CFT and String ampl.

* String action:

$$S = \frac{1}{4\pi\alpha'}\iint d\tau d\sigma \sqrt{-h}\, h^{ab}\, \partial_a X\cdot \partial_b X$$

- $ISO(1,D-1)$ invariance
- Weyl x Diff. invariance $\longrightarrow$ redundancy in $Z = \int \mathcal{D}h_{ab}\, \mathcal{D}X^\mu\, e^{iS[g,X]}$.

$\Downarrow$

i) LC gauge: use gauge redundancy to fix $X^+(\xi^+,\xi^-) \sim p^+$ and build the spectrum.

ii) fix $h_{ab} = e^{2\phi}\, \hat{h}_{ab} \to$ fixed

$\hookrightarrow$ e.g.: $\hat{h}_{ab} = \eta_{ab} \Rightarrow d\tau d\sigma \sqrt{-h}\, h^{ab} = d\tau d\sigma\, \eta^{ab}$

preserved by $\xi^\pm \mapsto f_\pm(\xi^\pm) \Rightarrow \infty$ symm charges / algebra!

Consider $\tau = i\tau_E \longrightarrow S_E = \frac{1}{2\pi\alpha'}\int dz d\bar{z}\, \partial_z X\cdot \partial_{\bar z} X$ where $z = e^{\tau_E + i\sigma}$, $\bar z = z^*$:

CONFORMAL INVARIANCE:

- $\eta_{ab} \to \delta_{ab} \Rightarrow$ fix $h_{ab} = e^{2\phi}\delta_{ab} \Rightarrow h_{zz} = h_{\bar z\bar z} = 0$, $h_{z\bar z} = h_{\bar z z} = e^{2\phi}$
- $T(z) = -\frac{1}{2}\partial X^\mu \cdot \partial X_\mu \longrightarrow\ :\partial X^\mu(z)\partial X^\nu(w): \ = \partial X^\mu(z)\,\partial X^\nu(w) - \frac{1}{(z-w)^2}$

$\hookrightarrow c = D$ (spacetime dimension)

$\Rightarrow$ consider $T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2} \Rightarrow L_n |phys\rangle = 0 \quad n>0$

$\downarrow$

$|phys\rangle = \lim_{\substack{z\to 0\\ \bar z\to 0}} V(z,\bar z)|0\rangle$ s.t.: $(h,\bar h) = (1,1)$

$\to$ define $\mathcal{V} = \iint dz d\bar z\, V(z,\bar z) = \int dz\, \tilde V(z) \int d\bar z\, \overline{\tilde V}(\bar z) \Rightarrow \mathcal{V}$ must reparam. invar.

We could try:

$V_1^{\mu\nu}(z;k) = \partial X^\mu \partial X^\nu e^{ik\cdot X}(z)$

$V_2^\mu(z;k) = \partial^2 X^\mu e^{ik\cdot X}(z)$

$\Rightarrow V_{phys}(z;k) = \xi_{1\,\mu\nu} V_1^{\mu\nu}(z;k) + \xi_{2\,\mu} V_2^\mu(z;k)$

$\Rightarrow n>0: L_n|phys\rangle = \lim_{\omega\to 0} L_n V_{phys}(\omega;k) = \lim_{\omega\to 0} \oint_{C_\omega} \frac{dz}{2\pi i} z^{n+1} R\big(T(z)V_{phys}(\omega;k)\big) = 0$

Then it implies:

$$\eta_{\mu\nu}\xi_1^{\mu\nu} + k_\mu \xi_2^\mu = 0$$
$$k_\mu \xi_1^{\mu\nu} + \xi_2^\mu = 0$$
$$k^2 = -2. \qquad\qquad (k^2 = -m^2).$$

NB: STATES AND OPERATORS ARE IN 1:1 CORRESPONDENCE

$$\partial X^\mu(z) = \sum_{n\in\mathbb{Z}} \alpha_n^\mu z^{-n-1} \Rightarrow |\partial X^\mu\rangle = \alpha_{-1}^\mu |0\rangle \quad \text{where} \quad \alpha_{-1}^\mu = \oint_0 \frac{dz}{2\pi i} z^n \partial X^\mu(z)$$

→ BRST formulation: path integral measure → Diff × Weyl invariance

- we chose: $h_{ab} = e^{2\phi}\,\widehat{h_{ab}}$

but we still have $\delta h_{ab} = \nabla_a \xi_b + \nabla_b \xi_a \Rightarrow \delta h_{zz} = \partial_z \xi_z \quad \delta h_{\bar z\bar z} = \partial_{\bar z}\xi_{\bar z}$

⇒ GHOSTS: $s h_{ab} = \nabla_a c_b \,, \quad s b_{ab} = \rho_{ab} \,, \quad s\rho_{ab} = 0$

$$\Rightarrow S' = S + S_{GF} = \frac{1}{4\pi\alpha'}\iint d\tau_E\, d\sigma \sqrt{-h}\left(h^{ab}\partial_a X^\mu \partial_b X_\mu + s\left[b_{ab}\, h^{ab}\right]\right) =$$

$$= \frac{1}{4\pi\alpha'}\iint d\tau_E\, d\sigma \sqrt{-h}\left[h^{ab}\partial_a X^\mu \partial_b X_\mu + \rho_{ab}\, e^{-2\phi}\delta^{ab} + b_{ab}\nabla^a c^b\right] =$$

$$= \frac{1}{4\pi\alpha'}\iint d\tau_E\, d\sigma \left(\partial_a X^\mu \partial^a X_\mu + b_{ab}\nabla^a c^b\right) =$$

$$= \frac{1}{2\pi\alpha'}\iint dz\, d\bar z \left(\partial X^\mu \bar\partial X_\mu + b\,\bar\partial c + \bar b\,\partial c\right)$$

s.t.: $\langle b(z) c(w)\rangle = \frac{1}{z-w} \Rightarrow$ b, c form a CFT (anticommuting)

$\rightarrow$ bc CFT (purely holomorphic: $\bar{\partial} c = \bar{\partial} b = 0$) $\quad S = \frac{1}{\pi}\iint d^2 z \; b(z)\, \bar{\partial} c(z)$

$$\begin{cases} h_b = \lambda \\ h_c = 1-\lambda \end{cases} \Rightarrow T_{GH}(z) = \partial b(z)\, c(z) - \lambda\, \partial\big(b(z) c(z)\big) \quad \left(c_{GH} = -3(2\lambda-1)^2 + 1\right)$$

$\longrightarrow$ String theory: $h_b = 2$, $h_c = -1$:

$$b(z) = \sum_{n\in\mathbb{Z}} b_n\, z^{-n-2}$$

$$c(z) = \sum_{n\in\mathbb{Z}} c_n\, z^{-n+1}$$

$$T_{GH}(z) = -\partial b(z)\, c(z) - 2\, b(z)\, \partial c(z) = c(z)\, \partial b(z) + 2\, \partial c(z)\, b(z).$$

$$c_{GH} = -26$$

Therefore $\Rightarrow T(z) = T_X(z) + T_{GH}(z) \longrightarrow \quad c = c_X + c_{GH} = D - 26 = 0 \Leftrightarrow D = 26.$

$$\Rightarrow j_{BRST}(z) = c(z)\underbrace{\left(T_X(z) + T_{GH}(z)\right)}_{\text{generators of the transf.}} + \frac{3}{2}\partial^2 c(z) \longrightarrow Q_{BRST} = \oint_0 \frac{dz}{2\pi i}\, j_{BRST}(z)$$

$$j_{BRST,X}(z) = c(z)\, T_X(z) = \sum_{n,m\in\mathbb{Z}} c_n L_m\, z^{-n-m-4} = \sum_n \left(j_{BRST,X}\right)_n z^{-n-4}$$

$\hookrightarrow Q^2_{BRST} = 0 \Leftrightarrow D = 26$

Ghosts do not interact with $X^\mu \Rightarrow \mathcal{H}_n = \left\{ |\{n\}; k\rangle \otimes c_1 |0\rangle_{GH} \right\}$:

$$Q_{BRST} |phys\rangle = 0 \Rightarrow \left[Q_{BRST}, V_{phys}\right] = 0.$$

(consider only non trivial BRST cohomology, since $\left[Q_{BRST}, V'_{phys}\right] = 0 \;\; \forall\, V'_{phys} = \left[Q_{BRST}, \mathcal{O}\right]$).

Since we have: $\delta h_{ab} = \nabla_a c_b \rightarrow \nabla_{\bar{z}} c^z \Rightarrow c_z$ is a $U(1)$ field coupled to $D=2$ spin-connection:

$$j(z) = c(z) b(z) = \sum_{n,m} c_n b_m\, z^{-n-m-1} = \sum_{n\in\mathbb{Z}} j_n\, z^{-n-1}, \qquad j_n = \sum_{m\in\mathbb{Z}} c_m b_{n-m}$$

$$\rightarrow \bar{\partial} j(z,\bar{z}) = \frac{1}{8} Q \sqrt{-h}\, R^{(2)} \Rightarrow \oint \frac{dz}{2\pi i} j(z) = j_0 = \sum_{m\in\mathbb{Z}} c_m b_{-m} = \frac{1}{2}\left(c_0 b_0 - b_0 c_0\right) + \sum_{m=1}^{\infty}\left(c_{-m} b_m - b_{-m} c_m\right) + \frac{1}{2} =$$
$$= Q(1-g)$$

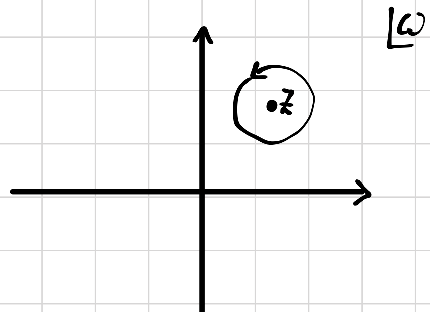
$\Rightarrow$ no. 0-modes of c $-$ no. 0-modes of b $= Q(1-g)$

→ sphere: 3 0-modes and no moduli.

↓

inequivalent surfaces of genus g. ⇒ integrate all over the moduli space

⇒ SPURIOUS STATES DECOUPLE:



$$V_{sp}(z) = \left[Q_{BRST}, \mathcal{O}(z)\right] = \oint_z \frac{dw}{2\pi i}\, j_{BRST}(w)\,\mathcal{O}(z)$$

↓

deform towards a path with no sing. → vanish

STATES: $|0\rangle_{GH}$ given by Fermi sea-level + SL_2 inv.

→ b,c alg: $c_1|0\rangle_{GH}$ highest weight → $c(0)|0\rangle_{GH}$ where $c(z) = \dots + c_{-1}z^2 + c_0 z + c_1 + \dots$

TACHYONIC VACUUM

then

$$L_0^{TOT}\left[|0\rangle_X \otimes c_1|0\rangle_{GH}\right] = -\left[|0\rangle_X \otimes c_1|0\rangle_{GH}\right]$$

and $(c_1|0\rangle_{GH})^\dagger = {}_{GH}\langle 0|c_{-1}c_0 \Rightarrow {}_{GH}\langle 0|c_{-1}c_0c_1|0\rangle_{GH} = 1$

↳ they multiply $L_{\pm1,0}$ in Q_{BRST} → conformal Killing vec.

→ fix SL_2 inv to fix gauge symm ⇒ 1) Drop integration for 3 points

2) Use $c(z)\,V_{phys}(z)$ for 3 points → SATURATION of the 0-modes

Superconformal Field Theory

→ CFT + SUSY: $T(z,\theta) = T_{z\theta}(z) + \theta T_{zz}(z)$

$$D = \partial_\theta + \theta\partial_z \longrightarrow D^2 = \partial_z$$

$$\bar{D} = \partial_{\bar\theta} + \bar\theta\partial_{\bar z} \longrightarrow \bar{D}^2 = \partial_{\bar z}$$

$$\Rightarrow T(z,\theta) = \sum_n z^{-n-\frac{3}{2}}\left(\tfrac{1}{2}\right)G_n + \theta\sum_n z^{-n-2}L_n$$

$$\phi(z,\theta) = \sum_n z^{-n-h}\phi_{0,n} + \theta\sum_n z^{-n-\frac{1}{2}-h}\phi_{1,n}$$

$\Downarrow$

double valued

NS: $\phi_F^{NS}(ze^{2\pi i}) = \phi_F^{NS}(z) \longrightarrow G_n,\ n\in\mathbb{Z}+\frac{1}{2}$

R: $\phi_F^{R}(ze^{2\pi i}) = -\phi_F^{R}(z) \longrightarrow G_n,\ n\in\mathbb{Z}$

$\Downarrow$

SUSY ALG:

- NS: $G_{-\frac{1}{2}}^2 = L_{-1}$
- R: $G_0^2 = L_0 - \frac{\hat c}{16}$

SUSY UNBROKEN if $\ldots$ $\mid$ $G_0|0\rangle_R = 0 \Rightarrow \left(L_0 - \frac{\hat c}{16}\right)|0\rangle_R = \left(h - \frac{\hat c}{16}\right)|0\rangle_R = 0 \Rightarrow h = \frac{\hat c}{16} \longleftarrow$

* R STATES ARE DOUBLE VALUED

$$\hookrightarrow \phi_F^R(z) = \sum_{n\in\mathbb{Z}} z^{-n-\frac{1}{2}-h}\phi_{F,n}^R$$

* VACUUM IS NS

$$\hookrightarrow |0\rangle_{SL_2} \longrightarrow |h\rangle = \phi_{F,h}^{NS}(0)|0\rangle_{SL_2}$$

→ include spin fields to FLIP the B.C. on fermion fields:

endpoint of branch cut → $|\pm\rangle = S^\pm(0)|0\rangle$

$$\rightarrow |-\rangle = G_0|+\rangle$$

$$G_0^2|+\rangle = \left(h - \frac{\hat c}{16}\right)|+\rangle = G_0|-\rangle$$

$$\rightarrow T_{z\theta}(z)S^\pm(w) \sim \frac{1}{2}\frac{a_\pm}{(z-w)^{3/2}}S^\mp(w), \qquad a_+ = 1,\ a_- = h - \frac{\hat c}{16}.$$

$\hookrightarrow$ if $h = \frac{\hat c}{16} \longrightarrow$ drop $|-\rangle$. $\Rightarrow$ we can recover $|+\rangle$ from OPE's.

NS: $|h\rangle = \phi_F(0)|0\rangle_{SL_2}$

R: $|h^\pm\rangle = \phi_F(0)|\pm\rangle = \phi_F(0)S^\pm(0)|0\rangle_{SL_2}$

→ Fermionic String:

- $S = \frac{1}{2}\int d^2\xi \left(\partial_a X^\mu \cdot \partial_b X_\mu \eta^{ab} - i \psi^\mu \rho^a \partial_a \psi_\mu \right) = \int d^2z\, d^2\theta\, \bar{D}X^\mu D X_\mu$, $\quad X^\mu = X^\mu + \theta\psi^\mu$

NS $\Rightarrow D X^\mu = \sum_{n\in\mathbb{Z}} z^{-n-1}\left(\psi^\mu_{n+\frac{1}{2}} + \theta \alpha^\mu_n\right)$

↓

we must add $q^\mu \to$ 0-modes of ψ^μ

R $\Rightarrow D X^\mu = \sum_{n\in\mathbb{Z}} z^{-n-1}\left(\psi^\mu_n + \theta\alpha^\mu_n\right)$

↳ $\{\psi^\mu_0, \psi^\nu_0\} = -g^{\mu\nu} \longrightarrow$ CLIFFORD ALG.

$[L_0, \psi^\mu_0] = 0 \longrightarrow$ degeneration $\Rightarrow$ R ground state is a Majorana Spinor

$|C\rangle = S_\alpha(0)|0\rangle_{SL_2}$

$|S\rangle = S^\beta(0)|0\rangle_{SL_2}$

corr. funct. of S_α and ψ_μ's

⇒ GO TO CURRENT SO(1,9) AFFINE ALGEBRA (gives prop. for vect rep ψ^μ and spin rep. S_α):

$$j^{\mu\nu}(z) = \psi^\mu\psi^\nu(z) \longrightarrow j^{\mu\nu}(z)\, j_{\mu\nu}(w) \sim \frac{-d(d-1)}{(z-w)^2} + 2(d-1)\,\partial\psi\cdot\psi(w)$$

↳ $T^{(z)}_\psi = -\frac{1}{4(d-1)} :j^{\mu\nu} j_{\mu\nu}(z): = -\frac{1}{2}\partial\psi\cdot\psi(z).$

⇒ build ~ Verma module for $j^{\mu\nu}(z) = \sum_{n\in\mathbb{Z}} z^{-n-1} j^{\mu\nu}_n$:

$$j^{\mu\nu}(z)\,\psi^\rho(w) \sim \frac{1}{z-w}\left(g^{\mu\rho}\psi^\nu(w) - g^{\nu\rho}\psi^\mu(w)\right)$$

$$j^{\mu\nu}(z)\, S_\alpha(w) \sim \frac{1}{z-w}\;\frac{1}{2}\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right)_{\alpha\beta} S^\beta(w)$$

follows

$\Rightarrow \psi^\mu(z) S_\alpha(w) \sim \frac{1}{z-w}(\gamma^\mu)_{\alpha\beta} S^\beta(w)$

↳ $\psi^\mu \sim$ generalised γ mat.

→ S_α has $h = \frac{5}{8}$ since $h = \frac{\hat{c}}{16} = \frac{D}{16} = \frac{10}{16} = \frac{5}{8}$.

Consider $SO(1,9) \to SO(10)$ current alg $\longrightarrow$ Maximal torus: $U(1)^5$ [SO(10) alg. has rank 5]

↳ dim so(10) = 45

* $\psi^j_\pm = \psi^{2j} \pm \psi^{2j+1} = e^{\pm i H_j}$

↳ $j^{2j,2j+1} = :\psi^{2j}_\pm \psi^{2j+1}_\pm: = \partial H^j \longrightarrow$ 5 currents

* $S_\alpha = \prod_{j=0}^{4} e^{\pm\frac{i}{2}H_j} \Longrightarrow$ weight $\frac{5}{8}$

Therefore:

$$S_\alpha(z)S_\beta(\omega) \sim \frac{1}{(z-\omega)^{3/4}}(\gamma^\mu)_{\alpha\beta}\,\psi_\mu$$

$$S^\alpha(z)S_\beta(\omega) \sim \frac{1}{(z-\omega)^{5/4}}\delta^\alpha{}_\beta + \frac{1}{(z-\omega)^{1/4}}\left(\frac{1}{2}\gamma^\mu\gamma^\nu\right)^\alpha{}_\beta\,\psi_\mu\psi_\nu$$

→ BRST:

$$S_{GH} = \int d^2z\, d^2\theta\; B\,\bar{D}\,C \quad \text{where} \quad B = \beta_{z\theta} + \theta\, b_{zz} \longrightarrow \frac{3}{2}$$

$$C = c^z + \theta\gamma^\theta \longrightarrow -1$$

⇒ GHOST + MATTER: $\hat{c} = D - 10 = 0 \iff D = 10.$

→ $[Q_{BRST}, V_{phys}] = 0$ for both NS and R vertex ops.

$$V_{phys} = \int d^2z\, d^2\theta\; \tilde{V}_{phys}(z,\theta)\; \tilde{\bar{V}}_{phys}(\bar{z},\bar{\theta})$$

$$(h,\bar{h}) = \left(\frac{1}{2},\frac{1}{2}\right)$$

NB: $|0\rangle_{SL_2} \longrightarrow V_{NS} \xrightarrow{S_\alpha} V_R \longrightarrow$ fermions

sth like: $u^\alpha \cdot \underbrace{\Sigma\; S_\alpha}_{\frac{1}{2}\text{ global}}\, e^{ik\cdot X}(z)$ with $k^2 = 0$

$\Sigma \to ?$, $S_\alpha \to \frac{5}{8}$, $e^{ik\cdot X} \to 0$

$\frac{1}{2}$ global $\Rightarrow ? = -\frac{3}{8}$

→ how to determine Σ ? → SPINOR GHOST SPIN FIELD

↳ determined from no. current: $j_{GH} = -\beta\gamma$.

⇒ define $j(z) = -bc(z) = \sum_n z^{-n-1} j_n \longrightarrow j_n = \sum_{m\in\mathbb{Z}} :c_m b_{n-m}:$

$$\Rightarrow j_0 = \sum_{m\in\mathbb{Z}} :c_m b_{-m}: \;=\; :c_{-1}b_1 + c_0 b_0 + c_1 b_{-1} + \sum_{m=2}^{\infty}\left(c_m b_{-m} + c_{-m} b_m\right): =$$

0-modes on the sphere !

$$j(z)c(\omega) \sim \frac{c(\omega)}{z-\omega} \rightarrow \oint \frac{dz}{2\pi i} j(z)c(\omega) = j_0 c(\omega) = c(\omega)$$

$$j(z)b(\omega) \sim -\frac{b(\omega)}{z-\omega} \rightarrow \oint \frac{dz}{2\pi i} j(z)b(\omega) = j_0 b(\omega) = -b(\omega)$$

→ j_0 counts $c = +1$ and $b = -1$ → anomalous!

↳ $j_0^\dagger = -(j_0 + Q)$

$Q = -3$ on the sphere for bc ghosts

$Q = 2$ 〃 for $\beta\gamma$ ghosts

⇒ only ops. which cancel background charge.

→ study through bosonization:

$$j(z) = \pm \partial\phi(z) \qquad (= -b(z)c(z))$$

→ take a look at $e^{q\phi(z)}$ in the presence of background charge:

WEIGHT: $\pm \frac{1}{2} q(q+Q)$ $\qquad (+F, -B)$

$[j_0, e^{q\phi(w)}] = q e^{q\phi(w)} \longrightarrow$ SHIFTS THE CHARGE BY q

$\Rightarrow$ $c(z) = e^{\phi(z)} \longrightarrow h = \frac{1}{2}(1-3) = -1$

$b(z) = e^{-\phi(z)} \longrightarrow h = -\frac{1}{2}(-1-3) = 2$

$\Rightarrow$ $\gamma(z) = e^{\phi(z)}\eta(z) \rightarrow h = -\frac{1}{2}(1+2) + 1 = -\frac{1}{2}$

$\underbrace{\beta(z) = e^{-\phi(z)}\partial\xi(z)} \rightarrow h = \frac{1}{2}(-1+2)+1 = \frac{3}{2} \longrightarrow$ ξ_0 does not appear! → $|\phi\rangle$ and $\xi_0|\phi\rangle$ are degen.

↳ we can use $\Sigma^{\pm} = e^{\pm\frac{\phi(z)}{2}}$ to build the spinor ghost spin fields.

→ the $|VAC.\rangle$ must be chosen properly → BOUNDED!

⇓

$|q\rangle$ → instead of $|0\rangle_{GH}$

$(|q\rangle^{\dagger} = \langle -q-Q|)$

$\Rightarrow |q\rangle = e^{q\phi(0)}|0\rangle$

and here it goes!!!

→ $V_{-\frac{1}{2}} = u^{\alpha} e^{-\frac{\phi(z)}{2}} S_{\alpha}(z) e^{ik\cdot X(z)}$ COVARIANT FERMION VERTEX (one of many...)

$$-\frac{3}{8} + \frac{5}{8} + 0 = \frac{1}{2}$$

SPINOR GHOST CHARGE $q = -\frac{1}{2}$ ⟶ MUST NOT SHIFT BACKGROUND CHARGE

$\Rightarrow$ we need $V_{\frac{1}{2}}$:

$$V'_{phys} = [Q_{BRST}, \xi V_{phys}] = \underbrace{[Q_{BRST}, \xi(z)]}\, V_{phys}(z) = \overset{\text{PICTURE CHANGING OP.}}{X(z)}\, V_{phys}(z)$$

ξ is not part of the irreducible alg of (β,γ).

(only $\phi, \eta, \partial\xi$)

Consider:

$$b = e^{-\sigma} \qquad\qquad c = e^{\sigma}$$

$$\beta = e^{-\phi}\,\partial\xi = e^{-\phi+\chi}\,\partial\chi \qquad\qquad \gamma = e^{\phi}\,\eta = e^{\phi-\chi}$$

$\Rightarrow$ SMALL $\mathcal{H}$:

$$|0\rangle_{SL_2}^{\dagger} = {}_{SL_2}\langle 0|\, e^{3\sigma-2\phi} \Rightarrow \left(Q_{BRST}|0\rangle\right)^{\dagger} = \langle 0|\, e^{3\sigma-2\phi}\, Q_{BRST}^{\dagger} = 0$$

LARGE $\mathcal{H}$:

$$|0\rangle_{SL_2}^{\dagger} \neq \langle 0|\, e^{3\sigma-2\phi+\chi} \Longrightarrow \text{NOT BRST}$$

$\rightarrow$ consider small/large $\mathcal{H}$ $\Longrightarrow$ only neutral comb of $V_{-\frac{1}{2}}, V_{\frac{1}{2}}$