Reality of $X(u,\bar{u})$: let $x_t < x < x_{t+1}$:

$$\partial_u X_L(x+i0^+) = U_{(t)}\, \partial_{\bar{u}} X_R(x-i0^+) \longrightarrow X_L(x+i0^+) = U_{(t)} X_R(x-i0^+) + \Delta_{(t)}$$

Define:

$$P_{\parallel(t)} = \frac{1}{2}(\mathbb{I} + U_{(t)}) \qquad\qquad \Rightarrow \quad P_{\parallel(t)} + P_{\perp(t)} = P_{\parallel(t)}^2 = P_{\perp(t)}^2 = \mathbb{I} \;; \quad P_{\parallel(t)} P_{\perp(t)} = 0$$
$$P_{\perp(t)} = \frac{1}{2}(\mathbb{I} - U_{(t)})$$

$\Rightarrow X(x+i0^+, x-i0^+) = 2\, P_{\parallel(t)} X_R(x-i0^+) + \Delta_{(t)}$ and $X(x_t, x_t) = f_{(t)}$

Then:

$$① \quad P_{\perp(t)} X(x_t, x_t) = \Delta_{(t)}^{\perp(t)} = f_{(t)}^{\perp(t)}$$
$$② \quad P_{\parallel(t)} X(x_t, x_t) = 2 X_R(x_t) + \Delta_{(t)}^{\parallel(t)} = f_{(t)}^{\parallel(t)}$$

$$③ \quad \operatorname{Im} X(x+i0^+, x-i0^+) = 2 \operatorname{Im} X_R^{\parallel(t)}(x-i0^+) + \operatorname{Im} \Delta_{(t)} = 0$$
$$④ \quad \operatorname{Re} X(x+i0^+, x-i0^+) = 2 \operatorname{Re} X_R^{\parallel(t)}(x-i0^+) + \operatorname{Re} \Delta_{(t)} = f_{(t)}$$

Then, since $X(u,\bar{u}) = X^*(u,\bar{u})$, we have

$$⑤ \quad X_L^*(u) = X_R(\bar{u}) + Y_{(t)} \quad \text{for} \quad x_t < \operatorname{Re} u < x_{t-1}$$

From ⑤ and the continuity property we have:

$$X_L^*(x_t^+) = X_R(x_t^+) + Y_{(t)}$$
$$= X_L^*(x_t^-) = X_R(x_t^-) + Y_{(t+1)} \quad \Longrightarrow Y_{(t)} = Y_{(t+1)} = Y \quad \forall t \quad ⑥$$

From ⑤, ⑥ and $X = X^*$, we have:

$$⑦ \quad X^* = X_L^* + X_R^* = X_R + Y + X_L - Y^* = X + 2i \operatorname{Im} Y \Rightarrow \operatorname{Im} Y = 0.$$

From ② we have:

$$⑧ \quad \operatorname{Re} \Delta_{(t)}^{\parallel(t)} = f_{(t)}^{\parallel(t)} - 2 \operatorname{Re} X_R(x_t) = f_{(t)}^{\parallel(t)} - X_R - X_R^* = f_{(t)}^{\parallel(t)} - X_L - X_R + Y =$$
$$= f_{(t)}^{\parallel(t)} - X(x_t, x_t) + Y = -f_{(t)}^{\perp(t)} + Y_{(t)}^{\perp(t)} + Y^{\parallel(t)}$$

Moreover from ④ we find:

$$\operatorname{Re} X(x_t, x_t) = f_{(t)} = 2 \operatorname{Re} X_R^{\parallel(t)}(x_t) + \operatorname{Re} \Delta_{(t)}^{\parallel(t)} + \operatorname{Re} \Delta_{(t)}^{\perp(t)} =$$

$$= 2\,\mathrm{Re}\, X_R^{\parallel(t)} + \mathrm{Re}\, \Delta_{(t)}^{\parallel(t)} + f_{(t)}^{\perp(t)} = f_{(t)}$$

$$\Leftrightarrow \quad (9) \quad \mathrm{Re}\, \Delta_{(t)}^{\parallel} = f_{(t)}^{\parallel} - 2\,\mathrm{Re}\, X_R^{\parallel(t)}$$

Then from $X = X_L + X_R$ we find:

$$X = X_R^{*} + X_R + Y \Rightarrow X^{\parallel(t)} = 2\,\mathrm{Re}\, X_R^{\parallel(t)} + Y^{\parallel(t)} = f_{(t)}^{\parallel(t)} \qquad (10)$$

From (9) and (10)

$$(11) \quad \mathrm{Re}\, \Delta_{(t)}^{\parallel(t)} = Y^{\parallel(t)}$$

Then from (8):

$$(12) \quad f_{(t)}^{\perp(t)} = Y^{\perp(t)} \longrightarrow f_{(t)}^{\perp(t)} \text{ is universal}$$

Therefore:

$$\Delta_{(t)} = f_{(t)}^{\perp(t)} + Y^{\parallel(t)} - 2i\,\mathrm{Im}\, X_R^{\parallel(t)} = \text{const.}$$