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# Boundary states for branes with non trivial homology in constant closed and open background.

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## Abstract

For the bosonic string on the torus we compute boundary states describing branes with not trivial homology class in presence of constant closed and open background. It turns out that boundary states with non trivial open background generically require the introduction of non physical “twisted” closed sectors, that only  $F$  and not  $\mathcal{F} = F + B$  determines the geometric embedding for  $Dp$  branes with  $p < 25$  and that closed and open strings live on different tori which are relatively twisted and shrunk. Finally we discuss the T-duality transformation for the open string in a non trivial background.

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# 1 Introduction.

Systems of interacting open- and closed-strings play important roles in several aspects of string theory. It has been found out that theories even formulated as pure closed-strings have open-string sectors which can be described as boundary states. One of the most important features of such open-closed mixed systems is a duality between open- and closed-strings. This duality becomes manifest, for instance, by seeing one-loop diagrams of open-string as tree propagations of closed-string through modular transformations on the string world-sheets. In the systems of D-branes, this duality should be a rationale for correspondence between gauge theory on the world-volumes (open-string sector) and gravity theory in the bulk space-time (closed-string sector). AdS/CFT correspondence may be regarded as one of the most remarkable examples of such bulk-boundary correspondence (see for example [1] for a review along these lines).

It is well-known that the so called boundary states provide a description of D-branes in closed string theory. Open and closed strings interact on the branes in space-time and therefore boundary states can be used to describe these interactions. In [2] this was achieved for tachyons and gluons in a non compact spacetime with a constant  $B$  field background while in [3] for all open states in a compact spacetime without background fields.

On the other hand intersecting branes on tori have been used as a building block for constructing (semi)realistic versions of the standard model (see for example [5, 6] for reviews). This construction requires considering branes with non trivial homology in presence of both open and closed background fields since the geometric embedding of these branes is governed by these “parameters”, i.e. background fields values.

In this article we want to build boundary states for bosonic boundary states describing branes with non trivial homology on tori in presence of constant open and closed background fields (see also [7, 8, 9, 10] for related construction). This paper is organized as follows. In section 2 we fix our conventions and review the closed string quantization on a torus in presence of background fields and the action of T-duality. In this section we describe also the quantization of open string describing homologically non trivial branes in this closed background in presence of a constant magnetic field. In section 3 we proceed to construct the boundary states for the  $D25$  brane: we find that it is necessary to introduce non physical “twisted” states to accomplish this task. In section 4 we discuss the action of T-duality on these states from

the closed string point of view and we sketch how this can be derived from the open string point of view ( see [4] for more details) Finally in section 5 we draw our conclusions.

## 2 Review of quantization of string and T-duality.

### 2.1 The quantization of closed string in a constant $G$ and $B$ background.

In order to fix our notation and conventions we start writing the usual action for the closed string

$$\begin{aligned} S &= -\frac{1}{4\pi\alpha'} \int d\tau \int_0^\pi d\sigma \left( \sqrt{-g} g^{\alpha\beta} G_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu - \epsilon^{\alpha\beta} B_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu \right) \\ &= \frac{1}{4\pi\alpha'} \int d\tau \int_0^\pi d\sigma \left( G_{\mu\nu} \left( \dot{X}^\mu \dot{X}^\nu - X'^\mu X'^\nu \right) - 2B_{\mu\nu} \dot{X}^\mu X'^\nu \right) \end{aligned}$$

where we have chosen the conformal gauge  $g_{\alpha\beta} \propto \eta_{\alpha\beta} = \text{diag}(-1, 1)$  and chosen  $\epsilon^{01} = -1$ . We suppose moreover that the dimensionful background fields  $G_{st} = ||G_{\mu\nu}||$  and  $B_{st} = ||B_{\mu\nu}||$  be constant,  $G_{i0} = B_{i0} = 0$  ( $i = 1, \dots, d$ ) and the spacial coordinates  $X = ||X^i||$  be periodic of period  $2\pi$

$$X^i \equiv X^i + 2\pi \quad (1)$$

and dimensionless. From the equations of motion  $\ddot{X} - X'' = 0$  and the closed string boundary condition  $X^\mu(\sigma + \pi) = X^\mu(\sigma)$  we get the string modes expansion which reads

$$\begin{aligned} X^\mu(z, \bar{z}) &= \frac{1}{2} (X_L^\mu(z) + X_R^\mu(\bar{z})) \\ X_L^\mu(z) &= x_L^\mu - 2\alpha' i p_L^\mu \ln(z) + i\sqrt{2\alpha'} \sum_{n \neq 0} \frac{\text{sgn}(n) a_n^\mu}{\sqrt{|n|}} z^{-n} \\ X_R^\mu(\bar{z}) &= x_R^\mu - 2\alpha' i p_R^\mu \ln(\bar{z}) + i\sqrt{2\alpha'} \sum_{n \neq 0} \frac{\text{sgn}(n) \tilde{a}_n^\mu}{\sqrt{|n|}} \bar{z}^{-n} \end{aligned}$$

where  $z = e^{2i(\tau+\sigma)} = e^{2(\tau_E+i\sigma)}$  ( $0 \leq \sigma \leq \pi$ ),  $x_L^0 = x_R^0$  and  $p_L^0 = p_R^0$ .

The canonical momentum density is given by

$$\mathcal{P}_\mu = \frac{1}{2\pi\alpha'} (G_{\mu\nu} \dot{X}^\nu - B_{\mu\nu} X'^\nu)$$

so that from the canonical quantization we find the non vanishing commutators

$$\begin{aligned} [x_L^\mu, p_L^\nu] &= [x_R^\mu, p_R^\nu] = iG^{\mu\nu} \\ [a_n^\mu, a_m^\nu] &= [\tilde{a}_n^\mu, \tilde{a}_m^\nu] = G^{\mu\nu} \operatorname{sgn}(n) \delta_{n+m,0} \end{aligned}$$

and the Hamiltonian density

$$\begin{aligned} \mathcal{H} &= \frac{1}{4\pi\alpha'} G_{\mu\nu} (\dot{X}^\mu \dot{X}^\nu + X'^\mu X'^\nu) \\ &= \frac{1}{4\pi\alpha'} (G^{\mu\nu} (2\pi\alpha' \mathcal{P}_\mu + B_{\mu\lambda} X'^\lambda) (2\pi\alpha' \mathcal{P}_\nu + B_{\nu\kappa} X'^\kappa) + G_{\mu\nu} X'^\mu X'^\nu) \end{aligned}$$

Using the previous quantization the energy momentum tensor

$$T(z) = -\frac{1}{4\alpha'} : \partial X_L^\mu G_{\mu\nu} \partial X_L^\nu : = \sum_k \frac{L_k}{z^{k+2}} \quad \bar{T}(\bar{z}) = -\frac{1}{4\alpha'} : \bar{\partial} X_R^\mu G_{\mu\nu} \bar{\partial} X_R^\nu : = \sum_k \frac{\bar{L}_k}{\bar{z}^{k+2}}$$

with in particular

$$\begin{aligned} L_0 &= \alpha' p_L^\mu G_{\mu\nu} p_L^\nu + \sum_{n=1}^{\infty} n a_n^{\mu\dagger} G_{\mu\nu} a_n^\nu \\ \tilde{L}_0 &= \alpha' p_R^\mu G_{\mu\nu} p_R^\nu + \sum_{n=1}^{\infty} n \tilde{a}_n^{\mu\dagger} G_{\mu\nu} \tilde{a}_n^\nu \end{aligned}$$

implies that the vacuum is defined by

$$p_L^\mu |0, \tilde{0}\rangle = p_R^\mu |0, \tilde{0}\rangle = a_n^\mu |0, \tilde{0}\rangle = \tilde{a}_n^\mu |0, \tilde{0}\rangle = 0 \quad n > 0 \quad (2)$$

and it is normalized as

$$\langle k_0, n, w | k'_0, n', w' \rangle = 2\pi \delta(k_0 - k'_0) \delta_{n,n'} \delta_{w,w'}$$

where we have defined  $p_L^\mu |k_L\rangle = G^{\mu\nu} k_{L\nu} |k_L\rangle$  with  $|k_L\rangle = e^{ik_{L\mu} x_L^\mu} |0, \tilde{0}\rangle$ ,  $\langle k_L| = |k_L\rangle^\dagger$ , similarly for the other momenta and the possible values of  $k_L, k_R$  together with the definitions of  $n, w$  are given in (3).

Because of the space periodicity the translation generator

$$T = ||T_i|| = \int_0^\pi d\sigma \mathcal{P} = E^T p_L + E p_R$$

along with eq. (1) implies that the operators  $p_L = ||p_L^i||$  and  $p_R = ||p_R^i||$  have spectrum

$$\begin{aligned} p_L &= G^{-1}k_L = \frac{1}{2}G^{-1}\left(n + E\frac{w}{\alpha'}\right) \\ p_R &= G^{-1}k_R = \frac{1}{2}G^{-1}\left(n - E^T\frac{w}{\alpha'}\right) \end{aligned} \quad (3)$$

where  $E = G + B$ ,  $E^T = G - B$  ( with  $G = ||G_{ij}||$ ,  $B = ||B_{ij}||$  ) and  $n = ||n_i||$ ,  $w = ||w^i||$  are integer valued vectors.

Following [11] we consider T-duality as a canonical transformation given by<sup>2</sup>

$$\begin{aligned} \mathcal{P}_a &= \frac{1}{2\pi\alpha'}\gamma_{ab}X^{t'b} & \mathcal{P}_a^t &= \frac{1}{2\pi\alpha'}X^{tb}\gamma_{ba} \\ \mathcal{P}_m &= \mathcal{P}_m^t & X^m &= X^{tm} \end{aligned} \quad (4)$$

where we have split  $i, j, \dots$  in  $a, b, \dots$  for the “perpendicular” directions which we T-dualize and  $m, n, \dots$  for the other “parallel” directions,  $X^{ti}, \mathcal{P}_j^t$  are the new T-dual coordinates, and  $||\gamma_{ab}|| \in \alpha' SL(d_\perp, \mathbb{Z})$ <sup>3</sup> is a constant matrix, usually taken proportional to the unity and  $d_\perp$  is the number of perpendicular directions. These relations imply the usual transformations

$$\partial X^t = (P_\perp \gamma^{-1} E^T + P_\parallel) \partial X \quad \bar{\partial} X^t = -(P_\perp \gamma^{-1} E - P_\parallel) \bar{\partial} X \quad (5)$$

where we have introduced the projector  $P_\perp$  ( $P_\parallel$ ) on the directions we (do not) T-dualize and defined  $\gamma_{mn} = \alpha' \delta_{mn}$ ,  $\gamma_{am} = \gamma_{ma} = 0$ . These relations can be extended to zero modes. The matrix  $E = G + B$  transforms then as

$$E^t = (P_\perp \gamma^T - P_\parallel E)(P_\perp \gamma^{-1} E - P_\parallel)^{-1}$$

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<sup>2</sup>For the time direction we have trivially

$$\mathcal{P}_0 = \mathcal{P}_0^t \quad X^0 = X^{t0}$$

<sup>3</sup> $\gamma_{ab} \in \alpha' \mathbb{Z}$  because we want that the new theory described by the new canonical variables be equivalent theory to the old one, this in particular means mapping integers  $n$  and  $w$  into new integers  $n^t$  and  $w^t$ .

The inverse relations can be trivially obtained exchanging  $\gamma \leftrightarrow \gamma^T$  as follows from eq. (4), i.e. for example

$$E = (P_\perp \gamma - P_\parallel E^t)(P_\perp \gamma^{-T} E^t - P_\parallel)^{-1} \quad (6)$$

## 2.2 The “neutral” string.

The action for a “neutral” string in constant open and closed background can be written as

$$S = \int d\tau \int_0^\pi d\sigma \left( G_{\mu\nu} \frac{\dot{X}^\mu \dot{X}^\nu - X'^\mu X'^\nu}{4\pi\alpha'} - \mathcal{F}_{\mu\nu} \dot{X}^\mu X'^\nu \right) - \int d\tau \left( e_{(0)} a_{(0)\mu} \dot{X}^\mu|_{\sigma=0} - e_{(\pi)} a_{(\pi)\mu} \dot{X}^\mu|_{\sigma=\pi} \right)$$

where  $\mathcal{F} = F + \frac{B}{2\pi\alpha'}$ ,  $e_{(0)}$  ( $e_{(\pi)}$ ) is the charge at  $\sigma = 0$  ( $\sigma = \pi$ ) and  $a_{(0)\mu}$  ( $a_{(\pi)\mu}$ ) is the constant gauge field felt by the string at  $\sigma = 0$  ( $\sigma = \pi$ ) and that cannot be gauged away on a circle. It is noteworthy to stress that the previous expression is valid when  $F = e_{(0)} F_{(0)} = e_{(\pi)} F_{(\pi)}$  ( $F_{(0)}$  being the field strength at  $\sigma = 0$  and similarly for  $F_{(\pi)}$ ) and does not imply that the open string ends on the same brane and has the same charges (even if charges can only be  $\pm 1$ ) but only the degeneracy of the force felt by the two string endpoints: this is the reason of the quotes around neutral in the title. We assume moreover that the two branes<sup>4</sup> where the string ends have non trivial but equal homology (similar and more general configurations are considered in [12]) described by

$$X(\sigma^0, \sigma^i + 2\pi s^i) = X(\sigma^0, \sigma^i) + 2\pi W s \quad \forall s \in Z^d \quad W_{\cdot i}^i \in N^* \quad (7)$$

where  $\sigma$  are the worldvolume coordinates of the D25 and we have fixed  $0 \leq \sigma^i < 2\pi$ . Notice that we write this and all following expressions as if we used a generic matrix  $W$  even if we actually use  $W = \text{diag}(W^1, W^2, \dots, W^d)$ . By this we mean that the fields living on the branes must have that periodicity, for example in the case of tachyons this means  $T_{(0)}(x^0, x^i) = T_{(0)}(x^0, x^i + 2\pi W_j^i s^j)$  with  $x = X(0, \tau)$ ,  $T_{(\pi)}(x^0, x^i) = T_{(\pi)}(x^0, x^i + 2\pi W_j^i s^j)$  with  $x = X(\pi, \tau)$ . More general situations are possible where the homology of the

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<sup>4</sup>When  $e_{(0)} = e_{(\pi)}$  and  $a_{(0)} = a_{(\pi)}$  the open string could actually have both endpoints on the same brane and therefore the system could be made of just one brane. This is a special case but all what follows goes through in the same way up to trivial modifications.

two branes are different but we will not discuss them here since they can be treated efficiently with the boundary state formalism.

The boundary conditions are now

$$G_{\mu\nu}X'^{\nu} + 2\pi\alpha' \dot{X}^{\nu} \mathcal{F}_{\nu\mu}|_{\sigma=0,\pi} = 0 \quad (8)$$

which implies the modes expansion

$$\begin{aligned} X^{\mu}(\sigma, \tau) &= x^{\mu} + 2\alpha'(\delta_{\nu}^{\mu}\tau + 2\pi\alpha' G^{\mu\kappa} \mathcal{F}_{\kappa\nu} \sigma) p^{\nu} \\ &+ i\sqrt{2\alpha'} \sum_{n \neq 0} \frac{\text{sgn}(n)e^{-in\tau}}{\sqrt{|n|}} (\delta_{\nu}^{\mu} \cos(n\sigma) - i2\pi\alpha' G^{\mu\kappa} \mathcal{F}_{\kappa\nu} \sin(n\sigma)) a_n^{\nu} \end{aligned} \quad (9)$$

The non vanishing commutation relations [13, 14] are

$$\begin{aligned} [x^0, p^0] &= i\mathcal{G}^{00} & [a_m^0, a_n^0] &= \mathcal{G}^{00} \text{sgn}(m) \delta_{m+n,0} \\ [x^i, p^j] &= i\mathcal{G}^{ij} & [x^i, x^j] &= i\theta^{ij} \\ [a_m^i, a_n^j] &= \mathcal{G}^{ij} \text{sgn}(m) \delta_{m+n,0} \end{aligned}$$

where we have defined  $\mathcal{E} = ||\mathcal{E}_{ij}|| = G + 2\pi\alpha' \mathcal{F} = E + 2\pi\alpha' F$ , the open string metric [15] as

$$\begin{aligned} \mathcal{G}_{00} &= G_{00} \quad \mathcal{G}_{0i} = 0 \\ \mathcal{G} &= ||\mathcal{G}_{ij}|| = G - (2\pi\alpha')^2 \mathcal{F} G^{-1} \mathcal{F} = \mathcal{E}^T G^{-1} \mathcal{E} \end{aligned} \quad (10)$$

and  $\mathcal{G}^{-1} = \mathcal{E}^{-T} G \mathcal{E}^{-1} = \mathcal{E}^{-1} G \mathcal{E}^{-T}$ ,  $\theta = (2\pi\alpha')^2 \mathcal{E}^{-T} \mathcal{F} \mathcal{E}^{-1}$ .

Using the previous quantization the energy momentum tensor

$$T(z) = -\frac{1}{\alpha'} : \partial X^{\mu} G_{\mu\nu} \partial X^{\nu} : = \sum_k \frac{L_k}{z^{k+2}}$$

with in particular

$$L_0 = \alpha' p^{\mu} \mathcal{G}_{\mu\nu} p^{\nu} + \sum_{n=1}^{\infty} n a_n^{\mu\dagger} \mathcal{G}_{\mu\nu} a_n^{\nu} \quad (11)$$

implies that the vacuum is defined by

$$p^{\mu}|0\rangle = a_n^{\mu}|0\rangle = 0 \quad n > 0 \quad (12)$$

and it is normalized as

$$\langle k_0, n | k'_0, n' \rangle = 2\pi \delta(k_0 - k'_0) \delta_{n, n'}$$

where we have defined  $p^\mu |k\rangle = \mathcal{G}^{\mu\nu} k_\nu |k\rangle$  with  $|k\rangle = |n\rangle = e^{ik_\mu x^\mu} |0\rangle$ ,  $\langle k| = |k\rangle^\dagger$  and the relation between  $k$  and  $n$  is given by eq. (13).

Since we have assumed the space be compact from the expression for the space translation generators

$$T = ||T_i|| = \int_0^\pi d\sigma \mathcal{P} = -(e_{(0)} a_{(0)} - e_{(\pi)} a_{(\pi)}) + \mathcal{G}p$$

where we have used  $\mathcal{G} = G - (2\pi\alpha')^2 \mathcal{F}G^{-1}\mathcal{F} = \mathcal{E}^T G^{-1} \mathcal{E}$ , we get the spectrum for the  $p$  operator to be<sup>5</sup>

$$p = \mathcal{G}^{-1}k = \mathcal{G}^{-1}(W^{-T}n + (e_{(0)}a_{(0)} - e_{(\pi)}a_{(\pi)})) \quad (13)$$

where  $W = \text{diag}(W^1, W^2, \dots, W^d)$  and  $0 \leq (e_{(0)}a_{(0)i} - e_{(\pi)}a_{(\pi)i}) < \frac{1}{W^i}$ <sup>6</sup>.

All the field strengths are quantized as follows from their first Chern class<sup>7</sup>

$$c_1 = \frac{1}{2\pi} \int_{\Sigma^{\bar{i}\bar{j}}} F = 2\pi (W^T F W)_{\bar{i}\bar{j}} = 2\pi F_{\bar{i}\bar{j}} W^{\bar{i}} W^{\bar{j}} \in \mathbb{Z} \quad \forall \bar{i}, \bar{j}$$

since the space is compact and where we have defined the cycle  $\Sigma^{\bar{i}\bar{j}} = \{X^i = 2\pi (W^{\bar{i}} t^{\bar{i}} + W^{\bar{j}} t^{\bar{j}}), 0 \leq t^{\bar{i}}, t^{\bar{j}} < 1\}$ .

What we have written until now is valid for a  $D25$  brane. In analogy to the superstring case where we actually can measure the various branes charges, we can assert that when all possible  $F_{ij}$  are turned on we generically describe a complex system of a  $D25$  brane together with all possible lower dimensional branes. While lower dimensional branes at angles can be obtained by T-duality and by switching off some  $F$  components, lower dimensional branes wrapping straight can formally obtained letting some  $F$  component go to infinity. In a more explicit way we start by splitting the space time indexes  $\mu, \nu, \dots$  as  $m, n, \dots$  for directions parallel to the brane ( $i, j, \dots$  for the spacial

<sup>5</sup>We write  $W^{-T}$  and not  $W^{-1}$  even if the  $W$  matrix we consider now is diagonal since this is the proper way of writing the momenta in presence of non diagonal  $W$ s.

<sup>6</sup>In the case where  $W$  were not diagonal, from the gauge invariance of the zero modes part of the “generalized” velocity matrix elements  $\langle \Psi | \frac{1}{2\pi\alpha'} (G_{\mu\nu} \dot{X}^\nu - B_{\mu\nu} X^\nu) | \Psi' \rangle$  we get  $W^T(e_{(0)}a_{(0)} - e_{(\pi)}a_{(\pi)}) \equiv W^T(e_{(0)}a_{(0)} - e_{(\pi)}a_{(\pi)}) + m \quad \forall m \mathbb{Z}^d$ .

<sup>7</sup>We have used  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ .



ones) and  $a, b, \dots$  for directions perpendicular to the brane, we set  $F_{am} = 0$  to simplify and  $F_{ab} = 0$ <sup>8</sup> in order to avoid higher dimensional branes in the system<sup>9</sup>. Then boundary conditions read

$$\begin{aligned} G_{mn}X'^m + G_{ma}X'^a - 2\pi\alpha' \mathcal{F}_{mn}\dot{X}^n|_{\sigma=0,\pi} &= 0 \\ X^a - x_0^a|_{\sigma=0} &= X^a - x_\pi^a|_{\sigma=\pi} = 0 \end{aligned} \quad (14)$$

which can formally be obtained from eq. (8) letting  $F_{ab} \rightarrow \infty$  (if the number of transverse directions is bigger than one) and noticing that the term  $-2\pi\alpha' \mathcal{F}_{ma}\dot{X}^a|_{\sigma=0,\pi}$  is then identically zero because  $\dot{X}^a|_{\sigma=0,\pi} = 0$ .

### 3 Boundary states.

#### 3.1 Boundary states for D25

In order not to kludge the notation we start with the simplest case, i.e. the D25 with non trivial homology given by eq. (7). We can now rewrite the open boundary condition (8) for the closed string as

$$(G_{\mu\nu}\dot{X}^\nu + 2\pi\alpha' X'^\nu \mathcal{F}_{\nu\mu})|_{\tau=0}|D25(E, F, W)\rangle = 0 \quad (15)$$

This can also be reexpressed using the closed string momentum which already incorporates all  $B$  as

$$(\mathcal{P}_\mu - F_{\mu\nu}X'^\nu)|_{\tau=0}|D25(E, F, W)\rangle = 0 \quad (16)$$

This expression is already interesting since it reveals that under T-duality  $B$  does not play the same role of  $F$ .

The previous expression can be analyzed in modes to give

$$(\mathcal{E}^T \alpha_n - \mathcal{E} \tilde{\alpha}_{-n})|D25(E, F, W)\rangle = 0$$

In particular the zero mode sector yields

$$(\hat{n} - 2\pi F \hat{w})|D25(E, F, W)\rangle = 0 \quad (17)$$

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<sup>8</sup>In sigma model under T-duality we roughly have  $A_a(X^\mu)\dot{X}^a|_{\sigma=0} \Rightarrow \frac{1}{2\pi\alpha'}\Phi_a(X^a, X^m)X'^a|_{\sigma=0}$  but equations of motions imply the constraints  $\Phi_a(X^a, X^m) = 0$  and hence we set  $F_{am} = F_{ab} = 0$ .

<sup>9</sup>It is also natural to assume  $W^a = 1$  but this is again not strictly necessary since  $W^a > 1$  would give special cases where  $W^a$  Dp branes are located at regular interval in direction  $x^a$  as it can be seen by the momentum quantization.

where  $\hat{n}$  and  $\hat{w}$  are the operators, obtained by a linear combination of  $p_L$  and  $p_R$ , which have the corresponding unhatted quantities defined in eq. (3) as eigenvalues. These expressions imply the possibility of rewriting the spectrum of  $p_L$ ,  $p_R$  as a function of  $w$  only as

$$p_L = \frac{1}{2}G^{-1}\mathcal{E}\frac{w}{\alpha'} \quad p_R = -\frac{1}{2}G^{-1}\mathcal{E}^T\frac{w}{\alpha'}$$

It is now immediate to deduce that all the entries of the matrix  $2\pi F = ||2\pi F_{ij}||$  must be rational otherwise the previous constraint would not have any solution.

Actually it turns out that also  $\hat{n}$  must have *rational* eigenvalues when acting on the boundary and not only integers as it comes from the closed string spectrum. To explore this point let us consider the simplest case where only  $F_{12} \neq 0$ , explicitly  $2\pi F_{12} = \frac{p}{q}$  ( $p, q \in \mathbb{Z}$ ) then the non trivial equations are

$$\begin{aligned} \left(\hat{n}_1 - \frac{p}{q}\hat{w}^2\right) |D25(E, F, W)\rangle &= 0 \\ \left(\hat{n}_2 + \frac{p}{q}\hat{w}^1\right) |D25(E, F, W)\rangle &= 0 \end{aligned} \quad (18)$$

If we insist on having only integer  $n$  then we must conclude that  $(w^1, w^2) = q(l^1, l^2)$  and  $(n_1, n_2) = p(l^2, -l^1)$  with arbitrary integers  $l^1$  and  $l^2$ . We can now take the simplest boundary given by the sum of all allowed  $|n, w\rangle$  with equal coefficients, i.e. the zero modes non trivial part of the boundary is given by  $|D25(E, F, W)_{(0)}\rangle \sim \sum_{(l^1, l^2) \in \mathbb{Z}^2} |n = p(l^2, -l^1), w = q(l^1, l^2)\rangle$ . If we now compute the boundary-boundary interaction as  $\langle D25(E, F, W) | e^{-t(L_0 + \tilde{L}_0)} | D25(E, F, W) \rangle$  we see immediately that we get a contribution from these zero modes of the form  $\sum_{(l^1, l^2)} e^{-\frac{t}{2}q^2(l^1, l^2)\frac{\mathcal{G}_2}{\alpha^2}(l^1, l^2)^T} (\mathcal{G}_2 \text{ being the } \mathcal{G} \text{ submatrix with indexes running on 1 and 2 only})$ . Performing a Poisson resummation<sup>10</sup> on the previous expression we find something proportional to  $\sum_{(m_1, m_2) \in \mathbb{Z}^2} e^{-\frac{2\pi^2}{t}\alpha'\frac{1}{q}(m_1, m_2)\mathcal{G}_2^{-1}\frac{1}{q}(m_1, m_2)^T}$  which we want to interpret in the open channel. We are therefore led to take

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<sup>10</sup>The Poisson formula is as follows

$$\sum_{n \in \mathbb{Z}^d} e^{-\frac{1}{2}n^T A n + i n^T B} = \frac{1}{\sqrt{\det A}} \sum_{m \in \mathbb{Z}^d} e^{-\frac{1}{2}(2\pi m + B)^T A^{-1} (2\pi m + B)}$$

the open string momenta of the form  $\mathcal{G}_2^{-1} \frac{1}{q} (m_1, m_2)^T$ . From the  $F$  quantization condition (14) we know that  $q = W^1 W^2$  and therefore we find a *wrong* open spectrum which should be of the form  $\mathcal{G}_2^{-1} (\frac{n_1}{W^1}, \frac{n_2}{W^2})$  as in eq. (13).

We notice that the problem becomes even worst when we consider the case where more  $2\pi F_{ij} \propto \frac{1}{W^i W^j}$  are turned on, since in this case in the generic situation  $q \propto \prod W^i$  (where the product is over all directions where there is a non vanishing  $F$ ) in order to have integer  $ns$ .

Let us now assume that the  $(n_1, n_2) = (\frac{l^2}{W^1}, -\frac{l^1}{W^2})$  then the possible windings are now  $(w^1, w^2) = (W^1 l^1, W^2 l^2)$  again with arbitrary integers  $l^1$  and  $l^2$ . Performing the same steps as before we find that the closed contribution is now  $\sum_{(l^1, l^2)} e^{-\frac{i}{2} (W^1 l^1, W^2 l^2) \frac{\mathcal{G}_2}{\alpha'} (W^1 l^1, W^2 l^2)^T}$  which gives the right expression when Poisson resummed, i.e. a contribution which is proportional to  $\sum_{(m_1, m_2) \in \mathbb{Z}^2} e^{-\frac{2\pi^2}{i} \alpha' (m_1, m_2) W^{-1} \mathcal{G}_2^{-1} W^{-1} (m_1, m_2)^T}$

This is not completely unexpected. In the “trivial” case where  $F = 0$  and branes still have a non trivial homology the zero modes contribution to open string partition function is  $\sum_{(m_1, m_2) \in \mathbb{Z}^2} e^{-\tau \alpha' (m_1, m_2) W^{-1} G_2^{-1} W^{-1} (m_1, m_2)^T}$  which can be reproduced in the closed string channel if we take a boundary state  $|D25(E, F, W)_{(0)}\rangle \sim \sum_{(l^1, l^2) \in \mathbb{Z}^2} |n = 0, w = (W^1 l^1, W^2 l^2)\rangle = \sum_{(l^1, l^2) \in \mathbb{Z}^2} |n = 0, w = W(l^1, l^2)\rangle$ , i.e. summing over some (not all!) of the possible closed string states which satisfy (17) for  $F = 0$ . But now consider the case of boundary states which describe open string interactions on parallel branes without background fields [3]<sup>11</sup> then we know that these boundaries are produced by acting on the usual boundaries by the open string vertex operators where the open string fields are substituted by the left moving closed string fields. In presence of a brane winding  $W^1, W^2$  times in  $x^1, x^2$  directions we expect that the exponential part of the open string vertexes being of the form  $V \sim e^{i(\frac{m_1}{W^1}, \frac{m_2}{W^2})(X^1, X^2)^T}$  and this explains why we can find non integer  $n$ <sup>12</sup>. We can reformulate this by saying that these non integer momenta are due to the fact that open string sees directions of length  $2\pi W^i$  while closed string sees a length of  $2\pi$  and therefore we need adding more (non physical) states

<sup>11</sup>The results in [3] are actually obtained in the case of trivial homology but can easily be extended to more complex cases where the open string momenta are rational since all results are obtained when arbitrary Wilson lines are turned on.

<sup>12</sup>There is actually a further subtlety since we have asserted that the open string vertexes are made using the left moving part of closed fields only. As shown in [3] there is actually a further contribution to these interacting boundaries whose aim is to divide equally the open string momenta in the left and right moving sector and this fixes completely all the details.

to closed string in order to describe open string states which explore shorter distances.

Because of this interpretation of these extra states it must be stressed that they do not represent new physical closed string excitations, i.e. they do not belong to the physical closed string Hilbert space but they are the closed string representations of open string physical states. Nevertheless they do propagate in space time even if they do not interact with closed string while away from the brane.

With this understanding we can now write the complete boundary state as

$$\begin{aligned}
|D25(E, F, W, y)\rangle &= \frac{T_d(E, F)}{2} |D25(E, F)\rangle_{time} |D25(E, F, W, y)\rangle_{space} |D\rangle_{ghost} \\
|D25(E, F)\rangle_{time} &= e^{-\sum_{n=1}^{\infty} a_n^{\dagger 0} G_{00} \tilde{a}_n^{\dagger 0}} |k^0 = 0\rangle \\
|D25(E, F, W, y)\rangle_{space} &= e^{-\sum_{n=1}^{\infty} a_n^{\dagger} G \mathcal{E}^{-T} \mathcal{E} \tilde{a}_n^{\dagger T}} \sum_{w \in I(F, W)} c_w(y) |n = 2\pi F w, w\rangle \\
|D\rangle_{ghost} &= e^{\sum_{n=1}^{\infty} c_{-n} \tilde{b}_{-n} + \tilde{c}_{-n} b_{-n}} \frac{c_0 + \tilde{c}_0}{2} |q = \tilde{q} = 1\rangle
\end{aligned} \tag{19}$$

where  $I(F, W)$  is the lattice given by  $I(F, W) = \{w \in \mathbb{Z}^d | \forall i \ \& \ w^i \propto W^i \ \& \ n = 2\pi F w\}$ <sup>13</sup>,  $c_w(y)$  are in principle arbitrary complex numbers but because of the open string channel interpretation they turn out to be  $c_w(y) = e^{2\pi i w^T y}$  with  $y_i = e_{(0)} a_{(0)i}$ , the tension  $T_d(E, F)$  can be obtained as in [16, 17] and in [2] We have also introduced the usual ghost sector with energy momentum tensor using the conventions of [17] as

$$\begin{aligned}
T^{(bc)} &= -2b\partial c - \partial bc = \sum \frac{L_n^{(bc)}}{z^{n+2}} \\
L_0^{bc} &= \sum_{n=0}^{\infty} n (b_n^{\dagger} c_n - c_n^{\dagger} b_n)
\end{aligned}$$

similarly for the right moving sector and defined the  $|q = 1, \tilde{q} = 1\rangle$  vacuum from the  $SL(2, \mathbb{C})$  invariant ghost vacuum as  $|q = \tilde{q} = 1\rangle = c_1 \tilde{c}_1 |q = \tilde{q} = 0\rangle$  such as  $c_n |q = \tilde{q} = 1\rangle = \tilde{c}_n |q = \tilde{q} = 1\rangle = 0$  for  $n \geq 1$ .

---

<sup>13</sup>Since  $2\pi F_{ij} = \frac{f_{ij}}{W^i W^j}$  with  $f_{ij} \in \mathbb{Z}$  and  $w^i = W^i s^i$  with  $s^i \in \mathbb{Z}$  this means that  $n_i = \frac{f_{ij} s^j}{W^i} \propto \frac{1}{W^i}$ .

### 3.2 Modular transformation

We can now compute the boundary-boundary interaction as usual as

$$\begin{aligned} & \langle D25(E, F_1, W_1, y_1) | e^{-t(L_0^{(X+bc)} + \tilde{L}_0^{(X+bc)})} (b_0 + \tilde{b}_0)(c_0 - \tilde{c}_0) | D25(E, F_2, W_2, y_2) \rangle = \\ & \left(\frac{T_d}{2}\right)^2 V_{time} e^{2t} \left( \prod_{n=1}^{\infty} (1 - e^{-2nt}) \right)^{2-1} \left( \prod_{n=1}^{\infty} \det(1 - \mathcal{E}_1^{-1} \mathcal{E}_1^T \mathcal{E}_2^{-T} \mathcal{E}_2 e^{-2nt}) \right)^{-1} \\ & \sum_{w \in I(F_1, W_1) \cap I(F_2, W_2)} c_w^*(y_1) c_w(y_2) \delta_{(F_1 - F_2)w, 0} e^{-\frac{t}{2} w^T \frac{g}{\alpha'} w} \end{aligned}$$

where we have inserted  $(b_0 + \tilde{b}_0)(c_0 - \tilde{c}_0)$  to account for the surviving part of the  $SL(2, \mathbb{C})$  symmetry. The complete boundary-boundary amplitude reads<sup>14</sup>

$$\begin{aligned} \mathcal{A}(E, F_1, W_1, y_1, F_2, W_2, y_2) = \\ \frac{\alpha'}{2} \int_0^\infty dt \langle D25(E, F_1, W_1, y_1) | e^{-t(L_0^{(X+bc)} + \tilde{L}_0^{(X+bc)})} | D25(E, F_2, W_2, y_2) \rangle \end{aligned}$$

which in special case where  $F = F_1 = F_2$  and  $W_1 = W_2$  becomes

$$\begin{aligned} \mathcal{A}(E, F, W, y_1, y_2) = & \frac{\alpha'}{2} \left(\frac{T_d}{2}\right)^2 V_{time} \int_0^\infty dt \left( \prod_{n=1}^{\infty} (1 - e^{-2nt}) \right)^{2-1-d} \\ & \sum_{w \in I(F, W)} c_w^*(y_1) c_w(y_2) e^{-\frac{t}{2} w^T \frac{g}{\alpha'} w} \end{aligned}$$

which can be transformed by a modular transformation of parameter  $\tau = \frac{2\pi^2}{t}$  into the corresponding open channel annulus free energy when  $D = 26$

$$F(E, F, W, a_{(0)} - a_{(\pi)}) = 2 \frac{1}{V_{time} \sqrt{\det W \mathcal{G} W}} \int_0^\infty \frac{d\tau}{2\tau} Tr'(e^{-\tau L_0^{(X+bc)}})$$

where  $Tr'$  means a trace over all oscillators but ghost zero modes and the factor  $V_{time} \sqrt{\det W \mathcal{G} W}$  is inserted for allowing the decompactification limit, the trace is over all modes but the ghost zero modes.

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<sup>14</sup>We use as in [17] the closed string propagator to be  $D = \frac{\alpha'}{2} \delta_{L_0 - \tilde{L}_0, 0} \int_0^\infty dt e^{-t(L_0^{(X+bc)} + \tilde{L}_0^{(X+bc)})}$ .

## 4 T-duality and boundary states for $Dp$ branes.

We want now to discuss the action of T-duality on the boundary state. We start our discussion from eq. (16) and not from eq. (15) because T-duality is better defined at the canonical level by eq.s (4) in a  $B$  independent way. Applying the T-duality transformation on eq. (16) written in term of  $X^t$  and  $\mathcal{P}^t$  we get

$$\begin{aligned} \mathcal{P}_m - F^t_{mn} X'^n - 2\pi\alpha' F^t_{ma} (\gamma^{-1})^{ab} \mathcal{P}_b|_{\tau=0} |D25(E^t, F^t, W^t, y^t)\rangle &= 0 \\ \frac{1}{2\pi\alpha'} (\gamma^T)_{ab} X'^b - F^t_{am} X'^m - 2\pi\alpha' F^t_{ab} (\gamma^{-1})^{bc} \mathcal{P}_c|_{\tau=0} |D25(E^t, F^t, W^t, y^t)\rangle &= 0 \end{aligned} \quad (20)$$

where  $a, b, \dots$  run on the  $d_\perp = d - p$  directions “perpendicular” to the brane, i.e. on the directions along which we T-dualize, and  $m, n, \dots$  on the  $p + 1$  directions parallel to the brane. The second equation clearly shows that the geometrical embedding properties (the “angles” of the brane) are only determined by  $F^t_{am}$  when  $F^t_{ab} = 0$  (the would be field strengths in the directions transverse to the brane which describe higher dimensional branes than our  $Dp$ , i.e. we describe a system without higher dimensional branes than our  $Dp$ ); in fact eq. (20) can be rewritten as

$$\left[ X^a - (\alpha' \gamma^{-T})^{ab} 2\pi F^t_{bm} X^m \right]' |_{\tau=0} |D25(E^t, F^t, W^t, y^t)\rangle = 0$$

which is interpreted in the open channel as the constraint that one open string endpoint must be on the hyperplane  $X^a - (\alpha' \gamma^{-T})^{ab} 2\pi F^t_{bm} X^m = \text{const}$  which, as it should, has rational “angles”  $(\alpha' \gamma^{-T})^{ab} 2\pi F^t_{bm}$ .

This is actually a special case which can be obtained as the (careful) limit  $F^t_{ab} \rightarrow 0$  of the generic case with  $\det(F^t_{ab}) \neq 0$ <sup>15</sup> in which case eq.s (20) can be recast in the same form of eq. (16) as

$$(\mathcal{P}_\mu - F_{\mu\nu} X'^\nu) |_{\tau=0} |D25(E^t, F^t, W^t, y^t)\rangle = 0$$

where now the field strength is given by

$$||F_{\mu\nu}|| = \begin{pmatrix} F^t_{mn} - F^t_{mc} (F_{\perp\perp}^{t-1})^{cd} F^t_{dn} & \frac{1}{2\pi\alpha'} F^t_{mc} (F_{\perp\perp}^{t-1})^{cd} (\gamma^T)_{db} \\ -\frac{1}{2\pi\alpha'} \gamma_{ac} (F_{\perp\perp}^{t-1})^{cd} F^t_{dn} & \frac{1}{(2\pi\alpha')^2} \gamma_{ac} (F_{\perp\perp}^{t-1})^{cd} (\gamma^T)_{db} \end{pmatrix} \quad (21)$$

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<sup>15</sup>This requires that we T-dualize at least 2 directions.

where  $|(F_{\perp\perp}^{t-1})^{cd}|$  is the inverse of the matrix  $F_{\perp\perp}^t = ||F_{ab}^t||$ . When we express the boundary  $|D25(E^t, F^t, W^t, y^t)\rangle$  using the operators  $a, \tilde{a}$  and the matrices  $E, F$  we get

$$\begin{aligned} |Dp(E, F, W, y)\rangle &= |D25(E^t, F^t, W^t, y^t)\rangle \\ &= \frac{T_d}{2} |D25(E^t, F^t)\rangle_{time} |D25(E^t, F^t, W^t, y^t)\rangle_{space} |D\rangle_{ghost} \end{aligned} \quad (22)$$

$$\begin{aligned} |D25(E^t, F^t, W^t, y^t)\rangle_{space} &= e^{-\sum_{n=1}^{\infty} a_n^\dagger G \mathcal{E}^{-T} \mathcal{E} \tilde{a}_n^{\dagger T}} \\ &\quad \sum_{w^t \in I(F^t, W^t)} c_{w^t}(y^t) |n = 2\pi F w, w > \end{aligned} \quad (23)$$

where the pieces not explicitly written are the same as in eq. (19). To get this result we have used  $n_a = \frac{\gamma_{ab}}{\alpha'} w^{tb}$ ,  $w^a = \alpha' (\gamma^{-T})^{ab} n_b^t$ . This has a somewhat dramatic consequence: now all  $n_a$  are *integer* while  $w^a$  are *fractional*, moreover  $W^{ta}$  are changed into a matrix  $||W^{ab}||$  as discussed below, see eq. (29) and  $y$  are changed too as in eq. (30)<sup>16</sup>.

#### 4.1 The non degenerate case: $\det(F^t_{ab}) \neq 0$

To understand what is going on we consider again the simplest case we considered in section 3.1, i.e. we take  $2\pi F_{12} = \frac{f_1 f_2}{W^1 W^2} = \frac{p}{q}$ . We rewrite the non trivial boundary equations (18) for the involved zero modes as

$$\begin{aligned} \left( \hat{n}_1 - \frac{f_1 f_2}{W^1 W^2} \hat{w}^2 \right) |B\rangle &= 0 \\ \left( \hat{n}_2 + \frac{f_1 f_2}{W^1 W^2} \hat{w}^1 \right) |B\rangle &= 0 \end{aligned} \quad (24)$$

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<sup>16</sup>Notice that we have still written  $c_{w^t}(y^t) = e^{2\pi i w^{tT} y^t}$  which must be reexpressed using the  $Dp$  quantities  $y, w$  and  $n$  as  $c_w(y) = c_{w^t}(y^t) = e^{2\pi i (w^m y_m + n_a y^a)}$ .

These equations have two different kinds of solution assuming that either  $n_1$  or  $w^1$  be integer and similarly for  $n_2$  and  $w^2$  <sup>17</sup>:

$$“D2” : \begin{cases} w^1 = W^1 l^1 & n_1 = \frac{f_1 f_2}{W^1} l^2 \\ w^2 = W^2 l^2 & n_2 = -\frac{f_1 f_2}{W^2} l^1 \end{cases} \quad l^1, l^2 \in \mathbb{Z} \quad (25)$$

$$“D0” : \begin{cases} n_1 = f_1 j_1 & w^1 = \frac{W_1 W_2}{f_1} j_2 \\ n_2 = f_2 j_2 & w^2 = -\frac{W_1 W_2}{f_2} j_1 \end{cases} \quad j_1, j_2 \in \mathbb{Z} \quad (26)$$

As we discuss below these two solutions have a clear different meaning: an integer  $w$  means a wound worldvolume direction while a fractional  $w$  and an integer  $n$  means a direction perpendicular to the worldvolume. It is important also to realize that given the defining equations for a boundary state (24) there can be many different solutions according to the interpretation we give to the defining equations, besides the difference between integer and rational  $w$  we stated above there is a further ambiguity associated with a given  $2\pi F_{12} = \frac{p}{q}$  since there can be many different ways of factorizing  $p = f_1 f_2$  and  $q = W^1 W^2$  and each of this corresponds to a different configuration.

The first solution given by eq.s (25) is the one already discussed in section 3.1 and describes two branes wrapped  $W^1$  ( $W^2$ ) times along  $X^1$  ( $X^2$ ) .

The second solution eq.s (26) can be seen as the T-dual solution of the previous where T-duality has been performed both along  $X^1$  and  $X^2$  and it has fractional windings and integer momenta. Given this viewpoint we can think that the branes worldvolumes are orthogonal to the plane  $X^1 X^2$ . Since the branes are at fixed position in the coordinate  $X^1$  and  $X^2$  it is not unreasonable to interpret this solution as due to the fact that the *open* string sees a fractional periodicity in these directions. From the values of momenta in eq. (26) we see that the boundary is invariant for translations which are multiple of  $\frac{2\pi}{f_1}$  in direction  $X^1$  and similarly for  $X^2$ . We suppose therefore that the open string has periodicity

$$X_{(open)}^1 \equiv X_{(open)}^1 + 2\pi \frac{N^1}{f_1} \quad X_{(open)}^2 \equiv X_{(open)}^2 + 2\pi \frac{N^2}{f_2}$$

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<sup>17</sup>There is another solution one would think about:

$$\begin{cases} w^1 = W^1 l^1 & n_2 = -\frac{f_1 f_2}{W^2} l^1 \\ n_1 = f_1 j_1 & w^2 = \frac{W_1 W_2}{f_2} j_1 \end{cases} \quad l^1, j_1 \in \mathbb{Z}$$

but now neither  $n_2$  nor  $w^2$  are integers. Moreover this solution is not the T-dual of the first solution.



and, as we show below, it turns out that

$$N^1 = N^2 = W^1 W^2. \quad (27)$$

This would clearly be unacceptable if the branes wound along these directions since fractional winding is meaningless. Given this interpretation we can check as in section 3.1 that the open string zero modes contribution to the one loop free energy which is proportional to  $\sum_{(m_1, m_2) \in \mathbb{Z}^2} e^{-\tau \alpha' (\frac{f_1}{N^1} m_1, \frac{f_2}{N^2} m_2) \mathcal{G}_2^{-1} (\frac{f_1}{N^1} m_1, \frac{f_2}{N^2} m_2)^T}$  is correctly reproduced after a Poisson resummation in the closed channel where the zero modes give in the boundary-boundary interaction a contribution proportional to  $\sum_{(l^1, l^2)} e^{-\frac{i}{2} (\frac{W_1 W_2}{f_1} l^1, \frac{W_1 W_2}{f_2} l^2) \frac{\mathcal{G}_2}{\alpha'} (\frac{W_1 W_2}{f_1} l^1, \frac{W_1 W_2}{f_2} l^2)^T}$  only when eq. (27) holds.

The open string sees a fractional periodicity in directions perpendicular to the branes but the closed string sees an integer periodicity therefore the picture from the closed string point of view is the existence of  $W^1 W^2$  identical copies of the basic system of two branes covering regularly the torus. Pictorially and naively the basic system of two branes can be portrayed by taking two  $Dp$  branes spaced along each direction  $X^a$   $a = 1, 2$  by  $2\pi \left( e_{(0)} a_{(0)a} - e_{(\pi)} a_{(\pi)a} \right)$ , inscribing them into a parallelogram with side lengths  $\frac{2\pi}{W^a}$  and then tiling the  $T^2$  torus with side lengths  $2\pi$  using this configuration<sup>18</sup>. All systems in this configuration are identical because they are the same system from the open string point of view. This picture is the T-dual of the open string point of view of the existence of closed strings which seem to be open when seen from the multiple wound open strings perspective.

The existence of  $W^1 W^2$  copies is also consistent with the computation of the local energy density in this picture and its T-dual.

In the general case the description is more complicated and involves a twisting of the torus on which the open string lives. To find the open string description of the boundary, actually the open string description of the boundary - boundary amplitude, we can proceed as before and again use the fact that the open string zero modes contribution to the one loop free energy must be reproduced by the closed string zero modes contribution to the boundary - boundary amplitude to fix the open string periodicity. We

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<sup>18</sup>Since either  $\frac{W_1 W_2}{f_1}$  or  $\frac{W_1 W_2}{f_2}$  or both can be greater than unity, we are actually tiling a multiple of  $T^2$ .

assume that the open string “homology”<sup>19</sup> is given by

$$X_{(open)} = X_{(open)} + 2\pi W s \quad \forall s \in Z^d \quad (28)$$

and we want to determine  $W$ . If we parametrize the possible values of  $n$  and  $w$  entering the zero modes part of the boundary (23) describing the  $Dp$  using the quantities of the  $D25$  boundary,

$$\begin{aligned} w^m &= W^t m l^t m & w^a &= \alpha' (\gamma^{-T})^{ac} (2\pi F_{cm}^t W^t m l^t m + 2\pi F_{cb}^t W^t b l^t b) \\ n_m &= 2\pi F_{cm} w^m & n_a &= 2\pi F_{ab} w^b \end{aligned}$$

where  $F$  is given by eq. (21),  $l^t \in Z^d$  and we write  $2\pi F^t = W^{t-T} c^t W^{t-1}$  with  $c^t \in Mat_d(Z)$ <sup>20</sup> for the  $D25$  field strength, we can then compare the open string zero modes contribution to the one loop free energy which is proportional to  $\sum_{n \in Z^d} e^{-\tau \alpha' (n + W^T \Delta(ea))^T W^{-1} g^{-1} W^{-T} (n + W^T \Delta(ea))}$  to the closed string contribution using the Poisson resummed open string contribution proportional to  $\sum_{l \in Z^d} e^{-l^T W^T \frac{g}{\alpha'} W l} e^{i 2\pi l^T W^T \Delta(ea)}$ .

From this we can read the  $Dp$  homology matrix which enters the definition of  $|Dp\rangle$  boundary state (22) as

$$\begin{aligned} W &= \begin{pmatrix} P_{\parallel} W^t P_{\parallel} & 0 \\ \alpha' P_{\perp} \gamma^{-T} W^{t-1} c^t P_{\parallel} & \alpha' P_{\perp} \gamma^{-T} W^{t-1} c^t P_{\perp} \end{pmatrix} \\ &= \begin{pmatrix} W^t m \delta_n^m & 0 \\ \alpha' (\gamma^{-T})^{ac} \frac{1}{W^{tc}} c_{cn}^t & \alpha' (\gamma^{-T})^{ac} \frac{1}{W^{tc}} c_{cb}^t \end{pmatrix} \end{aligned} \quad (29)$$

so that the windings entering the boundary are  $w = W l^t$  with  $l^t \in Z^d$  and the  $y$  is connected to  $y^t$  by

$$y = \alpha' \gamma^{-T} y^t \quad (30)$$

We can then conclude that the open string sees a  $d$ -dimensional torus defined by eq. (28) and eq. (29) which is different from the closed string torus defined by eq. (1). Using eq. (28) we can read that the two  $Dps$  wind  $W^m = W^t m$  times in the direction  $X^m$  parallel to the worldvolume exactly as the original  $D25$  while they see twisted and shrunk perpendicular directions. The shrinking is  $\frac{1}{W^{ta}}$  for any transverse direction  $X^a$  and therefore we get  $\prod_b W^{tb}$  identical images on the closed string torus.

<sup>19</sup>Actually  $W$  contains homological information only in the directions parallel to the worldvolume.

<sup>20</sup>The  $c_{ij}^t = -c_{ji}^t$  is the first Chern class along the 2-cycle parametrized by  $x^{ti}$  and  $x^{tj}$ .

## 4.2 The degenerate case: $F^t_{ab} = 0$ .

As done in the previous section we begin with the simplest case, the T-dual version of eq.s (18) along  $X^1$ :

$$\begin{aligned} \left( \hat{n}_1 - \frac{p}{q} \hat{n}_2 \right) |B\rangle &= 0 \\ \left( \hat{w}^2 + \frac{p}{q} \hat{w}^1 \right) |B\rangle &= 0 \end{aligned} \quad (31)$$

which can be derived from the boundary defining equations  $\mathcal{P}_1 - \frac{p}{q} \mathcal{P}_2 |B\rangle = \left( X^2 + \frac{p}{q} X^1 \right)' |B\rangle = 0$  which cannot be reduced to the standard form of eq. (16). These equations suggest to consider the following canonical transformation

$$\bar{X} = \Lambda X \quad \bar{\mathcal{P}} = \Lambda^{-T} \mathcal{P} \\ \Lambda = \begin{pmatrix} r & s \\ p & q \end{pmatrix} \quad , \quad \Lambda^{-T} = \begin{pmatrix} q & -p \\ -s & r \end{pmatrix} \in SL(2, \mathbb{Z})$$

We require  $\Lambda \in SL(2, \mathbb{Z})$  since we do not want to change the closed string theory and in particular its periodicity which is still  $\bar{X} \equiv \bar{X} + 2\pi s$  with  $s \in \mathbb{Z}^2$ . After this change of coordinate the boundary equations become trivially

$$\bar{\mathcal{P}}_1 |B\rangle = \bar{X}'^2 |B\rangle = 0$$

If we express the zero mode part of the boundary using the T-dual parameterization of the one given in eq. (25), i.e.  $\bar{n}_1 = n_1, \bar{w}^1 = w^1, \bar{n}_2 = w^2, \bar{w}^2 = n_2$  we find that the part of our interest can be written as

$$|B\rangle_{space \text{ zero modes}} \sim \sum_{l^t} |\bar{n}_1 = 0, \bar{n}_2 = -\frac{1}{W^{t1}} l^{t2}, \bar{w}^1 = \frac{1}{W^{t2}} l^{t1}, \bar{w}^2 = 0\rangle$$

from which we can deduce that

$$\bar{X}_{(open)} \equiv \bar{X}_{(open)} + 2\pi \bar{W} s \quad \bar{W} = \begin{pmatrix} \frac{1}{W^{t2}} & 0 \\ 0 & W^{t1} \end{pmatrix} \quad \forall s \in \mathbb{Z}^2$$

from which results clearly that the open string winds  $\bar{W}^2 = W^{t1} \in \mathbb{Z}$  times in the direction  $\bar{X}^2$  parallel to the world volume.

The general case is not treated since it is more difficult to treat since it is harder to deal with  $SL(d, \mathbb{Z})$  matrices which are needed to move the system in the proper coordinates.

Finally we notice that we can find the same results using the following reasoning (more details will be given elsewhere [4]). From the previous experience with the emission vertexes for closed string states in open string formalism [18, 19, 3] we expect that the closed string tachyon emission vertex can be written as  $W_{T_c(o)} \sim: e^{ik_L^T X_L(z)} :: e^{ik_R^T X_R(\bar{z})} :$  where  $X_L(z)$  and  $X_R(\bar{z})$  are the left and right moving part of the *open* string coordinate. Since we know how  $k_L$  and  $k_R$  transform under a T-duality as they are the closed string momenta and if we want a T-duality invariant formalism, we deduce immediately that the left and right moving part of the *open* string coordinates must transform as the closed string ones, i.e. as it follows from the extension of eq.s (5) to an expression involving zero modes too<sup>21</sup>:

$$X_{L(open)} = (P_\perp \gamma^{-T} E^{tT} + P_\parallel) X_{L(open)}^t \quad X_{R(open)} = -(P_\perp \gamma^{-T} E^t - P_\parallel) X_{R(open)}^t \quad (32)$$

which satisfy the boundary conditions we can read from the boundary state defining eq.s (20). In particular we get the same b.c. as in eq. (8)

$$G_{\mu\nu} X'^\nu + 2\pi\alpha' \dot{X}^\nu \mathcal{F}_{\nu\mu} |_{\sigma=0,\pi} = 0$$

in the non degenerate case where now  $F$  and  $E$  are related to  $F^t$  and  $E^t$  by eq.s (21) and (6) respectively and

$$\begin{aligned} \partial_\tau \left[ X^a - (\alpha' \gamma^{-T})^{ab} 2\pi F_{bm}^t X^m \right] |_{\sigma=0,\pi} &= 0 \\ \left( G_{mi} - 2\pi\alpha' F_{ma}^t (\gamma^{-1})^{ab} G_{bi} \right) X'^i + \left( B_{mi} - 2\pi\alpha' F_{ma}^t (\gamma^{-1})^{ab} B_{bi} \right) \dot{X}^i \\ - 2\pi\alpha' F_{mn}^t \dot{X}^n |_{\sigma=0,\pi} &= 0 \end{aligned}$$

in the case  $F_{ab}^t = 0$ . These equations reduce to (14) in the simpler case  $F_{am}^t = 0$ .

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<sup>21</sup>These transformation rules were almost obtained in [20] in open string formalism guessing the open string canonical transformations (but missing the somewhat important  $B$  dependence) and in [21] using path integral formalism in the special case of trivial homology and  $F_{ab} = 0$  where an isometry is present.

## 5 Conclusions.

In this article we have shown it is possible to describe branes with non trivial homology and non trivial but constant open background in a non trivial but constant closed background on tori. An interesting and somewhat unexpected consequence is that the physical Hilbert space of closed string theory is not enough and therefore it must be extended with some “twisted” non physical sectors in order to give a closed string representation of open string states. This extension is dictated by the knowledge of the open string spectrum and cannot be guessed, at least in an obvious way, just from the closed string spectrum. This fact can have non trivial consequences in the construction of branes in curved background using Ishibashi formalism. Infact also the boundary state in flat space can be understood as a weighted sum of Ishibashi states, therefore it can be necessary to construct Ishibashi states using non physical (for the closed string) representations.

Another point we discussed is the fact that open string T-duality does not just amount to an exchange  $\tau \leftrightarrow \sigma$  in presence of a non trivial  $B$  field but requires the open string fields to transform as the closed string ones. Because of this  $B$  and  $F$  behave differently under T-duality. This fact can also simply explained by noticing that  $B$  transformation is ruled by closed string which does not know anything about the open field  $F$ . As a further consequence it turns out that if we derive  $Dp$  branes by T-duality their geometrical embedding properties are only determined by  $F$ . In particular this is also true for type I string where only discrete values of  $B$  are allowed [22] since  $B$  is not in the spectrum because T duality rules are inherited from type IIB.

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## A Conventions.

We define:

- WS metric signature:  $\eta_{\alpha\beta} = (-, +)$ ;  $\epsilon^{01} = -1$ ;  $\xi^0 = \tau, \xi^1 = \sigma$   
Space-time metric signature:  $G_{\mu\nu} = (-, +, \dots, +)$ ;
- Indexes:  $D = 26$ ,  $\mu, \nu, \dots = 0, \dots, D-1$ ,  $i, j, \dots = 1, \dots, d = D-1$  are splitted into two sets of indexes:  $m, n, \dots$  for the directions along which we do not T-dualize,  $a, b, \dots$  for the directions which we T-dualize
- $z = e^{2(\tau_E + i\sigma)}$ ,  $\bar{z} = e^{2(\tau_E - i\sigma)}$ , with  $\sigma \in [0, \pi]$
- All matrices have spacial only indexes, i.e.  $X = ||X^i||$ ,  $p = ||p^i||$ ,  $k = ||k_i||$ ,  $\mathcal{G} = ||\mathcal{G}_{ij}||, \dots$
- $X^\mu$ ,  $F_{\mu\nu}$  are dimensionless while  $G_{\mu\nu}$ ,  $B_{\mu\nu}$  have the same dimension of  $\alpha'$

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