Operator Product Expansion of a Vertex Operator

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Consider:

$$\begin{aligned}
\left\langle X^{\alpha}\left(z\right) X^{\beta}\left(w\right)\right\rangle &= -\eta^{\alpha\beta} \ln\left(z-w\right), \\
\left\langle \partial_z X^{\alpha}\left(z\right) X^{\beta}\left(w\right)\right\rangle &= -\eta^{\alpha\beta} \frac{1}{z-w}, \\
\left\langle \partial_z X^{\alpha}\left(z\right) \partial_w X^{\beta}\left(w\right)\right\rangle &= -\eta^{\alpha\beta} \frac{1}{\left(z-w\right)^2}.
\end{aligned}$$

Then:

$$\begin{aligned}
\partial_z X^{\alpha}\left(z\right) : e^{ik_\beta X^{\beta}\left(w\right)} : \; &= \partial_z X^{\alpha}\left(z\right) : 1 : \; + \partial_z X^{\alpha}\left(z\right) \left(ik_\beta\right) : X^{\beta}\left(w\right) : + \\
&+ \frac{1}{2} \partial_z X^{\alpha}\left(z\right) \left(i^2 k_\beta k_\gamma\right) : X^{\beta}\left(w\right) X^{\gamma}\left(w\right) : \; + \ldots = \\
&= \partial_z X^{\alpha}\left(z\right) : 1 : \; + ik_\beta \left\langle \partial_z X^{\alpha}\left(z\right) X^{\beta}\left(w\right)\right\rangle - \\
&- \frac{1}{2} k_\beta k_\gamma \left(\left\langle \partial_z X^{\alpha}\left(z\right) X^{\beta}\left(w\right)\right\rangle X^{\gamma}\left(w\right) + \left\langle \partial_z X^{\alpha}\left(z\right) X^{\gamma}\left(w\right)\right\rangle X^{\beta}\left(w\right)\right) + \ldots = \\
&= \partial_z X^{\alpha}\left(z\right) + \frac{ik^{\alpha}}{z-w} - \frac{k^{\alpha}}{z-w} k \cdot X\left(w\right) + \ldots = \\
&= \frac{ik^{\alpha}}{z-w} : \sum_{n=0}^{+\infty} \frac{\left(ik \cdot X\left(w\right)\right)^n}{n!} = \\
&= \frac{ik^{\alpha}}{z-w} : e^{ik \cdot X\left(w\right)} : .
\end{aligned}$$

Therefore:

$$\partial_z X^{\alpha}\left(z\right) : e^{ik \cdot X\left(w\right)} : \; = \frac{ik^{\alpha}}{z-w} : e^{ik \cdot X\left(w\right)} : .$$