SEIBERG - WITTEN THEORY

- SUSY in $d = 4$

* Spinors: $\quad Q^i_\alpha \quad \tilde{Q}_{\dot\alpha i} \quad \alpha, \dot\alpha = 1,2 \quad i = 1, \dots, N$

SUSY charges transf. as N copies of a spinor

$\Rightarrow$ In Lorentzian sign. $\quad Q_\alpha = (\tilde{Q}_{\dot\alpha})^\dagger$

$\longrightarrow$ same no. left and right

* algebra:

$$\{ Q^i_\alpha, \tilde{Q}_{\dot\alpha j} \} = \delta^i_j \, \sigma^\mu_{\alpha\dot\alpha} P_\mu \qquad \sigma^\mu = (\mathbb{I}, \vec\sigma)$$

$$\{ Q^i_\alpha, Q^j_\beta \} = 0$$

$$\{ \tilde{Q}_{\dot\alpha i}, \tilde{Q}_{\dot\beta j} \} = 0$$

Bosonic symm $\Rightarrow$ Poincaré symm (+ conf. symm)
(+ commuting internal symm.)

$\hookrightarrow$ Coleman-Mandula: "the S-matrix is trivial if $\exists$ more bosonic symm. other than those (i.e.: free theory) if Lorentz invariant"

NB $\quad (\frac{1}{2}, 0) \otimes (0, \frac{1}{2}) = (\frac{1}{2}, \frac{1}{2}) \longrightarrow$ vector $\Longrightarrow P_\mu$ (+ K_μ if the case)

$\Rightarrow \{ Q, \tilde{Q} \} \sim P$

but $\quad (\frac{1}{2}, 0) \otimes (\frac{1}{2}, 0) = \underbrace{(1, 0)}_{\text{self-dual}} \oplus \underbrace{(0, 0)}_{\text{scalar}} \quad$ [similar for $(0, \frac{1}{2}) \otimes (0, \frac{1}{2})$]

not there if it is not conf. inv $\qquad$ see later (central charges)

IRREPS of SUSY ALGEBRA

* $\boxed{M > 0}$

- Rest frame: $P_\mu = (M, \vec{0})$

$$\{ Q^i_\alpha , \widetilde{Q}_{\beta j} \} = \delta^i_j \, \delta_{\alpha\beta} \, M$$

$\Rightarrow$ $2N$ fermion creation/annihilation op.

↳ irrep: dimension 2^{2N}

* $\boxed{M = 0}$

$\rightarrow$ $P_\mu = (E, 0, 0, E)$

$$\mathbb{I} + \sigma_z = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow \{ Q^i_\alpha , \widetilde{Q}_{\dot\alpha j} \} = \delta^i_j \, 2E \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{\alpha\dot\alpha}$$

Now we use $\widetilde{Q} = \overline{Q} = Q^\dagger$:

$\alpha = \dot\alpha = 2 \Rightarrow \{ Q^i_2 , \overline{Q}_{2i} \} = 0$ (no sum over i)

↓

the anticomm of an op. w/ its adjoint is POSITIVE SEMI-DEFINITE :

$\Rightarrow Q_2 = \widetilde{Q}_2 \equiv 0$ for $M = 0$

↳ N creation/annihilation op.

↳ irrep: dimension $= 2^N$
(w/ CPT: 2×2^N)

$\rightarrow$ unless N is rather small, we have $2^{2N} > 2 \times 2^N$.

Start by looking to small values of $\mathcal{N}$.

$\Rightarrow$ $\underline{\mathcal{N}=1}$:

* $\underline{M=0}$

- Helicity : $j = -3 \quad -\frac{5}{2} \quad -2 \ \dots \ -1 \dots \ 0 \dots \ 1 \ \dots$

for $\mathcal{N}=1$ $\exists$ 1 creat. + 1 annhil. op.

$\Rightarrow$ they are spinors transf. w/ ang. mom. $\pm\frac{1}{2}$ along the z-axis

$\hookrightarrow$ they rise/lower helicity

e.g.:

$$j = -2 \overset{\tilde{Q}}{\curvearrowright} -\frac{3}{2},\ \underbrace{\frac{3}{2},\ 2}_{CPT} \qquad Q: -\frac{3}{2} \to -2$$

$\Rightarrow$ for $\mathcal{N}=1$ ALWAYS 4 STATES

We shall cons. GAUGE TH.: $|j| \leq 1$

$$\frac{1}{-1} \quad \frac{1}{-\frac{1}{2}} \quad \frac{}{0} \quad \frac{1}{\frac{1}{2}} \quad \frac{1}{1} \Rightarrow \text{VECTOR MULTIPLET}$$

$$\frac{}{-1} \quad \frac{1}{-\frac{1}{2}} \quad \frac{2}{0} \quad \frac{1}{\frac{1}{2}} \quad \frac{}{1} \Rightarrow \text{CHIRAL MULTIPLET}$$

$\Rightarrow$ the chiral mult. can get a mass in SUSY fashion
the vector mult. needs a Higgs mechanism

$\Rightarrow \boxed{\mathcal{N}=2}$

$* \boxed{M=0}$

2 creation + 2 ann. op:

$$\begin{array}{cccc} 1 & 2 & 1 & \\ \_ & \_ & \_ & \_ \\ -j & -j+\frac{1}{2} & -j+1 & -j+\frac{3}{2} \end{array}$$

$\Rightarrow$ we still need the CPT inv. $\longrightarrow$ e.g. if $j=1$ then we need another multiplet

$\Rightarrow$ case of interest: $j=\frac{1}{2}$:

$$\begin{array}{ccc} 1 & 2 & 1 \\ \_ & \_ & \_ \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{array}$$

$\Rightarrow$ can be CPT self-conj.
(same spectrum of $\mathcal{N}=1$ CPT invariant rep w/ $j=\frac{1}{2}$)

If self-conj: HALF-HYPER (mult.)
If not: HYPER

$\Rightarrow$ is it possible to have $\mathcal{N}=2$ th. w/ a massless spectrum such as this?

$\longrightarrow$ It actually fails to be CPT invariant:

CPT = real quantum states
(the rep. of Hermitian op. is a real rep. on the quantum Hilb. space: commutes w/ CPT)

Consider the $2Q$, $2\tilde{Q}$ which are $\neq 0$.
They are Herm. op:

$$\begin{array}{l} Q+Q^\dagger \\ i(Q-Q^\dagger) \end{array} \quad \text{w/ 2 choices of } Q_s$$

Now consider an extension to the previous algebra:

$$\{ Q^i_\alpha , Q^j_\beta \} = \varepsilon_{\alpha\beta} \, \varepsilon^{ij} \, Z$$

$$\{ \tilde{Q}_{\dot\alpha i} , \tilde{Q}_{\dot\beta j} \} = \varepsilon_{\dot\alpha\dot\beta} \, \varepsilon_{ij} \, \bar{Z}$$

$\mathcal{N} = 2$ CENTRAL CHARGES

Let $A = Q_1^1$, $B = Q_2^2$ for $M > 0$ reps:

$$\{ A, A^\dagger \} = \{ B, B^\dagger \} = M$$
$$\{ A, B \} = Z$$
$$\{ A^\dagger, B^\dagger \} = \bar{Z}$$

Then consider a $M > 0$ rep. of the alg. in the rest frame:

$$\{ (A + \lambda B^\dagger), (A + \lambda B^\dagger)^\dagger \} = M(1 + \lambda\bar\lambda) + \bar\lambda Z + \lambda \bar{Z}$$

$\Rightarrow$ if $\exists \lambda \mid$ RHS $= 0 \Rightarrow A + \lambda B^\dagger = 0, \; (A + \lambda B^\dagger)^\dagger = 0$

$\hookrightarrow$ SMALLER REP. OF THE ALG.

Therefore:

$$\left.\begin{aligned} \lambda &= -\frac{\bar{Z}}{M} \\ \bar\lambda &= -\frac{Z}{M} \end{aligned}\right\} \quad \text{RHS} = M\left(1 - \frac{Z\bar{Z}}{M^2}\right)$$

$\Rightarrow$ we find $M \geq \sqrt{Z\bar{Z}} = |Z|$

(if $M = |Z|$ the irrep is smaller)

If $M > |Z| \longrightarrow$ 16 states AS IF $Z = 0$

$M = |Z| \longrightarrow$ 4 states + 4 CPT states $\Rightarrow$ 8 states

Given a single massless multiplet in $\mathcal{N} = 2$, it can become massive if we change the coupling const. preserving $\mathcal{N} = 2$ SUSY.

The deformed theory MUST HAVE A CENTRAL CHARGE Z, though. And

$$M = |Z|$$

$\Rightarrow$ its mass can be predicted by susy properties.

$\mathcal{N}=2$ THEORY w/ GAUGE FIELDS

- Field content:

complex scalar

$$A_\mu \quad \lambda_1 \quad \lambda_2 \quad \phi$$

$\mathcal{N}=1$ vector mul. $\qquad$ $\mathcal{N}=1$ chiral mul.

$\mathcal{N}=1$ action:

* $$\frac{1}{e^2}\int d^4x\, d^2\theta \; \mathrm{Tr}\, W_\alpha W^\alpha + \text{c.c.}$$

$$\downarrow$$

$$\frac{1}{e^2}\int d^4x \; \mathrm{Tr}\left(F_{\mu\nu}F^{\mu\nu} + i\bar{\lambda}^1 \not{D} \lambda^1 + D^2 \right) \Rightarrow \text{same rep as vector}$$

$\Rightarrow$ ADJ of G

* Consider then the chiral mult. Φ: it has to be in the adj. rep. as well since it's in the same $\mathcal{N}=2$ mult.

$$\frac{1}{e^2}\int d^4x\, d^4\theta \; \mathrm{Tr}\, \bar{\Phi}\, e^V \Phi$$

needed to show some symm. w/ the gauge field once we rescale

$$\downarrow$$

$$\frac{1}{e^2}\int d^4x \; \mathrm{Tr}\Big(D_\mu\phi D^\mu\phi + \bar{\lambda}^2 i\not{D}\lambda^2 + D[\phi,\bar{\phi}] +$$

$$+ \#\underbrace{[\bar{\lambda}^1,\bar{\lambda}^2]}\phi + \#\underbrace{[\lambda^1,\lambda^2]}\bar{\phi} \Big)$$

fixed by Lorentz inv. $\Rightarrow$ it does not fix ϕ or $\bar{\phi}$ but int. symm. do

GLOBAL SYMM.

* $U(1) \times U(1)_R$

$\Phi \to e^{i\alpha}\Phi$

$\Phi = \underbrace{\phi + \theta\lambda}_{\text{they all have charge } +1} + \ldots$

$\bar{\Phi} = \underbrace{\bar{\phi} + \bar{\theta}\bar{\lambda}}_{\text{charge } -1} + \ldots$

It does not affect the vector multiplet

$\Downarrow$

λ^2 must have $\bar{\phi}$

$\bar{\lambda}^2$ " ϕ

in the interaction terms.

R-symmetry:

$x \to x$

$\theta \to e^{i\beta}\theta$

$\bar{\theta} \to e^{-i\beta}\bar{\theta}$

$\Downarrow$

$\phi \to \phi$

$\lambda^2 \to e^{-i\beta}\lambda^2$

$\bar{\lambda}^2 \to e^{i\beta}\bar{\lambda}^2$

$\Downarrow$

$\lambda^1 \to e^{i\beta}\lambda^1$

$\bar{\lambda}^1 \to e^{-i\beta}\bar{\lambda}^1$

NB: Yukava coupl.: $\frac{1}{2}\mathrm{Tr}\left(\underbrace{\varepsilon_{ij}\varepsilon^{\alpha\beta}}_{\substack{\text{explicitly antisymmetric} \\ \text{in the indices}}}\lambda_\alpha^i\lambda_\beta^j\bar{\phi}\right)$

$\Rightarrow$ gauge ind. are antisymm. because they are couplings in the adj. rep. (gluon coupl. depend on comm.)

The Lagr. gets an extra symm. by accident:

$SU(2)$ acting on FERMIONS ONLY

$\begin{pmatrix}\lambda^1\\ \lambda^2\end{pmatrix} \to M\begin{pmatrix}\lambda^1\\ \lambda^2\end{pmatrix}$ $\quad \det M = 1$

We can relax $\det M = 1$ if we transform

$$\phi \longrightarrow (\det M)\,\phi$$

$\Rightarrow \quad M \in U(2)$.

We then have $U(2)$ symm.

Of course the obvious symm $U(1) \times U(1)$ is s.t.

$$U(1) \times U(1) \subset U(2)$$

$$\begin{pmatrix} * & \\ & * \end{pmatrix}$$

The realization by $\mathcal{N}=1$ SUSY enables us to save the diag. part of $U(2)$.

↳ the $U(2)$ does not commute w/ $\mathcal{N}=1$ SUSY

$$\left[\mathcal{N}=1 \text{ alg.}: \quad \delta A_\mu = \bar{\varepsilon}\,\gamma_\mu \lambda^1, \quad \text{etc.}\right]$$

The $U(2)$ symm. mixes λ^1 w/ λ^2 $\Rightarrow$ $\exists$ the same $\mathcal{L}$ w/ λ^1 replaced by λ^2.

↳ promotion of $\mathcal{N}=1$ SUSY to $\mathcal{N}=2$ SUSY

The $\mathcal{N}=2$ general. is

$$\delta A_\mu = \sum_{i=1}^{2} \bar{\varepsilon}_i\,\gamma_\mu \lambda^i, \quad \text{etc.}$$

DYNAMICS (Classical)

After integrating out D:

$$V(\phi, \bar{\phi}) = e^2 \, \mathrm{Tr}\left(i \left[\phi, \bar{\phi} \right] \right)^2$$

Then $V = 0 \iff [\phi, \bar{\phi}] = 0$

$\Downarrow$

they can be diagonalized simoultaneously

$\Downarrow$

by $SU(2)$ gauge transf:

$$\phi = \begin{pmatrix} a & \\ & -a \end{pmatrix}$$

However a is not gauge inv. but $\mathrm{Tr}\,\phi^2$ is:

$$\mathrm{Tr}\,\phi^2 = 2a^2 = u \in \mathbb{C}$$

It labels the possible classical/quantum vacua of the theory

Classically we have:

u

$u=0$ $\times$

$\phi = 0 \Rightarrow SU(2)$ gauge symm. is unbroken

When $u \neq 0 \Rightarrow \cancel{SU(2)} \longrightarrow U(1)$ $\longrightarrow$ Coulomb forces

strong interact. (in general $SU(3)$)

So, if $u \neq 0$ the classical picture works while approaching $u \to 0$ the gauge dynamics become strong.

Consider the spectrum for $u \neq 0$:

MASSLESS: same as for $U(1)$ gauge th.

1 2 1
1 2 1
$j = -1 \quad \dots \quad 1$

[$U(1)$ gauge field + superpartners]

$\Rightarrow$ massless $U(1)$ vector multiplet : 8 states

Now let's look at W bosons (the part which got mass):

$$W^+ = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$$

1 2 1
1 2 1
$j = -1 \quad \dots \quad 1$

$\Rightarrow$ the rep. didn't get bigger just because they're massive $\longrightarrow$ they're still 8 (massive) STATES

MASS + 8 HELICITY STATES

$\Rightarrow$ THERE HAS TO BE A CENTRAL CHARGE!

(short multiplet)

So, what is the central charge? What is the only possible bosonic piece we can insert?

$$\{ Q^i_\alpha , Q^j_\beta \} = \frac{1}{e^2} \int d^3x \; \partial^i \left(\phi F^{0j+} \right)$$

where

$$F^+_{0i} = \frac{1}{2} \left(F_{0i} + \frac{i}{2} \varepsilon_{ijk} F^{jk} \right)$$

$$\left(\vec{F}^+_{0i} = \frac{1}{2} \left(\vec{E}_i + i \vec{B}_i \right) \right)$$

We can write this as a surface term at ∞:

$\to$ the $\vec{E}$ at ∞ measures the electric charge and $\vec{B}$ the magn. charge:

$$\{ Q^i_\alpha , Q^j_\beta \} = a \left(Q_{el} + \frac{i}{e^2} Q_{mag} \right)$$

w/ the current normalization of $\mathcal{L}$ the electric field due to unit charge is $\propto e^2$ and cancels the $\frac{1}{e^2}$.

[Olive, Witten 1978]

<u>Classically</u> the central charge is $Z = a \left(Q_{el} + \frac{i}{e^2} Q_{mag} \right)$

$$\to \quad M \geq |a| \sqrt{Q_{el}^2 + \left(\frac{Q_{mag}}{e^2} \right)^2}$$

For small (BPS) rep:

$$M = |a| \sqrt{Q_{el}^2 + \left(\frac{Q_{mag}}{e^2} \right)^2}$$

and for W bosons, $Q_{mag} = 0$:

$$M = \underbrace{|a|}\, |Q_{el}|$$

Higgs formula
(the W boson mass is the absolute value of the Higgs field $\Rightarrow |a|$)

$$\langle \phi \rangle = \begin{pmatrix} a & \\ & -a \end{pmatrix}$$

The factor i in the central charge is important: Q is chiral $\Rightarrow$ self-dual field strength $\Rightarrow$ complex gauge field in Lorentzian signature $(E + iB)$.

Therefore we get

$$M \geq |a| \sqrt{Q_{el}^2 + \left(\frac{Q_{mag}}{e^2}\right)^2 + \underbrace{0}_{} \times Q_{el} \cdot Q_{mag}}$$

0 due to CP symmetry

→ we can break CP symm. adding a θ-angle and, while SUSY is untouched, the central charge has to change.

Here we built the $\mathcal{N}=2$ theory by hand. Now consider:

MINIMAL SYM in D dimensions:

A_μ, λ (fermion in the smallest rep)

$$\Rightarrow \quad \frac{1}{e^2} \int d^4x \, \mathrm{Tr}\left(F_{\mu\nu} F^{\mu\nu} + \bar{\lambda} \, i \not{D} \, \lambda \right)$$

(λ in adj. rep. for SUSY)

If SUSY:

$$\delta A_\mu = \bar{\epsilon} \, \Gamma_\mu \lambda$$

$$\delta \lambda = \Gamma^{\mu\nu} F_{\mu\nu} \epsilon$$

Is it susy? The term linear in λ CANCELS in ANY DIMENSION:

→ variation:
$$4 \int d^4x \, \mathrm{Tr}\left(F_{\mu\nu} D^\mu (i \bar{\epsilon} \Gamma^\nu \lambda) + \frac{1}{2} \bar{\lambda} \, i \Gamma^\mu D_\mu \left(\Gamma^{\alpha\beta} F_{\alpha\beta} \epsilon\right)\right)$$
$$= 0$$

since
$$\Gamma^\mu \Gamma^{\alpha\beta} = \underbrace{\Gamma^{\mu\alpha\beta}}_{\text{antisymm}} + \underbrace{g^{\mu\alpha} \Gamma^\beta - g^{\mu\beta} \Gamma^\alpha}_{\text{symm}}$$

Then $\Gamma^{\mu\alpha\beta} D_\mu F_{\alpha\beta} \equiv 0$ for the Bianchi identity.

The part w/ one Γ matrix can be carried out by int. by parts.

However there's a cubic part in λ in $\bar{\lambda}\, i \not{D} \lambda$ if we vary A_μ in D_μ :

$$\frac{\delta}{\delta A_\mu}\left(\bar{\lambda}\,\Gamma^\mu [A_\mu,\lambda]\right) = f^{abc}\, \lambda_a^\alpha\, \Gamma_{\mu\alpha\beta}\, \lambda_b^\beta (\Gamma^\mu)^{\gamma\delta}\, \lambda_{\gamma c}\, \varepsilon_\delta$$

it has to be completely antisymm. in the gauge indices

$\Rightarrow$ only hope to be 0 is that $\Gamma_{\mu\alpha\beta}\Gamma^\mu{}_{\gamma\delta} + (\text{sym } \alpha\beta\gamma) = 0$

↳ ONLY TRUE IN $D = 3, 4, 6, 10$

$(D=4 \rightarrow \mathcal{N}=1 \text{ SYM})$

Consider now $D=6$:

$$\int d^6x \, \mathrm{Tr}\left(F_{\mu\nu}F^{\mu\nu} + i\bar{\lambda}\,\Gamma^\mu D_\mu \lambda\right)$$

→ How do we build a $D=4$ $\mathcal{N}=2$ theory from this?

⇒ Brutally take the fields independent of the last two dimensions:

$$\underbrace{X^0, \dots, X^3}_{\text{Minkowski spacetime}},\ \underbrace{X^4, X^5}_{\text{drop dependence}}$$

Choose $\phi = A_4 + i A_5$ (complex scalar in adj. rep.)

$A_\mu \quad \mu = 0, \dots, 3$

$\lambda \rightarrow 2$ spinors in $D=4$ $\quad [D=6 \Rightarrow 2^{\frac{6}{2}} = 8 \text{ comp.}]$

It then becomes the same $\mathcal{L}$ as before:

$$\begin{matrix} \mathcal{N}=1 \text{ minimal SYM} \\ D=6 \end{matrix} \longrightarrow \begin{matrix} \mathcal{N}=2 \text{ SUSY} \\ D=4 \end{matrix}$$

$[\mathcal{N}=1 \ D=10 \longrightarrow \mathcal{N}=4 \ D=4]$

This gives new explanation on why there are central charges in the SUSY alg:

$$\{Q_\alpha, \tilde{Q}_\beta\} = \sum_{\mu=0}^{5} \sigma^\mu_{\alpha\beta} P_\mu =$$

$$= \sum_{\mu=0}^{3} \sigma^\mu_{\alpha\beta} P_\mu + \underbrace{\sum_{\mu=4}^{5} \sigma^\mu_{\alpha\beta} P_\mu}$$

$\Downarrow$

P_μ commutes w/ itself and the SUSY charges and w/ $D=4$ Lorentz transf. (all we got left when we dropped the dep on $\mu=4,5$)

They become CENTRAL CHARGES (electric) $\Longleftarrow$

What is λ, though? In $D=3,4,6,10$ they are ferm. in adj rep. w/ the min. possible no. of components. More specifically:

* $D=6$: we have 6 Γ mat.: $\underbrace{\Gamma^0, \dots, \Gamma^5}$

combine them in 3 creation and 3 annihilation operators:

CREATION: $\Gamma^0 + i\Gamma^1, \ \Gamma^2 + i\Gamma^3, \ \Gamma^4 + i\Gamma^5$

ANNHILATION: $\Gamma^0 - i\Gamma^1, \ \Gamma^2 - i\Gamma^3, \ \Gamma^4 - i\Gamma^5$

$\rightarrow$ Clifford algebra $\Rightarrow$ 8 states (2^3)

eigenval. : ± 1

In $D=6$ we have $\underbrace{\Gamma^0 \dots \Gamma^5}_{\text{chirality op.}} \lambda = +\lambda$ (since $(\Gamma^0 \dots \Gamma^5)^2 = \mathbb{I}$)

in $D=4$ $(\Gamma^0 \dots \Gamma^3)^2 = -1$

eigenv. : $\pm i$

in $D=4$, if we have $\leftarrow$ (exchanged by CPT) one kind of spinor then the Hermitian adj. is of the other kind.

In Lorentzian signature, in $D=6$ however we have chirality conditions:

$\rightarrow$ apparently : $\frac{1}{2} \times 2^3 = 4$ components

However $SO(1,5)$ has indeed a 4d Spin_+ rep BUT IT'S PSEUDOREAL : we need to double it $\Rightarrow$ 8 real comp.

The SUSY generator ε has 8 comp. as well [in $D=4$ it reduces to $\mathcal{N}=2$ SUSY $\Rightarrow 2 \times 4 = 8$]

in $D=4$: $(\Gamma^0 \dots \Gamma^3)\lambda = -\Gamma^4 \Gamma^5 \lambda$

$(\Gamma^0 \dots \Gamma^3)\varepsilon = -\Gamma^4 \Gamma^5 \varepsilon$

w/o internal symm. we have both chiralities

What about GLOB. symm.?

Now consider the fermion field:

x is to double its comp. to make it real

$\lambda^\alpha \longrightarrow \lambda^{ax}$ where $a = 1, \dots, 4$ [$SO(1,5)$ index]

$x \rightarrow SU(2)$ index

$\Rightarrow \varepsilon_{xy} \lambda^{ax} \Gamma^\mu_{ab} D_\mu \lambda^{by} \Rightarrow SO(1,5)$ AND $SU(2)$ invariant

$\Rightarrow \exists\ SU(2)_R$ global symm. that only acts on fermions $\Rightarrow$ exactly as before in manually constructing the model.

$\Rightarrow \exists\ U(1)_R$ symm. on bos. AND ferm. $\Rightarrow$ rotates A_4, A_5 of the 6d gauge field which are SCALARS in $D=4$:

λ has charge $\pm\frac{1}{2}$ of the charge of the bosons

$[\,A_4, A_5 \longrightarrow$ charge $\pm 2 \Rightarrow \lambda$: charge $\pm 1\,]$

How do we show that this is SUSY? We use:

$$\lambda^\alpha \lambda^\beta \lambda^\gamma \gamma^\mu_{\alpha\beta} \gamma_{\mu\gamma\delta} \varepsilon^\delta + (\text{symm. } \alpha\beta\gamma) = 0$$

In detail:

$$\lambda^{a_1 x_1} \lambda^{a_2 x_2} \lambda^{a_3 x_3} \dots \varepsilon^{a_4 x_4}$$

The only inv. in the rep. is $\varepsilon_{a_1 a_2 a_3 a_4}$ ($SO(1,5)$ inv.)
but $\nexists$ a way to antisymm the $SU(2)$ indices (since $a_1 \dots a_4$ can only be antisymm. then we need to antisym. the others, too) because they take only 2 values and cannot antisymm. 3 doublets. $\Rightarrow$ the comb. CAN ONLY BE 0.

$\hookrightarrow$ THE MODEL IS SUPERSYMMETRIC.

We still need to find the other term in the central charge.
Consider the case of WEAK COUPLING.

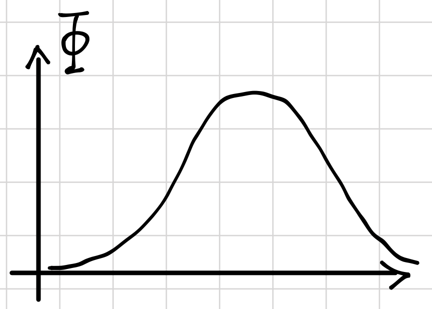
$\rightarrow$ two types of particles:
- quantize small oscillations
  $\hookrightarrow$ each field = one part.

• quantize classical "solitons"

→ Take non linear field eq. and find a solution

$\Phi(\vec{x})$ * indep. of time
* stable sol.

↳ "Quantization" (bosons):



→ assume <u>unique</u> (except for whatever we can deduce from symm.), <u>stable</u> sol.

⇒ ∃ family of class. sol.

$$\Phi_a(\vec{x}) = \Phi(\vec{x}+\vec{a})$$

We must build a quant. wave funct. that dep. on a : $\Psi(\vec{a})$ (a: central position of the lump). E.g.:

$$\Psi(a) = \exp(i\vec{p}\cdot\vec{a})$$

in mom. eigenstate. The energy the will be:

$$H = \int d^3x \left(\frac{e^2}{2}\dot{\phi}^2 + \underbrace{\frac{1}{2e^2}(\nabla\phi)^2 + V(\phi)}_{M\,:\ \text{mass of the lump}}\right)$$

CLASSICALLY to minimize the energy we take $\dot{\phi} = 0$

QUANT. MECH. $\frac{1}{2}\dot{\phi}^2$ it's going to turn into $-\frac{\hbar^2}{2}\left(\frac{\partial}{\partial\phi}\right)^2$ ⇒ the sol. can dep. on that

The important part of the sol. for the ground state are those classical var. which do not contribute to the energy → the ZERO MODES ⟹ $-\frac{\hbar^2}{2}\left(\frac{\partial}{\partial\phi}\right)^2 \longrightarrow -\frac{\hbar^2}{2}\cdot$ sth. $\frac{\partial^2}{\partial a^2}$.

$\Rightarrow \quad H\psi = \left(M + \frac{p^2}{2M}\right)\psi \longrightarrow$ it's the beginning of a relat. exp. $\sqrt{M^2+p^2}$

If we start w/ an action:

$$I = \frac{1}{e^2}\int (\dots)$$

(eg.: WEAK COUPLING $\Rightarrow e \ll 1$) then

$$M \sim \frac{1}{e^2}$$ $\Rightarrow$ near the classical limit the obj is heavy and slowly moving

Now look at SUSY:

$\mathcal{N}=2 \quad D=4 \longrightarrow 8$ SUSY charges:

$$Q^i_\alpha$$

$$\tilde{Q}_{\dot\alpha j}$$

where in $D=6$ notation: $\delta\psi = \Gamma^{\mu\nu}F_{\mu\nu}\,\varepsilon \qquad (\mu,\nu = 0,\dots,5)$

$$\Downarrow$$

$$\delta\psi = \Big(\sum_{\substack{a=0\\b=0}}^{3}\Gamma^{ab}F_{ab} + \sum_{a=0}^{3}\Gamma^{a4}D_aA_4 + (4\to5) + \\ + [A_4, A_5]\,\Gamma^{45}\Big)\varepsilon$$

For random solution $\longrightarrow$ 8 ferm. zero-modes $\lambda_1 \dots \lambda_8$

$$\Downarrow$$

$$\Lambda = \sum_{i=1}^{8} c_i\lambda_i \;+\; \text{non zero modes}$$

s.t. $I = \int \bar\Lambda\, i\not{D}\Lambda$.

For bosons we gave a a small time dep. and then we constructed wave functions depending on a.

For fermions we treat c_i in the same way:

$$I\big|_{c\ dep} = \sum_{i=1}^{8} \int dt \left(c_i \frac{dc_i}{dt} + 0 \cdot c^2 \right) +$$

$$+ \text{ non zero-modes}$$

⇒ no contr. from spatial derivatives because we took a classical sol. (time indep.), acted w/ SUSY and got another classical sol ⇒ the zero-modes (c_i) are solutions to the spatial part of the Dirac eqn.

Once we quantize we have $\{c_i, c_j\} = 2\delta_{ij}$

$\Downarrow$

$2^4 = 16$ states (generic supermult)

↓

[generic assumpt → gen. mult.] ← No lin. comb. of Qs acts as 0.

Now suppose

$A_5 \longrightarrow 0$ at ∞

$A_4 \neq 0$ 〃 (e.g.: for $SU(2)$ it's always possible)

and look for SUSY solutions to e.o.m. (Bogomolny eq.):

$$F_{ij} = \frac{1}{2} \varepsilon_{ijk} D_k A_4 \quad (\Rightarrow \text{Eul.-Lag. eq.})$$

For $SU(2)$ $\exists$ sol. w/ <u>magnetic charge</u> (the simplest sol. is a magnetic monopole)

NB: in this case, for half of the $\varepsilon \Rightarrow \delta\psi = 0$:

$$F_{12} = D_3 A_4 \Rightarrow \left(\Gamma^{12} F_{12} + \Gamma^{34} D_3 A_4\right)\varepsilon + \text{etc.} \quad (2\times 2 = 4 \text{ terms})$$

focus on these terms: $F_{12}\left(\Gamma^{12} + \Gamma^{34}\right)\varepsilon$

It is vanishing if:

$$\left(\Gamma^{12} + \Gamma^{34}\right)\varepsilon = 0$$

$$\Leftrightarrow \underbrace{\Gamma^{1234}}_{\substack{\text{traceless and}\\ \text{squares to } \mathbb{I}}}\varepsilon = \varepsilon \longrightarrow \text{the eq. is } SO(3) \text{ symm. for rotations of } 123.$$

traceless and squares to $\mathbb{I} \Rightarrow \begin{cases} 4 \text{ eigenv } +1 \\ 4 \quad '' \quad -1 \end{cases} \Rightarrow$ we only keep $+1$

The other half obeys $\Gamma^{1234}\tilde{\varepsilon} = -\tilde{\varepsilon}$.

$\Rightarrow$ Half of the ε obey $\delta\lambda = 0$

↳ 4 (not 8) 0-modes c_i:

$$I = \sum_{i=1}^{4} \int dt \; c_i \dot{c}_i \Rightarrow \{c_i, c_j\} = 2\delta_{ij}$$

(4-dim. Cliff. alg.: $2^{4/2} = 4$ st.)

MASSIVE HYPER. $\Leftarrow$ $\dfrac{1}{-\frac{1}{2}} \quad \dfrac{2}{0} \quad \dfrac{1}{\frac{1}{2}}$ (massive)

NB: CPT $\Rightarrow$ double the spectrum but magn. charge -1.

$\longrightarrow$ Monopole can carry el. charge $\longrightarrow$ classical sol is not inv. under $U(1)$ rot.

↳ $\cancel{SU(2)} \longrightarrow U(1) \Rightarrow$ new collective coord:

angular coord. $\longleftarrow \beta \in U(1) \; (S^1)$

The classical sol.:

$$\Phi = \Phi(x;\ \underset{\mathbb{R}^3}{\underset{\cap}{a}},\ \underset{S^1}{\underset{\cap}{\beta}})$$

$\Rightarrow$ wave function: $\Psi(a,\beta) = e^{ip\cdot a}\, \underbrace{e^{in\beta}}$

$\Psi(\beta) = \Psi(\beta+2\pi)$

eigenvalue of the rotation of the circle is the electric charge

monopoles w/ arbitr. integer electric charge $\leftarrow$

Now assume, more generically:

$$\Psi(\beta+2\pi) = \Psi(\beta)\, e^{i\theta} \Rightarrow \Psi(\beta) = e^{i(n+\frac{\theta}{2\pi})\beta}$$

"θ-angle"

$\Rightarrow$ in general $\int dt \int d^3x\ \mathrm{Tr}\,(\underbrace{F_{\mu\nu}F^{\mu\nu}}_{\vec{E}^2-\vec{B}^2} + \theta\, \underbrace{F_{\mu\nu}\tilde{F}^{\mu\nu}}_{\vec{E}\cdot\vec{B}})$

Evaluate as a funct. of β:

$$\int dt \left(\frac{1}{2}\dot{\beta}^2 + \theta\dot{\beta}\right)$$

$\Rightarrow$ suppose $\delta\beta = \text{const} \rightarrow \text{Symm} \Rightarrow Q_e$ conserved:

$$Q_e = \frac{\delta I}{\delta\dot{\beta}} = \dot{\beta} + \theta$$

$\Rightarrow$ we then find $Q_e = n_e + \frac{\theta}{2\pi}\underbrace{n_m}$

magn. charge that we assumed to be $= 1$.

The central charge is: $\left[Q_m = 1 \Rightarrow M = \frac{4\pi}{e^2} |a| \right]$

$$Z = a Q_e + i \frac{4\pi}{e^2} a Q_m =$$

$$= a n_e + n_m \underbrace{\left(\frac{\theta}{2\pi} + \frac{4\pi i}{e^2} \right)}_{\tau} a =$$

$$= a (n_e + n_m \tau).$$

This is the classical comp. What about the quantum corr?
(τ has to be renorm.)

QM $\Rightarrow$ asymptotically free $\Rightarrow$ β funct $\neq 0$ (poss. chiral anomaly?)
for large $u = \mathrm{Tr}\, \phi^2$

We have

$\phi = A_4 + i A_5$ charge 2 under $U(1)_R$

λ^i_α " 1 "

$\tilde{\lambda}_{\dot{\alpha} j}$ " -1 "

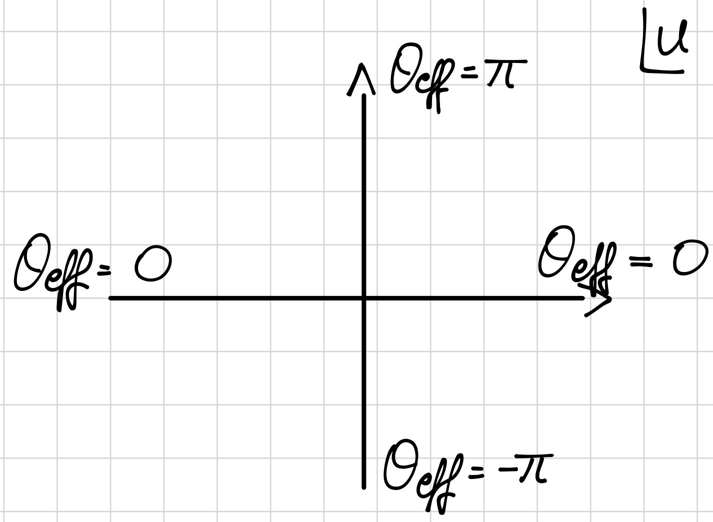
↳ Quantum anomaly in the instanton field:

8 λ_α 0-modes ($\mathcal{N} = 2$ in adj rep.)

$\Rightarrow$ $\cancel{U(1)_R} \to \mathbb{Z}_8$

and $u = \mathrm{Tr}\, \phi^2$ charge 4 and $u \overset{\mathbb{Z}_8}{\longleftrightarrow} -u$

We can perform a $U(1)_R$ transf. to rotate away θ but it also acts on u:



and we can take:

$$\theta_{eff} = 2 \arg(u)$$

Therefore everytime we go around the plane Q_e gets shifted by $2 \Rightarrow 2$ monopoles on positive u-axis:

1) positive even n_e
2) " odd n_e

↳ $\theta \to \theta + 2\pi$ does not interchange them

In the same way we can use superspace ($\mathcal{N} = 2$):

$$x^\mu \quad \theta^i_\alpha \quad \tilde{\theta}_{\dot{\alpha} j}$$

$$\Rightarrow Q_{\alpha i} = \frac{\partial}{\partial \theta^{\alpha i}} + i \tilde{\theta}^{\dot{\alpha}}_i \gamma^\mu_{\alpha \dot{\alpha}} \frac{\partial}{\partial x^\mu} \quad (\text{sim for } \tilde{Q}_{\dot{\alpha}}{}^j)$$

$$D_{\alpha i} = \frac{\partial}{\partial \theta^{\alpha i}} - i \tilde{\theta}^{\dot{\alpha}}_i \gamma^\mu_{\alpha \dot{\alpha}} \frac{\partial}{\partial x^\mu} \quad (\;" \quad \tilde{D}_{\dot{\alpha}}{}^j)$$

↳ Gauge field: $\mathcal{D}_{\alpha i} = D_{\alpha i} + A_{\alpha i}(x, \theta, \tilde{\theta})$

$$D_\mu = \frac{\partial}{\partial x^\mu} + A_\mu(x, \theta, \tilde{\theta})$$

$$\rightarrow \{\mathcal{D}_{\alpha i}, \tilde{\mathcal{D}}_{\dot{\alpha}}{}^j\} = \delta_i{}^j \sigma^\mu_{\alpha \dot{\alpha}} D_\mu = P_{\alpha \dot{\alpha}}{}^i{}_j$$

($= 0$ if $A \equiv 0 \Rightarrow$ gauge covariant)

SUSY condition $\Rightarrow P_{\alpha \dot{\alpha}}{}^i{}_j = 0$

Now suppose:

$$\{ \tilde{\mathcal{D}}_{\dot\alpha}{}^{i}, \tilde{\mathcal{D}}_{\dot\beta}{}^{j} \} = \Phi^{ij}_{\dot\alpha\dot\beta} = \varepsilon_{\dot\alpha\dot\beta}\, \varepsilon^{ij}\, \Phi$$

only antisymm. because we have

$$\Phi = A_4 + i A_5 = $$
$$= \phi + \theta\lambda + \theta^2 F + \dots$$

Moreover $\tilde{\mathcal{D}}_{\dot\alpha}{}^{i} \Phi = 0 \longrightarrow \Phi$ is chiral:

$$\Downarrow$$

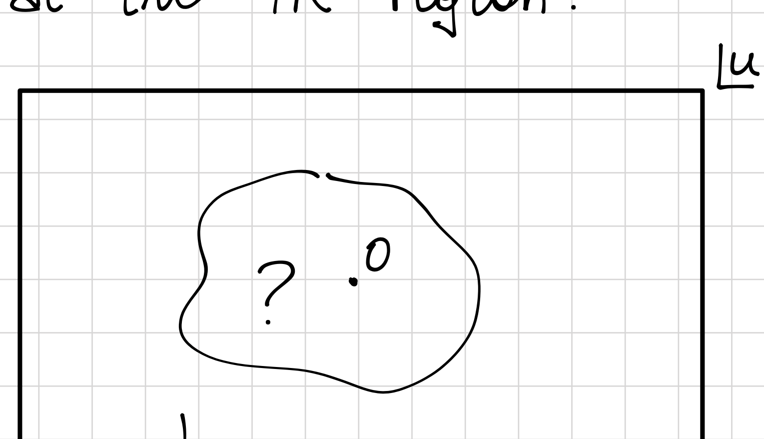
Pick a holomorphic funct. $\mathcal{F}(\Phi)$:

$$I = \mathrm{Im} \int d^4x \, d\theta^i_\alpha \, \mathcal{F}(\Phi)$$

Microscopic definition of $SU(2)$ theory (UV):

$$\mathcal{F}(\Phi) = \tau \, \mathrm{Tr}\, \Phi^2$$

Let's now look at the IR region:



large u $\longrightarrow$ classical picture: $\cancel{SU(2)} \to U(1)$

$$\Phi = a + \theta\lambda + \theta\sigma^{\mu\nu}\tilde\theta F_{\mu\nu} + \dots + \theta^4 \Box \bar a$$

$\hookrightarrow$ any holom. funct $\mathcal{F}(\Phi)$ makes sense

$\longrightarrow \quad I = \mathrm{Im} \int d^4x \, d^4\theta \, \mathcal{F}(\Phi) \Longrightarrow$ what $\mathcal{F}(\Phi)$ should we use ?

Let's consider the $SU(2)$ theory:

$$\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} a & \\ & -a \end{pmatrix} \Rightarrow \quad \mathcal{F}(\Phi) = \tau_{cl} \, a^2 \quad \text{for } U(1) \text{ class. th.}$$

There's the 1L correction, though:

$$\mathcal{F}(\phi) = \tau_{cl} \, a^2 + \underbrace{\frac{i}{2\pi} a^2 \ln \frac{a^2}{\Lambda_0^2}}_{\substack{\text{it incorporates} \\ \text{asymptotic freedom}}} = \frac{i}{2\pi} a^2 \ln \frac{a^2}{\Lambda^2}$$

Now consider the action and some of its terms:

$$I = \mathrm{Im} \int d^4x \Big(\underbrace{\frac{\partial \mathcal{F}}{\partial a} \Box \bar{a}}_{\text{kin. term for } a} + \frac{\partial^2 \mathcal{F}}{\partial a^2} F^+_{\mu\nu} F^{\mu\nu +} + \text{fermions} \Big)$$

$\longrightarrow$ IBP: $\mathrm{Im} \int d^4x \, \frac{\partial^2 \mathcal{F}}{\partial a^2} \partial_\mu a \, \partial^\mu \bar{a}$

$\to$ KÄHLER METRIC: $\underbrace{\mathrm{Im} \frac{\partial^2 \mathcal{F}}{\partial a^2}}_{\text{should be } > 0} da \otimes d\bar{a}$

In general:

$$\tau_{eff} := \tau = \frac{\partial^2 \mathcal{F}}{\partial a^2} = \frac{\theta_{eff}}{2\pi} + i \frac{4\pi}{e^2_{eff}}$$

$$\Rightarrow \mathrm{Im}\, \tau = \frac{4\pi}{e^2_{eff}} > 0 \quad \text{everywhere}$$

However the minimum modulus principle for holom. funct. does NOT allow it

In the 1L approx $\quad \tau_{eff} = \frac{\partial^2 \mathcal{F}}{\partial a^2} = \frac{i}{\pi} \ln \frac{a^2}{\Lambda^2}$ (+ subleading)

$$\longrightarrow \frac{4\pi}{e^2_{eff}} \sim \frac{1}{\pi} \ln \frac{|a|^2}{\Lambda^2} \Rightarrow \frac{1}{e^2_{eff}} \text{ grows logar. at } \infty$$

$\Rightarrow$ ASYMPTOTIC FREEDOM

$$\rightarrow \frac{\theta_{eff}}{2\pi} = \frac{2}{\pi} \, \mathrm{Im} \ln a$$

Therefore on the $u = a^2$ plane:

monodromy of u at $\infty \longrightarrow \frac{1}{2}$-monodromy for $a \longrightarrow i\pi$ for $\ln a$

$$\rightarrow \quad \frac{\theta_{eff}}{2\pi} \longrightarrow \frac{\theta_{eff}}{2\pi} + 2$$

$U(1)$ anomaly: the θ_{eff} changes at ∞ because of the shift in U

$\nexists$ no higher order pert. corrections: [Seiberg, Witten]

$\mathcal{F}(a; \tau_{cl})$ is HOLOMORP. in τ_{cl}

A L-loop contrib goes like $(g^2)^{L-1}$:

$(\tau_{cl})^{1-L}$
- $L = 0$: ✓ $\Rightarrow$ I_{cl} is linear in τ
- $L = 1$: ✓
- $L > 1 \Rightarrow$ not possible

$\Rightarrow$ It is not exact though $\Rightarrow$ $\frac{4\pi}{e^2_{eff}}$ IS NOT POSITIVE DEFINITE

$\downarrow$

$\exists$ NP corrections (INSTANTONS)

they can be holom in τ AND dep. on θ

- We already used the fact that the eff. action is NOT unique:

$$\tau \rightarrow \tau + 1 \quad [\text{or } \theta \rightarrow \theta + 2\pi]$$

We however focus on E-M duality:

$$QM: \quad \tau \to -\frac{1}{\tau}$$

Let's introduce:

$$a_D(a) = \frac{\partial \mathcal{F}}{\partial a} \qquad \tau = \frac{\partial^2 \mathcal{F}}{\partial a^2} = \frac{d a_D}{da}$$

$$\hookrightarrow I = \mathrm{Im} \int d^4x \; \partial_\mu a_D \, \partial^\mu \bar{a} \qquad \left(\partial_\mu a_D = \frac{\partial^2 \mathcal{F}}{\partial a^2} \partial_\mu a \right)$$

Then $\tau \to -\frac{1}{\tau} \Longrightarrow \begin{pmatrix} a_D \\ a \end{pmatrix} \to \pm \begin{pmatrix} a \\ -a_D \end{pmatrix}$

$$\Downarrow$$

$$\frac{d a_D}{da} \longrightarrow -\frac{da}{d a_D} = -\frac{1}{d a_D / da} \qquad \left[\tau \to -\frac{1}{\tau} \right]$$

There are 2 low en. descr:

i) $a, \; a_D, \tau \longrightarrow A_\mu$ (photon)

ii) $a_D, -a, -\frac{1}{\tau} \longrightarrow \tilde{A}_\mu$ (~ photon)

The central charge becomes:

$$Z = n_e a + n_m \tau_{cl} \, a = n_e a + n_m a_D \quad \text{(renorm. inv.)}$$

Therefore we have:

$$\tau \to \tau + 1 \Rightarrow a_D \to a_D + a \Rightarrow \begin{pmatrix} a_D \\ a \end{pmatrix} \to \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}}_{T} \begin{pmatrix} a_D \\ a \end{pmatrix}$$

$$\tau \to -\frac{1}{\tau} \Rightarrow \begin{pmatrix} a_D \\ a \end{pmatrix} \to \overbrace{\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}}^{S} \begin{pmatrix} a_D \\ a \end{pmatrix} \qquad \Rightarrow SL(2, \mathbb{Z}) \text{ ambiguity}$$

Therefore:

$$Z = n_e a + n_m a_D$$

inver. if:

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \to M \begin{pmatrix} a_D \\ a \end{pmatrix} \Rightarrow (n_m \ \ n_e) \to (n_m \ \ n_e) M^{-1}$$

for $M \in SL(2,\mathbb{Z})$.

Therefore the moduli space of vacua is:

CLASS: the only bad point is $u = 0$.

Everywhere else the $U(1)$ desc. is good

↳ the only monodromy is at ∞ → shift in θ angle (a is a good coord.):

$$u \to e^{2\pi i} u \Rightarrow a \to -a \Rightarrow a_D \to \underbrace{-a_D + 2\pi}_{\text{shift in } \tau \text{ at } \infty}$$

$$\Rightarrow \begin{pmatrix} a_D \\ a \end{pmatrix} \to \underbrace{\begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}}_{a^2 \text{ is invariant}} \begin{pmatrix} a_D \\ a \end{pmatrix}$$

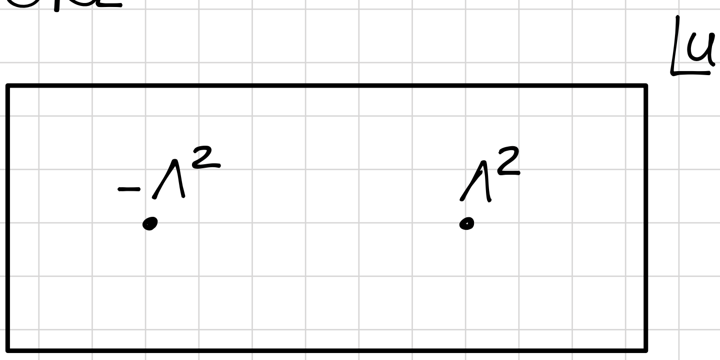
If only one bad point (= class) ⇒ the mon. at ∞ is the only mon. ⇒ a^2 is a good coord.

↳ $\text{Im}\,\tau$ cannot be pos. def.
⇒ bad for QM

→ ∃ MORE BAD POINTS:

* 2 points ⇒ since $u \leftrightarrow -u$ is a symm. ⇒ $u \neq 0$

→ MINIMAL PICTURE



⇒ [Seiberg, Witten] ⟶ the nature of the sing. is due to EXTRA MASSLESS PARTICLES

* classical ⟶ extra vector mult. at $u=0$

* QM ⟶ extra hypermult. (initially massive and then massless at $\pm\Lambda^2$)

⇒ the only hyperm. we have are magnetically charged (+ el. charge from Bogomolny eq.). Some of those magn. monopoles become massless.

take a monopole: $n_m = 1$, $n_e = 0$

What happens at 0-mass?

We use a_D and $\tilde{A}$ (instead of a, A) ⟹ we can treat the monopole as if it were el. charged ⟹ QED is NOT asym. free

$$-\frac{1}{\tau} = \tau_D \sim \ln a_D \ , \quad a_D \longrightarrow 0$$

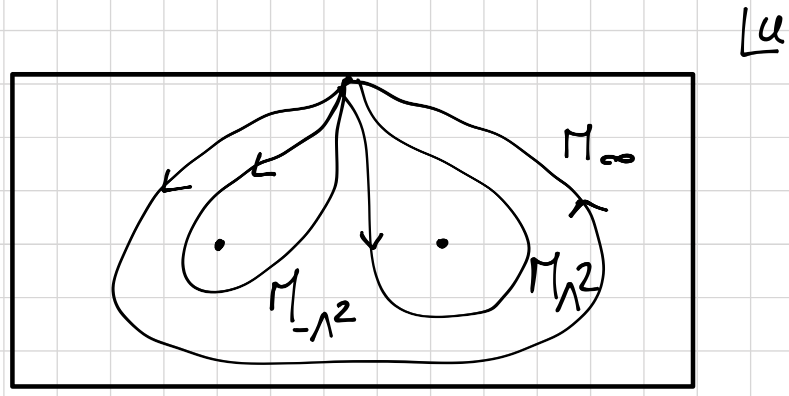
$$\frac{da}{da_D} = \tau_D = -\frac{i}{\pi} \ln a_D \qquad (\beta \text{ func. for QED})$$

Moreover QED has no inst. corrections (even though it doesn't need them because it's a continuous funct at $a_D = 0$):

NB: $a_D = 0 \Rightarrow \tau_D \to \tau_D - 2 \Rightarrow a \to a - 2a_D$ and $a_D \to a_D$

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \to \begin{pmatrix} 1 & \\ -2 & 1 \end{pmatrix} \begin{pmatrix} a_D \\ a \end{pmatrix}$$

Therefore:



$$\Rightarrow \quad M_\infty = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix} \qquad M_{\Lambda^2} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$

$$\Rightarrow M_\infty = M_{-\Lambda^2} M_{\Lambda^2} \Rightarrow M_{-\Lambda^2} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}$$

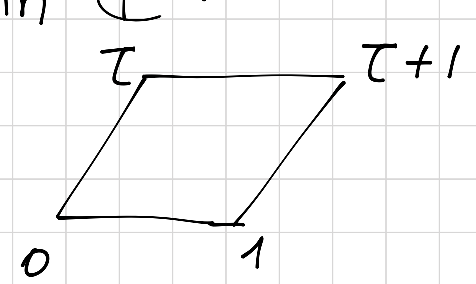
monodromy due to massless DYON $(n_m \ n_e) = (1 \ 1)$

$\Rightarrow$ a monopole becomes massless in Λ^2
a dyon " $-\Lambda^2$.

This a $SL(2,\mathbb{Z})$ bundle on u plane w/o $\{\Lambda^2, -\Lambda^2\}$:

$$\tau \longleftrightarrow \frac{a\tau + b}{x\tau + y} \qquad a,b,x,y \in \mathbb{Z}$$
$$\begin{vmatrix} a & b \\ x & y \end{vmatrix} = 1$$

The invariant info in τ is the same contained in a $g=1$ Riemann surf. $\Rightarrow$ "ELLIPTIC CURVE" got from a lattice in $\mathbb{C}$:



$\Rightarrow$ on the u plane w/ 2 punctures we have a family of elliptic curves which dep. on u and has these monodromies:

$$y^2 = (x^2 - \Lambda^2)(x+u).$$