LACES 2017

Berkovits

Superstring Theory

Outline:

1) BRST method (bosonic)
2) Spinning part. + string (RNS)
3) Superpart B-S (GS) + pure spinor
4) " see ICTP lecture notes
5) Superstring (GS + p.s.)
6) Applic. (ampl., $AdS_5 \times S^5$)

BRST

Review of QFT Hamilt. formalism:

→ point particle: $S = \int d\tau \left(P_m \dot{X}^m + e(P_m P^m - M^2) \right)$

$\rightsquigarrow M = 0$: $S = \int d\tau \left(P_m \dot{X}^m + \frac{1}{2} e P_m P^m \right)$

then $\delta X^m = \varepsilon P^m$ from $[P^m P_m, X_n] = i P_n$ the we have to consider $\delta e = -\dot{\varepsilon}(\tau)$

↳ $\delta(\dot{\varepsilon} P) P = \dot{\varepsilon} P^2 + \partial(\dots)$

Choose now a gauge fix: $e = 1 \longrightarrow$ FP GHOSTS! (b,c) $\int d\tau \, b \dot{c}$

$\Rightarrow S = \int d\tau \left(P_m \dot{X}^m + \frac{1}{2} P^2 + b\dot{c} \right) \overset{\text{e.om. } P}{=} \int d\tau \left(-\frac{1}{2} \dot{x}^m \dot{X}_m + b\dot{c} \right)$

s.t. $\exists Q = c \underbrace{P^m P_m}_{P_m P^m - M^2 \text{ in general}}$ BRST ops ⇒ physical states are in Q cohom.

b is the conj mom. of c $[P^m, P^n] \dots \{b, c\} = i$

no p or b because we could go to one or another trough canonical comm

⇒ choose $V = f(x,c)$ a state:

$\Rightarrow V = \varphi(x) + c\,\varphi^*(x) \Rightarrow$ phys if $QV = 0$ $\delta V = Q\Omega$ $\quad \Omega = \omega + c\omega^*$

$\rightarrow c\,\Box\varphi = 0 \quad \delta\varphi^* = \Box\omega \longrightarrow \Box\varphi = 0, \; \varphi^* \neq \Box\omega$

if $\Box\omega = 0$ then φ^* cannot be gauged away

eigenvalues eqn.

if non 0

but $\varphi^* \neq \Box\omega \Rightarrow \Box\varphi^* = 0$ (cohomology)

$\rightarrow \varphi(x)$: field
$c\,\varphi^*(x)$: antifield $\Big\}\ \neq$ ghost no.

⇝ **STRINGS** : $\int d\tau d\sigma \sqrt{(\dot{x})^2(\dot{x})^2 - (\dot{x}x')^2}$

Homework ✱ $P_m = \frac{\delta \mathcal{L}}{\delta \dot{x}}$ and show that it satisfies the Virasoro const

$$\Rightarrow S = \int d\tau d\sigma \left(P_m \dot{X}^m + e\left(P_m + \frac{\partial X_m}{\partial\sigma}\right)\left(P^m + \frac{\partial X^m}{\partial\sigma}\right) + \right.$$
$$\left. + \bar{e}\left(P_m - \frac{\partial X_m}{\partial\sigma}\right)\left(P^m - \frac{\partial X^m}{\partial\sigma}\right)\right)$$

CONFORMAL GAUGE

$\Rightarrow$ Gauge fix $e = \bar{e} = 1$ (conf on worlds., Lor. on spacetime)

$$\left[\tfrac{1}{2}\epsilon(P+X')^2(\sigma), X^m\right] \Rightarrow \delta_1 X^m = \epsilon(P+X')^m$$
$$\left[\tfrac{1}{2}\bar{\epsilon}(P-X')^2(\sigma), X^m\right] \Rightarrow \delta_2 X^m = \bar{\epsilon}(P-X')^m$$

"Gauge transf"

Similarly: $\delta_1 P^m = \frac{\partial}{\partial\sigma}\left(\varepsilon\,(P+X')^m\right)$ the transfs cancel in $P_m - X'_m$ (e.g.)

and $\delta e = \dot{\varepsilon} + \varepsilon' + e\,\partial_\sigma\varepsilon + \partial_\sigma(e\varepsilon)$ ⇝ in order to cancel the transf in $P_m + X'_m$

NB: these are ∞ constraints because they depend on σ !!! $\left[(P+X')^2(\sigma), (P+X')^2(\sigma')\right] = \delta(\sigma-\sigma')\,\partial_\sigma(P+X')^2 + \#\,\partial_\sigma\delta(\sigma-\sigma')\,(P+X')^2$

$\Rightarrow$ This is just $[L_m, L_n] = \delta_{m+n}(\dots)$

⇝ Q = GHOST x CONSTRAINTS $= \int d\sigma\left[c(P+X')^2 + \bar{c}(P-X')^2 + bcc' - \bar{b}\bar{c}\bar{c}'\right]$

$\Rightarrow Q^2 = \int d\sigma d\sigma'\, c(\sigma)\,(P+X')^2(\sigma)\, c(\sigma')(P+X')(\sigma') + \dots =$ in order for $Q^2 = 0$
$= \int d\sigma d\sigma'\, c(\sigma)\,\partial_\sigma c\,(P+X')^2(\sigma)$ (they account for the whole δe)
need to cancel $\Rightarrow$ need bcc' and $\bar{b}\bar{c}\bar{c}'$

GENERIC : $Q = \sum_j \left(c_j G_j + b^k c_j c_\ell f_k{}^{j\ell}\right)$ when $[G_j, G_\ell] = f^k{}_{j\ell}\, G_k$

The action in the gauge is:

$$S = \int d\tau d\sigma \left(P\dot{X} + \frac{1}{2}(P+X')^2 + \frac{1}{2}(P-X')^2 \right) =$$

$$= \int d\tau d\sigma \left(P\dot{X} + P^2 + (X')^2 \right) = \int d\tau d\sigma \left(-(\dot{X})^2 + (X')^2 \right) =$$

$$= -\int d\tau d\sigma \; \partial X \cdot \bar{\partial} X \qquad \partial = \partial_\tau + \partial_\sigma \,, \quad \bar{\partial} = \partial_\tau - \partial_\sigma$$

$\Rightarrow$ WITH GHOSTS: $$S = -\int d\tau d\sigma \left(\partial X \cdot \bar{\partial} X + b \bar{\partial} c + \bar{b} \partial \bar{c} \right)$$

$$Q = \int d\sigma \left[c (\partial X)^2 + bc\partial c + \bar{c} (\bar{\partial} X)^2 + \bar{b}\bar{c}\bar{\partial}\bar{c} \right]$$

Another gauge choiche would be to use ε and $\bar{\varepsilon}$ to gauge some of the X and $P \to \partial_\sigma X^+ = 0 \quad \partial_\sigma P^+ = 0 \quad \int d\sigma\, e = \int d\bar{\sigma}\, \bar{e} = 1$.

$\hookrightarrow$ LC gauge : no propagating ghosts

$$\int d\tau d\sigma \; e\,(P+X')^2 \qquad \frac{\delta \mathcal{L}}{\delta e} = 0 \Rightarrow \partial_\sigma (P+X')^2 = 0$$

$$\frac{\delta \mathcal{L}}{\delta \bar{e}} = 0 \Rightarrow \partial_\sigma (P-X')^2 = 0$$

$$\frac{\partial c}{\partial \sigma} = 0 \longleftarrow \text{but } \frac{\delta \mathcal{L}}{\delta P'} \Rightarrow \frac{\partial e}{\partial \sigma} = 0$$

$$\Rightarrow Q_{LC} = c \int d\sigma\, (P+X')^2 + \bar{c} \int d\sigma\, (P-X')^2$$

Then: $$S = \int d\tau d\sigma \left(\hat{P}^- \dot{\hat{X}}^+ + \hat{P}^+ \dot{\hat{X}}^- + (P+X')^j (P+X')^j + (P-X')^j (P-X')^j + - \hat{P}^- \hat{P}^+ + P^j(\sigma) \dot{X}^j(\sigma) \right)$$

the first part is σ indep. thanks to LC gauge

$$\to S = \int d\tau d\sigma \left(\dot{\hat{X}}^- \dot{\hat{X}}^+ + \partial X^j \bar{\partial} X^j + \hat{b}\dot{\hat{c}} + \bar{b}\dot{\hat{\bar{c}}} \right)$$ $\to$ non covariant, conformal inv.

$\rightsquigarrow$ COHOMOLOGY: $V = \varphi(x^j(\sigma), \hat{x}^+, \hat{x}^-, \hat{c})$ (choose Open str $\Rightarrow \hat{c} = \hat{\bar{c}}$)

$$V = \varphi(x^j, \hat{x}^+, \hat{x}^-) + \hat{c}\, \varphi^*(\dots)$$

$\hookrightarrow QV = 0 \Rightarrow \left[\int d\sigma\, (P+X')^2, \varphi \right] = 0 \Rightarrow \int d\sigma\, (P+X')^2 = \underset{= M^2}{P_0^2} + \underbrace{(P_1+X_1')(P_{-1}+X_{-1}')}_{\text{modes}}$

$\delta\varphi^* = \left[\int d\sigma\,(P+X')^2,\ \Omega\right] \to$ gauge away φ^* unless $P_0^2 = \ldots$

- Conformal invariance: (open)

$$Q = \int d\sigma \left(c\,\partial X\,\partial X + b c\,\partial c\right)$$

$$V = \varphi\left(x^m, c_{-1}, c_0, c_1, \ldots ; b_2, b_3, \ldots\right) \longrightarrow \text{conformal weights!}$$

$$\to \int dz\,d\bar{z}\,\left(\partial X\,\bar{\partial} X + b\,\bar{\partial} c + \bar{b}\,\partial\bar{c}\right)$$

$$\Rightarrow \quad c(z):\ (h,\bar{h}) = (-1,0) \longrightarrow \text{prove with def of conf transf}$$
$$b(z):\ (h,\bar{h}) = (2,0)$$

VERTEX OPS:

$$V = \varphi_0(x) + c\,\varphi_1(x) + c\,\partial c\,\varphi_2(x) + c\,\partial c\,\partial^2 c\,\varphi_3(x)$$

Cohomology condition: $\varphi_0(x) = \varphi_3(x) = 1$, $\varphi_1 = \varphi$, $\varphi_2 = \varphi^*$

↳ Tachyon: $T(x)$

Gluon: $A_\mu(x)$

Higher Spin: $B_{\mu\nu\ldots}(x)$

$$\Rightarrow \varphi(x) = T(x) + A_\mu(x)\,\partial^\mu X + B_{\mu\nu}(x)\,\partial X^\mu \partial X^\nu + B_\rho\,\partial^2 X^\rho + \ldots$$

phys states have ghost no. 1 $\quad \delta V = Q\,\Omega$

$\neq$ part.

$$\delta A_\mu = \partial_\mu \Lambda \Leftarrow \Omega = \Lambda + \ldots \to Q\Omega = \left[c\,\partial X\,\partial X, \Lambda\right] = c\,\partial X^\mu \frac{\partial \Lambda}{\partial X^\mu} = c\,\partial X^\mu\,\delta A_\mu$$

At ghost no. 0 $\Rightarrow QV = 0 \Rightarrow \left[c\,\partial X^\mu \partial X_\mu, \varphi_0\right] = 0 \to \partial_\mu \varphi_0 = 0$

$$\varphi_0 = \text{const}$$

General BRST quant. (review)

Consider 1st class constraints:

$$S = \int d\tau \left[P_m \dot{X}^m + e\, G^J(X,P) \right] \qquad \left[P_m, X^n\right] = -i\,\delta_m{}^n$$

with gauge transf: $\delta X^m = \left[\varepsilon_J G^J, X^m\right]$ and $\left[G^I, G^J\right] = f^{IJ}{}_K G^K$ → Lie alg.

$\delta P_m = \left[\varepsilon_J G^J, P_m\right]$

in order to have invariance: $\delta e_J = -i\,\dot{\varepsilon}_J - \varepsilon_K f^{KL}{}_J e_L$

* Gauge symmetry fixing:

$$e_J = h_J = \text{const.} \;\longrightarrow\; \int d\tau \left[P_m \dot{X}^m - h_J G^J + b^J\left(-i\,\dot{c}_J - c_K f^{KL}_J h_L\right)\right]$$

⇒ gauge (local): broken

global ⟶ BRST charge is intact:

$$Q = c_J G^J + f_L^{JK} c_J c_K b^L \longrightarrow \text{Noether charge}$$

Specifically:

PARTICLE: $S = \int d\tau \left[P_m \dot{X}^m + e P^2\right] \xrightarrow{\text{gauge } e = \frac{1}{2}} S = \int d\tau \left[\dot{X}^2 + b\dot{c}\right]$

$$Q = cP^2$$

$$V(X,c) = \varphi_0(X) + c\,\varphi_1(X)$$

$|phys\rangle \in \text{Cohom}(Q) \Rightarrow QV(X,c) = 0 \Rightarrow \Box\varphi_0 = 0$

$\delta V = Q\Omega \Rightarrow \delta\varphi_1 = \Box\omega \Rightarrow$ we can gauge away unless $\Box\varphi_1 = 0$

$(\Omega(X) = \omega(X) + c\,\omega(X))$

$|phys\rangle \in \dfrac{\text{"e.o.m."}}{\text{gauge}} = \dfrac{\ker Q}{\operatorname{Im} Q}$ ⟵ → $\varphi_0, \varphi_1 \in \dfrac{\ker Q}{\operatorname{Im} Q}$ $\quad \Box\varphi_0 = \Box\varphi_1 = 0$

STRING: $S = \int d\sigma \left[P_m \dot{X}_m + e\left(P + X'\right)^2 + \bar{e}\left(P - X'\right)^2\right]$

conf. gauge → $e = \bar{e} = 1$ ⟶ $S = \int d\sigma \left[\partial X^m \bar{\partial} X_m + b\,\bar{\partial} c + \bar{b}\,\partial\bar{c}\right]$

$$Q = \int d\sigma \left[c\left(P + X'\right)^2 + \bar{c}\left(P - X'\right)^2 + bc\,\partial c + \bar{b}\bar{c}\,\partial\bar{c}\right]$$

LC gauge fixing:

$$\int d\sigma\, e(\sigma) = \int d\sigma\, \bar{e}(\sigma) = 1 \,, \qquad \frac{\partial X^+}{\partial\sigma} = \frac{\partial P^+}{\partial\sigma} = 0 \qquad \Big[\underbrace{\mathcal{E}(\sigma) = \mathcal{E}_0}_{\substack{\text{this fixes}\\ \int d\sigma\, e = \int d\sigma\, \bar e = 1}} + \underbrace{\sum_{n\neq 0} \mathcal{E}_n e^{in\sigma}}_{\text{fixes } X^{+\prime} = P^{+\prime} = 0} \Big],$$

$$\Rightarrow \quad X^+ = \hat{X}^+$$
$$P^+ = \hat{P}^+ \longrightarrow \text{0-mode only}$$

$$\Rightarrow S = \int d\tau d\sigma \left[P_j \dot{X}_j + e^{-1}(P+X')^2 + \bar{e}^{-1}(P-X')^2 \right] +$$
$$+ \int d\tau \left[\hat{P}^+ \hat{\dot{X}}^- + \hat{P}^- \hat{\dot{X}}^+ + \hat{b}\hat{\dot{c}} + \hat{\bar{b}}\hat{\dot{\bar{c}}} \right]$$

$$Q = \hat{c} \int d\sigma\, (P+X')^2 + \hat{\bar{c}} \int d\sigma\, (P-X')^2$$

NB: the e.o.m. for e: $\frac{\delta \mathcal{L}}{\delta e} = 0 \Rightarrow$
$\Rightarrow \partial_\sigma \left[(P+X')^2 \right] = 0 \Rightarrow$ only 0 modes survive

$\frac{\partial}{\partial\sigma}\left((P^- + X'^-)\hat{P}^+ + (P^j + X'^j) \right) = 0$
$\Leftrightarrow$ etc. $\Leftrightarrow \frac{\partial}{\partial\sigma} P^- = \ldots$
$\hookrightarrow$ not indep $\partial_\sigma X = (\ldots)$

* COHOMOLOGY: for <u>open str.</u> $\hat{c} = \hat{\bar{c}}$

$$\Rightarrow V(\hat{c}, X_j(\sigma), \hat{X}^+, \hat{X}^-) = \varphi_0(X_j, \hat{X}^+, \hat{X}^-) + \hat{c}\, \varphi_1(X_j, \hat{X}^+, \hat{X}^-)$$

expand $X_j = \hat{X}_j + \sum_{N\neq 0} a_N^j e^{iN\sigma}$ $\qquad a_N^j |0\rangle = 0, \; N<0$

states: $V = e^{ik\cdot\hat{X}} \prod_{N,j} (a_N^j)^{M_{N,j}} |0\rangle$

over all x — only transv.

then $Q = \hat{c} \int d\sigma \left[(P+X')_0^2 + \sum_{N\neq 0} (P+X')_N (P+X')_{-N} \right]$

then $QV = 0 \Rightarrow \underbrace{k^2 + \sum_{N,j} N M_{N,j} + 1}\; = 0$ e.o.m. $\longrightarrow$ NORMAL ORD.

$$e^{ik\cdot\hat{X}}|0\rangle \longrightarrow T(x) \qquad k^2 = -1$$

$$e^{ik\cdot\hat{X}}\, a_{-1}^j(x)|0\rangle \longrightarrow A_j\, \partial X^j(x) e^{ik\cdot\hat{X}} \qquad k^2 = 0$$

$\rightsquigarrow$ in LC the gauge form A_j has 24 comp. $\rightarrow$ NON LORENTZ COVARIANT

$\rho = \tau + i\sigma = \ln z$

Go back to conformal gauge: $\boxed{e = \bar{e} = 1}$ $\quad X^m = \hat{X}^m + \sum_{N\neq 0} a_N^m z^N$

with this sign creat. op. have $\boxed{N > 0}$

$$V(c, X^m, b) \to V(c, a^m_{N>0}, b)$$

What about c, b?

$c(z) = c_{-1} + c_0 z + c_1 z^2 + \dots \Rightarrow$ conf weight $= -1 \longrightarrow$ diff exp.

$b(z) = b_2 + b_3 z + \dots \Rightarrow$ conf w. $= 2$

$$\Rightarrow k^2 + \sum_{N,m} N M_{N,m} + \underbrace{\sum N M'_c + \sum N M''_b}_{\text{GHOSTS!}} = 0$$

Now in general

$$V(b_2, b_3, \dots; c_{-1}, c_0, c_1, \dots; a_1^n, a_2^n, \dots; \hat{x}^m) =$$

MIND THE GHOST NUMBER!

$\to k^2 = -1 \Rightarrow c_{-1} e^{ik\cdot\hat{x}} |0\rangle$

$k^2 = 0 \Rightarrow e^{ik\cdot\hat{x}}|0\rangle,\ c_{-1} a_1^m e^{ik\cdot\hat{x}}|0\rangle,\ c_0 e^{ik\cdot\hat{x}}|0\rangle,$

$c_{-1}c_1 e^{ik\cdot\hat{x}}|0\rangle,\ c_0 c_{-1} a_1^m e^{ik\cdot\hat{x}}|0\rangle,\ c_{-1}c_0c_1 e^{ik\cdot\hat{x}}|0\rangle$

$k^2 = 1 \Rightarrow b_2 c_{-1} e^{ik\cdot\hat{x}}|0\rangle, \dots$

NB: $Q = \int d\sigma \left[c(P+X')^2 + b c \partial c \right] = c_0 (P+X')^2_0 + c_{-1}(P+X')^2_1 + \dots$

and compute the cohomology: fixes k of each state

$A_m(x) = u_m^{(k)} e^{ik\cdot X}$

all at $z=0$

$e^{ik\cdot\hat{x}}|0\rangle \longrightarrow k=0 \Rightarrow \Lambda$ (const)

$c_{-1} a_1^m e^{ik\cdot\hat{x}}|0\rangle \longrightarrow c\,\partial X^m A_m(x)$ (vector)

$c_0 e^{ik\cdot X}|0\rangle \longrightarrow \partial c\, B(x)$ (scal)

$c_{-1}c_1 e^{ik\cdot\hat{x}}|0\rangle \longrightarrow c\,\partial^2 c\, D(x)$ (scal)

$c_0 c_{-1} a_1^m e^{ik\cdot\hat{x}}|0\rangle \longrightarrow c\,\partial c\,\partial X^m E_m(x)$ (vect)

$c_{-1}c_0c_1 e^{ik\cdot\hat{x}}|0\rangle \longrightarrow c\,\partial c\,\partial^2 c\, F(x)$ (scal)

$\Rightarrow QV = 0 \longrightarrow \partial^m A_m = B \qquad \delta V = Q\Omega \longrightarrow \delta A_m = \partial_m \Lambda$

Homework $\qquad \Box A_m = \partial_m B \qquad \delta B = \Box \Lambda$

$\Rightarrow \partial^m(\partial_{[m} A_{n]}) = 0$

The cohomology leads to

$$\Lambda \qquad c\,\partial X^m A_m \qquad \partial c\, B(x) \qquad \underbrace{c\,\partial^2 c\, D(x) \qquad c\,\partial c\,\partial X^m E_m}_{B^*(x) \quad A^*_m(x)} \qquad \underbrace{c\,\partial c\,\partial^2 c\, F(x)}_{\Lambda^*}$$

(opposite statistic)

Usually we do not consider antifields but only ghost number 0 and 1.

SPINNING PARTICLE

$$S = \int d\tau \left[P_m \dot{X}^m + i\psi_m \dot{\psi}^m + e P^2 + \chi \underbrace{P^m \psi_m}_{= G} \right]$$

Spinning particle:

$$[P_m, X^n] = -i\delta_m^n \qquad \{G, G\} = P^m P_m$$

$$\{\psi_m, \psi_n\} = \eta_{mn}$$

$$\delta X^m = \xi [G, X^m] = \xi \psi^m$$

$$\delta P_m = 0$$

$$\delta \psi^m = \xi \{G, \psi^m\} = \xi P^m$$

$$\delta e = \xi \chi$$

$$\delta \chi = \dot{\xi}$$

$\rightarrow$ GAUGE FIX: $e = 1$, $\chi = 0$ — a gauge choice is fine when we can find a prop for ghosts

$$S = \int d\tau \left[(\dot{X})^2 + i\psi\dot{\psi} + \underbrace{b\dot{c}}_{\text{anticomm.}} + \underbrace{\beta\dot{\gamma}}_{\text{comm}} \right]$$

we can combine

$\mathbb{X}^m(\tau, \kappa) = X^m + \kappa\psi^m \Rightarrow S = \int d\tau d\kappa \left(D\mathbb{X}^m \dot{\mathbb{X}}_m + \mathbb{B}\dot{\mathbb{C}} + D\mathbb{B}D\mathbb{C} \right)$

$\mathbb{C} = c + \kappa\gamma$

$$Q = cP^2 + \gamma(P^m \psi_m) + \gamma^2 b \rightarrow \frac{\partial}{\partial c}$$

Spectrum?

$V(c, \gamma, X, \psi)$ — half of the ψ are coord and half are momenta: we cannot choose them without break. Lorentz!

$\Downarrow$

$$\{\psi_m, \psi_n\} \rightarrow \left\{ \frac{1}{\sqrt{2}}\gamma_m, \frac{1}{\sqrt{2}}\gamma_n \right\} = \eta_{mn}$$

Therefore

$V(c, \gamma, X, \psi) \rightarrow V^\alpha(c, \gamma, X)$ (spinor) where $\psi^m V^\alpha = \frac{1}{\sqrt{2}} (\Gamma^m)^{\alpha\beta} V_\beta \quad \alpha = 1, .., 32$

covariantly.

$\hookrightarrow \psi$ acts as a spinorial rep.

Otherwise we can break $SO(10)$:

$$\psi^1 + i\psi^2 \qquad\qquad \psi^1 - i\psi^2$$
$$\psi^3 + i\psi^4 \qquad\qquad \psi^3 - i\psi^4$$
$$\vdots \qquad\qquad\qquad \vdots$$

CREATION ANNHILATION

$$|0\rangle^{-----}$$
$$(\psi^1 + i\psi^2)|0\rangle^{-----} = |0\rangle^{+----}$$

$\Rightarrow$ in general : $|0\rangle^{\pm\pm\pm\pm\pm} \Rightarrow$ 32 different ground states

Then $V^\alpha = \underbrace{\varphi_0^\alpha(x)}_{\substack{\text{spin } \frac{1}{2} \\ \not{\partial}\psi_0 = 0}} + \underbrace{c\varphi_1^\alpha(x) + \gamma\tilde{\varphi}_1^\alpha(x)}_{\not{\partial}\psi_1 = 0} + \underbrace{c\gamma\psi_2(x) + (\infty \text{ no. of } \gamma_s)}_{\not{\partial}\psi_2 = 0}$

these are just copies of φ_0^α

$\rightarrow$ difficult because it doesn't show directly spacetime SUSY.

Now compute the cohomology: we're not cons. the Weyl decoup. $(\Gamma^m)^{\alpha\beta} = \begin{pmatrix} 0 & (\gamma^m)^{AB} \\ (\gamma_m)_{AB} & 0 \end{pmatrix}$

$$V^\alpha(x,\gamma,c) = \varphi_0^\alpha + c\varphi_1^\alpha + \gamma\tilde{\varphi}_1^\alpha + c\gamma\varphi_2^\alpha + \gamma^2\tilde{\varphi}_2^\alpha + \dots \qquad \alpha = 1,\dots,32$$

$$\Rightarrow QV^\alpha = c\Box\varphi_0^\alpha + \gamma(\not{\partial}\varphi_0)^\alpha + \gamma^2\varphi_1^\alpha + c\gamma(\not{\partial}\varphi_1)^\alpha + c\gamma\Box\tilde{\varphi}_1^\alpha + $$
$$+ \gamma^2(\not{\partial}\tilde{\varphi}_1)^\alpha$$

$$\gamma^2 b c\varphi_1^{\alpha\beta}|0\rangle = \dots + \gamma^2 c \cancel{\varphi_1^\alpha b|0\rangle}^{=0}$$

then $\Box\varphi_0^\alpha = (\not{\partial}\varphi_0)^\alpha = 0$

$$\varphi_1^\alpha = (\not{\partial}\tilde{\varphi}_1)^\alpha$$

$$\varphi_2^\alpha = (\not{\partial}\tilde{\varphi}_2)^\alpha$$

Moreover $\delta V^\alpha = Q\Omega = c\Box\omega_0^\alpha + \gamma(\not{\partial}\omega_0)^\alpha + c\gamma(\not{\partial}\omega_1)^\alpha + \gamma^2\omega_1^\alpha + c\gamma(\Box\tilde{\omega}_1)^\alpha +$
$+ \gamma^2(\not{\partial}\tilde{\omega}_1)^\alpha + \dots$

Therefore:

$$\delta\varphi_0^{\alpha} = 0$$
$$\delta\varphi_1^{\alpha} = \Box\omega_0^{\alpha}$$
$$\delta\tilde{\varphi}_1^{\alpha} = (\partial\!\!\!/\omega_0)^{\alpha}$$

$\Rightarrow$ e.o.m: $(\partial\!\!\!/\varphi_0)^{\alpha} = 0 \longrightarrow \varphi_0$ is a spin-$\frac{1}{2}$ particle

$\delta\tilde{\varphi}_1^{\alpha} = (\partial\!\!\!/\omega_0)^{\alpha} \longrightarrow$ suppose $P^2\tilde{\varphi}_1 \neq 0$, $P\!\!\!/\tilde{\varphi}_1 \neq 0$ $\Rightarrow \tilde{\varphi}_1 = P\!\!\!/\underbrace{\left(\frac{P\!\!\!/\tilde{\varphi}_1}{P^2}\right)}_{\omega_0} = \frac{P^2\tilde{\varphi}_1}{P^2}$

etc...

we want to prove that $\tilde{\varphi}_1$ has the same dimen. as φ_0 and we can gauge it away

- then $\delta\tilde{\varphi}_2^{\alpha} = (\partial\!\!\!/\omega_1)^{\alpha}$, ..., etc.!

STRING

$$S = \int d\tau d\sigma \Big[P_m \dot{X}^m + \tfrac{1}{2}\psi_m\dot{\psi}^m + e\left[(P+X')^2 + \psi\cdot\psi'\right] + \chi\psi\cdot(P+X') +$$
$$+ \tfrac{1}{2}\bar{\psi}_m\dot{\bar{\psi}}^m + \bar{e}\left[(P-X')^2 + \bar{\psi}\cdot\bar{\psi}'\right] + \bar{\chi}\bar{\psi}\cdot(P-X')\Big]$$

* Boundary conditions:

$$\left.\begin{array}{ll} \psi(\sigma+2\pi) = \pm\psi(\sigma) & \begin{smallmatrix}R\\NS\end{smallmatrix} \\ \bar{\psi}(\sigma+2\pi) = \pm\psi(\sigma) & \begin{smallmatrix}R\\NS\end{smallmatrix} \end{array}\right\} \text{closed}$$

$$\left.\begin{array}{ll} \psi^m(0) = \bar{\psi}^m(0) & \\ \psi(\pi) = \pm\bar{\psi}^m(\pi) & \begin{smallmatrix}R\\NS\end{smallmatrix} \end{array}\right\} \text{open}$$

gauge: $e = \bar{e} = 1$, $\chi = \bar{\chi} = 0 \Rightarrow S = \int d\tau d\sigma \big(\partial X\bar{\partial}X + \psi\bar{\partial}\psi + \bar{\psi}\partial\bar{\psi} + F^2 +$
$+ b\bar{\partial}c + \beta\bar{\partial}\gamma + \bar{b}\partial\bar{c} + \bar{\beta}\partial\bar{\gamma}\big)$

$\Rightarrow Q = \int d\sigma \big(c(\partial X\partial X - \psi\partial\psi) + \gamma(\partial X\psi) + \gamma^2 b + bc\partial c + c(\gamma\partial\beta + \partial(\gamma\beta)) + cc\big)$

Cohomology:

$P^2 \varphi \neq 0$
$\not{P} \varphi \neq 0$ $\Rightarrow$ $\varphi = \not{P} \Omega$? $\longrightarrow$ $\Omega = \frac{\not{P}\varphi}{P^2}$ $\longrightarrow$ NO COHOMOLOGY

$P^2 \varphi = 0$
$(\not{P}\varphi)^\alpha = 0$ $\Rightarrow$ $\varphi = \frac{\not{P}\not{k}\varphi}{2P\cdot k}$ where $k\cdot P \neq 0$ $\Rightarrow$ $\varphi = \frac{\{\not{P},\not{k}\}\varphi}{2P\cdot k} = \varphi \Rightarrow$ NO COHOM.

$P^2\varphi \neq 0$
$\not{P}\varphi = 0$ $\rightarrow$ impon.

$P^2\varphi = 0$
$\not{P}\varphi \neq 0$ $\Rightarrow$ $\varphi = \not{P}\Omega$? $\Rightarrow$ $\not{P}\varphi = P^2\Omega = 0$ $\Rightarrow$ contradicts hp

$\Rightarrow$ COHOMOLOGY !

these states are in the cohomology!

$\Rightarrow \chi = \not{P}\varphi$

What's the vacuum? $X^m = \tilde{X}^m + a_1^m z + \ldots$; $c = c_{-1} + c_0 z + \ldots$

$b = b_2 + b_3 z + \ldots$

$\psi^m = \frac{d_0}{\sqrt{z}} + d_1\sqrt{z} + \ldots \rightarrow R$

$= d_{1/2} + d_{3/2} z + \ldots \Rightarrow NS$

$a_N^m |0\rangle = 0, \quad N<0$

$c_N |0\rangle = 0, \quad N<-1$

$b_N |0\rangle = 0, \quad N<2$

R: $d_N |0\rangle = 0 \quad N<0 \rightarrow \{d_0^m, d_0^n\} = 2\eta^{mn}$

NS: $d_N |0\rangle = 0 \quad N<\frac{1}{2}$

Plus: R: $\gamma = \gamma_{-1}/\sqrt{z} + \gamma_0 \sqrt{z} + \ldots$

NS: $\gamma = \gamma_{-1/2} + z\gamma_{1/2} + \ldots$

NS $\gamma_N |0\rangle = 0 \quad N<-\frac{1}{2}$

same for $\beta \rightarrow \beta_N |0\rangle = 0 \quad N< 3/2$

$\Rightarrow$ $k^2 = -1$: $c_{-1}|0\rangle, \ (\gamma_{-1/2})^2|0\rangle$

$k^2 = -\frac{1}{2}$: $\gamma_{-1/2}|0\rangle, \ c_{-1} d_{1/2}^m |0\rangle$

NS

NS: none of the pure ghosts are in the cohomology

$k^2 = 0$: $|0\rangle$; $c_{-1} a_1^m |0\rangle$, $\gamma_{-1/2} d_{1/2}^m |0\rangle$, $c_0|0\rangle$,

$c_0 c_1 a_1^n |0\rangle, \ldots, c_{-1} c_0 c_1 |0\rangle$, $c_0 \gamma_{-1/2} \gamma_{\frac{1}{2}} |0\rangle$, $c_{-1} d_{\frac{1}{2}}^m d_{-\frac{1}{2}}^n |0\rangle$

$\rightarrow$ Write the ops.

$|0\rangle \longrightarrow \mathbb{I} \rightarrow \Lambda(x)$

$c_{-1} a_1^m |0\rangle \longrightarrow c\,\partial X^m A_m(x)$

$\gamma_{-\frac{1}{2}} d_{\frac{1}{2}}^m |0\rangle \rightarrow \gamma \psi^m A'_m(x)$

$c_0 |0\rangle \longrightarrow \partial c\, B(x)$

$c_0 c_{-1} a_1^m |0\rangle \longrightarrow c\,\partial c\,\partial X^m A''_m(x)$

$\gamma_{-\frac{1}{2}}|0\rangle \longrightarrow \gamma T(x)$

$c_{-1}|0\rangle \longrightarrow c\, \tilde{T}(x)$

And compute the cohomology:

$Q(c\tilde{T}(x)) = \gamma^2 \tilde{T}(x) = 0 \Rightarrow \tilde{T}(x) = 0$

$Q(\gamma T(x)) = c\gamma \Box T(x) + c\gamma \frac{1}{2} T(x) + \gamma^2 \psi^m \partial_m T(x)$

$Q(c\psi^m B_m(x)) = \gamma^2 \psi^m B_m + \cancel{c\gamma \psi^m \psi^n \partial_n B_m}^{=0}$

$\Big\} = 0$ if $B_m = \partial_m T$

Then $V = \gamma T(x) + c\psi^m \partial_m T = (\underbrace{\gamma + c\psi^m k_m}) T(x)$

$V = \int d\kappa \, \mathbb{C}\, T(\mathbb{X})$

in cohomol $\Rightarrow$ the tachyon is a superpos. of two states

$\rightarrow$ Massless states:

$$V = \gamma \psi^m A_m(x) + c\,\partial X^m A_m(x) + c\,\psi^m \psi^n \partial_m A_n(x)$$

$$= \int d\kappa \, \mathbb{C}\, \mathbb{X}^m A_m(\mathbb{X})$$

Now look at R sector

$V = c_{-1} |0\rangle^\alpha \longrightarrow$ $\left.\begin{array}{l} \gamma(z) V(0) \sim \sqrt{z} \\ \psi(z) V(0) \sim 1/\sqrt{z} \end{array}\right\}$ SPIN FIELDS !

$\rightarrow$ bosonization: $\gamma = \xi e^{\varphi}$, $\psi^m = e^{\pm \sigma^m}$

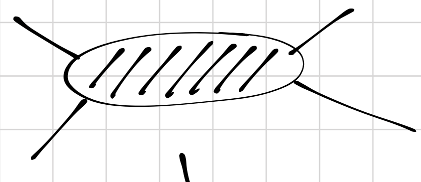
$\Rightarrow V = e^{\varphi/2} e^{\pm \frac{1}{2}\sigma_m}$

RNS - Spinning String

GSO $\Downarrow$

Superstring

How to decouple unphysical states?



SUM OVER (all) SPIN STRUCT.

SPACETIME SUSY

$D = 10 \implies m = 0, \dots, 9 \qquad X^m$

Weyl spinor $\implies \alpha = 1, \dots, 16 \qquad \theta^\alpha$

Spinor Lor. transf.: $\delta\chi^\alpha = \frac{1}{2}\Lambda_{mn}(\gamma^{mn}\chi)^\alpha$

$SO(10) \to U(5)$

32 comp $\begin{pmatrix} +++++ \\ \\ ----- \end{pmatrix} \longrightarrow$ Weyl: 16 Weyl $\begin{array}{c} \begin{pmatrix} +++++ \\ +++-- \\ \\ +---- \end{pmatrix} \end{array} \begin{array}{c} 1 \\ 10 \\ \\ \bar{5} \end{array}$

Anti Weyl: 16 AWeyl $\begin{pmatrix} ++++- \\ ++--- \\ ----- \end{pmatrix} \begin{array}{c} 5 \\ \overline{10} \\ 1 \end{array}$

$\downarrow$

eigenvalues of

$\gamma^1 \pm i\gamma^2$
$\vdots$
$\gamma^9 \pm i\gamma^{10}$ $\longrightarrow$ "(+ sum. +, − sum. −)"

e.g.: $(\gamma^1 + i\gamma^2)_{\alpha\beta}\,\psi^\beta$ s.t. $\beta = (+++++) \Rightarrow\ = 0$

$(\gamma^1 - i\gamma^2)\underbrace{(-++++ \quad +++++)}$

in this sense γs "change" Weyl into AntiWeyl

$\Rightarrow$ global spacetime SUSY:

$$\delta_q\theta^\alpha = \varepsilon^\alpha$$
$$\delta_q X^m = \frac{1}{2}\theta^\alpha\gamma^m_{\alpha\beta}\theta^\beta$$

$$\to \{\delta_{q_1}, \delta_{q_2}\}\theta^\alpha = 0$$

$$[\delta_{q_1}, \delta_{q_2}]X^m = \delta_{q_1}\left(\frac{1}{2}\theta^\alpha\gamma^m_{\alpha\beta}\theta^\beta\right) - \dots =$$

$$= \frac{1}{2}\epsilon_1^\alpha\gamma^m_{\alpha\beta}\epsilon_2^\beta - \frac{1}{2}\epsilon_2^\alpha\gamma^m_{\alpha\beta}\epsilon_1^\beta = \epsilon_2\gamma^m\epsilon_1$$

SUPERPARTICLE (Green - Schwarz)

$$S = \int d\tau \left(P_m \left(\dot{X}^m - \frac{1}{2} \dot{\theta} \gamma^m \theta \right) + e P^2 \right)$$

$\rightarrow D = 10$ SYM (gluon + gluino)

We have:

$$\Gamma^m = \begin{pmatrix} 0 & (\gamma^m)^{\alpha\beta} \\ (\gamma^m)_{\alpha\beta} & 0 \end{pmatrix} \Rightarrow \alpha, \beta = 1, \dots, 16$$

NB.: $(\gamma^m)^{\alpha\beta} = (\gamma^m)^{\beta\alpha}$

$\theta^\alpha \rightarrow$ Weyl $\begin{pmatrix} ----- \\ ---++ \\ -++++ \end{pmatrix} \begin{matrix} 1 \\ 10 \\ 5 \end{matrix}$

$\psi_\alpha \rightarrow$ Anti-Weyl $\begin{pmatrix} +---- \\ +++-- \\ +++++ \end{pmatrix} \begin{matrix} 5 \\ 10 \\ 1 \end{matrix}$

NB in $D=4$

Weyl θ^α $\begin{pmatrix} + + \\ - - \end{pmatrix}$

Anti-Weyl $\bar{\theta}^{\dot\alpha}$ $\begin{pmatrix} + - \\ - + \end{pmatrix}$ → complex conj (in $D=10$ θ^α is real)

s.t. $\Gamma^0 = \begin{pmatrix} 0 & \mathbb{I}_2 \\ -\mathbb{I}_2 & 0 \end{pmatrix}$, $\Gamma^j = \begin{pmatrix} 0 & \sigma^j \\ \sigma^j & 0 \end{pmatrix}$

$$\Rightarrow \Gamma^1 + i\Gamma^2 = \begin{pmatrix} & & & 2 \\ & & 0 & \\ & 0 & & \\ 0 & & & \end{pmatrix} \qquad \Gamma^1 - i\Gamma^2 = \begin{pmatrix} & & & 0 \\ & & 2 & \\ & 0 & & \\ 2 & & & \end{pmatrix}$$

$$\Gamma^0 + \Gamma^3 = \begin{pmatrix} & & 2 & \\ & & & 0 \\ 0 & & & \\ & -2 & & \end{pmatrix} \qquad \Gamma^0 - \Gamma^3 = \begin{pmatrix} & & 0 & \\ & & & 2 \\ -2 & & & \\ & 0 & & \end{pmatrix}$$

Then $\theta^{++} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $\theta^{--} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, $\theta^{+-} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, $\theta^{-+} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

→ first 2 comp. are Weyl, while the others are anti-Weyl

Let's try to quantize the action:

momentum w.r.t. θ : $\quad p_\alpha = \frac{\delta \mathcal{L}}{\delta \dot{\theta}^\alpha} = -\frac{i}{2} P^m (\gamma_m \theta)_\alpha \rightarrow \theta$ dependent

$\longrightarrow d_\alpha = p_\alpha + \frac{i}{2} P^m (\gamma_m \theta)_\alpha = 0$ is a constr.

NB: the no. of comp. α, β is related to $N=1$ or $N=2$ supersymm.

$\Rightarrow$ we have: $\quad \{ d_\alpha, P^2 \} = 0$

(NB: $\{ p_\alpha, \theta^\beta \} = -i \delta_\alpha{}^\beta$) $\quad \{ d_\alpha, d_\beta \} = \{ p_\alpha, p_\beta \} + \{ \frac{i}{2} P^m (\gamma_m \theta)_\alpha, \frac{i}{2} P^n (\gamma_n \theta)_\beta \} +$

$+ \{ p_\alpha, \frac{i}{2} P^n (\gamma_n \theta)_\beta \} + \{ \frac{i}{2} P^m (\gamma_m \theta)_\alpha, p_\beta \} =$

$= P_m (\gamma^m)_{\alpha\beta} \Rightarrow$ II class constraints ?

Consider the ref. frame $P^+ \neq 0 \Rightarrow P^+ (\gamma^-)_{\alpha\beta} = P^+ \begin{pmatrix} \mathbb{I}_8 & 0 \\ 0 & 0_8 \end{pmatrix}$

$\Downarrow$

8 first class

8 second class

$\left[\gamma^-_{\alpha\beta} = \begin{pmatrix} \mathbb{I}_8 & \\ & 0 \end{pmatrix}, \ \gamma^+_{\alpha\beta} = \begin{pmatrix} 0 & \\ & \mathbb{I}_{\alpha\beta} \end{pmatrix} \right]$

$\gamma^0 - \gamma^9 \qquad \gamma^0 + \gamma^9$

$\rightarrow$ I class constraints ? $\Rightarrow$ BRST

$\rightarrow$ II class constraints ? $\Rightarrow$ Dirac brackets: $\{ A, B \}_{DB} = \{ A, B \}_{PB} -$

$- \{ A, \varphi_k \} C^{k}_{ik} \{ \varphi_{ij}, B \}$

$\Rightarrow \{ A, \varphi_i \}_{DB} = 0$

II class

how to invert C^i_{jk} when it becomes operator ? $\Rightarrow$ NO USE

To remove I class constraints, the simplest way is to gauge fix the action:

symmetry: $\quad \delta \theta^\alpha = (\not{P} K(\tau))^\alpha$

$\delta X^m = -\frac{1}{2} (\theta \gamma^m \delta\theta)$

$\delta P^m = 0$

$\delta e = \dot{\theta}^\alpha K_\alpha(\tau)$

Homework: show that this is a symm.

local K-symmetry

$$\text{assume } P^+ \neq 0$$

Gauge fix the fermions θ^α: $(\gamma^+\theta)_\alpha = 0 \Rightarrow$ the last 8 comp. of θ_α will be 0.

$$\begin{pmatrix} 0 & \\ & \mathbb{I}_8 \end{pmatrix}\begin{pmatrix} \theta^\alpha \end{pmatrix}$$

$$\rightsquigarrow S = \int d\tau \left(P_m \dot{X}^m + e P^2 - \cancel{\tfrac{1}{2}\dot{\theta}^\alpha \gamma^j_{\alpha\beta}\theta^\beta P^j} - \cancel{\tfrac{1}{2}\dot{\theta}^\alpha \gamma^+_{\alpha\beta}\theta^\beta P^-} - \tfrac{1}{2}\dot{\theta}^\alpha \gamma^-_{\alpha\beta}\theta^\beta P^+ \right)$$

0 ; 0: gauge fix

$$\Rightarrow \dot{\theta}^\alpha \gamma^j_{\alpha\beta}\left(\frac{\gamma^+\gamma^- + \gamma^-\gamma^+}{2}\right)\theta^\beta = \dot{\theta}^\alpha \gamma^+ \gamma^j \theta = 0$$

$$\rightsquigarrow S = \int d\tau \left(P_m \dot{X}^m + e P^2 + \tfrac{1}{2}\dot{\theta}^a \theta^a P^+ \right)$$

$$\theta^\alpha = \begin{pmatrix} \theta^a \\ \theta^{\dot{a}} \end{pmatrix} \quad \begin{matrix} \theta^a \to SO(8) \text{ Weyl} \\ \theta^{\dot a} \to SO(8) \text{ Anti-Weyl} \end{matrix} \qquad a, \dot{a} = 1, \dots, 8$$

Now: $p_a = \frac{1}{2}\theta_a P^+ \Rightarrow S^a = \sqrt{\frac{P^+}{2}}\,\theta^a$

$$\Rightarrow S = \int d\tau \left(P_m \dot{X}^m + e P^2 + \dot{S}^a S^a \right) \qquad \{S^a, S^b\} = 2\delta^{ab}$$

as for $\{\psi^m, \psi^n\} = 2\eta^{mn} \Rightarrow |0\rangle^\alpha$ s.t. $\psi^m |0\rangle^\alpha \to (\Gamma^m)^{\alpha\beta}|0\rangle_\beta$

we find $\{\Gamma^j_{a\dot{b}}, \Gamma^k_{c\dot{b}}\} = 2\delta^{jk}\delta_{ac}$ $\left[\{\Gamma^j_{a\dot{b}}, \Gamma^k_{a\dot{c}}\} = 2\delta^{jk}\delta_{\dot{b}\dot{c}} \right.$ $\qquad a, \dot{a} = 1, \dots, 8$; $j = 1, \dots, 8$

$\left. \{\Gamma^j_{a\dot{b}}, \Gamma^j_{c\dot{d}}\} = 2\delta_{ac}\delta_{\dot{b}\dot{d}} \right]$

SO(8) triality

$$\Rightarrow \psi^j \to \begin{matrix} |0\rangle^a & \text{Weyl} \\ |0\rangle^{\dot{a}} & \text{Anti-Weyl} \end{matrix}$$

$$S^a \to \begin{matrix} |0\rangle^j & SO(8) \text{ vector} & \longrightarrow \text{GLUON } A_j \\ |0\rangle^{\dot{a}} & \text{anti-Weyl spinor } SO(8) & \to \text{GLUINO } \chi_{\dot{a}} \end{matrix} \qquad (\text{SYM})$$

i.e.: $S^a |0\rangle^j = (\Gamma^j)^{a\dot{a}} |0\rangle_{\dot{a}}$

unrelated to S^a

Add dof: $S = \int d\tau \left(P_m \dot{X}^m + eP^2 + \dot{S}^a S^a + p_\alpha \dot{\theta}^\alpha + f^\alpha \underbrace{\left(d_\alpha + \frac{1}{\sqrt{p}}(\not{P}\gamma^+ S)_\alpha\right)}_{\hat{d}_\alpha} \right)$

Lagrange mult. $\quad$ I class

$d_\alpha = p_\alpha + \frac{1}{2}(\not{P}\theta)_\alpha$

$(\alpha = 1, \dots, 16\ ;\ a = 1, \dots, 8)$

$\longrightarrow \{S_\alpha, S_\beta\} = \gamma^-_{\alpha\beta} \Leftrightarrow \{S_a, S_b\} = \delta_{ab}$

$$\rightarrow \{\hat{d}_\alpha, \hat{d}_\beta\} = (\not{P})_{\alpha\beta} + \frac{\not{P}\gamma^+\gamma^-\gamma^+\not{P}}{p^+} = \frac{P^2}{p^+}(\gamma^+)_{\alpha\beta}$$

NB: spacetime SUSY $\Rightarrow$

$$\delta p_\alpha = -\frac{1}{2}(\not{P}\epsilon)_\alpha$$
$$\delta x^m = \frac{1}{2}\delta\theta\gamma^m\theta \qquad \text{(global)}$$
$$\delta\theta^\alpha = \epsilon^\alpha$$

because of the I class constr.

LOCAL: $(\tilde{S} = \gamma^+ S)$

$$\delta\theta^\alpha = \varepsilon^\alpha, \quad \delta\tilde{S}^\beta = \varepsilon^\alpha\{\hat{d}_\alpha, \tilde{S}^\beta\} = \frac{1}{\sqrt{p^+}}(\not{P}\gamma^+)_\alpha{}^\beta \varepsilon^\alpha$$
$$\delta f^\alpha = \dot{\varepsilon}^\alpha \frac{\gamma^+_{\alpha\beta} f^\beta}{p^+}$$
$$\delta X^m = \varepsilon^\alpha\left(\frac{1}{2}(\gamma^m\theta)_\alpha + \frac{1}{\sqrt{p^+}}(\gamma^m\tilde{S})_\alpha - \frac{1}{2}\frac{\delta^m_{\cdot}}{(p^+)^{3/2}}(\not{P}\tilde{S})_\alpha\right)$$
$$\delta p_\alpha = (\not{P}\varepsilon)_\alpha$$

$\rightarrow \boxed{\text{Gauge } \theta^\alpha = 0}$

$\downarrow$

ORIGINAL ACTION

16 local symm. which will allow to recover the previous formulation!

Suppose instead:

$f^\alpha \rightarrow$ 16 components

gauge: $f^\alpha = 0 \Rightarrow \tilde{\lambda}^\alpha$ ghost, Majorana-Weyl spinor

$\rightarrow Q = \tilde{\lambda}^\alpha \hat{d}_\alpha + cP^2 + \frac{\tilde{\lambda}\gamma^+\tilde{\lambda} b}{p^+}$

$\rightarrow$ Cohom $\Rightarrow$ non Lorentz covariant!

fix 11 components of f^α to $0 \Rightarrow$ PURE SPINOR

$$\lambda^\alpha = 0$$
$$Q = \lambda^\alpha d_\alpha$$
$$d_\alpha = p_\alpha + \frac{1}{2}(\not{P}\theta)_\alpha$$

* what is λ?

symm.

$$\rightarrow D \text{ even: } \lambda^\alpha\lambda^\beta - (\gamma^{m_1 \dots m_{D/2}})^{\alpha\beta} f_{m_1 \dots m_{D/2}} + (\gamma^{m_1 \dots m_{\frac{D}{2}-4}})^{\alpha\beta} f_{m_1 \dots m_{\frac{D}{2}-4}} + \dots$$

$\Rightarrow$ "pure spinor" if <u>ONLY</u> $f_{m_1 \dots m_{D/2}} \neq 0$

projective pure spinor : $SO(D)/U(D/2) \Rightarrow \frac{D(D-1)}{2} - \left(\frac{D}{2}\right)^2 \longrightarrow D=10 \Rightarrow 10$ $\underline{\underline{\text{compl}}}$

pure spinor: $\mathbb{C} \times SO(D)/U(D/2) \Rightarrow D=10 \longrightarrow 11$ $\underline{\underline{\text{compl}}}$

NB: pure spinor is a BOSON!

Therefore we show that:

$$S = \int dz \left[\dot{X}^m P_m + \frac{1}{2} \dot{S}^a S^a + e P^2 + p_\alpha \dot{\theta}^\alpha + f^\alpha \hat{d}_\alpha \right]$$

can be studied as:

$$* \; S = \int dz \left[\dot{X}^m P_m + \frac{1}{2} \dot{S}^a S^a + P^2 + b\dot{c} \right] \quad \left(\theta^\alpha = 0,\; e = 1,\; Q = cP^2\right)$$

$$Q = cP^2$$

$$* \; S = \int dz \left[\dot{X}^m P_m + \frac{1}{2} \dot{S}^a S^a + p_\alpha \dot{\theta}^\alpha + b\dot{c} + \hat{\omega}_\alpha \dot{\hat{\lambda}}^\alpha \right] \quad \left(f^\alpha = 0,\; e = 1\right.$$

$$Q = \hat{\lambda}^\alpha \hat{d}_\alpha + cP^2 + \frac{b(\hat{\lambda}\gamma^+\hat{\lambda})}{P^+}$$

$$* \; S = \int dz \left(\dot{X}^m P_m + p_\alpha \dot{\theta}^\alpha + \omega_\alpha \dot{\lambda}^\alpha \right] \quad \left(\lambda\gamma^m\lambda = 0, \right.$$

$$Q = \lambda^\alpha d_\alpha \qquad \left. \text{gauge } f^\alpha = u_m (\gamma^m \bar{\lambda})^\alpha \right)$$

11 indep. solutions

PURE SPINOR

$\lambda\gamma^m\lambda = 0$ Non zero: λ^{+++++}

Take $U(1) \subset SO(10)$ rotations $\Rightarrow (x^i + x^{i+1})' = e^{i\alpha_i}(x^i + i x^{i+1})$

$\lambda^{+++++} \longrightarrow$ charged $\frac{5}{2}$ w.r.t. to $U(1)$ transf

$$SO(10) \longrightarrow U(5) \sim SU(5) \times U(1)$$

$$\begin{pmatrix} +++++ \\ +++-- \\ +---- \end{pmatrix} \begin{matrix} 1_{5/2} \\ 10_{1/2} \\ \bar{5}_{-3/2} \end{matrix}$$

We know:

$$X^m \begin{cases} \nearrow X^a \begin{pmatrix} x^1+ix^2 \\ x^3+ix^4 \\ \vdots \end{pmatrix} \longrightarrow 5_1 \\ \searrow X^{\bar a} = \begin{pmatrix} x^1-ix^2 \\ \vdots \end{pmatrix} \longrightarrow \bar 5_{-1} \end{cases}$$

$$\Rightarrow \lambda^{+++++}\lambda^{+++++} \longrightarrow \lambda\gamma^{(1+i2)(3+i4)\dots}\lambda$$

scale transf

$$\Rightarrow \text{PURE SPINOR} \in \frac{SO(10)}{SU(5)} \times \mathbb{R} = \frac{SO(10)}{U(5)} \times \mathbb{C}$$

$$\rightarrow \lambda^a = \begin{pmatrix} +++++ \\ +++-- \\ +---- \end{pmatrix} \begin{matrix} 1 \\ 10 \\ \bar 5 \end{matrix} \longrightarrow \begin{pmatrix} \lambda^+ \\ \lambda_{[ab]} \\ \lambda^a \end{pmatrix} \longrightarrow \text{NOT INDEP} \quad \lambda^a = \epsilon^{abcde}\frac{\lambda_{(bc)}\lambda_{(de)}}{8\lambda^+}$$

$$\lambda^{-----} \neq 0 \longrightarrow 1_{-5/2}$$

Now gauge $f^\alpha = (\bar\lambda\gamma^m)\,\epsilon_m \longrightarrow 11$ comp of f^α.

$*$ Remaining : $\epsilon^\alpha = (\lambda\gamma^m)^\alpha g_m \longrightarrow$ NB: cannot gauge fix comm. var. with anticomm:

$$\delta e = \frac{\epsilon^\alpha(\lambda\gamma)_\alpha}{p^+}$$

OK only for small var.

Now we have to study the cohomology of $Q = \lambda^\alpha d_\alpha \quad (\lambda\gamma^m\lambda = 0)$

$$V(\lambda^\alpha, x^m, \theta^\alpha) = \varphi(x,\theta) + \lambda^\alpha A_\alpha(x,\theta) + \lambda^\alpha\lambda^\beta B_{\alpha\beta}(x,\theta)$$

$$Q = \lambda^\alpha D_\alpha$$

$QV = 0$:

$$\lambda^\alpha D_\alpha\varphi = 0 \Rightarrow D_\alpha\varphi = 0 \Rightarrow \frac{\partial}{\partial x^m}\varphi = 0 \Rightarrow \varphi \text{ const.}$$

$$\lambda^\alpha\lambda^\beta D_\alpha A_\beta = 0 = \underbrace{(\lambda\gamma^m\lambda)}_{0}(D\gamma_m A) + (\lambda\gamma^{mnpqr}\lambda)(D\gamma_{mnpqr}A) \Rightarrow$$

$$\Rightarrow \gamma^{\alpha\beta}_{mnpqr} D_\alpha A_\beta = 0$$

$$\lambda^\alpha\lambda^\beta\lambda^\gamma D_{(\alpha}B_{\beta\gamma)} = 0$$

$$\delta V = Q\Sigma \Rightarrow \lambda^\alpha A_\alpha = \lambda^\alpha D_\alpha\Omega \Rightarrow \delta A_\alpha = D_\alpha\Omega$$

$$\lambda^\alpha\lambda^\beta B_{\alpha\beta} = \lambda^\alpha\lambda^\beta D_{(\alpha}X_{\beta)}$$

Therefore we can gauge:

$$A_\alpha(x,\theta) = (\gamma_m\theta)_\alpha a^m(x) + (\gamma_m\theta)_\alpha(\gamma^m\theta)_\beta \chi^\beta(x) + \dots$$

GLUON

where $a_m(x)$ satisfies $\partial^m \partial_{[m} a_{n]} = 0$ up to $\delta a_m = \partial_m \Lambda$

$\chi^\beta(x)$ " $(\partial\!\!\!/ \chi)_\beta = 0$

GLUINO

Plus:

$$B_{\alpha\beta}(x) = (\gamma^m\theta)_\alpha(\theta\gamma_{mnp}\theta)(\gamma^{np}\chi^*)_\beta + (\theta a^{*m})_\alpha$$

$$\delta\chi^*_\alpha = (\partial\!\!\!/\rho)_\alpha$$
$$\delta a^{*m} = \partial_n \rho^{mn}$$

NB: $\Omega(x,\theta) = a(x) + \theta^\alpha b_\alpha(x) + \theta^\alpha\theta^\beta c_{\alpha\beta}(x) + \dots$

$$D_\alpha\Omega(x,\theta) = b_\alpha + \theta^\beta(c_{\alpha\beta} + \partial\!\!\!/_{\alpha\beta} a) + \dots$$

$$A_\alpha(x,\theta) = f_\alpha + \theta^\beta g_{\alpha\beta} + \dots$$

$$g_{\alpha\beta} = g^m(\gamma_m)_{\alpha\beta} + g^{mnpqr}(\gamma_{mnpqr})_{\alpha\beta} + g^{mnp}(\gamma_{mnp})_{\alpha\beta}$$

$\Rightarrow$ STRING desccription

$$S = \int d\tau d\sigma \left[\partial X^m \bar\partial X_m + p_\alpha \bar\partial \theta^\alpha + \omega_\alpha \bar\partial \lambda^\alpha + \bar p^\alpha \partial \bar\theta_\alpha + \bar\omega^\alpha \partial \bar\lambda_\alpha \right]$$

IIA : θ^α Weyl , $\bar\theta_\alpha$ Anti-Weyl

IIB " $\bar\theta^\alpha$ Anti-Weyl

$\lambda^\alpha, \theta^\alpha$ conf weight 0

ω_α, p_α " 1

$\theta^\alpha(\sigma + 2\pi) = \theta^\alpha(\sigma)$

$\theta^\alpha = \bar\theta^\alpha (\sigma = 0)$

$\theta^\alpha = \bar\theta^\alpha \ (\sigma = \pi)$

OPEN STR.

$\lambda^\alpha A_\alpha(x, \theta) \quad k^2 = 0$

$\lambda^\alpha \partial X^m A_{\alpha m}(x, \theta) \quad k^2 = 1$

CLOSED STR.

$V^{\otimes}\left[c\bar c \, \partial X^m \bar\partial X^n (h_{mn} + b_{mn} + \varphi \eta_{mn}) \right]$

$\lambda^\alpha \bar\lambda^\beta A_{\alpha\beta}(x, \theta, \bar\theta) =$

$= (\lambda \gamma^m \theta)(\bar\lambda \gamma^n \bar\theta)\left[h_{mn} + b_{mn} + \varphi \eta_{mn} \right]$

$+ \ (\lambda \bar\lambda \theta^2 \bar\theta) \chi_{m\alpha} +$

$+ \ (\lambda \bar\lambda \theta^2 \bar\theta^2)^{\alpha\beta} F_{\alpha\beta}(x) + \ldots$

TREE AMPLITUDES

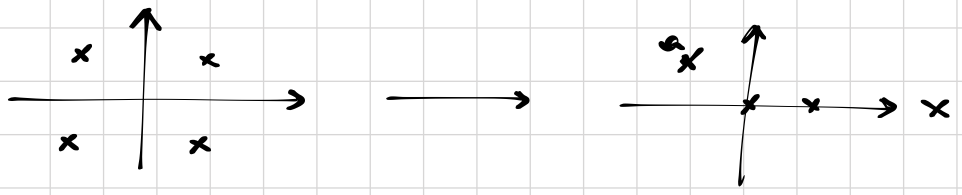
$\rightarrow$ easy to classify $\Rightarrow$ 2D Riemann surfaces

$\rightarrow$ CFT $\Rightarrow$ map 

if gauge fixed $\Rightarrow$ broken symmetry $\rightarrow$ more complicated

Use CFT : $z \mapsto \dfrac{az + b}{cz + d} \quad ad - bc = 1$

$$\int \mathcal{D}X \int \mathcal{D}c \, e^{-\int d^2 z (\partial X \bar\partial X + b \bar\partial c + ..)}$$



$\Rightarrow$ vertex ops : $\int \mathcal{D}X \int \mathcal{D}c \, e^{-S} \, V_1(z_1) V_2(z_2) V_3(z_3) V_4(z_4)$

e.g.: $V(z) = a_m(z) \partial X^m(z)$

$$\int \mathcal{D}X\, \mathcal{D}c\, e^{-S} \int dz_4\; V_1(0)\, V_2(1)\, V_3(\infty)\, V_4(z_4)$$

$$V_1(0) = c(0)\, a_m(z)\, \partial X^m(0) \quad \text{etc}$$

$\Rightarrow$ with $\int dz_4$ this is not conf. inv $\Rightarrow$ no ghost on V_4

$\downarrow$

$\alpha' \to 0 \Rightarrow$ Feynman diagram! not BRST inv ? $\to$ total derivative

e.g.:

$$\langle c(z_1)\, a_m(z_1)\, \partial X^m(z_1)\; c(z_2)\, a_p(z_2)\, \partial X^p(z_2)\; c(z_3)\, a_q(z_3)\, \partial X^q(z_3) \rangle$$

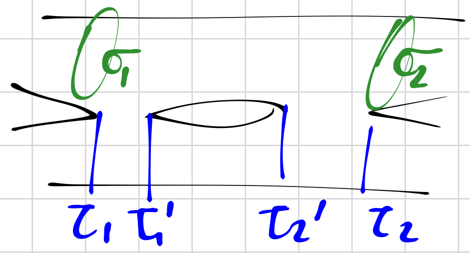
LOOP AMPLITUDES

$\to$ not all surfaces are the same :

SPHERE $\longrightarrow$ they're all related by radius rescaling

TORUS $\longrightarrow$ Teichmüller param

genus g $\longrightarrow$ $3g-3$ COMPLEX PARAMETERS



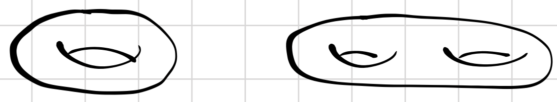
$\Rightarrow$ $\begin{matrix}\tau_2 - \tau_1 \\ \sigma_2, \sigma_1\end{matrix} \longrightarrow z_4, \bar{z}_4$

$\begin{matrix}\tau_1' - \tau_1 \\ \tau_2' - \tau_2\end{matrix} \longrightarrow \rho_1, \rho_2, \ldots$

$$\underline{\underline{\int (d^2\rho)^{3g-3}}} \prod_{i=1}^{N} \int (d^2 z_i) \int \mathcal{D}X\, \mathcal{D}c\; e^{-\int d^2z\, (\partial X \bar{\partial} X + b \bar{\partial} c)} \cdot$$

$$\cdot\, V_1(z_1) \ldots V_N(z_N) \int (db)^{3g-3}$$

Amplitudes

PERTURBATIVE:  4 graviton scatt.

NON PERT. DUALITIES: e.g.: type IIB $g_s \longleftrightarrow \frac{1}{g_s} \Rightarrow R + (\alpha')^3 R^4 + (\alpha')^5 \partial^2 R + \ldots$

→ Einstein gravity has stringy corrections

Non renorm. conjecture

→ related to 32 supersymm.

→ due to higher order derivatives ⇒ SUGRA is not finite

→ sum over diff. struct of same genus ⇒ allow periodic / anti-periodic cycles

⟿ Scattering eqn $\longrightarrow \alpha' \to 0$

$\alpha' = \infty$

nobody knows how to relate them...

How to derive the expansion?

BOS $$S_g = \left(\int d^2\tau\right)^{3g-3} \left(\int d^2 z\right)^N \int \mathcal{D}X \int \mathcal{D}b \int \mathcal{D}c \; e^{-S} \; V_1(z_1) \ldots V_N(z_N) \left(\int dz\, \mu(z)\, b\right)^{3g-3} \left(\int d\bar{z}\, \bar{\mu}(\bar{z})\, \bar{b}\right)^{3g-3}$$

SUPERST: $$S_g = \left(\int d^2\tau\right)^{3g-3} \int \mathcal{D}X \int \mathcal{D}\theta \mathcal{D}\bar{\theta} \int \mathcal{D}p \mathcal{D}\bar{p} \int \mathcal{D}\lambda \int \mathcal{D}\omega \; e^{S} \; V_1 \ldots V_N \left(\int dz\, \mu(z)\, b\right)^{3g-3} \left(\int d\bar{z}\, \bar{\mu}(\bar{z})\, \bar{b}\right)^{3g-3}$$

not indep of other modes!

BOSONIC: $\int b \rightarrow$ related to def $\int d^2\tau$

FERMIONIC: no b ghost but insert an operator such that it compensates for surface mod.

→ if only 3 vertex ops. $V_1 V_2 V_3 \Rightarrow$ not enough θs and ps unless $g=0$

⟿ always vanishes unless $g=0$ (sphere)

→ 3 gravitons scatt. contributes only at $g=0$

→ same for 4 operators up to $g=1$

↪ i.e.: not enough 0-modes

How to integrate over λ and ω?

$$\lambda \gamma^m \lambda = 0 \to \text{constraint!}$$

$\rightsquigarrow$ open problem at 3-loops $\Rightarrow$ divergences!!!

$\to$ τ integration might mess up the ampl. with non trivial dependencies!

Therefore

RNS: $S_g = \left(\int d^2\tau\right)^{3g-3} \left(\int d^2 z\right)^N \int \mathcal{D}X \int \mathcal{D}\psi \int \mathcal{D}b \int \mathcal{D}c \int \mathcal{D}\beta \int \mathcal{D}\gamma \; e^S \, V_1 \dots V_N \left(\int b\right)^{3g-3} \left(\int \bar{b}\right)^{3g-3}$

PS: $S_g = \left(\int d^2\tau\right)^{3g-3} \int \mathcal{D}X \int \mathcal{D}\theta \mathcal{D}\bar{\theta} \int \mathcal{D}p \mathcal{D}\bar{p} \int \mathcal{D}\lambda \int \mathcal{D}\omega \; e^S \, V_1 \dots V_N \left(\int b\right)^{3g-3} \left(\int \bar{b}\right)^{3g-3}$

$\to$ how to prove that they're equivalent is an open problem

$\Rightarrow$ $g_{mn}(x) \sim \eta_{mn} + h_{mn}(x)$

$\sqrt{g} R \to h \partial h \partial h + hh \partial h \partial h + \dots$ $\to$ compute each scatt ampl $\to$ not manif diff invariant

the simplest way:

non flat background

$$S = \int d\tau d\sigma \left(\partial X^m \bar{\partial} X^n g_{mn}(x) + b \bar{\partial} c + \bar{b} \partial \bar{c} \right)$$

$$Q = \int d\sigma \left(c \, g_{mn} \partial X^m \partial X^n + bc \partial c \right)$$

$\Rightarrow Q^2 \neq 0$ (no longer conf. invariant at quantum lev.)

$\Rightarrow 0 = Ric_{mn} - g_{mn} R + \alpha'(\dots)$

$\to$ g_{mn} must satisfy stringy correction!!!

Interpretation: $g_{mn}(x) = \eta_{mn} + h_{mn}(x)$

graviton vertex!

$$e^{S[g_{mn}]} = e^{S[\eta_{mn}]} \, e^{\int d\tau d\sigma \boxed{h_{mn}(x) \partial X^m \bar{\partial} X^n}} = e^{S[\eta_{mn}]} \left(1 + V_{grav} + \tfrac{1}{2} V_{grav}^2 + \dots\right)$$

such that $QV_{grav} = 0$ (on shell graviton)

$\Rightarrow$ Other background?

$$S = \int d\sigma dz \left[b\bar\partial c + \bar b \partial \bar c + \partial X^m \partial X^n g_{mn} + \partial X^m \partial X^n b_{mn} + \alpha' R^{(2)} \varphi(x) \right]$$

$$\Rightarrow \text{if } \varphi = \text{const} \longrightarrow \alpha' \int d^2\sigma\, R^{(2)} \varphi = 4\pi\alpha' (1-g)\varphi$$

$$\Rightarrow e^{S} = \dots e^{(2-2g)\varphi_0} = (\lambda_s)^{2g-g} \Longrightarrow \lambda_s = e^{-\varphi_0}$$

$\Rightarrow$ λ_s is NOT background indep!

$\Rightarrow$ it depends on φ VEV!

The description of R-R fields is more difficult:

$$V = F_{mnpqr}(x) (\gamma^{mnpqr})^{\alpha\beta} (\lambda \gamma^m \theta)(\gamma_m \theta)_\alpha (\bar\lambda \gamma^n \theta)(\gamma_n \theta)_\beta$$

$$\int V = F_{mnpqr}(x) (\gamma^{mnpqr})^{\alpha\beta} (p + \partial X^m \gamma_m \theta)_\alpha (\bar p + \bar\partial X^m \gamma_m \bar\theta)_\beta$$

in R-R background $p, \bar p$ can be int. out

$$\Rightarrow S = \int dz d\sigma \left\{ \partial X^m \bar\partial X^n g_{mn}(x) + (p + \partial X\theta)_\alpha (\bar p + \bar\partial X \bar\theta)_\beta F_{mnpqr}(x) (\gamma^{mnpqr})^{\alpha\beta} + \omega \bar\partial \lambda + \bar\omega \partial \bar\lambda \right\} + \alpha' \int dz d\sigma\, R^{(2)} \varphi(x)$$

$\Rightarrow PSU(2,2|4)$

$= SO(4,2) \oplus SO(4) \oplus 32$ SUSY

30 bosonic   32 fermionic

$\Rightarrow H = SO(1,1) \times SO(5)$

20 bosonic

$$\Rightarrow AdS_5 \times S^5 \longrightarrow PSU(2,2|4) / SO(1,1) \times SO(5) = G(X, \theta, \bar\theta)$$

$$\bar\nabla \lambda^a = \bar\partial \lambda^a + (G^{-1} \bar\partial G)^{mn} (\gamma_{mn} \lambda)^a$$

$\hookrightarrow$ left $PSU(2,2|4)$ invariant

$$\rightarrow S = \int dz d\sigma \left[(G^{-1}\partial G)^I (\bar G^{-1} \bar\partial G)^J \eta_{IJ} + \omega \bar\nabla \lambda + \bar\omega \nabla \bar\lambda \right]$$

$\hookrightarrow$ run over all dofs in the algebra of $AdS_5 \times S^5$

16 fermionic SUSY

$$Q = \int d\sigma \, \lambda^{\alpha} \left(G^{-1} \partial G\right)_{\alpha} \quad ; \quad \bar{Q} = \int d\sigma \, \bar{\lambda}^{\alpha'} \left(G^{-1} \bar{\partial} G\right)_{\alpha'}$$

→ find $V(x, \theta, \bar{\theta}, \lambda, \bar{\lambda})$ s.t. $QV = 0$

→ various approx.: (e.g.: $\frac{1}{2}$-BPS) ⇝ NO known full form however!

STRING FIELD THEORY

I quant. → $X(z)$

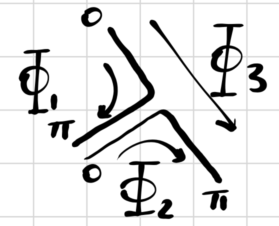
vertex ops

II quant. → $\Phi\left(X(z), b, c\right) = c A_m \partial X^m + \ldots$

e.o.m.: $Q\Phi + \Phi \times \Phi = 0$

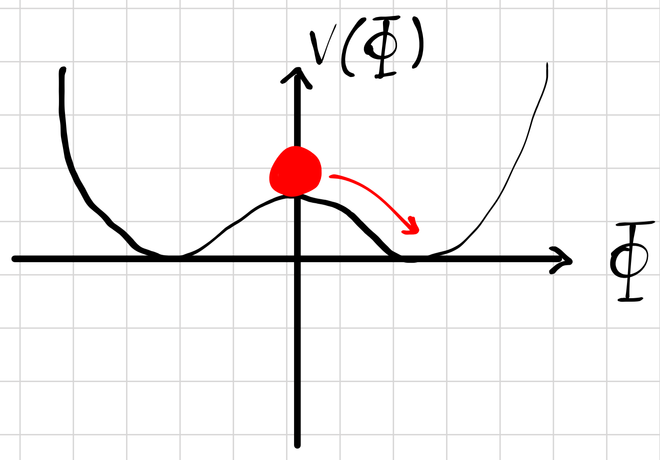
$$\delta\Phi = Q\Lambda + \Lambda \times \Phi - \Phi \times \Lambda$$

$\Rightarrow \quad \Phi_1 \times \Phi_2 = \Phi_3$



$$\rightarrow \boxed{S = \langle \Phi \times Q\Phi + \frac{2}{3} \Phi \times \Phi \times \Phi \rangle}$$

Tachyon ?



→ SFT is good for the study of tachyon decay ⇒ need off-shell amplitudes

TWISTORS

bosonic spinors ⇒ pure spinor!

instead of X → twistors → $S = \int dz d\sigma \, \mu_{\alpha} \bar{\partial} \Lambda^{\alpha} \longleftrightarrow \int dz d\sigma \left(\omega \bar{\partial} \lambda + \partial X \bar{\partial} X\right)$