Klein-Gordon product: properties

Consider

$$\left(\partial_x^2 + \partial_y^2\right) X_i(x,y) = \partial_u \partial_{\bar{u}} X_i(u,\bar{u}) = 0$$

Define:

$$J(X_1, X_2) = w^{\rho} * \left(X_1^T \overleftrightarrow{d} X_2\right)$$

Then we have:

$$\begin{aligned}
J(X_2, X_1) &= w^{\rho} * \left(X_2^T \overleftrightarrow{d} X_1\right) = w^{\rho} * \left(X_2^T dX_1 - dX_2^T X_1\right) = \\
&= -w^{\rho} * \left(dX_2^T X_1 - X_2^T dX_1\right) = \\
&= -w^{\rho} * \left(X_1^T dX_2 - dX_1^T X_2\right)^T = \\
&= -w^{\rho} * \left(X_1^T \overleftrightarrow{d} X_2\right)^T = \\
&= -J(X_1, X_2)^T.
\end{aligned}$$

Moreover

$$\begin{aligned}
J(X_1, X_2)^* &= w^{\rho *} * \left(X_1^\dagger dX_2^* - dX_1^\dagger X_2^*\right) = w^{\rho *} * \left(dX_2^\dagger X_1^* - X_2^\dagger dX_1^*\right)^T = \\
&= -w^{\rho *} * \left((X_2^*)^T dX_1^* - d(X_2^*)^T X_1^*\right)^T = \\
&= -\frac{w^{\rho *}}{w^{\rho}} \cdot w^{\rho} * \left((X_2^*)^T \overleftrightarrow{d} X_1^*\right)^T = \\
&= -\frac{w^{\rho *}}{w^{\rho}} J(X_2^*, X_1^*)^T.
\end{aligned}$$

$$\Rightarrow J(X_1, X_2)^\dagger = -\frac{w^{\rho *}}{w^{\rho}} J(X_2^*, X_1^*).$$