Inflation and the Theory of Cosmological Perturbations

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Abstract

These lectures provide a pedagogical introduction to inflation and the theory of cosmological perturbations generated during inflation which are thought to be the origin of structure in the universe.

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Notation

A few words on the metric notation. We will be using the convention $(-,+,+,+)$, even though we might switch time to time to the other option $(+,-,-,-)$. This might happen for our convenience, but also for pedagogical reasons. Students should not be shielded too much against the phenomenon of changes of convention and notation in books and articles. Also, references are provided in alphabetical order by the name of the first author.

Units

We will adopt natural, or high energy physics, units. There is only one fundamental dimension, energy, after setting $\hbar = c = k_{\rm b} = 1$,

$$[\text{Energy}] = [\text{Mass}] = [\text{Temperature}] = [\text{Length}]^{-1} = [\text{Time}]^{-1}\,.$$

The most common conversion factors and quantities we will make use of are

1 $\text{GeV}^{-1} = 1.97 \times 10^{-14}$ cm=6.59×10^{-25} sec,

1 Mpc= 3.08×10^{24} cm=1.56×10^{38} GeV^{-1},

$M_{\rm Pl} = 1.22 \times 10^{19}$ GeV,

H_0= $100\,h$ Km sec^{-1} Mpc^{-1}=$2.1\,h\,\times 10^{-42}$ GeV,

$\rho_{\rm c} = 1.87 h^2 \cdot 10^{-29} {\rm g\,cm}^{-3} = 1.05 h^2 \cdot 10^4\,{\rm eV\,cm}^{-3} = 8.1 h^2 \times 10^{-47}$ GeV^4,

$T_0 = 2.75$ K=2.3×10^{-13} GeV,

$T_{\rm eq} = 5.5(\Omega_0 h^2)$ eV,

$T_{\rm ls} = 0.26\,(T_0/2.75\,{\rm K})$ eV.

$s_0 = 2969\,{\rm cm}^{-3}$.

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Introduction

Inflation is a beautiful theoretical paradigm which explains why our universe looks the way we see it. It assumes that in the early universe an infinitesimally small patch underwent this period of rapid exponential expansion becoming the universe (or much larger portion of) we observe today. The observed universe is therefore so homogeneous and isotropic because inhomogeneities were wiped out [27]. In fact, the main subject of these lectures regards another incredible gift inflation give us [28, 33, 68]. The inflationary expansion of the universe quantum-mechanically excite quantum fields and stretches their perturbations from microphysical to cosmological scales. These vacuum fluctuations become classical on large scales and induce energy density fluctuations. When, after inflation, these fluctuations re-enter the observable universe, they generate temperature and matter anisotropies. It is believed that this is responsible for the observed cosmic microwave background (CMB) anisotropy and for the large-scale distribution of galaxies and dark matter. Inflation brings another bonus: it sources gravitational waves as a vacuum fluctuation, which may contribute to CMB anisotropy polarization and are now the subject of an intense search program. Therefore, a prediction of inflation is that all of the structure we see in the universe is a result of quantum-mechanical fluctuations during the inflationary epoch.

The goal of these lectures is to provide a pedagogical introduction to the inflationary paradigm and to the theory of cosmological perturbations generated during inflation and they are organized as follows.

The reader might find also useful consulting the following textbooks [20, 40, 43, 44, 52, 56, 69] (where some notes are taken from) and review [52] for more material.

Part I

Basics of the Big-Bang model

Our current understanding of the evolution of the universe is based upon the Friedmann-Robertson-Walker (FRW) cosmological model, or the hot big bang model as it is usually called. The model is so successful that it has become known as the standard cosmology. The FRW cosmology is so robust that it is possible to make sensible speculations about the universe at times as early as 10^{-43} sec after the Big Bang.

Our universe is, at least on large scales, homogeneous and isotropic. This observation gave rise to the so-called Cosmological Principle. The best evidence for the isotropy of the observed universe is the uniformity

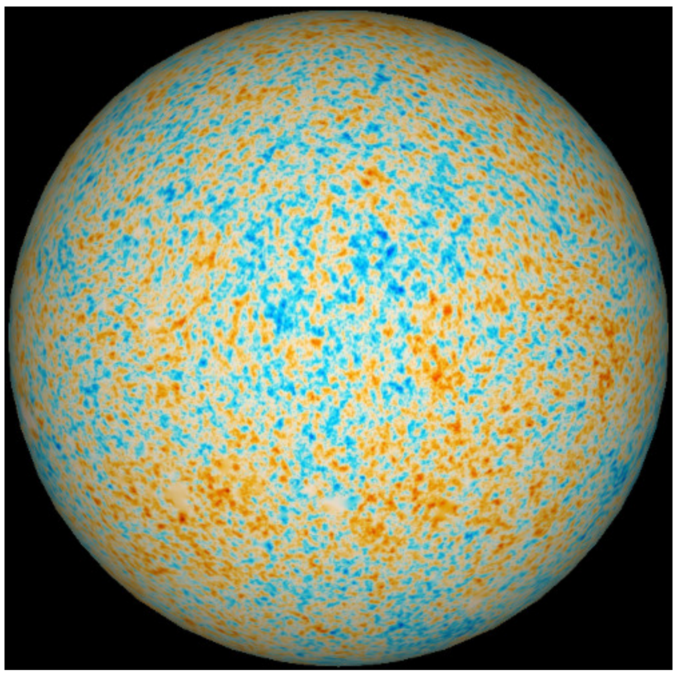

Figure 1: The CMB radiation projected onto a sphere.

of the temperature of the cosmic microwave background (CMB) radiation: intrinsic temperature anisotropies is smaller than about one part in 10^5. This uniformity signals that at the epoch of last scattering for the CMB radiation (about 200,000 yr after the bang) the universe was to a high degree of precision (order of 10^{-5} or so) isotropic and homogeneous. Homogeneity and isotropy is of course true if the universe is observed at sufficiently large scales. Indeed, our observed universe has a current size of about 4000 Mpc. The inflationary theory, as we shall see, suggests that the universe continues to be homogeneous and isotropic over distances larger than 4000 Mpc.

From now on we will work under the assumption that our observable universe is homogeneous and isotropic on large scales and that our spacetime is a maximally symmetric space satisfying the Cosmological Principle. This is the so-called Friedmann-Robertson-Walker metric, which can be written in the form

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right], \tag{1}$$

where (t, r, θ, ϕ) are comoving coordinates, $a(t)$ is the cosmic scale factor and k can be chosen to be $+1, -1$, or 0 by rescaling the coordinates for spaces of constant positive, negative, or zero spatial curvature, respectively. The coordinate r is dimensionless, *i.e.* $a(t)$ has dimensions of length and only relative ratios are physical, and r ranges from 0 to 1 for $k = +1$. The time coordinate is just the proper (or clock) time measured by an observer at rest in the comoving frame, *i.e.*, (r, θ, ϕ)=constant.

In order to illustrate the meaning of the constant k, consider the simpler case of a two spatial dimensions. Examples of two-dimensional spaces that are homogeneous and isotropic are the flat plane $\mathbb{R}^2$ (flat geometriy), the positively-curved closed two-sphere $\mathbb{S}^2$ (closed geometry) and the negatively-curved hyperbolic plane $\mathbb{H}^2$ (open geometry). Consider first a two-sphere $\mathbb{S}^2$ of radius a and embedded in a three-dimensional space $\mathbb{R}^3$. The equation of the sphere of radius a is

$$x_1^2 + x_2^2 + x_3^2 = a^2. \tag{2}$$

The element of length is the three-dimensional Euclidean space is

$$d\mathbf{x}^2 = dx_1^2 + dx_2^2 + dx_3^2. \tag{3}$$

Since x_3 is a fictitious coordinate, we can eliminate it in favour of the other two and write

$$d\mathbf{x}^2 = dx_1^2 + dx_2^2 + \frac{(x_1 dx_1 + x_2 dx_2)^2}{a^2 - x_1^2 - x_2^2}. \tag{4}$$

Now, let us introduce the polar coordinates

$$x_1 = r' \cos\theta, \;\; x_2 = r' \sin\theta, \tag{5}$$

in terms of which the infinitesimal line length becomes

$$d\mathbf{x}^2 = \frac{a^2 dr'^2}{a^2 - r'^2} + r'^2 d\theta^2. \tag{6}$$

Finally, with the definition of a dimensionless coordinate $r = r'/a$ $(0 \le r \le 1)$, the spatial metric becomes

$$d\mathbf{x}^2 = a^2\left[\frac{dr^2}{1-r^2} + r^2 d\theta^2\right]. \tag{7}$$

Note the similarity between this metric and the $k = 1$ FRW metric, which therefore has the global geometry of a sphere. The equivalent formulas for a space of constant negative curvature can be obtained with the replacement $a \to ia$. In this case the embedding is in a three-dimensional Minkowski space. The metric corresponding to the form for the negative curvature case $k = -1$ is

$$d\mathbf{x}^2 = a^2\left[\frac{dr^2}{1+r^2} + r^2 d\theta^2\right], \tag{8}$$

which resembles $k = -1$ FRW metric, which therefore has the global geometry of a hyperboloid. Finally, the spatially-flat model can be obtained from either of the above examples by taking the radius a to infinity. For the flat case the scale factor does not represent any physical radius as in the closed case, or an imaginary radius as in the open case, but merely represents how the physical distance between comoving points scales as the space expands or contracts.

Let us now summarize here the main passages leading to the FRW metric.

1. First of all, we write the FRW metric as

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\widetilde{g}_{ij}\mathrm{d}x^i\mathrm{d}x^j. \tag{9}$$

   From now on, all objects with a tilde will refer to three-dimensional quantities calculated with the metric $\widetilde{g}_{ij}$.

2. One can then calculate the Christoffel symbols in terms of $a(t)$ and $\widetilde{\Gamma}^i{}_{jk}$. Recalling from the GR course that the Christoffel symbols are

$$\boxed{\Gamma^\mu{}_{\nu\lambda} = \frac{1}{2}g^{\mu\rho}\left(\frac{\partial g_{\rho\nu}}{\partial x^\lambda} + \frac{\partial g_{\rho\lambda}}{\partial x^\nu} - \frac{\partial g_{\nu\lambda}}{\partial x^\rho}\right)}, \tag{10}$$

   we may compute the non vanishing components

$$\begin{aligned}
\Gamma^i{}_{jk} &= \widetilde{\Gamma}^i{}_{jk}, \\
\Gamma^i{}_{j0} &= \frac{\dot{a}}{a}\,\delta^i{}_j, \\
\Gamma^0{}_{ij} &= \frac{\dot{a}}{a}g_{ij} = \dot{a}a\widetilde{g}_{ij}.
\end{aligned} \tag{11}$$

3. The relevant components of the Riemann tensor

$$\boxed{R^\lambda{}_{\sigma\mu\nu} = \partial_\mu\Gamma^\lambda{}_{\sigma\nu} - \partial_\nu\Gamma^\lambda{}_{\sigma\mu} + \Gamma^\lambda{}_{\mu\rho}\Gamma^\rho{}_{\nu\sigma} - \Gamma^\lambda{}_{\nu\rho}\Gamma^\rho{}_{\mu\sigma}} \tag{12}$$

   for the FRW metric are

$$\begin{aligned}
R^i{}_{0j0} &= -\frac{\ddot{a}}{a}\delta^i{}_j, \\
R^0{}_{i0j} &= \ddot{a}a\widetilde{g}_{ij}, \\
R^k{}_{ikj} &= \widetilde{R}_{ij} + 2\dot{a}^2\widetilde{g}_{ij}.
\end{aligned} \tag{13}$$

4. Now we can use $\widetilde{R}_{ij} = 2k\widetilde{g}_{ij}$ (as a consequence of the maximally symmetry of $\widetilde{g}_{ij}$) to calculate $R_{\mu\nu}$. The nonzero components are

$$\begin{aligned}
R_{00} &= -3\frac{\ddot{a}}{a}, \\
R_{ij} &= \left(a\ddot{a} + 2\dot{a}^2 + 2k\right)\widetilde{g}_{ij}, \\
&= \left(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + 2\frac{k}{a^2}\right)g_{ij}.
\end{aligned} \tag{14}$$

5. The Ricci scalar is

$$R = \frac{6}{a^2}\left(a\ddot{a} + \dot{a}^2 + k\right), \tag{15}$$

and

6. the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ has the components

$$\begin{aligned} G_{00} &= 3\left(\frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right), \\ G_{0i} &= 0, \\ G_{ij} &= -\left(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right) g_{ij}. \end{aligned} \tag{16}$$

1 Standard cosmology

The dynamics of the expanding universe only appeared implicitly in the time dependence of the scale factor $a(t)$. To make this time dependence explicit, one must solve for the evolution of the scale factor using the Einstein equations

$$\boxed{R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G_{\rm N} T_{\mu\nu} - \Lambda g_{\mu\nu},} \tag{17}$$

where $T_{\mu\nu}$ is the stress-energy tensor for all the fields present (matter, radiation, and so on) and we have also included the presence of a cosmological constant. With very minimal assumptions about the right-hand side of the Einstein equations, it is possible to proceed without detailed knowledge of the properties of the fundamental fields that contribute to the stress tensor $T_{\mu\nu}$.

1.1 The stress-energy momentum tensor

To be consistent with the symmetries of the metric, the total stress-energy tensor tensor must be diagonal, and by isotropy the spatial components must be equal. The simplest realization of such a stress-energy tensor is that of a perfect fluid characterized by a time-dependent energy density $\rho(t)$ and pressure $P(t)$

$$\boxed{T^{\mu}_{\ \nu} = (\rho + P)u^{\mu}u_{\nu} + P\delta^{\mu}_{\ \nu} = \mathrm{diag}(-\rho, P, P, P),} \tag{18}$$

where $u^{\mu} = (1, 0, 0, 0)$ in a comoving coordinate system. This is precisely the energy-momentum tensor of a perfect fluid. The four-vector u_{μ} is known as the velocity field of the fluid, and the comoving coordinates are those with respect to which the fluid is at rest. In general, this matter content has to be supplemented by an equation of state. This is usually assumed to be that of a barytropic fluid, *i.e.* one whose pressure depends only on its density, $P = P(\rho)$. The most useful toy-models of cosmological fluids arise from considering a linear relationship between P and ρ of the type

$$\boxed{P = w\rho}, \tag{19}$$

where w is known as the equation of state parameter. Occasionally also more exotic equations of state are considered. For non-relativistic particles (NR) particles, there is no pressure, $p_{\rm NR} = 0$, *i.e.* $w_{\rm NR} = 0$, and such matter is usually referred to as dust. The trace of the energy-momentum tensor is

$$T^\mu_\mu = -\rho + 3P. \tag{20}$$

For relativistic particles, radiation for example, the energy-momentum tensor is (like that of Maxwell theory) traceless, and hence relativistic particles have the equation of state

$$P_{\rm r} = \frac{1}{3}\rho_{\rm r}, \tag{21}$$

and thus $w_{\rm r} = 1/3$. For physical (gravitating instead of anti-gravitating) matter one usually requires $\rho > 0$ (positive energy) and either $P > 0$, corresponding to $w > 0$ or, at least, $(\rho + 3P) > 0$, corresponding to the weaker condition $w > -1/3$. A cosmological constant, on the other hand, corresponds, as we will see, to a matter contribution with $w_\Lambda = -1$ and thus violates either $\rho > 0$ or $(\rho + 3P) > 0$.

Let us now turn to the conservation laws associated with the energy-momentum tensor,

$$\nabla_\mu T^{\mu\nu} = 0. \tag{22}$$

The spatial components of this conservation law give

$$\nabla_\mu T^{\mu i} = \nabla_0 T^{0i} + \nabla_j T^{ji} = 0 + \nabla_j T^{ji} = P \nabla_j g^{ij} = 0, \tag{23}$$

where the last passage has been made because the metric is covariantly conserved. The only interesting conservation law is thus the zero-component

$$\nabla_\mu T^{\mu 0} = \partial_\mu T^{\mu 0} + \Gamma^\mu_{\ \mu\nu} T^{\nu 0} + \Gamma^0_{\ \mu\nu} T^{\mu\nu} = 0, \tag{24}$$

which for a perfect fluid becomes

$$\dot\rho + \Gamma^\mu_{\ \mu 0}\rho + \Gamma^0_{\ 00}\rho + \Gamma^0_{\ ij} T^{ij} = 0. \tag{25}$$

Using the Christoffel symbols previously computed, see Eq. (11), we get

$$\boxed{\dot\rho + 3H(\rho + P) = 0}. \tag{26}$$

For instance, when the pressure of the cosmic matter is negligible, like in the universe today, and we can treat the galaxies (without disrespect) as dust, then one has

$$\boxed{\rho_{\rm NR}\, a^3 = \text{constant} \quad (\text{MATTER})}. \tag{27}$$

The energy (number) density scales like the inverse of the volume whose size is $\sim a^3$ On the other hand, if the universe is dominated by, say, radiation, then one has the equation of state $P = \rho/3$, then

$$\boxed{\rho_{\rm r}\, a^4 = \text{constant} \quad (\text{RADIATION}).} \tag{28}$$

The energy density scales like the inverse of the volume (whose size is $\sim a^3$) and the energy which scales like $1/a$ because of the red-shift: photon energies scale like the inverse of their wavelengths which in turn scale like $1/a$. More generally, for matter with equation of state parameter w, one finds

$$\boxed{\rho\, a^{3(1+w)} = \text{constant}.} \tag{29}$$

In particular, for $w = -1$, ρ is constant and corresponds, as we will see more explicitly below, to a cosmological constant vacuum energy

$$\boxed{\rho_\Lambda = \text{constant} \quad (\text{VACUUM ENERGY}).} \tag{30}$$

The early universe was radiation dominated, the adolescent universe was matter dominated and the adult universe is dominated by the cosmological constant. If the universe underwent inflation, there was again a very early period when the stress-energy was dominated by vacuum energy. As we shall see next, once we know the evolution of ρ and P in terms of the scale factor $a(t)$, it is straightforward to solve for $a(t)$. Before going on, we want to emphasize the utility of describing the stress energy in the universe by the simple equation of state $P = w\rho$. This is the most general form for the stress energy in a FRW space-time and the observational evidence indicates that on large scales the FRW metric is quite a good approximation to the space-time within our Hubble volume. This simple, but often very accurate, approximation will allow us to explore many early universe phenomena with a single parameter.

2 The Friedmann equations

After these preliminaries, we are now prepared to tackle the Einstein equations. We allow for the presence of a cosmological constant and thus consider the equations

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G_{\rm N} T_{\mu\nu}. \tag{31}$$

It will be convenient to rewrite these equations in the form

$$R_{\mu\nu} = 8\pi G_{\rm N}\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^\lambda_\lambda\right) + \Lambda g_{\mu\nu}. \tag{32}$$

Because of isotropy, there are only two independent equations, namely the 00-component and any one of the non-zero ij-components. Using Eqs. (14) we find

$$\begin{aligned} -3\frac{\ddot a}{a} &= 4\pi G_{\rm N}(\rho + 3P) - \Lambda\,, \\ \frac{\ddot a}{a} + 2\frac{\dot a^2}{a^2} + 2\frac{k}{a^2} &= 4\pi G_{\rm N}(\rho - P) + \Lambda. \end{aligned} \tag{33}$$

Using the first equation to eliminate $\ddot a$ from the second, one obtains the set of equations for the Hubble rate

$$\boxed{H^2 + \frac{k}{a^2} = \frac{8\pi G_{\rm N}}{3}\rho + \frac{\Lambda}{3}} \tag{34}$$

and for the acceleration

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G_{\rm N}}{3}(\rho + 3P) + \frac{\Lambda}{3}}. \tag{35}$$

Together, this set of equation is known as the Friedman equations. They are supplemented by the conservation equation (26). Note that because of the Bianchi identities, the Einstein equations and the conservation equations should not be independent, and indeed they are not. It is easy to see that (34) and (26) imply the second order equation (35) so that, a pleasant simplification, in practice one only has to deal with the two first order equations (34) and (26). Sometimes, however, (35) is easier to solve than (34), because it is linear in $a(t)$, and then (34) is just used to fix one constant of integration.

Notice that Eqs. (34) and (35) can be obtained, in the non-relativistic limit $P = 0$ from Newtonian physics. Imagine that the distribution of matter is uniform and its matter density is ρ. Put a test particle with mass m on a surface of a sphere of radius a and let gravity act. The total energy is constant and therefore

$$E_{\rm kin} + E_{\rm pot} = \frac{1}{2}m\dot{a}^2 - G_{\rm N}\frac{mM}{a} = \kappa = \text{constant}. \tag{36}$$

Since the mass M contained in a sphere of radius a is $M = (4\pi\rho a^3/3)$, we obtain

$$\frac{1}{2}m\dot{a}^2 - \frac{4\pi G_{\rm N}}{3}m\rho a^2 = \kappa = \text{constant}. \tag{37}$$

By dividing everything by $(ma^2/2)$ we obtain Eq. (34) with of course no cosmological constant and after setting $k = 2\kappa/m$. Eq. (35) can be analogously obtained from Newton's law relating the gravitational force and the acceleration (but still with $P = 0$).

The expansion rate of the universe is determined by the Hubble rate H which is not a constant and generically scales like t^{-1}. The Friedmann equation (34) can be recast as

$$\boxed{\Omega - 1 = \frac{\rho}{3H^2/8\pi G_{\rm N}} = \frac{k}{a^2H^2}}, \tag{38}$$

where we have defined the parameter Ω as the ratio between the energy density ρ and the critical energy density $\rho_{\rm c}$

$$\boxed{\Omega = \frac{\rho}{\rho_{\rm c}}, \quad \rho_{\rm c} = \frac{3H^2}{8\pi G_{\rm N}}}. \tag{39}$$

Since $a^2H^2 > 0$, there is a correspondence between the sign of k and the sign of $(\Omega - 1)$

$$\begin{aligned} k = +1 \quad &\Rightarrow \quad \Omega > 1 \quad \text{CLOSED}, \\ k = 0 \quad &\Rightarrow \quad \Omega = 1 \quad \text{FLAT}, \\ k = -1 \quad &\Rightarrow \quad \Omega < 1 \quad \text{OPEN}. \end{aligned} \tag{40}$$

Eq. (38) is valid at all times, note also that both Ω and $\rho_{\rm c}$ are not constant in time. At early times once has a radiation-dominated (RD) phase radiation and $H^2 \sim a^{-4}$ with $(\Omega - 1) \sim a^2$; during the matter-dominated phase (MD) one finds $H^2 \sim a^{-3}$ with $(\Omega - 1) \sim a$. These relations will be crucial when we will study the inflationary universe. From the FRW metric it is also clear that the effect of the curvature becomes important only at a comoving radius $r \sim |k|^{-1/2}$. So we define the physical radius of curvature of the universe $R_{\rm curv} = a(t)|k|^{-1/2} = (6/|{}^3\mathcal{R}|)^{1/2}$, related to the Hubble radius H^{-1} by

$$R_{\rm curv} = \frac{H^{-1}}{|\Omega - 1|^{1/2}}. \tag{41}$$

When $|\Omega - 1| \ll 1$, such a curvature radius turns out to be much larger than the Hubble radius and we can safely neglect the effect of curvature in the universe. Note also that for closed universes, $k = +1$, $R_{\rm curv}$ is just the physical radius of the three-sphere.

3 Equilibrium thermodynamics of the early universe

Because the early universe was to a good approximation in thermal equilibrium at very early epochs, we can assume that it was characterized by a RD phase and we will quickly review some basic thermodynamics of it.

The number density n, energy density ρ and pressure P of a dilute, weakly interacting gas of particles with g internal degrees of freedom can be written in terms of its phase space distribution function $f(\mathbf{p})$

$$\begin{aligned} n &= \frac{g}{(2\pi)^3} \int {\rm d}^3p \, f(\mathbf{p}), \\ \rho &= \frac{g}{(2\pi)^3} \int {\rm d}^3p \, E(\mathbf{p}) \, f(\mathbf{p}), \\ P &= \frac{g}{(2\pi)^3} \int {\rm d}^3p \, \frac{|\mathbf{p}|^2}{3E(\mathbf{p})} f(\mathbf{p}), \end{aligned} \tag{42}$$

where $E^2 = |\mathbf{p}|^2 + m^2$.The phase space f is given by the familiar Fermi-Dirac or Bose-Einstein distributions

$$\boxed{f(\mathbf{p}) = [\exp(E/T) \pm 1]^{-1}}, \tag{43}$$

where we have neglected a possible chemical potential, $+1$ refers to Fermi-Dirac species and -1 to Bose-Einstein species. From the equilibrium distributions, the number density n, energy density ρ and pressure P of a species of mass m, and temperature T are

$$\begin{aligned} n &= \frac{g}{2\pi^2} \int_m^\infty {\rm d}E \, E \, \frac{(E^2 - m^2)^{1/2}}{\exp(E/T) \pm 1}, \\ \rho &= \frac{g}{2\pi^2} \int_m^\infty {\rm d}E \, E^2 \, \frac{(E^2 - m^2)^{1/2}}{\exp(E/T) \pm 1}, \\ P &= \frac{g}{6\pi^2} \int_m^\infty {\rm d}E \frac{(E^2 - m^2)^{3/2}}{\exp(E/T) \pm 1}. \end{aligned} \tag{44}$$

In the relativistic limit $T \gg m$ we obtain

$$
\begin{aligned}
\rho &= \begin{cases} (\pi^2/30)gT^4 & \text{(BOSE)} \\ (7/8)(\pi^2/30)gT^4 & \text{(FERMI)} \end{cases} \\
n &= \begin{cases} (\zeta(3)/\pi^2)gT^3 & \text{(BOSE)} \\ (3/4)(\zeta(3)/\pi^2)gT^3 & \text{(FERMI)} \end{cases} \\
P &= \rho/3.
\end{aligned} \tag{45}
$$

Here $\zeta(3) \simeq 1.2$ is the Riemann zeta function of three. The total energy density and pressure of all species in equilibrium can be expressed in terms of the photon temperature T

$$
\begin{aligned}
\rho_{\rm r} &= T^4 \sum_{\text{all species}} \left(\frac{T_i}{T}\right)^4 \frac{g_i}{2\pi^2} \int_{x_i}^{\infty} {\rm d}u \, \frac{(u^2 - x_i^2)^{1/2} u^2}{\exp(u) \pm 1}, \\
P_{\rm r} &= T^4 \sum_{\text{all species}} \left(\frac{T_i}{T}\right)^4 \frac{g_i}{6\pi^2} \int_{x_i}^{\infty} {\rm d}u \, \frac{(u^2 - x_i^2)^{3/2}}{\exp(u) \pm 1},
\end{aligned} \tag{46}
$$

where $x_i = m_i/T$ and $u = E/T$ and we have taken into account the possibility that the species have a different temperature than the photons.

Since the energy density and pressure of non-relativistic species is exponentially smaller than that of relativistic species, it is a very good approximation to include only the relativistic species in the sums and we obtain

$$
\boxed{\rho_{\rm r} = 3P_{\rm r} = \frac{\pi^2}{30} g_*(T)\, T^4,} \tag{47}
$$

where

$$
\boxed{g_*(T) = \sum_{\text{bosons}} g_i \left(\frac{T_i}{T}\right)^4 + \frac{7}{8} \sum_{\text{fermions}} g_i \left(\frac{T_i}{T}\right)^4,} \tag{48}
$$

counts the effective total number of relativistic degrees of freedom in the plasma. For instance, for $T \ll$ MeV, the only relativistic species are the three neutrinos with $T_\nu = (4/11)^{1/3} T_\gamma$ and the photons. This gives

$$
g_*(\ll \text{MeV}) = \frac{7}{8} \cdot 3\,(=\text{families}) \cdot 2\,(=\text{Weyl d.o.f.}) \cdot \left(\frac{4}{11}\right)^{4/3} + 2\,(=\text{photon d.o.f.}) \simeq 3.36. \tag{49}
$$

For 100 MeV $\gtrsim T \gtrsim$ 1 MeV, the electron and positron are additional relativistic degrees of freedom and $T_\nu = T_\gamma$. We thus find

$$
\begin{aligned}
g_*(100\,\text{MeV} \gtrsim T \gtrsim 1\,\text{MeV}) &= \frac{7}{8} \cdot 3\,(=\text{families}) \cdot 2\,(=\text{Weyl d.o.f.}) + 2\,(=\text{photon d.o.f.}) \\
&+ \frac{7}{8} \cdot 4\,(=\text{Dirac d.o.f.}) \simeq 10.75.
\end{aligned} \tag{50}
$$

For $T \gtrsim$ 300 GeV, all the species of the Standard Model (SM) are in equilibrium: 8 gluons, $W^\pm$, Z, three generations of quarks and leptons and one complex Higgs field and $g_* \simeq 106.75$.

During early RD phase when $\rho \simeq \rho_{\rm r}$ and supposing that $g_* \simeq$ constant, we have that the Hubble rate is

$$H^2 = \frac{8\pi G_{\rm N}}{3}\rho_{\rm r} = \frac{8\pi}{3}\frac{\rho_{\rm r}}{M_{\rm Pl}^2} = \frac{8\pi}{3M_{\rm Pl}^2}\frac{\pi^2}{30}g_*(T)\,T^4, \tag{51}$$

where $G_{\rm N} = 1/M_{\rm Pl}^2$ and $M_{\rm Pl} \simeq 1.2 \cdot 10^{19}$ GeV is the Planck mass. We get therefore

$$\boxed{H \simeq 1.66\, g_*^{1/2}\,\frac{T^2}{M_{\rm Pl}}} \tag{52}$$

and the corresponding time is, being $H \simeq 1/2t^{-1}$,

$$\boxed{t \simeq 0.3\, g_*^{-1/2}\,\frac{M_{\rm Pl}}{T^2} \simeq \left(\frac{T}{\rm MeV}\right)^{-2}\ \text{sec.}} \tag{53}$$

In thermal equilibrium the entropy per comoving volume V, S, is conserved. It is useful to define the entropy density s as

$$s = \frac{S}{V} = \frac{\rho + P}{T}. \tag{54}$$

It is dominated by the relativistic degrees of freedom and to a very good approximation

$$\boxed{s = \frac{2\pi^2}{45} g_{*S} T^3,} \tag{55}$$

where

$$\boxed{g_{*S}(T) = \sum_{\rm bosons} g_i \left(\frac{T_i}{T}\right)^3 + \frac{7}{8}\sum_{\rm fermions} g_i \left(\frac{T_i}{T}\right)^3,} \tag{56}$$

For most of the history of the universe, all particles had the same temperature and one replace therefore g_{*S} with g_*. Note also that s is proportional to the number density of relativistic degrees of freedom and in particular it can be related to the photon number density n_γ

$$\boxed{s \simeq 1.8 g_{*S} n_\gamma.} \tag{57}$$

Today $s_0 \simeq 7.04 n_{\gamma,0}$. The conservation of S implies that $s \sim a^{-3}$ and therefore

$$\boxed{g_{*S} T^3 a^3 = \text{constant}} \tag{58}$$

during the evolution of the universe we are concerned with. The fact that $S = g_{*S}T^3a^3 =$ constant implies that the temperature of the universe evolves as

$$\boxed{T \sim g_{*S}^{-1/3} a^{-1}.} \tag{59}$$

When g_{*S} is constant one gets the familiar result $T \sim a^{-1}$. The factor $g_{*S}^{-1/3}$ enters because whenever a particle species becomes non-relativistic and disappears from the plasma, its entropy is transferred to the other relativistic particles in the thermal plasma causing T to decrease slightly less slowly (sometimes it is said, but in a wrong way, that the universe slightly reheats up).

Lastly, an important quantity is the entropy within a horizon volume: $S_{\rm HOR} \sim H^{-3}T^3$; during the RD epoch $H \sim T^2/M_{\rm Pl}$, so that

$$S_{\rm HOR} \sim \left(\frac{M_{\rm Pl}}{T}\right)^3 . \tag{60}$$

From this we will shortly conclude that at early times the comoving volume that encompasses all that we can see today (that is a region as large as our present horizon) was comprised of a very large number of causally disconnected regions.

4 The particle horizon and the Hubble radius

A fundamental question in cosmology that one might ask is: what fraction of the universe is in causal contact? More precisely, for a comoving observer with coordinates (r_0, θ_0, ϕ_0), for what values of (r, θ, ϕ) would a light signal emitted at $t = 0$ reach the observer at, or before, time t? This can be calculated directly in terms of the FRW metric. A light signal satisfies the geodesic equation $\mathrm{d}s^2 = 0$. Because of the homogeneity of space, without loss of generality we may choose $r_0 = 0$. Geodesics passing through $r = 0$ are lines of constant θ and ϕ, just as great circles passing from the poles of a two-sphere are lines of constant θ (*i.e.*, constant longitude), so $\mathrm{d}\theta = \mathrm{d}\phi = 0$. Of course, the isotropy of space makes the choice of direction (θ_0, ϕ_0) irrelevant. Thus, a light signal emitted from coordinate position $(r_{\rm H}, \theta_0, \phi_0)$ at time $t = 0$ will reach $r_0 = 0$ in a time t determined by

$$\int_0^t \frac{\mathrm{d}t'}{a(t')} = \int_0^{r_{\rm H}} \frac{\mathrm{d}r'}{\sqrt{1-kr'^2}}. \tag{61}$$

The proper distance to the horizon measured at time t is

$$\boxed{R_{\rm H}(t) = a(t)\int_0^t \frac{\mathrm{d}t'}{a(t')} = a(t)\int_0^a \frac{\mathrm{d}a'}{a'}\frac{1}{a'H(a')} = a(t)\int_0^{r_{\rm H}} \frac{\mathrm{d}r'}{\sqrt{1-kr'^2}} \ \text{(PARTICLE HORIZON).}} \tag{62}$$

If $R_{\rm H}(t)$ is finite, it sets the boundary between the visible universe and the part of the universe from which light signals have not reached us. The behavior of $a(t)$ near the singularity will determine whether or not the particle horizon is finite. We will see that in the standard cosmology $R_{\rm H}(t) \sim t$, that is the particle horizon is finite. The particle horizon should not be confused with the notion of the Hubble radius

$$\boxed{\frac{1}{H} = \frac{a}{\dot{a}} \ \text{(HUBBLE RADIUS).}} \tag{63}$$

The Hubble radius has the following meaning: it is the distance travelled by particles in the course of one expansion time, roughly the time which takes the scale factor to double (think of the distance as $\mathrm{d}t \sim (\mathrm{d}a/a)H^{-1}$) [20]. So the particle horizon and the Hubble radius are different quantities: particles separated by distances greater than $R_H(t)$ have never communicated with one another; on the contrary, if they are separated by an amount larger than Hubble radius H^{-1}, this means that they cannot communicate at the given time t. We shall see that in standard cosmology the distance to the horizon is finite, and up to numerical factors, equal to the Hubble radius, H^{-1}, but during inflation, for instance, they are drastically different.

One can also define a comoving particle horizon distance

$$\boxed{\tau_{\rm H} = \int_0^t \frac{{\rm d}t'}{a(t')} = \int_0^a \frac{{\rm d}a'}{H(a')a'^2} = \int_0^a {\rm d}\ln a' \left(\frac{1}{Ha'}\right) \text{ (COMOVING PARTICLE HORIZON).}} \tag{64}$$

Here, we have expressed the comoving horizon as the logarithmic integral of the comoving Hubble radius $(aH)^{-1}$

$$\boxed{\frac{1}{aH} \text{ (COMOVING HUBBLE RADIUS),}} \tag{65}$$

which will play a crucial role in inflation. Particles separated by comoving distances greater than $\tau_{\rm H}$ never talked to each other; if separated by comoving distances greater than $(aH)^{-1}$, they are not talking at some time τ. It is therefore possible that $\tau_{\rm H}$ could be much larger than the comoving Hubble radius at the present epoch, so that there could not be any communication today but there was at earlier epochs. As we shall see, this might happen if the comoving Hubble radius early on was much larger than it is now so that $\tau_{\rm H}$ got most of its contribution from early times. We will see that this can happen, although it did not happen during matter-dominated or radiation-dominated epochs. In those cases, the comoving Hubble radius increases with time, so typically we expect the largest contribution to $\tau_{\rm H}$ to come from the most recent times.

Recall that in a universe dominated by a fluid with equation of state $P = w/\rho$ we have $n = 2/3(1 + w)$. The comoving Hubble radius goes like

$$\frac{1}{aH} \sim \frac{t}{t^n} = t^{1-n} \tag{66}$$

In particular, for a MD universe $w = 0$ and $n = 2/3$, while for a RD universe $w = 1/3$ and $n = 1/2$. In both cases the comoving Hubble radius increases with time. We see that in the standard cosmology the particle horizon is finite, and up to numerical factors, equal to the Hubble radius, H^{-1}. For this reason, one can use the words horizon and Hubble radius interchangeably for standard cosmology. As we shall see, in inflationary models the horizon and Hubble radius are drastically different as the horizon distance grows exponentially relative to the Hubble radius; in fact, at the end of inflation they differ by e^N, where N is the number of e-folds of inflation.

Note also that a physical length scale λ is within the Hubble radius if $\lambda < H^{-1}$. Since we can identify the length scale λ with its wavenumber k, $\lambda = 2\pi a/k$, we will have the following rule

$$\boxed{\begin{aligned} \frac{k}{aH} &\ll 1 \Longrightarrow \text{SCALE } \lambda \text{ OUTSIDE THE HUBBLE RADIUS} \\ \frac{k}{aH} &\gg 1 \Longrightarrow \text{SCALE } \lambda \text{ WITHIN THE HUBBLE RADIUS} \end{aligned}}$$

Notice that in standard cosmology

$$\frac{\lambda}{\text{PARTICLE HORIZON}} = \frac{\lambda}{R_H} \sim \lambda\, H \sim \frac{aH}{k}. \tag{67}$$

This shows once more that Hubble radius and particle horizon can be used interchangeably in standard cosmology.

5 Some conformalities

Before launching ourselves into the description of the shortcomings of the Big-Bang model and inflation, we would like to go back to the concept of conformal time which will be useful in the next sections. The conformal time τ is defined through the following relation

$$\mathrm{d}\tau = \frac{\mathrm{d}t}{a}. \tag{68}$$

The metric $\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\mathrm{d}\mathbf{x}^2$ then becomes

$$\mathrm{d}s^2 = a^2(\tau)\left[-\mathrm{d}\tau^2 + \mathrm{d}\mathbf{x}^2\right]. \tag{69}$$

The reason why τ is called conformal is manifest from Eq. (69): the corresponding FRW line element is conformal to the Minkowski line element describing a static four dimensional hypersurface.

Any function $f(t)$ satisfies the rule

$$\dot{f}(t) = \frac{f'(\tau)}{a(\tau)}, \tag{70}$$

$$\ddot{f}(t) = \frac{f''(\tau)}{a^2(\tau)} - \mathcal{H}\frac{f'(\tau)}{a^2(\tau)}, \tag{71}$$

where a prime now indicates differentiation wrt to the conformal time τ and

$$\boxed{\mathcal{H} = \frac{a'}{a}}. \tag{72}$$

In particular we can set the following rules

$$\boxed{\begin{aligned}
H &= \frac{\dot{a}}{a} = \frac{a'}{a^2} = \frac{\mathcal{H}}{a},\\
\ddot{a} &= \frac{a''}{a^2} - \frac{\mathcal{H}^2}{a},\\
\dot{H} &= \frac{\mathcal{H}'}{a^2} - \frac{\mathcal{H}^2}{a^2},\\
H^2 &= \frac{8\pi G\rho}{3} - \frac{k}{a^2} \Longrightarrow \mathcal{H}^2 = \frac{8\pi G\rho a^2}{3} - k\\
\dot{H} &= -4\pi G\left(\rho + P\right) \Longrightarrow \mathcal{H}' = -\frac{4\pi G}{3}\left(\rho + 3P\right)a^2,\\
\dot{\rho} &+ 3H(\rho + P) = 0 \Longrightarrow \rho' + 3\,\mathcal{H}(\rho + P) = 0
\end{aligned}}$$

Finally, if the scale factor $a(t)$ scales like $a \sim t^n$, solving the relation (68) we find

$$a \sim t^n \Longrightarrow a(\tau) \sim \tau^{\frac{n}{1-n}}. \tag{73}$$

Therefore, for a RD era $a(t) \sim t^{1/2}$ one has $a(\tau) \sim \tau$ and for a MD era $a(t) \sim t^{2/3}$, that is $a(\tau) \sim \tau^2$.

Part II

The shortcomings of the standard Big-Bang Theory

This part is dedicated to a description of the drawbacks present in the cosmological Big-Bang theory. They are the origin for understanding why inflation is needed.

6 The flatness problem

Let us make a tremendous extrapolation and assume that Einstein equations are valid until the Planck era, when the temperature of the universe is $T_{\rm Pl} \sim 10^{19}$ GeV. From the equation for the curvature

$$\Omega - 1 = \frac{\kappa}{H^2 a^2}, \tag{74}$$

we read that if the universe is perfectly flat, then $(\Omega = 1)$ at all times. On the other hand, if there is even a small curvature term, the time dependence of $(\Omega - 1)$ is quite different.

During a RD period, we have that $H^2 \propto \rho_{\rm r} \propto a^{-4}$ and

$$\Omega - 1 \propto \frac{1}{a^2 a^{-4}} \propto a^2. \tag{75}$$

During MD, $\rho_{\rm NR} \propto a^{-3}$ and

$$\Omega - 1 \propto \frac{1}{a^2 a^{-3}} \propto a. \tag{76}$$

In both cases $(\Omega - 1)$ decreases going backwards with time. Since we know that today $(\Omega_0 - 1)$ is of order unity, we can deduce its value at $t_{\rm Pl}$ (the time at which the temperature of the universe is $T_{\rm Pl} \sim 10^{19}$ GeV)

$$\frac{\mid \Omega - 1 \mid_{T=T_{\rm Pl}}}{\mid \Omega - 1 \mid_{T=T_0}} \approx \left(\frac{a_{\rm Pl}^2}{a_0^2}\right) \approx \left(\frac{T_0^2}{T_{\rm Pl}^2}\right) \approx \mathcal{O}(10^{-64}). \tag{77}$$

where 0 stands for the present epoch, and $T_0 \sim 10^{-13}$ GeV is the present-day temperature of the CMB radiation. If we are not so brave and go back simply to the epoch of nucleosynthesis when light elements abundances were formed, at $T_N \sim 1$ MeV, we get

$$\frac{\mid \Omega - 1 \mid_{T=T_N}}{\mid \Omega - 1 \mid_{T=T_0}} \approx \left(\frac{a_N^2}{a_0^2}\right) \approx \left(\frac{T_0^2}{T_N^2}\right) \approx \mathcal{O}(10^{-16}). \tag{78}$$

In order to get the correct value of $(\Omega_0 - 1) \sim 1$ at present, the value of $(\Omega - 1)$ at early times have to be fine-tuned to values amazingly close to zero, but without being exactly zero. This is the reason why the flatness problem is also dubbed the ‘fine-tuning problem’.

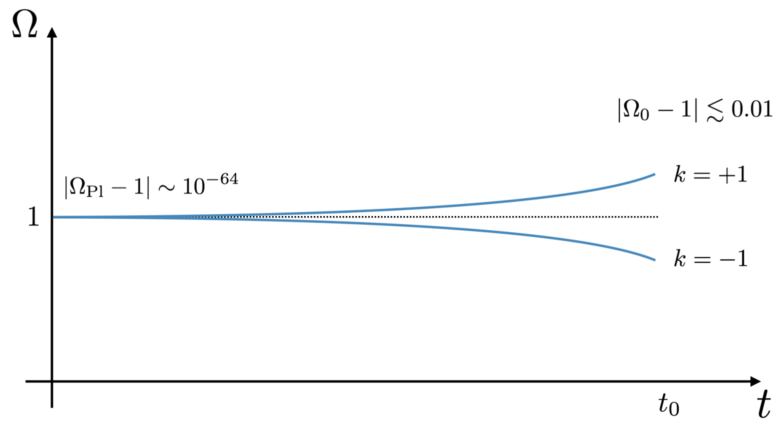

Figure 2: Illustration of the flatness problem in standard cosmology.

7 The entropy problem

Let us now see how the hypothesis of adiabatic expansion of the universe is connected with the flatness problem. From the Friedman equations we know that during a RD period

$$H^2 \simeq \rho_{\rm r} \simeq \frac{T^4}{M_{\rm Pl}^2}, \tag{79}$$

from which we deduce

$$\Omega - 1 = \frac{k M_{\rm Pl}^2}{a^2 T^4} = \frac{k M_{\rm Pl}^2}{S^{\frac{2}{3}} T^2}. \tag{80}$$

Under the hypothesis of adiabaticity, S is constant over the evolution of the universe and therefore

$$|\Omega - 1|_{t=t_{\rm Pl}} = \frac{M_{\rm Pl}^2}{T_{\rm Pl}^2} \frac{1}{S_{\rm U}^{2/3}} = \frac{1}{S_{\rm U}^{2/3}} \approx 10^{-60}, \tag{81}$$

where we have used the fact that the present horizon contains a total entropy

$$S_{\rm U} = \frac{4\pi}{3} H_0^{-3} s = \frac{4\pi}{3} H_0^{-3} \frac{2\pi^2 g_*(T_0) T_0^3}{45} \simeq 10^{90}. \tag{82}$$

We have discovered that $(\Omega - 1)$ is so close to zero at early epochs because the total entropy of our universe is so incredibly large. The flatness problem is therefore a problem of understanding why the (classical) initial conditions corresponded to a universe that was so close to spatial flatness. One would have indeed expected the most natural number for the total entropy of the universe to be of the order of unity at the Planckian temperature, when the horizon itself was of the order of the Planckian length. In a sense, the problem is one of fine–tuning and although such a balance is possible in principle, one nevertheless feels that it is unlikely. On the other hand, the flatness problem arises because the entropy in a comoving volume is conserved. It is possible, therefore, that the problem could be resolved if the cosmic expansion was non–adiabatic for some finite time interval during the early history of the universe.

8 The horizon problem

According to the standard cosmology, photons decoupled from the rest of the components (electrons and baryons) at a temperature of the order of 0.3 eV. This corresponds to the so-called surface of 'last-scattering' at a redshift of about 1100 and an age of about $180\,000\,(\Omega_0 h^2)^{-1/2}$ yrs.

From the epoch of last-scattering onwards, photons free-stream and reach us basically untouched. Detecting primordial photons is therefore equivalent to take a picture of the universe when the latter was about 300 000 yrs old. The spectrum of the cosmic background radiation is consistent with that of a black body at temperature 2.73 K over more than three decades in wavelength. The length corresponding to our present Hubble radius (which is approximately the radius of our observable universe) at the time of last-scattering was

$$\lambda_{\rm H}(t_{\rm ls}) = R_{\rm H}(t_0)\left(\frac{a_{\rm ls}}{a_0}\right) = R_{\rm H}(t_0)\left(\frac{T_0}{T_{\rm ls}}\right).$$

On the other hand, during the MD period, the Hubble length has decreased with a different law

$$H^2 \propto \rho_{\rm NR} \propto a^{-3} \propto T^3.$$

At last-scattering

$$H_{\rm ls}^{-1} = R_{\rm H}(t_0)\left(\frac{T_{\rm ls}}{T_0}\right)^{-3/2} \ll R_{\rm H}(t_0).$$

The length corresponding to our present Hubble radius was much larger that the horizon at that time. This can be shown comparing the volumes corresponding to these two scales

$$\frac{\lambda_H^3(T_{\rm ls})}{H_{\rm ls}^{-3}} = \left(\frac{T_0}{T_{\rm ls}}\right)^{-\frac{3}{2}} \approx 10^6. \tag{83}$$

There were $\sim 10^6$ casually disconnected regions within the volume that now corresponds to our horizon! It is difficult to come up with a process in the history of the universe that would lead to a precise black body [58] for a bath of photons which were causally disconnected the last time they interacted with the surrounding plasma.

The horizon problem is well represented by Fig. 3 where the blue line indicates the horizon scale and the red line any generic physical length scale λ. Suppose that, indeed, λ indicates the distance between two photons we detect today. From Eq. (83) we discover that at the time of emission (last-scattering) the two photons could not talk to each other, the red line is above the blue line.

There is another aspect of the horizon problem which is related to the problem of initial conditions for the cosmological perturbations.

The temperature difference measured between two points separated by a large angle ($\gtrsim 1^\circ$) is due to the so-called Sachs-Wolfe effect and is caused by the fact that these two points had a different value of the gravitational potential associated to it at the last-scattering surface. The temperature anisotropy is commonly expanded in spherical harmonics

$$\frac{\Delta T}{T}(x_0, \tau_0, \mathbf{n}) = \sum_{\ell m} a_{\ell m}(x_0) Y_{\ell m}(\mathbf{n}), \tag{84}$$

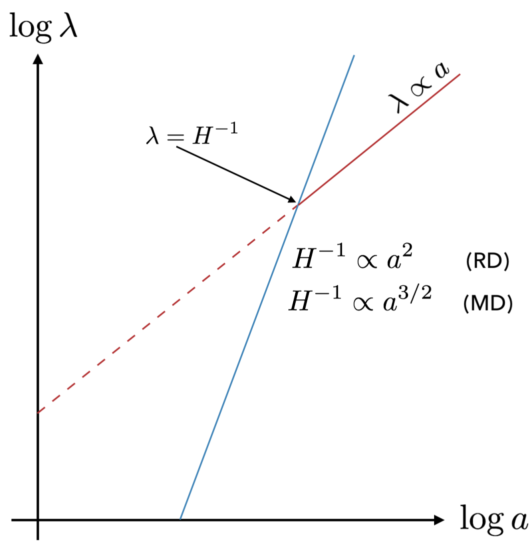

Figure 3: The horizon scale (blue line) and a physical scale λ (red line) as function of the scale factor a. From Ref. [41].

where x_0 and τ_0 are our position and the present time, respectively, $\mathbf{n}$ is the direction of observation, ℓ's are the different multipoles and[1]

$$\langle a_{\ell m} a^*_{\ell' m'} \rangle = \delta_{\ell \ell'} \delta_{m m'} C_\ell, \tag{85}$$

where the deltas are due to the fact that the process that created the anisotropy is statistically isotropic. The C_ℓ are the so-called CMB power spectrum [34]. For homogeneity and isotropy, the C_ℓ's are neither a function of x_0, nor of m. The two-point-correlation function is related to the C_ℓ's in the following way

$$\begin{aligned}\left\langle \frac{\delta T(\mathbf{n})}{T} \frac{\delta T(\mathbf{n}')}{T} \right\rangle &= \sum_{\ell \ell' m m'} \langle a_{\ell m} a^*_{\ell' m'} \rangle Y_{\ell m}(\mathbf{n}) Y^*_{\ell' m'}(\mathbf{n}') \\ &= \sum_\ell C_\ell \sum_m Y_{\ell m}(\mathbf{n}) Y^*_{\ell m}(\mathbf{n}') = \frac{1}{4\pi} \sum_\ell (2\ell+1) C_\ell P_\ell(\mu = \mathbf{n} \cdot \mathbf{n}') \end{aligned} \tag{86}$$

where we have used the addition theorem for the spherical harmonics, and P_ℓ is the Legendre polynomial of order ℓ. In expression (86) the expectation value is an ensemble average. It can be regarded as an average over the possible observer positions, but not in general as an average over the single sky we observe, because of the cosmic variance[2].

[1] An alternative definition is $C_\ell = \langle |a_{\ell m}|^2 \rangle = \frac{1}{2\ell+1} \sum_{m=-\ell}^{\ell} |a_{\ell m}|^2$.

[2] The usual hypothesis is that we observe a typical realization of the ensemble. This means that we expect the difference between the observed values $|a_{\ell m}|^2$ and the ensemble averages C_ℓ to be of the order of the mean-square deviation of $|a_{\ell m}|^2$ from C_ℓ. The latter is called cosmic variance and, because we are dealing with a Gaussian distribution, it is equal to $2C_\ell$ for each multipole ℓ. For a single ℓ, averaging over the $(2\ell+1)$ values of m reduces the cosmic variance by a factor $(2\ell+1)$, but it remains a serious limitation for low multipoles.

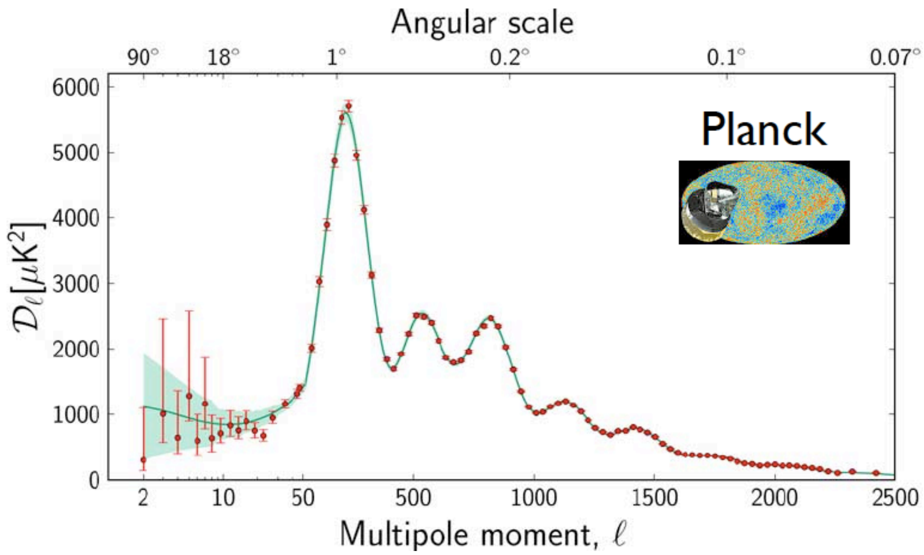

Figure 4: The CMBR anisotropy as function of ℓ from the recent Planck satellite data.

Let us now consider the last-scattering surface. In comoving coordinates the latter is 'far' from us a distance equal to

$$\int_{t_{\rm ls}}^{t_0} \frac{{\rm d}t}{a} = \int_{\tau_{\rm ls}}^{\tau_0} {\rm d}\tau = (\tau_0 - \tau_{\rm ls})\,. \tag{87}$$

A given comoving scale λ is therefore projected on the last-scattering surface sky on an angular scale

$$\theta \simeq \frac{\lambda}{(\tau_0 - \tau_{\rm ls})}, \tag{88}$$

where we have neglected tiny curvature effects. Consider now that the scale λ is of the order of the comoving sound horizon at the time of last-scattering, $\lambda \sim c_s \tau_{\rm ls}$, where $c_s \simeq 1/\sqrt{3}$ is the sound velocity at which photons propagate in the plasma at the last-scattering. This corresponds to an angle

$$\theta \simeq c_s \frac{\tau_{\rm ls}}{(\tau_0 - \tau_{\rm ls})} \simeq c_s \frac{\tau_{\rm ls}}{\tau_0}, \tag{89}$$

where the last passage has been performed knowing that $\tau_0 \gg \tau_{\rm ls}$. Since the universe is MD from the time of last-scattering onwards, the scale factor has the following behavior: $a \sim T^{-1} \sim t^{2/3} \sim \tau^2$, where we have made use of the relation (73). The angle $\theta_{\rm HOR}$ subtended by the sound horizon on the last-scattering surface then becomes

$$\theta_{\rm HOR} \simeq c_s \left(\frac{T_0}{T_{\rm ls}}\right)^{1/2} \sim 1^\circ, \tag{90}$$

where we have used $T_{\rm ls} \simeq 0.3$ eV and $T_0 \sim 10^{-13}$ GeV. This corresponds to a multipole $\ell_{\rm HOR}$

$$\ell_{\rm HOR} = \frac{\pi}{\theta_{\rm HOR}} \simeq 200. \tag{91}$$

From these estimates we conclude that two photons which on the last-scattering surface were separated by an angle larger than $\theta_{\rm HOR}$, corresponding to multipoles smaller than $\ell_{\rm HOR} \sim 200$ were not in causal contact. On the other hand, from Fig. 4 it is clear that small anisotropies, of the *same* order of magnitude $\delta T/T \sim 10^{-5}$ are present at $\ell \ll 200$. We conclude that one of the striking features of the CMB fluctuations is that they appear to be non causal. Photons at the last-scattering surface which were causally disconnected have the same small anisotropies! The existence of particle horizons in the standard cosmology precludes explaining the smoothness as a result of microphysical events: the horizon at decoupling, the last time one could imagine temperature fluctuations being smoothed by particle interactions, corresponds to an angular scale on the sky of about 1°, which precludes temperature variations on larger scales from being erased.

9 The solution to the standard cosmology shortcomings

From the considerations made so far, it appears that solving the shortcomings of the standard Big Bang theory requires two basic modifications of the assumptions made so far:

- The universe has to go through a non-adiabatic period. This is necessary to solve the entropy and the flatness problem. A non-adiabatic phase may give rise to the large entropy $S_{\rm U}$ we observe today.
- The universe has to go through a primordial period during which the physical scales λ evolve faster than the Hubble radius H^{-1}.

The second condition is obvious from Fig. 5. If there is period during which physical length scales grow

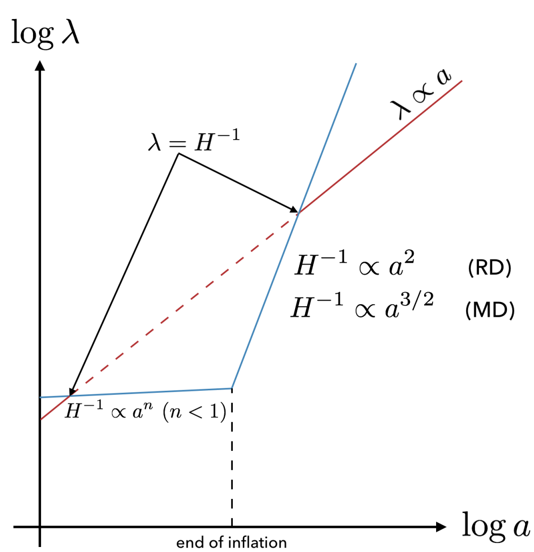

Figure 5: The behavior of a generic scale λ and the Hubble radius H^{-1} in the standard inflationary model. From Ref. [41].

faster than the Hubble radius H^{-1}, length scales λ which are within the horizon today, $\lambda < H^{-1}$ (such as the distance between two detected photons) and were outside the Hubble radius at some period, $\lambda > H^{-1}$ (for instance at the time of last-scattering when the two photons were emitted), had a chance to be within the Hubble radius at some primordial epoch, $\lambda < H^{-1}$ again. If this happens, the homogeneity and the isotropy of the CMB can be easily explained: photons that we receive today and were emitted from the last-scattering surface from causally disconnected regions have the same temperature because they had a chance to talk to each other at some primordial stage of the evolution of the universe. The solution to the horizon is based on the difference between the (comoving) particle horizon and the (comoving) Hubble radius: R_H is bigger than the Hubble radius now, so that particles are in causal contact early on, but not at later epochs.

The second condition can be easily expressed as a condition on the scale factor a. Since a given scale λ scales like $\lambda \sim a$ and the Hubble radius $H^{-1} = a/\dot{a}$, we need to impose that there is a period during which

$$\left(\frac{\lambda}{H^{-1}}\right)^{\cdot} > 0 \Rightarrow \ddot{a} > 0. \tag{92}$$

Notice that is equivalent to require that the ratio between the comoving length scales λ/a and the comoving Hubble radius

$$\left(\frac{\lambda}{H^{-1}}\right)^{\cdot} = \left(\frac{\lambda/a}{H^{-1}/a}\right)^{\cdot} = \left(\frac{\lambda/a}{1/aH}\right)^{\cdot} > 0 \tag{93}$$

increases with time. We can therefore introduced the following rigorous definition: an inflationary stage is a period of the universe during which the latter accelerates

$$\boxed{\text{INFLATION} \quad \Longleftrightarrow \quad \ddot{a} > 0.}$$

Comment: Let us stress that during such a accelerating phase the universe expands *adiabatically*. This means that during inflation one can exploit the usual FRW equations. It must be clear therefore that the non-adiabaticity condition is satisfied not during inflation, but during the phase transition between the end of inflation and the beginning of the RD phase. At this transition phase a large entropy is generated under the form of relativistic degrees of freedom: the Big Bang has taken place.

Part III

The standard inflationary universe

From the previous section we have learned that an accelerating stage during the primordial phases of the evolution of the universe might be able to solve the horizon problem. Therefore we learn that

$$\ddot{a} > 0 \Longleftrightarrow (\rho + 3P) < 0.$$

An accelerating period is obtainable only if the overall pressure P of the universe is negative: $P < -\rho/3$. Neither a RD phase nor a MD phase (for which $P = \rho/3$ and $P = 0$, respectively) satisfy such a condition. Let us postpone for the time being the problem of finding a 'candidate' able to provide the condition $P < -\rho/3$. For sure, inflation is a phase of the history of the universe occurring before the era of nucleosynthesis ($t \approx 1$ sec, $T \approx 1$ MeV) during which the light elements abundances were formed. This is because nucleosynthesis is the earliest epoch we have experimental data from and there is agreement with what the standard Big-Bang theory predicts. However, the thermal history of the universe before the epoch of nucleosynthesis is unknown.

In order to study the properties of the period of inflation, we assume the extreme condition $P = -\rho$ which considerably simplifies the analysis. A period of the universe during which $P = -\rho$ is called *de Sitter* stage. By inspecting the FRW equations and the energy conservation equation, we learn that during the de Sitter phase

$$\begin{aligned} \rho &= \text{constant}, \\ H_{\rm I} &= \text{constant}, \end{aligned}$$

where we have indicated by $H_{\rm I}$ the value of the Hubble rate during inflation. Correspondingly, we obtain

$$a = a_{\rm I}\, e^{H_{\rm I}(t - t_{\rm I})}, \tag{94}$$

where $t_{\rm I}$ denotes the time at which inflation starts. Let us now see how such a period of exponential expansion takes care of the shortcomings of the standard Big Bang Theory.[3]

10 Inflation as a solution to the standard cosmology problems

10.1 Inflation and the horizon problem

During the inflationary (de Sitter) epoch the Hubble radius $H_{\rm I}^{-1}$ is constant. If inflation lasts long enough, all the physical scales that have left the Hubble radius during the RD or MD phase can re-enter the Hubble

[3] Despite the fact that the growth of the scale factor is exponential and the expansion is *superluminal*, this is not in contradiction with what dictated by General Relativity. Indeed, it is the space-time itself which is propagating so fast and not a light signal in it.

radius in the past: this is because such scales are exponentially reduced. Indeed, during inflation the particle horizon grows exponential

$$R_H(t) = a(t)\int_{t_{\rm I}}^{t}\frac{{\rm d}t'}{a(t')} = a_{\rm I}\, e^{H_{\rm I}(t-t_{\rm I})}\left(-\frac{1}{H_{\rm I}}\right)\left[e^{-H_{\rm I}(t-t_{\rm I})}\right]_{t_{\rm I}}^{t} \simeq \frac{a(t)}{H_{\rm I}}, \tag{95}$$

while the Hubble radius remains constant

$$\text{HUBBLE RADIUS} = \frac{a}{\dot{a}} = H_{\rm I}^{-1}, \tag{96}$$

and points that our causally disconnected today could have been in contact during inflation. Notice that in comoving coordinates the comoving Hubble radius shrink exponentially

$$\text{COMOVING HUBBLE RADIUS} = H_{\rm I}^{-1} e^{-H_{\rm I}(t-t_{\rm I})}, \tag{97}$$

while comoving length scales remain constant. As we have seen in the previous section, this explains both the problem of the homogeneity of CMB and the initial condition problem of small cosmological perturbations. Once the physical length is within the horizon, microphysics can act, the universe can be made approximately homogeneous and the primeval inhomogeneities can be created.

Let us see how long inflation must be sustained in order to solve the horizon problem. Let $t_{\rm I}$ and $t_{\rm f}$ be, respectively, the time of beginning and end of inflation. We can define the corresponding number of e-foldings N

$$N = \ln\frac{a_{\rm e}}{a_{\rm I}} = \left[H_{\rm I}(t_{\rm e} - t_{\rm I})\right]. \tag{98}$$

A necessary condition to solve the horizon problem is that the largest scale we observe today, the present horizon H_0^{-1}, was reduced during inflation to a value $\lambda_{H_0}(t_{\rm I})$ smaller than the value of Hubble radius $H_{\rm I}^{-1}$ during inflation. This gives

$$\lambda_{H_0}(t_{\rm I}) = H_0^{-1}\left(\frac{a_{t_{\rm f}}}{a_{t_0}}\right)\left(\frac{a_{t_{\rm I}}}{a_{t_{\rm f}}}\right) = H_0^{-1}\left(\frac{T_0}{T_{\rm f}}\right)e^{-N} \lesssim H_{\rm I}^{-1},$$

where we have neglected for simplicity the short period of MD and we have called $T_{\rm f}$ the temperature at the end of inflation (to be identified with the reheating temperature $T_{\rm RH}$ at the beginning of the RD phase after inflation, see later). We get

$$N \gtrsim \ln\left(\frac{T_0}{H_0}\right) - \ln\left(\frac{T_{\rm f}}{H_{\rm I}}\right) \approx 67 + \ln\left(\frac{T_{\rm f}}{H_{\rm I}}\right).$$

Apart from the logarithmic dependence, we obtain $N \gtrsim 70$.

10.2 Inflation and the flatness problem

Inflation solves elegantly also the flatness problem. Since during inflation the Hubble rate is constant

$$\Omega - 1 = \frac{k}{a^2H^2} \propto \frac{1}{a^2}.$$

On the other end the condition (77) tells us that to reproduce a value of $(\Omega_0 - 1)$ of order of unity today the initial value of $(\Omega - 1)$ at the beginning of the RD phase must be $|\Omega - 1| \sim 10^{-60}$. Since we identify the beginning of the RD phase with the beginning of inflation, we require

$$|\Omega - 1|_{t=t_\mathrm{f}} \sim 10^{-60}.$$

During inflation

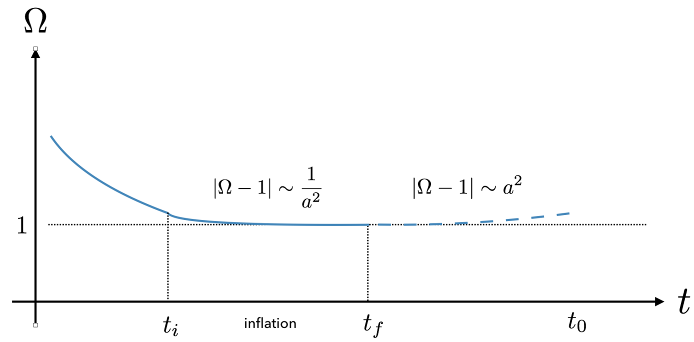

Figure 6: Illustration of the solution of the flatness problem in standard inflationary cosmology.

$$\frac{|\Omega - 1|_{t=t_\mathrm{f}}}{|\Omega - 1|_{t=t_\mathrm{I}}} = \left(\frac{a_\mathrm{I}}{a_\mathrm{f}}\right)^2 = e^{-2N}. \tag{99}$$

Taking $|\Omega - 1|_{t=t_\mathrm{I}}$ of order unity, it is enough to require that $N \approx 70$ to solve the flatness problem.

1. <u>*Comment*</u>: In the previous section we have written that the flatness problem can be also seen as a fine-tuning problem of one part over 10^{60}. Inflation ameliorates this fine-tuning problem, by explaining a tiny number $\sim 10^{-60}$ with a number N of the order 70.

2. <u>*Comment*</u>: The number $N \simeq 70$ has been obtained requiring that the present-day value of $(\Omega_0 - 1)$ is of order unity. For the expression (99), it is clear that, if the period of inflation lasts longer than 70 e-foldings, the present-day value of Ω_0 will be equal to unity with a great precision. One can say that a generic prediction of inflation is that

$$\boxed{\text{INFLATION} \quad \Longrightarrow \quad \Omega_0 = 1.}$$

This statement, however, must be taken *cum grano salis* and properly specified. Inflation does not change the global geometric properties of the space-time. If the universe is open or closed, it will always remain open or closed, independently from inflation. What inflation does is to magnify the radius of curvature R_curv so that locally the universe is flat with a great precision. The current data on the CMB anisotropies confirm this prediction.

10.3 Inflation and the entropy problem

In the previous section, we have seen that the flatness problem arises because the entropy in a comoving volume is conserved. It is possible, therefore, that the problem could be resolved if the cosmic expansion was non-adiabatic for some finite time interval during the early history of the universe. We need to produce a large amount of entropy $S_{\rm U} \sim 10^{90}$. Let us postulate that the entropy changed by an amount

$$S_{\rm f} = Z^3 \, S_{\rm I} \tag{100}$$

from the beginning to the end of the inflationary period, where Z is a numerical factor. It is very natural to assume that the total entropy of the universe at the beginning of inflation was of order unity, one particle per horizon. Since, from the end of inflation onwards, the universe expands adiabatically, we have $S_{\rm f} = S_{\rm U}$. This gives $Z \sim 10^{30}$. On the other hand, since $S_{\rm f} \sim (a_{\rm f} T_{\rm f})^3$ and $S_{\rm I} \sim (a_{\rm I} T_{\rm I})^3$, where $T_{\rm f}$ and $T_{\rm I}$ are the temperatures of the universe at the end and at the beginning of inflation, we get

$$\left(\frac{a_{\rm f}}{a_{\rm I}}\right) = e^N \approx 10^{30} \left(\frac{T_{\rm I}}{T_{\rm f}}\right), \tag{101}$$

which gives again $N \sim 70$ up to the logarithmic factor $\ln (T_{\rm I}/T_{\rm f})$. We stress again that such a large amount of entropy is not produced during inflation, but during the non-adiabatic phase transition which gives rise to the usual RD phase.

11 Inflation and the inflaton

In the previous subsections we have described the various advantages of having a period of accelerating phase. The latter required $P < -\rho/3$. Now, we would like to show that this condition can be attained by means of a simple scalar field. We shall call this field the *inflaton* ϕ.

The action of the inflaton field reads

$$S = \int {\rm d}^4x \sqrt{-g}\, \mathcal{L} = \int d^4x \sqrt{-g} \left[-\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right], \tag{102}$$

where $\sqrt{-g} = a^3$ for the FRW metric. From the Eulero-Lagrange equations

$$\partial^\mu \frac{\delta(\sqrt{-g}\mathcal{L})}{\delta\, \partial^\mu \phi} - \frac{\delta(\sqrt{-g}\mathcal{L})}{\delta \phi} = 0, \tag{103}$$

we obtain

$$\boxed{\ddot{\phi} + 3H\dot{\phi} - \frac{\boldsymbol{\nabla}^2 \phi}{a^2} + V'(\phi) = 0}, \tag{104}$$

where $V'(\phi) = ({\rm d}V(\phi)/{\rm d}\phi)$. Note, in particular, the appearance of the friction term $3H\dot{\phi}$: a scalar field rolling down its potential suffers a friction due to the expansion of the universe. Using the relation

$$\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} = -\frac{1}{2}\sqrt{-g}\, g_{\mu\nu},$$

we can write the energy-momentum tensor of the scalar field

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} = \partial_\mu \phi \partial_\nu \phi + g_{\mu\nu}\, \mathcal{L}.$$

The corresponding energy density ρ_ϕ and pressure density P_ϕ are

$$\rho_\phi = T_{00} = \frac{\dot{\phi}^2}{2} + V(\phi) + \frac{(\nabla\phi)^2}{2a^2}, \tag{105}$$

$$P_\phi = \frac{T^i_{\ i}}{3} = \frac{\dot{\phi}^2}{2} - V(\phi) - \frac{(\nabla\phi)^2}{6a^2}. \tag{106}$$

Notice that, if the gradient term were dominant, we would obtain $P_\phi = -\rho_\phi/3$, not enough to drive inflation. We can now split the inflaton field in

$$\phi(t) = \phi_0(t) + \delta\phi(\mathbf{x}, t),$$

where ϕ_0 is the 'classical' (infinite wavelength) field, that is the expectation value of the inflaton field on the initial isotropic and homogeneous state, while $\delta\phi(\mathbf{x}, t)$ represents the quantum fluctuation around ϕ_0. As for now, we will be only concerned with the evolution of the classical field ϕ_0. This separation is justified by the fact that quantum fluctuations are much smaller than the classical value and therefore negligible when looking at the classical evolution. The energy-momentum tensor becomes

$$T_{00} = \rho_\phi = \frac{\dot{\phi_0}^2}{2} + V(\phi_0), \tag{107}$$

$$\frac{T^i_{\ i}}{3} = P_\phi = \frac{\dot{\phi}_0^2}{2} - V(\phi_0). \tag{108}$$

If

$$V(\phi_0) \gg \dot{\phi}_0^2$$

we obtain the following condition

$$P_\phi \simeq -\rho_\phi.$$

From this simple calculation, we realize that a scalar field whose energy is dominant in the universe and whose potential energy dominates over the kinetic term drives inflation. Inflation is driven by the vacuum energy of the inflaton field.

11.1 Slow-roll conditions

Let us now quantify better under which circumstances a scalar field may give rise to a period of inflation. The equation of motion of the classical value of the field is

$$\ddot{\phi}_0 + 3H\dot{\phi}_0 + V'(\phi_0) = 0. \tag{109}$$

If we require that $\dot{\phi}_0^2 \ll V(\phi_0)$, the scalar field is slowly rolling down its potential. This is the reason why such a period is called *slow-roll*. We may also expect that, being the potential flat, $\ddot{\phi}_0$ is negligible as well. We will assume that this is true and we will quantify this condition soon. The FRW equation becomes

$$H^2 \simeq \frac{8\pi G_\mathrm{N}}{3} V(\phi_0), \tag{110}$$

where we have assumed that the inflaton field dominates the energy density of the universe. The new equation of motion becomes

$$3H\dot{\phi}_0 = -V'(\phi_0), \tag{111}$$

which gives $\dot{\phi}_0$ as a function of $V'(\phi_0)$. Using Eq. (111) slow-roll conditions then require

$$\boxed{\dot{\phi}_0^2 \ll V(\phi_0) \Longrightarrow \frac{(V')^2}{V} \ll H^2}$$

and

$$\boxed{|\ddot{\phi}_0| \ll |3H\dot{\phi}_0| \Longrightarrow |V''| \ll H^2}.$$

It is now useful to define the slow-roll parameters, ϵ and η in the following way

$$\boxed{\begin{aligned}
\epsilon &= -\frac{\dot{H}}{H^2} = 4\pi G_{\rm N}\frac{\dot{\phi}_0^2}{H^2} = \frac{\dot{\phi}_0^2}{2\overline{M}_{\rm Pl}^2 H^2} = \frac{1}{16\pi G_{\rm N}}\left(\frac{V'}{V}\right)^2,\\
\eta &= \frac{1}{8\pi G_{\rm N}}\left(\frac{V''}{V}\right) = \frac{1}{3}\frac{V''}{H^2},\\
\delta &= \eta - \epsilon = -\frac{\ddot{\phi}_0}{H\dot{\phi}_0},
\end{aligned}}$$

where we have indicated by $\overline{M}_{\rm Pl}$ the reduced Planck mass,

$$\overline{M}_{\rm Pl}^2 = \frac{1}{8\pi G_{\rm N}} = \frac{{M_{\rm Pl}}^2}{8\pi}. \tag{112}$$

It might be useful to have the same parameters expressed in terms of conformal time

$$\boxed{\begin{aligned}
\epsilon &= 1 - \frac{\mathcal{H}'}{\mathcal{H}^2} = 4\pi G_{\rm N}\frac{\phi_0'^2}{\mathcal{H}^2},\\
\delta &= \eta - \epsilon = 1 - \frac{\phi_0''}{\mathcal{H}\phi_0'}\,.
\end{aligned}}$$

The parameter ϵ quantifies how much the Hubble rate H changes with time during inflation. Notice that, since

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 = (1-\epsilon)\,H^2,$$

inflation can be attained only if $\epsilon < 1$:

$$\boxed{\text{INFLATION} \quad \Longleftrightarrow \quad \epsilon < 1.}$$

As soon as this condition fails, inflation ends. In general, slow-roll inflation is attained if $\epsilon \ll 1$ and $|\eta| \ll 1$. During inflation the slow-roll parameters ϵ and η can be considered to be approximately constant since the potential $V(\phi)$ is very flat.

<u>Comment</u>: In the following, we will work at *first-order* perturbation in the slow-roll parameters, that is we will take only the first power of them. Since, using their definition, it is easy to see that $\dot{\epsilon}, \dot{\eta} = \mathcal{O}\left(\epsilon^2, \eta^2\right)$, this amounts to saying that we will treat the slow-roll parameters as constant in time.

Within these approximations, it is easy to compute the number of e-foldings between the beginning and the end of inflation. If we indicate by $\phi_{\rm i}$ and $\phi_{\rm f}$ the values of the inflaton field at the beginning and at the end of inflation, respectively, we have that the *total* number of e-foldings is

$$\begin{aligned} N &\equiv \int_{t_{\rm i}}^{t_{\rm f}} H\,{\rm d}t \\ &\simeq \int_{\phi_{\rm i}}^{\phi_{\rm f}} {\rm d}\phi_0 \frac{H}{\dot{\phi}_0} \\ &\simeq -3\int_{\phi_{\rm i}}^{\phi_{\rm f}} {\rm d}\phi_0 \frac{H^2}{V'} \\ &\simeq -8\pi G_{\rm N}\int_{\phi_{\rm i}}^{\phi_{\rm f}} \frac{V}{V'}\,{\rm d}\phi_0 \\ &= -\frac{1}{M_{\rm Pl}^2}\int_{\phi_{\rm i}}^{\phi_{\rm f}} \frac{V}{V'}\,{\rm d}\phi_0 \end{aligned} \tag{113}$$

We may also compute the number of e-foldings ΔN which are left to go to the end of inflation

$$\Delta N \simeq 8\pi G_{\rm N}\int_{\phi_{\rm f}}^{\phi_{\Delta N}} \frac{V}{V'}\,{\rm d}\phi_0, \tag{114}$$

where $\phi_{\Delta N}$ is the value of the inflaton field when there are ΔN e-foldings to the end of inflation.

1. *Comment*: A given scale length $\lambda = a/k$ leaves the Hubble radius when $k = aH_{\bf k}$ where $H_{\bf k}$ is the the value of the Hubble rate at that time. One can compute easily the rate of change of H_k^2 as a function of k

$$\frac{{\rm dln}\,H_{\bf k}^2}{{\rm dln}\,k} = \left(\frac{{\rm dln}\,H_{\bf k}^2}{{\rm d}t}\right)\left(\frac{{\rm d}t}{{\rm dln}\,a}\right)\left(\frac{{\rm dln}\,a}{{\rm dln}\,k}\right) = 2\frac{\dot{H}}{H}\times\frac{1}{H}\times 1 = 2\frac{\dot{H}}{H^2} = -2\epsilon. \tag{115}$$

2. *Comment*: Take a given physical scale λ today which crossed the Hubble radius during inflation. This happened when

$$\lambda\left(\frac{a_{\rm f}}{a_0}\right)e^{-\Delta N_\lambda} = \lambda\left(\frac{T_0}{T_{\rm f}}\right)e^{-\Delta N_\lambda} = H_{\rm I}^{-1},$$

where ΔN_λ indicates the number of e-foldings from the time the scale crossed the Hubble radius during inflation to the end of inflation. This relation gives a way to determine the number of e-foldings to the end of inflation corresponding to a given scale

$$\Delta N_\lambda \simeq 65 + \ln\left(\frac{\lambda}{3000\ {\rm Mpc}}\right) + 2\ln\left(\frac{V^{1/4}}{10^{14}\ {\rm GeV}}\right) - \ln\left(\frac{T_{\rm f}}{10^{10}\ {\rm GeV}}\right).$$

Scales relevant for the CMB anisotropies correspond to $\Delta N \sim$60.

12 The last stage of inflation and reheating

Inflation ended when the potential energy associated with the inflaton field became smaller than the kinetic energy of the oscillating field. The process by which the energy of the inflaton field is transferred from the inflaton field to radiation has been dubbed *reheating*. In the old theory of reheating [1, 22] the comoving energy density in the zero mode of the inflaton decays into normal particles. The latter then scatter and thermalize to form a thermal background. Of particular interest is a quantity known usually as the reheat temperature, denoted as $T_{\rm RH}$. It is calculated by assuming an instantaneous conversion of the energy density in the inflaton field into radiation. The decay happens when the width of the inflaton energy, Γ_ϕ, is equal to H, the expansion rate of the universe.

The reheat temperature is calculated quite easily. After inflation, the inflaton field executes coherent oscillations about the minimum of the potential at some $\phi_0 \simeq \phi_{\rm m}$

$$V(\phi_0) \simeq \frac{1}{2}V''(\phi_{\rm m})(\phi_0 - \phi_{\rm m})^2 \equiv \frac{1}{2}m^2(\phi_0 - \phi_{\rm m})^2. \tag{116}$$

Indeed, the equation of motion for ϕ_0 is

$$\ddot{\phi}_0 + 3H\dot{\phi}_0 + m^2(\phi_0 - \phi_{\rm m}) = 0, \tag{117}$$

whose solution is

$$\phi_0(t) = \phi_{\rm i}\left(\frac{a_{\rm i}}{a}\right)^{3/2} \cos\left[m(t - t_{\rm i})\right], \tag{118}$$

where now the label $_{\rm i}$ denotes here the beginning of the oscillations. Since the period of the oscillation is much shorter than the Hubble time, $H \gg m$, we can compute the equation satisfied by the energy density stored in the oscillating field averaged over many oscillations

$$\begin{aligned}
\langle \dot{\rho}_\phi \rangle &= \left\langle \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\dot{\phi}_0^2 + V(\phi_0)\right)\right\rangle_{\text{many oscillations}} \\
&= \left\langle \dot{\phi}_0\left(\ddot{\phi}_0 + V'(\phi_0)\right)\right\rangle_{\text{many oscillations}} \\
&= \left\langle \dot{\phi}_0\left(-3H\dot{\phi}_0\right)\right\rangle_{\text{many oscillations}} \\
&= -3H\left\langle \dot{\phi}_0^2\right\rangle_{\text{many oscillations}} \\
&= -3H\left\langle \rho_\phi\right\rangle_{\text{many oscillations}},
\end{aligned} \tag{119}$$

where we have used the equipartition property of the energy density during the oscillations $\langle \dot{\phi}_0^2/2\rangle = \langle V(\phi_0)\rangle = \langle \rho_\phi/2\rangle$ and Eq. (109). The solution of Eq. (119) is (removing the symbol of averaging)

$$\rho_\phi = (\rho_\phi)_{\rm i}\left(\frac{a_{\rm i}}{a}\right)^3. \tag{120}$$

The energy density of an oscillating scalar field scales like a non-relativistic fluid, the reason being that the effective averaged pressure vanishes:

$$\begin{aligned}
\left\langle P\right\rangle_{\text{many oscillations}} &= \left\langle \frac{1}{2}\dot{\phi}_0^2 - V(\phi_0)\right\rangle_{\text{many oscillations}} \\
&= \left\langle \frac{1}{2}\dot{\phi}_0^2\right\rangle_{\text{many oscillations}} - \left\langle V(\phi_0)\right\rangle_{\text{many oscillations}} = 0.
\end{aligned} \tag{121}$$

The Hubble expansion rate as a function of a is

$$H^2(a) = \frac{8\pi}{3}\frac{(\rho_\phi)_{\rm i}}{M_{\rm Pl}^2}\left(\frac{a_{\rm i}}{a}\right)^3. \tag{122}$$

Equating $H(a)$ and Γ_ϕ leads to an expression for $(a_{\rm i}/a)$. Now if we assume that all available coherent energy density is instantaneously converted into radiation at this value of $(a_{\rm i}/a)$, we can find the reheat temperature by setting the coherent energy density, $\rho_\phi = (\rho_\phi)_{\rm i}(a_{\rm i}/a)^3$, equal to the radiation energy density, $\rho_{\rm r} = (\pi^2/30)g_* T_{\rm RH}^4$, where g_* is the effective number of relativistic degrees of freedom at temperature $T_{\rm RH}$. The result is

$$\boxed{T_{\rm RH} = \left(\frac{90}{8\pi^3 g_*}\right)^{1/4}\sqrt{\Gamma_\phi M_{\rm Pl}} = 0.2\left(\frac{200}{g_*}\right)^{1/4}\sqrt{\Gamma_\phi M_{\rm Pl}}.} \tag{123}$$

In some models of inflation reheating can be anticipated by a period of preheating [39] when the the classical inflaton field very rapidly decays into ϕ-particles or into other bosons due to broad parametric resonance.

12.1 Preheating

In preheating there is a new decay channel that is non-perturbative: stimulated emissions of bosonic particles into energy bands with large occupation numbers are induced due to the coherent oscillations of the inflaton field. The oscillations of the inflaton field induce mixing of positive and negative frequencies in the quantum state of the field it couples to because of the *time-dependent* mass of the quantum field. Let us take, for the sake of simplicity, the case of a massive inflaton ϕ with quadratic potential $V(\phi) = \frac{1}{2}m^2\phi^2$, coupled to a massless scalar field χ via the quartic coupling $g^2\phi_0^2\chi^2$.

Neglecting the Hubble rate in the frequency term (being smaller than the time-dependent term), the evolution equation for the Fourier modes of the χ field with momentum $\mathbf{k}$ is

$$\ddot{X}_{\mathbf{k}} + \omega_k^2 X_{\mathbf{k}} = 0, \tag{124}$$

with

$$\begin{aligned} X_{\mathbf{k}} &= a^{3/2}(t)\chi_{\mathbf{k}}, \\ \omega_k^2 &= k^2/a^2(t) + g^2\phi_0^2(t). \end{aligned} \tag{125}$$

This Klein-Gordon equation may be cast in the form of a Mathieu equation

$$X_{\mathbf{k}}'' + [A(k) - 2q\cos 2z]X_{\mathbf{k}} = 0, \tag{126}$$

where $z = mt$ and

$$\begin{aligned} A(k) &= \frac{k^2}{a^2m^2} + 2q, \\ q &= g^2\frac{\Phi^2}{4m^2}, \end{aligned} \tag{127}$$

where Φ is the amplitude and m is the frequency of inflaton oscillations, $\phi_0(t) = \Phi(t)\sin(mt)$. Notice that, at least initially, if $\Phi \gg M_{\rm Pl}$

$$g^2\frac{\Phi^2}{4m^2} \gg g^2\frac{M_{\rm Pl}^2}{m^2} \tag{128}$$

can be extremely large. If so, the resonance is broad. For certain values of the parameters (A, q) there are exact solutions $X_{\mathbf{k}}$ and the corresponding number density $n_{\mathbf{k}}$ grows exponentially with time because they belong to an instability band of the Mathieu equation

$$X_{\mathbf{k}} \propto e^{\mu_{\mathbf{k}} m t} \Rightarrow n_{\mathbf{k}} \propto e^{2\mu_{\mathbf{k}} m t}, \tag{129}$$

where the parameter $\mu_{\mathbf{k}}$ depends upon the instability band and, in the broad resonance case, $q \gg 1$, it is ~ 0.2.

These instabilities can be interpreted as coherent "particle" production with large occupation numbers. One way of understanding this phenomenon is to consider the energy of these modes as that of a harmonic oscillator, $E_{\mathbf{k}} = |\dot{X}_{\mathbf{k}}|^2/2 + \omega_{\mathbf{k}}^2 |X_{\mathbf{k}}|^2/2 = \omega_{\mathbf{k}}(n_{\mathbf{k}} + 1/2)$. The occupation number of level $\mathbf{k}$ can grow exponentially fast, $n_{\mathbf{k}} \sim \exp(2\mu_{\mathbf{k}} m t) \gg 1$, and these modes soon behave like classical waves. The parameter q during preheating determines the strength of the resonance. It is important to notice that, after the short period of preheating, the universe is likely to enter a long period of matter domination where the biggest contribution to the energy density of the universe is provided by the residual small amplitude oscillations of the classical inflaton field and/or by the inflaton quanta produced during the back-reaction processes. This period will end when the age of the universe becomes of the order of the perturbative lifetime of the inflaton field, $t \sim \Gamma_\phi^{-1}$. At this point, the universe will be reheated up to a temperature $T_{\rm RH}$ obtained applying the old theory of reheating described previously.

13 A brief survey of inflationary models

Even restricting ourselves to a simple single-field inflation scenario, the number of models available to choose from is large. It is convenient to define a general classification scheme, or "zoology" , for models of inflation. We divide models into three general types: *large-field*, *small-field*, and *hybrid* [21]. A generic single-field potential can be characterized by two independent mass scales: a "height" Λ^4, corresponding to the vacuum energy density during inflation, and a "width" μ, corresponding to the change in the field value $\Delta\phi$ during inflation:

$$V(\phi) = \Lambda^4 f\left(\frac{\phi}{\mu}\right). \tag{130}$$

Different models have different forms for the function f. Let us now briefly describe the different class of models.

13.1 Large-field models

Large-field models are potentials typical of the "chaotic" inflation scenario [47], in which the scalar field is displaced from the minimum of the potential by an amount usually of order the Planck mass. Such models are characterized by $V''(\phi) > 0$, and $-\epsilon < \delta \leq \epsilon$. The generic large-field potentials we consider are polynomial potentials $V(\phi) = \Lambda^4 (\phi/\mu)^p$, and exponential potentials, $V(\phi) = \Lambda^4 \exp(\phi/\mu)$. In the chaotic inflation scenario, it is assumed that the universe emerged from a quantum gravitational state with an energy density comparable to that of the Planck density. This implies that $V(\phi) \approx M_{\rm Pl}^4$ and results in a large friction term in the Friedmann equation. Consequently, the inflaton will slowly roll down its potential. The condition for

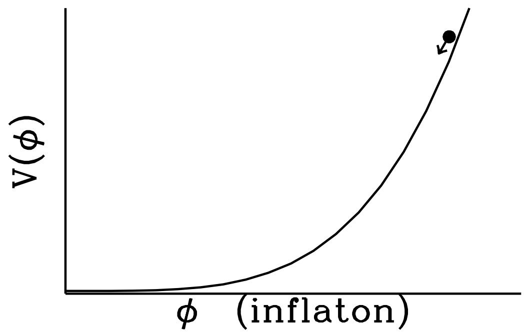

Figure 7: Large field models of inflation. From Ref. [41].

inflation is therefore satisfied and the scale factor grows as

$$a(t) = a_{\rm I} e^{\left(\int_{t_{\rm I}}^{t} {\rm d}t' H(t')\right)}. \tag{131}$$

The simplest chaotic inflation model is that of a free field with a quadratic potential, $V(\phi) = m^2\phi^2/2$, where m represents the mass of the inflaton. During inflation the scale factor grows as

$$a(t) = a_{\rm I} e^{2\pi G_{\rm N}(\phi_{\rm I}^2 - \phi^2(t))} \tag{132}$$

and inflation ends when $\phi = \mathcal{O}(M_{\rm Pl})$. If inflation begins when $V(\phi_{\rm i}) \approx M_{\rm Pl}^4$, the scale factor grows by a factor $\exp(4\pi M_{\rm Pl}^2/m^2)$ before the inflaton reaches the minimum of its potential. We will later show that the mass of the field should be $m \approx 10^{-6} M_{\rm Pl}$ if the microwave background constraints are to be satisfied. This implies that the volume of the universe will increase by a factor of $Z^3 \approx e^{3\times 10^{12}}$ and this is more than enough inflation to solve the problems of the hot big bang model.

In the chaotic inflationary scenarios, the present-day universe is only a small portion of the universe which suffered inflation. Notice also that the typical values of the inflaton field during inflation are of the order of $M_{\rm Pl}$, giving rise to the possibility of testing planckian physics [17].

13.2 Small-field models

Small-field models are the type of potentials that arise naturally from spontaneous symmetry breaking (such as the original models of "new" inflation [3, 46]) and from pseudo Nambu-Goldstone modes (natural inflation [25]). The field starts from near an unstable equilibrium (taken to be at the origin) and rolls down the potential to a stable minimum. Small-field models are characterized by $V''(\phi) < 0$ and $\eta < -\epsilon$. Typically ϵ is close to zero. The generic small-field potentials we consider are of the form $V(\phi) = \Lambda^4 \left[1 - (\phi/\mu)^p\right]$, which can be viewed as a lowest-order Taylor expansion of an arbitrary potential about the origin. See, for instance, Ref. [19].

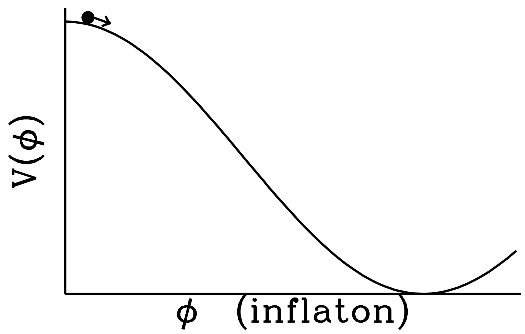

Figure 8: Small field models of inflation. From Ref. [41].

13.3 Hybrid models

The hybrid scenario [16, 48, 49] frequently appears in models which incorporate inflation into supersymmetry [61] and supergravity [50]. In a typical hybrid inflation model, the scalar field responsible for inflation evolves toward a minimum with nonzero vacuum energy. The end of inflation arises as a result of instability in a second field. Such models are characterized by $V''(\phi) > 0$ and $0 < \epsilon < \delta$. We consider generic potentials for hybrid inflation of the form $V(\phi) = \Lambda^4 \left[1 + (\phi/\mu)^p\right]$. The field value at the end of inflation is determined by some other physics, so there is a second free parameter characterizing the models.

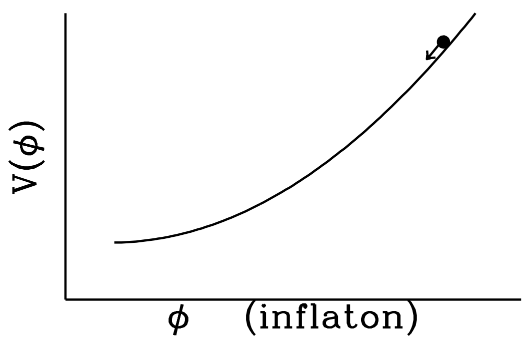

Figure 9: Hybrid field models of inflation. From Ref. [41].

This enumeration of models is certainly not exhaustive. There are a number of single-field models that do not fit well into this scheme, for example logarithmic potentials $V(\phi) \propto \ln(\phi)$ typical of supersymmetry [14, 23, 24, 32, 38, 51, 52, 62]. Another example is potentials with negative powers of the scalar field $V(\phi) \propto \phi^{-p}$ [7] used in intermediate inflation and dynamical supersymmetric inflation [35, 36]. Both of these cases require and auxiliary field to end inflation and are more properly categorized as hybrid models, but fall into the small-field class. However, the three classes categorized by the relationship between the slow-roll parameters as $-\epsilon < \delta \leq \epsilon$ (large-field), $\delta \leq -\epsilon$ (small-field) and $0 < \epsilon < \delta$ (hybrid) seems to be good enough for comparing theoretical expectations with experimental data.

Part IV

Inflation and the cosmological perturbations

As we have seen in the previous section, the early universe was made very nearly uniform by a primordial inflationary stage. However, the important caveat in that statement is the word 'nearly'. Our current understanding of the origin of structure in the universe is that it originated from small 'seed' perturbations, which over time grew to become all of the structure we observe. Once the universe becomes matter dominated, some seeds of the density inhomogeneities start growing thanks to the phenomenon of gravitational instabilities thus forming the structure we see today [57]. The gravitational instability is called Jeans instability.

The presence of the primordial inflationary seeds is also confirmed by detailed measurements of the CMB anisotropies; the temperature anisotropies at angular scales larger than 1° are caused by some inflationary inhomogeneities since causality prevents microphysical processes from producing anisotropies on angular scales larger than about 1°, the angular size of the horizon at last-scattering.

Our best guess for the origin of these perturbations is quantum fluctuations during an inflationary era in the early universe. Although originally introduced as a possible solution to the cosmological conundrums such as the horizon, flatness and entropy problems, by far the most useful property of inflation is that it generates spectra of both density perturbations and gravitational waves. These perturbations extend from extremely short scales to cosmological scales by the stretching of space during inflation.

Once inflation has ended, however, the Hubble radius increases faster than the scale factor, so the fluctuations eventually reenter the Hubble radius during the radiation- or matter-dominated eras. The fluctuations that exit around 60 e-foldings or so before reheating reenter with physical wavelengths in the range accessible to cosmological observations. These spectra provide a distinctive signature of inflation. They can be measured in a variety of different ways including the analysis of microwave background anisotropies.

The physical processes which give rise to the structures we observe today are well-explained in Fig. 10.

Since gravity talks to any component of the universe, small fluctuations of the inflaton field are intimately related to fluctuations of the space-time metric, giving rise to perturbations of the curvature $\mathcal{R}$ (which will be defined in the following; the reader may loosely think of it as a gravitational potential). The wavelengths λ of these perturbations grow exponentially and leave soon the Hubble radius when $\lambda > H^{-1}$. On super-Hubble scales, curvature fluctuations are frozen in and may be considered as classical. Finally, when the wavelength of these fluctuations reenters the horizon, at some radiation- or matter-dominated epoch, the curvature (gravitational potential) perturbations of the space-time give rise to matter (and temperature) perturbations via the Poisson equation. These fluctuations will then start growing giving rise to the structures we observe today.

In summary, two are the key ingredients for understanding the observed structures in the universe within the inflationary scenario:

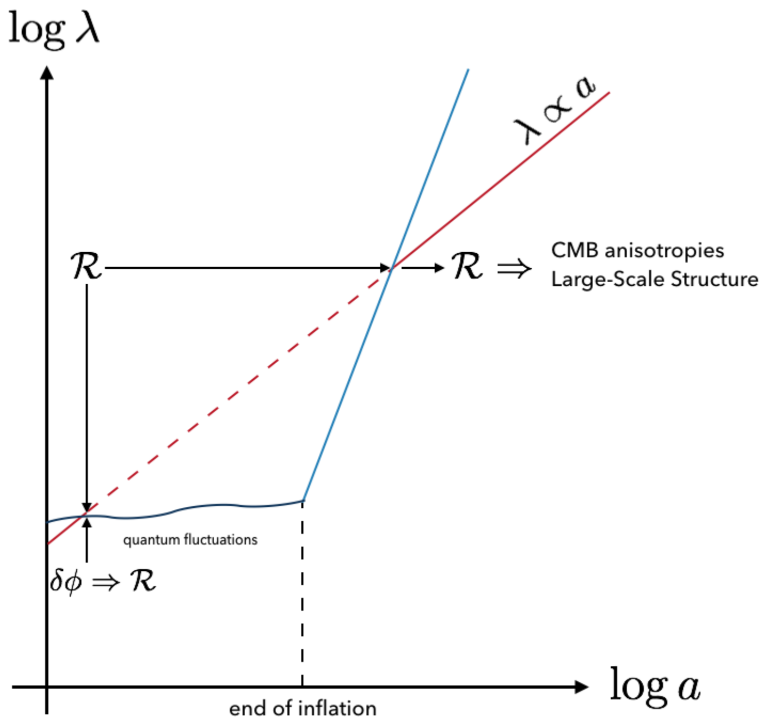

Figure 10: A schematic representation of the generation of quantum fluctuations during inflation. From Ref. [41].

- Quantum fluctuations of the inflaton field are excited during inflation and stretched to cosmological scales. At the same time, being the inflaton fluctuations connected to the metric perturbations through Einstein's equations, ripples on the metric are also excited and stretched to cosmological scales.
- Gravity acts a messenger since it communicates to baryons and photons the small seed perturbations once a given wavelength becomes smaller than the Hubble radius after inflation.

Let us now see how quantum fluctuations are generated during inflation. We will proceed by steps. First, we will consider the simplest problem of studying the quantum fluctuations of a generic scalar field during inflation: we will learn how perturbations evolve as a function of time and compute their spectrum. Then – since a satisfactory description of the generation of quantum fluctuations have to take both the inflaton and the metric perturbations into account – we will study the system composed by quantum fluctuations of the inflaton field and quantum fluctuations of the metric.

14 Quantum fluctuations of a generic massless scalar field during inflation

Let us first see how the fluctuations of a generic scalar field χ, which is *not* the inflaton field, behave during inflation. To warm up we first consider a de Sitter epoch during which the Hubble rate is constant.

14.1 Quantum fluctuations of a generic massless scalar field during a de Sitter stage

We assume this field to be massless. The massive case will be analyzed in the next subsection. Expanding the scalar field χ in Fourier modes

$$\delta\chi(\mathbf{x},t) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\, e^{i\mathbf{k}\cdot\mathbf{x}}\, \delta\chi_{\mathbf{k}}(t),$$

we can write the equation for the fluctuations as

$$\delta\ddot{\chi}_{\mathbf{k}} + 3H\,\delta\dot{\chi}_{\mathbf{k}} + \frac{k^2}{a^2}\,\delta\chi_{\mathbf{k}} = 0. \tag{133}$$

Let us study the qualitative behavior of the solution to Eq. (133).

- For wavelengths within the Hubble radius, $\lambda \ll H^{-1}$, the corresponding wavenumber satisfies the relation $k \gg a\,H$. In this regime, we can neglect the friction term $3H\,\delta\dot{\chi}_{\mathbf{k}}$ and Eq. (133) reduces to

$$\delta\ddot{\chi}_{\mathbf{k}} + \frac{k^2}{a^2}\,\delta\chi_{\mathbf{k}} = 0, \tag{134}$$

  which is – basically – the equation of motion of an harmonic oscillator. Of course, the frequency term k^2/a^2 depends upon time because the scale factor a grows exponentially. On the qualitative level, however, one expects that when the wavelength of the fluctuation is within the horizon, the fluctuation oscillates.

- For wavelengths above the Hubble radius, $\lambda \gg H^{-1}$, the corresponding wavenumber satisfies the relation $k \ll aH$ and the term k^2/a^2 can be safely neglected. Eq. (133) reduces to

$$\delta\ddot{\chi}_{\mathbf{k}} + 3H\,\delta\dot{\chi}_{\mathbf{k}} = 0, \tag{135}$$

  which tells us that on super-Hubble scales $\delta\chi_{\mathbf{k}}$ remains constant.

We have therefore the following picture: take a given fluctuation whose initial wavelength $\lambda \sim a/k$ is within the Hubble radius. The fluctuations oscillates till the wavelength becomes of the order of the horizon scale. When the wavelength crosses the Hubble radius, the fluctuation ceases to oscillate and gets frozen in.

Let us now study the evolution of the fluctuation in a more quantitative way. To do so, we perform the following redefinition

$$\delta\chi_{\mathbf{k}} = \frac{\delta\sigma_{\mathbf{k}}}{a}$$

and we work in conformal time $\mathrm{d}\tau = \mathrm{d}t/a$. For the time being, we solve the problem for a pure de Sitter expansion and we take the scale factor exponentially growing as $a \sim e^{Ht}$; the corresponding conformal factor reads (after choosing properly the integration constants)

$$a(\tau) = -\frac{1}{H\tau} \quad (\tau < 0).$$

In the following we will also solve the problem in the case of quasi de Sitter expansion. The beginning of inflation coincides with some initial time $\tau_{\mathrm{i}} \ll 0$. We find that Eq. (133) becomes

$$\delta\sigma_{\mathbf{k}}'' + \left(k^2 - \frac{a''}{a}\right)\delta\sigma_{\mathbf{k}} = 0. \tag{136}$$

We obtain an equation which is very 'close' to the equation for a Klein-Gordon scalar field in flat space-time, the only difference being a negative time-dependent mass term $-a''/a = -2/\tau^2$. Eq. (136) can be obtained from an action of the type

$$\delta S_{\mathbf{k}} = \int \mathrm{d}\tau \left[\frac{1}{2}\delta\sigma_{\mathbf{k}}'^2 - \frac{1}{2}\left(k^2 - \frac{a''}{a}\right)\delta\sigma_{\mathbf{k}}^2\right], \tag{137}$$

which is the canonical action for a simple harmonic oscillator with canonical commutation relations

$$\delta\sigma_{\mathbf{k}}^*\delta\sigma_{\mathbf{k}}' - \delta\sigma_{\mathbf{k}}\delta\sigma_{\mathbf{k}}^{*\prime} = -i. \tag{138}$$

The action (137) can be obtained also in a more straightforward way. The starting action for the scalar field χ in conformal time is

$$S = -\frac{1}{2}\int \mathrm{d}\tau \mathrm{d}^3x \, \sqrt{-g}\, g^{\mu\nu}\, \partial_\mu\delta\chi\partial_\nu\delta\chi = \frac{1}{2}\int \mathrm{d}\tau \mathrm{d}^3x\, a^2 \left[(\partial_\tau\delta\chi)^2 - (\partial_i\delta\chi)^2\right]. \tag{139}$$

We have

$$\delta\chi' \;=\; \frac{\delta\sigma'}{a} - \frac{\delta\sigma}{a^2}a', \tag{140}$$

and therefore

$$\begin{aligned} S &= \frac{1}{2}\int \mathrm{d}\tau \mathrm{d}^3x \left[(\partial_\tau\delta\sigma)^2 + \left(\frac{a'}{a}\right)^2\delta\sigma^2 - 2\frac{a'}{a}\partial_\tau\delta\sigma\delta\sigma - (\partial_i\delta\sigma)^2\right] \\ &= \frac{1}{2}\int \mathrm{d}\tau \mathrm{d}^3x \left[(\partial_\tau\delta\sigma)^2 + \left(\frac{a'}{a}\right)^2\delta\sigma^2 - \frac{a'}{a}\partial_\tau(\delta\sigma)^2 - (\partial_i\delta\sigma)^2\right] \\ &= \frac{1}{2}\int \mathrm{d}\tau \mathrm{d}^3x \left[(\partial_\tau\delta\sigma)^2 + \left(\frac{a'}{a}\right)^2\delta\sigma^2 + \partial_\tau\left(\frac{a'}{a}\right)\delta\sigma^2 - (\partial_i\delta\sigma)^2\right] \\ &= \frac{1}{2}\int \mathrm{d}\tau \mathrm{d}^3x \left[(\partial_\tau\delta\sigma)^2 + \left(\frac{a'}{a}\right)^2\delta\sigma^2 + \frac{a''}{a}\delta\sigma^2 - \left(\frac{a'}{a}\right)^2\delta\sigma^2 - (\partial_i\delta\sigma)^2\right] \\ &= \frac{1}{2}\int \mathrm{d}\tau \mathrm{d}^3x \left[(\partial_\tau\delta\sigma)^2 + \frac{a''}{a}\delta\sigma^2 - (\partial_i\delta\sigma)^2\right], \end{aligned} \tag{141}$$

which gives the equation of motion (136).

Let us study the behavior of this equation on sub-Hubble and super-Hubble scales. Since

$$\frac{k}{aH} = -k\,\tau,$$

on sub-Hubble scales $k^2 \gg a''/a$ Eq. (136) reduces to

$$\delta\sigma''_{\mathbf{k}} + k^2\,\delta\sigma_{\mathbf{k}} = 0,$$

whose solution is a plane wave

$$\delta\sigma_{\mathbf{k}} = A_1\frac{e^{-ik\tau}}{\sqrt{2k}} + A_2\frac{e^{-ik\tau}}{\sqrt{2k}} \quad (k \gg aH). \tag{142}$$

We find again that fluctuations with wavelength within the horizon oscillate exactly like in flat space-time. This does not come as a surprise. In the ultraviolet regime, that is for wavelengths much smaller than the Hubble radius scale, one expects that approximating the space-time as flat is a good approximation. To choice of the the coefficients A_1 and A_2 is equivalent to the choice of the vacuum and it is standard to choose the so-called Bunch-Davies vacuum, the zero-particle state as seen by a geodesic observer, that is, an observer who is in free fall in the expanding state. This corresponds to $A_1 = 1$ and $A_2 = 0$, therefore

$$\delta\sigma_{\mathbf{k}} = \frac{e^{-ik\tau}}{\sqrt{2k}} \quad (k \gg aH). \tag{143}$$

On super-Hubble scales, $k^2 \ll a''/a$ Eq. (136) reduces to

$$\delta\sigma''_{\mathbf{k}} - \frac{a''}{a}\delta\sigma_{\mathbf{k}} = 0,$$

which is satisfied by

$$\delta\sigma_{\mathbf{k}} = B(k)\,a \quad (k \ll aH). \tag{144}$$

where $B(k)$ is a constant of integration. Roughly matching the (absolute values of the) solutions (143) and (144) at $k = aH$ $(-k\tau = 1)$, we can determine the (absolute value of the) constant $B(k)$

$$|B(k)|\,a = \frac{1}{\sqrt{2k}} \Longrightarrow |B(k)| = \frac{1}{a\sqrt{2k}} = \frac{H}{\sqrt{2k^3}}.$$

Going back to the original variable $\delta\chi_{\mathbf{k}}$, we obtain that the quantum fluctuation of the χ field on super-Hubble scales is constant and approximately equal to

$$\boxed{|\delta\chi_{\mathbf{k}}| \simeq \frac{H}{\sqrt{2k^3}} \quad \text{(ON SUPER – HUBBLE SCALES).}}$$

In fact we can do much better, since Eq. (136) has an *exact* solution:

$$\delta\sigma_{\mathbf{k}} = \frac{e^{-ik\tau}}{\sqrt{2k}}\left(1 - \frac{i}{k\tau}\right). \tag{145}$$

This solution reproduces all what we have found by qualitative arguments in the two extreme regimes $k \ll aH$ and $k \gg aH$. The reason why we have performed the matching procedure is to show that the latter can be very useful to determine the behavior of the solution on super-Hubble scales when the exact solution is not known.

14.2 Quantum fluctuations of a generic massive scalar field during a de Sitter stage

So far, we have solved the equation for the quantum perturbations of a generic massless field, that is neglecting the mass squared term m_χ^2. Let us know discuss the solution when such a mass term is present. Eq. (136) becomes

$$\delta\sigma_{\mathbf{k}}'' + \left[k^2 + M^2(\tau)\right]\delta\sigma_{\mathbf{k}} = 0, \tag{146}$$

where

$$M^2(\tau) = \left(m_\chi^2 - 2H^2\right)a^2(\tau) = \frac{1}{\tau^2}\left(\frac{m^2}{H^2} - 2\right).$$

Eq. (146) can be recast in the form

$$\delta\sigma_{\mathbf{k}}'' + \left[k^2 - \frac{1}{\tau^2}\left(\nu_\chi^2 - \frac{1}{4}\right)\right]\delta\sigma_{\mathbf{k}} = 0, \tag{147}$$

where

$$\nu_\chi^2 = \left(\frac{9}{4} - \frac{m_\chi^2}{H^2}\right). \tag{148}$$

The generic solution to Eq. (146) for ν_χ *real* is

$$\delta\sigma_{\mathbf{k}} = \sqrt{-\tau}\left[c_1(k)\,H^{(1)}_{\nu_\chi}(-k\tau) + c_2(k)\,H^{(2)}_{\nu_\chi}(-k\tau)\right],$$

where $H^{(1)}_{\nu_\chi}$ and $H^{(2)}_{\nu_\chi}$ are the Hankel's functions of the first and second kind, respectively. If we impose again the Bunch-Davies vacuum, that is that in the ultraviolet regime $k \gg aH$ ($-k\tau \gg 1$) the solution matches the plane-wave solution $e^{-ik\tau}/\sqrt{2k}$, and knowing that

$$H^{(1)}_{\nu_\chi}(x \gg 1) \sim \sqrt{\frac{2}{\pi x}}\,e^{i\left(x - \frac{\pi}{2}\nu_\chi - \frac{\pi}{4}\right)} \quad , H^{(2)}_{\nu_\chi}(x \gg 1) \sim \sqrt{\frac{2}{\pi x}}\,e^{-i\left(x - \frac{\pi}{2}\nu_\chi - \frac{\pi}{4}\right)},$$

we set $c_2(k) = 0$ and $c_1(k) = \frac{\sqrt{\pi}}{2}\,e^{i\left(\nu_\chi + \frac{1}{2}\right)\frac{\pi}{2}}$. The exact solution becomes

$$\delta\sigma_{\mathbf{k}} = \frac{\sqrt{\pi}}{2}\,e^{i\left(\nu_\chi + \frac{1}{2}\right)\frac{\pi}{2}}\,\sqrt{-\tau}\,H^{(1)}_{\nu_\chi}(-k\tau). \tag{149}$$

On super-Hubble scales, since $H^{(1)}_{\nu_\chi}(x \ll 1) \sim \sqrt{2/\pi}\,e^{-i\frac{\pi}{2}}\,2^{\nu_\chi - \frac{3}{2}}\left(\Gamma(\nu_\chi)/\Gamma(3/2)\right)x^{-\nu_\chi}$, the fluctuation (149) becomes

$$\delta\sigma_{\mathbf{k}} = e^{i\left(\nu_\chi - \frac{1}{2}\right)\frac{\pi}{2}}\,2^{\left(\nu_\chi - \frac{3}{2}\right)}\,\frac{\Gamma(\nu_\chi)}{\Gamma(3/2)}\,\frac{1}{\sqrt{2k}}\,(-k\tau)^{\frac{1}{2} - \nu_\chi}.$$

Going back to the old variable $\delta\chi_{\mathbf{k}}$, we find that on super-Hubble scales, the fluctuation with nonvanishing mass is not exactly constant, but it acquires a tiny dependence upon the time

$$\boxed{|\delta\chi_{\mathbf{k}}| \simeq \frac{H}{\sqrt{2k^3}}\left(\frac{k}{aH}\right)^{\frac{3}{2} - \nu_\chi} \quad \text{(ON SUPER -- HUBBLE SCALES)}}$$

If we now define, in analogy with the definition of the slow roll parameters η and ϵ for the inflaton field, the parameter $\eta_\chi = (m_\chi^2/3H^2) \ll 1$, one finds

$$\frac{3}{2} - \nu_\chi \simeq \eta_\chi. \tag{150}$$

14.3 Quantum to classical transition

We have previously said that the quantum flactuations can be regarded as classical when their corresponding wavelengths cross the Hubble radius. To better motivate this statement, we should compute the number of particles $n_{\mathbf{k}}$ per wavenumber $\mathbf{k}$ on super-Hubble scales and check that it is indeed much larger than unity, $n_{\mathbf{k}} \gg 1$ (in this limit one can neglect the "quantum" factor 1/2 in the Hamiltonian $H_{\mathbf{k}} = \omega_{\mathbf{k}} \left(n_{\mathbf{k}} + \frac{1}{2}\right)$ where $\omega_{\mathbf{k}}$ is the energy eigenvalue). If so, the fluctuation can be regarded as classical. The number of particles $n_{\mathbf{k}}$ can be estimated to be of the order of $H_{\mathbf{k}}/\omega_{\mathbf{k}}$, where $H_{\mathbf{k}}$ is the Hamiltonian corresponding to the action

$$\delta S_{\mathbf{k}} = \int \mathrm{d}\tau \left[\frac{1}{2}\delta\sigma_{\mathbf{k}}'^2 + \frac{1}{2}\left(k^2 - M^2(\tau)\right)\delta\sigma_{\mathbf{k}}^2\right]. \tag{151}$$

One obtains on super-Hubble scales

$$n_{\mathbf{k}} \simeq \frac{\delta\sigma_{\mathbf{k}}'^2}{\omega_{\mathbf{k}}} \sim \left(\frac{H}{\sqrt{k^3}}\right)^2 \frac{a'^2}{k} \sim \left(\frac{k}{aH}\right)^{-4} \gg 1,$$

which confirms that fluctuations on super-Hubble scales may be indeed considered as classical.

14.4 The power spectrum

Let us define now the power spectrum, a useful quantity to characterize the properties of the perturbations. For a generic quantity $g(\mathbf{x}, t)$, which can expanded in Fourier space as

$$g(\mathbf{x}, t) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3}\, e^{i\mathbf{k}\cdot\mathbf{x}}\, g_{\mathbf{k}}(t),$$

the power spectrum can be defined as

$$\langle 0|g_{\mathbf{k}_1} g_{\mathbf{k}_2}|0\rangle \equiv (2\pi)^3 \delta^{(3)}\left(\mathbf{k}_1 + \mathbf{k}_2\right)\, |g_{\mathbf{k}}|^2, \tag{152}$$

where $|0\rangle$ is the vacuum quantum state of the system. This definition leads to the relation

$$\left\langle 0 \middle| g^2(\mathbf{x}, t) \middle| 0 \right\rangle = \int \frac{\mathrm{d}^3 k}{(2\pi)^3}\, g_{\mathbf{k}} g_{-\mathbf{k}} = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} |g_{\mathbf{k}}|^2 = \int \frac{\mathrm{d}k}{k}\, \mathcal{P}_g(k), \tag{153}$$

which defines the power spectrum of the perturbations of the field $g(\mathbf{x}, t)$ as

$$\boxed{\mathcal{P}_g(k) = \frac{k^3}{2\pi^2} |g_{\mathbf{k}}|^2}. \tag{154}$$

14.5 Quantum fluctuations of a generic scalar field in a quasi de Sitter stage

So far, we have computed the time evolution and the spectrum of the quantum fluctuations of a generic scalar field χ supposing that the scale factor evolves like in a pure de Sitter expansion, $a(\tau) = -1/(H\tau)$. However, during inflation the Hubble rate is not exactly constant, but changes with time as $\dot{H} = -\epsilon\, H^2$ (quasi de Sitter expansion), In this subsection, we will solve for the perturbations in a quasi de Sitter expansion. Using the definition of the conformal time, one can show that the scale factor for small values of ϵ becomes

$$a(\tau) = -\frac{1}{H}\frac{1}{\tau^{(1+\epsilon)}}.$$

Eq. (146) has now a squared mass term

$$M^2(\tau) = m_\chi^2 a^2 - \frac{a''}{a},$$

where

$$\frac{a''}{a} \simeq \frac{1}{\tau^2}\left(2 + 3\epsilon\right). \tag{155}$$

Taking $m_\chi^2/H^2 = 3\eta_\chi$ and expanding for small values of ϵ and η_χ we get Eq. (147) with

$$\nu_\chi \simeq \frac{3}{2} + \epsilon - \eta_\chi. \tag{156}$$

Armed with these results, we may compute the power spectrum of the fluctuations of the scalar field χ. Since we have seen that fluctuations are (nearly) frozen in on super-Hubble scales, a way of characterizing the perturbations is to compute the spectrum on scales larger than the Hubble radius

$$\boxed{\mathcal{P}_{\delta\chi}(k) \equiv \frac{k^3}{2\pi^2}\,|\delta\chi_{\mathbf{k}}|^2 = \left(\frac{H}{2\pi}\right)^2\left(\frac{k}{aH}\right)^{3-2\nu_\chi}.} \tag{157}$$

We may also define the *spectral index* $n_{\delta\chi}$ of the fluctuations as

$$\boxed{n_{\delta\chi} - 1 = \frac{d\ln\mathcal{P}_{\delta\phi}}{d\ln k} = 3 - 2\nu_\chi = 2\eta_\chi - 2\epsilon.}$$

The power spectrum of fluctuations of the scalar field χ is therefore *nearly flat*, that is is nearly independent from the wavelength $\lambda = \pi/k$: the amplitude of the fluctuation on super-Hubble scales does not (almost) depend upon the time at which the fluctuations crosses the Hubble radius and becomes frozen in. The small tilt of the power spectrum arises from the fact that the scalar field χ is massive and because during inflation the Hubble rate is not exactly constant, but nearly constant, where 'nearly' is quantified by the slow-roll parameters ϵ. Adopting the traditional terminology, we may say that the spectrum of perturbations is blue if $n_{\delta\chi} > 1$ (more power in the ultraviolet) and red if $n_{\delta\chi} < 1$ (more power in the infrared). The power spectrum of the perturbations of a generic scalar field χ generated during a period of slow roll inflation may be either blue or red. This depends upon the relative magnitude between η_χ and ϵ. For instance, in chaotic inflation with a quadric potential $V(\phi) = m^2\phi^2/2$, one can easily compute

$$n_{\delta\chi} - 1 = 2\eta_\chi - 2\epsilon = \frac{2}{3H^2}\left(m_\chi^2 - m^2\right),$$

which tells us that the spectrum is blue (red) if $m_\chi^2 > m_\phi^2$ ($m_\chi^2 < m^2$).

<u>*Comment*</u>*:* We might have computed the spectral index of the spectrum $\mathcal{P}_{\delta\chi}(k)$ by first solving the equation for the perturbations of the field χ in a di Sitter stage, with $H =$ constant and therefore $\epsilon = 0$,

and then taking into account the time-evolution of the Hubble rate introducing the subscript in $H_{\mathbf{k}}$ whose time variation is determined by Eq. (115). Correspondingly, $H_{\mathbf{k}}$ is the value of the Hubble rate when a given wavelength $\sim k^{-1}$ crosses the horizon (from that point on the fluctuations remains frozen in). The power spectrum in such an approach would read

$$\mathcal{P}_{\delta\chi}(k) = \left(\frac{H_{\mathbf{k}}}{2\pi}\right)^2 \left(\frac{k}{aH}\right)^{3-2\nu_\chi} \tag{158}$$

with $3 - 2\nu_\chi \simeq 2\eta_\chi$. Using Eq. (115), one finds

$$n_{\delta\chi} - 1 = \frac{\mathrm{d}\ln \mathcal{P}_{\delta\chi}}{\mathrm{d}\ln k} = \frac{\mathrm{d}\ln H_{\mathbf{k}}^2}{\mathrm{d}\ln k} + 3 - 2\nu_\chi = 2\eta_\chi - 2\epsilon,$$

which reproduces our previous findings.

<u>*Comment*</u>*:* Since on super-Hubble scales

$$\delta\chi_{\mathbf{k}} \simeq \frac{H}{\sqrt{2k^3}} \left(\frac{k}{aH}\right)^{\eta_\chi - \epsilon} \simeq \frac{H}{\sqrt{2k^3}} \left[1 + (\eta_\chi - \epsilon) \ln\left(\frac{k}{aH}\right)\right],$$

we discover that

$$|\delta\dot{\chi}_{\mathbf{k}}| \simeq |H\,(\eta_\chi - \epsilon)\ \delta\chi_{\mathbf{k}}| \ll |H\,\delta\chi_{\mathbf{k}}|\,, \tag{159}$$

that is on super-Hubble scales the time variation of the perturbations can be safely neglected.

15 Quantum fluctuations during inflation

As we have mentioned in the previous section, the linear theory of the cosmological perturbations represents a cornerstone of modern cosmology and is used to describe the formation and evolution of structures in the universe as well as the anisotropies of the CMB. The seeds were generated during inflation and stretched over astronomical scales because of the rapid superluminal expansion of the universe during the (quasi) de Sitter epoch.

In the previous section we have already seen that perturbations of a generic scalar field χ are generated during a (quasi) de Sitter expansion. The inflaton field is a scalar field and, as such, we conclude that inflaton fluctuations will be generated as well. However, the inflaton is special from the point of view of perturbations. The reason is very simple. By assumption, the inflaton field dominates the energy density of the universe during inflation. Any perturbation in the inflaton field means a perturbation of the stress energy-momentum tensor

$$\delta\phi \Longrightarrow \delta T_{\mu\nu}.$$

A perturbation in the stress energy-momentum tensor implies, through Einstein's equations of motion, a perturbation of the metric

$$\delta T_{\mu\nu} \Longrightarrow \left[\delta R_{\mu\nu} - \frac{1}{2}\delta\left(g_{\mu\nu}R\right)\right] = 8\pi G\delta T_{\mu\nu} \Longrightarrow \delta g_{\mu\nu}.$$

On the other hand, a pertubation of the metric induces a back reaction on the evolution of the inflaton perturbation through the perturbed Klein-Gordon equation of the inflaton field

$$\delta g_{\mu\nu} \Longrightarrow \delta\left(-\partial_\mu\partial^\mu\phi + \frac{\partial V}{\partial\phi}\right) = 0 \Longrightarrow \delta\phi.$$

This logic chain makes us conclude that the perturbations of the inflaton field and of the metric are tightly coupled to each other and have to be studied together

$$\boxed{\delta\phi \Longleftrightarrow \delta g_{\mu\nu}.}$$

As we will see shortly, this relation is stronger than one might think because of the issue of gauge invariance.

Before launching ourselves into the problem of finding the evolution of the quantum perturbations of the inflaton field when they are coupled to gravity, let us give a heuristic explanation of why we expect that during inflation such fluctuations are indeed present.

If we take Eq. (104) and split the inflaton field as its classical value ϕ_0 plus the quantum fluctuation $\delta\phi$, $\phi(\mathbf{x},t) = \phi_0(t) + \delta\phi(\mathbf{x},t)$, the quantum perturbation $\delta\phi$ satisfies the equation of motion

$$\delta\ddot{\phi} + 3H\,\delta\dot{\phi} - \frac{\nabla^2\delta\phi}{a^2} + V''\,\delta\phi = 0. \tag{160}$$

Differentiating Eq. (109) with respect to time and taking H constant (de Sitter expansion) we find

$$(\phi_0)^{\cdots} + 3H\ddot{\phi}_0 + V''\,\dot{\phi}_0 = 0. \tag{161}$$

Let us consider for simplicity the limit $k/a \ll H$ and let us disregard the gradient term. Under this condition we see that $\dot{\phi}_0$ and $\delta\phi$ solve the same equation. The solutions have therefore to be related to each other by a constant of proportionality which depends upon space only, that is

$$\delta\phi = -\dot{\phi}_0\,\delta t(\mathbf{x}). \tag{162}$$

This tells us that $\phi(\mathbf{x},t)$ will have the form

$$\boxed{\phi(\mathbf{x},t) = \phi_0\left(\mathbf{x}, t - \delta t(\mathbf{x})\right).}$$

This equation indicates that the inflaton field does not acquire the same value at a given time t in all the space. On the contrary, when the inflaton field is rolling down its potential, it acquires different values from one spatial point $\mathbf{x}$ to the other. The inflaton field is not homogeneous and fluctuations are present. These fluctuations, in turn, will induce fluctuations in the metric.

15.1 The metric fluctuations

The mathematical tool to describe the linear evolution of the cosmological perturbations is obtained by perturbing at the first-order the FRW metric $g^{(0)}_{\mu\nu}$,

$$g_{\mu\nu} \quad = \quad g^{(0)}_{\mu\nu}(t) \,+\, \delta g_{\mu\nu}(\mathbf{x},t)\,; \qquad \delta g_{\mu\nu} \,\ll\, g^{(0)}_{\mu\nu}\,. \tag{163}$$

The metric perturbations can be decomposed according to their spin with respect to a local rotation of the spatial coordinates on hypersurfaces of constant time. This leads to

- *scalar perturbations*,
- *vector perturbations*,
- *tensor perturbations*.

Tensor perturbations or gravitational waves have spin 2 and are the "true" degrees of freedom of the gravitational field in the sense that they can exist even in the vacuum. Vector perturbations are spin 1 modes arising from rotational velocity fields and are also called vorticity modes. Finally, scalar perturbations have spin 0.

Let us make a simple exercise to count how many scalar degrees of freedom are present. Take a space-time of dimensions $D = n + 1$, of which n coordinates are spatial coordinates. The symmetric metric tensor $g_{\mu\nu}$ has $\frac{1}{2}(n+2)(n+1)$ degrees of freedom. We can perform $(n+1)$ coordinate transformations in order to eliminate $(n+1)$ degrees of freedom, this leaves us with $\frac{1}{2}n(n+1)$ degrees of freedom. These $\frac{1}{2}n(n+1)$ degrees of freedom contain scalar, vector and tensor modes. According to Helmholtz's theorem we can always decompose a vector U_i $(i = 1, \cdots, n)$ as $U_i = \partial_i v + v_i$, where v is a scalar (usually called potential flow) which is curl-free, $v_{[i,j]} = 0$, and v_i is a real vector (usually called vorticity) which is divergence-free, $\nabla \cdot v = 0$. This means that the real vector (vorticity) modes are $(n-1)$. Furthermore, a generic traceless tensor Π_{ij} can always be decomposed as $\Pi_{ij} = \Pi^S_{ij} + \Pi^V_{ij} + \Pi^T_{ij}$, where $\Pi^S_{ij} = \left(-\frac{k_i k_j}{k^2} + \frac{1}{n}\delta_{ij}\right)\Pi$, $\Pi^V_{ij} = (-i/2k)\left(k_i\Pi_j + k_j\Pi_i\right)$ $(k_i\Pi_i = 0)$ and $k_i\Pi^T_{ij} = 0$. This means that the true symmetric, traceless and transverse tensor degrees of freedom are $\frac{1}{2}(n-2)(n+1)$.

The number of scalar degrees of freedom are therefore

$$\frac{1}{2}n(n+1) - (n-1) - \frac{1}{2}(n-2)(n+1) = 2,$$

while the degrees of freedom of true vector modes are $(n-1)$ and the number of degrees of freedom of true tensor modes (gravitational waves) are $\frac{1}{2}(n-2)(n+1)$. In four dimensions $n = 3$, meaning that one expects 2 scalar degrees of freedom, 2 vector degrees of freedom and 2 tensor degrees of freedom. As we shall see, to the two scalar degrees of freedom from the metric, one has to add an another one, the inflaton field perturbation $\delta\phi$. However, since Einstein's equations will tell us that the two scalar degrees of freedom from the metric are equal during inflation, we expect a total number of scalar degrees of freedom equal to 2.

At the linear order, the scalar, vector and tensor perturbations evolve independently (they decouple) and it is therefore possible to analyze them separately. Vector perturbations are not excited during inflation because there are no rotational velocity fields during the inflationary stage. We will analyze the generation of tensor modes (gravitational waves) in the following. For the time being we want to focus on the scalar degrees of freedom of the metric.

Considering only the scalar degrees of freedom of the perturbed metric, the most generic perturbed metric reads

$$g_{\mu\nu} \,=\, a^2 \begin{pmatrix} -(1\,+\,2\,\Phi) & \partial_i B \\ \partial_i B & (1\,-\,2\,\Psi)\,\delta_{ij}\,+\,D_{ij}E \end{pmatrix}, \tag{164}$$

while the line-element can be written as

$$\mathrm{d}s^2 \,=\, a^2\big[-(1+2\,\Phi)\mathrm{d}\tau^2 \,+\, 2\,\partial_i B\,\mathrm{d}\tau\,\mathrm{d}x^i \,+\, ((1-2\,\Psi)\delta_{ij}\,+\,D_{ij}E)\,\mathrm{d}x^i\,\mathrm{d}x^j\big], \tag{165}$$

where $D_{ij} \,=\big(\partial_i\partial_j \,-\, \frac{1}{3}\,\delta_{ij}\,\boldsymbol{\nabla}^2\big)$.

We now want to determine the inverse $g^{\mu\nu}$ of the metric at the linear order

$$g^{\mu\alpha}\,g_{\alpha\nu} \,=\, \delta^\mu_\nu. \tag{166}$$

We have therefore to solve the equations

$$\left(g^{\mu\alpha}_{(0)} \,+\, g^{\mu\alpha}\right)\left(g^{(0)}_{\alpha\nu} \,+\, g_{\alpha\nu}\right) \,=\, \delta^\mu_\nu\,, \tag{167}$$

where $g^{\mu\alpha}_{(0)}$ is simply the unperturbed FRW metric. Since

$$g^{\mu\nu}_{(0)} \,=\, \frac{1}{a^2}\begin{pmatrix} -1 & 0 \\ 0 & \delta^{ij} \end{pmatrix}, \tag{168}$$

we can write in general

$$\begin{aligned} g^{00} &= \frac{1}{a^2}\,(\,-1\,+\,X)\,; \\ g^{0i} &= \frac{1}{a^2}\,\partial^i Y\,; \\ g^{ij} &= \frac{1}{a^2}\,\big((1\,+\,2\,Z)\,\delta^{ij}\,+\,D^{ij}K\big)\,. \end{aligned} \tag{169}$$

Plugging these expressions into Eq. (167) we find for $\mu = \nu = 0$

$$(-1\,+\,X)(-1\,-\,2\,\Phi)\,+\,\partial^i Y\,\partial_i B \,=\, 1. \tag{170}$$

Neglecting the terms $-\,2\,\Phi\cdot X$ e $\partial^i Y\cdot\partial_i B$ because they are second-order in the perturbations, we find

$$1\,-\,X\,+\,2\,\Phi \,=\, 1 \qquad \Rightarrow \qquad X \,=\, 2\,\Phi\,. \tag{171}$$

Analogously, the components $\mu = 0$, $\nu = i$ of Eq. (167) give

$$(-1\,+\,2\,\Phi)(\partial_i B)\,+\,\partial^j Y\,[(1\,-\,2\,\Psi)\delta_{ji}\,+\,D_{ji}E] \,=\, 0. \tag{172}$$

At the first-order, we obtain

$$-\,\partial_i B\,+\,\partial_i Y \,=\, 0 \qquad \Rightarrow \qquad Y = B\,. \tag{173}$$

Finally, the components $\mu = i, \quad \nu = j$ give

$$\partial^i B\,\partial_j B\,+\,\big((1\,+\,2\,Z)\delta^{ik}\,+\,D^{ik}K\big)\,((1-2\,\Psi)\delta_{kj}\,+\,D_{kj}E) \,=\, \delta^i_j. \tag{174}$$

Neglecting the second-order terms, we obtain

$$(1\,-\,2\,\Psi\,+\,2\,Z)\delta^i_j\,+\,D^i_jE\,+\,D^i_jK \,=\, \delta^i_j \Rightarrow Z \,=\, \Psi\,; \qquad K \,=\, -\,E\,. \tag{175}$$

The metric $g^{\mu\nu}$ finally reads

$$g^{\mu\nu} \,=\, \frac{1}{a^2}\begin{pmatrix} -1\,+\,2\,\Phi & \partial^i B \\ \partial^i B & (1\,+\,2\,\Psi)\delta^{ij}\,-\,D^{ij}E \end{pmatrix}. \tag{176}$$

15.2 Perturbed affine connections and Einstein's tensor

In this subsection we provide the reader with the perturbed affine connections and Einstein's tensor. First, let us list the unperturbed affine connections

$$\Gamma^0_{00} \, = \, \frac{a'}{a}, \qquad \Gamma^i_{0j} \, = \, \frac{a'}{a}\,\delta^i_j, \qquad \Gamma^0_{ij} \, = \, \frac{a'}{a}\,\delta_{ij}, \tag{177}$$

$$\Gamma^i_{00} \quad = \quad \Gamma^0_{0i} \quad = \quad \Gamma^i_{jk} \quad = \quad 0\,. \tag{178}$$

The expression for the affine connections in terms of the metric is

$$\Gamma^\alpha_{\beta\gamma} \, = \, \frac{1}{2}\, g^{\alpha\rho} \left(\frac{\partial g_{\rho\gamma}}{\partial x^\beta} \, + \, \frac{\partial g_{\beta\rho}}{\partial x^\gamma} \, - \, \frac{\partial g_{\beta\gamma}}{\partial x^\rho} \right), \tag{179}$$

which implies

$$\begin{aligned} \delta\Gamma^\alpha_{\beta\gamma} \quad &= \quad \frac{1}{2}\,\delta g^{\alpha\rho} \left(\frac{\partial g_{\rho\gamma}}{\partial x^\beta} \, + \, \frac{\partial g_{\beta\rho}}{\partial x^\gamma} \, - \, \frac{\partial g_{\beta\gamma}}{\partial x^\rho} \right) \\ &+ \quad \frac{1}{2}\, g^{\alpha\rho} \left(\frac{\partial \delta g_{\rho\gamma}}{\partial x^\beta} \, + \, \frac{\partial \delta g_{\beta\rho}}{\partial x^\gamma} \, - \, \frac{\partial \delta g_{\beta\gamma}}{\partial x^\rho} \right), \end{aligned} \tag{180}$$

or in components

$$\delta\Gamma^0_{00} \, = \, \Phi', \tag{181}$$

$$\delta\Gamma^0_{0i} \, = \, \partial_i\,\Phi \, + \, \frac{a'}{a}\partial_i\,B, \tag{182}$$

$$\delta\Gamma^i_{00} \, = \, \frac{a'}{a}\,\partial^i B \, + \, \partial^i B' \, + \, \partial^i \Phi, \tag{183}$$

$$\delta\Gamma^0_{ij} = -\,2\,\frac{a'}{a}\,\Phi\,\delta_{ij} \, - \, \partial_i\partial_j B \, - \, 2\,\frac{a'}{a}\,\psi\,\delta_{ij} \, - \, \Psi'\,\delta_{ij} \, - \, \frac{a'}{a}\,D_{ij}E \, + \, \frac{1}{2}\,D_{ij}E',$$

$$\delta\Gamma^i_{0j} \, = \, -\,\Psi'\delta_{ij} \, + \, \frac{1}{2}\,D_{ij}E', \tag{184}$$

$$\delta\Gamma^i_{jk} = \partial_j\Psi\,\delta^i_k \, - \, \partial_k\Psi\,\delta^i_j \, + \, \partial^i\Psi\,\delta_{jk} \, - \, \frac{a'}{a}\,\partial^i B\,\delta_{jk} + \, \frac{1}{2}\,\partial_j D^i_k E \, + \, \frac{1}{2}\,\partial_k D^i_j E \, - \, \frac{1}{2}\,\partial^i D_{jk}E. \tag{185}$$

We may now compute the Ricci scalar, defined as

$$R_{\mu\nu} \, = \, \partial_\alpha\,\Gamma^\alpha_{\mu\nu} \, - \, \partial_\mu\,\Gamma^\alpha_{\nu\alpha} \, + \, \Gamma^\alpha_{\sigma\alpha}\,\Gamma^\sigma_{\mu\nu} \, - \, \Gamma^\alpha_{\sigma\nu}\,\Gamma^\sigma_{\mu\alpha}. \tag{186}$$

Its variation at the first-order reads

$$\begin{aligned} \delta R_{\mu\nu} \quad &= \quad \partial_\alpha\,\delta\Gamma^\alpha_{\mu\nu} \, - \, \partial_\mu\,\delta\Gamma^\alpha_{\nu\alpha} \, + \, \delta\Gamma^\alpha_{\sigma\alpha}\,\Gamma^\sigma_{\mu\nu} \, + \, \Gamma^\alpha_{\sigma\alpha}\,\delta\Gamma^\sigma_{\mu\nu} \\ &- \quad \delta\Gamma^\alpha_{\sigma\nu}\,\Gamma^\sigma_{\mu\alpha} \, - \, \Gamma^\alpha_{\sigma\nu}\,\delta\Gamma^\sigma_{\mu\alpha}. \end{aligned} \tag{187}$$

The background values are given by

$$R_{00} \, = \, -\,3\,\frac{a''}{a} \, + \, 3\left(\frac{a'}{a}\right)^2, \qquad R_{0i} \, = \, 0 \tag{188}$$

$$R_{ij} \, = \, \left(\frac{a''}{a} \, + \, \left(\frac{a'}{a}\right)^2\right)\delta_{ij}, \tag{189}$$

which give

$$\delta R_{00} = \frac{a'}{a}\partial_i\partial^i B + \partial_i\partial^i B' + \partial_i\partial^i\Phi + 3\Psi'' + 3\frac{a'}{a}\Psi' + 3\frac{a'}{a}\Phi', \tag{190}$$

$$\delta R_{0i} = \frac{a''}{a}\partial_i B + \left(\frac{a'}{a}\right)^2 \partial_i B + 2\partial_i \Psi' + 2\frac{a'}{a}\partial_i \Phi + \frac{1}{2}\partial_k D_i^K E', \tag{191}$$

$$\begin{aligned}
\delta R_{ij} &= \Big(- \frac{a'}{a}\Phi' - 5\frac{a'}{a}\psi' - 2\frac{a''}{a}\Phi - 2\left(\frac{a'}{a}\right)^2 \Phi \\
&- 2\frac{a''}{a}\Psi - 2\left(\frac{a'}{a}\right)^2 \Psi - \Psi'' + \partial_k\partial^k \Psi - \frac{a'}{a}\partial_k\partial^k B\Big)\delta_{ij} \\
&- \partial_i\partial_j B' + \frac{a'}{a}D_{ij}E' + \frac{a''}{a}D_{ij}E + \left(\frac{a'}{a}\right)^2 D_{ij}E \\
&+ \frac{1}{2}D_{ij}E'' + \partial_i\partial_j\Psi - \partial_i\partial_j\Phi - 2\frac{a'}{a}\partial_i\partial_j B \\
&+ \frac{1}{2}\partial_k\partial_i D_j^k E + \frac{1}{2}\partial_k\partial_j D_i^k E - \frac{1}{2}\partial_k\partial^k D_{ij}E,
\end{aligned} \tag{192}$$

The perturbation of the scalar curvature

$$R \,=\, g^{\mu\alpha}\, R_{\alpha\mu}\,, \tag{193}$$

for which the first-order perturbation is

$$\delta R \,=\, \delta g^{\mu\alpha}\, R_{\alpha\mu} \,+\, g^{\mu\alpha}\, \delta R_{\alpha\mu}\,. \tag{194}$$

The background value is

$$R \,=\, \frac{6}{a^2}\,\frac{a''}{a}, \tag{195}$$

while from Eq. (194) one finds

$$\begin{aligned}
\delta R &= \frac{1}{a^2}\Big(- 6\frac{a'}{a}\partial_i\partial^i B - 2\partial_i\partial^i B' - 2\partial_i\partial^i \Phi - 6\Psi'' \\
&- 6\frac{a'}{a}\Phi' - 18\frac{a'}{a}\Psi' - 12\frac{a''}{a}\Phi + 4\partial_i\partial^i\Psi + \partial_k\partial^i D_i^k E\Big).
\end{aligned} \tag{196}$$

Finally, we may compute the perturbations of the Einstein tensor

$$G_{\mu\nu} \,=\, R_{\mu\nu} \,-\, \frac{1}{2}\, g_{\mu\nu}\, R, \tag{197}$$

whose background components are

$$G_{00} \,=\, 3\left(\frac{a'}{a}\right)^2, \qquad G_{0i} \,=\, 0, \qquad G_{ij} \,=\, \left(-\, 2\,\frac{a''}{a} \,+\, \left(\frac{a'}{a}\right)^2 \right)\delta_{ij}. \tag{198}$$

At first-order, one finds

$$\delta G_{\mu\nu} \,=\, \delta R_{\mu\nu} \,-\, \frac{1}{2}\,\delta g_{\mu\nu}\, R \,-\, \frac{1}{2}\, g_{\mu\nu}\,\delta R\,, \tag{199}$$

or in components

$$\delta G_{00} = -2\frac{a'}{a}\,\partial_i\partial^i\, B - 6\frac{a'}{a}\,\Psi' + 2\,\partial_i\partial^i\,\Psi + \frac{1}{2}\,\partial_k\partial^i D_i^K E, \tag{200}$$

$$\delta G_{0i} = -2\frac{a''}{a}\,\partial_i B + \left(\frac{a'}{a}\right)^2 \,\partial_i B + 2\partial_i\,\Psi' + \frac{1}{2}\partial_k\, D_i^K E' + 2\,\frac{a'}{a}\,\partial_i\Phi, \tag{201}$$

$$
\begin{aligned}
\delta G_{ij} &= \Bigg(2\frac{a'}{a}\,\Phi' + 4\frac{a'}{a}\,\Psi' + 4\frac{a''}{a}\,\Phi - 2\left(\frac{a'}{a}\right)^2\Phi \\
&+ 4\frac{a''}{a}\,\Psi - 2\left(\frac{a'}{a}\right)^2\Psi + 2\Psi'' - \partial_k\partial^k\,\Psi \\
&+ 2\frac{a'}{a}\,\partial_k\partial^k B + \partial_k\partial^k B' + \partial_k\partial^k\Phi + \frac{1}{2}\partial_k\partial^m\,D^k_m\,E\Bigg)\delta_{ij} \\
&- \partial_i\partial_j B' + \partial_i\partial_j\Psi - \partial_i\partial_j A + \frac{a'}{a}\,D_{ij}E' - 2\,\frac{a''}{a}\,D_{ij}E \\
&+ \left(\frac{a'}{a}\right)^2 D_{ij}E + \frac{1}{2}D_{ij}E'' + \frac{1}{2}\partial_k\partial_i D^k_j E \\
&+ \frac{1}{2}\partial^k\partial_j D_{ik}E - \frac{1}{2}\partial_k\partial^k D_{ij}E - 2\,\frac{a'}{a}\partial_i\partial_j B.
\end{aligned} \tag{202}
$$

For convenience, we also give the expressions for the perturbations with one index up and one index down

$$
\begin{aligned}
\delta G^\mu_\nu &= \delta(g^{\mu\alpha}\,G_{\alpha\nu}) \\
&= \delta g^{\mu\alpha}\,G_{\alpha\nu} + g^{\mu\alpha}\,\delta G_{\alpha\nu},
\end{aligned} \tag{203}
$$

or in components

$$
\delta G^0_0 = \frac{1}{a^2}\left[6\left(\frac{a'}{a}\right)^2\Phi + 6\,\frac{a'}{a}\Psi' + 2\,\frac{a'}{a}\partial_i\,\partial^i B - 2\,\partial_i\,\partial^i\Psi - \frac{1}{2}\,\partial_k\partial^i\,D^K_i\,E\right], \tag{204}
$$

$$
\delta G^0_i = \frac{1}{a^2}\left[-2\,\frac{a'}{a}\partial_i\Phi - 2\,\partial_i\Psi' - \frac{1}{2}\,\partial_k D^K_i E'\right], \tag{205}
$$

$$
\begin{aligned}
\delta G^i_j &= \frac{1}{a^2}\Bigg[\Bigg(2\,\frac{a'}{a}\Phi' + 4\,\frac{a''}{a}\Phi - 2\left(\frac{a'}{a}\right)^2\Phi + \partial_i\,\partial^i\Phi + 4\,\frac{a'}{a}\Psi' + 2\,\Psi'' \\
&- \partial_i\,\partial^i\Psi + 2\,\frac{a'}{a}\partial_i\,\partial^i B + \partial_i\,\partial^i B' + \frac{1}{2}\partial_k\partial^m D^k_m E\Bigg)\delta^i_j \\
&- \partial^i\partial_j\Phi + \partial^i\partial_j\Psi - 2\,\frac{a'}{a}\partial^i\partial_j B - \partial^i\partial_j B' + \frac{a'}{a}D^i_j E' + \frac{1}{2}\,D^i_j E'' \\
&+ \frac{1}{2}\,\partial_k\partial^i\,D^k_j E + \frac{1}{2}\,\partial_k\partial_j\,D^{ik}E - \frac{1}{2}\,\partial_k\partial^k\,D^i_j E\Bigg].
\end{aligned} \tag{206}
$$

15.3 Perturbed stress energy-momentum tensor

As we have seen previously, the perturbations of the metric are induced by the perturbations of the stress energy-momentum tensor of the inflaton field

$$
T_{\mu\nu} = \partial_\mu\phi\,\partial_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}\,g^{\alpha\beta}\,\partial_\alpha\phi\,\partial_\beta\phi + V(\phi)\right), \tag{207}
$$

whose background values are (we are not going to put the subscript $_0$ any longer for the background quantities)

$$
\begin{aligned}
T_{00} &= \frac{1}{2}\,\phi'^2\, +\, V(\phi)\, a^2, \\
T_{0i} &= 0, \\
T_{ij} &= \left(\frac{1}{2}\,\phi'^2\, -\, V(\phi)\, a^2\right)\,\delta_{ij}.
\end{aligned} \tag{208}
$$

The perturbed stress energy-momentum tensor reads

$$
\begin{aligned}
\delta T_{\mu\nu} &= \partial_\mu \delta\phi\, \partial_\nu\phi\, +\, \partial_\mu\phi\, \partial_\nu\delta\phi\, -\, \delta g_{\mu\nu}\left(\frac{1}{2}\, g^{\alpha\beta}\, \partial_\alpha\phi\, \partial_\beta\phi\, +\, V(\phi)\right) \\
&- \; g_{\mu\nu}\left(\frac{1}{2}\delta g^{\alpha\beta}\, \partial_\alpha\phi\, \partial_\beta\phi\, +\, g^{\alpha\beta}\, \partial_\alpha\delta\phi\, \partial_\beta\phi\, +\, \frac{\partial V}{\partial\phi}\, \delta\phi\, +\, \frac{\partial V}{\partial\phi}\delta\phi\right).
\end{aligned} \tag{209}
$$

In components we have

$$
\delta T_{00} \,=\, \delta\phi'\, \phi'\, +\, 2\,\Phi\, V(\phi)\, a^2\, +\, a^2\, \frac{\partial V}{\partial\phi}\, \delta\phi, \tag{210}
$$

$$
\delta T_{0i} \,=\, \partial_i\, \delta\phi\, \phi'\, +\, \frac{1}{2}\, \partial_i B\, \phi'^2\, -\, \partial_i B\, V(\phi)\, a^2, \tag{211}
$$

$$
\begin{aligned}
\delta T_{ij} &= \left(\delta\phi'\, \phi'\, -\, \Phi\, \phi'^2\, -\, a^2\, \frac{\partial V}{\partial\phi}\, \delta\phi\, -\, \Psi\, \phi'^2\, +\, 2\,\Psi\, V(\phi)\, a^2\right)\delta_{ij} \\
&+ \; \frac{1}{2}\, D_{ij}E\, \phi'^2\, -\, D_{ij}E\, V(\phi)\, a^2.
\end{aligned} \tag{212}
$$

For convenience, we list the mixed components

$$
\begin{aligned}
\delta T^\mu_\nu &= \delta(g^{\mu\alpha}\, T_{\alpha\nu}) \\
&= \delta g^{\mu\alpha}\, T_{\alpha\nu}\, +\, g^{\mu\alpha}\, \delta T_{\alpha\nu},
\end{aligned} \tag{213}
$$

or

$$
\begin{aligned}
\delta T^0_0 &= \Phi\, \phi'^2\, -\, \delta\phi'\, \phi'\, -\, \delta\phi\, \frac{\partial V}{\partial\phi}\, a^2, \\
\delta T^i_0 &= \partial^i B\, \phi'^2\, +\, \partial^i\delta\phi\, \phi', \\
\delta T^0_i &= -\,\partial^i\delta\phi\, \phi', \\
\delta T^i_j &= \left(-\,\Phi\, \phi'^2\, +\, \delta\phi'\, \phi'\, -\, \delta\phi\, \frac{\partial V}{\partial\phi}\, a^2\right)\delta^i_j.
\end{aligned} \tag{214}
$$

15.4 Perturbed Klein-Gordon equation

The inflaton equation of motion is the Klein-Gordon equation of a scalar field under the action of its potential $V(\phi)$. The equation to perturb is therefore

$$
\frac{1}{\sqrt{-g}}\, \partial_\nu\left(\sqrt{-g}\, g^{\mu\nu}\, \partial_\nu\phi\right) \,=\, \frac{\partial V}{\partial\phi}, \tag{215}
$$

which at the zero-th order gives the inflaton equation of motion

$$
\phi''\, +\, 2\, \frac{a'}{a}\, \phi' \,=\, -\, \frac{\partial V}{\partial\phi}\, a^2. \tag{216}
$$

The variation of Eq. (215) is the sum of four different contributions corresponding to the variations of $\frac{1}{\sqrt{-g}}$, $\sqrt{-g}$, $g^{\mu\nu}$ and ϕ. For the variation of g we have

$$\delta g \;=\; g\, g^{\mu\nu}\delta g_{\nu\mu}, \tag{217}$$

which give at the linear order

$$\begin{aligned} \delta\sqrt{-g} \quad &= \quad -\frac{\delta g}{2\sqrt{-g}}, \\ \delta\frac{1}{\sqrt{-g}} \quad &= \quad \frac{\delta\sqrt{-g}}{g}. \end{aligned} \tag{218}$$

Plugging these results into the expression for the variation of Eq. (216)

$$\begin{aligned} \delta\partial_\mu\partial^\mu\,\phi \quad &= \quad -\,\delta\phi'' \,-\, 2\,\frac{a'}{a}\delta\phi' \,+\, \partial_i\,\partial^i\delta\phi \,+\, 2\,\Phi\,\phi'' \,+\, 4\,\frac{a'}{a}\Phi\,\phi' \,+\, \Phi'\phi' \\ &+ \quad 3\,\Psi'\phi' \,+\, \partial_i\,\partial^i B\,\phi' \\ &= \quad \delta\phi\,\frac{\partial^2 V}{\partial\phi^2}\,a^2. \end{aligned} \tag{219}$$

Using Eq. (216) to write

$$2\,\Phi\,\phi'' \,+\, 4\,\frac{a'}{a}\Phi\phi' \;=\; 2\,\Phi\,\frac{\partial V}{\partial\phi}a^2, \tag{220}$$

Eq. (219) becomes

$$\begin{aligned} \delta\phi'' \,+\, 2\,\frac{a'}{a}\delta\phi' \quad &- \quad \partial_i\,\partial^i\delta\phi \,-\, \Phi'\phi' \,-\, 3\,\Psi'\phi' \,-\, \partial_i\,\partial^i B\,\phi' \\ &= \quad -\delta\phi\,\frac{\partial^2 V}{\partial\phi^2}\,a^2 \,-\, 2\,\Phi\,\frac{\partial V}{\partial\phi}\,a^2. \end{aligned} \tag{221}$$

After having computed the perturbations at the linear order of the Einstein's tensor and of the stress energy-momentum tensor, we are ready to solve the perturbed Einstein's equations in order to quantify the inflaton and the metric fluctuations. We pause, however, for a moment in order to deal with the problem of gauge invariance.

15.5 The issue of gauge invariance

When studying the cosmological density perturbations, what we are interested in is to follow the evolution of a space-time which is neither homogeneous nor isotropic. This is done by following the evolution of the differences between the actual space-time and a well understood reference space-time. So we will consider small perturbations away from the homogeneous, isotropic space-time (see Fig. 11). The reference system in our case is the spatially flat Friedmann–Robertson–Walker space-time, with line element $\mathrm{d}s^2 = a^2(\tau)\left(-\mathrm{d}\tau^2 + \mathrm{d}\mathbf{x}^2\right)$. Now, the key issue is that general relativity is a gauge theory where the gauge transformations are the generic coordinate transformations from a local reference frame to another.

When we compute the perturbation of a given quantity, this is defined to be the difference between the value that this quantity assumes on the real physical space-time and the value it assumes on the unperturbed background. Nonetheless, to perform a comparison between these two values, it is necessary to compute the at the same space-time point. Since the two values "live" on two different geometries, it is necessary to specify

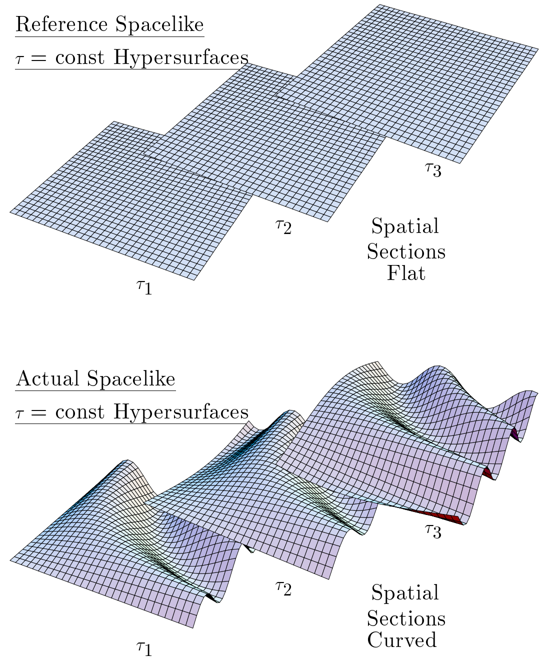

Figure 11: In the reference unperturbed universe, constant-time surfaces have constant spatial curvature (zero for a flat FRW model). In the actual perturbed universe, constant-time surfaces have spatially varying spatial curvature. From Ref. [41].

a map which allows to link univocally the same point on the two different space-times. This correspondence is called a gauge choice and changing the map means performing a gauge transformation.

Fixing a gauge in general relativity implies choosing a coordinate system. A choice of coordinates defines a *threading* of space-time into lines (corresponding to fixed spatial coordinates $\mathbf{x}$) and a *slicing* into hypersurfaces (corresponding to fixed time τ). A choice of coordinates is called a *gauge* and one should be aware that there is no preferred gauge

$$\boxed{\text{GAUGE CHOICE} \quad \Longleftrightarrow \quad \text{SLICING AND THREADING}}$$

Similarly, we can look at the change of coordinates either as an active transformation, in which we slightly alter the manifold or as a passive transformation, in which we do not alter the manifold, all the points remain fixed, and we just change the coordinate system. So this is tantamount to a relabeling of the points. From the passive point of view, in which a coordinate transformation represents a relabeling of the points of the space, one then compares a quantity, say the metric (or its perturbations), at a point P (with coordinates x^μ) with the new metric at the point P' which has the same values of the new coordinates as the point P had in the old coordinate system, $\widetilde{x^\mu}(P') = x^\mu(P)$. This is by the way an efficient way to detect symmetries (isometries if one is concerned with the metric), we only need to consider infinitesimal coordinate transformations.

From a more formal point of view, operating an infinitesimal gauge transformation on the coordinates

$$\widetilde{x^\mu} \;=\; x^\mu \;+\; \delta x^\mu \tag{222}$$

implies on a generic quantity Q a transformation on its perturbation

$$\widetilde{\delta Q} \;=\; \delta Q \;+\; \pounds_{\delta x}\, Q_0 \tag{223}$$

where Q_0 is the value assumed by the quantity Q on the background and $\pounds_{\delta x}$ is the Lie-derivative of Q along the vector δx^μ. Notice that for a scalar, the Lie derivative is just the ordinary directional derivative (and this is as it should be since saying that a function has a certain symmetry amounts to the assertion that its derivative in a particular direction vanishes).

Decomposing in the usual manner the vector δx^μ

$$\begin{aligned}
\delta x^0 &\;=\; \xi^0(x^\mu)\,; \\
\delta x^i &\;=\; \partial^i \beta(x^\mu) \;+\; v^i(x^\mu)\,; \qquad \partial_i v^i \;=\; 0\,,
\end{aligned} \tag{224}$$

we can easily deduce the transformation law of a scalar quantity f (like the inflaton scalar field ϕ and energy density ρ). Instead of applying the formal definition (223), we find the transformation law in an alternative (and more pedagogical) way. We first write $\delta f(x) = f(x) - f_0(x)$, where $f_0(x)$ is the background value. Under a gauge transformation we have $\widetilde{\delta f}(\widetilde{x^\mu}) = \widetilde{f}(\widetilde{x^\mu}) - \widetilde{f_0}(\widetilde{x^\mu})$. Since f is a scalar we can write $\widetilde{f}(\widetilde{x^\mu}) = f(x^\mu)$ (the value of the scalar function in a given physical point is the same in all the coordinate system). On the other side, on the unperturbed background hypersurface the background solution is the same, $\widetilde{f_0}(\widetilde{x^0}) = f_0(\widetilde{x^0})$. We have therefore

$$\begin{aligned}
\widetilde{\delta f}(\widetilde{x^\mu}) &\;=\; \widetilde{f}(\widetilde{x^\mu}) - \widetilde{f_0}(\widetilde{x^0}) \\
&\;=\; f(x^\mu) - f_0(\widetilde{x^0}) \\
&\;=\; f(x^\mu) - \delta x^0\, \frac{\partial f_0(x^0)}{\partial x^0} - f_0(x^0),
\end{aligned} \tag{225}$$

from which we finally deduce

$$\boxed{\widetilde{\delta f} = \delta f - f_0'\, \xi^0.}$$

For the spin zero perturbations of the metric, we can proceed analogously. We use the following trick. Upon a coordinate transformation $x^\mu \to \widetilde{x^\mu} = x^\mu + \delta x^\mu$, the line element is left invariant, $\mathrm{d}s^2 = \widetilde{\mathrm{d}s^2}$. This implies, for instance, that $a^2(\widetilde{x^0})\left(1 + 2\widetilde{\Phi}\right)\left(\widetilde{\mathrm{d}x^0}\right)^2 = a^2(x^0)\,(1+2\Phi)\,(\mathrm{d}x^0)^2$. Since $a^2(\widetilde{x^0}) \simeq a^2(x^0) + 2a\,a'\,\xi^0$ and $\widetilde{\mathrm{d}x^0} = \left(1 + \xi^{0\prime}\right)\mathrm{d}x^0 + \frac{\partial x^0}{\partial x^i}\,\mathrm{d}x^i$, we obtain $1 + 2\Phi = 1 + 2\widetilde{\Phi} + 2\,\mathcal{H}\xi^0 + 2\xi^{0\prime}$. A similar procedure leads to the following transformation laws

$$\boxed{\begin{aligned}
\widetilde{\Phi} &= \Phi - \xi^{0\prime} - \frac{a'}{a}\xi^0\,; \\
\widetilde{B} &= B + \xi^0 + \beta' \\
\widetilde{\Psi} &= \Psi - \frac{1}{3}\,\boldsymbol{\nabla}^2\beta + \frac{a'}{a}\xi^0\,; \\
\widetilde{E} &= E + 2\,\beta\,.
\end{aligned}}$$

The gauge problem stems from the fact that a change of the map (a change of the coordinate system) implies the variation of the perturbation of a given quantity which may therefore assume different values (all of them on a equal footing!) according to the gauge choice. To eliminate this ambiguity, one has therefore a double choice:

- Identify those combinations representing gauge invariant quantities;
- choose a given gauge and perform the calculations in that gauge.

Both options have advantages and drawbacks. Choosing a gauge may render the computation technically simpler with the danger, however, of including gauge artifacts, *i.e.* gauge freedoms which are not physical. Performing a gauge-invariant computation may be technically more involved, but has the advantage of treating only physical quantities.

Let us first indicate some gauge-invariant quantities. They are the so-called gauge invariant potentials or Bardeen's [6]

$$\Phi_{\rm GI} \,=\, \Phi - \frac{1}{a}\left[\left(-B + \frac{E'}{2}\right)a\right]'\,, \tag{226}$$

$$\Psi_{\rm GI} \,=\, \Psi + \frac{1}{6}\,\boldsymbol{\nabla}^2 E - \frac{a'}{a}\left(B - \frac{E'}{2}\right). \tag{227}$$

Analogously, one can define a gauge invariant quantity for the perturbation of the inflaton field. Since ϕ is a scalar field $\widetilde{\delta\phi} = \left(\delta\phi - \phi'\,\xi^0\right)$ and therefore

$$\boxed{\delta\phi_{\rm GI} \,=\, \delta\phi - \phi'\left(\frac{E'}{2} - B\right).}$$

is gauge-invariant. Analogously, one can define a gauge-invariant energy-density perturbation

$$\boxed{\delta\rho_{\rm GI} \,=\, \delta\rho - \rho'\left(\frac{E'}{2} - B\right).}$$

We now want to pause to introduce in details some gauge-invariant quantities which play a major role in the computation of the density perturbations. In the following we will be interested only in the coordinate transformations on constant time hypersurfaces and therefore gauge invariance will be equivalent to independence from the slicing.

15.6 The comoving curvature perturbation

The intrinsic spatial curvature of hypersurfaces on constant conformal time τ and for a flat universe is given by

$$^{(3)}R = \frac{4}{a^2}\boldsymbol{\nabla}^2\,\Psi.$$

The quantity Ψ is usually referred to as the *curvature perturbation.* We have seen, however, that the the curvature potential Ψ is *not* gauge invariant, but is defined only on a given slicing. Under a transformation on constant time hypersurfaces $\tau \to \tau + \xi^0$ (change of the slicing)

$$\Psi \to \Psi + \mathcal{H}\,\xi^0.$$

We now consider the *comoving slicing* which is defined to be the slicing orthogonal to the world lines of comoving observers. The latter are free-falling and the expansion defined by them is isotropic. In practice, what this means is that there is no flux of energy measured by these observers, that is $T_{0i} = 0$. During inflation this means that these observers measure $\delta\phi_{\rm com} = 0$ since T_{0i} goes like $\partial_i\delta\phi(\mathbf{x},\tau)\phi'(\tau)$.

Since $\delta\phi \to \delta\phi - \phi'\delta\xi^0$ for a transformation on constant time hypersurfaces, this means that

$$\delta\phi \to \delta\phi_{\rm com} = \delta\phi - \phi'\,\xi^0 = 0 \Longrightarrow \xi^0 = \frac{\delta\phi}{\phi'},$$

that is $\xi^0 = \frac{\delta\phi}{\phi'}$ is the time-displacement needed to go from a generic slicing with generic $\delta\phi$ to the comoving slicing where $\delta\phi_{\rm com} = 0$. At the same time the curvature pertubation Ψ transforms into

$$\Psi \to \Psi_{\rm com} = \Psi + \mathcal{H}\,\xi^0 = \Psi + \mathcal{H}\frac{\delta\phi}{\phi'}.$$

The quantity

$$\boxed{\mathcal{R} = \Psi + \mathcal{H}\frac{\delta\phi}{\phi'} = \Psi + H\frac{\delta\phi}{\dot{\phi}}}$$

is the *comoving curvature perturbation.* This quantity is gauge invariant by construction and is related to the gauge-dependent curvature perturbation Ψ on a generic slicing to the inflaton perturbation $\delta\phi$ in that gauge. By construction, the meaning of $\mathcal{R}$ is that it represents the gravitational potential on comoving hypersurfaces where $\delta\phi = 0$

$$\mathcal{R} = \left.\Psi\right|_{\delta\phi=0}.$$

15.7 The curvature perturbation on spatial slices of uniform energy density

We now consider the *slicing of uniform energy density* which is defined to be the slicing where there is no perturbation in the energy density, $\delta\rho = 0$.

Since $\delta\rho \to \delta\rho - \rho'\,\xi^0$ for a transformation on constant time hypersurfaces, this means that

$$\delta\rho \to \delta\rho_{\rm unif} = \delta\rho - \rho'\,\xi^0 = 0 \Longrightarrow \xi^0 = \frac{\delta\rho}{\rho'},$$

that is $\xi^0 = \frac{\delta\rho}{\rho'}$ is the time-displacement needed to go from a generic slicing with generic $\delta\rho$ to the slicing of uniform energy density where $\delta\rho_{\rm unif} = 0$. At the same time the curvature pertubation Ψ transforms into

$$\Psi \to \Psi_{\rm unif} = \Psi + \mathcal{H}\,\xi^0 = \Psi + \mathcal{H}\frac{\delta\rho}{\rho'}.$$

The quantity

$$\boxed{\zeta = \Psi + \mathcal{H}\frac{\delta\rho}{\rho'} = \Psi + H\frac{\delta\rho}{\dot\rho}}$$

is the *curvature perturbation on slices of uniform energy density.* This quantity is gauge invariant by construction and is related to the gauge-dependent curvature perturbation Ψ on a generic slicing and to the energy density perturbation $\delta\rho$ in that gauge. By construction, the meaning of ζ is that it represents the gravitational potential on slices of uniform energy density

$$\zeta = \left.\Psi\right|_{\delta\rho=0}.$$

Notice that, using the energy-conservation equation $\rho' + 3\,\mathcal{H}(\rho+P) = 0$, the curvature perturbation on slices of uniform energy density can be also written as

$$\zeta = \Psi - \frac{\delta\rho}{3(\rho+P)}.$$

During inflation $\rho + P = \dot\phi^2$. Furthermore, on super-Hubble scales from what we have learned in the previous section (and will be rigorously shown in the following) the inflaton fluctuation $\delta\phi$ is frozen in and $\delta\dot\phi =$ (slow roll parameters) $\times\, H\,\delta\phi$. This implies that $\delta\rho = \dot\phi\delta\dot\phi + V'\delta\phi \simeq V'\delta\phi \simeq -3H\dot\phi\delta\phi$, leading to

$$\zeta \simeq \Psi + \frac{3H\dot\phi}{3\dot\phi^2}\delta\phi = \Psi + H\frac{\delta\phi}{\dot\phi} = \mathcal{R} \qquad ({\rm ON\ SUPER-HUBBLE\ SCALES})$$

The comoving curvature pertubation and the curvature perturbation on uniform energy density slices are equal on super-Hubble scales during inflation.

15.8 Scalar field perturbations in the spatially flat gauge

We now consider the *spatially flat gauge* which is defined to be the the slicing where there is no curvature $\Psi_{\rm flat} = 0$.

Since $\Psi \to \Psi + \mathcal{H}\,\xi^0$ for a transformation on constant time hypersurfaces, this means that

$$\Psi \to \Psi_{\rm flat} = \Psi + \mathcal{H}\,\xi^0 = 0 \Longrightarrow \xi^0 = -\frac{\Psi}{\mathcal{H}},$$

that is $\xi^0 = -\Psi/\mathcal{H}$ is the time-displacement needed to go from a generic slicing with generic Ψ to the spatially flat gauge where $\Psi_{\rm flat} = 0$. At the same time the fluctuation of the inflaton field transforms as

$$\delta\phi \to \delta\phi - \phi'\,\xi^0 = \delta\phi + \frac{\phi'}{\mathcal{H}}\,\Psi.$$

The quantity

$$\boxed{Q = \delta\phi + \frac{\phi'}{\mathcal{H}}\,\Psi = \delta\phi + \frac{\dot\phi}{H}\Psi \equiv \frac{\dot\phi}{H}\,\mathcal{R}}$$

is the inflaton perturbation on spatially flat gauges. This quantity is often referred to as the Sasaki or Mukhanov variable [55, 66]. This quantity is gauge invariant by construction and is related to the inflaton perturbation $\delta\phi$ on a generic slicing and to the curvature perturbation Ψ in that gauge. By construction, the meaning of Q is that it represents the inflaton potential on spatially flat slices

$$Q = \left.\delta\phi\right|_{\Psi=0}.$$

Notice that $\delta\phi = -\phi'\delta\tau = -\dot\phi\delta t$ on flat slices, where $\delta t = \xi^0$ is the time displacement going from flat to comoving slices. This relation makes somehow rigorous the expression (162). Analogously, going from flat to comoving slices one has $\mathcal{R} = -H\,\delta t$.

15.9 Adiabatic and isocurvature perturbations

Let us comment now on the type of perturbation we may have. Arbitrary cosmological perturbations can be decomposed into:

- *adiabatic or curvature perturbations* are along the same trajectory in phase-space of the background solution. Given a generic scalar quantity X, its perturbations can be described by a unique perturbation in expansion with respect to the background

  $$H\,\delta t = H\,\frac{\delta X}{\dot X} \qquad \text{FOR EVERY } X.$$

  In particular, this holds for the energy density and the pressure

  $$\frac{\delta\rho}{\dot\rho} = \frac{\delta P}{\dot P}$$

  which implies that $P = P(\rho)$. This explains why they are called adiabatic. They are called curvature perturbations because a given time displacement δt causes the same relative change $\delta X/\dot X$ for all quantities. In other words the perturbations is democratically shared by all components of the universe.

- *isocurvature perturbations* perturb the solution away from the background solution

$$\frac{\delta X}{\dot{X}} \neq \frac{\delta Y}{\dot{Y}} \quad \text{FOR SOME } X \text{ AND } Y.$$

One can specify a generic isocurvature perturbation δX by giving its value on uniform-density slices, related to its value on a different slicing by the gauge-invariant definition

$$H \left.\frac{\delta X}{\dot{X}}\right|_{\delta\rho=0} = H\left(\frac{\delta X}{\dot{X}} - \frac{\delta\rho}{\dot{\rho}}\right).$$

For a set of fluids with energy density ρ_i, the isocurvature perturbations are conventionally defined by the gauge invariant quantities

$$\boxed{S_{ij} = 3H\left(\frac{\delta\rho_i}{\dot{\rho}_i} - \frac{\delta\rho_j}{\dot{\rho}_j}\right) = 3\left(\zeta_i - \zeta_j\right).}$$

One simple example of isocurvature perturbations is the baryon-to-photon ratio

$$S = \delta(n_{\rm b}/n_\gamma) = \frac{\delta n_{\rm b}}{n_{\rm b}} - \frac{\delta n_\gamma}{n_\gamma}. \tag{228}$$

1. Comment:

From the definitions above, it follows that the cosmological perturbations generated during inflation are of the adiabatic type *if* the inflaton field is the only field driving inflation. However, if inflation is driven by more than one field, isocurvature perturbations are expected to be generated (and they might even be cross-correlated to the adiabatic ones [8, 9, 11]). In the following we will give one example of the utility of generating isocurvature perturbations.

2. Comment: The perturbations generated during inflation are nearly *Gaussian*, *i.e.* the two-point correlation functions (like the power spectrum) suffice to define all the higher-order even correlation functions, while the odd correlation functions (such as the three-point correlation function) vanish. This conclusion is drawn by the very same fact that cosmological perturbations are studied *linearizing* Einstein's and Klein-Gordon equations. This turns out to be a good approximation because we know that the inflaton potential needs to be very flat in order to drive inflation and the interaction terms in the inflaton potential might be present, but they are small. Non-Gaussian features are therefore suppressed since the non-linearities of the inflaton potential are suppressed too. The same argument applies to the metric perturbations; non-linearities appear only at the second-order in deviations from the homogeneous background solution and are therefore small. This expectation is confirmed by a direct computation of the cosmological perturbations generated during inflation up to second-order in deviations from the homogeneous background solution which fully accounts for the inflaton self-interactions as well as for the second-order fluctuations of the background metric [2].

15.10 The next steps

After all these technicalities, it is useful to rest for a moment and to go back to physics. Up to now we have learned that during inflation quantum fluctuations of the inflaton field are generated and their wavelengths are stretched on large scales by the rapid expansion of the universe. We have also seen that the quantum fluctuations of the inflaton field are in fact impossible to disentangle from the metric perturbations. This happens not only because they are tightly coupled to each other through Einstein's equations, but also because of the issue of gauge invariance. Take, for instance, the gauge invariant quantity $Q = \delta\phi + \frac{\phi'}{\mathcal{H}}\,\Psi$. We can always go to a gauge where the fluctuation is entirely in the curvature potential Ψ, $Q = \frac{\phi'}{\mathcal{H}}\,\Psi$, or entirely in the inflaton field, $Q = \delta\phi$. However, as we have stressed at the end of the previous section, once ripples in the curvature are frozen in on super-Hubble scales during inflation, it is in fact gravity that acts as a messenger communicating to baryons and photons the small seeds of perturbations once a given scale reenters the horizon after inflation. This happens thanks to Newtonian physics; a small perturbation in the gravitational potential Ψ induces a small perturbation of the energy density ρ through Poisson's equation $\boldsymbol{\nabla}^2\Psi = 4\pi G_{\rm N}\delta\rho$. Similarly, if perturbations are adiabatic/curvature perturbations and, as such, treat democratically all the components, a ripple in the curvature is communicated to photons as well, giving rise to a nonvanishing $\delta T/T$.

These considerations make it clear that the next steps of these lectures will be

- Compute the curvature perturbation generated during inflation on super-Hubble scales. As we have seen we can either compute the comoving curvature perturbation $\mathcal{R}$ or the curvature on uniform energy density hypersurfaces ζ. They will tell us about the fluctuations of the gravitational potential.
- See how the fluctuations of the gravitational potential are transmitted to photons, baryons and matter in general.

We now intend to address the first point. As stressed previously, we are free to follow two alternative roads: either pick up a gauge and compute the gauge-invariant curvature in that gauge or perform a gauge-invariant calculation. We take both options.

15.11 Computation of the curvature perturbation using the longitudinal gauge

The longitudinal (or conformal newtonian) gauge is a convenient gauge to compute the cosmological perturbations. It is defined by performing a coordinate transformation such that $B = E = 0$. This leaves behind two degrees of freedom in the scalar perturbations, Φ and Ψ. As we have previously seen, these two degrees of freedom fully account for the scalar perturbations in the metric.

First of all, we take the non-diagonal part ($i \neq j$) of the (ij)-Einstein equation. Since the stress energy-momentum tensor does not have any non-diagonal component (no stress), we have

$$\partial_i\partial_j\left(\Psi - \Phi\right) = 0 \Longrightarrow \Psi = \Phi$$

and we can now work only with one variable, let it be Ψ. The ($0i$)-component of Einstein equation gives

$$\Psi' + \mathcal{H}\,\Psi = 4\pi G_{\rm N}\phi'\,\delta\phi = \epsilon\,\mathcal{H}^2\frac{\delta\phi}{\phi'}, \tag{229}$$

while the (00)- and the diagonal part (ii)-component ($i = j$) component of Einstein equations give respectively

$$3\,\mathcal{H}\,(\Psi' + \mathcal{H}\Psi) - \boldsymbol{\nabla}^2\Psi \;=\; -4\pi G_{\rm N}\left(\phi'\delta\phi' - \phi'^2\Psi + a^2 V'\delta\phi\right), \tag{230}$$

$$\left(2\frac{a''}{a} - \left(\frac{a'}{a}\right)^2\right)\Psi + 3\,\mathcal{H}\Psi' + \Psi'' \;=\; 4\pi G_{\rm N}\left(\phi'\delta\phi' - \phi'^2\Psi - a^2 V'\delta\phi\right). \tag{231}$$

If we now use the fact that $a''/a = (\mathcal{H}' + \mathcal{H}^2)$, sum the two equations above and use the background Klein-Gordon equation to eliminate V', we arrive at the equation for the gravitational potential

$$\Psi''_{\mathbf{k}} + 2\left(\mathcal{H} - \frac{\phi''}{\phi'}\right)\Psi'_{\mathbf{k}} + 2\left(\mathcal{H}' - \mathcal{H}\frac{\phi''}{\phi'}\right)\Psi_{\mathbf{k}} + k^2\,\Psi_{\mathbf{k}} = 0 \tag{232}$$

and in terms of the slow-roll parameters ϵ and η

$$\Psi''_{\mathbf{k}} + 2\,\mathcal{H}\,(\eta - \epsilon)\,\Psi'_{\mathbf{k}} + 2\,\mathcal{H}^2\,(\eta - 2\epsilon)\,\Psi_{\mathbf{k}} + k^2\,\Psi_{\mathbf{k}} = 0. \tag{233}$$

Using the same logic leading to Eq. (159), we can infer that on super-Hubble scales the gravitational potential Ψ is nearly constant (up to a mild logarithmic time-dependence proportional to slow-roll parameters), that is $\dot{\Psi}_{\mathbf{k}} \sim$ (slow-roll parameters)$\times\Psi_{\mathbf{k}}$. This is hardly surprising, we know that fluctuations are frozen in on super-Hubble scales.

Using Eq. (229), we can therefore relate the fluctuation of the gravitational potential Ψ to the fluctuation of the inflaton field $\delta\phi$ on super-Hubble scales

$$\Psi_{\mathbf{k}} \simeq \epsilon\,H\frac{\delta\phi_{\mathbf{k}}}{\dot{\phi}} \quad \text{(ON SUPER – HUBBLE SCALES)}. \tag{234}$$

This gives us the chance to compute the gauge-invariant comoving curvature perturbation $\mathcal{R}_{\mathbf{k}}$

$$\mathcal{R}_{\mathbf{k}} = \Psi_{\mathbf{k}} + H\,\frac{\delta\phi_{\mathbf{k}}}{\dot{\phi}} = (1+\epsilon)\,H\frac{\delta\phi_{\mathbf{k}}}{\dot{\phi}} \simeq H\frac{\delta\phi_{\mathbf{k}}}{\dot{\phi}}. \tag{235}$$

The power spectrum of the the comoving curvature perturbation $\mathcal{R}_{\mathbf{k}}$ then reads on super-Hubble scales

$$\mathcal{P}_{\mathcal{R}} = \frac{k^3}{2\pi^2}\frac{H^2}{\dot{\phi}^2}\left|\delta\phi_{\mathbf{k}}\right|^2 = \frac{k^3}{4\overline{M}^2_{\rm Pl}\epsilon\,\pi^2}\left|\delta\phi_{\mathbf{k}}\right|^2.$$

What is left to evaluate is the time evolution of $\delta\phi_{\mathbf{k}}$. To do so, we consider the perturbed Klein-Gordon equation (221) in the longitudinal gauge (in cosmic time)

$$\delta\ddot{\phi}_{\mathbf{k}} + 3H\delta\dot{\phi}_{\mathbf{k}} + \frac{k^2}{a^2}\delta\phi_{\mathbf{k}} + V''\delta\phi_{\mathbf{k}} = -2\Psi_{\mathbf{k}}V' + 4\dot{\Psi}_{\mathbf{k}}\dot{\phi}.$$

Since on super-Hubble scales $\left|4\dot{\Psi}_{\mathbf{k}}\dot{\phi}\right| \ll |\Psi_{\mathbf{k}}V'|$, using Eq. (234) and the relation $V' \simeq -3H\dot{\phi}$, we can rewrite the perturbed Klein-Gordon equation on super-Hubble scales as

$$\delta\ddot{\phi}_{\mathbf{k}} + 3H\delta\dot{\phi}_{\mathbf{k}} + \left(V'' - 6\epsilon H^2\right)\delta\phi_{\mathbf{k}} = 0.$$

We now introduce as usual the field $\delta\chi_{\mathbf{k}} = \delta\phi_{\mathbf{k}}/a$ and go to conformal time τ. The perturbed Klein-Gordon equation on super-Hubble scales becomes, using Eq. (155),

$$\begin{aligned}
\delta\chi''_{\mathbf{k}} &- \frac{1}{\tau^2}\left(\nu^2 - \frac{1}{4}\right)\delta\chi_{\mathbf{k}} = 0, \\
\nu^2 &= \frac{9}{4} + 9\epsilon - 3\eta, \\
\nu &\simeq \frac{3}{2} - \eta + 3\epsilon.
\end{aligned} \tag{236}$$

Using what we have learned in the previous section, we conclude that

$$|\delta\phi_{\mathbf{k}}| \simeq \frac{H}{\sqrt{2k^3}}\left(\frac{k}{aH}\right)^{\frac{3}{2}-\nu} \quad \text{(ON SUPER} - \text{HUBBLE SCALES)}$$

which justifies our initial assumption that both the inflaton perturbation and the gravitational potential are nearly constant on super-Hubble scale.

We may now compute the power spectrum of the comoving curvature perturbation on super-Hubble scales

$$\boxed{\mathcal{P}_{\mathcal{R}}(k) = \frac{4\pi G_{\rm N}}{\epsilon}\left(\frac{H}{2\pi}\right)^2\left(\frac{k}{aH}\right)^{n_{\mathcal{R}}-1} = \frac{1}{2M_{\rm Pl}^2\epsilon}\left(\frac{H}{2\pi}\right)^2\left(\frac{k}{aH}\right)^{n_{\mathcal{R}}-1} \equiv A_{\mathcal{R}}^2\left(\frac{k}{aH}\right)^{n_{\mathcal{R}}-1},}$$

where we have defined $A_{\mathcal{R}}^2$ as the amplitude of the comoving curvature perturbation and $n_{\mathcal{R}}$ its *spectral index*

$$\boxed{n_{\mathcal{R}} - 1 = \frac{{\rm d}\ln\mathcal{P}_{\mathcal{R}}}{{\rm d}\ln k} = 3 - 2\nu = 2\eta - 6\epsilon.}$$

We conclude that inflation is responsible for the generation of adiabatic/curvature perturbations with an almost scale-independent spectrum.

From the curvature perturbation we can easily deduce the behavior of the gravitational potential $\Psi_{\mathbf{k}}$ from Eq. (229). The latter is solved by

$$\Psi_{\mathbf{k}} = \frac{A(k)}{a} + \frac{4\pi G_{\rm N}}{a}\int^t {\rm d}t'\, a(t')\,\dot{\phi}(t')\,\delta\phi_{\mathbf{k}}(t') \simeq \frac{A(k)}{a} + \epsilon\,\mathcal{R}_{\mathbf{k}}.$$

We find that during inflation and on super-Hubble scales the gravitational potential is the sum of a decreasing function plus a nearly constant in time piece proportional to the curvature perturbation. Notice in particular that in an exact de Sitter stage, that is $\epsilon = 0$, the gravitational potential is not sourced and any initial condition in the gravitational potential is washed out as a^{-1} during the inflationary stage.

15.12 Computation of the curvature perturbation using the flat gauge

We might have computed the spectrum $\mathcal{P}_{\mathcal{R}}(k)$ by first solving the equation for the perturbation $\delta\phi_{\mathbf{k}}$ in the flat slice in a de Sitter stage, with $H = $ constant ($\epsilon = \eta = 0$) and then taking into account the time-evolution of the Hubble rate and of ϕ introducing the subscript in $H_{\mathbf{k}}$ and $\dot{\phi}_{\mathbf{k}}$. The time variation of the latter is determined by

$$\frac{\mathrm{d}\ln\dot{\phi}_{\mathbf{k}}}{\mathrm{d}\ln k} = \left(\frac{\mathrm{d}\ln\dot{\phi}_{\mathbf{k}}}{\mathrm{d}t}\right)\left(\frac{\mathrm{d}t}{\mathrm{d}\ln a}\right)\left(\frac{\mathrm{d}\ln a}{\mathrm{d}\ln k}\right) = \frac{\ddot{\phi}_{\mathbf{k}}}{\dot{\phi}_{\mathbf{k}}} \times \frac{1}{H} \times 1 = -\delta = \epsilon - \eta. \tag{237}$$

Correspondingly, $\dot{\phi}_{\mathbf{k}}$ is the value of the time derivative of the inflaton field when a given wavelength $\sim k^{-1}$ crosses the horizon (from that point on the fluctuations remains frozen in). Now, remember that in the flat gauge the Klein-Gordon equation reads

$$\delta\ddot{\phi}_{\mathbf{k}} + 3H\delta\dot{\phi}_{\mathbf{k}} + \frac{k^2}{a^2}\delta\phi_{\mathbf{k}} + V''\delta\phi_{\mathbf{k}} = -2\Phi_{\mathbf{k}}V' + \dot{\Phi}_{\mathbf{k}}\dot{\phi}. \tag{238}$$

It is clear that, up to slow-roll corrections, the scalar field perturbation in the flat gauge obeys the Klein-Gordon equation of a massless scalar field in de Sitter since the gravitational potential is switched on only when one is slightly away from de Sitter (for instance on super-Hubble scales the gravitational potential is given by $-2\Phi = \delta\rho/\rho \simeq (V'/V)\delta\phi$ and $\dot{\Phi} \simeq 0$). Therefore, the curvature perturbation in such an approach would read

$$\mathcal{R}_{\mathbf{k}} = \frac{H_{\mathbf{k}}}{\dot{\phi}_{\mathbf{k}}}\,\delta\phi_{\mathbf{k}} \simeq \frac{H_{\mathbf{k}}^2}{2\pi\dot{\phi}_{\mathbf{k}}},$$

which reproduces our previous findings. Correspondingly

$$n_{\mathcal{R}} - 1 = \frac{\mathrm{d}\ln\mathcal{P}_{\mathcal{R}}}{\mathrm{d}\ln k} = \frac{\mathrm{d}\ln H_{\mathbf{k}}^4}{\mathrm{d}\ln k} - \frac{\mathrm{d}\ln\dot{\phi}_{\mathbf{k}}^2}{\mathrm{d}\ln k} = -4\epsilon + (2\eta - 2\epsilon) = 2\eta - 6\epsilon.$$

During inflation the curvature perturbation is generated on super-Hubble scales with a spectrum which is nearly scale invariant, that is nearly independent from the wavelength $\lambda = \pi/k$: the amplitude of the fluctuation on super-Hubble scales does not (almost) depend upon the time at which the fluctuations crosses the Hubble radius and then becomes frozen in after a few Hubble times. The small tilt of the power spectrum arises from the fact that the inflaton field is massive, giving rise to a nonvanishing η and because during inflation the Hubble rate is not exactly constant, but nearly constant, where 'nearly' is quantified by the slow-roll parameters ϵ.

15.13 A proof of time-independence of the comoving curvature perturbation for adiabatic modes: linear level

From what have found so far, we may conclude that on super-Hubble scales the comoving curvature perturbation $\mathcal{R}$ and the uniform-density gauge curvature ζ satisfy on super-Hubble scales the relation

$$\mathcal{R}_{\mathbf{k}} \simeq \zeta_{\mathbf{k}}.$$

We now describe an argument showing that, in the presence of only the adiabatic mode, the curvature perturbation remains *exactly* constant on super-Hubble scales. The general argument follows from energy-momentum conservation [43].

Let us consider a generic fluid with energy-momentum tensor $T^{\mu\nu} = (\rho+P)u^\mu u^\nu + g^{\mu\nu}P$. The four-velocity u^μ is subject to the constraint $u^\mu u_\nu = -1$. Since it can be decomposed as

$$u^\mu = \frac{1}{a}\left(\delta_0^\mu + v^\mu\right), \tag{239}$$

we get

$$v^0 = -\Psi. \tag{240}$$

Similarly, we obtain

$$\begin{aligned} u_0 &= a(-1-\Phi), \\ u_i &= a v_i. \end{aligned} \tag{241}$$

Notice that, since we will work on super-Hubble scales we have only taken the gravitational potentials in the metric. The associated perturbation of the energy-momentum tensor is

$$\begin{aligned} \delta T_0^0 &= -(\delta\rho+\delta P) + (\bar\rho+\bar P)(1-\Psi)(-1-\Phi) + \delta P \simeq -\delta\rho, \\ \delta T_0^i &\simeq 0, \\ \delta T_j^i &= \delta P \delta_j^i, \end{aligned} \tag{242}$$

The associated continuity equation

$$\boldsymbol{\nabla}_\mu T_\nu^\mu = \partial_\mu T_\nu^\mu + \Gamma^\mu{}_{\mu\lambda} T_\nu^\lambda - \Gamma^\lambda{}_{\mu\nu} T_\lambda^\mu \tag{243}$$

gives

$$\begin{aligned} &\partial_0 T_0^0 + \partial_i T_0^i + \Gamma^\mu{}_{\mu\lambda} T_0^\lambda - \Gamma^\lambda{}_{\mu 0} T_\lambda^\mu \\ &= \partial_0 T_0^0 + \Gamma^\mu{}_{\mu\lambda} T_0^\lambda - \Gamma^\lambda{}_{\mu 0} T_\lambda^\mu \\ &= \partial_0 T_0^0 + \Gamma^\mu{}_{\mu 0} T_0^0 - \Gamma^\lambda{}_{00} T_\lambda^0 - \Gamma^\lambda{}_{i0} T_\lambda^i \\ &= \partial_0 T_0^0 + \Gamma^0{}_{00} T_0^0 + \Gamma^i{}_{i0} T_0^0 - \Gamma^0{}_{00} T_0^0 - \Gamma^j{}_{i0} T_j^i. \end{aligned} \tag{244}$$

This expression, using the Christoffel symbols in subsection 15.2 gives

$$\delta\dot\rho = -3H\left(\delta\rho + \delta P\right) + 3\dot\Psi\left(\bar\rho + \bar P\right).$$

We write $\delta P = \delta P_{\rm nad} + c_s^2\delta\rho$, where $\delta P_{\rm nad}$ is the non-adiabatic component of the perturbation of the pressure and $c_s^2 = \delta P_{\rm ad}/\delta\rho$ is the adiabatic one. In the uniform-density gauge $\Psi = \zeta$ and $\delta\rho = 0$ and therefore $\delta P_{\rm ad} = 0$. The energy conservation equation implies

$$\boxed{\dot\zeta = \frac{H}{\bar P + \bar\rho}\,\delta P_{\rm nad}.}$$

If perturbations are adiabatic, the curvature on uniform-density gauge is constant on super-Hubble scales

$$\boxed{\dot{\zeta} = 0 \text{ FOR ADIABATIC MODES ON SUPER} - \text{HUBBLE SCALES.}}$$

The same holds for the comoving curvature $\mathcal{R}$ as the latter and ζ are equal on super-Hubble scales. Let us check that this is true, *e.g.* in the longitudinal gauge. There we have

$$\mathcal{R}_{\mathbf{k}} \simeq \frac{H}{\dot{\phi}}\,\delta\phi_{\mathbf{k}} \propto \frac{H^2}{2\pi\dot{\phi}}(-\tau)^{\eta-3\epsilon}. \tag{245}$$

Therefore

$$\begin{aligned}
\frac{1}{H}\frac{\mathrm{d}\ln\mathcal{R}_{\mathbf{k}}}{\mathrm{d}t} &= \frac{1}{H}\frac{\mathrm{d}\ln H}{\mathrm{d}t} - \frac{1}{H}\frac{\mathrm{d}\ln\dot{\phi}}{\mathrm{d}t} + (\eta-3\epsilon)\frac{1}{H}\frac{\mathrm{d}\ln(-\tau)}{\mathrm{d}t} \\
&= -2\epsilon + (\eta-\epsilon) + (\eta-3\epsilon)\left(\frac{-1}{H\tau}\right)\frac{\mathrm{d}(-\tau)}{\mathrm{d}t} \\
&= -2\epsilon + (\eta-\epsilon) + (\eta-3\epsilon)\left(\frac{1}{H\tau}\right)\frac{1}{a} \\
&= -2\epsilon + (\eta-\epsilon) + (\eta-3\epsilon)\left(\frac{1}{H\tau}\right)(-H\tau) \\
&= -2\epsilon + (\eta-\epsilon) - (\eta-3\epsilon) \\
&= 0.
\end{aligned} \tag{246}$$

15.14 A proof of time-independence of the comoving curvature perturbation for adiabatic modes: all orders

We prove now that the comoving curvature perturbation is conserved at all orders in perturbation theory for adiabatic models on scales larger than the horizon [31]. To do so, at momenta $k \ll Ha$ the universe looks like a collection of separate almost homogeneous universes. We choose a threading of spatial coordinates comoving with the fluid

$$u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}t},\ v^i = \frac{u^i}{u^0} = \frac{\mathrm{d}x^i}{\mathrm{d}t} = 0. \tag{247}$$

The rate of the expansion is

$$\Theta = \boldsymbol{\nabla}_\mu u^\mu = \frac{1}{\mathcal{N}}\partial_0\, e^{3\alpha}, \tag{248}$$

where $g_{00} = \mathcal{N}^2$, $g_{ij} = e^{2\alpha}\gamma_{ij}$, with $\det\gamma_{ij} = 1$. The energy conservation equation

$$u_\nu \boldsymbol{\nabla}_\mu T^{\mu\nu} = 0 \Rightarrow \frac{\mathrm{d}}{\mathrm{d}\tau}\rho + (\rho+P)\Theta = 0, \tag{249}$$

where $\mathrm{d}t/\mathrm{d}\tau = u^0 = 1/\mathcal{N}$. Therefore, we obtain

$$\dot{\rho} + 3(\rho+P)\dot{\alpha} = 0. \tag{250}$$

Upon defining

$$a(t)e^{-\Psi} = e^{\alpha}, \tag{251}$$

we obtain

$$3\left(\frac{\dot a}{a} - \dot\Psi\right) = 3\dot\alpha = -\frac{\dot\rho}{\rho + P}. \tag{252}$$

This implies that the number of e-folds of expansion along an integral curve of the four-velocity comoving with the fluid is

$$N(t_2, t_1, x^i) = \frac{1}{3}\int_{\tau_1}^{\tau_2} \mathrm{d}\tau\,\Theta = \frac{1}{3}\int_{t_1}^{t_2} \mathrm{d}t\,\mathcal{N}\,\Theta = -\frac{1}{3}\int_{t_1}^{t_2} \mathrm{d}t\ \left.\frac{\dot\rho}{\rho + P}\right|_{x^i}. \tag{253}$$

This implies that

$$\Psi(t_2, x^i) - \Psi(t_1, x^i) = -N(t_2, t_1, x^i) + \ln\frac{a(t_2)}{a(t_1)}, \tag{254}$$

that is the change in Ψ from one slice to another equals the difference of the actual number of e-folds and the background. In particular, in a flat slice

$$N(t_2, t_1, x^i) = \ln\frac{a(t_2)}{a(t_1)}, \tag{255}$$

From (254) we find therefore

$$-\Psi(t_2, x^i) + \Psi(t_1, x^i) = -\frac{1}{3}\int_{\rho(t_1,x^i)}^{\rho(t_2,x^i)} \frac{\mathrm{d}\rho}{\rho + P} - \ln\frac{a(t_2)}{a(t_1)}. \tag{256}$$

If the perturbation are adiabatic, that is if $P = P(\rho)$, then we conclude that

$$\boxed{\zeta(x^i) = \Psi(t, x^i) - \frac{1}{3}\int_{\rho(t)}^{\rho(t,x^i)} \frac{\mathrm{d}\rho}{\rho + P}} \tag{257}$$

is constant and this holds at any order in perturbation theory. This is the non-linear generalization of the comoving curvature perturbation.

Consider now two different slices A and B which coincide at $t = t_1$. From (254) we have that

$$-N_A(t_2, t_1, x^i) + N_B(t_2, t_1, x^i) = \Psi_A(t_2, x^i) - \Psi_B(t_2, x^i). \tag{258}$$

Now, choose the slice A such that it is flat at $t = t_1$ and ends on a uniform energy slice at $t = t_2$ and B to be flat both at t_1 and t_2

$$\boxed{-\Psi_A(t_2, x^i) = N_A(t_2, t_1, x^i) - N_0(t_2, t_1) \equiv \delta N}, \tag{259}$$

since B is flat. This means that $-\Psi_A(t_2, x^i)$ is the difference in the number of e-folds (from $t = t_1$ to $t = t_2$) between the uniform-density slicing and the flat slicing. Therefore, by choosing the initial slice at the t_1 to be the flat slice and the slice at generic time t to have uniform energy density, the curvature perturbation on that slice is the difference in the number of e-folds between the uniform energy density slice and the flat slice from t_1 to t

$$\boxed{-\zeta = \delta N = \delta N(\phi(\mathbf{x},t)) \Rightarrow \zeta = \frac{\partial N}{\partial \phi}\delta\phi = \frac{\partial N}{\partial t}\frac{\delta\phi}{\dot{\phi}} = H\frac{\delta\phi}{\dot{\phi}}.} \tag{260}$$

This is indeed the easiest way of computing the comoving curvature perturbation and is dubbed the δN formalism. In general

$$\boxed{\zeta(x^i) = -\delta N - \frac{1}{3}\int_{\rho(t)}^{\rho(t,x^i)} \frac{\mathrm{d}\rho}{\rho + P}} \tag{261}$$

where δN must be interpreted ad the amount of expansion along the world line of a comoving observer from a spatially flat $\Psi = 0$ slice at time t_1 to a generic slice at time t.

15.15 Gauge-invariant computation of the curvature perturbation

In this subsection we would like to show how the computation of the curvature perturbation can be performed in a gauge-invariant way. We first rewrite Einstein's equations in terms of Bardeen's potentials (226) and (227)

$$\begin{aligned}
\delta G^0_0 &= \frac{2}{a^2}\left(-3\,\mathcal{H}\,(\mathcal{H}\,\Phi_{\rm GI} + \Psi'_{\rm GI}) + \boldsymbol{\nabla}^2\Psi_{\rm GI} + 3\,\mathcal{H}\left(-\mathcal{H}' + \mathcal{H}^2\right)\left(\frac{E'}{2} - B\right)\right), &(262)\\
\delta G^0_i &= \frac{2}{a^2}\,\partial_i\left(\mathcal{H}\,\Phi_{\rm GI} + \Psi'_{\rm GI} + \left(\mathcal{H}' - \mathcal{H}^2\right)\left(\frac{E'}{2} - B\right)\right), &(263)\\
\delta G^i_j &= -\frac{2}{a^2}\left(\left(\left(2\,\mathcal{H}' + 2\,\mathcal{H}^2\right)\Phi_{\rm GI} + \mathcal{H}\,\Phi'_{\rm GI} + \Psi''_{\rm GI} + 2\,\mathcal{H}\,\Psi'_{\rm GI} + \frac{1}{2}\,\boldsymbol{\nabla}^2 D_{\rm GI}\right)\delta^i_j\right.\\
&+ \left.\left(\mathcal{H}'' - \mathcal{H}\,\mathcal{H}' - \mathcal{H}^3\right)\left(\frac{E'}{2} - B\right)\delta^i_j - \frac{1}{2}\,\partial^i\partial_j D_{\rm GI}\right), &(264)
\end{aligned}$$

with $D_{\rm GI} = \Phi_{\rm GI} - \Psi_{\rm GI}$. These quantities are not gauge-invariant, but using the gauge transformations described previously, we can easily generalize them to gauge-invariant quantities

$$\begin{aligned}
\delta G^{({\rm GI})0}_0 &= \delta G^0_0 + (G^0_0)'\left(\frac{E'}{2} - B\right), &(265)\\
\delta G^{({\rm GI})0}_i &= \delta G^0_i + \left(G^0_i - \frac{1}{3}\,T^k_k\right)\partial_i\left(\frac{E'}{2} - B\right), &(266)\\
\delta G^{({\rm GI})i}_j &= \delta G^i_j + (G^i_j)'\left(\frac{E'}{2} - B\right) &(267)
\end{aligned}$$

and

$$\delta T_0^{(\mathrm{GI})0} \;=\; \delta T_0^0 \,+\, (T_0^0)' \left(\frac{E'}{2} \,-\, B\right) = -\delta\rho^{(\mathrm{GI})}\,, \tag{268}$$

$$\delta T_i^{(\mathrm{GI})0} \;=\; \delta T_i^0 \,+\, \left(T_i^0 \,-\, \frac{1}{3}\,T_k^k\right)\partial_i \left(\frac{E'}{2} \,-\, B\right) = \left(\bar\rho + \bar P\right) a^{-1} v_i^{(\mathrm{GI})}\,, \tag{269}$$

$$\delta T_j^{(\mathrm{GI})i} \;=\; \delta T_j^i \,+\, (T_j^i)' \left(\frac{E'}{2} \,-\, B\right) = \delta P^{(\mathrm{GI})}, \tag{270}$$

where we have written the stress energy-momentum tensor as $T^{\mu\nu} = (\rho + P)\,u^\mu u^\nu + P\eta^{\mu\nu}$ with $u^\mu = (1, v^i)$. Barred quantities are to be intended as background quantities. Einstein's equations can now be written in a gauge-invariant way

$$- \;\; 3\,\mathcal{H}\,(\mathcal{H}\,\Phi_{\mathrm{GI}} \,+\, \Psi'_{\mathrm{GI}}) \,+\, \boldsymbol{\nabla}^2\,\Psi_{\mathrm{GI}} \tag{271}$$

$$= \;\; 4\,\pi\,G_{\mathrm{N}}\left(-\Phi_{\mathrm{GI}}\,\phi'^2 \,+\, \delta\phi^{(\mathrm{GI})}\,\phi' \,+\, \delta\phi^{(\mathrm{GI})}\,\frac{\partial V}{\partial\phi}\,a^2\right)\,,$$

$$\partial_i\,\left(\mathcal{H}\,\Phi \,+\, \Psi'_{\mathrm{GI}}\right) \,=\, 4\,\pi\,G_{\mathrm{N}}\left(\partial_i\,\delta\phi^{(\mathrm{GI})}\,\phi'\right)\,,$$

$$\left(\left(2\,\mathcal{H}' \,+\, \mathcal{H}^2\right)\Phi_{\mathrm{GI}} \,+\, \mathcal{H}\,\Phi'_{\mathrm{GI}} \,+\, \Psi''_{\mathrm{GI}} \,+\, 2\,\mathcal{H}\,\Psi'_{\mathrm{GI}} \,+\, \frac{1}{2}\,\boldsymbol{\nabla}^2 D_{\mathrm{GI}}\right)\delta^i_j \,-\, \frac{1}{2}\partial^i\partial_j D_{\mathrm{GI}},$$

$$= \;\; -\,4\,\pi\,G_{\mathrm{N}}\left(\Phi_{\mathrm{GI}}\,\phi'^2 \,-\, \delta\phi^{(\mathrm{GI})}\,\phi' \,+\, \delta\phi^{(\mathrm{GI})}\,\frac{\partial V}{\partial\phi}\,a^2\right)\delta^i_j\,. \tag{272}$$

Taking $i \neq j$ from the third equation, we find $D_{\mathrm{GI}} = 0$, that is $\Psi_{\mathrm{GI}} = \Phi_{\mathrm{GI}}$ and from now on we can work with only the variable Φ_{GI}. Using the background relation

$$2\left(\frac{a'}{a}\right)^2 \,-\, \frac{a''}{a} \,=\, 4\,\pi\,G_{\mathrm{N}}\,\phi'^2 \tag{273}$$

we can rewrite the system of Eqs. (272) in the form

$$\begin{aligned}
\boldsymbol{\nabla}^2\,\Phi_{\mathrm{GI}} \,-\, 3\,\mathcal{H}\,\Phi'_{\mathrm{GI}} \,-\, \left(\mathcal{H}' \,+\, 2\,\mathcal{H}^2\right)\Phi_{\mathrm{GI}} \;&=\; 4\,\pi\,G_{\mathrm{N}}\left(\delta\phi^{(\mathrm{GI})}\,\phi' \,+\, \delta\phi^{(\mathrm{GI})}\,\frac{\partial V}{\partial\phi}\,a^2\right)\,;\\
\Phi'_{\mathrm{GI}} \,+\, \mathcal{H}\,\Phi_{\mathrm{GI}} \;&=\; 4\,\pi\,G_{\mathrm{N}}\left(\delta\phi^{(\mathrm{GI})}\,\phi'\right)\,;\\
\Phi''_{\mathrm{GI}} \,+\, 3\,\mathcal{H}\,\Phi'_{\mathrm{GI}} \,+\, \left(\mathcal{H}' \,+\, 2\,\mathcal{H}^2\right)\Phi_{\mathrm{GI}} \;&=\; 4\,\pi\,G_{\mathrm{N}}\left(\delta\phi^{(\mathrm{GI})}\,\phi' \,-\, \delta\phi^{(\mathrm{GI})}\,\frac{\partial V}{\partial\phi}\,a^2\right)\,.
\end{aligned} \tag{274}$$

Subtracting the first equation from the third, using the second equation to express $\delta\phi^{(\mathrm{GI})}$ as a function of Φ_{GI} and Φ'_{GI} and using the Klein-Gordon equation one finally finds the

$$\Phi''_{\mathrm{GI}} \,+\, 2\left(\mathcal{H} \,-\, \frac{\phi''}{\phi'}\right)\Phi'_{\mathrm{GI}} \,-\, \boldsymbol{\nabla}^2\,\Phi_{\mathrm{GI}} \,+\, 2\left(\mathcal{H}' \,-\, \mathcal{H}\,\frac{\phi''}{\phi'}\right)\Phi_{\mathrm{GI}} \,=\, 0\,, \tag{275}$$

for the gauge-invariant potential Φ_{GI}. We now introduce the gauge-invariant quantity

$$u \;\equiv\; a\,\delta\phi^{(\mathrm{GI})} \,+\, z\,\Psi_{\mathrm{GI}}\,, \tag{276}$$

$$z \;\equiv\; a\,\frac{\phi'}{\mathcal{H}} = a\,\frac{\dot\phi}{H}. \tag{277}$$

Notice that the variable u is equal to $-a\,Q$, the gauge-invariant inflaton perturbation on spatially flat gauges. Eq. (275) becomes

$$u'' \,-\, \boldsymbol{\nabla}^2\,u \,-\, \frac{z''}{z}\,u \,=\, 0\,, \tag{278}$$

while the two remaining equations of the system (274) can be written as

$$\boldsymbol{\nabla}^2\,\Phi_{\rm GI} \quad = \quad 4\,\pi\,G_{\rm N}\,\frac{\mathcal{H}}{a^2}\,(z\,u' \,-\, z'u)\;, \tag{279}$$

$$\left(\frac{a^2\,\Phi_{\rm GI}}{\mathcal{H}}\right)' \quad = \quad 4\,\pi\,G_{\rm N}\,z\,u\,, \tag{280}$$

which allow to determine the variables Φ and $\delta\phi^{\rm (GI)}$.

We have now to solve Eq. (278). First, we have to evaluate z''/z in terms of the slow-roll parameters

$$\frac{z'}{\mathcal{H}z} = \frac{a'}{\mathcal{H}a} + \frac{\phi''}{\mathcal{H}\phi'} - \frac{\mathcal{H}'}{\mathcal{H}^2} = \epsilon + \frac{\phi''}{\mathcal{H}\phi'}.$$

We then deduce that

$$\delta \equiv 1 - \frac{\phi''}{\mathcal{H}\phi'} = 1 + \epsilon - \frac{z'}{\mathcal{H}z}.$$

Keeping the slow-roll parameters constat in time (as we have mentioned, this corresponds to expand all quantities to first-order in the slow-roll parameters), we find

$$0 \simeq \delta' = \epsilon'(\simeq 0) - \frac{z''}{\mathcal{H}z} + \frac{z'\,\mathcal{H}'}{z\,\mathcal{H}^2} + \frac{(z')^2}{\mathcal{H}z^2},$$

from which we deduce

$$\frac{z''}{z} \simeq \frac{z'\,\mathcal{H}'}{z\,\mathcal{H}} + \frac{(z')^2}{z^2}.$$

Expanding in slow-roll parameters we find

$$\frac{z''}{z} \simeq (1+\epsilon-\delta)\,(1-\epsilon)\;\mathcal{H}^2 + (1+\epsilon-\delta)^2\;\mathcal{H}^2 \,\simeq\, \mathcal{H}^2\,(2+2\epsilon-3\delta)\,.$$

If we set

$$\frac{z''}{z} = \frac{1}{\tau^2}\left(\nu^2 - \frac{1}{4}\right),$$

this corresponds to

$$\nu \simeq \frac{1}{2}\left[1 + 4\frac{(1+\epsilon-\delta)\,(2-\delta)}{(1-\epsilon)^2}\right]^{1/2} \simeq \frac{3}{2} + (2\epsilon-\delta) \simeq \frac{3}{2} + 3\epsilon - \eta.$$

On sub-Hubble scales ($k \gg aH$), the solution of equation (278) is obviously $u_{\mathbf{k}} \simeq e^{-ik\tau}/\sqrt{2k}$. Rewriting Eq. (280) as

$$\Phi_{\mathbf{k}}^{\mathrm{GI}} = -\frac{4\pi G_{\mathrm{N}} a^2}{k^2} \frac{\dot{\phi}^2}{H} \left(\frac{H}{a\dot{\phi}} u_{\mathbf{k}} \right)^{\cdot},$$

we infer that on sub-Hubble scales

$$\Phi_{\mathbf{k}}^{\mathrm{GI}} \simeq i \, \frac{4\pi G_{\mathrm{N}} \dot{\phi}}{\sqrt{2k^3}} \, e^{-i\frac{k}{a}}.$$

On super-Hubble scales ($k \ll aH$), one obvious solution to Eq. (278) is $u_{\mathbf{k}} \propto z$. To find the other solution, we may set $u_{\mathbf{k}} = z\,\widetilde{u}_{\mathbf{k}}$, which satisfies the equation

$$\frac{\widetilde{u}''_{\mathbf{k}}}{\widetilde{u}'_{\mathbf{k}}} = -2\frac{z'}{z},$$

which gives

$$\widetilde{u}_{\mathbf{k}} = \int^{\tau} \frac{\mathrm{d}\tau'}{z^2(\tau')}.$$

On super-Hubble scales therefore we find

$$u_{\mathbf{k}} = c_1(k) \frac{a\dot{\phi}}{H} + c_2(k) \frac{a\dot{\phi}}{H} \int^{t} \mathrm{d}t' \, \frac{H^2}{a^3 \dot{\phi}^2}. \simeq c_1(k) \frac{a\dot{\phi}}{H} - c_2(k) \frac{1}{3a^2\dot{\phi}},$$

where the last passage has been performed supposing a de Sitter epoch, $H =$ constant. The first piece is the constant mode $c_1(k)z$, while the second is the decreasing mode. To find the constant $c_1(k)$, we apply what we have learned previously. We know that on super-Hubble scales the exact solution of equation (278) is

$$u_{\mathbf{k}} = \frac{\sqrt{\pi}}{2} \, e^{i\left(\nu+\frac{1}{2}\right)\frac{\pi}{2}} \, \sqrt{-\tau} \, H_{\nu}(-k\tau). \tag{281}$$

On super-Hubble scales, since $H_{\nu}(x \ll 1) \sim \sqrt{2/\pi}\, e^{-i\frac{\pi}{2}}\, 2^{\nu-\frac{3}{2}}\, (\Gamma(\nu_{\chi})/\Gamma(3/2))\, x^{-\nu}$, the fluctuation (281) becomes

$$u_{\mathbf{k}} = e^{i\left(\nu-\frac{1}{2}\right)\frac{\pi}{2}} 2^{\left(\nu-\frac{3}{2}\right)} \frac{\Gamma(\nu)}{\Gamma(3/2)} \frac{1}{\sqrt{2k}} \, (-k\tau)^{\frac{1}{2}-\nu}.$$

Therefore

$$c_1(k) = \lim_{k\to 0} \left| \frac{u_{\mathbf{k}}}{z} \right| = \frac{H}{a\dot{\phi}} \frac{1}{\sqrt{2k}} \left(\frac{k}{aH} \right)^{\frac{1}{2}-\nu} = \frac{H}{\dot{\phi}} \frac{1}{\sqrt{2k^3}} \left(\frac{k}{aH} \right)^{\eta-3\epsilon} \tag{282}$$

The last steps consist in relating the variable u to the comoving curvature $\mathcal{R}$ and to the gravitational potential Φ_{GI}. The comoving curvature takes the form

$$\mathcal{R} \;\equiv\; \Psi_{\mathrm{GI}} + \frac{H}{\phi'} \, \delta\phi^{(\mathrm{GI})} = \frac{u}{z}. \tag{283}$$

Since $z = a\dot{\phi}/H = a\sqrt{2\epsilon}M_{\mathrm{Pl}}$, the power spectrum of the comoving curvature can be expressed on super-Hubble scales as

$$\mathcal{P}_{\mathcal{R}}(k) \;=\; \frac{k^3}{2\,\pi^2} \left| \frac{u_{\mathbf{k}}}{z} \right|^2 = \frac{1}{2\overline{M}_{\mathrm{Pl}}^2 \epsilon} \left(\frac{H}{2\pi} \right)^2 \left(\frac{k}{aH} \right)^{n_{\mathcal{R}}-1} \equiv A_{\mathcal{R}}^2 \left(\frac{k}{aH} \right)^{n_{\mathcal{R}}-1} \tag{284}$$

with

$$n_{\mathcal{R}} - 1 = 3 - 2\nu = 2\eta - 6\epsilon. \tag{285}$$

These results reproduce those found in the previous subsection. The last step is to find the behavior of the gauge-invariant potential $\Phi_{\rm GI}$ on super-Hubble scales. If we recast equation (280) in the form

$$u_{\mathbf{k}} = \frac{1}{4\pi G_{\rm N}} \frac{H}{\dot{\phi}} \left(\frac{a}{H} \Phi^{\rm GI}_{\mathbf{k}} \right)^{\dot{}}, \tag{286}$$

we can infer that on super-Hubble scales the nearly constant mode of the gravitational potential during inflation reads

$$\Phi^{\rm GI}_{\mathbf{k}} = c_1(k) \left[1 - \frac{H}{a} \int^t {\rm d}t'\, a\,(t') \right] \simeq -c_1(k) \frac{\dot{H}}{H^2} = \epsilon\, c_1(k) \simeq \epsilon \frac{u_{\mathbf{k}}}{z} \simeq \epsilon\, \mathcal{R}_{\mathbf{k}}. \tag{287}$$

Indeed, plugging this solution into Eq. (286), one reproduces $u_{\mathbf{k}} = c_1(k) \frac{a\dot{\phi}}{H}$.

16 Gravitational waves

Quantum fluctuations in the gravitational fields are generated in a similar fashion of that of the scalar perturbations discussed so far. A gravitational wave may be viewed as a ripple of space-time in the FRW background metric and in general the linear tensor perturbations may be written as

$$ds^2 = a^2(\tau) \left[-d\tau^2 + (\delta_{ij} + h_{ij})\, dx^i dx^j \right],$$

where $|h_{ij}| \ll 1$. The tensor h_{ij} has six degrees of freedom, but, as we studied previously, the tensor perturbations are traceless, $\delta^{ij} h_{ij} = 0$, and transverse $\partial^i h_{ij} = 0$ $(i = 1, 2, 3)$. With these 4 constraints, there remain 2 physical degrees of freedom, or polarizations, which are usually indicated $\lambda = +, \times$. More precisely, we can write

$$h_{ij} = h_+\, e^+_{ij} + h_\times\, e^\times_{ij},$$

where e^+ and $e^\times$ are the polarization tensors which have the following properties

$$e^\lambda_{ij} = e^\lambda_{ji}, \quad k^i e^\lambda_{ij} = 0, \quad e^\lambda_{ii} = 0,$$
$$e^\lambda_{ij}(-\mathbf{k}) = \left[e^\lambda_{ij}(\mathbf{k}) \right]^*, \quad \sum_{ij} \left(e^\lambda_{ij} \right)^* e^{\lambda'}_{ij} = \delta_{\lambda\lambda'}.$$

Notice also that the tensors h_{ij} are gauge-invariant and therefore represent physical degrees of freedom.

If the stress-energy momentum tensor is diagonal, as the one provided by the inflaton potential $T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi + g_{\mu\nu} \mathcal{L}$, the tensor modes do not have any source and their action can be written as

$$S_h = \frac{1}{32\pi G_{\rm N}} \int d^3x d\tau\, a^2 \frac{1}{2} \left[\left(h'_{ij} \right)^2 - (\nabla h_{ij})^2 \right] = \frac{\overline{M}^2_{\rm Pl}}{4} \int d^3x d\tau\, a^2 \frac{1}{2} \left[\left(h'_{ij} \right)^2 - (\nabla h_{ij})^2 \right], \tag{288}$$

that is the action of two (not yet canonically normalized) independent massless scalar fields. The gauge-invariant tensor amplitude of each of these modes is (we drop for the time being the indices $_{ij}$)

$$v_{\mathbf{k}} = \frac{a\overline{M}_{\rm Pl}}{2}\, h_{\mathbf{k}}$$

and it satisfies the equation

$$v''_{\mathbf{k}} + \left(k^2 - \frac{a''}{a}\right) v_{\mathbf{k}} = 0,$$

which is the equation of motion of a massless scalar field in a quasi-de Sitter epoch. We can therefore make use of the results learnt previously to conclude that on super-Hubble scales the tensor modes scale like

$$|v_{\mathbf{k}}| = \left(\frac{H}{2\pi}\right)\left(\frac{k}{aH}\right)^{\frac{3}{2}-\nu_T},$$

where

$$\nu_T \simeq \frac{3}{2} - \epsilon.$$

Since fluctuations are (nearly) frozen in on super-Hubble scales, a way of characterizing the tensor perturbations is to compute the spectrum on scales larger than the horizon

$$\mathcal{P}_T(k) = \frac{k^3}{2\pi^2}\sum_{ij}\left|h_{ij\,\mathbf{k}}\right|^2 = 4\sum_{ij}\frac{k^3}{2\pi^2}\left|v_{ij\,\mathbf{k}}\right|^2 = 4\sum_{ij}\frac{k^3}{2\pi^2}\left(\left|v^{+}_{\mathbf{k}}\right|^2 + \left|v^{\times}_{\mathbf{k}}\right|^2\right). \tag{289}$$

This gives the power spectrum on super-Hubble scales

$$\boxed{\mathcal{P}_T(k) = \frac{8}{\overline{M}^2_{\rm Pl}}\left(\frac{H}{2\pi}\right)^2\left(\frac{k}{aH}\right)^{n_T} \equiv A_T^2\left(\frac{k}{aH}\right)^{n_T}}$$

where *spectral index* n_T is the spectral index of the tensor perturbations

$$\boxed{n_T = \frac{d\ln\mathcal{P}_T}{d\ln k} = 3 - 2\nu_T = -2\epsilon.}$$

The tensor perturbation is almost scale-invariant. Notice that the amplitude of the tensor modes depends only on the value of the Hubble rate during inflation. This amounts to saying that it depends only on the energy scale $V^{1/4}$ associated to the inflaton potential. A detection of gravitational waves from inflation is therefore a direct measurement of the energy scale associated to inflation.

16.1 The consistency relation

The results obtained so far for the scalar and tensor perturbations allow to predict a *consistency relation* which holds for the models of inflation addressed in these lectures, *i.e.* the models of inflation driven by one-single field ϕ. We define tensor-to-scalar amplitude ratio to be

$$r = \frac{A_T^2}{A_{\mathcal{R}}^2} = \frac{8\left(\frac{H}{2\,\pi\,\overline{M}_{\rm Pl}}\right)^2}{(2\epsilon)^{-1}\left(\frac{H}{2\,\pi\,\overline{M}_{\rm Pl}}\right)^2} = 16\epsilon.$$

This means that

$$\boxed{r = -8n_T.}$$

One-single models of inflation predict that during inflation driven by a single scalar field, the ratio between the amplitude of the tensor modes and that of the curvature perturbations is equal to minus one-half of the tilt of the spectrum of tensor modes. If this relation turns out to be falsified by the future measurements of the CMB anisotropies, this does not mean that inflation is wrong, but only that inflation has not been driven by only one field. Generalizations to two-field models of inflation can be found for instance in Refs. [9, 11].

Furthermore, using the consistency relation $r = 16\epsilon$ and the definition of ϵ, one deduces that [29]

$$\frac{\Delta\phi}{M_{\rm Pl}} \simeq \left(\frac{r}{2\times 10^{-2}}\right)^{1/2}, \tag{290}$$

meaning that a future measurement of the B-mode of CMB polarization will imply an inflaton excursus of Planckian values. Therefore, future measurements of the B-mode polarization of the CMB will allow a determination of the value of the energy scale of inflation. This explains the utility of CMB polarization measurements as probes of the physics of inflation. They are fundamental to prove or disprove that inflation took place at high-energy and to see if super-Planckian ranges were probed.

17 Transferring the perturbation to radiation during reheating

When the inflaton decays, the comoving curvature perturbation associated to the inflaton field are transferred to radiation. Let us see how this works [8].

Let us consider the system composed by the oscillating scalar field ϕ and the radiation fluid. Each component has energy-momentum tensor $T^{\mu\nu}_{(\phi)}$ and $T^{\mu\nu}_{(\gamma)}$. The total energy momentum $T^{\mu\nu} = T^{\mu\nu}_{(\phi)} + T^{\mu\nu}_{(\gamma)}$ is covariantly conserved, but allowing for an interaction between the two fluids

$$\begin{aligned} \boldsymbol{\nabla}_\mu T^{\mu\nu}_{(\phi)} &= Q^\nu_{(\phi)}\,, \\ \boldsymbol{\nabla}_\mu T^{\mu\nu}_{(\gamma)} &= Q^\nu_{(\gamma)}\,, \end{aligned} \tag{291}$$

where $Q^{\nu}_{(\phi)}$ and $Q^{\nu}_{(\gamma)}$ are the generic energy-momentum transfer to the scalar field and radiation sector respectively and are subject to the constraint

$$Q^{\nu}_{(\phi)} + Q^{\nu}_{(\gamma)} = 0\,. \tag{292}$$

The energy-momentum transfer $Q^{\nu}_{(\phi)}$ and $Q^{\nu}_{(\gamma)}$ can be decomposed for convenience as

$$\begin{aligned} Q^{\nu}_{(\phi)} &= \hat{Q}_{\phi} u^{\nu} + f^{\nu}_{(\phi)}\,, \\ Q^{\nu}_{(\gamma)} &= \hat{Q}_{\gamma} u^{\nu} + f^{\nu}_{(\gamma)}\,, \end{aligned} \tag{293}$$

where the f^{ν}'s are required to be orthogonal to the the total velocity of the fluid u^{ν}. The energy continuity equations for the scalar field and radiation can be obtained from $u_{\nu}\nabla_{\mu}T^{\mu\nu}_{(\phi)} = u_{\nu}Q^{\nu}_{(\phi)}$ and $u_{\nu}\nabla_{\mu}T^{\mu\nu}_{(\gamma)} = u_{\nu}Q^{\nu}_{(\gamma)}$ and hence from Eq. (293)

$$\begin{aligned} u_{\nu}\nabla_{\mu}T^{\mu\nu}_{(\phi)} &= \hat{Q}_{\phi}\,, \\ u_{\nu}\nabla_{\mu}T^{\mu\nu}_{(\gamma)} &= \hat{Q}_{\gamma}\,. \end{aligned} \tag{294}$$

In the case of an oscillating scalar field decaying into radiation the energy transfer coefficient $\hat{Q}_{\phi}$ is given by

$$\begin{aligned} \hat{Q}_{\phi} &= -\Gamma\rho_{\phi}, \\ \hat{Q}_{\gamma} &= \Gamma\rho_{\phi}, \end{aligned} \tag{295}$$

where Γ is the decay rate of the scalar field into radiation.

The equations of motion for the curvature perturbations ζ_{ϕ} and ζ_{γ} can be obtained perturbing at first order the continuity energy equations (294) for the scalar field and radiation energy densities, including the energy transfer. Expanding the transfer coefficients $\hat{Q}_{\phi}$ and $\hat{Q}_{\gamma}$ up to first order in the perturbations around the homogeneous background as

$$\hat{Q}_{\phi} = Q_{\phi} + \delta Q_{\phi}\,, \tag{296}$$

$$\hat{Q}_{\gamma} = Q_{\gamma} + \delta Q_{\gamma}\,, \tag{297}$$

Eqs. (294) give on wavelengths larger than the horizon scale

$$\begin{aligned} &\delta\rho'_{\phi} + 3\,\mathcal{H}\,(\delta\rho_{\phi} + \delta P_{\phi}) - 3\,(\rho_{\phi} + P_{\phi})\,\Psi' \\ &= a\,Q_{\phi}\Phi + a\,\delta Q_{\phi}\,, \end{aligned} \tag{298}$$

$$\begin{aligned} &\delta\rho'_{\gamma} + 3\,\mathcal{H}\,(\delta\rho_{\gamma} + \delta P_{\gamma}) - 3\,(\rho_{\gamma} + P_{\gamma})\,\Psi' \\ &= a\,Q_{\gamma}\Phi + a\,\delta Q_{\gamma}\,. \end{aligned} \tag{299}$$

Notice that the oscillating scalar field and radiation have fixed equations of state with $\delta P_{\phi} = 0$ and $\delta P_{\gamma} = \delta\rho_{\gamma}/3$ (which correspond to vanishing intrinsic non-adiabatic pressure perturbations). Using the perturbed $(0-0)$-component of Einstein's equations for super-horizon wavelengths $\Psi' + \mathcal{H}\Phi = -\mathcal{H}(\delta\rho/\rho)/2$, we can rewrite Eqs. (298) and (299) in terms of the gauge-invariant curvature perturbations ζ_ϕ and ζ_γ

$$\zeta_\phi' = \frac{a\,\mathcal{H}}{\rho_\phi'}\left[\delta Q_\phi - \frac{Q_\phi'}{\rho_\phi'}\delta\rho_\phi + Q_\phi\frac{\rho'}{2\rho}\left(\frac{\delta\rho_\phi}{\rho_\phi'} - \frac{\delta\rho}{\rho'}\right)\right], \tag{300}$$

$$\zeta_\gamma' = \frac{a\,\mathcal{H}}{\rho_\gamma'}\left[\delta Q_\gamma - \frac{Q_\gamma'}{\rho_\gamma'}\delta\rho_\gamma + .Q_\gamma\frac{\rho'}{2\rho}\left(\frac{\delta\rho_\gamma}{\rho_\gamma'} - \frac{\delta\rho}{\rho'}\right)\right], \tag{301}$$

where $\delta Q_\gamma = -\delta Q_\phi$ from the constraint in Eq (292). If the energy transfer coefficients $\hat{Q}_\phi$ and $\hat{Q}_\gamma$ are given in terms of the decay rate Γ as in Eq. (295), the first order perturbation are respectively

$$\delta Q_\phi = -\Gamma\delta\rho_\phi, \tag{302}$$

$$\delta Q_\gamma = \Gamma\delta\rho_\phi. \tag{303}$$

Plugging the expressions (302-303) into Eqs. (300-301), the first order curvature perturbations for the scalar field and radiation obey on large scales

$$\zeta_\phi' = \frac{a\Gamma}{2}\frac{\rho_\phi}{\rho_\phi'}\frac{\rho'}{\rho}\left(\zeta_\phi - \zeta\right), \tag{304}$$

$$\zeta_\gamma' = \frac{a}{\rho_\gamma'}\left[\Gamma\rho'\frac{\rho_\phi'}{\rho_\gamma'}\left(1 - \frac{\rho_\phi}{2\rho}\right)(\zeta - \zeta_\phi)\right]. \tag{305}$$

From the total comoving curvature perturbation

$$\zeta = \frac{\dot\rho_\phi}{\dot\rho}\zeta_\phi + \frac{\dot\rho_\gamma}{\dot\rho}\zeta_\gamma, \;\; \rho = \rho_\phi + \rho_\gamma. \tag{306}$$

it is thus possible to find the equation of motion for the total curvature perturbation ζ using the evolution of the individual curvature perturbations in Eqs. (304) and (305)

$$\begin{aligned}\zeta' &= f'\left(\zeta_\phi - \zeta_\gamma\right) + f\zeta_\phi' + (1-f)\zeta_\gamma' \\ &= \mathcal{H}f(1-f)\left(\zeta_\phi - \zeta_\gamma\right) = -\mathcal{H}f\left(\zeta - \zeta_\phi\right),\end{aligned} \tag{307}$$

where $f = (\dot\rho_\phi/\dot\rho)$. Notice that during the decay of the scalar field into the radiation fluid, ρ_γ' in Eq. (305) may vanish. So it is convenient to close the system of equations by using the two equations (304) and (307) for the evolution of ζ_ϕ and ζ. These equations say that $\zeta = \zeta_\phi$ is a fixed point: during the reheating phase the comoving curvature perturbation stored in the inflaton field is transferred to radiation smoothly.

18 Comoving curvature perturbation from isocurvature perturbation

Let us give one example of how the fact that the comoving curvature perturbation is not constant when there are isocurvature perturbation can be useful. The paradigm we will describe goes under the name of the curvaton mechanism.

Suppose that during inflation there is another field σ, the curvaton, which is supposed to give a negligible contribution to the energy density and to be an almost free scalar field, with a small effective mass $m_\sigma^2 = |\partial^2 V/\partial\sigma^2| \ll H^2$.

The unperturbed curvaton field satisfies the equation of motion

$$\sigma'' + 2\,\mathcal{H}\sigma' + a^2 \frac{\partial V}{\partial \sigma} = 0. \tag{308}$$

It is also usually assumed that the curvaton field is very weakly coupled to the scalar fields driving inflation and that the curvature perturbation from the inflaton fluctuations is negligible. Thus, if we expand the curvaton field up to first-order in the perturbations around the homogeneous background as $\sigma(\mathbf{x},\tau) = \sigma_0(\tau) + \delta\sigma(\mathbf{x},\tau)$, the linear perturbations satisfy on large scales

$$\delta\sigma'' + 2\,\mathcal{H}\delta\sigma' + a^2 \frac{\partial^2 V}{\partial \sigma^2}\,\delta\sigma = 0. \tag{309}$$

As a result on super-Hubble scales its fluctuations $\delta\sigma$ will be Gaussian distributed and with a nearly scale-invariant spectrum given by

$$\mathcal{P}^{\frac{1}{2}}_{\delta\sigma}(k) \approx \frac{H_*}{2\pi}, \tag{310}$$

where the subscript $*$ denotes the epoch of horizon exit $k = aH$. Once inflation is over the inflaton energy density will be converted to radiation (γ) and the curvaton field will remain approximately constant until $H^2 \sim m_\sigma^2$. At this epoch the curvaton field begins to oscillate around the minimum of its potential which can be safely approximated to be quadratic $V \approx \frac{1}{2}m_\sigma^2\sigma^2$. During this stage the energy density of the curvaton field just scales as non-relativistic matter $\rho_\sigma \propto a^{-3}$. The energy density in the oscillating field is

$$\rho_\sigma(\tau,\mathbf{x}) \approx m_\sigma^2 \sigma^2(\tau,\mathbf{x}), \tag{311}$$

and it can be expanded into a homogeneous background $\rho_\sigma(\tau)$ and a first-order perturbation $\delta\rho_\sigma$ as

$$\rho_\sigma(\tau,\mathbf{x}) = \rho_\sigma(\tau) + \delta\rho_\sigma(\tau,\mathbf{x}) = m_\sigma^2\sigma + 2m_\sigma^2\,\sigma\,\delta\sigma. \tag{312}$$

As it follows from Eqs. (308) and (309) for a quadratic potential the ratio $\delta\sigma/\sigma$ remains constant and the resulting relative energy density perturbation is

$$\frac{\delta\rho_\sigma}{\rho_\sigma} = 2\left(\frac{\delta\sigma}{\sigma}\right)_*, \tag{313}$$

Such perturbations in the energy density of the curvaton field produce in fact a primordial density perturbation well after the end of inflation. The primordial adiabatic density perturbation is associated with a perturbation in the spatial curvature Ψ and it is, as we have shown, characterized in a gauge-invariant manner by the curvature perturbation ζ on hypersurfaces of uniform total density ρ. We recall that at linear order the quantity ζ is given by the gauge-invariant formula

$$\zeta = \Psi + \mathcal{H}\frac{\delta\rho}{\rho'}, \tag{314}$$

and on large scales it obeys the equation of motion

$$\zeta' = \frac{\mathcal{H}}{\rho + P}\,\delta P_{\rm nad}, \tag{315}$$

In the curvaton scenario the curvature perturbation is generated well after the end of inflation during the oscillations of the curvaton field because the pressure of the mixture of matter (curvaton) and radiation produced by the inflaton decay is not adiabatic. A convenient way to study this mechanism is to consider the curvature perturbations ζ_i associated with each individual energy density components, which to linear order are defined as

$$\zeta_i \equiv \Psi + \mathcal{H}\left(\frac{\delta\rho_i}{\rho_i'}\right). \tag{316}$$

Therefore, during the oscillations of the curvaton field, the total curvature perturbation can be written as a weighted sum of the single curvature perturbations

$$\zeta = \frac{\dot{\rho}_\gamma}{\dot{\rho}}\zeta_\gamma + \frac{\dot{\rho}_\sigma}{\dot{\rho}}\zeta_\sigma = (1-f)\zeta_\gamma + f\zeta_\sigma, \tag{317}$$

where the quantity

$$f = \frac{3\rho_\sigma}{4\rho_\gamma + 3\rho_\sigma} \tag{318}$$

defines the relative contribution of the curvaton field to the total curvature perturbation. From now on we shall work under the approximation of sudden decay of the curvaton field. Under this approximation the curvaton and the radiation components ρ_σ and ρ_γ satisfy separately the energy conservation equations

$$\begin{aligned} \rho_\gamma' &= -4\,\mathcal{H}\rho_\gamma\,, \\ \rho_\sigma' &= -3\,\mathcal{H}\rho_\sigma, \end{aligned} \tag{319}$$

and the curvature perturbations ζ_i remains constant on super-Hubble scales until the decay of the curvaton. Therefore from Eq. (317) it follows that the first-order curvature pertubation evolves on large scales as

$$\zeta' = f'(\zeta_\sigma - \zeta_\gamma) = \mathcal{H}f(1-f)(\zeta_\sigma - \zeta_\gamma), \tag{320}$$

and by comparison with Eq. (315) one obtains the expression for the non-adiabatic pressure perturbation at first order

$$\delta P_{\rm nad} = \rho_\sigma(1-f)(\zeta_\sigma - \zeta_\gamma). \tag{321}$$

Since in the curvaton scenario it is supposed that the curvature perturbation in the radiation produced at the end of inflation is negligible, then

$$\zeta_\gamma = \Psi - \frac{1}{4}\frac{\delta\rho_\gamma}{\rho_\gamma} = 0. \tag{322}$$

Similarly the value of ζ_σ is fixed by the fluctuations of the curvaton during inflation

$$\zeta_\sigma = \Psi - \frac{1}{3}\frac{\delta\rho_\sigma}{\rho_\sigma} = \zeta_{\sigma\rm I}, \tag{323}$$

where I stands for the value of the fluctuations during inflation. From Eq. (317) the total curvature perturbation during the curvaton oscillations is given by

$$\zeta = f\zeta_\sigma. \tag{324}$$

As it is clear from Eq. (324) initially, when the curvaton energy density is subdominant, the density perturbation in the curvaton field ζ_σ gives a negligible contribution to the total curvature perturbation, thus corresponding to an isocurvature (or entropy) perturbation. On the other hand during the oscillations $\rho_\sigma \propto a^{-3}$ increases with respect to the energy density of radiation $\rho_\gamma \propto a^{-4}$, and the perturbations in the curvaton field are then converted into the curvature perturbation. Well after the decay of the curvaton, during the conventional radiation and matter dominated eras, the total curvature perturbation will remain constant on super-Hubble scales at a value which, in the sudden decay approximation, is fixed by Eq. (324) at the epoch of curvaton decay

$$\zeta = f_{\rm D}\,\zeta_\sigma, \tag{325}$$

where D stands for the epoch of the curvaton decay.

Going beyond the sudden decay approximation it is possible to introduce a transfer parameter r defined as

$$\zeta = r\zeta_\sigma, \tag{326}$$

where ζ is evaluated well after the epoch of the curvaton decay and ζ_σ is evaluated well before this epoch. Numerical studies of the coupled perturbation equations has been performed and show that the sudden decay approximation is exact when the curvaton dominates the energy density before it decays ($r = 1$), while in the opposite case

$$r \approx \left(\frac{\rho_\sigma}{\rho}\right)_{\rm D}. \tag{327}$$

19 Symmetries of the de Sitter geometry and its consequences

Before launching ourselves into the computation of the post-inflationary evolution of the cosmological perturbations, we wish to summarize the symmetries of the de Sitter geometry to understand better the properties of the inflationary perturbations.

The four-dimensional de Sitter space-time of radius H^{-1} is described by the hyperboloid

$$\eta_{AB}X^A X^B = -X_0^2 + X_i^2 + X_5^2 = \frac{1}{H^2} \qquad (i = 1, 2, 3), \tag{328}$$

embedded in five-dimensional Minkowski space-time $\mathbb{M}^{1,4}$ with coordinates X^A and flat metric $\eta_{AB} = {\rm diag}(-1, 1, 1, 1, 1)$. A particular parametrization of the de Sitter hyperboloid is provided by

$$\begin{aligned}
X^0 &= \frac{1}{2H}\left(H\tau - \frac{1}{H\tau}\right) - \frac{1}{2}\frac{x^2}{\tau},\\
X^i &= \frac{x^i}{H\tau},\\
X^5 &= -\frac{1}{2H}\left(H\tau + \frac{1}{H\tau}\right) + \frac{1}{2}\frac{x^2}{\tau},
\end{aligned} \tag{329}$$

which may easily be checked that satisfies Eq. (328). The de Sitter metric is the induced metric on the hyperboloid from the five-dimensional ambient Minkowski space-time

$$\mathrm{d}s_5^2 = \eta_{AB}\mathrm{d}X^A\mathrm{d}X^B. \tag{330}$$

For the particular parametrization (329), for example, we find

$$\mathrm{d}s^2 = \frac{1}{H^2\tau^2}\left(-\mathrm{d}\tau^2 + \mathrm{d}\mathbf{x}^2\right). \tag{331}$$

The group SO(1,4) acts linearly on $\mathbb{M}^{1,4}$. Its generators are

$$J_{AB} = X_A\frac{\partial}{\partial X^B} - X_B\frac{\partial}{\partial X^A} \qquad A,B = (0,1,2,3,5) \tag{332}$$

and satisfy the SO(1,4) algebra

$$[J_{AB}, J_{CD}] = \eta_{AD}J_{BC} - \eta_{AC}J_{BD} + \eta_{BC}J_{AD} - \eta_{BD}J_{AC}. \tag{333}$$

We may split these generators as

$$J_{ij}, \quad P_0 = J_{05}\,, \quad \Pi_i^+ = J_{i5} + J_{0i}\,, \quad \Pi_i^- = J_{i5} - J_{0i}, \tag{334}$$

which act on the de Sitter hyperboloid as

$$\begin{aligned}
J_{ij} &= x_i\frac{\partial}{\partial x_j} - x_j\frac{\partial}{\partial x_i},\\
P_0 &= \tau\frac{\partial}{\partial\tau} + x^i\frac{\partial}{\partial x^i},\\
\Pi_i^- &= -2H\tau x^i\frac{\partial}{\partial\tau} + H\left(x^2\delta_{ij} - 2x_ix_j\right)\frac{\partial}{\partial x_j} - H\tau^2\frac{\partial}{\partial x_i},\\
\Pi_i^+ &= \frac{1}{H}\frac{\partial}{\partial x_i}
\end{aligned} \tag{335}$$

and satisfy the commutator relations

$$\begin{aligned}
[J_{ij}, J_{kl}] &= \delta_{il}J_{jk} - \delta_{ik}J_{jl} + \delta_{jk}J_{il} - \delta_{jl}J_{ik},\\
[J_{ij}, \Pi_k^\pm] &= \delta_{ik}\Pi_j^\pm - \delta_{jk}\Pi_i^\pm,\\
[\Pi_k^\pm, P_0] &= \mp\Pi_k^\pm,\\
[\Pi_i^-, \Pi_j^+] &= 2J_{ij} + 2\delta_{ij}P_0.
\end{aligned} \tag{336}$$

This is nothing else that the conformal algebra. Indeed, by defining

$$L_{ij} = iJ_{ij}\,, \quad D = -iP_0\,, \quad P_i = -i\Pi_i^+\,, \quad K_i = i\Pi_i^-\,, \tag{337}$$

we get

$$\begin{aligned}
P_i &= -\frac{i}{H}\partial_i,\\
D &= -i\left(\tau\frac{\partial}{\partial\tau} + x^i\partial_i\right),\\
K_i &= -2iHx_i\left(\tau\frac{\partial}{\partial\tau} + x^i\partial_i\right) - iH(-\tau^2 + x^2)\partial_i,\\
L_{ij} &= i\left(x_i\frac{\partial}{\partial x_j} - x_j\frac{\partial}{\partial x_i}\right).
\end{aligned} \tag{338}$$

These are also the Killing vectors of de Sitter space-time corresponding to symmetries under space translations (P_i), dilitations (D), special conformal transformations (K_i) and space rotations (L_{ij}). They satisfy the conformal algebra in its standard form

$$[D, P_i] = iP_i, \tag{339}$$

$$[D, K_i] = -iK_i, \tag{340}$$

$$[K_i, P_j] = 2i\Big(\delta_{ij} D - L_{ij}\Big) \tag{341}$$

$$[L_{ij}, P_k] = i\Big(\delta_{jk} P_i - \delta_{ik} P_j\Big), \tag{342}$$

$$[L_{ij}, K_k] = i\Big(\delta_{jk} K_i - \delta_{ik} K_j\Big), \tag{343}$$

$$[L_{ij}, D] = 0, \tag{344}$$

$$[L_{ij}, L_{kl}] = i\Big(\delta_{il} L_{jk} - \delta_{ik} L_{jl} + \delta_{jk} L_{il} - \delta_{jl} L_{ik}\Big). \tag{345}$$

The de Sitter algebra SO(1,4) has two Casimir invariants

$$\mathcal{C}_1 = -\frac{1}{2} J_{AB} J^{AB}\,, \tag{346}$$

$$\mathcal{C}_2 = W_A W^A\,, \qquad W^A = \epsilon^{ABCDE} J_{BC} J_{DE}. \tag{347}$$

Using Eqs. (334) and (337), we find that

$$\mathcal{C}_1 = D^2 + \frac{1}{2}\{P_i, K_i\} + \frac{1}{2} L_{ij} L^{ij}, \tag{348}$$

which turns out to be, in the explicit representation Eq. (338),

$$H^{-2}\mathcal{C}_1 = -\frac{\partial^2}{\partial\tau^2} - \frac{2}{\tau}\frac{\partial}{\partial\tau} + \boldsymbol{\nabla}^2. \tag{349}$$

As a result, $\mathcal{C}_1$ is the Laplace operator on the de Sitter hyperboloid and for a scalar field $\phi(x)$ we have

$$\mathcal{C}_1 \phi(x) = \frac{m^2}{H^2}\phi(x). \tag{350}$$

Let us now consider the case $H\tau \ll 1$. The parametrization (329) turns out then to be

$$\begin{aligned} X^0 &= -\frac{1}{2H^2\tau} - \frac{1}{2}\frac{x^2}{\tau}, \\ X^i &= \frac{x^i}{H\tau}, \\ X^5 &= -\frac{1}{2H^2\tau} + \frac{1}{2}\frac{x^2}{\tau} \end{aligned} \tag{351}$$

and we may easily check that the hyperboloid has been degenerated to the hypercone

$$-X_0^2 + X_i^2 + X_5^2 = 0. \tag{352}$$

We identify points $X^A \equiv \lambda X^A$ (which turns the cone (352) into a projective space). As a result, τ in the denominator of the X^A can be ignored due to projectivity condition. Then, on the cone, the conformal group acts linearly, whereas induces the (non-linear) conformal transformations $x_i \to x'_i$ with

$$x'_i = a_i + M_i^j x_j, \tag{353}$$

$$x'_i = \lambda x_i, \tag{354}$$

$$x'_i = \frac{x_i + b_i x^2}{1 + 2b_i x_i + b^2 x^2}. \tag{355}$$

on Euclidean $\mathbb{R}^3$ with coordinates x^i. These transformations correspond to translations and rotations (generated by P_i, L_{ij}), dilations (generated by D) and special conformal transformations (generated by K_i), respectively, acting now on the constant time hypersurfaces of de Sitter space-time. It should be noted that special conformal transformations can be written in terms of inversion

$$x_i \to x'_i = \frac{x_i}{x^2} \tag{356}$$

as inversion×translation×inversion.

19.1 Representations

The representations of the SO(1,4) algebra are constructed by employing the method of induced representations. Let us consider the stability subgroup at $x^i = 0$ which is the group G generated by (L_{ij}, D, K_i). It is easy to see from the conformal algebra, that P_i and K_i are actually raising and lowering operators for the dilation operator D. Therefore there should be states which will be annihilated by K_i. Every irreducible representation will then be specified by an irreducible representation of the rotational group SO(3) (*i.e.* its spin) and a definite conformal dimension annihilated by K_i. Representations $\phi_s(\mathbf{0})$ of the stability group at $\mathbf{x} = \mathbf{0}$ with spin s and dimension Δ are specified by

$$\begin{aligned}
&[L_{ij}, \phi_s(\mathbf{0})] = \Sigma_{ij}^{(s)} \phi_s(\mathbf{0}), \\
&[D, \phi_s(\mathbf{0})] = -i\Delta \phi_s(\mathbf{0}), \\
&[K_i, \phi_s(\mathbf{0})] = 0,
\end{aligned} \tag{357}$$

where $\Sigma_{ij}^{(s)}$ is a spin-s representation of SO(3). Those representations $\phi_s(\mathbf{0})$ that satisfy the relations (357) are called primary fields. Once the primary fields are known, all other fields, the descendants, are constructed by taking derivatives of the primaries $\partial_i \cdots \partial_j \phi_s(\mathbf{0})$. For scalars in particular, the momentum P_i generates translations so that for a scalar $\phi(\mathbf{x})$ we have

$$[P_i, \phi(\mathbf{x})] = -i\partial_i \phi(\mathbf{x}). \tag{358}$$

Denoting collectively any generator of the stability subgroup G as $J = (L_{ij}, D, K_i)$ and taking into account that $\phi(\mathbf{x}) = e^{i\mathbf{P}\cdot\mathbf{x}} \phi(\mathbf{0}) e^{-i\mathbf{P}\cdot\mathbf{x}}$, we find that

$$[J, \phi(\mathbf{x})] = e^{i\mathbf{P}\cdot\mathbf{x}} [\hat{J}, \phi(\mathbf{0})] e^{-i\mathbf{P}\cdot\mathbf{x}} \tag{359}$$

where

$$\hat{J} = e^{-i\mathbf{P}\cdot\mathbf{x}} J e^{i\mathbf{P}\cdot\mathbf{x}} = \sum_n \frac{(-i)^n}{n!} x^{i_1} x^{i_2} \ldots x^{i_n} [P_{i_1} [P_{i_2} \ldots [P_{i_n}, J], \ldots]] \tag{360}$$

and $\phi(\mathbf{0})$ is a representation of the stability subgroup. Specifying for $J = L_{ij}, D$ and and $J = K_i$ we find

$$\hat{L}_{ij} = L_{ij} + x_i P_j - x_j P_i, \tag{361}$$

$$\hat{D} = D + x^i P_i, \tag{362}$$

$$\hat{K}_i = K_i + 2(x_i D - x^j L_{ij}) + 2x_i x^j P_j - x^2 P_i. \tag{363}$$

For a scalar, the right-hand side of the first equation in (357) vanishes, therefore we find that for a scalar $\phi(\mathbf{x})$

$$i[L_{ij}, \phi(\mathbf{x})] = (x_i\partial_j - x_j\partial_i)\,\phi(\mathbf{x}), \tag{364}$$

$$i[K_i, \phi(\mathbf{x})] = \left(2\Delta x_i + 2x_i x^j\partial_j - x^2\partial_i\right)\phi(\mathbf{x}), \tag{365}$$

$$i[D, \phi(\mathbf{x})] = \left(x^i\partial_i + \Delta\right)\phi(\mathbf{x}), \tag{366}$$

$$i[P_i, \phi(\mathbf{x})] = \partial_i\phi(\mathbf{x}). \tag{367}$$

For example,

$$[\mathcal{C}_1, \phi(\mathbf{0})] = -\Delta(\Delta - 3)\phi(\mathbf{0}), \tag{368}$$

which implies that

$$m^2 = -\Delta(\Delta - 3)H^2. \tag{369}$$

It can be shown that the scalar representations of the de Sitter group SO(1,4) actually splits into three distinct series: the principal series with masses $m^2 \geq 9H^2/4$, the complementary series with masses in the range $0 < m^2 < 9H^2/4$ and the discrete series. It is the principal representations which survive the Winger-Inonü contraction $(H \to 0)$ to the Poincáre group.

The method of the induced representations used above for the scalar can be employed to include higher-spin fields as well. For a higher-spin field described by a symmetric-traceless tensor $\phi_{i_1\ldots i_s}$ we get

$$i[L_{ij}, \phi_{k_1\ldots k_s}] = \left(x_i\partial_j - x_j\partial_i + i\Sigma^{(s)}_{ij}\right)\phi_{k_1\ldots k_s}, \tag{370}$$

$$i[K_i, \phi_{k_1\ldots k_s}] = \left(2\Delta x_i + 2x_i x^j\partial_j - x^2\partial_i + 2ix^j\Sigma^{(s)}_{ji}\right)\phi_{k_1\ldots k_s}, \tag{371}$$

$$i[D, \phi_{k_1\ldots k_s}] = \left(x^i\partial_i + \Delta_s\right)\phi_{k_1\ldots k_s}, \tag{372}$$

$$i[P_i, \phi_{k_1\ldots k_s}] = \partial_i\phi_{k_1\ldots k_s}, \tag{373}$$

where the spin operator $\Sigma^{(s)}_{ij}$ acts as

$$\Sigma^{(s)}_{ij}\phi_{k_1\ldots k_s} = \sum_{\{a\}}(\phi_{k_1\ldots k_{a-1} i k_{a+1}\ldots k_s}\delta_{jk_a} - \phi_{k_1\ldots k_{a-1} j k_{a+1}\ldots k_s}\delta_{ik_a}). \tag{374}$$

It is then easy to verify that

$$\mathcal{C}_1 = \frac{m^2}{H^2} = -\Delta_s(\Delta_s - 3) - s(s+1) \;\; \text{since} \;\; \frac{1}{2}\Sigma^{(s)}_{ij}\Sigma^{(s)}_{ij} = s(s+1). \tag{375}$$

19.2 Killing vectors of the de Sitter space

We have seen that the essential kinematical feature of a vacuum dominated de Sitter universe is that the conformal group of certain embeddings of three dimensional hypersurfaces in de Sitter space-time may be mapped (either one-to-one or multiple-to-one) to the geometric isometry group of the full four dimensional space-time into which the hypersurfaces are embedded. The first example of such an embedding of three dimensional hypersurfaces is that of flat Euclidean $\mathbb{R}^3$ in de Sitter space-time in coordinates. The conformal group of the three dimensional spatial $\mathbb{R}^3$ sections is in fact identical (isomorphic) to the isometry group SO(4,1) of the four dimensional de Sitter space-time, as we now review.

Since (eternal) de Sitter space is maximally symmetric, it posseses the maximum number of isometries for a space-time in $n = 4$ dimensions, namely $\frac{n(n+1)}{2} = 10$, corresponding to the 10 solutions of the Killing equation,

$$\nabla_\mu \epsilon_\nu^{(\alpha)} + \nabla_\nu \epsilon_\mu^{(\alpha)} = 0\,, \qquad \mu,\nu = 0,1,2,3\,; \quad \alpha = 1,\ldots,10\,. \tag{376}$$

Each of the 10 linearly independent solutions to this equation (labelled by α) is a vector field in de Sitter space corresponding to an infinitesimal coordinate transformation, $x^\mu \to x^\mu + \epsilon^\mu(x)$ that leaves the de Sitter geometry and line element invariant. These are the 10 generators of the de Sitter isometry group, the non-compact Lie group SO(4,1).

The isomorphism with conformal transformations of $\mathbb{R}^3$ is that each of these 10 solutions of (376) may be placed in one-to-one correspondence with the 10 solutions of the conformal Killing equation of three dimensional flat space $\mathbb{R}^3$, *i.e.*

$$\partial_i \xi_j^{(\alpha)} + \partial_j \xi_i^{(\alpha)} = \frac{2}{3}\,\delta_{ij}\,\partial_k \xi_k^{(\alpha)}\,, \qquad i,j,k = 1,2,3; \quad \alpha = 1,\ldots 10\,. \tag{377}$$

In (376) the space-time indices μ,ν range over 4 values and ∇_ν is the covariant derivative with respect to the full four dimensional metric of de Sitter space-time, whereas in (377), i,j are three dimensional spatial indices of the three Cartesian coordinates x^i of Euclidean $\mathbb{R}^3$ of one dimension lower with flat metric δ_{ij}. Solutions to the conformal Killing Eq. (377) are transformations of $x^i \to x^i + \xi^i(\vec{x})$ which preserve all angles in $\mathbb{R}^3$. This isomorphism between geometric isometries of $(3+1)$ dimensional de Sitter space-time and conformal transformations of 3 dimensional flat space embedded in it is the origin of conformal invariance of correlation functions generated in a de Sitter phase of the universe.

The 10 solutions of (377) for vector fields in flat $\mathbb{R}^3$ are easily found. They are of two kinds. First there are 6 solutions of (377) with $\partial_k \xi_k = 0$, corresponding to the strict isometries of $\mathbb{R}^3$, namely 3 translations and 3 rotations. Second, there are also 4 solutions of (377) with $\partial_k \xi_k \neq 0$. These are the 4 conformal transformations of flat space that are not strict isometries but preserve all angles. They consist of one global dilation and three special conformal transformations. The Killing Eq. (376) can be rewritten as

$$g_{\nu\lambda}\partial_\mu \epsilon^\lambda + g_{\mu\lambda}\partial_\nu \epsilon^\lambda + \partial_\sigma g_{\mu\nu}\epsilon^\sigma = 0\,, \tag{378}$$

which, for de Sitter space, provide

$$\partial_t \epsilon_t = 0\,, \tag{379}$$

$$\partial_t \epsilon_i + \partial_i \epsilon_t = 2H\epsilon_i\,, \tag{380}$$

$$\partial_i \epsilon_j + \partial_j \epsilon_i = 2Ha^2 \delta_{ij} \epsilon_t\,. \tag{381}$$

Its solutions of can be catalogued as follows. For $\epsilon_t = 0$ we have the three translations,

$$\epsilon_t^{(\mathrm{T}j)} = 0\,, \qquad \epsilon_i^{(\mathrm{T}j)} = a^2 \delta_i{}^j\,, \qquad j = 1, 2, 3\,, \tag{382}$$

and the three rotations,

$$\epsilon_\tau^{(\mathrm{R}\ell)} = 0\,, \qquad \epsilon_i^{(\mathrm{R}\ell)} = a^2 \epsilon_{i\ell m} x^m\,, \qquad \ell = 1, 2, 3\,. \tag{383}$$

The spatial $\mathbb{R}^3$ sections also have four conformal Killing vectors which satisfy the Killing vector equations with $\epsilon_t \neq 0$. They are the three special conformal transformations of $\mathbb{R}^3$,

$$\epsilon_t^{(\mathrm{C})} = -2Hx^n\,, \qquad \epsilon_i^{(\mathrm{C})} = H^2 a^2 (\delta_i{}^n \delta_{jk} x^j x^k - 2\delta_{ij} x^j x^n) - \delta_i^n\,, \qquad n = 1, 2, 3\,, \tag{384}$$

and the dilation,

$$\epsilon_t^{(\mathrm{D})} = 1\,, \qquad \epsilon_i^{(\mathrm{D})} = Ha^2\, \delta_{ij} x^j\,. \tag{385}$$

This last dilational Killing vector is the infinitesimal form of the finite dilational symmetry,

$$\mathbf{x} \to \lambda \mathbf{x}\,, \tag{386}$$

$$a(t) \to \lambda^{-1} a(t)\,, \tag{387}$$

$$t \to t - H^{-1} \ln \lambda \tag{388}$$

of de Sitter space. Since the maximum number of Killing isometries in 4 dimensions is 10, there are no other solutions and de Sitter space, being a fully symmetric space, possesses the maximum number of symmetries.

We can understand the issue of scale-invariance rather easily looking at the symmetries of de Sitter. In conformal time the metric during inflation reads approximately de Sitter

$$\mathrm{d}s^2 = \frac{1}{H^2 \tau^2}(-\mathrm{d}\tau^2 + \mathrm{d}\mathbf{x}^2)\,, \tag{389}$$

whose isometry group is SO(4,1). The time-evolving inflaton background is homogeneous and rotationally invariant, so that translations and rotations are good symmetries of the whole system. The dilation isometry

$$\tau \to \lambda \tau, \quad \mathbf{x} \to \lambda \mathbf{x}\,, \tag{390}$$

is also an approximate symmetry of the inflaton background in the limit in which its dynamics varies slowly in time. It is this isometry which guarantees a scale invariant spectrum, independently of the inflaton dynamics. In the limit of infinite-slow roll the action for the inflaton field is the one of a free scalar field. Written in conformal time it becomes

$$S = -\frac{1}{2} \int \mathrm{d}\tau \mathrm{d}^3 x \sqrt{-g}\, g^{\mu\nu}\, \partial_\mu \phi \partial_\nu \phi = \frac{1}{2} \int \frac{\mathrm{d}\tau \mathrm{d}^3 x}{H^2 \tau^2} \left[(\partial_\tau \phi)^2 - (\partial_i \phi)^2 \right]. \tag{391}$$

The dilatation symmetry (390) says that if $\phi(\tau, \mathbf{x})$ is a solution, then also $\phi'(\tau, \mathbf{x}) = \phi(\lambda\tau, \lambda\mathbf{x})$ is a solution. Indeed, indicating by $\tau' = \lambda\tau$ and $\mathbf{x}' = \lambda\mathbf{x}$, the action is left invariant

$$S' = \frac{1}{2} \int \frac{\mathrm{d}\tau \mathrm{d}^3 x}{H^2 \tau^2} \left[(\partial_\tau \phi')^2 - (\partial_i \phi')^2 \right] = \frac{1}{2} \int \frac{\lambda^{-4} \mathrm{d}\tau \mathrm{d}^3 x}{\lambda^{-2} 2H^2 \tau^2} \frac{1}{\lambda^{-2}} \left[(\partial_\tau \phi)^2 - (\partial_i \phi)^2 \right] = S, \tag{392}$$

where in the last passage we have changed the variables from $\lambda\tau$ to τ and $\mathbf{x}' = \lambda\mathbf{x}$ to $\mathbf{x}$. In Fourier space dilations act on a scalar field $\phi(\mathbf{x},\tau)$ on large scales, where there is no time dependence, as

$$\phi_{\mathbf{k}} \to \lambda^{-3}\phi_{\mathbf{k}/\lambda}\,. \tag{393}$$

Indeed, consider a transformation $\mathbf{x} \to \lambda\mathbf{x}$. Then, in real space $\phi(\mathbf{x}) \to \phi_\lambda(\mathbf{x}) = \phi(\lambda\mathbf{x})$. Expressing this in terms of the Fourier transform of $\phi(\mathbf{x})$ gives how the rescaling acts in Fourier space

$$\phi(\lambda\mathbf{x}) = \int \mathrm{d}^3k\, e^{-i\mathbf{k}\cdot\lambda\mathbf{x}}\phi(\mathbf{k}) = \lambda^{-3}\int \mathrm{d}^3p\, e^{-i\mathbf{p}\cdot\mathbf{x}}\phi(\mathbf{p}/\lambda)\,, \tag{394}$$

where, in the last step, we have made a change in the variable of integration with $\mathbf{p} = \lambda\mathbf{k}$. Therefore, the two-point function is constrained to have the form

$$\langle\phi_{\mathbf{k}_1}\phi_{\mathbf{k}_2}\rangle = (2\pi)^3\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2)\frac{F(k_1\tau)}{k_1^3}. \tag{395}$$

In such a way

$$\langle\phi_{\mathbf{k}_1}\phi_{\mathbf{k}_2}\rangle \to \lambda^{-6}\langle\phi_{\mathbf{k}_1/\lambda}\phi_{\mathbf{k}_2/\lambda}\rangle = \lambda^{-6}(2\pi)^3\delta^{(3)}(\mathbf{k}_1/\lambda+\mathbf{k}_2/\lambda)\frac{F(k_1\tau/\lambda)}{(k_1/\lambda)^3} = (2\pi)^3\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2)\frac{F(k_1\tau/\lambda)}{k_1^3}. \tag{396}$$

If perturbations become time independent when out of the Hubble radius, the function F must be a constant in this limit and this gives a scale invariant spectrum. As we have seen, de Sitter possesses three additional isometries

$$\tau \to \tau - 2\tau(\mathbf{b}\cdot\mathbf{x}), \quad \mathbf{x} \to \mathbf{x} + \mathbf{b}(-\tau^2+x^2) - 2\mathbf{x}(\mathbf{b}\cdot\mathbf{x}), \tag{397}$$

where $\mathbf{b}$ is an infinitesimal vector. Indeed,

$$\begin{aligned}
\mathrm{d}\tau &\to \mathrm{d}\tau - 2\mathrm{d}\tau(\mathbf{b}\cdot\mathbf{x}) - 2\tau b_i\mathrm{d}x^i,\\
\mathrm{d}x^i &\to \mathrm{d}x^i + 2b^ix_j\mathrm{d}x^j - 2\mathrm{d}x^i(\mathbf{b}\cdot\mathbf{x}) - 2x^ib_j\mathrm{d}x^j - 2\tau b^i\mathrm{d}\tau,
\end{aligned} \tag{398}$$

so that

$$\begin{aligned}
\mathrm{d}\tau^2 &\to \mathrm{d}\tau^2(1-4\mathbf{b}\cdot\mathbf{x}) - 4\tau\mathrm{d}\tau b_j\mathrm{d}x^j,\\
\mathrm{d}\mathbf{x}^2 &\to \mathrm{d}\mathbf{x}^2(1-4\mathbf{b}\cdot\mathbf{x}) - 4\tau\mathrm{d}\tau b_j\mathrm{d}x^j,
\end{aligned} \tag{399}$$

and

$$\frac{1}{H^2\tau^2}(-\mathrm{d}\tau^2+\mathrm{d}\mathbf{x}^2) \to \frac{(1-4\mathbf{b}\cdot\mathbf{x})}{(1-4\mathbf{b}\cdot\mathbf{x})H^2\tau^2}(-\mathrm{d}\tau^2+\mathrm{d}\mathbf{x}^2) = \frac{1}{H^2\tau^2}(-\mathrm{d}\tau^2+\mathrm{d}\mathbf{x}^2). \tag{400}$$

The point is that for late time correlation functions of scalars, that is for $\tau \to 0$, with a mass much smaller than H, when the wavelengths are much larger than the Hubble radius, the transformation of the spatial coordinates in the isometry (397) becomes

$$\mathbf{x} \to \mathbf{x} + \mathbf{b}\, x^2 - 2\mathbf{x}(\mathbf{b}\cdot\mathbf{x}), \tag{401}$$

which is a special conformal transformation of the spatial coordinates of $\mathbb{R}^3$ on future constant-time hypersurfaces. A scalar $\phi(\tau, \mathbf{x})$ with mass m on super-Hubble scales evolves as

$$\phi \sim \tau^{\Delta}, \quad \Delta = \frac{3}{2}\left(1 - \sqrt{1 - \frac{4}{9}\frac{m^2}{H^2}}\right) \tag{402}$$

and the transformation in (397) acts the same way a conformal transformation would act on a primary field of dimension Δ. Indeed the generators of the de Sitter algebra

$$\begin{aligned} D &= -i\left(\tau\partial_\tau + x^i\partial_i\right), \\ K_i &= -2ix_i\left(\tau\partial_\tau + x^j\partial_j\right) - i(\tau^2 - x^2)\partial_i, \end{aligned} \tag{403}$$

when applied to the scalar field with scaling as in Eq. (402) reduce to

$$\begin{aligned} D &= -i\left(\Delta + x^i\partial_i\right), \\ K_i &= -2ix_i\left(\Delta + x^j\partial_j\right) + ix^2\partial_i, \end{aligned} \tag{404}$$

We conclude that the late time correlation functions, at equal time, are conformal invariant with conformal weight Δ. As an application, suppose now inflation has generated a long mode ζ_L for the comoving curvature perturbation such that the perturbed metric reads

$$\mathrm{d}s^2 = \frac{1}{H^2\tau^2}\left(\mathrm{d}\tau^2 - e^{2\zeta_L}\mathrm{d}\mathbf{x}^2\right). \tag{405}$$

Under a dilatation symmetry $\tau \to e^{\lambda}\tau$ and $\mathbf{x} \to \mathbf{x}' = e^{\lambda}\mathbf{x}$, the long mode transforms non-linearly as a Nambu-Goldstone mode, $\zeta_L \to \zeta_L - \lambda$ to leave the metric invariant. Furthermore, one can approximate the effect of such a constant long-wavelength mode on an n-point function as a rescaling of the coordinates

$$\langle\zeta(\mathbf{x}_1)\cdots\zeta(\mathbf{x}_n)\rangle_{\zeta_L} = \langle\zeta(\mathbf{x}'_1)\cdots\zeta(\mathbf{x}'_n)\rangle. \tag{406}$$

Let us take the infinitesimal version of the dilatation symmetry

$$\tau \to (1+\lambda)\tau, \qquad x_i \to (1+\lambda)x_i, \qquad (i = 1, 2, 3) \tag{407}$$

and let us see what the invariance implies. We write down the n-point correlator of the field ζ. The variation of the latter is

$$
\begin{aligned}
\delta\langle\zeta(\mathbf{x}_1,\tau)\cdots\zeta(\mathbf{x}_n,\tau)\rangle &= \left(\lambda\tau\partial_\tau+\lambda\sum_{a=1}^n x_{ai}\partial_{ai}\right)\langle\zeta(\mathbf{x}_1,\tau)\cdots\zeta(\mathbf{x}_n,\tau)\rangle \\
&= \int\frac{\mathrm{d}^3k_1}{(2\pi)^3}\cdots\int\frac{\mathrm{d}^3k_n}{(2\pi)^3}\sum_{a=1}^n\delta^{(3)}(\mathbf{k}_1+\cdots+\mathbf{k}_n)\langle\zeta_{\mathbf{k}_1}\cdots\zeta_{\mathbf{k}_n}\rangle \\
&\times \lambda k_{ai}\left(\partial_{k_{ai}}\,e^{i\mathbf{k}_1\cdot\mathbf{x}_1+\cdots+i\mathbf{k}_n\cdot\mathbf{x}_n}\right) \\
&+ n\lambda\Delta\int\frac{\mathrm{d}^3k_1}{(2\pi)^3}\cdots\int\frac{\mathrm{d}^3k_n}{(2\pi)^3}\sum_{a=1}^n\delta^{(3)}(\mathbf{k}_1+\cdots+\mathbf{k}_n)\langle\zeta_{\mathbf{k}_1}\cdots\zeta_{\mathbf{k}_n}\rangle\,e^{i\mathbf{k}_1\cdot\mathbf{x}_1+\cdots+i\mathbf{k}_n\cdot\mathbf{x}_n} \\
&= -\int\frac{\mathrm{d}^3k_1}{(2\pi)^3}\cdots\int\frac{\mathrm{d}^3k_n}{(2\pi)^3}\sum_{a=1}^n\delta^{(3)}(\mathbf{k}_1+\cdots+\mathbf{k}_n)\langle\zeta_{\mathbf{k}_1}\cdots\zeta_{\mathbf{k}_n}\rangle \\
&\times \lambda\left(\partial_{k_{ai}}k_{ai}\right)\,e^{i\mathbf{k}_1\cdot\mathbf{x}_1+\cdots+i\mathbf{k}_n\cdot\mathbf{x}_n} \\
&- \lambda\int\frac{\mathrm{d}^3k_1}{(2\pi)^3}\cdots\int\frac{\mathrm{d}^3k_n}{(2\pi)^3}\sum_{a=1}^n\delta^{(3)}(\mathbf{k}_1+\cdots+\mathbf{k}_n)k_{ai} \\
&\times \left(\partial_{k_{ai}}\langle\zeta_{\mathbf{k}_1}\cdots\zeta_{\mathbf{k}_n}\rangle\right)\,e^{i\mathbf{k}_1\cdot\mathbf{x}_1+\cdots+i\mathbf{k}_n\cdot\mathbf{x}_n}, \\
&+ n\lambda\Delta\int\frac{\mathrm{d}^3k_1}{(2\pi)^3}\cdots\int\frac{\mathrm{d}^3k_n}{(2\pi)^3}\sum_{a=1}^n\delta^{(3)}(\mathbf{k}_1+\cdots+\mathbf{k}_n)\langle\zeta_{\mathbf{k}_1}\cdots\zeta_{\mathbf{k}_n}\rangle\,e^{i\mathbf{k}_1\cdot\mathbf{x}_1+\cdots+i\mathbf{k}_n\cdot\mathbf{x}_n}, \\
&- \lambda\int\frac{\mathrm{d}^3k_1}{(2\pi)^3}\cdots\int\frac{\mathrm{d}^3k_n}{(2\pi)^3}\sum_{a=1}^n\left(k_{ai}\partial_{k_{ai}}\delta^{(3)}(\mathbf{k}_1+\cdots+\mathbf{k}_n)\right) \\
&\times \langle\zeta_{\mathbf{k}_1}\cdots\zeta_{\mathbf{k}_n}\rangle\,e^{i\mathbf{k}_1\cdot\mathbf{x}_1+\cdots+i\mathbf{k}_n\cdot\mathbf{x}_n},
\end{aligned}
\tag{408}
$$

Indicating by $\mathbf{k}=\mathbf{k}_1+\cdots+\mathbf{k}_n$, we find

$$
\begin{aligned}
k_{ai}\partial_{k_{ai}}\delta^{(3)}(\mathbf{k}_1+\cdots+\mathbf{k}_n) &= k_{ai}\frac{\partial k_j}{\partial k_{ai}}\partial_{k_j}\delta^{(3)}(\mathbf{k}_1+\cdots+\mathbf{k}_n) \\
&= k_i\partial_{k_i}\delta^{(3)}(\mathbf{k}_1+\cdots+\mathbf{k}_n) \\
&= \frac{\partial}{\partial_{k_i}}\left[k_i\delta^{(3)}(\mathbf{k}_1+\cdots+\mathbf{k}_n)\right]-3\,\delta^{(3)}(\mathbf{k}_1+\cdots+\mathbf{k}_n) \\
&= -3\,\delta^{(3)}(\mathbf{k}_1+\cdots+\mathbf{k}_n)
\end{aligned}
\tag{409}
$$

and we have introduced the scaling dimension Δ for the field ζ for completeness even though ζ is constant on super-Hubble scales and therefore $\Delta=0$ for it. Imposing that the variation vanishes, we obtain the condition

$$
\left[3(n-1)-n\Delta+\sum_{a=1}^n\mathbf{k}_a\cdot\vec{\nabla}_{k_a}\right]\langle\zeta_{\mathbf{k}_1}\cdots\zeta_{\mathbf{k}_n}\rangle'=0,
\tag{410}
$$

where the primes on the correlators stand for the standard notation indicating that correlators are evaluated without the π's and the Dirac delta function Taking $\lambda=\zeta_L$, this argument implies also that in that case the squeezed limit of the $(n+1)$-point function would be

$$
\langle\zeta_{\mathbf{q}}\zeta_{\mathbf{k}_1}\cdots\zeta_{\mathbf{k}_n}\rangle'_{q\to 0}=P_\zeta(q)\left[3(n-1)-n\Delta+\sum_{a=1}^n\mathbf{k}_a\cdot\vec{\nabla}_{k_a}\right]\langle\zeta_{\mathbf{k}_1}\cdots\zeta_{\mathbf{k}_n}\rangle'.
\tag{411}
$$

For $n = 2$ and taking into account that the $\langle \zeta_{\mathbf{k}} \zeta_{-\mathbf{k}} \rangle$ scales like $k^{-3+n_\zeta-1}$, the relation above provides the famous Maldacena's consistency relation for the three-point correlator of the comoving curvature perturbation in the squeezed limit stating that its size is proportional to the deviation of the two-point function from scale invariance and therefore proportional to the slow-roll parameters.

Similar arguments lead to the so called conformal consistency relation where the long mode of the curvature perturbation in the metric can be removed not only at the level of the constant zero mode, but also at its first constant gradient. This is achieved simply by a special conformal transformations

$$\mathbf{x} \to \mathbf{x} + \mathbf{b}\, x^2 - 2\mathbf{x}(\mathbf{b} \cdot \mathbf{x}), \tag{412}$$

which can be neutralized by transforming the long mode ζ_L as

$$\zeta_L \to \zeta_L + 2\mathbf{b} \cdot \mathbf{x}, \tag{413}$$

and taking $x_i = -1/2\partial_i \zeta_L$. Consequently, the effect a constant long-wavelength gradient mode acts on the n-point function as a rescaling of the coordinates (412)

$$\langle \zeta(\mathbf{x}_1) \cdots \zeta(\mathbf{x}_n) \rangle_{\vec{\nabla}\zeta_L} = \langle \zeta(\mathbf{x}'_1) \cdots \zeta(\mathbf{x}'_n) \rangle, \tag{414}$$

which in momentum space becomes

$$\langle \zeta_{\mathbf{q}} \zeta_{\mathbf{k}_1} \cdots \zeta_{\mathbf{k}_n} \rangle'_{q \to 0} = -\frac{1}{2} P_\zeta(q) q^i \sum_{a=1}^{n} \left(6\vec{\nabla}_{k_{ai}} - k_{ai} \nabla^2_{k_a} + 2\mathbf{k}_a \cdot \vec{\nabla}_{k_a} \right) \langle \zeta_{\mathbf{k}_1} \cdots \zeta_{\mathbf{k}_n} \rangle'. \tag{415}$$

Part V

The post-inflationary evolution of the cosmological perturbations

The primordial perturbations set up during inflation manifest themselves in the radiation as well as in the matter distribution. By understanding the evolution of the photon perturbations we can make therefore predictions about the expected CMB anisotropy spectrum today. The evolution is determined by Einstein equations and Boltzmann equations. Perturbations to photons evolved completely different before and after the epoch of last scattering at $z_{\rm ls} \sim 1100$. Before recombination, photons were tightly coupled to electrons and protons; all together they can be described by a single fluid, dubbed the baryon-photon fluid. After recombination, photons free-streamed from the surface of last scattering to us today. This means that detecting these photons is like taking a picture of the universe when it was about 300.000 yr old.

Inflation provides the initial conditions for the perturbations once the latter re-enter the horizon. Let us turn again to the longitudinal gauge. On super-Hubble scales, from Eq. (204) we have

$$6\mathcal{H}^2\Phi_{\mathbf{k}} = -4\pi G_{\rm N}a^2\frac{\delta\rho_{\mathbf{k}}}{\bar{\rho}} \Rightarrow \frac{\delta\rho_{\mathbf{k}}}{\bar{\rho}} = -2\Phi, \tag{416}$$

on super-Hubble scales, where $\mathcal{H}^2 = (8\pi G_{\rm N}/3)\bar{\rho}a^2$ defines the average energy density. Recalling now that $\Psi = \Phi$ and that $\zeta = \Psi + \mathcal{H}\delta\rho/\rho'$ and $\rho' = -3\,\mathcal{H}(\rho + P) = -3\,\mathcal{H}(1+w)\rho$, where we have defined $P/\rho = w$, we find that on super-Hubble scales

$$\zeta_{\mathbf{k}} = \Phi_{\mathbf{k}} - \frac{\delta\rho_{\mathbf{k}}}{3(1+w)\bar{\rho}} = \left(1 + \frac{2}{3(1+w)}\right)\Phi_{\mathbf{k}} = \frac{5+3w}{3(1+w)}\Phi_{\mathbf{k}}. \tag{417}$$

This means that during the RD phase one has ($w = 1/3$)

$$\boxed{\Phi_{\mathbf{k}}^{\rm RD} = \frac{2}{3}\zeta_{\mathbf{k}}\ (\text{RD})}, \tag{418}$$

and during the MD phase

$$\boxed{\Phi_{\mathbf{k}}^{\rm MD} = \frac{3}{5}\zeta_{\mathbf{k}}\ (\text{MD})}, \tag{419}$$

In particular, notice that

$$\boxed{\Phi_{\mathbf{k}}^{\rm MD} = \frac{9}{10}\Phi_{\mathbf{k}}^{\rm RD}}. \tag{420}$$

One of the last steps we wish to take is now fixing the amplitude of the density perturbation in the CMB through inflation. As we have seen, on large scales and during matter-domination we have at last scattering

$$\frac{\delta\rho_{\rm m}}{\bar{\rho}_{\rm m}} = -2\Phi^{\rm MD}(\tau_{\rm ls}), \tag{421}$$

and, if the adiabatic condition holds,

$$\frac{1}{3}\frac{\delta\rho_{\rm m}}{\bar{\rho}_{\rm m}}=\frac{1}{4}\Delta_0(\tau_{\rm ls}), \tag{422}$$

where $\Delta=\delta\rho_{\rm r}/\bar{\rho}_{\rm r}$ and the lower index $_0$ stands for the monopole. The CMB anisotropy has an oscillating structure (the famous Doppler peaks) because the baryon-photon fluid oscillations. However, The overall amplitude of the CMB anisotropy can be fixed at large angular scales (super-Hubble modes) where there is no evolution and therefore one can match the amplitude with the theoretical prediction from inflation. A standard computation implies that the observed CMB anisotropy on large scales at the last scattering epoch should be the Sachs-Wolfe term [20, 65]

$$\boxed{\left(\frac{\Delta}{4}+\Phi^{\rm MD}\right)_{\rm SW}(\tau_{\rm ls})=\left(-\frac{2}{3}+1\right)\Phi^{\rm MD}=\frac{1}{3}\Phi^{\rm MD}(\tau_{\rm ls}).} \tag{423}$$

The temperature anisotropy is commonly expanded in spherical harmonics

$$\frac{\Delta T}{T}(\mathbf{x}_0,\tau_0,\mathbf{n})=\sum_{\ell m}a_{\ell m}(\mathbf{x}_0)Y_{\ell m}(\mathbf{n}), \tag{424}$$

where $\mathbf{x}_0$ and τ_0 are our position and the present time, respectively, $\mathbf{n}$ is the direction of observation, ℓ's are the different multipoles and

$$\langle a_{\ell m}a^*_{\ell' m'}\rangle=\delta_{\ell\ell'}\delta_{mm'}C_\ell, \tag{425}$$

where the deltas are due to the fact that the process that created the anisotropy is statistically isotropic. The C_ℓ are the so-called CMB power spectrum. For homogeneity and isotropy, the C_ℓ's are neither a function of $\mathbf{x}_0$, nor of m. We get therefore that

$$a_{\ell m}(\mathbf{x}_0,\tau_0)=\int\frac{{\rm d}^3k}{(2\pi)^3}e^{i\mathbf{k}\cdot\mathbf{x}_0}\int{\rm d}\Omega\, Y^*_{\ell m}(\mathbf{n})\Theta(\mathbf{k},\mathbf{n},\tau_0), \tag{426}$$

where we have made use the orthonormality property of the spherical harmonics

$$\int{\rm d}\Omega\, Y^*_{\ell m}(\mathbf{n})\, Y_{\ell' m'}(\mathbf{n})=\delta_{\ell\ell'}\delta_{mm'}. \tag{427}$$

The C_ℓ are given by

$$\begin{aligned}
C_\ell &= \int\frac{{\rm d}^3k}{(2\pi)^3}\int\frac{{\rm d}^3p}{(2\pi)^3}e^{i(\mathbf{k}-\mathbf{p})\cdot\mathbf{x}_0}\int{\rm d}\Omega\, Y^*_{\ell m}(\mathbf{n})\int{\rm d}\Omega'\, Y_{\ell m}(\mathbf{n}')\Big\langle\Theta(\mathbf{k},\mathbf{n},\tau_0)\Theta^*(\mathbf{p},\mathbf{n}',\tau_0)\Big\rangle\\
&= \sum_{\ell'\ell''}(-i)^{\ell'+\ell''}(2\ell'+1)(2\ell''+1)\int\frac{{\rm d}^3k}{(2\pi)^3}\int\frac{{\rm d}^3p}{(2\pi)^3}e^{i(\mathbf{k}-\mathbf{p})\cdot\mathbf{x}_0}\\
&\times \int{\rm d}\Omega\, Y^*_{\ell m}(\mathbf{n})P_{\ell'}(\mathbf{k}\cdot\mathbf{n})\int{\rm d}\Omega'\, Y_{\ell m}(\mathbf{n}')P_{\ell''}(\mathbf{p}\cdot\mathbf{n}')\Big\langle\Theta_{\ell'}(\mathbf{k})\Theta^*_{\ell''}(\mathbf{p})\Big\rangle.
\end{aligned} \tag{428}$$

where we have decomposed the temperature anisotropy in multipoles as usual

$$\Theta(\mathbf{k},\mathbf{n},\tau_0)=\sum_\ell(-i)^\ell(2\ell+1)P_\ell(\mathbf{k}\cdot\mathbf{n})\Theta_\ell(k). \tag{429}$$

In the SW limit we have

$$\Theta_\ell^{\rm SW}(\mathbf{k}) \simeq \frac{1}{3}\Phi^{\rm MD}(\mathbf{k},\tau_{\rm ls})j_\ell(k\tau_0), \tag{430}$$

with the spectrum of the gravitational potential defined as

$$\left\langle \Phi^{\rm MD}(\mathbf{k},\tau_{\rm ls})\Phi^{\rm MD}(\mathbf{p},\tau_{\rm ls})\right\rangle = (2\pi)^3\delta^{(3)}(\mathbf{k}+\mathbf{p})P_{\Phi^{\rm MD}}(k). \tag{431}$$

Therefore we obtain

$$\begin{aligned}
C_\ell^{\rm SW} &= \frac{1}{9}\int\frac{{\rm d}^3k}{(2\pi)^3}P_{\Phi^{\rm MD}}(k)j_\ell^2(k\tau_0)\\
&\times \sum_{\ell'\ell''}(-i)^{\ell'-\ell''}(2\ell'+1)(2\ell''+1)\int {\rm d}\Omega\, Y_{\ell m}^*(\mathbf{n})P_{\ell'}(\mathbf{k}\cdot\mathbf{n})\int {\rm d}\Omega'\, Y_{\ell m}(\mathbf{n}')P_{\ell''}(\mathbf{p}\cdot\mathbf{n}')\\
&= \frac{1}{9}\int\frac{{\rm d}^3k}{(2\pi)^3}P_{\Phi^{\rm MD}}(k)j_\ell^2(k\tau_0)\\
&\times \sum_{\ell'\ell''}(-i)^{\ell'-\ell''}(2\ell'+1)(2\ell''+1)\frac{4\pi}{(2\ell+1)}\delta_{\ell\ell'}Y_{\ell m}(\mathbf{k})\frac{4\pi}{(2\ell+1)}\delta_{\ell\ell''}Y_{\ell m}^*(\mathbf{k})\\
&= \frac{2}{9\pi}\int {\rm d}k\, k^2P_{\Phi^{\rm MD}}(k)j_\ell^2(k\tau_0)\int {\rm d}\Omega_{\mathbf{k}}\, |Y_{\ell m}(\mathbf{k})|^2\\
&= \frac{2}{9\pi}\int\, dk\, k^2P_{\Phi^{\rm MD}}(k)j_\ell^2(k\tau_0),
\end{aligned} \tag{432}$$

where we have made use of the property

$$P_\ell(\mathbf{n}\cdot\mathbf{n}') = \frac{4\pi}{2\ell+1}\sum_{m=-\ell}^{\ell} Y_{\ell m}(\mathbf{n})Y_{\ell m}^*(\mathbf{n}'). \tag{433}$$

If we generically indicate by

$$\left\langle \left|\Phi_{\mathbf{k}}^{\rm MD}\right|^2\right\rangle k^3 = A^2\,(k\tau_0)^{n-1}, \tag{434}$$

we can perform the integration knowing that

$$\int_0^\infty {\rm d}x\, j_\ell^2(x)x^{n-2} = 2^{n-4}\frac{\Gamma(3-n)\Gamma\left(\frac{2\ell+n-1}{2}\right)}{\Gamma^2\left(\frac{4-n}{2}\right)\Gamma\left(\frac{2\ell+5-n}{2}\right)}. \tag{435}$$

We obtain

$$C_\ell^{\rm SW} = \frac{2^{n-3}A^2}{9\pi}\left(\frac{H_0}{2}\right)^{n-1}\frac{\Gamma(3-n)\Gamma(\ell+\frac{n-1}{2})}{\Gamma^2\left(\frac{4-n}{2}\right)\Gamma\left(\ell+\frac{5-n}{2}\right)} \tag{436}$$

For $n\simeq 1$ and $100 \gg \ell \gg 1$, we can approximate this expression knowing that $\Gamma(\ell)/\Gamma(\ell+2) = [\ell(\ell+1)]^{-1}$ and $\Gamma(2)/\Gamma^2(3/2) = 4/\pi$, and we get

$$\ell(\ell+1)C_\ell^{\rm SW} = \frac{A^2}{9\pi^2}. \tag{437}$$

This result shows that inflation predicts a very flat spectrum for low ℓ. This prediction has been confirmed by CMB anisotropy measurements. Furthermore, since inflation predicts $\Phi_{\mathbf{k}}^{\rm MD} = \frac{3}{5}\zeta_{\mathbf{k}}$, we find that

$$\ell(\ell+1)C_\ell^{\rm SW} = \frac{A_\zeta^2}{25\pi^2} = \frac{1}{25\pi^2}\frac{1}{2\,\overline{M}_{\rm Pl}^2\,\epsilon}\left(\frac{H}{2\pi}\right)^2. \tag{438}$$

Assuming that

$$\ell(\ell+1)C_\ell^{\rm SW} \simeq 10^{-10}, \tag{439}$$

we find

$$\boxed{\left(\frac{V}{\epsilon}\right)^{1/4} \simeq 6.7\times 10^{16}\,{\rm GeV}.}$$

Take for instance a model of chaotic inflation with quadratic potential $V(\phi) = \frac{1}{2}m^2\phi^2$. Using Eq. (114) one easily computes that when there are ΔN e-foldings to go, the value of the inflaton field is $\phi^2_{\Delta N} = (\Delta N/2\pi G_{\rm N})$ and the corresponding value of ϵ is $1/(2\Delta N)$. Taking $\Delta N \simeq 60$ (corresponding to large-angle CMB anisotropies), one finds that COBE normalization imposes $m \simeq 10^{13}$ GeV.

Part VI

Comments on non-Gaussianity in the cosmological perturbations

Up to now, we have been describing the cosmological perturbations at the linear level. Is that a correct assumption? After all we know that gravity is non-linear, so some amount of non-linearities should be expected. This is true both during inflation and after inflation: gravity is always present. Furthermore, the inflaton potential may be characterized by self-interactions.

Non-Gaussianities (NG's) in cosmology are reviewed in Ref. [12] and we wish to illustrate a simple computation illustrating why NG arises due to the gravitational interactions, which are non-linear [13]. We use the metric in the Poisson gauge (we only focus on the scalar degrees of freedom)

$$\mathrm{d}s^2 = -e^{2\Phi}\mathrm{d}t^2 + a^2(t)e^{-2\Psi}\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j \,. \tag{440}$$

The choice of adopting exponentials is a technical and convenient one, as we shall see. We consider only perturbations with wavelengths larger than the horizon at the last scattering surface. A local observer perceives them as classical a background. We write the gravitational potential Φ as

$$\Phi = \Phi_\ell + \Phi_s \,, \tag{441}$$

where

$$\begin{aligned}
\Phi_\ell &= \int \frac{\mathrm{d}^3k}{(2\pi)^3}\,\theta\left(aH - k\right)\,\Phi_{\mathbf{k}}\;e^{i\mathbf{k}\cdot\mathbf{x}}\,,\\
\Phi_s &= \int \frac{\mathrm{d}^3k}{(2\pi)^3}\,\theta\left(k - aH\right)\,\Phi_{\mathbf{k}}\;e^{i\mathbf{k}\cdot\mathbf{x}}\,,
\end{aligned} \tag{442}$$

An analogous definition is adopted for the gravitational potential Ψ. For Φ_ℓ and Ψ_ℓ we can neglect the spatial gradients and can be considered only as functions of time. By the redefinition

$$\mathrm{d}\bar{t} = e^{\Phi_\ell}\mathrm{d}t\,, \tag{443}$$

$$\bar{a} = a\,e^{-\Psi_\ell}\,, \tag{444}$$

we can therefore absorb the long modes and obtain a new metric describing a homogeneous and isotropic Universe

$$\mathrm{d}s^2 = -\mathrm{d}\bar{t}^2 + \bar{a}^2\delta_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j \,. \tag{445}$$

The Universe now is a a collection of regions of size of the Hubble radius evolving like unperturbed patches.

Take now a photon moving from an emission point $\mathcal{E}$ to the observer point $\mathcal{O}$. It suffers a redshift given by the ration between the frequency $\overline{\omega}_{\mathcal{E}}$ and the observed one $\overline{\omega}_{\mathcal{O}}$

$$\overline{T}_{\mathcal{O}} = \overline{T}_{\mathcal{E}}\,\frac{\overline{\omega}_{\mathcal{O}}}{\overline{\omega}_{\mathcal{E}}}\,. \tag{446}$$

This expression tells us that large-scale anisotropies get contributions: the intrinsic inhomogeneities at the last scattering surface and the anisotropies in the frequencies. The photon frequency goes like the inverse of a time period and therefore

$$\frac{\overline{\omega}_{\mathcal{E}}}{\overline{\omega}_{\mathcal{O}}} = \frac{\omega_{\mathcal{E}}}{\omega_{\mathcal{O}}} e^{-\Phi_{\ell\mathcal{E}}+\Phi_{\ell\mathcal{O}}} . \tag{447}$$

As for the intrinsic fluctuation, we relate the photon energy density $\overline{\rho}_\gamma$ to the energy density of the non-relativistic matter $\overline{\rho}_{\rm m}$ by using the adiabaticity condition. Using the energy continuity equation on large scales $\partial\overline{\rho}/\partial\overline{t} = -3\overline{H}(\overline{\rho}+\overline{P})$, where $\overline{H} = {\rm d}\ln\overline{a}/{\rm d}\overline{t}$ and $\overline{P}$ is the pressure of the fluid, we have shown previously in the lectures that the quantity

$$-\zeta \equiv \ln\overline{a} + \frac{1}{3}\int^{\overline{\rho}} \frac{d\overline{\rho'}}{(\overline{\rho'}+\overline{P'})} \tag{448}$$

is conserved in time at any order in perturbation theory. At the non-linear level the adiabaticity condition becomes to

$$\frac{1}{3}\int \frac{{\rm d}\overline{\rho}_{\rm m}}{\overline{\rho}_{\rm m}} = \frac{1}{4}\int \frac{{\rm d}\overline{\rho}_\gamma}{\overline{\rho}_\gamma} , \tag{449}$$

or

$$\ln\overline{\rho}_{\rm m} = \ln\overline{\rho}_\gamma^{3/4} . \tag{450}$$

The (0-0) component of Einstein equations in the matter-dominated era with the "barred" metric (445) gives

$$\overline{H}^2 = \frac{8\pi G_N}{3}\overline{\rho}_{\rm m} . \tag{451}$$

Using Eqs. (443) and (444) the Hubble parameter $\overline{H}$ reads

$$\overline{H} = \frac{1}{\overline{a}}\frac{{\rm d}\overline{a}}{{\rm d}\overline{t}} = e^{-\Phi_\ell}(H - \dot{\Psi}_\ell) , \tag{452}$$

where $H = {\rm d}\ln a/{\rm d}t$ is the Hubble parameter in the "unbarred" metric. We then obtain

$$\overline{\rho}_{\rm m} = \rho_{\rm m} e^{-2\Phi_\ell} , \tag{453}$$

having dropped the negligible piece $\dot{\Psi}_\ell$ on large scales. Finally, we obtain

$$\overline{T}_{\mathcal{E}} = T_{\mathcal{E}}\, e^{-2\Phi_\ell/3} , \tag{454}$$

and

$$\overline{T}_{\mathcal{O}} = \left(\frac{\omega_{\mathcal{O}}}{\omega_{\mathcal{E}}}\right) T_{\mathcal{E}}\, e^{\Phi_\ell/3} . \tag{455}$$

It allows to find the Sachs-Wolfe effect at all orders

$$\frac{\delta_{\rm np}\overline{T}_{\mathcal{O}}}{T_{\mathcal{O}}} = e^{\Phi_\ell/3} - 1 . \tag{456}$$

Eq. (456) represents the extension of the linear Sachs-Wolfe effect. At first order one gets

$$\frac{\delta^{(1)}T_{\mathcal{O}}}{T_{\mathcal{O}}} = \frac{1}{3}\Phi^{(1)} , \tag{457}$$

and at second order

$$\frac{1}{2}\frac{\delta^{(2)}T_{\mathcal{O}}}{T_{\mathcal{O}}} = \frac{1}{6}\Phi^{(2)} + \frac{1}{18}\left(\Phi^{(1)}\right)^2 . \tag{458}$$

This result shows that the CMB anisotropies are nonlinear on large scales and that a source of NG is inevitably sourced by gravity and that the corresponding nonlinearities are order unity in units of the linear gravitational potential.

20 Primordial non-Gaussianity

One of the first ways to parameterize non-Gaussianity phenomenologically was via a non-linear correction to a Gaussian primordial perturbation $\zeta_{\rm g}$,

$$\boxed{\zeta(\mathbf{x}) = \zeta_{\rm g}(\mathbf{x}) + \frac{3}{5} f_{\rm NL}^{\rm local}\left[\zeta_{\rm g}(\mathbf{x})^2 - \langle \zeta_{\rm g}(\mathbf{x})^2\rangle\right]}\,. \tag{459}$$

This definition is local in real space and therefore called local NG. Experimental constraints on non-Gaussianity are often set on the parameter $f_{\rm NL}^{\rm local}$ defined via Eq. (459). The factor of 3/5 in Eq. (459) is historical because NG was initially parametrized by using the Newtonian potential, $\Phi(\mathbf{x}) = \Phi_{\rm g}(\mathbf{x}) + f_{\rm NL}^{\rm local}\left[\Phi_{\rm g}(\mathbf{x})^2 - \langle \Phi_{\rm g}(\mathbf{x})^2\rangle\right]$, which during the matter era is related to ζ by a factor of 3/5. Despite the absence of primordial NG in the CMB data [60]

$$f_{\rm NL}^{\rm loc} = 0.8 \pm 5 \;\text{ at }\; 68\%\,\text{CL}, \tag{460}$$

the detecting some level of NG is still an important target of many current experiments in cosmology. What size do we expect as far as primordial NG is concerned?

20.1 Non-Gaussianity in single-field models

Let us consider now a period of inflation and that there are a number of light fields σ^I which are quantum mechanically excited. As we have previously seen, by the δN formalism, the comoving curvature perturbation ζ on a uniform energy density hypersurface at time $t_{\rm f}$ is, on sufficiently large scales, equal to the perturbation in the time integral of the local expansion from an initial flat hypersurface ($t = t_*$) to the final uniform energy density hypersurface. On sufficiently large scales, the local expansion can be approximated quite well by the expansion of the unperturbed Friedmann universe. Hence the curvature perturbation at time $t_{\rm f}$ can be expressed in terms of the values of the relevant scalar fields $\sigma^I(t_*, \vec{x})$ at t_* (notice the change of an irrelevant sign with respect to the previous definition of ζ (261))

$$\zeta(t_{\rm f}, \vec{x}) = N_I \sigma^I + \frac{1}{2} N_{IJ}\sigma^I\sigma^J + \cdots\,, \tag{461}$$

where N_I and N_{IJ} are the first and second derivative, respectively, of the number of e-folds

$$N(t_{\rm f}, t_*, \vec{x}) = \int_{t_*}^{t_{\rm f}} {\rm d}t\, H(t, \vec{x})\,. \tag{462}$$

with respect to the field σ^I. From the expansion (461) one can read off the n-point correlators. For instance, the three-point correlator of the comoving curvature perturbation, the so-called bispectrum and trispectrum respectively, is given by

$$B_\zeta(k_1, k_2, k_3) = N_I N_J N_K B^{IJK}_{k_1 k_2 k_3} + N_I N_{JK} N_L \left(P^{IK}_{k_1} P^{JL}_{k_2} + 2 \text{ permutations}\right) \tag{463}$$

Let us consider single-field inflation. Using the fact that $N_\phi = H/\dot\phi$, one gets that

$$\frac{N_{\phi\phi}}{N_\phi^2} = \frac{1}{N_\phi^2}\frac{\rm d}{\dot\phi\,{\rm d}t}N_\phi = \left(\frac{\dot H}{\dot\phi} - \frac{H\ddot\phi}{\dot\phi^2}\right)\times\frac{1}{\dot\phi}\times\frac{\dot\phi^2}{H^2} = \left(\frac{\dot H}{H^2} - \frac{\ddot\phi}{H\dot\phi}\right) = (-\epsilon + \eta - \epsilon) = \eta - 2\epsilon\,. \tag{464}$$

This incomplete result makes intuitive sense since the slow-roll parameters characterize deviations of the inflaton from a free field. To get the the full result behavior, let us consider Eq. (463) restricting ourselves to the one-single field case. Then

$$B_\zeta(k_1,k_2,k_3) = N_\phi^3 B^\phi_{k_1k_2k_3} + N_\phi^2 N_{\phi\phi}\left(P^\phi(k_1)P^\phi(k_2) + 2 \text{ permutations}\right) . \tag{465}$$

At first-order we have $\delta\phi^{(1)}_{\mathbf{k}} \simeq (H/2\pi)$. However at second-order there is a local correction to the amplitude of vacuum fluctuations at Hubble exit due to first-order perturbations in the local Hubble rate $\widetilde{H}(\phi)$. This is determined by the local scalar field value due to longer wavelength modes that have already left the horizon

$$\widetilde{H}(\phi) = H(\phi) + H'(\phi)\int_0^{k_c}\frac{\mathrm{d}^3k}{(2\pi)^3}\delta\phi_{\mathbf{k}} \, , \tag{466}$$

where k_c is the cut-off wavenumber which selects only long wavelength perturbation at horizon crossing. Thus for a mode $k_1 \simeq k_2 \gg k_3$ one can write at second-order

$$\delta\phi^{(2)}_{\mathbf{k}_1} \simeq \frac{H'}{H}\int_0^{k_c}\frac{\mathrm{d}^3k'}{(2\pi)^3}\delta\phi^{(1)}_{\mathbf{k}'}\delta\phi^{(1)}_{\mathbf{k}_1-\mathbf{k}'} \, , \tag{467}$$

where $k_1 \simeq k_2 \gg k_c$. The bispectrum for the inflation field therefore reads in the squeezed limit

$$\begin{aligned}
B^\phi_{k_1k_2k_3} &\simeq \langle\delta\phi^{(2)}_{\mathbf{k}_1}\delta\phi^{(1)}_{\mathbf{k}_2}\delta\phi^{(1)}_{\mathbf{k}_3}\rangle + \langle\delta\phi^{(1)}_{\mathbf{k}_1}\delta\phi^{(2)}_{\mathbf{k}_2}\delta\phi^{(1)}_{\mathbf{k}_3}\rangle \simeq (2\pi)^3\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)2\frac{H'}{H}P^\phi(k_3)P^\phi(k_1) \\
&\simeq -2\epsilon\left(\frac{H}{\dot\phi}\right)(2\pi)^3\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)P^\phi(k_3)P^\phi(k_1) \\
&= -2\epsilon N_\phi(2\pi)^3\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)P^\phi(k_3)P^\phi(k_1) \, .
\end{aligned} \tag{468}$$

Using Eq. (464) we then get

$$\begin{aligned}
B_\zeta(k_1,k_2,k_3) &= (2\pi)^3\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\left[-2\epsilon N_\phi^4 P^\phi(k_3)P^\phi(k_1) + 2(\eta-2\epsilon)P^\zeta(k_3)P^\zeta(k_1)\right] \\
&= (2\eta-6\epsilon)P^\zeta(k_3)P^\zeta(k_1) \\
&= (n_\zeta-1)P^\zeta(k_3)P^\zeta(k_1)
\end{aligned} \tag{469}$$

and we have obtained a $(n_\zeta - 1)$ suppression. In fact this result is more general is valid for any type of single-field models (for instance even for those without a canonical kinetic term) as long as the inflaton is the only degree of freedom present

$$\boxed{\lim_{k_3\to 0}\langle\zeta_{\mathbf{k}_1}\zeta_{\mathbf{k}_2}\zeta_{\mathbf{k}_3}\rangle = (2\pi)^3\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\,(n_\zeta-1)\,P^\zeta(k_1)P\zeta(k_3)} \, . \tag{470}$$

Eq. (470) states that the squeezed limit of the three-point function for single-field inflation,as well as the corresponding $f_{\mathrm{NL}}^{\mathrm{loc}}$, is always suppressed by $(1-n_\zeta)$. A detection of a large enough non-Gaussianity in the squeezed limit can therefore rule out single-field inflation altogether. let us see how to get the result (470) in general. Let us take a mode with long wavelength k_L. The corresponding curvature perturbation $\zeta_{\mathbf{k}_L}$ rescales the spatial coordinates (or changes the effective scale factor) within a given Hubble patch

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)e^{-2\zeta}\mathrm{d}\mathbf{x}^2 \, . \tag{471}$$

The two-point function $\langle\zeta_{\mathbf{k}_1}\zeta_{\mathbf{k}_2}\rangle$, with $k_1 \simeq k_2 \gg k_3 = k_L$, will depend on the value of the background fluctuations $\zeta_{\mathbf{k}_\mathrm{L}}$ already frozen outside the horizon. In position space the variation of the two-point function given by the long-wavelength fluctuations ζ_L is at linear order

$$\frac{\partial}{\partial\zeta_\mathrm{L}}\langle\zeta(x)\zeta(0)\rangle\cdot\zeta_\mathrm{L} = -x\frac{\mathrm{d}}{\mathrm{d}x}\langle\zeta(x)\zeta(0)\rangle\cdot\zeta_\mathrm{L}\,. \tag{472}$$

To get the three-point function one multiplies Eq. (472) by ζ_L and average over it. Going to Fourier space gives Eq. (470).

20.2 Non-Gaussianity in multiple field models

If we depart from single-field inflation models, where the interactions are limited by the requirement that the flatness of the potential is not spoiled, one can construct models where non-Gaussianity can be sizable. In models like the curvaton mechanism or inhomogeneous reheating non-Gaussian fluctuations are generated thanks to an extra field other than the inflaton. Let us describe in some detail the curvaton case. We expand the curvaton field up to first-order in the perturbations around the homogeneous background as $\sigma(\tau,\mathbf{x}) = \sigma_0(\tau) + \delta\sigma$, the linear perturbations satisfy on large scales

$$\delta\sigma'' + 2\,\mathcal{H}\delta\sigma' + a^2\frac{\partial^2 V}{\partial\sigma^2}\,\delta\sigma = 0\,. \tag{473}$$

As a result on super-Hubble scales its fluctuations $\delta\sigma$ will be Gaussian distributed and with a nearly scale-invariant spectrum given by

$$\mathcal{P}^{\frac{1}{2}}_{\delta\sigma}(k) \approx \frac{H_*}{2\pi}\,, \tag{474}$$

where the subscript $*$ denotes the epoch of horizon exit $k = aH$. Once inflation is over the inflaton energy density will be converted to radiation (γ) and the curvaton field will remain approximately constant until $H^2 \sim m_\sigma^2$. At this epoch the curvaton field begins to oscillate around the minimum of its potential which can be safely approximated to be quadratic $V \approx \frac{1}{2}m_\sigma^2\sigma^2$. During this stage the energy density of the curvaton field just scales as non-relativistic matter $\rho_\sigma \propto a^{-3}$. The energy density in the oscillating field is

$$\rho_\sigma(\tau,\mathbf{x}) \approx m_\sigma^2\sigma^2(\tau,\mathbf{x})\,, \tag{475}$$

and it can be expanded into a homogeneous background $\rho_\sigma(\tau)$ and a second-order perturbation $\delta\rho_\sigma$ as

$$\rho_\sigma(\tau,\mathbf{x}) = \rho_\sigma(\tau) + \delta\rho_\sigma(\tau,\mathbf{x}) = m_\sigma^2\sigma + 2m_\sigma^2\,\sigma\,\delta\sigma + m_\sigma^2\delta\sigma^2\,. \tag{476}$$

The ratio $\delta\sigma/\sigma$ remains constant and the resulting relative energy density perturbation is

$$\frac{\delta\rho_\sigma}{\rho_\sigma} = 2\left(\frac{\delta\sigma}{\sigma}\right)_* + \left(\frac{\delta\sigma}{\sigma}\right)^2_*\,, \tag{477}$$

where the $*$ stands for the value at horizon crossing. Such perturbations in the energy density of the curvaton field produce in fact a primordial density perturbation well after the end of inflation and a potentially large NG.

During the oscillations of the curvaton field, the total curvature perturbation can be written as a weighted sum of the single curvature perturbations

$$\zeta = (1-f)\zeta_\gamma + f\zeta_\sigma \,, \tag{478}$$

where the quantity

$$f = \frac{3\rho_\sigma}{4\rho_\gamma + 3\rho_\sigma} \tag{479}$$

defines the relative contribution of the curvaton field to the total curvature perturbation. Working under the approximation of sudden decay of the curvaton field. Under this approximation the curvaton and the radiation components ρ_σ and ρ_γ satisfy separately the energy conservation equations

$$\begin{aligned} \rho'_\gamma &= -4\,\mathcal{H}\rho_\gamma \,, \\ \rho'_\sigma &= -3\,\mathcal{H}\rho_\sigma \,, \end{aligned} \tag{480}$$

and the curvature perturbations ζ_i remains constant on super-Hubble scales until the decay of the curvaton. In the curvaton scenario it is supposed that the curvature perturbation in the radiation produced at the end of inflation is negligible. From Eq. (478) the total curvature perturbation during the curvaton oscillations is given by

$$\zeta = f\zeta_\sigma \simeq \frac{f}{3}\frac{\delta\rho_\sigma}{\rho_\sigma} \simeq \frac{f}{3}\left[2\left(\frac{\delta\sigma}{\sigma}\right)_* + \left(\frac{\delta\sigma}{\sigma}\right)_*^2\right] , \tag{481}$$

from which we deduce that

$$\zeta = \zeta_{\rm g} + \frac{3}{4f}(\zeta_{\rm g}^2 - \langle\zeta_{\rm g}^2\rangle) \,, \;\; \zeta_{\rm g} = (2f/3)(\delta\sigma/\sigma)_* \,, \tag{482}$$

and therefore

$$f_{\rm NL}^{\rm loc} = \frac{5}{4f} \,. \tag{483}$$

We discover that the NG can be very large if $f \ll 1$. Furthermore, the NG is of the local type. This is because it is generated not at horizon-crossing, but when the fluctuations are already outside the horizon.

It is nice to reproduce the same result with the δN formalism [67]. In the absence of interactions, fluids characterized by well-defined equation of state (for radiation $P_\gamma = \rho_\gamma/3$) or for the non-relativistic curvaton ($P_\sigma = 0$), there are the conserved curvature perturbations (notice again a change of an irrelevant sign from Eq. (261))

$$\zeta_i = \delta N + \frac{1}{3}\int_{\bar\rho_i}^{\rho_i} \frac{{\rm d}\tilde\rho_i}{\tilde\rho_i + P_i(\tilde\rho_i)} \,. \tag{484}$$

If the curvaton decays on a uniform-total density hypersurface corresponding to $H = \Gamma$, *i.e.*, when the local Hubble rate equals the decay rate for the curvaton (assumed constant), then on this s hypersurface we have

$$\rho_\gamma(t_{\rm dec}, \mathbf{x}) + \rho_\sigma(t_{\rm dec}, \mathbf{x}) = \bar\rho(t_{\rm dec}) \,. \tag{485}$$

The quantity ζ is conserved after the curvaton decay since the total pressure is simply $P = \rho/3$.

However, the local curvaton and radiation densities on the decay surface are perturbed

$$\zeta_\gamma = \zeta + \frac{1}{4}\ln\left(\frac{\rho_\gamma}{\bar{\rho}_\gamma}\right), \tag{486}$$

$$\zeta_\sigma = \zeta + \frac{1}{3}\ln\left(\frac{\rho_\sigma}{\bar{\rho}_\sigma}\right), \tag{487}$$

or, equivalently,

$$\rho_\gamma = \bar{\rho}_\gamma e^{4(\zeta_\gamma - \zeta)}, \tag{488}$$

$$\rho_\sigma = \bar{\rho}_\sigma e^{3(\zeta_\sigma - \zeta)}. \tag{489}$$

Imposing that total density is not perturbed on the decay surface, we obtain the relation

$$(1-\Omega_{\sigma,\mathrm{dec}})e^{4(\zeta_\gamma-\zeta)} + \Omega_{\sigma,\mathrm{dec}}e^{3(\zeta_\sigma-\zeta)} = 1\,, \tag{490}$$

where $\Omega_{\sigma,\mathrm{dec}} = \bar{\rho}_\sigma/(\bar{\rho}_\gamma + \bar{\rho}_\sigma)$ is the dimensionless density parameter for the curvaton at the decay time.

Let us consider the simplest possible scenario, where the curvature perturbation in the radiation fluid before the curvaton decays is negligible, *i.e.*, $\zeta_\gamma = 0$. Eq. (490) reads

$$e^{4\zeta} - \left[\Omega_{\sigma,\mathrm{dec}}e^{3\zeta_\sigma}\right] e^{\zeta} + \left[\Omega_{\sigma,\mathrm{dec}} - 1\right] = 0\,. \tag{491}$$

At first-order Eq. (490) gives

$$4(1-\Omega_{\sigma,\mathrm{dec}})\zeta^{(1)} = 3\Omega_{\sigma,\mathrm{dec}}(\zeta_\sigma^{(1)} - \zeta^{(1)})\,, \tag{492}$$

and hence we can write

$$\zeta^{(1)} = f\zeta_\sigma^{(1)}\,, \tag{493}$$

where

$$f = \frac{3\Omega_{\sigma,\mathrm{dec}}}{4-\Omega_{\sigma,\mathrm{dec}}} = \left.\frac{3\bar{\rho}_\sigma}{3\bar{\rho}_\sigma + 4\bar{\rho}_\gamma}\right|_{t_{\mathrm{dec}}}. \tag{494}$$

At second order Eq. (490) gives

$$4(1-\Omega_{\sigma,\mathrm{dec}})\zeta^{(2)} - 16(1-\Omega_{\sigma,\mathrm{dec}})\zeta^{(1)2} = 3\Omega_{\sigma,\mathrm{dec}}(\zeta_\sigma^{(2)} - \zeta^{(2)}) + 9\Omega_{\sigma,\mathrm{dec}}(\zeta_\sigma^{(1)} - \zeta^{(1)})^2\,, \tag{495}$$

and hence

$$\zeta^{(2)} = \frac{3}{4f}\zeta^{(1)2}\,, \tag{496}$$

which gives again Eq. (483).

Conclusions

Along these lectures, we have learned that a stage of inflation during the early epochs of the evolution of the universe solves many drawbacks of the standard Big-Bang cosmology, such as the flatness or entropy problem and the horizon problem. Luckily, despite inflation occurs after a tiny bit after the bang, it leaves behind some observable predictions:

- *The universe should be flat*, that is the total density of all components of matter should sum to the critical energy density and $\Omega_0 = 1$. The current data on the CMB anisotropies offer a spectacular confirmation of such a prediction. The universe appears indeed to be spatially flat.

- *Primordial perturbations are adiabatic.* Inflation provides the seeds for the cosmological perturbations. In one-single field models of inflation, the perturbations are *adiabatic* or curvature perturbations, *i.e.* they are fluctuations in the total energy density of the universe or, equivalently, scalar perturbations to the spacetime metric. Adiabaticity implies that the spatial distribution of each species in the universe is the same, that is the ratio of number densities of any two of these species is everywhere the same. Adiabatic perturbations in excellent agreement with the CMB data [4].

- *Primordial perturbations are almost scale-independent.* The primordial power spectrum predicted by inflation has a characteristic feature, it is almost scale-independent, that is the spectral index n_ζ is very close to unity. Possible deviations from exact scale-independence arise because during inflation the inflaton is not massless and the Hubble rate is not exactly constant.

- *Primordial perturbations are nearly gaussian.* The fact that cosmological perturbations are tiny allow their analysis in terms of linear perturbation theory. Non-gaussian features are therefore suppressed since the non-linearities of the inflaton potential and of the metric perturbations are suppressed. Non-gaussian features are indeed present, but may appear only at the second-order in deviations from the homogeneous background solution and are therefore small. This is rigorously true only for one-single field models of inflation. Many-field models of inflation may give rise to some level of NG [10]. If the next generation of observations we will detect a non-negligible amount of primordial NG, this will rule out one-single field models of inflation.

- *Production of gravitational waves.* A stochastic background of gravitational waves is produced during inflation in the very same way classical perturbations to the inflaton field are generated. The spectrum of such gravitational waves is again flat, *i.e.* scale-independent and the tensor-to-scalar amplitude ratio r is, at least in one-single models of inflation, related to the spectral index n_T by the consistency relation $r = -8n_T$. A confirmation of such a relation would be a spectacular proof of one-single field models of inflation. The detection of the primordial stochastic background of gravitational waves from inflation is challenging, but would not only set the energy scale of inflation, but would also give the opportunity of discriminating among different models of inflation [5, 37].

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