, the action of supersymmetry on the vertices can be expressed in an elegant picture-independent form:

$$\left\{\Lambda^{A}_{r},V^{\mu}_{s,k}\right\}=-\tfrac{1}{2}V^{B}_{r+s,k}(\not{k}\gamma^{\mu})_{B}{}^{A},$$

$$\left[\Lambda^{A}_{r},V^{b}_{s,k}\right]=\left[\Lambda^{A}_{r},V^{c}_{s,k}\right]=0,$$

$$\left[\Lambda^{A}_{r},V^{B}_{s,k}\right]=-i\sqrt{\tfrac{1}{2}}\,(C\gamma_{\mu})^{AB}V^{\mu}_{r+s,k}. \tag{3.37}$$

Note that the action of Λ^{A} on V^{μ} vanishes when $k=0$. This is one of several reasons why a regulator momentum is essential.

Given this algebra, the action of the supercharge on the boundary state is easy to work out:

$$\left(\Lambda_r^A+\tilde{\Lambda}_r^A\right)|B_0\rangle_{\mathrm{NS}}=\tfrac{1}{4}\kappa\sum_s\Big\{-V^B_{r+s,k_{\mathrm{L}}}\left(\not{k}_{\mathrm{L}}\gamma^\mu\right)_B{}^A\tilde{V}^\mu_{-2-s,\tilde{k}_{\mathrm{R}}}$$

$$+V^\mu_{s,k_{\mathrm{L}}}\tilde{V}^B_{r-2-s,\tilde{k}_{\mathrm{R}}}\left(\tilde{\not{k}}_{\mathrm{R}}\gamma^\mu\right)_B{}^A\Big\}\left(c_0+\tilde{c}_0\right)|\Omega\rangle, \qquad (3.38)$$

$$\left(\Lambda_r^A+\tilde{\Lambda}_r^A\right)|B_0\rangle_{\mathrm{R}}=-\frac{i}{2^{3/2}}\sum_s\Big\{\left(C\gamma_\mu L^s\right)^A{}_B V^\mu_{r+s,k_{\mathrm{L}}}\tilde{V}_B^{-2-s,\tilde{k}_{\mathrm{R}}}$$

$$+\left(L^sC\gamma_\mu\right)_B{}^A V^B_{s,k_{\mathrm{L}}}\tilde{V}^\mu_{r-2-s,\tilde{k}_{\mathrm{R}}}\Big\}\left(c_0+\tilde{c}_0\right)|\Omega\rangle. \quad (3.39)$$

The sum of these two expressions vanish if

$$\left(L^{s+r}C\gamma_\mu\right)_A{}^B=i\sqrt{\tfrac{1}{2}}\,\kappa\left(\not{k}_{\mathrm{L}}\gamma_\mu\right)_A{}^B, \qquad (3.40)$$

$$\left(C\gamma_\mu L^{s-r}\right)^A{}_B=-i\sqrt{\tfrac{1}{2}}\,\kappa\left(\tilde{\not{k}}_{\mathrm{R}}\gamma^\mu\right)_B{}^A. \qquad (3.41)$$

These two equations are consistent if and only if the regulator momenta satisfy (3.34) and the matrix L_{AB} satisfies

$$L^s_{AB}=i\sqrt{\tfrac{1}{2}}\,\kappa\left(\not{k}_{\mathrm{L}}C\right)_{AB}. \qquad (3.42)$$

Thus our peculiar choice of "chiral" regulator momentum is needed for consistency with spacetime supersymmetry. Note also that twist invariance would normally forbid the coupling of left- and right-moving fermion vertices via a single gamma matrix (see (3.10)). The extra minus sign in the left-right flip coming from $\tilde{k}_{\mathrm{R}}=-k_{\mathrm{L}}$ handily defeats this selection rule. Our original choice of $s_{\mathrm{L}}+s_{\mathrm{R}}=-2$ was arbitrary and it is in fact more convenient for later work to picture change the RR boundary state to $s_{\mathrm{L}}+s_{\mathrm{R}}=-1$. Our final result for the regulated RR boundary state is therefore

$$\tfrac{1}{2}\left(\mathscr{X}+\tilde{\mathscr{X}}\right)|B_0\rangle_{\mathrm{R}}=i\frac{\kappa}{2^{3/2}}\sum_s V^A_{s,k_{\mathrm{L}}}\left(\not{k}_{\mathrm{L}}C\right)_{AB}\tilde{V}^B_{-1-s,\tilde{k}_{\mathrm{R}}}|\Omega\rangle. \qquad (3.43)$$

The expression of the boundary state as a sum over infinitely many pictures requires some comment. When it is inserted in any given tree-level process, and specific choices are made for the pictures of the other vertices, only one term in the infinite series for the bosonized boundary state will contribute. In principle one should show that answers to physical questions are independent of this sort of picture choice. Precisely because we will be cancelling loop-generated BRS anomalies by moving the worldsheet sigma model away from its conformal fixed point, the usual arguments for the equivalence of vertex operators in different pictures do not work for us. Our procedure will allow us to show that the spacetime background field equations needed to cancel loop-level BRS anomalies (at least a subset of all the physical information in the theory!) are picture-independent. We believe, but have not yet shown, that our loop-corrected background field equations actually guarantee picture-independence of all physical quantities. Some comments and speculations on this matter are presented in the conclusion.

This construction identifies the zero-mass component of the zero-field RR sector boundary operator. To compare with the result of sect. 2, we must include the super-Teichmüller insertions there. Therefore, we will compare (3.43) with (2.60) for $F=0$. The required identification is

$$i2^{-3/2}\not{k}\delta(\beta_0)\{|0;+\rangle-|0;-\rangle\}\sim\tfrac{1}{2}(\mathscr{X}+\tilde{\mathscr{X}})|B_0\rangle_{\mathrm{R}}$$
$$=i\kappa 2^{-3/2}\sum_s V_s^A(\not{k}_{\mathrm{L}})_{AB}\tilde{V}^B_{-1-s}(c_0+\tilde{c}_0)|\Omega\rangle\,. \quad (3.44)$$

On $|B_0\rangle$, $\mathscr{X}=\tilde{\mathscr{X}}$ and $G_0\sim i\not{k}$, so we have the identifications

$$G_0\delta(\beta_0)\sim\mathscr{X}\,,$$
$$|0;+\rangle-|0;-\rangle\sim|B_0\rangle\,. \quad (3.45)$$

The first of these has already been noted, and is motivated in appendix C.

We may note that the r.h.s. of (3.44) is even under twists, since V^A and $\tilde{V}^B$ anticommute, $\not{k}_{AB}$ is symmetric, $|\Omega\rangle$ is odd under twists, and $k_{\mathrm{L}}\to\tilde{k}_{\mathrm{R}}=-k_{\mathrm{L}}$. Twisting interchanges ψ and $\tilde{\psi}$ as well, so the vacua $|0;+\rangle$ and $|0;-\rangle$ are exchanged, up to a phase. This phase is fixed to be -1 by the NS-NS sector and this choice makes both sides of (3.44) twist-invariant. The chirality projection on $|B_0\rangle_{\mathrm{GSO}}$ noted at the end of sect. 2.2 applies as well to (3.44), since the V_s^A have positive chirality.

The identification $\psi_0^\mu\equiv\gamma^\mu$ is imposed by the short-distance expansion (3.36). Writing

$$\psi_0^\mu=\oint\frac{\mathrm{d}z}{2\pi i}z^{-1/2}\psi^\mu(z)$$

and using (3.36) implies

$$\psi_0^\mu V_s^A\tilde{V}_{A,-2-s}|\Omega\rangle=-2^{-1/2}V_s^A(\gamma^\mu)_A{}^B\tilde{V}_{B,-2-s}|\Omega\rangle\,. \quad (3.46)$$

Thus, a fermionic state of the form

$$\psi_0^{\mu_1} \cdots \psi_0^{\mu_n}(|0; +\rangle - |0; -\rangle)$$

must be identified with a bosonized state of the form

$$\left(-\sqrt{\tfrac{1}{2}}\right)^n \sum_s V_s^A (\gamma^{\mu_1} \cdots \gamma^{\mu_n})_A{}^B \tilde{V}_{B,-2-s}|\Omega\rangle .$$

A special case of this is seen in (3.44). Only statements which are independent of spin-structure are easily checked in the bosonized formalism. Therefore, to convert a fermionic expression into one which is to be compared to a bosonized result, it is advisable to first use the boundary conditions in each spin-structure separately to convert $\tilde{\psi}$'s to ψ's, and then apply the GSO projection. Finally, use (3.46) to interpret the fermion zero-modes as γ-matrices. The resulting statements will then be independent of spin structure, and thus comparable.

3.3. BACKGROUND GAUGE FIELDS

Now we turn on a constant abelian gauge field $F_{\mu\nu}$. The path integral calculation of sect. 2 taught us that this introduces two very simple changes in the NS-sector boundary operator: the overall normalization changes by a factor of $\sqrt{\det(1+F)}$ and spacetime vector operators built out of left-moving fields are Lorentz-rotated relative to those built out of right-moving fields by the orthogonal matrix $\left(\frac{1-F}{1+F}\right)_{\mu\nu}$. Taking both features over to our bosonized boundary operator, we are led to replace (3.28) by

$$|F\rangle_{\mathrm{NS}} = \kappa[\det(1+F)]^{1/2} \sum_{s=-\infty}^{\infty} \left\{ V_s^\mu \left(\frac{1-F}{1+F}\right)_{\mu\nu} \tilde{V}^\nu_{-2-s} - V_s^b \tilde{V}^c_{-2-s} + V_s^c \tilde{V}^b_{-2-s} \right\} \tfrac{1}{2}(c_0 + \tilde{c}_0)|\Omega\rangle . \quad (3.47)$$

This state is twist-invariant, as it was when $F=0$, since a twist changes the orientation of the boundary, causing F to change sign.

A small digression on boundary conditions will help to justify this simple recipe. The relative Lorentz rotation between the left and right vector operators found in sect. 2 came ultimately from the effect of the gauge field on the boundary conditions. As was mentioned there, a worldsheet boundary carrying constant gauge field $F_{\mu\nu}$ imposes the following boundary conditions on spacetime scalar and vector worldsheet fields of conformal weight d:

$$O_d(z) = (-1)^d \tilde{O}_d(1/\bar{z}),$$

$$O_d^\mu(z) = (-1)^d \left(\frac{1-F}{1+F}\right)_{\mu\nu} \tilde{O}_d^\nu(1/\bar{z}). \tag{3.48}$$

The vertex operators in our bosonized boundary operator are composites of the worldsheet fields, and must satisfy appropriate composites of the above conditions. For instance, the vector operator $V^\mu_{-1} = \psi^\mu e^{-\phi} c$ defined in (3.29) will satisfy

$$\psi^\mu e^{-\phi} c(z) = \left(\frac{1-F}{1+F}\right)_{\mu\nu} \tilde{\psi}^\nu e^{\tilde{\phi}} \tilde{c}(\bar{z}^{-1}). \tag{3.49}$$

This is just the boundary condition for a vector operator of conformal weight zero, which is the weight of V^μ_{-1}. Note that, as promised in sect. 2, the factors of i which afflicted the boundary conditions of the worldsheet fermions do not afflict the V^μ, which are automatically GSO-projected and directly create physical states. On the other hand, the same field-dependent Lorentz rotation does appear in the V_μ boundary condition, and therefore in the boundary state (3.47).

Although a background gauge field breaks Lorentz invariance, a fixed rotation of the right-moving fields merely redefines the Lorentz generators. This redefinition affects the hole but not the crosscap, and the violation only occurs when hole and crosscaps are combined, because their Lorentz symmetries are incompatible. The same remark applies to space-time supersymmetry.

The problem is now to find the analog of (3.47) in the R-R sector. Since spinor and vector operators are related to each other by the Lorentz-covariant short-distance expansion (3.36), consistency requires that if we rotate right vectors by a certain angle, we must rotate right spinors by the same angle, wherever they appear. Thus the supersymmetry constraint (3.33) must change to

$$\left\{\Lambda_{A,r} + M(F)_A{}^B \tilde{\Lambda}_{B,r}\right\}|F\rangle = 0, \tag{3.50}$$

where $M(F)$ is the *spinor* representation of the same Lorentz rotation that occurs in (3.49). By the same token, the constraint (3.34) on the regulating momentum must be replaced by

$$k^L_\mu = -\left(\frac{1-F}{1+F}\right)_{\mu\nu} \tilde{k}^\nu_{\mathrm{R}}. \tag{3.51}$$

The matrix $M(F)$ is unitary and satisfies

$$\left(\frac{1-F}{1+F}\right)_{\mu\nu} (\gamma^\nu)_A{}^B = \left[M(F)^{-1}\gamma_\mu M(F)\right]_A{}^B. \tag{3.52}$$

It is easily calculated if we choose the spacetime basis such that $F_{\mu\nu}$ can be written as five 2×2 blocks along the diagonal

$$F_{\mu\nu} = \bigoplus_{j=1}^{5} \begin{pmatrix} 0 & f_j \\ -f_j & 0 \end{pmatrix}. \tag{3.53}$$

Then

$$\left(\frac{1-F}{1+F}\right)_{\mu\nu} \equiv \bigoplus_{y=1}^{5} \begin{pmatrix} \cos\vartheta_j & \sin\vartheta_j \\ -\sin\vartheta_j & \cos\vartheta_j \end{pmatrix}$$

represents rotations in the Cartan subalgebra through angles

$$\vartheta_j = -2\tan^{-1} f_j, \qquad j = 1,\ldots,5. \tag{3.54}$$

The spinor representation will be the diagonal matrix

$$\begin{aligned} M_A{}^B &= e^{i(A\cdot\vartheta)}\delta_A{}^B \\ &= \delta_A{}^B \prod_{j=1}^{5}\left[1 - 2iA_j f_j\right]\left(1+f_j^2\right)^{-1/2}, \\ &= \left[\det(1+F)\right]^{-1/2}\delta_A{}^B \prod_{j=1}^{5}\left[1 - 2iA_j f_j\right], \end{aligned} \tag{3.55}$$

where $A_j = \pm\frac{1}{2}$ are the weights. The appearance of the determinant which multiplies the boundary state in the *denominator* of (3.55) is significant, and will provide an alternate route to the cancellation of determinants of the R-R sector which was noted in sect. 2.

In the spinor representation, the five diagonal generators of O(10) are

$$\sigma_3^{(j)} = 2A_j\delta_A{}^B, \tag{3.56}$$

so the numerator polynomial from (3.55) is

$$\prod_{j=1}^{5}\left[1 - i\sigma_3^{(j)} f_j\right]. \tag{3.57}$$

To express this covariantly, we use the notation $Æ(-\frac{1}{2}\not{F})$, which was introduced in sect. 2, for an antisymmetrized exponential of $-\frac{1}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}$. The γ-matrix products in the expansion of the exponential are to be antisymmetrized term by term. This means that all terms in the expansion with repeated Lorentz indices are to be dropped, leaving a polynomial in $F_{\mu\nu}$ of order $\frac{1}{2}D$. Thus for $D=2$

$$Æ\{-\tfrac{1}{2}F_{12}\sigma^1\sigma^2 - \tfrac{1}{2}F_{21}\sigma^2\sigma^1\} = 1 - i\sigma^3 f\,,$$

which is one factor of (3.57). Since (3.57) was obtained by choosing $F_{\mu\nu}$ to lie in the Cartan subalgebra of O(10), we see that its covariant form is

$$\prod_{j=1}^{5}\left[1 - i\sigma_3^{(j)} f_j\right]_A{}^B = \left[Æ(-\tfrac{1}{2}\not{F})\right]_A{}^B .$$

Therefore, we have

$$M(F)_A{}^B = \left[\det(1+F)\right]^{-1/2}\left[Æ(-\tfrac{1}{2}\not{F})\right]_A{}^B . \tag{3.58}$$

This is the desired covariant representation of $M(F)$.

The matrices L_s in (3.30) will now acquire a dependence upon F. A direct analog of the calculation which led to (3.40) and (3.41) gives the equations (in matrix notation)

$$\gamma_\mu L_s(F) = -2^{-1/2} i\kappa\left[\det(1+F)\right]^{1/2} M(F)\left(\frac{1-F}{1+F}\right)_{\mu\nu}\gamma^\nu \tilde{\not{k}}_{\mathrm{R}} C\,,$$

$$L_{-2-s}(F) C\gamma_\mu M(F)^{\mathrm{T}} = 2^{-1/2} i\kappa\left[\det(1+F)\right]^{1/2} \not{k}_{\mathrm{L}}\gamma^\nu\left(\frac{1-F}{1+F}\right)_{\nu\mu} . \tag{3.59}$$

The matrix relations

$$\left[CM(F)\right]^{\mathrm{T}} = -CM(-F)\,, \qquad \left[M(-F)\right]^{-1} = M(F) \tag{3.60}$$

follow easily from (3.58) and (3.9), and can be used together with (3.51) and the unitarity of $M(F)$ to show that the two equations (3.59) are identical. From (3.52), it follows that

$$\tilde{\not{k}}_{\mathrm{R}} = -\left[M(F)\right]^{-1}\not{k}_{\mathrm{L}} M(F)\,. \tag{3.61}$$

Applying this to the first equation of (3.59) shows that

$$L_s(F) = L_{-2-s}(F) = 2^{-1/2} i\kappa \not{k}_{\mathrm{L}}\, Æ(-\tfrac{1}{2}\not{F})C\,.$$

Therefore, the zero-mass part of the F-dependent R-R boundary state is

$$|F\rangle_{\mathrm{R}} = 2^{-3/2} i\kappa \sum_s V_s^A\left[\not{k}_{\mathrm{L}}\, Æ(-\tfrac{1}{2}\not{F})\right]_A{}^B \tilde{V}_{B,-2-s}(c_0+\tilde{c}_0)|\Omega\rangle\,. \tag{3.62}$$

Since the determinants have cancelled, this result is a fifth-order polynomial in F. This gives an independent check of the zeta-function miracle in sect. 2.

We can easily check the equivalence of this with the fermionic result (2.60). That result has already been expressed in terms of γ-matrices and written in a GSO-projected form, so the identifications at the end of sect. 3.2 can be applied directly. Again we must include the appropriate super-Teichmüller and picture-changing insertions to match the vacua. Then we find that the fermionic and bosonized boundary states obey the expected relation

$$\not{k}\delta(\beta_0)\,\mathcal{A}\!\!E\left(-\tfrac{1}{2}\not{F}\right)(|0;+\rangle-|0;-\rangle)$$
$$\sim\kappa\sum_s V_s^A\left[\not{k}_{\mathrm{L}}\,\mathcal{A}\!\!E\left(-\tfrac{1}{2}\not{F}\right)C\right]_{AB}\tilde{V}^B_{-1-s}\tfrac{1}{2}(c_0+\tilde{c}_0)|\Omega\rangle\,, \tag{3.63}$$

up to normalization. Using the fact that $F\to -F$ under a twist, it can be shown that (3.62) is left-right symmetric. The l.h.s. of (3.63) is also symmetric, since a twist replaces $\theta_0^\mu\sim\gamma^\mu$ by $\pm 2i\pi_0^\mu\sim\pm i\gamma^\mu$, so that the signs of the γ-matrices in $\mathcal{A}\!\!E(-\frac{1}{2}\not{F})$ change to compensate the sign-change of F.

Finally, we have to discuss the generalization of these results to nonabelian gauge groups and nonconstant background fields. The considerations of appendix B lead to the following conclusions: As long as $F_{\mu\nu}$ is constant and all its components commute with each other, the correct generalization of eqs. (3.47) and (3.62) is simply to take the group trace of all expressions. For constant, but noncommuting fields, some of our expressions require modification. The general boundary state will be expressible as a sum of group and Lorentz traces of powers of $F_{\mu\nu}$, but new terms, which vanish in the "abelian limit", may appear. In particular, the normalization of the NS boundary state is not, in general, given by a simple determinant of the background field. It turns out, however, that our expression for the massless part of the R-R sector boundary state is exact, provided the products of group generators are symmetrized before tracing. Thus, our expressions for the loop corrections to the antisymmetric tensor equations of motion are exact as they stand if we neglect terms in ∇F. This will turn out to be important for understanding how loop corrections generate spacetime gauge anomaly cancellations. Beyond this, if the gauge field, abelian or not, is not constant, there will be corrections to all our formulas which can be computed as power series expansions in ∇F. We have not pursued this subject here, and will only discuss the properties of loop-corrected equations in the "low-energy" limit where ∇F may be neglected. Where confusion will not arise, we will use the expressions (3.47) and (3.62) to symbolize the general result.

For zero gauge field, the tr operation multiplies the simple abelian boundary anomaly by N for an O(N) gauge group and it is well-known that the boundary and crosscap anomalies in the R-R sector will cancel for O(32) [13, 4, 10]. The gauge field will change the boundary state, but not the crosscap, since the latter cannot couple to an open string photon (it is not a boundary). So adding the crosscap to (3.47) and (3.62) just subtracts their $F_{\mu\nu}=0$ value. Because $Æ(-\frac{1}{2}\not{F})$ is a polynomial, this removes the entire 0-form from (3.62). Odd terms in $F_{\mu\nu}$ are cancelled by the O(32) trace, so in the R-R sector we are left with 4-form and 8-form anomalies. Since our $D=10$ spinors are chiral, the 8-form is dual to a 2-form. These are precisely the forms which appear in the field equations for the antisymmetric tensor and, as we will now show, lead to crucial loop corrections to its equations of motion.

4. Field equations

4.1. LOWER GENUS

In this section we deduce the loop-modified field equations for the superstring in a background gauge field. They are inhomogeneous differential equations which cancel the BRST anomaly from a surface with a small hole against a BRST anomaly on a surface with no hole, induced by deliberately evaluating the sigma model away from its conformal fixed point. We consider the lower genus surface first.

The zero mass bosons of the closed superstring are the graviton, dilaton, and antisymmetric tensor. We give each a space-dependent vacuum expectation value. We then have a σ-model on the worldsheet, whose interaction lagrangian $\mathscr{L}_{\mathrm{I}}$ describes this classical background. We limit ourselves to first order perturbation theory in the σ model and will therefore only need to calculate correlation functions of the external vertices with a single insertion of $\mathscr{L}_{\mathrm{I}}$, which we can fix at $z=0$. For the superstring there is a choice of pictures [14,22] for each vertex, differing by choices of superghost numbers. On a sphere, the sum of both the left and right superghost numbers $(s_{\mathrm{L}}, s_{\mathrm{R}})$ of all closed string vertices must be each -2, but the picture choice is otherwise arbitrary. We will find it convenient to use the $s_{\mathrm{L}}=s_{\mathrm{L}}=-1$ picture for the NS-sector fields (graviton and dilaton). Had we chosen the $s_{\mathrm{L}}=s_{\mathrm{R}}=0$ picture, the NS-sector graviton vertex would have been the familiar interaction term of a supersymmetric σ model [27,12]. In the $s_{\mathrm{L}}=s_{\mathrm{R}}=-1$ picture, however, it takes the simpler, if less familiar form

$$\mathscr{L}_{\mathrm{I}}=\tfrac{1}{2}\psi^{\mu}\mathrm{e}^{-\phi}\tilde{\psi}^{\nu}\mathrm{e}^{-\tilde{\phi}}h_{\mu\nu}(x)\,. \tag{4.1}$$

The antisymmetric tensor lies in the R-R sector and its vertex is the product of two Ramond fermion vertices, whose superghost numbers must be half-integers. We will find it convenient to choose the picture $s_{\mathrm{L}}=s_{\mathrm{R}}=-\frac{1}{2}$. Since $\mathscr{L}_{\mathrm{I}}$ is taken as a Koba-Nielsen fixed point at $z=0$, we must also remember to include the canonical ghost factor $c(0)\tilde{c}(0)$ in the vertices.

The graviton, dilaton, and antisymmetric tensor wave-functions are then outer products of four kinds of left-moving vertices with the corresponding right-moving vertices. The left-moving vertices are

$$V^{\mu}_{-1}=\psi^{\mu}\mathrm{e}^{-\phi}c\,\mathrm{e}^{ik\cdot x_+}\,, \tag{4.2}$$

$$V^{b}_{-1}=2\,\partial\xi\,\mathrm{e}^{-2\phi}c\mathrm{e}^{ik\cdot x_+}\,, \tag{4.3}$$

$$V^{c}_{-1}=\tfrac{1}{2}\eta c\mathrm{e}^{ik\cdot x_+}\,, \tag{4.4}$$

$$V^{A}_{-1/2}=S^{A}\mathrm{e}^{-\phi/2}c\mathrm{e}^{ik\cdot x_+}\,, \tag{4.5}$$

and the right-moving vertices are built out of right-moving fields in a corresponding way. The coefficients of these products are the momentum-space wavefunctions for the various backgrounds. Where confusion will not arise, we will often assemble the $e^{ik\cdot x}$ factors in the vertices and the momentum-space wave functions into spacetime fields. We have already used these objects in our construction of the boundary state. A particular advantage of the $s=-1$ picture has to do with moving the dilation. The $c\tilde{c}$ factor must be removed if the position of the vertex is integrated. This is trivial for our choice of picture but, as can be seen from (3.26), is not so trivial for the $s=0$ picture superstring dilaton or for the bosonic string dilaton. Progress in moving the $s=0$ dilaton has recently been made by Polchinski [9].

Our first step is to compute the action of the bosonized BRST operator on these left-moving vertices [22]. Explicitly,

$$Q_{\mathrm{BRST}}=Q_0+Q_1+Q_2\,, \tag{4.6}$$

$$Q_0=\oint\frac{\mathrm{d}z}{2\pi i}\mathrm{e}^{\sigma}T\,, \tag{4.7}$$

$$Q_1=\oint\frac{\mathrm{d}z}{2\pi i}\left(\tfrac{1}{2}i\eta\psi_{\mu}\,\partial x^{\mu}\,\mathrm{e}^{\phi}\right), \tag{4.8}$$

$$Q_2=\oint\frac{\mathrm{d}z}{2\pi i}\left(-\tfrac{1}{4}\mathrm{e}^{-\sigma}\eta\,\partial\eta\mathrm{e}^{2\phi}\right), \tag{4.9}$$

where the bosonized ghosts were defined in (3.2), and T is the total stress tensor

$$T=-\tfrac{1}{2}\,\partial x\cdot\partial x-\tfrac{1}{2}\psi\cdot\partial\psi+\text{ghosts}\,. \tag{4.10}$$

Using short-distance expansions one can easily show that

$$[Q_0,V]_{\pm}=\tfrac{1}{2}k^2\,\partial cV\,, \tag{4.11}$$

$$[Q_2,V]_{\pm}=0\,, \tag{4.12}$$

where V is any of the four expressions (4.2)–(4.5). For Q_1 we get

$$\{Q_1, V^{\mu}_{-1}\} = k_\mu V^c_{-1},$$

$$[Q_1, V^c_{-1}] = 0,$$

$$[Q_1, V^b_{-1}] = k_\mu V^{\mu}_{-1},$$

$$[Q_1, V^A_{-1/2}] = U^B_\eta (\not{k})_B{}^A, \tag{4.13}$$

where, in the last equation,

$$U^B_\eta \equiv -2^{-3/2} \eta S^B e^{\phi/2} c e^{ik\cdot x_+}. \tag{4.14}$$

Note the (anti)commutator structure of the above equations: In the bosonized formalism, moving vertices have correct space-time statistics, but the c-ghost factors in the fixed-point vertices invert the statistics. The right-moving operators satisfy the same set of equations. The product of left- and right-moving vertices contains a factor of $e^{ik\cdot x}$, coming from the product of $e^{ik\cdot x}$ and $e^{ik\cdot x_-}$.

If we use $e^{ik\cdot x}$ to construct the position-space wave functions, the σ-model interaction is found to be

$$\mathscr{L}_\mathrm{I} = \tfrac{1}{2} h_{\mu\nu}(x) V^{\mu}_{-1} \tilde{V}^{\nu}_{-1} + \tfrac{1}{2}\Phi(x)\left[V^b_{-1}\tilde{V}^c_{-1} - V^c_{-1}\tilde{V}^b_{-1}\right]$$

$$+2^{-1/2}[\not{H}(x)C]_{AB} V^A_{-1/2} \tilde{V}^B_{-1/2}. \tag{4.15}$$

The spacetime tensor

$$h_{\mu\nu}(x) = g_{\mu\nu}(x) - \delta_{\mu\nu} \tag{4.16}$$

is the graviton wave-function while Φ is Siegel's dilaton [28]. By performing a BRST gauge transformation (see (4.52) below), one finds that the combination

$$\varphi(x) = \Phi(x) - \tfrac{1}{2} h^\mu{}_\mu(x) \tag{4.17}$$

is invariant under linearized general coordinate transformations, and can therefore be identified with the spacetime scalar dilaton coupled to world-sheet curvature [29]. These two terms are in the NS-NS sector, and were discussed in our previous paper [4] using the unbosonized formalism for the superghosts. In contrast to the U(1) string considered in [4], the O(N) string is non-orientable, and there is no antisymmetric counterpart of $h_{\mu\nu}$ in the NS-NS sector. The last term in (4.15) corresponds to the R-R-sector antisymmetric tensor and has the form derived in sect. 3.1.

The spacetime background fields are of course not arbitrary. At tree level, they must satisfy equations of motion which guarantee the conformal invariance of the worldsheet sigma model. The most efficient way to derive these equations is to require that $\mathscr{L}_{\mathrm{I}}$ be the BRST-invariant:

$$[Q+\tilde{Q}, \mathscr{L}_{\mathrm{I}}]=0. \tag{4.18}$$

Using (4.13) and (4.15) and using the notation

$$k_\mu=-i\partial/\partial x^\mu, \tag{4.19}$$

where $k_\mu=\tilde{k}_\mu$ is an ordinary momentum, unlike the regulator (3.34), we find that

$$[Q_0+\tilde{Q}_0, \mathscr{L}_{\mathrm{I}}]=\tfrac{1}{2}k^2\mathscr{L}_{\mathrm{I}}(\partial c+\bar{\partial}\tilde{c}), \tag{4.20}$$

$$[Q_1+\tilde{Q}_1, \mathscr{L}_{\mathrm{I}}^{\mathrm{NS}}]=-\tfrac{1}{2}i(\partial^\nu g_{\nu\mu}-\partial_\mu\Phi)[V^c_{-1}\tilde{V}^\mu_{-1}-V^\mu_{-1}\tilde{V}^c_{-1}], \tag{4.21}$$

$$[Q_1, \mathscr{L}_{\mathrm{I}}^{\mathrm{R}}]=2^{-1/2}(\not{k}\not{H}C)_{AB}U^A_\eta\tilde{V}^B_{-1/2}, \tag{4.22a}$$

$$[\tilde{Q}_1, \mathscr{L}_{\mathrm{I}}^{\mathrm{R}}]=-2^{-1/2}(\not{H}\not{k}C)_{AB}V^A_{-1/2}\tilde{U}^B_\eta, \tag{4.22b}$$

(in deriving those results, it is important to remember the peculiar statistics $\tilde{Q}_1V^\mu_{-1}=-V^\mu_{-1}\tilde{Q}_1$!). Using the γ-matrix notation of (2.41), we can derive the identity

$$\begin{aligned}\gamma^\lambda\gamma^{\mu\nu\rho}-\gamma^{\lambda\mu\nu\rho}&=\gamma^{\mu\nu\rho}\gamma^\lambda+\gamma^{\lambda\mu\nu\rho}\\&=(\delta^{\lambda\mu}\gamma^{\nu\rho}-\delta^{\mu\nu}\gamma^{\mu\rho}+\delta^{\lambda\rho}\gamma^{\mu\nu}),\end{aligned} \tag{4.23}$$

which can be used to reduce (4.22) to

$$\begin{aligned}[Q_1, \mathscr{L}_{\mathrm{I}}^{\mathrm{R}}]&=2^{-1/2}U^A_\eta\left(\gamma^{\lambda\mu\nu\rho}k_\lambda H_{\mu\nu\rho}+3\gamma^{\nu\rho}k^\mu H_{\mu\nu\rho}\right)_A{}^B\tilde{V}_{B,-1/2}\\&=2^{-1/2}U^A_\eta\left(\gamma^{\lambda\mu\nu\rho}k_\lambda H_{\mu\nu\rho}+3\gamma^{\nu\rho}\gamma^{11}k^\mu H_{\mu\nu\rho}\right)_A{}^B\tilde{V}_{B,-1/2},\end{aligned} \tag{4.24}$$

$$\begin{aligned}[\tilde{Q}_1, \mathscr{L}_{\mathrm{I}}^{\mathrm{R}}]&=2^{-1/2}V^A_{-1/2}\left(\gamma^{\lambda\mu\nu\rho}k_\lambda H_{\mu\nu\rho}-3\gamma^{\nu\rho}k^\mu H_{\mu\nu\rho}\right)_A{}^B\tilde{U}^\eta_B\\&=2^{-1/2}V^A_{-1/2}\left(\gamma^{\lambda\mu\nu\rho}k_\lambda H_{\mu\nu\rho}+3\gamma^{\nu\rho}\gamma^{11}k^\mu H_{\mu\nu\rho}\right)_A{}^B\tilde{U}^\eta_B.\end{aligned} \tag{4.25}$$

To obtain the second equalities in (4.24)–(4.25), we have used the fact that $\tilde{V}_{B,-1/2}$ has positive chirality and $\tilde{U}^\eta_B$ has negative chirality to insert a γ^{11}. Then we can combine these equations to obtain

$$\left[Q_1 + \tilde{Q}_1, \mathscr{L}_{\mathrm{I}}^{\mathrm{R}}\right] = 2^{-1/2}\left(\gamma^{\lambda\mu\nu\rho}k_\lambda H_{\mu\nu\rho} + 3\gamma^{\nu\rho}\gamma^{11}k^\mu H_{\mu\nu\rho}\right)_A{}^B \times\left[U_\eta^A\tilde{V}_{B,-1/2} + V_{-1/2}^A\tilde{U}_{B,\eta}\right]. \tag{4.26}$$

The coefficients of every distinct composite field in (4.18) must vanish, so at tree level (4.20) gives Klein-Gordon equations (with $\nabla^2 = \partial^\lambda\partial_\lambda$)

$$\nabla^2 g_{\mu\nu} = 0, \qquad \nabla^2\Phi = 0, \qquad \nabla^2 H_{\mu\nu\rho} = 0, \tag{4.27}$$

while (4.21) is a gauge condition for the graviton

$$\partial^\nu g_{\nu\mu} = \partial_\mu\Phi, \tag{4.28}$$

and (4.24), (4.25) are first order field equations for the antisymmetric tensor, which we write in wedge product notation [30] as

$$\partial\wedge H = 0, \qquad \partial\cdot H = 0. \tag{4.29}$$

These are analogous to Maxwell's equation for $F_{\mu\nu}$. The first equation of (4.29) implies that $H_{\mu\nu\rho}$ is the field strength of a second rank potential $B_{\mu\nu}$

$$H = \partial\wedge B. \tag{4.30}$$

The third equation of (4.27) is a consequence of the two first order equations (4.29). Below, we will show how (4.27)–(4.30) all receive string loop corrections.

4.2. HIGHER GENUS

In sect. 3, we constructed the complete massless part of the closed-string state which represents a small hole in the worldsheet. In the presence of a background gauge field, the NS-NS and R-R components were given by (3.47) and (3.62). These must be multiplied by the propagator and ghost insertions necessary to attach the hole to a lower-genus surface, which we will take to be a sphere. We will center the hole at a Koba-Nielsen fixed point, so that its radius e^τ varies, while its center is fixed at $z \equiv 0$. This means that [4] a closed string propagator, with its accompanying Teichmüller insertions [31],

$$\Pi \equiv \tfrac{1}{2}\left(b_0 + \tilde{b}_0\right)\int_{-\infty}^{0} \mathrm{d}\tau\, e^{\tau(L_0+\tilde{L}_0)} = \tfrac{1}{2}\left(b_0 + \tilde{b}_0\right)\left(L_0 + \tilde{L}_0\right)^{-1},$$

sits in front of the entire boundary state. In the small-hole limit, this diverges when acting on the zero-mass part of the boundary state [4]. We regulate by allowing a small momentum k^μ to flow through the boundary, so that the divergence from the massless part of the boundary state $|B_0\rangle$ is

$$\Pi|B_0\rangle = \tfrac{1}{2}\left(b_0+\tilde{b}_0\right)k^{-2}|B_0\rangle. \tag{4.31}$$

The regulating momentum has $k_{\mathrm{L}}^\mu \neq \tilde{k}_{\mathrm{R}}^\mu$ as in (3.51), but satisfies $k_{\mathrm{L}}^2 = \tilde{k}_{\mathrm{R}}^2$. The ghost insertions are needed because the ghost vacuum is topology-dependent. They turn the "hole-vacuum" $|Z\rangle$ that appears in the boundary state into the SL_2-invariant vacuum $|\Omega\rangle$ of the sphere. In both the NS-NS and R-R sectors, the boundary state is multiplied by the divergent propagator factor Π (4.31), while in the R-R sector there is an additional picture-changing insertion $\mathscr{X}+\tilde{\mathscr{X}}$.

Our procedure of continuing the boundary state off-shell by allowing a small momentum to flow through it makes this anomaly calculation formally very similar to the tree-level calculations of the previous section. This will allow the higher-genus anomalies to be cancelled against corresponding anomalies at tree-level, obtaining loop-corrected equations of motion for the background fields. Before it is multiplied by Π, the boundary state is BRST-invariant because of the boundary conditions (2.2), i.e.

$$\left(Q+\tilde{Q}\right)|F\rangle_{\mathrm{NS}} = \left(Q+\tilde{Q}\right)|F\rangle_{\mathrm{R}} = 0. \tag{4.32}$$

This statement is not affected by a constant background. However, it is destroyed by the off-shell continuation needed to regulate the propagator. Spurious states can be written as BRST transforms, so BRST invariance is necessary and sufficient for their decoupling [14, 31]. We therefore have to cancel this BRST anomaly against the lower genus worldsheet. This will generate the loop-corrected equations of motion.

The boundary state is a sum over all pictures, but only one will contribute to any given tree-level process, as noted in sect. 3. The anomaly in that picture must cancel the anomaly from a corresponding sigma-model state in the same picture. On shell, all pictures are equivalent [14], and the choice is one of algebraic convenience. Off shell, this is no longer true, so in choosing a picture, we may be implicitly constraining the regularization. For our present calculations, we will choose the picture $s_{\mathrm{L}} = s_{\mathrm{R}} = -1$ for the NS-NS states, and $s_{\mathrm{L}} = s_{\mathrm{R}} = -\frac{1}{2}$ for the R-R states, in agreement with sect. 4.1.

To find the NS-NS anomaly, we first compute the divergence resulting from multiplying $|F\rangle_{\mathrm{NS}}$ by Π. The b-ghosts in Π just cancel $(c_0+\tilde{c}_0)$ in (3.47) and, acting on the zero-mass part of the off-shell boundary state, we have

$$\left(L_0+\tilde{L}_0\right)|F\rangle = k_{\mathrm{L}}^2|F\rangle = \tilde{k}_{\mathrm{R}}^2|F\rangle. \tag{4.33}$$

Therefore, our final result for the small hole divergence in the NS-NS sector is

$$|D\rangle_{\mathrm{NS}} = \frac{\kappa}{2k^2}\left[\det(1+F)\right]^{1/2}\left\{\left(\frac{1-F}{1+F}\right)_{\mu\nu} V^{\mu}_{-1}\tilde{V}^{\nu}_{-1} - V^{b}_{-1}\tilde{V}^{c}_{-1} + V^{c}_{-1}\tilde{V}^{b}_{-1}\right\}|\Omega\rangle . \tag{4.34}$$

This has the same form as the sphere wave-function (4.15), and multiplies a BRST invariant vacuum, so the action of the BRST operator can be immediately calculated from (4.20) and (4.21). As we shall see in the next paragraphs, k^{-2} in (4.34) cancels k^2 in (4.20), generating an anomaly in the $k \to 0$ limit.

At this point, it is important to recall that for a nonorientable theory, it is also possible to add crosscaps to the worldsheet. This can be done by cutting a hole in a sphere and identifying diametrically opposite points. The resulting non-orientable closed string worldsheet has the same order in the loop expansion as the disc. The radius of the crosscap is a conformal modulus, and there is a divergence when it shrinks to a point. The closed string state which represents the crosscap is easily constructed [4, 10] using the methods of sect. 2. Since the crosscap is not a boundary, it does not couple to F. The divergence produces an anomaly which cancels the zero-field hole anomaly precisely when the gauge group is O(32). Thus, we can include the effect of the crosscap by subtracting the $F=0$ value of (4.34) [4, 10].

The action of $Q+\tilde{Q}$ on $|D\rangle_{\mathrm{NS}}$ is then given by (4.20) if we substitute

$$h_{\mu\nu} \to \frac{\kappa}{k^2}\left\{\left[\det(1+F)\right]^{1/2}\left(\frac{1-F}{1+F}\right)_{\mu\nu} - \delta_{\mu\nu}\right\}, \tag{4.35}$$

$$\Phi \to -\frac{\kappa}{k^2}\left\{\left[\det(1+F)\right]^{1/2} - 1\right\}. \tag{4.36}$$

Thus (4.27) becomes

$$\nabla^2 g_{\mu\nu} = \kappa\left\{\left[\det(1+F)\right]^{1/2}\left(\frac{1-F}{1+F}\right)_{\mu\nu} - \delta_{\mu\nu}\right\}, \tag{4.37}$$

$$\nabla^2 \Phi = -\kappa\left\{\left[\det(1+F)\right]^{1/2} - 1\right\}. \tag{4.38}$$

The loop correction to the graviton gauge condition (4.28) vanishes by (4.47) below.

Now we calculate the anomaly in the R-R sector. The relevant pieces of the zero-mass boundary state (3.62) are, in the $(-\frac{1}{2}, -\frac{1}{2})$ picture which matches the sphere as in (3.43),

$$\tfrac{1}{2}(\mathscr{X}+\tilde{\mathscr{X}})|F\rangle_{\mathrm{R}} = 2^{-3/2} i\kappa\left[\not{k}_{\mathrm{L}}\,\text{Æ}\left(-\tfrac{1}{2}\not{F}\right)\right]_{A}{}^{B} V^{A}_{-1/2}\tilde{V}_{B,-1/2}(c_0+\tilde{c}_0)|\Omega\rangle . \tag{4.39}$$

Multiplying by Π has the same effect as in the NS-NS sector. We find the divergence

$$|D\rangle_{\mathrm{R}} = 2^{-3/2} i\kappa k^{-2}\left[\not{k}_{\mathrm{L}} Æ\left(-\tfrac{1}{2}\not{F}\right)\right]_A{}^B V^A_{-1/2}\tilde{V}_{B,-1/2}|\Omega\rangle. \tag{4.40}$$

Applying Q_1 and $\tilde{Q}_1$ to the divergence, and using (4.13), (3.61) and the relation $(\not{k}\not{k})_A{}^B = k^2\delta_A{}^B$, gives

$$Q_1|D\rangle_{\mathrm{R}} = i2^{-3/2}\kappa\left[Æ\left(-\tfrac{1}{2}\not{F}\right)\right]_A{}^B U^A_\eta \tilde{V}_{B,-1/2}|\Omega\rangle, \tag{4.41}$$

$$\begin{aligned}\tilde{Q}_1|D\rangle_{\mathrm{R}} &= -i2^{-3/2}\kappa k^{-2}\left[\not{k}_{\mathrm{L}} Æ\left(-\tfrac{1}{2}\not{F}\right)\tilde{\not{k}}_{\mathrm{R}}\right]_A{}^B V^A_{-1/2}\tilde{U}_{B,\eta}|\Omega\rangle \\ &= i2^{-3/2}\kappa\left[Æ\left(-\tfrac{1}{2}\not{F}\right)\right]_A{}^B V^A_{-1/2}\tilde{U}_{B,\eta}|\Omega\rangle. \end{aligned}\tag{4.42}$$

Note that, as promised, the factors of regulator momentum k_{L} have cancelled from the final result and the vanishing regulator momentum limit can be taken.

Now comes the payoff. Recall that $Æ(-\frac{1}{2}\not{F})$ is a polynomial. It was defined to be the expansion of $\exp(-\frac{1}{2}F_{\mu\nu}\gamma^\mu\gamma^\nu)$ with all terms with any repeated Lorentz indices dropped. Expanding it gives 0, 2, 4, 8, and 10-forms, each totally antisymmetric. At this point it is appropriate to remember that we are really dealing with an O(32) gauge theory and to trace all our expressions over group indices. Since the 2-form and 6-form involve odd powers of $F_{\mu\nu}$, they will cancel when we take the gauge trace. Following the convention of ref. [30], so that our wedge products are not divided by factorials, and using the tr notation for the O(32) trace, we have

$$Æ\left(-\tfrac{1}{2}\not{F}\right) = 1 + \tfrac{1}{2}\mathrm{tr}(F\wedge F)_{\mu\nu\rho\sigma}\gamma^{\mu\nu\rho\sigma} + \tfrac{1}{24}\mathrm{tr}(F\wedge F\wedge F\wedge F)_{\mu_1\cdots\mu_8}\gamma^{\mu_1\cdots\mu_8}. \tag{4.43}$$

The last term dualizes to

$$\tfrac{1}{24} i\,\mathrm{tr}^*(F\wedge F\wedge F\wedge F)_{\mu\nu}\gamma^{\mu\nu}\gamma_{11}. \tag{4.44}$$

Therefore, (4.41), (4.42) imply

$$\begin{aligned}\left(Q_1+\tilde{Q}_1\right)|D\rangle_{\mathrm{R}} &= i2^{-3/2}\kappa\left[U^A_\eta\tilde{V}_{B,-1/2} + V^A_{-1/2}\tilde{U}_{B,\eta}\right]|SL_2\rangle \\ &\quad\times\left[1+\tfrac{1}{2}\mathrm{tr}(F\wedge F)_{\mu\nu\rho\sigma}\gamma^{\mu\nu\rho\sigma} + \tfrac{1}{24} i\,\mathrm{tr}^*(F\wedge F\wedge F\wedge F)_{\mu\nu}\gamma^{\mu\nu}\gamma^{11}\right]_A{}^B. \end{aligned}\tag{4.45}$$

We of course want to interpret (4.45) as a source term for the lower-genus BRST invariance condition (4.26). There appears to be a problem, since (4.45) contains a 0-form, while there is no corresponding term for the lower genus which would imply that the loop anomaly cannot be cancelled [10]. Once again, it is important to include the crosscap contribution. Just as in the NS sector, for an O(32) gauge group, the $F=0$ value of the anomaly cancels between the hole and the crosscap [4, 14]. To cancel the BRST anomaly for nonzero F requires that the 0-form in the R-R boundary state be F-independent, and (4.45) shows that it is. Here we see the crucial role of the cancellation of the $[\det(1+F)]^{-1/2}$ factors in the R-R sector. But for this cancellation, the 0-form would have acquired F-dependence, and it would have been impossible to cancel the anomalies for nonzero backgrounds.

If we subtract the crosscap contribution from (4.45), the remaining anomaly is easily cancelled against (4.26), and the loop-modified equations in the R-R sector are found to be

$$\partial \wedge H = \tfrac{1}{4}\kappa \operatorname{tr}(F \wedge F), \tag{4.46}$$

$$\partial^{\lambda} H_{\lambda\mu\nu} = -\tfrac{1}{144} i\kappa \operatorname{tr}^{*}(F \wedge F \wedge F \wedge F)_{\mu\nu}, \tag{4.47}$$

where the traces are over O(32) generators, κ is the string loop coupling constant and the wedge product and dual are not divided by factorials. As noted in appendix B, the O(32) traces are to be symmetrized over all orderings of the generators in the nonabelian case.

For $F_{\mu\nu}=0$, the hole correction to the harmonic gauge condition (4.21) is by (4.34) and (4.13)

$$\left(Q_1+\tilde{Q}_1\right)|D\rangle_{\mathrm{NS}} = \tfrac{1}{2}\kappa k_{\mathrm{L}}^{-2}\left(k_{\mathrm{L}}^{\mu}+\tilde{k}_{\mathrm{R}}^{\mu}\right)\left[V^{c}\tilde{V}^{\mu}+V^{\mu}\tilde{V}^{c}\right]|\Omega\rangle, \tag{4.48}$$

which vanishes for our regularization $\tilde{k}_{\mathrm{R}}=-k_{\mathrm{L}}$. Turning on the electric field just rotates this zero vector, as one can check using (3.49) and (3.51). So (4.28) is unchanged.

For $F_{\mu\nu}=0$, the correction to the second-order R-R equation (4.27) is, using (4.39) and (4.11),

$$\left(Q_0+\tilde{Q}_0\right)|D\rangle_{\mathrm{R}} = i\kappa(c_0+\tilde{c}_0)2^{-5/2}V^{A}_{-1/2}(\not{k}_{\mathrm{L}})_{AB}\tilde{V}^{B}_{-1/2}|\Omega\rangle, \tag{4.49}$$

This vanishes when the regulator is removed ($k_{\mathrm{L}}^{\mu}\to 0$) and Lorentz rotation of $\tilde{V}^{B}_{-1/2}$ by (3.58) cannot affect this. A second order equation for H can also be derived by differentiating and combining the first order equations (4.46), (4.47). Provided $F_{\mu\nu}$ is constant, the r.h.s. will vanish trivially. Our regularization (3.51) cannot consistently be applied otherwise.

4.3. EFFECTIVE LAGRANGIAN

What we have done so far shows at least that available tree-level equations of motion have the right structure to cancel the loop BRST anomalies. We now show that these loop-corrected field equations are consistent by showing that they can be derived from a spacetime effective action for the massless particles. The physics content of this action will turn out to be very illuminating.

A preliminary technical point to unravel concerns the spacetime tensor character of the dilaton. If the σ-model action $\mathscr{L}_{\mathrm{I}}$ is BRST-invariant (obeys (4.18)), then so is $\mathscr{L}_{\mathrm{I}}' = \mathscr{L}_{\mathrm{I}} + \{Q + \tilde{Q}, E\}$ for any E. The new action corresponds to the background fields related by a spacetime gauge transformation to those defined by the original action. Consider the example

$$E(x) \equiv \tfrac{1}{2} i \varepsilon_{\mu}(x)\left[V^{b}_{-1}\tilde{V}^{\mu}_{-1} - V^{\mu}_{-1}\tilde{V}^{b}_{-1}\right], \tag{4.50}$$

where the vertices are defined in (4.2), (4.3). By choosing $k^2 E = 0$, we guarantee that $\{Q_0 + \tilde{Q}_0, E\} = 0$, while by (4.3)

$$\{Q_1 + \tilde{Q}_1, E(x)\} = \tfrac{1}{2}\left[\partial_{\mu}\varepsilon_{\nu} + \partial_{\nu}\varepsilon_{\mu}\right] V^{\mu}_{-1}\tilde{V}^{\nu}_{-1} + \tfrac{1}{2}\partial^{\mu}\varepsilon_{\mu}\left[V^{b}_{-1}\tilde{V}^{c}_{-1} - V^{c}_{-1}\tilde{V}^{b}_{-1}\right]. \tag{4.51}$$

This means that (4.15) has the on-shell gauge invariance

$$\begin{aligned} h_{\mu\nu} &\to h_{\mu\nu} + \partial_{\mu}\varepsilon_{\nu} + \partial_{\nu}\varepsilon_{\mu}\,, \\ \Phi &\to \Phi + \partial^{\mu}\varepsilon_{\mu}\,, \end{aligned} \tag{4.52}$$

which, as far as the metric is concerned, is just an infinitesimal general coordinate transformation. At the same time, it shows that Φ is not a coordinate scalar and shows that the true spacetime dilaton φ is related to Φ by [28]

$$\varphi(x) - \bar{\varphi} + \Phi(x) - \tfrac{1}{2}h^{\mu}{}_{\mu}(x)\,. \tag{4.53}$$

We subtracted the constant v.e.v. $\bar{\varphi}$ because it is related to the string loop coupling by [32]

$$\kappa = \mathrm{e}^{-\bar{\varphi}/2}\,, \tag{4.54}$$

and can therefore be large, while Φ and $h_{\mu\nu}$ are supposed small. Written in terms of the true dilaton, the gauge condition (4.28) takes on the standard harmonic gauge form

$$\partial^{\nu} g_{\nu\mu} - \tfrac{1}{2}\partial_{\mu} g_{\nu\nu} = \partial_{\mu}\varphi\,. \tag{4.55}$$

Although we have not done so explicitly, it is possible to interpret our calculation as an expansion about some arbitrary point x in spacetime, about which coordinates have been chosen for which the metric at x is flat (normal coordinates). To low orders in derivatives of the fields, a normal-coordinate expansion [27, 12, 3] about x would lead to the results we have found. This procedure was followed for the bosonic string in [3], and leads to a simple prescription for covariantizing our results: replace $\delta_{\mu\nu}$ by the metric $g_{\mu\nu}(x)$, and use the substitution

$$\kappa \to e^{-\varphi(x)/2} \tag{4.56}$$

instead of (4.54). The latter replacement is justified by the fact that φ couples to the curvature in the sigma model. To lowest order in the normal coordinate expansion [3], the dilaton contribution to the Polyakov integral on a surface of Euler characteristic n is $e^{-n\varphi(x)/2}$. Since adding a boundary to the sphere changes the Euler characteristic by 1, we obtain the prescription (4.56).

This new dilaton dependence has an interesting effect. It turns out that it implies a special form for the some of the higher order dilaton dependence of the tree-level equations of motion. To see this, note that our new version of (4.46), $\partial \wedge H \sim e^{-\varphi/2}\mathrm{tr}(F \wedge F)$, is inconsistent. Taking the exterior derivative of the r.h.s. and using the fact that $\mathrm{tr}(F \wedge F)$ is closed gives $0 = \mathrm{tr}(\partial\varphi \wedge F \wedge F)$, which has no reason to be true. (The dilaton-completed versions of (4.37), (4.38) do not suffer from this problem.) The problem can be solved by modifying the left-hand or tree-level, side of the equation. The correct dilaton-completed version of (4.46) is thus found to be

$$\partial_\varphi \wedge H \equiv \partial \wedge H + \tfrac{1}{2}(\partial\varphi) \wedge H = \tfrac{1}{4}e^{-\varphi/2}\mathrm{tr}(F \wedge F). \tag{4.57}$$

We have defined a dilaton-extended exterior derivative ∂_ϕ, which is useful for expressing the dilaton dependence. Similar considerations apply to the dual of (4.47), and lead to the modified form

$$\partial^\lambda H_{\lambda\mu\nu} + \tfrac{1}{2}(\partial^\lambda\varphi) H_{\lambda\mu\nu} = \tfrac{1}{24} i\, e^{-\varphi/2}\mathrm{tr}^*(F \wedge F \wedge F \wedge F)_{\mu\nu}. \tag{4.58}$$

The new terms in $\partial\varphi$ are contributions to the tree-level background field equations. They would have been missed by the lowest-order considerations presented earlier in this section, but could easily be verified by a more thorough sigma model beta function calculation.

To derive these equations from an action, we need a 2-form potential $B_{\mu\nu}$ from which the antisymmetric tensor field strength $H_{\mu\nu\rho}$ can be derived. The relation between the two is obtaining by solving (4.57) and involves the familiar Chern-Simons 3-form [33,13] $\omega_{\mu\nu\rho}$ in a slightly unfamiliar (because of the φ-dependence) way:

$$H = \partial_\varphi \wedge B + \tfrac{1}{4}e^{-\varphi/2}\omega_3,$$

$$\omega_3 = \mathrm{tr}\left\{A \wedge F - \tfrac{1}{3}A \wedge A \wedge A\right\}. \tag{4.59}$$

H is invariant under the gauge transformation $B \to B + \partial_\varphi \wedge \Lambda$ of B alone. It is also invariant under a gauge transformation of A_μ with gauge parameter α if, at the same time, B transforms as

$$B \to B - \tfrac{1}{4}e^{-\varphi/2}\omega_2^1 = B - \tfrac{1}{4}e^{-\varphi/2}\mathrm{tr}(\alpha\partial\wedge A). \tag{4.60}$$

The only unfamiliar feature of this gauge transformation law is the appearance of the dilaton field. It is really a string loop coupling constant, and reflects the fact that the coupling of the antisymmetric tensor to spacetime gauge fields is generated at loop level, rather than tree level, in a theory of open strings. It should be noted that our wedge products are defined without factorials, so that

$$\partial\wedge\omega_3 = \mathrm{tr}(F\wedge F).$$

It is now possible to show that the loop-corrected field equations for all the fields we have been discussing can be derived from the following simple spacetime effective action:

$$\begin{aligned}\sqrt{g}\mathscr{L} = &-\sqrt{g}\,e^{\varphi}\left\{-R + (\nabla\varphi)^2 + 2\nabla^2\varphi + \tfrac{1}{12}H^2\right\} + e^{\varphi/2}\mathrm{tr}\left\{\sqrt{g} - \sqrt{\det(g+F)}\right\}\\ &- \tfrac{1}{144}ie^{\varphi/2}\mathrm{tr}\{{}^*(B\wedge F\wedge F\wedge F\wedge F)\}.\end{aligned} \tag{4.61}$$

The relation between H and B is given by (4.59). The e^{φ} terms are the standard σ-model result [12, 29, 34] and represent the contribution of lower genus world sheets. The $e^{\varphi/2}$ terms are the upper genus contributions, generated by the BRST anomaly cancellation procedure. The factor of i in the last term arises because we have implicitly been working with a euclidean spacetime metric (our gamma matrices are those of O(10), not O(9, 1)). The structure is similar to the topological charge term in QCD and the i disappears on continuation to Minkowski space.

It is rather straightforward to show that the loop-corrected equations of motion for the antisymmetric tensor (4.58) are generated by varying this action with respect to $B_{\mu\nu}$, while the constraint equation (4.57) is already contained in the definition of H. To verify the graviton and dilaton equations we use (4.53) and (4.55) to expand the tree part of the action to quadratic order in small quantities, obtaining

$$-\sqrt{g}\,e^{\varphi}\left\{-R + (\nabla\varphi)^2 + 2\nabla^2\varphi\right\} \approx e^{\bar\varphi}\left\{-\tfrac{1}{4}(\partial_\lambda h_{\mu\nu})^2 + \tfrac{1}{2}(\partial_\lambda\Phi)^2\right\}. \tag{4.62}$$

In writing this, we take the dilaton field to have some constant background value $\bar\varphi$ around which the dynamical dilaton fluctuates and which will fix the string loop coupling. The expansion of the upper genus side of the action to linear order in $h_{\mu\nu}$ and Φ gives

$$\begin{aligned}e^{\varphi/2}\left\{\sqrt{g} - \sqrt{\det(g+F)}\right\} \approx\ & e^{\bar\varphi/2}\left(1 + \tfrac{1}{2}\Phi + \tfrac{1}{4}h^{\mu}{}_{\mu}\right)\\ &- e^{\bar\varphi/2}\sqrt{\det(1+F)}\left[1 + \tfrac{1}{2}\Phi + \tfrac{1}{4}h_{\mu\nu}\left(\frac{1-F}{1+F}\right)^{\mu\nu}\right].\end{aligned} \tag{4.63}$$

With the identification (4.54), the Euler-Lagrange equations for the effective action given by the sum of (4.62) and (4.63) are easily seen to be identical to our NS-sector BRST anomaly-cancelling equations (4.37)–(4.38). We note that we find fully consistent equations for the graviton and dilaton and no trace of the inconsistency claimed by Fischler et al. [6].

The effective action (4.61) actually contains one bit of information which we have not verified. The *overall* coefficient of the B-dependent terms has not been fixed by our calculations. The $\frac{1}{12}$ in the H^2 term was chosen to match the result of ref. [12]. However, in that paper, the B-field appeared in the sigma model as an antisymmetric counterpart of the metric coupling. Our model is defined instead by (4.15), and could give a different result. The issue could be settled, if desired, by calculating the H^2 contribution to the graviton beta function. This can be found by examining the singularity in the correlation function of two H-vertices from (4.15), one fixed at 0 and the other integrated over z (the moving vertex will not contain c-ghosts). The coefficient of $(z\bar{z})^{-1}$ in the operator product will, upon integration, produce a divergence in the correlation function which is logarithmic in the cutoff. (The H^2 contribution to the graviton beta function of the bosonic string can be found similarly, but the H-vertex is then of the form $H_{\lambda\mu\nu}x\,\partial x\,\bar{\partial}x$.) It is easy to check that the residue of $(z\bar{z})^{-1}$ is the short distance expansion of two H-vertices from (4.15) is proportional to

$$H_{\mu\lambda\rho}H_{\nu}^{\lambda\rho}V_{-1}^{\mu}\tilde{V}_{-1}^{\nu},$$

but determining the coefficient is tedious. We simply note that this singularity has the right form to reproduce the necessary contribution to the graviton beta function.

The effective action can be put into a somewhat more familiar form by making the following field redefinitions:

$$g_{\mu\nu} \to \mathrm{e}^{-2\varphi/(D-2)}g_{\mu\nu},$$
$$B_{\mu\nu} \to \mathrm{e}^{-\varphi/2}B_{\mu\nu}. \tag{4.64}$$

Then (4.61) becomes

$$\begin{aligned}\sqrt{g}\mathscr{L} = \sqrt{g}\left\{R - (D-2)^{-1}(\nabla\varphi)^2 - \tfrac{1}{12}\mathrm{e}^{-(D-6)\varphi/(D-2)}H^2\right\} \\ + \mathrm{e}^{-(D+2)\varphi/2(D-2)}\mathrm{tr}\left\{\sqrt{g} - \left[\det\left(g + \mathrm{e}^{2\varphi/(D-2)}F\right)\right]^{1/2}\right\} \\ - \tfrac{1}{144}i\,\mathrm{tr}^*\{B\wedge F\wedge F\wedge F\wedge F\},\end{aligned} \tag{4.65}$$

and the field-strength H is given by the φ-*independent* relation

$$H = \partial\wedge B + \tfrac{1}{4}\omega_3. \tag{4.66}$$

A closer resemblance to the usual Chapline-Manton lagrangian is obtained by defining a new dilaton

$$\Upsilon \equiv e^{(D-6)\varphi/2(D-2)}.$$

Then, in $D = 10$ dimensions, (4.65) can be rewritten as

$$\begin{aligned}\sqrt{g}\mathscr{L} = \sqrt{g}\Big\{ & R - 2(\Upsilon^{-1}\nabla\Upsilon)^2 - \tfrac{1}{12}\Upsilon^{-2}H^2 \\ & + \tfrac{1}{4}\Upsilon^{-1}\mathrm{tr}(F^2) + \tfrac{1}{8}\Upsilon\,\mathrm{tr}(F^4) - \tfrac{1}{32}\Upsilon\left[\mathrm{tr}(F^2)\right]^2 - \cdots \Big\} \\ & - \tfrac{1}{144} i\,\mathrm{tr}^*\{B\wedge F\wedge F\wedge F\wedge F\},\end{aligned} \tag{4.67}$$

where we have expanded the determinant to order F^4. As noted in appendix B, (4.67) should be true in the full nonabelian O(32) theory to this order. Intrinsically nonabelian corrections to the determinant factor can first appear at order F^5.

From the point of view of the graviton and dilaton, perhaps the key qualitative feature of the effective action (4.61) is that the loop correction terms incorporate the energy-momentum tensor of the gauge field, and hence its coupling to gravity. From the point of view of the antisymmetric tensor, the key feature is that the term linear in B incorporates the mechanism of spacetime gauge anomaly cancellation. While H is invariant under a gauge transformation of A_μ if B transforms according to (4.60), the last term in the effective action (4.61) is not. Using (4.60), we find the variation to be

$$\delta_\alpha(\sqrt{g}\mathscr{L}) = \tfrac{1}{576} i^*\{\mathrm{tr}(\alpha\partial\wedge A)\wedge\mathrm{tr}(F\wedge F\wedge F\wedge F)\}. \tag{4.68}$$

Note that the factors of e^φ have cancelled, so that this gauge variation is of loop order corresponding to genus zero worldsheets, just the loop order at which the chiral fermion gauge anomaly appears. Furthermore, the functional form of (4.68) is precisely what is needed, according to familiar arguments [13], to cancel the anomaly. These are all rather essential features of the low-energy theory and that they are reproduced by our procedure is strong evidence that it is correct and consistent.

5. Conclusions

At the tree level, vacuum stability is equivalent to conformal invariance of the 2D sigma model. By analogy with ordinary field theory, one would expect string loop corrections to vacuum stability conditions, so the initial sigma model would not be conformal invariant in many cases. There has been considerable doubt whether this is consistent. The present paper, we think, definitively resolves the question.

The open O(32) superstring is known to be a consistent theory of photons and gravitons. It is physically obvious that it should give a consistent theory of gravitons in a constant electric field. However, this apparently trivial extension forces one to abandon conformal invariance of the sigma model and to cancel BRST anomalies between string trees and loops. This takes us into unknown territory, where independence of amplitudes from the 2D metric, from the divergence regulation scheme, and from the superghost picture can no longer be assumed. We have made the simplest choices and found that everything works. A general framework, rationalizing our method and extending it to all orders in string loop perturbation theory, remains to be constructed.

It is already known [29] that the open string photon interacts with itself through a nonlinear Born-Infeld lagrangian. We found, reassuringly, that it interacts with the graviton through the energy-momentum tensor derived in the standard way:

$$\sqrt{g}\,T^{\mu\nu} = \mathrm{e}^{\varphi/2}\left\{\sqrt{g}\,g^{\mu\nu} - \sqrt{\det(g+F)}\left(\left[g+F\right]^{-1}\right)^{\mu\nu}\right\}. \tag{5.1}$$

We emphasize that this is an anomaly. The two terms in (5.1) are the residues of the string loop divergences from a small crosscap and from a small hole assuming an O(32) gauge group. For $F_{\mu\nu}=0$ they are known to cancel [13]. The electric field changes the hole term but not the crosscap, forcing one to cancel the divergence against a sigma model with $\beta \neq 0$. This just gives the l.h.s. of Einstein's equation.

The dependence on the dilaton field φ is also consistent. By the Euler number argument, each boundary should contribute $\mathrm{e}^{-\varphi/2}$ to the effective lagrangian. The resulting field equation is exactly what one gets from computing the contribution of the ghost determinant to the BRST anomaly. Here we disagree with Fischler et al. [6]. The appendices of ref. [4] show that the boundary state determined by our methods correctly reproduces the whole cylinder amplitude of the bosonic string, including the ghost determinant. The divergent part of the ghost contribution plays a crucial role in renormalizing the dilaton coupling. Such effects were neglected in ref. [6].

Most of the complexity of the present paper was caused by the antisymmetric tensor in the R-R sector. Here the sigma model consists of left and right spin fields [14], coupled through $\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}H_{\mu\nu\rho}$, where $H_{\mu\nu\rho}$ is the totally antisymmetric field strength. It has first-order field equations involving the 4-form $\partial \wedge H$ and the 2-form $\partial \cdot H$. As shown by Polchinski and Cai [10], the 6-photon hexagon anomaly is carried by a 0-form in the closed string channel. This cancels between the hole and the crosscap for O(32) only. Comparing to (5.1) we see a potential disaster: if the antisymmetric tensor coupling also contained a factor of $[\det(g+F)]^{1/2}$, this would contribute to the 0-form and destroy the anomaly cancellation.

Miraculously, this does not happen. The hole anomaly in the R-R sector is a *polynomial* in $F_{\mu\nu}$,

$$i2^{-3/2}\mathrm{e}^{\varphi/2}\left\{1 + \tfrac{1}{2}\gamma^{\mu\nu\rho\sigma}\mathrm{tr}(F\wedge F)_{\mu\nu\rho\sigma} + \tfrac{1}{144}i\gamma^{\mu\nu}\gamma_{11}\mathrm{tr}^{*}(F\wedge F\wedge F\wedge F)_{\mu\nu}\right\}. \tag{5.2}$$

The 0-form is independent of $F_{\mu\nu}$, and is therefore still cancelled by the crosscap. The 4-form and the 2-form correspond to $\partial \wedge H$ and $\partial \cdot H$ in the lower-genus conformal anomaly. Cancelling the two generates the expected Chern-Simons and anomalous 4-photon couplings [13].

We have two separate proofs of the crucial result that $[\det(g+F)]^{1/2}$ in (5.1) cancels from (5.2). One (sect. 2) depends on ζ-function regulation. This leads to different numbers for infinite sums over integer and as opposed to half-integer modes. The other (sect. 3) uses the fact that a constant electric field at the boundary induces a relative Lorentz rotation between the right- and left-moving modes of the closed string. Differences between the O(10) rotation matrices for vectors and spinors then lead to the same determinant cancellation as before. In string theory, remarkable cancellations generating unexpected consistency are evidence that one is on the right track.

We appeal to string field theory experts to develop a more complete theory of picture changing. Sums over all possible pictures occur already on the cylinder, and they severely strain existing technology. This was discussed in sect. 3 and appendix C. In the present paper, we were able to evade the issue by inelegant contortions, but a more systematic approach will surely be needed to deal with fermionic multiloop amplitudes.

Finally, we should say something about our hopes for the future development of this subject. We have shown in a simple context, that contains most of the essential features of the realistic string theories, that it is possible to renormalize consistently string loop amplitudes The price one pays is that the two-dimensional nonlinear sigma model underlying the Polyakov path integral approach has to be evaluated away from its conformal fixed point. Although we have emphasized the point of view that string loop corrections break strict sigma-model conformal invariance, we in fact believe that our loop-corrected beta functions have the effect of imposing a new kind of conformal invariance which is realized, not on individual worldsheets, but rather on the complete Polyakov sum over worldsheets of all genera. In the body of the paper we have discussed the mechanics and consistency of cancelling the standard field theory divergences of the sigma model on a genus-g worldsheet against divergences in the integration over the moduli of a genus-$(g+1)$ worldsheet. In particular, we studied the case of a hole degenerating into a puncture in a lower genus worldsheet and showed that the modular divergence could be cancelled against the insertion of a standard sigma-model counterterm interaction at the puncture. (Strictly speaking, we showed that BRST anomalies could be cancelled in this way, but that also implies divergence cancellation.) Now, a crucial point is that a cutoff is needed to define the divergences, and that cutoff should treat both types of divergence in a compatible way. Field theory regulators are naturally based on length and can only be defined if we pick a metric for our genus-g worldsheet. It is then quite natural to pick a metric for the genus-$(g+1)$ surface and define its moduli in terms of lengths (radius of a hole, etc.). If the theory on individual worldsheets were going to be conformally invariant, it would not matter which metrics we chose, but since we know that string loop corrections destroy naive conformal invariance, more care is needed. Given the nature of our divergence-cancelling scheme, it is very natural to restrict the choice of metrics on the genus-$(g+1)$ worldsheet so that, at the boundary of moduli space, where the higher genus worldsheet degenerates into the lower, the genus-g and genus-$(g+1)$ metrics coincide. This restriction ties together the choices of slice through conformal equivalence classes of metrics on worldsheets of different genus. We believe that, with this restriction on what one means by slice, the loop-corrected beta functions guarantee that the partition function summed over worldsheets is slice independent. A very attractive framework for proving this conjecture, at least up to lowest loop order and for the bosonic string is provided by Polchinski's recent work on moving dilatons on curved worldsheets [9]. The existence of a restricted form of conformal invariance, which survives loop corrections, is no doubt related to the fact that the relevant non-conformal sigma model must retain a very high degree of symmetry to decouple the infinite set of unphysical string states. The challenge is to identify the detailed algebraic structure of the new symmetry and to understand it well enough to make nonperturbative statements about quantum string theory, just as we are now able to use conformal invariance to make some exact statements about classical string theory. We hope to discuss these issues in greater detail in a future publication.

Appendix A

COMPLETENESS

Here we verify that the position eigenstates (2.21) and (2.30) satisfy the completeness equations (2.20) and (2.31). Let $:f(a^\dagger, a):$ denote normal ordering with $|0\rangle\langle 0|$ inserted between $a^\dagger$ and a. Then the completeness relation for a single harmonic oscillator,

$$1=\sum_{n=0}^{\infty}\frac{1}{n!}(a^\dagger)^n|0\rangle\langle 0|a^n, \tag{A.1}$$

can be abbreviated as

$$1= :\mathrm{e}^{a^\dagger a}:. \tag{A.2}$$

Using the round bracket notation of (2.22), we thus have for our two families of harmonic oscillators

$$\begin{aligned}1&= :\mathrm{e}^{(a^\dagger|a)+(\tilde a^\dagger|\tilde a)}:\\&= :\exp\left\{\left(\tilde a+a^\dagger|a+\tilde a^\dagger\right)-\left(a^\dagger|\tilde a^\dagger\right)-\left(\tilde a|a\right)\right\}:.\end{aligned} \tag{A.3}$$

Next, we factorize this expression by writing it as a functional integral:

$$:\exp(\tilde{a}+a^{\dagger}|a+\tilde{a}^{\dagger}): = \left[\mathrm{Det}(2\pi i)\right]^{-1}\int \mathscr{D}\bar{x}\,\mathscr{D}x$$

$$\times :\exp\left\{-(\bar{x}|x)+(\bar{x}|a)+(\bar{x}|\tilde{a}^{\dagger})+(\tilde{a}|x)+(a^{\dagger}|x)\right\}:. \tag{A.4}$$

Finally, we normal order explicitly and insert $|0\rangle\langle 0|$ to get (2.20), (2.21).

In the fermion case, because of the minus sign in (2.26), one must start from

$$1 = :\mathrm{e}^{(\psi^{\dagger}|\psi)-(\tilde{\psi}^{\dagger}|\tilde{\psi})}:. \tag{A.5}$$

Then (2.30) and (2.31) can be proved by the same tricks, remembering that fermions anticommute inside a normal product. The bra is

$$\langle\theta,\bar{\theta};\pm| = \langle 0|\exp\left\{-\tfrac{1}{2}(\bar{\theta}|\theta)\mp i(\tilde{\psi}|\psi)+(\bar{\theta}|\psi)\pm i(\tilde{\psi}|\theta)\right\}. \tag{A.6}$$

For a single pair of fermions there are only four states and (2.31) can be checked explicitly and algebraically.

Appendix B

NONABELIAN BACKGROUND

In sect. 2 we obtained a functional integral for the boundary state. It is valid for arbitrary open string backgrounds. In general it leads to a one-dimensional field theory on the boundary. This is exactly soluble for a constant abelian $F_{\mu\nu}$, as discussed in the body of the paper. Here we consider more general situations.

The vanishing of the boundary beta-function β_A is necessary for conformal invariance. If it does not vanish, we must expect BRST anomalies from the boundary state itself, and not just from the divergent closed string propagator. The condition $\beta_A = 0$ is a field equation for A_μ. To lowest order in the sigma-model expansion, it reduces to Maxwell's equation in the abelian case [18], and to the Yang-Mills equation in the nonabelian case [35]. In the abelian case, a constant field-strength $F_{\mu\nu}$ satisfies Maxwell's equation, and BRST invariance of the boundary state is automatic. This can be seen directly from the fact that $F_{\mu\nu}$ cancels from the Virasoro and Ramond generators, as discussed after (2.48). However, even a constant $F_{\mu\nu}$ does not satisfy the Yang-Mills equation [36] unless it is either abelian $[F_{\mu\nu}, F_{\rho\sigma}] = 0$, or can be derived from a constant potential $F_{\mu\nu} = [A_\mu, A_\nu]$. In general we must therefore expect BRST anomalies from the boundary state itself, and not just from the divergent closed string propagator. These should come from the higher order terms in the normal-coordinate expansion of the Wilson line (2.5) in symmetrized covariant derivatives of F, and should be absent when β_A vanishes.

In general the 1D field theory on the boundary can only be solved perturbatively, but one can still hope to prove theorems. In particular, the vacuum amplitude should be the effective lagrangian for the massless particles of the open string, while the loop source terms for the β-functions of the massless particles of the closed string should be its variational derivatives with respect to the corresponding background fields. Another general result which looks accessible is the cancellation of determinants in the R-R sector. This should follow from supersymmetry of the boundary field theory.

To see whether these conjectures are plausible, and to exhibit the technical obstacles to a rigorous proof, we now examine perturbation theory for a nonabelian background. We will work at zeroth order in derivatives of F, so the interaction is given by (2.46), and the constraint $\beta_A = 0$ does not yet appear. For an $O(N)$ gauge group, the generators are

$$(\lambda^{ab})_{ij} = \tfrac{1}{2}\left(\delta_i^a \delta_j^b - \delta_i^b \delta_j^a\right), \tag{B.1}$$

and $F_{\mu\nu} \equiv F_{\mu\nu}^{ab}\lambda_{ab}$. These vertices must be ordered around the boundary according to σ, and then traced, giving

$$F_{\mu_1\nu_1}^{a_1a_2} F_{\mu_2\nu_2}^{a_2a_3} \cdots F_{\mu_k\nu_k}^{a_ka_1}\,. \tag{B.2}$$

In the abelian case, only even products of F contribute. However, in the nonabelian case, there is no such restriction.

Using the mode expansions (2.16) and (2.25), the interaction (2.46) can be written

$$\tfrac{1}{2}F_{\mu\nu}^{ab} e^{i(n-m)\sigma}\left[\sqrt{\frac{n}{m}}\,\bar{x}_m^\mu x_n^\nu + \bar{\theta}_m^\mu \theta_n^\nu\right]. \tag{B.3}$$

The frequencies m of the string oscillators may be interpreted as discrete momenta conjugate to the cyclic variable σ. The functional integral (2.44) will now join these vertices by propagators

$$\begin{aligned}\langle \bar{x}_m^\mu x_n^\nu\rangle &= 2\delta_{mn}\delta^{\mu\nu}\,,\\ \langle \bar{\theta}_m^\mu \theta_n^\nu\rangle &= 2\delta_{mn}\delta^{\mu\nu}\,.\end{aligned} \tag{B.4}$$

There will also be external lines arising from the exponents

$$(a^\dagger|x), \qquad (\bar{x}|\tilde{a}^\dagger), \qquad (\psi^\dagger|\theta), \qquad \mp i(\bar{\theta}|\tilde{\psi}^\dagger)\,. \tag{B.5}$$

These carry the closed string creation operators on which the boundary state depends.

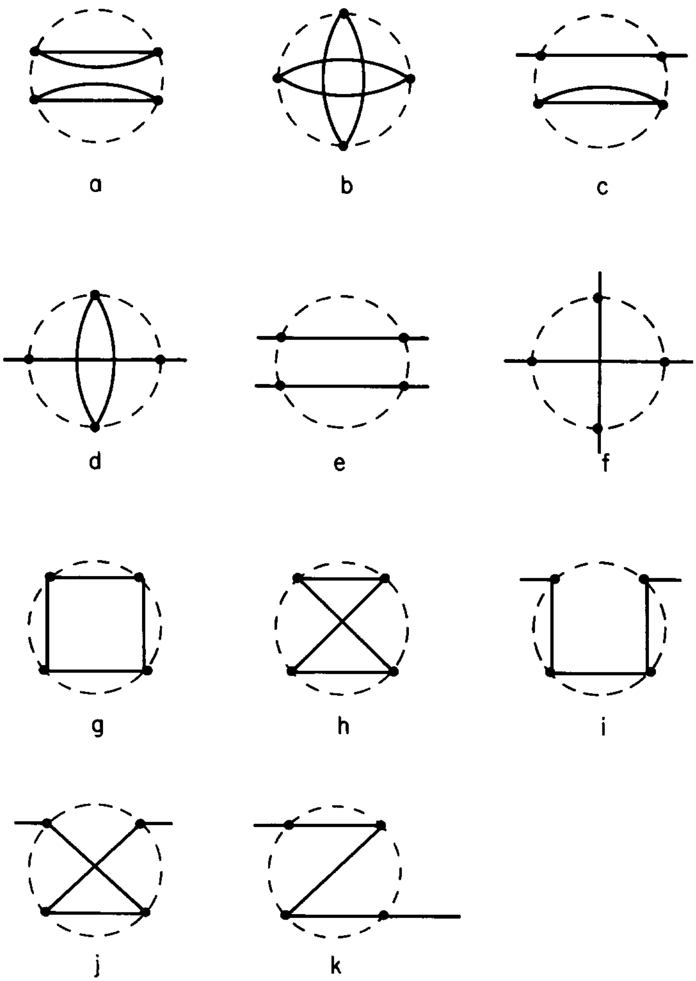

Fig. 1. Fourth-order graphs for the nonabelian boundary state.

What makes the problem nontrivial is the path ordering of the integral

$$(2\pi)^{-k}\int_0^{2\pi}\mathrm{d}\sigma_1\int_0^{\sigma_1}\mathrm{d}\sigma_2\cdots\int_0^{\sigma_{k-1}}\mathrm{d}\sigma_k, \tag{B.6}$$

which cannot be permuted because it must match (B.2). This ordering comes from the nonabelian Wilson loop round the boundary in (2.43). To symbolize this, we place the vertices on a dashed circle representing the boundary, and then join them by solid lines representing propagators. For example, all topologically distinct fourth order graphs are shown in fig. 1. These graphs illustrate the obstruction to an exact nonabelian solution. If we could combine figs. 1g and h for example, then all vertices could be integrated independently, giving

$$\prod_{j=1}^{k} \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\sigma_j \mathrm{e}^{i\sigma_j(m_j-n_j)} = \prod_{j=1}^{k} \delta_{m_j, n_j}. \tag{B.7}$$

Unfortunately figs. 1g and h have different group theoretic factors, leaving us with ordered integrals

$$\prod_{j=1}^{k} \frac{1}{2\pi} \int_0^{\sigma_{j-1}} \mathrm{d}\sigma_j \mathrm{e}^{i\sigma_j(m_j-n_j)}. \tag{B.8}$$

Doing the σ integrals now leads to complicated m, n sums. Doing the m, n sums first leads to complicated σ integrals.

However, we can still make some qualitative observations. In the abelian case, the determinant factor in (2.48) comes from the loops in fig. 1, while the exponent comes from the open lines. Some graphs require regularization, since even in the abelian case the loops give

$$\sum_{m>0} 1 = \lim_{s\to 0} \sum_{m>0} m^{-s}. \tag{B.9}$$

We can regulate any graph by multiplying the propagators (B.4) by m^{-s}, and taking $s \to 0$ in the end. Now (B.3) is supersymmetric, so we expect cancellations between boson and fermion loops. In the R-R sector, where m_j, n_j have the same integer values for bosons and fermions, this supersymmetry is preserved by the regularization (B.9). So the vital determinant cancellation in the R-R sector is valid even for a nonabelian background. The complete R-R contribution comes from the fermion zero-modes, and is given by (2.57), symmetrized over path-orderings. In the NS-NS sector, however, the oscillator frequencies are different for bosons and fermions. The formal supersymmetry of (B.3) is then broken by the regularization, and the bosonic and fermionic loops will not cancel. The sum of the NS-NS vacuum graphs gives the effective action (2.53) in the abelian case, and some generalization of this in the nonabelian case.

Although evaluation of the nonabelian NS-NS graphs is complicated, the connection between the vacuum amplitudes and the effective action suggests the form they should take. An effective action for the O(N) superstring has been constructed from scattering amplitudes through order F^5 [37–39]. The results show that the F^3 contribution cancels between bosons and fermions, and that the F^4 result is the same as the abelian result, symmetrized over path-orderings. Nontrivial nonabelian corrections first occur at order F^5.

Another qualitative feature follows from (B.5). The open lines in fig. 1 carry *closed* string oscillators, which have no λ-matrices. So Fig. 1i is derived from Fig. 1g by cutting open one line in a very simple way represented by $\delta/\delta g_{\mu\nu}$. The effective photon Lagrangian is the sum of vacuum graphs, while the graviton anomaly is the coefficient of $a_1^{\mu\dagger}\tilde{a}_1^{\nu\dagger}$ or $\psi_{1/2}^{\mu\dagger}\tilde{\psi}_{1/2}^{\nu\dagger}$, and therefore has one open line. Our result that the anomaly in the graviton equation is the energy-momentum tensor of the photon should therefore also generalize.

Appendix C

THE HEXAGON ANOMALY

In the body of the paper we have discussed the cancellation of BRST anomalies. The type I superstring has a spacetime gauge anomaly which, as is well-known, cancels for an O(32) gauge group [13]. In this appendix, we would like to show how the two kinds of anomaly cancellation are related. We will redo the construction by Polchinski and Cai [10] of the open string loop diagram with external gauge bosons, including the superghosts which were not considered in [10].

The superghosts of the open Ramond string satisfy

$$[\gamma_m, \beta_n] = \delta_{m,-n}. \tag{C.1}$$

The superghost zero modes define two possible vacuum states,

$$\beta_0|\downarrow\rangle = 0, \qquad \gamma_0|\uparrow\rangle = 0. \tag{C.2}$$

The conformal ghost zero modes, which *anti*commute, define a similar pair of vacua which can easily be transformed into each other by action of the zero mode algebra:

$$b_0|\uparrow\rangle = |\downarrow\rangle, \qquad c_0|\downarrow\rangle = |\uparrow\rangle.$$

By a famous identity for Grassmann variables, these equations may equivalently be written

$$\delta(b_0)|\uparrow\rangle = |\downarrow\rangle, \qquad \delta(c_0)|\downarrow\rangle = |\uparrow\rangle.$$

This suggests that the corresponding formulae for the commuting superghosts are

$$\delta(\beta_0)|\uparrow\rangle = |\downarrow\rangle, \qquad \delta(\gamma_0)|\downarrow\rangle = |\uparrow\rangle, \tag{C.3}$$

which implies the useful, if heuristic, formula

$$\delta(\beta_0)\,\delta(\gamma_0) = 1. \tag{C.4}$$

In what follows, we will assume this sort of delta function identity to hold for other pairs of bosons satisfying canonical commutation relations.

It would be useful to make these delta functions of superghost zero modes BRST invariant. The Ramond generators

$$\sum_m G_m z^{-m-3/2} = i\psi_\mu \partial x^\mu(z) + \text{ghosts}, \tag{C.5}$$

satisfy

$$[Q, \beta_m] = G_m, \tag{C.6}$$

and

$$G_0^2 = L_0, \tag{C.7}$$

so that

$$\mathscr{X} \equiv G_0 \delta(\beta_0) = \{Q, \beta_0\} \delta(\beta_0) \tag{C.8}$$

will be BRST-invariant on-shell (i.e. so long as $Q^2 = 0$). $\mathscr{X}$ is of course just the picture-changing operator. Comparing (C.5) and (C.8) to (3.24) suggests the heuristic identifications [40]

$$\delta(\beta_0) \sim e^{\phi}, \qquad \delta(\gamma_0) \sim e^{-\phi} \tag{C.9}$$

where ϕ is the bosonized superghost field (at least the superghost charges of the right- and left-hand sides of these relations are clearly right!). So $\mathscr{X}$, defined in (C.8), is the picture-changing operator in the $\beta\gamma$ formalism (it is more often written in the bosonized formalism).

As was explained in sect. 2, on the boundary of a worldsheet, the boundary conditions

$$\begin{aligned} \gamma_m |B; \pm\rangle &= \mp i\tilde{\gamma}_{-m} |B; \pm\rangle, \\ \beta_m |B; \pm\rangle &= \mp i\tilde{\beta}_{-m} |B; \pm\rangle, \end{aligned} \tag{C.10}$$

must be imposed. Superghost vacua satisfying the zero mode parts of these conditions can be constructed by arguments similar to those which led to (C.3). In particular, the state

$$|0; \pm\rangle = \delta(\gamma_0 \pm i\tilde{\gamma}_0) |\downarrow\downarrow\rangle, \tag{C.11}$$

satisfies $\gamma_0 \pm i\tilde{\gamma}_0 = 0$. The same state satisfies $\beta_0 \pm i\tilde{\beta}_0 = 0$ by virtue of (C.2) and the commutation relation

$$\left[\beta_0 \pm i\tilde{\beta}_0, \gamma_0 \pm i\tilde{\gamma}_0\right] = 0.$$

The canonical commutator,

$$\left[\beta_0 \mp i\tilde{\beta}_0, \gamma_0 \pm i\tilde{\gamma}_0\right] = -2$$

suggests that the analog of (C.4) is

$$\delta\left(\beta_0 \mp i\tilde{\beta}_0\right) \delta(\gamma_0 \pm i\tilde{\gamma}_0) \sim 1. \tag{C.12}$$

The associated left-vacua are defined by

$$\langle 0; \pm| = \langle\uparrow\uparrow|\delta\left(\beta_0 \pm i\tilde{\beta}_0\right), \tag{C.13}$$

and the usual ghost-vacuum orthogonality relations,

$$\langle\uparrow\uparrow|\downarrow\downarrow\rangle = 1,$$

then imply that

$$\langle 0; \mp|0; \pm\rangle = 1,$$

$$\langle 0; \pm|0; \pm\rangle = 0. \tag{C.14}$$

Now we construct the various spin structures [41] of the one-loop open string amplitude by gluing together two of these boundary states. We have constructed the boundary states in the R-R sector of the closed string going along the cylinder. The open string going around the cylinder is in either its R(NS) sectors according as the signs in the boundary conditions at the two ends are taken to be equal (opposite). These correspond to the PP or AP spin structures of the cylinder amplitude in the functional integral formalism. From (C.14) we see that the cylinder with equal boundary conditions requires superghost zero mode insertions to yield a non-zero matrix element. By (C.11)–(C.13) the appropriate non-vanishing matrix elements are

$$\langle 0; \pm|\delta(\gamma_0 \mp i\tilde{\gamma}_0) \cdot \delta\left(\beta_0 \mp i\tilde{\beta}_0\right)|0; \pm\rangle = 1. \tag{C.15}$$

The delta functions are the needed insertions for commuting zero modes.

The counting of zero modes agrees with the functional formalism, since the PP spin structure on the cylinder has one superconformal Killing spinor and one superconformal super-Teichmüller zero mode [19]. Interestingly, it is harmless to incorporate the superghost insertions into the boundary states of (C.15), even in the AP case, where they are not needed. In that case, (C.12) shows that they simply cancel. This is actually an important consistency condition, since the superghost insertions are important in sect. 2 for obtaining the $\not{k}$ in the fermionic boundary state. In the bosonized formalism, this $\not{k}$ is seen to contribute to the beta function loop corrections, which must be local on the worldsheet. Therefore, the superghost insertions must be associated with the boundary state in a way that is independent of the particular amplitude being constructed.

The boundary conditions (2.24) on the ψ's imply

$$\left(\psi_m^\mu \mp i\tilde{\psi}_{-m}^\mu\right)|B, \pm\rangle = 0. \tag{C.16}$$

For equal signs, *both* the boundary operators are also annihilated by $\psi_0^\mu \mp i\tilde{\psi}_0^\mu$ with $\mu = 1,\dots,10$, so a non-vanishing matrix element in the PP sector will require the insertion of

$$\prod_{\mu=1}^{10}\left(\psi_0^\mu \pm i\tilde{\psi}_0^\mu\right) \sim \prod_{\mu=1}^{10}\psi_0^\mu \sim \gamma_{11}\,. \tag{C.17}$$

The boundary conditions on the Ramond generators (C.5) are [4]

$$\left(G_m \pm i\tilde{G}_{-m}\right)|B;\pm\rangle = 0\,, \tag{C.18}$$

so we can turn the β insertion in (C.15) into a picture-changing operator

$$\mathscr{X}_{\pm} = \left(G_0 \mp i\tilde{G}_0\right)\delta\left(\beta_0 \mp i\tilde{\beta}_0\right), \tag{C.19}$$

by multiplying and dividing by $(G_0 \mp i\tilde{G}_0)$. The sign of $\tilde{G}_0$ changes when we commute it through the fermion zero modes (C.17), giving the inverse picture-changing operator

$$\begin{aligned}\mathscr{Y}_{\mp} &= \delta(\gamma_0 \mp i\tilde{\gamma}_0)\left[G_0 \pm i\tilde{G}_0\right]^{-1}\\ &= (\mathscr{X}_{\mp})^{-1},\end{aligned} \tag{C.20}$$

by (C.12). So the minimal non-zero amplitude for the PP spin structure is finally

$$\langle B;\pm|\mathscr{Y}_{\mp}\prod_{\mu=1}^{10}\psi_0^\mu\mathscr{X}_{\pm}|B;\pm\rangle\,. \tag{C.21}$$

Now we are ready to construct the hexagon graph. There are six photons on the left boundary of the cylinder. The two standard forms of the photon vertex are [22, 21]

$$V_0^\mu = \left[i\,\partial x^\mu + (k\cdot\psi)\psi^\mu\right]\mathrm{e}^{ik\cdot x}\,, \tag{C.22}$$

$$V_{-1}^\mu = \psi^\mu \mathrm{e}^{-\phi}\mathrm{e}^{ik\cdot x}\,. \tag{C.23}$$

They differ by a picture-changing operator

$$V_0^\mu = \mathscr{X}V_{-1}^\mu\,, \tag{C.24}$$

where $\mathscr{X}$ is equivalent to (C.8). The superghost numbers of the insertions on a cylinder must add up to zero [14]. If we start with six V_0^μ vertices, the inverse picture-changing operator $\mathscr{Y}$ in (C.21) will convert one of them to V_{-1}^μ (this is the superconformal analog of a Koba-Nielsen fixed point). The other five vertices provide the ten factors of ψ_0^μ called for in (C.17). Including the propagator needed to connect the two boundary states together to make a cylinder, we get the hexagon amplitude

$$\langle B; \pm | V_{-1}^{\mu}[\varepsilon \cdot V_0]^5 \big(L_0 + \tilde{L}_0\big)^{-1} \mathscr{X}_{\pm} | B; \pm \rangle . \tag{C.25}$$

Now V_0^μ is "identically" transverse, but V_{-1}^μ is only transverse by virtue of BRST invariance: by (4.8),

$$\begin{aligned} k_\mu V_{-1}^{\mu} &= (k \cdot \psi)\, \mathrm{e}^{-\phi} \mathrm{e}^{ik\cdot x} \\ &= \big[Q, W_{-2}^{S}\big], \end{aligned} \tag{C.26}$$

with

$$W_{-2}^{S} = 2\,\mathrm{e}^{-2\phi}\,\partial\xi\,\mathrm{e}^{ik\cdot x}. \tag{C.27}$$

When we couple open strings to a closed string, the fields inside the open string vertices become the closed string combinations that are nonzero on the boundary. The open string BRS operator is transformed by the vertex [42] into the closed string operator $Q + \tilde{Q}$. We peel this off from (C.26) and commute it to the right-hand end of (C.25), getting

$$\big(Q + \tilde{Q}\big)\big[L_0 + \tilde{L}_0\big]^{-1} \mathscr{X}_{\pm} | B; \pm \rangle . \tag{C.28}$$

This is a state on the empty boundary of the cylinder, opposite the one where the six photons enter.

Now the hexagon gauge anomaly can be related to the BRST anomaly calculated in sect. 4. The zero-mass piece of $|B; \pm\rangle$ causes $(L_0 + \tilde{L}_0)^{-1}$ to diverge. We regulate it by allowing a small momentum k to enter the empty boundary. Then the Ramond generator in the picture-changing operator (C.19) becomes $\not{k}$, the one in the BRST operator (4.8) becomes another $\not{k}$, and $(L_0 + \tilde{L}_0)^{-1}$ becomes k^{-2}. The various factors of k cancel, even in the limit $k \to 0$, leaving us with an anomaly

$$\big(Q + \tilde{Q}\big)\big[L_0 + \tilde{L}_0\big]^{-1} \mathscr{X}_{\pm} | B; \pm \rangle \neq 0 . \tag{C.29}$$

For an O(32) gauge group, it is possible to show that this anomaly cancels against the diagram in which a boundary is glued to a crosscap [10]. This is of course necessary for the unphysical longitudinal photon to decouple.

The various elements of (C.29) match the terms appearing in the evaluation of the R-R-sector anomaly presented in the body of the paper. The correctness of the off-shell continuation used in sect. 4.2 and the structural significance of the picture-changing operator in (3.43) and (4.39) should now be much clearer. We presented this argument in $\beta\gamma$ language because there are still some difficulties associated with bosonizing superghosts on the cylinder: Each boundary state must contain an infinite sum over equivalent pictures like (3.28) and to avoid an infinite result for the cylinder, there must be some "picture-fixing" insertion or projection. This must reproduce the phenomena arising from the zero modes in the PP spin structure, even though the bosonization by and large obliterates the spin structures.

Appendix D

SUPERSYMMETRY AND PICTURE-CHANGING

In the text we used a simple picture-independent form of the action of spacetime supersymmetry on the vertices,

$$\{ \Lambda_r^A, V_s^\mu \} = -\tfrac{1}{2} V_{r+s}^B (\not{k}\gamma^\mu)_B{}^A, \tag{D.1}$$

$$[\Lambda_r^A, V_s^b] = [\Lambda_r^A, V_s^c] = 0, \tag{D.2}$$

$$[\Lambda_r^A, V_s^B] = -i\sqrt{\tfrac{1}{2}}\,(C\gamma_\mu)^{AB} V_{r+s}^\mu. \tag{D.3}$$

to derive the R-R boundary state. To write the algebra in this form, we dropped a variety of picture-dependent extra longitudinal and BRS transform pieces. It is well known that neglect of such terms is legitimate in "on-shell" applications, but, since we take the boundary state off-shell at intermediate stages of the calculation, it is not obvious that such neglect is legitimate here. This appendix is devoted to showing that our construction of the BRST anomaly is (rather miraculously) insensitive to such off-shell ambiguities in supersymmetry.

To make clear what we are talking about we list the "extra" terms (denoted by Δ) in the supersymmetry action for a few specific choices of picture for the supercharges and vertices. The commutators are computed using the definitions of vertices and supercharges as well as the short distance expansions given in the text. Although the vertices are all evaluated "off-shell" at momentum k, for compactness of notation we have dropped the momentum subscript from our vertex operator notation. For the vector vertex we find

$$\Delta\{ \Lambda_{1/2}^A, V_{-1}^\mu \} = V_{-1/2}^A k^\mu$$

$$\Delta\{ \Lambda_{-1/2}^A, V_{-1}^\mu \} = \{ Q, W^B (\gamma^\mu)_B{}^A \} \tag{D.3}$$

with

$$W^B = \sqrt{2}\, e^{-7\phi/2}\, \partial\xi\, \partial^2\xi\, S^B e^{ik\cdot x} + \partial cc. \tag{D.4}$$

For the c ghost we find

$$\Delta\left[\Lambda^A_{1/2}, V^c_{-1}\right] = -\left[Q_1, V^A_{-1/2}\right]$$

$$= -\left[Q, V^A_{-1/2}\right] + \mathrm{O}(k^2),$$

$$\Delta\left[\Lambda^A_{-1/2}, V^c_{-1}\right] = 0. \tag{D.5}$$

The terms of $\mathrm{O}(k^2)$ arise from the action of the Q_0 piece of the total BRS charge (4.6)–(4.9). For the b ghost we find

$$\Delta\left[\Lambda^A_{1/2}, V^b_{-1}\right] = 0,$$

$$\Delta\left[\Lambda^A_{-1/2}, V^b_{-1}\right] = \left[Q, W^A_b\right] - W^B(\not{k})_B{}^A + \mathrm{O}(k^2) \tag{D.6}$$

with

$$W^A_b = -\tfrac{2}{3} S^A e^{-9\phi/2}\,\partial\xi\,\partial^2\xi\,\partial^3\xi\,\partial cc\, e^{ik\cdot x_+}. \tag{D.7}$$

For the fermion vertices we find

$$\Delta\left[\Lambda^A_{1/2}, V^B_{-1/2}\right] = \Delta\left[\Lambda^A_{-1/2}, V^B_{1/2}\right] = 0,$$

$$\Delta\left[\Lambda^A_{1/2}, V^B_{-3/2}\right] = -\sqrt{2}\,(C\not{k})^{AB}\,\partial\xi\, e^{-2\phi} e^{ik\cdot x_+}\partial cc. \tag{D.8}$$

Some of the extra terms vanish when the off-shell momentum is taken to zero and some are BRS transforms. They are all in some sense spurious states and in the usual on-shell S-matrix applications can be argued to make no contribution [14, 22]. For reasons we will now explain, it is less evident that these spurious states can be dropped from our calculation.

The crucial point is that certain off-shell information about the boundary states is needed in order to evaluate the BRS anomaly generated by a boundary. As discussed in the text the contribution of the R-R boundary state is multiplied by a $\not{k}/k^2$ Dirac propagator so that in the limit $k \to 0$ one is entitled to neglect only terms of $\mathrm{O}(k^2)$ or higher. Since our method of calculation generates the R-R boundary state from the action of supersymmetry on the NS boundary state, it is easy to see that we need to evaluate the action of supersymmetry on boundary states correct to $\mathrm{O}(k)$. Since, as shown in the preceding paragraph, there are many picture-dependent extra pieces of $\mathrm{O}(k)$ in the action of supersymmetry on the vertices, a naive use of (D.1) in the calculation is not guaranteed to be right. To justify the naive computation one has to show that, in any given picture, the potentially dangerous extra terms in the action of supersymmetry on the boundary state either cancel or organize themselves into a spurious state. Since the boundary state is a source term for the closed string, only closed string spurious states (i.e. states of the form $[Q+\tilde{Q}, X]$) can be disposed of in this fashion. We have verified for two pictures that all extra terms of $O(k)$ in the action of supersymmetry on the NS boundary state can be eliminated and we present the detailed argument below. Because it turns out that the R-R boundary state is already of $O(k)$, even before it is acted on by the supersymmetry generators (see (3.43)), the above subtleties are of concern only for the NS sector.

First consider the effect of $(\Lambda^A_{1/2}+\tilde{\Lambda}^A_{1/2})$ on the zero-field NS boundary state in the $s_L = s_R = -1$ picture,

$$|B_0\rangle_{NS} = \left[V^{\mu}_{-1}\tilde{V}^{\mu}_{-1} - V^{b}_{-1}\tilde{V}^{c}_{-1} + V^{c}_{-1}\tilde{V}^{b}_{-1}\right]\tfrac{1}{2}(c_0+\tilde{c}_0)|\Omega\rangle . \qquad \text{(D.9)}$$

Using the specific enumeration of extra terms in the action of supersymmetry on the vertices given earlier, we can calculate the "extra" terms arising from the action of $\Lambda^A + \tilde{\Lambda}^A$ on the vector vertices:

$$\begin{aligned}\Delta\left[\left(\Lambda^A_{1/2}+\tilde{\Lambda}^A_{1/2}\right), V^{\mu}_{-1}\tilde{V}^{\mu}_{-1}\right] &= V^A_{-1/2}k^{\mu}_{L}\tilde{V}_{-1} - V^{\mu}_{-1}\tilde{k}^{\mu}_{R}\tilde{V}^A_{-1/2}\\ &= V^A_{-1/2}\tilde{k}^{\mu}_{R}\tilde{V}^{\mu}_{-1} + k^{\mu}_{L}V^{\mu}_{-1}\tilde{V}^A_{-1/2}\\ &= -V^A_{-1/2}\left[\tilde{Q},\tilde{V}^b_{-1}\right] + \left[Q, V^b_{-1}\right]\tilde{V}^A_{-1/2} + O(k^2)\\ &= \left[Q, V^A_{-1/2}\right]\tilde{V}^b_{-1} - V^b_{-1}\left[\tilde{Q},\tilde{V}^A_{-1/2}\right] .\end{aligned} \qquad \text{(D.10)}$$

The first equality follows from (D.3), the next from our standard choice of chiral regulator momentum and the third follows from the easy BRS identity

$$k_{\mu}V^{\mu}_{-1} = \left[Q_1, V^b_{-1}\right] = \left[Q, V^b_{-1}\right] + O(k^2) . \qquad \text{(D.11)}$$

The final equality follows from a BRS "integration by parts" identity

$$\left[Q+\tilde{Q}, X\tilde{Y}\right] = [Q, X]_{\mp}\tilde{Y} \pm X\left[\tilde{Q},\tilde{Y}\right]_{\mp} \simeq 0 , \qquad \text{(D.12)}$$

which holds by virtue of our neglect of closed string BRS transforms. We will use this argument without comment in the future. Using (D.5) and (D.6) we find the extra pieces in the action of supersymmetry on the ghost part of the boundary state to be

$$\begin{aligned}&\Delta\left[\left(\Lambda^A_{1/2}+\tilde{\Lambda}^A_{1/2}\right),\left(V^c_{-1}\tilde{V}^b_{-1} - V^b_{-1}\tilde{V}^c_{-1}\right)\right]\\ &\quad = -\left[Q, V^A_{-1/2}\right]\tilde{V}^b_{-1} + V^b_{-1}\left[\tilde{Q},\tilde{V}^A_{-1/2}\right] + O(k^2) .\end{aligned} \qquad \text{(D.13)}$$

This cancels (D.10) showing that if we neglect closed string BRS transforms and terms of $O(k^2)$ the naive supersymmetry algebra gives the correct result in this picture.

Next consider the action of $(\Lambda^A_{-1/2}+\tilde{\Lambda}^A_{-1/2})$ on the same boundary state. Using (D.3) we find

$$\begin{aligned}
&\Delta\left[\Lambda^A_{-1/2}+\tilde{\Lambda}^A_{-1/2}, V^\mu_{-1}\tilde{V}^\mu_{-1}\right]\\
&\quad=\left\{Q, W^B(\gamma^\mu)_B{}^A\right\}\tilde{V}^\mu_{-1}-V^\mu_{-1}\left\{\tilde{Q},\tilde{W}^B(\gamma^\mu)_B{}^A\right\}+O(k^2)\\
&\quad=W^B(\tilde{\not{k}}_{\mathrm{R}})_B{}^A\tilde{V}^c_{-1}-(\not{k}_{\mathrm{L}})_B{}^A V^c_{-1}\tilde{W}^B+O(k^2),
\end{aligned}\tag{D.14}$$

where the second equality is obtained by using the integration by parts trick and the easy BRS identity

$$\left\{Q, V^\mu_{-1}\right\}=k^\mu V^c_{-1}+O(k^2).\tag{D.15}$$

Next, using (D.5) and (D.6), we find

$$\begin{aligned}
&\Delta\left[\left(\Lambda^A_{-1/2}+\tilde{\Lambda}^A_{-1/2}\right), V^c_{-1}\tilde{V}^b_{-1}-V^b_{-1}\tilde{V}^c_{-1}\right]\\
&\quad=V^c_{-1}\left[\tilde{\Lambda}^A_{-1/2},\tilde{V}^b_{-1}\right]-\left[\Lambda^A_{-1/2},V^b_{-1}\right]\tilde{V}^c_{-1}\\
&\quad=V^c_{-1}\left(\left[\tilde{Q},\tilde{W}^A_b\right]-\tilde{W}^B(\tilde{\not{k}}_{\mathrm{R}})_B{}^A\right)-\left(\left[Q,W^A_b\right]-W^B(\not{k}_{\mathrm{L}})_B{}^A\right)\tilde{V}^c_{-1}\\
&\quad=-V^c_{-1}\tilde{W}^B(\tilde{\not{k}}_{\mathrm{R}})_B{}^A+W^B(\not{k}_{\mathrm{L}})_B{}^A\tilde{V}^c_{-1}+O(k^2).
\end{aligned}\tag{D.16}$$

The last equality follows from the integration by parts trick and the easy BRS identity $[Q, V^c_{-1}]=O(k^2)$. Using the chiral regulator momentum condition $k^\mu_{\mathrm{L}}=-\tilde{k}^\mu_{\mathrm{R}}$, we see that this cancels against (D.14). Once again, up to terms of $O(k^2)$ and closed string BRS transforms, the extra terms are seen to cancel.

Although the details of the argument are rather different in the two pictures, and the argument is certainly not elegant, we believe that these calculations give strong evidence that the naive supersymmetry algebra gives the boundary state, correct to the off-shell order we need, in any picture. A general argument would be useful, but we do not have one.

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