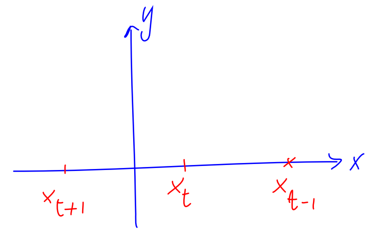
1 TO DO

- T and twisted fields in the non trivial vacuum
- Green function
- Say it is a 2d CFT "time dependent".



2 The generic formalism

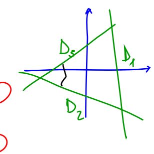
2.1 Boundary conditions and discontinuities

Any solution of the string eom is required to satisfy the b.c.

$X_{(t)} = R_{(t)} X - g_{(t)}$

$\hookrightarrow \partial_y X_{(t)}|_{y=0^+} = 0$

$X_{(t)}|_{y=0^+} = 0$



$R_{(t)} - R_{(t)}\begin{pmatrix} O^{\parallel(t)} & \\ & O^{\perp(t)} \end{pmatrix}$

$$\begin{aligned} N \longrightarrow\ & R_{(t)}{}^{\parallel}\partial_y X(u,\bar u)|_{u=x+i0^+} = 0\\ D \longrightarrow\ & R_{(t)}{}^{\perp} X(u,\bar u)|_{u=x+i0^+} = g_{(t)}{}^{\perp} \qquad x \in (x_t, x_{t-1}). \end{aligned} \tag{1}$$

that is for $x \in (x_t, x_{t-1})$ the string boundary is on $D_{(t)}$. Here $\parallel = 1,\dots p$ and $\perp = p+1,\dots q$ are the indexes in the well adapted basis. These are universal to all branes since we suppose all of them are Dp branes along the $\parallel = 1,\dots p$ of the well adapted frame. The matrix $R_{(t)}$ is in principle in $SO(q)$ but it is actually $\boxed{R_{(t)} \in SO(q)/[SO(p)\times SO(q-p)]}$ because of the preserved rotational symmetries. **?? Moreover we have $g_{(t)}{}^{\perp} > 0$ since we can use the elements of $O(p)\times O(q-p)\cap SO(q)$ to choose the signs of the $g_{(t)}$s without changing the embedding. ??**Notice therefore that $\parallel$ is an index while the symbol $^{\perp(t)}$ introduced afterwards as for example in $X^{\perp(t)}$ is a projection.

Using the solution to e.o.m from these conditions we deduce that

$$\boxed{X'_L(x+i0^+) = U_{(t)} X'_R(x-i0^+)} \qquad x \in (x_t, x_{t-1}) \tag{2}$$

where we defined[1]

$$\frac{SO(q)}{SO(p)\times SO(q-p)} \ni U_{(t)} = R_{(t)}{}^{-1}\mathcal{S}R_{(t)}, \Rightarrow U_{(t)}{}^2 = \mathbb{I}, \tag{3}$$

and the non vanishing entries of $\mathcal{S}$ are $\mathcal{S}_{\parallel}^{\parallel} = +\mathbb{I}_{\parallel}^{\parallel}$ and $\mathcal{S}_{\perp}^{\perp} = -\mathbb{I}_{\perp}^{\perp}$.

The previous set of equations (2) is of local nature and it is not equivalent to the original ones since it only contains information on the directions but not on the location of space-time intersection between two branes hence they must be supplemented with this information

"global" conditions

$$\boxed{X(x_t, x_t) = f_{(t)},} \tag{4}$$

which is of global nature

[1] We write $R_{(t)}{}^{-1}$ because the relation $R_{(t)}{}^{-1} = R_{(t)}{}^{T}$ is true only with a metric g which is diagonal. This is not the case when using complex coordinates, the true relation is $R_{(t)}{}^{T} R_{(t)} = g$.

In particular from ${U_{(t)}}^2 = \mathbb{I}$ it follows that

$$P_{\|_{(t)}} = \frac{1+U_{(t)}}{2}, \qquad P_{\perp_{(t)}} = \frac{1-U_{(t)}}{2} \tag{5}$$

are projectors along the directions parallel and perpendicular to $D_{(t)}$. Notice that they do not generically commute for different t since generically $U_{(t)}$ do not commute, i.e. for example $P_{\|_{(t)}} P_{\|_{(t+1)}} \neq P_{\|_{(t+1)}} P_{\|_{(t)}}$ In the following we use the notation

$$X^{\perp_{(t)}}(u,\bar{u}) = P_{\perp_{(t)}} X(u,\bar{u}), \qquad X^{\|_{(t)}}(u,\bar{u}) = P_{\|_{(t)}} X(u,\bar{u}). \tag{6}$$

The previous eq. (2) can be integrated into

$$X_L(x+i0^+) = U_{(t)} X_R(x-i0^+) + \Delta_{(t)} \qquad x \in (x_t, x_{t-1}), \tag{7}$$

where $\Delta_{(t)}$ is a constant dependent on $D_{(t)}$.

As consequence we can write the boundary value of the X field as

$$X(x+i0^+, x-i0^+) = \boxed{2P_{\|_{(t)}} X_R(x-i0^+)} + \Delta_{(t)} = (1+U_{(t)}) X_R(x-i0^+) + \Delta_{(t)} \quad x \in (x_t, x_{t-1}). \tag{8}$$

2.1.1 Continuity of string coordinates at b.c. discontinuities

Because of the continuity of $X(u,\bar{u})$ at any $u = x_t$ we get the system of equations

$$\begin{pmatrix} {R_{(t)}}^{\perp} f_{(t)} \\ {R_{(t+1)}}^{\perp} f_{(t)} \end{pmatrix} = \begin{pmatrix} {g_{(t)}}^{\perp} \\ {g_{(t+1)}}^{\perp} \end{pmatrix} \Big\} \text{ system of embedding equations} \tag{9}$$

since the intersection point $f_{(t)}$ lies in both $D_{(t)}$ ~~for~~ $\cancel{u} = {x_t}^+$ and $D_{(t+1)}$ for $u = {x_t}^-$. Notice that this system determines just a point in $\mathbb{R}^q$ whenever the sum of codimesionalities of the two branes equal q, i.e. when $2(q-p) = q$ or $p = \frac{1}{2}q$. We assume this is always the case in the rest of the paper.

eg. D2-branes $\longrightarrow$ point in $\mathbb{R}^4$, line in $\mathbb{R}^3$

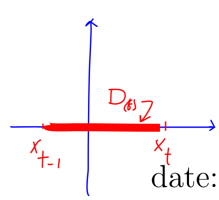
Therefore giving the $R_{(t)}$ and the ${g_{(t)}}^{\perp}$ is equivalent to give $U_{(t)}$ and f_t.?? **In fact we can recover $R_{(t)}$ from $U_{(t)}$ up to $SO(p) \times SO(q-p)$ rotations ??**

From the continuity of $X(u,\bar{u})$ and eq. (8) by projecting in the directions perpendicular to the $D_{(t)}$ we can determine the value of $\Delta_{(t)}$ in the directions perpendicular to the brane $D_{(t)}$ as

$${\Delta_{(t)}}^{\perp_{(t)}} = {f_{(t)}}^{\perp_{(t)}}. \tag{10}$$

Notice that $f_{(t-1)} = X({x_{t-1}}^-, \bar{x}_{t-1}^-) \in D_{(t)}$ and $f_{(t)} = X({x_t}^+, \bar{x}_t^+) \in D_{(t)}$ hence the vector $f_{(t-1)} - f_{(t)}$ belongs to the brane $D_{(t)}$ therefore it is parallel to it and therefore

$$\boxed{{f_{(t-1)}}^{\perp_{(t)}} = {f_{(t)}}^{\perp_{(t)}}.} \tag{11}$$



2.1.2 Reality in Euclidean space

We can now explore the consequences of the reality of $X(u, \bar{u})$ in Euclidean space where $u = e^{\tau_E + i\sigma}$ and $\bar{u} = e^{\tau_E - i\sigma}$ so that $u^* = \bar{u}$ [2]. It follows immediately that we determine the value of $\Im\Delta_{(t)}$ in the directions parallel to the brane $D_{(t)}$ as

from $\underline{\underline{\operatorname{Im} X = 0}}$
$$\Im\Delta_{(t)}{}^{\|(t)} = -2\Im X_R^{\|(t)}(x - i0^+), \qquad x \in (x_t, x_{t-1}). \tag{12}$$

In particular follows that $\Im X_R^{\|(t)}(x - i0^+)$ is constant on any interval $x \in (x_t, x_{t-1})$.

We can hence write

$$\Delta_{(t)} = f_{(t)}{}^{\perp(t)} + \Re\Delta_{(t)}{}^{\|(t)} - 2i\Im X_R^{\|(t)}(x - i0^+). \tag{13}$$

Also from the reality of $X(u, \bar{u}) = X_L(u) + X_R(\bar{u})$ we can write

$$[X_L(u)]^* = X_R(\bar{u}) + Y, \tag{14}$$

where Y is a constant which must be *real* by consistency. The presence of Y can be justified by thinking of the usual expansion with $X_L = x_0 + y_0 + \dots$ and $X_R = x_0 - y_0 + \dots$. The reality of Y can be shown by considering the reality of X on the boundary, i.e.

$$X(x, x) = X_L(x) + X_R(x) = (X_R + Y)^* + (X_L^* - Y) = X^*(x, x) - 2i\Im Y. \tag{15}$$

Notice that shifting $X_{L,R} \to X_{L,R} \pm \frac{1}{2}Y$ we can set $Y = 0$.

If we consider the reality of X on the boundary it may seem strange that there is only one Y. It is in fact natural to write

$$[X_L(x + i0^+)]^* = X_R(x - i0^-) + Y_{(t)} \qquad x \in (x_t, x_{t-1}), \tag{16}$$

with a constant $Y_{(t)}$ which depends on the brane $D_{(t)}$ where the string is attached. However if we suppose the continuity of $X_R(x_t)$ then we can write

$$[X_L(x_t^+ + i0^+)]^* = X_R(x_t^+ - i0^-) + Y_{(t)} = [X_L(x_t^- + i0^+)]^* = X_R(x_t^- - i0^-) + Y_{(t+1)} \tag{17}$$

since $x_t{}^- \in D_{(t+1)}$ and $x_t{}^+ \in D_{(t)}$. It follows immediately the same result as before $Y = Y_{(t)}$ for all ts. From the continuity of $X_R(x_t)$ follows also that

$$\Im\Delta_{(t)}{}^{\|(t)} = -2\Im X_R^{\|(t)}(x_t{}^+ - i0^+) = -2\Im X_R^{\|(t)}(x_t{}^- - i0^+) = \Im\Delta_{(t-1)}{}^{\|(t-1)}, \tag{18}$$

and the constance of $\Im X_R^{\|(t)}(x - i0^+)$.

[2] As we discuss later for the quantum part there is also the concept of Euclidean Hermitian which is the Minkowskian "reality" transported to Euclidean. The reality in Euclidean space is the relevant concept when dealing with instantons.

We want now to show that

$$\Re\Delta_{(t)}{}^{\parallel(t)} = \Re Y^{\parallel(t)} = Y^{\parallel(t)}, \qquad \Im\Delta_{(t)}{}^{\perp(t)} = \Im Y^{\perp(t)} = \Im f_{(t)}{}^{\perp(t)} = 0. \tag{19}$$

From the previous eq.s (14) and (7) we can compute $[X_L(x+i0^+)]^*$ in two different ways and write

$$\begin{aligned}[X_L(x+i0^+)]^* &= X_R(x-i0^-) + Y \\ &= U_{(t)}X_R^*(x-i0^+) + \Delta_{(t)}{}^* = U_{(t)}[X_L(x+i0^+) - Y^*] + \Delta_{(t)}{}^* \\ &\Rightarrow (\Delta_{(t)} - Y^*) + U_{(t)}(\Delta_{(t)}{}^* - Y) = 0,\end{aligned} \tag{20}$$

the considering the real and imaginary part we get the desired result.

Finally we can summarize the results by writing

$$\boxed{\Delta_{(t)} = f_{(t)}{}^{\perp(t)} + (Y - 2iI_{(t)})^{\parallel(t)},} \tag{21}$$

where we have introduced the *constant* $I_{(t)} = \Im X_R^{\parallel(t)}(x - i0^+)$ for $x \in (x_t, x_{t-1})$.

2.1.3 Fixing $\Re X_{R,L}(x_t)$

From the continuity of $X_{L,R}$ and the real part of eq. (7) we can also fix $\Re X_R(x_t - i0^+)$ and $\Re X_L(x_t + i0^+)$, in fact using eq. (11) for $t \to t+1$ we find

$$\begin{aligned}\boxed{\Re X_R(x_t - i0^+)} &= -\frac{1}{2}Y + (U_{(t)} - U_{(t+1)})^{-1}(f_{(t+1)}{}^{\perp(t+1)} - f_{(t)}{}^{\perp(t)}) = -\frac{1}{2}Y + \frac{1}{2}f_{(t)} \\ \boxed{\Re X_L(x_t + i0^+)} &= +\frac{1}{2}Y + (U_{(t)} - U_{(t+1)})^{-1}(f_{(t+1)}{}^{\perp(t+1)} - f_{(t)}{}^{\perp(t)}) = +\frac{1}{2}Y + \frac{1}{2}f_{(t)}.\end{aligned} \tag{22}$$

$Re\, X_R(x_{t-1} - i0^+) - Re\, X_R(x_t - i0^+) = \frac{1}{2}(f_{(t-1)} - f_{(t)})$

2.2 Double fields

Whenever X_L and X_R are not constant and therefore eq. (2) is not empty[3] we can define the double field defined on $(\mathbb{C} \setminus \mathbb{R}) \cup (x_{\bar{t}}, x_{\bar{t}-1})$, i.e. on all $\mathbb{C}$ but almost all the real axis as

glue the field on a brane

$$\mathcal{X}_{(\bar{t})}(z) = \begin{cases} X_L(u) & z = u \in H \cup (x_{\bar{t}}, x_{\bar{t}-1}) \\ U_{(\bar{t})}X_R(\bar{u}) + \Delta_{(\bar{t})} & z = \bar{u} \in H^- \cup (x_{\bar{t}}, x_{\bar{t}-1}) \end{cases}. \tag{23}$$

Using the double filed we can express the string field as

$$X(u, \bar{u}) = \mathcal{X}_{(\bar{t})}(u) + U_{(\bar{t})}{}^{-1}\mathcal{X}_{(\bar{t})}(\bar{u}) - U_{(\bar{t})}{}^{-1}\Delta_{(\bar{t})}, \tag{24}$$

?? since we can always set $\Delta_{(\bar{t})} = 0$ by shifting X. ??

[3] In the case where the only solution is a constant this may generically not be put in a double field as discussed in section 4.1.

The double field has the following boundary conditions on the cuts

$$\mathcal{X}_{(\bar{t})}(x+i0^+) = U_{(t)}U_{(\bar{t})}{}^{-1}\mathcal{X}_{(\bar{t})}(x-i0^+) + \Delta_{(t)} - U_{(t)}U_{(\bar{t})}{}^{-1}\Delta_{(\bar{t})}, \quad x \in (x_t, x_{t-1}). \tag{25}$$

This expression clearly shows that there is not discontinuity on the glued segment, as it should be. $\longrightarrow t=\bar{t} \Rightarrow \Delta_{(\bar{t})} - \mathbb{I}\Delta_{(\bar{t})} \equiv 0$: no discontinuity!

The previous discontinuities imply the following monodromies around the points x_t $(0 < \delta < min(x_t - x_{t+1}, x_{t-1} - x_t))$

$$\mathcal{X}_{(\bar{t})}(x_t + \delta e^{i2\pi^-}) = U_{(t+1)}U_{(t)}{}^{-1}\mathcal{X}_{(\bar{t})}(x_t + \delta e^{i0^+}) + \Delta_{(t+1)} - U_{(t+1)}U_{(t)}\Delta_{(t)}. \tag{26}$$

At the derivative level we do not see the shift and we get the cleaner expression

$$\boxed{\partial\mathcal{X}_{(\bar{t})}(x_t + \delta e^{i2\pi^-}) = U_{(t+1)}U_{(t)}{}^{-1}\partial\mathcal{X}_{(\bar{t})}(x_t + \delta e^{i0^+})}. \tag{27}$$

e.g.: for the classical solution

2.3 Radially conserved product

Given two *real* solutions ${}^*F^I$ and G^I of the *bulk* problem we can define a conserved current[4]. In the following we do not impose that ${}^*F$ and G have equal boundary conditions but we search the condition for a well defined product under the only assumption that they satisfy the bulk equation of motion. This is the reason of the notation which should remind the reader that the basic field is G while ${}^*F$ is the dual wrt the product we are defining.

The conserved current reads

$$\boxed{J({}^*F, G) = \mathcal{N} * ({}^*F^T g\, dG - d{}^*F^T g\, G)} \tag{28}$$

is conserved even if ${}^*F$ and G have different boundary conditions. We will choose

$$\mathcal{N} = \frac{i}{\pi} \tag{29}$$

for later convenience.

This current is obviously antisymmetric

$$J({}^*F, G) = -J(G, {}^*F). \tag{30}$$

?? and not assuming the reality of the solutions (which is not the our case) we get

$$\boxed{J({}^*F, G)^* = J(G^*, {}^*F^*).} \tag{31}$$

see notes!

?? If the following relations

$$\begin{aligned} \mathcal{N}\int_{r_0}^{r_1} -({}^*F^T g\, \partial_y G - \partial_y {}^*F^T g\, G)\, dx &= \mathcal{B}_{(+)}(r_1) - \mathcal{B}_{(+)}(r_0) \\ \mathcal{N}\int_{-r_1}^{-r_0} -({}^*F^T g\, \partial_y G - \partial_y g\, {}^*F^T G)\, dx &= -\mathcal{B}_{(-)}(-r_0) + \mathcal{B}_{(-)}(-r_1) \end{aligned} \tag{32}$$

[4]Our conventions are $*dx = dy$, $*dy = -dx$ which are equivalent to $*dr = r d\theta$ and $*d\theta = -dr/r$.

$\begin{cases} *dx = dy \\ *dy = -dx \end{cases} \Rightarrow \begin{cases} dx = \cos\theta\, dr - r\sin\theta\, d\theta = \frac{x}{r} dr - y\, d\theta \\ dy = \sin\theta\, dr + r\cos\theta\, d\theta = \frac{y}{r} dr + x\, d\theta \end{cases} \Rightarrow \begin{cases} *dx = -y *d\theta + \frac{x}{r} *dr = dy \\ *dy = x *d\theta + \frac{y}{r} *dr = -dx \end{cases} \longrightarrow \begin{matrix} *dr = r\, d\theta \\ *d\theta = -\frac{dr}{r} \end{matrix}$

MUST DEPEND ONLY ON BOUNDARY POSITIONS

hold then we can defined a conserved product as

$$\langle {}^*F,\ G\rangle = \int_{|u|=r_0} \mathcal{N} * ({}^*F^T g\, dG - d{}^*F^T g\, G) + \boxed{\mathcal{B}_{(+)}(r_0) + \mathcal{B}_{(-)}(-r_0)} = -\langle G,\ {}^*F\rangle \quad (33)$$

whose expression simplifies when proper b.c. are chosen as discussed in the following. Moreover, and more importantly, the previous expression gives a truly radial independent product only when certain conditions (37) are satisfied. These further conditions follows from the discontinuity of the boundary conditions as we now discuss.

Let us now exam under what conditions eq.s (32) hold. Let us start with the case $\boxed{(r_0, r_1) \subset (x_t, x_{t-1}) \in D_{(t)}}$. In this case with the use of eq. (8) we can write

i.e.: all on one brane

$\partial_y F_L(x+i0^+) + \partial_y F_R(x-i0^+) = i\left[\partial_x F_L - \partial_x F_R\right]$

$$\begin{aligned}
&\mathcal{N}\int_{r_0}^{r_1} - ({}^*F^T g\ \partial_y G - \partial_y {}^*F^T g\ G)dx \\
&\quad = -\, i\mathcal{N}\left[-\Delta_{(t)}[{}^*F]^T g\ (\mathbb{I} - U_{(t)})\ G_R(x) + {}^*F_R(x)^T g\ (\mathbb{I} - U_{(t)})\ \Delta_{(t)}[G]\right]\Big|_{r_0}^{r_1} \\
&\quad = -\, 2i\mathcal{N}\left[\boxed{f_{(t)}[G]^{\perp_{(t)}}}^T g\ {}^*F_R(r_0) - \boxed{f_{(t)}[{}^*F]^{\perp_{(t)}}}^T g\ G_R(r_0)\right]\Big|_{r_0}^{r_1}, \qquad (34)
\end{aligned}$$

in principle they can be different but it's not a problem!

where in the last line we used eq. (21) and we have *not* imposed that ${}^*F$ and G have the same boundary conditions, i.e. it may be that $f_{(t)}{}^{\perp_{(t)}}[{}^*F] \neq f_{(t)}{}^{\perp_{(t)}}[G]$. We read therefore the boundary contribution to be

$$\boxed{\mathcal{B}_{(+)}(r_0) = -2i\mathcal{N}\left(f_{(t)}{}^{\perp_{(t)}T}[G]{}^*F_R(r_0) - f_{(t)}{}^{\perp_{(t)}T}[{}^*F]G_R(r_0)\right),} \quad r_0 \in D_{(t)} = (x_t, x_{t-1}), \qquad (35)$$

and

$$\boxed{\mathcal{B}_{(-)}(-r_0) = 2\left(f_{(N+1)}{}^{\perp_{(N+1)}T}[G]{}^*F_R(-r_0) - f_{(N+1)}{}^{\perp_{(N+1)}T}[{}^*F]G_R(-r_0)\right),} \quad -r_0 \in (-\infty, 0) \subset D_{(N+1)}, \qquad (36)$$

where we have not written $f_{(1)}$ but $f_{(N+1)}$ since $u = 0$ may be a source of discontinuity despite the fact that $D_{(N+1)} \equiv D_{(1)}$. This can be understood physically as the fact that $D_{(N+1)}$ and $D_{(1)}$ may be two *different* but parallel branes, eventually separated by a distance.

In the more general case we can split the interval (r_0, r_1) into pieces overlapping different branes. In order to get a contribution which does not depend on the intermediate branes we need for all ts

$$\boxed{\mathcal{B}_{(+)}(x_t{}^+) = \mathcal{B}_{(+)}(x_t{}^-).} \qquad (37)$$

continuity between branes!

This is not an issue for the interval $(-r_1, -r_0)$ since it always lies on $D_{(N+1)} \equiv D_{(1)}$.

Now if both ${}^*F_R$ and G_R are continuous at $x = x_t$ **?? both ${}^*F_R$ and G_R have the same *real* value at $x = x_t$ but differ in the imaginary part therefore ??** the previously stated conditions read

$$(f_{(t-1)}{}^{\perp_{(t-1)}}[G] - f_{(t)}{}^{\perp_{(t)}}[G]){}^*F_R(x_t) = (f_{(t-1)}{}^{\perp_{(t-1)}}[{}^*F] - f_{(t)}{}^{\perp_{(t)}}[{}^*F])G_R(x_t). \qquad (38)$$

We immediately see that if we identify G with X and impose the desired generic boundary conditions $f_{(t)}[G] \neq 0$ which implies $\Delta_{(t)}[G] \neq 0$ we are obliged to require

$$ {}^{*}\!F_R(x_t) = f_{(t)}[{}^{*}\!F] = 0 \qquad \forall t. \tag{39} $$

?? We immediately see that if we impose the quantum boundary conditions $f_{(t)} = 0$ on both G and ${}^{*}\!F$ the previous condition is identically satisfied. It turns however out that they are too restrictive despite the fact that they are necessary for the self-adjointness of the worldsheet Laplacian. ??

When eq.s (37) are satisfied for all ts then we can safely introduce the boundary contributions (35,36). Notice however that they do depend on the brane $D_{(t)}$ on which we perform the computation but *not* on the sequence of branes on which we perform the integration.

2.3.1 Simplifying the product

Let us now exam how we can simplify the "bulk" contribution which can be written as

$$ \begin{aligned} \int_{|u|=r_0} \mathcal{N} * ({}^{*}\!F^T g\, dG - d{}^{*}\!F^T g\, G) &= \mathcal{N} r_0 \int_0^\pi d\theta ({}^{*}\!F^T g\, \partial_r G - \partial_r {}^{*}\!F^T g\, G) \\ &= -2i\mathcal{N} \left[\int_0^\pi d\theta\, {}^{*}\!F_L^T(r_0 e^{i\theta}) g\, \partial_\theta G_L(r_0 e^{i\theta}) + \int_{-\pi}^0 d\theta\, {}^{*}\!F_R^T(r_0 e^{i\theta}) g\, \partial_\theta G_R(r_0 e^{i\theta}) \right] \\ &+ i\mathcal{N} {}^{*}\!\widetilde{F}(r,r)^T g\, G(r,r) \Big|_{r=-r_0}^{r=r_0}, \end{aligned} \tag{40} $$

where we have written $G(u,\bar u) = G_L(u) + G_R(\bar u)$, ${}^{*}\!\widetilde{F}(u,\bar u) = {}^{*}\!F_L(u) - {}^{*}\!F_R(\bar u)$ with $u = re^{i\theta}$ so that $ir\partial_r F(u,\bar u) = \partial_\theta F_L(u) - \partial_\theta F_R(\bar u)$ and then performed an integration by parts on some $\partial_\theta F$ terms.e To simplify further this expression we can use eq. (7) for the boundary contributions

$$ \begin{aligned} {}^{*}\!\widetilde{F}(r_0,r_0)^T g\, G(r_0,r_0) = &- \left((1-U_{(t)}) {}^{*}\!F_R(r_0)\right)^T g\, \Delta_{(t)}[G] + \Delta_{(t)}[{}^{*}\!F]^T g\, (1+U_{(t)}) G_R(r_0) \\ &+ \Delta_{(t)}[{}^{*}\!F]^T g\, \Delta_{(t)}[G], \end{aligned} \tag{41} $$

when $r_0 \in D_{(t)} = (x_t, x_{t-1})$ and the definition of the double field as in eq. (23) to write

$$ \int_0^\pi d\theta\, {}^{*}\!F_L^T(r_0 e^{i\theta}) g\, \partial_\theta G_L(r_0 e^{i\theta}) + \int_{-\pi}^0 d\theta\, {}^{*}\!F_R^T(r_0 e^{i\theta}) g\, \partial_\theta G_R(r_0 e^{i\theta}) = \oint_{|z|=r_0;-\pi}^{\pi} dz\, {}^{*}\!\mathcal{F}_{(\bar t)}{}^T(z) g\, \partial \mathcal{G}_{(\bar t)}(z) \tag{42} $$

where the path in u may go through two cuts.

Putting all together we get the final expression for the product

$$ \begin{aligned} \langle {}^{*}\!F, G \rangle = &- 2i\mathcal{N} \oint_{|z|=r_0;-\pi}^{\pi} dz\, {}^{*}\!\mathcal{F}_{(\bar t)}{}^T(z) g\, \partial \mathcal{G}_{(\bar t)}(z) \\ &+ i\mathcal{N} \left[\Delta_{(t)}[{}^{*}\!F]^T g\, (\Delta_{(t)}[G] + 2U_{(t)} G_R(r_0))\right] \\ &+ i\mathcal{N} \left[\Delta_{(N+1)}[{}^{*}\!F]^T g\, (\Delta_{(N+1)}[G] + 2U_{(N+1)} G_R(-r_0))\right]. \end{aligned} \tag{43} $$

If we consider the previous expression from the Hamiltonian point of view and take G to be the string coordinate X it is clear that the appearance of G_R is problematic. In fact $\bar\partial_{\bar u} X(u,\bar u) = \bar\partial_{\bar u} X_R(\bar u)$ can be expressed using the momentum $\mathcal{P} \sim \partial_r X$ and $\partial_\theta X$ but to obtain an expression for $X_R(\bar u)$ requires an integration. Since we work within the Hamiltonian radial formalism we could write

$$X_R(\bar u) = \int_{r_0}^{r_0 \exp(-i\theta)} d\bar u|_{\bar u = r_0 \exp(-i\theta)} \bar\partial_{\bar u} X(u,\bar u) = \int_0^\theta r_0 d e^{-i\theta} \bar\partial_{\bar u} X(u = r_0 e^{i\theta}, \bar u = r_0 e^{-i\theta}), \tag{44}$$

as “constant time” integration is an integration at constant radial time r_0. However X_R computed at different times r_0 will differ by a constant $-\delta$. We have therefore require that the product is invariant under the shift $G_R \to G_R - \delta$. This requires

$$\Delta_{(t)}({}^*F) = f_{(t)}{}^{\perp(t)}[{}^*F] = 0, \qquad \forall t. \tag{45}$$

Changing the origin of the coordinates it is always possible to set $f_{(t)}{}^{\perp(t)}[{}^*F] = 0$ for a given t and a given ${}^*F$ but we need to satisfy the condition for all t and all ${}^*F$ hence we really have to require the previous conditions. As a matter of facts we require the **?? stronger ??** condition

$${}^*F_R(x_t) = 0, \tag{46}$$

so that also the equations (38) are satisfied for all t and all ${}^*F$.

2.3.2 The final expression for the radially conserved product

Upon the identification $G = X$ we get the product

$$\langle {}^*F, X\rangle = -\,2i\mathcal{N} \oint_{|z|=r_0;-\pi}^{\pi} dz\ {}^*\mathcal{F}_{(\bar t)}{}^T(z) g\ \partial \mathcal{X}_{(\bar t)}(z) \tag{47}$$

whenever the string coordinates satisfy the wanted b.c. and the dual functions satisfy

$${}^*F_R(x_t) = f_{(t)}[{}^*F] = 0 \qquad \forall t. \tag{48}$$

Moreover the previous expression would suggest to choose the normalization $\mathcal{N} = \mathcal{N} = \frac{1}{4\pi}$ but we will actually choose

$$\mathcal{N} = \frac{1}{2\pi} \tag{49}$$

since we are working with space complex coordinates where $g_{z\bar z} = \frac{1}{2}$.

It is worth commenting why we have chosen this precise expression for the product. The reason being that the most general expression for X requires the use of basis elements which are expressed through integrals and therefore we want them to be derived.

2.4 Radial quantization

We start from the Euclidean action written in radial coordinates

$$S_E = T \int_{r_0}^{r_1} dr\ r \int_0^\pi d\theta\ \frac{1}{2} \left[\partial_r X^T g \partial_r X + \frac{1}{r^2} \partial_\theta X^T g \partial_\theta X \right], \tag{50}$$

and interpret r as time. This action is twofold "time" dependent since it depends on time through the boundary conditions and through the explicit time dependence r. Using the usual Hamiltonian approach we get the momentum density at constant time

$$P_I(\sigma) = Tr\ g_{IJ} \partial_r X^J(\sigma), \tag{51}$$

which is "time" dependent because of the explicit r. We can also compute the "time" dependent Hamiltonian

$$H_E = \int_0^\pi d\theta\ \frac{1}{2r_0} \left[\frac{1}{T} P^T g^{-1} P - T \frac{1}{r^2} \partial_\theta X^T g \partial_\theta X \right]. \tag{52}$$

The non vanishing canonical Poisson brackets read

$$\{X^I(\sigma), P_J(\sigma')\} = \delta^I_J\ \delta_{b.c.}(\sigma - \sigma'), \tag{53}$$

where $\delta_{b.c.}$ is the delta function with the proper boundary conditions a fixed "time" r.

These yield the expected eom, in fact from

$$\begin{aligned} \partial_r X^I(\sigma) &= \{X^I(\sigma), H_E\} = \frac{1}{Tr} g^{IJ} P_J(\sigma) \\ \partial_r P_I(\sigma) &= \{P_I(\sigma), H_E\} = -\frac{T}{r} g_{IJ} \partial_\sigma X^J(\sigma), \end{aligned} \tag{54}$$

we get

$$\partial_r (r \partial_r X^I) + \frac{1}{r} \partial_\theta^2 X^I = 0. \tag{55}$$

Using the fact that $\mathcal{B}_{(+)} = \mathcal{B}_{(-)} = 0$ we can write the product as

$$\begin{aligned} \langle {}^*F, X \rangle &= \mathcal{N} \int_0^\pi d\theta\ r\ \left[{}^*F^T g \partial_r X - \partial_r {}^*F^T g X \right] \\ &= \mathcal{N} \int_0^\pi d\theta \left[\frac{1}{T} {}^*F^T P - r \partial_r {}^*F^T g X \right], \end{aligned} \tag{56}$$

from which follows that the Poisson brackets

$$\{\langle {}^*F_1, X \rangle, \langle {}^*F_2, X \rangle\} = \frac{\mathcal{N}}{T} \langle {}^*F_1, {}^*F_2 \rangle, \tag{57}$$

where the last expression follows from the fact that boundary conditions for ${}^*F$ are more restrictive than those for X. The we the get the commutation relation

$$[\langle {}^*F_1, X\rangle, \langle {}^*F_2, X\rangle] = \frac{\mathcal{N}}{T}\langle {}^*F_1, {}^*F_2\rangle. \tag{58}$$

Since we are working in the Euclidean there is no i in the commutation relation as it can also be verified in the very simple example of the Euclidean free particle with $H_{Eu} = \frac{1}{2}p^2$ where the Euclidean time τ evolution of the position operator $x(\tau)$ is given by $x(\tau) = e^{+H_{Eu}\tau} x(0) e^{-H_{Eu}\tau}$ and goes over the proper Minkowskian result when $-it = \tau$.

3 The explicit case of $\mathbb{R}^2$: the setup

In this section we want to make explicit the previous generic construction in the case of $\mathbb{R}^2$ and consider the case for which the in- and out- strings are twisted.

In this case we are able to write explicitly both the string expansion and its dual basis. The main point to build the base is to recognize that only the final expression for X which is given by a sum must satisfy the global boundary conditions and that any summand does not.

Then after defining the in- and out- vacua we can show that the system is conformal and that the vacua are built on the $SL(2,\mathbb{R})$ vacuum using the desired twist fields. Finally we compute the Green function.

3.1 The radially conserved product in $\mathbb{R}^2$

Any solution of the string eom is required to satisfy the b.c.s (1) which in the case of $\mathbb{R}^2$ and in complex coordinates read

$$\begin{aligned} 2\Re\left[e^{-i\pi\alpha_{(t)}}\partial_y X^z(u,\bar u)|_{u=x+i0^+}\right] &= e^{-i\pi\alpha_{(t)}}\partial_y X^z(u,\bar u)|_{u=x+i0^+} + e^{i\pi\alpha_{(t)}}\partial_y X^{\bar z}(u,\bar u)|_{u=x+i0^+} = 0 \\ 2i\Im\left[e^{-i\pi\alpha_{(t)}} X^z(u,\bar u)|_{u=x+i0^+}\right] &= e^{-i\pi\alpha_{(t)}} X^z(u,\bar u)|_{u=x+i0^+} - e^{i\pi\alpha_{(t)}} X^{\bar z}(u,\bar u)|_{u=x+i0^+} = 2ig_{(t)} \qquad x_t < x < x \end{aligned} \tag{59}$$

that is for $x \in (x_t, x_{t-1})$ the boundary is on D_t and $g_{(t)} \in \mathbb{R}$. Moreover we choose $-1 < \alpha_{(t)} < 1$ and $g_{(t)} \in \mathbb{R}^+$, i.e. we take the generic rotation but we interpret $g_{(t)}$ as a distance [5].

In this case the well adapted indexes are $\| = p = 1$ and $\perp = p+1 = q = 2$ so that the matrix $R_{(t)}$ is given by

$$R_{(t)} = \frac{1}{2}\begin{pmatrix} e^{-i\pi\alpha_{(t)}} & e^{+i\pi\alpha_{(t)}} \\ ie^{-i\pi\alpha_{(t)}} & -ie^{+i\pi\alpha_{(t)}} \end{pmatrix}, \tag{60}$$

[5] There is another possible choice $0 \le \alpha_{(t)} < 1$ and $g_{(t)} \in \mathbb{R}$.

and it is an orthogonal matrix which satisfies the relation is ${R_{(t)}}^T R_{(t)} = g$. which follows from $ds^2 = dX^T \mathbb{I}_2 dX = dX^T_{(z)} g\ dX_{(z)} = dz d\bar{z}$ with $g = \frac{1}{2}\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$ and $dX_{(z)} = \begin{pmatrix} dz \\ d\bar{z} \end{pmatrix}$. The matrix $U_{(t)}$ defined in eq. (3) is given by

$$U_{(t)} = \begin{pmatrix} 0 & e^{+i2\pi\alpha_{(t)}} \\ e^{-i2\pi\alpha_{(t)}} & 0 \end{pmatrix}. \tag{61}$$

so that eq.s (2) and (4) become explicitly

$$\begin{aligned} \partial X^z_L(x+i0^+) &= e^{+i2\pi\alpha_{(t)}} \partial X^{\bar{z}}_R(x-i0^+), \qquad x_t < x < x_{t-1} \\ \partial X^{\bar{z}}_L(x+i0^+) &= e^{-i2\pi\alpha_{(t)}} \partial X^z_R(x-i0^+), \qquad x_t < x < x_{t-1}, \end{aligned} \tag{62}$$

a set of local conditions and

$$\begin{aligned} X^z_L(x_t+i0^+) + X^z_R(x_t+i0^-) &= f_{(t)}, \\ X^{\bar{z}}_L(x_t+i0^+) + X^{\bar{z}}_R(x_t+i0^-) &= {f_{(t)}}^*, \end{aligned} \tag{63}$$

a set of global conditions. Eq. (7) which is the integration of the first two previous equations reads

$$\begin{aligned} X^z_L(x+i0^+) &= e^{i2\pi\alpha_{(t)}} X^{\bar{z}}_R(x+i0^-) + {\Delta_{(t)}}^z[X(u,\bar{u})] \\ X^{\bar{z}}_L(x+i0^+) &= e^{-i2\pi\alpha_{(t)}} X^z_R(x+i0^-) + {\Delta_{(t)}}^{\bar{z}}[X(u,\bar{u})], \qquad x_t < x < x_{t-1}, \end{aligned} \tag{64}$$

with ${\Delta_{(t)}}^I[X(u,\bar{u})]$ constant. In the following we do not indicate the functional dependence of ${\Delta_{(t)}}^I$ unless it is required for avoiding confusion. The values of the intersection point between $D_{(t)}$ and $D_{(t+1)}$ is

$$f_{(t)} = 2i \frac{e^{-i\pi\alpha_{(t)}} g_{(t)} - e^{-i\pi\alpha_{(t+1)}} g_{(t+1)}}{e^{-i2\pi\alpha_{(t)}} - e^{-i2\pi\alpha_{(t+1)}}} = \frac{e^{i\pi\alpha_{(t+1)}} g_{(t)} - e^{i\pi\alpha_{(t)}} g_{(t+1)}}{\sin\pi(\alpha_{(t+1)} - \alpha_{(t)})}, \tag{65}$$

which follows from eq. (9) which now is

$$\frac{1}{2i}\begin{pmatrix} e^{-i\pi\alpha_{(t)}} X^z({x_t}^+, \bar{x}_t^+) - e^{i\pi\alpha_{(t)}} X^{\bar{z}}({x_t}^+, \bar{x}_t^+) \\ e^{-i\pi\alpha_{(t+1)}} X^z({x_t}^-, \bar{x}_t^-) - e^{i\pi\alpha_{(t+1)}} X^{\bar{z}}({x_t}^-, \bar{x}_t^-) \end{pmatrix} = \begin{pmatrix} g_{(t)} \\ g_{(t+1)} \end{pmatrix}. \tag{66}$$

In particular this set derives from the continuity of X, i.e. $X^z({x_t}^-, {x_t}^-) = X^z({x_t}^+, {x_t}^+)$ as for eq.s (9).

The projectors (5) act as

$$\begin{aligned} P_{\|_{(t)}} F &= \Re(e^{-i\pi\alpha_{(t)}} F^z) \begin{pmatrix} +e^{+i\pi\alpha_{(t)}} \\ e^{-i\pi\alpha_{(t)}} \end{pmatrix}, \\ P_{\perp_{(t)}} F &= \Im(e^{-i\pi\alpha_{(t)}} F^z) \begin{pmatrix} +ie^{+i\pi\alpha_{(t)}} \\ -ie^{-i\pi\alpha_{(t)}} \end{pmatrix}, \end{aligned} \tag{67}$$

on a generic vector $F = \begin{pmatrix} F^z \\ F^{\bar z} \end{pmatrix}$ so that eq. (10) is equivalent to

$$\Im(e^{-i\pi\alpha_{(t)}} \Delta_{(t)}{}^z) = g_{(t)}, \tag{68}$$

and eq. (10) is nothing else but a way to reinterpret eq. (66) for $x = x_t{}^+$ and $x = x_{t-1}{}^-$

$$g_{(t)} = \Im(e^{-i\pi\alpha_{(t)}} f_{(t)}) = \Im(e^{-i\pi\alpha_{(t)}} f_{(t-1)}). \tag{69}$$

The double fields are then defined as in eq. (23) and satisfy the monodromy conditions (27)

$$\begin{aligned} \partial\mathcal{X}_{(\bar t)}{}^z(x_t + \epsilon e^{i0^+}) &= e^{i2\pi\epsilon_{(t)}} \partial\mathcal{X}_{(\bar t)}{}^z(x_t + \epsilon e^{i2\pi^-}) \\ \partial\mathcal{X}_{(\bar t)}{}^{\bar z}(x_t + \epsilon e^{i0^+}) &= e^{-i2\pi\epsilon_{(t)}} \partial\mathcal{X}_{(\bar t)}{}^{\bar z}(x_t + \epsilon e^{i2\pi^-}), \end{aligned} \tag{70}$$

with $\epsilon_{(t)} = \alpha_{(t+1)} - \alpha_{(t)} + \theta(\alpha_{(t)} - \alpha_{(t+1)})$ and $\bar\epsilon_t = 1 - \epsilon_t$ so that $0 < \epsilon_t, \bar\epsilon_t < 1$.

As discussed in ([]) there are different sectors characterized by

$$M = \sum_{t=1}^N \epsilon_{(t)} \Leftrightarrow \bar M = N - M = \sum_{t=1}^N \bar\epsilon_{(t)}. \tag{71}$$

Finally the radially conserved product reads

$$\langle {}^*F, X\rangle = -\, i\mathcal{N} \oint_{|z|=r_0;-\pi}^{\pi} dz \ \left({}^*\mathcal{F}_{(\bar t)}{}^z(z) \partial\mathcal{X}_{(\bar t)}{}^{\bar z}(z) + {}^*\mathcal{F}_{(\bar t)}{}^{\bar z}(z) \partial\mathcal{X}_{(\bar t)}{}^z(z) \right), \tag{72}$$

when the dual fields satisfy eq.s (48)

$$\begin{aligned} {}^*\mathcal{F}_{(\bar t)}{}^z(x_t) &= {}^*\mathcal{F}_{(\bar t)}{}^{\bar z}(x_t) = 0 \\ \Delta_{(t)}{}^z[{}^*\mathcal{F}_{(\bar t)}] &= \Delta_{(t)}{}^{\bar z}[{}^*\mathcal{F}_{(\bar t)}] = 0, \end{aligned} \tag{73}$$

where the first equation comes about because double fields are expressed using the chiral parts.

4 The explicit case of $\mathbb{R}^2$: the basic $(N, M) = (2, 1)$ twisted string

4.1 The basic twisted string: $(N, M) = (2, 1)$

We consider a string with a twist $\epsilon_{(2)}$ in $x_N = x_2 = 0$ and a twist $\bar\epsilon_{(2)} = 1 - \epsilon_{(2)}$ in $x_1 = \infty$. The brane $D_{(2)}$ is mapped to $\mathbb{R}^+$ and is rotated $\pi\alpha_{(2)}$ with respect to the x^1 axis. We consider the double fields glued along $D_{(2)}$, i.e. we take $\bar t = N = 2$.

In this case there are no issues with the boundary conditions of the duals and in fact they are the same functions used to expand the string fields.

The canonical eingenmodes can be written using the double field formalism as

$$\mathcal{X}_n(z) = \begin{pmatrix} e^{+i\pi\alpha_{(2)}} z^{-n+\epsilon_{(2)}} \\ 0 \end{pmatrix}, \quad \tilde{\mathcal{X}}_n(z) = \begin{pmatrix} 0 \\ e^{-i\pi\alpha_{(2)}} z^{-n+\bar{\epsilon}_{(2)}} \end{pmatrix}, \quad z \in \mathbb{C} - \mathbb{R}^-, \quad n \in \mathbb{Z}, \tag{74}$$

with $\epsilon_{(2)} = \alpha_{(1)} - \alpha_{(2)} + \theta(\alpha_{(2)} - \alpha_{(1)})$. The overall phases, e.g. $e^{+i\pi\alpha_{(2)}}$ for $\mathcal{X}$ are determined by the request that $\mathbb{R}^+$ is mapped onto $D_{(2)}$ and on the way we have chosen the cut. Since we are looking for a basis of solutions which satisfy the local boundary conditions, i.e. the monodromies in eq.s (70) we could also consider the constant solutions $X^z_* = 1$ and $X^{\bar z}_* = 1$. However these solutions cannot be put in a double field form, like $\mathcal{X}_* = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\tilde{\mathcal{X}}_* = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ since these would imply that $\Im(e^{-i\pi\alpha_{(2)}} x_*) = 0$ i.e. that $D_{(2)}$ should go through the origin which is too restrictive.

Expanding the string coordinates on this basis we get

$$\begin{aligned} X^z(u,\bar u) &= \hat{x}_* + e^{i\pi\alpha_{(2)}} \sum_{n\in\mathbb{Z}} \left[\hat{x}_n u^{-n+\epsilon_{(2)}} + \hat{\tilde{x}}_n \bar{u}^{-n+\bar{\epsilon}_{(2)}} \right], \\ X^{\bar z}(u,\bar u) &= \hat{x}^*_* + e^{-i\pi\alpha_{(2)}} \sum_{n\in\mathbb{Z}} \left[\hat{x}_n \bar{u}^{-n+\epsilon_{(2)}} + \hat{\tilde{x}}_n u^{-n+\bar{\epsilon}_{(2)}} \right]. \end{aligned} \tag{75}$$

The previous expansion can be also written in double field formalism as

$$\mathcal{X}^I(z) = \hat{x}^I_* + \sum_{n\in\mathbb{Z}} \left[\hat{x}_n \mathcal{X}^I_n(z) + \hat{\tilde{x}}_n \tilde{\mathcal{X}}^I_n(z) \right] \Rightarrow \begin{cases} \mathcal{X}^z(z) = \hat{x}^z_* + \sum_{n\in\mathbb{Z}} \hat{x}_n \mathcal{X}^z_n(z) \\ \mathcal{X}^z(z) = \hat{x}^{\bar z}_* + \sum_{n\in\mathbb{Z}} \hat{\tilde{x}}_n \mathcal{X}^{\bar z}_n(z) \end{cases},$$

where the right expression follows from the fact that $\mathcal{X}^I_n(z)$ ($\tilde{\mathcal{X}}^I_n(z)$) are only non vanishing when $I = z$ ($I = \bar z$) and it is right despite its strange appearance because $\mathcal{X}^I$ has both a left and right moving part as follows from the definition (23).

Using the radial canonical formalism and the products

$$\langle \tilde{X}_n, X_m \rangle = 2\pi\mathcal{N}(n - \bar{\epsilon}_{(2)}) \delta_{n+m,1}, \tag{76}$$

and

$$\langle \tilde{X}_n, X \rangle = 2\pi\mathcal{N}(n - \bar{\epsilon}_{(2)}) \hat{x}_{1-n}, \qquad \langle X_n, X \rangle = 2\pi\mathcal{N}(n - \epsilon_{(2)}) \hat{\tilde{x}}_{1-n}, \tag{77}$$

we find the non vanishing commutation relations

$$[\hat{x}_n, \hat{\tilde{x}}_m] = \frac{1}{2\pi T} \frac{1}{n - \epsilon_{(2)}} \delta_{n+m,1}. \tag{78}$$

4.1.1 Classical solution

We want to impose the conditions on the intersecting points eq. (4). Since $X(u,\bar u)$ may be divergent both at $x_2 = 0$ and $x_1 = \infty$ we need to set all $\hat{x}$ and $\hat{\tilde{x}}$ to zero but $\hat{x}_*$. Then these conditions imply $\hat{x}_* = f_{(2)}$.

Another way of getting the same result is the following. We impose the original boundary conditions (1) which in $D=2$ become eq.s (59). Since $\hat{x}_n$ and $\hat{\bar{x}}_n$ are real and $\Delta_{(t)}[X_n(u,\bar{u})] = \Delta_{(t)}[\tilde{X}_n(u,\bar{u})] = 0$ the original conditions (59) become

$$\Im(e^{-i\pi\alpha_{(t)}} X^z(x,\bar{x})) = \Im(e^{-i\pi\alpha_{(t)}} \hat{x}_*) = g_{(t)}, \tag{79}$$

when x is in the proper interval. Explicitly these conditions become $\Im(e^{-i\pi\alpha_{(2)}}\hat{x}_*) = g_{(2)}$ and $\Im(e^{-i\pi\alpha_{(1)}}\hat{x}_*) = g_{(1)}$ which imply $\hat{x}_* = f_{(2)}$. The $\hat{x}$ and $\hat{\bar{x}}$ are then fixed to zero by the request of a finite action.

Another point to discuss is the Euclidean reality condition $(X^z(u,\bar{u}))^* = X^{\bar{z}}(u,\bar{u})$ which implies that $\hat{x}_n^* = \hat{x}_n$ and $\hat{\bar{x}}_n^* = \hat{\bar{x}}_n$.

4.1.2 Quantum theory

Because of the divergences of the modes it is not possible to impose the boundary conditions in the form of intersection points as strong constraints.

We need to impose them in a weak sense. Therefore the constraints can be imposed only when we have defined the Fock space where we represent the commutation relations despite the fact that the operatorial algebra is conceptually completely distinguished from its representation.

We turn now to the discussion of the Fock space.

The use the asymptotics of the modes to define the $SL(2,\mathbb{R})$ in-vacuum as

$$\hat{x}_n|T_{in}\rangle = \hat{\bar{x}}_n|T_{in}\rangle = 0, \qquad n \geq 1, \tag{80}$$

since X_n and $\bar{X}_n$ diverge for $n \geq 1$ and $u \to 0$. Similarly the asymptotics of the modes for the out-vacuum gives

$$\langle\langle T_{out}|\hat{x}_n = \langle\langle T_{out}|\hat{\bar{x}}_n = 0, \qquad n \leq 0. \tag{81}$$

Notice that we write $\langle\langle T_{out}|$ and not $\langle T_{out}|$ since this state is defined using the asymptotics of the modes and not by taking the hermitian conjugate of $|T_{in}\rangle$. In this case, however, it turns out that the in- and out- vacua are actually hermitian conjugate and equal to the vacuum $|T\rangle = |T_{in}\rangle = \langle\langle T_{out}|^\dagger$. One naive mode to reach the conclusion is to notice that the operators $\hat{x}_n$ and $\hat{\bar{x}}_n$ are either in-creator and out-annihiliator or in-annihilator and out-creator, so looking to the indeces we can guess that $\hat{x}_n^\dagger \sim \hat{\bar{x}}_{1-n}$.

To discuss the previous statement we have to distinguish between two concepts. The Euclidean Hermiticity and the Minkowskian Hermiticity in Euclidean space. The Euclidean hermiticity is simply defined as $[O(u,\bar{u})]_E^\dagger = O(u,\bar{u})$. On the other side the Minkowskian Hermiticity in Euclidean space is defined as $\left[O(\frac{1}{\bar{u}},\frac{1}{u})\right]_M^\dagger = \left(\frac{1}{u^2}\right)^h \left(\frac{1}{\bar{u}^2}\right)^{\bar{h}} O(u,\bar{u})$ for a conformal operator of weight $(h,\bar{h})$. The reason is that

We can now build the Fock space as

$$|\{N_n,\tilde N_n\}_{n\le 0}\rangle = \prod_{n\le 0}\left[\hat x_n^{N_n}\hat{\tilde x}_n^{\tilde N_n}\right]|T\rangle,$$
$$\langle\{N_n,\tilde N_n\}_{n\le 0}| = \langle T|\prod_{n\le 0}\left[\hat x_{1-n}^{N_n}\hat{\tilde x}_{1-n}^{\tilde N_n}\right], \tag{82}$$

so that the we have the products

$$\langle\{M_n,\tilde M_n\}_{n\le 0}|\{N_n,\tilde N_n\}_{n\le 0}\rangle \propto \prod_{n\le 0}\left[\delta_{N_n,M_n}\delta_{\tilde N_n,\tilde M_n}\right]. \tag{83}$$

?? How does hermitian conjugation go well with reality of $\hat x$??

The question is then whether we should impose the constraints on the vacuum only as

$$\langle T|X(x_N=0,\bar x_N=0)|T\rangle = \langle T|X(x_1=0,\bar x_1=0)|T\rangle = f_N\langle T|T\rangle, \tag{84}$$

or for a broader set of states as

$$\langle phys|X(x_N=0,\bar x_N=0)|phys'\rangle = \langle phys|X(x_1=0,\bar x_1=0)|phys'\rangle = f_N\langle phys|phys'\rangle, \tag{85}$$

for any physical states $|phys\rangle$, $|phys'\rangle$.

The question and the answer are important since we can suppose the same answer can be utilizied for more general cases.

It is then clear that only the strict case works since for example

$$\lim_{u\to 0}\langle\{N_n+\delta_{n,k},\tilde N_n\}_{n\le 0}|X^z(u,\bar u)|\{N_n,\tilde N_n\}_{n\le 0}\rangle \propto \lim_{u\to 0}\mathcal{X}^z_k(u)=\infty,\ k\le 0. \tag{86}$$

An intuitive reason of why it is so is that only the vacuum corresponds to the classical finite Euclidean action.

5 The explicit case of $\mathbb{R}^2$: the general (N,M) case

5.1 The dual basis

Given the monodromies in eq. (70) we can try to set up a basis. As before we consider the case where there are discontinuities at $x_N=0$ and $x_1=\infty$, i.e. in- and out- strings are twisted strings. Moreover we write the double fields with $\bar t=N$, i.e. the upper and lower half planes are joined along the segment $(x_N=0,x_{N-1})$. We will not write this $(\bar t)$ explicitly as done before in order to make the notation lighter.

It turns out that the most naive ansatz corresponds to a dual basis. Using the double field formalism we can write

$$\mathcal{Y}_n(z) = \begin{pmatrix} e^{i\pi\alpha_{(N)}} z^{-n+\epsilon_{(N)}} \prod_{t=2}^{N-1} \left(1 - \frac{z}{x_t}\right)^{\epsilon_{(t)}} \\ 0 \end{pmatrix},$$
$$\tilde{\mathcal{Y}}_n(z) = \begin{pmatrix} 0 \\ e^{-i\pi\alpha_{(N)}} z^{-n+\bar{\epsilon}_{(N)}} \prod_{t=2}^{N-1} \left(1 - \frac{z}{x_t}\right)^{\bar{\epsilon}_{(t)}} \end{pmatrix}, \tag{87}$$

as the $N = 2$ case the overall phase is dictated by the cuts and the mapping of the real line on the branes.

Let us see the properties this basis has

1. There are the proper number of in-modes and they have the proper $z \to 0$ asymptotic, i.e. $\mathcal{Y}^z_n \sim_{z\to 0} z^{-n+\epsilon_{(N)}}$ and $\tilde{\mathcal{Y}}^z_n \sim_{z\to 0} z^{-n+\bar{\epsilon}_{(N)}}$.

2. There are the proper number of out-modes and they have the proper $z \to \infty$ asymptotic, i.e. $\mathcal{Y}^z_n \sim_{z\to\infty} z^{-n+M-\epsilon_{(1)}}$.

3. They are linearly independent.

4. $\mathcal{Y}_n(x_t) = \tilde{\mathcal{Y}}_n(x_t) = 0$ for $2 \le t \le N - 1$. We do not consider the $t = 1, N$ cases because in the configuration we consider they do not give contribution to $\mathcal{B}$s.

5. $\Delta_{(t)}[Y_n(u, \bar{u})] = \Delta_{(t)}[\bar{Y}_n(u, \bar{u})] = 0$ for $1 \le t \le N + 1$ where $t = N + 1$ refers to the the negative axis where $D_{(N+1)} \equiv D_{(1)}$ is mapped.

   Therefore we could be tempted to expand $X(u, \bar{u})$ on this basis as

$$X^z(u, \bar{u}) = \sum_{n\in\mathbb{Z}} \left[\hat{x}_n \mathcal{Y}^z_n(u) + \hat{\tilde{x}}_n U_{(\bar{t})}{}^z_{\bar{z}} \tilde{\mathcal{Y}}^{\bar{z}}_n(\bar{u})\right],$$
$$X^{\bar{z}}(u, \bar{u}) = \sum_{n\in\mathbb{Z}} \left[\hat{x}_n U_{(\bar{t})}{}^{\bar{z}}_z \mathcal{Y}^z_n(\bar{u}) + \hat{\tilde{x}}_n \tilde{\mathcal{Y}}^{\bar{z}}_n(u)\right]. \tag{88}$$

   There is however an immediate issue since we cannot get the string fields with $f_{(t)} \neq 0$. But even if we could solve this point and add to the previous solution the classical solution then there is another insoluble issue. The boundary conditions for $f_{(t)} = 0$ requires $X(x_t, \bar{x}_t) = 0$ *only* but the previous expansion has $X_L(x_t) = X_R(\bar{x}_t) = 0$ which is too restrictive but these boundary conditions are well suited for the dual basis.

5.2 A natural basis for a bigger space.

As in the previous section we can use the monodromies in eq. (70) to set up a basis, this means that we will take care of the non derivative boundary conditions later and hence that the proposed basis is for bigger space. This time we take as suggestion the fact that the expression for the classical solution is given by a sum of integrals and that only the sum satisfies the global boundary conditions at the same time any summand does not. As before we consider the case where there are discontinuities at $x_N = 0$ and $x_1 = \infty$, i.e. in- and out- strings are twisted strings. Moreover we write the double fields with $\bar{t} = N$, i.e. the upper and lower half planes are joined along the segment $(x_N = 0, x_{N-1})$. We will not write this $(\bar{t})$ explicitly as done before in order to make the notation lighter.

It turns out that this ansatz is a basis for a bigger space since it can accommodate solutions with boundary conditions other than the desired ones. Using the double field formalism we can write

$$
\begin{aligned}
\mathcal{I}_n(z) &= \begin{pmatrix} e^{i\pi\alpha_{(N)}} \int_{x_{N-1}+i0^+}^{z} dw\ w^{-n+\epsilon_{(N)}-1} \prod_{t=2}^{N-1} \left(1 - \frac{w}{x_t}\right)^{\epsilon_{(t)}-1} \\ 0 \end{pmatrix}, \\
\tilde{\mathcal{I}}_n(z) &= \begin{pmatrix} 0 \\ e^{-i\pi\alpha_{(N)}} \int_{x_{N-1}+i0^+}^{z} dw\ w^{-n+\bar{\epsilon}_{(N)}} \prod_{t=2}^{N-1} \left(1 - \frac{w}{x_t}\right)^{\bar{\epsilon}_{(t)}} \end{pmatrix},
\end{aligned}
\tag{89}
$$

where the integration is performed on the cut complex plane, i.e. $(\mathbb{C} \setminus \mathbb{R}) \cup (0, x_{N-1})$ and therefore the integrals have not monodromies nevertheless $\mathcal{I}(x_t + i0^+) \neq \mathcal{I}(x_t - i0^+)$ since these points lie above and under the cuts.

Before discussing the properties of this basis we introduce the constants

$$
\begin{aligned}
I_{(t)n} &= \int_{x_t}^{x_{t-1}} dx\ x^{-n+\epsilon_{(N)}-1} \prod_{u=2}^{N-1} \left|1 - \frac{x}{x_u}\right|^{\epsilon_{(u)}-1}, \\
\tilde{I}_{(t)n} &= \int_{x_t}^{x_{t-1}} dx\ x^{-n+\bar{\epsilon}_{(N)}-1} \prod_{u=2}^{N-1} \left|1 - \frac{x}{x_u}\right|^{\bar{\epsilon}_{(u)}-1} \qquad , 3 \leq t \leq N-1.
\end{aligned}
\tag{90}
$$

It is also possible to introduce the following other sets of finite constants

$$
\begin{aligned}
I_{(N)n} &= \int_{x_N=0}^{x_{N-1}} dx\ x^{-n+\epsilon_{(N)}-1} \prod_{u=2}^{N-1} \left|1 - \frac{x}{x_u}\right|^{\epsilon_{(u)}-1}, \qquad n \leq 0 \\
I_{(2)n} &= \int_{x_2}^{x_1=\infty} dx\ x^{-n+\epsilon_{(N)}-1} \prod_{u=2}^{N-1} \left|1 - \frac{x}{x_u}\right|^{\epsilon_{(u)}-1}, \qquad n \geq 2 - \bar{M},
\end{aligned}
\tag{91}
$$

where the ranges over n are fixed because we want a finite result.

Let us enumerate the properties this basis has and then discuss some of them in more details.

1. Its linear span is bigger than that of $\mathcal{Y}, \tilde{\mathcal{Y}}$, i.e. $span\{\mathcal{Y}, \tilde{\mathcal{Y}}\} \subset span\{\mathcal{I}, \tilde{\mathcal{I}}\}$.

2. We can get all in-modes up to a constant and they have the proper $z \to 0$ asymptotic, i.e. they can diverge

$$
\mathcal{I}_n^z =_{z\to 0} e^{+i\pi\alpha_{(N)}} \frac{z^{-n+\epsilon_{(N)}}}{-n+\epsilon_{(N)}} \left(1 + O(z)\right), \qquad n \geq 1,
\tag{92}
$$

or be finite

$$\mathcal{I}^z_n =_{z\to 0} e^{+i\pi\alpha_{(N)}} \left[-I_{(N)n} + \frac{z^{-n+\epsilon_{(N)}}}{-n+\epsilon_{(N)}} (1+O(z)) \right], \quad n \le 0 \tag{93}$$

where $I_{(N)}$ is defined in eq. (91) and similarly for $\tilde{\mathcal{I}}^{\bar z}_n$ with $I \to \bar I$, $\alpha \to -\alpha$ and $\epsilon \to \bar\epsilon$.

3. We can get all out-modes up to a constant and they have the proper $z \to \infty$ asymptotic, i.e.

$$\mathcal{I}^z_n =_{z\to\infty} e^{+i\pi\alpha_{(N)}} \frac{z^{-n+2+\bar\epsilon_{(1)}-\bar M}}{-n+2+\bar\epsilon_{(1)}-\bar M} (1+O(z^{-1})), \quad n \le 1-\bar M, \tag{94}$$

and

$$\mathcal{I}^z_n =_{z\to\infty} J_{(2)n} - e^{+i\pi\alpha_{(N)}} \frac{z^{-n+2+\bar\epsilon_{(1)}-\bar M}}{-n+2+\bar\epsilon_{(1)}-\bar M} (1+O(z^{-1})) \\ , \quad n \ge 2-\bar M \tag{95}$$

where $J_{(2)n} = \mathcal{I}^z_n(x_1+i0^+) = \sum_{t=N-1}^{2} e^{i\pi\alpha_{(t)}} (-)^{N-t+\sum_{u=2}^{N-1}\theta(\alpha_{(u)}-\alpha_{(u+1)})} I_{(t)n}$ as given in eq. (96) and similarly for $\tilde{\mathcal{I}}^{\bar z}_n$ with $J \to \bar J$, $\alpha \to -\alpha$ and $\epsilon \to \bar\epsilon$.

4. They are linearly independent since their derivatives are.

5. They assume the following non vanishing values at the interaction points

$$\mathcal{I}^z{}_n(x_{N-1}+i0^+) = 0, \\ \mathcal{I}^z{}_n(x_{t-1}+i0^+) = \mathcal{I}^z{}_n(x_t+i0^+) + e^{i\pi\alpha_{(t)}} \cdot e^{i\pi\left[N-t+\sum_{u=t}^{N-1}\theta(\alpha_{(u)}-\alpha_{(u+1)})\right]} I_{(t)n}, \quad t \le N-1 \tag{96}$$

where $e^{i\pi\left[N-1-t+\sum_{u=t-1}^{N-1}\theta(\alpha_{(u)}-\alpha_{(u+1)})\right]} I_{(t)n}$ is real but without a definite sign and

$$\mathcal{I}^z{}_n(x_t-i0^+) = e^{i2\pi\alpha_{(N)}} [\mathcal{I}^z{}_n(x_t+i0^+)]^* \\ = \mathcal{I}^z{}_n(x_{t+1}-i0^+) + e^{-i\pi\alpha_{(t)}} \cdot e^{i\pi\left[N-t+\sum_{u=t}^{N-1}\theta(\alpha_{(u)}-\alpha_{(u+1)})\right]} I_{(t)n}. \tag{97}$$

Similarly we get

$$\tilde{\mathcal{I}}^z{}_n(x_{N-1}+i0^+) = 0, \\ \tilde{\mathcal{I}}^z{}_n(x_{t-1}+i0^+) = \tilde{\mathcal{I}}^z{}_n(x_t+i0^+) + e^{-i\pi\alpha_{(t)}} \cdot e^{i\pi\left[\sum_{u=t}^{N-1}\theta(\alpha_{(u)}-\alpha_{(u+1)})\right]} \tilde I_{(t)n}, \quad t \le N-1. \tag{98}$$

6. $\Delta_{(t)}[I_n(u,\bar{u})] = \Delta_{(t)}[\bar{I}_n(u,\bar{u})] = 0$ for $1 \le t \le N+1$ where $t = N+1$ refers to the the negative axis where $D_{(N+1)} \equiv D_{(1)}$ is mapped. The previous relations are true because for any path in the upper half plane from the starting point x_{N-1} to any final point $x + i0^+$ we can consider the complex conjugate path in the lower half plane to $x - i0^-$ so that the integrand along the two paths are complex conjugate because the the cuts are chosen.

   This would seem at variance with eq. (10) but it is not since $\mathcal{I}(x+i0^+)$ moves exactly on the brane to which x is mapped and does not have a transverse part as eq. (97) shows.

We can now proof the first property, i.e. we want to show that

$$\mathcal{Y}_n(z) = \sum_{m=n-(N-2)}^{n} c_{nm}\mathcal{I}_m(z) + c_n,$$
$$\tilde{\mathcal{Y}}_n(z) = \sum_{m=n-(N-2)}^{n} \bar{c}_{nm}\tilde{\mathcal{I}}_m(z) + \bar{c}_n, \qquad z \in (\mathbb{C}\setminus\mathbb{R}) \cup (0, x_{N-1}), \tag{99}$$

where the constants c_n and $\bar{c}_n$ are zero because both $\mathcal{Y}$ and $\mathcal{I}$ vanish at $z = x_{N-1}$. As shown later the coefficients c and $\bar{c}$ are related as

$$c_{n,m} = -\bar{c}_{1-m,1-n}. \tag{100}$$

To show the previous relations (99) we take the derivative of both sides w.r.t. z and simplify the result to get for example

$$z\prod_{t=2}^{N-1}\left(1-\frac{z}{x_t}\right)\left[\frac{-n+\epsilon_{(N)}}{z} + \sum_{u=2}^{N-1}\frac{\epsilon_{(u)}}{z-x_u}\right] = \sum_m c_{n,m} z^{n-m}, \tag{101}$$

then the range of m follows from the fact the the lhs is a polynomial of order $N-2$ hence $0 \le n-m \le N-2$. Comparing explicitly the two sides we get

$$\begin{aligned}
c_{n,n} &= -n + \epsilon_{(N)} \\
c_{n,n-1} &= \sum_{u=2}^{N-1}\frac{n-\epsilon_{(u)}-\epsilon_{(N)}}{x_u} \\
&\dots \\
c_{n,n-(N-2)} &= -\sum_{t=2}^{N-1}\frac{1}{x_u}\left[\sum_{u=2}^{N-1}\epsilon_{(u)} + \epsilon_{(N)} - n\right].
\end{aligned} \tag{102}$$

The coefficients $\bar{c}$ are obtained with the usual substitution $\epsilon \to \bar{\epsilon}$. It is then easy to verify the relation (100) on the previous expressions.

Notice then that the matrices $c = \parallel c_{n,m} \parallel$ and $\bar{c} = \parallel \bar{c}_{n,m} \parallel$ have zero eigenvectors $I_{(t)} = \parallel I_{(t)m} \parallel$ and $\tilde{I}_{(t)} = \parallel \tilde{I}_{(t)m} \parallel$ respectively

$$
\begin{aligned}
& c I_{(t)} = 0, \\
& \bar{c} \tilde{I}_{(t)} = 0, \qquad 3 \le t \le N-2,
\end{aligned} \tag{103}
$$

as follows from the fact that $\mathcal{Y}(x_t) = \tilde{\mathcal{Y}}(x_t) = 0$ $(t = 2, \dots N-1)$ which imply $\mathcal{Y}(x_t) - \mathcal{Y}(x_{t-1}) = \tilde{\mathcal{Y}}(x_t) - \tilde{\mathcal{Y}}(x_{t-1}) = 0$ $(t = 3, \dots N-1)$ since $\mathcal{Y}(x_t) = \tilde{\mathcal{Y}}(x_t) = 0$ for $t = 2$ has trivial information. Actually there exist other zero eigenvectors, one for each of the two matrices. We call them $I_{(N2)}$ and $\tilde{I}_{(N2)}$. They arise because $\mathcal{Y}_n(0) = 0$ for $n \le 0$ and $\mathcal{Y}_n(\infty) = 0$ for $n \ge M$, and similarly $\tilde{\mathcal{Y}}_n(0) = 0$ for $n \le 0$ and $\tilde{\mathcal{Y}}_n(\infty) = 0$ for $n \ge \bar{M} = N - M$. The idea is to extend the I_m from the range implied from the previous relations to the whole $\mathbb{Z}$. Let us discuss the case of $\mathcal{Y}$ and then quote the result for $\tilde{\mathcal{Y}}$. The fact that $\mathcal{Y}_n(x_N) = 0$ for $n \le 0$ implies that we know the component $I_{(N2)m} = I_{(N)m}$ for $m \ge 2 - N$. Then using the fact that $c_{n,m} \ne 0$ for $n + 2 - N \le m \le m$ we can impose $0 = (c I_{(N2)})_1 = c_{1,1} I_{(N2)1} + \dots c_{1,3-N} I_{(N2)3-N}$ and compute $I_{(N2)3-N}$. Then imposing all the remaining $0 = (c I_{(N2)})_{n \ge 2} = 0$ we can compute all the components of $I_{(N2)}$.

In a similar way we could compute an apparently different zero eigenvector starting from $\mathcal{Y}_n(\infty) = 0$ for $n \ge M$. As discussed in appendix (**??**) we can show that for $M = 1$ the two constructed zero eigenvectors are actually the same. For the other cases we assume to be so.

The same procedure applies for $\tilde{\mathcal{Y}}_n(0) = 0$ for $n \le 0$ and $\tilde{\mathcal{Y}}_n(\infty) = 0$ for $n \ge \bar{M} = N - M$. and gives rise to the zero eigenvector $\tilde{I}_{(N2)}$.

5.3 Products

It is now an easy matter to compute the following products

$$
\begin{aligned}
& \langle Y_m, I_n \rangle = 0, \\
& \langle \tilde{Y}_m, I_n \rangle = \pi \mathcal{N} \delta_{n+m,1}, \\
& \langle Y_m, \tilde{I}_n \rangle = \pi \mathcal{N} \delta_{n+m,1}, \\
& \langle Y_m, I_n \rangle = 0.
\end{aligned} \tag{104}
$$

Using these results and the expansions in eq.s (99) we get the results

$$
\begin{aligned}
& \langle Y_m, Y_n \rangle = 0, \\
& \langle \tilde{Y}_m, Y_n \rangle = \pi \mathcal{N} c_{n,1-m}, \\
& \langle Y_m, \tilde{Y}_n \rangle = \pi \mathcal{N} \bar{c}_{n,1-m}, \\
& \langle Y_m, Y_n \rangle = 0.
\end{aligned} \tag{105}
$$

Since Ys and $\tilde{Y}$s satisfy the more restricted boundary conditions we have $\langle \tilde{Y}_m, Y_n \rangle = -\langle Y_n, \tilde{Y}_m \rangle$ and therefore eq. (100) follows.

5.4 String mode expansion and creation and annihilation operators

We can now expand $X(u, \bar{u})$ on the basis given by I and $\tilde{I}$ as

$$\begin{aligned} X^z(u,\bar{u}) =& \hat{x}_* + \sum_{n\in\mathbb{Z}} \left[\hat{x}_n \mathcal{I}^z_n(u) + \hat{\bar{x}}_n U_{(\bar{t})}{}^z{}_{\bar{z}} \tilde{\mathcal{I}}^{\bar{z}}_n(\bar{u}) \right], \\ X^{\bar{z}}(u,\bar{u}) =& \hat{x}^*_* + \sum_{n\in\mathbb{Z}} \left[\hat{x}_n U_{(\bar{t})}{}^{\bar{z}}{}_z \mathcal{I}^z_n(\bar{u}) + \hat{\bar{x}}_n \tilde{\mathcal{I}}^{\bar{z}}_n(u) \right]. \end{aligned} \tag{106}$$

The coefficients $\hat{x}_*$ and $\hat{\bar{x}}_*$ are fixed later to be $f_{(N-1)}$ and $\bar{f}_{(N-1)}$ respectively using the boundary conditions. The coefficients $\hat{x}_n$ and $\hat{\bar{x}}_n$ can instead be extracted using the product as

$$\begin{aligned} \langle \tilde{Y}_{1-n}, X \rangle &= \pi \mathcal{N} \hat{x}_n, \\ \langle Y_{1-n}, X \rangle &= \pi \mathcal{N} \hat{\bar{x}}_n, \end{aligned} \tag{107}$$

and using eq. (58) they satisfy the non trivial commutation relations

$$[\hat{x}_n, \hat{\bar{x}}_m] = \frac{1}{\pi T} c_{1-m,n} = -\frac{1}{\pi T} \bar{c}_{1-n,m}. \tag{108}$$

Because the matrices c and $\bar{c}$ have zero eigenvectors the previous commutation relations are constrained as

$$[I_{(t)n} \hat{x}_n, \hat{\bar{x}}_m] = [\hat{x}_n, \hat{\bar{x}}_m \tilde{I}_{(t)m}] = 0, \qquad t \in \{N2, 3, \dots N-1\}. \tag{109}$$

This means that the quantities $Q_{(t)} = I_{(t)n}\hat{x}_n$ and $\tilde{Q}_{(t)} = \tilde{I}_{(t)m}\hat{\bar{x}}_m$ commute with everything. One could be tempted to identify them with c-numbers but in doing so we would "lose" some modes. We will therefore impose their conservation in a weak sense. When discussing the in- and out- vacuum we will see that imposing them in a strong sense is incompatible with the naive expectation on the way we can divide $\hat{x}$ and $\hat{\bar{x}}$ as creator or annihilator according to the behavior of the corresponding mode.

5.5 Classical solution

To better understand the meaning of these conserved quantities we now turn our attention to the classical solution. It turns out that these quantities are constrained by the boundary conditions.

Requiring at the classical level that $X(x_N, \bar{x}_N) = f_N$ and that $X(x_1, \bar{x}_1) = f_1$ sets to zero almost all $\hat{x}$ and $\hat{\bar{x}}$. X and $\bar{X}$ be finite at $x_N = 0$ and $x_1 = \infty$ explicitly implies

$$\begin{aligned} &\hat{x}_{aa} = 0, \quad aa \geq 1; \quad \hat{x}_{ca} = 0, \quad ca \leq 1 - \bar{M} \\ &\hat{\bar{x}}_{\bar{a}c} = 0, \quad \bar{a}c \geq 1; \quad \hat{\bar{x}}_{\bar{c}a} = 0, \quad \bar{c}a \leq 1 - M, \end{aligned} \tag{110}$$

therefore the possibly non vanishing coefficients are

$$\hat{x}_{cc} \lesseqgtr 0, \quad 2-\bar{M} \le cc \le 0,$$
$$\hat{\bar{x}}_{\bar{c}\bar{c}} \lesseqgtr 0, \quad 2-M \le \bar{c}\bar{c} \le 0. \tag{111}$$

In the previous formula we have introduced a notation for the indexes which is rooted in the quantum formulation and, in particular, to the fact that $\hat{x}$ or $\hat{\bar{x}}$ behave as creator or annihilator. Explicitly cc is an index associated with $\hat{x}$ which is both in- and out- creator, ca is an index associated with $\hat{x}$ which is in- creator and out- annihilator and finally aa is an index associated with $\hat{x}$ which is in- annihilator and out- creator. Similarly for $\hat{\bar{x}}$. We have distinguished between cc and $\bar{c}\bar{c}$ and the other sets of indexes since they may have different ranges.

Let us now spell out the constraints on the coefficients $\hat{x}_{cc}$ and $\hat{\bar{x}}_{\bar{c}\bar{c}}$ and see how to determine these $N-2$ unknowns.

There are obviously the two previously used conditions $X(x_N, \bar{x}_N) = f_N$ and $X(x_1, \bar{x}_1) = f_1$. To these for $t = 2, \dots, N-1$ we can use the conditions on the other intersecting points eq. (4) which can be written as

$$\begin{aligned} f_{(N-1)} &= \hat{x}_* \\ f_{(t-1)} - f_{(t)} &= e^{i\pi\alpha_{(t)}} (-1)^{\sum_{u=t}^{N-1} \theta(\alpha_{(u)}-\alpha_{(u+1)})} \sum_n \left[(-)^{N-t} I_{(t)n} \hat{x}_n + \tilde{I}_{(t)n} \hat{\bar{x}}_n \right] \\ &= e^{i\pi\alpha_{(t)}} (-1)^{\sum_{u=t}^{N-1} \theta(\alpha_{(u)}-\alpha_{(u+1)})} \left[(-)^{N-t} Q_{(t)} + \tilde{Q}_{(t)} \right], \quad 3 \le t \le N-1. \end{aligned} \tag{112}$$

These are further $N-3$ equations on $\hat{x}_{cc}$ and $\hat{\bar{x}}_{\bar{c}\bar{c}}$ for a total of $N-1$ equations which form an apparently sovra determined set. That it is not so can be understood geometrically. Given the N angles $\alpha_{(t)}$ and the $N-1$ lengths of the sides of a N-agon then the length of remaning side is completely determined.

Having determined the $N-2$ coefficients $\hat{x}_{cc}$ and $\hat{\bar{x}}_{\bar{c}\bar{c}}$ we can then fix the $2(N-2)$ charges.

?? These are fully fixed when we ask for a finite Euclidean action, i.e. when we fix the classical solution. On the other side for the points $t = 1, N$ we cannot because $X(u, \bar{u})$ may diverge as it happened for $(N, M) = (2, 1)$. Exactly as in that case we impose the original boundary conditions (59). Since $\hat{x}_n$ and $\hat{\bar{x}}_n$ are real and $\Delta_{(t)}[I_n(u, \bar{u})] = \Delta_{(t)}[\bar{I}_n(u, \bar{u})] = 0$ as discussed above the original conditions (59) become

$$\Im(e^{-i\pi\alpha_{(t)}} X^z(x, \bar{x})) = \Im(e^{-i\pi\alpha_{(t)}} \hat{x}_*) = g_{(t)}, x \in (x_t, x_{t+1}). \tag{113}$$

For $t = N, N{+}1 \equiv 1$ these conditions are explicitly $\Im(e^{-i\pi\alpha_{(N)}} \hat{x}_*) = g_{(N)}$ and $\Im(e^{-i\pi\alpha_{(1)}} \hat{x}_*) = g_{(1)}$ which imply again $\hat{x}_* = f_{(N-1)}$. ??

5.6 Quantum field and in- and out- vacua

We want now to determine the in- and out- vacua which corresponds to the $SL(2, \mathbb{R})$ vacuum in absence of twists and the way we have to implement the constraints form the conserved charges.

The first observation is that somewhat like the classical case the $x_N = 0$ and $x_1 = \infty$ teach something. In particular they show that it is not possible to impose $X(x_N, \bar{x}_N) = f_N$ and $X(x_1, \bar{x}_1) = f_1$ as strong constraints since $X(u, \bar{u})$ diverges at that points. We can implement them as weak constraints as

$$\begin{aligned}
\langle phys|X(x_N = 0, \bar{x}_N = 0)|phys'\rangle &= f_N \langle phys|phys'\rangle \\
\langle phys|X(x_1 = \infty, \bar{x}_1 = \infty)|phys'\rangle &= f_1 \langle phys|phys'\rangle.
\end{aligned} \tag{114}$$

If we consider the first constraint in the case of the in- vacuum $|phys'\rangle = |T_{in}\rangle$ and use the asymptotic behavior for $u \to 0$ as given by eq. (92) it is natural to demand that all operator whose modes are diverging are annihilation operators of the in-vacuum we find

$$\hat{x}_{aa}|T_{in}\rangle = \hat{\tilde{x}}_{\bar{a}c}|T_{in}\rangle = 0, \qquad aa, \bar{a}c \geq 1. \tag{115}$$

Similarly from (94) for the out-vacuum we get

$$\begin{aligned}
\langle T_{out}|\hat{x}_n &= 0, \qquad n \leq 1 - \bar{M} \\
\langle T_{out}|\hat{\tilde{x}}_n &= 0, \qquad n \leq 1 - M.
\end{aligned} \tag{116}$$

Differently from what it is usual, i.e. that an operator is either an in-annihilator and an out-creator or an in-creator and out-annihilator here it generically happens that there are operators which are in- and out- creators. Explicitly we get

$$\begin{aligned}
\hat{x}_n|T_{in}\rangle, \langle T_{out}|\hat{x}_n &\neq 0, \qquad -(\bar{M} - 2) \leq n \leq 1 \\
\hat{\tilde{x}}_n|T_{in}\rangle, \langle T_{out}|\hat{\tilde{x}}_n &\neq 0, \qquad -(M - 2) \leq n \leq 1.
\end{aligned} \tag{117}$$

Notice that in- and out- creators are compatible with a representation of the commutation relations while in- and out- annihilators are not as it is immediately seen by sandwiching a non vanishing commutation relation with an in- and out- annihilator between the out- and in- vacua.

5.7 Energy-momentum tensor and the vacua

We want to show that the theory is conformal and that the vacua discussed in the previous section are obtained from a $SL(2, \mathbb{R})$ vacuum by acting with twist fields.

5.8 Green function