D BRANES IN STRING THEORY, I

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Abstract. In these lectures we present a detailed description of the origin and of the construction of the boundary state that is now widely used for studying the properties of D branes.

1. Introduction

The existence of Dp-branes in string theories has been an essential ingredient for concluding that the five consistent and perturbatively inequivalent supersymmetric string theories in ten dimensions belong to a unique eleven dimensional theory that is called M-theory. In the framework of string theories their existence was required by T-duality in theories with both open and closed strings 1 . On the other hand classical solutions of the low-energy

¹See Ref. [1] and references therein.

string effective action coupled to graviton, dilaton and (p+1)-form R-R potential were later constructed ². Since their tension is proportional to the inverse of the string coupling constant they correspond to new nonperturbative states of string theory. At the end of 1995 Polchinski [3] provided strong arguments for identifying this new states with the Dp-branes required by T-duality opening the way to study their properties in string theory. In particular their interaction can be computed through the oneloop open string annulus diagram. On the other hand, since the very early days of string theory it is known that this one-loop open string diagram can be equivalently rewritten as a tree diagram in the closed string theory in which a closed string is generated from the vacuum, propagates for a while and then annihilates again in the vacuum. The state that describes the creation of closed string from the vacuum is called the boundary state, that first appeared in the literature [4] in the early days of string theory for factorizing the planar and non-planar loops of open string in the closed string channel. In the middle of the eighties after that the BRST invariant formulation of string theory became available the boundary states was considered again in a series of beautiful papers by Callan et al. [5], where, among other things, the ghost contribution was added and the boundary state with an external abelian gauge field was constructed. It was also used for deriving the gauge group of open string theories by requiring the tadpole cancellation [6]. Its extension to the case of Dirichlet boundary conditions was given in a series of beautiful papers written by M.Green et al. [7] for studying Dp-branes before it became clear that they were new states of string theory corresponding to the classical solution of the low-energy string effective action. In the last few years the boundary state has been widely used for studying properties of D branes in string theory.

In these lectures after a review of the main properties of perturbative string theory we discuss in detail T-duality for both open and closed string theories and we show how the requirement of T-duality in presence of open strings implies the existence of Dp-branes that are then identified with the new non-perturbative states obtained as classical solutions of the low-energy string effective action. Then, by requiring that the interaction between two Dp-branes gives the same result if we compute it in the open or in the closed string channel, we construct the boundary state that provides a stringy description of the simplest Dp brane solutions. Finally we show how to connect the boundary state to the supergravity classical solutions.

²See Ref. [2] and references therein.

2. Perturbative String Theory

The action of the bosonic string theory is

$$S = -\frac{T}{2} \int_{M} d^{2}\xi \sqrt{-h} \ h^{\alpha\beta} \partial_{\alpha} X^{\mu} \partial_{\beta} X_{\mu} \ , \tag{2.1}$$

where T is the string tension, M is the world-sheet of the string described by the world sheet coordinate $\xi^{\alpha} \equiv (\tau, \sigma)$, $h^{\alpha\beta}$ is the world sheet metric tensor and $h = \det h_{\alpha\beta}$. The string tension is related to the Regge slope by $T = (2\pi\alpha')^{-1}$.

The action in eq.(2.1) is invariant under local reparametrizations of the world sheet coordinates corresponding to $\xi^{\alpha} \to f^{\alpha}(\xi)$ and under local Weyl transformations corresponding to a local rescaling of the metric tensor $h_{\alpha\beta} \to \Lambda(\xi)h_{\alpha\beta}$. These symmetries allow one to bring the metric tensor in the form $h_{\alpha\beta} = e^{\phi}\eta_{\alpha\beta}$. This choice is referred to as the conformal gauge choice.

In this gauge the string action is still invariant under some residual local symmetries. It is in fact invariant under a combination of a Weyl rescaling and a local reparametrization $\xi^{\alpha} \to \xi^{\alpha} + \varepsilon^{\alpha}$ ($f^{\alpha} = 1 + \epsilon^{\alpha}$) satisfying the following condition

$$\partial^{\alpha} \varepsilon^{\beta} + \partial^{\beta} \varepsilon^{\alpha} = \Lambda(\sigma) \eta^{\alpha\beta}, \tag{2.2}$$

which corresponds to an infinitesimal conformal transformation. String theory in the conformal gauge is then conformal invariant.

The equation of motion for the string coordinate X^{μ} following from the action in eq.(2.1) in the conformal gauge is given by

$$(\partial_{\sigma}^{2} - \partial_{\tau}^{2})X^{\mu} = 0$$
 , $\mu = 0, ..., d - 1,$ (2.3)

while that for the metric implies the vanishing of the world sheet energy-momentum tensor

$$T_{\alpha\beta} \equiv \frac{2}{T\sqrt{-h}} \frac{\delta S}{\delta h^{\alpha\beta}} = \partial_{\alpha} X \cdot \partial_{\beta} X - \frac{1}{2} \eta_{\alpha\beta} \partial_{\gamma} X \cdot \partial^{\gamma} X = 0, \tag{2.4}$$

where d is the number of dimensions of the embedding space-time. By varying the action in eq.(2.1) in the conformal gauge, in addition to the previous eqs. of motion, we must also impose the following boundary conditions:

$$\int d\tau \left(\partial_{\sigma} X \cdot \delta X |_{\sigma=\pi} - \frac{1}{2} \partial_{\sigma} X \cdot \delta X |_{\sigma=0} \right) = 0, \tag{2.5}$$

where we have taken $\sigma \in [0, \pi]$

The previous boundary conditions can be satisfied in two different ways leading to two different theories. By imposing the periodicity condition

$$X^{\mu}(\tau,0) = X^{\mu}(\tau,\pi),$$
 (2.6)

we obtain a closed string theory, while requiring

$$\partial_{\sigma} X_{\mu} \delta X^{\mu}|_{0,\pi} = 0, \tag{2.7}$$

separately at both $\sigma = 0$ and $\sigma = \pi$ we obtain an open string theory. In this latter case eq.(2.7) can be satisfied in either of the two ways

$$\begin{cases} \partial_{\sigma} X_{\mu}|_{0,\pi} = 0 \to \text{Neumann boundary conditions} \\ \delta X^{\mu}|_{0,\pi} = 0 \to \text{Dirichlet boundary conditions.} \end{cases}$$
 (2.8)

If the open string satisfies Neumann boundary conditions at both its endpoints (N-N boundary conditions) the general solution of the eqs.(2.3) and (2.5) is equal to

$$X^{\mu}(\tau,\sigma) = q^{\mu} + 2\alpha' p^{\mu} \tau + i\sqrt{2\alpha'} \sum_{n \neq 0} \left(\frac{\alpha_n^{\mu}}{n} e^{-in\tau} \cos n\sigma \right), \qquad (2.9)$$

where n is an integer. In order to have more compact expressions without any distinction between the zero and non-zero modes it is convenient to introduce $\alpha_0^{\mu} = \sqrt{2\alpha'}p^{\mu}$. For D-D boundary conditions we have

$$X^{\mu}(\tau,\sigma) = \frac{c^{\mu}(\pi-\sigma) + d^{\mu}\sigma}{\pi} - \sqrt{2\alpha'} \sum_{n \neq 0} \left(\frac{\alpha_n^{\mu}}{n} e^{-in\tau} \sin n\sigma\right). \tag{2.10}$$

Finally for mixed boundary conditions we have

$$X^{\mu}(\tau,\sigma) = c^{\mu} - \sqrt{2\alpha'} \sum_{r \in Z + \frac{1}{2}} \left(\frac{\alpha_r^{\mu}}{r} e^{-ir\tau} \sin r\sigma \right), \qquad (2.11)$$

in the case of D-N boundary conditions and

$$X^{\mu}(\tau,\sigma) = d^{\mu} + i\sqrt{2\alpha'} \sum_{r \in Z + \frac{1}{2}} \left(\frac{\alpha_r^{\mu}}{r} e^{-ir\tau} \cos r\sigma \right), \qquad (2.12)$$

for N-D boundary conditions. c^{μ} and d^{μ} are two constant vectors describing the position of the two endpoints of the string in the embedding spacetime. Among the four solutions in eqs. (2.9)-(2.12) the only one which is Poincaré invariant is the one correponding to N-N boundary conditions. In the following, unless explicitly mentioned, we will refer to this case.

Passing to the case of a closed string the most general solution of the eqs. of motion and of the periodicity condition in eq.(2.6) can be written as follows

$$X^{\mu}(\tau,\sigma) = q^{\mu} + 2\alpha' p^{\mu} \tau + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \left(\frac{\alpha_n^{\mu}}{n} e^{-2in(\tau-\sigma)} + \frac{\widetilde{\alpha}_n^{\mu}}{n} e^{-2in(\tau+\sigma)} \right), \tag{2.13}$$

Also here it is convenient to introduce the notation $\alpha_0^{\mu} = \widetilde{\alpha}_0^{\mu} = p^{\mu} \sqrt{\frac{\alpha'}{2}}$.

The world sheet energy-momentum tensor given in eq.(2.4) is conserved if the string eqs. of motion are satisfied and is also traceless as a consequence of the invariance under Weyl rescaling. It is useful to rewrite it in the light cone coordinates

$$\xi_{+} = \tau + \sigma \qquad ; \qquad \xi_{-} = \tau - \sigma, \tag{2.14}$$

where its two independent components are

$$T_{++} = \partial_+ X \cdot \partial_+ X \qquad ; \qquad T_{--} = \partial_- X \cdot \partial_- X. \tag{2.15}$$

They are both vanishing as a consequence of the eq. of motion for the metric in eq.(2.4).

Inserting in the previous eqs. the mode expansion for a closed string we get

$$T_{++} \sim \sum_{n \in Z} \tilde{L}_n e^{-2in(\tau + \sigma)}$$
 ; $T_{--} \sim \sum_{n \in Z} L_n e^{-2in(\tau - \sigma)}$, (2.16)

where L_n and \widetilde{L}_n are given by

$$L_n = \frac{1}{2} \sum_{m \in \mathbb{Z}} \alpha_{-m} \cdot \alpha_{n+m} \qquad ; \qquad \widetilde{L}_n = \frac{1}{2} \sum_{m \in \mathbb{Z}} \widetilde{\alpha}_{-m} \cdot \widetilde{\alpha}_{n+m}. \tag{2.17}$$

and α_0 and $\tilde{\alpha}_0$ are defined after eq.(2.13) in terms of the momentum. In the case of an open string we have only one set of Virasoro generators:

$$L_n = \frac{1}{2} \sum_{m \in \mathbb{Z}} \alpha_{-m} \cdot \alpha_{n+m} . \tag{2.18}$$

where α_0 is defined after eq.(2.9) in terms of the momentum. The theory can be quantized by imposing equal time canonical commutation relations

$$[\dot{X}^{\mu}(\sigma,\tau), X^{\nu}(\sigma',\tau)] = -i\delta(\sigma - \sigma')\eta^{\mu\nu}, \tag{2.19}$$

$$[X^{\mu}(\sigma,\tau), X^{\nu}(\sigma',\tau)] = [\dot{X}^{\mu}(\sigma,\tau), \dot{X}^{\nu}(\sigma',\tau)] = 0 , \qquad (2.20)$$

which require the following commutation relations on the oscillators and the zero modes

$$[\alpha_m^{\mu}, \alpha_n^{\nu}] = [\widetilde{\alpha}_m^{\mu}, \widetilde{\alpha}_n^{\nu}] = m\delta_{m+n,0}\eta^{\mu\nu} \quad ; \quad [\widehat{q}^{\mu}, \widehat{p}^{\nu}] = i\eta^{\mu\nu} , \qquad (2.21)$$

$$[\alpha_m^{\mu}, \tilde{\alpha}_n^{\nu}] = [\hat{q}^{\mu}, \hat{q}^{\nu}] = [\hat{p}^{\mu}, \hat{p}^{\nu}] = 0 . \tag{2.22}$$

In the quantum theory the Virasoro generators given in eqs. (2.17) and (2.18) are defined by normal ordering the oscillators. But the only operators for which this normal ordering matters are L_0 and \widetilde{L}_0 , because they are the only ones containing products of non-commuting oscillators. We get therefore:

$$L_0 = \frac{\alpha'}{4}\hat{p}^2 + \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n \qquad ; \qquad \widetilde{L}_0 = \frac{\alpha'}{4}\hat{p}^2 + \sum_{n=1}^{\infty} \widetilde{\alpha}_{-n} \cdot \widetilde{\alpha}_n , \quad (2.23)$$

for closed strings, and

$$L_0 = \alpha' \hat{p}^2 + \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n \quad , \tag{2.24}$$

for open strings. The commutation relations for the L_n operators give rise to the Virasoro algebra with central extention

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{d}{12}m(m^2 - 1)\delta_{m+n,o} , \qquad (2.25)$$

The central extension of Virasoro algebra is a consequence of the fact that we have defined L_0 with the normal ordering. In the case of a closed string we also have the operators \tilde{L}_m that commute with all L_m operators and satisfy the same Virasoro algebra as in eq.(2.25).

In the quantum theory the oscillators α_n and $\tilde{\alpha}_n$ become creation and annihilation operators acting on a Fock space. The vacuum state $|0\rangle_{\alpha}|0\rangle_{\alpha}|p\rangle$ with momentum p is defined by the conditions

$$\alpha_n^{\mu}|0\rangle_{\alpha}|0\rangle_{\widetilde{\alpha}}|p\rangle = \tilde{\alpha}_n^{\mu}|0\rangle_{\alpha}|0\rangle_{\widetilde{\alpha}}|p\rangle = 0 \quad \forall n > 0 \quad ,$$
$$\hat{p}^{\mu}|0\rangle_{\alpha}|0\rangle_{\widetilde{\alpha}}|p\rangle = p^{\mu}|0\rangle_{\alpha}|0\rangle_{\widetilde{\alpha}}|p\rangle \quad , \tag{2.26}$$

Because of the Lorentz metric the Fock space defined by the commutation relations in eqs.(2.21) and (2.22) contains states with negative norm. The physical states in the closed string case are characterized by the following conditions:

$$\begin{cases}
L_m |\psi_{\text{phys}}\rangle = \tilde{L}_m |\psi_{\text{phys}}\rangle = 0 & m > 0 \\
(L_0 - 1)|\psi_{\text{phys}}\rangle = (\tilde{L}_0 - 1)|\psi_{\text{phys}}\rangle = 0 & ,
\end{cases}$$
(2.27)

where the intercept -1 appearing in the second equations is a consequence of the normal ordering of L_0 and \tilde{L}_0 . In the open string we have to impose only one set of the previous conditions.

From the lowest eqs. in (2.27) and from eqs.(2.23) and (2.24) we can read the expression for the mass operator. For an open string one gets

$$M^{2} = \frac{1}{\alpha'} \left(\sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_{n} - 1 \right), \tag{2.28}$$

while for a closed string one gets

$$M^{2} = \frac{2}{\alpha'} \left[\sum_{n=1}^{\infty} \left(\alpha_{-n} \cdot \alpha_{n} + \widetilde{\alpha}_{-n} \cdot \widetilde{\alpha}_{n} \right) - 2 \right], \tag{2.29}$$

together with the level matching condition:

$$(\widetilde{L}_0 - L_0)|\psi_{\text{phys}}\rangle = 0. \tag{2.30}$$

The action of superstring in the superconformal gauge is

$$S = -\frac{T}{2} \int_{M} d\tau d\sigma \left(\eta^{\alpha\beta} \partial_{\alpha} X^{\mu} \partial_{\beta} X_{\mu} - i \bar{\psi}^{\mu} \rho^{\alpha} \partial_{\alpha} \psi_{\mu} \right), \tag{2.31}$$

where ψ is a world sheet Majorana spinor and the matrices

$$\rho^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \rho^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \tag{2.32}$$

provide a representation of the Clifford algebra in two dimensions. The previous action is invariant under the following supersymmetry transformations

$$\delta X^{\mu} = \bar{\varepsilon}\psi^{\mu} \quad \delta\psi^{\mu} = -i\rho^{\alpha}\partial_{\alpha}X^{\mu}\varepsilon \quad , \tag{2.33}$$

where ε is a constant Majorana spinor. The Nöther current corresponding to the previous invariance is the supercurrent

$$J_{\alpha} = \frac{1}{2} \rho^{\beta} \rho_{\alpha} \psi^{\mu} \partial_{\beta} X_{\mu}. \tag{2.34}$$

It is useful to write the equations of motion for the fermionic degrees of freedom in the light cone coordinates

$$\partial_{+}\psi_{-}^{\mu} = 0 \quad ; \quad \partial_{-}\psi_{+}^{\mu} = 0,$$
 (2.35)

where

$$\psi_{\pm}^{\mu} = \frac{1 \mp \rho^3}{2} \psi^{\mu} \quad \text{with } \rho^3 \equiv \rho^0 \rho^1.$$
 (2.36)

The boundary conditions are

$$\int d\tau \left(\psi_{+} \delta \psi_{+} - \psi_{-} \delta \psi_{-} \right) \Big|_{\sigma=0}^{\sigma=\pi} = 0.$$
 (2.37)

As before, also these boundary conditions can be fulfilled in two different ways. In the case of an open string eqs. (2.37) are satisfied if we require

$$\begin{cases} \psi_{-}(0,\tau) = \eta_{1}\psi_{+}(0,\tau) \\ \psi_{-}(\pi,\tau) = \eta_{2}\psi_{+}(\pi,\tau) \end{cases},$$
 (2.38)

where η_1 and η_2 can take the values ± 1 . In particular if $\eta_1 = \eta_2$ we get what is called the Ramond (R) sector of the open string, while if $\eta_1 = -\eta_2$ we get the Neveu-Schwarz (NS) sector. In the case of a closed string the fermionic coordinates ψ_{\pm} are independent from each other and they can be either periodic or anti-periodic. This amounts to impose the following conditions:

$$\psi_{-}^{\mu}(0,\tau) = \eta_3 \psi_{-}^{\mu}(\pi,\tau) \qquad \psi_{+}^{\mu}(0,\tau) = \eta_4 \psi_{+}^{\mu}(\pi,\tau), \tag{2.39}$$

that satisfy the boundary conditions in eq.(2.37). In this case we have four different sectors according to the two values that η_3 and η_4 take

$$\begin{cases} \eta_{3} = \eta_{4} = 1 \Rightarrow (R - R) \\ \eta_{3} = \eta_{4} = -1 \Rightarrow (NS - NS) \\ \eta_{3} = -\eta_{4} = 1 \Rightarrow (R - NS) \\ \eta_{3} = -\eta_{4} = -1 \Rightarrow (NS - R) \end{cases}$$
(2.40)

The general solution of eq.(2.35) satisfying the boundary conditions in eqs.(2.38) is given by

$$\psi^{\mu}_{\mp} \sim \sum_{t} \psi^{\mu}_{t} e^{-it(\tau \mp \sigma)}$$
 where
$$\begin{cases} t \in Z + \frac{1}{2} \to \text{NS sector} \\ t \in Z \to \text{R sector} \end{cases}$$
, (2.41)

while the ones satisfying the boundary conditions in eq.(2.39) are given by

$$\psi_{-}^{\mu} \sim \sum_{t} \psi_{t}^{\mu} e^{-2it(\tau - \sigma)}$$
 where $\begin{cases} t \in Z + \frac{1}{2} \to \text{NS sector} \\ t \in Z \to \text{R sector} \end{cases}$, (2.42)

$$\psi_{+}^{\mu} \sim \sum_{t} \widetilde{\psi}_{t}^{\mu} e^{-2it(\tau+\sigma)}$$
 where
$$\begin{cases} t \in Z + \frac{1}{2} \to \widetilde{NS} \text{ sector} \\ t \in Z \to \widetilde{R} \text{ sector} \end{cases}$$
 (2.43)

The energy-momentum tensor, in the light cone coordinates, has two non zero components

$$T_{++} = \partial_{+}X \cdot \partial_{+}X + \frac{i}{2}\psi_{+} \cdot \partial_{+}\psi_{+} \quad ; \quad T_{--} = \partial_{-}X \cdot \partial_{-}X + \frac{i}{2}\psi_{-} \cdot \partial_{-}\psi_{-}, \quad (2.44)$$

while the supercurrent defined in eq. (2.34) reduces to

$$J_{-} = \psi_{-} \cdot \partial_{-} X$$
 ; $J_{+} = \psi_{+} \cdot \partial_{+} X$. (2.45)

From the energy-momentum tensor we can get the Virasoro generators using again the mode expansion in eq.(2.16) and one gets

$$L_n = \frac{1}{2} \sum_{m \in \mathbb{Z}} \alpha_{-m} \cdot \alpha_{n+m} + \frac{1}{2} \sum_{t} \left(\frac{n}{2} + t \right) \psi_{-t} \cdot \psi_{t+n}, \tag{2.46}$$

for an open string and

$$L_n = \frac{1}{2} \sum_{m \in \mathbb{Z}} \alpha_{-m} \cdot \alpha_{n+m} + \frac{1}{2} \sum_{t} \left(\frac{n}{2} + t \right) \psi_{-t} \cdot \psi_{t+n},$$

$$\widetilde{L}_n = \frac{1}{2} \sum_{m \in \mathbb{Z}} \widetilde{\alpha}_{-m} \cdot \widetilde{\alpha}_{n+m} + \frac{1}{2} \sum_{t} \left(\frac{n}{2} + t \right) \widetilde{\psi}_{-t} \cdot \widetilde{\psi}_{t+n}, \tag{2.47}$$

for a closed string. The index t used in the previous expressions and the index v that will be used later on refer both to the NS sector where $t \in Z + \frac{1}{2}$ and to the R sector where $t \in Z$.

The Fourier components of the supercurrent that we denote with G_t and \tilde{G}_t are given by the following expressions in terms of the oscillators

$$G_t = \sum_{n = -\infty}^{\infty} \alpha_{-n} \cdot \psi_{t+n} \quad ; \quad \widetilde{G}_t = \sum_{n = -\infty}^{\infty} \widetilde{\alpha}_{-n} \cdot \widetilde{\psi}_{t+n}. \tag{2.48}$$

The superstring can be quantized by imposing the canonical commutation relations in eqs.(2.19) and (2.20) for the bosonic coordinates and the following canonical anticommutation relations for the fermionic ones

$$\{\psi_A^{\mu}(\sigma,\tau),\psi_B^{\mu}(\sigma'\tau)\} = \pi\delta(\sigma-\sigma')\eta^{\mu\nu}\delta_{AB}.$$
 (2.49)

In terms of the oscillators, together with eqs.(2.21) and (2.22) we have

$$\{\psi_t^{\mu}, \psi_v^{\nu}\} = \eta^{\mu\nu} \delta_{v+t,0}. \tag{2.50}$$

Also in the supersymmetric case the quantum Virasoro generators are defined with a normal ordered product of the oscillators, and again the normal ordering affects only the L_0 operator that becomes

$$L_0 = \alpha' \hat{p}^2 + \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n + \sum_{t>0} t \psi_{-t} \cdot \psi_t . \qquad (2.51)$$

in the case of an open string, while in the case of a closed string the operators L_0 and \tilde{L}_0 are given by

$$L_0 = \frac{\alpha'}{4}\hat{p}^2 + \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n + \sum_{t>0} t\psi_{-t} \cdot \psi_t$$
 (2.52)

and

$$\tilde{L}_0 = \frac{\alpha'}{4}\hat{p}^2 + \sum_{n=1}^{\infty} \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n + \sum_{t>0} t\tilde{\psi}_{-t} \cdot \tilde{\psi}_t$$
 (2.53)

The (anti)commutation relations for the operators given in eqs. (2.46), for $n \neq 0$ and (2.51) for n = 0 and the operators (2.48) give rise to the super Virasoro algebra with central extention

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{d}{8}m(m^2 - 1)\delta_{m+n,o}$$

$$[L_m, G_r] = (\frac{1}{2}m - r)G_{r+m}$$

$$\{G_r, G_s\} = 2L_{r+s} + \frac{d}{2}(r^2 - \frac{1}{4})\delta_{r+s,o}$$

$$(NS) (2.54)$$

for the NS sector and

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{d}{8}m^3\delta_{m+n,o} \\
[L_m, G_n] &= (\frac{1}{2}m-n)G_{n+m} \\
\{G_m, G_n\} &= 2L_{m+n} + \frac{d}{2}n^2\delta_{m+n,o}
\end{aligned} (R)$$
(2.55)

in the R sector. Notice that only the c-number terms in the r.h.s. of the previous equations are different in the two sectors. The algebra of the Ramond sector can be brought into the same form as the one in the NS sector by a redefinition of $L_0 \to L_0 + d/16$. This observation will be used later on to determine the dimension of the spin field operator in the Ramond sector.

Also in superstring the spectrum contains unphysical states with negative norm. The conditions which select the physical states are

$$\begin{cases}
L_m |\psi_{\text{phys}}\rangle = 0 & m > 0 \\
(L_0 - a_0) |\psi_{\text{phys}}\rangle = 0 & , \\
G_t |\psi_{\text{phys}}\rangle = 0 & \forall t \ge 0
\end{cases}$$
(2.56)

where

$$\begin{cases} a_0 = \frac{1}{2} & \text{for the NS sector} \\ a_0 = 0 & \text{for the R sector} \end{cases}$$
 (2.57)

In the case of a closed string we should add to the previous conditions the analogous ones involving the tilded sector. From the middle condition in eq. (2.56) we can read the expression of the mass operator, that in the open string case is equal to

$$M^{2} = \frac{1}{\alpha'} \left(\sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_{n} + \sum_{t>0} t \psi_{-t} \cdot \psi_{t} - a_{0} \right). \tag{2.58}$$

In the case of a closed string we get instead

$$M^2 = \frac{1}{2} \left(M_+^2 + M_-^2 \right) \quad , \tag{2.59}$$

where

$$M_{-}^{2} = \frac{4}{\alpha'} \left(\sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_{n} + \sum_{t} t \psi_{-t} \cdot \psi_{t} - a_{0} \right), \tag{2.60}$$

$$M_{+}^{2} = \frac{4}{\alpha'} \left(\sum_{n=1}^{\infty} \widetilde{\alpha}_{-n} \cdot \widetilde{\alpha}_{n} + \sum_{t} t \widetilde{\psi}_{-t} \cdot \widetilde{\psi}_{t} - \widetilde{a}_{0} \right), \tag{2.61}$$

and the values of a_0 and \tilde{a}_0 are given in eq.(2.57) depending if we are in the NS or in R sector. In the closed string case we should also add the level matching condition

$$(L_0 - \widetilde{L}_0 - a_0 + \widetilde{a}_0)|\psi_{\text{phys}}\rangle = 0.$$
 (2.62)

3. Conformal Field Theory Formulation

As mentioned in Sect. 2, string theories in the conformal gauge are two-dimensional conformal field theories. Thus, instead of the operatorial analysis that we have discussed until now, one can give an equivalent description by using the language of conformal field theory in which one works with the OPE rather then commutators or anticommutators and that contributes to simplify many calculations. In the case of a closed string it is convenient to introduce the variables z and \bar{z} that are related to the world sheet variables τ and σ through a conformal transformation:

$$z = e^{2i(\tau - \sigma)}$$
 ; $\bar{z} = e^{2i(\tau + \sigma)}$, (3.63)

In the case of an euclidean world sheet $(\tau \to -i\tau)$ z and \bar{z} are complex conjugate of each other. In terms of them we can write the bosonic coordinate X^{μ} as follows:

$$X^{\mu}(z,\bar{z}) = \frac{1}{2} \left[X^{\mu}(z) + \tilde{X}^{\mu}(\bar{z}) \right]$$
 (3.64)

where

$$X^{\mu}(z) = \hat{q}^{\mu} - i\sqrt{2\alpha'}\log z\alpha_0^{\mu} + i\sqrt{2\alpha'}\sum_{n\neq 0} \frac{\alpha_n^{\mu}}{n}z^{-n}$$
 (3.65)

and

$$\widetilde{X}^{\mu}(\bar{z}) = \hat{q}^{\mu} - i\sqrt{2\alpha'}\log\bar{z}\widetilde{\alpha}_{0}^{\mu} + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{\widetilde{\alpha}_{n}^{\mu}}{n}\bar{z}^{-n}$$
(3.66)

with $\alpha_0^{\mu} = \tilde{\alpha}_0^{\mu} = \sqrt{\alpha'/2} \; \hat{p}^{\mu}$. In the case of an open string theory one can introduce the variables:

$$z = e^{i(\tau - \sigma)}$$
 ; $\bar{z} = e^{i(\tau + \sigma)}$ (3.67)

and the string coordinate can be written as

$$X^{\mu}(z,\bar{z}) = \frac{1}{2} \left[X^{\mu}(z) + X^{\mu}(\bar{z}) \right] , \qquad (3.68)$$

where X^{μ} is given in eq. (3.65) and $\sqrt{2\alpha'}\hat{p}^{\mu} = \alpha_0^{\mu}$. In superstring theory we must also introduce a conformal field with conformal dimension equal to 1/2 corresponding to the fermionic coordinate. In the closed string case we have two independent fields for the holomorphic and anti-holomorphic sectors which are obtained from eqs. (2.42) and (2.43) through the Wick rotation $\tau \to -i\tau$ and the conformal transformation $(\tau, \sigma) \to (z, \bar{z})$

$$\Psi^{\mu}(z) \sim \sum_{t} \psi_{t} z^{-t-1/2} \quad ; \quad \widetilde{\Psi}^{\mu}(\bar{z}) \sim \sum_{t} \widetilde{\psi}_{t} \bar{z}^{-t-1/2}$$
(3.69)

In the open string case, starting from eq.(2.41) and applying the same operations we get again eqs.(3.69), but this time with the same oscillators.

In what follows we will explicitly consider only the holomorphic sector for the closed string. Analogous considerations hold for the antiholomorphic sector. In the case of an open string it is sufficient to consider the string coordinate at the string endpoint $\sigma = 0$. In both cases it is convenient to introduce a bosonic dimensionless variable:

$$x^{\mu}(z) \equiv X^{\mu}(z)/(\sqrt{2\alpha'}) = \tilde{q}^{\mu} - i\alpha_0^{\mu} \log z + i\sum_{n\neq 0} \frac{\alpha_n}{n} z^{-n}$$
, (3.70)

where $\tilde{q} = \hat{q}/\sqrt{2\alpha'}$ and a fermionic one:

$$\psi^{\mu}(z) = -i\sum_{t} \psi_{t} z^{-t-1/2} \tag{3.71}$$

The theory can be quantized by imposing the following OPEs

$$x^{\mu}(z)x^{\nu}(w) = -\eta^{\mu\nu}\log(z-w) + \dots; \quad \psi^{\mu}(z)\psi^{\nu}(w) = -\frac{\eta^{\mu\nu}}{z-w} + \dots , \quad (3.72)$$

where the dots denote finite terms for $z \to w$. Notice that these OPEs coincide with the 2-points Green's functions except for the Ramond case where the Green's function is equal to:

$$<\psi^{\mu}(z)\psi^{\nu}(w)> = -\frac{\eta^{\mu\nu}}{z-w}\frac{1}{2}\left[\sqrt{\frac{z}{w}} + \sqrt{\frac{w}{z}}\right]$$
 (3.73)

Since, however, its singular behaviour when $z \to w$ is the same as in eq.(3.72), we use the contractions in eqs.(3.72) for both the NS and the R sector. In terms of the previous conformal fields we can define the generators of superconformal transformations:

$$G(z) = -\frac{1}{2}\psi \cdot \partial x$$
 ; $T(z) = T^{x}(z) + T^{\psi}(z) = -\frac{1}{2}(\partial x)^{2} - \frac{1}{2}\partial \psi \cdot \psi$. (3.74)

Their mode expansion is given by:

$$L_n = \frac{1}{2\pi i} \oint dz z^{n+1} T(z)$$
 ; $G_t = \frac{1}{2\pi i} \oint dz z^{t+1/2} G(z)$. (3.75)

The conformal fields in eq.(3.74) satisfy the following OPEs:

$$T(z)T(w) = \frac{\frac{d}{dw}T(w)}{z-w} + 2\frac{T(w)}{(z-w)^2} + \frac{c/2}{(z-w)^4} + \dots , \qquad (3.76)$$

$$T(z)G(w) = \frac{\partial/\partial w G(w)}{z - w} + \frac{3}{2} \frac{G(w)}{(z - w)^2} + \dots$$
 (3.77)

$$G(z)G(w) = \frac{2T(z)}{z-w} + \frac{d}{(z-w)^3} + \dots$$
 (3.78)

Using eqs.(3.75) it is easy to see that the previous OPEs imply the super Virasoro algebra in eq.(2.54) for both the NS and the R sector. But then the superconformal algebra that we get in the R sector differs from the one given in eq.(2.55). However, as we have noticed in the previous section, eq.(2.55) can be reduced to eq. (2.54) by translating L_0 in the R sector in eq.(2.55) by a constant:

$$L_0 \to L_0^{conf} \equiv L_0 + \frac{d}{16} = \sum_{n=1}^{\infty} (\alpha_{-n} \cdot \alpha_n + n\psi_{-n} \cdot \psi_n) + \alpha' p^2 + \frac{d}{16}$$
 (3.79)

Therefore in the R sector we have two L_0 operators that are related by eq.(3.79). L_0 determines the spectrum of superstring through eq.(2.56), while L_0^{conf} that satisfies the algebra in eq.(2.54) encodes the correct conformal properties of the R sector.

In conformal field theory one introduces the concept of conformal or primary field $\Phi(z)$ with dimension h as the object that satisfies the following OPE with the energy-momentum tensor:

$$T(z)\Phi(w) = \frac{\partial_w \Phi(w)}{z - w} + h \frac{\Phi(w)}{(z - w)^2} + \dots$$
 (3.80)

From it one can compute the corresponding highest weight state $|\Phi\rangle$ by means of the following limiting procedure

$$|\Phi\rangle = \lim_{z \to 0} \Phi(z)|0\rangle$$
 , $\langle \Phi| = \lim_{z \to 0} \langle 0|\Phi^{\dagger}(z) \sim \lim_{z \to \infty} \langle 0|(z^2)^h \Phi(z)$ (3.81)

The hermitian conjugate field Φ^{\dagger} in the previous expression has been defined as the field transformed under the conformal transformation $z \to 1/z$ apart from possibly a phase factor. Using the previous definition and the

expression for L_0 given in eq.(3.75), it is easy to see that, if the conformal field has conformal dimension h, then the corresponding state is an eigenstate of L_0 with eigenvalue h and it is annihilated by all the Virasoro operators with n > 0, namely

$$L_0|\Phi\rangle = h|\Phi\rangle$$
 ; $L_n|\Phi\rangle = 0$. (3.82)

In the bosonic case the physical conditions on the states given in eq.(2.27) imply that the vertex operators of the bosonic string theory $V_{\alpha}(z)$ are conformal fields with conformal dimension equal to 1. This insures that the quantity $dzV_{\alpha}(z)$ is invariant under conformal transformations. In the supersymmetric case the physical conditions in eq.(2.56) imply $L_0|\psi_{phys}\rangle = 1/2|\psi_{phys}\rangle$ in the NS-sector and $L_0|\psi_{phys}\rangle = 0$ in the R-sector. Therefore the corresponding vertex operators are not conformal fields with conformal dimension equal to 1 as in the bosonic string. On the other hand in the case of the R sector we have seen that, in order to determine the correct dimension of a vertex operator, we should use the operator $L_0^{\rm conf}$ that, for d=10, acts on the spinorial ground state $|A\rangle$ as follows

$$L_0^{conf}|A\rangle = \frac{5}{8}|A\rangle , \qquad (3.83)$$

This does not help, however, to get a vertex operator that has dimension equal to 1. In the case of superstring we will see that, in order to get the correct physical vertex operators in both NS and R sectors, one must add the contribution of the superghost degrees of freedom that will be discussed later on. The R vacuum state $|A\rangle$ in eq.(3.83) can be written in terms of the NS vacuum by introducing the spin field operator $S^A(z)$ satisfying the eq.:

$$\lim_{z \to 0} S^A(z)|0\rangle = |A\rangle \quad , \tag{3.84}$$

where $|0\rangle$ is the NS vacuum. Thus from eq. (3.83) we see that the spin field $S^A(z)$, which maps the NS vacuum into the R one, must have confromal dimension 5/8.

One can show that the spin field $S^A(z)$ satisfies the following OPEs [8]:

$$\psi^{\mu}(z)S_A(w) = (z - w)^{-1/2}(\Gamma^{\mu})_{AB}S^B(w) + \dots$$

$$S_A(w)S_B(w) = (z - w)^{-3/2}(\Gamma)_{AB}\psi^{\mu} + \dots$$
(3.85)

Until now we have completely disregarded the analysis of the ghost and superghost degrees of freedom that, however, must be included in a correct Lorentz covariant quantization of string theory. They arise from the exponentiation of the Faddev-Popov determinant that is obtained when the

string is quantized through the path-integral quantization. In particular in the bosonic case, choosing the conformal gauge, one gets the following ghost action [9]

$$S_{\text{ghosts}} \sim \int d^2 z [b\bar{\partial}c + \text{c.c.}] ,$$
 (3.86)

where b and c are fermionic fields with conformal dimension equal respectively to 2 and -1. The ghost system of the bosonic string is a particular case of the fermionic bc system described in the Appendix corresponding to a screening charge Q = -3 and a central charge of the Virasoro operator c = -26.

With the introduction of ghosts the string action in the conformal gauge becomes invariant under the BRST transformations and the physical states are characterized by the fact that they are annihilated by the BRST charge that in the bosonic case is given by

$$Q \equiv \oint \frac{dz}{2\pi i} J_{BRST}(z) \equiv \oint \frac{dz}{2\pi i} c(z) \left[T^x(z) + \frac{1}{2} T^{bc}(z) \right] , \qquad (3.87)$$

where $T^x(z) = -1/2(\partial x)^2$, and T^{bc} is given in eq.(A.7) for $\lambda = 2$. It can be shown that Q is nilpotent if the space-time dimension d = 26. The physical states are annihilated by the BRST charge

$$Q|\psi_{\rm phys}\rangle = 0 \tag{3.88}$$

This implies that the vertex operators corresponding to the physical states must satisfy the condition

$$[Q, \mathcal{W}(w)]_{\eta} = \oint_{w} \frac{dz}{2\pi i} J_{BRST}(z) \,\mathcal{W}(w) = 0 \quad , \tag{3.89}$$

where $[,]_{\eta}$ means commutator $(\eta = -1)$ [anticommutator $(\eta = 1)$] when the vertex operator is a bosonic [fermionic] quantity. By using the OPE it can be shown that in the bosonic string the most general BRST invariant vertex operator has the following form

$$\mathcal{W}(z) = c(z)\mathcal{V}_{\alpha}^{x}(z) \tag{3.90}$$

where \mathcal{V}_{α}^{x} is a conformal field with dimension equal to 1 that depends only on the string coordinate x^{μ} .

In the supersymmetric case one must add to the ghost action in eq.(3.86) the superghost one:

$$S_{sghost} \sim \int d^2 z \left(\beta \bar{\partial} \gamma + c.c\right) ,$$
 (3.91)

where β and γ are bosonic fields with conformal dimensions equal respectively to 3/2 and -1/2. This is a particular case of the bosonic bc system described in the Appendix with $\epsilon = -1$, $\lambda = 3/2$ and $\mathcal{Q} = 2$, corresponding to a central charge of the Virasoro algebra c = 11. The BRST charge can be conveniently defined by using a world-sheet superfield formulation where one introduces

$$Z = (z, \theta)$$
 ; $\hat{X}(Z) = x + \theta \psi$; $D = \partial_{\theta} + \theta \partial_{z}$ (3.92)

and defines the BRST-supercharge as

$$Q = \oint \frac{dzd\theta}{2\pi i} C(Z) \left[\hat{T}^m(Z) + \frac{1}{2} \hat{T}^g(Z) \right]$$
 (3.93)

where

$$\hat{T}^m(Z) = -\frac{1}{2}D\hat{X}\partial\hat{X} = G(z) + \theta T(z) \quad , \tag{3.94}$$

$$C(Z) = c(z) + \theta \gamma(z) \quad ; \quad B(Z) = \beta(z) + \theta b(z) \tag{3.95}$$

with T(z) and G(z) defined in eq.(3.74) and

$$\hat{T}^{g}(Z) \equiv G^{g}(z) + \theta T^{g}(z) = -C\partial B + \frac{1}{2}DCDB - \frac{3}{2}\partial CB =$$

$$= -c\partial \beta + \frac{1}{2}\gamma\beta - \frac{3}{2}\partial c\beta + \theta(T^{bc} + T^{\beta\gamma}) . \tag{3.96}$$

Performing the Grassmann integration over θ one gets

$$Q \equiv \oint dz J_{BRST}(z) = Q_0 + Q_1 + Q_2 \quad , \tag{3.97}$$

where

$$Q_0 = \oint \frac{dz}{2\pi i} c(z) \left[T(z) + T^{\beta\gamma}(z) + \partial c(z) b(z) \right]$$
 (3.98)

and

$$Q_1 = \frac{1}{2} \oint \frac{dz}{2\pi i} \gamma(z) \psi(z) \cdot \partial X(z) \quad ; \quad Q_2 = -\frac{1}{4} \oint \frac{dz}{2\pi i} \gamma^2(z) b(z) \quad (3.99)$$

A vertex operator corresponding to a physical state must be BRST invariant, i.e.

$$[Q, \mathcal{W}(Z)]_n = 0 \tag{3.100}$$

Before the introduction of ghosts and superghosts, the vertex operators for the NS sector in the superfield formalism can be written as

$$\mathcal{V}(Z) = \mathcal{V}_0(z) + \theta \mathcal{V}_1(z) \tag{3.101}$$

where V_0 and V_1 are two conformal fields with dimension 1/2 and 1 respectively. For example in the massless NS sector the two fields are given by

$$\mathcal{V}_0(z) = \epsilon \cdot \psi(z)e^{ik\cdot X(z)}$$
; $\mathcal{V}_1(z) = (\epsilon \cdot \partial X(z) + ik \cdot \psi \epsilon \cdot \psi)e^{ik\cdot X(z)}$. (3.102)

But they are not BRST invariant. In order to construct a BRST invariant version of the vertex $\mathcal{V}_0(z)$ we must add the contribution of the ghosts and superghosts. This can be easily done and one gets

$$W_{-1}(z) = c(z)e^{-\varphi(z)}V_0(z)$$
(3.103)

In the case of the massless vertex in eq.(3.102) the vertex in eq.(3.103) is BRST invariant if $k^2 = \epsilon \cdot k = 0$. We can proceed in an analogous way in the R sector and obtain the following BRST invariant vertex operator for the massless fermionic state of open superstring [8]:

$$W_{-1/2}(z) = u_A(k)c(z)S^A(z)e^{-\frac{1}{2}\varphi(z)}e^{ik \cdot X(z)}$$
(3.104)

It is BRST invariant if $k^2 = 0$ and $u_A (\Gamma^{\mu})_B^A k_{\mu} = 0$. Both vertices in eqs.(3.103) and (3.104) have conformal dimension equal to zero as in the case of the bosonic string (see eq.(3.90)).

In superstring, however, unlike the bosonic string, for each physical state we can construct an infinite tower of equivalent physical vertex operators all (anti)commuting with the BRST charge and characterized according to their superghost picture P that is equal to the total ghost number of the scalar field φ and of the $\eta\xi$ system that appear in the "bosonization" of the $\beta\gamma$ system (see the Appendix for details):

$$P = \oint \frac{dz}{2\pi i} \left(-\partial \varphi + \xi \eta \right) \tag{3.105}$$

Notice that the vertex in eq.(3.103) is in the picture -1, while the one in eq.(3.104) is in the picture -1/2. Vertex operators in different pictures are related through the picture changing procedure that we are now going to describe. Starting from a BRST invariant vertex W_t in the picture t (characterized by a value of P equal to t), where t is integer (half-integer) in the NS (R) sector, one can construct another BRST invariant vertex operator W_{t+1} in the picture t+1 through the following operation [8]

$$W_{t+1}(w) = [Q, 2\xi(w)W_t(w)]_{\eta} = \oint_w \frac{dz}{2\pi i} J_{BRST}(z) \ 2\xi(w)W_t(w) \ . \ (3.106)$$

Using the Jacobi identity and the fact that $Q^2 = 0$ one can easily show that the vertex $W_{t+1}(w)$ is BRST invariant:

$$[Q, \mathcal{W}_{t+1}]_n = 0 (3.107)$$

On the other hand the vertex $W_{t+1}(w)$ obtained through the construction in eq.(3.106) is not BRST trivial because the corresponding state contains the zero mode ξ_0 that is not contained in the Hilbert space of the $\beta\gamma$ system (see eq.(A.27)). In conclusion all the vertices constructed through the procedure given in eq.(3.106) are BRST invariant and non trivial in the sense that all give a non-vanishing result when inserted for instance in a tree-diagram correlator provided that the total picture number is equal to -2. Using the picture changing procedure from the vertex operator in eq.(3.103) we can construct the vertex operator in the 0 superghost picture which is given by [10]

$$W_0(z) = c(z)V_1(z) - \frac{1}{2}\gamma(z)V_0(z)$$
 (3.108)

Analogously starting from the massless vertex in the R sector in eq.(3.104) one can construct the corresponding vertex in an arbitrary superghost picture t.

In the closed string case the vertex operators are given by the product of two vertex operators of the open string. Thus for the massless NS-NS sector in the superghost picture (-1, -1) we have

$$W_{(-1,-1)} = \epsilon_{\mu\nu} V_{-1}^{\mu}(k/2, z) \tilde{V}_{-1}^{\nu}(k/2, \bar{z}) , \qquad (3.109)$$

where $\mathcal{V}_{-1}^{\mu}(k/2,z) = c(z)\psi^{\mu}(z)e^{-\varphi(z)}e^{i\frac{k}{2}\cdot X(z)}$ and $\widetilde{\mathcal{V}}_{-1}^{\nu}$ is equal to an analogous expression in terms of the tilded modes. This vertex is BRST invariant if $k^2=0$ and $\epsilon_{\mu\nu}k^{\nu}=k^{\mu}\epsilon_{\mu\nu}=0$.

In the R-R sector the vertex operator for massless states in the $(-\frac{1}{2}, -\frac{1}{2})$ superghost picture is

$$\mathcal{W}_{(-1/2,-1/2)} = \frac{(C\Gamma^{\mu_1\dots\mu_{m+1}})_{AB} F_{\mu_1\dots\mu_{m+1}}}{2\sqrt{2}(m+1)!} \mathcal{V}_{-1/2}^A(k/2,z) \widetilde{\mathcal{V}}_{-1/2}^B(k/2,\bar{z})$$
(3.110)

where $\mathcal{V}^A_{-1/2}(k/2,z)=c(z)S^A(z)e^{-\frac{1}{2}\varphi(z)}e^{i\frac{k}{2}\cdot X(z)}$ and

$$F_{\mu_1...\mu_{m+1}} = \frac{(-1)^{m+1}}{2^5} u_D(k) (\Gamma_{\mu_1...\mu_{m+1}} C^{-1})^{DE} \widetilde{u}_E(k) . \tag{3.111}$$

It is BRST invariant if $k^2 = 0$ and $F_{\mu_1...\mu_m}$ is a field strength satisfying both the Maxwell equation (dF = 0) and the Bianchi identity $(d^*F = 0)$. The two Weyl-Majorana spinors u_A and \tilde{u}_B may have the same or opposite chirality. In the first case one obtains type IIB theory while in the second case one obtains the type IIA theory. From eq.(3.111) one can see that the only field strengths which are allowed are those for (m+1) odd in IIB theory

and those for (m+1) even in IIA theory. Moreover, from eq.(3.111) one can show that the field strengths with values of m related by Hodge duality are not independent and one can restrict oneself to the values (m+1) = 1, 3, 5 in type IIB and (m+1) = 2, 4 in type IIA string theory.

Since the physical state corresponding to the symmetric vertex given in eq.(3.110) cannot be used to compute its coupling with a D- brane because the boundary state that we will construct in Sect. 7 is in an asymmetric picture, in the following we will explicitly write the vertex operator of a physical R-R state in the asymmetric picture (-1/2, -3/2). It is given by [11]:

$$\mathcal{W}_{(-1/2,-3/2)} = \sum_{M=0}^{\infty} \frac{a_M}{2\sqrt{2}} \left(C \mathcal{A}^{(m)} \Pi_M \right)_{AB} \mathcal{V}_{-1/2+M}^A(k/2,z) \tilde{\mathcal{V}}_{-3/2-M}^B(k/2,\bar{z})$$
(3.112)

where

$$(CA^{(m)})_{AB} = \frac{(C\Gamma^{\mu_1...\mu_m})}{m!} A_{\mu_1...\mu_m} , \quad \Pi_q = \frac{1 + (-1)^q \Gamma_{11}}{2}$$
 (3.113)

and

$$\mathcal{V}^{A}_{-1/2+M}(k/2,z) = \partial^{M-1}\eta(z)...\eta(z)c(z)S^{A}(z)e^{\left(-\frac{1}{2}+M\right)\varphi(z)}e^{i\frac{k}{2}\cdot X(z)} \ \ (3.114)$$

$$\widetilde{\mathcal{V}}^{B}_{-3/2-M}(k/2,\bar{z}) = \bar{\partial}^{M}\widetilde{\xi}(\bar{z})...\bar{\partial}\widetilde{\xi}(\bar{z})\widetilde{c}(\bar{z})\widetilde{S}^{A}(\bar{z})e^{\left(-\frac{3}{2}-M\right)\widetilde{\varphi}(\bar{z})}e^{i\frac{k}{2}\cdot\widetilde{X}(\bar{z})} \quad (3.115)$$

It can be shown that the vertex operator in eq.(3.112) is BRST invariant if $k^2 = 0$ and the following two conditions are satisfied

$$a_M = \frac{(-1)^{M(M+1)}}{[M!(M-1)!...1]^2} , \quad d^*A^{(m)} = 0 .$$
 (3.116)

By acting with the picture changing operator on the vertex in eq.(3.112) it can be shown that one obtains the vertex in the symmetric picture in eq.(3.110). In particular one can show that only the first term in the sum in eq.(3.112) reproduces the symmetric vertex, while all the other terms give BRST trivial contributions.

4. T-Duality

The compactification of a dimension in string theory is characterised by the appearance of new interesting phenomena with respect to those already present in field theory. In fact, in the case of a closed string, together with the Kaluza-Klein (K-K) excitations, a new kind of states called winding states appear in the spectrum. It turns out that the bosonic closed string theory is invariant under the exchange of the winding modes with the K-K modes according to a transformation that is called T-duality. In the supersymmetric case, instead, this transformation is in general not a symmetry anymore but brings from a certain string theory to another string theory. For instance T-duality along a certain direction acts interchanging the IIA with the IIB theory. In the case of an open string, instead, this analysis naturally leads to the existence of other objects called Dp-branes.

Let us discuss in some detail the compactification and the T-duality invariance, starting with the bosonic closed string.

The most general solution of the eqs. of motion for the bosonic closed string in eq.(2.3) can be written as:

$$X^{\mu}(\tau,\sigma) = q^{\mu} + \sqrt{2\alpha'} \left(\alpha_0^{\mu} + \widetilde{\alpha}_0^{\mu}\right) \tau - \sqrt{2\alpha'} \left(\alpha_0^{\mu} - \widetilde{\alpha}_0^{\mu}\right) \sigma + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \left(\frac{\alpha_n^{\mu}}{n} e^{-2in(\tau - \sigma)} + \frac{\widetilde{\alpha}_n^{\mu}}{n} e^{-2in(\tau + \sigma)}\right) , \qquad (4.117)$$

where the momentum of the string is given by

$$p^{\mu} = \frac{1}{\sqrt{2\alpha'}} \left(\alpha_0^{\mu} + \tilde{\alpha}_0^{\mu} \right) . \tag{4.118}$$

In the uncompactified case the two zero modes must be identified because the string coordinate must be invariant under $\sigma \to \sigma + \pi$ and the expression for the momentum in eq.(4.118) reduces to the one obtained just after eq.(2.13).

Let us compactify one of the space dimensions along a circle with radius R. This means that the string coordinate corresponding to this direction that we denote for simplicity with X without any index must be periodically identified as:

$$X \sim X + 2\pi R$$
 (4.119)

As in the point particle case, the conjugate momentum corresponding to the compactified direction must be quantized as

$$p = \frac{n}{R} \quad \text{with} \quad n \in Z \ . \tag{4.120}$$

This is simply a consequence of the fact that the generator of the translations along the compact direction e^{ipa} must reduce to the identity for $a=2\pi R$. Moreover in the compactified case the string coordinate X must be invariant under $\sigma \to \sigma + \pi$ apart from a factor $2\pi Rw$ (w is an integer) as follows from eq.(4.119). This implies that

$$\pi\sqrt{2\alpha'}(\alpha_0 - \tilde{\alpha}_0) = 2\pi wR \quad \text{with} \quad w \in Z ,$$
 (4.121)

where w corresponds to the number of times that the closed string winds around the compact direction.

Eqs. (4.118) and (4.120) together with eq. (4.121) imply that the zero modes for the compact direction must have the following expression

$$\alpha_0 = \sqrt{\frac{\alpha'}{2}} \left(\frac{n}{R} + \frac{wR}{\alpha'} \right)$$
 and $\widetilde{\alpha}_0 = \sqrt{\frac{\alpha'}{2}} \left(\frac{n}{R} - \frac{wR}{\alpha'} \right)$. (4.122)

Inserting eq.(4.122) in eq.(2.23) and writing also the contribution of the uncompactified directions we get

$$L_0 = \frac{\alpha'}{4}\hat{p}^2 + \frac{1}{2}\alpha_0^2 + \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n = \frac{\alpha'}{4}\hat{p}^2 + \frac{\alpha'}{4}\left(\frac{n}{R} + \frac{wR}{\alpha'}\right)^2 + \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n ,$$
(4.123)

and

$$\widetilde{L}_0 = \frac{\alpha'}{4}\widehat{p}^2 + \frac{\alpha'}{4}\left(\frac{n}{R} - \frac{wR}{\alpha'}\right)^2 + \sum_{n=1}^{\infty} \widetilde{\alpha}_{-n} \cdot \widetilde{\alpha}_n , \qquad (4.124)$$

The mass operator becomes

$$M^{2} = \frac{2}{\alpha'} \left[\sum_{n=1}^{\infty} \left(\alpha_{-n} \cdot \alpha_{n} + \widetilde{\alpha}_{-n} \cdot \widetilde{\alpha}_{n} \right) - 2 \right] + \left(\frac{n}{R} \right)^{2} + \left(\frac{wR}{\alpha'} \right)^{2} . \quad (4.125)$$

From the previous expression we see that the spectrum of the closed string has been enriched by the appearance of two kinds of particles: the usual K-K modes which contribute to the energy with $\frac{n}{R}$ together with some new excitations that are called winding modes because they can be thought of as generated by the winding of the closed string around the compact direction which in fact contributes to the energy of the system as

$$T \ 2\pi Rw = \frac{wR}{\alpha'} \ , \tag{4.126}$$

where $T=1/(2\pi\alpha')$ is the string tension. All previous formulas can be trivially generalized to the case of a toroidal compactified theory in which more then one coordinate X^{ℓ} is compactified on circles with radii $R^{(\ell)}$. Of course in this case we will have K-K and winding modes corresponding to all the compactified directions.

From eq.(4.125) we see that the spectrum of the theory is invariant under the exchange of KK modes with winding modes together with an inversion of the radius of compactification:

$$w \leftrightarrow n$$
 ; $R \leftrightarrow \hat{R} \equiv \frac{\alpha'}{R}$. (4.127)

This is called a T-duality transformation and \hat{R} is the compactification radius of the T-dual theory. It can also be shown that both the partition function and the correlators are invariant under T-duality. This means that T-duality is a symmetry of the bosonic closed string theory. As a consequence of this invariance, whenever we have to consider compactified theories, we can limit ourselves to the case $R \geq \sqrt{\alpha'}$. That is the reason why $\sqrt{\alpha'}$ is often called the minimal length of the string theory.

Substituting eq.(4.127) into eq.(4.122) we obtain the action of T-duality on the zero modes

$$\alpha_0 \to \alpha_0$$
 ; $\widetilde{\alpha}_0 \to -\widetilde{\alpha}_0$. (4.128)

This transformation, however, changes the operators \tilde{L}_n in eq.(2.17) and then does not leave invariant the physical subspace. In order to keep the physical subspace invariant we must extend the previous transformation property to all the non-zero oscillators:

$$\alpha_n \to \alpha_n$$
 ; $\widetilde{\alpha}_n \to -\widetilde{\alpha}_n \quad \forall n \in \mathbb{Z}$. (4.129)

This transformation leaves obviously also invariant the mass of the states given in eq.(4.125).

Eqs. (4.128) and (4.129) allow us to define the action of T-duality directly on the string coordinate X. In fact writing

$$X = \frac{1}{2} (X_{-} + X_{+}), \qquad (4.130)$$

where

$$X_{-} = q + 2\sqrt{2\alpha'}(\tau - \sigma)\alpha_0 + i\sqrt{2\alpha'}\sum_{n \neq 0} \frac{\alpha_n}{n} e^{-2in(\tau - \sigma)} , \qquad (4.131)$$

and

$$X_{+} = q + 2\sqrt{2\alpha'}(\tau + \sigma)\widetilde{\alpha}_{0} + i\sqrt{2\alpha'}\sum_{n \neq 0} \frac{\widetilde{\alpha}_{n}}{n} e^{-2in(\tau + \sigma)} , \qquad (4.132)$$

we see from eqs. (4.128) and (4.129) that the T-dual coordinate \hat{X} must satisfy the conditions

$$\partial_{\tau}X \to \partial_{\tau}\hat{X} = -\partial_{\sigma}X \qquad ; \qquad \partial_{\sigma}X \to \partial_{\sigma}\hat{X} = -\partial_{\tau}X.$$
 (4.133)

They are satisfied if the T-dual coordinate is equal to

$$\hat{X} = \frac{1}{2} (X_{-} - X_{+}) . {(4.134)}$$

Therefore the T-duality transformation acts on the right sector as a parity operator changing sign of the right moving coordinate X_+ and leaving unchanged the left moving one X_- .

In an open string theory the string coordinate does not satisfy any periodicity requirement on σ . This implies that in its compactified version there are only K-K modes, while the winding modes are absent. This could suggest that T-duality is not a symmetry of the open string theory. Such a conclusion, however, leads to some problem when we remember that theories with open strings also contain closed strings. Let us consider a theory with open and closed strings with d-p-1 directions compactified on circles with radii R^{ℓ} and take the limit

$$R^{\ell} \to 0 \quad \forall \text{ compact direction} ,$$
 (4.135)

In this limit the open string theory loses effectively d-p-1 directions because all the K-K modes become infinitely massive decoupling from the spectrum and because there cannot be any open string oscillation along the directions with zero radii. Therefore in this limit the open string will appear to only be living in a p+1-dimensional subspace of the entire ddimensional target space. Let us analyze what happens in the same limit in the closed string sector. When the radii of the compact dimensions are vanishing the K-K modes again decouple, while the winding modes will appear as a continuum of states. This is a first indication that for closed strings the compact directions do not disappear from the theory as it happened for open strings! More precisely, in the closed string sector we can perform a T-duality transformation, that is allowed because it is a symmetry of this sector, and in so doing we can completely restore all the d space-time dimensions, as a consequence of the fact that in the limit in eq.(4.135) the T-dual radii go to infinity. But in this way we would end up with a theory in which open strings live in a p+1-dimensional subspace of the entire space-time, while closed strings live in the entire d-dimensional target space. This mismatch can be solved by requiring that, in the T-dual picture, open string still can oscillate in d dimensions, while their endpoints are fixed on a p+1-dimensional hyperplane that we call Dp-brane. Open strings with their endpoints fixed on these hyperplanes satisfy Dirichlet boundary conditions in the d-p-1 transverse directions. They are allowed boundary conditions as we have already seen in eq.(2.8) although they destroy the Poincarè invariance of the theory.

In conclusion, in order to avoid a different behaviour between the closed and the open sector of a string theory, we must require that the action of T-duality on an open string theory consists in transforming Neumann boundary conditions into Dirichlet ones. This can, in fact, be very naturally obtained if we extend the definition of the T-dual coordinate given in eq.(4.134) to the open string case. In this way we obtain the following T-dual open string coordinate:

$$\hat{X}^{\ell} = \frac{1}{2} \left[X_{-}^{\ell} - X_{+}^{\ell} \right] , \qquad (4.136)$$

where now the left and right movers contain the same set of oscillators

$$X_{-}^{\ell} = q^{\ell} + c^{\ell} + \sqrt{2\alpha'}(\tau - \sigma)\alpha_{0}^{\ell} + i\sqrt{2\alpha'}\sum_{n \neq 0} \frac{\alpha_{n}^{\ell}}{n} e^{-in(\tau - \sigma)} , \qquad (4.137)$$

and

$$X_{+}^{\ell} = q^{\ell} - c^{\ell} + \sqrt{2\alpha'}(\tau + \sigma)\alpha_{0}^{\ell} + i\sqrt{2\alpha'}\sum_{n \neq 0} \frac{\alpha_{n}^{\ell}}{n} e^{-in(\tau + \sigma)} , \qquad (4.138)$$

From eqs.(4.136), (4.137) and (4.138) one can immediately see that T-duality has transformed a string coordinate satisfying Neumann boundary conditions and given by $1/2 \left[X_-^\ell + X_+^\ell \right]$ into a T-dual one satisfying Dirichlet boundary conditions and given in eq.(4.136). Of course it is also true that, if we had started with a string coordinate satisfying Dirichlet boundary conditions we would have obtained a T-dual coordinate satisfying Neumann ones.

The fact that open strings satisfy Dirichlet boundary conditions implies the existence in the theory of objects, called the Dp-branes, that are characterized by the fact that open strings have their endpoints attached to them. From the three previous equations it also follows that

$$\hat{X}^{\ell}(\pi) - \hat{X}^{\ell}(0) = -2\pi\alpha' p^{\ell} = -\frac{2\pi\alpha' n^{(\ell)}}{R^{(\ell)}} = -2\pi n^{\ell} \hat{R}^{(\ell)} \Rightarrow \hat{X}^{\ell}(\pi) \sim \hat{X}^{\ell}(0) ,$$
(4.139)

This means that in the T-dual theory the two endpoints of the open string are attached to the same D-brane.

From the previous construction it also follows that each Dp-brane can be transformed into a Dp'-brane through an appropriate sequence of compactifications and T-duality transformations. In fact let us start with a Dp-brane embedded in a d-dimensional space-time and let us compactify one of the space directions X^{α} that lies in its world-volume on a circle with radius $R^{(\alpha)}$. The action of a T-duality transformation on this coordinate has the effect that an open string attached to the brane changes from Neumann to Dirichlet boundary conditions in that direction. Therefore, the brane 'loses' one longitudinal coordinate which becomes transverse and is transformed into a D(p-1)-brane embedded into a space-time which has

the direction \hat{X}^{α} compactified on a circle of radius $\hat{R}^{(\alpha)} = \alpha'/R^{(\alpha)}$. To obtain a D(p-1)-brane living in the d-dimensional uncompactified space-time we need to make the decompactification limit in the T-dual theory, namely

$$\hat{R}^{(\alpha)} \to \infty$$
, or equivalently $R^{(\alpha)} \to 0$. (4.140)

In the same way, if instead of compactifying one of the space-time directions which are longitudinal to the Dp-brane, we compactify one of the directions transverse to the brane, and then we act with a T-duality transformation on this coordinate we will get a D(p+1)-brane embedded into a space-time with one compact direction. Then taking the limit in eq.(4.140) we get again the uncompactified theory.

To conclude we observe that open strings satisfying Neumann boundary conditions in all the directions can be thought as being attached to a space-filling brane that is a D25-brane in the bosonic string or a D9-brane in the superstring. Therefore, as a consequence of the previous discussion, starting with a space-filling brane through a T-duality transformation we can obtain an arbitrary Dp-brane. More precisely, a Dp-brane can be obtained from a space-filling brane by first compactifying d-p-1 directions, then performing a T-duality transformation and finally taking the decompactification limit.

Up to now we have treated a Dp-brane as a pure geometrical hyperplane to which open strings are attached and we have completely disregarded the excitations of the attached open strings. But we will see that, as soon as we let them come into play, they provide dynamical degrees of freedom to the Dp-brane.

Among all possible excitations of an open string the massless ones have the peculiarity of not changing the energy of the Dp-brane to which the open string is attached. Therefore from the brane point of view they can be interpreted as collective coordinates of the brane.

In absence of Chan-Paton factors, the massless excitations of an open string with Neumann boundary conditions in all directions are described by a d-dimensional abelian gauge potential A^{μ} . From the previous discussion it can be thought as a gauge field living on the space-filling brane. Then by compactifying $d_{\perp} = d - p - 1$ dimensions and making a T-duality transformation in each of the compact directions, followed by a decompactification limit in the T-dual theory, we see that the vector potential \hat{A}^{μ} splits in a (p+1)-dimensional vector \hat{A}^{α} with $\alpha \in \{0,...,p\}$ and d_{\perp} scalars fields. The most natural interpretation of the T-dual version of the abelian gauge field is that the longitudinal coordinates \hat{A}^{α} still describe a gauge field living on the Dp-brane while the d_{\perp} scalars coming from the transverse components \hat{A}^{ℓ} , with $\ell \in \{p+1,...,d-1\}$, appear as the transverse coordinates of the Dp-brane.

This interpretation becomes more clear as soon as we introduce a non-abelian U(N) gauge group in the open string theory through the Chan-Paton factors and in addition we turn on Wilson lines. The Chan-Paton procedure for introducing non-abelian gauge degrees of freedom on an open string consists in adding non-dynamical degrees of freedom at each of its two endpoints. A generic string state will therefore be denoted with a ket $|\alpha,i,\bar{j}\rangle$ where α describes the usual degrees of freedom of a string, while the indices i and \bar{j} refer to the gauge degrees of freedom. In the case of a U(N) gauge group i transforms according to the fundamental representation N, while \bar{j} according to the complex conjugate representation \bar{N} :

$$|i'\rangle = U_{i'i}|i\rangle \quad ; \quad |\bar{j}'\rangle = |\bar{j}\rangle U_{jj'}^{+} .$$
 (4.141)

If we now introduce a basis of $N \times N$ matrices λ_{ij}^a , expand an open string state as

$$|\alpha, a\rangle = \sum_{i,j=1}^{N} |\alpha, i, \bar{j}\rangle \lambda_{ij}^{a},$$
 (4.142)

and use eq. (4.141) we see that the transformations in eq.(4.141) act on the matrices λ_{ij}^a as follows

$$\lambda_{ij}^a \to U_{i'i} \lambda_{ij}^a U_{jj'}^+ = (U \lambda^a U^+)_{i'j'} .$$
 (4.143)

This means that the matrix λ_{ij}^a transforms according to the adjoint representation $(N \times \bar{N})$ of U(N) which is in fact the appropriate representation for a gauge field.

Let us now consider the effect of compactification in the presence of Chan-Paton factors. For the sake of simplicity we compactify just one coordinate that we denote with X without any index and turn on a pure gauge field of the form

$$(A)_{ij} = \frac{1}{2\pi R} \operatorname{diag}(\theta_1, ..., \theta_N) .$$
 (4.144)

This corresponds to a pure gauge configuration generated by the matrix:

$$U = \operatorname{diag}\left(e^{iX\frac{\theta_1}{2\pi R}}, ..., e^{iX\frac{\theta_N}{2\pi R}}\right) , \qquad (4.145)$$

because the gauge field configuration in eq.(4.144) can be written as

$$A = -iU^{-1}\partial U . (4.146)$$

But in the case of a compact coordinate the presence of a pure gauge field affects the parallel transport along the compact dimension and we get non-zero Wilson lines:

$$e^{i\int_0^{2\pi R} Adx} = \text{diag}(e^{i\theta_1}, ..., e^{i\theta_N})$$
 (4.147)

In particular the parallel transport around the compact coordinate transforms $|i\rangle$ and $|\bar{j}\rangle$ as follows

$$|i\rangle \to e^{i\theta_i}|i\rangle \quad ; \quad |\bar{j}\rangle \to e^{-i\theta_j}|\bar{j}\rangle , \qquad (4.148)$$

and therefore the open string state transforms as

$$|\alpha, a\rangle = \sum_{i,j=1}^{N} e^{i(\theta_i - \theta_j)} |\alpha, i, \bar{j}\rangle \lambda_{ij}^a . \tag{4.149}$$

The presence of Wilson lines changes the possible values of the momentum of the state $|\alpha, i, \bar{j}\rangle$. In fact in this case a translation of $2\pi R$ acts both on the string and the gauge degrees of freedom that are located at its endpoints. Requiring that this combined action leaves the state invariant:

$$e^{2\pi i R\hat{p}}e^{i(\theta_i - \theta_j)}|\alpha, i, \bar{j}\rangle = |\alpha, i, \bar{j}\rangle$$
, (4.150)

we get that the momentum of the state, that we call p, is equal to

$$p = \frac{n}{R} + \frac{(\theta_j - \theta_i)}{2\pi R} \ . \tag{4.151}$$

Let us now see what are the consequences of the presence of Wilson lines in the T-dual theory. Inserting eq.(4.151) in eq.(4.139) we get

$$\hat{X}(\pi) - \hat{X}(0) = -(2\pi n + \theta_j - \theta_i)\hat{R} \sim -(\theta_j - \theta_i)\hat{R} . \tag{4.152}$$

This means that the open string is stretching between two Dp-branes whose coordinates are $\theta_i \hat{R}$ and $\theta_j \hat{R}$. Moreover remembering eq.(4.144) we immediately see that

$$\theta_i \hat{R} = 2\pi \alpha'(A)_{ii}$$
 ; $\theta_j \hat{R} = 2\pi \alpha'(A)_{jj}$. (4.153)

and we can conclude that turning on U(N) Wilson lines in a theory of open strings along a compactified direction corresponds, in the T-dual theory, to introduce N Dp-branes located respectively at

$$X_1 = -2\pi\alpha'(A)_{11}, ..., X_N = -2\pi\alpha'(A)_{NN}$$
(4.154)

In this way the transverse components of a U(N) gauge field carried by an open string are correctly interpreted as the coordinates of N Dp-branes.

In superstring theory the effect of T-duality on the bosonic coordinates is exactly the same as discussed for the bosonic string, namely T-duality acts as a parity transformation over the tilded sector

$$X = \frac{1}{2}(X_{-} + X_{+}) \rightarrow \hat{X} = \frac{1}{2}(X_{-} - X_{+})$$
 (4.155)

For the fermionic coordinates the transformations under T-duality can be fixed by requiring the superconformal invariance of the theory which imposes

$$\psi_{+} \to -\psi_{+} \quad ; \quad \psi_{-} \to \psi_{-} , \qquad (4.156)$$

or in terms of the oscillators

$$\widetilde{\psi}_t \to -\widetilde{\psi}_t \quad ; \quad \psi_t \to \psi_t ,$$
 (4.157)

This transformation propriety of the fermionic coordinates can also be understood as due to the requirement that the subspace of the physical states of the superstring, defined by eqs. (2.56) and by the analogous ones for the right sector, is left invariant by T-duality. Therefore, looking at the structure of the operator \tilde{G}_t given in eq. (2.48) and taking into account eq. (4.129), we obtain again eq. (4.157).

5. Classical Solutions Of The Low-Energy String Effective Action

In the previous sections we have seen that the requirement of invariance under T-duality transformations in presence of open strings implies the existence of p-dimensional objects called Dp-branes to which open strings can be attached determining their dynamics. Although these objects are required by T-duality their meaning is still rather obscure in the present framework. On the other hand, following a completely different line of research with the aim to get some non-perturbative information about string theories some people were investigating classical solutions of the low-energy string effective action. The underlying idea was in fact that, as the construction of 't Hooft-Polyakov monopoles in non abelian gauge theories teaches us many things about the non-perturbative structure of non-abelian gauge theories, so from the study of classical solutions of the low-energy string effective action one could learn a great deal on non-perturbative aspects of string theories. It turns out that starting from the low-energy string effective action one finds solutions of the classical equations of motion corresponding to p-dimensional objects. In the following we just want to remind their main properties.

The starting point is the low-energy string effective action containing the graviton, the dilaton and only one n-form potential, that written in the Einstein frame is given by:

$$S = \frac{1}{2\kappa^2} \int d^d x \sqrt{-g} \left[R - \frac{1}{2} (\nabla \varphi)^2 - \frac{1}{2(n+1)!} e^{-a\varphi} (F_{n+1})^2 \right] , \quad (5.158)$$

where n = p + 1 and $F_{n+1} = d\mathcal{C}$. For simplicity we have neglected all fermionic fields and the other NS-NS and R-R fields. An electric Dp-brane

corresponds to the following ansatz:

$$ds^{2} = [H(r)]^{2A} \left(\eta_{\alpha\beta} dx^{\alpha} dx^{\beta} \right) + [H(r)]^{2B} \left(\delta_{ij} dx^{i} dx^{j} \right) , \qquad (5.159)$$

for the metric g, and

$$e^{-\varphi(x)} = [H(r)]^{\tau}$$
 , $C_{01...p}(x) = \pm \sqrt{2\sigma} [H(r)]^{-1}$, (5.160)

for the dilaton φ and for the R-R (p+1)-form potential \mathcal{C} respectively. The two signs in \mathcal{C} correspond to the brane and the anti-brane case and H(r) is assumed to be only a function of the square of the transverse coordinates $r=x_{\perp}^2=x_ix^i$. If the parameters are chosen as

$$A = -\frac{d-p-3}{2(d-2)}$$
 , $B = \frac{p+1}{2(d-2)}$, $\tau = \frac{a}{2}$, $\sigma = \frac{1}{2}$, (5.161)

with a obeying the equation

$$\frac{2(p+1)(d-p-3)}{d-2} + a^2 = 4 \quad , \tag{5.162}$$

then the function H(r) satisfies the flat space Laplace equation. An extremal p-brane solution is constructed by introducing in the right hand side of the eqs. of motion following from the action in eq.(5.158) a δ -function source term in the transverse directions. If we restrict ourselves to the simplest case of just one p-brane, we obtain the following expression for H(r)

$$H(r) = 1 + 2\kappa T_p G(r)$$
 , (5.163)

where

$$G(r) = \begin{cases} \left[(d-p-3)r^{(d-p-3)}\Omega_{d-p-2} \right]^{-1} & p < d-3 , \\ -\frac{1}{2\pi}\log r & p = d-3 , \end{cases}$$
 (5.164)

with

$$\Omega_q = 2\pi^{(q+1)/2}/\Gamma((q+1)/2)$$
 (5.165)

being the area of a unit q-dimensional sphere S_q . For future use it is convenient to introduce the quantity:

$$Q_p = \mu_p \frac{\sqrt{2} \kappa}{(d-p-3)\Omega_{d-p-2}} \; ; \; \mu_p \equiv \sqrt{2}T_p \; .$$
 (5.166)

and (if p < d - 3) to rewrite H(r) in eq.(5.163) as follows:

$$H(x) = 1 + \frac{Q_p}{r^{d-3-p}} \quad , \tag{5.167}$$

The classical solution has a mass per unit p-volume, M_p and an electric charge with respect to the R-R field, μ_p , given respectively by

$$M_p = \frac{T_p}{\kappa} , \qquad \mu_p = \pm \sqrt{2}T_p . \qquad (5.168)$$

The fundamental observation made by Polchinski [3] has been to identify the Dp-branes required by T-duality with the p-branes obtained as classical solutions of the low-energy string effective action. Therefore, on the one hand the p-branes are new non-perturbative states of string theory and on the other hand have the important property that open string have their endpoints attached on them. The latter property will allow one to compute their interactions and more in general to study their properties by computing open string one-loop diagrams. On the other hand we should not be worried that the Dirichlet boundary conditions break Poincarè invariance because this happens in presence of any kind of solitonic state. In the next chapter we will introduce the boundary state and we will show that it provides a stringy description of these new states.

6. Bosonic Boundary State

As discussed in the previous section Dp-branes are extended p dimensional objects characterized by the fact that open strings can have their endpoints attached to them. In general they are dynamical and non rigid objects, that can fluctuate in shape and on which external fields can live. In these lectures we limit ourselves to treat them as static and rigid objects.

The open string with the endpoint at $\sigma = 0$ attached to a D*p*-brane satisfies the usual Neumann boundary conditions along the directions longitudinal to the world volume of the brane

$$\partial_{\sigma} X^{\alpha}|_{\sigma=0} = 0 \qquad \qquad \alpha = 0, 1, \dots, p , \qquad (6.169)$$

and Dirichlet boundary conditions along the directions transverse to the brane

$$X^{i}|_{\sigma=0} = y^{i}$$
 $i = p + 1, ...d - 1$, (6.170)

where y^i are the coordinates of the brane and d is the dimension of the Minkowski space-time, that in the case of the bosonic string is equal to d = 26.

As the interaction between two superconducting plates is obtained by computing the vacuum fluctuation of the electromagnetic field that gives rise to the Casimir effect, so the interaction between two Dp-branes is given by the vacuum fluctuation of an open string that is stretching between them. This means that their interaction is simply given by the one-loop

open string "free-energy" which is usually represented by the annulus or equivalently by the cylinder diagram. From either of those two diagrams it is easy to see that by exchanging the variables σ and τ the one-loop open string amplitude can also be viewed as a tree diagram of a closed string created from the vacuum, propagating for a while and then annihilating again into the vacuum. These two equivalent descriptions of the same diagram are called respectively the 'open-channel' and the 'closed-channel'. We want to stress that the physical content of the two descriptions is a priori completely different. In the first case we describe the interaction between two Dp-branes as a one-loop amplitude of open strings, which is the amplitude of a quantum theory of open strings, while in the second case we describe the same interaction as a tree-level amplitude of closed strings, which is instead a classical amplitude in a theory of closed strings. The fact that these two descriptions are equivalent is a consequence of the conformal symmetry of string theory that allows one to connect the two apriori different descriptions.

To show that, let us consider a one-loop diagram with an open string circulating in it and stretching between two parallel Dp-branes with coordinates respectively given by $(y^{p+1},...,y^{d-1})$ and $(w^{p+1},...,w^{d-1})$. The open string satisfies the boundary conditions in eq.(6.169) both at $\sigma = 0$ and $\sigma = \pi$ along the world-volume directions of the brane, while along the transverse directions satisfies the following equations:

$$X^{i}|_{\sigma=0} = y^{i}$$
 $X^{i}|_{\sigma=\pi} = w^{i}$ $i = p+1, ..., d-1$, (6.171)

where we take σ and τ in the two intervals $\sigma \in [0, \pi]$ and $\tau \in [0, T]$.

We now want to find a conformal transformation acting on the previous open string boundary conditions in order to transform them into the boundary conditions for a closed string propagating between the two Dp-branes. In terms of the complex coordinate $\zeta \equiv \sigma + i\tau$, a conformal transformation simply transforms $\zeta \to f(\zeta)$, where $f(\zeta)$ is an arbitrary holomorphic function of ζ . Let us consider the following conformal transformation

$$\zeta = \sigma + i\tau \rightarrow -i\zeta = \tau - i\sigma$$
 (6.172)

After the inversion $\sigma \to -\sigma$ the previous conformal transformation simply amounts to exchange σ with τ and viceversa

$$(\sigma, \tau) \to (\tau, \sigma)$$
 . (6.173)

Finally in order to have the closed string variables σ and τ to vary in the intervals $\sigma \in [0, \pi]$ and $\tau \in [0, \hat{T}]$ corresponding to a closed string propagating between the two D branes one must perform the following conformal rescaling

$$\sigma \to \frac{\pi}{T}\sigma \qquad \qquad \tau \to \frac{\pi}{T}\tau , \qquad (6.174)$$

with $\hat{T}=\pi^2/T$. We have therefore constructed a conformal transformation that brings us from the open string to the closed string channel. In the closed string channel we need to construct the two boundary states $|B_X\rangle$ that describe the two Dp-branes respectively at $\tau=0$ and $\tau=\hat{T}$. The equations that characterize these states are obtained by applying the conformal transformation previously constructed to the boundary conditions for the open string given in eqs.(6.169) and (6.171). At $\tau=0$ we get the following conditions:

$$\partial_{\tau} X^{\alpha}|_{\tau=0} |B_X\rangle = 0 \qquad \alpha = 0, ..., p , \qquad (6.175)$$

$$X^{i}|_{\tau=0}|B_{X}\rangle = y^{i}$$
 $i = p+1, ..., d-1$. (6.176)

Analogous conditions can be obtained for the Dp-brane at $\tau = \hat{T}$.

The previous equations can be easily written in terms of the closed string oscillators by making use of the expansion in eq.(2.13), obtaining

$$(\alpha_n^{\alpha} + \widetilde{\alpha}_{-n}^{\alpha})|B_X\rangle = 0 \quad ; \quad (\alpha_n^i - \widetilde{\alpha}_{-n}^i)|B_X\rangle = 0 \quad \forall n \neq 0$$
$$\hat{p}^{\alpha}|B_X\rangle = 0 \qquad (\hat{q}^i - y^i)|B_X\rangle = 0 . \tag{6.177}$$

Introducing the matrix

$$S^{\mu\nu} = (\eta^{\alpha\beta}, -\delta^{ij}) , \qquad (6.178)$$

the equations for the non-zero modes can be rewritten as

$$(\alpha_n^{\mu} + S^{\mu}_{\nu} \widetilde{\alpha}_{-n}^{\nu})|B_X\rangle = 0 \qquad \forall \ n \neq 0 \ . \tag{6.179}$$

The state satisfying the previous equations can easily be determined to be

$$|B_X\rangle = N_p \delta^{d-p-1} (\hat{q}^i - y^i) \left(\prod_{n=1}^{\infty} e^{-\frac{1}{n}\alpha_{-n}S \cdot \widetilde{\alpha}_{-n}} \right) |0\rangle_{\alpha} |0\rangle_{\widetilde{\alpha}} |p = 0\rangle , \quad (6.180)$$

where N_p is a normalization constant to be fixed.

The previous boundary state describes only the degrees of freedom corresponding to the string coordinate X. In order to have a BRST invariant boundary state we have to supplement it with the boundary state for the ghost degrees of freedom obtaining the full boundary state

$$|B\rangle = |B_X\rangle |B_{gh}\rangle . (6.181)$$

We will later on write the ghost part of the boundary state.

The overlap conditions for the conjugate boundary state can be easily obtained by taking the adjoint of eqs. (6.177) and (6.179) and are given by

$$\langle B_X | (\alpha_{-n}^{\mu} + S^{\mu}_{\nu} \tilde{\alpha}_{n}^{\nu}) ; \langle B_X | \hat{p}^{\alpha} = 0 ; \langle B_X | (\hat{q}^{j} - y^{j}) = 0 , (6.182)$$

that imply

$$\langle B_X | = \langle p = 0 | \alpha \langle 0 | \widetilde{\alpha} \langle 0 | N_p \delta^{d-p-1} (\hat{q}^i - y^i) \left(\prod_{n=1}^{\infty} e^{-\frac{1}{n} \alpha_n S \cdot \widetilde{\alpha}_n} \right) , \quad (6.183)$$

In the following we will compute the interaction between two parallel Dp-branes both in the open and in the closed string channel. By comparing the two results we can determine the normalization factor N_p appearing in the boundary state in eq.(6.180). Let us start by computing the interaction in the closed string channel. For the sake of simplicity we perform this calculation considering only the part of the boundary state containing the string coordinate X and then adding by hand the contribution of the ghosts. With this simplification the free energy reads as

$$F = \langle B_X | D | B_X \rangle , \qquad (6.184)$$

where D is the bosonic closed string propagator

$$D = \frac{\alpha'}{4\pi} \int_{|z|<1} \frac{d^2z}{|z|^2} z^{L_0 - 1} \bar{z}^{\widetilde{L}_0 - 1} , \qquad (6.185)$$

 L_0 and \widetilde{L}_0 are the usual Virasoro operators of the closed bosonic string given in eq. (2.23).

Inserting eqs.(6.180), (6.183) and (6.185) into eq. (6.184) we get

$$\langle B_X | D | B_X \rangle = (N_p)^2 \frac{\alpha'}{4\pi} \int_{|z| \le 1} \frac{d^2 z}{|z|^4} \langle 0 |_{\alpha} \langle 0 |_{\widetilde{\alpha}} \langle p = 0 | \prod_{n=1}^{\infty} \left(e^{-\frac{1}{n} \alpha_n \cdot S \cdot \widetilde{\alpha}_n} \right)$$
$$\delta^{d_{\perp}}(\hat{q}_i) \ z^{L_0} \bar{z}^{\widetilde{L}_0} \ \delta^{d_{\perp}}(\hat{q}_i - y_i) \ \prod_{n=1}^{\infty} \left(e^{-\frac{1}{n} \alpha_{-n} \cdot S \cdot \widetilde{\alpha}_{-n}} \right) |0\rangle_{\alpha} |0\rangle_{\widetilde{\alpha}} |p = 0\rangle \ (6.186)$$

where $d_{\perp} \equiv d-p-1$ and |y| is the distance between the two Dp-branes. The matrix element in the previous expression can be factorized in two parts containing respectively the contribution of the zero and non-zero modes. The contribution of the zero modes is given by:

$$\begin{split} \langle p = 0 | \delta^{d_{\perp}}(\hat{q}_{i}) | z |^{\frac{\alpha'}{2}\hat{p}^{2}} \delta^{d_{\perp}}(\hat{q}_{i} - y_{i}) | p = 0 \rangle = \\ = \int \frac{d^{d_{\perp}}Q}{(2\pi)^{d_{\perp}}} \int \frac{d^{d_{\perp}}Q'}{(2\pi)^{d_{\perp}}} \langle p = 0 | e^{iQ \cdot \hat{q}} | z |^{\frac{\alpha'}{2}\hat{p}^{2}} e^{iQ' \cdot (\hat{q} - y)} | p = 0 \rangle = \\ = \int \frac{d^{d_{\perp}}Q}{(2\pi)^{d_{\perp}}} \int \frac{d^{d_{\perp}}Q'}{(2\pi)^{d_{\perp}}} | z |^{\frac{\alpha'}{2}Q'^{2}} e^{-iQ' \cdot y} \langle p_{\perp} = -Q | p_{\perp} = Q' \rangle \langle p_{\parallel} = 0 | p_{\parallel} = 0 \rangle = \end{split}$$

$$= V_{p+1} \int \frac{d^{d_{\perp}} Q}{(2\pi)^{d_{\perp}}} |z|^{\frac{\alpha'}{2}Q^2} e^{iQ \cdot y} , \qquad (6.187)$$

where we have used the following normalization for each component of the momentum

$$\langle k|k'\rangle = 2\pi\delta(k-k') , \qquad (6.188)$$

with

$$(2\pi)^d \delta^d(0) \equiv V_d \ . \tag{6.189}$$

Performing the gaussian integral, eq. (6.187) becomes

$$V_{p+1}e^{-y^2/(2\pi\alpha't)} \left(2\pi^2t\alpha'\right)^{-d_{\perp}/2} , \quad |z| = e^{-\pi t} .$$
 (6.190)

The contribution of the non-zero modes is instead given by

$$_{\alpha}\langle 0|_{\widetilde{\alpha}}\langle 0|\ \prod_{m=1}^{\infty}\left(e^{-\frac{1}{m}\alpha_{m}\cdot S\cdot\widetilde{\alpha}_{m}}\right)z^{N}\bar{z}^{\widetilde{N}}\prod_{n=1}^{\infty}\left(e^{-\frac{1}{n}\alpha_{-n}\cdot S\cdot\widetilde{\alpha}_{-n}}\right)|0\rangle_{\alpha}|0\rangle_{\widetilde{\alpha}}=$$

$$= {}_{\alpha}\langle 0|_{\widetilde{\alpha}}\langle 0| \prod_{m=1}^{\infty} e^{-\frac{1}{m}\alpha_{m} \cdot S \cdot \widetilde{\alpha}_{m}} \prod_{n=1}^{\infty} e^{-\frac{1}{n}\alpha_{-n} \cdot S \cdot \widetilde{\alpha}_{-n}|z|^{2n}} |0\rangle_{\alpha}|0\rangle_{\widetilde{\alpha}}, \qquad (6.191)$$

where we have defined

$$N \equiv \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n \quad ; \quad \widetilde{N} \equiv \sum_{n=1}^{\infty} \widetilde{\alpha}_{-n} \cdot \widetilde{\alpha}_n , \quad (6.192)$$

and we have used the following relations:

$$z^N e^{\alpha_{-n}} z^{-N} = e^{\alpha_{-n} z^n} \quad \text{ and } \quad \bar{z}^N e^{\alpha_{-n}} \bar{z}^{-N} = e^{\alpha_{-n} \bar{z}^n} \quad \forall n \neq 0 \ . \ (6.193)$$

By explicitly evaluating the contractions among the oscillators in eq.(6.191) we get

$$\prod_{n=1}^{\infty} \left(\frac{1}{1 - |z|^{2n}} \right)^{d-2} . \tag{6.194}$$

To be more precise the previous calculation leads to a power d instead of d-2 as we have written in eq.(6.194). The extra power (-2) comes from the ghost contribution that, for the sake of simplicity, we are not presenting here.

Inserting eqs.(6.187) and (6.194) in eq.(6.186) after having changed variables according to

$$|z| = e^{-\pi t}$$
 $d^2z = -\pi e^{-2\pi t} dt d\varphi$, (6.195)

we get

$$\langle B_X | D | B_X \rangle =$$

$$= (N_p)^2 V_{p+1} \frac{\alpha' \pi}{2} \int_0^\infty dt \, \left(2\pi^2 \alpha' t \right)^{-\frac{d_\perp}{2}} e^{-\frac{y^2}{2\pi\alpha' t}} e^{2\pi t} \prod_{n=1}^\infty \left(\frac{1}{1 - e^{-2\pi t n}} \right)^{d-2} =$$

$$= (N_p)^2 V_{p+1} \frac{\alpha' \pi}{2} \left(2\pi^2 \alpha' \right)^{-\frac{d_\perp}{2}} \int_0^\infty dt \, t^{-\frac{d_\perp}{2}} e^{-\frac{y^2}{2\pi\alpha' t}} [f_1(e^{-\pi t})]^{-24} \quad (6.196)$$

where we have introduced the function

$$f_1(q) \equiv q^{\frac{1}{12}} \prod_{n=1}^{\infty} (1 - q^{2n}) \; ;$$
 (6.197)

The factor $\alpha'\pi/2 = \alpha'/(4\pi)\pi(2\pi)$ in eq.(6.196) comes from the product of the factor $\alpha'/(4\pi)$ present in the propagator in eq.(6.185), the factor π obtained in eq.(6.195) and the factor 2π obtained by performing the trivial integration over the angular variable φ .

Let us now proceed to the calculation of the interaction between two Dp-branes in the open string channel. The one-loop planar free-energy for an open string with d-p-1 Dirichlet boundary conditions is equal to ³

$$F = -\frac{1}{2}Tr\log[L_0 - 1] = \int_0^\infty \frac{d\tau}{2\tau} Tr\left[e^{-2\pi(L_0 - 1)\tau}\right] , \qquad (6.198)$$

where

$$L_0 = \alpha' k^2 + \frac{y^2}{(2\pi)^2 \alpha'} + \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n , \qquad (6.199)$$

y represents the distance between the two parallel Dp-branes and k the momentum lying along the world volume of the two branes. Here the L_0 operator differs from the one in eq. (2.24) because in this case the open string satisfies Dirichlet boundary conditions in the d-p-1 transverse directions and Neumann boundary conditions in the longitudinal ones.

The trace in eq.(6.198) must be understood as an integration over the longitudinal loop momentum and a trace over the oscillators, namely

$$F = 2\frac{V_{p+1}}{2} \int \frac{d^{p+1}k}{(2\pi)^{p+1}} \times$$

$$\times \int_0^\infty \frac{d\tau}{\tau} e^{2\pi\tau} e^{-2\pi\tau\alpha'k^2} e^{-\frac{y^2\tau}{2\pi\alpha'}} e^{2\pi\tau} Tr \left(\prod_{n=1}^\infty e^{-2\pi\tau\alpha_{-n}\cdot\alpha_n} \right) =$$

³Note that here we use the regularized expression $\log(x) = -\lim_{\varepsilon \to 0^+} \int_{\varepsilon}^{\infty} \frac{d\tau}{\tau} e^{-\tau x}$

$$= V_{p+1} \int_0^\infty \frac{d\tau}{\tau} (8\pi^2 \alpha' \tau)^{-\frac{p+1}{2}} e^{-\frac{y^2 \tau}{2\pi \alpha'}} Tr \left(\prod_{n=1}^\infty e^{-2\pi \tau \alpha_{-n} \cdot \alpha_n} \right) , \quad (6.200)$$

where we have performed the Gaussian integral over the longitudinal momentum circulating in the loop and we have inserted a factor 2 coming from the freedom of exchanging the two endpoints of the string. Evaluating explicitly the trace over the oscillators we get

$$Tr\left(\prod_{n=1}^{\infty} e^{-2\pi\tau\alpha_{-n}\cdot\alpha_n}\right) = \prod_{n=1}^{\infty} \left(\frac{1}{1 - e^{-2\pi\tau n}}\right)^{d-2} . \tag{6.201}$$

Notice that also in this case we have introduced by hand the information that the ghosts contribution amounts to change the exponent from d to d-2. Finally, inserting eq. (6.201) into eq. (6.200) and writing it in terms of the function f_1 defined in eq.(6.197), the one-loop free-energy becomes

$$F = V_{p+1} (8\pi^2 \alpha')^{-\frac{p+1}{2}} \int_0^\infty \frac{d\tau}{\tau} \tau^{-\frac{p+1}{2}} e^{-\frac{y^2 \tau}{2\pi \alpha'}} \left(f_1(e^{-\pi \tau}) \right)^{-24} =$$

$$= V_{p+1} (8\pi^2 \alpha')^{-\frac{p+1}{2}} \int_0^\infty \frac{d\tau}{\tau} \tau^{12 - \frac{p+1}{2}} e^{-\frac{y^2 \tau}{2\pi \alpha'}} \left(f_1(e^{-\frac{\pi}{\tau}}) \right)^{-24} , \qquad (6.202)$$

where we have taken d = 26 and we have used the modular transformation property of the function f_1

$$f_1(e^{-\frac{\pi}{t}}) = \sqrt{t}f_1(e^{-\pi t}) \tag{6.203}$$

In order to compare eq.(6.202) with eq.(6.196) we must perform in the second one the change of variable $t = \frac{1}{\tau}$. In this way eq.(6.196) becomes

$$\langle B_X | D | B_X \rangle =$$

$$= (N_p)^2 V_{p+1} \frac{\alpha' \pi}{2} \left(2\pi^2 \alpha' \right)^{-\frac{d_\perp}{2}} \int_0^\infty \frac{d\tau}{\tau} \, \tau^{12 - \frac{p+1}{2}} e^{-\frac{y^2 \tau}{2\pi \alpha'}} [f_1(e^{-\frac{\pi}{\tau}})]^{-24} \, . \tag{6.204}$$

By comparing eqs.(6.202) and (6.204) we can determine the normalization factor of the boundary state:

$$N_p = \frac{T_p}{2}$$
 , $T_p = \frac{\sqrt{\pi}}{2^{\frac{d-10}{4}}} (2\pi\sqrt{\alpha}')^{\frac{d}{2}-2-p}$. (6.205)

In conclusion, by performing the calculation of F in the closed string channel with the normalization factor given in eq.(6.205) we get:

$$F = V_{p+1} \left(8\pi^2 \alpha' \right)^{-\frac{p+1}{2}} \int_0^\infty \frac{dt}{t} \ t^{\frac{p+1}{2} - 12} e^{-\frac{y^2}{2\pi\alpha't}} \left(f_1(e^{-\pi t}) \right)^{-24} , \quad (6.206)$$

while performing the calculation in the open string channel we get:

$$F = V_{p+1} \left(8\pi^2 \alpha'\right)^{-\frac{p+1}{2}} \int_0^\infty \frac{d\tau}{\tau} \tau^{-\frac{p+1}{2}} e^{-\frac{y^2 \tau}{2\pi \alpha'}} \left(f_1(e^{-\pi \tau})\right)^{-24} , \qquad (6.207)$$

The two expressions are manifestly identical as one can see if one changes variable $\tau = 1/t$ in eq.(6.207) and uses the modular transformation in eq.(6.203).

Notice that the function $[f_1]^{-24}$ has the following expansion for large value of the argument $(x \to \infty)$:

$$[f_1(e^{-\pi x})]^{-24} = \sum_{n=0}^{\infty} c_n e^{-2\pi x(n-1)} = e^{2\pi x} + 24 + 0(e^{-2\pi x}).$$
 (6.208)

In the open string channel the *n*th term of the previous expansion corresponds to the contribution in the loop of open string states with mass $\alpha' M^2 = n-1$, while in the closed string channel corresponds to the exchange between the two branes of closed string states with mass $\frac{\alpha'}{2}M^2 = 2(n-1)$. In particular from eq.(6.206) we can see that the dominant contribution to F at large distance $(y \to \infty)$ comes from light closed string states:

$$F = V_{p+1} \left(8\pi^2 \alpha' \right)^{-\frac{p+1}{2}} \int_0^\infty \frac{dt}{t} t^{\frac{p+1}{2} - 12} e^{-\frac{y^2}{2\pi\alpha't}} \left(e^{2\pi t} + 24 + \dots \right) , \quad (6.209)$$

where the first term corresponds to the exchange between the two branes of the closed string tachyon, the second term to the exchange of the massless closed string states and the additional terms to the exchange of closed string states with higher mass. The first term is obviously divergent, but it is due to the presence of the tachyon that will disappear in superstring. The second term, that is called massless tadpole, can be cancelled if we add the contribution of the non-orientable Moebius diagram and we choose a particular gauge group $(SO(2^{13}))$ for the bosonic string or SO(32) for the type I superstring). The requirement of tadpole cancellation is a convenient way of fixing the gauge group besides the one of anomaly cancellation. Both ways fix the gauge group in the type I theory to be SO(32).

In the field theory limit $(\alpha' \to 0)$ it is more convenient to use the expression for F written in the open string channel because in this case the dominant contribution comes from the lowest open string states circulating in the loop. This limit can be conveniently done by introducing in eq.(6.207) the dimensional Schwinger proper time s related to the modular parameter τ by the relation $s = \alpha'\tau$. Then using the expansion in eq.(6.208) we can rewrite eq.(6.207) as follows:

$$F = V_{p+1}(8\pi^2)^{-\frac{p+1}{2}} \int_0^\infty \frac{ds}{s} \ s^{-\frac{p+1}{2}} e^{-\frac{y^2 s}{2\pi(\alpha')^2}} \left(e^{2\pi s/\alpha'} + 24 + 0(e^{-2\pi s/(\alpha')^2} \right) . \tag{6.210}$$

The first term corresponds to the open string tachyon that will not be present in superstring and the second term corresponds to the open-string massless states. Finally the additional terms correspond to states with higher mass in open string theory that are negligible for $\alpha' \to 0$. Notice that, if we neglect the tachyon contribution that is absent in superstring, the massless states give a non vanishing contribution to F only if the distance between the two branes $y \to 0$.

As we have seen in eq.(6.181) the BRST invariant boundary state is the product of the boundary state $|B_X\rangle$ for the bosonic coordinate, that we have constructed in this section, and of $|B_{gh}\rangle$ that we are now going to construct. BRST invariance requires that the total boundary state satisfies the equation

$$(Q + \widetilde{Q})|B\rangle = 0 , \qquad (6.211)$$

where the BRST charge, given in eq.(3.87), is equal to

$$Q = \sum_{n} c_{n} L_{-n}^{X} + \sum_{n=-1}^{\infty} c_{-n} L_{n}^{gh} + \sum_{n=2}^{\infty} L_{-n}^{gh} c_{n}$$
 (6.212)

 \tilde{Q} is given by an analogous expression in terms of the tilded variables. The overlap conditions in eq.(6.177) imply that the boundary state for the bosonic coordinate satisfies the following eqs.:

$$(L_m^X - \tilde{L}_{-m}^X)|B_X\rangle = 0$$
 (6.213)

Inserting the expression for Q and the corresponding expression for \tilde{Q} in eq.(6.211) and using eq.(6.213) we can see that eq.(6.211) implies the following overlap conditions for the ghost boundary state

$$(c_n + \tilde{c}_{-n})|B_{gh}\rangle = 0$$
 ; $(b_n - \tilde{b}_{-n})|B_{gh}\rangle = 0$. (6.214)

The second overlap condition in the previous equation follows from the first one and from the analogous of eq.(6.213) for the ghost boundary state:

$$(L_m^{gh} - \tilde{L}_{-m}^{gh})|B_g h\rangle = 0$$
 (6.215)

Eqs. (6.214) are satisfied by the state

$$|B_{gh}\rangle_{gh} = e^{\sum_{n=1}^{\infty} (c_{-n}\widetilde{b}_{-n} - b_{-n}\widetilde{c}_{-n})} \left(\frac{c_0 + \widetilde{c}_0}{2}\right) |q = 1\rangle |\widetilde{q} = 1\rangle \tag{6.216}$$

where $|q=1\rangle$ is the state that is annihilated by the following oscillators

$$c_n|q=1\rangle = 0 \quad \forall n \ge 1; \quad ; \quad b_m|q=1\rangle = 0 \quad \forall m \ge 0. \quad (6.217)$$

7. Fermionic Boundary State

In this section we want to generalize the previous construction to the superstring case where, together with the boundary state $|B_X\rangle$ corresponding to the bosonic coordinate X that we have already constructed we also need to determine the boundary state $|B_{\psi}\rangle$ corresponding to the fermionic coordinate ψ . The procedure that we follow for determining $|B_{\psi}\rangle$ is precisely the same used in the previous section. We perform on the boundary conditions for an open string stretching between two Dp-branes the conformal transformation that brings from the open string to the closed string channel. In this way we obtain the equations that the fermionic boundary state must satisfy. We then solve them finding the explicit expression for $|B_{\psi}\rangle$.

Let us consider the boundary conditions of an open superstring stretching between two Dp-branes and circulating in the planar loop describing the interaction between two parallel branes. If the bosonic degrees of freedom satisfy Neumann boundary conditions in all the directions we have the following boundary conditions for the fermionic coordinate:

$$\begin{cases} \psi_{-}(0,\tau) = \eta_{1}\psi_{+}(0,\tau) \\ \psi_{-}(\pi,\tau) = \eta_{2}\psi_{+}(\pi,\tau) \end{cases}$$
 (7.218)

where η_1 and η_2 can take the values ± 1 . If $\eta_1 = \eta_2$ we get the Ramond (R) sector, while if $\eta_1 = -\eta_2$ we get instead the Neveu-Schwarz (NS) sector.

In order to understand how they change when the bosonic coordinate satisfies Dirichlet boundary conditions in some of the directions, we compactify them and apply T-duality. A T-duality transformation along a direction i of an open string theory transforms Neumann into Dirichlet boundary conditions for the bosonic coordinate and, as discussed in sect. 4, changes the sign of the fermionic coordinate in the right sector leaving that of the left sector unchanged, i.e.

$$\psi_-^i \to \psi_-^i \qquad \qquad \psi_+^i \to -\psi_+^i \tag{7.219}$$

Therefore the boundary conditions in eq. (7.218) are generalized to the case of an open superstring satisfying Neumann boundary conditions in the directions longitudinal to the world-volume of the Dp-brane and Dirichelet boundary conditions in the transverse directions, as follows

$$\begin{cases} \psi_{-}^{\mu}(0,\tau) = \eta_{1} S_{\nu}^{\mu} \psi_{+}^{\nu}(0,\tau) \\ \psi_{-}^{\mu}(\pi,\tau) = \eta_{2} S_{\nu}^{\mu} \psi_{+}^{\nu}(\pi,\tau) \end{cases}$$
(7.220)

where the matrix S has been defined in eq.(6.178).

But, together with the assignment of the usual boundary conditions that connect the left and right modes at the endpoints of the open superstring, we must also give the periodicity or anti periodicity conditions for the fermionic degrees of freedom in going around the loop. These are chosen to be

$$\begin{cases}
\psi_{-}(\sigma,0) = \eta_3 \psi_{-}(\sigma,T) \\
\psi_{+}(\sigma,0) = \eta_4 \psi_{+}(\sigma,T)
\end{cases}$$
(7.221)

where η_3 and η_4 can take the values ± 1 . From the boundary conditions in eqs.(7.220) and (7.221) we get

$$\psi_{-}^{\mu}(0,0) = \eta_1 S^{\mu}_{\nu} \psi_{+}^{\nu}(0,0) = \eta_1 \eta_4 S^{\mu}_{\nu} \psi_{+}^{\nu}(0,T)$$
 (7.222)

and

$$\psi_{-}^{\mu}(0,0) = \eta_3 \psi_{-}^{\mu}(0,T) = \eta_3 \eta_1 S_{\nu}^{\mu} \psi_{+}^{\nu}(0,T) \tag{7.223}$$

But the two set of boundary conditions in eqs. (7.218) and (7.221) must be consistent with each other. This implies $\eta_3 = \eta_4$.

Let us now perform the conformal transformation given in eq.(6.172) on the previous open string boundary conditions in order to pass to the closed string channel. Since the right and left fermionic coordinates ψ_- and ψ_+ are two-dimensional conformal fields with conformal weight $h=\frac{1}{2}$ with respect to the their variables ζ and $\bar{\zeta}$ respectively, then under the conformal transformation

$$\zeta \to f(\zeta) = -i\zeta$$
 and $\bar{\zeta} \to \bar{f}(\bar{\zeta}) = i\bar{\zeta}$ (7.224)

they transform as

$$\psi_{-}(\zeta) \to \psi'_{-}(\zeta) = \left(\frac{\partial f(\zeta)}{\partial \zeta}\right)^{1/2} \psi_{-}(\zeta') = (-i)^{\frac{1}{2}} \psi_{-}(f(\zeta))$$
 (7.225)

and

$$\psi_{+}(\bar{\zeta}) \to \psi'_{+}(\bar{\zeta}) = \left(\frac{\partial f(\bar{\zeta})}{\partial \bar{\zeta}}\right)^{1/2} \psi_{+}(\bar{\zeta}) = (i)^{\frac{1}{2}} \psi_{+}(\bar{f}(\bar{\zeta})) \tag{7.226}$$

This implies that, performing the previous transformation on eq. (7.220), we get a relative factor i between the right and left modes. More specifically from the boundary conditions given in eqs. (7.220) and (7.221) after the conformal rescaling in eq. (6.174) we get

$$\begin{cases} \psi_{-}^{\mu}(0,\sigma) = i\eta_{1}S^{\mu}_{\nu}\psi_{+}^{\nu}(0,\sigma) \\ \psi_{-}^{\mu}(\hat{T},\sigma) = i\eta_{2}S^{\mu}_{\nu}\psi_{+}^{\nu}(\hat{T},\sigma) \end{cases}$$
(7.227)

and

$$\begin{cases} \psi_{-}^{\mu}(0,\tau) = \eta_{3}\psi_{-}^{\mu}(\pi,\tau) \\ \psi_{+}^{\mu}(0,\tau) = \eta_{3}\psi_{+}^{\mu}(\pi,\tau) \end{cases}$$
(7.228)

where we have explicitly put $\eta_4 = \eta_3$.

If we now compare the usual boundary conditions for the closed superstring theory given in eqs. (2.39) and (2.40) with eq. (7.228) we see that, as a consequence of the identity between η_3 and η_4 , the fermionic boundary state has only the R-R and the NS-NS sectors.

As for the bosonic coordinate the boundary state for the fermionic coordinate at $\tau = 0$ is defined, from the first equation in (7.227), as the state that satisfies the equation:

$$(\psi_{-}^{\mu}(0,\sigma) - i\eta S^{\mu}_{\nu\nu}\psi_{+}^{\nu}(0,\sigma))|B_{\psi},\eta\rangle = 0 \tag{7.229}$$

where $\eta = \pm 1$. Using the mode expansion given in eqs. (2.42) and (2.43) we get the overlap conditions for the fermionic boundary state

$$\left(\psi_t^{\mu} - i\eta S^{\mu}_{\ \nu} \widetilde{\psi}_{-t}^{\nu}\right) |B_{\psi}, \eta\rangle = 0 \tag{7.230}$$

where the index t is integer [half-integer] in the R-R [NS-NS] sector.

In the case of the NS-NS-sector the determination of the fermionic boundary state satisfying eq.(7.230) is straightforward and leads to the following expression:

$$|B_{\psi}, \eta\rangle = -i \prod_{r=1/2}^{\infty} \left(e^{i\eta\psi_{-r} \cdot S \cdot \widetilde{\psi}_{-r}} \right) |0\rangle$$
 (7.231)

In the R-R sector the boundary state has the same form as in the NS-NS sector for what the non-zero modes is concerned, but with integer instead of half-integer modes. We get therefore 4

$$|B_{\psi}, \eta\rangle = -\prod_{m=1}^{\infty} e^{i\eta\psi_{-m}\cdot S\cdot\widetilde{\psi}_{-m}} |B_{\psi}, \eta\rangle^{(0)}$$
 (7.232)

where the zero mode contribution $|B_{\psi},\eta\rangle^{(0)}$ must satisfy the condition

$$\left(\psi_0^{\mu} - i\eta S^{\mu}_{\ \nu} \widetilde{\psi}_0^{\nu}\right) |B_{\psi}, \eta\rangle^{(0)} = 0 \tag{7.233}$$

The previous equation is satisfied by the state

$$|B_{\psi}, \eta\rangle^{(0)} = \mathcal{M}_{AB}|A\rangle|\widetilde{B}\rangle$$
 (7.234)

⁴The unusual phases introduced in Eqs. (7.231) and (7.232) will turn out to be convenient to study the couplings of the massless closed string states with a D-brane and to find the correspondence with the classical D-brane solutions obtained from supergravity. Note that these phases are instead irrelevant when one computes the interactions between two D-branes.

where

$$\mathcal{M}_{AB} = \left(C\Gamma^0 \dots \Gamma^p \frac{1 + i\eta \Gamma^{11}}{1 + i\eta}\right)_{AB} \tag{7.235}$$

where C is the charge conjugation matrix and Γ^{μ} are the Dirac Γ matrices in the 10-dimensional space (see Ref. [12] for some detail about the derivation of eqs.(7.234) and (7.235)).

The overlap conditions for the conjugate boundary state can be obtained from eq.(7.230) by taking the adjoint of this equation namely

$$\langle B_{\psi}, \eta | \left(\psi_{-t}^{\mu} + i \eta S^{\mu}_{\nu} \widetilde{\psi}_{t}^{\nu} \right) = 0 \tag{7.236}$$

and are solved by

$$\langle B_{\psi}, \eta |_{\text{NS}} = i \langle 0 | \prod_{r=1/2}^{\infty} \left(e^{i\eta\psi_r \cdot S \cdot \widetilde{\psi}_r} \right)$$
 (7.237)

in the NS-NS sector and by

$$\langle B_{\psi}, \eta |_{\mathcal{R}} = -\langle B_{\psi}, \eta |_{R}^{(0)} \prod_{m=1}^{\infty} e^{i\eta\psi_{m} \cdot S \cdot \widetilde{\psi}_{m}}$$
 (7.238)

in the R-R sector, where the zero mode is given by

$$\langle B_{\psi}, \eta |_{\mathcal{R}}^{(0)} = \langle A | \langle \widetilde{B} | \mathcal{N}_{AB}$$
 (7.239)

with

$$\mathcal{N}_{AB} = (-1)^p \left(C\Gamma^0 \dots \Gamma^p \frac{1 + i\eta \Gamma^{11}}{1 - i\eta} \right)_{AB}$$
 (7.240)

Notice that the previous overlap conditions for the conjugate boundary state differ from those given in Ref. [11] by the exchange $\eta \to -\eta$. Our choice of η corresponds to keep also in the closed channel the same η_1, η_2 appearing in the open string boundary conditions (see eq.(7.220)).

As in the bosonic string we must also in this case introduce a boundary state for the reparametrization ghosts b, c. Moreover we must also add the boundary state for the superghosts β, γ . The complete boundary state for both the NS-NS and R-R sectors is given by:

$$|B,\eta\rangle_{R,NS} = \frac{T_p}{2}|B_{mat},\eta\rangle|B_g,\eta\rangle \tag{7.241}$$

where

$$|B_{mat}\rangle = |B_X\rangle|B_{\psi},\eta\rangle$$
 ; $|B_g\rangle = |B_{gh}\rangle|B_{sgh},\eta\rangle$ (7.242)

The matter part of the boundary state consists of the boundary state for the bosonic coordinate X given in eq.(6.180) without the normalization factor N_p and of the one for the fermionic coordinate ψ given in eq.(7.231) for the NS-NS sector and in eq.(7.232) for the R-R sector. The ghost part $|B_g\rangle$ contains the boundary state corresponding to the ghosts (b,c) given in eq.(6.216) and the one corresponding to the superghosts (β, γ) that we now want to determine.

It is not difficult to check that the identifications (6.177) and (7.230) imply that $|B_{\rm mat}, \eta\rangle$ is annihilated by the following linear combinations of left and right generators of the super Virasoro algebra

$$\left(L_n^{\text{mat}} - \tilde{L}_{-n}^{\text{mat}}\right) |B_{\text{mat}}, \eta\rangle = 0 \quad , \quad \left(G_m^{\text{mat}} + i\eta \tilde{G}_{-m}^{\text{mat}}\right) |B_{\text{mat}}, \eta\rangle = 0 \quad . \quad (7.243)$$

The boundary state $|B,\eta\rangle$ must be BRST invariant, that is

$$\left(Q + \tilde{Q}\right)|B,\eta\rangle = 0 \quad , \tag{7.244}$$

where the BRST charge introduced in eq.(3.93) is equal to

$$Q = \oint \frac{dz}{2\pi i} \left[c(z) \left(T^{mat}(z) + \frac{1}{2} T^g(z) \right) - \gamma(z) \left(G^{mat}(z) + \frac{1}{2} G^g(z) \right) \right]. \tag{7.245}$$

If we write the previous expression in the following form

$$Q = Q^{(1)} + Q^{(2)} , (7.246)$$

where

$$Q^{(1)} = \sum_{n} c_n L_{-n} - \sum_{t} \gamma_t G_{-t}$$
 (7.247)

and

$$2Q^{(2)} = \sum_{n=2}^{\infty} L_{-n}^g c_n + \sum_{n=-1}^{\infty} c_{-n} L_n^g - \sum_{t>3/2} G_{-t}^g \gamma_t - \sum_{t>-1/2} \gamma_{-t} G_t^g , \quad (7.248)$$

it is easy to show that eqs. (7.243) and (7.244) imply

$$(c_n + \tilde{c}_{-n}) |B_{\rm gh}\rangle = 0$$
 , $(b_n - \tilde{b}_{-n}) |B_{\rm gh}\rangle = 0$, $(\gamma_t + i\eta\tilde{\gamma}_{-t}) |B_{\rm sgh}, \eta\rangle = 0$, $(\beta_t + i\eta\tilde{\beta}_{-t}) |B_{\rm sgh}, \eta\rangle = 0$. (7.249)

Those equations imply that the relations in eqs. (7.243) must be supplemented by the analogous ones in the ghost sector, namely

$$\left(L_n^{\rm g} - \tilde{L}_{-n}^{\rm g}\right)|B_{\rm g},\eta\rangle = 0 \quad , \quad \left(G_m^{\rm g} + \mathrm{i}\eta \tilde{G}_{-m}^{\rm g}\right)|B_{\rm g},\eta\rangle = 0 \quad . \tag{7.250}$$

The overlap equations involving the ghost fields b and c can be solved and one obtains the boundary state for the bc system given in eq.(6.216). On the other hand the overlap equations for the superghosts determine the superghost boundary state to be:

$$|B_{\rm sgh}, \eta\rangle_{\rm NS} = \exp\left[i\eta \sum_{r=1/2}^{\infty} (\gamma_{-r}\tilde{\beta}_{-r} - \beta_{-r}\tilde{\gamma}_{-r})\right] |P = -1\rangle |\tilde{P} = -1\rangle , (7.251)$$

in the NS sector in the picture (-1, -1) and

$$|B_{\rm sgh}, \eta\rangle_{\rm R} = \exp\left[i\eta \sum_{m=1}^{\infty} (\gamma_{-m}\tilde{\beta}_{-m} - \beta_{-m}\tilde{\gamma}_{-m})\right] |B_{\rm sgh}, \eta\rangle_{\rm R}^{(0)}, \quad (7.252)$$

in the R sector in the (-1/2, -3/2) picture. The superscript ⁽⁰⁾ denotes the zero-mode contribution that, if $|P = -1/2\rangle |\tilde{P} = -3/2\rangle$ denotes the superghost vacuum that is annihilated by β_0 and $\tilde{\gamma}_0$, is given by [13]

$$|B_{\rm sgh}, \eta\rangle_{\rm R}^{(0)} = \exp\left[i\eta\gamma_0\tilde{\beta}_0\right] |P = -1/2\rangle |\tilde{P} = -3/2\rangle .$$
 (7.253)

The conjugate boundary state for the superghost is equal to

$$_{\text{NS}}\langle B_{\text{sgh}}, \eta | = \langle P = -1 | \langle \tilde{P} = -1 | \exp \left[-i\eta \sum_{m=1/2}^{\infty} (\beta_m \tilde{\gamma}_m - \gamma_m \tilde{\beta}_m) \right]$$
(7.254)

in the NS sector and

$$_{\rm R}\langle B_{\rm sgh}, \eta | = \langle P = -3/2 | \langle \tilde{P} = -1/2 | \exp\left[-i\eta\beta_0\tilde{\gamma}_0\right] \times \\ \times \exp\left[-i\eta\sum_{m=1}^{\infty} (\beta_m\tilde{\gamma}_m - \gamma_m\tilde{\beta}_m)\right]$$
 (7.255)

in the R-R sector.

We would like to stress that the boundary states $|B\rangle_{\rm NS,R}$ are written in a definite picture (P,\tilde{P}) of the superghost system, where P is given in eq.(3.105) and $\tilde{P}=-2-P$ in order to soak up the anomaly in the superghost number. In particular we have chosen P=-1 in the NS sector and P=-1/2 in the R sector, even if other choices would have been in principle possible [13]. Since P is half-integer in the R sector, the boundary state $|B\rangle_{\rm R}$ has always $P\neq\tilde{P}$, and thus it can couple only to R-R states in the asymmetric picture (P,\tilde{P}) . However, as we have seen in section 3 the massless R-R states in the (-1/2, -3/2) picture contain a part that is proportional to the R-R potentials [14], as opposed to the standard

massless R-R states in the symmetric picture (-1/2, -1/2) that are always proportional to the R-R field strengths.

The boundary state in eq.(7.241) depends on the two values of $\eta = \pm 1$. Actually, as we will now show, we have to take a combination of the two values of η corresponding to the GSO projection. Let us start with the NS sector. In the NS-NS sector the GSO projected boundary state is

$$|B\rangle_{\rm NS} \equiv \frac{1 + (-1)^{F+G}}{2} \frac{1 + (-1)^{\widetilde{F}+\widetilde{G}}}{2} |B, +\rangle_{\rm NS} ,$$
 (7.256)

where F and G are the fermion and superghost number operators

$$F = \sum_{m=1/2}^{\infty} \psi_{-m} \cdot \psi_m - 1 \quad , \quad G = -\sum_{m=1/2}^{\infty} (\gamma_{-m}\beta_m + \beta_{-m}\gamma_m) \quad . \quad (7.257)$$

Their action on the boundary state corresponding to the fermionic coordinate ψ and to the superghosts can easily be computed and one gets:

$$(-1)^F |B_{\psi}, \eta\rangle = -|B_{\psi}, -\eta\rangle \quad ; \quad (-1)^{\widetilde{F}} |B_{\psi}, \eta\rangle = -|B_{\psi}, -\eta\rangle \quad (7.258)$$

$$(-1)^G |B_{sqh}, \eta\rangle = |B_{sqh}, -\eta\rangle \quad ; \quad (-1)^{\widetilde{G}} |B_{sqh}, \eta\rangle = |B_{sqh}, -\eta\rangle \quad (7.259)$$

Using the previous expressions after some simple algebra we get

$$|B\rangle_{\rm NS} = \frac{1}{2} \left(|B, +\rangle_{\rm NS} - |B, -\rangle_{\rm NS} \right) \tag{7.260}$$

Passing to the R-R sector the GSO projected boundary state is

$$|B\rangle_{\rm R} \equiv \frac{1 + (-1)^p (-1)^{F+G}}{2} \frac{1 - (-1)^{\widetilde{F}+\widetilde{G}}}{2} |B, +\rangle_{\rm R} .$$
 (7.261)

where p is even for Type IIA and odd for Type IIB, and

$$(-1)^{F} = \psi_{11}(-1)^{\sum_{m=1}^{\infty} \psi_{-m} \cdot \psi_{m}} , \quad G = -\gamma_{0}\beta_{0} - \sum_{m=1}^{\infty} \left[\gamma_{-m}\beta_{m} + \beta_{-m}\gamma_{m} \right] .$$

$$(7.262)$$

From the previous expressions it is easy to see after some calculation that the action of the fermion number operators is given by:

$$(-1)^F |B_{\psi}, \eta\rangle = (-1)^p |B_{\psi}, -\eta\rangle \quad ; \quad (-1)^{\widetilde{F}} |B_{\psi}, \eta\rangle = |B_{\psi}, -\eta\rangle \quad (7.263)$$

and

$$(-1)^G |B_{sqh}, \eta\rangle = |B_{sqh}, -\eta\rangle \quad ; \quad (-1)^{\widetilde{G}} |B_{sqh}, \eta\rangle = -|B_{sqh}, -\eta\rangle \quad (7.264)$$

Using the previous expressions after some straightforward manipulations, one gets

$$|B\rangle_{\mathcal{R}} = \frac{1}{2} \left(|B, +\rangle_{\mathcal{R}} + |B, -\rangle_{\mathcal{R}} \right) . \tag{7.265}$$

8. Classical Solutions From Boundary State

In this section we want to connect the boundary state introduced in the previous sections to the Dirichlet branes intended as electric R-R charged p-brane solutions of the low-energy string effective action. In particular we will show that the large distance behaviour of the graviton, dilaton and R-R p+1-form fields that one obtains from the boundary state exactly agrees with that obtained from the classical solution in sect. 5.

The long distance behaviour of the classical massless fields generated by a Dp-brane can be determined by computing the projection of the boundary state along the various fields after having inserted a closed string propagator. This amounts to compute the following matrix element

$$\langle P_x | D | B \rangle \tag{8.266}$$

where P_x runs over all the projectors of the closed superstring massless sector listed in Ref. [15], D is the propagator in eq.(6.185) if we perform the calculation in the bosonic string or is given by

$$D = \frac{\alpha'}{4\pi} \int \frac{d^2z}{|z|^2} z^{L_0 - a} \bar{z}^{\tilde{L}_0 - a}$$
 (8.267)

if we more correctly perform the calculation in superstring, where the constant a=1/2 in the NS-NS sector and a=0 in the R-R sector.

Let us start by computing the expression for the generic NS-NS massless field which is given by

$$J^{\mu\nu} \equiv {}_{-1}\langle \widetilde{0}|_{-1}\langle 0|\psi^{\nu}_{1/2} \ \widetilde{\psi}^{\mu}_{1/2}|D|B\rangle_{NS} = -\frac{T_p}{2k_{\parallel}^2} V_{p+1} S^{\nu\mu}$$
 (8.268)

This equation is exactly the same of the one that one gets in the bosonic string (except that in this case d=10 and not d=26) if we use the propagator in eq.(6.185), the boundary state in eq.(6.180) and the bosonic massless closed string state $\langle \tilde{0} | \langle 0 | \alpha_1^{\nu} \tilde{\alpha}_1^{\mu}$. Because of this we keep the value of the space-time dimension d arbitrary in such a way that our calculation is valid in both cases. Specifing the different polarizations corresponding to the various fields (see Refs. [12, 15] for details) we get

$$\delta\phi = \frac{1}{\sqrt{d-2}} \left(\eta^{\mu\nu} - k^{\mu}\ell^{\nu} - k^{\nu}\ell^{\mu} \right) J_{\mu\nu} = \frac{d-2p-4}{2\sqrt{2(d-2)}} \mu_p \frac{V_{p+1}}{k_{\perp}^2}$$
 (8.269)

for the dilaton,

$$\delta h_{\mu\nu}(k) = \frac{1}{2} \left(J_{\mu\nu} + J_{\nu\mu} \right) - \frac{\delta \phi}{\sqrt{d-2}} \, \eta_{\mu\nu} =$$

$$= \sqrt{2} \mu_p \frac{V_{p+1}}{k_\perp^2} \operatorname{diag} \left(-A, A \dots A, B \dots B \right) \quad , \tag{8.270}$$

for the graviton, where A and B are given in eq. (5.161); and

$$\delta B_{\mu\nu}(k) = \frac{1}{\sqrt{2}} \left(J_{\mu\nu} - J_{\nu\mu} \right) = 0 \tag{8.271}$$

for the antisymmetric tensor. In the R-R sector we get instead

$$\delta C_{01...p}(k) \equiv \langle P_{01\cdots p}^{(C)} | D | B \rangle_{\mathcal{R}} = \mp \mu_p \frac{V_{p+1}}{k_{\perp}^2} . \qquad (8.272)$$

Expressing the previous fields in configuration space using the following Fourier transform valid for p < d - 3

$$\int d^{(p+1)}x \, d^{(d-p-1)}x \frac{e^{ik_{\perp} \cdot x_{\perp}}}{(d-p-3) \, r^{d-p-3} \, \Omega_{d-p-2}} = \frac{V_{p+1}}{k_{\perp}^2} \,, \tag{8.273}$$

remembering the expression Q_p defined in eq.(5.166) and rescaling the fields according to

$$\varphi = \sqrt{2}\kappa\phi$$
 , $\tilde{h}_{\mu\nu} = 2\kappa h_{\mu\nu}$, $C_{01...p} = \sqrt{2}\kappa C_{01...p}$, (8.274)

we get the following large distance behaviour

$$\delta\varphi(r) = \frac{d - 2p - 4}{2\sqrt{2(d - 2)}} \frac{Q_p}{r^{d - p - 3}}$$
(8.275)

for the dilaton,

$$\delta \tilde{h}_{\mu\nu}(r) = 2 \frac{Q_p}{rd-p-3} \operatorname{diag}(-A, \dots A, B \dots B) \quad , \tag{8.276}$$

for the graviton and

$$\delta \mathcal{C}_{01\dots p} = \mp \frac{Q_p}{r^{d-p-3}} \tag{8.277}$$

for the R-R form potential.

The previous equations reproduce exactly the behavior for $r \to \infty$ of the metric in eq.(5.159) and of the R-R potential given in eq.(5.160). In fact at large distance their fluctuations around the background values are

exactly equal to $\delta \tilde{h}_{\mu\nu}$ and $\delta C_{01...p}$. In the case of the dilaton, in order to find agreement between the boundary state and the classical solution, we have to take d=10. This strongly suggests that, as expected, the calculation has to be performed in superstring. As a matter of fact, a comparison between the p-brane solution of the classical eqs. of motion that follow from the action in (5.158) and a string calculation, does make sense only in the superstring case where the graviton, dilaton and Kalb-Ramond field come from the NS-NS sector and the antisymmetric gauge potentials like $C_{\mu_1...\mu_n}$ from the R-R sector. Nonetheless, the bosonic case we have also considered in this section already tells us what are the distinctive features of the boundary state and how the long-distance behavior of the massless fields is encoded in it.

9. Interaction Between a p and a p' Brane

In this section we study the static interaction between a Dp-brane located at y_1 , and a Dp'-brane located at y_2 , with $NN \equiv \min\{p,p'\} + 1$ directions common to the brane world-volumes, $DD \equiv \min\{d-p-1,d-p'-1\}$ directions transverse to both, and $\nu = (d-NN-DD)$ directions of mixed type. We will not consider instantonic D-branes, hence also $NN \geq 1$. The two D-branes simply interact via tree-level exchange of closed strings whose propagator is

$$D = \frac{\alpha'}{4\pi} \int \frac{d^2z}{|z|^2} z^{L_0} \bar{z}^{\tilde{L}_0} , \qquad (9.278)$$

so that indicating with $|B_1\rangle$ and $|B_2\rangle$ the boundary states describing the two D-branes the static amplitude is given by

$$A = \langle B_1 | D | B_2 \rangle = \frac{T_p T_{p'}}{4} \frac{\alpha'}{4\pi} \int_{|z| < 1} \frac{d^2 z}{|z|^2} \mathcal{A} \mathcal{A}^{(0)} , \qquad (9.279)$$

where we have indicated with \mathcal{A} and $\mathcal{A}^{(0)}$ respectively the non zero mode and the zero mode contribution in which the previous amplitude can be factorized. We do not have any intercept as we had in eq.(6.185) for the bosonic string because we assume that both L_0 and \tilde{L}_0 contain the ghost degrees of freedom. The details of the computation of the quantity in eq.(9.279) can be found in Ref. [11]. Here we just give the results of the various terms starting from the non-zero modes. In the NS-NS sector after the GSO projection we get

$$\mathcal{A}_{\text{NS-NS}} = \frac{1}{2} \left[\left(\frac{f_3}{f_1} \right)^{8-\nu} \left(\frac{f_4}{f_2} \right)^{\nu} - \left(\frac{f_4}{f_1} \right)^{8-\nu} \left(\frac{f_3}{f_2} \right)^{\nu} \right] , \qquad (9.280)$$

In the R-R sector instead before the GSO projection we get

$$\mathcal{A}_{R-R}(\eta_1, \eta_2) = \left[2^{\nu-4} \left(\frac{f_2}{f_1} \right)^{8-2\nu} \delta_{\eta_1 \eta_2, 1} + \delta_{\eta_1 \eta_2, -1} \right] , \qquad (9.281)$$

where the functions f_i are equal to

$$f_1 \equiv q^{\frac{1}{12}} \prod_{n=1}^{\infty} (1 - q^{2n})$$
 ; $f_2 \equiv \sqrt{2}q^{\frac{1}{12}} \prod_{n=1}^{\infty} (1 + q^{2n})$; (9.282)

$$f_3 \equiv q^{-\frac{1}{24}} \prod_{n=1}^{\infty} (1 + q^{2n-1})$$
 ; $f_4 \equiv q^{-\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^{2n-1})$. (9.283)

They transform as follows under the modular transformation $t \to 1/t$ ($q = e^{-\pi t}$):

$$f_1(e^{-\frac{\pi}{t}}) = \sqrt{t} f_1(e^{-\pi t}) \; ; \; f_2(e^{-\frac{\pi}{t}}) = f_4(e^{-\pi t}) \; ; f_3(e^{-\pi t}) = f_3(e^{-\frac{\pi}{t}}) \; .$$

$$(9.284)$$

The zero modes contribution in the NS-NS sector comes only from the bosonic coordinate and can be obtained by following the same procedure outlined in eqs. (6.187)-(6.190) for the bosonic string. Also in this way one gets eq. (6.190). If we insert the contributions in eqs. (9.280) and (6.190) in eq. (9.279) we get the total NS-NS contribution to the interaction between a Dp and a Dp' brane

$$A_{\rm NS-NS} = V_{NN} (8\pi^2 \alpha')^{-\frac{NN}{2}} \int_0^\infty dt \left(\frac{1}{t}\right)^{\frac{DD}{2}} e^{-y^2/(2\alpha'\pi t)} \times \frac{1}{2} \left[\left(\frac{f_3}{f_1}\right)^{8-\nu} \left(\frac{f_4}{f_2}\right)^{\nu} - \left(\frac{f_4}{f_1}\right)^{8-\nu} \left(\frac{f_3}{f_2}\right)^{\nu} \right] , (9.285)$$

where V_{NN} is the common world-volume of the two D-branes, |y| is the transverse distance between them.

It is interesting to notice that the two terms in the square brackets of Eq. (9.285) come respectively from the NS-NS and the NS-NS $(-1)^{(F+G)}$ sectors of the exchanged closed string, which, under the transformation $t = 1/\tau$, are mapped into the NS and R sectors of the open string suspended between the branes. Notice that $A_{\text{NS-NS}} = 0$ if $\nu = 4$.

The evaluation of the zero mode contribution in the R-R sector requires more care due to the presence of zero modes both in the fermionic matter fields and the bosonic superghosts. Inserting eq. (9.281) into eq. (9.279) we can write the total R-R contribution as

$$A_{\rm R-R}(\eta_1, \eta_2) = V_{NN} (8\pi^2 \alpha')^{-\frac{NN}{2}} 2^{-\frac{\nu}{2}} \int_0^\infty dt \left(\frac{1}{t}\right)^{\frac{DD}{2}} e^{-y^2/(2\pi\alpha' t)}$$

$$\times \left[2^{\nu-4} \left(\frac{f_2}{f_1} \right)^{8-2\nu} \delta_{\eta_1 \eta_2, +1} + \delta_{\eta_1 \eta_2, -1} \right] \stackrel{(0)}{R} \langle B^1, \eta_1 | B^2, \eta_2 \rangle_{\mathbf{R}}^{(0)} , \qquad (9.286)$$

where

$$|B,\eta\rangle_{\mathbf{R}}^{(0)} = |B_{\psi},\eta\rangle_{\mathbf{R}}^{(0)} |B_{\mathrm{sgh}},\eta\rangle_{\mathbf{R}}^{(0)}$$
 (9.287)

Note that in Eq. (9.286) it is essential not to separate the matter and the superghost zero-modes. In fact, a naïve evaluation of $_{\rm R}^{(0)}\langle B^1,\eta_1|B^2,\eta_2\rangle_{\rm R}^{(0)}$ would lead to a divergent or ill defined result: after expanding the exponentials in $_{\rm R}^{(0)}\langle B_{\rm sgh}^1,\eta_1|B_{\rm sgh}^2,\eta_2\rangle_{\rm R}^{(0)}$, all the infinite terms with any superghost number contribute, and yield the divergent sum $1+1+1+\dots$ if $\eta_1\eta_2=-1$, or the alternating sum $1-1+1-\dots$ if $\eta_1\eta_2=1$. This problem has already been addressed in Ref. [13] and solved by introducing a regularization scheme for the pure Neumann case (NN=10). This method has been extended to the most general case with D-branes in Ref. [11]. Here, we give the final result for the fermionic zero mode part of the R-R sector:

$${}_{R}^{(0)}\langle B^{1}, \eta_{1}|B^{2}, \eta_{2}\rangle_{R}^{(0)} = -16\,\delta_{\nu,0}\,\delta_{\eta_{1}\eta_{2},1} + 16\,\delta_{\nu,8}\,\delta_{\eta_{1}\eta_{2},-1} \quad . \tag{9.288}$$

We can now write the final expression for the R-R amplitude. Inserting Eq. (9.288) into Eq. (9.286), after performing the GSO projection we get

$$A_{\rm R-R} = V_{NN} (8\pi^2 \alpha')^{-\frac{NN}{2}}.$$

$$\int_0^\infty dt \left(\frac{1}{t}\right)^{\frac{DD}{2}} e^{-y^2/(2\pi\alpha't)} \frac{1}{2} \left[-\left(\frac{f_2}{f_1}\right)^8 \delta_{\nu,0} + \delta_{\nu,8} \right] . \tag{9.289}$$

The $\nu=0$ and $\nu=8$ terms in Eq. (9.289) come respectively from the R-R and the R-R(-1)^(F+G) sectors of the exchanged closed string, which, under the transformation $t\to 1/t$, are mapped into the NS(-1)^(F+G) and R(-1)^(F+G) sectors of the open string suspended between the branes. Due to the "abstruse identity", the total D-brane amplitude

$$A = A_{\rm NS-NS} + A_{\rm R-R} \tag{9.290}$$

vanishes if $\nu = 0, 4, 8$; these are precisely the configurations of two D-branes which break half of the supersymmetries of the Type II theory and satisfy the BPS no-force condition.

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Appendix A

In this appendix we will describe the properties of bosonic and fermionic bc systems that enter in the covariant quantization of string theories. Their dynamics is described by the action

$$S[b,c] \sim \int d^2z \ [b\bar{\partial}c + \bar{b}\partial\bar{c}] \ ,$$
 (A.1)

which implies the equations of motion

$$\bar{\partial}b = 0$$
 ; $\bar{\partial}c = 0$, (A.2)

and their conjugate ones. Thus, the fields b and c are functions only of z and they admit the following holomorphic expansions 5

$$b(z) = \sum b_n z^{-n-\lambda}$$
 ; $c(z) = \sum c_n z^{-n+\lambda-1}$. (A.3)

where the variable n is integer for integer values of the conformal dimension λ , while for half-integer values of λ different spin structures are possible. In particular for periodic boundary conditions (R-sector) n is integer while for anti-periodic boundary conditions (NS-sector) n is half-integer. The oscillators in eq. (A.3) satisfy the following hermiticity properties

$$c_n^{\dagger} = c_{-n} \quad ; \quad b_n^{\dagger} = \epsilon b_{-n} \quad , \tag{A.4}$$

where $\epsilon = 1$ for fermions and $\epsilon = -1$ for bosons.

The theory can be quantized by either requiring canonical commutation relations that on the mode expansion in eq.(A.3) read as

$$[c_n, b_m]_{\epsilon} = \delta_{n+m,0}$$
 ; $[b_n, b_m]_{\epsilon} = [c_n, c_m]_{\epsilon} = 0$, (A.5)

where $[\ ,\]_{\epsilon}$ means commutator [anticommutator] for bosonic [fermionic] fields or by imposing the OPE

$$c(z)b(w) = \frac{1}{z - w}$$
 , $b(z)c(w) = \frac{\epsilon}{z - w}$. (A.6)

The energy-momentum tensor T(z) and the ghost number current j(z) of the theory are given by

$$T(z) =: [-\lambda b\partial c + (1-\lambda)\partial bc] := \sum_{n} L_n z^{-n-2} , \qquad (A.7)$$

 5 To avoid repetition, we write all definitions for the holomorphic sector of the theory only; similar expressions hold for the antiholomorphic sector.

$$j(z) = -: b(z)c(z) := \epsilon : c(z)b(z) := \sum_{n} j_n z^{-n-1}$$
, (A.8)

where the normal ordering is explicitly given by

$$: b_n c_{-n} := \begin{cases} b_n c_{-n} & \text{if } n < 1 - \lambda \\ -\epsilon c_{-n} b_n & \text{if } n \ge 1 - \lambda \end{cases}$$
 (A.9)

The Fourier coefficients L_n and j_n take the form

$$L_{n} = \oint \frac{dz}{2\pi i} T(z) z^{n+1} = \sum_{m} (\lambda n - m) : b_{m} c_{n-m} :$$

$$j_{n} = \oint \frac{dz}{2\pi i} j(z) z^{n} = -\sum_{m} : b_{m} c_{n-m} : , \qquad (A.10)$$

and as a consequence of eq. (A.4) they satisfy the following hermiticity properties

$$L_n^{\dagger} = L_{-n} \quad ; \quad j_n^{\dagger} = -j_{-n} \quad . \tag{A.11}$$

From the ghost number current we can obtain the ghost number j_0 as

$$j_0 = \oint dz j(z) = -\sum_{n=-\infty}^{\infty} : b_n c_{-n} : ,$$
 (A.12)

By using the OPE in eq.(A.6) one gets

$$T(z)b(w) = \frac{\lambda b(w)}{(z-w)^2} + \frac{\partial_w b(w)}{(z-w)} + \dots; \ T(z)c(w) = \frac{(1-\lambda)c(w)}{(z-w)^2} + \frac{\partial_w c(w)}{(z-w)} + \dots,$$
(A.13)

that are consistent with the fact that b and c are conformal fields with conformal weights λ and $1 - \lambda$ respectively and

$$j(z)b(w) = -\frac{b(w)}{(z-w)} + \dots \quad ; \quad j(z)c(w) = \frac{c(w)}{(z-w)} + \dots \quad , \qquad (\text{A}.14)$$

that imply that b and c have ghost charge -1 and 1 respectively. Moreover one can see that T(z) and j(z) satisfy the OPE

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} + \cdots$$

$$T(z)j(w) = \frac{Q}{(z-w)^3} + \frac{j(w)}{(z-w)^2} + \frac{\partial_w j(w)}{z-w} + \cdots$$

$$j(z)j(w) = \frac{\epsilon}{(z-w)^2} + \cdots , \qquad (A.15)$$

where the "screening charge" Q and the c-number of the Virasoro algebra are respectively given by

$$Q = \epsilon(1 - 2\lambda) \quad , \quad c = \epsilon(1 - 3Q^2) \quad . \tag{A.16}$$

Using eqs. (A.10) the OPEs given in eq. (A.15) imply the commutation relations

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{1-3Q^2}{12}n(n^2-1)\delta_{n+m,0} , \qquad (A.17)$$

$$[L_n, j_m] = -mj_{n+m} + \frac{Q}{2}n(n+1)\delta_{n+m,0}$$
, (A.18)

$$[j_n, j_m] = n\delta_{n+m,0} \quad . \tag{A.19}$$

We observe that the \mathcal{Q} -dependent term in eq.(A.18) (or equivalently in the second OPE in eq.(A.15)), which makes the current j(w) not quite a good conformal primary field, is a consequence of the anomaly appearing in the conservation law of the ghost number current [8]

$$\bar{\partial}j(z) = \frac{1}{8}\mathcal{Q}\sqrt{g}R^{(2)} \quad , \tag{A.20}$$

where g and $R^{(2)}$ are respectively, the determinant of the metric and the scalar curvature of the two dimensional world-sheet Σ , on which the theory is defined.

Comparing the two equations obtained from (A.18) for n=-m=1 and n=-m=-1 and using the hermiticity properties in eq.(A.11) we get that j_0 is neither hermitian nor antihermitian, but satisfies the following property

$$j_0 + j_0^{\dagger} + \mathcal{Q} = 0$$
 (A.21)

Let us introduce the q-vacuum, |q>, defined by the relations

$$b_n|q>=0$$
 if $n>\epsilon q-\lambda$,
 $c_n|q>=0$ if $n\geq -\epsilon q+\lambda$. (A.22)

Then, from eq.(A.10) one can easily show that $|q\rangle$ is an eigenstate of both j_0 and L_0 with eigenvalues given respectively by the following equations

$$j_0|q> = q|q>$$
 , $L_0|q> = \frac{1}{2}\epsilon q(q+Q)|q>$, (A.23)

and that, as a consequence of eq. (A.21), it is normalized as follows

$$\langle q'|q\rangle = \delta(q'+q+Q)$$
 (A.24)

We observe that the state $|q=0\rangle$ is the only SL(2,R) invariant vacuum since it is the only one which is annihilated simultaneously by L_0, L_1 and L_{-1} .

Because of eq.(A.24), in order not to get a vanishing result when we compute correlation functions involving b and c, we must make sure that the total ghost number of the correlator be equal -Q. For instance the following correlation function

$$<-q-\mathcal{Q}|c(z)b(w)|q> = \left(\frac{z}{w}\right)^{\epsilon q} \frac{1}{z-w}$$
, (A.25)

is different from zero. The contraction given in eq.(A.6) can be obtained from the previous equation by choosing the SL(2,R) invariant vacuum $|q=0\rangle$.

By using the mode expansion in eq.(A.3) and the anticommutation relations in eq. (A.5), or equivalently the contraction in eq.(A.6) together with the Wick theorem, one can very easily compute any correlation function of b and c fields on the sphere.

A fermionic bc system can be bosonized in terms of a scalar field with a background charge Q through the following relations

$$b(z) =: e^{-\varphi(z)}:$$
 $c(z) =: e^{\varphi(z)}:$, (A.26)

while a bosonic bc system can be "bosonized" in terms of a scalar field φ with background charge Q and a fermionic bc system with $\lambda = 1$, that we call $\xi \eta$ system, through the following relations

$$\beta(z) = \partial \xi(z)e^{-\phi(z)} \qquad \gamma(z) = e^{\phi(z)}\eta(z) \quad . \tag{A.27}$$

where we have called β , γ the bosonic b, c fields.

Let us give some detail about this bosonization procedure. The action of a scalar field with background charge Q is given by

$$S[\varphi] \sim \int_{\Sigma} d^2 z \left[-\epsilon \bar{\partial} \varphi \partial \varphi - \frac{1}{4} \mathcal{Q} \sqrt{g} R^{(2)} \varphi \right]$$
 (A.28)

where g and $R^{(2)}$ are respectively, the determinant of the metric and the scalar curvature of the two dimensional world-sheet Σ , on which the theory is defined. The equation of motion for this field is

$$\partial \bar{\partial} \varphi(z) = \frac{1}{8} \epsilon Q \sqrt{g} R^{(2)}$$
 (A.29)

Notice that $\epsilon \partial \varphi(z)$ satisfies exactly the same equation as the anomalous current j(z) of the previous system (see eq. (A.20)). This system is invariant

under the conformal transformations generated by the energy-momentum tensor

$$T(z) =: \frac{1}{2} [\epsilon(\partial \varphi)^2 - Q \partial^2 \varphi] : (z) , \qquad (A.30)$$

and under a U(1) Kac-Moody algebra generated by the current

$$j(z) = \epsilon \partial \varphi(z)$$
 . (A.31)

The theory can be quantized by requiring the standard OPE for a free scalar field, namely

$$\varphi(z)\varphi(w) = \epsilon \log(z - w)$$
 (A.32)

By using it one can easily check that T(z) and j(z) satisfy the following OPEs

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + 2\frac{T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} + \cdots , \qquad (A.33)$$

$$T(z)j(w) = \frac{Q}{(z-w)^3} + \frac{j(w)}{(z-w)^2} + \frac{\partial_w j(w)}{z-w} + \cdots ,$$
 (A.34)

$$j(z)j(w) = \frac{\epsilon}{(z-w)^2} + \cdots , \qquad (A.35)$$

where the central charge c of the Virasoro algebra is equal to

$$c = 1 - 3\epsilon \mathcal{Q}^2 \quad . \tag{A.36}$$

The presence of a third-order pole in (A.34) is a signal of the fact that j(z) is not really a good conformal field of weight 1, when there is a non-vanishing background charge Q. Notice that eqs. (A.33) - (A.35) reproduce the OPE given in eq. (A.15) except for the value of the central charge c.

The field φ admits the following expansion

$$\varphi(z) = x + N \log z + \sum_{n \neq 0} \frac{\alpha_n}{n} z^{-n} \quad , \tag{A.37}$$

where the harmonic oscillators satisfy the usual commutation relations

$$[\alpha_n, \alpha_m] = n\epsilon \delta_{n+m,0}$$
 $[x, N] = -\epsilon$. (A.38)

Using eq.(A.37) in eqs.(A.30) and (A.31), one can easily obtain the oscillator expressions for the Virasoro generators L_n and for the Fourier components j_n of the current j(z), namely

$$L_n = \frac{1}{2} \sum_m : \alpha_m \alpha_{n-m} : -\frac{1}{2} \mathcal{Q}(n+1) \alpha_n ,$$

$$j_n = -\epsilon \alpha_n \quad , \tag{A.39}$$

with $\alpha_0 = -N$ and where the symbol: : is the usual normal ordering of harmonic oscillators. The OPEs in eqs.(A.33), (A.34) and (A.35) are then equivalent to the following commutation relations

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{1-3Q^2}{12}n(n^2-1)\delta_{n+m,0} ,$$

$$[L_n, j_m] = -mj_{n+m} - \frac{\mathcal{Q}}{2}n(n+1)\delta_{n+m,0} \; ; \; [j_n, j_m] = n\delta_{n+m,0} \; . \quad (A.40)$$

As in the case of a bc system the zero mode of the fermionic number current is not hermitian, but satisfies the following hermiticity properties

$$j_0 + j_0^{\dagger} + \mathcal{Q} = 0$$
 , (A.41)

which implies

$$\langle q | q' \rangle = \delta(q + q' + Q)$$
 (A.42)

where $|q\rangle$ and $|q'\rangle$ are eigenstates of N with eigenvalues q and q' respectively

Due to the presence of the zero mode logarithmic term in eq.(A.37), the field $\varphi(z)$ does not transform properly under a conformal transformation, whereas : $e^{q\varphi(z)}$: behaves as a primary conformal field of weight $\frac{1}{2}\epsilon q(q+Q)$. In addition it transforms as a field with charge q under the ghost number current generated by j(z). This can be checked by computing the following OPE

$$T(z): e^{q\varphi(w)} := \frac{1}{2}\epsilon q(q+Q)\frac{e^{q\varphi(w)}}{(z-w)^2} + \frac{\partial_w: e^{q\varphi(w)}}{(z-w)} + \cdots,$$
 (A.43)

$$j(z): e^{q\varphi(w)} := q \frac{e^{q\varphi(w)}}{z - w}$$
 (A.44)

Introducing the corresponding highest weight state according to

$$|q> = \lim_{z \to 0} : e^{q\varphi(z)} : |0>$$
, (A.45)

it is easy to see that $|q\rangle$ is an eigenstate of the ghost number j_0 and of L_0 with eigenvalues given respectively by

$$L_0|q> = \frac{1}{2}\epsilon q(q+Q)|q> \quad , \quad j_0|q> = q|q> \quad .$$
 (A.46)

If we consider the case $\epsilon = 1$ and we takes $Q = 1 - 2\lambda$ we immediately see that the central charge in eq. (A.36) reproduces exactly the one given in

eq. (A.16) and that the OPEs in eqs.(A.15) and in eqs.(A.33), (A.34) and (A.35) are coincident. Moreover if we consider eqs. (A.43) and (A.44) for $\epsilon = 1$ and $Q = 1-2\lambda$ and put $q = \pm 1$ they reproduce eqs. (A.13) and (A.14) respectively for b and c. This is consistent with the fact that a fermionic bc system is completely equivalent to a scalar field with a background charge $Q = 1 - 2\lambda$ and with $\epsilon = 1$. The fields b and c can be expressed in terms of the scalar field through the bosonization eqs.(A.26) and the current j(z) in eq.(A.31) turns out to be the bosonized version of the fermionic number current in eq.(A.8). Consequently the zero mode N in eq.(A.37) is just the bosonized version of the fermionic number, as one can see from

$$N = \oint dz j(z) = j_0 \quad . \tag{A.47}$$

In the case of a bosonic bc system the central charge of the Virasoro algebra in eq.(A.16) can be written as:

$$c = -1 + 3Q^2 = (1 + 3Q^2) - 2$$
, (A.48)

that corresponds to the sum of the central charges of a scalar field with $\epsilon = -1$ given by $c = 1 + 3Q^2$ and of a fermionic bc system with $\lambda = 1$ given by c = -2. In this case the "bosonization" rules are given in eqs.(A.27). Introducing the new energy momentum tensor as

$$T(z) = T_{\varphi}(z) + T_{\eta\xi}(z) \quad ; \tag{A.49}$$

where T_{φ} is given in eq. (A.30) for $\epsilon = -1$, $Q = (-1 + 2\lambda)$ and $T_{\eta\xi}$ is given in eq. (A.7) for $\epsilon = 1$ and $\lambda = 1$, it is easy to verify that the fields in the r.h.s. of eqs.(A.27) have exactly the same conformal weights of a bosonic (b,c) system. Moreover, if we introduce the sum of U(1) number currents of the scalar field φ and of the fermionic ξ, η system:

$$j(z) = j_{\varphi}(z) + j_{\eta \xi}(z) = -\partial \varphi(z) + \xi(z)\eta(z) \quad , \tag{A.50}$$

it is easy to verify that the OPE of j(z) with $\beta(z)$ and $\gamma(z)$ has no simple pole term implying that both $\beta(z)$ and $\gamma(z)$ have charge zero with respect to the total U(1) number given by

$$P = (j_0)_{\varphi} + (j_0)_{\eta\xi} = \oint \frac{dz}{2\pi i} (-\partial \phi + \xi \, \eta) \tag{A.51}$$

On the other hand the U(1) current for the bosonic b, c system given in eq.(A.12) is instead reproduced in the "bosonized" system by only the term $(j_0)_{\varphi}$ in eq.(A.51).

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