LACES 2017

BRST Symmetry and Field Theory

1) Maxwell: $A_\mu(x) \longrightarrow F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$

s.t.: $\delta A_\mu = \partial_\mu \lambda(x) \Rightarrow F'_{\mu\nu}(x) = F_{\mu\nu}(x)$

$\hookrightarrow \mathcal{L}_M = -\frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x) \longrightarrow$ 2 dof of A_μ propagate!

Gauge inv. $\Rightarrow$ correct no. of dofs

Then gauge fix $\longrightarrow$ how to recover the correct no. of dofs?

There are others gauge inv. operators:

$T_{\mu\nu} = F_{\mu\rho} F_\nu{}^\rho - \frac{1}{4} \eta_{\mu\nu} F^2$ and others from $F_{\mu\nu}, \partial_\rho F_{\mu\nu}, \dots, \partial_\rho \partial_\sigma \dots \partial_\lambda F_{\mu\nu}$

Quantization:

1) HAMILTONIAN APPROACH: fields + momenta

$\hookrightarrow$ break manifest Lorentz invariance [Itzykson-Zuber]

2) PATH INTEGRAL (LAGRANGIAN) APPROACH:

$$Z[A] = \int \mathcal{D}A \, e^{-S[A]} \Rightarrow \text{ill defined!}$$

GAUGE ORBITS

$$\Big(\mathcal{D}A \sim \underset{\substack{\uparrow \\ \infty \text{ vol.}}}{\mathcal{D}A_{G.O.}} \; \underset{\substack{\uparrow \\ \text{OK}}}{\mathcal{D}A_\perp} \Big)$$

$\rightarrow$ add regulator (the result must NOT depend on it):

$$Z[A] = \int \mathcal{D}A \, e^{-(S[A] + S_{GF}[A])}$$

$\hookrightarrow$ I'm changing the theory!

$\Rightarrow$ I must add sth "irrelevant" for the theory!

e.g.: $S = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) = \frac{1}{2} \int d^4x \left(A_\mu \Box A^\mu - (\partial A)^2 \right)$

gauge invar!

Now suppose to add $S_{GF} = \frac{1}{2}\int d^4x \, (\partial A)^2$:

$$S = \frac{1}{2}\int d^4x \, A_\mu \Box A^\mu \longrightarrow \text{not gauge invariant}$$

How to quantize:

$$\left[\dot{A}_\mu(\vec{x},t), A_\nu(\vec{y},t)\right] = i\hbar \eta_{\mu\nu} \, \delta^3(\vec{x}-\vec{y})$$

↳ NOT def. pos.

↳ build the states of the Hilbert space :

$$A^\dagger_\mu(x)|0\rangle \longrightarrow A^\dagger_0|0\rangle \text{ has NEGATIVE NORM!}$$

⇒ This is not really a Hilbert space! ⟶ How to solve the problem?

* EFFECTIVE ACTION $\Gamma[A]$

- WARD symmetries ⟶ manifestation of gauge sym. in Green func.

$$\delta\Gamma = 0$$
$$= \int d^4x \, \delta A_\mu(x) \frac{\delta\Gamma}{\delta A_\mu(x)} = \int d^4x \, \partial_\mu\lambda \frac{\delta\Gamma}{\delta A_\mu} = -\int d^4x \, \lambda \, \partial_\mu \frac{\delta\Gamma}{\delta A_\mu}$$

$$\longrightarrow -\partial_\mu \frac{\delta\Gamma}{\delta A_\mu} = 0 \quad \text{(WI for "free Maxwell")}$$

⟶ Corel. funct. (2 point) : $\left.\frac{\delta^2\Gamma}{\delta A_\mu(x)\,\delta A_\nu(y)}\right|_{A=0} = \Gamma_{\mu\nu}(x,y) \Rightarrow -\partial^\mu \Gamma_{\mu\nu}(x,y) = 0$

$$p^\mu \, \Gamma_{\mu\nu}(x,y) = 0$$

transverse operator

Now reconsider

$$\delta A_\mu = \partial_\mu \lambda \xrightarrow{\text{fermionize}} s A_\mu = \partial_\mu c$$

antiderivation ⇒ ANTICOMMUTING

GHOST

⇒ $s^2 = 0$ nilpotent op. [coboundary op.]

↳ $s^2 A_\mu = s(\partial_\mu c) = \partial_\mu (sc) = 0 \Rightarrow \boxed{sc = 0}$

ghosts are invariant under s op. (BRST)

$$\Rightarrow \begin{cases} sA_\mu = \partial_\mu c \\ sc = 0 \end{cases} \qquad \begin{matrix} s^2 A_\mu = 0 \\ s^2 c = 0 \end{matrix}$$

consider

$s^2 = 0 \longrightarrow s\omega = 0 \Rightarrow$ ω is s-closed, $\omega = s\Lambda$

$$\Rightarrow \omega = \omega_{\text{non-trivial}} + \underbrace{s\Lambda}$$

s-trivial term

Therefore:

$$\mathcal{L}_M \longrightarrow \mathcal{L}^{GF} = \mathcal{L}_M + s\Omega \qquad \Rightarrow s\mathcal{L}^{GF} = s\mathcal{L}_M + \cancel{s^2\Omega}^{\,0}$$

NB: $\mathcal{L}_M$ is gauge invariant ⇒ $\mathcal{L}_M$ is s-invariant → $\underline{s\mathcal{L}_M = 0}$

⇒ add the <u>NON MINIMAL SECTOR</u>

add another field: $\bar{c}$ (totally indep.)

↳ must create sth non physical

add another <u>commuting</u> variable: ρ (NAKANISHI-LAUTRUP)

↳ it will be a Lagrange multipl.

NB: their cohom. is empty

s.t.: $s\bar{c} = \rho$ ← TOPOLOGICAL QUARTET (together with conjugates)

$s\rho = 0 \longrightarrow s^2\bar{c} = 0,\ s^2\rho = 0$

↳ BRST doublets

DYNAMICS:

$$\int \mathcal{D}\rho \, e^{i\rho\,\partial A} = \delta(\partial^\mu A_\mu) \quad [\text{Dirac Delta}]$$

$$\mathcal{L}_M + \rho(\partial^\mu A_\mu) \longrightarrow \frac{\delta S}{\delta \rho} = \partial^\mu A_\mu = 0$$

(LANDAU-FERMI gauge)

However it is not of the form $\mathcal{L}_M + s\Omega$:

$$s(\rho\,\partial^\mu A_\mu) = \rho\,\partial^\mu\partial_\mu c = \rho\,\Box c \neq 0$$

Therefore consider:

$$\mathcal{L}^{GF} = \mathcal{L}_M + \rho(\partial^\mu A_\mu) - \bar{c}\,\Box c$$

s.t.: $s[-\bar{c}\,\Box c] = -\bar{\rho}\,\Box c \longrightarrow \bar{c}$ has $\sim$ role of ρ

$$\Rightarrow \mathcal{L}^{GF} = \mathcal{L}_M + \rho(\partial^\mu A_\mu) - \bar{c}\,\Box c = \mathcal{L}_M + s[c\,\partial^\mu A_\mu]$$

$\Rightarrow$ exactly what you want!

For a particle

$$\mathcal{L}_p = P_m \dot{X}^m + \frac{e}{2} P^2$$

$\bar{c}$ in previous $\mathcal{L}$

$$\begin{cases} sb = \rho \\ s\rho = 0 \end{cases}$$

$$\begin{cases} s X_m = c P_m \\ s P_m = 0 \\ sc = 0 \\ se = \dot{c} \end{cases} \longrightarrow \mathcal{L}_p + s[b(e-1)] = \mathcal{L}_p + \rho(e-1) - b\dot{c}$$

$\Rightarrow \dfrac{\delta S}{\delta \rho} = e - 1 = 0 \Rightarrow$ now I can insert it into the action:

$$\mathcal{L}_p^{GF} = P_m \dot{X}^m + \frac{1}{2} P^2 - b\dot{c}$$

Now I can write:

$$\mathcal{L}^{GF} = \underbrace{-\frac{1}{2} A_\mu \Box A^\mu}_{+4} \; \underbrace{- \bar{c}\,\Box c}_{-2} \; = 2 \text{ dof (exactly the polarization)}$$

Now suppose Feynman gauge:

$$\Omega = \bar{c}\left(\partial^\mu A_\mu + \frac{\xi}{2}\rho\right)$$

$$\Rightarrow s\Omega = \rho\left(\partial^\mu A_\mu + \frac{\xi}{2}\rho\right) - \bar{c}\,\partial^\mu\partial_\mu c$$

$$\Rightarrow \mathcal{L}^{GF} = \mathcal{L}_M + s\Omega$$

$$\frac{\delta S}{\delta \rho} = \partial^\mu A_\mu + \xi\rho = 0 \longrightarrow \mathcal{L}^{GF} = \mathcal{L}_M - \frac{1}{2\xi}(\partial A)^2 - \bar{c}\,\Box c$$

s.t.: $s\bar{c} = \rho = -\frac{1}{\xi}(\partial A)$

↳ not nihilpotent

NON ABELIAN GAUGE THEORIES

$$A_\mu = A_\mu^a t_a \quad , \quad [t_a, t_b] = i f_{ab}{}^c t_c$$

$$\Rightarrow F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + ig f_{ab}{}^c A_\mu^a A_\nu^b t_c$$

then $\delta A_\mu = \nabla_\mu \lambda = \partial_\mu \lambda^a t_a + ig[A_\mu, \lambda] \rightarrow \delta A_\mu^a = \partial_\mu \lambda^a - g f_{bc}{}^a A_\mu^b \lambda^c$

$$\rightarrow [\delta_1, \delta_2] A_\mu = \delta_3 A_\mu = \nabla_\mu [\lambda_1, \lambda_2]$$

$$\delta F_{\mu\nu} = ig[F_{\mu\nu}, \lambda]$$

$tr(t_a t_b) = \frac{1}{2}\eta_{ab}$ (Cartan-Killing)

$$\Rightarrow \mathcal{L}_{YM} = -\frac{1}{2}\mathrm{Tr}\left(F_{\mu\nu}F^{\mu\nu}\right) = -\frac{1}{4}F_{\mu\nu}^a F_a^{\mu\nu}$$

→ WARD IDENTITIES

$\partial_\mu \lambda^a - g f_{bc}{}^a A_\mu^b \lambda^c$

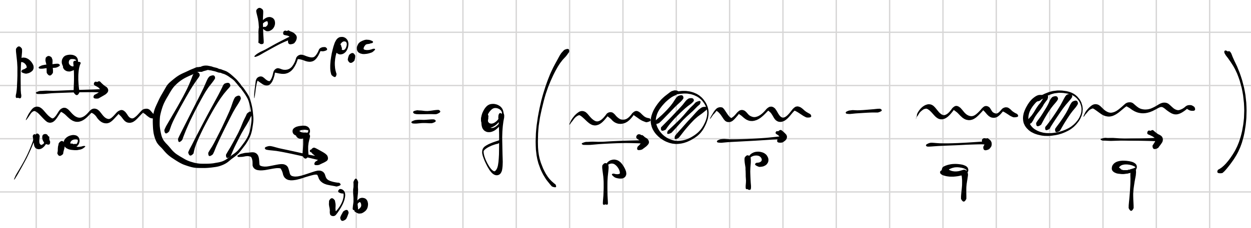
$$\delta\Gamma = \int d^4x\, \delta A_\mu^a \frac{\delta\Gamma}{\delta A_\mu^a} = \int d^4x\, (\nabla_\mu \lambda)^a \frac{\delta\Gamma}{\delta A_\mu^a} \overset{IBP}{=} \int d^4x \left(-\lambda^a\left(\nabla_\mu \frac{\delta\Gamma}{\delta A_\mu^a}\right)\right) = 0$$

$$\Rightarrow -\nabla_\mu \frac{\delta\Gamma}{\delta A_\mu^a} = 0 \longrightarrow -\partial_\mu \frac{\delta\Gamma}{\delta A_\mu^a} + g f_{ab}{}^c A_\mu^b \frac{\delta\Gamma}{\delta A^c} = 0$$

$$\Rightarrow \Gamma_{\mu\nu}^{ab}(x,y) = \left.\frac{\delta^2\Gamma}{\delta A_\mu^a(x)\,\delta A_\nu^b(y)}\right|_{A=0} \Longrightarrow \partial^\mu \Gamma_{\mu\nu}^{ab}(x,y) = 0 \longrightarrow p^\mu \Gamma_{\mu\nu}^{ab}(x,y) = 0$$

(there are no 1-pt funct)

$$\Longrightarrow \text{3-pts funct} \longrightarrow i(p+q)^\mu \Gamma^{abc}_{\mu\nu\rho}(p,q) + g f^{e}_{bd}\,\Gamma^{dc}_{\nu\rho}(p) - g f^{e}_{dc}\,\Gamma^{bd}_{\nu\rho}(q) = 0$$



GHOSTS ? $\quad sA = dc + ig[A,c]$

$$\nabla_\mu c = \partial_\mu c + ig[A_\mu, c]$$

$$\delta A^a_\mu = \nabla_\mu \lambda^a \longrightarrow sA^a_\mu = \nabla_\mu c^a = \partial_\mu c + ig[A_\mu, c]$$

$$\longrightarrow s\mathcal{L}_{YM} = 0$$

$$s^2 A_\mu = 0 \longrightarrow s\left[\partial_\mu c^a + g f^a_{bc} A^b_\mu c^c\right] = \partial_\mu(sc^a) + g f^a_{bc}\left(\partial_\mu c^b + g f^b_{de} A^d_\mu c^e\right)c^c + g f^a_{bc} A^b_\mu sc^c =$$

$$= \left(\nabla_\mu(sc)\right)^a + g f^a_{[bc]}\,\partial_\mu c^{[b} c^{c]} - g^2 \underbrace{f^a_{bc} f^b_{de}}_{\text{antysym.}} c^{[e} c^{c]}$$

$$= \nabla_\mu\left[sc^a + g\frac{1}{2} f^a_{bc}\, c^b c^c\right] = 0$$

$$\longrightarrow sc^a = -\frac{g}{2} f^a_{bc}\, c^b c^c \Rightarrow sc = -\frac{1}{2}[c,c]$$

$$s^2 c^a = 0 = \left(-\frac{1}{2}g\right)^2 f^a_{bc}\left(f^b_{de}\, c^d c^e c^c\right) - c^b f^c_{de}\, c^d c^e = 0 \quad \text{by Jacobi id.}$$

Summary:

$$\begin{cases} \delta A_\mu = \partial_\mu \lambda \\ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \\ \mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} \end{cases} \longrightarrow \begin{array}{l} sA_\mu = \partial_\mu c \\ sc = 0 \\ s^2 = 0 \end{array}$$

Add non minimal sector:

$$\begin{array}{l} s\bar{c} = \rho \\ s\rho = 0 \end{array} \longrightarrow GF:\ \mathcal{L}^{GF} = \mathcal{L}^{INV} + s\left[\bar{c}\left(\partial^\mu A_\mu + \xi\rho\right)\right]$$

$$\Rightarrow s\mathcal{L}^{GF} = 0$$

Now add the non minimal sector:

$$\bar{c}, \rho \longrightarrow s\bar{c}^a = \rho^a, \quad s\rho^a = 0 \qquad \left[s^2\bar{c}^a = 0, \; s^2\rho^a = 0 \right]$$

$\Rightarrow$ Gauge fixing: $\mathcal{L}^{GF} = \mathcal{L}^{INV} + s\,\mathrm{Tr}\left[\bar{c}\,\partial^\nu A_\nu\right]$

NB:	A_μ	c	$\bar{c}$	ρ	s
dim	1	1	1	1	1
	(1	0	2	2	0)
ghost no.	0	+1	-1	0	+1

2 choices:

1) $s \to 0$, $\partial \to 1 \Rightarrow c \to 0$ OK for massless fields

2) $s \to 1$, $\partial \to 1 \Rightarrow c \to 2$ OK for massive fields

Then:

$$\mathcal{L}^{GF} = \mathcal{L}^{INV} + \rho\,\mathrm{Tr}\left(\partial^\mu A_\mu\right) - \mathrm{Tr}\left(\bar{c}\,\partial^\mu \nabla_\mu c\right) =$$

$$= -\frac{1}{2}\mathrm{Tr}\left[\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)^2\right] + \mathrm{Tr}\left(\rho\,\partial\cdot A\right) - \mathrm{Tr}\left(\bar{c}\,\Box c\right) + \underline{\text{interactions}}$$

Now add a gauge param:

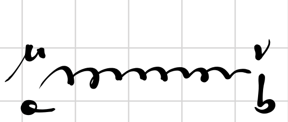
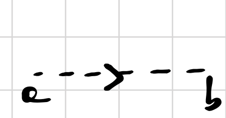
$$\mathcal{L}^{GF} = \mathcal{L}^{INV} + s\left[\bar{c}\left(\partial^\mu A_\mu + \frac{\xi}{2}\rho\right)\right]$$

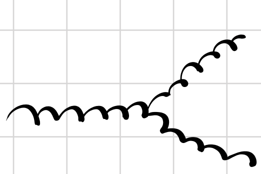
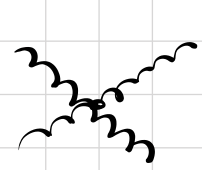
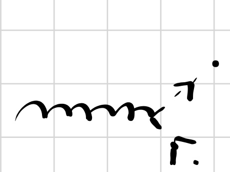
$$\Rightarrow \mathcal{L}^{GF} = -\frac{1}{2}\mathrm{Tr}\left(\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)^2\right) + \mathrm{Tr}\left(\rho\,\partial\cdot A\right) - \mathrm{Tr}\left(\bar{c}\,\Box c\right) + \frac{1}{2}\mathrm{Tr}\left(\xi\rho^2\right) + \underline{\text{int}}.$$

$\xi = 0 \to$ LANDAU

$\xi = 1 \to$ Feynman

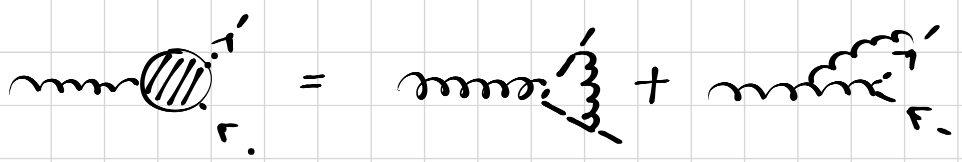
$\xi \neq 1 \to R_\xi$ gauge

NB: $\langle A^a_\mu A^b_\nu \rangle$  $\langle c\bar{c} \rangle$ 

Interactions?   

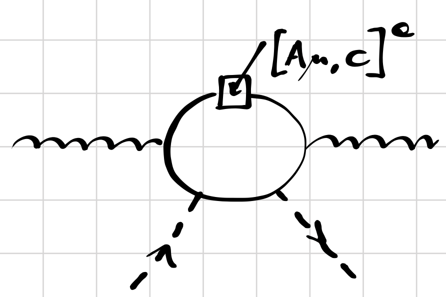
2-pts func $\longrightarrow \Gamma^{ab}_{\mu\nu}(p)$

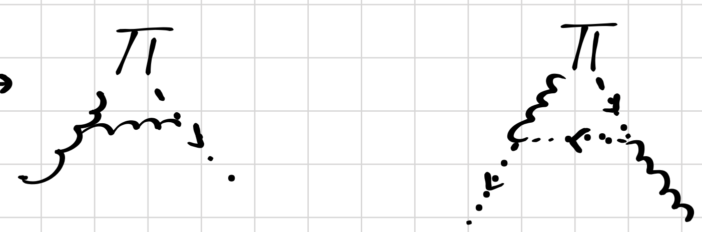
3-pts func $\longrightarrow \Gamma^{abc}_{\mu}$



NB: $sA_\mu,\ sc \Rightarrow$ NON LINEAR $\longrightarrow$ these obj. get corrections!

$s\bar{c} = \rho,\ s\rho = 0 \Rightarrow$ LINEAR

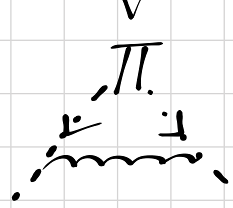


$\rightarrow$ 

$\Rightarrow$ DIVERGENT! (exactly as the other 1L factors but with sth more $\Rightarrow$ need to add other terms)

The same goes for $[c,c]^a = f^a_{bc}\, c^b c^c$

$\Downarrow$



$\longrightarrow$ DIVERGENT as well!

$\Rightarrow$ BRST is NON LINEAR (composite operators) $\Rightarrow$ HOW TO RENORMALIZE?

$\longrightarrow$ <u>ANTIFIELDS</u>:

introduce a <u>classical</u> source A^*_μ (with same quantum no. as A_μ) and c^* (with same quantum no. as c)

[generally we don't need $\bar{c}^*$, ρ^* because their transf. is simply linear]

$$\Rightarrow \mathcal{L} = -\frac{1}{2}\mathrm{Tr}\left((\partial_\mu A_\nu - \partial_\nu A_\mu)^2\right) + \mathrm{Tr}(\rho\, \partial\cdot A) - \mathrm{Tr}(\bar{c}\,\Box c) + \text{interactions} +$$

$$+ \frac{1}{2}\mathrm{Tr}(\xi\rho^2) + \mathrm{Tr}\left[A^*_\mu\, sA^\mu + c^* sc\right] + \mathrm{Tr}\left(\rho^* s\rho + \bar{c}^* s\bar{c}\right)$$

↳ A_μ commuting, A^*_μ anti-commu. $\quad$ c anticomm, c^* commut

$\Rightarrow sA^*_\mu = 0$

$sc^* = 0 \quad (s\bar{c}^* = s\rho^* = 0)$

$$\rightarrow \mathcal{L}^{GF} = \mathcal{L}^{INV} + s\left[\bar{c}(\partial^\mu A_\mu + \xi\rho) - A^*_\mu A^\mu + c^* c - \rho^*\rho + \bar{c}^* c\right]$$

	A_μ	c	$\bar{c}$	ρ	A^*_μ	c^*	$\bar{c}^*$	ρ^*
ghost no.	0	1	-1	0	-1	-2	0	-1
antifield no.	0	0	0	0	1	1	1	1
statist	B	F	F	B	F	B	B	F

NB: $A^a_\mu \rightarrow$ 1 form, Lie alg. vector

$(A^*)^\mu_a \rightarrow$ vector, Lie alg. vect (with inverse index)

$\Rightarrow (A^*)^\mu_a\, sA^a_\mu \Rightarrow$ THERE ARE NO METRICS INVOLVED

WARD IDENTITIES

Γ — generating func of 1PI diag $\left(\Gamma|_{\text{tree lev}} = S = \int d^4x\, \mathcal{L}^{GF}\right)$

$$\frac{\delta\Gamma}{\delta\rho} = \partial^\mu A_\mu + \xi\rho$$

→ if we add more non linear GF terms then we need more sources in $\mathcal{L}$

$$\rightarrow s\Gamma = \int d^4x \left\{ sA_\mu \frac{\delta\Gamma}{\delta A_\mu} + sc\frac{\delta\Gamma}{\delta c} + s\bar{c}\frac{\delta\Gamma}{\delta\bar{c}} + s\rho\frac{\delta\Gamma}{\delta\rho}\right\}$$

Plus: $sS = 0 \Leftrightarrow \int d^4x \left[\frac{\delta S}{\delta A^{*\mu}_a}\frac{\delta S}{\delta A^a_\mu} + \frac{\delta S}{\delta c^*}\frac{\delta S}{\delta c} + \rho\frac{\delta S}{\delta\bar{c}}\right] = 0$

$\downarrow S \rightarrow \Gamma$

$$S(\Gamma) = \int d^4x \left[\frac{\delta\Gamma}{\delta A^*}\frac{\delta\Gamma}{\delta A} + \frac{\delta\Gamma}{\delta c^*}\frac{\delta\Gamma}{\delta c} + \rho\frac{\delta\Gamma}{\delta\bar{c}}\right] = 0 \rightarrow \text{NON LINEAR!}$$

SLAVNOV-TAYLOR IDENTITIES

↳ We can expand this order by order ⇒ we have to renormalize <u>each</u> of them

Consider the diff operator:

$$\mathcal{S} = \frac{1}{2}\int d^4x \left\{ \frac{\delta\Gamma}{\delta A^*}\frac{\delta}{\delta A} + \frac{\delta\Gamma}{\delta c^*}\frac{\delta}{\delta c} + 2\rho\frac{\delta}{\delta \bar{c}} \right\} \quad \text{s.t.} \quad \mathcal{S}(\Gamma) = S(\Gamma)$$

$$\Rightarrow \quad \frac{\delta\Gamma}{\delta\bar{c}} + \partial^\mu \frac{\delta\Gamma}{\delta A^{*a}_\mu} = 0$$ (GHOST EQN) (consistency between $S(\Gamma)=0$ and $\frac{\delta\Gamma}{\delta\rho} = \dots$) ($[\mathcal{S}, \frac{\delta}{\delta\rho}]$ and Frobenius])

$\Rightarrow$ $\begin{cases} \text{GHOST eqn} \\ \delta\Gamma/\delta\rho = \partial^\mu A_\mu + \xi\rho \\ \text{SLAVNOV-TAYLOR IDS.} \end{cases}$ $\longrightarrow$ they form a closed alg. of diff. ops.

$\Downarrow$

the solution to these + quantum no. is exactly the action with the antifields!

Define: $\hat{\Gamma} = \Gamma - (\partial A + \frac{\xi}{2}\rho)\rho$

$$\Rightarrow \frac{\delta\hat{\Gamma}}{\delta\rho} = 0$$

$$\hat{A}^{*a}_\mu = A^{*a}_\mu - \partial_\mu \bar{c}^a$$

$$\Rightarrow \hat{\Gamma} = \hat{\Gamma}(A, c, \bar{c}, \hat{A}^*, c^*) \longrightarrow \frac{\delta\hat{\Gamma}}{\delta\bar{c}} = 0$$

$$\Rightarrow S(\hat{\Gamma}) = \int d^4x \left[\frac{\delta\hat{\Gamma}}{\delta A^*}\frac{\delta\Gamma}{\delta A} - \frac{\delta\hat{\Gamma}}{\delta c^*}\frac{\delta\hat{\Gamma}}{\delta c} \right] = 0$$ (no linear eqn)

$\hookrightarrow$ MASTER EQUATION

Introduce BRACKET (ANTI BR.)

$$\hookrightarrow (F,G) = \int d^4x \left(\frac{\delta\Gamma}{\delta A^*}\frac{\delta G}{\delta A} + \frac{\delta G}{\delta A^*}\frac{\delta F}{\delta A} + \frac{\delta F}{\delta c^*}\frac{\delta G}{\delta c} + \frac{\delta G}{\delta c^*}\frac{\delta F}{\delta c} \right)$$

s.t.: $(\hat{\Gamma}, \hat{\Gamma}) = S(\hat{\Gamma})$

$((\Gamma, G), H) = 0$

NB: $(S,S) = 0 \longrightarrow$ CLASSICAL MASTER EQN

$\downarrow$

$(\hat{\Gamma}, \hat{\Gamma}) = 0 \longrightarrow$ QUANTUM MASTER EQUATION

$\longrightarrow = \hbar\Delta\hat{\Gamma}$ (there can be a deformation)

$$s = (S, \cdot) \qquad \text{e.g.} \quad sA_\mu^a = (S, A_\mu^a) \qquad s^2 = 0 \qquad \xrightarrow{} \; (S,S) = 0$$

$$\Rightarrow sA_\mu^a = (S, A_\mu^a) = \frac{\delta S}{\delta A_\mu^{*a}} = sA_\mu^a \,, \qquad \text{where } S = \int d^4x \left(\mathcal{L}^{\mathrm{inv}} + \hat{A}^* sA + c^* sc \right)$$

$$\Rightarrow s\hat{A}^{*\mu}_a = (S, \hat{A}^{*\mu}_a) = \frac{\delta S}{\delta A_\mu^a} = \underbrace{-\nabla^\nu F_{\mu\nu}^a}_{\delta} + \underbrace{ig[A_\nu^*, c]^a}_{\gamma}$$

$$\Rightarrow sc^a = (S, c^a) = \frac{\delta S}{\delta c_a^*} = sc^a$$

$$\Rightarrow sc_a^* = (S, c_a^*) = \frac{\delta S}{\delta c^a} = \underbrace{\nabla_\mu \hat{A}^{*\mu}_a}_{\delta} + \underbrace{ig[c^*, c]^a}_{\gamma}$$

$\Rightarrow$ sA_μ^a, sc^a are the BRST symm.
$s\hat{A}^{*\mu}_a$, sc_a^* are the e.o.m. + sth else

$$\Rightarrow \quad s = (S, \cdot) = \gamma + \delta \qquad (\gamma^2 = 0, \; \gamma\delta + \delta\gamma = 0)$$

γ: BRST symmetry

δ: Koszul-Tate operator
$\Rightarrow$ implementation of e.o.m.

$\rightarrow$ the MASTER EQN. knows both symmetry and e.o.m.

NB: gauge symmetry $\to$ group $\to$ struct const: $f^a{}_{bc} \mapsto f^a_{bc}(\phi)$ (struct. funct.)

[SFT includes naturally all these constructions]

Gauge Symmetry: $\begin{cases} \delta A^a_\mu = (\nabla_\mu \lambda)^a \\ F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + ig f^a_{bc} A^b_\mu A^c_\nu \end{cases}$

$\to$ study gauge invariant operators: $\mathcal{L}^{inv}$, $T_{\mu\nu}$, ..., $\langle T_{\mu\nu}(A) \dots T_{\rho\sigma}(A) \rangle$

But what are the states?

Hilbert space $\longrightarrow$ FOCK SPACE $\longleftarrow$ $\boxed{\text{BRST Symmetry!}}$

that is:

1) $\begin{cases} sA^a_\mu = \nabla_\mu c^a \\ s^2 A = 0 \end{cases}$, $\quad sc^a = -i \frac{g}{2} f^a_{bc} c^b c^c$

2) Non minimal sector: $\begin{cases} s\bar{c} = \rho \\ s\rho = 0 \end{cases}$

3) gauge fixing

4) add antifields $\to$ Renormalization!

5) $s \longrightarrow S = (S[A, c, A^*, c^*, \bar{c}, \rho], \cdot)$

$$(F, G) = \int d^4x \left(\frac{\delta F}{\delta \phi^*} \frac{\delta G}{\delta \phi} + \frac{\delta G}{\delta \phi^*} \frac{\delta F}{\delta \phi} \right) \qquad \phi = (A, c, \dots) \quad \phi^* = (A^*, c^*, \dots)$$

$\Rightarrow (S, S) = 0 \Rightarrow s^2 = 0$

$\hookrightarrow$ the symmetry is completely encoded into the operator s
$s^2 = 0$ (coboundary operator)

$\Rightarrow$ study the cohomology of

functional of the fields

$H(s, \Omega[A, c, \dots])$

(NB: $\mathcal{L}[A(x), \dots] \to$ local functional
$S = \int d^4x \, \mathcal{L}[\dots] \to$ integrated functional)

NB it's a graded space!

ghost no.

antifields can have $gh < 0$.

$\Omega = \bigoplus_{gh=-\infty}^{\infty} \Omega_{gh}$ $\to$ important for $s: \Omega_{gh} \longrightarrow \Omega_{gh+1}$

Then $\Omega = s\Lambda \Rightarrow s\Omega = 0$:

$$H(s,\Omega) = \frac{\ker s}{\operatorname{Im} s} \qquad \text{(cohomology of BRST operator)}$$

NB: $\begin{cases} \text{BRST operator} \longrightarrow \text{classical fields} \\ \text{BRST charge} \longrightarrow \text{quantum field} \end{cases}$

$\Rightarrow$ Consider

$$gh = 0 \longrightarrow \Omega_0(A,\dots) = \{ \mathcal{L}^{inv} + \overset{\text{BRST exact}}{s\Sigma}, \dots \}$$

$\mathcal{L}^{inv}$ $\hookrightarrow$ gauge inv. $\Rightarrow$ THIS $\in H(s,\Omega)$ [contains/coincides with gauge inv. operators!]

QUANTUM:

$\rightarrow \langle O_1(A,\dots)\, O_2(A,\dots)\dots\rangle$ (s.t. $sO_i = 0 \ \forall i$)

$\hookrightarrow$ suppose $\langle (O_1'(A,\dots) + s\Sigma)\, O_2(A,\dots)\dots\rangle = \langle O_1'(A,\dots) O_2(A,\dots)\dots\rangle + \langle \overset{=0}{\cancel{s(\Sigma\, O_2(A,\dots))}}\rangle$

similar to:

$$\omega \in H(\omega, M) \Rightarrow \int_M \omega + d\omega = \int_M \omega^k + \cancel{\int_{\partial M} \omega^{k-1}}^{\,0}$$

$\hookrightarrow \partial M = \varnothing$

in other words:

$$\int \mathcal{D}A\mathcal{D}c\dots\, e^{-S[A,\dots]} (s\Sigma)\, O_2\dots = \int \mathcal{D}A\dots\, s\Big(\overset{s \text{ closed}}{e^{-S[A]}}\, \Sigma\, O_2\dots\Big) = 0$$

Clearly $gh = 0 \Rightarrow \Omega_0 \equiv$ gauge inv.

But

$$\Omega_1[A,c,\dots] = c^a f_a(A,\dots) + \dots \qquad \left.\begin{aligned} sA_\mu^a &= \nabla_\mu c^a \\ sc^a &= -ig f^a_{bc} c^b c^c \end{aligned}\right\}$$

$\hookrightarrow s\Omega_1 = 0 \longrightarrow$ e.g.: consider $\rightarrow \Omega_1 = c^a f_a(A, \overset{\text{just deriv.}}{\partial A}, \dots)$

$$s\Omega_1 = -i\frac{g}{2} f^a_{bc} c^b c^c f_a(\dots) - c^a\left[(\nabla_\mu c)^b \frac{\partial f_a(\dots)}{\partial A_\mu^b} + \partial_\nu \nabla_\mu c^e \frac{\partial f_a(\dots)}{\partial(\partial_\nu A_\mu^e)} + \dots\right]$$

$\Downarrow$

WZ consistency relation!

$\rightarrow f_a(A) = d_{abc}\, \varepsilon^{\mu\nu\rho\sigma} \partial_\mu A_\nu^b \partial_\rho A_\sigma^c + \dots$ (where $d_{(abc)}$ is totally symmetric)

$\hookrightarrow \Omega_1 = c^a f_a(A) \longrightarrow$ ABJ anomaly

$\Rightarrow$ this is the representative of the anomaly $\Rightarrow$ need counterterms!

$\rightsquigarrow$ What happens to ghost no. -1 ?

$$\Omega_{-1} = \bar{c}^a g_a [A, \ldots] + \ldots$$

$\hookrightarrow$ Parametrizes all the conserved charges!

$\Omega_2 \longrightarrow$ parametrizes the diff. from a pure Lie Algebra structure!

(deformed Lie Alg. $\to$ SUGRA)

$\Rightarrow$ Consider $B^{(2)} = B_{\mu\nu}\, dx^\mu \wedge dx^\nu$

$$\Rightarrow sS = 0 \longrightarrow \quad sA_\mu^a = (\nabla_\mu c)^a$$

$$\delta A_\mu^a = \xi(x) (\nabla_\mu c)^a$$

$\hookrightarrow$ GRASSMANN

$$\Rightarrow \partial^\mu J_\mu^{BRST} = 0 \longrightarrow sS = \int \partial^\mu \xi\, J^{BRST} \longrightarrow Q^{BRST} \quad (\text{BRST charge})$$

ACTING ON FOCK SPACE! $\Rightarrow$ quantum fields

$Q_{BRST}|\psi\rangle$

physical Fock space

$Q_{BRST}|\psi\rangle = 0 \quad | \quad |\psi\rangle \neq Q|\Omega\rangle$

$\hookrightarrow |\psi\rangle \in H(Q_{BRST}, \mathcal{F})$

Consider CS theory 3D:

$$\mathcal{L}^{INV} = \varepsilon^{\mu\nu\rho}\left(A_\mu^a \partial_\nu A_\rho^a + \frac{2}{3} A_\mu^a A_\nu^b A_\rho^c f_{abc}\right)$$

$$\Rightarrow S = \# \int_{M^3} \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right) \quad \longrightarrow \quad \frac{\delta S}{\delta A_\mu^a} = F_{\mu\nu}^a = 0$$

$\rightsquigarrow$ BRST symmetry: $sA_\mu^a = \nabla_\mu c^a \longrightarrow sc^a = -ig f^a_{bc} c^b c^c$

$\Rightarrow$ Antifields: $(A^*)^\mu_a$, c^*_a

in terms of diff form $\Rightarrow$ $A_\mu^a \to A^a$ 1-form $\quad gh = 0$

$c^a \longrightarrow c^a$ 0-form $\quad gh = 1$

Cartan-Killing

$$A^{*\mu}_{a} \longrightarrow \eta_{eb}\, \varepsilon^{\mu\nu\rho} A^{*b}_{[\nu\rho]} \longrightarrow A^{*e} \quad \text{2-form} \quad gh=-1$$

$$c^{*}_{e} \rightarrow \eta_{eb}\, c^{*b} \rightarrow \text{0-form} \longleftrightarrow \boxed{\text{3-form}} \quad gh=-2$$

generalizing: $\mathcal{A}^{a} = c^{a} + A^{e} + A^{*e} + c^{*a} \Rightarrow gh=-1$

$\hookrightarrow sA^{a}_{\mu} \longrightarrow sA^{e} = -dc^{e} + [A,c]^{e} \Longrightarrow sA = -dc + [A,c]$

$\rightarrow \{s,d\}=0 \quad s^2=0 \quad d^2=0 \rightarrow \hat{s} = s+d$ "double complex"

$$\Rightarrow \hat{s}\mathcal{A} + \tfrac{1}{2}[\mathcal{A},\mathcal{A}] = 0 \qquad \left(\text{see } dA + \tfrac{1}{2}[A,A]=0\right)$$

$$\Rightarrow (s+d)(c+A+A^*+c^*) + \tfrac{1}{2}\left[c+A+A^*+c^*, c+A+A^*+c^*\right] =$$

$$= sc + sA + sA^* + sc^* + dc + dA + dA^* + \cancel{dc^*}^{\to 0 \text{ (d of a 3 form in 3D)}} +$$

$$+ \tfrac{1}{2}\left([c,c] + 2[A,c] + 2[c,A^*] + 2[c,c^*] + [A,A] + 2[A,A^*] + 2[A,c^*]\right) = 0$$

$\Rightarrow sc = -\frac{1}{2}[c,c] \Rightarrow$ the BRST transf.!

$sA^* = -\left(dA + \frac{1}{2}[A,A]\right) + (c, A^*) \qquad \Rightarrow$ BRST + e.o.m.!

$sc^* = (dA^* + A \wedge A^*) + (c, c^*) \Rightarrow$ BRST + e.o.m.!

Is there an action such that $\hat{s}\mathcal{A} + \frac{1}{2}[\mathcal{A},\mathcal{A}] = 0$ is the e.o.m.?

$$\hookrightarrow \int_{\Pi} \mathrm{Tr}\left[\mathcal{A}(s+d)\mathcal{A} + \tfrac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\right]$$

$\big\downarrow$ promoted to

$$\int \mathrm{Tr}\left[\Psi Q \Psi + \tfrac{2}{3}\Psi * \Psi * \Psi\right] \longrightarrow$$

(bosonic open) STRING FIELD THEORY

e.o.m.: $Q\Psi + \Psi * \Psi = 0$