Inflationary Cosmology and Cosmological perturbation

$1\,\text{Mpc} \sim 3 \times 10^{24}$ cm
$1\,\text{GeV}^{-1} \sim 10^{-14}$ cm

* FRW metric ← COSMOLOGICAL PRINCIPLE [homogeneous and isotropic]

→ Killing vectors: rotations and transl. (spacial)

$$ds^2 = -dt^2 + a^2(t)\, d\vec{x}^{\,2}$$

$a = a(t)$: scale factor

Polar coord $(r, \theta, \phi) \longrightarrow ds^2 = \dfrac{dr^2}{1 - Kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2$

$K = 0, \pm 1 \longrightarrow$ GLOBAL GEOMETRY

$K=0 \to$ flat
$K=1 \to$ sphere
$K=-1 \to$ hyperb.

[evolution cannot change K but only local geom]

⇝ EINSTEIN EQUATION:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G_N T_{\mu\nu}$$

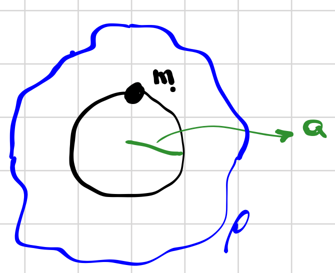
fluid: $T_{\mu\nu} = \text{diag}(\rho, P, P, P)$
(according to cosmological princ.)

↳ Hubble rate: $H = \dfrac{\dot{a}}{a} \to \boxed{H^2 + \dfrac{K}{a^2} = \dfrac{8\pi G_N \rho}{3}}$ I Friedman equation

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4}{3}\pi G_N (\rho + 3P)} \quad \text{II Friedman eq.}$$

$$\dot{\rho} + 3H(\rho + P) = 0 \qquad [\text{from } \nabla_\mu T^{\mu\nu} = 0]$$

Newtonian cosmology:



⇒ energy conservation:

$M = \frac{4}{3}\pi \rho a^3$

$$\frac{1}{2} m \dot{a}^2 - \frac{G_N m M}{a} = C = \text{const}$$

$$\hookrightarrow \left(\frac{\dot{a}}{a}\right)^2 - G_N \frac{8}{3}\pi\rho = \frac{C}{ma^2} = -\frac{K}{a^2} \Rightarrow \text{I F. eq.}$$

What about $\dot{\rho} + 3H(\rho + P) = 0$?

i) non-relat fluid: $P \simeq 0 \longrightarrow \rho \sim a^{-3}$

ii) relativistic: $P = \frac{1}{3}\rho \longrightarrow \rho \sim a^{-4}$

iii) Vacuum: $P = -\rho \longrightarrow \rho = \text{const}$

Therefore

i) $a \sim t^{2/3}$

ii) $a \sim t^{1/2}$

iii) $a \sim e^{H_0 t}$

Particle !

$$ds^2 = -dt^2 + a(t)\, d\vec{x}^2 = 0 \Rightarrow |d\vec{x}| = \frac{dt}{a} \rightsquigarrow R_H = a(t) \int_0^t \frac{dt'}{a(t')}$$

① COS

$$a(t) = a_0 \left(\frac{t}{t_0}\right)^n \qquad n = \begin{cases} 2/3 \to \text{matter dominated (MD)} \\ 1/2 \to \text{radiation " (RD)} \end{cases}$$

$\hookrightarrow R_H(t) = \dots = \frac{t}{1-n} \to$ FINITE HORIZON!

decelerating universe

$$R_H(t) = a(t) \int_0^t \frac{dt'}{a(t')} = a(t) \int_0^a \frac{dt'}{da'} \frac{da'}{a(t')} =$$

$= d\ln a'$

(Hubble radius $= \begin{cases} H^{-1}(t) \\ \frac{1}{aH(t)} \text{ (COMOVING)} \end{cases}$) $= a(t) \int_0^a \frac{da'}{a'} \cdot \frac{1}{H' a'} =$

suppose $\exists$ (irrelevant) BB end

Particle horizon: how far a photon can travel at a given time (from the 0)

Hubble radius : units of expans. at given ~~time~~ expans. par.

NB: ~~If~~ $a \sim t^n \to \dot{a} \sim n t^{n-1} \to \frac{\dot{a}}{a} \approx \frac{n}{t} \to H^{-1} \sim t$

FLATNESS PROBLEM

we can measure!

$$\begin{cases} H^2 + \frac{K}{a^2} = \frac{8\pi G_N}{3} \\ \rho_c = \frac{3H^2}{8\pi G_N} \to \Omega = \rho/\rho_c \end{cases} \Rightarrow \Omega - 1 = \frac{K}{a^2 H^2} \to \text{CLOSE/OPEN/FLAT univ.}$$

Suppose $|\Omega - 1| \sim 1$ and RD universe $\to H^2 \sim a^{-4}$:

$$|\Omega - 1| \sim a^2 \longrightarrow \frac{|\Omega-1|(t)}{|\Omega-1|(t_0)} = \frac{a^2(t)}{a^2(t_0)}$$

Suppose the univ is adiabatically expanding:

S = total entropy = const $= s \cdot a^3$ (volume)

$$s = \frac{2\pi^2}{45} g_{*,s} T^3 \to S \sim T^3 a^3 = \text{const} \Rightarrow T \sim \frac{1}{a}$$

Then: $\dfrac{|\Omega-1|(T)}{|\Omega-1|(T_0)} = \left(\dfrac{T_0}{T}\right)^2 \rightarrow$ we know $T_0 \sim 10^{-13}$ GeV

Suppose $T \sim M_p \sim 10^{18}$ GeV $\Rightarrow \left(\dfrac{T_0}{T}\right)^2 = 10^{-64} \Rightarrow$ in order to have $|\Omega-1| \sim 1$ TODAY $\Rightarrow$ back in time it had to be extremely FINE TUNED

FLATNESS PROBLEM

$$\Omega(T_p) \sim 1 + 10^{-64}$$

ENTROPY PROBLEM

$$H^2 = \frac{8\pi G_N}{3} \cdot \frac{\pi^2}{30} g_* T^4$$

($G_N = \frac{1}{M_p^2}$)

$$\Omega - 1 = \frac{K}{a^2 H^2} \sim \frac{K}{a^2 T^4} M_p^2 = \frac{K M_p^2}{S^{2/3} T^2} \xrightarrow[\text{EXPANS.}]{\text{ADIABATIC}} \frac{K M_p^2}{S_0^{2/3} T^2}\Bigg|_{T=M_p} \sim \frac{K}{S_0^{2/3}} \sim 10^{-60}$$

$S = \text{const}$

$\hookrightarrow S_0 \sim T_0^3 H_0^{-3} = (10^{-13}\,\text{GeV})^3 (10^{42}\,\text{GeV}^{-1})^3 \sim 10^{90}$ part.

they lead to the FINE TUNING problem

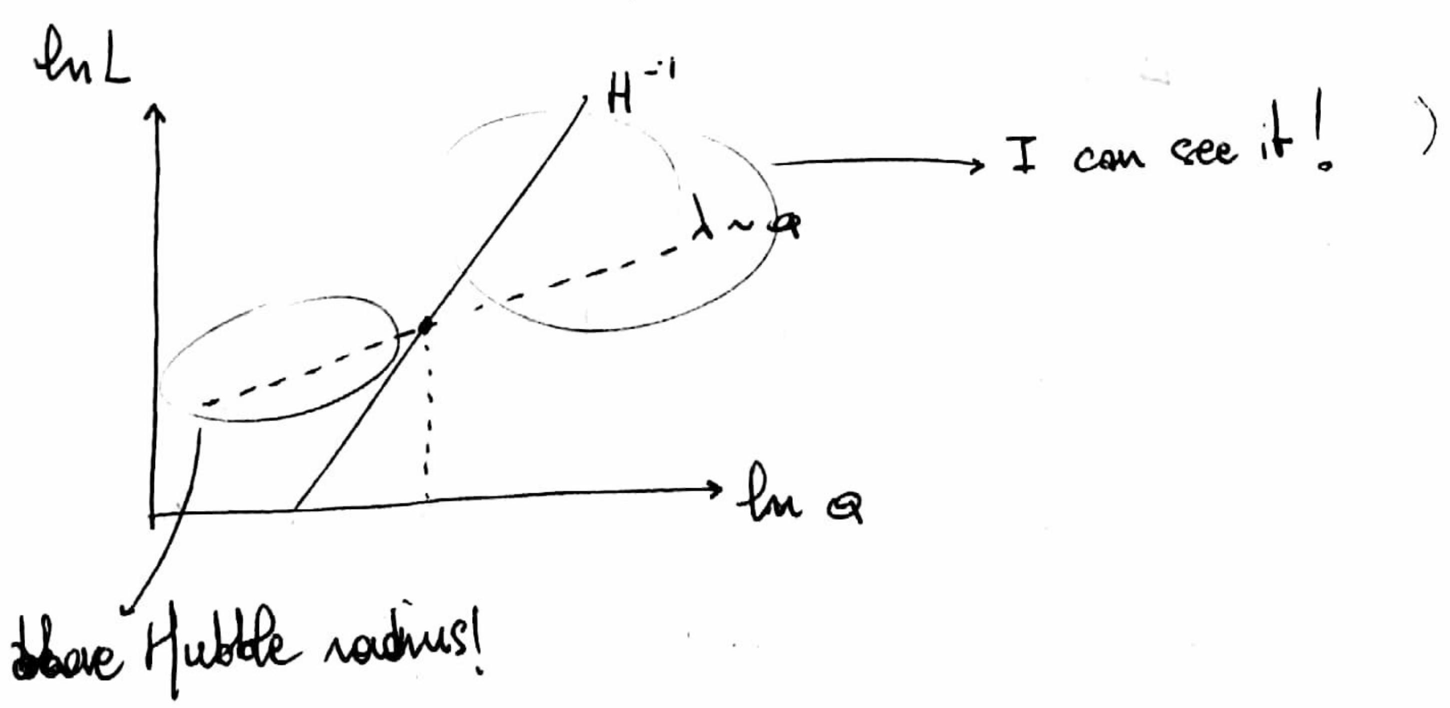
the universe was born with a huge amount particles!

$\Rightarrow$ solution: $\exists$ time where ADIAB. EXP. breaks down

HORIZON PROBLEM

Scale $L \rightarrow$ certain distance I observe

$$H^2 \sim \rho = \begin{cases} a^{-3} & \text{MD} \\ a^{-4} & \text{RD} \end{cases} \Rightarrow H^{-1} \sim \begin{cases} a^{3/2} & \text{MD} \\ a^{2} & \text{RD} \end{cases}$$



Suppose $\exists$ photons:

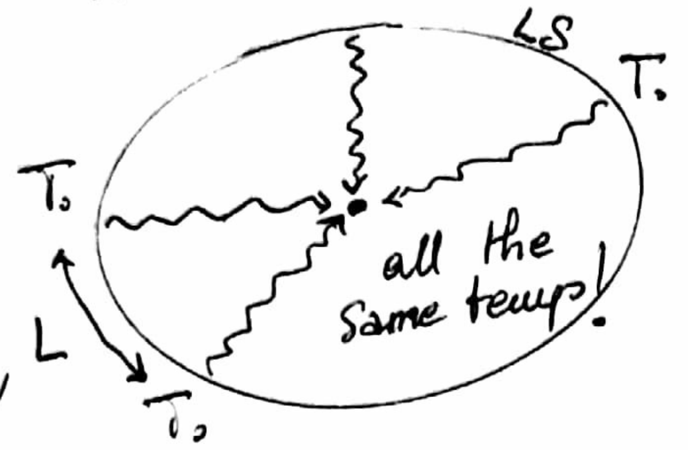
$$p + e^- \rightleftarrows H + \gamma$$

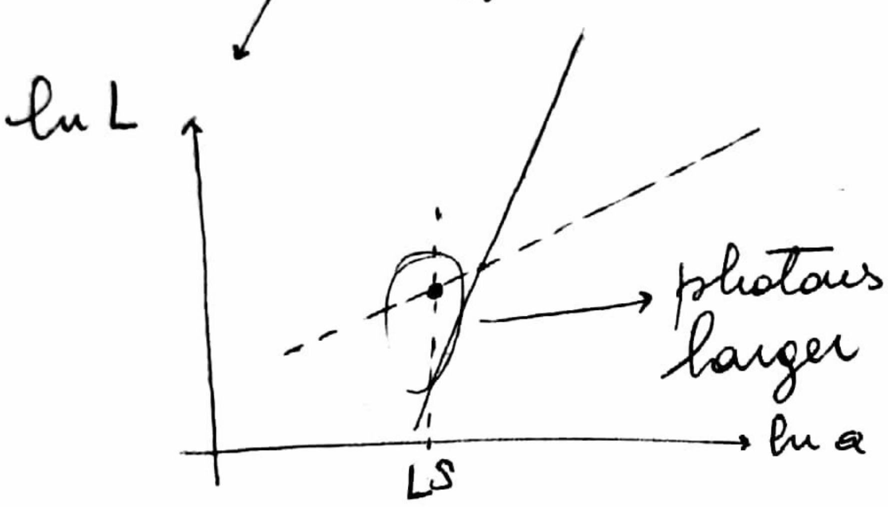
↳ LAST SCATTERING [redacted] SURFACE

$T_{LS} \sim 0.3\,eV \rightarrow$ [redacted] only $p + e^- \longrightarrow H + \gamma$

$\Downarrow$

UNIVERSE TRANSPARENT TO PHOTONS





photons were separated by a dist larger than the CAUSAL horizon!

What was H_{LS}?

$H_0^{-1} \cdot \left(\frac{a_{LS}}{a_0}\right)$ vs H_{LS}^{-1}

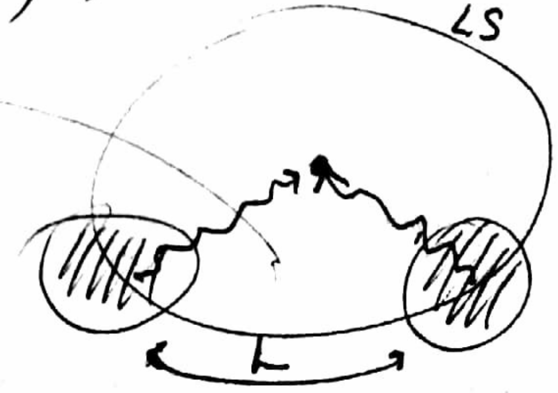
"shrink" the current univ.

$$\left(H^{-1} \sim a^{3/2}\ (MD)\right) \Rightarrow H_{LS}^{-1} = H_0^{-1}\left(\frac{a_{LS}}{a_0}\right)^{\frac{3}{2}}$$

$$↳ \frac{H_0^{-1}\left(\frac{a_{LS}}{a_0}\right)^{3/2}}{H_0^{-1}\left(\frac{a_{LS}}{a_0}\right)} = \left(\frac{a_0}{a_{LS}}\right)^{1/2} \approx \left(\frac{T_{LS}}{T_0}\right)^{1/2} \sim 10^2$$

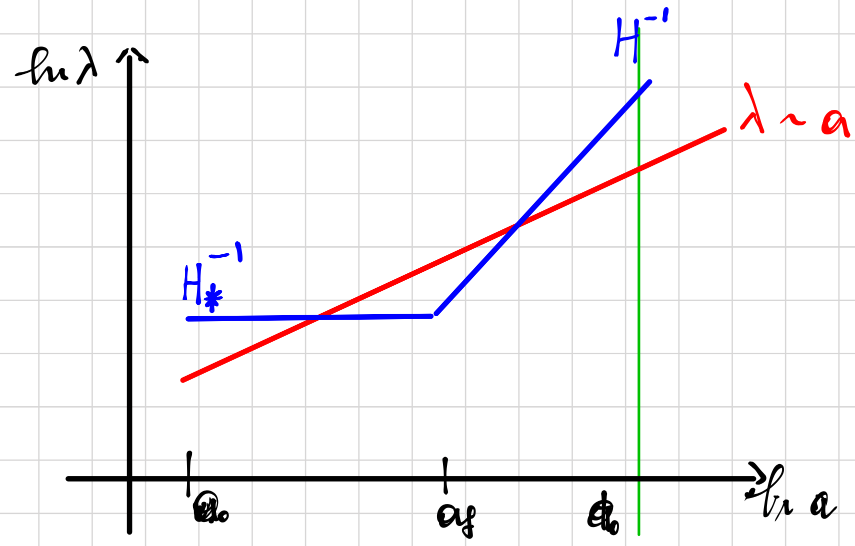
$(10^2)^3 = 10^6$ Hubble regions on LS

they were not in causal contact!



We can try to solve:

* cannot change physical scales
* change H^{-1}



$$\Rightarrow \frac{d}{dt}\left(\frac{\lambda}{H^{-1}}\right) > 0 \Leftrightarrow \ddot{a} > 0 \quad \text{INFLATION} \qquad \left[\text{contrary to: } a \sim \begin{cases} t^{1/2} & \text{RD} \\ t^{2/3} & \text{MD} \end{cases}\right.$$

Now suppose:

$$\ddot{a} > 0$$

$$H = \frac{\dot{a}}{a} = \text{const (assumption)} \longrightarrow a(t) = a_* e^{H_*(t-t_*)} \quad \text{(De Sitter)}$$

Define: $N = \ln \frac{a_f}{a_i}$ = tot no. of e-folds (a_f: end of infl., a_i: begin of infl.)

$$\Rightarrow H_0^{-1} \frac{a_f}{a_0} \cdot \frac{a_i}{a_f} \lesssim H_*^{-1} \longrightarrow H_0^{-1} \frac{T_0}{T_f} e^{-N} \lesssim H_*^{-1}$$

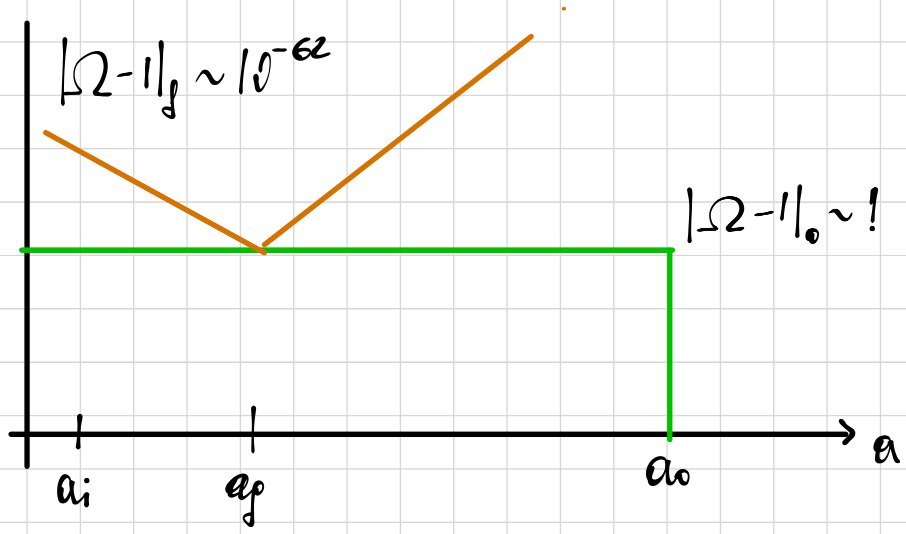
T_f: temp. at the end of infl.

$$\Rightarrow e^{-N} \gtrsim \underbrace{(H_0^{-1} \times T_0)}_{\text{I can compute}} \frac{H_*}{T_f} \Rightarrow N \gtrsim \ln(H_0^{-1} T_0) + \ln \frac{H_*}{T_f}$$

($H_0^{-1} \sim 10^{42}\,\text{GeV}^{-1}$, $T_0 \sim 10^{-13}\,\text{GeV}$)

$$\Rightarrow N \gtrsim 60 + \ln \frac{H_*}{T_f} \longrightarrow \text{SOLVES HORIZON PROBLEM!}$$

→ FLATNESS PROBLEM ?



$$\rightarrow \Omega - 1 = \frac{K}{a^2 H_*^2} \longrightarrow \frac{|\Omega-1|_f}{|\Omega-1|_i} = \left(\frac{a_i}{a_f}\right)^2 = e^{-2N} \rightarrow \sim 10^{-62} \Rightarrow \text{SOLVE FLATNESS PROBLEM}$$

Suppose: $\underset{(\sim 10^{12})}{N \gg 60} \Rightarrow |\Omega - 1|_f \ll 10^{-62} \longrightarrow |\Omega-1|_0 = 10^{62} \cdot 10^{-x} = 0$

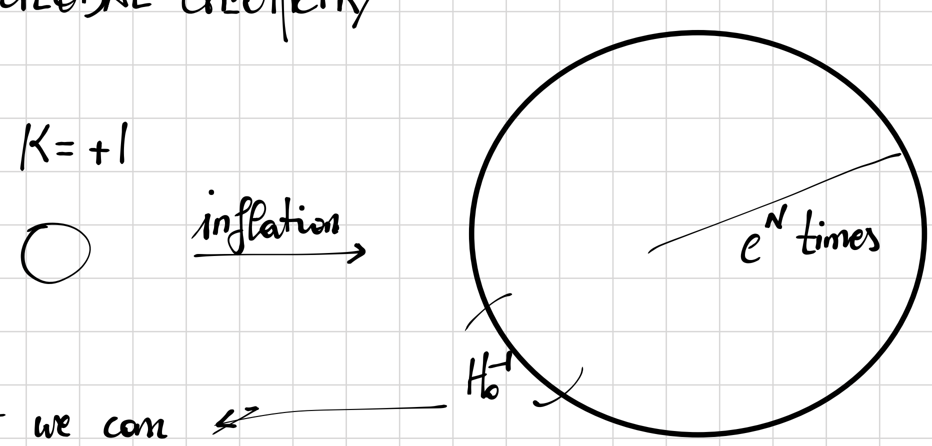
$\boxed{x \gg 62}$!!!

→ inflation lasted much more than the minimum required.

$$\Omega_0 \sim 1 \longrightarrow \text{FLAT UNIVERSE}$$

⇝ however INFLATION DOES NOT THE GLOBAL GEOMETRY

→ local geometry: $K = +1$



this is what we can see! ⇒ locally flat!

Particle horizon:

$$R_H(t) = a(t)\int_{t_i}^{t} \frac{dt'}{a(t')} = a_i\, e^{H_*(t-t_i)} \int_{t_i}^{t} \frac{dt'}{a_i}\, e^{-H_*(t'-t_i)} =$$

$$= -\frac{1}{H_*}\left\{1 - e^{H_*(t-t_i)}\right\} \sim \frac{1}{H_*}\, e^{H_*(t-t_i)}$$

→ now we can motivate $H_* = \text{const}$.

NB: $\ddot{a} > 0 \Rightarrow \quad \frac{\ddot{a}}{a} = -\frac{4}{3}\pi G_N (\rho + 3P) > 0 \Leftrightarrow P < -\frac{\rho}{3}$

$$\text{RD}: \quad P = \frac{\rho}{3}$$
$$\text{MD}: \quad P \simeq 0$$

→ SCALAR FIELD: $\quad \phi = \Phi(\vec{x}, t)$: inflaton

$$\rightarrow S[\phi] = \int d^4x \sqrt{-g}\left[-\frac{1}{2} g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - V(\phi)\right]$$

$$\Rightarrow T_{\mu\nu} = \partial_\mu\phi\,\partial_\nu\phi + g_{\mu\nu}\mathcal{L}$$

Consider $\phi_0(t) \rightarrow$ <u>classical value</u> of ϕ:

$$T_{\mu\nu} = \mathrm{diag}(\rho, P, P, P)$$

$$\Rightarrow \begin{cases} \rho = \frac{1}{2}\dot{\phi}_0^2 + V(\phi_0) \\ P = \frac{1}{2}\dot{\phi}_0^2 - V(\phi_0) \end{cases} \longrightarrow \frac{P}{\rho} = \frac{\frac{1}{2}\dot{\phi}_0^2 - V(\phi_0)}{\frac{1}{2}\dot{\phi}_0^2 + V(\phi_0)} \sim -1 \text{ if } \dot{\phi}_0^2 \ll V(\phi_0) \quad [\text{slowly moving}]$$

$$\Rightarrow \text{e.o.m.}: \quad \ddot{\phi}_0 + 3H\dot{\phi}_0 + V'(\phi) = 0$$

SLOW-ROLL INFLATION

$$3H\dot{\phi}_0 = -V'(\phi_0)$$

NB: what's the consequence of $\quad \dot{\phi}_0^2 \ll V(\phi_0)$?

$$\frac{(V'(\phi_0))^2}{9H^2} \ll V(\phi_0) \Longrightarrow \frac{(V'(\phi_0))^2}{V(\phi_0)} \ll H^2$$

↳ SLOW-ROLL PARAM.: $\quad \varepsilon = -\frac{\dot{H}}{H^2}$

$$\dot{H} = -4\pi G_N(\rho + P) = -4\pi G_N \dot{\phi}_0^2$$

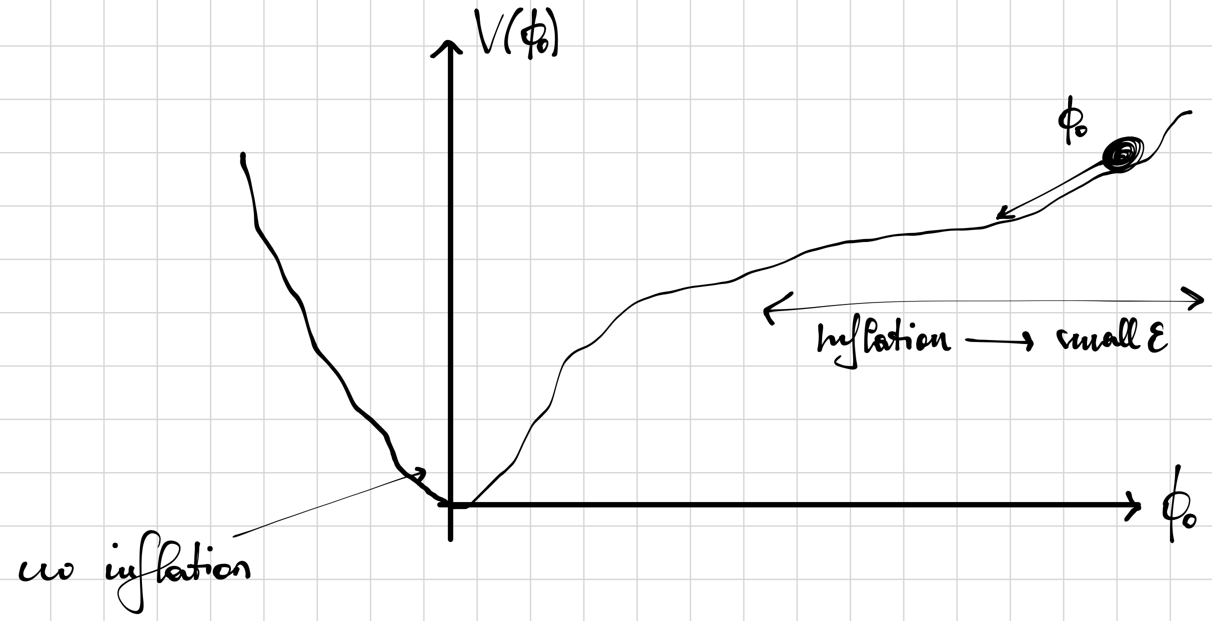
$$\left[H^2 = \frac{8\pi G_N}{3} V\right]$$

$$\rightarrow \varepsilon = \ldots = \frac{3}{2}\frac{\dot{\phi}_0^2}{V} \longrightarrow \dot{H} = \frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} \Rightarrow \frac{\ddot{a}}{a} = H^2 + \dot{H} = (1-\varepsilon)H^2$$

Define: $\delta = -\frac{\ddot{\phi}_0}{H\dot{\phi}_0} = \eta - \varepsilon$ where $\eta = \frac{1}{3}\frac{V''}{H}$

Then

$$\dot{\varepsilon} = \frac{4\pi G_N}{H^2}\frac{2\dot{\phi}^2\ddot{\phi}}{\dot{\phi}} = O(\varepsilon\,\delta)$$ we can neglect the time dep. of ε [at first approx]



Number of e-folds?

$$N = \int_{t_i}^{t_f} dt'\, H(t') = \int_{t_i}^{t_f} H\,\frac{dt'}{d\phi_0}\,d\phi_0 = \int_{\phi_i}^{\phi_f}\frac{H}{\dot{\phi}_0}\,d\phi_0$$

$$\Rightarrow 3H\dot{\phi}_0 = -V' \longrightarrow N = \int_{\phi_i}^{\phi_f} d\phi_0\,\frac{3H^2}{-V'} = \int_{\phi_i}^{\phi_f} d\phi_0\,\frac{3\cdot\frac{8\pi G_N}{3}V}{V'} =$$

$$= 8\pi G_N\int_{\phi_i}^{\phi_f} d\phi_0\,\frac{V(\phi_0)}{V'(\phi_0)}$$ ϕ_f is easy since $\varepsilon = 1 \Rightarrow$ END OF INFL.

e.g.: $V(\phi_0) = \frac{m^2}{2}\phi_0^2$

$$N = 8\pi G_N\int_{\phi_i}^{\phi_f} d\phi_0\,\frac{m^2/2\,\phi_0^2}{m^2\phi_0} = 2\pi G_N\left[\phi_i^2 - \phi_f^2\right]$$

$$\Rightarrow \varepsilon = \frac{4\pi G_N\dot{\phi}_0^2}{H^2} = \dots = \frac{4\pi G_N (V')^2}{64\pi^2 G_N^2 V^2} = \frac{4\pi}{64\pi^2 G_N}\cdot\frac{4}{\phi_0} \Rightarrow \phi_f^2 = \frac{1}{4\pi G_N}$$

$3H\dot{\phi}_0 = -V'$

$H^2 = \frac{8\pi G_N}{3}V$

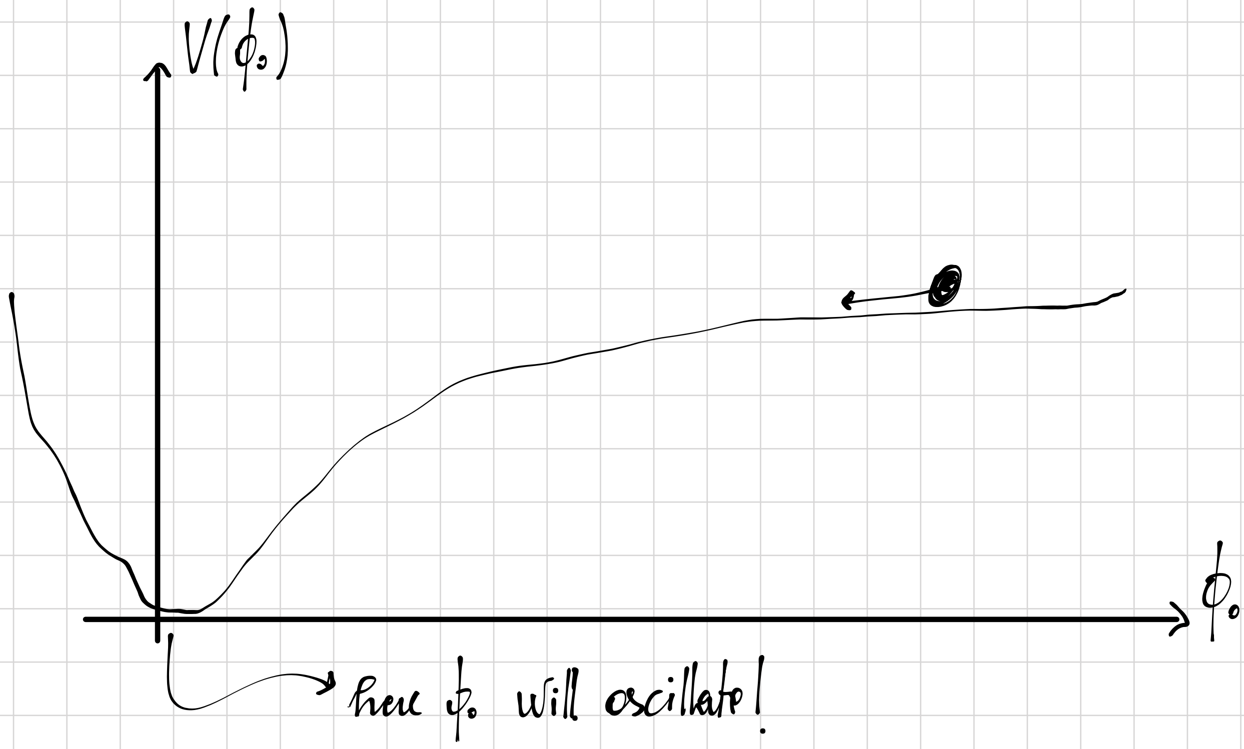
Suppose $V(\phi_0) = \frac{m^2}{2}\phi_0^2 \sim M_{pl}^4$

$$\Rightarrow \phi_i^2 \sim \frac{M_{pl}^4}{m^2} \longrightarrow m \sim 10^{12}\,\text{GeV (obs)} \Rightarrow \phi_i^2 \gg \phi_f^2$$

$$\Rightarrow N \sim 2\pi G_N \phi_i^2 \sim \frac{M_{pl}^2}{m^2} \sim 10^{12} \gg 60$$

$$\rightarrow \boxed{\Omega_0 \sim 1}$$

Suppose:



$$\rightarrow \rho = \frac{1}{2}\dot{\phi}_0^2 + V(\phi_0)$$

$$\ddot{\phi} + 3H\dot{\phi}_0 + V'(\phi_0) = 0$$

$$\dot{\rho} = \dot{\phi}_0\ddot{\phi}_0 + V'\dot{\phi}_0 = \dot{\phi}_0(\ddot{\phi}_0 + V') = -3H\dot{\phi}_0^2$$

let $m^2 \gg H^2 \longrightarrow \langle\dot{\rho}\rangle_{os} = -3H\langle\dot{\phi}_0^2\rangle_{os} = -3H\langle\rho\rangle_{os}$

$$\rho \sim a^{-3}$$

$$P = \frac{1}{2}\dot{\phi}^2 - V \longrightarrow \langle P\rangle_{os} = 0.$$

$\Rightarrow$ IT WILL DECAY into lighter part

Γ_ϕ : decay rate (model dep)

$\Rightarrow \Gamma_\phi = \frac{1}{\tau_\phi} = H \longrightarrow \tau_\phi \sim H^{-1} \longrightarrow$ call T_{RH} the temp. after thermalization

$$\Gamma_\phi^2 = H^2 = \frac{8\pi G_N}{3}\rho_R \longrightarrow \frac{\pi^2}{30} g_* T_{RH}^4$$

$$\Rightarrow T_{RH} \sim \sqrt{M_{pl}\Gamma_\phi}$$

Cosmological Perturbations

→ study perturbations during inflation

⇒ INFLATON FIELD $\phi_0(t)$:

$$S[\phi] = -\int d^4x \sqrt{-g}\left[\frac{1}{2} g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi + V(\phi)\right]$$

• e.o.m. (full scalar): $\ddot{\phi} + 3H\dot{\phi} - \frac{\nabla^2\phi}{a^2} + V'(\phi) = 0$ $\qquad$ ∇: comoving

Now consider: $\phi(\vec{x},t) = \phi_0(t) + \delta\phi(\vec{x},t)$:

$$\begin{cases} \ddot{\phi}_0 + 3H\dot{\phi}_0 + V'(\phi_0) = 0 \\ \delta\ddot{\phi} + 3H\delta\dot{\phi} - \frac{\nabla^2\delta\phi}{a^2} + V''\delta\phi = 0 \longrightarrow \delta\ddot{\phi} + 3H\delta\dot{\phi} + V''\delta\phi = 0 \end{cases}$$

Suppose perturb $\lambda \gg H^{-1}$ ⟶ neglect $\nabla^2\delta\phi$

Take a deriv of the first eq.:

$$\dddot{\phi}_0 + 3H\ddot{\phi}_0 + V''\dot{\phi} = 0 \qquad (H = \text{cost at 1st approx})$$

↳ The eq for $\dot{\phi}_0$ and $\delta\phi$ are the same:

$$\delta\phi(\vec{x},t) = -\dot{\phi}_0(t)\,\tau(\vec{x}) \qquad [\text{solut. must be prop.}]$$

$$\phi(\vec{x},t) = \phi_0 + \delta\phi(\vec{x},t) =$$
$$= \phi_0(t - \tau(\vec{x})) \qquad [\text{1st order approx: each point has diff. time}]$$

→ MASSLESS SCALAR FIELD IN DE SITTER

Take a generic scalar field: $\chi(\vec{x},t)$

$$S[\chi] = -\int d^4x \sqrt{-g}\, g^{\mu\nu}\partial_\mu\chi\,\partial_\nu\chi \longrightarrow \ddot{\chi} + 3H\dot{\chi} - \frac{\nabla^2\chi}{a^2} = 0$$

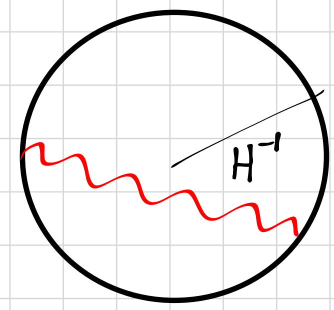
$$\chi(\vec{x},t) = \int \frac{d^3k}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\vec{x}} \chi_k(t)\, a_k + \text{h.c.} \qquad \text{s.t. } \chi'\chi^{*} - \chi^{*\prime}\chi = -i$$

$$\Rightarrow \ddot{\chi}_k + 3H\dot{\chi}_k + \frac{k^2}{a^2}\chi_k = 0$$

We suppose a pure De Sitter period: $H = \text{const}$

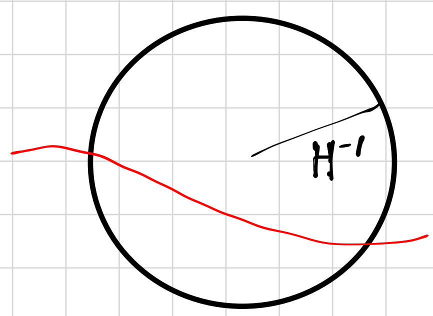
$\rightarrow$ Hubble radius H^{-1}:

1) $\lambda \ll H^{-1}$: $\frac{k}{aH} \gg 1$



$\Rightarrow \ddot{\chi}_k + \frac{k^2}{a^2}\chi_k = 0 \rightarrow$ propagating wave

2) $\lambda \gg H^{-1}$: $\frac{k}{aH} \ll 1$



$\Rightarrow \ddot{\chi}_k + 3H\dot{\chi}_k = 0 \rightarrow \chi_k = \text{const}$

$\Downarrow$

freezed perturb.

1) We always start from: $ds^2 = -dt^2 + a^2(t)\, d\vec{x}^2 = a^2(\tau)\left[-d\tau^2 + d\vec{x}^2\right]$

$d\tau = \frac{dt}{a}$: CONFORMAL TIME

$$\tau - \tau_i = \int_{t_i}^{t} \frac{dt'}{a(t')} = -\frac{1}{H a_*}\left[e^{-H_*(t'-t_*)}\right]_{t_i}^{t}$$

$\Rightarrow a = -\frac{1}{H\tau}$ $\quad (\tau < 0$

2) $$S[\chi] = -\frac{1}{2}\int d^3x\, d\tau\; a^4 g^{\mu\nu}\partial_\mu\chi\partial_\nu\chi = \frac{1}{2}\int d^3x\, d\tau\; a^2\left[\chi'^2 - (\nabla\chi)^2\right] =$$

$$= \int d^4x\; a^2\left[\frac{1}{2}\chi'^2 - \frac{(\nabla\chi)^2}{2}\right] \qquad \chi' = \frac{\partial\chi}{\partial\tau}$$

Call $\chi(\vec{x},t) = \frac{\sigma(\vec{x},t)}{a} \rightarrow \chi' = \frac{\sigma'}{a} - \sigma\frac{a'}{a^2}$

$= -(\sigma^2)'\frac{a'}{a^3}$

$$\Rightarrow S[\sigma] = \int d^4x\, \frac{a^2}{2}\left[(\sigma')^2\frac{1}{a^2} + \sigma^2\left(\frac{a'}{a^2}\right)^2 - 2\sigma'\sigma\frac{a'}{a^3} - \frac{(\nabla\sigma)^2}{a^2}\right] =$$

$$= \int d^4x\, \frac{1}{2}\left[(\sigma')^2 + \sigma^2\left(\frac{a'}{a}\right)^2 - (\sigma^2)'\frac{a'}{a} - (\nabla\sigma)^2\right] =$$

$$\overset{IBP}{=} \int d^4x\, \frac{1}{2}\left[(\sigma')^2 + \sigma^2\left(\frac{a'}{a}\right)^2 + \sigma^2\frac{a''}{a} - \sigma^2\frac{(a')^2}{a^2} - (\nabla\sigma)^2\right] =$$

$$= \int d^4x\, \frac{1}{2}\left[(\sigma')^2 - (\nabla\sigma)^2 + \sigma^2\frac{a''}{a}\right]$$

Go to momentum space:

$$\sigma_k'' + \left(k^2 - \frac{a''}{a}\right)\sigma_k = 0$$

and compute $\frac{k}{aH} = \frac{k}{-\frac{1}{H\tau} H} = -k\tau > 0$

$$a' = \frac{1}{H\tau^2} \longrightarrow a'' = -\frac{2}{H\tau^3}$$

$$\Rightarrow \sigma_k'' + \left(k^2 - \frac{2}{\tau^2}\right)\sigma_k = 0$$

Then:

1) $\lambda \ll H^{-1} \rightarrow (-k\tau) \gg 1 \longrightarrow \sigma_k'' + k^2\sigma_k = 0 \rightarrow$ WAVE

2) $\lambda \gg H^{-1} \rightarrow (-k\tau) \ll 1 \longrightarrow \sigma_k'' = \frac{a''}{a}\sigma_k \longrightarrow \frac{\sigma_k''}{\sigma_k} = \frac{a''}{a} \rightarrow \sigma_k \sim a \Rightarrow \chi = \frac{\sigma}{a} \sim$ Const

Solutions: 1) $\sigma_k(\tau) = \frac{A_1}{\sqrt{2k}} e^{-ik\tau} + \frac{A_2}{\sqrt{2k}} e^{ik\tau}$

$\hookrightarrow$ choosing the initial (vacuum) state selects A_1 and A_2:

BUNCH-DAVIES (vacuum with no part.)

$$A_1 = 1, \quad A_2 = 0$$

2) $(-k\tau) \ll 1 \rightarrow \sigma_k(\tau) = B(k)\, a$ [not interested in decay mode]

$\hookrightarrow$ MATCHING: $\frac{1}{\sqrt{2k}} = B(k)\, a \Big|_{\frac{k}{aH}=1} \Leftrightarrow B(k) = \frac{H}{\sqrt{2k^3}}$

glue the two solutions

$$\Rightarrow \chi_k = \frac{H}{\sqrt{2k^3}}$$

Now solve exactly:

$$\sigma_k'' + \left(k^2 - \frac{2}{\tau^2}\right)\sigma_k = 0$$
$$\text{with } \sigma_k(-k\tau \gg 1) = e^{-ik\tau}\frac{1}{\sqrt{2k}} \rightarrow \sigma_k(\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}}\left(1 - \frac{i}{k\tau}\right)$$

POWER SPECTRUM

$$P_\chi(k) := \frac{k^3}{2\pi^2}|\chi_k|^2 \rightarrow \langle\chi^2\rangle = \int\frac{d^3k}{(2\pi)^3}|\chi_k|^2 = \int d\ln k\, P_\chi$$

PURE DE SITTER $\rightarrow P_\chi(k) = \left(\frac{H}{2\pi}\right)^2 \rightarrow$ indep. of k! Depends only on H scale!

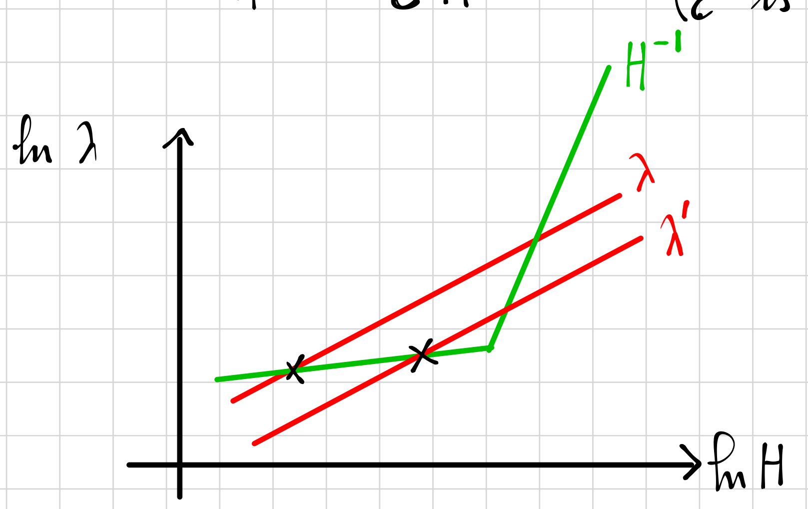
$\Downarrow$

"FLAT POWER SPECTRUM"

Parameterize: $P(k) = A(k_*)\left(\frac{k}{k_*}\right)^{n-1} \longrightarrow n=1$: massless field in De Sitter.

Now consider: Hubble radius SLOWLY changing in time [QUASI-DE SITTER]

$$\dot{H} = -\varepsilon H^2 \qquad (\varepsilon \text{ is one of the slow roll param})$$



$\rightarrow P_\chi(k) = \left(\frac{H_k}{2\pi}\right)^2$ $\longrightarrow$ the exact point when the wavelength exits the Hubble radius.

$\downarrow$

$$P_\chi(k) = A(k_*)\left(\frac{k}{k_*}\right)^{n-1} \longrightarrow n-1 = \frac{d\ln P_\chi}{d\ln k} = \frac{d\ln H_k^2}{d\ln k} = \frac{d\ln H_k^2}{dt}\frac{dt}{d\ln a}\frac{d\ln a}{d\ln k}$$
$$= \frac{2}{H_k}\cdot\frac{\dot{H}_k}{H_k}\cdot 1 = 2\frac{\dot{H}_k}{H_k^2} \sim -2\varepsilon \text{ (at 1st order)}$$

$\Rightarrow \boxed{n = 1-2\varepsilon} \longrightarrow$ it's deviating from unity by a very small amount.

$\rightarrow$ the spectral index can be measured:

$$\boxed{n = 0.96} \longrightarrow \text{very good prediction!}$$

We found:

$$\frac{\ddot{a}}{a} = (1-\varepsilon)H^2 \quad \text{with } a \sim t^{\alpha} \Rightarrow \alpha = \frac{1}{\varepsilon}$$

$$a(\tau) \sim \tau^{\frac{\alpha}{1-\alpha}} \rightarrow -\frac{1}{H}\frac{1}{\tau^{1+\varepsilon}}$$

$$\begin{cases} a(\tau)_{RD} \sim \tau \\ a(\tau)_{MD} \sim \tau^2 \end{cases}$$

Then $\sigma_k'' + \left(k^2 - \frac{a''}{a}\right)\sigma_k = 0 \Rightarrow \frac{a''}{a} = \frac{2+3\varepsilon}{\tau^2}$

$$\rightarrow \sigma_k'' + \left[k^2 - \frac{1}{\tau^2}\left(\nu^2 - \frac{1}{4}\right)\right]\sigma_k = 0 \qquad \nu^2 - \frac{1}{4} = 2 + 3\varepsilon$$

$$\hookrightarrow \sigma_k(\tau) = \sqrt{-\tau}\left[C_1 H_\nu^{(1)}(-k\tau) + C_2 H_\nu^{(2)}(-k\tau)\right]$$

Hankel

If $-k\tau \ll 1 \longrightarrow \sigma_k(\tau) \sim \frac{1}{\sqrt{-k\tau}}(-k\tau)^{\frac{1}{2}-\nu} \longrightarrow \chi_k \sim (-k\tau)^{3/2-\nu}$

$$P_\chi(k) \sim k^{-2\varepsilon}$$

Set $\varepsilon = 0 \Rightarrow$ FLAT SPECTRUM! $\longrightarrow$ explained by symmetries

$$ds^2 = a^2(\tau)\left[-d\tau^2 + d\vec{x}^2\right] = \frac{1}{H^2\tau^2}\left[-d\tau^2 + d\vec{x}^2\right]$$

$\Rightarrow$ 10 symm: CONFORMAL GROUP!

* take $\vec{x} \rightarrow \lambda\vec{x}$ and $\tau \rightarrow \lambda\tau$ $\qquad \lambda \in \mathbb{R}$

Consider the action in conformal time:

$$S[\chi] = \frac{1}{2}\int \frac{d^4x}{H^2\tau^2}\left[\chi'^2 - (\nabla\chi)^2\right] \qquad \chi(\vec{x},t) \text{ sol} \Rightarrow \underset{\tilde{\chi}(\vec{x},t)}{\chi(\lambda\vec{x},\lambda\tau)} \text{ is also a sol.}$$

$$\Rightarrow S[\tilde{\chi}] = \frac{1}{2}\int \frac{d^4x}{H^2\tau^2}\left[\tilde{\chi}'^2 - (\nabla\chi)^2\right]$$

plug in $\chi(\lambda\vec{x},\lambda\tau) = \tilde{\chi}(\vec{x},t) \longrightarrow S[\tilde{\chi}] = S[\chi]$

$\to$ Fourier transf: $\chi(\lambda\vec{x}) = \int \frac{d^3k}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\lambda\vec{x}} \chi_{\vec{k}} \qquad \vec{k}\cdot\lambda = \vec{p}$

$$\Rightarrow \chi_{\vec{k}} \to \frac{1}{\lambda^3}\chi_{\vec{k}/\lambda}$$

I know it from conformal invariance

$$\rightsquigarrow \langle \chi_{\vec{k}_1}\chi_{\vec{k}_2}\rangle = (2\pi)^3\delta^3(\vec{k}_1+\vec{k}_2)\frac{\#}{k_1^3} \longrightarrow (2\pi)^3\delta^3\left(\frac{\vec{k}_1}{\lambda}+\frac{\vec{k}_2}{\lambda}\right)\frac{\#}{(k_1/\lambda)^3}\frac{1}{\lambda^6}$$

$\Rightarrow |\chi_k|^2 \sim k^{-3}$ by conformal invariance

Now look at the GRAVITY coupling

$$\phi(\vec{x},t) \text{ inflaton} \longrightarrow \delta\phi \longrightarrow \delta T_{\mu\nu}$$

KG EQ, EINSTEIN EQ, $\delta g_{\mu\nu}$

Consider

$$g_{\mu\nu} = g_{\mu\nu}^{FRW}(t) + \delta g_{\mu\nu}(\vec{x},t) \qquad \text{(linear perturb. theory)}$$

$\to$ how many scalars do we have?

- dimension $D = n+1 \Rightarrow g_{\mu\nu}$ has $\frac{1}{2}(n+1)(n+2)$ comp.

$-(n+1)$ dofs by invariance

we know $V_i = \partial_i V + w_i$ s.t. $\vec{\nabla}\cdot\vec{w} = 0$

$-(n-1)$ dofs vector dofs

tensor $\frac{1}{2}n(n+1) - n - 1$ $\hookrightarrow$ traceless

$-\left[\frac{n}{2}(n+1) - n - 1\right]$

---

2 SCALAR DOFs

Now choose:

$$ds^2 = a^2(\tau)\left[-(1+2\phi)d\tau^2 + (1-2\psi)d\vec{x}^2\right]$$

and define gauge (diff.) invariant transf.:

choose a scalar $f(\vec{x},t)$ (e.g.: $\delta\phi(\vec{x},t)$ or $\delta\rho(\vec{x},t)$) no ~ on background!

$$\hookrightarrow \delta f(\vec{x},t) = f(\vec{x},t) - f_0(t) \xrightarrow[\text{SCALAR!}]{x^\mu \mapsto \tilde{x}^\mu = x^\mu + \delta x^\mu} \widetilde{\delta f} = \tilde{f}(\tilde{x}^\mu) - f_0(\tilde{x}^0)$$

$$\Rightarrow \widetilde{\delta f} = f(x^\mu) - f_0(x^0) - \dot{f}_0\,\delta x^0 = \delta f - \dot{f}\,\delta x^0$$

$\rightarrow$ f is gauge inv $\Leftrightarrow \dot{f} = 0$ (eg: $\delta\phi$ is NOT gauge invariant !!!)
$\hookrightarrow$ the potential is slowly varying the value!

$$\Rightarrow x^\mu \mapsto x^\mu + \delta x^\mu \Rightarrow x^i \mapsto x^i$$

$$x^0 \mapsto x^0 + \delta x^0 = \tilde{x}^0$$

$$\hookrightarrow ds^2 = \widetilde{ds}^2 \rightarrow a^2(x^0)(1-2\psi) = a^2(\tilde{x}^0)(1-2\tilde{\psi}) \rightsquigarrow \tilde{\psi} = \psi + H\delta x^0$$

Then:

$$\begin{aligned} \psi &\longrightarrow \psi + H\,\delta x^0 \\ \delta\rho &\longrightarrow \delta\rho - \dot{\rho}_0\,\delta x^0 \end{aligned} \quad \zeta = \psi + H\frac{\delta\rho}{\dot{\rho}} \quad \text{IS GAUGE INVARIANT!}$$

$$\zeta \mapsto \tilde{\zeta} = \psi + H\delta x^0 + H\frac{\delta\rho}{\dot{\rho}_0} - H\delta x^0 = \zeta$$

ζ: COMOVING CURVATURE PERTURBATION

ζ can be computed in any gauge:

$$\psi = 0 \rightarrow \zeta = H\left.\frac{\delta\rho}{\dot{\rho}_0}\right|_{\psi=0} : \text{ FLAT GAUGE}$$

$$\delta\rho = 0 \rightarrow \zeta = \psi|_{\delta\rho=0} : \text{ UNIFORM ENERGY DENSITY}$$

We can choose:

$$\mathcal{R} = \psi + H \frac{\delta\phi}{\dot{\phi}_0}$$

$$\zeta = \psi + H \frac{\delta\rho}{\dot{\rho}_0}$$

$$\dot{\rho}_0 + 3H(\rho_0 + P_0) = 0 \longrightarrow \rho_0 + P_0 = \dot{\phi}_0^2$$

$$\Rightarrow \zeta = \psi - \frac{\delta\rho}{3\dot{\phi}_0^2} \longrightarrow \underbrace{\delta\rho \simeq - 3H\dot{\phi}_0\,\delta\phi}_{\text{Slow Roll}} \longrightarrow \zeta = \psi + H\frac{\delta\phi}{\dot{\phi}_0} = \mathcal{R}$$

→ <u>Power spectrum</u>

$$P_\zeta = ?$$

In flat gauge: $\zeta\big|_{\psi=0} = \ldots = H\frac{\delta\phi}{\dot{\phi}_0}\Big|_{\psi=0}$ no gravitational potential

$$\hookrightarrow \langle \zeta^2 \rangle = \frac{H^2}{\dot{\phi}_0^2}\left(\frac{H}{2\pi}\right)^2 \longrightarrow P_\zeta = \frac{1}{2M_p^2\varepsilon}\left(\frac{H}{2\pi}\right)^2 \qquad \varepsilon = 4\pi G_N \frac{\dot{\phi}_0^2}{H^2}$$

→ spectral index? $P_\zeta = \left(\frac{H_k}{2\pi}\right)\frac{H_k^2}{\dot{\phi}_0^2}$ ↳ when they go out of the Hubble radius

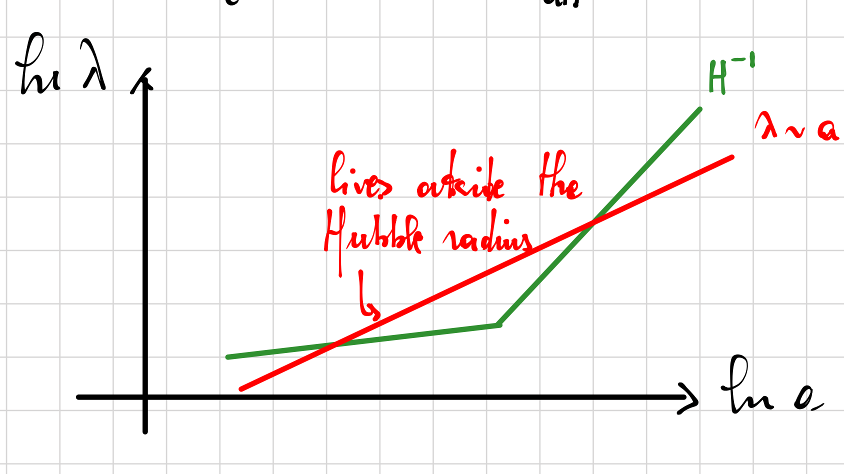
$$n_\zeta - 1 = \frac{d\ln P_\zeta}{d\ln k} = -4\varepsilon + 2\eta - 2\varepsilon = 2\eta - 6\varepsilon \neq 1 \text{ by slow-roll param.}$$

ζ is important for 2 reasons.

1) gauge inv.

2) in single-field models ζ is const on super-Hubble scales:

$$\dot{\zeta}_k = 0 \quad \text{when} \quad \frac{k}{aH} < 1$$



$$\zeta\big|_{\psi=0} = H\frac{\delta\rho}{\dot{\rho}_0}$$

→ sth is happening at quantum level → INFLATION → galaxies, anisotropies, etc.

$\hookrightarrow \nabla_\mu T^{\mu\nu} = 0 \longrightarrow \delta[\nabla_\mu T^{\mu\nu} = 0] \longrightarrow \delta\rho = ?$

Suppose H^{-1}: Hubble radius and perturb on super-Hubble scales:

$$ds^2 = -(1+2\phi)\,dt^2 + (1-2\psi)\,a^2 d\vec{x}^2 =$$
$$= -e^{2\phi}\,dt^2 + e^{-2\psi}\,a^2 d\vec{x}^2 =$$
$$= -\bar{dt}^2 + \bar{a}^2 \vec{x}^2 \longrightarrow \text{def. coord. to "localise" physics}$$

$$\bar{dt} = e^{\phi} dt \qquad \bar{a} = a e^{-\psi}$$

Now: (where $\dot{} = \frac{d}{d\bar{t}}$)

$$\dot{\rho} + 3\bar{H}(\rho + P) = 0 \Longrightarrow \bar{H} = \frac{\dot{\bar{a}}}{\bar{a}} = (H - \dot{\psi})e^{-\phi}$$

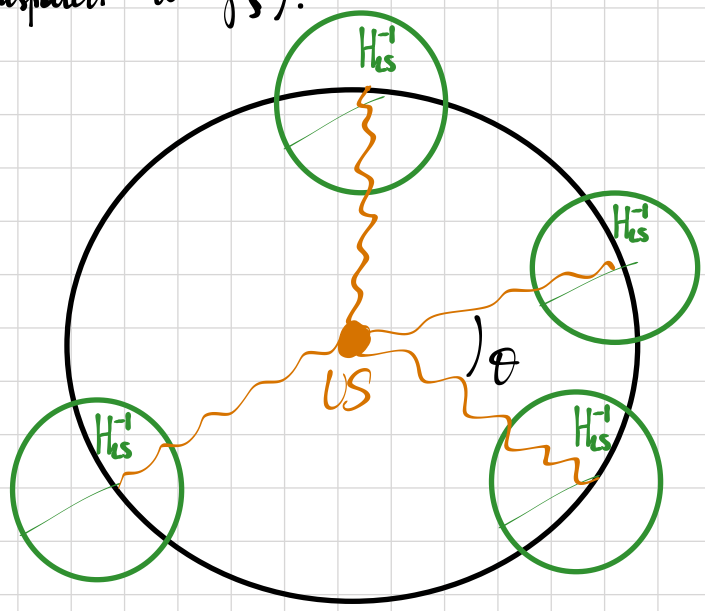
$$\hookrightarrow \delta\dot{\rho} + 3H(\delta\rho + \delta P) - 3\dot{\psi}(\rho_0 + P_0) = 0$$

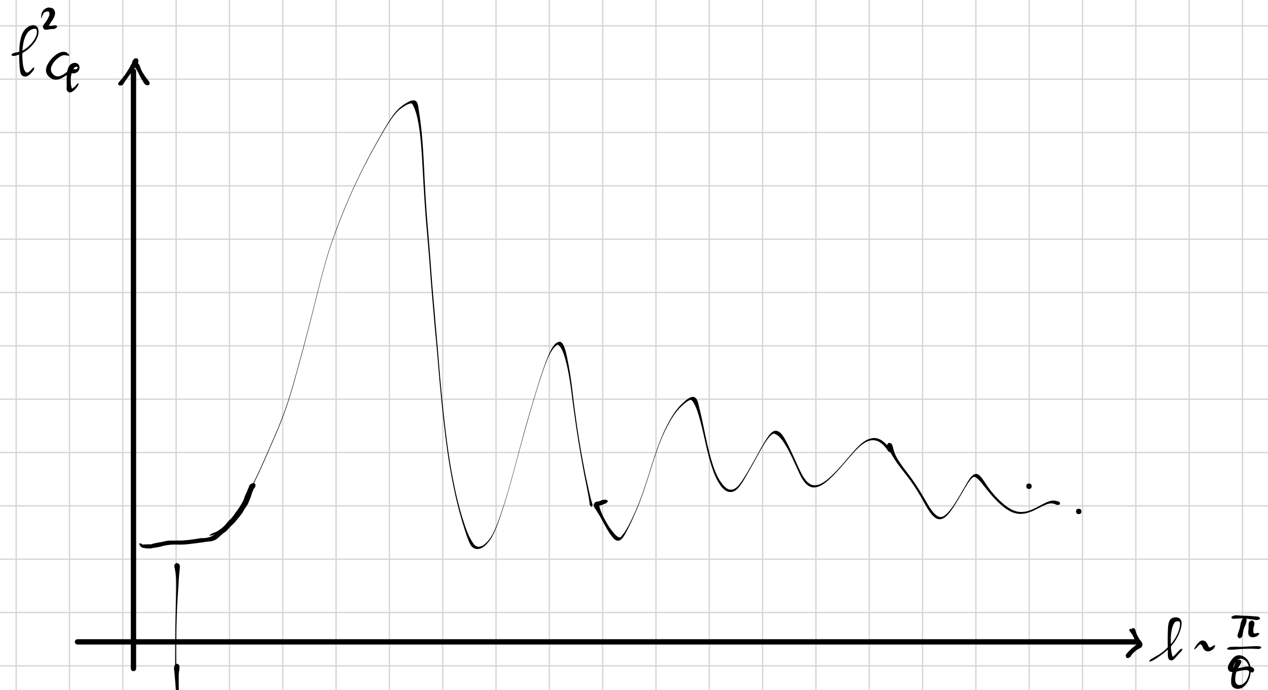
$$\delta P = c_s^2\,\delta\rho + \delta P_{\substack{NON\\ADIAB}} \qquad c_s^2 = \frac{\delta P}{\delta\rho}$$

→ in $\delta\rho = 0$ GAUGE: $\zeta|_{\delta\rho=0} = \psi \longrightarrow \dot{\zeta} = \frac{H\,\delta P_{NAD}}{\rho_0 + P_0}$

if only 1 dof (only 1 scalar field): $\delta P_{NAD} = 0 \longrightarrow \dot{\zeta} = 0$

After LS surface (universe transparent to γ's):





$\sim$ FLAT $\longrightarrow \sim$ small perturb. $\Rightarrow$ θ longer than angular dist. between 2 Hubble vol.

$\downarrow$

$\frac{\delta T}{T} \sim 10^{-5}$

$\Rightarrow$ OBSERVER: $\quad T_0 = T_E \frac{\omega_0}{\omega_E} \longrightarrow$ interested in very large wavelenghts

$T_0 \qquad T_E$

$$\omega_E \sim \frac{1}{dt} \sim e^{-\phi}$$

$$\rightarrow \frac{\delta T_0}{T_0} = \frac{\delta T_E}{T_E} + \Phi_E$$

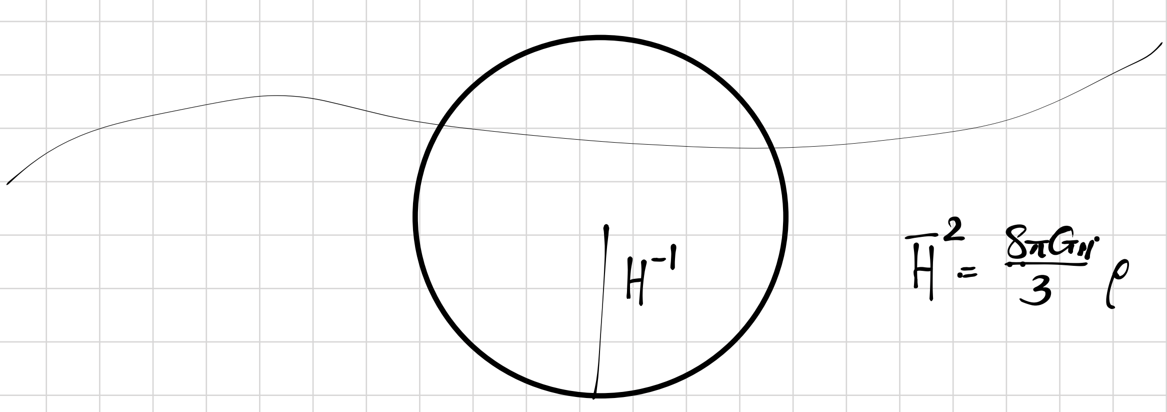
Now: $\rho_\gamma \sim T^4 \longrightarrow \frac{\delta T}{T} = \frac{1}{4}\frac{\delta\rho_\gamma}{\rho_\gamma} \Rightarrow \zeta\big|_{\psi=0} = H\frac{\delta\rho}{\dot\rho}$

$$\hookrightarrow \zeta_{rad} = H\frac{\delta\rho_r}{\dot\rho_r} \Rightarrow \dot\rho_{rad} + 4H\rho_{rad} = 0 \Rightarrow \zeta = \zeta_{rad} = -\frac{1}{4}\frac{\delta\rho_{rad}}{\rho_{rad}} = \zeta_m = -\frac{1}{3}\frac{\delta\rho_m}{\rho_m}$$

$$\Rightarrow \frac{1}{4}\frac{\delta\rho_r}{\rho_r} = \frac{1}{3}\frac{\delta\rho_m}{\rho_m}$$

$$\frac{1}{4}\frac{\delta\rho_r}{\rho_r} = \frac{\delta T_E}{T_E}$$

Now $\frac{\delta T_0}{T_0} = \frac{\delta T_E}{T_E} + \Phi_E$



$$H^2 = \frac{8\pi G_N}{3}\rho \longrightarrow \frac{\delta\rho_m}{\rho_m} = -2\phi$$

$$\Rightarrow \frac{\delta T_0}{T_0} = \Phi_E + \frac{1}{3}\frac{\delta\rho_m}{\rho_m} = \frac{1}{3}\Phi_E \longrightarrow \text{the perturbed universe is reflected in the temperature fluctuations}$$

SACHS-WOLFE EFFECT

$$\dot{\rho} = -3H(\rho + P)$$
$$P = w\rho$$

$$\rightarrow \zeta = \psi + H\frac{\delta\rho}{\dot{\rho}} = \psi - \frac{1}{3}\frac{\delta\rho}{(1+w)\rho} = \psi + \frac{2}{3}\frac{\phi}{(1+w)} = \qquad (\psi = \phi)$$

$$= \frac{5+3w}{3(1+w)}\phi$$

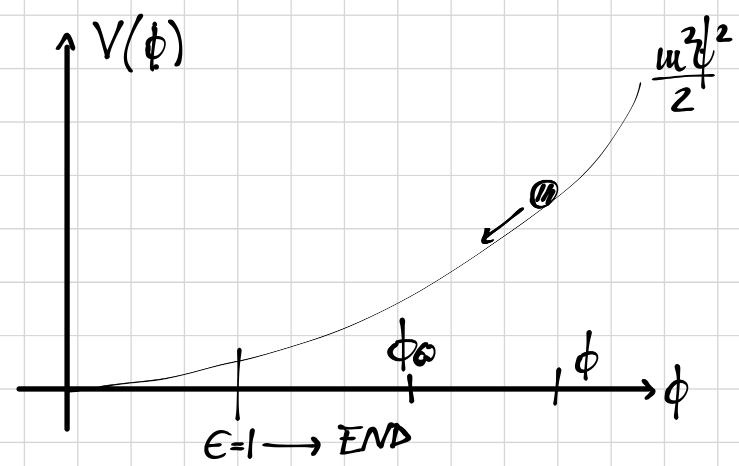
$$\rightarrow \text{MD: } \zeta = \frac{5}{3}\phi \quad (w=0) \rightarrow \boxed{\frac{\delta T}{T} = \frac{1}{5}\zeta}$$

MEASUREMENT

PREDICTION

We know:

$$P_\zeta = \left(\frac{H}{2\pi}\right)^2 \frac{1}{2\epsilon M_{pl}^2}$$



no. of e-folds till the end of inflation:

$$\Delta N = \frac{1}{M_p^2}\int_{\phi_{END}}^{\phi_{\Delta N}} d\phi \, \frac{V}{V'} \longrightarrow \phi_{\Delta N}^2 = 4\Delta N M_{pl}^2$$

$$\rightarrow \epsilon_{\Delta N} = \frac{M_p^2}{2}\left(\frac{V'}{V}\right)^2_{\Delta N} = \frac{1}{2\Delta N}$$

$$\eta_{\Delta N} = \frac{1}{3}\left(\frac{V''}{H^2}\right)_{\Delta N} = \epsilon_{\Delta N} = \frac{1}{2\Delta N}$$

$\Rightarrow$ spectral index: $n_\zeta - 1 = 2\eta_{\Delta N} - 6\epsilon_{\Delta N} = -\frac{2}{\Delta N}$

$$\Delta N = 60 \longrightarrow \underline{\underline{n_\zeta = 0.97}} \quad (\text{obs. } 0.96)$$

$$\rightarrow H^2 = \frac{1}{3}\frac{V}{M_p^2} = \frac{1}{3}\frac{m^2}{2}\frac{\phi_{\Delta N}^2}{M_p^2} \longrightarrow P_\zeta \sim \frac{m^2}{M_p^2} \Rightarrow \frac{m}{M_p} \sim 10^{-5} \longrightarrow m \sim 10^{13}\,\text{GeV}$$

Now let's look at tensor modes $\longrightarrow$ grav. waves :

$$ds^2 = -dt^2 + a^2\left(\delta_{IJ} + h_{IJ}\right) dx^I dx^J$$

$$\rightarrow S[h] = \frac{1}{32\pi G_N} \int d^4x \sqrt{-g} \left[\frac{\dot{h}_{ij}^2}{2} - \frac{(\nabla h_{ij})^2}{2} \right] \qquad h_{ij} = 2\frac{V_{ij}}{M_P}$$

$$\hookrightarrow \text{dof's: } 2 \text{ in } h_{ij} \longrightarrow P_h = \frac{\overset{\text{COUNTING}}{\downarrow}}{} \!\!\! \frac{8}{M_P}\left(\frac{H}{2\pi}\right)^2 \sim k^{n_T}$$

$$\boxed{n_T = -2\epsilon}$$

$$\Rightarrow \frac{P_h}{P_\zeta} = 16\epsilon = -8n_T \longrightarrow r = \boxed{\frac{P_h}{P_\zeta} = -8n_T}$$

$\longrightarrow$ to be verified if single field inflation is true!