Introduction to Localization

* GAUGE THEORIES on $\mathbb{R}^{1,3}$ $\quad$ [4D $\mathcal{N}=1$]

→ SUSY alg. : $\{ Q_\alpha, \bar{Q}_{\dot\beta}, P_m \}$

s.t.

$$\{Q_\alpha, \bar{Q}_{\dot\beta}\} = \sigma^m_{\alpha\dot\beta} P_m$$
$$\{Q_\alpha, Q_\beta\} = 0$$
$$\{(-)^F, Q_\alpha\} = \{(-1)^F, \bar{Q}_{\dot\beta}\} = [(-)^F, P_m] = 0$$

$$(-)^F |F\rangle = -|F\rangle \; ; \quad (-)^F |B\rangle = |B\rangle$$

→ Spinor convention:

$$m, n = 0, 1, 2, 3$$
$$\alpha, \beta, \dot\alpha, \dot\beta = 1, 2$$
$$\eta_{mn} = (-1, +1, +1, +1)$$
$$\{\gamma^m, \gamma^n\} = -2\eta^{mn}$$

choose $\gamma^m = \begin{pmatrix} 0 & \sigma^m \\ \bar\sigma^m & 0 \end{pmatrix}$ $\qquad \sigma^m = (-\mathbb{I}_2, \vec\sigma)$, $\quad \bar\sigma^m = (-\mathbb{I}_2, -\vec\sigma)$

$$\Gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

→ Dirac spinor: $\quad \psi = \begin{pmatrix} \chi_\alpha \\ \bar\psi^{\dot\alpha} \end{pmatrix}$

Majorana spinor: $\quad \psi = \begin{pmatrix} \chi_\alpha \\ \bar\chi^{\dot\alpha} \end{pmatrix}$ $\qquad \chi_\alpha, \psi_\alpha$ Weyl spinor $[\Gamma^5 = 1]$

→ free Dirac spinor action $[\bar\psi = \psi^\dagger \gamma^0]$:

$$\mathcal{L} = \bar\psi (i\gamma^m \partial_m - m)\psi =$$
$$= -i\left(\bar\chi_{\dot\alpha} \bar\sigma^{m\dot\alpha\alpha} \partial_m \chi_\alpha + \psi^\alpha \sigma^m_{\alpha\dot\alpha} \partial_m \bar\psi^{\dot\alpha}\right) - m\left(\bar\chi_{\dot\alpha} \bar\psi^{\dot\alpha} + \psi^\alpha \chi_\alpha\right) =$$
$$= -i\left(\bar\chi \bar\sigma^m \partial_m \chi + \psi \sigma^m \partial_m \bar\psi\right) - m\left(\bar\chi\bar\psi + \psi\chi\right)$$

where $(\psi^\alpha)^\dagger = \bar\psi^{\dot\alpha}$; $\quad \psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta$; $\quad \psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta$

→ free Weyl (= Majorana) spinor action:

$$\mathcal{L} = i(\partial_m\bar\psi)\bar\sigma^m\psi - \frac{1}{2}m(\psi\psi + \bar\psi\bar\psi)$$

* Simplest SUSY QFT

→ free massless complex scalar + Weyl spinor:

$$\mathcal{L} = \bar\phi\,\Box\phi + i(\partial_m\bar\psi)\bar\sigma^m\psi$$

transf. as total derivative!

⇝ SUSY transf: $\begin{cases} \delta_\xi\phi = \sqrt{2}\,\xi\psi \\ \delta_\xi\psi = i\sqrt{2}\,\sigma^m\bar\xi\,\partial_m\phi \end{cases} \Rightarrow \delta_\xi\mathcal{L}_0 = \partial_m(\text{sth})$

[NB ξ is a spinor parameter]

→ Adding mass term:

$$\mathcal{L}_m = -m^2\bar\phi\phi - \frac{1}{2}m(\psi\psi + \bar\psi\bar\psi) \longrightarrow \delta_\xi\mathcal{L}_m = -\sqrt{2}\,m^2\phi\bar\xi\bar\psi - i\sqrt{2}\,m\psi\sigma^m\bar\xi\,\partial_m\phi + \text{h.c.} =$$

$$= -\sqrt{2}(\Box\phi)\bar\xi\bar\psi + \sqrt{2}\,\partial_n\bar\psi\bar\sigma^n\sigma^m\bar\xi\,\partial_m\phi + \text{h.c.} + \partial_m(\text{sth}) + (\text{e.o.m.})$$

e.o.m.:

$$\begin{cases} \Box\phi - m^2\phi = 0 \\ i\,\partial_m\bar\psi\,\bar\sigma_m - m\psi = 0 \end{cases}$$

$$\boxed{\begin{gathered}\delta_\xi(\mathcal{L}_0 + \mathcal{L}_m) = \partial_m(\text{sth}) + (\text{e.o.m.}) \\ \Downarrow \\ \text{invariant on shell}\end{gathered}}$$

→ SUSY symm ⇒ Noether charge Q

→ SUSY alg: $(\delta_\eta\delta_\xi - \delta_\xi\delta_\eta)X =$

$$= -2i(\eta\sigma^m\bar\xi - \xi\sigma^m\bar\eta)\,\partial_m X + (\text{e.o.m.})$$

where $X = X(\phi, \psi, \bar\phi, \bar\psi)$

⇒ $[\xi Q, \bar\xi\bar Q] = 2\xi\sigma^m\bar\xi P_m \longrightarrow$ ON SHELL FORMALISM

↓

not convenient

⟶ AUXILIARY CONDITIONS

* OFF SHELL FORMALISM

$$\delta_\xi \phi = \sqrt{2}\, \xi \psi$$

$$\delta_\xi \psi = i\sqrt{2}\, \sigma^m \bar{\xi}\, \partial_m \phi + \sqrt{2}\, \xi F \longrightarrow \left(\delta_\eta \delta_\xi - \delta_\xi \delta_\eta\right) X = -2i\left(\eta \sigma^m \bar{\xi} - \xi \sigma^m \bar{\eta}\right) \partial_m X$$

$$\delta_\xi F = i\sqrt{2}\, \bar{\xi}\, \bar{\sigma}^m\, \partial_m \psi$$

$$\rightarrow \mathcal{L}_0 = i\, \partial_m \bar{\psi}\, \bar{\sigma}^m \psi + \bar{\phi}\, \Box \phi + \bar{F} F$$

$$\mathcal{L}_m = m\left(\phi F + \bar{\phi} F - \tfrac{1}{2}\psi\psi - \tfrac{1}{2}\bar{\psi}\bar{\psi}\right)$$

$\longrightarrow$ dofs:

	ON SHELL	OFF SHELL
ϕ	2	2
F	/	2
ψ_α	$4-2$	4
	$2-2=0$	$4-4=0$

From the POV of modes:

"$2 \times \infty - 2 \times \infty = \text{finite}$"

→ SUSY invariant action including interaction, superspace and superfields:

MORE GENERAL

- on shell ⟶ INDEX (in operator formalism)
- off shell ⟶ LOCALIZATION
- superfield ⟶ HOLOMORPHY

MORE USEFUL

→ SUPERFIELDS and EXACT RESULTS

rep of $\{P_m\}$ ⟶ fields $\phi(x)$ on space

rep of $\{P_m, Q, \bar{Q}\}$ ⟶ superfields $\phi(x, \theta, \bar{\theta})$ on superspace $\{x^m, \theta^\alpha, \bar{\theta}_{\dot{\alpha}}\}$

$$\Rightarrow P_m = -i \frac{\partial}{\partial x^m}$$

$$Q_\alpha = \frac{\partial}{\partial \theta^\alpha} - i\, \sigma^m_{\alpha\dot{\alpha}}\, \bar{\theta}^{\dot{\alpha}} \frac{\partial}{\partial x^m}$$

$$\bar{Q}_{\dot{\alpha}} = -\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} + i\, \theta^\alpha \sigma^m_{\alpha\dot{\alpha}} \frac{\partial}{\partial x^m}$$

→ infinitesimal translation:

$$x \to x + c \implies \delta\phi(x) = i\, c^m P_m \phi(x)$$

→ SUSY transf.:

$$\delta_\xi \Phi(x,\theta,\bar\theta) = \left(\xi Q + \bar\xi \bar Q\right) \Phi(x,\theta,\bar\theta)$$

$$\Rightarrow \left(\delta_\eta \delta_\xi - \delta_\xi \delta_\eta\right) \Phi(x,\theta,\bar\theta) = -2\left(\xi\sigma^m\bar\eta - \eta\sigma^m\bar\xi\right) P_m \Phi(x,\theta,\bar\theta)$$

→ Taylor expansion:

$$\Phi(x,\theta,\bar\theta) = A(x) + \theta\psi(x) + (\theta\theta)(\bar\theta\bar\theta) D(x)$$

[A, ψ, D are ordinary fields]

→ $\int d^4x\, d^2\theta\, d^2\bar\theta\; K\left[\Phi(x,\theta,\bar\theta)\right]$ is SUSY invar. (as $\int d^4x\, \mathcal{L}[\phi(x)]$ in transl. inv.)

Taylor ↳ $\int d^4x\; \tilde{\mathcal{L}}\left[A(x), \psi(x), D(x)\right]$ is SUSY inv!

→ THERE ARE TOO MANY FIELDS!

↳ we need an irrep! Not just any rep!

* How?

1) $\bar D_{\dot\alpha} \Phi(x,\theta,\bar\theta) = 0$ → chiral superfield

2) $V(x,\theta,\bar\theta)^\dagger = V(x,\theta,\bar\theta)$ → vector superfield

where
$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\sigma^m_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\frac{\partial}{\partial x^m}$$
$$\bar D_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} - i\theta^\alpha\sigma^m_{\alpha\dot\alpha}\frac{\partial}{\partial x^m}$$

$$\Rightarrow \left\{\bar D_{\dot\alpha}, Q_\beta\right\} = \left\{\bar D_{\dot\alpha}, \bar Q^{\dot\beta}\right\} = 0$$

$\Rightarrow$ CHIRAL SUPERFIELD $\boxed{\bar{D}_{\dot\alpha}\Phi = 0}$

define: $y^m = x^m + i\theta\sigma^m\bar\theta \longrightarrow \bar{D}_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}}$

so that: $\bar{D}_{\dot\alpha}\Phi(x,\theta,\bar\theta) = 0 \Rightarrow \Phi = \Phi(y,\theta) =$
$$= \phi(y) + \sqrt{2}\,\psi_\alpha(y)\theta^\alpha + (\theta\theta)F(y)$$

$\longrightarrow$ then the SUSY transf:

$$\delta_\xi\Phi = (\xi Q + \bar\xi\bar Q)\Phi \Leftrightarrow \begin{cases}\delta_\xi\phi = \sqrt{2}\,\xi\psi \\ \vdots\end{cases}$$ (same as OFF SHELL FORMALISM)

$\longrightarrow$ SUSY invariant action:

$$S = \int d^4x\, d^2\theta\, d^2\bar\theta\; \underbrace{K(\Phi,\Phi^\dagger)}_{\text{Kähler pot.}} + \int d^4x\, d^2\theta\; \underbrace{W(\Phi(y,\theta))}_{\text{Superpotential}} + h.c.$$

$\Rightarrow$ $W(z)$ is a holomorphic funct of z (there is no $\bar z$):

e.g.: Wess-Zumino model:

$$\mathcal{L} = \int d^2\theta\, d^2\bar\theta\; \Phi\Phi^\dagger + \int d^2\theta\left(\frac{m}{2}\Phi\Phi + \frac{\lambda}{3}\Phi^3\right) + h.c. =$$

$$= \underbrace{-\partial_m\bar\phi\,\partial_m\phi + i\,\partial_m\bar\psi\bar\sigma^m\psi + F\bar F}_{\text{Kähler potential}} +$$

Yukawa interactions

$$+ \underbrace{\left(m(\phi F + \psi\psi) + \lambda(\phi^2 F + \phi\psi\psi) + h.c.\right)}_{\text{Superpotential}}$$

NB: $\Phi(y,\theta) = \Phi(x) + i\theta\sigma^m\bar\theta\,\partial_m\phi(x) + \theta\theta F(x) +$ ferm.

- We can integrate $F \Rightarrow \bar{F} = -(m\phi + \lambda\phi^2)$
- potential: $V = \frac{1}{2} F \bar{F}$

$\rightarrow$ VACUA : $\boxed{V = 0} \Longleftrightarrow F = 0$

For general Kähler potential:

NON LINEAR σ MODEL

$$\int d^2\theta d^2\bar\theta \; K(\Phi, \Phi^\dagger) = g_{i\bar{j}} \left(-\partial_m \phi^i \partial_m \bar{\phi}^{\bar{j}} + F^i \bar{F}^{\bar{j}} \right) + \text{fermion}$$

where $g_{i\bar{j}} = \frac{\partial}{\partial\phi^i} \frac{\partial}{\partial\bar{\phi}^{\bar{j}}} K(\phi, \phi^\dagger)$

$\hookrightarrow$ Kähler manifold

(Target space is Kähler)

Holomorphy and superpotential

$\rightarrow$ Consider mass m and coupling λ can be regarded as VEV of CHIRAL SUPERF.

$$M(y, \theta) = m(y) + \ldots + \theta\theta F_m(y)$$

$$L \sim \alpha \int d^2\theta d^2\bar\theta \; M\bar{M}$$

$\alpha \to \infty$ limit $\longrightarrow$ $m' = \frac{m}{\sqrt{\alpha}}$ (other fields decouple and only $m(y)$ survives and remains frozen)

$F_m = 0$ $\qquad$ = const

$\Downarrow$

$m(y)$ is decoupled.

$\sim$ chiral superf.

$\longrightarrow$ $W_{eff}(\Phi, m, \lambda)$ is holom. function in its variables [no $\bar{m}$ nor $\bar{\lambda}$]

$\sim \langle M(y,\theta) \rangle$

$\longrightarrow$ Kähler pot. depend on $\bar{m}, \bar{\lambda}$ as well.

$\Rightarrow$ Wilsonian low energy effective action should be:

$$\mathcal{L}_{eff} = \int d^2\theta d^2\bar\theta \; K_{eff} + \int d^2\theta \; W_{eff}(\Phi, m, \lambda)$$

$\rightarrow$ exact results on W_{eff} will be obtained $\qquad$ $\rightarrow$ Quant. vacua: $\frac{\partial W_{eff}}{\partial\phi} = 0$

e.g.: WZ - model

$$W = \frac{m}{2}\Phi^2 + \frac{\lambda}{3}\Phi^3$$

* $U(1)$ $\begin{cases} \Phi \mapsto e^{i\alpha}\Phi \\ m \mapsto e^{-2i\alpha} m \\ \lambda \mapsto e^{-3i\alpha}\lambda \end{cases}$

* $U(1)_R$ $\begin{cases} \Phi(y,\theta) \mapsto \Phi(y, e^{i\beta}\theta) \\ m \mapsto e^{-2i\beta} m \\ \lambda \mapsto e^{-2i\beta}\lambda \end{cases}$

→ invariance under symmetry: $U(1) \times U(1)_R$

$$\frac{\lambda\Phi}{m} \text{ inv} \Rightarrow W_{eff} = m\Phi^2 f\left(\frac{\lambda\Phi}{m}\right) \qquad \Rightarrow f\left(\frac{\lambda\Phi}{m}\right)$$ is invariant (and arbitrary in principle)

⇒ WEAK COUPLING LIMIT: $\left(\lambda \to 0,\ m \to 0,\ \frac{\lambda}{m} = \text{fixed}\right)$

no neg. powers of m allowed

$$W_{eff} \to \frac{m}{2}\Phi^2 + \frac{\lambda}{3}\Phi^3 \Rightarrow f(t) \sim 1 + \frac{1}{3}t \qquad \left(t = \frac{\lambda\Phi}{m^2}\right)$$

⇒ SUPERPOT does not have quantum correct (protected)

↳ NOT renormalized! ⇒ exact result!

* If W has flat dimens (= continuous vacua and massless particles) then

IT IS FLAT INCLUDING QUANTUM EFFECTS!

[Not Goldstone theorem]

→ ∞ many vacua!

Define a MODULI space of all the vacua:

Now consider the vector superfield $V^\dagger = V$

$G = U(1)$ case $\longrightarrow W_\alpha = \bar{D}_{\dot\alpha} \bar{D}^{\dot\alpha} D_\alpha V \Rightarrow \bar{D}_{\dot\alpha} W_\alpha = 0 \Rightarrow$ chiral!

$\rightarrow \int d^2\theta\, W^\alpha W_\alpha$ is SUSY invariant

gauge symm.

$\left[W_\alpha \text{ is invariant under } V \longrightarrow V + \Phi + \Phi^\dagger \right]$

$$\Rightarrow V = -\theta\sigma^m\bar\theta\, v_m(x) + i(\theta\theta)(\bar\theta\bar\lambda) - i(\bar\theta\bar\theta)(\theta\lambda(x)) + \tfrac{1}{2}(\theta\theta)(\bar\theta\bar\theta) D(x)$$

(WZ gauge) $\quad \hookrightarrow v_m, D$ are <u>REAL</u>

s.t.: $v_m \longmapsto v_m + \partial_m a$

$$\Longrightarrow W_\alpha = -i\lambda_\alpha + \theta D - \frac{i}{2}(\sigma^m\bar\sigma^n)_\alpha{}^\beta \theta_\beta F_{mn} + \theta\theta\, \sigma^m_{\alpha\dot\alpha} \partial_m \bar\lambda^{\dot\alpha}$$

$F_{mn} = \partial_m v_n - \partial_n v_m$ [i.e. $U(1)$ field str.]

Then the SUSY invariant action is:

$$\mathcal{L} = -\frac{1}{16\pi} \int d^2\theta \left(i\tau W^\alpha W_\alpha \right) + \text{h.c.} =$$

$$= -\frac{1}{4g^2}\left(F_{mn}^2 + 4i\lambda\sigma^m D_m\bar\lambda - 2D^2 \right) + \frac{\theta}{64\pi^2} \varepsilon^{mnpq} F_{mn} F_{pq}$$

$\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{g^2} \Rightarrow \theta \to \theta + 2\pi$ (quant. of instanton no.)

$\Rightarrow$ LOW ENERGY EFFECTIVE ACTION

$\int d^2\theta\, \tau_{eff} W^\alpha W_\alpha + \text{h.c.} + \ldots$

preservation of SUSY!

$\hookrightarrow \tau_{eff} = \tau_{eff}(\tau, \cancel{\bar\tau})$ [holomorphic funct.]

$\rightarrow$ β funct. for τ: $\quad \beta(\tau) = \dfrac{d\tau_{eff}(\mu)}{d\ln\mu}$

$\hookrightarrow \beta$ should be invariant under $\tau \mapsto \tau + 1$ (i.e.: $\theta \to \theta + 2\pi$)

$$\Rightarrow \beta(\tau) = \sum_{n=0}^{\infty} e^{2\pi i \tau n} f_n(\mu)$$

Thus perturbatively $\beta(\tau)$ is τ-indep:

$\Rightarrow$ τ comes from instanton effects (not tree level)

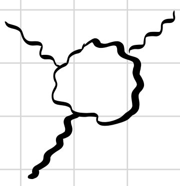
$\hookrightarrow$ $\beta(\tau)$ is $1L$ exact $\quad (n=0)$
(1-loop)

$$\rightarrow \beta = \frac{d}{d\ln\mu}\left(\frac{\theta}{2\pi} + i\frac{4\pi}{g^2}\right) = -8\pi i \frac{1}{g^3}\frac{dg}{d\ln\mu}$$

where $\dfrac{dg}{d\ln\mu} = -\underbrace{\dfrac{b}{16\pi^2}g^3}_{1L} + \underbrace{\# g^4}_{2L} + \dots$

$1L \to g^3$ simplify (τ indep!)

$2L \to O(g)$



We now define Λ st.: (analogous of QCD scale)

$$\frac{\theta}{2\pi} + \frac{4\pi}{ig^2(\mu)} = i\frac{b}{2\pi}\ln\frac{\mu}{\Lambda}$$

$$\Rightarrow g^2(\mu = |\Lambda|) = \infty \qquad (\Lambda \to \text{dynamic scale})$$

$$\rightarrow \left(\frac{\Lambda}{\mu}\right)^b = e^{2\pi i \tau_{eff}^{(pert)}}$$

Non abelian case

$W_\alpha = -\frac{1}{4}\bar{D}\bar{D}\left(e^{-V}D_\alpha e^V\right)$ with gauge transf. $e^V \to e^{-i\Lambda^\dagger}e^V e^{i\Lambda}$

$W_\alpha \to e^{-i\Lambda}W_\alpha e^{i\Lambda}$

$$\Rightarrow \text{SUSY inv}: \mathcal{L} = -\frac{1}{8\pi}\int d^2\theta\, \mathrm{Tr}(i\tau W^\alpha W_\alpha) + \text{h.c.}$$

Gauge field + matter

$$\Phi \to \Phi' = e^{-i\Lambda}\Phi$$

$\Rightarrow \quad \Phi^\dagger e^V \Phi$ is gauge inv. $\longrightarrow$ Kähler pot.

$$\Rightarrow \mathcal{L} = \int d^4\theta \ \Phi^\dagger e^V \Phi = -D_m\phi^\dagger D_m\phi - i\bar\psi\bar\sigma^m D_m\psi + F^\dagger F + i\sqrt{2}\left(\phi^\dagger T^a \psi\lambda^a - \right.$$
$$\left. - \bar\lambda^a\bar\psi T^a\phi\right) + D^a\left(\phi^\dagger T^a\phi\right)$$

$\longrightarrow SU(N)$

$\Rightarrow$ Total Lagr. includes: $\frac{1}{2g^2}\left(D^a\right)^2 + \underbrace{D^a}_{\text{auxiliary field}}\left(\phi^\dagger T^a\phi\right)$

$\to D^a = -g^2\left(\phi^\dagger T^a\phi\right)$

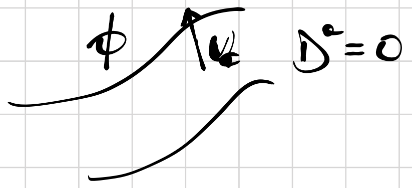
potential: $V = \frac{g^2}{2}\left(\phi^\dagger T^a\phi\right)^2 = \frac{1}{2g^2}\left(D^a\right)^2$

$\Rightarrow$ classical vacua $\quad V = 0 \iff \begin{cases} D^a = 0 \\ F = \frac{\partial W}{\partial\phi} = dW = 0 \end{cases}$

$\to$ space of $\{D^a = 0\}$ is inv. under gauge transf. $e^{i\alpha_a T^a}$, $\alpha \in \mathbb{R}$

but orthogonal to complexified gauge transf:

$$U_{\mathbb{C}} = e^{\alpha_a T^a}$$



$\Rightarrow D^{(a)} = 0$ as gauge fixing conditions of complexified gauge transf

i.e.: $\quad \{D^a = 0\}\big/_{\text{gauge transf}} = \dfrac{\{\text{all conf}\}}{\{\text{compl. gauge trans.}\}}$

$\to$ gauge inv. oper. including Φ only (i.e. not $\phi^\dagger$) are inv under $U_{\mathbb{C}}$ also

$\Rightarrow$ classical moduli space of vacua is parametrized by VEV of invariant chiral superf. (= holomorphic) restricted by $F=0$

→ $\mathcal{N}=1$ SQCD (hep-th/9509066)

* gauge group: $SU(N_c)$
* matter: N_f flavours
* superfields: W_α vector (adj)
  $Q^i, \tilde{Q}_i$ chiral $(N_c)\ (\bar{N}_c)$ $i = 1, \dots, N_f$

Action:

$$\mathcal{L} = -\frac{1}{8\pi} Tr\left(\int d^2\theta \, \tau (W_\alpha W^\alpha) \right) + \int d^4\theta \left(Q^i e^V (Q^i)^\dagger + \tilde{Q}_i e^V (\tilde{Q}_i)^\dagger \right) +$$

$$+ \int d^2\theta \left(m_i{}^j Q^i \tilde{Q}_j \right) + \text{h.c.}$$

Moduli space of vacua param by GAUGE INVARIANT CHIRAL SUPERFIELD

"meson": $M^i{}_j = Q_r^i \tilde{Q}^r_j$ $r = 1, \dots, N_c$

"baryon": $B^{i_1 \dots i_{N_c}} = Q^{i_1}_{r_1} \dots Q^{i_{N_c}}_{r_{N_c}} \epsilon^{r_1 \dots r_{N_c}}$

$\tilde{B}_{i_1 \dots i_{N_c}} = \tilde{Q}^{r_1}_{i_1} \dots \tilde{Q}^{r_{N_c}}_{i_{N_c}} \epsilon_{r_1 \dots r_{N_c}}$ (for $N_f \geq N_c$)

Classically they are not indep: e.g.: $N_f = N_c \Rightarrow \det M = \tilde{B} B$

$$\beta_g = \frac{dg}{d \ln \mu} = -\frac{b}{16\pi^2} g^3, \quad b = 3N_c - N_f \quad \text{(exact)}$$

→ define Λ s.t. $\left(\frac{\Lambda}{\mu}\right)^{3N_c - N_f} = e^{2\pi i \tau_{eff}}$

⇒ GLOBAL SYMM.:

$$U(1)_X : \begin{cases} Q(\theta) \longrightarrow Q(e^{-i\alpha}\theta) \\ \tilde{Q}(\theta) \longrightarrow Q(e^{-i\alpha}\theta) \end{cases}$$

$$U(1)_B : \begin{cases} Q \longrightarrow e^{i\alpha} Q \\ \tilde{Q} \longrightarrow e^{-i\alpha} \tilde{Q} \end{cases}$$

→ these can have anomalies

$$U(1)_A : \begin{cases} Q \longrightarrow e^{i\alpha} Q \\ \tilde{Q} \longrightarrow e^{i\alpha} \tilde{Q} \end{cases}$$

$$\int \mathcal{D}\psi\mathcal{D}\bar\psi \, e^{\int \bar\psi \not{D} \psi} \to \int \mathcal{D}\psi\mathcal{D}\bar\psi \, e^{\int \psi \not{D} \psi + \dots \int F\wedge F}$$

$$\Rightarrow U(1)_R: \quad J^\mu_R = J^\mu_X + \frac{N_f - N_c}{N_f} J^\mu_A \Rightarrow \partial_\mu J^\mu_R = 0 \quad \checkmark$$

$$U(1)_A: \quad \theta \to \theta + \alpha \Rightarrow e^{\dots\theta \int F\wedge F} \longrightarrow e^{\dots \alpha \int F\wedge F}$$

WE CAN CANCEL THE ANOMALY TERMS BY SUMMING THESE!

we also need: $\Lambda^{3N_c - N_f} \to e^{2N_f i\alpha}\Lambda^{3N_c-N_f}$

Therefore:

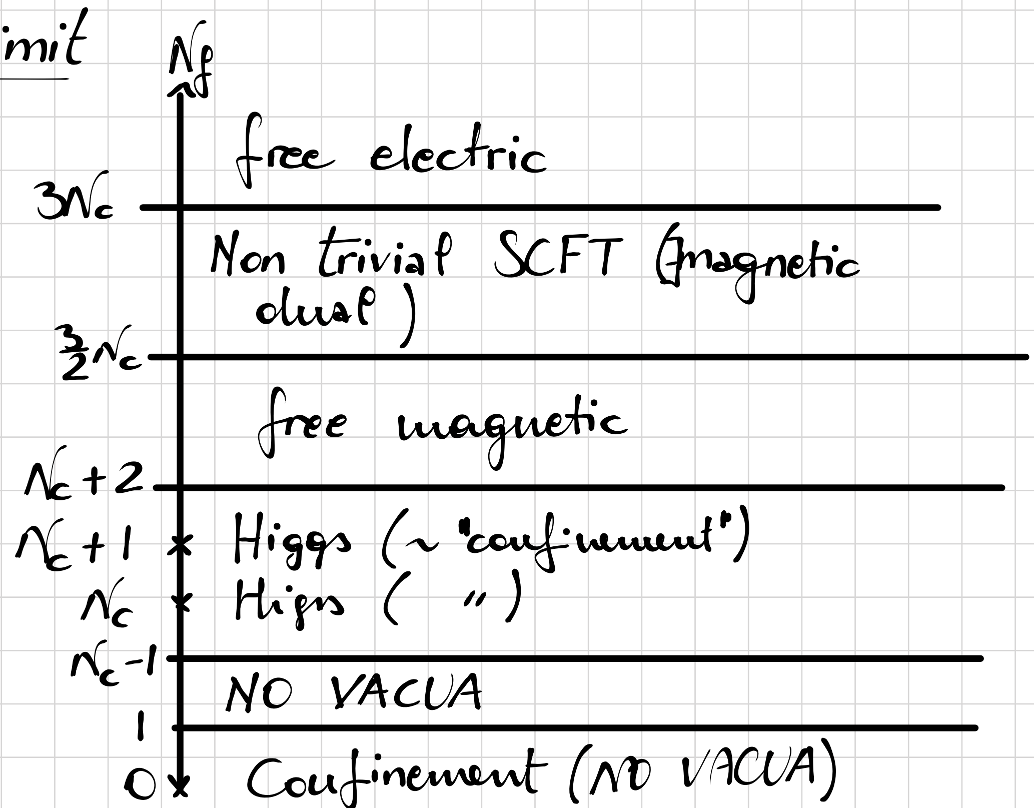
	$SU(N_f)_L$	$\times\ SU(N_f)_R$	$\times\ U(1)_A$	$\times\ U(1)_B$	$\times\ U(1)_R$
Q	N_f	1	1	1	$\frac{N_f-N_c}{N_f}$
$\tilde Q$	1	$\bar N_f$	1	-1	$\frac{N_f-N_c}{N_f}$
$\Lambda^{3N_c-N_f}$	1	1	$2N_f$	0	0
m	$\bar N_f$	N_f	-2	0	$2-2\frac{N_f-N_c}{N_f}$ (s.t.: $m_i^{\ j} M_j^{\ i}$ invariant)
M	N_f	$\bar N_f$	2	0	$\frac{2(N_f-N_c)}{N_f}$

Therefore:

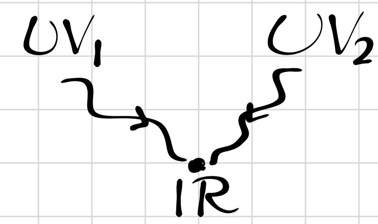
$$W_{eff} = \# \left(\frac{3\Lambda^{3N_c-N_f}}{\underset{\overset{\shortparallel}{M}}{\det(Q\tilde Q)}} \right)^{\frac{1}{N_c-N_f}} + m^i_{\ j} M_i^{\ j} \qquad (N_c < N_f)$$

$$\Rightarrow \langle M \rangle = m^{-1}\left(\Lambda^{3N_c-N_f} \det m\right)^{1/N_c} \longrightarrow N_c \text{ vacua}$$

Low Energy Limit



Seiberg dual (conjecture)



$\Rightarrow$ 2 UV theories with same IR completion

$\exists$ "magnetic theory" whose low en. limit is the SCFT $\left(SU(N_c)\ N_f = 3N_c\right)$

$\hookrightarrow$ $SU(N_c')$, N_f flavours $\oplus$ N_f^2 singlets
$N_c' = 2N_c$

$\tilde{W}_\alpha$ (adj) $\qquad q_i\ (N_c')$ $\qquad \tilde{q}_i\ (N_c')$ $\qquad M^i{}_j \quad i = 1, \dots, N_f$

$$W = \frac{1}{\hat{\Lambda}} M_i{}^j q^i \tilde{q}_j \qquad \left(\frac{\hat{\Lambda}}{\mu}\right)^{3N_f - N_c} = e^{2\pi i \tilde{\tau}_{eff}}$$

$\Rightarrow$ gauge groups are diff but they are dual theories
$\rightarrow$ global symm <u>MUST MATCH</u>

electric $\rightarrow$ $G = SU(N_c)$ $\ N_f$ flavours SQCD $\ m = 0$
magnetic $\rightarrow$ $G = SU(N_f - N_c)$ $\ N_f$ flavours $\oplus$ N_c^2 singlets (gauge)

$$W = \frac{1}{\hat{\mu}} M^i{}_j q_i \tilde{q}^j$$

$$\hat{\mu}^{N_f} = (-1)^{N_c - N_f} \Lambda^{3N_c - N_f} \tilde{\Lambda}^{3(N_f - N_c)/N_f} \qquad \text{for } 3N_c \geq N_F \geq N_c$$

Plus

EL.	MAG.
$Q^i \tilde{Q}_j$	$M^i{}_j$
$B^{i_1 \dots i_{N_c}}$	$b_{j_1 \dots j_{N_c}} \epsilon^{i_1 \dots i_{N_c} j_1 \dots j_{N_c}}$
	$\frac{\partial W}{\partial M^i_j} = q_i \tilde{q}^j = 0$

Now consider the mass term for one flavour

electric. $\delta W = m\, Q_{N_f} \tilde{Q}_{N_f}$ (decoupled @ low energy)

$\hookrightarrow SU(N_c)\ (N_f - 1)$ flavours

magnetic. $SU(N_f - N_c)$ N_f flavours

$$\hookrightarrow W = \frac{1}{\tilde{\mu}} M q \tilde{q} + m M^{N_f}{}_{N_f}$$

low en $\rightarrow$ int out $M^{N_f}{}_{N_f} \longrightarrow \frac{\partial W}{\partial M} = \frac{1}{\tilde{\mu}} q_{N_f} \tilde{q}^{N_f} + m = 0$

$$\Rightarrow q_{N_f} \tilde{q}^{N_f} \longrightarrow \frac{1}{\tilde{\mu}} M^{N_f} \tilde{q}^j = 0$$

$$M^i{}_{N_f} \longrightarrow q^i \tilde{q}^{N_f} = 0$$

$M^{N_f}{}_j = M^i{}_{N_f} = 0$ $\qquad q^{N_f} \sim \left.\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}\right\} N_c$

$$\tilde{q}_{N_f} \sim \underbrace{(1, \dots, 0)}_{N_c}$$

Higgs effect: $SU(N_c') \longrightarrow SU(N_c' - 1)$

$$N_f \longrightarrow N_f - 1$$

$$W = \frac{1}{\tilde{\mu}} M^i_j\, q_i \tilde{q}^j$$

$\Rightarrow$ Magnetic th. is IR free for $N_f > 3N_c' \Longleftrightarrow N_f < \frac{3}{2} N_c$

$\Rightarrow \mathcal{N}=1$ $SU(N_c)$ SQCD of N_f flavours becomes free non Abelian theory in low energy for $N_c + 2 \leq N_f < \frac{3}{2} N_c$

$\mathcal{N}=2$ SUSY YM (hep-th/9901069)

→ ALGEBRA:

$$\{Q^A_\alpha, (Q^B_\beta)^\dagger\} = 2\sigma^m_{\alpha\dot\beta} P_m \delta^A{}_B \qquad A,B=1,2$$

$$\{Q^A_\alpha, Q^B_\beta\} = 2\varepsilon_{\alpha\beta}\bar{Z}\varepsilon^{AB}$$

$$\{(Q^A_\alpha)^\dagger, (Q^B_\beta)^\dagger\} = 2\varepsilon_{\dot\alpha\dot\beta} Z\varepsilon^{AB}$$

Z is CENTRAL CHARGE (i.e.: $[Z,Q]=[Z,Q^\dagger]=0$)

→ massive case: $P^0=-2M$, $P^i=0$

$$a_\alpha = \frac{1}{\sqrt2}\left(\frac{1}{\sqrt Z} Q^1_\alpha + \varepsilon_{\alpha\beta}\frac{1}{\sqrt{\bar Z}}(Q^2_\beta)^\dagger\right)$$

$$b_\alpha = \frac{1}{\sqrt2}\left(\frac{1}{\sqrt Z} Q^1_\alpha - \varepsilon_{\alpha\beta}\frac{1}{\sqrt{\bar Z}}(Q^2_\beta)^\dagger\right)$$

$$\Rightarrow \{a_\alpha,(a_\beta)^\dagger\} = 2\delta^\alpha_\beta\left(\frac{M}{|Z|}+1\right)$$
$$\{b_\alpha,(b_\beta)^\dagger\} = 2\delta^\alpha{}_\beta\left(\frac{M}{|Z|}-1\right)$$

$\Rightarrow$ must be $\geq 0 \longrightarrow \boxed{M \geq |Z|}$ (BPS bound)

if $M=|Z| \Rightarrow b_\alpha = 0$ i.e. SUSY of b_α IS NOT broken!

$\Rightarrow$ # of states is $1/2 \longrightarrow \frac12$ <u>BPS states</u>

⇝ $\mathcal{N}=2$ SU(N) SYM

in <u>$\mathcal{N}=1$ superfield</u> notation — the only interaction is GAUGE int.

$\Rightarrow$ SU(N) vector multiplet $W_\alpha \supset \{\lambda(x), V_m(x)\}$ (adj rep.)

$\Rightarrow$ chiral multiplet $\Phi \supset \{\psi(x), \phi(x)\}$ (adj. rep.)

→ λ and ψ has same action $\Rightarrow$ exchange $\lambda \longleftrightarrow \psi$ is a symmetry

i.e.:

$$\begin{array}{ccc} \lambda & & \phi \\ \text{susy}\updownarrow & \searrow & \updownarrow\text{susy} \\ V_m & & \psi \end{array}$$

→ this is $\mathcal{N}=2$ susy! ($\sim$ 2 copies of sth.)

→ CLASSICAL MODULI SPACE

$$D^{\alpha} = -\frac{g^2}{2}\left(\phi^{\dagger} T^{a} \phi\right) = 0$$

N_c^2-1 vector

$(N_c^2-1)\times(N_c^2-1)$ matrix in adj rep.

we will denote $\phi = \phi^{a} T_N^{a}$ → fields are $N\times N$ matrix

$$\Rightarrow [\phi, \phi^{\dagger}] = 0$$

→ Hermitian combination:

$$\phi + \phi^{\dagger} = U D U^{\dagger}$$
$$i(\phi - \phi^{\dagger}) = U E U^{\dagger}$$

(D is diagonal, U is unitary, E is diag)

$$\hookrightarrow [\phi, \phi^{\dagger}] = 0 \Leftrightarrow [D, E] = 0$$

$$\Rightarrow \phi = U \begin{pmatrix} a_1 & & & 0 \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_N \end{pmatrix} U^{\dagger} \longrightarrow a \text{ is complex}$$

$\Longleftrightarrow \mathrm{Tr}(\phi^n)$ $n = 2, \dots, N_c$ are moduli.

$\Rightarrow$ In almost all vacua: $SU(N) \xrightarrow{\text{broken to}} U(1)^{N_c-1} \Rightarrow$ Coulomb phase

→ Below $N=2$ → $\phi = \frac{1}{2} U \begin{pmatrix} a & \\ & -a \end{pmatrix} U^{\dagger} \Rightarrow u = \mathrm{tr}\,\phi^2 = \frac{1}{2}a^2$

Semi-classically we can compute $Q, \bar{Q}$ and $Z = \{Q, Q\}$

$$\longrightarrow Z = \int d^2x\ \partial_n \left(\mathrm{Tr}\left(\phi E^{n} + i\frac{1}{g^2}\phi B^{n}\right)\right) \quad \text{i.e.: } SU(2) \xrightarrow{\text{broken to}} U(1)$$

$E^n \to F^{on}$, $B^n \to \varepsilon^{okln} F_{kl}$

$$= n_e a + n_m (\tau a)$$

n_e: electric charge, n_m: magnetic charge

Quantum mechanically: define a s.t.: $Z = a n_e + n_m a_D(a)$

$\Rightarrow$ mass of BPS state is $M = |Z| = |a|$ with $n_e = 1$.

In $\mathcal{N}=2$ superspace $\{\theta^{(1)}_\alpha, \theta^{(2)}_\alpha, \bar\theta^{(1)}_{\dot\alpha}, \bar\theta^{(2)}_{\dot\alpha}\}$

in the sense → there are no $\bar{\mathcal{A}}$

$\mathcal{N}=2$ vector multiplet $\mathcal{A}$ described by one holomorphic func $\mathcal{F}(\mathcal{A})$ ⇒ PREPOTENTIAL

"comes before the Kähler pot"

$$\rightarrow \mathcal{L} = \frac{1}{4\pi}\,\mathrm{Im}\left\{ \int d^4\theta\, \frac{\partial \mathcal{F}(\mathcal{A})}{\partial \mathcal{A}}\,\bar{\mathcal{A}} + \int d^2\theta\, \frac{1}{2}\frac{\partial^2 \mathcal{F}(\mathcal{A})}{\partial \mathcal{A}^2}\, W^\alpha W_\alpha \right\} \quad \text{i.e.: } K = \mathrm{Im}\left(\frac{\partial \mathcal{F}(\mathcal{A})}{\partial \mathcal{A}}\bar{\mathcal{A}}\right) \quad \tau(a) - \frac{\partial^2 \mathcal{F}(\mathcal{A})}{\partial \mathcal{A}^2}$$

For $a \gg \Lambda$, instanton expansion:

$$\mathcal{F} = \underbrace{\frac{i}{2\pi}\mathcal{A}^2 \ln\left(\frac{\mathcal{A}^2}{\Lambda^2}\right)}_{1L} + \underbrace{\sum_{n=1}^{\infty} F_n \left(\frac{\Lambda}{\mathcal{A}}\right)^{4n} \mathcal{A}^2}_{\text{instanton}}$$

$$\Rightarrow a_D = \frac{\partial \mathcal{F}(a)}{\partial a} = \frac{i}{\pi} a\left(\ln\frac{\mathcal{A}^2}{\Lambda^2} + 1\right) + \sum_{n=1}^{\infty} F_n\,(2-4n)\left(\frac{\Lambda}{\mathcal{A}}\right)^{4n} a$$

$$\Rightarrow \tau = \frac{\partial a_D}{\partial a}\,; \quad K = \mathrm{Im}\,(a_D \bar{a})$$

→ $U(1)$ gauge theory is electromagn dual:

$$E \longleftrightarrow B \Rightarrow \underline{g^2 \longleftrightarrow \frac{1}{g^2}} \qquad (\text{i.e.: strong} \longleftrightarrow \text{weak coupling})$$

→ with θ-term, it becomes $\begin{cases} \tau \to -1/\tau & \longrightarrow S = \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix} \\ \tau \to \tau+1 & \longrightarrow T = \begin{pmatrix} 1 & 1 \\ 0 & 1\end{pmatrix}\end{cases}$

i.e.: $\tau \mapsto \frac{a\tau + b}{c\tau + d} \rightarrow \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in SL(2,\mathbb{Z})$

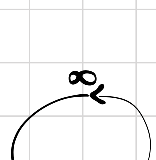
* in this case: $\tau = \frac{\partial a_D}{\partial a} \Rightarrow \begin{pmatrix} a_D \\ a \end{pmatrix}' = \begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix} a_D \\ a \end{pmatrix}$

plus $(n_m \;\; n_e) \longrightarrow (n_m \;\; n_e)\begin{pmatrix} a & b \\ c & d\end{pmatrix}^{-1}$

➡ $Z = n_e a + n_m a_D$ <u>is</u> $SL(2,\mathbb{Z})$ invariant

Now consider moving in moduli space

Lu



$u = R e^{i\varphi}$, $R \to \Lambda$

$\varphi = 0 \longrightarrow \varphi = -2\pi$

$\Rightarrow u \sim \frac{1}{2}a^2 \Rightarrow a \to -a$

$$a_D = \frac{i}{\pi} a\left(\ln \frac{a^2}{\Lambda^2} + 1\right) + \dots \longrightarrow a_D' = a_D + 2a$$

$$\Rightarrow \begin{pmatrix} a_D \\ a \end{pmatrix} \to M_\infty \begin{pmatrix} a_D \\ a \end{pmatrix} \qquad M_\infty = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$$

$M_\infty \neq \mathbb{I} \longrightarrow$ other singularities in upper plane

MASSLESS BPS STATE

$\longrightarrow$ Symm $u \to -u \rightsquigarrow \mathbb{Z}$ singular ($u \neq 0$ not allowed)

$\Lambda^2 \quad -\Lambda^2 \quad \infty$ $\longrightarrow$ as a convention at $u = \Lambda^2$ monopole $(n_m, n_e) = (1, 0)$ becomes massless

$\longrightarrow \mathcal{N}=2$ SQED $\Longrightarrow \mathcal{N}=2$ vector $(A_0, W_{0\alpha})$

hypmul $(M, \tilde{M})$ $\qquad \mathcal{N} = \sqrt{2} A_0 M \tilde{M}$

$$* \; M_1 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \qquad M_\infty = M_1 M_{-1}$$

$$M_{-1} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} \longrightarrow \text{dyon } (1,-1) \text{ becomes MASSLESS}$$

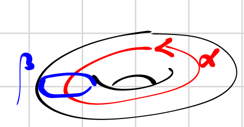
$\Rightarrow$ Singularities are determined using u:

$\mathcal{F}(\mathcal{A})$ is also det $\Longleftrightarrow \begin{cases} a(u) \\ a_D(u) \end{cases}$ are det.

Consider a torus $\quad y^2 = (x + \Lambda^2)(x - \Lambda^2)(x - u) \qquad$ SW curve

$$\rightarrow \lambda = \frac{1}{\sqrt{2}\pi} \frac{x-u}{y} dx \qquad \text{SW 1-form} \qquad \left(\frac{\partial \lambda}{\partial u} \text{ regular}\right)$$

$$a = \int_\alpha \lambda$$



$$\tau = \frac{\partial a_D}{\partial a} = \frac{\int_\beta \partial\lambda/\partial u}{\int_\alpha \partial\lambda/\partial u} = \tau_{TORUS} \quad !$$

$$a_D = \int_\beta \lambda$$

these tori appear by chance in $\mathcal{N}=2$ theories
we don't know why yet!

NB: IIA on CY_3
M5 on Σ $\Big\} \Longrightarrow$ exactly this kind of torus!

Index and Localization

(arXiv: 1608.02952)

→ Witten index.

Assuming $H = \{Q, Q^\dagger\}$

$[H, Q] = 0\,,\quad Q^2 = 0$

$\{(-1)^F, Q\} = 0 \qquad ((-1)^F)^2 = 1$

$\Rightarrow \mathrm{Tr}\left((-1)^F e^{-tH}\right)$ is t-indepent [Witten index]

Why?

$Q_B = Q + Q^\dagger \qquad Q_B^\dagger = Q_B \Rightarrow Q_B^2 = H \geq 0$

* consider $H|i\rangle = E_i |i\rangle$ subspace

→ $(-1)^F |b\rangle = |b\rangle$

$(-1)^F |f\rangle = -|f\rangle$

For $E_i \neq 0$, $\forall |b\rangle \overset{1:1\ \text{rel}}{\longleftrightarrow} Q_B|b\rangle = |f\rangle \quad \left(\frac{Q_B}{\sqrt{E_i}}|f\rangle = |b\rangle\right)$

⇒ They DO NOT CONTRIBUTE TO INDEX ⇒ $\mathrm{Tr}(\ldots)|b,f\rangle = 0$

↳ index is t-independent!

For $E_i = 0$, $\underbrace{Q_B^2}_{H}|i\rangle = 0 \longrightarrow \langle i|Q_B^2|i\rangle = |Q_B|i\rangle|^2 = 0 \Leftrightarrow Q_B|i\rangle = 0$

⇒ $|i\rangle$ is <u>Q_B-closed</u> (there are no Q_B-exact states for $H|i\rangle = 0$ space, because if $|i\rangle = Q_B|j\rangle$ then $H|j\rangle = Q_B|j\rangle = 0$)

⇒ INDEX COUNTS THE NO. OF 0-MODES (it is quantized and invariant under (small) deformations of the theory)

↳ we can compute this in any case (weak, strong coupling, etc...)

→ Generalized WITTEN INDEX

$$Z(t) = Tr\left((-)^F \prod_\alpha \theta_\alpha \, e^{-t\{Q_B, V\}}\right) \qquad \text{where } [Q_B^2, V] = [Q_B, \theta_\alpha] = 0$$

$$\Rightarrow \frac{d}{dt} Z(t) = 0 = Tr\left(\left((-)^F \prod_\alpha \theta_\alpha \left(-\{Q_B, V\}\right) e^{-t\{Q_B,V\}}\right)\right) =$$

$$= - Tr\left((-1)^F \left\{Q_B, \prod_\alpha \theta_\alpha V e^{-t\{Q_B,V\}}\right\}\right) = 0$$

} Jacobi identity

$$[Q_B, \{Q_B, V\}] = Q_B^2 V + Q_B V Q_B - Q_B V Q_B - V Q_B^2 = [Q_B^2, V] = 0$$

because $Tr\left((-1)^F \{Q_B, A\}\right) = Tr\left((-1)^F Q_B A + Q_B (-1)^F A\right) = 0$

⇒ OPERATOR FORMALISM:

(SUSY) QFT on $S^1 \times \mathcal{M}_{d-1}$ (S^1 → TIME)

→ In path integral we can consider compact manifold ⇒ LOCALIZATION TECNIQUE

→ δ is a symmetry of the theory (i.e.: $\delta S = 0$) → NO ANOMALIES
(⇒ δ^2 is as well)

Consider

$$Z(t) = \int \mathcal{D}\phi \, e^{-S - t\delta I} O_1 \dots O_n$$

↳ $Z(0) = \langle O_1 \dots O_n \rangle$ where $\delta O_i = 0$

δI s.t. $\begin{cases} \delta^2 I = 0 \\ \delta I\big|_{\text{bos. part}} > 0 \end{cases}$ the ferm. $\int$ acts as as deriv → we don't need the bound

$$\Rightarrow \frac{dZ(t)}{dt} = -\int \mathcal{D}\phi \, \delta I \, e^{-S - t\delta I} O_1 \dots O_n = -\int \mathcal{D}\phi \, \delta\Big(\underbrace{I e^{-S - t\delta I} O_1 \dots O_n}_{A(\phi)}\Big) = 0$$

(O_i are inv.)

because $\int \mathcal{D}\phi \, \delta A(\phi) = \int \mathcal{D}\phi \, A(\phi + \delta\phi) - \int \mathcal{D}\phi \, A(\phi) = \int \mathcal{D}\phi' \, A(\phi') - \int \mathcal{D}\phi \, A(\phi) = 0$
$\left(\phi' = \phi + \delta\phi \, ; \, \mathcal{D}\phi' = \mathcal{D}\phi\right)$

Then $Z(t=0) = \langle O_1 \dots O_n \rangle = \lim_{t\to\infty} Z(t)$ (because t-indep):

$Z(t)$ contains $e^{-t\delta I}$ ($t \to \infty$) ⇒ the path integral is localized on saddle points ϕ_0 s.t. $\delta I(\phi_0) = 0$

$\delta I > 0$ is key to the localization tecnique

The saddle point "approx." is EXACT in this case $(t \to \infty)$!

includes 1L factor for $\delta I(\phi)$

$$\Rightarrow \langle O_1 O_2 \dots O_n \rangle = \int_{\substack{\text{saddle}\\ \text{point}}} \mathcal{D}\phi_0 \, e^{-S(\phi_0)} \times (\text{1L factor}) \longrightarrow \int_{\substack{\text{saddle}\\ \text{point}}} d\phi_0 \, e^{-S(\phi_0)} \times (\text{1L factor})$$

physical oper (BRST closed)

"normal" integration measure

$\rightsquigarrow e^{-t\delta I} \sim$ "gauge fixing term"

Let us consider a SUSY theory and choose a SUSY generator δ ($\delta^2 =$ bosonic symmetry)

↳ Then define $I = \int \sum_i (\delta\lambda_i)^T \lambda_i$ (λ_i are fermions)

↳ any kind of indices!

$\Rightarrow \delta^2 I = 0$ because all indices i are contracted, thus I is invariant under bosonic symmetry

$$\longrightarrow \delta I\big|_{\text{bosonic}} = \int \sum_i (\delta\lambda_i)^\dagger (\delta\lambda_i) \geq 0$$

there is a subtelty for Euclidean Space

$\rightarrow$ For any SUSY theory we can apply localization tecnique to compute BPS correlators EXACTLY

SUSY QFT on $\mathbb{R}^{1,10}$ ($\mathcal{N}=2$ SUSY QM or Witten model)

$$\rightarrow \mathcal{L} = \frac{1}{2}(\partial_t q)^2 + \frac{1}{2}F^2 + i\bar\eta \frac{\partial\eta}{\partial t} + \underbrace{\frac{\partial W}{\partial q}}_{W'} F + \underbrace{\frac{\partial^2 W}{\partial q \partial q}}_{W''} \bar\eta \eta$$

* q, F are real
* $\eta, \bar\eta$ are independent

$\rightarrow$ SUSY transformation is:

$$\delta q = \epsilon\bar\eta + \eta\epsilon$$
$$\delta F = -\eta\left(\varepsilon\frac{\partial\bar\eta}{\partial t} - \frac{\partial\eta}{\partial t}\bar\varepsilon\right)$$
$$\delta\eta = \varepsilon\left(i\frac{\partial q}{\partial t} + F\right)$$
$$\delta\bar\eta = \bar\varepsilon\left(-i\frac{\partial q}{\partial t} + F\right)$$

$$\boxed{[\delta_\varepsilon, \delta_{\varepsilon'}]q = 2i(\varepsilon'\bar\varepsilon - \varepsilon\bar\varepsilon')\frac{\partial q}{\partial t}}$$

⇒ Nöther charge

$$Q = \left(\dot{q} - i\frac{\partial W}{\partial q}\right)\eta \quad \longrightarrow \quad (\hat{p} - iW'(\hat{q}))\,\hat{\eta}$$
$$Q^\dagger = \left(\dot{q} + i\frac{\partial W}{\partial q}\right)\bar{\eta} \quad \longrightarrow \quad (\hat{p} + iW'(\hat{q}))\,\hat{\eta}^\dagger$$

↳ <u>operator formalism</u>

$$\hat{p} = \hat{\dot{q}}\,, \quad \hat{\pi}_\eta = -i\hat{\eta}^\dagger \qquad \{\hat{p}, \hat{q}\} = -i$$
$$\{-i\eta^\dagger, \hat{\eta}\} = -i \Longleftrightarrow \{\hat{\eta}^\dagger, \hat{\eta}\} = 1$$

$$\Rightarrow \hat{H} = \frac{1}{2}\hat{p}^2 - \frac{1}{2}(W'(q))^2 - \frac{1}{2}W''(q)\left(\hat{\eta}^\dagger\eta - \hat{\eta}\hat{\eta}^\dagger\right) = \frac{1}{2}\{Q^\dagger, Q\}$$

$$\Rightarrow \hat{\eta}^2 = (\hat{\eta}^\dagger)^2 = 0\,, \quad \{\hat{\eta}^\dagger, \hat{\eta}\} = 1 \Longleftrightarrow \begin{cases} \hat{\eta} = \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ \hat{\eta}^\dagger = \sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{cases}$$

$$\hookrightarrow \hat{H} = \frac{1}{2}\left(-\frac{\partial^2}{\partial q^2} + \frac{1}{2}W'(q)^2\right)\mathbb{I}_2 - \frac{1}{2}\sigma_3 W''(q)$$

$$(-)^F = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_3$$

then
$$\hat{Q} = \sigma^-\left(-i\frac{\partial}{\partial q} - iW'(q)\right)$$
$$\hat{Q}^\dagger = \sigma^+\left(-i\frac{\partial}{\partial q} + iW'(q)\right)$$

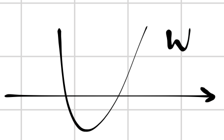
→ compute the index: $0 = (\hat{Q} + \hat{Q}^\dagger)\begin{pmatrix} \psi_b(Q) \\ \psi_f(\hat{Q}) \end{pmatrix} \Longleftrightarrow \begin{matrix} \psi_b(q) \sim e^{-W(q)} \\ \psi_f(q) \sim e^{W(q)} \end{matrix}$

→ 3 CASES:

1) $\mathrm{Tr}\,(-)^F = 1$

$\int dq\, |e^W|^2 < \infty$
$\int dq\, |e^{-W}|^2 = \infty$ → zero mode $= \begin{pmatrix} e^W \\ 0 \end{pmatrix}$ ✓

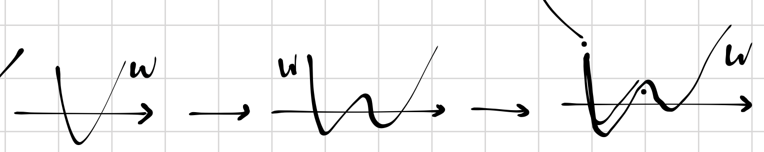
2) $\mathrm{Tr}\,(-)^F = -1$ $\quad \int dq\, |e^W| = \infty \longrightarrow \begin{pmatrix} 0 \\ e^{-W} \end{pmatrix}$ ✓



3) $\mathrm{Tr}\,(-)^F = 0$ $\quad \int dq\, |e^W| = \int dq\, |e^{-W}| = \infty \longrightarrow$ no 0 modes

SUSY breaking case!

index is invariant under deform of W (



)

Localization computation in path integral

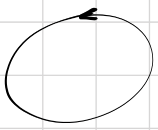
→ Euclidean action $(t \to -i\tau,\ L \to L_E,\ F \to iF)$

$$L_E = \frac{1}{2}\left(\frac{\partial q}{\partial \tau}\right)^2 + \frac{1}{2}F^2 + \bar\eta \frac{d}{d\tau}\eta - iW'(q)F - W'(q)\bar\eta\eta$$

"holom" in the sense that there are no $\eta^\dagger$ or $\bar\eta^\dagger$ and η and $\bar\eta$ are indep. var.

$(\eta, \bar\eta$ and $\varepsilon, \bar\varepsilon$ are indep$)$

S^1



→ $\mathrm{Tr}\left((-)^{\hat F} e^{-\beta \hat H}\right) = \int dq\, d\eta\, d\bar\eta\; e^{-\int_0^\beta d\tau\, L_E}$ → where $q, \eta, \bar\eta$ are periodic (i.e. $q(\tau=0) = q(\tau=\beta)$)

~ finite temp. part. funct.

LOCALIZATION

→ add $-\alpha \int dt\, \delta V = -\alpha\, \delta I$,

$$V = \eta\, \delta\bar\eta = \eta\bar\varepsilon\left(\frac{dq}{d\tau} + iF\right)$$

$$\delta V = -(\varepsilon\bar\varepsilon)\left(\left(\frac{dq}{d\tau}\right)^2 + F^2\right) + \eta\bar\varepsilon\left(\varepsilon\frac{d\bar\eta}{d\tau} + \varepsilon\frac{d\eta}{d\tau}\right)$$

$$= (\bar\varepsilon\varepsilon)\left(\left(\frac{dq}{d\tau}\right)^2 + F^2 + 2\bar\eta\frac{d\eta}{d\tau} + \cancel{\frac{d}{d\tau}(*)}\right) \qquad (\bar\varepsilon\varepsilon = 1)$$

arbitrary choice

Then $\lim_{\alpha\to\infty} \int dq\, d\eta\, d\bar\eta\, dF\, \exp\left(-\int_0^\beta d\tau\left(L_E + \alpha\,\delta I\right)\right)$

→ saddle point ⇒ zero of δV ⇒ $q = \text{const}$, $F = 0$:

$$\begin{cases} q = q_0 + \sum_{n\neq 0}\left(\sin\left(2\pi\frac{n}{\beta}\tau\right) q_n^1 + \cos\left(2\pi\frac{n}{\beta}\tau\right) q_n^2\right) \\ \eta = \eta_0 + \sum_{n\neq 0}(\dots) \quad (\text{same for } \bar\eta) \\ F = F_0 + \sum_{n\neq 0}(\dots) \end{cases}$$

CONST!

In $\alpha \to 0$ limit, localization to $\delta V = 0$ ⟺ zero modes of $q, \eta, \bar\eta$:

$$\int_0^\beta d\tau\, \delta V = \sum_{n\neq 0}\left[\left((q_n^1)^2 + (q_n^2)^2\right)\left(2\pi\frac{n}{\beta}\right)^2 + 2\left(\bar\eta_n^1\eta_n^2 - \bar\eta_n^2\eta_n^1\right)\left(2\pi\frac{n}{\beta}\right) + (F_n^1)^2 + (F_n^2)^2\right]\frac{\beta}{2} + \beta F_0^2$$

$$= \sum_{n\neq 0}\left(\frac{2\pi^2 n^2}{\beta}\left((q_n^1)^2 + (q_n^2)^2\right) + 2\pi n\left(\bar\eta_n^1\eta_n^2 - \bar\eta_n^2\eta_n^1\right) + \frac{\beta}{2}\left((F_n^1)^2 + (F_n^2)^2\right) + \beta F_0^2\right.$$

take $\sqrt{\alpha}\, q_n^a = q_n^{a\prime}$, $\sqrt{\alpha}\, \eta_n^a = \eta_n^{a\prime}$, $\sqrt{\alpha}\, F_n^a = F_n^{a\prime}$ for $n \neq 0$ ⇒ In $L_E(\eta, \bar\eta, F)$ we can neglect 0-modes

$q \to \frac{1}{\sqrt{\alpha}} q' \to 0$ in the limit

$= 2\pi\, dq_0\, d\eta_0\, d\bar\eta_0\, dF_0 \,[\quad]_{non\ zero}$

Gaussian integral

$$\int [dq\, d\eta\, d\bar\eta\, dF]_{non\ 0\ modes}\; e^{-\alpha \int_0^\beta d\tau\, \delta V|_{non\ zero\ modes}} = \prod_{n\neq 0}\left(\frac{\beta}{2\pi n^2 \alpha}\cdot (2\pi n \alpha)^2 (-1)\frac{2\pi}{\beta\alpha}\right) =$$

$$= \prod_{n\neq 0}(-4\pi^2) = e^{2\ln(4\pi^2)\underbrace{\sum_{n=1}^{\infty} 1}_{\zeta(0) = -\frac{1}{2}}} = -\frac{1}{4\pi^2}$$

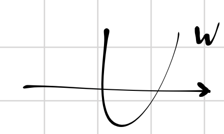
$$\to \text{index} = -\frac{1}{2\pi}\int dq_0\, d\eta_0\, d\bar\eta_0\, dF_0 \exp\left(-\beta\left(\alpha F_0^2 + \tfrac{1}{2}F_0^2 - i W'(q_0) F_0 - W''(q_0)\bar\eta_0 \eta_0\right)\right) =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} dq_0\, W''(q_0)\int dF_0 \exp\left(-\alpha\beta\left(F_0 + \frac{i}{2\alpha\beta}W'(q_0)\right)^2 - \frac{1}{4\alpha\beta}\left(W'(q_0)\right)^2\right) =$$

$$= \frac{1}{2\pi}\sqrt{\frac{\pi}{\alpha\beta}}\int_{-\infty}^{+\infty} dq_0\, W''(q_0)\exp\left(-\frac{1}{4\alpha\beta}\left(W'(q_0)\right)^2\right) =$$

↳ $W' = x \Rightarrow dx = W''(q_0)\, dq_0$

① $q_0 \to {+\infty \atop -\infty}$ $\begin{matrix}\Rightarrow W' \to \infty \\ \Rightarrow W' \to -\infty\end{matrix}$ $\longrightarrow = \frac{1}{2\pi}\sqrt{\frac{\pi}{\alpha\beta}}\sqrt{\pi 4\alpha\beta} = 1$



② $q_0 \longrightarrow {+\infty \atop -\infty}$ $\begin{matrix}\Rightarrow W' \to -\infty \\ \Rightarrow W' \to +\infty\end{matrix}$ $\longrightarrow = \ldots = -1$

③ $q_0 \longrightarrow {+\infty \atop -\infty}$ $\begin{matrix}\Rightarrow W' \to \pm\infty \\ \Rightarrow W' \to \pm\infty\end{matrix}$ $\longrightarrow \ldots = 0$

3d $\mathcal{N}=2$ SUSY QFT on S^3

On $\mathbb{R}^3$ (euclidean): $\quad$ 3d $\mathcal{N}=2 \longleftrightarrow$ dim. reduction of 4d $\mathcal{N}=1$ $\quad SO(4) \sim SU(2)\times SU(2)$

$SO(3) \sim SU(2)$

γ^a: Dirac mat (Pauli) $\quad \longrightarrow \gamma_m = e^a_m \gamma^a$ (vielbein)

$\varepsilon\psi = \varepsilon^\alpha C_{\alpha\beta}\psi^\beta$

$\varepsilon\gamma^a\psi = \varepsilon^\alpha C_{\alpha\beta}(\gamma^a)^\beta{}_\gamma \psi^\gamma$

$\Rightarrow$ CHIRAL MULTIPLET $\quad (\phi, \bar\phi, \psi, \bar\psi, F, \bar F)$

$\delta\phi = \varepsilon\psi \qquad\qquad \delta F = i\bar\varepsilon\gamma^m D_m\psi - i\bar\varepsilon(\sigma\psi + \bar\lambda\phi)$

$\delta\bar\phi = \bar\varepsilon\bar\psi \qquad\qquad \delta\bar F = i\varepsilon\gamma^m D_m\bar\psi - i(\varepsilon\bar\psi\sigma - \bar\phi\lambda)$

$\delta\psi = i\gamma^m\bar\varepsilon D_m\phi + \varepsilon F$

$\delta\bar\psi = i\gamma^m\varepsilon D_m\bar\phi + \bar\varepsilon\bar F$

$\Rightarrow$ VECTOR MULTIPLET $\quad (A_\mu, \lambda_\alpha, \bar{\lambda}_{\dot\alpha}, D)$ $\hookrightarrow$ in 4D

$D=4 \to D=3 \downarrow$ "ADJOINT" SCALAR $\longrightarrow$ COULOMB BRANCH

$(A_m, A_4 = \sigma, \lambda_\alpha, \bar{\lambda}_\alpha, D)$ $\hookrightarrow$ 3d

$$\to \delta A_m = -\frac{i}{2}(\varepsilon \partial_m \bar{\lambda} + \bar{\varepsilon}\gamma_m \lambda)$$

$$\delta\sigma = \frac{1}{2}(\varepsilon\bar{\lambda} - \bar{\varepsilon}\lambda)$$

$$\delta\lambda = \frac{1}{2}\gamma^{mn}\varepsilon F_{mn} - \varepsilon D - i\gamma^m \varepsilon D_m \sigma$$

$$\delta\bar{\lambda} = \frac{1}{2}\gamma^{mn}\bar{\varepsilon} F_{mn} + \bar{\varepsilon} D + i\gamma^m \bar{\varepsilon} D_m \sigma$$

$$\delta D = \frac{i}{2}\varepsilon(\gamma^m D_m \bar{\lambda} + [\sigma, \bar{\lambda}]) - \frac{i}{2}\bar{\varepsilon}(\gamma^m D_m \lambda - [\sigma, \lambda])$$

* On S^3: $\quad x_1^2 + x_2^2 + x_3^2 + x_4^2 = \ell^2 - \frac{1}{(2M)^2}$

→ How to build SUSY on curved manifold? Expand by curvature (M) and add terms order by order s.t. δ satisfies algebra (Noether method)

$\Rightarrow$ $D_m \epsilon$ does not vanish! $\left(= (\partial_m + \frac{1}{4}\omega_m^{ab}\gamma^{ab})\epsilon\right)$ on curved space!

$\hookrightarrow$ we may need Killing eqn: $\quad D_m \varepsilon = \pm i M \gamma_m \varepsilon$

$$D_m \bar{\varepsilon} = \pm i M \gamma_m \bar{\varepsilon}$$

→ consider the eqns with + sign: $\mathcal{N}=2$ SUSY transf on S^3

$$\delta D = \dots + \frac{M}{2}(-\varepsilon\bar{\lambda} + \bar{\varepsilon}\lambda)$$

$$\delta F = \dots + \frac{i}{3}(2r-1)(D_m \bar{\varepsilon})\gamma^m \psi$$

$$\delta \bar{F} = \dots + \frac{i}{3}(2r-1)(D_m \varepsilon)\gamma^m \bar{\psi}$$

r is the R-charge of ϕ:

	A_m	σ	λ	$\bar{\lambda}$	D	ϕ	$\bar{\phi}$	ψ	$\bar{\psi}$	F	$\bar{F}$
R-charge:	0	0	1	-1	0	r	-r	r-1	1-r	r-2	2-r

$r = \frac{1}{2}$ conformally chiral

→ we can map from $\mathbb{R}^4$ to S^3 the conf. transf!

$\Rightarrow$ SUSY ACTION on S^3:

$$\mathcal{L}_g = \mathrm{Tr}\left(\frac{1}{2}F^{mn}F_{mn} + D_m\sigma D^m\sigma + D^2 + i\bar\lambda\gamma^m D_m\lambda - i\bar\lambda[\sigma,\lambda] - \underline{\underline{M\bar\lambda\lambda}}\right)$$

$$\mathcal{L}_m = D_m\bar\phi D^m\phi + \bar\phi\sigma^2\phi - i\bar\phi D\phi - i\bar\psi\gamma^m D_m\psi + i\bar\psi\sigma\psi + i\bar\psi\bar\lambda\phi - i\bar\phi\lambda\psi + \bar F F +$$
$$+ 4i(r-1)M\bar\phi\sigma\phi + 4r(2-r)M^2\bar\phi\phi - 2(2r-1)M\bar\psi\psi$$

THE CURVATURE GIVES MASS !!!

$$\mathcal{L}_{CS} = \frac{ik}{4\pi}\mathrm{Tr}\left(\varepsilon^{mnl}A_m\partial_n A_l - \frac{2i}{3}A_mA_nA_l - \bar\lambda\lambda - 2\sigma D - 4M^2\sigma^2\right)$$

$\rightarrow$ Superpotential W is SUSY invariant if $R(W) = 2$
without ANY corrections

$\rightarrow$ need term $\langle\sigma\rangle = m$ to make it MANIFESTLY inv.

SUSY Wilson loop

$$W(C) = \mathrm{Tr}_R\left(\mathcal{P}\exp\oint_C (iA + \sigma\, dl)\right)$$

$S^3 = S^2$, S^1 — Hopf fibration

C; loop winds along Killing vector $\bar\varepsilon\gamma_m\varepsilon$

$$Z(t) = \int \mathcal{D}A_m\,\mathcal{D}\lambda\,\mathcal{D}\bar\lambda\,\mathcal{D}\phi\,\mathcal{D}\bar\phi\,\mathcal{D}\psi\,\mathcal{D}\bar\psi\,\mathcal{D}F\,\mathcal{D}\bar F\,\exp\left(-\int d^3x\,(\mathcal{L}_g + \mathcal{L}_m + \dots) + tSV\right) \quad t\to\infty$$

$\Rightarrow SV = S(\lambda S\lambda^\dagger)$ is ok

$$(\bar\varepsilon\varepsilon)\mathcal{L}_g = \delta_\varepsilon\left(\delta_{\bar\varepsilon}\mathrm{Tr}\left(\bar\lambda\lambda + 4D\sigma + 8M\sigma^2\right)\right)$$
$=1$
$$(\bar\varepsilon\varepsilon)\mathcal{L}_m = \delta_\varepsilon\left(\delta_{\bar\varepsilon}\left(\bar\psi\psi - 2i\bar\phi\sigma\phi + 4M(r-1)\bar\phi\phi\right)\right) \qquad \Rightarrow SV \sim -(\mathcal{L}_g + \mathcal{L}_m)$$

$\varepsilon, \bar\varepsilon$ "bosonic" spinor

$\Rightarrow t\to\infty \Rightarrow \mathcal{L}_g = 0 \Leftrightarrow F_{mn} = 0 \quad (D_m\sigma) = 0 \quad D = 0$

$\mathcal{L}_m = 0 \Leftrightarrow \phi = \bar\phi = F = \bar F = 0$

cocycle

$\Rightarrow \sigma(x) = a$ is a const remain

Plus:

$$t(\mathcal{L}_g + \mathcal{L}_m) + \mathcal{L}_{tr} = t(\underbrace{\text{free particle}}_{\sim (\partial_m \phi)^2} + \underbrace{\text{interactions}}_{\sim A\partial_m\phi}) + \underbrace{\mathcal{L}_{te}}_{\sim \phi}$$

$\sigma = a$, others $= 0$

$$\rightarrow \phi\sqrt{t} = \hat{\phi} \Rightarrow (\text{free} + \text{int.}) + \mathcal{L}_{te}\left(\frac{\hat{\phi}}{\sqrt{t}}\right) \xrightarrow{t\to\infty} \text{free particle} + \mathcal{L}_{te}|_{\text{saddle}}$$

PATH INTEGRAL of free part (other gauge fixing $\partial^m \hat{A}_m = 0$)

$\sigma \to$ Higgs

$$\mathcal{L}_g|_{\text{free}} = \mathrm{Tr}\left[\hat{A}\left(-\Delta_{S^3}^{\text{vec}} + 4\pi^2 a_{\text{adj}}^2\right)\hat{A} - \bar{\lambda}\left(-i\gamma^m D_m + \pi + 2i\pi a_{\text{adj}}\right)\lambda + D^2\right]$$

$\Rightarrow SO(4) \sim SU(2) \times SU(2)$

			#	Δ_{S^3}
Scalar	$\bigoplus_{j\geq 0}$	$\left(\frac{j}{2},\frac{j}{2}\right)$	$(j+1)^2$	$4\pi^2 j(j+2)$
Spinor	$\bigoplus_{j\geq 0}$	$\left(\frac{j}{2},\frac{j+1}{2}\right)\oplus\left(\frac{j+1}{2},\frac{j}{2}\right)$	$(j+1)(j+2)$	$\pm 2\pi\left(j+\frac{3}{2}\right)$
dimensionless vector	$\bigoplus_{j\geq 0}$	$\frac{1}{2}\left(\frac{j}{2},\frac{j+2}{2}\right)\oplus\left(\frac{j+2}{2},\frac{j}{2}\right)$	$(j+1)(j+2)$	$4\pi^2(j+1)^2$

e.g.: $U(N) \to \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}$

$$Z^{1L}_{\text{vec}} = \prod_{j\geq 0} \frac{\det_{\text{ADJ}}(-ia-j-2)\det_{\text{ADJ}}(-ia+j+1)}{\left|\det_{\text{ADJ}}\left(a^2+(j+2)^2\right)\right|^{(j+1)(j+3)}} = \prod_{\alpha\in\text{adj}}\prod_{j\geq 1}\frac{(j+ia\cdot\alpha)^{j+1}}{(j-ia\cdot\alpha)^{j-1}} = \prod_{\alpha\in\Delta_+}\left(\frac{2\sinh(\pi a\,\alpha)}{a\alpha}\right)^2$$

$\hookrightarrow$ Cartan part

$$\rightarrow [da] = \prod_i da_i \left(\prod_{\alpha\in\Delta_+}(\alpha\cdot a)\right)^2$$

$\Rightarrow$ chiral multiplet: $$Z^{1L}_{\text{matt}} = \frac{\det_\psi\left(\frac{1}{2} + i\alpha - \gamma - i\gamma^m D_m\right)}{\det_\phi\left(-g^{mn}D_m D_n + 1 - (r-1-ia)^2\right)} = \prod_{n=1}^{\infty}\prod_w \frac{(n+1-r+i\alpha\cdot a)^n}{(n-1+r-i\alpha\cdot w)^n} = \prod_w S_{b=1}(i(1-r) - a\cdot w)$$

$$\rightarrow S_b(x) = \prod_{m,n\in\mathbb{Z}} \frac{mb + nb^{-1} + \frac{Q}{2} - ix}{mb + nb^{-1} + \frac{Q}{2} + ix} \qquad Q \doteq b + \frac{1}{b}$$

$\rightsquigarrow$ at saddle point: $$Z = \int da_i\, e^{i\pi k \mathrm{Tr}(a^2) + 4\pi i \xi a}\, \mathrm{Tr}_R\, e^{2\pi a} \prod_{\alpha\in\Delta^+}\left(2\sinh(\pi\alpha\cdot a)\right)^2 \prod_w S_{b=1}(i(1-r) - a\cdot w)$$

$\rightarrow$ Large N limit of ABJM $\rightarrow$ 3d $\mathcal{N}=2$ $\rightarrow$ $U(N) \times U(N')$ $\rightarrow$ 4 bifund chiral multiplets ϕ^4

$$\mathcal{L} = i\frac{k}{4\pi} CS - i\frac{k}{4\pi} CS' + \text{superpot} \sim \phi^6 \longrightarrow \mathcal{N}=6 \text{ SUSY CFT}$$

Then

$$\begin{tikzcd}
\text{M2 branes} \arrow[r, "N\to\infty"] \arrow[dr, "\text{low energy}"'] & \text{ABJM} \arrow[d, bend left, "\text{AdS/CFT}"] \\
& M \text{ on } AdS_4 \times S^7/\mathbb{Z}_k
\end{tikzcd}$$

$$\longrightarrow Z \sim e^{\pi\frac{\sqrt{2k}}{3} N^{3/2}}$$

With localization:

$$Z(N) = \prod_{i=1}^{N} \int d\lambda_i \, d\tilde{\lambda}_i \, e^{-f(\lambda,\tilde{\lambda})}$$

$$\rightarrow f(\lambda,\tilde{\lambda}) = \pi i k \left(\sum_{i\geq 1} \lambda_i^2 - \sum_{i\geq 1} \tilde{\lambda}_i^2 \right) - \sum_{i>j} \log \sinh^2(\pi(\lambda_i - \lambda_j)) - \sum_{i>j} \log \sinh^2(\pi(\tilde{\lambda}_i - \tilde{\lambda}_j)) + \sum_{i,j\geq 1} \ln \cosh \pi(\lambda_i - \tilde{\lambda}_j)$$

$\rightarrow$ Large N limit $\Rightarrow$ saddle point of $f(\lambda,\tilde{\lambda})$