Klein-Gordon product: conservation

Consider

$$(\partial_x^2 + \partial_y^2)\, X_i(x,y) = \partial_u \partial_{\bar u} X_i(u,\bar u) = 0$$

Define:

$$J(X_1, X_2) = \mathcal{N} * (X_1^T \overleftrightarrow{d} X_2)$$

where:

$$\begin{cases} *dx = \varepsilon^{xy}\, dy = dy \\ *dy = \varepsilon^{yx}\, dx = -dy \end{cases} \longrightarrow \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \Rightarrow \begin{cases} dx = \frac{x}{r}\, dr - y\, d\theta \\ dy = \frac{y}{r}\, dr + x\, d\theta \end{cases}$$

then:

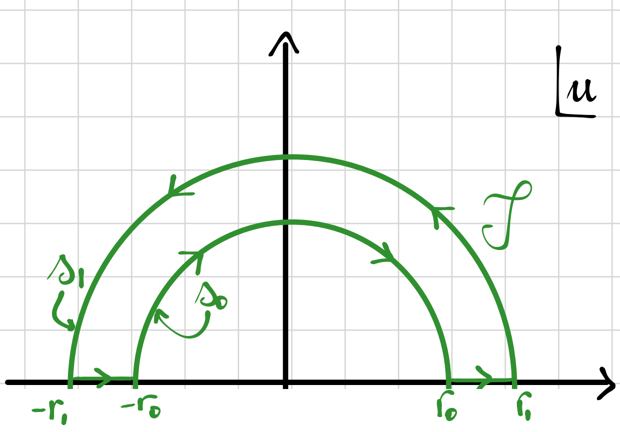
$$\begin{cases} *dx = -y * d\theta + \frac{x}{r} * dr = dy = \frac{y}{r}\, dr + x\, d\theta \\ *dy = x * d\theta + \frac{y}{r} * dr = -dy = -\frac{x}{r}\, dr + y\, d\theta \end{cases}$$

$$\Rightarrow \quad *dr = r\, d\theta \qquad *d\theta = -\frac{1}{r}\, dr$$

Since

$$\begin{aligned} dJ(X_1, X_2) &= \mathcal{N}\, d * \left(X_1 \overleftrightarrow{\partial_x} X_2\, dx + X_1 \overleftrightarrow{\partial_y} X_2\, dy\right) = \\ &= \mathcal{N}\, d\left(-X_1 \overleftrightarrow{\partial_y} X_2\, dx + X_1 \overleftrightarrow{\partial_x} X_2\, dy\right) = \\ &= \mathcal{N}\left[-\partial_y X_1\, \partial_y X_2\, dy\, dx + X_1 \partial_y^2 X_2\, dy\, dx + \partial_x X_1\, \partial_x X_2\, dx\, dy - X_1 \partial_x^2 X_2\, dx\, dy\right] = \\ &\overset{\text{IBP}}{=} \mathcal{N}\left[X_1 \overleftrightarrow{(\partial_x^2 + \partial_y^2)} X_2\right] dx\, dy = 0. \end{aligned}$$

Therefore choose:



Then:

$$0 = \oint_\gamma dJ(X_1, X_2) = \int_{r_0}^{r_1} J(X_1, X_2) + \int_{-r_1}^{-r_0} J(X_1, X_2) + \int_{s_1} J(X_1, X_2) - \int_{s_0} J(X_1, X_2).$$

$\rightarrow$ conserved if $\int_{r_0}^{r_1} J(X_1, X_2) + \int_{-r_1}^{-r_0} J(X_1, X_2)$ does not depend on the radius r.

Remember $\mathcal{U}_{(t)}^2=\mathbb{I}\Rightarrow \mathcal{U}_{(t)}=\mathcal{U}_{(t)}^{-1}=\mathcal{U}_{(t)}^T \Rightarrow P^T_{\parallel,\perp(t)}=P_{\parallel,\perp(t)}$.

Therefore, we can use:

$$\begin{aligned} X_i(x+i0^+,x-i0^+) &= 2X_R^{\parallel(t)}(x-i0^+)+\Delta_{(t)} = \\ &= 2P_{\parallel(t)}X_R(x-i0^+) + f_{(t)}^{\perp(t)} + Y^{\parallel(t)} - 2i\,\mathrm{Im}\,X_R^{\parallel(t)} \end{aligned}$$

(1) BOUNDARY CONTRIBUTION: $(r_0,r_1)\subset(x_t,x_{t-1})$; $\begin{cases}\partial_x f(x\pm iy)=\frac{\partial u}{\partial x}f'(u)=f'(u)\\ \partial_y f(x\pm iy)=\frac{\partial u}{\partial y}f'(u)=\pm i f'(u)\end{cases}\to \partial_y f=\pm i\partial_x f$

$$-\mathcal{N}\int_{r_0}^{r_1}dx\left[X_1^T\partial_y X_2 - \partial_y X_1^T X_2\right] = -\mathcal{N}\int_{r_0}^{r_1}dx\left[(X_{1L}+X_{1R})^T\partial_y(X_{2L}+X_{2R}) - \partial_y(X_{1L}+X_{1R})^T(X_{2L}+X_{2R})\right]=$$

$$=-i\mathcal{N}\int_{r_0}^{r_1}dx\left[(X_{1L}+X_{1R})^T(X'_{2L}-X'_{2R}) - (X'_{1L}-X'_{1R})^T(X_{2L}+X_{2R})\right]=$$

$$=-i\mathcal{N}\int_{r_0}^{r_1}dx\left[X_{1L}^TX'_{2L} - X'^T_{1L}X_{2L} - X^T_{1L}X'_{2R} - X'^T_{1L}X_{2R} + X^T_{1R}X'_{2L} + X'^T_{1R}X_{2L} - X^T_{1R}X'_{2R} + X'^T_{1R}X_{2R}\right]=$$

$$\begin{aligned}=-i\mathcal{N}\int_{r_0}^{r_1}dx\Big\{&[X_{1R}^T\mathcal{U}_{(t)}+\Delta^T_{(t)1}]\mathcal{U}_{(t)}X'_{2R} - X'^T_{1R}\mathcal{U}_{(t)}[\mathcal{U}_{(t)}X_{2R}+\Delta_{(t)2}] - [X^T_{1R}\mathcal{U}_{(t)}+\Delta^T_{(t)1}]X'_{2R} - X'^T_{1R}\mathcal{U}_{(t)}X_{2R} + \\ &+ X^T_{1R}\mathcal{U}_{(t)}X'_{2R} + X'^T_{1R}[\mathcal{U}_{(t)}X_{2R}+\Delta_{(t)2}] - X^T_{1R}X'_{2R} + X'^T_{1R}X_{2R}\Big\}=\end{aligned}$$

$$\begin{aligned}=-i\mathcal{N}\int_{r_0}^{r_1}dx\Big\{&\cancel{X^T_{1R}X'_{2R}} + \Delta^T_{(t)1}\mathcal{U}_{(t)}X'_{2R} - \cancel{X'^T_{1R}X_{2R}} - X'^T_{1R}\mathcal{U}_{(t)}\Delta_{(t)2} - \cancel{X^T_{1R}\mathcal{U}_{(t)}X'_{2R}} - \Delta^T_{(t)1}X'_{2R} - \\ &- \cancel{X'^T_{1R}\mathcal{U}_{(t)}X_{2R}} + \cancel{X^T_{1R}\mathcal{U}_{(t)}X'_{2R}} + \cancel{X'^T_{1R}\mathcal{U}_{(t)}X_{2R}} + X'^T_{1R}\Delta_{(t)2} - \cancel{X^T_{1R}X'_{2R}} + \cancel{X'^T_{1R}X_{2R}}\Big\}=\end{aligned}$$

$$=-i\mathcal{N}\int_{r_0}^{r_1}dx\left\{\Delta^T_{(t)1}(\mathcal{U}_{(t)}-\mathbb{I})X'_{2R} - X'^T_{1R}(\mathcal{U}_{(t)}-\mathbb{I})\Delta_{(t)2}\right\}=$$

$$=-2i\mathcal{N}\int_{r_0}^{r_1}dx\left\{X'^T_{1R}P_{\perp(t)}\Delta_{(t)2} - \Delta^T_{(t)1}P_{\perp(t)}X'_{2R}\right\}=$$

$$=-2i\mathcal{N}\int_{r_0}^{r_1}dx\left\{X'^T_{1R}f^{\perp(t)}_{(t)2} - f^{\perp(t)T}_{(t)1}X'_{2R}\right\}=$$

$$=-2i\mathcal{N}\left[X^T_{1R}(x)f^{\perp(t)}_{(t)2} - f^{\perp(t)T}_{(t)1}X_{2R}(x)\right]_{x=r_0}^{x=r_1}$$

And:

$$\left[\int_{r_0}^{r_1}+\int_{-r_1}^{-r_0}\right]J(X_1,X_2) = -2i\mathcal{N}\left\{\left[X^T_{1R}(x)f^{\perp(t)}_{(t)2} - f^{\perp(t)T}_{(t)1}X_{2R}(x)\right]_{x=r_0}^{x=r_1} + \left[X^T_{1R}(x)f^{\perp(t)}_{(t)2} - f^{\perp(t)T}_{(t)1}X_{2R}(x)\right]_{x=-r_1}^{x=-r_0}\right\}$$

$$= \mathcal{B}_+(r_1)-\mathcal{B}_+(r_0)+\mathcal{B}_-(-r_0)-\mathcal{B}_-(-r_1).$$

NB: if $(r_0,r_1)\supset(x_t,x_{t-1})\to\int_{r_0}^{r_1}=\int_{r_0}^{x_t}+\int_{x_t}^{x_{t-1}}+\int_{x_{t-1}}^{r_1}$ s.t.: $\mathcal{B}_+(x_t^-)=\mathcal{B}_+(x_t^+)$.

② BULK CONTRIBUTION:

$$\left.\begin{aligned}\partial_r f(re^{\pm i\theta}) &= \frac{\partial u}{\partial r} f'(u) = \frac{u}{r} f'(u)\\ \partial_\theta f(re^{\pm i\theta}) &= \frac{\partial u}{\partial \theta} f'(u) = \pm i u f'(u)\end{aligned}\right\} \Rightarrow \partial_r f = \mp \frac{i}{r}\partial_\theta f$$

$$\mathcal{N}\int_{0,r=r_1}^{\pi} d\theta\, r\left(X_1^T \partial_r X_2 - \partial_r X_1^T X_2\right) = \mathcal{N}\int_0^{\pi} d\theta\, r\left[(X_{1L}^T + X_{1R}^T)\partial_r(X_{2L}+X_{2R}) - \partial_r(X_{1L}+X_{1R})^T(X_{2L}+X_{2R})\right] =$$

$$= -i\mathcal{N}\int_{0,r=r_1}^{\pi} d\theta \left[(X_{1L}^T + X_{1R}^T)\partial_\theta(X_{2L}-X_{2R}) - \partial_\theta(X_{1L}^T - X_{1R}^T)(X_{2L}+X_{2R})\right] =$$

$$= -i\mathcal{N}\int_{0,r=r_1}^{\pi} d\theta\left\{X_{1L}^T\partial_\theta X_{2L} - \underbrace{\partial_\theta X_{1L}^T}_{\text{ADD}} X_{2L} - X_{1L}^T\partial_\theta X_{2R} - \underbrace{\partial_\theta X_{1L}^T}_{\text{SUB}} X_{2R} + X_{1R}^T\partial_\theta X_{2L} + \underbrace{\partial_\theta X_{1R}^T}_{\text{SUB}} X_{2L} - X_{1R}^T\partial_\theta X_{2R} + \underbrace{\partial_\theta X_{1R}^T}_{\text{ADD}} X_{2R}\right\}$$

$$= -2i\mathcal{N}\int_{0,r=r_1}^{\pi} d\theta\left\{X_{1L}^T\partial_\theta X_{2L} - X_{1R}^T\partial_\theta X_{2R}\right\} - i\mathcal{N}\left[-X_{1L}^T X_{2L} - X_{1L}^T X_{2R} + X_{1R}^T X_{2L} + X_{1R}^T X_{2R}\right]_{-r_1}^{r_1} =$$

$$= -2i\mathcal{N}\left[\int_0^{\pi} d\theta\, X_{1L}^T(r_1e^{i\theta})\,\partial_\theta X_{2L}(r_1e^{i\theta}) + \int_{-\pi}^{0} d\theta\, X_{1R}^T(r_1e^{i\theta})\,\partial_\theta X_{2R}(r_1e^{i\theta})\right] + i\mathcal{N}\left[X_1^T(r,r)\, X_2(r,r)\right]_{-r_1}^{r_1}$$

$$= -2i\mathcal{N}\oint_{|z|=r_1} dz\, X_1^T(z)\frac{d}{dz}X_2(z) + i\mathcal{N}\left[X_1^T(r,r)X_2(r,r)\right]_{-r_1}^{r_1}$$

And:

$$\left[\int_{S_1} - \int_{S_0}\right] J(X_1,X_2) = -2i\mathcal{N}\left[\oint_{|z|=r_1} - \oint_{|z|=r_0}\right] X_1^T(z)\frac{d}{dz}X_2(z) + i\mathcal{N}\left[X_1^T(r,r)X_2(r,r)\right]_{-r_1}^{r_1} - i\mathcal{N}\left[\dots\right]_{-r_0}^{r_0}$$

Since $\int_S dJ(X_1,X_2)$ is conserved, we define the KLEIN-GORDON PRODUCT:

$$(X_1,X_2)_{KG} = -2i\mathcal{N}\oint_{|z|=r} dz\, X_1^T(z)\frac{dX_2(z)}{dz} + i\mathcal{N}\left[X_1^T(r',r')X_2(r',r')\right]_{r'=-r}^{r'=r} - 2i\mathcal{N}\left[X_{1R}^T(r)\, f_{(t)2}^{\perp(t)} - f_{(t)1}^{\perp(t)T} X_2(r)\right] + 2i\mathcal{N}\left[X_{1R}^T(-r)\, f_{(t')2}^{\perp(t')} - f_{(t')1}^{\perp(t')T} X_2(-r)\right].$$