Normal Ordering of a Free Boson

Riccardo Finotello

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Consider the holomorphic field $\partial_z X(z)$ (conformal weight $(1,0)$):

$$\begin{aligned}\partial_z X(z) = -i\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}} \alpha_n z^{-n-1} = \\ = -i\sqrt{\frac{\alpha'}{2}} \sum_{n=0}^{+\infty} \left(\alpha_n z^{-n-1} + \alpha_{-n-1} z^n\right),\end{aligned}$$

and the commutation relation:

$$[\alpha_n, \alpha_m] = n\delta_{n+m,0},$$

where $\delta_{i,j}$ is the Kronecker delta.

Then the radially ordered product between $\partial_z X(z)$ and $\partial_w X(w)$ ($|z| > |w|$) is:

$$\begin{aligned}
&\mathrm{R}\left(\partial_z X(z)\, \partial_w X(w)\right) = \\
&= -\frac{\alpha'}{2} \sum_{n=0}^{+\infty} \sum_{m=0}^{+\infty} \big(\alpha_n \alpha_m z^{-n-1} w^{-m-1} + \\
&+ \alpha_n \alpha_{-m-1} z^{-n-1} w^m + \alpha_{-n-1} \alpha_m z^n w^{-m-1} + \alpha_{-n-1}\alpha_{-m-1} z^n w^m\big) = \\
&=: \partial_z X(z)\, \partial_w X(w) : \; - \frac{\alpha'}{2} \sum_{n=0}^{+\infty} \sum_{m=0}^{+\infty} \left[\alpha_n, \alpha_{-m-1}\right] z^{-n-1} w^m = \\
&=: \partial_z X(z)\, \partial_w X(w) : \; - \frac{\alpha'}{2} \sum_{n=0}^{+\infty} n z^{-n-1} w^{n-1} = \\
&=: \partial_z X(z)\, \partial_w X(w) : \; + \frac{1}{zw} \frac{\alpha'}{2} \sum_{n=1}^{+\infty} n \left(\frac{w}{z}\right)^n = \\
&=: \partial_z X(z)\, \partial_w X(w) : \; - \frac{1}{zw} \frac{\alpha'}{2} \frac{w/z}{\left(1 - w/z\right)^2} = \\
&=: \partial_z X(z)\, \partial_w X(w) : \; - \frac{\alpha'}{2} \frac{1}{\left(z-w\right)^2}.
\end{aligned}$$

Then, since:

$$\mathrm{R}\left(\partial_z X(z)\, \partial_w X(w)\right) =: \partial_z X(z)\, \partial_w X(w) : + \left\langle \partial_z X(z)\, \partial_w X(w) \right\rangle,$$

we find:

$$\left\langle \partial_z X(z)\, \partial_w X(w) \right\rangle = -\frac{\alpha'}{2} \frac{1}{\left(z-w\right)^2}.$$

Then:

$$\begin{aligned}
T(z) = \frac{2}{\alpha'} : \partial_z X(z)\, \partial_z X(z) : = \\
= \lim_{w\to z} \left[\frac{2}{\alpha'} \mathrm{R}\left(\partial_z X(w)\, \partial_z X(z)\right) + \frac{1}{\left(w-z\right)^2} \right].
\end{aligned}$$