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Supersymmetry

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[Bilal, hep-th/0101055
Quevedo (2016)
Bertolini (2018)]

* Poincaré

Notation: $\eta_{\mu\nu} = \mathrm{diag}(+1,-1,-1,-1)$

Lorentz transf.: $O(1,3) \Rightarrow x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$ s.t. $\Lambda^{T}\eta\Lambda = \eta$ $[\eta_{\mu\nu} = \Lambda_{\mu}{}^{\rho}\Lambda^{\sigma}{}_{\nu}\eta_{\rho\sigma}]$

3 boosts and 3 rotations: $K_i = L_{0i}$ $\quad R_i = \varepsilon_{ijk} L^{jk}$ $\quad i,j,k = 1,2,3$

$$\Rightarrow \Lambda = \mathbb{I} - \frac{1}{2}\omega_{[\mu\nu]} L^{[\mu\nu]}$$

s.t.: $[L_{\mu\nu}, L_{\rho\sigma}] = i\left(L_{\mu\rho}\eta_{\nu\sigma} + L_{\nu\sigma}\eta_{\mu\rho} - L_{\nu\rho}\eta_{\mu\sigma} - L_{\mu\rho}\eta_{\nu\sigma}\right)$

Poicaré: $P_\mu = -i\partial_\mu$ s.t. $[P_\mu, P_\nu] = 0$; $[L_{\mu\nu}, P_\rho] = i(P_\mu \eta_{\nu\rho} - P_\nu \eta_{\mu\rho})$

→ 2 Casimir operators: $[C_i, P_\mu] = [C_i, L_{\mu\nu}] = 0$.

$C_1 = P^{\mu}P_{\mu}$

$C_2 = W^{\mu}W_{\mu}$, where $W^{\mu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}P_\nu L_{\rho\sigma}$ and $[W^{\mu}, P^{\nu}] = 0$

$[W^{\mu}, W^{\nu}] = -i\varepsilon^{\mu\nu\rho\sigma}W_\rho W_\sigma$.

* The Little Group:

Massive	Massless
↓	↓
$P^{\mu}P_{\mu} = m^2$	$P^{\mu}P_{\mu} = 0$
Go to rest frame: $P^{\mu} = (m, \vec{0})$	↓ $P^{\mu} = (E, 0, 0, E)$
⇓	⇓
SO(3) Little Group	SO(2) Little Group
~ SU(2) (see QM)	~ helicity

MASSIVE

$\Downarrow$

$W^2 = -m^2 s(s+1)$

[s eigenv. of L_z and

$\vec{J}^2 = s(s+1)$]

MASSLESS

$\Downarrow$

$W^\mu = \underbrace{L_{12}}\, P^\mu$

$\lambda = \hat{p} \cdot \vec{L}$ [helicity]

$\Rightarrow$ Different fields in diff. Lorentz reps:

* SCALAR : $\varphi(x) \longrightarrow \varphi(x')$

REAL $\Rightarrow$ 1 DOF

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\varphi\,\partial^\mu\varphi - \frac{m^2}{2}\varphi^2 + \dots \qquad [\varphi] = \frac{d-2}{2}$$

mass dimensions.

* DIRAC SPINOR: $\Psi(0) \rightarrow \Lambda\Psi(0), \qquad \Lambda = \exp\left(-\frac{i}{2}\omega^{\mu\nu}\Sigma_{\mu\nu}\right)$

$$\mathcal{L} = -i\,\overline{\Psi}\gamma^\mu\partial_\mu\Psi - M\overline{\Psi}\Psi + \dots \qquad [\Psi] = \frac{d-1}{2}$$

COMPLEX $\longrightarrow$ 4 $\mathbb{C}$ dof OFF-SHELL

2 $\mathbb{C}$ dof ON-SHELL

NB: $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ Clifford algebra

$\Sigma_{\mu\nu} = \frac{i}{4}[\gamma_\mu, \gamma_\nu]$

$\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$ s.t.: $(\gamma_5)^2 = \mathbb{I}$, $\{\gamma_5, \gamma_\mu\} = 0$, $\gamma_5^\dagger = \gamma_5$, $[\gamma_5, \Sigma_{\mu\nu}] = 0$

Define $P_\pm = \frac{\mathbb{I} \pm \gamma_5}{2} \longrightarrow$ $P_+\Psi$ L-handed WEYL spinor

$P_-\Psi$ R-handed WEYL spinor

$\Rightarrow \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \qquad \sigma^\mu = (\mathbb{I}, \vec{\sigma}) \qquad \bar{\sigma}^\mu = (\mathbb{I}, -\vec{\sigma}).$

$\gamma_5 = \begin{pmatrix} \mathbb{I} & \\ & -\mathbb{I} \end{pmatrix} \qquad \Sigma^{\mu\nu} = \begin{pmatrix} \sigma^{\mu\nu} & \\ & \bar{\sigma}^{\mu\nu} \end{pmatrix}$

where $\sigma^{\mu\nu} = \frac{i}{4}\left(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu\right)$

$\bar{\sigma}^{\mu\nu} = \frac{i}{4}\left(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu\right)$

$\Rightarrow \Sigma^{\mu\nu}, \sigma^{\mu\nu}, \bar{\sigma}^{\mu\nu}$ all satisfy Lorentz algebra

NB: Weyl spinor $\Rightarrow$ 2 $\mathbb{C}$ dof OFF-SHELL
2 $\mathbb{R}$ dof ON-SHELL

[Weyl = irreps of Lorentz]

→ SPINOR NOTATION:

$$\psi = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot\alpha} \end{pmatrix} < \begin{matrix} (\sigma^\mu)_{\alpha\dot\alpha} \\ (\bar\sigma^\mu)^{\dot\alpha\alpha} \end{matrix}$$ since indices act $(^\alpha{}_\alpha)$ $(_{\dot\alpha}{}^{\dot\alpha})$

$$\Downarrow$$

$$(\sigma^{\mu\nu})_\alpha{}^\beta \quad \text{and} \quad (\bar\sigma^{\mu\nu})^{\dot\alpha}{}_{\dot\beta}$$

$$P_+\psi = \begin{pmatrix} \psi_\alpha \\ 0 \end{pmatrix}$$

$$P_-\psi = \begin{pmatrix} 0 \\ \bar\chi^{\dot\alpha} \end{pmatrix}$$

The transf. acts as: $\psi_\alpha \longrightarrow \exp\left(-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\right)_\alpha{}^\beta \psi_\beta$

$$\bar\chi^{\dot\alpha} \longrightarrow \exp\left(-\frac{i}{2}\omega_{\mu\nu}\bar\sigma^{\mu\nu}\right)^{\dot\alpha}{}_{\dot\beta}\, \bar\chi^{\dot\beta}$$

NB: $\sigma^{\mu\nu} = \frac{1}{2i}\varepsilon^{\mu\nu\rho\sigma}\sigma_{\rho\sigma}$ and $\bar\sigma^{\mu\nu} = -\frac{1}{2i}\varepsilon^{\mu\nu\rho\sigma}\bar\sigma_{\rho\sigma}$

$\downarrow$ self-dual $\qquad \downarrow$ anti self-dual

$\Rightarrow$ use it to decompose an antisymmetric tensor to self-dual and antiself dual part

* VECTOR FIELD

$A_\mu \longrightarrow$ 3 $\mathbb{R}$ dof OFF-SHELL (gauge inv.)
2 $\mathbb{R}$ dof ON-SHELL (gauge inv. + e.o.m.)

$\Rightarrow$ $[F_{\mu\nu}] = 2$ since $F_{\mu\nu} \sim [D_\mu, D_\nu]$

$\hookrightarrow$ $\frac{1}{g_{YM}^2}\int d^4x\, \partial_\mu A_\nu \Rightarrow [g_{YM}^{-2}] = d-4$

* MAJORANA SPINOR $\longrightarrow$ reality condition on ψ_D:

$$\psi^c = \psi \quad \text{or} \quad \psi^* = B\psi \quad \text{where} \quad \psi^c = C\overline{\psi} = \begin{pmatrix} \chi_\alpha \\ \overline{\psi}^{\dot\alpha} \end{pmatrix} \text{ and } C = i\gamma^0\gamma^2,\ \overline{\psi} = \psi^\dagger\gamma^0$$
$$B = \gamma_3\gamma_5$$

$\Rightarrow$ MAJORANA REP: $\psi = \begin{pmatrix} \psi_\alpha \\ \overline{\psi}^{\dot\alpha} \end{pmatrix}$.

$\Rightarrow$ ALGEBRA: $so(1,3) \sim su(2)_L + su(3)_R$

$$\Rightarrow R_i = \varepsilon_{ijk} L^{jk}, \quad K_i = L_{0i} \Rightarrow J_i^\pm = \frac{1}{2}(R_i \pm iK_i) \text{ and } (J_i^\pm)^\dagger = J_i^\mp$$

$$\Rightarrow [J_i^\pm, J_j^\pm] = i\varepsilon_{ijk} J_k^\pm \rightarrow \text{two } su(2) \text{ algebras}$$

$$[J_i^\pm, J_j^\mp] = 0.$$

At group level: $SL(2,\mathbb{C}) \simeq SO(1,3)$

$\Rightarrow$ there is a map (homeomorphism):

$SL(2,\mathbb{C}) = \{ 2\times 2 \text{ matrices } M_\alpha{}^\beta \in \mathbb{C} \text{ s.t. } \det M = 1\}$

$\forall\, M_1, M_2 \in SL(2,\mathbb{C}) \quad \exists\, \Lambda(M_1), \Lambda(M_2) \in SO(1,3) \text{ s.t. } \Lambda(M_1 M_2) = \Lambda(M_1)\Lambda(M_2)$

<u>Define</u>: $X_{\alpha\dot\beta} = X_\mu(\sigma^\mu)_{\alpha\dot\beta} = \begin{pmatrix} X_0 + X_3 & X_1 - iX_2 \\ X_1 - iX_2 & X_0 - X_3 \end{pmatrix}$

$\hookrightarrow$ it transforms $X'_{\alpha\dot\beta} = M_\alpha{}^\beta X_{\beta\dot\alpha} \overline{M}^{\dot\alpha}{}_\beta$ where $\overline{M} = M^\dagger \in SL(2,\mathbb{C})$

$\Rightarrow \det X' = \det M \det X \det \overline{M} = \det X \Rightarrow \det X$ is $\overset{SL(2,\mathbb{C})}{\vee}$ invariant:

$$\det X = X_0^2 - X_1^2 - X_2^2 - X_3^2 = X^\mu X_\mu \Rightarrow \text{Lorentz invariant}$$

NB: $(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} = \varepsilon^{\dot{\alpha}\dot{\beta}}\varepsilon^{\alpha\beta}(\sigma^\mu)_{\beta\dot{\beta}}$, $\quad \bar{\sigma}^\mu = (\mathbb{I}, -\vec{\sigma})$ $\quad [\varepsilon^{12} = -\varepsilon_{12} = 1]$

↳ $\bar{\sigma}^\mu$ used to map back:

$$\mathrm{tr}(\sigma^\mu \bar{\sigma}^\nu) = 2\eta^{\mu\nu}$$

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \rightarrow [\sigma^\mu\bar{\sigma}^\nu + \sigma^\nu\bar{\sigma}^\mu]_\alpha{}^\beta = 2\eta^{\mu\nu}\delta_\alpha{}^\beta$$

$$\gamma^\mu\gamma_\mu = 4\,\mathbb{I}_4 \Rightarrow \sigma^\mu_{\alpha\dot{\beta}}\,\bar{\sigma}_\mu^{\dot{\gamma}\delta} = 2\,\delta_\alpha{}^\delta\,\delta_{\dot{\beta}}{}^{\dot{\gamma}}$$

Then:

$$\frac{1}{2}\mathrm{tr}(X\bar{\sigma}^\mu) = \frac{1}{2}X_\nu \underbrace{\mathrm{tr}(\sigma^\nu\bar{\sigma}^\mu)}_{2\eta^{\nu\mu}} = X^\mu$$

we can write explicitly $\Lambda^\mu{}_\nu(M) = \frac{1}{2}\mathrm{tr}(\bar{\sigma}^\mu M \sigma_\nu M^\dagger)$

NB: Both M and $-M$ give the same Λ

↳ the homeomorphism is a 2:1 map

Therefore we have

$$X^\mu \rightarrow \Lambda^\mu{}_\nu X^\nu \qquad\qquad X \rightarrow M X M^\dagger$$

$$X^\mu \longrightarrow \frac{1}{2}\mathrm{tr}(M X M^\dagger\bar{\sigma}^\mu) = \frac{1}{2}\mathrm{tr}(M\sigma_\nu M^\dagger\bar{\sigma}^\mu)\,X^\nu = \Lambda^\mu{}_\nu X^\nu$$

⇒ SL(2,ℂ) SPINORS:

$$X_{\alpha\dot{\beta}} \overset{?}{=} \psi_\alpha \bar{\chi}_{\dot{\beta}} \quad \text{st} \quad \psi \rightarrow M\psi \text{ and } \bar{\chi} \rightarrow \bar{\chi}\bar{M}$$

NB: $(\sigma^{\mu\nu})^\dagger = \left(-\frac{i}{4}\right)\left(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu\right)^\dagger = -\frac{i}{4}\left(\bar{\sigma}^\nu\sigma^\mu - \bar{\sigma}^\mu\sigma^\nu\right) = \bar{\sigma}^{\mu\nu}$

The fundamental rep. of SL(2,ℂ):

$$\psi_\alpha \rightarrow \exp\left(-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\right)_\alpha{}^\beta\psi_\beta = M_\alpha{}^\beta\psi_\beta \qquad \left(\tfrac{1}{2},0\right): \text{ L-handed}$$

$$\bar{\chi}^{\dot{\alpha}} \rightarrow \exp\left(-\frac{i}{2}\omega_{\mu\nu}\bar{\sigma}^{\mu\nu}\right)^{\dot{\alpha}}{}_{\dot{\beta}}\bar{\chi}^{\dot{\beta}} = \bar{M}^{\dot{\alpha}}{}_{\dot{\beta}}\bar{\chi}^{\dot{\beta}} \qquad \left(0,\tfrac{1}{2}\right): \text{ R-handed}$$

$[\bar{M} = M^*$ on the left$]$

Now we can write: $M_\alpha{}^\beta M_\gamma{}^\delta \varepsilon_{\beta\delta} = \det(M)\,\varepsilon_{\alpha\gamma} = \varepsilon_{\alpha\gamma}$

$\Rightarrow$ $\varepsilon_{\alpha\beta}$ is an INVARIANT TENSOR of $SL(2,\mathbb{C})$
(use it to raise and lower indices)

Since $X^\mu X_\mu$ is $SO(1,3)$ invariant, $\psi^\alpha \psi_\alpha$ is invariant under $SL(2,\mathbb{C})$ (and $\bar{\chi}_{\dot\alpha}\bar{\chi}^{\dot\alpha}$)

$$\psi^\alpha = \varepsilon^{\alpha\beta}\psi_\beta \Rightarrow \psi^{\alpha\prime} = \psi^\beta (M^{-1})_\beta{}^\alpha$$

$$\bar{\chi}_{\dot\alpha} = \varepsilon_{\dot\alpha\dot\beta}\bar{\chi}^{\dot\beta} \Rightarrow \bar{\chi}_{\dot\alpha}{}' = \bar{\chi}_{\dot\beta}(M^{-1})^{\dot\beta}{}_{\dot\alpha}$$

Then $(\psi_\alpha)^* = \bar{\psi}^{\dot\alpha}$ (conjugate rep.) and $(\psi_\alpha)^\dagger = \bar{\psi}_{\dot\alpha}$

Recall: Spinors are Grassmann valued: $\psi_1\psi_2 = -\psi_2\psi_1$

$$\Rightarrow \psi\chi = \psi^\alpha\chi_\alpha = \varepsilon^{\alpha\beta}\psi_\beta\chi_\alpha = -\varepsilon^{\beta\alpha}\psi_\beta\chi_\alpha = -\varepsilon^{\alpha\beta}\psi_\alpha\chi_\beta = \varepsilon^{\alpha\beta}\chi_\beta\psi_\alpha = \chi^\alpha\psi_\alpha = \chi\psi$$

$$\bar{\chi}\bar{\psi} = \bar{\psi}\bar{\chi}$$

$$\Rightarrow (\psi\chi)^\dagger = (\psi^\alpha\chi_\alpha)^\dagger = (\chi_\alpha)^\dagger(\psi^\alpha)^\dagger = \bar{\chi}_{\dot\alpha}\bar{\psi}^{\dot\alpha} = \bar{\chi}\bar{\psi} = \bar{\psi}\bar{\chi}$$

Prove $(\chi\sigma^\mu\bar{\psi})^\dagger = \psi\sigma^\mu\bar{\chi}$ and $\chi\sigma^\mu\bar{\psi} = -\bar{\psi}\bar{\sigma}^\mu\chi$

1) $(\chi^\alpha\sigma^\mu_{\alpha\dot\alpha}\bar{\psi}^{\dot\alpha})^\dagger = (\bar{\psi}^{\dot\alpha})^\dagger(\sigma^\mu_{\alpha\dot\alpha})^\dagger(\chi^\alpha)^\dagger = \psi^\alpha\sigma^\mu_{\alpha\dot\alpha}\bar{\chi}^{\dot\alpha} = \psi\sigma^\mu\bar{\chi}$ $\square$

2) $\chi^\alpha\sigma^\mu_{\alpha\dot\alpha}\bar{\psi}^{\dot\alpha} = \varepsilon^{\alpha\beta}\chi_\beta\sigma^\mu_{\alpha\dot\alpha}\varepsilon^{\dot\alpha\dot\beta}\bar{\psi}_{\dot\beta} = \varepsilon^{\alpha\beta}\varepsilon^{\dot\alpha\dot\beta}\sigma^\mu_{\alpha\dot\alpha}(-\bar{\psi}_{\dot\beta}\chi_\beta) = (\bar{\sigma}^\mu)^{\dot\beta\beta}(-\bar{\psi}_{\dot\beta}\chi_\beta) = -(\bar{\psi}\bar{\sigma}^\mu\chi).$

Now we can build higher rep. of $SL(2,\mathbb{C})$ / Lorentz as products of $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$:

* $(\frac{1}{2},0)\otimes(0,\frac{1}{2})=(\frac{1}{2},\frac{1}{2})$ VECTOR REP:

$$\psi_\alpha \bar{\chi}_{\dot\alpha}=\frac{1}{2}\left(\psi\sigma_\mu\bar{\chi}\right)\sigma^\mu_{\alpha\dot\alpha}$$

* $(\frac{1}{2},0)\otimes(\frac{1}{2},0)=(0,0)\oplus(1,0)$ SCALAR + ANTISELF-DUAL REP:

$$\psi_\alpha\chi_\beta=\frac{1}{2}\varepsilon_{\alpha\beta}(\psi\chi)+\frac{1}{2}\left(\psi\sigma_{\mu\nu}\chi\right)(\sigma^{\mu\nu})_\alpha{}^\gamma\varepsilon_{\gamma\beta}$$

* $(0,\frac{1}{2})\otimes(0,\frac{1}{2})=(0,0)\oplus(0,1)$ SCALAR + SELF-DUAL REP:

MINIMAL ($\mathcal{N}=1$) SUSY ALGEBRA

$$[L_{\mu\nu}, Q_\alpha] = ①$$
$$[Q_\alpha, P_\mu] = ②$$
$$\{Q_\alpha, Q_\beta\} = ③$$
$$\{Q_\alpha, \bar{Q}_{\dot\beta}\} = ④$$
$$[Q_\alpha, F] = ⑤$$

Then:

① Q_α is a $SL(2,\mathbb{C})$ spinor and an oper. on $\mathcal{H}$: $Q|\psi\rangle$:

$$U = e^{-\frac{i}{2}\omega_{\mu\nu}L^{\mu\nu}} \qquad U|\psi\rangle \longrightarrow \langle\psi'|Q_\alpha|\psi'\rangle = \langle\psi|\underbrace{U^\dagger Q_\alpha U}_{=Q'_\alpha}|\psi\rangle$$

Lorentz

and $Q'_\alpha = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\right)_\alpha{}^\beta Q_\beta$

$$\Rightarrow \left(\mathbb{I} + \frac{i}{2}\omega_{\mu\nu}L^{\mu\nu}\right)Q_\alpha\left(\mathbb{I} - \frac{i}{2}\omega_{\mu\nu}L^{\mu\nu}\right) = \left(\mathbb{I} - \frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\right)_\alpha{}^\beta Q_\beta$$

$$\Leftrightarrow [Q_\alpha, L^{\mu\nu}] = (\sigma^{\mu\nu})_\alpha{}^\beta Q_\beta$$
$$[\bar{Q}^{\dot\alpha}, L^{\mu\nu}] = (\bar\sigma^{\mu\nu})^{\dot\alpha}{}_{\dot\beta}\bar{Q}^{\dot\beta}$$

$$\hookrightarrow \left(\tfrac{1}{2},0\right) \otimes \left((1,0)\oplus(0,1)\right) = \left(\tfrac{1}{2},0\right)$$

only allowed object!

② $[Q_\alpha, P^\mu] = c\,(\sigma^\mu)_{\alpha\dot\alpha}\bar{Q}^{\dot\alpha} \longrightarrow \left(\tfrac{1}{2},0\right)\otimes\left(\tfrac{1}{2},\tfrac{1}{2}\right) = \left(0,\tfrac{1}{2}\right)$

$\hookrightarrow$ use Jacobi id.: $0 = [P^\mu,[P^\nu,Q_\alpha]] + [P^\nu,[Q_\alpha,P^\mu]] + [Q_\alpha,\cancelto{0}{[P^\mu,P^\nu]}]$

$$\Leftrightarrow c\,\sigma^\nu_{\alpha\dot\alpha}[P^\mu,\bar{Q}^{\dot\alpha}] - c\,\sigma^\mu_{\alpha\dot\alpha}[P^\nu,\bar{Q}^{\dot\alpha}] = 0$$
$$\Leftrightarrow cc^*\,[\underbrace{\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu}]_\alpha{}^\beta Q_\beta = 0 \Leftrightarrow c = 0$$
$$|c|^2(\sigma^{\mu\nu})_\alpha{}^\beta Q_\beta = 0$$

③ $\{Q_\alpha, Q_\beta\} = c'(\sigma^{\mu\nu})_\alpha{}^\gamma\varepsilon_{\gamma\beta}L_{\mu\nu} \longrightarrow \left(\tfrac{1}{2},0\right)\otimes\left(\tfrac{1}{2},0\right) = (1,0)$

$\hookrightarrow [P,\{Q,Q\}] = 0 \rightarrow c' = 0.$

④ $\{Q_\alpha, \bar{Q}_{\dot\beta}\} = 2\,\sigma^\mu_{\alpha\dot\beta} P_\mu$ ← by convention!

$\hookrightarrow \left(\tfrac{1}{2},0\right)\otimes\left(0,\tfrac{1}{2}\right) = \left(\tfrac{1}{2},\tfrac{1}{2}\right)$

⑤ $[F, Q] = 0$ by Coleman-Mandula theorem.

NB: $\exists$ outer automorphism of the SUSY algebra

$$\begin{aligned} Q_\alpha &\to e^{iq\theta} Q_\alpha \\ \bar{Q}_{\dot\alpha} &\to e^{-iq\theta} \bar{Q}_{\dot\alpha} \end{aligned} \quad \Rightarrow \quad U(1)_R \text{ R-symmetry}$$

$\Downarrow$

$$U_R^\dagger Q_\alpha U_R = e^{iq\theta} Q_\alpha \quad \text{where} \quad U_R = e^{-i\theta R}$$

$$\Rightarrow (\mathbb{I} + i\theta R) Q_\alpha (\mathbb{I} - i\theta R) = (\mathbb{I} + iq\theta) Q_\alpha$$

$$\hookrightarrow \begin{aligned} [R, Q_\alpha] &= q Q_\alpha \\ [R, \bar{Q}_{\dot\alpha}] &= -q \bar{Q}_{\dot\alpha} \end{aligned}$$

EXTENDED SUSY ALGEBRA ($\mathcal{N} > 1$)

$\Rightarrow$ More SUSY charges: $Q^I_\alpha \quad I = 1, \dots, \mathcal{N}$

$\bar{Q}_{\dot\alpha I} \quad$ 〃

$$\begin{aligned}
\Rightarrow [L^{\mu\nu}, Q^I_\alpha] &= -(\sigma^{\mu\nu})_\alpha{}^\beta Q^I_\beta \\
[L^{\mu\nu}, \bar{Q}^{\dot\alpha}_I] &= -(\bar\sigma^{\mu\nu})^{\dot\alpha}{}_{\dot\beta} \bar{Q}^{\dot\beta}_I \\
[Q^I_\alpha, P_\mu] &= 0 \\
[\bar{Q}^{\dot\alpha}_I, P_\mu] &= 0 \\
\{Q^I_\alpha, \bar{Q}_{\dot\beta J}\} &= 2\sigma^\mu_{\alpha\dot\beta} P_\mu \delta^I{}_J \\
\{Q^I_\alpha, Q^J_\beta\} &= \varepsilon_{\alpha\beta} Z^{IJ} + L_{\mu\nu} (\sigma^{\mu\nu})_\alpha{}^\gamma \varepsilon_{\gamma\beta} \underbrace{K^{IJ}}_{=0 \text{ by Jacobi}}
\end{aligned}$$

"CENTRAL CHARGES" $\leftarrow Z^{IJ} = -Z^{JI}$ to have $\varepsilon_{\alpha\beta} Z^{IJ}$ symm.

$\Rightarrow$ use Jacobi to show $[Z, J] = 0$ (J is any gen)

Now there is a bigger automorphism:

$$Q^{I\prime} = R^I{}_J Q^J$$
$$\bar{Q}'_I = R^{*J}_I \bar{Q}_J$$

s.t. $RR^\dagger = \mathbb{I} \Rightarrow$ at most $U(\mathcal{N})$ R-symmetry.

SUSY REP.

1) $W^2 = W_\mu W^\mu$ IS NO LONGER A CASIMIR: $[Q, W^2] \neq 0$

$\Rightarrow$ since $W^2 = -m^2 s(s+1) \longrightarrow$ the spin will change whitin a multiplet

2) Every SUSY state has positive energy:

$$\langle\psi|\{Q^I_\alpha, \bar{Q}_{\dot\beta J}\}|\psi\rangle = 2\sigma^\mu_{\alpha\dot\beta}\,\delta^I{}_J \langle\psi|P_\mu|\psi\rangle$$

$$\parallel \leftarrow \bar{Q}_{\dot\alpha} = (Q_\alpha)^\dagger$$

$$\langle\psi|QQ^\dagger + Q^\dagger Q|\psi\rangle = \|Q^\dagger|\psi\rangle\|^2 + \|Q|\psi\rangle\|^2 \geq 0$$

$$\Rightarrow P_\mu = (E, \vec{p}) \longrightarrow E_\psi \geq 0 \ \forall |\psi\rangle \in \mathcal{H}$$

3) no. of bosons $= n_B = n_F =$ no. of fermions in a multiplet (unless $E=0$)

$\hookrightarrow (-)^F|B\rangle = |B\rangle$
$(-)^F|F\rangle = -|F\rangle$ $\longrightarrow \mathrm{tr}((-)^F) = \sum_{|\psi\rangle}\langle\psi|(-)^F|\psi\rangle = 0$ "Witten index"

$\hookrightarrow (-)^F Q|F\rangle = (-)^F|B\rangle = |B\rangle = Q|F\rangle = -Q(-)^F|F\rangle$

$$\Rightarrow \{Q, (-)^F\} = 0$$

$$\Rightarrow \mathrm{tr}(\{Q_\alpha, (-)^F\}\bar{Q}_{\dot\beta}) = \mathrm{tr}(Q_\alpha(-)^F\bar{Q}_{\dot\beta} + (-)^F Q_\alpha \bar{Q}_{\dot\beta}) =$$
$$= \mathrm{tr}((-)^F\{Q^I_\alpha, \bar{Q}_{\dot\beta J}\}) =$$
$$= 2P_{\alpha\dot\alpha}\,\delta^I{}_J\,\mathrm{tr}((-)^F) = 0$$

$$\rightarrow \mathrm{tr}((-)^F) = 0 \text{ unless } E = 0.$$

Therefore: $\mathrm{tr}(-)^F = \sum_{|B\rangle} \langle B|(-)^F|B\rangle + \sum_{|F\rangle} \langle F|(-)^F|F\rangle =$

$$= \sum_{|B\rangle} \underbrace{\langle B|B\rangle}_{=1} - \sum_{|F\rangle} \underbrace{\langle F|F\rangle}_{=1} = n_B - n_F = 0$$

MASSLESS REPRESENTATIONS

- $P_{\alpha\dot\alpha} = \begin{pmatrix} P_0+P_3 & P_1-iP_2 \\ P_1+iP_2 & P_0-P_3 \end{pmatrix} \longrightarrow$ rest frame $P_\mu = (E, \underbrace{0, 0,}_{} E)$

SO(2) little group

$\Rightarrow$ helicity label states

$$\{Q^I_\alpha, \bar{Q}_{\dot\alpha J}\} = 4E \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \delta^I{}_J$$

$Q^I_2, \bar{Q}_{\dot 2 J}$ do not play role for massless states

$\Rightarrow Q_2|\psi\rangle = 0 \;;\; \bar{Q}_{\dot 2}|\psi\rangle = 0$

since $\langle\psi|\{Q_2, \bar{Q}_{\dot 2}\}|\psi\rangle = 0$.

$\rightarrow$ Define $a^I = \frac{1}{2\sqrt{E}} Q^I_1 \;;\; a^\dagger_I = \frac{1}{2\sqrt{E}} \bar{Q}_{\dot 1 I} \Rightarrow \{a_I, a^\dagger_J\} = \delta_{IJ}$

$\Rightarrow$ this is the algebra of fermionic osc.

$\Rightarrow \{a_I, a_J\} = 0 = \{a^\dagger_I, a^\dagger_J\} \Rightarrow$ central charges needs to be 0

$\Rightarrow [L_{12}, Q^I_1] = -\frac{1}{2} Q^I_1 \qquad [L_{12}, \bar{Q}_{\dot 1 I}] = \frac{1}{2}\bar{Q}_{\dot 1 I}$

a_I lowers the helicity

$a^\dagger_I$ raises the hel.

Define a Clifford algebra:

$$a_I|\lambda\rangle = 0 \quad \forall I$$

Now act with $a^\dagger_I$: $\quad |\lambda\rangle$

$a^\dagger_I|\lambda\rangle = |\lambda+\frac{1}{2}\rangle_I \longrightarrow$ fundam. rep. labeled by I

$a^\dagger_I a^\dagger_J|\lambda\rangle = |\lambda+1\rangle_{[I,J]}$

$\vdots$

$a^\dagger_1 \dots a^\dagger_{\mathcal{N}}|\lambda\rangle = |\lambda + \frac{\mathcal{N}}{2}\rangle \rightarrow$ singlet!

How many states ?

$$|\lambda + \tfrac{k}{2}\rangle \longrightarrow \binom{\mathcal{N}}{k}$$

$$\downarrow$$

$$\text{no. of states} = \sum_{k=0}^{\mathcal{N}} \binom{\mathcal{N}}{k} = 2^{\mathcal{N}} = \underbrace{2^{\mathcal{N}-1}}_{\text{BOS.}} + \underbrace{2^{\mathcal{N}-1}}_{\text{FER.}}$$

NB: CPT transf. flips λ ! $\Rightarrow$ for physical multiplets I must add the CPT conjugate !

$\mathcal{N}=1$: $\lambda = 0$: $|0\rangle, |\tfrac{1}{2}\rangle \oplus |-\tfrac{1}{2}\rangle, |0\rangle \Rightarrow$ WZ multiplet ($\mathcal{N}=1$ matter)

ϕ : complex scalar

ψ_α : Weyl fermion on-shell

$\lambda = \tfrac{1}{2}$: $|\tfrac{1}{2}\rangle, |1\rangle \oplus |-1\rangle, |-\tfrac{1}{2}\rangle \Rightarrow$ $\mathcal{N}=1$ vector mult. (A_μ, ψ_α)

$\lambda = 1$: $|1\rangle, |\tfrac{3}{2}\rangle \oplus |-\tfrac{3}{2}\rangle, |-1\rangle \Rightarrow$ $\mathcal{N}=1$ gravitino (A_μ, χ_μ^α)

$\lambda = \tfrac{3}{2}$: $|\tfrac{3}{2}\rangle, |2\rangle \oplus |-2\rangle, |-\tfrac{3}{2}\rangle \Rightarrow$ $\mathcal{N}=1$ graviton ($h_{\mu\nu}$, ψ_μ^α)

$\mathcal{N}=2$: $\lambda = -\tfrac{1}{2}$: $|-\tfrac{1}{2}\rangle, |0\rangle_I, |\tfrac{1}{2}\rangle \oplus \underbrace{|-\tfrac{1}{2}\rangle, |0\rangle_I, |\tfrac{1}{2}\rangle}_{\text{"HALF-HYPER"}} \Rightarrow$ 2 doub. of compl scal + 2 Weyl sp. — HYPERMUL.

$\lambda = 0$: $|0\rangle, |\tfrac{1}{2}\rangle_I, |1\rangle \oplus |-1\rangle, |-\tfrac{1}{2}\rangle_I, |0\rangle \Rightarrow A_\mu$, 2_R of ψ_α, ϕ_c — VECTOR MUL

$\vdots$

$\lambda = 1$: $|1\rangle, |\tfrac{3}{2}\rangle_I, |2\rangle \oplus |-2\rangle, |-\tfrac{3}{2}\rangle_I, |-1\rangle \Rightarrow$ $\mathcal{N}=2$ SUGRA MUL graviton, gravitino doublet, gauge bos.

$\Rightarrow$ $\mathcal{N}=2$ Hypermult. $= 2 \times$ ($\mathcal{N}=1$ WZ mult.)

$\mathcal{N}=2$ Vector mul. $= 1\times$($\mathcal{N}=1$ vector mul) $+ 1$ ($\mathcal{N}=1$ WZ mult)

$\mathcal{N}=2$ SUGRA mult $= 1\times$ ($\mathcal{N}=1$ SUGRA) $+ 1\times$ ($\mathcal{N}=1$ gravitino)

CPT self-conj: no need to add anyth.

$\mathcal{N}=4$:

$$\lambda=-1 \Rightarrow |-1\rangle, |-\tfrac{1}{2}\rangle_I, |0\rangle_{[IJ]}, |\tfrac{1}{2}\rangle_I, |1\rangle$$

for any $|\lambda\rangle$ $(\lambda \neq 1)$ $\Rightarrow$ we get $\lambda > 1$ $\Rightarrow$ NOT A GAUGE TH.

$\Rightarrow$ $\mathcal{N}=4$ vector mult is the max. SUSY vector

Q: what happens with $\mathcal{N}=3$?

$$\lambda=-\tfrac{1}{2}: \quad |-\tfrac{1}{2}\rangle, |0\rangle_I, |\tfrac{1}{2}\rangle_{IJ}, |1\rangle \ \oplus \ |-1\rangle, |-\tfrac{1}{2}\rangle_I, |0\rangle_{IJ}, |\tfrac{1}{2}\rangle$$

$\Rightarrow$ same dof as $\mathcal{N}=4$ vector mult.

$\Rightarrow$ MAX. SUSY Gravity in 4D: $\mathcal{N}=8$

$$|-2\rangle \quad |-\tfrac{3}{2}\rangle_I \quad |-1\rangle_{[IJ]} \quad |-\tfrac{1}{2}\rangle_{[IJK]} \quad |0\rangle_{[IJKL]} \quad |\tfrac{1}{2}\rangle_{[IJK]} \quad |1\rangle_{[IJ]} \quad |\tfrac{3}{2}\rangle_I \quad |2\rangle$$

$$1 + \binom{8}{1}=8 + \binom{8}{2}=28 + \binom{8}{3}=28 + \binom{8}{4}=70 + \ldots$$

MASSIVE REPRES $[\mathcal{N}=1]$

→ Go to rest frame: $P_\mu = (m, \underbrace{\vec{0}}_{SO(3) \text{ little group}})$

↳ $\{Q_\alpha, \bar{Q}_{\dot\beta}\} = 2\sigma^\mu_{\alpha\dot\beta} P_\mu = 2m \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$

$\{Q_\alpha, Q_\beta\} = 0 = \{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\}$

Now define the vacuum: $|\Omega\rangle = |m; s, s_3\rangle$

and remember

$\varepsilon_{ijk} L^{ij}$

$$\begin{aligned}
[Q_\alpha, R_k] &= (\sigma_{ij})_\alpha{}^\beta Q_\beta \\
[Q_\alpha, R_k] &= \tfrac{1}{2}(\sigma_k)_\alpha{}^\beta Q_\beta \\
[\bar{Q}^{\dot\alpha}, R_k] &= \tfrac{1}{2}(\bar{\sigma}_k)^{\dot\alpha}{}_{\dot\beta} \bar{Q}^{\dot\beta}
\end{aligned} \Bigg\} \quad \begin{matrix} Q_1 & Q_2 \\ \bar{Q}_{\dot 1} & \bar{Q}_{\dot 2} \\ \uparrow & \uparrow \end{matrix}$$

lower / raise spin

$\Rightarrow \quad a_\alpha = \frac{1}{\sqrt{2m}} Q_\alpha \; ; \quad a^\dagger_\alpha = \frac{1}{\sqrt{2m}} \bar{Q}^{\dot\alpha} \quad \Rightarrow$ <u>two</u> ferm. oscillators

↳ $|\Omega\rangle = |m; s=0, s_3=0\rangle$

$\Downarrow$

$$a^\dagger_\alpha |\Omega\rangle = |m; \tfrac{1}{2}, \pm\tfrac{1}{2}\rangle$$
$$a^\dagger_1 a^\dagger_2 |\Omega\rangle = |m; 0, 0\rangle$$

Now: $|\Omega\rangle = |m; j=s, j_3=-s,\dots,s\rangle \Rightarrow 2s+1$ states

↳ $a^\dagger_\alpha |\Omega\rangle = |m; j = s+\frac{1}{2}, \dots\rangle \Rightarrow \begin{cases} 2(s+\frac{1}{2})+1 & \text{states} \\ 2(s-\frac{1}{2})+1 & \text{states} \end{cases}$

$a^\dagger_1 a^\dagger_2 |\Omega\rangle = |m; j=s, j_3=\dots\rangle \Rightarrow 2s+1$ states

Comments:

For $s=0$ multiplet (the $\mathcal{N}=1$ massive matter mult.)

$\hookrightarrow$ $2_B + 2_F$ is the same as the $\mathcal{N}=1$ massless mult.

$\Downarrow$

we can add a mass term

w/o adding d.o.f.'s

Then we have these dofs — starting Clifford vacua

SPIN	matter $\|0\rangle$	vector $\|\frac{1}{2}\rangle$	$\|1\rangle$	$\|\frac{3}{2}\rangle$
0	2	1		
½	1 (1 Weyl)	2 (2 Weyl)	1	
1		1 (mass. vector)	2	1
3/2			1	2
2				1

$\Rightarrow$ NB massive vector mult:

$(1_B + 4_F + 3_B)$ dof =

= 1 MASSLESS VECTOR

+

1 MASSLESS CHIRAL

[Higgs Mechanism]

$\Rightarrow$ Massive $\mathcal{N}>1$ multiplets:

$$\{Q^I_\alpha, \bar{Q}_{\dot\beta J}\} = 2m \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \delta^I{}_J$$

$$\left.\begin{aligned}\{Q^I_\alpha, Q^J_\beta\} &= \varepsilon_{\alpha\beta} Z^{IJ} \\ \{\bar{Q}^{\dot\alpha}{}_I, \bar{Q}^{\dot\beta}{}_J\} &= \varepsilon^{\dot\alpha\dot\beta} \bar{Z}^{IJ}\end{aligned}\right\} \text{ use } U(\mathcal{N}) \text{ to have } Z^{IJ} = \begin{pmatrix} 0 & z_1 & & & \\ -z_1 & 0 & & & \\ & & 0 & z_2 & \\ & & -z_2 & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix}$$

If $\mathcal{N}$ odd, we have an extra 0.

Define: $a^1_\alpha = \frac{1}{\sqrt{2}}\left(Q^1_\alpha + \varepsilon_{\alpha\beta}(Q^2_\beta)^\dagger\right)$ $\quad a^2_\alpha = \frac{1}{\sqrt{2}}\left(Q^3_\alpha + \varepsilon_{\alpha\beta}(Q^4_\beta)^\dagger\right)$

$b^1_\alpha = \frac{1}{\sqrt{2}}\left(Q^1_\alpha - \varepsilon_{\alpha\beta}(Q^2_\beta)^\dagger\right)$ $\quad b^2_\alpha = \frac{1}{\sqrt{2}}\left(Q^3_\alpha - \varepsilon_{\alpha\beta}(Q^4_\beta)^\dagger\right)$ $\quad \dots$

s.t. a^r_α, b^r_α $\quad r,s = 1, \dots, \frac{\mathcal{N}}{2}$.

Then: $\left.\begin{array}{l}\{a^r_\alpha, (a^s_\beta)^\dagger\} = (2m - Z_r)\,\delta_{rs}\,\delta_{\alpha\beta} \\ \{b^r_\alpha, (b^s_\beta)^\dagger\} = (2m + Z_r)\,\delta_{rs}\,\delta_{\alpha\beta}\end{array}\right\}$ $2 \times \mathcal{N}$ HARM. OSCILL.

$$\{a, a\} = \{b, b\} = \{a^\dagger, a^\dagger\} = \{b^\dagger, b^\dagger\} = 0$$

COMMENT

no negative norm states.

For quant. consistence: $2m \geq |Z_r|$ BPS bound

* If all Z_r | $2m > |Z_r| \Rightarrow 2\mathcal{N}$ fermionic:

→ total no. of states (start with $|\Omega\rangle = |m; s\rangle$)

$$= (2s+1) \sum_{k=0}^{2\mathcal{N}} \binom{2\mathcal{N}}{k} = 2^{2\mathcal{N}}(2s+1)$$

$$\hookrightarrow n_B = n_F = 2^{2\mathcal{N}-1}(2s+1)$$

Let's do $\mathcal{N}=2$:

⇒ start with $\boxed{s=0}$:

$\|0\rangle$	→	1	spin 0
$A\|0\rangle$	→	$\binom{4}{1} = 4$	spin $\frac{1}{2}$
$AA\|0\rangle$	→	$\binom{4}{2} = 6$	spin 1 or 0
$AAA\|0\rangle$	→	$\binom{4}{3} = 4$	
$AAAA\|0\rangle$	→	$\binom{4}{4} = 1$	

⇓

VECTOR (NON BPS) MULTIPLET

$$\underset{\displaystyle 3_B}{\underset{\Downarrow}{1 \times (s=1)}} + \underset{\displaystyle 4 \times 2_F}{\underset{\Downarrow}{4 \times (s=\tfrac{1}{2})}} + \underset{\displaystyle 5_B}{\underset{\Downarrow}{5 \times (s=0)}} = 8_B + 8_F$$

⇒ $\mathcal{N}=2$ Higgs mechanism for Non BPS vector:

$$8_B + 8_F = 1 \times (\mathcal{N}=2 \text{ massless vector}) + 1 \times (\mathcal{N}=2 \text{ massless hyper})$$

→ For $\mathcal{N}=2$ BPS: $\quad 2m = Z \quad$ (BPS multiplet)

We will have only $\{ b_\alpha, (b_\beta)^\dagger \} \neq 0 \Rightarrow 2$ oscillators

↳ we can make $2^{\mathcal{N}}(2s+1)$ states

⇒ $\mathcal{N}=2$ BPS vector multiplet:

$$|\tfrac{1}{2}\rangle \quad b^\dagger |\tfrac{1}{2}\rangle \quad b_1^\dagger b_2^\dagger |\tfrac{1}{2}\rangle \rightarrow 1\times(s=1) + 2\times(s=\tfrac{1}{2}) + 1\times(s=0) = 4_B + 4_F$$

↳ BPS bound shortens the multiplet

→ These $4_B + 4_F$ dof = dof of a massless $\mathcal{N}=2$ vector mult.

↳ 2nd version of BPS Higgs mechanism:

THE MASSLESS VECTOR ATE THE ONE
REAL SCALAR OF ITS OWN MULTIPLET!!!

NB: If you want to be able to add a mass to a massless hyper you need to have a central charge $Z \neq 0$ and $Z = 2m$. If not, there will be too many dofs.

⇒ For $\mathcal{N}=4$ vector multiplet there is NO possible mass term!
($\mathcal{N}=4$ vector is self-conjugate)

SUPERSPACE

Realise SUSY rep. on fields. $\Rightarrow$ SUPERFIELDS

$\hookrightarrow$ enlarge Minkowski space x^μ by adding Grassmann valued $\theta_\alpha, \bar{\theta}^{\dot{\alpha}}$

generated by P_μ — generated by $Q_\alpha, \bar{Q}_{\dot{\alpha}}$

$\Downarrow$

$4_B + 4_F$ coords for $\mathcal{N}=1$ superspace

Properties of Grassmann Variables:

We have a single variable η (recall $\eta^2 = 0$)

$$\hookrightarrow f(\eta) = \sum_{n=0}^{\infty} f^{(n)}(\eta)\Big|_{\eta=0} \eta^n = \underbrace{f(0)}_{f_0} + \underbrace{f'(0)}_{f_1}\eta + \cancel{f''(0)\eta^2}^{\,0} + \dots$$

* derivatives: $\dfrac{df(\eta)}{d\eta} = f_1$

* integration: $\displaystyle\int d\eta \frac{df}{d\eta} = 0 \Rightarrow \int d\eta = 0$ and $\displaystyle\int d\eta\, \eta = 1$

$$\hookrightarrow \int d\eta\, f(\eta) = \int d\eta \left(f_0 + \eta f_1\right) = f_1 = \frac{df(\eta)}{d\eta}\Big|_{\eta=0}$$

Now we have 4 Grassmann coordinates:

$$\theta\theta = \theta^\alpha\theta_\alpha = \varepsilon^{\alpha\beta}\theta_\beta\theta_\alpha = 2\,\theta_2\theta_1 = -2\,\theta_1\theta_2$$

$$\bar{\theta}\bar{\theta} = \bar{\theta}_{\dot{\alpha}}\bar{\theta}^{\dot{\alpha}} = \varepsilon_{\dot{\alpha}\dot{\beta}}\bar{\theta}^{\dot{\beta}}\bar{\theta}^{\dot{\alpha}} = 2\bar{\theta}_{\dot{1}}\bar{\theta}_{\dot{2}} = -2\bar{\theta}_{\dot{2}}\bar{\theta}_{\dot{1}}$$

$$\hookrightarrow \theta^\alpha\theta^\beta = -\frac{1}{2}\varepsilon^{\alpha\beta}\theta\theta \qquad \theta_\alpha\theta_\beta = \frac{1}{2}\varepsilon_{\alpha\beta}\theta\theta$$

$$\bar{\theta}^{\dot{\alpha}}\bar{\theta}^{\dot{\beta}} = \frac{1}{2}\varepsilon^{\dot{\alpha}\dot{\beta}}\bar{\theta}\bar{\theta} \qquad \bar{\theta}_{\dot{\alpha}}\bar{\theta}_{\dot{\beta}} = -\frac{1}{2}\varepsilon_{\dot{\alpha}\dot{\beta}}\bar{\theta}\bar{\theta}$$

$$(\theta\sigma^\mu\bar{\theta})(\theta\sigma^\nu\bar{\theta}) = \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}\,\eta^{\mu\nu}$$

$$(\theta\psi)(\theta\chi) = -\frac{1}{2}\theta\theta(\psi\chi)$$

Derivatives: $\partial_\alpha\theta^\beta = \dfrac{\partial\theta^\beta}{\partial\theta^\alpha} = \delta_\alpha{}^\beta$

The most general superfield:

- arbitrary f. of $x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}}$:

$$\Phi(x,\theta,\bar{\theta}) = \phi(x) + \theta\psi(x) + \bar{\theta}\bar{\chi}(x) + \theta\theta M(x) + \bar{\theta}\bar{\theta} N(x) +$$
$$+ \theta\sigma^\mu\bar{\theta} A_\mu(x) + \theta\theta\bar{\theta}\bar{\lambda}(x) + \bar{\theta}\bar{\theta}\theta\mu(x) + \theta\theta\bar{\theta}\bar{\theta} D(x)$$

$\Rightarrow$ The superfields terminates with 4 fermion

A superfield is a collection (= a multiplet) of ordinary fields

$\Rightarrow$ Φ has too many dofs to be an irrep of $\mathcal{N}=1$ alg.

etc.

ii) $\partial_\alpha \Phi$ is not a good superf.:

$$\delta(\partial_\alpha \Phi) = i(\varepsilon Q + \bar{\varepsilon}\bar{Q})\,\partial_\alpha \neq i\,\partial_\alpha(\varepsilon Q + \bar{\varepsilon}\bar{Q})\Phi = \partial_\alpha(\delta\Phi)$$

$\Rightarrow$ Define a COVARIANT DERIVATIVE:

$$D_\alpha = \partial_\alpha + i(\sigma^\mu \bar{\theta})_\alpha \partial_\mu \;\rightarrow\; \{D_\alpha, Q_\beta\} = 0 = \{D_\alpha, \bar{Q}_{\dot\beta}\}$$
$$\bar{D}_{\dot\alpha} = -\partial_{\dot\alpha} - i(\theta\sigma^\mu)_{\dot\alpha}\partial_\mu \;\rightarrow\; \{\bar{D}_{\dot\alpha}, Q_\beta\} = 0 = \{\bar{D}_{\dot\alpha}, \bar{Q}_{\dot\beta}\}$$

and $\{D_\alpha, \bar{D}_{\dot\alpha}\} = 2i(\sigma^\mu)_{\alpha\dot\alpha}\partial_\mu$ $\quad [\{D_\alpha, D_\beta\} = 0 = \{\bar{D}_{\dot\alpha}, \bar{D}_{\dot\beta}\}]$

Note that $(D_\alpha)^\dagger = \bar{D}_{\dot\alpha}$.

SHORTENING THE SUPERFIELD

* CHIRAL SUPERFIELD: $\quad \bar{D}_{\dot\alpha}\Phi = 0 \quad$ (ANTICH.: $D_\alpha \bar{\Phi} = 0$)

i) $\bar{D}_{\dot\alpha}\theta_\alpha = 0$

ii) def $y^\mu = x^\mu + i\,\theta\sigma^\mu\bar{\theta} \Rightarrow \bar{D}_{\dot\alpha} y^\mu = 0.$

$$\Phi(y,\theta) = \phi(y) + \sqrt{2}\,\theta\psi(y) - \theta\theta F(y)$$

$$\Phi(x,\theta,\bar\theta) = \phi(x) + \sqrt{2}\,\theta\psi(x) - \theta\theta F(x) + i\,\theta\sigma^\mu\bar\theta\,\partial_\mu\phi(x) - \frac{i}{\sqrt{2}}\theta\theta\,\partial_\mu\psi(x)\sigma^\mu\bar\theta - \frac{1}{4}\theta\theta\bar\theta\bar\theta\,\Box\phi(x).$$

$$\Rightarrow \quad \delta\phi = \sqrt{2}\,\varepsilon\psi$$
$$\delta\psi_\alpha = \sqrt{2}\,i(\sigma^\mu\bar\varepsilon)_\alpha\partial_\mu\phi - \sqrt{2}\,\varepsilon_\alpha F$$
$$\delta F = i\sqrt{2}\,\partial_\mu\psi\sigma^\mu\bar\varepsilon$$

NB: $[\phi] = 1 \rightarrow [\Phi] = 1$

$[\psi] = \frac{3}{2} \Rightarrow [\theta] = -\frac{1}{2}$

$[F] = 2$ — auxiliary field: $\mathcal{L}_{KIN} = F\bar{F}$

What if we set $\underline{F = 0}$?

$\delta F = 0 \Rightarrow \bar{\partial}\psi = 0$ e.o.m. of Weyl fermion

We are left w/ :

$$\left.\begin{array}{l} \phi \longrightarrow 2\ \mathbb{R}\ \text{bos} \\ \psi_\alpha \xrightarrow{\text{on shell}} 2\ \mathbb{R}\ \text{ferm} \end{array}\right\}$$ MASSLESS MATTER MULTIPLET

Integration: $\underbrace{\int d\theta^1 d\theta^2\ \theta^2\theta^1}$ $= 1$

$$\int d^2\theta\ \theta\theta = 1 \quad \text{and} \quad \int d^2\bar\theta\ \bar\theta\bar\theta = 1$$

$$\int d^2\theta = \frac{1}{4}\varepsilon^{\alpha\beta}\partial_\alpha\partial_\beta \qquad \int d^2\bar\theta = -\frac{1}{4}\varepsilon^{\dot\alpha\dot\beta}\partial_{\dot\alpha}\partial_{\dot\beta}$$

$$\Rightarrow \int d^2\theta d^2\bar\theta\ \theta\theta\,\bar\theta\bar\theta = 1 = \int d^4\theta\ \theta\theta\bar\theta\bar\theta = 1$$

SUSY INVARIANT ACTIONS

$$\delta_{susy}\left[\int d^4x \int d^2\theta d^2\bar\theta\ F(x,\theta,\bar\theta)\right] = \int d^4x\, d^2\theta d^2\bar\theta\ \delta_{susy} F =$$

$$= \int d^4x\, d^2\theta\, d^2\bar\theta\ i\left(\varepsilon^\alpha\partial_\alpha(\dots) + \bar\varepsilon_{\dot\alpha}\partial^{\dot\alpha}(\dots) + \partial_\mu(\dots)\right) =$$

$= 0\ ! \Rightarrow$ SUPERS. IS GUARANTEED!

→ WESS-ZUMINO MODEL:

$$\bar D_{\dot\alpha}\Phi = 0 \Rightarrow [\Phi] = 1\quad [\theta] = -\frac{1}{2} \longrightarrow [d\theta] = \frac{1}{2} \longrightarrow [d^4\theta] = 2$$

$$\hookrightarrow S = \int d^4x\, d^2\theta d^2\bar\theta\ \underbrace{\bar\Phi(y,\theta)\,\Phi(y,\theta)}$$

i) $\mathbb{R}$
ii) $[\bar\Phi\phi] = 2$
iii) susy

$$\Rightarrow \int d\theta_\alpha = \partial_\alpha \longrightarrow D_\alpha = \partial_\alpha + \dots \cancel{\partial_\mu(\dots)}$$ no boundary terms

$$\longrightarrow \int d^4x\, d^2\bar\theta\ \bar\Phi D^2 \Phi$$

NB $D_\alpha\bar\Phi = 0$!

Then $\longrightarrow \int d^4x\, d^2\bar\theta\ \Phi \left[-F(x) - \frac{i}{\sqrt{2}}\,\partial_\mu \psi(x)\sigma^\mu\bar\theta - \frac{1}{4}\bar\theta\theta\,\Box\,\phi(x)\right] =$

$= \int d^4x\ \bar D^2 \left\{ \Phi [\dots] \right\} =$

$= \int d^4x \left[\underbrace{\bar F(x) F(x)}_{\text{auxiliary derivative}} + \underbrace{i\,\partial_\mu \psi(x)\,\sigma^\mu\,\bar\psi(x)}_{\text{Weyl fermion}} - \underbrace{\bar\phi(x)\,\Box\,\phi(x)}_{\text{scalar field}} \right] = S_{kin}$

What about interactions ?

$[d^4\theta] = 2$
$[\Phi] = 1$ $\longrightarrow \int d^4x\ K(\phi,\bar\phi)$

$K(\phi,\bar\phi) = \sum_{n,m} c_{nm}\, \Phi^n \bar\Phi^m \longrightarrow c_{nm} = \Lambda^{2-n-m}$

NOT interaction of a renorm. th. !!!

NB : $\Phi + \bar\Phi$ leads to $\int d^4x\ \Box\phi(x) \rightarrow$ total derivative.

Suppose though:

$\underbrace{K(\Phi,\bar\Phi)}_{\text{"KÄHLER POT."}} + \underbrace{\Lambda(\Phi) + \bar\Lambda(\bar\Phi)}_{\text{total derivatives} \Rightarrow \text{SYMMETRY !!!}}$ "KÄHLER TRANSF."

NB: $\int d^4x\, d^2\theta\, d^2\bar\theta\ \Psi(x,\theta,\bar\theta) = \int d^4x\, d^2\theta\ \underbrace{\bar D^2 \Psi(x,\theta,\bar\theta)}$

$\bar D_{\dot\alpha}(\bar D^2\Psi) = 0 \Rightarrow \bar D^2\Psi$ is a chiral superf.

$\Rightarrow$ we can just use CHIRAL SUPERF !

$\hookrightarrow S = \int d^4x\, d^2\theta\ W(\Phi)$ $\longrightarrow$ funct. of chiral superf.

$\bar D_{\dot\alpha} W(\Phi) = 0 = \frac{\partial W}{\partial \Phi}\underbrace{(\bar D_{\dot\alpha}\Phi)}_{=0} + \frac{\partial W}{\partial \bar\Phi}\underbrace{(\bar D_{\dot\alpha}\bar\Phi)}_{=0}$

$\Leftrightarrow \frac{\partial W}{\partial \bar\Phi} = 0 \Rightarrow$ HOLOMORPHIC

$W(\Phi) :=$ "superpotential". It includes all the renormalizable interactions

$$\downarrow$$

$$\mathcal{L}_{INT} = \int d^2\theta \, W(\Phi) + \int d^2\bar{\theta} \, \overline{W}(\bar{\Phi})$$

NB: $\left[\int d^2\theta\right] = 1 \Rightarrow \left[W(\Phi)\right] = 3 \Rightarrow W(\Phi) = \ldots + \ldots \Phi^3$ to be renorm.

e.g.: $W(\Phi) = \frac{1}{2} m \Phi^2 + \frac{1}{3!} g \Phi^3$

$$\int d^2\theta \, W(\Phi) = -m \phi F - m \psi\psi - \frac{1}{2} g \phi^2 F - g \phi \psi\psi$$

$$= -\left(\frac{\partial W}{\partial \phi}\right) F - \frac{1}{2} \frac{\partial^2 W}{\partial\phi\partial\phi} \psi\psi$$

Therefore:

$$\mathcal{L}_{TOT} = \mathcal{L}_{KIN} + \mathcal{L}_{INT} = \ldots + F\bar{F} - \left(\frac{\partial W}{\partial \phi}\right) F - \frac{1}{2} \frac{\partial^2 W}{\partial\phi\partial\phi} \psi\psi - \left(\frac{\partial \overline{W}}{\partial \bar{\phi}}\right) \bar{F} -$$

$$- \frac{1}{2} \frac{\partial^2 \overline{W}}{\partial\bar{\phi}\partial\bar{\phi}} \overline{\psi\psi} =$$

$$= \ldots + \left| F + \frac{\partial W}{\partial \phi} \right|^2 - \left| \frac{\partial W}{\partial \phi} \right|^2 + \mathcal{L}_{YUKAWA}$$

<u>e.o.m.</u>:

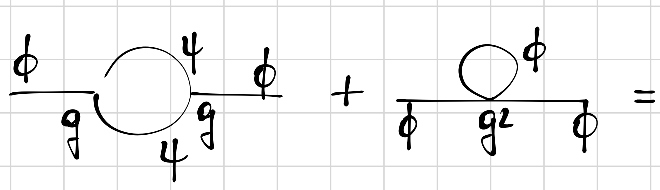
$$\bar{F} = -\frac{\partial W}{\partial \phi} \qquad F = -\frac{\partial \overline{W}}{\partial \bar{\phi}} \quad \longrightarrow \quad \mathcal{L} = \partial_\mu \phi \, \partial^\mu \bar{\phi} + \bar{\psi} \bar{\sigma}^\mu \partial_\mu \psi - V_{scalar} + \mathcal{L}_{YUK}$$

$$\hookrightarrow V_{scalar} = F\bar{F} = \left| \frac{\partial W}{\partial \phi} \right|^2 \geq 0 \quad \text{SUSY!}$$

NB: Let's use $W(\phi) = \frac{1}{2} m \Phi^2 + \frac{1}{3!} g \Phi^3$

$\hookrightarrow$ $|m|^2 \phi\bar{\phi} + m \psi\psi + \bar{m} \overline{\psi\psi} \Rightarrow$ fields in the same mult. have the same MASS!

Moreover: $g \psi\psi\phi + g^2 (\phi\bar{\phi})^2 \longrightarrow$ SAME COUPLING CONST!

 $= 0 \cdot \Lambda^2 + \# \ln \Lambda$

$\Rightarrow$ THEOREM: no Λ^2 in SUSY!

R-Symmetry ?

$$R[\theta] = 1 \longrightarrow R[d\theta] = -1$$

$$R[\mathcal{L}] = 0 \Rightarrow \boxed{R[W] = 2}$$ → leads to NON RENORM. THEOREM.

$$\hookrightarrow \int d^2\theta\, d^2\bar{\theta}\; K(\Phi, \bar{\Phi}) \longrightarrow R[K] = 0 \Rightarrow C_{nm} = \delta_{nm}.$$

Then we have:

$$\mathcal{L}_{MAT} = \int d^4\theta\; K(\Phi, \bar{\Phi}) + \int d^2\theta\; W(\Phi) + h.c.$$

$\Rightarrow$ allow for more Φ_i , $i = 1, \dots, n$ $\Rightarrow$ NON LINEAR SIGMA MODEL

* $V_{scalar}(\phi_i, \bar{\phi}_j) = \sum_{i=1}^{n} \left| \frac{\partial W}{\partial \phi_i} \right|^2 \geq 0$

* $\mathcal{L}_{KIN} = g_{i\bar{j}}(\phi, \bar{\phi})\, \partial \phi^i \partial \bar{\phi}^{\bar{j}} + \dots$ where $g_{i\bar{j}} = \frac{\partial^2 K(\Phi, \bar{\Phi})}{\partial \phi^i \partial \bar{\phi}^{\bar{j}}}$: "KÄHLER METRIC"

GAUGE INTERACTIONS

* REAL (VECTOR) SUPERF. : $V(x, \theta, \bar{\theta}) = V^\dagger(x, \theta, \bar{\theta}) = \bar{V}(x, \theta, \bar{\theta})$

$$\begin{aligned} V(x, \theta, \bar{\theta}) = {} & C(x) + i\theta \chi(x) - i\bar{\theta}\bar{\chi}(x) + \theta\sigma^\mu\bar{\theta} A_\mu(x) - \\ & - \frac{i}{2}\theta\theta\left(M(x) + iN(x)\right) - \frac{i}{2}\bar{\theta}\bar{\theta}\left(M(x) - iN(x)\right) + \\ & + i\theta\theta\bar{\theta}\left(\bar{\lambda}(x) + \frac{i}{2}\bar{\sigma}^\mu \partial_\mu \chi(x)\right) - i\bar{\theta}\bar{\theta}\theta\left(\lambda(x) + \frac{i}{2}\sigma^\mu \partial_\mu \bar{\chi}(x)\right) + \\ & + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}\left(D(x) - \frac{1}{2}\Box C(x)\right). \end{aligned}$$

$$\Downarrow$$

$8_F + 8_B$ dof → too many! (I would expect $2_B + 2_F$)

We can gauge fix it $\longrightarrow 4_B + 4_F \xrightarrow{\text{on-shell}} 2_B + 2_F$

NOTE: $\Phi + \bar{\Phi}$ is a real superf. $\Rightarrow V \to V + I + \bar{I}$ where $\bar{D}_{\dot{\alpha}} I = 0$, $D_\alpha \bar{I} = 0$

$$\Rightarrow I = \zeta + \theta \psi_\zeta + \theta\theta F_\zeta$$

$$\hookrightarrow \quad \begin{aligned} C &\to C + 2\,\mathrm{Re}\,\zeta \\ \chi &\to \chi + i\sqrt{2}\,\psi_\zeta \\ M &\to M - 2\,\mathrm{Im}\,F_\zeta \\ N &\to N + 2\,\mathrm{Re}\,F_\zeta \\ D &\to D \\ \lambda &\to \lambda \\ A_\mu &\to \boxed{A_\mu - 2\,\partial_\mu(\mathrm{Im}\,\zeta)} \end{aligned}$$

$\longrightarrow$ Choose I s.t. $C = 0$, $\chi = 0$, $M = 0$, $N = 0$

$\longrightarrow$ it's a gauge transform!

Choose then a gauge (WZ gauge) s.t.:

$$V_{WZ}(x,\theta,\bar{\theta}) = -i\,\theta\theta\,\bar{\theta}\bar{\lambda}(x) + i\,\bar{\theta}\bar{\theta}\,\theta\lambda(x) + \theta\sigma^\mu\bar{\theta}\,A_\mu(x) + \frac{1}{2}\,\theta\theta\bar{\theta}\bar{\theta}\,D(x)$$

$$\Rightarrow 4_F + 4_B \text{ dof !}$$

$$\hookrightarrow \quad V^2_{WZ}(x,\theta,\bar{\theta}) = \frac{1}{2}\,\theta\theta\bar{\theta}\bar{\theta}\,A_\mu(x)A^\mu(x)$$

$$V^{n>2}_{WZ}(x,\theta,\bar{\theta}) \equiv 0.$$

Susy transf: $\delta_{susy} V_{WZ}(x,\theta,\bar{\theta}) \not\sim V_{WZ} \longrightarrow$ not in the same gauge

$\Rightarrow$ FIELD STRENGHT

Abelian: $\quad \sim F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \Leftrightarrow A_\mu \to A_\mu + \partial_\mu \zeta \Rightarrow F_{\mu\nu} \to F_{\mu\nu}$

$$\Rightarrow W_\alpha(x,\theta,\bar{\theta}) = -\frac{1}{4}\,\bar{D}\bar{D}\,D_\alpha V(x,\theta,\bar{\theta}) \qquad \left[\bar{W}_{\dot{\alpha}} = -\frac{1}{4}\,DD\bar{D}_{\dot{\alpha}} V\right]$$

Gauge inv. for $V \to V + I + \bar{I}$?

$D_\alpha \bar{I} = 0$

$\{\bar{D}_{\dot{\alpha}}, D_\alpha\} \sim \partial_{\dot{\alpha}\alpha}$

$$W_\alpha \longrightarrow -\frac{1}{4}\,\bar{D}\bar{D}\,D_\alpha(V + I + \bar{I}) = W_\alpha - \frac{1}{4}\,\bar{D}\bar{D}\,D_\alpha I = W_\alpha - \frac{1}{4}\,\bar{D}^{\dot{\alpha}}\,\partial_{\dot{\alpha}\alpha} I =$$

$$= W_\alpha.$$

NB: $\bar{D}_{\dot\alpha} W_\alpha = 0 \longrightarrow W_\alpha$ is CHIRAL!
$(D_\alpha \bar{W}_{\dot\alpha} = 0)$

Remember that $y^\mu = x^\mu + i\,\theta\sigma^\mu\bar\theta \Rightarrow \bar{D}_{\dot\alpha} y^\mu = 0$

$$\Rightarrow W_\alpha(y,\theta) = -i\lambda_\alpha(y) + \theta_\alpha D(y) + i(\sigma^{\mu\nu}\theta)_\alpha F_{\mu\nu} + \theta\theta(\sigma^\mu\partial_\mu\bar\lambda)_\alpha(y)$$

$$[\lambda] = \frac{3}{2} \Rightarrow [W] = \frac{3}{2}.$$

NB: $\left[\int d^2\theta\right] = -1 \Rightarrow \int d^2\theta\, W^\alpha(y,\theta) W_\alpha(y,\theta) + h.c.$

$$\Rightarrow \int d^2\theta\, W^\alpha(y,\theta)W_\alpha(y,\theta) = -\frac{1}{2}F_{\mu\nu}(x)F^{\mu\nu}(x) + \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}(x)\tilde{F}_{\rho\sigma}(x) - 2i\,\lambda\sigma^\mu\partial_\mu\bar\lambda(x) + D^2(x).$$

NB: $tr[\sigma^{\mu\nu}\sigma^{\rho\sigma}] = -\frac{1}{2}(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}) - \frac{i}{2}\varepsilon^{\mu\nu\rho\sigma}$

Non Abelian:

Colour group G, $r = \mathrm{rank}(G)$

$\hookrightarrow F_{\mu\nu} = F_{\mu\nu}^a T^a$, $\quad a = 1,\dots,r$

$\Rightarrow F_{\mu\nu} \to U^{-1}F_{\mu\nu}U$ transforms covariantly
$tr(F_{\mu\nu}F^{\mu\nu})$ is invariant

$\Rightarrow$ gauge transf.: $e^V \to e^{i\bar\Lambda}e^V e^{-i\Lambda} \Rightarrow V \to V + \frac{i}{2}(\Lambda - \bar\Lambda) + \frac{i}{2}[V, \Lambda + \bar\Lambda]$

NB: $W_\alpha = -\frac{1}{4}\bar{D}\bar{D}(e^{-V}D_\alpha e^V)$
$\bar{W}_{\dot\alpha} = -\frac{1}{4}DD(e^V\bar{D}_{\dot\alpha}e^{-V})$

$\rightarrow$ COVARIANT: $W_\alpha \longrightarrow -\frac{1}{4}\bar{D}\bar{D}\left((e^{i\Lambda}e^{-V}e^{-i\bar\Lambda})D_\alpha(e^{i\bar\Lambda}e^V e^{-i\Lambda})\right) =$

$\bar{D}^{\dot\alpha}\Lambda = 0 \qquad D_\alpha\bar\Lambda = 0$

$= -\frac{1}{4}\bar{D}\bar{D}(e^{i\Lambda}e^{-V}D_\alpha e^V e^{-i\Lambda}) =$

$= -\frac{e^{i\Lambda}}{4}\bar{D}\bar{D}(e^{-V}D_\alpha(e^V e^{-i\Lambda})) = -\frac{1}{4}\bar{D}\bar{D}((e^{-V}D_\alpha e^V)e^{-i\Lambda} + D_\alpha e^{-i\Lambda})$

$$\Rightarrow \text{ then } W_\alpha \longrightarrow e^{i\Lambda} W_\alpha e^{-i\Lambda} \qquad \text{since} \quad \bar{D}\bar{D} D_\alpha \Lambda = \bar{D}^{\dot\alpha} D_\alpha \bar{D}_{\dot\alpha}\Lambda + \bar{D}^{\dot\alpha}\partial_{\alpha\dot\alpha}\Lambda = 0$$

$$\bar{W}_{\dot\alpha} \longrightarrow e^{i\bar\Lambda}\bar{W}_{\dot\alpha}\, e^{-i\bar\Lambda}$$

$$\Downarrow$$

COVARIANT TRANSF.

Moreover $\operatorname{tr}(W^\alpha W_\alpha)$ IS GAUGE INV. for the ciclic. of $\operatorname{tr}(\cdot)$

$$\Rightarrow \operatorname{tr}(WW) + \operatorname{tr}(\bar{W}\bar{W}) \ \checkmark$$

In WZ gauge:

$$e^V = 1 + V + \tfrac{1}{2}V^2$$

$$\hookrightarrow W_\alpha = -\tfrac{1}{4}\bar{D}\bar{D}\left((1-V+\tfrac{1}{2}V^2)D_\alpha(1+V+\tfrac{1}{2}V^2)\right) =$$

$$= -\tfrac{1}{4}\bar{D}\bar{D}D_\alpha V - \tfrac{1}{8}\bar{D}\bar{D}\underbrace{D_\alpha V^2}_{VD_\alpha V + (D_\alpha V)V} + \tfrac{1}{4}\bar{D}\bar{D}\, V D_\alpha V =$$

$$= -\tfrac{1}{4}\bar{D}\bar{D}D_\alpha V + \tfrac{1}{8}\bar{D}\bar{D}[V, D_\alpha V] =$$

$$= W_\alpha^{ABEL.} + \tfrac{1}{8}\bar{D}\bar{D}[V, D_\alpha V]$$

$$\tfrac{1}{8}\bar{D}\bar{D}[V, D_\alpha V] = \tfrac{1}{2}(\sigma^{\mu\nu}\theta)_\alpha [A_\mu, A_\nu] - \tfrac{i}{2}\theta\theta\,\sigma^\mu_{\alpha\dot\beta}[A_\mu, \bar\lambda^{\dot\beta}]$$

$$\Rightarrow W_\alpha = -i\lambda_\alpha(y) + \theta_\alpha D(y) + i(\sigma^{\mu\nu}\theta)_\alpha \underbrace{F_{\mu\nu}(y)}_{} + \theta\theta(\sigma^\mu D_\mu\bar\lambda(y))_\alpha$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \tfrac{i}{2}[A_\mu, A_\nu].$$

All in all:

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2_{YM}} \longrightarrow \text{holomorphic coupling}$$

$$\mathcal{L}^{\mathcal{N}=1}_{SYM} = \frac{1}{32\pi}\operatorname{Im}\left(\tau\int d^2\theta\, W^\alpha W_\alpha\right) = \frac{1}{g^2_{YM}}\operatorname{tr}\left[-\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} - i\lambda\sigma^\mu D_\mu\bar\lambda + \tfrac{1}{2}D^2\right]$$

$$+ \frac{\theta}{32\pi^2}\operatorname{tr}\left(F_{\mu\nu}\tilde{F}^{\mu\nu}\right) \qquad \left\{\tilde{F}_{\mu\nu} = \tfrac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}\right\}$$

$\longrightarrow$ to compute Feyn. diag it's best to rescale $V \to 2g_{YM}V$

Then $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i g_{YM} [A_\mu, A_\nu]$ $\quad D_\mu = \partial_\mu - i g_{YM} [A_\mu, \cdot\,]$

* $\mathcal{N}=1$ GAUGE-MATTER INTERACTIONS

Φ^i $\quad i = 1, \dots, n = \dim(R)$ $\qquad R$: rep. of G

Then $T^a \to (T^a_R)^i{}_j$ and $\Phi \to e^{i\Lambda}\Phi \to R$ because it's CHIRAL

$\bar\Phi \to \bar\Phi e^{-i\bar\Lambda} \to R^*$

$$\Rightarrow \bar\Phi\Phi \longrightarrow \bar\Phi \underbrace{e^{-i\bar\Lambda} e^{i\Lambda}}_{\neq 1} \Phi$$

$\downarrow$

We have to use $\bar\Phi e^V \Phi = \bar\Phi e^{-i\bar\Lambda}(e^{i\bar\Lambda} e^V e^{-i\Lambda}) e^{i\Lambda}\Phi =$

$= \bar\Phi e^V \Phi$

Therefore:

this is renorm. Otherwis $K(\bar\Phi, e^V\Phi)$

$$\mathcal{L}_{MAT} = \int d^4\theta\, \bar\Phi e^V \Phi + \int d^2\theta\, W(\Phi) + \int d^2\bar\theta\, \bar W(\bar\Phi)$$

as long as gauge inv.

NB: $W(\Phi)$ gauge inv. + renorm:

i) $G = SU(N)$, $N > 3 \Rightarrow R = \square \to$ we cannot write any $\mathcal{L}$

ii) $N = 3 \Rightarrow \exists!\ \varepsilon_{ijk}\Phi^i\Phi^j\Phi^k$.

iii) $N = 2 \Rightarrow \dots$

NB: $\bar\Phi e^{V_R}\Phi = \bar\Phi\Phi + \bar\Phi V \Phi + \frac{1}{2}\bar\Phi V^2 \Phi$

$\to \bar\Phi V \Phi\,\big|_{\theta\theta\bar\theta\bar\theta} = \frac{i}{2}\bar\phi A_\mu \partial_\mu \phi - \frac{i}{2}(\partial^\mu\bar\phi) A_\mu \phi - \frac{1}{2}\bar\psi\bar\sigma^\mu A_\mu \psi +$

YUKAWA $\longrightarrow + \frac{i}{\sqrt{2}}\bar\phi\lambda\psi - \frac{i}{2}\bar\psi\bar\lambda\phi + \frac{1}{2}\bar\phi D \phi$

$$\bar{\Phi} V^2 \Phi \Big|_{\theta\theta\bar\theta\bar\theta} = \frac{1}{2} \bar{\phi} A^\mu A_\mu \phi$$

$\Rightarrow$ Full Lagrangian: $\int d^4\theta\, \bar{\Phi} e^V \Phi = \left(\overline{D_\mu \phi}\right)\left(D^\mu \phi\right) - i \bar{\psi} \bar{\sigma}^\mu D_\mu \psi + \bar{F} F +$

$$+ \frac{i}{\sqrt{2}} \bar{\phi} \lambda \psi - \frac{i}{\sqrt{2}} \bar{\psi} \bar{\lambda} \phi + \frac{1}{2} \bar{\phi} D \phi$$

where $D_\mu = \partial_\mu - \frac{i}{2} A^a_\mu T^a_R$ and $\bar{\phi} \lambda \psi = \bar{\phi}_i \left(T^a_R\right)^i{}_j \lambda_a \psi^j$.

$\hookrightarrow$ NB $V \to 2gV \Rightarrow D_\mu = \partial_\mu - ig A^a_\mu \left(T^a_R\right)$ and $i\sqrt{2}\, g\, \bar{\phi} \lambda \psi + g \bar{\phi} D \phi$

FAYET - ILIOPOULOS TERM

$V \to V - \frac{i}{2}\left(\Lambda - \bar{\Lambda}\right) \Rightarrow \xi_A \int d^4\theta\, V^*$ IS GAUGE INV. !

Eventually: $\qquad$ A: no. of $U(1)$s in G.

$\mathcal{L}^{\mathcal{N}=1} = \mathcal{L}^{\mathcal{N}=1}_{SYM} + \mathcal{L}^{\mathcal{N}=1}_{MAT} + \mathcal{L}^{U(1)}_{FI} =$, colour group G, matter in rep R of G

$$= \frac{1}{32\pi} \operatorname{tr}\left[\operatorname{Im}\left(\tau \int d^2\theta\, W^\alpha W_\alpha\right)\right] + \int d^4\theta\, \bar{\Phi} e^V \Phi + \left(\int d^2\theta\, W(\Phi) + \text{h.c.}\right) + \sum_A \xi_A \int d^4\theta\, V_A$$

$$= \frac{1}{g^2_{YM}} \operatorname{tr}\left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - i \lambda \sigma^\mu \partial_\mu \bar{\lambda} + \frac{1}{2} D^2\right] + \frac{\theta_{YM}}{32\pi^2} \operatorname{tr}\left(\tilde{F}_{\mu\nu} F^{\mu\nu}\right) + \sum_A \xi_A D_A +$$

$$+ \left(\overline{D_\mu \phi_i}\right)\left(D^\mu \phi^i\right) - i \psi^i \sigma^\mu D_\mu \bar{\psi}_i + \bar{F}_i F^i + i\sqrt{2}\, \bar{\phi}_i \lambda \psi^i - i\sqrt{2}\, \bar{\psi}_i \bar{\lambda} \phi^i$$

$$+ \bar{\phi}_i D \phi^i + \frac{\partial W}{\partial \phi^i} F^i + \frac{\partial \bar{W}}{\partial \bar{\phi}_i} \bar{F}_i + \frac{1}{2} \frac{\partial^2 W}{\partial \phi \partial \phi} \psi^i \psi^j + \frac{1}{2} \frac{\partial^2 \bar{W}}{\partial \bar\phi \partial \bar\phi} \bar{\psi}_i \bar{\psi}_i .$$

e.o.m.: $\quad F: \quad \bar{F}_i = -\frac{\partial W}{\partial \phi^i} \qquad D: \quad D^a = -\bar{\phi}_i T^a \phi^i - \xi^a \quad (\xi^a = 0 \text{ if } a \neq A)$

$\quad \bar{F}: \quad F^i = -\frac{\partial \bar{W}}{\partial \bar{\phi}_i} \qquad \hookrightarrow D = D^a T^a$

Scalar potential: $V_{scalar}(\phi, \bar{\phi}) = F\bar{F} + \frac{1}{2} D^2 \geq 0$

$$= \sum_i \left|\frac{\partial W}{\partial \phi^i}\right|^2 + \sum_a \left|\bar{\phi}_i (T^a)^i{}_j \phi^j + \xi^a\right|^2$$

EXTENDED SUSY

$\mathcal{N}=2$ SUPERSYMMETRY THEORIES

$R \in$ Cartan

$\Rightarrow$ $\mathcal{N}=2$ <u>massless</u> multiplets: $\quad [\,h:$ helicity; $\;U(2) = SU(2)_R \times U(1)_r\,]$

- HYPERMULTIPLET:

$$|\Omega\rangle = |h=-\tfrac{1}{2};\, R=0\rangle_{r=1} \qquad I=1,2 \text{ for } SU(2)_R$$

$$a_I^\dagger\,|\Omega\rangle = |h=0;\, R=\pm\tfrac{1}{2}\rangle_{r=0} = |h=0;\, 2_R\rangle_{r=0} \Rightarrow Q^I$$

$$\varepsilon^{IJ} a_I^\dagger a_J^\dagger |\Omega\rangle = |h=+\tfrac{1}{2};\, R=0\rangle_{r=-1}$$

$\tilde{\bar{\psi}}_{\dot\alpha}$, ψ_α

$(Q^I)^\dagger = \bar{Q}_I = \varepsilon_{IJ} Q^J$

(but because of ε_{IJ} this is NOT enough)

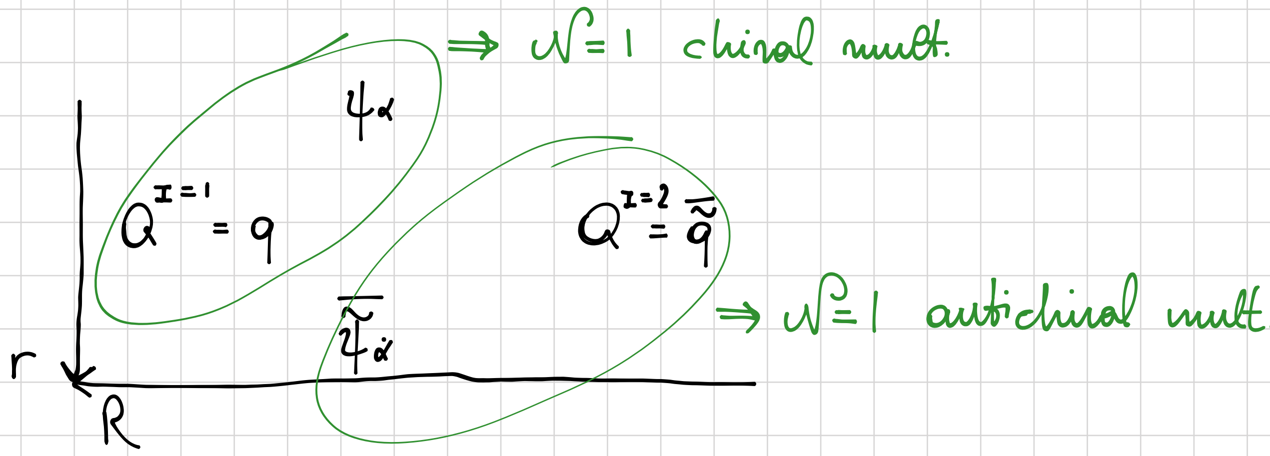
This is not CPT complet, however ($SU(2)_R$ in rep 2_R is PSEUDOREAL)

Solve it in 2 ways:

1) just add the CPT conjugate: $\quad Q^I,\ \bar{Q}_I,\ \psi_\alpha,\ \tilde{\psi}_\alpha,\ \bar{\psi}_{\dot\alpha},\ \tilde{\bar{\psi}}_{\dot\alpha}$

2) only if colour group $G = SU(2) \Rightarrow 2_R \times 2_G \Rightarrow 2$ pseudoreal $= 1$ real

$\Rightarrow$ you are allowed to work a half-hyper.



$$Q = q + \theta\psi + f\theta^2 \qquad \text{in } R \text{ of } G$$

$$\tilde{Q} = \tilde{q} + \theta\tilde{\psi} + \tilde{f}\theta^2 \qquad \text{in } R^* \text{ of } G$$

$$\Downarrow$$

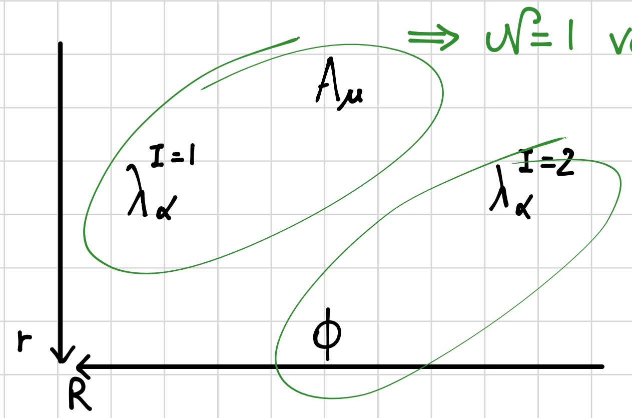
$$Q^I = \begin{pmatrix} q \\ \bar{\tilde{q}} \end{pmatrix} \qquad \bar{Q}_I = \begin{pmatrix} \tilde{q} \\ -\bar{q} \end{pmatrix} \qquad [SU(2), G] = 0.$$

- VECTOR

$$|\Omega\rangle = |h=0;\ R=0\rangle_{r=1} \qquad \varphi_1$$
$$a_I^\dagger|\Omega\rangle = |h=\tfrac{1}{2};\ R=\pm\tfrac{1}{2}\rangle_{r=1} \qquad \lambda_I$$
$$\varepsilon^{IJ}a_I^\dagger a_J^\dagger|\Omega\rangle = |h=1;\ R=0\rangle_{r=0} \qquad A_+$$

Add CPT conjugate: $|\Omega_{-1}\rangle = |h=-1;\ R=0\rangle_{r=0} \qquad A_-$

$$a_I^\dagger|\Omega_{-1}\rangle = |h=-\tfrac{1}{2},\ R=\pm\tfrac{1}{2}\rangle_{r=-1} \qquad \bar\lambda_I$$
$$\varepsilon^{IJ}a_I^\dagger a_J^\dagger|\Omega_{-1}\rangle = |h=0;\ R=0\rangle_{r=-2} \qquad \varphi_2$$



$\Rightarrow \mathcal{N}=1$ vector

$\Rightarrow \mathcal{N}=1$ chiral

$$V = \ldots + \theta\sigma^\mu\bar\theta A_\mu + i\,\theta\theta\bar\theta\bar\lambda^1 + \ldots + \tfrac{1}{2}\theta\theta\bar\theta\bar\theta D$$
$$\Phi = \phi + \theta\lambda_2 + \theta^2 F$$

$\Rightarrow$ Same mult $\rightarrow$ They are all in ADJ of the colour group

$\Rightarrow F, D, \bar F$ form a triplet under $SU(2)_R$

$\mathcal{N}=2$ SUPERSPACE for the vector field:

$x^\mu, \theta_\alpha, \bar\theta_{\dot\alpha}, \tilde\theta_\alpha, \bar{\tilde\theta}_{\dot\alpha}$ s.t. $\theta^I = (\theta_\alpha, \tilde\theta_\alpha)$ etc

$\hookrightarrow y^\mu = x^\mu + \theta\sigma^\mu\bar\theta$ in $\mathcal{N}=1$ supersp. $\longrightarrow \tilde y^\mu = y^\mu + \tilde\theta\sigma^\mu\bar{\tilde\theta}$ in $\mathcal{N}=2$ supersp.

$\rightarrow \bar D_{\dot\alpha}\tilde y^\mu = 0;\ \bar{\tilde D}_{\dot\alpha}\tilde y^\mu = 0$

$\Rightarrow W(\tilde y, \theta, \tilde\theta) = \Phi(\tilde y, \theta) + \tilde\theta^\alpha W_\alpha(\tilde y, \theta) + \tilde\theta^2 G(\tilde y, \theta)$

In components:

$$W = \phi + \theta^I\lambda_I + \theta^{\alpha[I}\theta_\alpha^{J]}F_{IJ} + \theta^{(\alpha I}\theta_I^{\beta)}\mathcal{F}_{\alpha\beta} + \ldots$$

where $F_{\mu\nu}(\sigma^{\mu\nu})_\alpha{}^\gamma\varepsilon_{\gamma\beta} = \mathcal{F}_{\alpha\beta}$ $\quad [F_{11}=F,\ F_{12}=D,\ F_{22}=\bar F]$

LAGRANGIANS for $\mathcal{N}=2$ theories using $\mathcal{N}=1$ supersp.

$\mathcal{N}=1$ supersp. makes $\mathcal{N}=1$ SUSY manifest but $SU(2)_R$ is <u>not</u> visible!

To get the full $\mathcal{N}=2$ impose $SU(2)_R$ and impose the correct gauge invariance.

$$\left.\begin{aligned} \Rightarrow\ W_\alpha &\to e^{i\Lambda} W_\alpha e^{-i\Lambda} \\ \Phi &\to e^{i\Lambda}\,\Phi\, e^{-i\Lambda} \\ \left(\bar{\Phi}\right. &\to \left. e^{i\bar{\Lambda}}\,\bar{\Phi}\, e^{-i\bar{\Lambda}}\right) \end{aligned}\right\}$$ they are in the same rep. (ADJ) of G

$$\left(e^V \longrightarrow e^{i\bar{\Lambda}} e^V e^{-i\Lambda}\right)$$

$$\Rightarrow \operatorname{tr}\left(\bar{\Phi} e^V \Phi e^{-V}\right) \longrightarrow \operatorname{tr}\left((e^{i\bar{\Lambda}}\bar{\Phi} e^{-i\bar{\Lambda}})(e^{i\bar{\Lambda}} e^V e^{-i\Lambda})(e^{i\Lambda}\Phi e^{-i\Lambda})(e^{i\Lambda} e^{-V} e^{-i\bar{\Lambda}})\right) =$$

$$= \operatorname{tr}\left(\bar{\Phi} e^V \Phi e^{-V}\right) \text{ gauge invariant } \checkmark$$

$$\mathcal{L}^{\mathcal{N}=2}_{\substack{\text{pure}\\ \text{SYM}}} = \frac{1}{32\pi} \operatorname{Im}\left[\tau \int d^2\theta\, W^\alpha W_\alpha\right] + C \int d^4\theta \operatorname{tr}\left(\bar{\Phi} e^V \Phi e^{-V}\right)$$

and there is nothing else (because of $SU(2)_R$)!

How to check the construction?

- gaugino kin. term: $\frac{1}{g^2_{YM}}(-i)\bar{\lambda}^1 \not{D} \lambda_1 \quad + \quad C(-i)\bar{\lambda}^2 \not{D}\lambda_2$

  $\Rightarrow C = \frac{1}{g^2_{YM}}$ and $-i\frac{1}{g^2_{YM}} \bar{\lambda}^I \not{D} \lambda_I$ is $SU(2)_R$ invariant! ✓

- Yukawa: $\operatorname{tr}(\bar{\phi}\lambda_1\lambda_2) - \operatorname{tr}(\bar{\phi}\lambda_2\lambda_1)\ldots$

  $\Downarrow$

  $\varepsilon^{IJ} \operatorname{tr}(\lambda_I \lambda_J \bar{\phi})$ is $SU(2)_R$ inv.! ✓

If I add $\int d^2\theta\, W(\Phi)$, I'll find $\propto \lambda_2\lambda_2\phi \Rightarrow \cancel{SU(2)_R} \to$ I cannot write this!

$$\Rightarrow \operatorname{tr}\left(\tfrac{1}{2}D^2\right) + \operatorname{tr}\left(\bar{\phi} D \phi - \bar{\phi}\phi D\right) \longrightarrow D = -[\bar{\phi}, \phi] \text{ and } F = 0$$

$$\Rightarrow V_{scalar}(\phi, \bar{\phi}) = \left|[\bar{\phi}, \phi]\right|^2.$$

$\longrightarrow$ choose $\phi = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_r \end{pmatrix}$, $r = \operatorname{rank} G$

and $\Rightarrow V_{scalar}(\phi, \bar{\phi}) = 0$.

In $\mathcal{N}=2$ superspace: $\mathcal{L}^{\mathcal{N}=2}_{\substack{pure\\ YM}} = \underbrace{\int d^2\theta\, d^2\tilde{\theta}}_{[d^2\theta d^2\tilde{\theta}]=2}\ \underbrace{\mathcal{W}\,\mathcal{W}}_{\mathcal{F}(\mathcal{W})}$ "PREPOTENTIAL"

$[\mathcal{W}]=1$

holomorphic prepotential

Recap:

$\mathcal{N}=2$ pure SYM: Field content: i) $\mathcal{N}=1$ vector $V = \ldots + \theta\sigma^\mu\bar{\theta} A_\mu + i\,\theta\theta\bar{\theta}\bar{\lambda}' - i\,\bar{\theta}\bar{\theta}\theta\lambda_1 + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta} D$

ii) $\mathcal{N}=1$ chiral mult $\Phi = \phi + \theta\lambda_2 + \theta\theta F$

$\Rightarrow$ $\lambda_I = \begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix}$ $SU(2)_R$ doublet

$(F, D, \bar{F}) = F_{IJ}$ $SU(2)_R$ triplet

$\Rightarrow$ all field in same ADJ REP of colour group G

$$\left.\begin{aligned} W_\alpha &\to e^{i\Lambda} W_\alpha e^{-i\Lambda} \\ \Phi &\to e^{i\Lambda}\Phi e^{-i\Lambda}\end{aligned}\right\} \qquad \begin{aligned} e^V &\to e^{+i\bar{\Lambda}} e^V e^{-i\Lambda} \\ W_\alpha &= -\frac{1}{4}\bar{D}\bar{D}\left(e^{-V} D_\alpha e^V\right)\end{aligned}$$

$\mathcal{N}=2$ superspace: multiplets packed in one $\mathcal{N}=2$ chiral superf.:

$$\bar{D}_{\dot{\alpha}} \mathcal{W} = 0 \qquad \bar{\tilde{D}}_{\dot{\alpha}} \mathcal{W} = 0$$

$$y^\mu = x^\mu + \theta\sigma^\mu\bar{\theta} \qquad \tilde{y}^\mu = y^\mu + \tilde{\theta}\sigma^\mu\bar{\tilde{\theta}} \qquad \left(\bar{D}_{\dot{\alpha}} y^\mu = \bar{\tilde{D}}_{\dot{\alpha}}\tilde{y}^\mu = 0\right)$$

$$\Rightarrow \mathcal{W}(\tilde{y},\theta,\tilde{\theta}) = \Phi(\tilde{y},\theta) + \tilde{\theta}^\alpha W_\alpha(\tilde{y},\theta) + \tilde{\theta}\tilde{\theta}\, G(\tilde{y},\theta).$$

Then $\mathcal{W} = \phi(\tilde{y}) + \theta^I \lambda_I(\tilde{y}) + \theta^{\alpha\{I}\theta_\alpha^{J\}} F_{\{IJ\}}(\tilde{y}) + \theta^{\{\alpha}_I \theta^{\beta\}}_I F_{\{\alpha\beta\}}(\tilde{y}) + \dots$

Now consider the Lagrangians:

$$= \int d^2\theta\, d^2\tilde{\theta}\, \mathcal{W}\mathcal{W}$$

$$\mathcal{L}^{\mathcal{N}=2}_{SYM} = \frac{1}{32\pi} \mathrm{Im}\left(\tau \int d^2\theta\, W^\alpha W_\alpha\right) + C \int d^4\theta\, \mathrm{tr}\left(\bar{\Phi} e^V \Phi e^{-V}\right) \; ; \qquad \tau = \frac{\theta_{YM}}{2\pi} + \frac{4\pi i}{g^2_{YM}}$$

$\Rightarrow$ Use $\mathcal{N}=1$ superspace $\rightarrow$ $\mathcal{N}=1$ SUSY manifest to get $\mathcal{N}=2$ susy:

1) the right gauge transform / gauge invariance
2) impose $SU(2)_R$ $\hookrightarrow C \sim \frac{1}{g^2_{YM}}$ $\; \begin{cases} \bar{\lambda}^I \not{D} \lambda_I \\ \bar{\Phi}\, \lambda_I \lambda_J \varepsilon^{IJ} \end{cases}$

NB: take $\int d^2\theta d^2\tilde{\theta}\, \mathcal{W}\mathcal{W}$ and gener. to HOLOMORPHIC functions of $\mathcal{W}$ (prepot.):

$$\mathcal{L}^{\mathcal{N}=2}_{pure\ SYM} = \int d^2\theta\, d^2\tilde{\theta}\, \mathcal{F}(\mathcal{W}) \Rightarrow \text{may contain non renorm terms}$$

$\mathcal{N}=2$ MATTER (HYPER)

The field content of a hyper is packed in 2 $\mathcal{N}=1$ massless chiral:

$$Q = q + \theta^\alpha \psi_\alpha + f\,\theta\theta \quad \text{in } R \text{ of } G$$

$$\tilde{Q} = \tilde{q} + \theta^\alpha \tilde{\psi}_\alpha + \tilde{f}\,\theta\theta \quad \text{in } R^* \text{ of } G$$

$\hookrightarrow Q^I = \begin{pmatrix} q \\ \bar{\tilde{q}} \end{pmatrix}$ doublet of $SU(2)_R$

$\Rightarrow$ $Q \rightarrow e^{i\Lambda} Q$ and $\bar{Q} \rightarrow \bar{Q} e^{-i\bar{\Lambda}}$
$\tilde{Q} \rightarrow \tilde{Q} e^{-i\Lambda}$ and $\bar{\tilde{Q}} \rightarrow e^{i\bar{\Lambda}} \bar{\tilde{Q}}$

where $\Lambda = \Lambda_a T^a$

Now consider $\mathcal{N}=2$ MATTER + GAUGE:

$$\mathcal{L}_{MAT}^{\mathcal{N}=2} = \int d^4\theta \left(\overline{Q} e^V Q + \tilde{Q} e^{-V} \overline{\tilde{Q}} \right) + \left[\int d^2\theta \, \tilde{Q} \Phi Q + h.c. \right] + \left[\int d^2\theta \, m \tilde{Q} Q + h.c. \right]$$

$\bar{q} \lambda_1 \psi$ $\quad - \tilde{\psi} \lambda_1 \tilde{q}^*$ $\quad \tilde{\psi} \lambda_2 q$ and $\tilde{q} \lambda_2 \psi$

$\tilde{\psi} \lambda_I Q^I$

$\overline{Q}^I \lambda_I \psi$

(where $\overline{Q}_I = \begin{pmatrix} \bar{q} \\ -\tilde{q} \end{pmatrix}$)

$\mathcal{N}=2$ massive and massless have same dof if BPS is saturated!

$\Rightarrow$ we can freely add a MASS TERM

We still need the SUPERPOT. for $SU(2)_R$!

$$\Rightarrow \mathcal{L}_{SYM}^{\mathcal{N}=2} + \mathcal{L}_{MAT}^{\mathcal{N}=2} = \ldots + \frac{1}{2} D^2 + F\overline{F} + \ldots + D[\phi, \bar{\phi}] + f\bar{f} + \bar{\tilde{f}}\tilde{f} + \ldots + \tilde{f} \phi q + \tilde{q} \phi f + h.c. + \tilde{q} F q + h.c.$$

$(V \to 2gV;\ \Phi \to 2g\Phi)$

a) w/o MAT $\left(\langle q \rangle = \langle \tilde{q} \rangle = 0 \right)$: $F = \overline{F} = 0$;

$D = [\phi, \phi]$

$V_{scalar} = \left| [\phi, \phi] \right|^2$

$\Rightarrow \phi = \mathrm{diag}(\phi_1, \ldots, \phi_r)$ $\quad r = \mathrm{rk}(G)$

SUSY Vacua

r-dimensional MODULI SPACE of vacua

"COULOMB BRANCH"

b) w/ MAT:

$\overline{F} = q\tilde{q}$ $\quad F = \bar{\tilde{q}}\bar{q}$ $\quad D = q\bar{q} - \bar{\tilde{q}}\tilde{q} + [\phi, \bar{\phi}]$ $\quad f = \tilde{q}\phi$ $\quad \bar{\tilde{f}} = \phi q$ and h.c.

- $\langle \phi \rangle = 0 \longrightarrow$ "HIGGS BRANCH" $\left[F_{IJ} = (\overline{F}, D, F) = \overline{Q}_{\{I} Q_{J\}} \right]$

What is the relation between the $U(1)$ R-sym. of $\mathcal{N}=1$ and the $U(1)_r$ and $SU(2)_R$ of $\mathcal{N}=2$?

$$\frac{2}{3} r_{\mathcal{N}=1} = 2R + 2r$$

$$W(\Phi, Q, \tilde{Q}) = \tilde{Q} \Phi Q$$

	R	r	$r_{\mathcal{N}=1}$
Q	$\square$	0	$2/3$
$\tilde{Q}$	$\overline{\square}$	0	$2/3$
Φ	1	1	$2/3$

$\mathcal{N}=4$ SYM (only max SUSY gauge th. in $d=4$)

- There is ONLY one possible massless mult. with $\mathcal{N}=4$ susy and helicity ≤ 1:

$$|\Omega_{-1}\rangle = |\lambda=-1;\ R=0\rangle \Rightarrow A_\mu \rightarrow \text{must be a singlet}$$
$$a_I^\dagger|\Omega_{-1}\rangle = |\lambda=-\tfrac{1}{2};\ \bar{4}_R\rangle \Rightarrow \lambda_I \rightarrow \text{anti-fundam of } SU(4)_R$$
$$a_I^\dagger a_J^\dagger|\Omega_{-1}\rangle = |\lambda=0;\ [4\times4]_A=6\rangle \rightarrow \phi^{[I,J]}$$
$$a_I^\dagger a_J^\dagger a_K^\dagger|\Omega_{-1}\rangle = |\lambda=\tfrac{1}{2};\ 4_R\rangle \rightarrow a_I^\dagger a_J^\dagger a_K^\dagger \varepsilon^{IJKL} = a^{L\dagger} \Rightarrow \bar{\lambda}^L$$
$$a_I^\dagger a_J^\dagger a_K^\dagger a_L^\dagger|\Omega_{-1}\rangle = |\lambda=1;\ 0\rangle \rightarrow \varepsilon_{IJKL}\, a^{I\dagger}a^{J\dagger}a^{K\dagger}a^{L\dagger} = \text{singlet of } SU(4)_R$$

FIELD CONTENT:

$\Phi^{[IJ]}$: 6 real scalars $\longrightarrow$ $\bar{\Phi}_{[IJ]} = \frac{1}{2}\varepsilon_{IJKL}\Phi^{[KL]}$ (self duality condit.)

λ^I : 4 Weyl spinors

$\bar{\lambda}_I$: 4 anti-Weyl spinors

A_μ : $SU(4)_R$ singlet

Therefore we have:

$$\mathcal{N}=4 \text{ vector} \Rightarrow 1\times(\mathcal{N}=2 \text{ vector}) + 1\times(\mathcal{N}=2 \text{ hyper}) \Rightarrow SU(4)_R \rightarrow \underbrace{SU(2)_R\times U(1)_r}_{\mathcal{N}=2\ R\text{-Symm.}} \times SU(2)_L$$
$$\Rightarrow 1\times(\mathcal{N}=1 \text{ vector}) + 3\times(\mathcal{N}=1 \text{ chiral}) \Rightarrow SU(4)_R \rightarrow \underbrace{U(1)_R}_{\mathcal{N}=1\ R\text{-sym.}}\times \underbrace{SU(2)_L}_{\text{extra flavour}} \times \underbrace{SU(3)}_{\text{extra global symmetry}}$$

NB: $\mathcal{N}=4$ vect. $\longrightarrow$ 3 ($\mathcal{N}=1$ chiral) in $SU(3)$ triplet:

$$\Phi^a = \phi^a + \theta\psi^a + \theta^2 F^a \Rightarrow \Phi^a \rightarrow e^{i\Lambda}\Phi^a e^{-i\Lambda}$$

NB: $\lambda_I = \begin{pmatrix}\lambda_1\\ \vdots \\ \lambda_4\end{pmatrix} \longrightarrow$ 1) $\mathcal{N}=2$ breaking $\left(\frac{SU(2)_R\times U(1)}{}\Big| \, SU(2)_L\times U(1)^*\right)$

2) $\mathcal{N}=1$ breaking $\begin{pmatrix}\lambda_1\\ \vdots \\ \lambda_4\end{pmatrix} \rightarrow \begin{pmatrix}\lambda\\ \psi_1\\ \vdots \\ \psi_3\end{pmatrix}$ $\left(\frac{U(1)}{}\Big| \, SU(3)\right)$

NB: $SU(4)_R \simeq SO(6) \longrightarrow$ 't Hooft symbols $\eta_{IJ}^{m=1,\dots,6}$

$$\Rightarrow \Phi_{[IJ]} = \left(\begin{array}{cc|cc} 0 & \phi_1 & \phi_2 & \phi_3 \\ -\phi_1 & 0 & \phi_3^* & -\phi_2^* \\ \hline -\phi_2 & -\phi_3^* & 0 & \phi_1^* \\ -\phi_3 & \phi_2^* & -\phi_1^* & 0 \end{array}\right) \qquad \Phi_i = \frac{X^i + i X^{i+1}}{\sqrt{2}} \quad \text{(3 } \mathbb{C} \text{ scalars instead of 6 } \mathbb{R}\text{)}$$

LAGRANGIAN FORMULATION:

$$\mathcal{L}_{SYM}^{\mathcal{N}=4} = \frac{1}{32\pi} \operatorname{Im}\left(\tau \int d^2\theta \, W^\alpha W_\alpha\right) + C \int d^4\theta \, \operatorname{tr}\left(\bar{\Phi}_a e^V \Phi^a e^{-V}\right) +$$

$$+ \left[C' \varepsilon_{abc} \int d^2\theta \, \operatorname{tr}\left(\Phi^a \Phi^b \Phi^c\right) + h.c.\right]$$

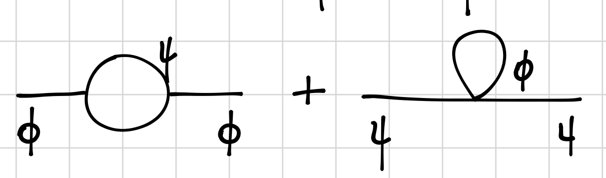
If I need λ_I in $4_{su(4)}$, then $C = \frac{1}{g_{YM}^2}$. I also need $\phi^{[IJ]} \lambda_I \lambda_J \Rightarrow$ fixes $C' \sim \frac{1}{3!}$

$\rightarrow$ there is NO OTHER POSSIBLE TERM (no way to write a mass term).

We have 7 $\mathbb{R}$ auxiliary fields: F^a, $\bar{F}_a$, $D \Rightarrow$ OCTONIONS! (the imag. part)

$\Rightarrow V_{scalar} \sim |[X^I, X^J]|^2$ six $\mathbb{R}$ scalars

NON RENORM. THEOREM

 $\simeq 0\,\Lambda^2 + \#\ln\Lambda \longleftarrow$ THEOREM

In perturbation theory: THE SUPERPOTENTIAL DOES NOT RECEIVE CORRECT. AT ANY ORDER!

The Kähler terms do receive corrections! $\Longleftarrow$!!!

Consequences:

1) $\mathcal{N}=1$ theories: $\quad \Phi \rightarrow Z_\Phi^{1/2} \Phi \;; \quad m \rightarrow Z_m m \;; \quad \lambda \rightarrow Z_\lambda \lambda$

WZ model: $\quad W(\Phi) = m\Phi^2 + \lambda\Phi^3 \Rightarrow Z_m Z_\Phi = 1$ and $Z_\lambda Z_\Phi^{3/2} = 1$.

gauge: $\bar{\Phi} e^{gV} \Phi \longrightarrow Z_\Phi \bar{\Phi} e^{g_{YM} V} \Phi \longrightarrow$ gauge inv. IMPLIES $Z_{g_{YM}} Z_V^{1/2} = 1$

$\longrightarrow$ at most I'll have to compute Z_Φ and $Z_{g_{YM}}$ (2 functions)

2) $\underline{\mathcal{N}=2 \text{ theories}}$: $\circledast$ $\mathcal{W} = \Phi + \tilde{\theta} W_\alpha + \ldots \Longrightarrow Z_V^{1/2} = Z_\Phi^{1/2} = Z_{g_{YM}}^{-1}$

w/ hyper $\circledast$ $\mathcal{W} = g \tilde{Q} \Phi Q \to Z_{\tilde{Q}} Z_Q \mathcal{W}$

$\downarrow$

$Z_{\tilde{Q}} Z_Q = 1$ impose $SU(2)_R \Rightarrow Z_Q = Z_{\tilde{Q}} = 1.$

3) $\underline{\mathcal{N}=4 \text{ theories}}$: $Z_{\Phi_a} = 1$ (because of $SU(3)$ symm.)

$\hookrightarrow Z_V = Z_\Phi = Z_{g_{YM}} = 1 \Leftrightarrow \beta^{\mathcal{N}=4}(g_{YM}) = 0.$

NO UV diverg.!

$\Rightarrow \mathcal{W}(\Phi)$: a holomorphic function is determined by asymptotics and singularities

$\hookrightarrow$ i) $\mathcal{W}_{eff} = \mathcal{W}(\underbrace{\Phi, m, \lambda}; \Lambda)$ $\to$ scale

uplift them all to $\mathcal{N}=1$ superfields / holomorphic for all

$\hookrightarrow \ln\left(\frac{\Lambda}{\mu}\right) \sim \frac{8\pi^2}{g^2(\mu)}$

$\Rightarrow$ The quantum corrections are constrained by

i) SYMMETRY

ii) HOLOMORPHICITY

iii) various limits $(g \to 0,\ m \to 0, \ldots)$

$\to$ Uniquely fix the Superpotential!

e.g.: [Seiberg, Intilligator] Non renorm. for WZ model:

$\mathcal{W}_{tree}(\Phi) = m\Phi^2 + \lambda\Phi^3$

	$U(1)$	$U(1)_R$
Φ	1	1
m	-2	0
λ	-3	-1
	✓	✓

$\to R[\mathcal{W}] = 2$

Now suppose a general holom. funct. obeying these symmetries:

$$W_{eff} = \underbrace{m\Phi^2}_{\downarrow} f\Big(\underbrace{\frac{\lambda\Phi}{m}}\Big) = \sum_{n=-\infty}^{+\infty} c_n \lambda^n \Phi^{n+2} m^{1-n}$$

$R[m\Phi^2] = 2$ $\quad$ $R\left[\frac{\lambda\Phi}{m}\right] = 0$ and also $U(1)$ inv.!

Now I can use the "various asymptotics":

- $\lambda \to 0 \Rightarrow n \geq 0$ (for W not to $\to \infty$)
- $m \to 0 \Rightarrow n \leq 1$ (") $\quad (c_0 = c_1 = 1)$

$$\hookrightarrow W_{eff}(\Phi) = m\Phi^2 + \lambda\Phi^3 = W_{tree}(\Phi).$$

<u>NSVZ β function:</u> (Novikov-Shifman-Vainstein-Zakharov)

$$\beta(g) = \frac{dg}{d\ln\mu} = -\frac{g^3}{16\pi^2}\cdot\frac{3T(\text{adj}) - \sum_j T(R_j)(1-\gamma_j)}{1 - T(\text{adj})\frac{g^2}{8\pi^2}}$$

where $\gamma_i = -\frac{dZ_i}{d\ln\mu}$; $Z_i = Z_{\Phi_i}$ and $T(R)$: Dynkin index of the rep R
$\left(T(\square) = \frac{1}{2}\ ;\ T(\text{adj}) = N \text{ for } SU(N)\right)$

$\Rightarrow$ For $\mathcal{N}=2$: $Z_\Phi^{1/2} = Z_V^{1/2} = Z_g^{-1}$ and $Z_Q = Z_{\tilde{Q}} = 1 \rightarrow \gamma_Q = \gamma_{\tilde{Q}} = 0$

$$\beta(g_{YM}) = 2 g_{YM}\gamma_\Phi \Rightarrow \left(1 - \cancel{T(\text{adj})}\right)\frac{g^2}{8\pi^2}\beta(g_{YM}) = -\frac{g^3}{16\pi^2}\Big(3T(\text{adj}) - T(\text{adj})\Big(1-\cancel{\tfrac{2\beta}{g}}\Big)$$
$$- n_{Hyper} T(R) \times 2\Big) \quad \hookrightarrow Q, \tilde{Q}$$

$$\Downarrow$$

$$\beta(g) = -\frac{g^3}{16\pi^2}\left(2T(\text{adj}) - 2n_{Hyper}T(R)\right)$$

$\downarrow$

IT'S JUST A ONE LOOP CONTRIBUTION!!!

$\Rightarrow$ e.g.: $\mathcal{N}=2$ SQCD $SU(N)$ colour and N_f fundam hyper

$$\beta(g) = -\frac{g_{YM}^3}{16\pi^2}(2N - N_f) \rightarrow \text{pick } N_f = 2N \Rightarrow \text{Superconformal!}$$

(Much more in general we can have it with $T(\text{adj}) = n_{Hyper}T(R)$ for Lagrangians).

$\Rightarrow$ $\underline{\mathcal{N}=4}$: i) vector = 1 ($\mathcal{N}=2$ vec) + 1 ($\mathcal{N}=2$ hyper)

ii) $T(R) = T(\text{adj})$ (all of them in same rep):

$$\beta(g) = -\frac{g^3}{16\pi^2} T(\text{adj}) = 2 - 2 = 0.$$

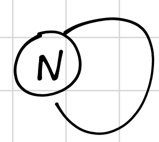
THE LANDSCAPE OF 4D SUSY THEORIES

* $\mathcal{N}=4$ SYM $\to$ unique up to the choice of G
  $\to$ conformal $(\beta(g) \equiv 0) \longrightarrow$ Superconformal algebra
  $\Rightarrow$ multiplet: $\hat{\mathcal{B}}^{(\frac{1}{2},\frac{1}{2})}_{[0,p,0]}$ $\quad \Delta = p$ primary $\text{tr}(X^{\{m_1}\ldots X^{m_p\}})$

* $\mathcal{N}=3$ $\to$ adding CPT uplifts to $\mathcal{N}=4$
  $\to$ $\exists$ non-$\mathcal{L}$ th.

* $\mathcal{N}=2$ $\to$ $\exists$ (non-)conf. $\mathcal{L}$ and non-$\mathcal{L}$ th.
  $\to$ $\exists$ classification for conf + $\mathcal{L}$ th (2013)
  $\to$ read $\mathcal{L}$ from QUIVER DIAGRAMS:

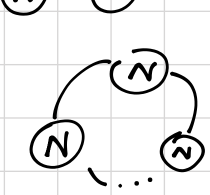
  (N) colour $SU(N)$
  — hypers
  $\boxed{N_f}$ no. of flavours

e.g.:  : $\mathcal{N}=4$ SYM / $\mathcal{N}=2^*$ (if hyper gets a mass)

(N)—$\boxed{N_f}$: SQCD [$G = SU(N)$ with N_f flavours]

$\boxed{N}$—(N)—(N)—$\cdots$—(N)—$\boxed{N}$

$\Rightarrow$ CONFORMAL TH.



$\hookrightarrow$ from string orbifold

$\Rightarrow$ SCFT + relevant deform. (mass)

SCFT $\longrightarrow$

↳ at any point we pick $\mathcal{N}=2$ th.

TH: $\mathcal{N}=2$ classif. from rep. th. (mass: $m\tilde{Q}Q$).

TH: ($\mathcal{N}=2$) all possible marginal deform. are class. in a specific multipl. $E_2 \longrightarrow Q^4\,\mathrm{tr}(\phi^2)$

↳ conformal manifold

* $\mathcal{N}=1$ th. $\longrightarrow$ much more unexplored

CONFORMAL SYMM. IN 4D

$\Rightarrow$ consider $\mathbb{R}^{1,3} \longrightarrow \mathbb{R}^4 \Rightarrow \eta^{\mu\nu} \to \delta^{\mu\nu}$

↳ Lorentz: $SO(4) \sim SU(2) \times SU(2)$

$\downarrow$

$(Q_\alpha)^\dagger \neq \bar{Q}_{\dot\alpha}$: INDEPENDENT

$\rightarrow$ Algebra:

$$[M^{\mu\nu}, M^{\rho\sigma}] = \dots$$

$$[M^{\mu\nu}, P^{\rho}] = \dots$$

$$[M^{\mu\nu}, K^{\rho}] = i\left(\delta^{\mu\rho}K^{\nu} - \delta^{\nu\rho}K^{\mu}\right)$$

$$[P^{\mu}, K^{\nu}] = -2i\left(\delta^{\mu\nu}D + M^{\mu\nu}\right)$$

$$[D, P^{\mu}] = iP^{\mu}$$

$$[D, K^{\mu}] = -iK^{\mu}$$

Think $\mathbb{R}^4 \longrightarrow S^3 \times \mathbb{R} \quad \rightarrow \quad |\Delta; j_1, j_2\rangle$

S^3: $SO(4)$ rotations (spin (j_1, j_2))

$\mathbb{R}$: "time" (gener. by D)

$$\Rightarrow P_\mu |\Delta; \dots\rangle = |\Delta+1; \dots\rangle_\mu$$
$$K_\mu |\Delta; \dots\rangle = |\Delta-1; \dots\rangle_\mu$$

Build mult.:

$$K_\mu |\psi\rangle = 0 \Rightarrow |\psi\rangle : \text{"lowest weight state"}$$

etc.

$$|\Delta+2; \dots\rangle = P_\mu P_\nu |\Delta; \dots\rangle$$
$$|\Delta+1; j_1 \pm \tfrac{1}{2}, j_2 \pm \tfrac{1}{2}\rangle$$
$$|\Delta; j_1, j_2\rangle$$

P_ν, P_μ

∞ dim. rep. because it's not compact

OPERATOR - STATE MAP

$$|\psi\rangle \longleftrightarrow \mathcal{O}$$

If $|\psi\rangle$ lowest weight $\Rightarrow \mathcal{O}$ is conformal primary (CP)

conf. dim. of the CP

$$\mathcal{A}^{\Delta}_{(j_1,j_2)} = \mathrm{span}\left\{ P_{\mu_1} \dots P_{\mu_m} \mathcal{O}^{\Delta}_{(j_1,j_2)} \right\}$$

multiplet

spin

To make unitary rep of the conf. gr. we impose $\| |\psi\rangle \|^2 \geq 0$ $\forall$ descendants

i.e.: $\| |\psi_i\rangle \|^2 \geq 0$ for all elem in mult.

$\rightarrow |\psi\rangle = |\Delta; j_1, j_2\rangle \longrightarrow P_\mu |\psi\rangle = |\Delta+1; j_1 \pm \tfrac{1}{2}, j_2 \pm \tfrac{1}{2}\rangle$

$\Rightarrow$ call ψ_1 the smallest norm : $\dfrac{\|\psi_1\|^2}{2} = \Delta - j_1 - 1 + \delta_{j_1,0} - j_2 - 1 + \delta_{j_2,0}$

$$\frac{\|\psi_1\|^2}{2} = \Delta - f(j_1, j_2)$$

↳ ⊛ if $\Delta > f(j_1, j_2) \longrightarrow$ this is a generic long multiplet $\mathcal{A}^{\Delta}_{(j_1,j_2)}$

⊛ if $\Delta = f(j_1, j_2) \longrightarrow \Delta = j_1 + j_2 + 2$ (for $j_1, j_2 \neq 0$) $\longleftarrow \mathcal{C}$

$\Delta = j_{1,2} - 1$ (for $j_1 = 0$ or $j_2 = 0$) $\longleftarrow \mathcal{B}_L, \mathcal{B}_R$

$\Delta = 0$ (for $j_1 = j_2 = 0$)

0-norm state $\longrightarrow$ null vectors $(\|\cdot\| = 0) \Rightarrow$ out of the mult.

SHORTER MULTIPLETS

The bounds come from P_μ on CF. What if we act twice on $j_1 = j_2 = 0$?

$$|\psi_2\rangle = P^2 |\Delta; 0, 0\rangle \Rightarrow \|\psi_2\|^2 \propto \Delta(\Delta - 1)$$

if $\Delta \in (0,1)$ then $\|\psi_2\|^2 < 0$

NOT GOOD

$\Rightarrow$ GAP:

~~OK~~ OK

~~OK~~ $\Delta = 1$

NOT OK

~~OK~~ $\Delta = 0 \Rightarrow \mathbb{I}$ operator

$\Delta = 1 \Leftrightarrow P^2 = 0$ i.e.: FREE SCALAR (KG eqn.)

$P^2 |\psi\rangle = 0$

$\Rightarrow \Delta[\phi] = 1 \longrightarrow \Box\phi = 0$

(null state cond = e.o.m. of free scal.)

Allow to turn on some interaction $(\sim \lambda \ldots)$

$\Rightarrow \Delta = 1 + \gamma_\phi(\lambda) \Rightarrow$ I'm moving from $\Delta = 1 \Rightarrow |\psi\rangle$ is no longer a null state $\Rightarrow \Box\phi \sim \lambda \ldots \neq 0$!!!

$$\Delta = f(j_1, j_2) + \gamma(\lambda)$$

Short multiplets obey $\Delta = f(j_1, j_2)$ and can only aquire anom. dimension if they recombine w/ some other mult. and become long multiplets.

- Free fermion:

$$\Delta = j_1 + 1 \quad \text{or} \quad j_2 + 1 \qquad \left(\Delta = \frac{3}{2}\right)$$

$$j_1 + 1 \downarrow \psi_\alpha \to B_L \qquad j_2 + 1 \downarrow \bar{\psi}_{\dot\alpha} \to B_R$$

$\Rightarrow$ must obey Weyl eqn. of motion : $\partial_{\alpha\dot\alpha} \bar{\psi}^{\dot\alpha} = 0 = \partial_{\dot\alpha\alpha} \psi^\alpha$ $\to$ null vectors

$\Rightarrow P_\mu |\psi\rangle$ descendant at lev. 1

- Vector field $\left(\frac{1}{2}, \frac{1}{2}\right) \Rightarrow \Delta = 3 \longrightarrow$ currents! (J_μ)

$C_{(\frac{1}{2},\frac{1}{2})}$ $\quad \Delta = 3 \to J_\mu \Longrightarrow P^\mu J_\mu = 0 \longrightarrow$ CURRENT CONSERVATION

$$\partial^\mu J_\mu = 0$$

$$|\Delta = 4, j_1 = 1, j_2 = 1\rangle \longrightarrow P^\mu |\ldots\rangle = 0 \Rightarrow \partial^\mu T_{\mu\nu} = 0$$

$$C_{(1,1)}$$

Example of recomb:

$$\lim_{\gamma \to 0} \left[\mathcal{A}^{\Delta = j_1 + j_2 + 2 + \gamma}_{(j_1, j_2)} \right] = C_{(j_1, j_2)} \oplus \mathcal{A}^{j_1 + j_2 + 3}_{(j_1 - \frac{1}{2}, j_2 - \frac{1}{2})}$$

SUPERCONFORMAL ALG

$$\{Q, \bar{Q}\} = P$$
$$\{\bar{S}, S\} = K$$
$$\{Q, S\} = D + L + R$$

R-symm (INSIDE THE ALG.)

$$[D, Q] = \frac{1}{2} Q \quad \rightarrow Q \text{ raises } \Delta$$
$$[D, S] = -\frac{1}{2} S \quad \rightarrow S \text{ lowers } \Delta$$

Superconformal primary: $S, \bar{S} |\psi\rangle = 0$

$\downarrow$

build acting with Q

labelling: $|\Delta ; j_1, j_2 ; R\rangle$

Note: act with symm. comb of Qs $\rightarrow \{Q, \tilde{Q}\} |\psi\rangle = P|\psi\rangle \rightarrow$ conformal desc.

" antisys. " $\rightarrow$ NEW! [SCFT multiplet IS FINITE]

Ask $\| Q |\psi\rangle \| \geq 0$? $\Rightarrow$ BPS : $\Delta \geq E_i$

$\rightarrow$ long mult. $\mathcal{A}$ if $\Delta > E_i$

$\rightarrow$ short " $\mathcal{C}$ if $\Delta = E_i$

$\Delta = E_i - 2$

$\rightarrow$ short mult. $\mathcal{B}$ if $\Delta = E_i - 2$

only if $j_i = 0$

What are E_i ?

$\mathcal{N} = 1$:
$$E_1 = 2 + j_1 + \frac{3}{2} r$$
$$E_2 = 2 + j_2 - \frac{3}{2} r$$

$\mathcal{N} = 2$:
$$E_1 = 2 + j_1 + 2R + r$$
$$E_2 = 2 + j_2 + 2R - r$$

e.g. study $\mathcal{B}$ in $\mathcal{N}=2$ th. $(j=0)$:

$$\underbrace{Q_\alpha}_{2\ Q\text{s killing } |\psi\rangle} |\psi\rangle = 0 \longrightarrow |\psi\rangle = |\Delta ; 0, j_2\rangle$$

$\longrightarrow \frac{1}{2}$ BPS $\Rightarrow \Delta = \frac{3}{2} r$

When $|\psi\rangle$ has spin:

$$Q^{\alpha_1}|\psi\rangle_{\alpha_1 \dots} = 0 \longrightarrow \text{only one linear comb. of } Q\text{s} \Rightarrow \frac{1}{4} \text{ BPS shortening}$$

C-type: $\Delta = 2 + 2j_1 + \frac{3}{2}r$

$C \cap \bar{C} = \hat{C} \Rightarrow \Delta = 2 + j_1 + j_2 \Rightarrow \hat{C}_{(\frac{1}{2},\frac{1}{2})} = (\text{R-sym curr., supercurr, stress-energy})$

$B \cap \bar{C}$: $\frac{3}{2}r = j_2 + 1$ set $j_2 = 0 \Rightarrow$ multiplet of Φ

$j_2 = \frac{1}{2} \Rightarrow$ " W_α

NB: for $\mathcal{N}=2$ th. $\exists\, \mathcal{E}_r$ cannot recombine: $\Delta = r \longrightarrow \mathrm{tr}\, \phi^\ell \rightarrow$ 1:1 w/ Coulomb branch