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Open and Closed String Vertices for branes with magnetic field and T-duality.

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Abstract

We discuss carefully the vertices which describe the dipole open strings and closed strings on a D-brane with magnetic flux on a torus. Translation invariance along closed cycles forces surprisingly closed string vertices written in open string formalism to acquire Chan-Paton like matrices. Moreover the one loop amplitudes have a single trace for the part of gauge group with the magnetic flux. These peculiarities are also required by consistency of the action of T-duality in the open string sector. In this way we can show to all orders in perturbation theory the equivalence of the T-dual open string theories, gravitational interactions included.

We provide also a new and direct derivation of the bosonic boundary state in presence of constant magnetic and Kalb-Ramond background based on Sciuto-Della Selva-Saito vertex formalism.

1 Introduction and conclusions.

In few years from now LHC will probably and hopefully have given a better picture of the physics beyond the standard model. For example it will probably be clear whether large extra dimensions are a feature of Nature or not. In the case, somewhat unexpected, that the string scale is around 1 TeV we could be able to see KK states and Regge resonances. This makes worth studying the string interactions with KK momenta and winding in the bottom-up approach.

In fact most of the current literature has focused in computing effective actions for the light states which have not momentum and/or winding in the compact directions. The first steps in a better understanding of the interactions among string states with momentum and winding were taken in [1] where both the open string and closed string vertices for branes with equal magnetic field wrapped on a torus were given. This description was up to cocycles. The main result was that Chan-Paton factors do depend on momenta along the directions where there is a non trivial magnetic field. In the simplest case of a T^2 with coordinates x^1, x^2 and adimensional magnetic field $2\pi\alpha' q F_{12} = \frac{f}{N}\mathbb{I}_N \in u(N)$ this dependence is the same of a non trivial section of a $U(N)$ gauge bundle transforming in the adjoint. For example the tachyonic vertex of a dipole string is given by

$$V_T(x;k) =: e^{ik\cdot X(x)} : \ \Lambda(k_1, k_2) \tag{1}$$

where both the compact momentum components $k_{1,2}$ have a spectrum given by $\frac{1}{\sqrt{\alpha'}}\frac{n_{1,2}}{N}$ ($n \in \mathbb{Z}$) and we have introduced the momentum dependent Chan-Paton matrices

$$\Lambda_{N\ IJ}(\frac{1}{\sqrt{\alpha'}}\frac{n_1}{N}, \frac{1}{\sqrt{\alpha'}}\frac{n_2}{N}) = \Lambda_{N\ IJ}(k) \ = \ \frac{1}{\sqrt{N}} e^{-i\frac{\pi}{N}\hat{h} n_1 n_2} \left(Q_N^{\hat{h} n_2} P_N^{-n_1}\right)_{IJ}, \quad 0 \le I, J < N \tag{2}$$

with $\hat{h} f \equiv -1 \ \ mod\ N$ (we will be more precise on requirements later in eq. (58)) and $P_{IJ} \propto \delta_{I+1,J}$ and $Q_{I,J} \propto e^{i\frac{2\pi}{N}I}\delta_{I,J}$ are the 't Hooft matrices. In the same work [1] the transformations for the open string momenta, for the open string metric and for other physical quantities under T-duality were discussed.

At the end of the discussion we were left with a puzzle on the true equivalence under the full T-duality group of a bound state of $D2$ and $D0$ branes with a single $D2$ branes. In fact immediately after the discovery of the description of D-branes in string theory it was realized that a bound state of N $D2$ and f $D0$ is T-dual to a single $D2$ when $gcd(N, f) = 1$. In particular in 1997 Guralnik and Ramgoolam in [2] showed the previous equivalence using the following chain of dualities

$$
\begin{aligned}
bound(N\ D2\ +\ fD0) &\xrightarrow{T\ duality\ on\ y} && bound(N\ D1_x\ +\ f\ D1_y) \\
&= && G\left(bound(\frac{N}{G}\ D1_x\ +\ \frac{f}{G}\ D1_y)\right) \\
&\xrightarrow{SL(2,\mathbb{Z})\ coordinate\ rotation} && G\ D1_{x'} \\
&\xrightarrow{T\ duality\ on\ y'} && G\ D2
\end{aligned}
\tag{3}
$$

where $bound(\ldots)$ means bound state made of $\ldots$, G is equal to $gcd(N,f)$ and "$SL(2,\mathbb{Z})$ coordinate rotation" means that we perform a $SL(2,\mathbb{Z})$ transformation on T^2 so that the $D1$ brane wrapped $\frac{N}{G}$ times along x and $\frac{f}{G}$ times along y becomes a $D1$ brane wrapped once along a new x' direction and with "perpendicular" direction y'.

In [1] we started from a tachyonic vertex for a single brane ($N=1$) wrapped on the T^2

$$V_T(x;k^t) =: e^{ik^t\cdot X(x)} : \tag{4}$$

with $k^t_{1,2} = \frac{1}{\sqrt{\alpha'}} n_{1,2}$ ($n_{1,2}\in\mathbb{Z}$). Studying the action of T-duality it was possible to recover the same vertex as in eq. (1) up to the Chan-Paton matrices, i.e. we found as expected that the T-duality mapped the momenta spectrum correctly. The presence or absence of the momentum dependent Chan-Paton matrices is not a problem as long as tree level and planar amplitudes are considered since the tree level contribution of these momentum dependent Chan-Paton matrices is given by

$$tr\left(\Lambda(k_{(1)})\ldots\Lambda(k_{(M)})\right) \propto \delta^{[1]}_{\sqrt{\alpha'}\sum_{o=1}^{M} k_{(o)1}}\delta^{[1]}_{\sqrt{\alpha'}\sum_{o=1}^{M} k_{(o)2}} \tag{5}$$

where $\delta^{[M]}_x$ means $x\equiv 0\ \ mod\ M$ and therefore these deltas are automatically satisfied upon the use of the momentum conservation.

The problems arise when we consider mixed open-closed amplitudes and, equivalently, non planar amplitudes.

Consider for example the one open string tachyon - one closed string tachyon amplitude in fig. 1. In the computation with the bundle we expect the amplitude to be

$$A_{bundle} \sim tr\left(\Lambda(k)\right)\ \delta_{n+n_c-\hat{F}m_c,0} \sim\ \delta^{[N]}_{n_1}\delta^{[N]}_{n_2}\ \delta_{n_1+n_{c\ 1}-\frac{f}{N}m_{c\ 2},0}\delta_{n_2+n_{c\ 2}+\frac{f}{N}m_{c\ 1},0} \tag{6}$$

where n_c and m_c are respectively the closed string momentum and winding. We have explicitly written the non trivial contribution from the Chan-Paton factor $tr\left(\Lambda(k)\right)\propto \delta^{[1]}_{\sqrt{\alpha'}k_1}\delta^{[1]}_{\sqrt{\alpha'}k_2} = \delta^{[N]}_{n_1}\delta^{[N]}_{n_2}$ and the contribution from zero modes in the T^2 directions $\delta_{n+n_c-\hat{F}m_c,0}$.

On the other side in the computation of the same diagram with a single brane without magnetic field we expect

$$A^t_{single\ with\ \hat{F}^t=0} \sim \delta_{n^t+n^t_c,0} \tag{7}$$

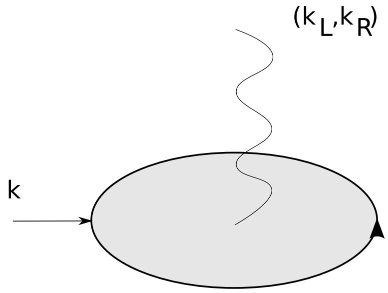

Figure 1: The simplest mixed amplitude: one open string and one closed string.

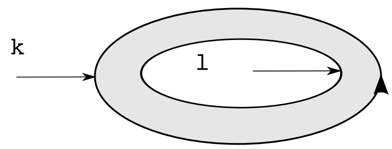

Figure 2: The simplest non planar amplitude: one open string on one border and one open string on the other border of the annulus.

which becomes after the T-duality

$$A_{T-dual\ of\ a\ single\ with\ \hat{F}=\frac{f}{N}} \sim \ \delta_{n+n_c-\hat{F}m_c,0} \sim \delta_{n_1+n_{c\ 1}-\frac{f}{N}m_{c\ 2},0}\delta_{n_2+n_{c\ 2}+\frac{f}{N}m_{c\ 1},0} \qquad (8)$$

which is the same amplitude as in eq. (6) *without* the constraint from the Chan-Paton factor. This happens since T-duality is nothing but rewriting an amplitude using different variables. Therefore the two amplitudes in eq.s (6) and (8) are *not* the same and it seems we have two different branes: those obtained by T-duality and the magnetized ones.

A similar result holds for the non planar amplitude depicted in fig. 2 where the bundle computation is expected to give

$$A_{bundle} \sim tr\left(\Lambda(k)\right)\ tr\left(\Lambda(l)\right) \quad \delta_{k+l,0} \sim \ \delta^{[N]}_{n_1}\delta^{[N]}_{n_2}\ \delta^{[N]}_{m_1}\delta^{[N]}_{m_2}\ \delta_{n_1+m_1,0}\delta_{n_2+m_2,0} \qquad (9)$$

where we have written only the Chan-Paton and zero modes contributions from the directions where there is a non trivial bundle and we have defined the integers m as in $l = \frac{1}{\sqrt{\alpha'}}\frac{m}{N}$. The T-dual transformation of the single brane one loop non planar amplitude is

$$A_{T-dual\ of\ a\ single\ with\ \hat{F}=\frac{f}{N}} \sim \ \delta_{k+l,0} \qquad (10)$$

again missing the Chan-Paton contribution.

On the other side in ([1],[3]) the boundary state for the bundle in presence of magnetic field, see fig. 3, was computed directly and shown to be equal to the T-dual of the boundary state of the single brane. As a consequence the 2 loop vacuum amplitude is equal in both theories since it can be computed in the closed string

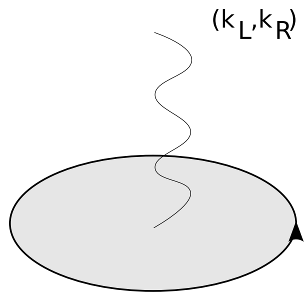

Figure 3: The amplitude of 1 closed string on a disk, i.e. the boundary state.

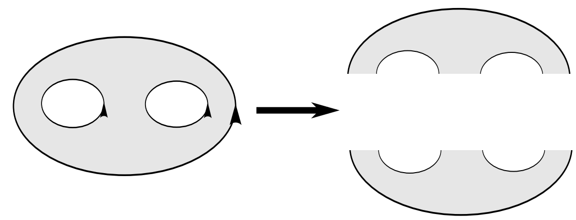

Figure 4: Two vacuum loop factorizes into two three points tree amplitudes.

channel by using boundary states. This two loop vacuum amplitude can then be factorized in two three point tree amplitudes as in fig. 4 ([3]) but this factorization seems unique and therefore independent of whether we compute them using the bundle picture of using the T-duality.

The aim of this article is to solve the previous puzzle. The results are somewhat at variance with the naive expectations derived from the common lore. Both the computations with the bundle in eq. (6) and in eq. (9) are *wrong*!

The 1 open string - 1 closed string in eq. (6) is wrong because *the closed string vertex (in open string formalism) has a Chan-Paton like factor* proportional to $\Lambda\left(-\frac{\hat{F}m}{\sqrt{\alpha'}}\right)$ so that the proper computation is

$$\begin{aligned} A_{bundle} &\sim tr\left(\Lambda(k)\ \Lambda\left(-\frac{\hat{F}m}{\sqrt{\alpha'}}\right)\right)\ \delta_{n+n_c-\hat{F}m_c,0} \\ &\sim \delta_{n_1+n_{c\ 1}-\frac{f}{N}m_{c\ 2},0}\delta_{n_2+n_{c\ 2}+\frac{f}{N}m_{c\ 1},0}\ \delta^{[N]}_{n_1-fm_{c\ 1}}\delta^{[N]}_{n_2+fm_{c\ 1}} \\ &\sim \delta_{n_1+n_{c\ 1}-\frac{f}{N}m_{c\ 2},0}\delta_{n_2+n_{c\ 2}+\frac{f}{N}m_{c\ 1},0} \end{aligned} \tag{11}$$

which reproduces the T-dual result of the single brane since the Chan-Paton contribution is now compatible with momentum conservation. Notice that there is no ambiguity in the ordering of the open and the closed vertices in the amplitude with the proper cocycles since then the open and closed vertices commute.

The 1 loop non planar amplitude in eq. (9) is wrong too because there is a *single trace*, so that the right computation is roughly

$$\begin{aligned} A_{bundle} &\sim tr\left[\Lambda(k)\ \Lambda(l)\right]\quad \delta_{k+l,0} \sim \delta_{n_1+m_1,0}\delta_{n_2+m_2,0}\ \delta^{[N]}_{n_1+m_1}\delta^{[N]}_{n_2+m_2} \\ &\sim \delta_{n_1+m_1,0}\delta_{n_2+m_2,0} \end{aligned} \tag{12}$$

and again the Chan-Paton contribution is compatible with momentum conservation. Notice however that the single trace is only for the part of gauge group where the magnetic field is switched on. If we consider a gauge group $U(NM)$ where we have a magnetic field embedded in a $U(N)$ subgroup the Chan-Paton has the factorized form $\Lambda(k) \otimes \Lambda_0$ where Λ_0 is the usual Chan-Paton matrix then in this case the contribution is $tr\,[\Lambda(k)\ \Lambda(l)]\ \ tr[\Lambda_0^{outer}]\ tr[\Lambda_0^{inner}]$

Having the proper prescription to compute the amplitudes with dipole strings allows to show that these amplitudes are actually the T-dual of a system with branes without magnetic field turned on as suggested but not proven by the chain of dualities in eq. (3).

The plain of this paper is the following. In section 2 we review our conventions for open and closed string and how we define the physical states on a bundle. In section 3 we discuss the spectral decomposition of unity and its consequence for the 1 loop non planar amplitude. Then in section 4 we discuss how the request of translational invariance of the closed string states along a cycle implies the presence of Chan-Paton like factors for the closed string vertices in open string formalism and we discuss how this is nevertheless true for other simpler cases even if gone unnoticed. To confirm the previous results in section 5 we discuss the closed string cocycles and the open string ones and we show that everything works nicely when we consider the operatorial product of vertices . To further confirm the results in 6 we compute various amplitudes such as the mixed N open tachyons 1 closed tachyon amplitude which we use to check the factorization of the 1 loop non planar in the closed string channel. In the same section we propose a new direct derivation of the boundary state in presence of constant magnetic and Kalb-Ramond background based on Sciuto-Della Selva-Saito vertex formalism. In section 7 we show that the amplitudes for a dipole in presence of constant magnetic and Kalb-Ramond background are T-dual to the amplitudes for a dipole with a vanishing constant magnetic. Finally in the appendices we give the details on the computations of cocycles and of the amplitudes.

2 A short review of closed string on torus and open string on non trivial bundles.

2.1 Closed string conventions.

In the following we use the notations used in ([1], [4]). In particular the closed string expansion in a metric background $E_{ij} = G_{ij} + B_{ij}$ on a generic flat space $R^{D-d} \otimes T^d$ $(D = 26)$ is given by

$$X^i(z,\bar{z}) = \frac{1}{2}\left(\tilde{X}_R^{(c)i}(\bar{z}) + X_L^{(c)i}(z)\right), \tag{13}$$

where

$$\begin{aligned}\tilde{X}_R^{(c)i}(\bar{z}) &= x_R^i - 2\alpha' p_R^i \ i \ \ln(\bar{z}) + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^i \bar{z}^{-n} \\ X_L^{(c)i}(z) &= x_L^i - 2\alpha' p_L^i \ i \ \ln(z) + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^i z^{-n} \end{aligned} \tag{14}$$

and $^{(c)}$ has been added to stress that these fields are the closed string ones, $z = e^{2(\tau_E + i\sigma)} \in \mathbb{C}$ $(0 \leq \sigma \leq \pi)$ and $i, j, \dots = 1, 2, \dots d$. For non compact directions $\mu, \nu = 0, d+1, \dots D$ we have the same expansions with the identification $p_R^\mu = p_L^\mu = \frac{1}{2}p^\mu$.

The closed string Hamiltonian can then be written as follows:

$$\begin{aligned}\frac{H_c - 4}{2} &= L_0 + \tilde{L}_0 = \frac{\alpha'}{4\pi}\int_0^\pi d\sigma \left[P_L^2 + P_R^2\right], \\ &= N + \tilde{N} + \frac{1}{2}\left[G_{ij}m^i m^j + (n_i - B_{ik}m^k)G^{ij}(n_j - B_{jh}m^h)\right] + \frac{\alpha'}{2}G^{\mu\nu}k_\mu k_\nu \end{aligned} \tag{15}$$

where the explicit expressions of L_0 and $\tilde{L}_0$ are given by

$$\begin{aligned} L_0 &= \alpha' G^{\mu\nu}\frac{k_\mu}{2}\frac{k_\nu}{2} + \alpha' G_{ij}p_R^i p_R^j + N \quad ; \quad N = \sum_{n=1}^{\infty} G_{\mu\nu}\alpha_{-n}^\mu \alpha_n^\nu + G_{ij}\alpha_{-n}^i \alpha_n^j \\ \tilde{L}_0 &= \frac{\alpha'}{4}G^{\mu\nu}k_\mu k_\nu + \alpha' G_{ij}p_L^i p_L^j + \tilde{N} \quad ; \quad \tilde{N} = \sum_{n=1}^{\infty} G_{\mu\nu}\tilde{\alpha}_{-n}^\mu \tilde{\alpha}_n^\nu + G_{ij}\tilde{\alpha}_{-n}^i \tilde{\alpha}_n^j, \end{aligned} \tag{16}$$

The spectrum of the compact momenta given by

$$\begin{cases} k_{Li} = G_{ij}\ p_L^j = \frac{1}{2\sqrt{\alpha'}}\left(n + E^T m\right)_i \\ k_{Ri} = G_{ij}\ p_R^j = \frac{1}{2\sqrt{\alpha'}}\left(n - E m\right)_i \end{cases} \tag{17}$$

In computing amplitudes and OPEs we make use of the following non vanishing commutators

$$[x_L^i, p_L^j] = i\ G^{i,j} \quad , \quad [\alpha_{Ln}^i, \alpha_{Lm}^j] = n\ \delta_{n+m,0}\ G^{i,j} \tag{18}$$

and similarly for the right movers and for the non compact directions. The normalization of the zero modes is taken to be

$$\langle k_\mu, n_i, m^i | k'_\mu, n'_i, m^{i'}\rangle = (2\pi)^{D-d}\delta^{D-d}(k_\mu - k'_\mu)\ (2\pi\sqrt{\alpha'})^d \delta_{n,n'}\delta_{m,m'}\ . \tag{19}$$

The vertex for a closed string state described by momentum (k_L, k_R) and quantum numbers (β_L, β_R) is written as

$$W^{(c)}_{\beta_L,\beta_R}(z,\bar{z};k_L,k_R) = V_{\beta_L}(z;k_L)\ \tilde{V}_{\beta_R}(\bar{z};k_R) \tag{20}$$

up to cocycles which we discuss later in section 5.1 and where the right moving part of the vertex $\tilde{V}_{\beta_R}$ is a normal ordered functional of the closed string right moving part $X_R^{(c)}$ and similarly for the left moving part. The previous description is however not completely exact for the non compact directions since in this case the zero modes are common to left and right moving sectors. Because of this the non compact left and right part of the vertex are not separately normal ordered but only the non zero modes parts are normal ordered separately while the common zero modes are normal ordered together.

Finally for later use we write our conventions for an element of the T-duality group Λ acting on the d compact direction which is described by

$$\Lambda \quad = \quad \begin{pmatrix} \hat{\mathcal{A}} & \hat{\mathcal{B}} \\ \hat{\mathcal{C}} & \hat{\mathcal{D}} \end{pmatrix} \in O(d,d,\mathbb{Z}) \qquad \Lambda \begin{pmatrix} 0 & \mathbb{I}_d \\ \mathbb{I}_d & 0 \end{pmatrix} \Lambda^T = \begin{pmatrix} 0 & \mathbb{I}_d \\ \mathbb{I}_d & 0 \end{pmatrix} \tag{21}$$

and acts on zero modes and operators as

$$\begin{cases} m^t = & \hat{\mathcal{A}}m + \hat{\mathcal{B}}n \\ n^t = & \hat{\mathcal{C}}m + \hat{\mathcal{D}}n \end{cases} \qquad \begin{cases} \alpha_n^t = & \left(\hat{\mathcal{A}} + \hat{\mathcal{B}}E\right)\alpha_n \\ \tilde{\alpha}_n^t = & \left(\hat{\mathcal{A}} - \hat{\mathcal{B}}E^T\right)\tilde{\alpha}_n \end{cases} \qquad n \in \mathbb{Z}^* \tag{22}$$

and on the closed string background as

$$E^{tT} = (-\hat{\mathcal{C}} + \hat{\mathcal{D}}E^T)(\hat{\mathcal{A}} - \hat{\mathcal{B}}E^T)^{-1} \quad E^t = (\hat{\mathcal{C}} + \hat{\mathcal{D}}E)(\hat{\mathcal{A}} + \hat{\mathcal{B}}E)^{-1} \tag{23}$$

where the quantities with a ^{t} are the T-duality transformed ones.

2.2 Gauge bundles.

On the torus all quantities such as for example the gauge field A_i have only to be periodic up to a gauge transformation:

$$A_i(x^j + 2\pi\sqrt{\alpha'}\delta_l^j) = \Omega_l(x)\ A_i(x^j)\ \Omega_l^{-1}(x) - i\frac{1}{q}\Omega_l(x)\ \partial_i\Omega_l^{-1}(x) \tag{24}$$

where q is the gauge coupling constant and $\Omega_l(x) \equiv \Omega_l(x^{j\neq l})$ is the gauge transition function. From now we mean by gauge bundle the assignment of a background field, together with a transition function which fixes the periodicity property of the gauge field. Notice that, if we perform a gauge transformation

$$A_i^\omega(x) \quad = \quad \omega(x)\ A_i(x)\ \omega^{-1}(x) - \frac{i}{q}\omega(x)\ \partial_i\omega^{-1}(x) \tag{25}$$

the transition functions transform as

$$\Omega_j^\omega(x) = \omega(x^1,\ldots,x^j + 2\pi\sqrt{\alpha'},\ldots,x^d)\ \Omega_j(x)\ \omega^{-1}(x^1,\ldots,x^d). \tag{26}$$

Furthermore, these transition functions have also to satisfy the *cocycle condition*, which simply means that the gauge fields must be unchanged when translated along a closed path:

$$\Omega_j(x^k + 2\pi\sqrt{\alpha'}\delta_i^k)\Omega_i(x^k) = \Omega_i(x^k + 2\pi\sqrt{\alpha'}\delta_j^k)\Omega_j(x^k). \tag{27}$$

Some examples of the previous constructions are the followings.

1. The $U(N)$ flat gauge bundle on T^d

$$A_i = \| a_i^I \delta_{IJ} \|, \quad \Omega_i = \mathbb{I}_N, \tag{28}$$

for generic a^I ($I = 1, \dots N$ are the color indices), breaks the symmetry down to $U(1)^N$. Using this surviving symmetry we can always choose

$$0 \leq \sqrt{\alpha'} q a_i^I < 1 \tag{29}$$

by a big gauge transformation given by $\omega = \| e^{i n_i^I x^i/\sqrt{\alpha'}} \delta_{IJ} \|$ where $n_i^I \in \mathbb{Z}$. The same conclusion can be reached by performing the gauge transformation $\omega = \| e^{i q a_i^I x^i} \delta_{IJ} \|$ so that the previous bundle is gauge equivalent to

$$A_i^\omega = 0, \quad \Omega_i^\omega = \| e^{i2\pi\theta_i^I} \delta_{IJ} \| \tag{30}$$

where $\theta_i^I \equiv \sqrt{\alpha'} q a_i^I$.

2. Another less trivial $U(N)$ bundle on T^d which has a constant magnetic field

$$\hat{\mathbb{F}}_{ij} = 2\pi\alpha' q \mathbb{F}_{ij} = 2\pi\alpha' q F_{ij} \mathbb{I}_N = \hat{F}_{ij} \mathbb{I}_N \tag{31}$$

is obtained for example with the choice of the gauge field

$$A_i = -\frac{1}{2} F_{ij} x^j \mathbb{I}_N = -\frac{1}{2} \frac{2\pi}{(2\pi\sqrt{\alpha'})^2} \frac{n_{ij}}{N} x^j \mathbb{I}_N \ , \tag{32}$$

along with the gauge transition functions given by

$$\Omega_i(x) = e^{-i\pi\sqrt{\alpha'} q F_{ij} x^j} \omega_i \tag{33}$$

where the constant matrices ω_i satisfy

$$\omega_i \omega_j = e^{i(2\pi\sqrt{\alpha'})^2 q F_{ij}} \omega_j \omega_i \ . \tag{34}$$

In particular on a T^2 when $\hat{F}_{12} \equiv 2\pi\alpha' F_{12} = \frac{f}{N}$ we can write

$$\omega_1 = Q_N e^{i2\pi\theta_1}, \quad \omega_2 = P_N^{-f} e^{i2\pi\theta_2} \tag{35}$$

and we can always choose

$$0 \leq \theta_{1,2} < \frac{1}{N} \tag{36}$$

since we can use a global transformation given by $\omega = Q_N^{k_2} P_N^{-k_1}$ with $k_i = [N\theta_i]$ to move the θ values into the desired range.

3. The previous T^2 case is more general than one could guess at first sight since, given a field strength proportional to the unity in color space, it is always possible to find a $SL(d,\mathbb{Z})$ transformation (see for example [3] for a demonstration) such that the constant magnetic field strength has only the following non vanishing components

$$\hat{F}_{2a-1,2a} = 2\pi\alpha' q F_{2a-1,2a} \quad , \quad a = 1, \dots r \tag{37}$$

The block diagonal field strength (37) and the cocycle conditions (27) imply that we are actually working on a torus $T^d = \prod_{a=1}^r T^2_{(a)} \otimes T^{d-2r}$ as far the bundle is concerned[1] since the transition functions (33) on different torii $T^2_{(a)}$ must commute, for example

$$\omega_{2a}\omega_{2b} = \omega_{2b}\omega_{2a} \qquad a \neq b \tag{38}$$

Therefore the cocycle conditions for the transition functions oblige to consider the transition functions to be factorized as $\otimes_{a=1}^r U(L_{(a)}) \otimes U(N_1)$ and the $U(N)$ bundle to be actually a $U(\prod_{b=1}^r L_{(b)} N_1)$ bundle on $T^{2r} \otimes T^{d-2r}$ with $N = \prod_{b=1}^r L_{(b)} N_1$ and with a constant background field

$$\hat{F}_{2a-1,2a} = 2\pi\alpha' q F_{2a-1,2a} = \frac{f_a}{L_{(a)}} \mathbb{I}_{\prod_{b=1}^r L_{(b)} N_1} \quad , \quad a = 1, \dots r \tag{39}$$

with all the other components vanishing. This does clearly *not* happen on a non compact surface and it is responsible for the rank reduction of the lower dimensional effective theory from $\prod_{b=1}^r L_{(b)} N_1$ to N_1 as we can see from the physical states in eq. (56) and we discuss after eq. (65). This fact has been used to explain the rank reduction in orientifold theories in presence of a discrete B ([5], [6]). The background field in eq. (39) is obtained from the gauge field (up to gauge choices)

$$A_i = -\frac{1}{2} F_{ij} x^j \mathbb{I}_{\prod_{b=1}^r L_{(b)} N_1} \tag{40}$$

along with the transition functions which we take to be

$$\begin{aligned}
\Omega_1 &= e^{i2\pi\theta_1} e^{-i\frac{f_1}{L_{(1)}}\frac{x^2}{2\sqrt{\alpha'}}} Q_{L_1} \otimes \mathbb{I}_{L_2} \dots \mathbb{I}_{N_1} \quad , \quad && \Omega_2 = e^{i2\pi\theta_2} e^{i\frac{f_1}{L_{(1)}}\frac{x^1}{2\sqrt{\alpha'}}} P_{L_1}^{-f_1} \otimes \mathbb{I}_{L_2} \dots \mathbb{I}_{N_1} \\
\Omega_3 &= e^{i2\pi\theta_3} \mathbb{I}_{L_1} \otimes e^{-i\frac{f_2}{L_{(2)}}\frac{x^4}{2\sqrt{\alpha'}}} Q_{L_2} \dots \mathbb{I}_{N_1} \quad , \quad && \Omega_4 = e^{i2\pi\theta_4} \mathbb{I}_{L_1} \otimes e^{i\frac{f_2}{L_{(2)}}\frac{x^3}{2\sqrt{\alpha'}}} P_{L_2}^{-f_2} \dots \mathbb{I}_{N_1} \\
&\vdots \\
\Omega_{2r+1} &= e^{i2\pi\theta_{2r+1}} \mathbb{I}_{L_1} \otimes \mathbb{I}_{L_2} \dots \mathbb{I}_{N_1} \quad , \dots \quad && \Omega_d = e^{i2\pi\theta_d} \mathbb{I}_{L_1} \otimes \mathbb{I}_{L_2} \dots \mathbb{I}_{N_1}
\end{aligned} \tag{41}$$

where $e^{i2\pi\theta_i}$ are the abelian Wilson lines with $0 \leq \theta_{2a-1}, \theta_{2a} < \frac{1}{L_{(a)}}$ and $0 \leq \theta_{2r+1}, \dots \theta_d < 1$.

[1] This factorized form is only true for the gauge field but it is not necessarily true for the metric and the other background fields.

4. The previous background in eq.s (39) and (41) can be further generalized allowing more general Wilson lines in the $U(N_1)$ as $e^{i2\pi\theta_i}\mathbb{I}_{N_1}$ can be generalized to $e^{i2\pi\Theta_i} = \| e^{i2\pi\theta_i^I}\delta_{IJ} \|$ (which further break $U(N_1)$ down to $U(1)^{N_1}$), explicitly

$$\Omega_1 = e^{-i\frac{f_1}{L_{(1)}}\frac{x^2}{2\sqrt{\alpha'}}}Q_{L_1}\otimes\mathbb{I}_{L_2}\cdots\otimes e^{i2\pi\Theta_1} \qquad ,\ldots \qquad \Omega_d = \mathbb{I}_{L_1}\otimes\mathbb{I}_{L_2}\cdots\otimes e^{i2\pi\Theta_d} \tag{42}$$

where Θ_i are diagonal $N_1\times N_1$ matrices.

2.3 Dipole open strings.

Let us now consider the open string in a metric background given by $E_{ij} = G_{ij}+B_{ij}$ and in presence of a constant background field $\hat{F}_{ij}$ defined in eq. (39) and (41) in the last subsection, i.e. a background where the original group $U(N)$ is broken to $\otimes_{a=1}^r U(L_{(a)})\otimes U(N_1)$ by a constant magnetic field with Wilson lines.

On this background and along the compact directions the open dipole string field expansion[2] is given by

$$X^i(z,\bar{z}) \;=\; \frac{1}{2}\left(X^i_L(z)+X^i_R(\bar{z})\right) \tag{43}$$

where $z = e^{\tau_E+i\sigma}\in\mathbb{H}$ $(0\le\sigma\le\pi)$ and

$$\begin{aligned} X^i_L(z) &= (G^{-1}\mathcal{E})^i_j\hat{X}^j_{L(0)}(z)+y^i_0 \\ X^i_R(\bar{z}) &= (G^{-1}\mathcal{E}^T)^i_j\hat{X}^j_{R(0)}(\bar{z})-y^i_0 \end{aligned} \tag{44}$$

where we have defined the following quantities

$$\mathcal{B}_{ij} = B_{ij}-\hat{F}_{ij} \tag{45}$$

$$\mathcal{E}_{ij} = G_{ij}-\mathcal{B}_{ij} = G_{ij}-B_{ij}+\hat{F}_{ij} \tag{46}$$

and the open string metric given by

$$\mathcal{G}_{ij} = G_{ij}-\mathcal{B}_{ik}G^{kh}\mathcal{B}_{hj} = \mathcal{E}^T_{ik}G^{kh}\mathcal{E}_{hj} \tag{47}$$

along with the non commutativity parameter Θ as

$$(\mathcal{E}^{-1})^{ij} = (\mathcal{G}^{-1})^{ij}-\Theta^{ij}\ . \tag{48}$$

Moreover we have also defined the fields $\hat{X}^i_{L(0)}$ and $\hat{X}^i_{R(0)}$ which have the usual field expansion

$$\begin{aligned} \hat{X}^i_{L(0)}(z) &= x^i-2\alpha' p^i\ i\ln(z)+i\sqrt{2\alpha'}\sum_{n\neq0}\frac{sgn(n)}{\sqrt{|n|}}a^i_n z^{-n} \\ \hat{X}^i_{R(0)}(\bar{z}) &= x^i-2\alpha' p^i\ i\ln(\bar{z})+i\sqrt{2\alpha'}\sum_{n\neq0}\frac{sgn(n)}{\sqrt{|n|}}a^i_n \bar{z}^{-n} \end{aligned} \tag{49}$$

[2] The open string is a dipole string since the magnetic field is proportional to the unity in color space.

but have a different set of commutation relations since the metric G is replaced by the open string metric $\mathcal{G}$ and the zero modes have a non trivial commutation relation, explicitly

$$[x^i, x^j] = i\ 2\pi\alpha'\ \Theta^{ij} \qquad [x^i, p^j] = i\mathcal{G}^{ij} \qquad [a^i_m, a^j_n] = \mathcal{G}^{ij}\ sgn(m)\ \delta_{n+m,0} \tag{50}$$

The previous expansion (49) looks as the usual one because we have used x^i with non vanishing commutator and not x^i_0 defined as $x^i = x^i_0 - \pi\alpha'\Theta^{il}\mathcal{G}_{lm}p^m$ with the usual commutator

$$[x^i_0, x^j_0] = 0 \tag{51}$$

Finally $y^i_0 = \sqrt{\alpha'}G^{ij}\theta_j$ are constants and proportional to the Wilson lines θ on the brane at $\sigma = 0$ ([7],[8] and [9]). Notice the y^i_0 do not enter the expansion of $X(z, \bar{z})$ and therefore they do enter the open string vertices but they do enter the closed string vertices in the open string formalism where they are necessary to reproduce from the open string side the phases in the boundary state due to Wilson lines or positions.

On this background the dipole string Hamiltonian is then given by

$$H_o - 1 = L_0 \ \ = \ \ \alpha' G^{\mu\nu}k_\mu k_\nu + \alpha' p^i \mathcal{G}_{ij} p^j + \sum_{n=1}^{\infty} n G_{\mu\nu}\alpha^{\dagger\mu}_n \alpha^\nu_n + \mathcal{G}_{ij} a^{\dagger i}_n a^j_n \tag{52}$$

where the spectrum of the momentum operators in compact directions is given in eq. (55).

The spectrum in compact directions can be deduced as follows. In order to define the physical states we need to introduce the conserved generalized translation operator $\mathcal{T}^{(i)}_{2\pi\sqrt{\alpha'}}$ ([10],[11],[1]) along the compact direction i by $2\pi\sqrt{\alpha'}$ so that the physical states can be defined to satisfy

$$\forall i \quad \mathcal{T}^{(i)}_{2\pi\sqrt{\alpha'}}|phys\rangle = |phys\rangle \tag{53}$$

The action of these operators is given by[3]

$$\mathcal{T}^{(i)}_{2\pi\sqrt{\alpha'}}|\Phi;\ I, J\rangle = e^{2\pi\sqrt{\alpha'}\, i\, \mathcal{G}_{ij}p^j}({\omega_i}^\dagger)_{MI}\ (\omega_i)_{JL}|\Phi;\ M, L\rangle \tag{54}$$

where $|I, J\rangle = |I\rangle_{\sigma=0}\ |J\rangle_{\sigma=\pi}$ (with hermitian conjugate $\langle I, J| = {}_{\sigma=0}\langle I|\ {}_{\sigma=\pi}\langle J|$) is an element of basis for the color indexes (see ([12]) for more details). The color index I is actually a collection of indices corresponding to the different factors into which the $U(N)$ bundle transition functions split $\otimes_{a=1}^r U(L_{(a)}) \otimes U(N_1) \subset U(N)$ as $I = (I_1, I_2, \dots I_{r+1})$ ($1 \le I_a \le L_{(a)}$, $1 \le I_{r+1} \le N_1$) and similarly for J.

[3] With respect to ([1],[5]) we have slightly changed notation as $|I, J\rangle_{here} = |J, I\rangle_{there} = |I\rangle_{\sigma=0}\ |J\rangle_{\sigma=\pi}$.

Since $\omega_{2a-1}^{L_{(a)}} = \omega_{2a}^{L_{(a)}} = \mathbb{I}_{L_{(a)}}$ we can iterate eq. (54) L_i times and get the spectrum of p_i; the compact momenta have therefore spectrum[4]

$$p^i = \mathcal{G}^{ij} \frac{1}{\sqrt{\alpha'}} \frac{n_i}{L_i} \tag{55}$$

where we have defined $L_{2a} = L_{2a-1} = L_{(a)}$ for $1 \leq a \leq r$ and $L_i = 1$ for $2r < i \leq d$.

It then follows that the normalized string states are given by ([4],[5])

$$\begin{aligned} |\chi; k_\mu, n_i; u\rangle &= \frac{1}{(2\pi\sqrt{\alpha'})^{d/2}} |\chi\rangle \otimes |k_\mu\rangle \\ &\otimes \Lambda_{L_{(1)};I_1 J_1}(n_1, n_2)\ |\frac{n_1}{\sqrt{\alpha'}L_{(1)}}, \frac{n_2}{\sqrt{\alpha'}L_{(1)}}\rangle_p\ |I_1, J_1\rangle \\ &\otimes \Lambda_{L_{(2)};I_2 J_2}(n_3, n_4)\ |\frac{n_3}{\sqrt{\alpha'}L_{(2)}}, \frac{n_4}{\sqrt{\alpha'}L_{(2)}}\rangle_p\ |I_2, J_2\rangle \dots \\ &\otimes\ T_{u\ \ N_1; I_{r+1} J_{r+1}}\ |\frac{n_{2r+1}}{\sqrt{\alpha'}}, \dots \frac{n_d}{\sqrt{\alpha'}}\rangle_p\ |I_{r+1}, J_{r+1}\rangle \end{aligned} \tag{56}$$

where $|\chi\rangle$ is the collective name for the quantum numbers associated with the non zero modes, $|\frac{n_1}{\sqrt{\alpha'}L_{(1)}}, \frac{n_2}{\sqrt{\alpha'}L_{(1)}}\rangle_p$ is the momentum eigenvector in the first torus of coordinates x^1, x^2 and similarly for the others. The physical meaning of writing $\Lambda_{L_{(1)};I_1 J_1}(n_1, n_2)\ |I_1, J_1\rangle$ is that for a given momentum $\left(\frac{n_1}{\sqrt{\alpha'}L_{(1)}}, \frac{n_2}{\sqrt{\alpha'}L_{(1)}}\right)$ not all the possible $L_{(1)}^2$ $|I_1, J_1\rangle$ color index combinations are possible, as it is usual with the trivial bundle, but only one.

In the eq. (56) T_u are the usual ${N_1}^2$ hermitian $u(N_1)$ generators which are normalized to unity as $tr(T_u\ T_v) = \delta_{u,v}$ and can be traded for the ${N_1}^2$ color states $|I_{r+1}, J_{r+1}\rangle$.

The Λ are the hermitian momentum dependent Chan-Paton matrices given by[5]

$$\begin{aligned} \Lambda_{L;IJ}(n_1, n_2) &= \Lambda_{L;IJ}(\frac{n_1}{L\sqrt{\alpha'}}, \frac{n_2}{L\sqrt{\alpha'}}) = \Lambda_{L;IJ}(k) \\ &= \frac{1}{\sqrt{L}} e^{-i\frac{\pi}{L}\hat{h}n_1 n_2} \left(Q_L^{\hat{h}n_2} P_L^{-n_1}\right)_{IJ}, \quad 0 \leq I, J < L \end{aligned} \tag{57}$$

[4] If we consider the more general bundle given in eq. (42) of the previous section, we find that the momenta depend on the the Wilson lines, i.e. they depend on both the starting brane and the ending one. For a string with color I_{r+1} at $\sigma = 0$ and color J_{r+1} at $\sigma = \pi$ we get $p^i_{I_{r+1}J_{r+1}} = \mathcal{G}^{ij} \frac{1}{\sqrt{\alpha'}} (\frac{n_i}{L_i} + \theta_i^{I_{r+1}} - \theta_i^{J_{r+1}})$. In the main text we consider only the case with $\theta = 0$ while in appendix we consider the more general case.

[5] We use a double notation for the dependence of Λ on momenta: by giving either their true values $k = (\frac{n_1}{L\sqrt{\alpha'}}, \frac{n_2}{L\sqrt{\alpha'}})$ or by the integers associated with them (n_1, n_2). We can easily pass from one notation to the the other. Notice however that in presence of Wilson lines the Λ_Ls still depend on the integers associated with the momenta and do not depend on Wilson lines.

where n_1 and n_2 are two arbitrary momenta not necessarily in directions 1 and 2 and[6]

$$P_{I,J} = \delta_{I+1,J}, \qquad Q_{I,J} = e^{i\frac{2\pi}{L}I}\delta_{I,J}$$
$$\hat{h}f = -1 + \tilde{f}L \equiv -1 \ \ mod\ L$$
$$f\tilde{f} \in 2\mathbb{Z} \tag{58}$$

which have the following properties:

- commutation with non abelian Wilson lines

$$\omega_i \Lambda_L(n_1,n_2)\omega_i^\dagger = e^{i\frac{2\pi}{L}n_i}\Lambda_L(n_1,n_2) \quad \Rightarrow \omega_i^\dagger \Lambda_L(n_1,n_2)\omega_i = e^{-i\frac{2\pi}{L}n_i}\Lambda_L(n_1,n_2) \tag{59}$$

- Chan-Paton matrices algebra

$$\begin{aligned}\Lambda_L(n_1,n_2)\Lambda_L(m_1,m_2) &= e^{-i\pi\frac{\hat{h}}{L}(n_1 m_2 - n_2 m_1)}\frac{1}{\sqrt{L}}\Lambda_L(n_1+m_1,n_2+m_2) \\ &= e^{-i\pi\alpha'(\frac{n_1}{\sqrt{\alpha'}L},\frac{n_2}{\sqrt{\alpha'}L})\Theta_{CP}(\frac{m_1}{\sqrt{\alpha'}L},\frac{m_2}{\sqrt{\alpha'}L})^T}\frac{1}{\sqrt{L}}\Lambda_L(n_1+m_1,n_2+m_2)\end{aligned} \tag{60}$$

  with

$$\Theta_{CP} = L\hat{h}\epsilon = L\hat{h}\begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}, \tag{61}$$

  and

$$\Theta_{CP}\hat{F} = (1-\tilde{f}L)\ \mathbb{I} = -\hat{h}f\ \mathbb{I}. \tag{62}$$

- hermitian conjugation

$$\Lambda_L^\dagger(n_1,n_2) = \Lambda_L(-n_1,-n_2) \tag{63}$$

- normalization

$$tr\left(\Lambda_L^\dagger(n_1,n_2)\Lambda_L(m_1,m_2)\right) = \delta_{n,m} \tag{64}$$

To show that the states in eq. (56) are normalized we perform computations like

$$\begin{aligned}&\langle M,K|\ {}_p\langle\frac{n_1}{\sqrt{\alpha'}L},\frac{n_2}{\sqrt{\alpha'}L}|\ \Lambda^*_{L;MK}(n_1,n_2)\ \Lambda_{L;IJ}(m_1,m_2)\ |\frac{m_1}{\sqrt{\alpha'}L},\frac{m_2}{\sqrt{\alpha'}L}\rangle_p\ |I,J\rangle \\ &= (2\pi\sqrt{\alpha'})^2\delta_{n,m}\ \delta_{K,J}\delta_{M,I}\ \left(\Lambda_L^\dagger\right)_{KM}(n_1,n_2)\ \Lambda_{L;IJ}(m_1,m_2) \\ &= (2\pi\sqrt{\alpha'})^2\delta_{n,m}\ tr\left(\Lambda_L^\dagger(n_1,n_2)\ \Lambda_L(m_1,m_2)\right) = (2\pi\sqrt{\alpha'})^2\delta_{n,m}\end{aligned} \tag{65}$$

We notice that even if we start with $\prod_{b=1}^r L_{(b)}N_1$ branes the number of massless states $k_\mu^2 = 0$ is only ${N_1}^2$. This does not mean that we have not $(\prod_{b=1}^r L_{(b)}N_1)^2$

[6] We can always choose $f\tilde{f}$ even since it f were odd we can shift $\hat{h} \to \hat{h} + L$ and $\tilde{f} \to \tilde{f} + f$ in order to get $\tilde{f}$ even.

states as we naively would expect but that some of them become massive, with a mass of order $\frac{1}{L}$: it is essentially a Scherk-Schwarz reduction mechanism and is the key idea of the explanation of the rank reduction.

Finally the vacuum is given by

$$|0\rangle = |0\rangle_p \sum_I |I, I\rangle \tag{66}$$

therefore the physical state (56) in the case of the tachyon (and disregarding a possible cocycle which we discuss later) is associated to the $\sigma = 0$ vertex and to the $\sigma = \pi$ vertex by

$$|T_o\rangle = \lim_{x \to 0^+} V_T(x; k)|0\rangle = \lim_{y \to 0^-} V_T(y; k)|0\rangle \tag{67}$$

The $\sigma = 0$ vertex of a tachyon with polarization $t_u(k)$ is given by

$$V_T(x; k) = t_u : e^{ik^T X(x)} : \ \Lambda_{L_{(1)}; I_1 J_1}(n_1, n_2) \otimes \ldots \ T^u_{I_{r+1} J_{r+1}} \tag{68}$$

or, in a more unconventional way which makes clear how it acts on indices (again up to possible cocycles)

$$\begin{aligned} V_T(x; k) \ &= \ t_u : e^{ik^T \hat{X}_{L(0)}(x)} : \ |I_1\rangle_{\sigma=0} \ \Lambda_{L_{(1)}; I_1 J_1}(n_1, n_2) \ {}_{\sigma=0}\langle J_1| \otimes \ldots \\ &\otimes \ |I_{r+1}\rangle_{\sigma=0} \ T^u_{I_{r+1} J_{r+1}} \ {}_{\sigma=0}\langle J_{r+1}| \end{aligned} \tag{69}$$

where we have written $X_{L(0)}(x)$ and not $X(x)$ to stress the T-duality invariant nature of this expression ([13]). The analogous vertex for the emission from the $\sigma = \pi$ boundary ($y < 0$) is then given by

$$\begin{aligned} V_T(y; k) \ &= \ t_u : e^{ik^T \hat{X}_{L(0)}(y)} : \ |I_1\rangle_{\sigma=\pi} \ (\Lambda_{L_{(1)}})^T_{I_1 J_1}(n_1, n_2) \ {}_{\sigma=\pi}\langle J_1| \otimes \ldots \\ &\otimes \ |I_{r+1}\rangle_{\sigma=\pi} \ (T^{u\ T})_{I_{r+1} J_{r+1}} \ {}_{\sigma=\pi}\langle J_{r+1}| \end{aligned} \tag{70}$$

where the transpose is necessary because is the $\sigma = \pi$ boundary is traveled in the opposite direction of the $\sigma = 0$ one.

3 Spectral decomposition of unity and unique trace in one loop amplitudes.

Since we know that the physical states are given by eq. (56) we can write the spectral decomposition of the unity.

Here and in the following we consider the case where the magnetic field is switched on only on a T^2, i.e. $r = 1$ since it is easier to write the formulae and it is easy to consider the more general case $r > 1$.

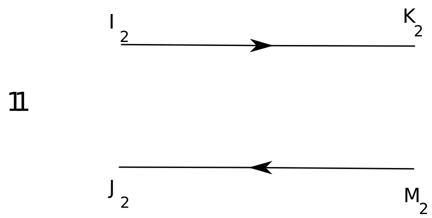

Figure 5: The spectral decomposition of unity with the "free" color indices made explicit.

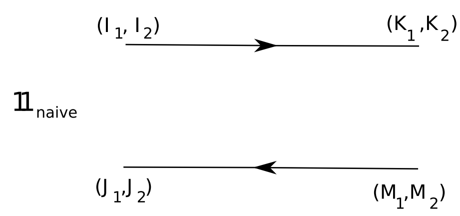

Figure 6: The usual color spectral decomposition of unity with the color indices made explicit which is not the right one in presence of a magnetic background.

The spectral decomposition of the unity is then given by

$$
\begin{aligned}
\mathbb{I} &= \int \frac{d^{D-d}k_\mu}{(2\pi)^{D-d}\,(2\pi\sqrt{\alpha'})^d} \sum_{\chi,n_i,u} |\chi;k_\mu,n_i;u\rangle\ \langle\chi;k_\mu,n_i;u| \\
&= \int \frac{d^{D-d}k_\mu}{(2\pi)^{D-d}\,(2\pi\sqrt{\alpha'})^d} \sum_{\chi,n_i,u} |\chi\rangle\otimes|k_\mu\rangle\otimes\Lambda_{L;I_1J_1}(n_1,n_2)\ |\frac{n_1}{\sqrt{\alpha'}L},\frac{n_2}{\sqrt{\alpha'}L}\rangle_p\ |I_1,J_1\rangle \\
&\qquad \otimes T_{u\ N_1;I_2J_2}\ |\frac{n_3}{\sqrt{\alpha'}},\ldots\frac{n_d}{\sqrt{\alpha'}}\rangle\ |I_2,J_2\rangle\ \ \langle M_2,K_2|\ {}_p\langle\frac{n_3}{\sqrt{\alpha'}},\ldots\frac{n_d}{\sqrt{\alpha'}}|\ T^*_{u\ N_1;K_2M_2} \\
&\qquad \langle K_1,M_1|\ {}_p\langle\frac{n_1}{\sqrt{\alpha'}L},\frac{n_2}{\sqrt{\alpha'}L}|\ \Lambda^*_{L;K_1M_1}(n_1,n_2)\otimes\langle k_\mu|\otimes\langle\chi|
\end{aligned}
\tag{71}
$$

where we have simplified the notation setting $L = L_{(1)}$ and I_1, I_2, K_1, K_2 are the color indices at $\sigma = 0$ boundary while J_1, J_2, M_1, M_2 are the color indices at $\sigma = \pi$ boundary. As previously noticed the only color indices which are "free" and can be summed in the spectral decomposition of the unity are those associated with the unbroken part of the gauge group. The structure of the "free" color indices is pictured in fig. 5 and is different from what we naively we would expect and is pictured in fig. 6. The very reason of this phenomenon is nothing but the fact that a momentum eigenstate is a unique fixed superposition of strings with different colors (in the part of the group which feels the magnetic moment).

It is then obvious to define the trace for a generic operator O as

$$
Tr(O) = \int \frac{d^{D-d}k_\mu}{(2\pi)^{D-d}\,(2\pi\sqrt{\alpha'})^d} \sum_{\chi,n_i,u} \langle\chi;k_\mu,n_i;u|\ O|\chi;k_\mu,n_i;u\rangle \tag{72}
$$

This definition has proven the right one to compute the normalization of the Moebius amplitude necessary for the tadpole cancellation when a discrete B is present in an orientifold ([5]).

Let us now see some consequences of this definition. In order to stress the main point, i.e. the presence in the one loop amplitudes of a unique trace for the part of the gauge algebra where the magnetic field is turned on, we sketch now the annulus and the simplest non planar one loop amplitudes whose generalization we compute in detail in sec 6.5.

The simplest one loop amplitude is the annulus which is given by

$$\begin{aligned} Z \;\; & \propto \;\; \int \frac{d\tau}{\tau} \; Tr_{nzm} e^{-\tau L_{0,nzm}^{X+bc}} \int \frac{d^{D-d} k_\mu}{(2\pi)^{D-d} \; (2\pi\sqrt{\alpha'})^d} \; e^{-\tau\alpha' G^{\mu\nu} k_\mu k_\nu} \; \delta^{D-d}(0) \\ & \sum_{n_i} e^{-\tau\left[\left(\frac{n_1}{L_1},\frac{n_2}{L_2},n_3,\ldots n_d\right)\mathcal{G}^{-1}\left(\frac{n_1}{L_1},\frac{n_2}{L_2},n_3,\ldots n_d\right)^T\right]} tr\left(\Lambda_L^\dagger(n_1,n_2)\Lambda_L(n_1,n_2)\right) \sum_u tr(T_u T_u^\dagger) \\ & \propto \;\; {N_1}^2 \int \frac{d\tau}{\tau} \; Tr_{nzm} e^{-\tau L_{0,nzm}^{X+bc}} \; \int \frac{d^{D-d} k_\mu}{(2\pi)^{D-d} \; (2\pi\sqrt{\alpha'})^d} \; e^{-\tau\alpha' G^{\mu\nu} k_\mu k_\nu} \sum_{(n_i)} e^{-\tau \sum \mathcal{G}^{ij} \frac{n_i}{L_i} \frac{n_j}{L_j}} \end{aligned} \tag{73}$$

where we have used the Λ normalization given in eq. (64) and we have not supposed the metric to be factorized. This result clearly shows how the amplitude is proportional only to the product of traces of the part of the group left unbroken by the magnetic field.

This point becomes even clearer in the simplest non planar one loop amplitude pictured in fig. 2. This amplitude in the case of two tachyon vertices given in eq.s (69) and (70) becomes

$$\begin{aligned} & Tr(V_T(x=1;k_1)\;\Delta\; V_T(y=-1;k_2)\;\Delta) = \\ & \quad = t_u t_v \int \frac{d\tau_1}{\tau_1} \int \frac{d\tau_2}{\tau_2} \; \int \frac{d^{D-d} k_\mu}{(2\pi)^{D-d} \; (2\pi\sqrt{\alpha'})^d} \sum_{(n_i)} \\ & \quad \times \langle k, \frac{n}{\sqrt{\alpha'} L} | \; Tr_{nzm}(e^{i k_1^T X(e^{-\tau_1})} e^{i k_2^T X(-e^{-(\tau_1+\tau_2)})}) \; | k, \frac{n}{\sqrt{\alpha'} L}\rangle \\ & \quad \times tr(\Lambda^\dagger(n)\Lambda(k_1)\Lambda(n)\Lambda(k_2)) \;\; tr(T_u) \; Tr(T_v) \end{aligned} \tag{74}$$

4 Translational invariant operators and Chan-Paton factors for closed string vertices.

We want now check that closed string vertices are invariant for a translation along a closed curve on the torus.

Since the closed string vertices cannot be applied directly to the open string vacuum to generate open string states we need an explicit expression for the conserved generalized translation operator $\mathcal{T}^{(i)}_{2\pi\sqrt{\alpha'}}$ along the x^i torus cycle which can be written as

$$\mathcal{T}^{(i)}_{2\pi\sqrt{\alpha'}} = e^{2\pi\sqrt{\alpha'}\, i\,\mathcal{G}_{ij}p^j} ({\omega_i}^{\dagger})_{MI}\ (\omega_i)_{JL}\ |M, L\rangle\langle I, J| \tag{75}$$

so that any operator O which is invariant along the cycle parametrized by x^i satisfies

$$\mathcal{T}^{(i)}_{2\pi\sqrt{\alpha'}}\ O\ \mathcal{T}^{(i)\ -1}_{2\pi\sqrt{\alpha'}} = O \qquad \forall i \tag{76}$$

It is then natural to assume that all physical operators are invariant for a translation along all cycles. We can now see what happens if we apply the previous translation operators to closed string vertices written in open string formalism. The *naive* closed string vertex operators can be written in open string formalism as ([14], [8], [9]) and up to cocycles as

$$W_{T_c,naive}(z, \bar{z}; k_L, k_R) =: e^{ik_L^T X_L(z)} :\ \ : e^{ik_R^T X_R(\bar{z})} :\ \sum_{I,J} |I, J\rangle\ \langle I, J| \tag{77}$$

where $X_L(z)$ and $X_R(\bar{z})$ are the open string fields in eq. (43) and the left and right moving part are separately normal ordered.

If we apply the translation operator along the x^i cycle on the previous naive vertex we get

$$\begin{aligned}
\mathcal{T}^{(i)}_{2\pi\sqrt{\alpha'}}\ W_{T_c,naive}(z, \bar{z}; k_L, k_R)\ \mathcal{T}^{(i)\ -1}_{2\pi\sqrt{\alpha'}} &= e^{i2\pi\sqrt{\alpha'}(k_L^T G^{-1}\mathcal{E} + k_R^T G^{-1}\mathcal{E}^T)_i}\ W_{T_c,naive}(z, \bar{z}; k_L, k_R) \\
&= e^{i2\pi(n-\hat{F}m)_i}\ W_{T_c,naive}(z, \bar{z}; k_L, k_R)
\end{aligned} \tag{78}$$

which is clearly not translational invariant when $\hat{F}_{ij} \in \mathbb{Q}$.

To cure this problem we propose to define the closed string vertex operators up to cocycles as

$$\begin{aligned}
W_{T_c}(z, \bar{z}; k_L, k_R) &= : e^{ik_L^T X_L(z)} :\ \ : e^{ik_R^T X_R(\bar{z})} : \\
&\quad \times \sqrt{L} \sum |(I_1, I_2), J\rangle\ \Lambda_{L;I_1K_1}\left(-L\ \hat{F}m\right)\ \langle (K_1, I_2), J|
\end{aligned} \tag{79}$$

Since $\hat{F}_{12} = \frac{f}{L}$ the action of the matricial part of the generalized translation operator $\mathcal{T}^{(i)}_{2\pi\sqrt{\alpha'}}$ is now

$$\begin{aligned}
({\omega_i}^{\dagger})_{M_0 I_0}\ (\omega_i)_{J_0 L_0}\ |M_0, L_0\rangle\langle I_0, J_0| &\times \sum |I_1, J\rangle\ \Lambda_{L;(I_1)_1(I_2)_1}\left(-L\ \hat{F}m\right)\ \langle I_2, J| \\
&\times ({\omega_i}^{T})_{M_3 I_3}\ (\omega_i^*)_{J_3 L_3}\ |M_3, L_3\rangle\langle I_3, J_3| = \\
= |M_0, L_0\rangle &\quad \left({\omega_i}^{\dagger}\Lambda_L\left(-L\ \hat{F}m\right)\omega_i\right)_{(M_0)_1(I_3)_1} \left(\omega_i{\omega_i}^{\dagger}\right)_{M_0 J_3} \langle I_3, J_3| \\
\Rightarrow \omega_i^{\dagger}\ \Lambda_L\left(-L\ \hat{F}m\right)\omega_i &= \omega_i^{\dagger}\ \Lambda_L\left(-fm^2, fm^1\right)\omega_i \\
&= e^{-i\frac{2\pi}{L}(-L\ \hat{F}m)_i}\Lambda_L\left(-fm^2, fm^1\right)
\end{aligned} \tag{80}$$

as follows from eq. (59). The additional phase then cancels exactly the one due to the operatorial part in eq. (78). While this is contrary to the common lore we will show that when cocycles are taken into account everything works as expected, i.e. for example two closed string vertices have the same OPE as in a pure closed string theory. Before discussing the cocycles which are more technical we would like to make a series of comments:

- the previous vertex is not in contrast with the derivation of the same from the open string σ model ([9]) since only the massless states which have vanishing winding $m = 0$ couple to the world-sheet action and in this case we have $\sqrt{L}\Lambda_L(0,0) = \mathbb{I}_L$;
- one could have canceled the unwanted phase in eq. (78) by using a matrix on the $\sigma = \pi$ boundary. A reason to prefer the solution given is that y_0 which is defined in eq. (44) enters the vertex as $e^{im^T G y_0/\sqrt{\alpha'}}$ and is identified (up to constant) with the commuting part of the Wilson lines on the brane at $\sigma = 0$ and therefore we can associate with $\Lambda_L\left(-L\ \hat{F}m\right)$ the non abelian part of the Wilson lines coded into ω_i;
- finally we notice that a dependence on Chan-Paton in closed string vertices is present in other systems even if not clearly stated. Consider a flat $U(2)$ bundle on S^1 defined on two branes with Wilson lines turned on as $\Omega_1 = \begin{pmatrix} e^{i2\pi\theta_1^1} & \\ & e^{i2\pi\theta_1^2} \end{pmatrix}$. Then any closed string vertex operator contains a (trivial) matrix dependence $\begin{pmatrix} e^{i2\pi\theta_1^1 m^1} & \\ & e^{i2\pi\theta_1^2 m^1} \end{pmatrix}$ associated to the fact that the factor $e^{im^T G y_0/\sqrt{\alpha'}}$ feels two different Wilson lines of the two branes. In fact consider the coupling of a closed string state described by W to the system made by the two branes, then the coupling is given very roughly by the sum $\langle W\rangle_{brane\ 1} + \langle W\rangle_{brane\ 2}$. If we make explicit the dependence on y_0, i.e. on the Wilson lines we can write $\langle e^{im^T\theta_1} W\rangle_{brane\ 1} + \langle e^{im^T\theta_2} W\rangle_{brane\ 2}$ But then if we want to describe the previous sum as due to a unique system described by a bundle we must write $Tr\langle\begin{pmatrix} e^{i2\pi\theta_1^1 m^1} & \\ & e^{i2\pi\theta_1^2 m^1} \end{pmatrix} W\rangle$.

5 Cocycles for a well defined open/closed string theory.

Until now we have discussed the vertices without explicitly writing the cocycles. The cocycles are necessary to have a well-defined open and closed theory especially when Chan-Paton like factors are introduced for closed string vertices.

5.1 Cocycles for the closed string theory and framing.

The first step is to determine the cocycles in the closed string formalism since we want to reproduce in open string formalism the closed string result for the product of two vertices. A well defined closed strings (which is a good CFT plus a proper treatment of zero modes) must satisfy at least the following criteria

1. closed string vertices commute;
2. a proper behavior under Hermitian conjugation.

The first request is necessary in order to ensure the mutual locality of closed vertices ([15]), or in other words, that vertices obey the spin-statistics theorem. For implementing the wanted properties we look for a solution of the form ([15])

$$\mathcal{W}^{(c)}_{\beta_L,\beta_R}(z,\bar{z};k_L,k_R) \;=\; c(k_L,k_R;p_L,p_R)V_{\beta_L}(z;k_L)\,\tilde{V}_{\beta_R}(\bar{z};k_R) \tag{81}$$

where the cocycles are given by

$$c(k_L,k_R;p_L,p_R) \;=\; e^{i\pi\left(n_i A_c^{ij}\hat{n}_j+n_i B_c^{i}{}_j\hat{m}^j+m^i C_c{}_i{}^j\hat{n}_j+m^i D_{c\;ij}\hat{m}^j\right)} \tag{82}$$

We choose the coefficients A_c, B_c, C_c, D_c to be matrices in order to be general. They must be determined, as said before, so that two arbitrary vertices are mutually local, i.e. commute. This is also equivalent to the fact that the radial ordering of a product of vertices is given by a unique expression which can be derived by analytically continuing whichever particular ordering of the vertices is chosen to perform the computation.

In order to compute these matrices we consider the ordering of the product of two arbitrary vertices as

$$\begin{aligned}\mathcal{W}^{(c)}_{\beta_L,\beta_R}(z,\bar{z};k_L,k_R)\mathcal{W}^{(c)}_{\alpha_L,\alpha_R}(w,\bar{w};l_L,l_R) \;&=\; e^{i\Phi_{(c)}(k,l)}c(k_L+l_L,k_R+l_R;p_L,p_R)\\ &\times\; V_{\beta_L}(z;k_L)V_{\alpha_L}(w;l_L)\;\;\tilde{V}_{\beta_R}(\bar{z};k_R)\tilde{V}_{\alpha_R}(\bar{w};l_R)\end{aligned} \tag{83}$$

where we have defined the phase

$$e^{i\Phi_c(k,l)} = e^{-i\pi\left[n_l^T A_c n_k+n_l^T B_c m_k+m_l^T C_c n_k+m_l^T D_c m_k\right]} \tag{84}$$

In the previous expression we have used the compact momenta

$$k_L = \frac{1}{2\sqrt{\alpha'}}\left(n_k+E^T m_k\right), \qquad k_R = \frac{1}{2\sqrt{\alpha'}}\left(n_k-E m_k\right) \tag{85}$$

and similarly for l. The general solution of the constraints listed at the beginning of the section is discussed in appendix (B.2) and is given by

$$\begin{aligned}A_c \;&=\; Z_{A\;A}+2[Z_{A0}]_S+2Z_A, \qquad Z^T_{A\;A}=-Z_{A\;A},\\ D_c \;&=\; Z_{D\;A}+2[Z_{D0}]_S+2Z_D, \qquad Z^T_{D\;A}=-Z_{D\;A},\\ B_c \;&=\; \frac{1}{2}\mathbb{I}+Z_B\\ C_c \;&=\; B_c^T-\mathbb{I}+2Z_C=-\frac{1}{2}\mathbb{I}+Z_B^T+2Z_C\end{aligned} \tag{86}$$

where all matrices are integer valued and we have defined $2[M]_S = M + M^T$[7] We notice that there is a kind of "gauge invariance": matrices A_c, B_c, C_c and D_c are in the same class of equivalence when they yield the same phase (84) and this happens for any choice of the Z_A, Z_D and Z_C matrices. Moreover because the phase (84) is actually only a sign we can always take a representative of the previous matrices with all entries in $\mathbb{Z}_2$ and we can always choose $[A_c]_A$ and $[D_c]_A$ as we want since $-1 \equiv 1 \quad mod\ 2$. This last possibility turns out to be fundamental for having a consistent theory of open and closed string interaction.

In particular the hermiticity condition implies

$$e^{i\Phi_c(k,k)} = 1 \tag{87}$$

For future reference we write the effect of a T-duality transformation defined by a $\Lambda \in O(d,d,\mathbb{Z})$ matrix in eq. (21) on the cocycles matrices as

$$M_c = \begin{pmatrix} D_c & C_c \\ B_c & A_c \end{pmatrix} \quad = \quad \Lambda^T \begin{pmatrix} D_c^t & C_c^t \\ B_c^t & A_c^t \end{pmatrix} \Lambda \tag{88}$$

As it were first discussed in ([9]) neither the previous values are T-duality invariant nor the amplitudes are, even if it was shown that different solutions differ by momentum dependent phases which are the same at all loops and therefore the transition probabilities are the same. In ([16], [3]) in the framework of the Reggeon approach it was proposed to keep fixed the the Reggeon vertex but change the states represented by bra vectors by a phase necessary to compensate the change of the cocycle. This approach suffers from an asymmetry since one must fix a default cocycle and then keep this cocycle fixed for the Reggeon while changing the states. Despite of this we will show in sec. 7 that this procedure extends correctly to the open string sector.

We will see that open string formalism has not any preferred frame but has a preferred gauge.

There is actually a further constraint we can impose. We would like the naive condition

$$|B(\hat{F})\rangle = tr(Pe^{i\oint A})|B\rangle \tag{89}$$

to be true. We have used the adjective naive because the previous equation is what can be deduced from path integral and usually the results from path integral can suffer from some subtleties. In any case, as we show in section (6.3) it is possible to show that eq. (89) holds if we restrict the possible values of the cocycles to

$$[M_c]_S = \begin{pmatrix} 2[Z_{A0} + Z_A]_S & 2(Z_B^T + Z_C) \\ 2(Z_B + Z_C^T) & 2[Z_{D0} + Z_D]_S \end{pmatrix} \equiv 0 \quad mod\ 4 \Rightarrow e^{\frac{i}{2}\Phi_c(k,k)} = 1 \tag{90}$$

The constraint implies $e^{\frac{i}{2}\Phi_c(k,k)} = 1$ but it is not true the converse nevertheless the previous choice is T-duality invariant and is therefore better. A particular

[7] This last definition implies therefore that $2[Z_{A0}]_S$ is symmetric matrix with even diagonal entries and arbitrary integer off diagonal entries and similarly for the other cases.

choice satisfying both the previous constraints and the ones from the open string construction given in eq. (97) is

$$A_c = 0, \quad B_c = -C_c = \frac{1}{2}\mathbb{I}, \quad D_c = f\tilde{f}\ \epsilon \ \leftrightarrow\ M_{c0} = \begin{pmatrix} f\tilde{f}\ \epsilon & -\frac{1}{2}\mathbb{I} \\ \frac{1}{2}\mathbb{I} & 0 \end{pmatrix} \tag{91}$$

and corresponds to the closed string tachyonic vertex

$$\mathcal{W}_{T_c}(z,\bar{z};k_L,k_R) = e^{-i\frac{1}{2}\pi(n_i\hat{m}^i - m^i\hat{n}_i) + i\pi\ f\tilde{f}(m^1\hat{m}^2 - m^2\hat{m}^1)} : e^{i\left[\frac{1}{2}k_\mu X^\mu_{L(c)}(z) + k_{Li}X^i_{L(c)}(z)\right]} :: e^{i\left[\frac{1}{2}k_\mu \tilde{X}^\mu_{R(c)}(\bar{z}) + k_{Ri}\tilde{X}^i_{R(c)}(\bar{z})\right]} : \tag{92}$$

which does depend on the magnetic field in a way which may or may be not be removed by a "gauge" choice of Z_D. This gauge transformation is possible when $f\tilde{f} \in 2\mathbb{Z}$ which can always be chosen and we have chosen in eq. (58) therefore we can always choose M_c to be gauge equivalent to

$$M_{c0} \equiv \begin{pmatrix} 0 & -\frac{1}{2}\mathbb{I} \\ \frac{1}{2}\mathbb{I} & 0 \end{pmatrix} \tag{93}$$

5.2 Cocycles for the open string theory.

We consider now the open-closed string theory. In particular a well defined string theory, which amounts to a proper treatment of zero modes besides a well behaved CFT, must satisfy the following 5 constraints:

1. the open string vertices at $\sigma = 0$ and $\sigma = \pi$ commute;
2. the open string emission vertex from $\sigma = 0$ commutes with the closed string vertices;
3. in a similar way the open string emission vertex from $\sigma = \pi$ commutes with the closed string vertices;
4. the closed string vertices in open string formalism must have the same product (OPE) as in closed string formalism, as a consequence we want the closed string vertices to commute;
5. a proper behavior under Hermitian conjugation.

To these constraints we want to add

1. the open closed string mixed amplitudes are sensitive to the value of y_0 which is periodic since it can be identified with the Wilson lines therefore these amplitudes, but not necessarily the the vertexes, must be invariant under this periodic identification;
2. the naive relation $|B(\hat{F})\rangle = tr(Pe^{i\oint A})|B\rangle$ holds.

On a compact space the previous constraints are not at all trivial and they are discussed in depth in appendix B. Here we summarize the non trivial results in the simplest case as before where the magnetic field is turned on only within a 2-torus.

Since the cocycles depend only on momenta the ones in tachyonic vertex operators are the same of all the other vertices, therefore we write only the tachyonic vertices which read:

$$\mathcal{V}_{(0)T}(x;k) = :e^{ik_M X^M(x)}: \ \Lambda_L(n_1,n_2)\otimes T_u \tag{94}$$

$$\mathcal{V}_{(\pi)T}(y;k) = :e^{ik_M X^M(y)}: \ \Lambda_L^T(n_1,n_2)\otimes T_u^T \tag{95}$$

$$\begin{aligned}\mathcal{W}_{T_c}(z,\bar z;k_L,k_R,y_0) &= e^{\frac{i}{2}\Phi_{(c)}(k,k)} e^{-i\pi\alpha' \ k_{RM}(\mathcal{E}^{-1}\mathcal{G}\mathcal{E}^{-1})^{MN}k_{LN}} \times \sqrt{L}\Lambda_L\left(-fm^2, fm^1\right)\otimes \mathbb{I}_{N_1} \\ &\quad \times :e^{ik_{LM}X_L^M(z)}::e^{ik_{RM}X_R^M(\bar z)}: \end{aligned}\tag{96}$$

where $x=|x|e^{i0}$, $y=|y|e^{i\pi}$ and z in the upper complex plane. Other expressions for the vertices which are more useful in the computations are given in eq.s (222).

A certain number of comments are worth doing.

1. It is important to stress that the previous vertices satisfy all the wanted constraints iff we use the following gauge for the closed string cocycle
$$[A_c]_A = 0 \quad [D_c]_A = \tilde f \ L\hat F = f\tilde f\epsilon, \quad C_c - B_c^T = \mathbb{I} \tag{97}$$
where we have defined $\tilde f$ in eq. (58).

2. The phase in the closed string vertex $e^{-i\pi\alpha' \ k_R^T\mathcal{E}^{-1}\mathcal{G}\mathcal{E}^{-1}k_L^T}$ is needed to get a vertex independent on the ordering of the left and right part since it let us to perform the substitution $\ln(z-\bar z)\to \ln|z-\bar z|$ in the amplitudes.

3. The closed string vertices are determined up to signs. In fact from the hermiticity of the closed string vertices we know $e^{i\Phi_{(c)}(k,k)} = 1$ therefore the associated phase in open string formalism is a sign $e^{\frac{i}{2}\Phi_{(c)}(k,k)}\in\{-1,1\}$ which can depend on k nevertheless the choice $e^{i\frac{1}{2}\Phi_{(c)}(k,k)}=1$ is always possible and it is the best one.

In the case of the more general gauge bundle described in eq. (42) where the Wilson lines are turned on to break the gauge group to $U(1)^{N_1}$ the previous vertices must be supplemented by the effect of the change of the Wilson lines. Consider to the emission of an open string from the $\sigma=0$ boundary with color J if the momentum of the emitted string is $\sqrt{\alpha'}k_{i,JL} = \left(\frac{n_i}{L_i}+\theta_i^J-\theta_i^L\right)$ then the final string has color L on the $\sigma=0$ boundary and the change in the Wilson lines implies that

$$\Delta y_0^i = -2\pi\alpha' G^{ij}(\theta_j^J-\theta_j^L) \tag{98}$$

since $G\frac{y_0}{\sqrt{\alpha'}}=\theta$ (see appendix B for further details). In particular the previous results reduce to what found in the simpler case $B=F=0$ in [9].

6 Some examples of amplitudes.

In this section we would like to compute some amplitudes which can be used to check the picture proposed in the previous sections. In particular the computation in 6.1 comments on the

The computation in sec. 6.2 is a warming up to the one in sec. 6.3 which shows that the momentum dependent Chan-Paton-like matrix in the closed vertices written in open string formalism is necessary in order to reproduce the boundary state from the open string point of view. This Chan-Paton-like matrix gives raise to a "trivial" momentum dependent sign which has been shown ([3]) essential for the correct factorization of the open string two loops vacuum amplitude as shown in fig. 4 and the same sign turns out to be fundamental for the T-duality to hold as discussed in sec. 7. In sec. 6.4 we compute the one closed string and N open string tachyonic amplitudes where the would-be closed string Chan-Paton gives again a momentum dependent sign. Using these results we can finally check that the one loop non planar amplitudes factorize correctly in the closed string channel in sec. 6.5.

6.1 Product of two open string vertices.

It is easy to check that the OPEs of two open string vertices are given by

$$\begin{aligned} \mathcal{V}_{(0)T}(x_1;k)\mathcal{V}_{(0)T}(x_2;l) &= \frac{1}{\sqrt{L}} e^{-i\pi\alpha'\ k_M(\Theta+\Theta_{CP})^{MN} l_N} (x_1-x_2)^{2\alpha' k_M \mathcal{G}^{MN} l_N} \\ &\quad : e^{ik_M X^M(x_1)+il_M X^M(x_2)} :\ \Lambda_L(k+l)\otimes T_u T_v \end{aligned} \tag{99}$$

and

$$\begin{aligned} \mathcal{V}_{(\pi)T}(y_1;k)\mathcal{V}_{(\pi)T}(y_2;l) &= \frac{1}{\sqrt{L}} e^{i\pi\alpha'\ k_M(2\mathcal{G}^{-1}+\Theta+\Theta_{CP})^{MN} l_N} (y_1-y_2)^{2\alpha' k_M \mathcal{G}^{MN} l_N} \\ &\quad e^{-i\pi\alpha'\ (k+l)_M \mathcal{G}^{MN}(k+l)_N} : e^{ik_M X^M(y_1)+il_M X^M(y_2)} :\ \Lambda_L^T(k+l)\otimes (T_v T_u)^T \end{aligned} \tag{100}$$

where despite the apparent asymmetry due to the phase $e^{i2\pi\alpha' k_M \mathcal{G}^{MN} l_N}$ the amplitudes are completely symmetric.

Notice how the nice feature of being able of writing the OPEs in function of $\Theta_{tot} = \Theta + \Theta_{CP}$ is not anymore true when Wilson lines are turned on as discussed in section C.1.

6.2 1 closed string emission amplitude from the disk.

We start by computing the amplitude given in fig. 3 which must give the result already found in the closed string channel ([1],[3]). The complete amplitude for a closed tachyon is then given by

$$\mathcal{A}(1T_c) = \mathcal{C}_0(E,\hat{F})\ \tilde{\mathcal{N}}_0(E)\ \frac{1}{2\pi}\ \langle 0|\ \mathcal{W}_{T_c}(z,\bar{z};k_L,k_R)\ |0\rangle\ \ |_{z=i} \tag{101}$$

where $\mathcal{C}_0(E,\hat{F})$ is the normalization of the disk amplitude, $\tilde{\mathcal{N}}_0(E)$ is the normalization of the closed string vertices and is independent of the open string background, $\frac{1}{2\pi}$ is the left over of the $SL(2,\mathbb{R})$ gauge fixing at $z=i$, $|0\rangle$ is the opens string vacuum given in eq. (66) and finally $\mathcal{W}_{T_c}$ is the closed string vertex in open string formalism given in eq. (96). An easy computation gives

$$\begin{aligned}\mathcal{A}(1T_c) &= \mathcal{C}_0(E,\hat{F})\ \tilde{\mathcal{N}}_0(E)\ \frac{1}{2\pi} \times N_1 L\ e^{i\frac{\pi}{L}\hat{h}f^2 m^1 m^2}\ \delta^{[L]}_{-L\hat{F}m} \\ &\quad \times e^{\frac{i}{2}\Phi_{(c)}(k,k)} e^{im^T G\frac{y_0}{\sqrt{\alpha'}}}\delta(k_\mu)(2\pi\sqrt{\alpha'})^d \delta_{\mathcal{E}^T G^{-1}k_L+\mathcal{E}G^{-1}k_R,0}|z-\bar{z}|^{2\alpha' k_R^T\mathcal{E}^{-1}\mathcal{G}\mathcal{E}^{-1}k_L}\end{aligned} \tag{102}$$

where in the first line we have the Chan-Paton contribution and in the second the operatorial one. It is worth noticing that the absolute value $|z-\bar{z}|$ which ensures a well defined phase for the amplitude is due to the non operatorial cocycle in the closed string vertex exactly as in the case without magnetic field ([9]). The conservation of the compact momentum $\delta_{\mathcal{E}^T G^{-1}k_L+\mathcal{E}G^{-1}k_R,0} = \delta_{(n-\hat{F}m)/\sqrt{\alpha'},0}$ can be rewritten as $\delta_{n,\hat{F}m}$ which implies that $\hat{F}m = \frac{f}{L}(m^2,-m^1)$ is actually integer as also imposed by the Chan-Paton trace $\delta^{[L]}_{-L\hat{F}m} = \delta^{[L]}_{fm^1}\delta^{[L]}_{fm^2}$. The exponent of $|z-\bar{z}|$ becomes $2\alpha' k_R^T\mathcal{E}^{-1}\mathcal{G}\mathcal{E}^{-1}k_L = -2\alpha' k_L^T G^{-1}k_L = -2$ due to the mass shell condition $\alpha'(G^{00}k_0^2 + k_L^T G^{-1}k_L) = 1$ therefore we get

$$\begin{aligned}\mathcal{A}(1T_c) &= \mathcal{C}_0(E,\hat{F})\ \tilde{\mathcal{N}}_0(E)\ \frac{1}{2\pi}N\ e^{\frac{i}{2}\Phi_{(c)}(k,k)+i\pi\frac{\hat{h}f^2}{L}m^1m^2} e^{im^T G\frac{y_0}{\sqrt{\alpha'}}} \\ &\quad \times(2\pi)^{D-d}\delta^{D-d}(k_\mu)(2\pi\sqrt{\alpha'})^d\delta_{n,\hat{F}m}\times\frac{1}{|z-\bar{z}|^2}\end{aligned} \tag{103}$$

Given our choice $e^{\frac{i}{2}\Phi_{(c)}(k,k)} = 1$ (90) it is then obvious that, if we can identify $G_{ij}\frac{y_0^j}{\sqrt{\alpha'}} = 2\pi\sqrt{\alpha'}qa_{0\ i}$, we reproduce exactly the closed string result, phases included obtained with the boundary state formalism. With our choices (58) and (91) we have both $\frac{\hat{h}f^2}{L} = \frac{f}{L}(-1+\tilde{f}) \equiv -\frac{f}{L}\ \ mod\ 2$ and $e^{\frac{i}{2}\Phi_c(k,k)} = 1$ so that we can reproduce the amplitude obtained from the boundary state.

6.3 Boundary state.

We want to generalize the previous computation to all the closed string states, i.e we want to derive the generating function in the closed string Hilbert space of all the one point closed string coupling to the disk: this is nothing else but the boundary state. We compute therefore the amplitude

$$
\begin{aligned}
\langle B(F);V_L,V_R| &= \frac{\mathcal{C}_0(E,\hat{F})\ \tilde{\mathcal{N}}_0(E)}{2\pi} \\
&\quad \langle x_L=x_R=0;0_{a_{(c)}},0_{\tilde{a}_{(c)}}|\ e^{-\frac{i}{2}\Phi_{(c)}(Gp_{(c)},Gp_{(c)})}\ e^{-i\pi\alpha'\ p_R^M(\mathcal{E}^T\mathcal{G}^{-1}\mathcal{E}^T)_{MN}p_L^N} \\
&\quad \times Tr\left(\sqrt{L}\Lambda_L\left(-L\hat{F}m\right)\otimes\mathbb{I}_{N_1}\right)\ \times {}_p\langle 0|\mathcal{S}_L(z;V_L)\ \mathcal{S}_R(\bar{z};V_R)|0\rangle_p
\end{aligned}
\tag{104}
$$

where we have introduced the closed string state $\langle x_L=x_R=0|$ normalized as $\langle x_L = x_R=0|k_L,k_R\rangle=1$ and the Sciuto-Della Selva-Saito vertex $\mathcal{S}_L(z;V_L)$ ($\mathcal{S}_R(\bar{z};V_R)$) as discussed in ([17], [18]) which acts on both the open string Hilbert space and the left (right) closed string Hilbert space. These vertices are given for arbitrary local $SL(2,\mathbb{C})$ coordinates $V_L(u;z)$ and $V_R(u;\bar{z})$ defined as

$$
V_L(u;z) = \frac{a_Lu+b_Lz}{c_Lu+b_L}, \quad b_L(a_L-c_Lz)=1, \quad V_L(0;z)=z \tag{105}
$$

and similarly for V_R.

Performing explicitly the computation the amplitude (104) can then be written as

$$
\begin{aligned}
\langle B(F);V_L,V_R| &= N\ \frac{\mathcal{C}_0(E,\hat{F})\ \tilde{\mathcal{N}}_0(E)}{2\pi} \\
\langle k_\mu=0|\ &\sum_{s\in\mathbb{Z}^d}\frac{1}{(2\pi\sqrt{\alpha'})^d}\langle n=L\ \hat{F}\ s, m=L\ s|e^{-\frac{1}{2}i\Phi_{(c)}(k,k)+i\pi\frac{\hat{h}f^2}{L}m^1m^2}\ e^{i\ m^M\ G_{MN}\ \frac{y_0^N}{\sqrt{\alpha'}}} \\
&\times\ \langle 0_a,0_{\tilde{a}}|\ \exp\left(-\sum_{n,m=0}^{\infty}a^N_{(c)n}(\mathcal{E}\mathcal{G}^{-1}\mathcal{E})_{NM}\tilde{a}^M_{(c)m}D_{nm}(U_LV_R)\right)
\end{aligned}
\tag{106}
$$

where D_{nm} is a (pseudo)representation of the $SL(2,\mathbb{C})$ group as explained in appendix C.2. This expression allows to compute any one point closed string amplitude by

$$
\mathcal{A}(\{\beta_L,\beta_R\}) = \langle B(F);V_L,V_R|\left(\left.\frac{dV_L}{du}\right|_{u=0}\right)^{-\Delta_{\beta_L}}\ \left(\left.\frac{dV_R}{du}\right|_{u=0}\right)^{-\Delta_{\beta_R}}|\beta_L,\beta_R\rangle \tag{107}
$$

where Δ_{β_L} is the conformal weight of the closed string left moving state $|\beta_L\rangle$ and similarly for the right moving part. This result is independent on V_L and V_R for the physical states since any closed string physical state $|\beta_L,\beta_R\rangle$ is annihilated by the $SL(2,\mathbb{C})$ generators.

The previous expression for the boundary state (106) is nevertheless not physical, i.e.

$$
\langle B(F);V_L,V_R|(L_n-\tilde{L}_{-n})\neq 0 \tag{108}
$$

We can now fix the local coordinates in such a way we get the usual boundary, i.e. we require $D_{nm}(U_L V_R) = \delta_{n,m}$ and therefore we impose

$$
\begin{aligned}
u &= U_L V_R(u) = \frac{1}{b_L} \frac{(a_L c_R - a_R c_L) u + b_R (a_L - c_L \bar{z})}{(a_R - c_R z) u - b_R (z - \bar{z})} \\
&\Rightarrow \ V_L(u; z) = \frac{c\bar{z}\ u + z}{cu + 1}, \qquad V_R(u; \bar{z}) = \frac{\frac{1}{c} z\ u + \bar{z}}{\frac{1}{c} u + 1},
\end{aligned} \tag{109}
$$

so that we get the final result

$$
\begin{aligned}
\langle B(F)| &= N\ \frac{\mathcal{C}_0(E, \hat{F})\ \tilde{\mathcal{N}}_0(E)}{2\pi}\ \langle k_\mu = 0| \\
& \sum_{s \in \mathbb{Z}^d} \frac{1}{(2\pi\sqrt{\alpha'})^d} \langle n = L\ \hat{F}\ s, m = L\ s| e^{-\frac{1}{2} i \Phi_{(c)}(k,k) + i\pi \frac{\hat{h} f^2}{L} m^1 m^2}\ e^{i\ m^M\ G_{MN}\ \frac{y_0^N}{\sqrt{\alpha'}}} \\
& \langle 0_a, 0_{\tilde{a}} | e^{-\sum_{n=1}^\infty a^N_{(c)n} (\mathcal{E}\mathcal{G}^{-1}\mathcal{E})_{NM} \tilde{a}^M_{(c)n}}.
\end{aligned} \tag{110}
$$

With our choices (58) and (91) we have both $\frac{\hat{h} f^2}{L} = \frac{f}{L}(-1 + \tilde{f}) \equiv -\frac{f}{L}\ \ mod\ 2$ and $e^{\frac{i}{2}\Phi_c(k,k)} = 1$. Hence we can reproduce the boundary already found in ([1],[3])[8].

6.4 N open tachyons and 1 closed tachyon string amplitude on the disk.

We can now compute the amplitude given in fig. 1 which is necessary to show the proper factorization of the non planar amplitude in the closed channel in the next section.

The complete amplitude for one closed tachyon and N open tachyon is given by the sum of all non cyclically equivalent permutations of the external legs, i.e the sum of all the possible permutations P of the $N-1$ open string vertices other than $\mathcal{V}_{(0)T}(x_1)$

$$
\mathcal{A}(t_1, \dots t_N, t_c) = \sum_P A(t_1, \dots t_{P(N)}, t_c) \tag{112}
$$

A piece[9] of the partial amplitude associated with the ordering $1, 2, \dots N$ and all

[8] Notice that the previous constraints are sufficient but not necessary since we need only to impose the equality of the the sign in eq. (110) and the one obtained in the computation $|B(F)\rangle = tr(P e^{i \oint A}) |B(F = 0)\rangle$ which implies

$$
e^{-\frac{1}{2} i \Phi_{(c)}(k,k) + i\pi \frac{\hat{h} f^2}{L} m^1 m^2} \Big|_{n = \hat{F} m; m = Ls} = e^{-\frac{1}{2} i \Phi_{(c)}(k,k) + i\pi \frac{f}{L} m^1 m^2} \Big|_{n = 0; m = Ls}. \tag{111}
$$

This equation has more solutions than the one we have chosen but many of them are not T-duality invariant and this is the reason why we have singled out the previous one.

[9] The other pieces are obtained by moving some vertices on the $\sigma = \pi$ boundary while keeping the cyclical ordering,

vertices at $\sigma = 0$ is given by

$$A(NT_o, 1T_c) = \mathcal{C}_0(E, \hat{F})\ \tilde{\mathcal{N}_0}(E)\ \mathcal{N}_0(E, \hat{F})^N$$
$$\int \frac{\prod_{r=1}^N dx_r\ dz\ d\bar{z}}{dV_{Killing}} \langle 0|\ \mathcal{V}_{(0)T}(x_1; k_1) \ldots\ \mathcal{V}_{(0)T}(x_N; k_N) \mathcal{W}_{T_c}(z, \bar{z}; k_L, k_R)|0\rangle \tag{113}$$

where $\mathcal{N}_0(E, \hat{F})$ is the normalization of the open string vertices which is dependent of both the open and the closed string background, $dV_{Killing}$ is the volume of the $SL(2, \mathbb{R})$ gauge invariance which we discuss later. We notice that the order of the open vertices w.r.t the closed one is not important since they commute when both the operatorial and the Chan-Paton parts are considered.

An easy computation gives

$$\begin{aligned}
A(NT_o, 1T_c) \quad = \quad & \mathcal{C}_0(E, \hat{F})\ \tilde{\mathcal{N}_0}(E)\ \mathcal{N}_0(E, \hat{F})^N\ e^{\frac{i}{2}\Phi(k,k)}\ e^{im^T G \frac{y_0}{\sqrt{\alpha'}}} \\
& \times\ t_{u_1}(k_N) \ldots t_{u_1}(k_N)\ tr(T_{u_1} \ldots T_{u_N})\ e^{i\pi\hat{h}\left(f(n_1 m^1 + n_2 m^2) - L n_1 n_2\right)} \\
& \times \left(\frac{1}{\sqrt{L}}\right)^{N-1} e^{-i\pi\alpha' \sum_{r<s} k_r^T (\Theta + \Theta_{CP}) k_s} \\
& \times\ (2\pi)^{D-d} \delta(\sum_r k_{r\mu} + k_\mu)\ (2\pi\sqrt{\alpha'})^d \delta_{\sum_r k_r + \mathcal{E}^T G^{-1} k_L + \mathcal{E} G^{-1} k_R, 0} \\
& \times \int \frac{\prod_{r=1}^N dx_r\ dz\ d\bar{z}}{dV_{Killing}} \prod_{r<s} (x_r - x_s)^{2\alpha' k_r^T \mathcal{G}^{-1} k_s} \\
& \times \prod_r \left[(x_r - z)^{2\alpha' k_L^T \mathcal{E}^{-T} k_r} (x_r - \bar{z})^{2\alpha' k_R^T \mathcal{E}^{-1} k_r}\right] |z - \bar{z}|^{2\alpha' k_R^T \mathcal{E}^{-1} \mathcal{G} \mathcal{E}^{-1} k_L}
\end{aligned} \tag{114}$$

where we have used the non operatorial cocycle in the closed string vertex to write the modulus of the difference $z - \bar{z}$ and momentum conservation to simplify the phase coming from the trace of the Λ matrices. In particular the sign $e^{i\pi\hat{h}\left(f(n_1 m^1 + n_2 m^2) - L n_1 n_2\right)}$ is due to the interaction among the open string Chan-Paton factors and the would-be closed one. We need now the full amplitude since we later want to compare it with the result of the factorization of the non planar amplitude. Very roughly it is convenient to proceed as follows to obtain the full amplitude (details are given in appendix). We fix the $SL(2, \mathbb{R})$ gauge invariance so that $z = i$ and $x_N = 0$. Then we can change variable of the integral to $w = \frac{z-i}{z+i}$ so that the point $z = i$ is mapped into $w = 0$ and real axis gets mapped into the circle $|w| = 1$. In particular for the real number we have $w = e^{i\phi}$ with $\phi = \arccos \frac{x}{\sqrt{1+x^2}}$ so that the positive real axis gets mapped into the lower semicircle in clockwise direction. Differently from the gauge fixing we use with pure open string amplitudes we have only fixed one open string at $x_N = 0$ therefore we must also consider the partial amplitudes with open string emitted on the $\sigma = \pi$ boundary. For example given the same ordering on the unit circle $|w| = 1$ as in eq. (113) we have to consider the amplitudes

$$
\begin{aligned}
\mathcal{C}_0(E,\hat{F})\ \tilde{\mathcal{N}}_0(E)\ \mathcal{N}_0(E,\hat{F})^N \qquad & \int \frac{\prod_{r=1}^{l-1} dy_r\ \prod_{r=l}^{N} dx_r\ dz\ d\bar{z}}{dV_{Killing}} \\
& \langle 0|R\Big[\ \mathcal{V}_{(0)T}(x_k;k_k)\ldots\ \mathcal{V}_{(0)T}(x_l;k_l) \\
& \quad \mathcal{V}_{(0)T}(y_{k-1};k_{k-1})\ldots\ \mathcal{V}_{(0)T}(y_1;k_1) \\
& \quad \mathcal{V}_{(0)T}(y_N;k_N)\ldots\ \mathcal{V}_{(0)T}(y_{l+1};k_{l+1})\ \mathcal{W}_{T_c}(z,\bar{z};k_L,k_R)\Big]|0\rangle
\end{aligned}
\tag{115}
$$

with $x_k > \cdots > x_l$ and $|y_{k-1}| > \ldots |y_1| > |y_N| > \ldots |y_{l+1}|$ for all ls and ks ($k < l$) and where R is the radial ordering. Because the vertices on the two boundaries commute summing over all the possible ls and ks and possible radial orderings amounts to integrate over the whole $|w| = 1$ circle with all $0 < \phi < 2\pi$ and $\phi_i > \phi_{i+1}$. Details on how the different pieces join together are given in appendix (C.3): it is nevertheless noteworthy that the whole procedure works because of some phase contributed by the would-be closed Chan-Paton factors.

Summing all the previous amplitudes allows to extend the ϕ_r integrations to the full range $[0, 2\pi]$ we can therefore write

$$
\begin{aligned}
\mathcal{A}^{same\ ordering}(NT_o, 1T_c) \quad = \quad & \mathcal{C}_0(E,\hat{F})\ \tilde{\mathcal{N}}_0(E)\ \mathcal{N}_0(E,\hat{F})^N\ e^{\frac{i}{2}\Phi(k,k)}\ e^{im^T G \frac{y_0}{\sqrt{\alpha'}}} \\
& \times\ t_{u_1}(k_1)\ldots t_{u_N}(k_N)\ tr(T_{u_1}\ldots T_{u_N})\ e^{i\pi\hat{h}\left(f(n_1 m^1 + n_2 m^2) - L n_1 n_2\right)} \\
& \times \left(\frac{1}{\sqrt{L}}\right)^{N-2} e^{-i\pi\alpha' \sum_{r<s} k_r^T(\Theta+\Theta_{CP})k_s} \\
& \times\ (2\pi)^{D-d}\delta(\sum_r k_{r\mu} + k_\mu)\ (2\pi\sqrt{\alpha'})^d \delta_{\sum_r k_r + \mathcal{E}^T G^{-1} k_L + \mathcal{E} G^{-1} k_R, 0} \\
& \times \int_0^{2\pi} \prod_{r=1}^{N-1} d\phi_r\ \theta(\phi_r - \phi_{r+1}) \prod_{1\le r<s\le N} (2\sin\frac{\phi_r - \phi_s}{2})^{2\alpha' k_r^T \mathcal{G}^{-1} k_s} \\
& \times \prod_{r=1}^{N-1} e^{i\phi_r\ \alpha' k_r^T(\mathcal{E}^{-1}k_L - \mathcal{E}^{-T}k_R)}
\end{aligned}
\tag{116}
$$

with $\phi_N = 0$.

6.5 The 1 loop non planar amplitude.

Finally we want to compute the non planar 1 loop amplitude with N_0 tachyons on one border and N_π ones on the other. The full amplitude is given by the sum over all cyclically nonequivalent amplitudes. This can be also be written as

$$\mathcal{A}\,(t_1,\dots t_{N_0};t_{N_0+1}t_{N_0+N_\pi}) = \sum_{P_0}\sum_{P_\pi}\mathcal{A}(P_0(1),\dots P_0(N_0);P_\pi(N_0{+}1),\dots P_\pi(N_0{+}N_\pi)) \tag{117}$$

where P_0 is any of the permutations of vertices at $\sigma=0$, P_π is any of the permutations of vertices at $\sigma=\pi$ which keeps the index N_0+N_π fixed and

$$\begin{aligned}
&\mathcal{A}(P_0(1),\dots P_0(N_0);P_\pi(N_0+1),\dots P_\pi(N_0+N_\pi)) = \\
&= \hat{\mathcal{A}}(P_0(1),\dots P_0(N_0),P_\pi(N_0+1),\dots P_\pi(N_0+N_\pi)) \\
&\ \ + \hat{\mathcal{A}}(P_0(1),\dots N_0+1,P_0(N_0),\dots P_\pi(N_0+N_\pi)) + \\
&\ \ \dots + \hat{\mathcal{A}}(P_\pi(N_0+1),\dots P_\pi(N_0+N_\pi-1),P_0(1),\dots P_0(N_0),N_0+N_\pi)
\end{aligned} \tag{118}$$

is the sum over the permutations Q of vertices at $\sigma=0$ relative to those at $\sigma=\pi$ which keep fixed the ordering of vertices on both boundaries.

Let us consider the orderings $1\dots N_0$ at $\sigma=0$ boundary and $N_0+1\dots N_0+N_\pi$ at $\sigma=\pi$. For these orderings we compute first the amplitude corresponding to the simplest relative ordering on the two boundaries and then we discuss the effect of summing over all the other relative orderings obtained by applying any permutation Q. We use the old formalism and therefore we compute

$$\begin{aligned}
&\hat{\mathcal{A}}\ \ (1,\dots N_0,N_0+1,\dots N_0+N_\pi) \\
&= \mathcal{C}_1\ \mathcal{N}_0(E,\hat{F})^{N_0+N_\pi}\ Tr\Big(\Delta\ \mathcal{V}_{(0)T}(1;k_1)\dots\mathcal{V}_{(0)T}(1;k_{N_0})\Delta\mathcal{V}_{(\pi)T}(-1;k_{N_0+1})\dots\mathcal{V}_{(\pi)T}(-1;k_{N_0+N_\pi})\Big) \\
&= \mathcal{C}_1\ \mathcal{N}_0(E,\hat{F})^{N_0+N_\pi}\ t_{u_1}(k_1)\dots t_{u_{N_0+N_\pi}}(k_{N_0+N_\pi}) \\
&\times\ \int\frac{d^{D-d}k_\mu}{(2\pi)^{D-d}}\ \frac{1}{(2\pi\sqrt{\alpha'})^d}\sum_{n_1,\dots n_d}\ \int_0^1 dx_1\dots\int_0^1 dx_{N_0+N_\pi} \\
&\times tr\left(\Lambda^\dagger(\frac{L^{-1}n}{\sqrt{\alpha'}})\Lambda(k_1)\dots\Lambda(k_{N_0})\Lambda(\frac{L^{-1}n}{\sqrt{\alpha'}})\Lambda(k_{N_0+N_\pi})\dots\Lambda(k_{N_0+1})\right) \\
&\times tr\left(T_{u_1}\dots T_{u_{N_0+1}}\right)\ tr\left(T_{u_{N_0+N_\pi}}\dots T_{u_{N_0+1}}\right) \\
&\times\langle k_\mu,\frac{(L^{-1}n)_i}{\sqrt{\alpha'}}|\ Tr_{nzm}\Big(x_1^{L_0-2}\ :e^{ik_{1M}X^M(1)}:\dots \\
&\qquad\qquad x_{N_0+N_\pi}^{L_0-2}e^{i\pi\alpha' k_{N_0+N_\pi N}\mathcal{G}^{NM}k_{N_0+N_\pi M}}\ :e^{ik_{N_0+N_\pi M}X^M(-1)}:|k_\mu,\frac{(L^{-1}n)_i}{\sqrt{\alpha'}}\rangle
\end{aligned} \tag{119}$$

where $\Delta=(L_0-1)^{-1}$ is the open string propator. With a straightforward computation, after including the ghost contribution and performing a Poisson resummation on the discrete momenta we find the following result

$$
\begin{aligned}
&\hat{\mathcal{A}}\ (1,\dots N_0, N_0+1,\dots N_0+N_\pi) = \\
&\quad = \ \mathcal{C}_1\ (\mathcal{N}_0(E,\hat{F}))^{N_0+N_\pi}\ \left[\det G_{\mu\nu}\ \det(L\mathcal{G}L)_{ij}\right]^{\frac{1}{2}} \\
&\qquad t_{1u_1}(k_1)\dots t_{N_0+N_\pi u_{N_0+N_\pi}}(k_{N_0+N_\pi})\ \left(\frac{1}{\sqrt{L}}\right)^{N_0+N_\pi}\quad tr\left(T_{u_1}\dots T_{u_{N_0+1}}\right)\ tr\left(T_{u_{N_0+N_\pi}}\dots T_{u_{N_0+1}}\right) \\
&\qquad \delta^{D-d}\left(\sqrt{\alpha'}\sum k_{r\mu}\right)\ \delta_{\sum k_{ri}} \\
&\qquad \prod_{1\le r<s\le N_0} e^{-i\pi\alpha' k_{ri}\Theta^{ij}_{tot}k_{sj}} \qquad \prod_{N_0+1\le r<s\le N_0+N_\pi} e^{+i\pi\alpha' k_{ri}\Theta^{ij}_{tot}k_{sj}} \\
&\qquad \int_0^1 \frac{dw}{w^2}\ \int_0^1 \prod_{r=1}^{N_o+N_\pi-1}\frac{d\rho_r}{\rho_r}\theta(\rho_r-\rho_{r+1})\ \left[\frac{-\ln w}{\pi}\right]^{-D/2}\ \left[\frac{1}{\prod_{n=1}^{\infty}(1-w^n)}\right]^{D-2} \\
&\qquad \prod_{1\le r<s\le N_0} e^{2\alpha' k_{rM}\mathcal{G}^{MN}k_{sN}\ \ln\psi_{rs}} \qquad \prod_{N_0+1\le r<s\le N_0+N_\pi} e^{2\alpha' k_{rM}\mathcal{G}^{MN}k_{sN}\ \ln\psi_{rs}} \\
&\qquad \prod_{1\le r\le N_0<s\le N_0+N_\pi} e^{2\alpha' k_{rM}\mathcal{G}^{MN}k_{sN}\ \ln\psi^T_{rs}} \\
&\qquad \sum_{(m_0^i)\in\mathbb{Z}^d} e^{\frac{\pi^2}{\ln w}\ \alpha'\left(\frac{Lm_0}{\sqrt{\alpha'}}+\Theta_{tot}\sum_{r=1}^{N_0}k_{ri}\right)^i\mathcal{G}_{ij}\left(\frac{Lm_0}{\sqrt{\alpha'}}+\Theta_{tot}\sum_{s=1}^{N_0}k_{si}\right)^j}\ e^{-i2\pi\alpha'\sum_{r=1}^{N_0+N_\pi}\frac{\ln\rho_r}{\ln w}\ k_{ri}\left(\frac{Lm_0}{\sqrt{\alpha'}}+\Theta_{tot}\sum_{s=1}^{N_0}k_s\right)^i}
\end{aligned}
\tag{120}
$$

where we have defined the new integration variables $\rho_r = x_1\dots x_r$, $w = \rho_{N_0+N_\pi}$ and, for $s>r$, the ratios $c_{sr} = \rho_s/\rho_r = x_{r+1}\dots x_s$. We have also defined the quantities $\psi_{rs} = \psi(c_{sr}, w)$ given by exponential of the annulus propagators as

$$
\begin{aligned}
\psi(c,w) &= c^{-\frac{1}{2}}\exp\left(\frac{(\ln c)^2}{2\ln w}\right)\exp\left(-\sum_{n=1}^{\infty}\frac{c^m+(w/c)^m-2w^m}{m(1-w^n)}\right) \\
\psi^T(c,w) &= c^{-\frac{1}{2}}\exp\left(\frac{(\ln c)^2}{2\ln w}\right)\exp\left(-\sum_{n=1}^{\infty}\frac{(-c)^m+(-w/c)^m-2w^m}{m(1-w^n)}\right)
\end{aligned}
\tag{121}
$$

In eq. (120) the contribution $\left[\det G_{\mu\nu}\ \det(L\mathcal{G}L)_{ij}\right]^{\frac{1}{2}}$ in the first line comes from Poisson resummation and is fundamental in fixing the normalization of the amplitudes as we discuss later.

If we now compute any other partial amplitude with a different relative ordering of the vertices on the two boundary we get almost the same result as before because the vertices on the boundaries commute: the only difference is given by the relative ordering of the ρ of the vertices on the two boundaries. Therefore when we sum over all possible relative ordering with $|y_{N_0+N_\pi}|$ less than all the other y and x as shown in eq. (117) we get the same result as above where the ordering of the ρ on the two boundaries are independent, explicitly

$$\int_0^1 \prod_{r=1}^{N_o+N_\pi-1} \frac{d\rho_r}{\rho_r}\theta(\rho_r-\rho_{r+1}) \Rightarrow \quad \int_0^1 \prod_{r=1}^{N_o-1} \frac{d\rho_r}{\rho_r}\theta(\rho_r-\rho_{r+1})\frac{d\rho_{N_0}}{\rho_{N_0}}\theta(\rho_r-\rho_{N_0+N_\pi})$$
$$\times \int_0^1 \prod_{r=N_0+1}^{N_o+N_\pi-1} \frac{d\rho_r}{\rho_r}\theta(\rho_r-\rho_{r+1}) \tag{122}$$

since the vertex located in $\rho_{N_0+N_\pi}=w$ is always the last and we keep the ordering on the two boundaries fixed.

We can now perform a modular transformation in order to express the amplitude using the closed string variables q and ν_r ($r=1,\dots N_0+N_\pi-1$) defined as

$$\ln q = \frac{2\pi^2}{\ln w}, \qquad \nu_r = \frac{\ln\rho_r}{\ln w} \tag{123}$$

so that the quantities which enter the amplitude become

$$\begin{aligned}
\psi(\nu,q) &= \psi(\rho,w) = \frac{2\pi}{-\ln q}\ \sin(\pi\nu)\ \prod_{n=1}^{\infty}\frac{(1-e^{i2\pi\nu}q^{2n})(1-e^{-i2\pi\nu}q^{2n})}{(1-q^{2n})^2}\\
\psi^T(\nu,q) &= \psi^T(\rho,w) = \frac{\pi}{-\ln q}\ q^{-1/4}\ \prod_{n=1}^{\infty}\frac{(1-e^{i2\pi\nu}q^{2n-1})(1-e^{-i2\pi\nu}q^{2n-1})}{(1-q^{2n})^2}\\
\prod_{n=1}^{\infty}(1-w^n) &= e^{-\frac{\pi^2}{12\log q}}\ \left(\frac{-\log q}{\pi}\right)^{\frac{1}{2}}\ q^{\frac{1}{12}}\prod_{n=1}^{\infty}(1-q^n)
\end{aligned} \tag{124}$$

The partial amplitude (118) obtained by summing over the subclass of permutations

Q can be rewritten for $D = 26$ as[10]

$$
\begin{aligned}
&\mathcal{A}\ (1,\dots N_0; N_0+1,\dots N_0+N_\pi) = \\
&= \quad 2^{1-\frac{D}{2}}(2\pi)^{N_0+N_\pi-1}\,\mathcal{C}_1\ (\mathcal{N}_0(E,\hat{F}))^{N_0+N_\pi}\ \ [\det G_{\mu\nu}\ \det(L\mathcal{G}L)_{ij}]^{\frac{1}{2}} \\
&\qquad t_{1u_1}(k_1)\dots t_{N_0+N_\pi u_{N_0+N_\pi}}(k_{N_0+N_\pi})\ \left(\frac{1}{\sqrt{L}}\right)^{N_0+N_\pi}\ tr\left(T_{u_1}\dots T_{u_{N_0+1}}\right)\ tr\left(T_{u_{N_0+N_\pi}}\dots T_{u_{N_0+1}}\right) \\
&\qquad \delta^{D-d}\left(\sqrt{\alpha'}\sum k_{r\mu}\right)\ \delta_{\sum k_{ri}} \\
&\qquad \prod_{1\le r<s\le N_0} e^{-i\pi\alpha' k_{ri}\Theta^{ij}_{tot}k_{sj}} \prod_{N_0+1\le r<s\le N_0+N_\pi} e^{+i\pi\alpha' k_{ri}\Theta^{ij}_{tot}k_{sj}} \\
&\qquad \sum_{(m_0^i)\in\mathbb{Z}^d}\int_0^1 dq\ q^{-3+\frac{1}{2}\alpha'\left\{(\sum_{r=1}^{N_0}k_r)^T\mathcal{G}^{-1}(\sum_{s=1}^{N_0}k_s)+\left(\frac{Lm_0}{\sqrt{\alpha'}}+\Theta_{tot}\sum_{r=1}^{N_0}k_r\right)^T\mathcal{G}\left(\frac{Lm_0}{\sqrt{\alpha'}}+\Theta_{tot}\sum_{s=1}^{N_0}k_s\right)\right\}} \\
&\qquad \left[\prod(1-q^{2n})^2\right]^{-24+2(N_0+N_\pi)} \\
&\qquad \int_0^1\prod_{r=1}^{N_0-1}d\nu_r\ \theta(\nu_{r+1}-\nu_r)\ d\nu_{N_0}\ \prod_{r=1}^{N_0}e^{-i2\pi\nu_r\alpha'\ k_r^T\left(\frac{Lm_0}{\sqrt{\alpha'}}+\Theta_{tot}\sum_{s=1}^{N_0}k_s\right)} \\
&\qquad\qquad \prod_{1\le r<s\le N_0}\left[2\sin\pi\nu_{sr}\ \prod_{n=1}^{\infty}\frac{(1-e^{i2\pi\nu_{sr}}q^{2n})(1-e^{-i2\pi\nu_{sr}}q^{2n})}{(1-q^{2n})^2}\right]^{2\alpha' k_r\mathcal{G}^{-1}k_s} \\
&\qquad \int_0^1\prod_{r=N_0+1}^{N_0+N_\pi-1}d\nu_r\ \theta(\nu_{r+1}-\nu_r)\ \prod_{r=N_0+1}^{N_0+N_\pi}e^{-i2\pi\nu_r\alpha'\ k_r^T\left(\frac{Lm_0}{\sqrt{\alpha'}}+\Theta_{tot}\sum_{s=1}^{N_0}k_s\right)} \\
&\qquad\qquad \prod_{1\le r<s\le N_0}\left[2\sin\pi\nu_{sr}\ \prod_{n=1}^{\infty}\frac{(1-e^{i2\pi\nu_{sr}}q^{2n})(1-e^{-i2\pi\nu_{sr}}q^{2n})}{(1-q^{2n})^2}\right]^{2\alpha' k_r\mathcal{G}^{-1}k_s} \\
&\qquad \prod_{1\le r\le N_0<s}\left[\ \prod_{n=1}^{\infty}\frac{(1-e^{i2\pi\nu_{sr}}q^{2n-1})(1-e^{-i2\pi\nu_{sr}}q^{2n-1})}{(1-q^{2n})^2}\right]^{2\alpha' k_r\mathcal{G}^{-1}k_s}
\end{aligned}
\tag{125}
$$

where we have set $\nu_{N_0+N_\pi} = 1$.

Summing over all the cyclically equivalent configurations on the $\sigma = 0$ boundary[11] and redefining in a proper way the integration variables as explained in ap-

[10] To make formlae more compact we define $m^M = \Theta^{MN} = 0$ when $M, N \neq i, j$.

[11] These permutations are a subset of the P_0 permutations which are non trivial since the P_0 permutations are not required to keep N_0 fixed.

pendix C.4 we finally get

$$
\begin{aligned}
&\sum_{k=1}^{N_0} \mathcal{A}(k,\dots N_0,1,\dots k-1;N_0+1,\dots N_0+N_\pi) = \\
&\quad = 2\pi\ 2^{1-\frac{D}{2}}\ \mathcal{C}_1\ (\mathcal{N}_0(E,\hat{F}))^{N_0+N_\pi}\ \left[\det G_{\mu\nu}\ \det(L\mathcal{G}L)_{ij}\right]^{\frac{1}{2}} \\
&\qquad t_{1u_1}(k_1)\dots t_{N_0+N_\pi u_{N_0+N_\pi}}(k_{N_0+N_\pi})\ \left(\frac{1}{\sqrt{L}}\right)^{N_0+N_\pi}\ tr\left(T_{u_1}\dots T_{u_{N_0+1}}\right)\ tr\left(T_{u_{N_0+N_\pi}}\dots T_{u_{N_0+1}}\right) \\
&\qquad \delta^{D-d}\left(\sqrt{\alpha'}\sum k_{r\mu}\right)\ \delta_{\sum k_{ri}} \\
&\qquad \prod_{1\le r<s\le N_0} e^{-i\pi\alpha' k_{ri}\Theta_{tot}^{ij}k_{sj}} \prod_{N_0+1\le r<s\le N_0+N_\pi} e^{+i\pi\alpha' k_{ri}\Theta_{tot}^{ij}k_{sj}} \\
&\qquad \sum_{(m_0^i)\in\mathbb{Z}^d}\int_0^1 dq\ q^{-3+\frac{1}{2}\alpha'\left\{(\sum_{r=1}^{N_0}k_r)^T\mathcal{G}^{-1}(\sum_{s=1}^{N_0}k_s)+\left(\frac{Lm_0}{\sqrt{\alpha'}}+\Theta_{tot}\sum_{r=1}^{N_0}k_r\right)^T\mathcal{G}\left(\frac{Lm_0}{\sqrt{\alpha'}}+\Theta_{tot}\sum_{s=1}^{N_0}k_s\right)\right\}} \\
&\qquad \left[\prod(1-q^{2n})^2\right]^{-24+2(N_0+N_\pi)} \\
&\qquad \int_0^1 d\nu_{N_0}\ e^{-i2\pi\nu_{N_0}\ \alpha'\sum_{r=1}^{N_0}k_r^T\frac{Lm_0}{\sqrt{\alpha'}}} \\
&\qquad \int_0^{2\pi}\prod_{r=1}^{N_0-1}d\phi_r\ \theta(\phi_r-\phi_{r+1})\ \prod_{r=1}^{N_0}e^{+i\phi_r\alpha'\ k_r^T\left(\frac{Lm_0}{\sqrt{\alpha'}}+\Theta_{tot}\sum_{s=1}^{N_0}k_s\right)} \\
&\qquad\qquad \prod_{1\le r<s\le N_0}\left[2\sin\frac{\phi_{rs}}{2}\ \prod_{n=1}^{\infty}\frac{(1-e^{i\phi_{rs}}q^{2n})(1-e^{-i\phi_{rs}}q^{2n})}{(1-q^{2n})^2}\right]^{2\alpha' k_r\mathcal{G}^{-1}k_s} \\
&\qquad \int_0^{2\pi}\prod_{r=N_0+1}^{N_0+N_\pi-1}d\phi_r\ \theta(\phi_{r+1}-\phi_r)\ \prod_{r=N_0+1}^{N_0+N_\pi}e^{-i\phi_r\alpha'\ k_r^T\left(\frac{Lm_0}{\sqrt{\alpha'}}+\Theta_{tot}\sum_{s=1}^{N_0}k_s\right)} \\
&\qquad\qquad \prod_{N_0+1\le r<s\le N_0+N_\pi}\left[2\sin\frac{\phi_{sr}}{2}\ \prod_{n=1}^{\infty}\frac{(1-e^{i\phi_{sr}}q^{2n})(1-e^{-i\phi_{sr}}q^{2n})}{(1-q^{2n})^2}\right]^{2\alpha' k_r\mathcal{G}^{-1}k_s} \\
&\qquad \prod_{1\le r\le N_0<s}\left[\prod_{n=1}^{\infty}\frac{(1-e^{i(\phi_r+\phi_r-2\pi\nu_{N_0})}q^{2n-1})(1-e^{-i(\phi_s+\phi_r-2\pi\nu_{N_0})}q^{2n-1})}{(1-q^{2n})^2}\right]^{2\alpha' k_r\mathcal{G}^{-1}k_s}
\end{aligned}
\tag{126}
$$

with $\phi_{N_0}=0$, $\phi_{N_0+N_\pi}=2\pi$.

We can now expand in powers of q, integrate over ν_{N_0}, shift $\bar{\phi}_r=\phi_r+2\pi-\phi_{N_0+1}$ for $r>N_0$ so that $\bar{\phi}_{N_0+1}=0$ and then we can compare with the N open tachyons - 1 closed tachyon amplitude. If we consider in particular the terms $e^{i\phi_r\alpha'\ k_r^T\left(\frac{Lm_0}{\sqrt{\alpha'}}+\Theta_{tot}\sum_{s=1}^{N_0}k_s\right)}$ we must identify

$$\left(\frac{Lm_0}{\sqrt{\alpha'}}+\Theta_{tot}\sum_{r=1}^{N_0}k_r\right)=(\mathcal{E}^{-1}k_L-\mathcal{E}^{-T}k_R) \tag{127}$$

which implies

$$m_0 = \hat{h}\ \epsilon n + \tilde{f} m \tag{128}$$

so that we have

$$\sum_{k=1}^{N_0} \mathcal{A}(k, \dots N_0, 1, \dots k-1; N_0+1, \dots N_0+N_\pi) \sim$$
$$\int \frac{d^{D-d} k_C}{(2\pi)^{D-d}\ (2\pi\sqrt{\alpha'})^d} \sum_{n_C} \mathcal{A}(1, \dots N_0; C)\ \mathcal{A}(N_0+N_\pi, \dots N_0+1; -C)\ \frac{1}{k_C^T \mathcal{G}^{-1} k_C - \frac{4}{\alpha'}} \tag{129}$$

where C stands for the closed string tachyon appearing in the mixed amplitude (116), $-C$ the closed string tachyon with opposite momentum and the disk amplitude associated to the $\sigma = \pi$ boundary is run in the opposite direction w.r.t. the one associated to the $\sigma = 0$ boundary as a simple picture shows it is the case.

Making the previous equation more precise and comparing the overall coefficients we can then write

$$(\mathcal{C}_0(E, \hat{F})\ \mathcal{N}_0(E, \hat{F}))^2\ (2\pi\sqrt{\alpha'})^D = 2\pi\ 2^{1-\frac{D}{2}}\ \mathcal{C}_1\ [\det G_{\mu\nu}\ \det \mathcal{G}_{ij}]^{\frac{1}{2}}\ \frac{2}{\alpha'} \tag{130}$$

which matches the result from the annulus given in appendix (C.5) and together the sewing relations ([19] [12])

$$\mathcal{C}_0(E, \hat{F})\ \mathcal{N}_0(E, \hat{F})^2\ \alpha' = \tilde{\mathcal{C}}_0(E)\ \tilde{\mathcal{N}}_0(E)^2\ \frac{\alpha'}{2} = 1 \tag{131}$$

allow to fix the normalization.

7 T-duality action on vertices.

In this section we would like to show that all the previous amplitudes for a theory of $N = N_1 L$ wrapped $D25$ branes with magnetic field $\hat{F}_{12} = \frac{f}{L}$ which breaks the gauge group to $U(N_1)$ can be obtained by a T-duality transformation of the amplitudes for a theory of N_1 $D25$ branes with vanishing magnetic field.

In doing so we prove the equivalence of the two theories also when gravitational interactions are considered. This happens because the phases depend on momenta only and we can use sewing techniques to argue that all amplitudes are invariant once we we have shown that pure open string amplitudes and mixed amplitudes with one closed string are invariant.

We can always choose the abelian field strength block diagonal in space-time and, because of that, in the previous sections we have considered the case where

[12] In [19] the open string normalization was $\mathcal{C}_0(E, \hat{F})\ \mathcal{N}_0(E, \hat{F})^2\ \alpha'$ because the gauge generators were normalized as $tr(T_u T_v) = \frac{1}{2}\delta_{uv}$ while here they are normalized as $tr(T_u T_v) = \delta_{uv}$.

the magnetic field is turned on only in two directions within a non factorized torus. We consider therefore only T-duality transformations which act on x^1 and x^2 whose non trivial part of the Λ matrix is given by

$$\Lambda \supset \begin{pmatrix} \tilde{f}\mathbb{I}_2 & \hat{h}\epsilon \\ -f\epsilon & L\mathbb{I}_2 \end{pmatrix} \in O(2,2,\mathbb{Z}) \quad L\tilde{f} - f\hat{h} = 1 \quad f\tilde{f} \in 2\mathbb{Z} \tag{132}$$

This T-duality transformation acts on zero modes as

$$\begin{pmatrix} m^1 \\ n_2 \end{pmatrix} = \begin{pmatrix} L & -\hat{h} \\ -f & \tilde{f} \end{pmatrix} \begin{pmatrix} m^{t1} \\ n^t_2 \end{pmatrix} \quad \begin{pmatrix} m^2 \\ n_1 \end{pmatrix} = \begin{pmatrix} L & \hat{h} \\ f & \tilde{f} \end{pmatrix} \begin{pmatrix} m^{t2} \\ n^t_1 \end{pmatrix} \tag{133}$$

The previous T-duality can be understood as the result of a T-duality, followed by a rotation and then by another T-duality as shown in eq. (3) therefore the closed string vertices must be transformed in order to have invariant amplitudes as shown in ([16], [3]) as

$$\mathcal{W}^{(c)}_{\beta_L,\beta_R}(n,m;[M_{c0}]) \to e^{i\pi m^2 n_2}\ e^{i\pi m^{t2} n^t_2}\ \mathcal{W}^{(c)}_{\beta^t_L,\beta^t_R}(n^t,m^t;[M_{c0}]) \tag{134}$$

where we have written the explicit dependence on the cocycle through the equivalence class of the matrix M_{c0} (93). It is then possible to check that the phase $e^{i\Phi_{(c)}(k,l)}$ in eq. (83) is the same of the one obtained by computing the product

$$e^{i\pi m^2_k n_{k2}}\ e^{i\pi m^{t2}_k n^t_{2k}} \mathcal{W}^{(c)}_{\beta^t_L,\beta^t_R}(k^t_L,k^t_R;[M_{c0}]) \times e^{i\pi m^2_l n_{l2}}\ e^{i\pi m^{t2}_l n^t_{2l}} \mathcal{W}^{(c)}_{\alpha^t_L,\alpha^t_R}(l^t_L,l^t_R;[M_{c0}]) \tag{135}$$

upon the use of the transformations (133). Notice that the same result holds if we perform the T-dualities along the x-axes.

We can now consider what happens in the open sector. In [1] we discussed how the T-duality transformations can be implemented in open string formalism. The starting point was to impose the same transformations in eq. (134) on the closed string vertices written in open string formalism, i.e.

$$\mathcal{W}_{\beta_L,\beta_R}(n,m;[M_{c0}]) \to e^{i\pi m^2 n_2}\ e^{i\pi m^{t2} n^t_2}\ \mathcal{W}_{\beta^t_L,\beta^t_R}(n^t,m^t;[M_{c0}]) \tag{136}$$

where the closed string vertices are the generalization of the one in eq. (96) to an arbitrary closed string state (β_L, β_R). In our particular case the first vertex has winding dependent "Chan-Paton" factors while the second has not (since we have only a stack of branes with equal Wilson lines). The results of the discussion in [1] can be summarized in the following transformation rules for the open string quantities

$$\begin{aligned} k^t &= T^{-T}(F)\ k & \theta &= T^T(F)\ \theta^t \\ \mathcal{G} &= T^T(F)\ \mathcal{G}^t\ T(F) & \Theta^t &= T(F)\ \Theta\ T^T(F) + \hat{\mathcal{B}} T^T(F) \\ \hat{X}_{L(0)}(z) &= \hat{X}^t_{L(0)}(z) & \hat{X}_{R(0)}(\bar{z}) &= \hat{X}^t_{R(0)}(\bar{z}) \end{aligned} \tag{137}$$

where we have defined $T(F) = \hat{\mathcal{A}} + \hat{\mathcal{B}}\hat{F}$. In the case at hand where $\hat{F}^t = 0$ it was shown that

$$T(F) = \hat{\mathcal{D}}^{-T} = L^{-1}\mathbb{I} \tag{138}$$

We are now in the position of showing that the product of two open string vertices in eq. (99,100) is invariant. This amounts to show that

$$\begin{aligned} k^T \mathcal{G}^{-1} l &= k^{tT} \mathcal{G}^{t-1} l^t \\ e^{-i\pi\alpha' k^T (\Theta + \Theta_{CP}) l} &= e^{-i\pi\alpha' k^{tT} \Theta^t l^t} \end{aligned} \tag{139}$$

This first equation is trivially satisfied. We can now use the explicit expression for $\hat{\mathcal{B}} = \hat{h}\epsilon$ and $\Theta_{CP} = L\hat{h}\epsilon$ to explicitly show that the second one holds:

$$\Theta^t = T\ (\Theta + T^{-1}\hat{h}\epsilon)\ T^T = T\ (\Theta + \Theta_{CP})\ T^T \tag{140}$$

Next we can consider the N open tachyons - 1 closed tachyon amplitude given in eq. (116). To show that it is invariant we notice that the previous closed string vertex transformation (136) can be written as

$$\mathcal{W}_{\beta_L,\beta_R}(n, m; [M_{c0}]) \to e^{i\pi\hat{h} f (m^1 n_1 + m^2 n_2)}\ e^{i\pi\hat{h} L n_1 n_2}\ \mathcal{W}_{\beta_L,\beta_R}(n^t, m^t; [M_{c0}]) \tag{141}$$

The sign in this expression reproduces exactly the one from the would-be closed string Chan-Paton in the second line of eq. (116). Momentum conservation and all products of two momenta can be easily checked to be invariant. It is then immediate to check that this amplitude is T-duality invariant when the normalization satisfies

$$\mathcal{C}_0(E, \hat{F})\ \mathcal{N}_0(E, \hat{F})^N\ \tilde{\mathcal{N}_0}(E) = \mathcal{C}_0(E^t, F^t)\ \mathcal{N}_0(E^t, F^t)^N\ \tilde{\mathcal{N}_0}(E^t) \tag{142}$$

but this is was verified in [1].

A Conventions.

- Indices:
  Compact $i, j, \dots = 1, \dots d$; non compact $\mu, \nu, \dots = 0, d+1 \dots D$; general $M, N, \dots = 0, \dots D$;
  Color $a, b, \dots$;
- $\delta^{[N]}_{m,n}$ means $m \equiv n \quad mod\ N$;
- Given a matrix Q, we use $[Q]_S$, $[Q]_A$ to mean respectively the symmetric and the antisymmetric part and $[Q]_>$ to denote the upper diagonal part, i.e. the matrix where we set $Q_{ij} = 0$ when $i < j$, and similarly for $[Q]_<$;
- 't Hooft matrices P_N and Q_N: $Q_N P_N = e^{-2\pi i \frac{1}{N}} P_N Q_N$.

- Background matrices:

$$
\begin{aligned}
E &= \parallel E_{ij} \parallel = G + B \\
\mathcal{E} &= \parallel \mathcal{E}_{ij} \parallel = E^T + 2\pi\alpha' q_0 F = G - \mathcal{B}
\end{aligned}
\tag{143}
$$

and

$$
\begin{aligned}
\hat{F} &= 2\pi\alpha' q_0 F \\
\mathcal{B} &= B - 2\pi\alpha' q_0 F = B - \hat{F} \\
\mathcal{E}^{-1} &= \mathcal{G}^{-1} - \Theta
\end{aligned}
\tag{144}
$$

from which we deduce that

$$
\begin{aligned}
&\mathcal{E}\mathcal{G}^{-1}\mathcal{E}^T = \mathcal{E}^T\mathcal{G}^{-1}\mathcal{E} = G \\
&\Theta = \frac{1}{2}\left(\mathcal{E}^{-T} - \mathcal{E}^{-1}\right) = -\mathcal{E}^{-1}\mathcal{B}\mathcal{E}^{-T}
\end{aligned}
\tag{145}
$$

Moreover we can extend all the previous quantities to both compact and non compact indices by setting:

$$
G_{i0} = B_{i0} = F_{i0} = 0. \tag{146}
$$

B Vertices and cocycles for dipole strings.

B.1 Useful formula.

Since we are using the open string formalism we define the logarithm as

$$
\ln(z-w) = \ln(z) - \sum_1^\infty \frac{1}{n}\left(\frac{w}{z}\right)^n \qquad |w| < |z|, 0 \le arg(z), arg(w) < \pi \tag{147}
$$

as suggested by the operatorial formalism and then we analytically continue it to the whole complex plane. Given the previous range for the arguments we find $(arg(\bar{z}) = -arg(z))$

$$
\begin{aligned}
\ln(z-\bar{w}) &= \ln(\bar{w}-z) + i\pi \\
\ln(z-w) &= \ln(w-z) + i\pi \; sgn\left(arg\left(\frac{z}{w}\right)\right) \\
&\Rightarrow \ln(\bar{z}-\bar{w}) = \ln(\bar{w}-\bar{z}) + i\pi \; sgn\left(arg\left(\frac{\bar{z}}{\bar{w}}\right)\right) = \ln(\bar{w}-\bar{z}) - i\pi \; sgn\left(arg\left(\frac{z}{w}\right)\right)
\end{aligned}
\tag{148}
$$

but if we write $y = |y|e^{i\pi}$ then we are out of our range and we actually find

$$
\ln(y-\bar{w}) = \ln(\bar{w}-y) + i\pi \tag{149}
$$

In the following we need also the following expressions. If we fix $z = |z|e^{i\zeta}$ with $0 < \zeta < \pi$, we get for $x > 0 > y$

$$\begin{array}{lll} \ln(x-z) = \ln|x-z| + i\psi \quad , & \ln(x-\bar{z}) = \ln|x-z| - i\psi, & -\pi + \zeta < \psi < 0 \\ \ln(y-z) = \ln|y-z| + i\psi \quad , & \ln(y-\bar{z}) = \ln|y-z| + i(2\pi - \psi), & \pi < \psi < \pi + \zeta \end{array} \tag{150}$$

The ranges can be obtained by comparing the $x, |y| \to \infty$ and $x = |y| = |z|$ values with the definition given in eq. (147) which is valid exactly for $x, |y| > |z|$. The expression for $\ln(y - \bar{z})$ is obtained since $\ln(y - \bar{z}) = i2\pi + (\ln(y - z))^*$ because $y = |y|e^{i\pi} = y^* e^{2i\pi}$.

B.2 Cocycles for closed string vertexes in presence of a B background.

Before computing the closed vertices OPEs in open string formalism we need to determine the closed string cocycles in closed string formalism so that any two closed string vertices commute and then compute their OPEs which we must reproduce in open string formalism. The compact part of the closed string vertex for the emission of a state with right (left) quantum number β_L (β_R) we consider is of the form

$$\mathcal{W}_{\beta_L\beta_R}(z,\bar{z};k_L,k_R) = c(k_L,k_R;p)W_{\beta_L\beta_R}(z,\bar{z};k_L,k_R) \tag{151}$$

$$c(k_L,k_R;p) = e^{i\pi\left(n_i A_c^{ij}\hat{n}_j + n_i B_c^{i}{}_{j}\hat{m}^j + m^i C_c{}_{i}^{\ j}\hat{n}_j + m^i D_{c\ ij}\hat{m}^j\right)} \tag{152}$$

where $W_{\beta_L\beta_R}(z,\bar{z};k_L,k_R) = V_{\beta_L}(z,k_L)\,\tilde{V}_{\beta_R}(\bar{z},k_R)$ is the usual closed string vertex without cocycle, n, m are the momentum and the winding associated to k_L, k_R and $\hat{n}$, $\hat{m}$ are the operator given by

$$\hat{n} = \parallel \hat{n}_i \parallel = \sqrt{\alpha'}\left(Ep_L + E^T p_R\right) \qquad \hat{m} = \parallel \hat{m}^i \parallel = \sqrt{\alpha'}\left(p_L - p_R\right)$$

and we want to fix the constant matrices A_c, B_c, C_c, D_c.

To compute these matrices we compare the analytic continuation from $|z| > |w|$ to $|z| < |w|$ of the expression $[\mathcal{W}(z,\bar{z};k_L,k_R)\mathcal{W}(w,\bar{w};l_L,l_R)]_{an.cont}$ with the other expression $\mathcal{W}(w,\bar{w};l_L,l_R)\mathcal{W}(z,\bar{z};k_L,k_R)$:

$$\begin{aligned} \mathcal{W}(z,\bar{z};k_L,k_R)\mathcal{W}(w,\bar{w};l_L,l_R) &= e^{i\Phi_c(k,l)}c(k_L+l_L,k_R+l_R;p) \\ &\qquad V_L(z,k_L)V_L(w,l_L)\,V_R(\bar{z},k_R)V_R(\bar{w},l_R) \qquad |z| > |w| \end{aligned}$$

with

$$\begin{aligned} e^{i\Phi_c(k,l)} &= e^{-i\pi\left[n_l^T A_c n_k + n_l^T B_c m_k + m_l^T C_c n_k + m_l^T D_c m_k\right]} \\ &= e^{-i\pi\alpha'\left[l_L^T G^{-1}(E^T A_c E + E^T B_c + C_c E + D_c)G^{-1}k_L + l_R^T G^{-1}(E A_c E^T - E B_c - C_c E^T + D_c)G^{-1}k_R\right]} \\ &\quad \times e^{-i\pi\alpha'\left[l_L^T G^{-1}(E^T A_c E^T - E^T B_c + C_c E^T - D_c)G^{-1}k_R + l_R^T G^{-1}(E A_c E + E B_c - C_c E - D_c)G^{-1}k_L\right]} \end{aligned} \tag{153}$$

where we have used $k_L = \frac{1}{2\sqrt{\alpha'}}\left(n_k + E^T m_k\right)$, $k_R = \frac{1}{2\sqrt{\alpha'}}\left(n_k - E m_k\right)$ and similarly for l. This expressions shows that the matrices A_c and so on are defined up to arbitrary even integer valued matrices, i.e. for example $A_c \equiv A_c + 2Z_A$ with Z_A an arbitrary integer valued matrix.

Now we commute the vertexes and we make use of $[\ln(z-w)]_{an.cont} = ln(w-z) + i\pi S$ and $[\ln(\bar z - \bar w)]_{an.cont} = ln(\bar w - \bar z) - i\pi S$ with $S \in 2\mathbb{Z}+1$[13] where the opposite sign is due to the fact that $\ln(\bar z - \bar w) = \overline{\ln(z-w)}$ as follows from the operatorial method where $\ln(z-w) = \ln(z) - \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{w}{z}\right)^n$ and $\ln(\bar z) = \overline{\ln(z)}$, hence the equation we need solving is

$$e^{i\Phi_c(k,l)} e^{i2\pi\alpha' S(k_L^T G^{-1} l_L - k_R^T G^{-1} l_R)} = e^{i\Phi_c(l,k)} \tag{154}$$

The previous expression can be rewritten as

$$\begin{aligned} e^{-i\pi(n_l^T(A_c - A_c^T)n_k + n_l^T(B_c - C_c^T - S\mathbb{I})m_k + m_l^T(C_c - B_c^T - S\mathbb{I})n_k + m_l^T(D_c - D_c^T)m_k)} = 1 \Rightarrow \\ n_l^T(A_c - A_c^T)n_k + n_l^T(B_c - C_c^T - S\mathbb{I})m_k + m_l^T(C_c - B_c^T - S\mathbb{I})n_k + m_l^T(D_c - D_c^T)m_k \equiv 0 \quad mod\ 2 \end{aligned} \tag{155}$$

and immediately solved as

$$\begin{aligned} A_c &= [A_c]_S + Z_{A\ A} \quad [A_c]_A \equiv Z_{A\ A} = -Z_{A\ A}^T \\ D_c &= [D_c]_S + Z_{D\ A} \quad [D_c]_A \equiv Z_{D\ A} = -Z_{D\ A}^T \\ C_c &= B_c^T - \mathbb{I} + 2Z_C \end{aligned} \tag{156}$$

by choosing respectively $n_{k,l} \neq 0, m_{k,l} = 0$, $m_{k,l} \neq 0, n_{k,l} = 0$, and $n_k, m_l \neq 0, m_k = n_l = 0$ where $[A_c]_S, [D_c]_S$ are arbitrary symmetric complex matrices, B_c is an arbitrary complex matrix, $Z_{A\ A}, Z_{D\ A}$ are arbitrary antisymmetric integer valued matrices and Z_C is an arbitrary integer valued matrix.

There is actually another constraint. It comes from the request of having hermitian vertices. If we compute using the vertex in eq. (151) and we suppose A_c, B_c, C_c and D_c are real , we get

$$\begin{aligned} (\mathcal{W}_{\beta_L\beta_R}(z,\bar z; k_L, k_R))^\dagger &= \frac{1}{|z|^4} W_{\beta_L\beta_R}(\frac{1}{\bar z}, \frac{1}{z}; -k_L, -k_R) c(-k_L, -k_R; p) \\ &= e^{i\Phi_c(k,k)} \frac{1}{|z|^4} \mathcal{W}_{\beta_L\beta_R}(\frac{1}{\bar z}, \frac{1}{z}; -k_L, -k_R) \end{aligned} \tag{157}$$

hence we get the constraint

$$e^{i\Phi_c(k,k)} = e^{-i\pi n^T [A_c]_S n}\, e^{-i\pi m^T [D_c]_S m}\, e^{-i\pi m^T(2B_c^T - \mathbb{I} + 2Z_C)n} = 1 \tag{158}$$

which can be solved by setting $[A_c]_S = Z_{A0} + Z_{A0}^T$ i.e. choosing a symmetric matrix with even integer diagonal entries and arbitrary integer entries otherwise, $[D_c]_S =$

[13] We have that $S = 2(n_z - n_w) + sgn([\phi_z] - [\phi_w])$ with $arg(z) = [\phi_z] + 2\pi n_z$, $-\pi < [\phi_z] \leq \pi$.

$Z_{D0} + Z_{D0}^T$ and $B_c = \frac{1}{2}\mathbb{I} + Z_B$ where all these Z matrices are integer valued. The final solution for the matrices A_c, B_c, C_c and D_c is

$$\begin{aligned} A_c &= Z_{A\ A} + 2[Z_{A0}]_S + 2Z_A, \qquad Z_{A\ A}^T = -Z_{A\ A}, \\ D_c &= Z_{D\ A} + 2[Z_{D0}]_S + 2Z_D, \qquad Z_{D\ A}^T = -Z_{D\ A}, \\ B_c &= \frac{1}{2}\mathbb{I} + Z_B \\ C_c &= B_c^T - \mathbb{I} + 2Z_C = -\frac{1}{2}\mathbb{I} + Z_B^T + 2Z_C \end{aligned} \tag{159}$$

where all matrices are integer valued and all matrices A_c, B_c, C_c and D_c are in the same class of equivalence for any choice of the Z_A, Z_D and Z_C matrices, i.e. they yield the same phase (153).

Four points are worth noticing:

- a change in Z_B gives rise to a different phase;
- the previous equations (159) and (153) mean that all the entries are actually equivalent either to 0 or 1;
- we can always set $Z_{AA} = Z_{DA} = 0$ by choosing $Z_A = -[Z_{AA}]_<$ so that $A_c = [[Z_{AA}]_>]_S + 2[Z_{A0}]_S = [A_c]_S$ and this possible "gauge" is important for solving the conditions which allow the product of two closed string vertices in open string formalism to reproduce the same product in closed string formalism;
- more generally we can always choose Z_A so that $[A_c]_A = \hat{Z}_A$ where $\hat{Z}_A$ is an arbitrary antisymmetric integer valued matrix by choosing $Z_A = [\hat{Z}_A - Z_{AA}]_<$ and similarly for Z_D.

We could try to fix the undetermined matrices requiring the phase $e^{i\Phi_c(k,l)}$ to be invariant under T-duality, but from the transformation properties $m \to n$ and $n \to n$ we see that the best we can do is to set $A_c = D_c = C_c = 0 \mod 2$ and $B_c = 1 \mod 2$ which nevertheless does not give neither a T-duality invariant phase nor a proper hermitian conjugation. The simplest choice compatible with hermitian conjugation is

$$A_c = D_c \equiv 0, \quad B_c = -C_c \equiv \frac{1}{2}\mathbb{I} \qquad mod\ 2 \tag{160}$$

so that

$$\begin{aligned} e^{i\Phi_c(k,l)} &= e^{-i\frac{1}{2}\pi(n_l^T m_k - m_l^T n_k)} \\ &= e^{i\pi\alpha'\left[l_L^T G^{-1} B G^{-1} k_L + l_R^T G^{-1} B G^{-1} k_R + l_L^T G^{-1} k_R - l_R^T G^{-1} k_L\right]} \\ &\quad \times e^{-i2\pi\alpha'\left[l_L^T G^{-1}(E^T Z_A E + E^T Z_B + Z_C E + Z_D) G^{-1} k_L + l_R^T G^{-1}(E Z_A E^T - E Z_B - Z_C E^T + Z_D) G^{-1} k_R\right]} \\ &\quad \times e^{-i2\pi\alpha'\left[l_L^T G^{-1}(E^T Z_A E^T - E^T Z_B + Z_C E^T - Z_D) G^{-1} k_R + l_R^T G^{-1}(E Z_A E + E Z_B - Z_C E - Z_D) G^{-1} k_L\right]} \end{aligned} \tag{161}$$

As in the case $B = 0$ [9] the existence of different possible cocycles does not mean that we have different theories since we get an overall common phase to a given amplitudes independently of the genus of the Riemann surface they are computed on.

B.3 Dipole vertices.

In this subsection we work on $R \otimes S^{D-1}$ and we use the usual $X_{L(0)}(z)$ expansion which contains the commuting x_0^i instead of $x^i = x_0^i - \pi\alpha'\Theta^{il}\mathcal{G}_{lm}p^m$ as given in eq. (49) of the main text, i.e. in this subsection we use

$$X^i_{L(0)} = x_0^i - i2\alpha' p^i \ln(z) + n.z.m. = \hat{X}^i_{L(0)} + \pi\alpha'\Theta^{il}\mathcal{G}_{lm}p^m \tag{162}$$

with OPEs

$$\begin{aligned} X_{L(0)}(z)X^T_{L(0)}(w) &\sim -2\alpha' \ln(z-w)\mathcal{G}^{-1} \\ X_{L(0)}(z)X^T_{R(0)}(\bar{w}) &\sim -2\alpha' \ln(z-\bar{w})\mathcal{G}^{-1} \\ X_{R(0)}(\bar{z})X^T_{R(0)}(\bar{w}) &\sim -2\alpha' \ln(\bar{z}-\bar{w})\mathcal{G}^{-1} \end{aligned} \tag{163}$$

We expect therefore a non vanishing p dependent cocycle due to the shift by p in the relation between x and x_0.

We consider the bundle given in eqs. (40) and (42) where there are also Wilson lines in the part of the group without magnetic field so that the surviving group is $U(1)^N$ hence we have to take into account that the momenta do depend on the the Wilson lines as described in footnote 4, i.e. the spectrum of the momentum $k_{M;IJ}$ which can be obtained imposing the periodicity of the vertices as given in eq. (76) is given by

$$\sqrt{\alpha'} k_{i;IJ} = \frac{n_i}{L} + \theta_i^I - \theta_i^J \tag{164}$$

with $0 \leq \theta_i^I < \frac{1}{L}$ as discussed around eq.(36).

In order to simplify the notation we consider as in the main text the simplest non trivial case where $r = 2$, i.e. the magnetic field is turned on in $U(L) \subset U(L) \otimes U(N_1) \subset U(LN_1)$ therefore the color indices have two components e.g. $I = (I_1, I_2)$. Notice however that when $r = 2$ the momentum $k_{i;IJ}$ depends only on I_2 and J_2 and not on I_1, J_1 since their dependence cancels

Since all problems associated with phases are captured by tachyonic vertex operators we consider only the vertices for the emission of an open tachyon from $\sigma = 0$, $\sigma = \pi$ boundaries and of a closed string tachyon.in eq. (164). The most general form of these vertices is

$$
\begin{aligned}
\mathcal{V}_{(0)T}(x;k) &= t_{I_2J_2}(k)\ e^{i2\alpha'\left(k_{N;IJ}D_0^{NM}k_{M;IJ}+k_{M;IJ}(C_0\mathcal{G})^M_{\ N}p^N\right)}\ e^{ik_{M;IJ}H^M_{0.N}y_0^N} \\
&\quad \times :e^{ik_{M;IJ}X^M_{L(0)}(x)}: \\
&\quad \times\Lambda_{L;I_1J_1}(n_1,n_2)|(I_1,I_2)\rangle_{\sigma=0}\ {}_{\sigma=0}\langle(J_1,J_2)| \\
\mathcal{V}_{(\pi)T}(y;k) &= t_{J_2I_2}(k)\ e^{i2\alpha'\left(k_{M;IJ}D_\pi^{MN}k_{N;IJ}+k_{M;IJ}(C_\pi\mathcal{G})^M_{\ N}p^N\right)}\ e^{ik_{M;IJ}H^M_{\pi.N}y_0^N} \\
&\quad \times :e^{ik_{M;IJ}X^M_{L(0)}(y)}: \\
&\quad \times\Lambda_{L;J_1I_1}(n_1,n_2)|(I_1,I_2)\rangle_{\sigma=\pi}\ {}_{\sigma=\pi}\langle(J_1,J_2)| \\
\mathcal{W}_{T_c}(z,\bar z;k_L,k_R,y_0) &= t_{(c)}(k)\ e^{i\left(n_MA_o^{MN}n_N+m^MD_{o\ MN}m^N+m^MC_{o\ M}^{\ \ N}n_N\right)}\ e^{i\sqrt{\alpha'}\left(n_M(B_1\mathcal{G})^M_N+m^M(B_2\mathcal{G})_{MN}\right)p^N} \\
&\quad \times e^{i(n_M(I_1)^M_N+m^M(I_2)_{MN})\frac{y_0^N}{\sqrt{\alpha'}}}\ :e^{ik_{LM}(G^{-1}\mathcal{E})^M_{.N}X^N_{L(0)}(z)}:\ \ :e^{ik_{RM}(G^{-1}\mathcal{E}^T)^M_{.N}X^N_{R(0)}(\bar z)}: \\
&\quad \times\Lambda_{L;I_1J_1}(-fm_2,fm_1)\otimes\mathbb{I}_{I_2J_2}|(I_1,I_2),K\rangle\ \langle(J_1,I_2),K| \qquad (165)
\end{aligned}
$$

where $|(I_1,I_2)\rangle_{\sigma=0}$ (${}_{\sigma=0}\langle(I_1,I_2)|$) is the ket (bra) for the state with color (I_1,I_2) at $\sigma=0$, similarly for $|(I_1,I_2)\rangle_{\sigma=\pi}$ and $|(I_1,I_2),K\rangle=|(I_1,I_2)\rangle_{\sigma=0}\ |(K_1,K_2)\rangle_{\sigma=\pi}$. We have also used $x=|x|e^{i0}$, $y=|y|e^{i\pi}$.

It is worth stressing that the implicit sum over I_2 and J_2 which also label the momenta $k_M=\parallel k_{M;IJ}\parallel=\parallel k_{M;(I_1,I_2)(J_1,J_2)}\parallel$ gives the vertices a non trivial, i.e. non factorized in operatorial and color space, matricial structure.

These vertices contain cocycles which are determined by the constant matrices $C_{0,\pi}$, $D_{0,\pi}$, $H_{0,\pi}$ and A_o, D_o, C_o, $B_{1,2}$, $I_{1,2}$ which we want to determine in order to satisfy the requirements needed for a proper CFT as discussed in section 5.2[14]. For the non compact direction our conventions are

$$
k_0=\frac{1}{2}k_{L0}=\frac{1}{2}k_{R0}=\frac{n_0}{\sqrt{\alpha'}},\qquad m_0=0 \tag{168}
$$

along with the hypothesis that mixed compact - non compact components of all matrices vanish, as for example

$$
\parallel C_0^{\mu\nu}\parallel=\begin{pmatrix} C_0^{00} & \\ & C_0\end{pmatrix},\qquad C_0=\parallel C_0^{ij}\parallel \tag{169}
$$

[14] We could also have expressed the vertices using $\hat X$ (for the compact part of the closed string emission vertex) as

$$
\begin{aligned}
\mathcal{W}_{T_c,compact}(z,\bar z;k_L,k_R,y_0) &= e^{i\left(n_i\tilde A_o^{ij}n_j+m^i\tilde D_{o\ ij}m^j+m^i\tilde C_{o\ i}^{\ \cdot j}n_j\right)}e^{i\sqrt{\alpha'}\left(n_i(B_1\mathcal{G})^i_j+m^j(B_2\mathcal{G})_{ij}\right)p^j} \\
&\quad \times e^{i(n_i(\tilde I_1)^i_j+m^i(\tilde I_2)_{ij})\frac{y_0^j}{\sqrt{\alpha'}}}:e^{ik_{Li}\hat X^i_L(z)}::e^{ik_{Ri}\hat X^i_R(\bar z)}: \qquad (166)
\end{aligned}
$$

but this amounts to a redefinition of the matrices:

$$
\begin{aligned}
A_o=\tilde A_o,\qquad D_o &= \tilde D_o+[(I_2+\frac{1}{2})\theta G]_S,\qquad C_o=\tilde C_o-G\theta(I_1-\frac{1}{2}) \\
I_1=\tilde I_1,\qquad\qquad & I_2=\tilde I_2+G \qquad (167)
\end{aligned}
$$

where θ is defined in eq. (171).

and similarly for all the other matrices. We take also

$$D_o^{00} = C_o^{00} = 0. \tag{170}$$

In the following subsection we will also assume that

$$[y_0^i, y_0^j] = i \; 2\pi\alpha' \; \theta^{ij} \tag{171}$$

even if our aim is to stick as close as possible with the naive form of the vertices by choosing

$$D_0 = C_0 = H_0 = 0, \quad H_\pi = 0, \quad I_1 = 0, \quad I_2 = G, \quad \theta^{ij} = 0 \tag{172}$$

Generalizing previous results obtained in ([9],[20],[23]) to this case we assume also that y_0 gets shifted after the emission of an open string with momentum k from the $\sigma = 0$ boundary as

$$\Delta_0(k_{IJ}) y_0^i \;\; = \;\; 2\sqrt{\alpha'} \nu_0^{ij} \Theta_j(\sqrt{\alpha'} k_{IJ}) \tag{173}$$

where k_{IJ} is the momentum for a string starting at $\sigma = 0$ with color (I_1, I_2) and ending at $\sigma = \pi$ with color $(J_1 J_2)$ and we have defined

$$\sqrt{\alpha'} k_{i;IJ} = \frac{n_i}{L} + \theta_i^I - \theta_i^J \Rightarrow I_i(\sqrt{\alpha'} k_{IJ}) = \frac{n_i}{L} \delta_{I,J}, \quad \Theta_i(\sqrt{\alpha'} k_{IJ}) = \theta_i^I - \theta_i^J \tag{174}$$

For use in the computations we recall that the product of two momentum dependent Chan-Paton is

$$\Lambda_L(n_1, n_2) \Lambda_L(m_1, m_2) = e^{-i\pi\alpha' (\frac{n_1}{\sqrt{\alpha'} L}, \frac{n_2}{\sqrt{\alpha'} L}) \Theta_{CP} (\frac{m_1}{\sqrt{\alpha'} L}, \frac{m_2}{\sqrt{\alpha'} L})^T} \frac{1}{\sqrt{L}} \Lambda_L(n_1 + m_1, n_2 + m_2) \tag{175}$$

where we have defined for later convenience

$$\Theta_{CP} = L\hat{h}\epsilon = L\hat{h} \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix} \tag{176}$$

with $\hat{h} f \equiv -1 \quad mod \; L$.

B.4 Requirements for a proper CFT.

In order to have a well defined CFT we want all operators to be relatively local, i.e. we do not want them to create cuts in the surface where they are inserted therefore we check the following properties

1. The open string vertices on the opposite boundaries commute;
2. The open string emission vertex from $\sigma = 0$ commutes with the closed string vertex;

3. In a similar way the open string emission vertex from $\sigma = \pi$ commutes with the closed string vertex;

4. The product of two closed string vertices in open string formalism must reproduce the result in the closed string formalism.

5. Proper behavior under Hermitian conjugation.

In details the constraints listed above become:

1. The open string vertices on the opposite boundaries commute:

$$\begin{aligned} &\left[\mathcal{V}_{(0)T}(x;k,y_0+\Delta_\pi(l)y_0)\ \mathcal{V}_{(\pi)T}(y;l,y_0)\right]_{an.\ cont.} \\ &\quad = \mathcal{V}_{(\pi)T}(y;l,y_0+\Delta_0(k)y_0)\ \mathcal{V}_{(0)T}(x;k,y_0) \end{aligned} \tag{177}$$

where *an. cont.* means analytically continued from the region $|x| > |y!$ to $|x| < |y|$ and we have introduced a shift in the y_0 as in eq. (173)

$$\Delta_0(k)y_0 = 2\alpha'\nu_0\Theta(k) \tag{178}$$

and we have also introduced for generality the analogous shifts in the y_0 for the emission from the $\sigma = \pi$ boundary:

$$\Delta_\pi(l)y_0 = 2\alpha'\nu_\pi\Theta(l). \tag{179}$$

even if it can be argued to be zero.

The constraint we get is **independent** of the Chan-Paton factors since when we commute the two vertices on the two borders we do not change the matrix ordering but only the radial ordering. If we denote the $\sigma = 0$ indices as I and J and $\hat{I}$ and $\hat{J}$ the $\sigma = \pi$ ones we get the constraint

$$\begin{aligned} & e^{i2\alpha' k_0\left(-C_\pi^T+C_0-\pi\mathcal{G}^{-1}\right)^{00} l_0} \\ \times\quad & e^{i2\alpha' k_{IJ}^T\ H_0\nu_\pi\ \Theta(l_{\hat{I}\hat{J}})} \\ \times\quad & e^{-i2\alpha' k_{IJ}^T\ C_\pi^T\ l_{\hat{I}\hat{J}}} \\ \times\quad & e^{-i2\pi\alpha' k_{IJ}^T\ H_0\theta H_\pi^T\ l_{\hat{I}\hat{J}}} \\ \times\quad & e^{-i2\pi\alpha' k_{IJ}^T\ \mathcal{G}\ l_{\hat{I}\hat{J}}} e^{i2\alpha' k_{IJ}^T\ C_0^T\ l_{\hat{I}\hat{J}}} \\ =\quad & e^{i2\alpha' l_{\hat{I}\hat{J}}^T\ H_\pi\nu_0\ \Theta(k_{IJ})} \end{aligned} \tag{180}$$

where the second contribution is from the y_0 shift, the third one from the p cocycle, the fourth one from the y_0 cocycle, and the fifth from the usual operatorial vertex. The previous expression can be rewritten as

$$e^{i2\alpha'\left[k_{IJ}^T\ \left(-C_\pi^T+C_0-\pi\mathcal{G}^{-1}-\pi H_0\theta H_\pi^T\right)\ l_{\hat{I}\hat{J}}+k_{IJ}^T\ H_0\nu_\pi\ \Theta(l_{\hat{I}\hat{J}})-\Theta^T(k_{IJ})\ \nu_0^T H_\pi^T\ l_{\hat{I}\hat{J}}\right]} = 1 \tag{181}$$

for the compact directions and for the time direction as

$$e^{i2\alpha' k_0\left(-C_\pi^{00}+C_0^{00}-\pi G^{00}\right)^{00} l_0} = 1. \tag{182}$$

2. The open string emission vertex from $\sigma = 0$ commutes with the closed string vertex:

$$\begin{aligned}&\left[\mathcal{V}_{(0)T}(x;k,y_0)\ \mathcal{W}_{T_c}(z,\bar{z};k_L,k_R,y_0)\right]_{an.\ cont.}\\&\quad= \mathcal{W}_{T_c}(z,\bar{z};k_L,k_R,y_0+\Delta_0(k)y_0)\ \mathcal{V}_{(0)T}(x;k,y_0)\end{aligned} \tag{183}$$

where the emission of a closed string does not change the "carrier" string.

As before we denote the $\sigma = 0$ indices as I and J then we find the constraint for the compact directions

$$\begin{aligned}
&e^{-2i\pi\ I(\sqrt{\alpha'}k_{IJ})^T\ \Theta_{CP}\ (-1)\hat{F}m}\\
&e^{-i\sqrt{\alpha'}\ k_{IJ}^T\ (B_1^T n+B_2^T m)}\\
&e^{-i2\pi\sqrt{\alpha'}\ k_{IJ}^T\ H_0\theta\ (I_1^T n+I_2^T m)}\\
&e^{-i2\alpha'\pi\ k_{IJ}^T\ \mathcal{G}^{-1}\mathcal{E}^T G^{-1}\ k_L}\ e^{i2\alpha'\ k_{IJ}^T\ C_0\mathcal{E}^T G^{-1}\ k_L}\\
&e^{i2\alpha'\pi\ k_{IJ}^T\ \mathcal{G}^{-1}\mathcal{E}G^{-1}\ k_R}\ e^{i2\alpha' k_{IJ}^T\ C_0\mathcal{E}G^{-1}\ k_R}\\
=\ &e^{i(n^T I_1+m^T I_2)\frac{1}{\sqrt{\alpha'}}\Delta_0(k_{IJ})y_0}
\end{aligned} \tag{184}$$

where the first line is due to the commuting of the Chan-Paton factors[15], the second one to the commuting of the p dependent closed string cocycle, the third one to $X_L(z)$, the fourth one to $X_R(\bar{z})$ and the last one to the shift in y_0 induced by the emission of an open string.

This constraint can be rewritten as

$$\begin{aligned}
&e^{i\sqrt{\alpha'}\ k_{IJ}^T\left(-B_1^T+2\pi\Theta+2C_0-2\pi H_0\theta I_1^T-2\nu_0^T I_1^T\right)n}\ e^{i\sqrt{\alpha'}\ \Theta(k_{IJ})^T\left(-2\nu_0^T I_1^T\right)n}\\
&e^{i\sqrt{\alpha'}k_{IJ}^T\left(-B^T-2\pi\mathbb{I}-2\pi\Theta\hat{F}-2C_0\hat{F}-2\pi H_0\theta I_2^T\right)m}\ e^{i\sqrt{\alpha'}\ \Theta(k_{IJ})^T\left(-2\nu_0^T I_2^T\right)m}\\
&e^{i\sqrt{\alpha'}I(k_{IJ})^T\left(2\pi\Theta_{CP}\hat{F}\right)m} = 1
\end{aligned} \tag{185}$$

where we have used $\mathcal{E}^{-T} = G^{-1}\mathcal{E}\mathcal{G}^{-1}$, $\mathcal{E}^{-1} = G^{-1}\mathcal{E}^T\mathcal{G}^{-1}$ and

$$\begin{aligned}
\mathcal{E}^{-1}E^T + \mathcal{E}^{-T}E &= 2\mathbb{I} + 2\Theta\hat{F}\\
\mathcal{E}^{-1}E^T - \mathcal{E}^{-T}E &= -2\mathcal{G}^{-1}\hat{F}
\end{aligned} \tag{186}$$

For the non compact direction the constraint in eq. (185) becomes

$$e^{i\alpha' k_0(-B_1^0+2C_0^{00})k_0^{(c)}} = 1 \tag{187}$$

3. In a similar way the open string emission vertex from $\sigma = \pi$ commutes with the closed string vertex:

$$\begin{aligned}&\left[\mathcal{V}_{(\pi)T}(y;k,y_0)\mathcal{W}_{T_c}(z,\bar{z};k_L,k_R,y_0)\right]_{an.\ cont.}\\&\quad= \mathcal{W}_{T_c}(z,\bar{z};k_L,k_R,y_0+\Delta_\pi(k)y_0)\ \mathcal{V}_{(\pi)T}(y;k,y_0)\end{aligned} \tag{188}$$

[15] We have written $I(\sqrt{\alpha'}k_{IJ})$ since the open Chan-Paton matrices $\Lambda(k)$ in eq.(57) depend on the "integer" part of k_{IJ} given in eq. (164).

We find therefore the constraint for the compact directions

$$
\begin{aligned}
& e^{-i\sqrt{\alpha'}\ k^T_{\hat{I}\hat{J}}\ (B_1^T n + B_2^T n)} \\
& e^{-i2\pi\sqrt{\alpha'}\ k^T_{\hat{I}\hat{J}}\ H_\pi \theta\ (I_1^T n + I_2^T m)} \\
& e^{i2\alpha'\pi\ k^T_{\hat{I}\hat{J}}\ \mathcal{G}^{-1}\mathcal{E}^T G^{-1}\ k_L} e^{i2\alpha'\ k^T_{\hat{I}\hat{J}}\ C_\pi \mathcal{E}^T G^{-1}\ k_L} \\
& e^{i2\alpha'\pi\ k^T_{\hat{I}\hat{J}}\ \mathcal{G}^{-1}\mathcal{E} G^{-1}\ k_R} e^{i2\alpha'\ k^T_{\hat{I}\hat{J}}\ C_\pi \mathcal{E} G^{-1}\ k_R} \\
= \ & e^{i(n^T I_1 + m^T I_2)\frac{1}{\sqrt{\alpha'}}\Delta_\pi(k_{\hat{I}\hat{J}}) y_0}
\end{aligned}
\tag{189}
$$

It is worth noticing that the previous expression is independent on Θ_{CP} since the closed string vertex has a "Chan-Paton factor" which acts on $\sigma = 0$. In performing this computation it is also necessary to be careful in computing the phases since $y = |y| e^{i\pi}$ and $y^* = e^{-2i\pi} y$, explicitly we have

$$
\ln(y - z) = \ln(z - y) + i\pi, \qquad \ln(y - \bar{z}) = \ln(\bar{z} - y) + i\pi \tag{190}
$$

which can be rewritten as

$$
\begin{aligned}
& e^{i\sqrt{\alpha'}\ k^T_{\hat{I}\hat{J}}\ \left(-B_1^T + 2\pi\mathcal{G}^{-1} + 2C_\pi - 2\pi H_\pi \theta I_1^T - 2\nu_\pi^T I_1^T\right)\ n} \\
& e^{i\sqrt{\alpha'}\ k^T_{\hat{I}\hat{J}}\ \left(-B_2^T - 2\pi\mathcal{G}^{-1}\hat{F} - 2C_\pi \hat{F} - 2\pi H_\pi \theta I_2^T - 2\nu_\pi^T I_2^T\right)\ m} = 1
\end{aligned}
\tag{191}
$$

while the time constraint reads

$$
e^{i\alpha'\ k_0\ (-B_1^0 + 2\pi G^{00} + 2C_\pi^{00})\ k_0^{(c)}} = 1 \tag{192}
$$

4. The product of two closed string vertices in open string formalism must reproduce the result in the closed string formalism. An easy computation yields

$$
\begin{aligned}
\mathcal{W}_{T_c}(z, \bar{z}; k_L, k_R, y_0) \ \times \ & \mathcal{W}_{T_c}(w, \bar{w}; l_L, l_R, y_0) \\
= \ & e^{i\Phi_o(k,l)}\ (z - w)^{2\alpha'\ k_L^T G^{-1} l_L} (\bar{z} - \bar{w})^{2\alpha'\ k_R^T G^{-1} l_R} \\
& \times N_o \left[\mathcal{W}_{T_c}(z, \bar{z}; y_0) \mathcal{W}_{T_c}(w, \bar{w}; y_0)\right]
\end{aligned}
\tag{193}
$$

In this expression the phase is given by

$$
\begin{aligned}
e^{i\Phi_o(k,l)} = \ & e^{i\left[-2n_l^T A_o n_k - 2m_l^T D_o m_k - m_l^T C_o n_k - n_l^T C_o^T m_k\right]} \\
& e^{-i\sqrt{\alpha'}\left(k_L^T G^{-1}\mathcal{E} + k_R^T G^{-1}\mathcal{E}^T\right)\left(B_1^T n_l + B_2^T m_l\right)} \\
& e^{-i\pi\left(n_k^T I_1 + m_k^T I_2\right)\theta\left(I_1^T n_l + I_2^T m_l\right)} \\
& e^{-i\pi 2\alpha' k_R^T \mathcal{E}^{-1}\mathcal{E}^T G^{-1} l_L} \\
& e^{-i\pi m_k^T \hat{F}^T \Theta_{CP} \hat{F} m_l} \\
& e^{i\alpha' l_0(-2A_0^{00} - B_1^{00} - \frac{\pi}{2} G^{00}) k_0}
\end{aligned}
\tag{194}
$$

where the first line is due to the numerical cocycles, the second one to the p dependent cocycle, the third one to the y_0 dependent cocycle, the fourth one to the commuting of $X_{(0)L}(w)$ with $X_{(0)R}(\bar{z})$ and the last but one to the "Chan-Paton" factors.

The open string equivalent of the closed string normal ordering $:\mathcal{W}_{T_c}(z,\bar{z})\mathcal{W}_{T_c}(w,\bar{w}):$ is then given by

$$\begin{aligned} N_o\left[\mathcal{W}_{T_c}(z,\bar{z})\mathcal{W}_{T_c}(w,\bar{w})\right] &= e^{i\left((n_k+n_m)^T A_o (n_k+n_m)+(m_k+m_l)^T D_o (m_k+m_l)+(m_k+m_l)^T C_o\ (n_k+n_l)\right)} \\ & \quad e^{\sqrt{\alpha'}\left((n_k+n_l)^T B_1+(m_k+m_l)^T B_2\right)\mathcal{G}p}\ e^{i\left((n_k+n_m)^T I_1+(m_k+m_l)^T I_2\right)\frac{y_0}{\sqrt{\alpha'}}} \\ & \quad : e^{ik_L^T G^{-1}\mathcal{E}X_{L(0)}(z)+il_L^T G^{-1}\mathcal{E}X_{L(0)}(w)} : \\ & \quad : e^{ik_R G^{-1}\mathcal{E}^T X_{R(0)}(\bar{z})+il_R G^{-1}\mathcal{E}^T X_{R(0)}(\bar{w})} : \Lambda_L(-L\ \hat{F}(m_k+m_l)) \end{aligned} \tag{195}$$

We can now explicitly write down the constraint by equating the phases given in eq. (194) and eq. (153)

$$\begin{aligned} e^{i\Phi_c(k,l)} &= e^{i\Phi_o(k,l)} \\ &= e^{in_l^T\left(-2A_o+\pi I_1\theta I_1^T-\frac{\pi}{2}\mathcal{E}^{-T}\mathcal{G}\mathcal{E}^{-T}-B_1\right)n_k} \\ & \quad e^{in_l^T\left(-C_o^T+\pi I_1\theta I_2^T+\frac{\pi}{2}\mathcal{E}^{-T}\mathcal{G}\mathcal{E}^{-T}E+B_1\hat{F}\right)m_k} \\ & \quad e^{im_l^T\left(-C_o+\pi I_2\theta I_1^T-\frac{\pi}{2}E\mathcal{E}^{-T}\mathcal{G}\mathcal{E}^{-T}-B_2\right)n_k} \\ & \quad e^{im_l^T\left(-2D_o+\pi I_2\theta I_2^T+\frac{\pi}{2}E\mathcal{E}^{-T}\mathcal{G}\mathcal{E}^{-T}E+B_2\hat{F}-\pi\hat{F}^T\Theta_{CP}\hat{F}\right)m_k} \end{aligned} \tag{196}$$

along with the constraint for the non compact direction

$$e^{i\alpha' k_0(-B_1^0+2\pi G^{00}+2C_\pi^{00})k_0^{(c)}} = 1 \tag{197}$$

5. Proper behavior under Hermitian conjugation.

   We start computing the Hermitian of the open string vertices under the assumption that all Chan-Paton factors are *Hermitian*[16]

$$[\Lambda_{CP}(k)]^\dagger = \Lambda_{CP}(-k) \tag{198}$$

[16] For example in the case of twisted bundle with degeneration this is not true and we have $[\Lambda_{CP}(k)]^\dagger = e^{ih\cdot k}\Lambda_{CP}(-k)$ where h is a vector which does not depend on the momentum k. In this case it is not possible to redefine Λ_{CP} to reabsorb the phase $e^{ih\cdot k}$ which shows also up in the Zamolodchikov metric. This phase cannot hence be removed from the Hermitian conjugate of a vertex, i.e. $\left[\mathcal{V}_{(0)T}(x;k)\right]^\dagger \propto e^{ih\cdot k}\mathcal{V}_{(0)T}(\frac{1}{x^*};-k)$.

then we easily get

$$\begin{aligned}
\left[\mathcal{V}_{(0)T}(x;k)\right]^{\dagger} &= e^{-i2\alpha'\ k^T(D_0+D_0^{\dagger}+C_0^{\dagger})k} e^{i2\alpha'\ k^T(C_0-C_0^*)\mathcal{G}p} \\
&\quad \times e^{-ik^T(H_0^*-H_0)y_0} e^{-i\sqrt{\alpha'}\pi k^T H_0^*\theta H_0^T k^T} \frac{\mathcal{V}_{(0)T}(\frac{1}{x^*};-k)}{x^{*\ 2\alpha' k^T\mathcal{G}^{-1}k}} \\
&\quad \Rightarrow C_0 = C_0^*, \qquad H_0 = H_0^*, \qquad \left[D_0 + D_0^{\dagger} + C_0^{\dagger}\right]_S = \pi n_0 \mathcal{G}^{-1} \\
\left[\mathcal{V}_{(\pi)T}(y;k)\right]^{\dagger} &= e^{-i2\alpha'\ k^T(D_\pi+D_\pi^{\dagger}+C_\pi^{\dagger})k} e^{i2\alpha'\ k^T(C_\pi-C_\pi^*)\mathcal{G}p} \\
&\quad \times e^{-ik^T(H_\pi^*-H_\pi)y_0} e^{-i\sqrt{\alpha'}\pi k^T H_\pi^*\theta H_\pi^T k^T} \frac{\mathcal{V}_{(\pi)T}(\frac{1}{y^*};-k)}{y^{*\ 2\alpha' k^T\mathcal{G}^{-1}k}} \\
&\quad \Rightarrow C_\pi = C_\pi^*, \qquad H_\pi = H_\pi^*, \qquad \left[D_\pi + D_\pi^{\dagger} + C_\pi^{\dagger}\right]_S = \pi n_\pi \mathcal{G}^{-1}
\end{aligned} \tag{199}$$

where we have used the fact that $\alpha' k^T \mathcal{G}^{-1} k \in \mathbb{Z}$, $[Q]_S$ means the symmetric part of the matrix Q and it is necessary to write y^* because the need to keep track of the phase due to the logarithm. Despite the fact that C_0 contains an antisymmetric part the symmetrization of the previous expression is such that fortunately we can satisfy the constraints with D_0 and D_π. This can be seen decomposing D_0 into the real and imaginary part as $D_0 = D_0^T = D_{0\ R} + iD_{0\ I}$, then the (Hermitian conjugate of the) constraint can be formally solved as

$$D_{0\ R} = -\frac{1}{2}\left[C_0\right]_S + \frac{1}{2}\pi n_0 \mathcal{G}^{-1} \tag{200}$$

and similarly for D_π:

$$D_{\pi\ R} = -\frac{1}{2}\left[C_\pi\right]_S = \frac{n_\pi + 1}{2}\pi\mathcal{G}^{-1} - \frac{1}{2}\left[C_0\right]_S \tag{201}$$

where we have already used eq. (209). In the following we choose

$$n_0 = 0, \qquad n_\pi = -1 \tag{202}$$

Once $n_0 = 0$ has been chosen then n_π is also fixed by the request of the cyclicity of amplitudes which, for example, requires that the four open strings amplitude can be computed either by fixing $x_4 = 0, x_2 = 1, x_1 = +\infty$ and integrating over $0 \leq x_3 \leq 1$ or by fixing $x_3 = 0, x_2 = 1, x_1 = +\infty$ and integrating over $x_4 \leq 0$.

We can now exam what happens to the closed string vertex where with the help of $arg\left(\frac{1}{z^*}\right) = arg(z)$ and $arg\left(\frac{1}{\bar{z}^*}\right) = arg(\bar{z})$ we find

$$
\begin{aligned}
[\mathcal{W}_{T_c}(z,\bar{z};k_L,k_R,y_0)]^\dagger &= e^{i\left[-n_k^T(A_o^*+A_o)n_k - m_k^T(D_o^*+D_o)m_k - m_k^T(C_o^*+C_o)n_k\right]} \\
&\quad e^{-i\sqrt{\alpha'}\left(k_L^T G^{-1}\mathcal{E}+k_R^T G^{-1}\mathcal{E}^T\right)\left(B_1^{*T}n_k+B_2^{*T}m_k\right)} e^{i\sqrt{\alpha'}[n^T(B_1-B_1^*)+m^T(B_2-B_2^*)]\mathcal{G}p} \\
&\quad e^{i(n^T(I_1-I_1^*)+m^T(I_2-I_2^*))\frac{y_0}{\sqrt{\alpha'}}} e^{-i\pi(n^T(I_1-I_1^*)+m^T(I_2-I_2^*))\theta(I_1^T n+I_2^T m)} \\
&\quad e^{-i\pi 2\alpha' k_R^T \mathcal{E}^{-1}\mathcal{E}^T G^{-1}k_L} \\
&\quad \times \frac{1}{z^{*\;2\alpha' k_L^T G^{-1}k_L}} \frac{1}{\bar{z}^{*\;2\alpha' k_R^T G^{-1}k_R}} \mathcal{W}_{T_c}\left(\frac{1}{\bar{z}^*},\frac{1}{z^*};-k_L,-k_R,y_0\right)
\end{aligned}
\tag{203}
$$

where the first contribution is due to the non operatorial cocycle, the second one to the p dependent cocycle, the third one to to the y_0 dependent cocycle and the last ones to the reordering of the naive vertices. From the operatorial part we get the constraints

$$
B_{1,2}^* = B_{1,2}, \qquad I_{1,2}^* = I_{1,2} \tag{204}
$$

and if we assume that

$$
A_o^* = A_o, \qquad D_o^* = D_o, \qquad C_o^* = C_o, \tag{205}
$$

we can immediately write

$$
\begin{aligned}
[\mathcal{W}_{T_c}(z,\bar{z};k_L,k_R,y_0)]^\dagger &= e^{i\Phi_o(k,k)} \\
&\quad \times \frac{1}{z^{*\;2\alpha' k_L^T G^{-1}k_L}} \frac{1}{\bar{z}^{*\;2\alpha' k_R^T G^{-1}k_R}} \mathcal{W}_{T_c}\left(\frac{1}{\bar{z}^*},\frac{1}{z^*};-k_L,-k_R,y_0\right)
\end{aligned}
\tag{206}
$$

Because of the constraint in eq. (196) and because $e^{i\Phi_c(k,k)} = 1$ as in eq. (158) we do not get any further constraints.

B.5 The solution of the constraints.

We now impose the constraints (172) to keep the solution as close as possible to the naive vertices.

B.5.1 Solving open-closed string constraints.

The constraints for the non compact direction can be easily solved as

$$
B_1^{00} = C_0^{00} = 0, \qquad C_o^{00} = G^{00}, \qquad A_o^{00} = -\frac{\pi}{4}G^{00} \tag{207}
$$

as the case of the trivial background.

Since we consider the general background given in eq. (42) with Wilson lines in the part of the group without magnetic fields we generically have that $\Theta(k) \neq 0$ therefore we get

$$\nu_0 = -\pi G^{-1} \tag{208}$$

in agreement with the naive expectation. Eq.s (185) and (191) together with the constraint (181) can be easily solved to yield

$$\begin{aligned} C_0 &= [C_0]_S - \frac{\pi}{2}\Theta \\ C_\pi &= [C_0]_S - \pi \mathcal{G}^{-1} + \frac{\pi}{2}\Theta \\ B_1 &= 2[C_0]_S - \pi\Theta \\ B_2 &= \hat{F}\ B_1 = \hat{F}\ (2[C_0]_S - \pi\Theta) \end{aligned} \tag{209}$$

A closer look to eq. (185) reveals that we are left with a term proportional $I(k)$ after using the previous equations. This term reads

$$\begin{aligned} & e^{i\sqrt{\alpha'}I(k)^T\left(-B_2^T - 2\pi\mathbb{I} - 2\pi\Theta\hat{F} - 2C_0\hat{F} - 2\pi H_0 \theta I_2^T\right)m}\ e^{i\sqrt{\alpha'}I(k)^T\left(2\pi\Theta_{CP}\hat{F}\right)m} \\ = \quad & e^{i\sqrt{\alpha'}I(k)^T\left(2\pi\Theta_{CP}\hat{F} + 2\nu_0^T I_2^T\right)m} = e^{i\frac{1}{L}n_k^T\left(2\pi\Theta_{CP}\hat{F} + 2\nu_0^T I_2^T\right)m} = 1 \end{aligned} \tag{210}$$

when we use eq. (176) we can write

$$\Theta_{CP}\hat{F} = (1 - \tilde{f}L)\mathbb{I} \tag{211}$$

and check that it does not imply any further constraint.

B.5.2 Consistency of the closed string constraints.

We can now write explicitly the constraint (196) using eq. (153) as

$$\begin{aligned} -\pi A_c &= -2A_o - \frac{\pi}{2}\mathcal{E}^{-T}\mathcal{G}\mathcal{E}^{-T} - B_1 \\ -\pi B_c &= -C_o^T + \frac{\pi}{2}\mathcal{E}^{-T}\mathcal{G}\mathcal{E}^{-T}E + B_1\hat{F} \\ -\pi C_c &= -C_o - \frac{\pi}{2}E\mathcal{E}^{-T}\mathcal{G}\mathcal{E}^{-T} - B_2 \\ -\pi D_c &= -2D_o + \frac{\pi}{2}E\mathcal{E}^{-T}\mathcal{G}\mathcal{E}^{-T}E + B_2\hat{F} - \pi\hat{F}\Theta_{CP}\hat{F} \end{aligned} \tag{212}$$

One could think these constraints just give the matrices A_o, D_o and C_o: this is not so because

$$A_o = A_o^T, \quad D_o = D_o^T, \tag{213}$$

moreover C_o enters two equations and we have to remember eq.s (86).

Because both A_o and D_o are symmetric we have to consider what happens to the antisymmetric part of the first and fourth equation; the first equation becomes

$$\begin{aligned} -\pi[A_c]_A &= -\pi(Z_{AA} + 2[Z_A]_A) \\ &= -\frac{\pi}{2}[\mathcal{E}^{-T}\mathcal{G}\mathcal{E}^{-T}]_A - [B_1]_A \\ &= -\pi\Theta - [B_1]_A = 0 \end{aligned} \tag{214}$$

where in the last line we have used eq. (209) from the previous section and

$$[\mathcal{E}^{-T}\mathcal{G}\mathcal{E}^{-T}]_A = \frac{1}{2}\left((\mathcal{E}^{-1} + 2\Theta)\mathcal{G}\mathcal{E}^{-T} - \mathcal{E}^{-1}\mathcal{G}\mathcal{E}^{-1}\right) = 2\Theta \tag{215}$$

This constraint is nothing else but a choice of "gauge": $[A_c]_A = 0$.

The antisymmetric part of the fourth equation can be now written as

$$\begin{aligned} -\pi[D_c]_A &= -\pi(Z_{DA} + 2[Z_D]_A) \\ &= +\frac{\pi}{2}[E\mathcal{E}^{-T}\mathcal{G}\mathcal{E}^{-T}E]_A + [B_2\hat{F}]_A - \pi\hat{F}\Theta_{CP}\hat{F} \\ &= +\pi\hat{F} + \pi\hat{F}\Theta\hat{F} + [B_2\hat{F}]_A - \pi\hat{F}\Theta_{CP}\hat{F} \\ &= -\pi\ L\tilde{f}\hat{F} \end{aligned} \tag{216}$$

where we have used eq. (211) and

$$[E\mathcal{E}^{-T}\mathcal{G}\mathcal{E}^{-T}E]_A = [(\mathcal{E}^T + \hat{F})\mathcal{E}^{-T}\mathcal{G}\mathcal{E}^{-T}(\mathcal{E}^T + \hat{F})]_A = 2\hat{F} + 2\hat{F}\Theta\hat{F} \tag{217}$$

This constraint is again nothing else but a choice of "gauge" on the antisymmetric part: $[D_c]_A = \tilde{f}\ L\hat{F} \in \mathbb{Z}$.

Finally summing the opposite of the transpose of the second equation with the third we eliminate C_o and we get

$$\begin{aligned} \pi(B_c^T - C_c) &= \pi(\mathbb{I} - 2Z_C) \\ &= -\frac{\pi}{2}E^T\mathcal{E}^{-1}\mathcal{G}\mathcal{E}^{-1} - \frac{\pi}{2}E\mathcal{E}^{-T}\mathcal{G}\mathcal{E}^{-T} + \hat{F}B_1^T - B_2 \\ &= -\pi\mathbb{I} - 2\pi\hat{F}\Theta + \hat{F}B_1^T - B_2 = -\pi\mathbb{I} \end{aligned} \tag{218}$$

where we have used

$$\begin{aligned} E^T\mathcal{E}^{-1}\mathcal{G}\mathcal{E}^{-1} + E\mathcal{E}^{-T}\mathcal{G}\mathcal{E}^{-T} &= (\mathcal{E} - \hat{F})\mathcal{E}^{-1}\mathcal{G}\mathcal{E}^{-1} + (\mathcal{E}^T + \hat{F})\mathcal{E}^{-T}\mathcal{G}\mathcal{E}^{-T} \\ &= 2\mathbb{I} - \hat{F}(\mathcal{E}^{-1}\mathcal{E}^T - \mathcal{E}^{-T}\mathcal{E})G^{-1} \\ &= 2\mathbb{I} - \hat{F}((\mathcal{E}^{-T} - 2\Theta)\mathcal{E}^T - (\mathcal{E}^{-1} + 2\Theta)\mathcal{E})G^{-1} = 2\mathbb{I} + 4\hat{F}\Theta \end{aligned} \tag{219}$$

which is again a choice of "gauge" ($Z_C = \mathbb{I}$).

B.5.3 Solving the closed string constraints.

When we have solved the previous constraints we can then determine the remaining matrices A_o, D_o and C_o as

$$
\begin{aligned}
A_o &= \frac{\pi}{2}[A_c]_S - \frac{\pi}{4}[\mathcal{E}^{-T}\mathcal{G}\mathcal{E}^{-T}]_S - \frac{1}{2}[B_1]_S \\
D_o &= \frac{\pi}{2}[D_c]_S + \frac{\pi}{4}[E\mathcal{E}^{-T}\mathcal{G}\mathcal{E}^{-T}E]_S + \frac{1}{2}[B_2\hat{F}]_S \\
C_o &= \frac{\pi}{2}(C_c + B_c^T) + \frac{\pi}{4}(E^T\mathcal{E}^{-1}\mathcal{G}\mathcal{E}^{-1} - E\mathcal{E}^{-T}\mathcal{G}\mathcal{E}^{-T}) - \frac{1}{2}(B_2 + \hat{F}B_1^T)
\end{aligned}
\tag{220}
$$

where the first and second equations are obtained by taking the symmetric part of the first and last equations in (212), while the last is obtained by summing the transpose of the second with the third one.

Notice that the matrices $[Z_A]_S$, $[Z_D]_S$ and Z_B are completely arbitrary but integers and hence the the numerical cocycle of closed string vertices in open string formalism will be determined up to powers of i, actually we can write

$$
\begin{aligned}
& e^{i\left(n_i A_o^{ij} n_j + m^i D_{o\ ij} m^j + m^i C_o{}^{\cdot j}{}_i n_j\right)} e^{ik_0 A_o^{00} k_0} = \\
& = e^{-i\pi\alpha' k_L^T \mathcal{E}^{-T}\mathcal{G}\mathcal{E}^{-T} k_R}\ e^{-i(n-\hat{F}m)^T [C_\pi]_S (n-\hat{F}m)} \\
& \times\ e^{i\frac{\pi}{2}(n^T A_c n + n^T B_c m + m^T C_c n + m^T D_c m)}
\end{aligned}
\tag{221}
$$

Notice that

1. the dependence on $A_c, B_c, \ldots$ is like a square root of the closed string dependence but it can be fixed to give a trivial result as done in eq. (90).

2. there is also another arbitrary matrix $[C_0]_S$ which can be and will be fixed to the trivial value $[C_0]_S = 0$.

Finally we can summarize our discussion by writing

$$
\begin{aligned}
\mathcal{V}_{(0)T}(x;k) &= t_{I_2J_2}(k)\ e^{-i\alpha' k_{N;IJ}[C_0]_S^{NM}k_{M;IJ}}\ e^{i2\alpha' k_{M;IJ}\left([C_0]_S\mathcal{G}-\frac{\pi}{2}\Theta\mathcal{G}\right)^M{}_N p^N} \\
&\quad \times :e^{ik_{M;IJ}X^M_{L(0)}(x)}: \\
&\quad \times\Lambda_{L;I_1J_1}(n_1,n_2)\ |(I_1,I_2)\rangle_{\sigma=0}\ \ {}_{\sigma=0}\langle(J_1,J_2)| \\
&= t_{I_2J_2}(k)\ e^{-i\alpha' k_{N;IJ}[C_0]_S^{NM}k_{M;IJ}}\ e^{i2\alpha' k_{M;IJ}([C_0]_S\mathcal{G})^M{}_N p^N} \\
&\quad \times :e^{ik_{M;IJ}\hat{X}^M_{L(0)}(x)}: \\
&\quad \times\Lambda_{L;I_1J_1}(n_1,n_2)\ |(I_1,I_2)\rangle_{\sigma=0}\ \ {}_{\sigma=0}\langle(J_1,J_2)|
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{V}_{(\pi)T}(y;k) &= t_{J_2I_2}(k)\ e^{i\alpha' k_{M;IJ}(-[C_0]_S)^{MN}k_{N;IJ}} e^{i2\alpha' k_{M;IJ}\left(-\pi\mathbb{I}+[C_0]_S\mathcal{G}+\frac{\pi}{2}\Theta\mathcal{G}\right)^M{}_N p^N} \\
&\quad \times :e^{ik_{M;IJ}X^M_{L(0)}(y)}: \\
&\quad \times\ \Lambda_{L;J_1I_1}(n_1,n_2)\ |(I_1,I_2)\rangle_{\sigma=\pi}\ \ {}_{\sigma=\pi}\langle(J_1,J_2)| \\
&= t_{J_2I_2}(k)\ e^{i\alpha' k_{M;IJ}(-[C_0]_S)^{MN}k_{N;IJ}}\ e^{i2\alpha' k_{M;IJ}(-\pi\mathbb{I}+\pi\Theta\mathcal{G}+[C_0]_S\mathcal{G})^M{}_N p^N} \\
&\quad \times :e^{ik_{M;IJ}\hat{X}^M_{L(0)}(y)}: \\
&\quad \times\ \Lambda_{L;J_1I_1}(n_1,n_2)\ |(I_1,I_2)\rangle_{\sigma=\pi}\ {}_{\sigma=\pi}\langle(J_1,J_2)| \\
&= t_{J_2I_2}(k)\ e^{-i\alpha' k_{M;IJ}(+[C_0]_S)^{MN}k_{N;IJ}}\ e^{i2\alpha' k_{M;IJ}([C_0]_S\mathcal{G})^M{}_N p^N}\times :e^{ik_{M;IJ}X^M(y)}: \\
&\quad \times\ \Lambda_{L;J_1I_1}(n_1,n_2)\ |(I_1,I_2)\rangle_{\sigma=\pi}\ {}_{\sigma=\pi}\langle(J_1,J_2)|
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{W}_{T_c}(z,\bar{z};k_L,k_R,y_0) &= e^{\frac{i}{2}\Phi_{(c)}(k,k)}\ e^{-i\pi\alpha' k_{L;M}(\mathcal{E}^{-T}\mathcal{G}\mathcal{E}^{-T})^{MN}k_{R;N}} e^{-i\pi\sqrt{\alpha'}\left(n_M+m^N\hat{F}_{NM}\right)(\Theta\mathcal{G})^M{}_L p^L} \\
&\quad \times e^{im^M G_{MN}\frac{y_0^N}{\sqrt{\alpha'}}} :e^{ik_{LM}(G^{-1}\mathcal{E})^M_{.N}X^N_{L(0)}(z)}::e^{ik_{RM}(G^{-1}\mathcal{E}^T)^M_{.N}X^N_{R(0)}(\bar{z})}: \\
&\quad \times\sqrt{L}\ \Lambda_{L;I_1J_1}(-fm_2,fm_1)\otimes\mathbb{I}_{I_2J_2}\ |(I_1,I_2),K\rangle\ \langle(J_1,J_2),K| \\
&= e^{\frac{i}{2}\Phi_{(c)}(k,k)}\ e^{i\pi\alpha' k_{L;M}(G^{-1}\mathcal{E}^T\Theta\mathcal{E}^T G^{-1})^{MN}k_{R;N}} \\
&\quad \times :e^{ik_{LM}X^M_L(z)}::e^{ik_{RM}X^M_R(\bar{z})}: \\
&\quad \times\sqrt{L}\ \Lambda_{L;I_1J_1}(-fm_2,fm_1)\otimes\mathbb{I}_{I_2J_2}\ |(I_1,I_2),K\rangle\ \langle(J_1,J_2),K|
\end{aligned}
\tag{222}
$$

where the different expressions are obtained using

$$
\begin{aligned}
X(x,x) &= \hat{X}_{L(0)}(x) = X_{L(0)}(x) - \pi\alpha'\Theta\mathcal{G}p, \\
X(y,y) &= \hat{X}_{L(0)}(y) + 2\pi\alpha'(-\mathbb{I}+\Theta\mathcal{G})p = X_{L(0)}(y) + 2\pi\alpha'(-\mathbb{I}+\frac{1}{2}\Theta\mathcal{G})p
\end{aligned}
\tag{223}
$$

where the $\hat{X}_{L(0)}$ fields are whose expansion contains the non commuting x and the $X_{L,R}$ in the case of the closed string.

C Details on different amplitude computations.

C.1 Open OPEs with Wilson lines.

It is easy to check that the OPEs of two open string vertices in presence of Wilson lines are given by

$$\begin{aligned}
\mathcal{V}_{(0)T}(x_1; \| k_{IL} \|)\ \mathcal{V}_{(0)T}(x_2; \| l_{LJ} \|) &= \frac{1}{\sqrt{L}} e^{-i\pi\alpha'\ I(k_{M;IL})\Theta_{CP}^{MN} I(l_{N;LJ})} \\
& \sum_L e^{-i\pi\alpha'\ k_{M;IL}\Theta^{MN} l_{N;LJ}} (x_1 - x_2)^{2\alpha' k_{M;IL}\mathcal{G}^{MN} l_{N;LJ}} \\
& : e^{ik_{M;IL}X^M(x_1) + il_{M;LJ}X^M(x_2)} : \\
& \Lambda_{L;I_1J_1}(k+l) \otimes (T_u T_v)_{I_2J_2}\ |(I_1, I_2)\rangle_{\sigma=0}\ \ {}_{\sigma=0}\langle(J_1, J_2)|
\end{aligned} \tag{224}$$

when we set $[C_0]_S = 0$ and where we have used the fact that $\| k_{IL} \|$ depends only on I_2 and L_2, and that $I(k_{M;IL})$ does not depend on I, L at all.

C.2 Details on the boundary derivation from reggeon vertices.

In this appendix we want to give some details on the computations performed for obtaining the boundary state from the reggeon formalism. In particular we want to start from eq. (104)

$$\begin{aligned}
\langle B(F); V_L, V_R| &= \frac{\mathcal{C}_0(E, \hat{F})\ \tilde{\mathcal{N}}_0(E)}{2\pi} \\
& \langle x_L = x_R = 0; 0_{a_{(c)}}, 0_{\tilde{a}_{(c)}}|\ e^{-\frac{i}{2}\Phi_{(c)}(Gp_{(c)}, Gp_{(c)})}\ e^{-i\pi\alpha'\ p_R^M (\mathcal{E}^T\mathcal{G}^{-1}\mathcal{E}^T)_{MN} p_L^N} \\
& \times Tr\left(\sqrt{L}\Lambda_L\left(-L\hat{F}m\right) \otimes \mathbb{I}_{N_1}\right)\ \ \times {}_p\langle 0|\mathcal{S}_L(z; V_L)\ \mathcal{S}_R(\bar{z}; V_R)|0\rangle_p
\end{aligned} \tag{225}$$

where we have introduced the state $\langle x_L = x_R = 0|$ normalized as $\langle x_L = x_R = 0|k_L, k_R\rangle = 1$ and the Sciuto-Della Selva-Saito vertices as discussed in ([17], [18], [22])

$$\begin{aligned}
\mathcal{S}_L(z; V_L) &= : \exp\left(-\frac{1}{2\alpha'}\oint_{u=0}\frac{du}{2\pi i}\partial X^M_{L(c)}(u)\ G_{MN}\ X^N_L(V_L(u;z))\right) : \\
& \times : \exp\left(\frac{i}{2\sqrt{2\alpha'}}\oint_{u=0}\frac{du}{2\pi i}\partial X^M_{L(c)}(u)\ G_{MN}\ \alpha^N_{(c)0}\ \log\frac{dV_L}{du}\right) : \\
&= : \exp\left(\frac{i}{\sqrt{2\alpha'}}\sum_{n=0}^{\infty}\alpha^M_{(c)n}\ \mathcal{E}_{MN}\frac{1}{n!}\ \partial^n_u \hat{X}^N_L(V_L(u;z))|_{u=0}\right) :\ e^{\frac{i}{\sqrt{2\alpha'}}\alpha^M_{(c)0}\ G_{MN}\ y^N_0} \\
& \times : \exp\left(\frac{1}{2}\sum_{n=0}^{\infty}\alpha^M_{(c)n}\ G_{MN}\ \alpha^N_{(c)0}\ \frac{1}{n!}\ \partial^n_u \log\frac{dV_L}{du}|_{u=0}\right) :
\end{aligned} \tag{226}$$

and

$$\begin{aligned}
\mathcal{S}_R(\bar z;V_R) &= :\exp\left(-\frac{1}{2\alpha'}\oint_{u=0}\frac{du}{2\pi i}\partial X^M_{R(c)}(u)\ G_{MN}\ X^N_R(V_R(u;\bar z))\right): \\
&\quad\times :\exp\left(\frac{i}{2\sqrt{2\alpha'}}\oint_{u=0}\frac{du}{2\pi i}\partial X^M_{R(c)}(u)\ G_{MN}\ \tilde\alpha^N_{(c)0}\ \log\frac{dV_R}{du}\right): \\
&= :\exp\left(\frac{i}{\sqrt{2\alpha'}}\sum_{n=0}^{\infty}\tilde\alpha^M_{(c)n}\ (\mathcal{E}^T)_{MN}\ \frac{1}{n!}\partial^n_u\hat X^N_R(V_R(u;\bar z))|_{u=0}\right):\ e^{-\frac{i}{\sqrt{2\alpha'}}\tilde\alpha^M_{(c)0}\ G_{MN}\ y^N_0} \\
&\quad\times :\exp\left(\frac{1}{2}\sum_{n=0}^{\infty}\tilde\alpha^M_{(c)n}\ G_{MN}\ \tilde\alpha^N_{(c)0}\ \frac{1}{n!}\ \partial^n_u\log\frac{dV_R}{du}|_{u=0}\right): \qquad (227)
\end{aligned}$$

where the zero modes are defined as $a^M_{(c)0}=\alpha^M_{(c)0}=\sqrt{2\alpha'}p^M_L$, $\sqrt{n}a^M_{(c)n}=\alpha^M_{(c)0}$ $(n>0)$ and similarly for the right moving operators. These vertices are given for arbitrary $SL(2,\mathbb{C})$ local coordinates $V_L(u;z)$ and $V_R(u;\bar z)$:

$$V_L(u;z) = \frac{a_L u+b_L z}{c_L u+b_L},\qquad b_L(a_L-c_L z)=1,\qquad V_L(0;z)=z \qquad (228)$$

and similarly for V_R.

Performing explicitly the computation we get therefore[17]

$$\begin{aligned}
&e^{-i\frac{1}{2}\pi\alpha^N_{(c)0}(\mathcal{E}\mathcal{G}^{-1}\mathcal{E})_{NM}\ \tilde\alpha^N_{(c)0}}{}_p\langle 0|\mathcal{S}_L(z;V_L)\ \mathcal{S}_R(\bar z;V_R)|0\rangle_p \\
&= \exp\left(\sum_{n,m=0}^{\infty}\alpha^N_{(c)n}(\mathcal{E}\mathcal{G}^{-1}\mathcal{E})_{NM}\tilde\alpha^M_{(c)m}\ \frac{\partial^n_u}{n!}\frac{\partial^m_v}{m!}\log(V_L(u;z)-V_R(v;\bar z))|_{u=v=0}\right) \\
&\quad \exp\left(\frac{1}{2}\sum_{n=0}^{\infty}\alpha^M_{(c)n}\ G_{MN}\ \alpha^N_{(c)0}\ \frac{1}{n!}\ \partial^n_u\log\frac{dV_L}{du}|_{u=0}\right) \\
&\quad \exp\left(\frac{1}{2}\sum_{n=0}^{\infty}\tilde\alpha^M_{(c)n}\ G_{MN}\ \tilde\alpha^N_{(c)0}\ \frac{1}{n!}\ \partial^n_u\log\frac{dV_R}{du}|_{u=0}\right) \\
&\quad \exp\left(i\frac{1}{2}\pi\ \alpha^N_{(c)0}(\mathcal{E}\Theta\mathcal{E})_{NM}\tilde\alpha^M_{(c)0}\right)\ \exp\left(-i\frac{1}{2}\pi\alpha^N_{(c)0}(\mathcal{E}\mathcal{G}^{-1}\mathcal{E})_{NM}\ \tilde\alpha^N_{(c)0}\right) \\
&\quad \exp\left(\frac{i}{\sqrt{2\alpha'}}(\alpha^M_{(c)0}-\tilde\alpha^M_{(c)0})\ G_{MN}\ y^N_0\right) \\
&\quad (2\pi\sqrt{\alpha'})^{D-d}\delta^{D-d}(\alpha_{0\mu})\ (2\pi\sqrt{\alpha'})^d\delta_{\mathcal{E}^T\alpha_{(c)0}+\mathcal{E}\tilde\alpha_{(c)0},0}
\end{aligned} \qquad (229)$$

[17] Notice that the phase $e^{-i\pi\alpha'\ p^M_R(\mathcal{E}^T\mathcal{G}^{-1}\mathcal{E}^T)_{MN}p^N_L}$ which enters the definition of the closed string vertex (96) and is present in eq. (229) is not anymore present in eq. (230). The reason is that the phase $e^{-i\pi\alpha'\ p^M_R(\mathcal{E}^T\mathcal{G}^{-1}\mathcal{E}^T)_{MN}p^N_L}$ is required in order to write the expression (230) with the use D_{00} given in eq.s (231) since $\frac{d}{du}(U_L V_R)|_{u=0}=\frac{e^{-i\pi}}{|z-\bar z|^2}\frac{d}{du}(V_L)|_{u=0}\frac{d}{du}(V_R)|_{u=0}$ where we have used the first of eq.s (148) to fix the phase. Another way to check this is to plug the special choice (109) in order to get the final result (110).

$$
\begin{aligned}
&= \exp\left(-\sum_{n,m=0}^{\infty} a^N_{(c)n}(\mathcal{E}\mathcal{G}^{-1}\mathcal{E})_{NM}\tilde{a}^M_{(c)m} D_{nm}(U_L V_R)\right) \\
&\quad \exp\left(-i\frac{\pi}{2}\ a^N_{(c)0}(\mathcal{E}\Theta\mathcal{E}^T)_{NM}\tilde{a}^M_{(c)0}\right) \quad \exp\left(\frac{i}{\sqrt{2\alpha'}}(a^M_{(c)0}-\tilde{a}^M_{(c)0})\ G_{MN}\ y_0^N\right) \\
&\quad (2\pi)^{D-d}\delta^{D-d}(k_\mu)\ (2\pi\sqrt{\alpha'})^d \delta_{\mathcal{E}^T a_{(c)0}+\mathcal{E}\tilde{a}_{(c)0},0}
\end{aligned}
\tag{230}
$$

where $U_L(u) = \Gamma V_L^{-1}(u) = \frac{1}{V_L^{-1}(u)}$. In the previous equation we have also introduced $D_{nm}(\gamma)$ which is a representation of the $SL(2,\mathbb{R})$ group given by

$$
\begin{aligned}
D_{nm}(\gamma) &= \sqrt{\frac{m}{n}}\frac{1}{m!}\partial^m[\gamma^n(u)]|_{u=0}, \qquad & D_{00}(\gamma) &= \frac{1}{2}\log\gamma'(0) \\
D_{n0}(\gamma) &= \sqrt{\frac{1}{n}}[\gamma^n(u)]|_{u=0}, \qquad & D_{0n}(\gamma) &= \frac{1}{2}\sqrt{m}\frac{1}{m!}\partial^m\log\gamma'(0)
\end{aligned}
\tag{231}
$$

with $n, m > 0$.

The amplitude (225) can then be written as

$$
\begin{aligned}
\langle B(F); V_L, V_R| &= N\ \frac{\mathcal{C}_0(E,\hat{F})\ \tilde{\mathcal{N}}_0(E)}{2\pi} \\
&\quad \langle k_\mu = 0|\sum_{s\in\mathbb{Z}^d}\frac{1}{(2\pi\sqrt{\alpha'})^d}\langle n = L\ \hat{F}\ s, m = L\ s| e^{-\frac{1}{2}i\Phi_{(c)}(k,k)+i\pi\frac{\hat{h}f^2}{L}m^1m^2}\ e^{i\ m^M\ G_{MN}\ \frac{y_0^N}{\sqrt{\alpha'}}} \\
&\quad \times\ \langle 0_a, 0_{\tilde{a}}|\ \exp\left(-\sum_{n,m=0}^{\infty} a^N_{(c)n}(\mathcal{E}\mathcal{G}^{-1}\mathcal{E})_{NM}\tilde{a}^M_{(c)m}D_{nm}(U_LV_R)\right)
\end{aligned}
\tag{232}
$$

where we used the definition of $\langle x_L = x_R = 0|$ and the momentum conservation. In particular the term with Θ. is canceled due to momentum conservation.

This expression allows to compute any one point closed string amplitude as given in eq. (107).

C.3 Details on N open - 1 closed tachyon amplitudes.

We want to compute the amplitudes with strings on the $\sigma = \pi$ border and show that they are necessary to complete the integration over the whole circle. In doing so a lot of care must be used in the treatment of phases, in particular eq.s (150) must be used. All these problems arises because we have been working with coordinates in closure of the upper complex plane where there is a special point ∞ for the ordering of the vertices; this would not happen with a disk formulation where the cyclically equivalent orderings are obvious.

We start therefore from our formulation in the closure of the upper complex plane and want to perform the change of variable

$$
x = \frac{\bar{z}\ e^{i\phi} - z}{e^{i\phi} - 1}
\tag{233}
$$

which maps the circle $0 \le \phi < 2\pi$ into the real axis. The key point is to correctly connect ϕ with the phase ψ given in eq.s (150). This can be done comparing the phase of $\ln(x-z)$, explicitly we have

$$x - z = e^{i(\pi+\frac{1}{2}\phi+2\pi k)}\frac{Im\ z}{\sin\frac{1}{2}\phi} \to \ln(x-z) = \ln\left|\frac{Im\ z}{\sin\frac{1}{2}\phi}\right| + i(\pi + \frac{1}{2}\phi) + i2\pi k \quad (234)$$

where we have used the modulus even if $Im\ z,\ \sin\frac{1}{2}\phi > 0$. Comparing with the ranges given in eq.s (150) we get

$$\frac{1}{2}\phi = \left\{ \begin{array}{lll} \psi + \pi, & x > 0, & \zeta < \frac{1}{2}\phi < \pi \\ \psi - \pi, & x < 0, & 0 < \frac{1}{2}\phi < \zeta \end{array} \right. \quad (235)$$

We have therefore to consider the cyclically equivalent correlators. For $k > l$ we have

$$\begin{aligned} A_{kl} &= \langle 0|R\Big[\ \mathcal{V}_{(0)T}(x_k;k_k)\ldots\ \mathcal{V}_{(0)T}(x_l;k_l) \\ &\quad \mathcal{V}_{(0)T}(y_{k-1};k_{k-1})\ldots\ \mathcal{V}_{(0)T}(y_1;k_1) \\ &\quad \mathcal{V}_{(0)T}(y_N;k_N)\ldots\ \mathcal{V}_{(0)T}(y_{l+1};k_{l+1})\ \mathcal{W}_{T_c}(z,\bar z;k_L,k_R)\Big]|0\rangle \end{aligned} \quad (236)$$

where $R[..]$ is the radial ordering and we have chosen $x_k > \cdots > x_l$ and $|y_{k-1}| > \ldots |y_1| > |y_N| > \ldots |y_{l+1}|$ in order to consider cyclically equivalent configurations and for $k < l$ we write

$$\begin{aligned} A_{kl} &= \langle 0|R\Big[\ \mathcal{V}_{(0)T}(x_k;k_k)\ldots\ \mathcal{V}_{(0)T}(x_1;k_1) \\ &\quad \mathcal{V}_{(0)T}(x_N;k_N)\ldots\ \mathcal{V}_{(0)T}(x_l;k_l) \\ &\quad \mathcal{V}_{(0)T}(y_{l-1};k_{l-1})\ldots\ \mathcal{V}_{(0)T}(y_{k+1};k_{k+1})\ \mathcal{W}_{T_c}(z,\bar z;k_L,k_R)\Big]|0\rangle \end{aligned} \quad (237)$$

with $x_l > \cdots > x_1 > x_N > \ldots x_k$ and $|y_{k-1}| > |y_k| > \ldots |y_{k-1}|$ in order to keep the desired cyclical ordering.

We start compute the cyclically equivalent correlators A_{1l} and then we discuss the others in order to show that they cover different ranges of the circle which together all the others A_{kl} correlators cover all the circle. Explicitly we have

$$\begin{aligned} A_{1l} &= \langle 0|\ \mathcal{V}_{(0)T}(x_1;k_1)\ldots\ \mathcal{V}_{(0)T}(x_l;k_l) \\ &\quad \mathcal{V}_{(0)T}(y_N;k_N)\ldots\ \mathcal{V}_{(0)T}(y_{l+1};k_{l+1})\ \mathcal{W}_{T_c}(z,\bar z;k_L,k_R)|0\rangle \end{aligned} \quad (238)$$

for all[18] $0 \leq l \leq N$ with $x_k > \dots x_l > |y_N| > \dots |y_{l+1}| > |z|$ which a simple computation shows to be

$$\begin{aligned}
A_{1l} \quad = \quad & e^{i2\pi\alpha'(\sum_{u=l+1}^{N} k_u)^T \ \Theta \ (\hat{k}_L+\hat{k}_R)} \ e^{i2\pi\alpha'(\sum_{u=l+1}^{N} k_u)^T \ \Theta_{CP} \ (-\frac{\hat{F}m}{\sqrt{\alpha'}})} \ e^{-i\pi\hat{h}L(\sum_r k_r - \frac{\hat{F}m}{\sqrt{\alpha'}})_1(\sum_r k_r - \frac{\hat{F}m}{\sqrt{\alpha'}})_2} \\
& e^{-i2\pi\alpha' \sum_{l+1\leq u<v} k_u\cdot k_v} \ e^{-i2\pi\alpha' \sum_{u=l+1}^{N} k_u\cdot(\hat{k}_L+\hat{k}_R)} \\
& e^{\frac{i}{2}\Phi(k,k)} \ e^{im^T G \frac{y_0}{\sqrt{\alpha'}}} \ e^{-i\pi\alpha' \sum_{1\leq r<s\leq N} k_r^T(\Theta+\Theta_{CP})k_s} \\
& \left(\frac{1}{\sqrt{L}}\right)^{N-2} \quad t_{u_1}(k_N)\dots t_{u_1}(k_N) \ tr(T_{u_1}\dots T_{u_N}) \\
& \prod_{1\leq r<s\leq l} (x_r - x_s)^{2\alpha' k_r\cdot k_s} \quad \prod_{r<\leq l<u} (x_r - y_u)^{2\alpha' k_r\cdot k_u} \quad \prod_{l+1\leq u<v\leq l} (y_v - y_u)^{2\alpha' k_v\cdot k_u} \\
& \prod_{1\leq r\leq l} \left[(x_r - z)^{2\alpha' k_r\cdot\hat{k}_L} \ (x_r - \bar{z})^{2\alpha' k_r\cdot\hat{k}_R}\right] \quad \prod_{l+1\leq u\leq N} \left[(y_u - z)^{2\alpha' k_u\cdot\hat{k}_L} \ (y_u - \bar{z})^{2\alpha' k_u\cdot\hat{k}_R}\right] \\
& |z-\bar{z}|^{2\alpha'\hat{k}_L\cdot\hat{k}_R} \times \ (2\pi)^{D-d}\delta(\sum_r k_{r\mu} + k_\mu) \ (2\pi\sqrt{\alpha'})^d \delta_{\sum_r k_r + \hat{k}_L + \hat{k}_R, 0}
\end{aligned} \tag{239}$$

where we have written in the first and second line all the terms from cocycles and Chan-Paton which change with l. We have also defined $\hat{k}_L = \mathcal{E}^T \mathcal{G}^{-1} k_L$, $\hat{k}_R = \mathcal{E}\mathcal{G}^{-1} k_R$ and $k \cdot l = k^T \mathcal{G}^{-1} l$. By writing $y_v - y_u = |y_v - y_u| e^{i\pi}$ we cancel the first term of the second line. Writing $y_u - \bar{z} = |y_u - z| e^{i(\pi - \frac{1}{2}\phi_u)}$, $y_u - z = |y_u - z| e^{i(\pi + \frac{1}{2}\phi_u)}$ with $0 < \phi_u < \zeta$ and $x_r - z = (x_r - \bar{z})^* = |x_r - z| e^{i(-\pi + \frac{1}{2}\phi_r)}$ with $\zeta < \phi_r < 2\pi$ we cancel all the remaining terms in the first two lines with the help of eq. (62). The final result is

$$\begin{aligned}
A_{1l} \quad = \quad & e^{\frac{i}{2}\Phi(k,k)} \ e^{im^T G \frac{y_0}{\sqrt{\alpha'}}} \ e^{-i\pi\alpha' \sum_{1\leq r<s\leq N} k_r^T(\Theta+\Theta_{CP})k_s} \ e^{i\pi\hat{h}\left(f(n_1 m^1 + n_2 m^2) - L n_1 n_2\right)} \\
& \left(\frac{1}{\sqrt{L}}\right)^{N-2} \quad t_{u_1}(k_N)\dots t_{u_1}(k_N) \ tr(T_{u_1}\dots T_{u_N}) \\
& \prod_{1\leq r<s\leq N} |x_r - x_s|^{2\alpha' k_r\cdot k_s} \quad \prod_{1\leq r\leq N} |x_r - z|^{2\alpha' k_r\cdot(\hat{k}_L+\hat{k}_R)} \quad \prod_{1\leq r\leq N} e^{i\phi_r \alpha' k_r\cdot(\hat{k}_L - \hat{k}_R)} \\
& |z-\bar{z}|^{2\alpha'\hat{k}_L\cdot\hat{k}_R} \times \ (2\pi)^{D-d}\delta(\sum_r k_{r\mu} + k_\mu) \ (2\pi\sqrt{\alpha'})^d \delta_{\sum_r k_r + \hat{k}_L + \hat{k}_R, 0}
\end{aligned} \tag{240}$$

where ϕ_rs are such that $\phi_r > \phi_{r+1}$ and have two different ranges according whether $r \leq l$ or $l < r$ as discussed above, i.e. $2\pi > \phi_1 > \dots {}_l > \zeta > \phi_{l+1} > \dots \phi_N > 0$.

To get the amplitude we change variables according to eq. (233), we multiply for the measure

$$\prod_{r=1}^{N} dx_r \ dz \ d\bar{z} = (\frac{1}{2} Im \ z)^N \prod_{r=1}^{N} \frac{d\phi_r}{\sin^2 \frac{\phi_r}{2}} \ dz \ d\bar{z} \tag{241}$$

[18] A_{10} is the correlator where all the vertices are on the $\sigma = \pi$ border.

and divide for the gauge fixing

$$dV_{Killing} = \frac{1}{4}(Im\ z)^{-2} d\phi_N\ dz\ d\bar{z} \tag{242}$$

which can simply be obtained by requiring that the $N = 1$ amplitude is independent on z and ϕ.

Finally fixing $\phi_N = \alpha$ we can sum the A_{1l} correlators as $\sum_{l=0}^{N} A_{1l}$ to eliminate the dependence on ζ and then we get a partial amplitude given by

$$\begin{aligned}
\mathcal{A}_1 \;=\; & \mathcal{C}_0(E,\hat{F})\ \mathcal{N}_0(E,\hat{F})^N\ \tilde{\mathcal{N}_0}(E)\ e^{\frac{i}{2}\Phi(k,k)}\ e^{im^T G \frac{y_0}{\sqrt{\alpha'}}}\ e^{-i\pi\alpha' \sum_{1\le r<s\le N} k_r^T(\Theta+\Theta_{CP})k_s} \\
& \left(\frac{1}{\sqrt{L}}\right)^{N-2} t_{u_1}(k_N)\ldots t_{u_1}(k_N)\ tr(T_{u_1}\ldots T_{u_N})\ e^{i\pi\hat{h}\left(f(n_1 m^1+n_2 m^2)-L n_1 n_2\right)} \\
& (2\pi)^{D-d}\delta(\sum_r k_{r\mu}+k_\mu)\ (2\pi\sqrt{\alpha'})^d \delta_{\sum_r k_r+\hat{k}_L+\hat{k}_R,0} \\
& \int_\alpha^{2\pi}\prod_{r=1}^{N-1} d\phi_r\theta(\phi_r-\phi_{r+1}) \prod_{1\le r<s\le N}(2\sin\frac{\phi_r-\phi_s}{2})^{2\alpha' k_r\cdot k_s} \prod_{1\le r\le N} e^{i\phi_r\alpha' k_r\cdot(\hat{k}_L-\hat{k}_R)}
\end{aligned} \tag{243}$$

where we still have a dependence on $\phi_N = \alpha$. To get rid of it we need consider all the others amplitudes.

Let us now exam all the other correlators A_{kl}. The previous discussion shows that all A_{1l} correlators are the same of A_{1N} up to the ranges where the ϕ_r are defined, i.e. we can freely move all the vertices from the $\sigma = \pi$ boundary to the $\sigma = 0$ one while keeping the cyclical order. We deduce therefore that all A_{kl} are the analytic continuation of the correlators

$$\begin{aligned}
A_{k\ k-1} \;=\; & \langle 0|R\Big[\ \mathcal{V}_{(0)T}(x_k;k_k)\ldots\ \mathcal{V}_{(0)T}(x_N;k_N) \\
& \mathcal{V}_{(0)T}(x_1;k_1)\ldots\ \mathcal{V}_{(0)T}(x_{k-1};k_{k-1})\ \mathcal{W}_{T_c}(z,\bar{z};k_L,k_R)\Big]|0\rangle
\end{aligned} \tag{244}$$

If we compare the explicit expression (239) for the usual ordering A_{1N} with the corresponding one for the previous correlators A_{kk-1} which can be obtained mutata mutandis we find that

$$\begin{aligned}
A_{kl} \;&=\; e^{2i\pi\alpha' \sum_{1\le r\le k-1} k_r^T(\Theta+\Theta_{CP})\sum_{k\le s\le N} k_s}\ e^{-2\pi i\alpha' \sum_{1\le r\le k-1} k_r\cdot(\hat{k}_L-\hat{k}_R)}\ A_{1N} \\
&=\; e^{2i\pi L\sqrt{\alpha'}\sum_{1\le r\le k-1} k_r^T(\hat{h}\epsilon n+\tilde{f}m)}\ A_{1N} = A_{1N}
\end{aligned} \tag{245}$$

where the first phase is due to the difference with the overall phase in the first line of eq (243) and depends crucially on the presence of the would-be closed string Chan-Paton, the second one is due to the shift of the range $\phi_r \to \phi_r - 2\pi$ for

$1 \leq r \leq k-1$ so that the new ϕ_r are ordered as in the A_{1l} amplitudes even if with a different range $\phi_N < \ldots \phi_k < 2\pi < \phi_{k-1} < \ldots \phi_1 < 2\pi + \phi_N$. To perform the computations we have used the momentum conservation, the definition of Θ_{CP} and the fact that $L\sqrt{\alpha'}k_r \in \mathbb{Z}$.

Notice that there is not a contribution from $\sin\frac{\phi_r-\phi_s}{2}$ since it can be written as $|\sin\frac{\phi_r-\phi_s}{2}|$ because of the ordering $\theta(\phi_r - \phi_{r+1})$.

Changing variables, multiplying for the measure, dividing for the gauge group and fixing $\phi_N = \alpha$ we can sum all A_{kl} correlators with fixed k to get a partial amplitude given by

$$\begin{aligned}
\mathcal{A}_k \;=\;& \mathcal{C}_0(E,\hat{F})\ \mathcal{N}_0(E,\hat{F})^N\ \tilde{\mathcal{N}_0}(E)\ e^{\frac{i}{2}\Phi(k,k)}\ e^{im^T G\frac{y_0}{\sqrt{\alpha'}}}\ e^{-i\pi\alpha'\sum_{1\leq r<s\leq N} k_r^T(\Theta+\Theta_{CP})k_s} \\
& \left(\frac{1}{\sqrt{L}}\right)^{N-2}\ t_{u_1}(k_N)\ldots t_{u_1}(k_N)\ tr(T_{u_1}\ldots T_{u_N})\ e^{i\pi\hat{h}\left(f(n_1m^1+n_2m^2)-Ln_1n_2\right)} \\
& (2\pi)^{D-d}\delta(\sum_r k_{r\mu}+k_\mu)\ (2\pi\sqrt{\alpha'})^d\delta_{\sum_r k_r+\hat{k}_L+\hat{k}_R,0} \\
& \int_\alpha^{2\pi}\prod_{r=k}^{N-1}d\phi_r\ \int_{2\pi}^{2\pi+\alpha}\prod_{r=1}^{k-1}d\phi_r\ \theta(\phi_r-\phi_{r+1})\prod_{1\leq r<s\leq N}(2\sin\frac{\phi_r-\phi_s}{2})^{2\alpha' k_r\cdot k_s}\prod_{1\leq r\leq N}e^{i\phi_r\alpha' k_r\cdot(\hat{k}_L-\hat{k}_R)}
\end{aligned} \tag{246}$$

Finally summing over all k we get the previous amplitude with integration range $[\alpha, 2\pi+\alpha]$ for all ϕ_r ($1 \leq r \leq N-1$) variables, we can then shift all the variables as $\phi \to \phi + \alpha$ to get final result described in the main text. Notice that the shift is allowed since using the momentum conservation we get

$$e^{i\alpha\alpha'\sum_r k_r\cdot(\hat{k}_L-\hat{k}_R)} = e^{-i\alpha\alpha'(\hat{k}_L+\hat{k}_R)\cdot(\hat{k}_L-\hat{k}_R)} = e^{-i\alpha\alpha'(k_L^TG^{-1}k_L-k_R^TG^{-1}k_LR)} = 1. \tag{247}$$

as long as $k_L^TG^{-1}k_L = k_R^TG^{-1}k_R$ which is granted the the σ rotational invariance of closed string.

C.4 Details on factorization of non planar one loop amplitudes.

In this section we want to describe the details of the computations necessary to derive the final expression in eq. (126). This amounts essentially to follow the steps as in ([21]) taking care of some more factors.

We start showing that

$$\begin{aligned}
&\mathcal{A}(k,\ldots N_0,1,\ldots k-1;N_0+1,\ldots N_0+N_\pi) \\
&''=''\prod_{k<r<s\leq N_0}e^{i2\pi\alpha' k_{ri}\Theta^{ij}_{tot}k_{sj}}\prod_{k<r<s\leq N_0}e^{-i2\pi\alpha' k_{ri}\Theta^{ij}_{tot}k_{sj}}\ \mathcal{A}(1,\ldots N_0;N_0+1,\ldots N_0+N_\pi) \\
&= \mathcal{A}(1,\ldots N_0;N_0+1,\ldots N_0+N_\pi)
\end{aligned} \tag{248}$$

where we have written $'' ='' $ since the lhs and the rhs differ in the range of integration variables only. The first phase in the second line is obtained while rewriting the non commutative phase in a canonical form, i.e. as in the $\mathcal{A}(1,\dots N_0; N_0+1,\dots N_0+N_\pi)$ amplitude. The second contribution arises because we redefine the integration variables from ν to $\bar\nu$ as $(k>1)$

$$\nu_r = \begin{cases} \bar\nu_r & r=1,\dots k-1 \\ \bar\nu_r - 1 & r=k,\dots N_0-1 \end{cases} \tag{249}$$

which gives a contribution

$$\prod_{s=k}^{N_0} e^{i2\pi\ \alpha' k_s^T(\frac{L m_0}{\sqrt{\alpha'}}+\Theta_{tot}\sum_{r=1}^{N_0} k_r)} = \prod_{k<r<s\le N_0} e^{-i2\pi\alpha' k_{ri}\Theta^{ij}_{tot} k_{sj}} \tag{250}$$

when we use that $L\sqrt{\alpha'}k_r \in \mathbb{Z}$. The reason why we redefine the integration variables is that the original integration variables ν in the $\mathcal{A}(k,\dots N_0,1,\dots k-1; N_0+1,\dots N_0+N_\pi)$ amplitude have range

$$0<\nu_k<\dots\nu_{N_0}<\nu_1<\dots\nu_{k-1}<1 \tag{251}$$

and a different ordering w.r.t. the $\mathcal{A}(1,\dots N_0; N_0+1,\dots N_0+N_\pi)$ amplitude while the new ones $\bar\nu$ have range

$$\nu_{N_0}<\bar\nu_1<\dots\bar\nu_{k-1}<1<\bar\nu_k<\dots\bar\nu_{N_0-1}<1+\nu_{N_0} \tag{252}$$

but the same ordering as in the amplitude $\mathcal{A}(1,\dots N_0; N_0+1,\dots N_0+N_\pi)$. If we now redefine the integration variables $\nu_r=\bar\nu_r-1$ for $r=1,\dots N_0$ in the $k=1$ amplitude $\mathcal{A}(1,\dots N_0; N_0+1,\dots N_0+N_\pi)$ the amplitude is left invariant. We can then perform the sum $\sum_{k=1}^{N_0}\mathcal{A}(k,\dots N_0,1,\dots k-1; N_0+1,\dots N_0+N_\pi)$ since the $\bar\nu$ variables cover different ranges of integration which can be joined together into a bigger range. The sum has then the same functional expression as the amplitude $\mathcal{A}(1,\dots N_0; N_0+1,\dots N_0+N_\pi)$ but with a different integration range, explicitly

$$\begin{aligned}
&\sum_{k=1}^{N_0}\mathcal{A}(k,\dots N_0,1,\dots k-1; N_0+1,\dots N_0+N_\pi) \\
&\quad = \quad \dots \int_0^1 d\nu_{N_0}\ e^{-i2\pi(\nu_{N_0}+1)\ \alpha' k_r^T(\frac{L m_0}{\sqrt{\alpha'}}+\Theta_{tot}\sum_{s=1}^{N_0}k_s)} \\
&\qquad\qquad \int_{\nu_{N_0}}^{\nu_{N_0}+1}\prod_{r=1}^{N_0-1} d\bar\nu_r\ \theta(\bar\nu_r-\bar\nu_{r+1})\ e^{-i2\pi\bar\nu_r\ \alpha' k_r^T(\frac{L m_0}{\sqrt{\alpha'}}+\Theta_{tot}\sum_{s=1}^{N_0}k_s)}\dots
\end{aligned} \tag{253}$$

where we have written only the pieces which differ from the original amplitude. Next we can change variables as

$$\phi_r = \begin{cases} 2\pi(1+\nu_{N_0}-\bar\nu_r) & r=1,\dots N_0-1 \\ 2\pi\bar\nu_r & r=N_0+1,\dots N_0+N_\pi \end{cases} \qquad \phi_{N_0}\equiv 0 \tag{254}$$

and get the result given in the main text whose main pieces are

$$\sum_{k=1}^{N_0} \mathcal{A}(k, \dots N_0, 1, \dots k-1; N_0+1, \dots N_0+N_\pi)$$
$$= \dots \frac{1}{(2\pi)^{N_0+N_\pi-2}} \int_0^1 d\nu_{N_0}\ e^{-i2\pi\nu_{N_0}\ \alpha' k_r^T \frac{Lm_0}{\sqrt{\alpha'}}}$$
$$\int_0^{2\pi} \prod_{r=1}^{N_0-1} d\phi_r\ \theta(\phi_r - \phi_{r+1})\ e^{+i\phi_r\ \alpha' k_r^T (\frac{Lm_0}{\sqrt{\alpha'}} + \Theta_{tot} \sum_{s=1}^{N_0} k_s)} \dots$$
$$\int_0^{2\pi} \prod_{r=N_0+1}^{N_0+N_\pi-1} d\phi_r\ \theta(\phi_{r+1} - \phi_r)\ e^{-i\phi_r\ \alpha' k_r^T (\frac{Lm_0}{\sqrt{\alpha'}} + \Theta_{tot} \sum_{s=1}^{N_0} k_s)} \dots \tag{255}$$

where we have $\phi_{N_0} = 0$ and $\phi_{N_0+N_\pi} = 2\pi$.

C.5 The annulus amplitude.

In this section we check the equation (130) which gives the normalization of the amplitudes by computing the annulus, explicitely we compute

$$\begin{aligned} Z &= -2 \times \frac{1}{2} Tr\left(\log(L_0 - 1)\right) = \mathcal{C}_1 \int_0^1 \frac{dw}{\ln w} Tr(w^{L_0-2}) \\ &= \mathcal{C}_1\ N_1^2\ \frac{\delta^{D-d}(0)}{(\sqrt{\alpha'})^{D-d}}\ [\det G_{\mu\nu}\ \det(L\mathcal{G}L)_{ij}]^{\frac{1}{2}} \\ &\times \int_0^1 \frac{dw}{w^2 \ln w}\ \frac{1}{[\prod_1^\infty (1-w^n)]^{D-2}}\ \left(-\frac{\ln w}{\pi}\right)^{D/2} \sum_{(m^i)\in\mathbb{Z}^d} e^{\frac{\pi^2}{\ln w} m^T L\mathcal{G}Lm} \end{aligned} \tag{256}$$

Changing variable as in eq. (123) and using the modular transformations given in eq. (124) we can rewrite the previous amplitude for $D = 26$ as

$$Z = \frac{\mathcal{C}_1}{2\pi}\ 2^{-D/2}\ N^2\ \frac{\delta^{D-d}(0)}{(\sqrt{\alpha'})^{D-d}}\ [\det \mathcal{G}_{MN}]^{\frac{1}{2}} \int_0^1 \frac{dq}{q^3}\ \frac{1}{[\prod_1^\infty (1-q^n)]^{D-2}} \sum_{(m^i)\in\mathbb{Z}^d} e^{\frac{1}{2}\ln q\ m^T L\mathcal{G}Lm} \tag{257}$$

which can be factorized on the tachyons as

$$Z \sim \frac{\mathcal{C}_1}{2\pi}\ 2^{-D/2}\ N^2\ \frac{\delta^{D-d}(0)}{(\sqrt{\alpha'})^{D-d}}\ [\det \mathcal{G}_{MN}]^{\frac{1}{2}} \sum_{(m^i)\in\mathbb{Z}^d} \frac{1}{\frac{1}{2} m^T L\mathcal{G}Lm - 2} \tag{258}$$

When we compare with the expected form

$$Z \sim \quad \int \frac{d^{D-d} k_C}{(2\pi)^{D-d} \, (2\pi\sqrt{\alpha'})^d} \sum_{n_C} \mathcal{A}(C) \, \mathcal{A}(-C) \, \frac{1}{k_C^T \mathcal{G}^{-1} k_C - \frac{4}{\alpha'}} \tag{259}$$

where C stands for the closed string tachyon appearing in the mixed amplitude (103), $-C$ the closed string tachyon with opposite momentum, we get

$$\frac{\mathcal{C}_1}{2\pi} \, 2^{-D/2} \, \left[\det \mathcal{G}_{MN}\right]^{\frac{1}{2}} = \left(\frac{\mathcal{C}_0(E, \hat{F}) \tilde{\mathcal{N}}_0(E)}{2\pi}\right)^2 \, (2\pi\sqrt{\alpha'})^D \frac{\alpha'}{2} \tag{260}$$

which reproduces the result given in the main text.

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