BRST QUANTIZATION

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- S matrix:

→ S-matrix elem. ⇒ computed from LSZ reduction formula → FUNCTIONAL FEYNMAN INTEGRAL

i) Green function:

$$\Delta^{i_1\dots i_n}(x) = \langle \phi^{i_1}(x_1)\dots\phi^{i_n}(x_n)\rangle = \frac{\delta}{\delta J_{i_1}(x_1)}\cdots\frac{\delta}{\delta J_{i_n}(x_n)}\Gamma[J_{i_1},\dots,J_{i_n}]\Big|_{J_i=0}$$

$$= \Delta^{i_1\dots i_n}_{ds}(x) + \underbrace{R^{i_1\dots i_n}(x)}_{\text{finite}}$$

where $\underbrace{e^{i\Gamma[J]}}_{\text{connected diagrams}} = Z[J] = \int \underbrace{D\phi}_{\text{"all the fields"}}\, e^{iS[\phi] + i\int d^dx\, J_i(x)\,\phi^i(x)}$.

2) LSZ → physical poles $(p^2 = m^2)$

→ "Quantum Effective action": (see Stat. mechanics)

- define: $Z[J] = e^{iE[J]}$ (E-J conjugated variables)

↳ then $\frac{\delta E[J]}{\delta J(x)} = \langle\Omega|\phi(x)|\Omega\rangle = \phi_{cl}(x)$

- Legendre transform: $\Gamma[\phi_{cl}] = E[J] - \int d^dx\, J(x)\,\phi_{cl}(x)$

$\Rightarrow \frac{\delta\Gamma[\phi_{cl}]}{\delta\phi_{cl}(x)}\Big|_{J=0} = 0.$ → the solutions are:

$\phi_{cl} = \langle\phi\rangle$ in the STABLE QUANTUM STATES of the theory.

↓

Γ is the generator of 1PI diagrams!

- Gauge theory: preliminary

→ Gauge invariance = redundant dof's → UNPHYSICAL STATES

e.g.: YM theory in canonical quant ⇒ $[A_\mu(x), A_\nu(y)]_{x^0=y^0} = i\eta_{\mu\nu}\,\delta^3(x-y)$

↳ $A_0(x) = A_0^{(+)}(x) + A_0^{(-)}(x) \Rightarrow A_0^{(+)}(0)|0\rangle$ has negative norm!

- Differential geom. and gauge orbits:

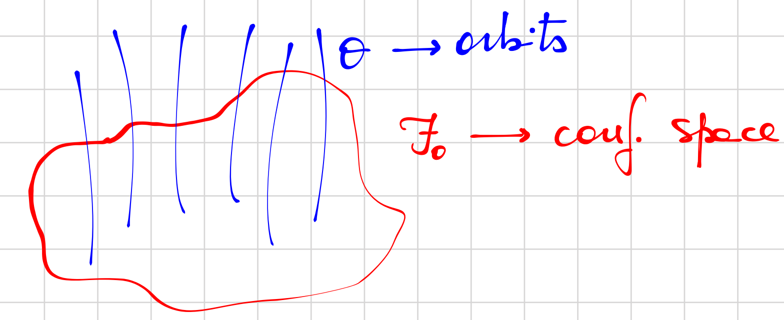
  - define: $\mathcal{F}_0 :=$ "configuration space" (i.e.: space of the fields $\neq$ coord. space)

    $\downarrow$

    if gauge theory $\longrightarrow$ invariance on gauge orbits $\left[G\cdot\phi = \{g\cdot\phi \mid g\in G\}, \phi\in\mathcal{F}_0\right]$

    $\Downarrow$

    $\mathcal{F}_0$ is fibered over the gauge orbits $\mathcal{O} = \{G\cdot\phi\}$



  $\rightarrow$ define translations over $\mathcal{O} \Rightarrow$ vectors on $\mathcal{F}_0$:

$$X_I = P_I^{\alpha}[\phi]\,\frac{\delta}{\delta\phi^{\alpha}(x)} \in T_\phi\mathcal{F}_0 \text{ where } \begin{array}{l}\alpha: \text{counts the fields}\\ I: \text{spacetime + discrete ind}\end{array}$$

e.g.: non abelian Lie group:

$$X_I(x) = \partial_\mu \frac{\delta}{\delta A^I_\mu(x)} - f_{IJ}{}^{K} A^J_\mu(x)\,\frac{\delta}{\delta A^K_\mu(x)}$$

In general: $[X_I, X_J] = C_{IJ}{}^{K}[\phi]\,X_K$

$\downarrow$

if Lie Algebra $\Rightarrow \infty$ dim. in this notation

$\Downarrow$

$C_{IJ}{}^{K}[\phi]$ decoup. $f_{IJ}{}^{K}\cdot$(others)

$\downarrow$

struct const.

- Path integral and gauge orbits:

  $\Rightarrow$ REMARK: the vacuum functional integral is const. over $\mathcal{O}$

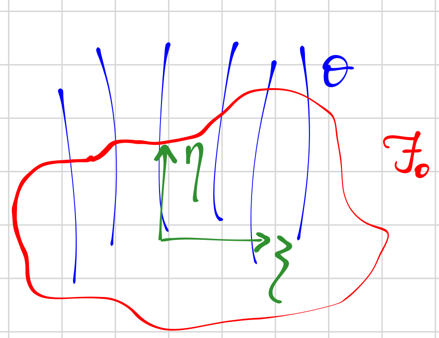
  $\downarrow$

  the PATH INT. IS ILL-DEF

  $\rightarrow$ consider the trivialization (local): $\mathcal{F}_0 \ni \phi \xrightarrow{\text{triv.}} (\xi, \eta)$

  $\xi \downarrow$ const along orbits

  $\eta \searrow$ "vertical" coords



Now add invariant measure elem. s.t.:

- the measure is still invariant
- indep of ξ (i.e.: of the orbit)
- chooses a specific section of $\mathcal{F}_0$

$$\Rightarrow \mathcal{D}\phi\, e^{iS[\phi]} \longrightarrow \mathcal{D}\phi\, \delta_{inv}(\eta-\bar{\eta})\, e^{iS[\phi]}$$

where
$$\delta_{inv}(\eta-\bar{\eta}) = \delta(\eta-\bar{\eta})\, |\det X_I \eta^J|$$

This can be inserted into the path integral such as:

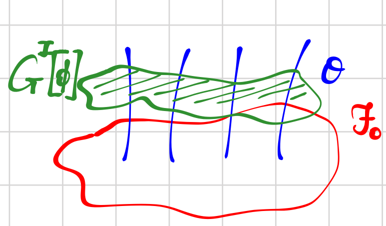
$$Z = \int \mathcal{D}\phi\, \delta(\eta-\bar{\eta})\, |\det X_I \eta^J|\, e^{iS[\phi]} =$$
$$= \int \mathcal{D}\phi\, e^{iS[\phi]} \int \mathcal{D}\rho\, \mathcal{D}c\, \mathcal{D}\bar{c}\, e^{i\int d^dx \left(\rho_I(\eta-\bar{\eta})^I - \bar{c}_J c^I X_I \eta^J\right)}$$

where $c, \bar{c}$ are the GHOST FIELDS (Grassmann variables).

Now we can generalise to:

i) $$c^I X_I \longrightarrow d_V = c^I X_I - \frac{1}{2} c^I c^J C_{IJ}{}^K[\phi] \frac{\partial}{\partial c^K}$$

$$\Rightarrow d_V^2 = 0 \quad := \text{"BRST operator"}$$



ii) choose a GLOBAL SECTION: $G^I[\phi]$ (intersecting the orbits only once)

$$\Rightarrow iS_{GF}[\rho, c, \bar{c}, \phi] = i\int d^dx \left(\rho_I G^I[\phi] - \bar{c}_J\, d_V G^J[\phi]\right)$$

and $$S[\phi, \rho, c, \bar{c}] = S_{inv}[\phi] + S_{GF}[\rho, c, \bar{c}, \phi]$$

$$\Rightarrow \frac{\delta S}{\delta \rho} = G[\phi] = 0 \Rightarrow \text{e.o.m. for } \rho := \text{"gauge condition"}$$

e.g.: YM theory:

$$G^a[A_\mu] = \partial^\mu A_\mu^a \Rightarrow X_a G^b[A_\mu] = \Box\, \delta_a{}^b - g f_{ac}{}^b A_\mu^c \partial^\mu = \partial^\mu (D_\mu)_a{}^b$$

$$\Rightarrow S_{GF}[c, \bar{c}, \rho, A_\mu] = \rho_a \partial^\mu A_\mu^a + \underbrace{\bar{c}_a \partial^\mu (D_\mu)^a{}_b c^b}$$

$$\hookrightarrow \bar{c}\,\Box\, c - g f^a{}_{bc}\, \bar{c}_a \partial^\mu A_\mu^c\, c^b$$

This FADDEV-POPOV METHOD can be translated in a simpler form:

- introduce a new <u>exterior derivative</u> (s):

$$\delta A^I_\mu = D_\mu \lambda^I \longrightarrow s A^I_\mu = D_\mu C^I = \partial_\mu C^I - g f^I_{JK} A^J_\mu C^K$$

such that

$$s^2 A^I_\mu = 0 \Leftrightarrow \partial_\mu (sC^I) - g f^I_{JK} (s A^J_\mu) C^K - g f^I_{JK} A^J_\mu (sC^K) =$$
$$= \partial_\mu (sC^I) - g f^I_{JK} (\partial_\mu C^J - g f^J_{LM} A^L_\mu C^M) C^K - g f^I_{JK} A^J_\mu (sC^K) =$$
$$= \partial_\mu (sC^I) - g f^I_{JK} \partial_\mu C^J C^K + g^2 f^I_{JK} f^J_{LM} A^L_\mu C^M C^K - g f^I_{JK} A^J_\mu (sC^K) =$$
$$= D_\mu (sC^I) - g f^I_{JK} (\partial_\mu C^J - g f^J_{LM} A^L_\mu C^M) C^K =$$
$$= D_\mu \left(sC^I + \frac{g}{2} f^I_{JK} C^J C^K \right) = 0 \Leftrightarrow sC^I = -\frac{1}{2} [C, C]^I$$

- introduce the NON MINIMAL SECTOR:

$$s\bar{C}^I = \rho^I \quad \text{and} \quad s\rho^I = 0 \qquad (\text{"topological quartet"})$$

- hence define:

$$s = d_V + \rho_I \frac{\partial}{\partial \bar{C}_I} \Rightarrow S_{GF} = \rho_I G^I[\phi] + \bar{C}_I d_V G^I[\phi] =$$
$$= s\left(\bar{C}_I G^I[\phi] \right)$$
$$(\Rightarrow s S_{GF} \equiv 0)$$

e.g.: free particle (linearized)

$$\mathcal{L}_0 = P_\mu \dot{X}^\mu + \frac{e}{2} P^2$$

where

$$\delta e = \dot{\lambda} \longrightarrow se = \dot{c} \qquad \left(\dot{\lambda} = \frac{d\lambda}{dt},\ t \text{ affine parameter} \right)$$

$$\Rightarrow \text{introduce}: \begin{cases} s\bar{c} = \rho \\ s\rho = 0 \end{cases} \longrightarrow \mathcal{L} = \mathcal{L}_0 + s[\bar{c}(e-1)] = P_\mu \dot{X}^\mu + \frac{e}{2} P^2 + \rho(e-1) + \bar{c}\dot{c}$$

$$(\text{traditionally: } \bar{c} \equiv b) \Rightarrow \mathcal{L}\Big|_{\rho \text{ on-shell}} = P_\mu \dot{X}^\mu + \frac{1}{2} P^2 + b\dot{c}$$

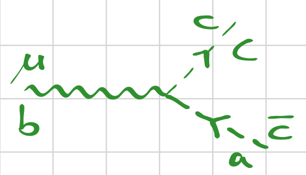
e.g.: YM theory:

$$\delta A^a_\mu = D_\mu \lambda^a \longrightarrow s A^a_\mu = D_\mu c^a$$

$\rightarrow$ introduce $s(s\bar{c}_a) = s\rho_a = 0$ with gauge $\partial^\mu A^a_\mu + \frac{\xi}{2}\rho^a = 0$:

$$S_{GF} = s\left(\bar{c}_a\left(\partial^\mu A^a_\mu + \frac{\xi}{2}\rho^a\right)\right) = \rho_a\left(\partial^\mu A^a_\mu + \frac{\xi}{2}\rho^a\right) + \bar{c}_a \partial^\mu D_\mu c^a =$$

$$= \mathrm{Tr}\left(\rho\, \partial^\mu A_\mu + \frac{\xi}{2}\rho^2\right) + \bar{c}\,\Box\, c - g f^a_{bc}\, \bar{c}_a \partial^\mu A^b_\mu c^c.$$

$\hookrightarrow$ 

$\hookrightarrow$ $\xi = 0$: Landau gauge

$\xi = 1$: Feynman gauge

$\xi \neq 0,1$: R_ξ-gauge

- Renormalization issues and Antifields:

  - notice:

    - sA_μ, $sc \longrightarrow$ action is NON LINEAR
    - $s\bar{c}$, $s\rho \longrightarrow$ action is LINEAR

$\Rightarrow$ non linear contrib. will get corrections (counterterms) at LOOP LEVEL.

$\longrightarrow$ We want to retrieve the SLAVNOV-TAYLOR identities (non abelian Ward-Takahashi)

i.e.: impose gauge invariance of field functionals (Green functions, etc.):

$$\int \mathcal{D}\phi\; e^{i(S[\phi] + S_{GF}[c,\bar{c},\rho,\phi])}\; s\left(\mathcal{F}[\phi]\right) = 0$$

again "all fields" BRST variation

One particular way to show the ST id. is to introduce the ANTIFIELD formalism

$\Downarrow$

INTRODUCE SOURCES FOR BRST INVARIANTS

$(s\phi,\ sc,\ s\rho, \dots)$

- introduce these fields:

FIELD	ϕ^I	c^I	$\bar{c}_I$	ρ^I	ϕ^*_I	c^*_I	$\bar{c}^{*I}$	ρ^*_I
GHOST No.	0	1	-1	0	-1	-2	0	-1
ANTIF. No.	0	0	0	0	1	1	1	1
STATISTIC	B	F	F	B	F	B	B	F

where $\phi^*, c^*, \bar{c}^*, \rho^*$ are treated as sources:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{GF} + \overbrace{\mathrm{Tr}\left[\phi^* s\phi + c^* sc + \bar{c}^* s\bar{c} + \rho^* s\rho\right]}^{\mathcal{L}^*}$$

such that: $s\phi^* = 0$; $sc^* = s\bar{c}^* = 0$; $s\rho^* = 0$. Therefore:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{GF} - s\,\mathrm{Tr}\left(\phi^*\phi - c^* c - \bar{c}^*\bar{c} + \rho^*\rho\right)$$

NB: ϕ^I is usually (in YM): A^a_μ $a \longrightarrow$ algebra vector, $\mu \longrightarrow$ spacetime form

ϕ^*_I " : $A^{*\mu}_a$ $\mu \longrightarrow$ spacetime vector, $a \longrightarrow$ algebra form

↳ in the contraction $A^{*\mu}_a\, s A^a_\mu$ there are no metrics $\Rightarrow$ TOPOLOGY

We now define a new measure of integration:

$$e^{i\Gamma[\phi^*, c^*, \bar{c}^*, \rho^*]} = Z[\phi^*, c^*, \bar{c}^*, \rho^*] = \int \mathcal{D}\phi\,\mathcal{D}c\,\mathcal{D}\bar{c}\,\mathcal{D}\rho\; e^{i\int d^dx\,(\mathcal{L}_0 + \mathcal{L}_{GF} + \mathcal{L}^*)}$$

and choose

$$\mathcal{F}[\phi, \ldots] = e^{i\int d^dx\,\left[J_I\phi^I + R_I\rho^I + \bar{D}_I c^I + D^I \bar{c}_I\right]}$$

We can now impose the ST identity to find:

$$\int \mathcal{D}\phi\,\mathcal{D}c\,\mathcal{D}\bar{c}\,\mathcal{D}\rho\; e^{iS + iS_{GF} + iS^*} \int d^4x\left(J_I\, s\phi^I - \bar{D}_I\, sc^I - D^I\rho_I\right)\mathcal{F}[\phi \ldots] = 0$$

$$\Rightarrow \int d^4x\left(J_I \frac{\delta\Gamma}{\delta\phi^*_I} - \bar{D}_I \frac{\delta\Gamma}{\delta c^*_I} - D^I \frac{\delta\Gamma}{\delta\bar{c}^I}\right) = 0 \Rightarrow \int d^4x\left(\frac{\delta\Gamma}{\delta\phi^I}\frac{\delta\Gamma}{\delta\phi^*_I} + \frac{\delta\Gamma}{\delta c^I}\frac{\delta\Gamma}{\delta c^*_I} + \rho^I\frac{\delta\Gamma}{\delta\bar{c}^I}\right) = 0$$

Moreover: consider $S = S_0 + S_{GF} + S^*$:

$$\int d^4x \left(\frac{\delta S}{\delta \phi^*_I} \frac{\delta S}{\delta \phi^I} + \frac{\delta S}{\delta c^*_I} \frac{\delta S}{\delta c^I} + \rho \frac{\delta \Gamma}{\delta \bar{c}} \right) = 0$$

Define:

$$\mathcal{S} = \frac{1}{2} \int d^dx \left(\frac{\delta \Gamma}{\delta \phi^*} \frac{\delta}{\delta \phi} + \frac{\delta \Gamma}{\delta c^*} \frac{\delta}{\delta c} + 2\rho \frac{\delta}{\delta \bar{c}} \right) \quad \text{s.t. } \mathcal{S}[\Gamma] = 0.$$

and consider:

- $\frac{\delta \Gamma}{\delta \rho^I} = G_I[\phi]$ ("gauge choice")
- $\frac{\delta \Gamma}{\delta \bar{c}_I} + G^I\left[\frac{\delta \Gamma}{\delta \phi^*}\right] = 0$ (GHOST EQUATION)

$\hookrightarrow$ NB: G^I is a diff. operator applied to $\frac{\delta \Gamma}{\delta \phi^*}$.

Now define:

$$\hat{\Gamma} = \Gamma - \rho_I G^I[\phi] \qquad \left(\text{and, e.g.: } \hat{A}^{*\mu}_a = A^{*\mu}_a - \partial^\mu \bar{c}_a\right)$$

from which define: (Batalin-Vilkoviski formalism)

$$(F, G) = \int d^dx \left(\frac{\delta F}{\delta \phi^*} \frac{\delta G}{\delta \phi} + \frac{\delta F}{\delta c^*} \frac{\delta G}{\delta c} + (F \leftrightarrow G) \right) \qquad \text{"ANTIBRACKET"}$$

Then we can state:

$(S, S) = 0 \longrightarrow$ "CLASSICAL MASTER EQUATION"

$(\hat{\Gamma}, \hat{\Gamma}) = 0 \longrightarrow$ "QUANTUM MASTER EQUATION"

[we can also have $(\hat{\Gamma}, \hat{\Gamma}) = \hbar \Delta_\Gamma$ as quantum correction (NB: not an anomaly)]

NB: $(S, S) = 0$ encodes fully $s^2 = 0$, i.e.: BRST nihilpotency.

Now we define:

$$s = (S, \cdot)$$

such that:

$$sA_\mu = (S, A_\mu) = \int d^4x \frac{\delta S}{\delta A^{*\mu}} = \underbrace{s A_\mu}_{BRST}$$

$$s\widehat{A^{*\mu}} = (S, \widehat{A^{*\mu}}) = \int d^4x \frac{\delta S}{\delta A_\mu} = \underbrace{-D^\nu F_{\mu\nu}}_{EOM} + \underbrace{ig\left[A^{*\nu}, c\right]}_{BRST}$$

$$sc = (S, c) = \int d^4x \frac{\delta S}{\delta c^*} = \underbrace{sc}_{BRST}$$

$$sc^* = (S, c^*) = \int d^4x \frac{\delta S}{\delta c} = \underbrace{D_\mu \widehat{A^{*\mu}}}_{EOM} + \underbrace{ig\left[c^*, c\right]}_{BRST}$$

which shows that the master eqn. knows both BRST SYM. and EOM:

Koszul-Tate reduction for BRST $\Rightarrow$ $s = \underbrace{\gamma}_{BRST} + \underbrace{\delta}_{EOM}$ Koszul-Tate operator

The BRST operator therefore acts as a COBOUNDARY OPERATOR s.t. $s^2 = 0$ acting on functional of the fields $\Omega(\phi, c, \bar{c}, \dots)$ (including ghosts and antifields).

NB: $\Omega(\phi, c, \bar{c}, \dots)$ contains ALL GHOST NUMBERS in general:

Space of Ω $\leftarrow$ $V_\Omega = \bigoplus_{h=-\infty}^{+\infty} V_\Omega^{(h)}$

where h = PURE GHOST NO. $-$ ANTIFIELD NO.

Moreover

$$s : V_\Omega^{(h)} \longrightarrow V_\Omega^{(h+1)}$$

i.e.: in general we can consider sth more complicated than simply fields-antifields.

Since $\Omega' = s\Omega \Rightarrow s\Omega' \equiv 0$, then we can restrict our study:

PHYSICAL STATES $\longrightarrow$ BRST COHOMOLOGY [i.e.: BRST-closed]

$$H(s,\Omega) = \frac{\ker s(\Omega)}{\operatorname{Im} s(\Omega)}$$

each $\mathcal{V}_{phys} \in H(s,\Omega)$ is an equiv. class of COCYCLES.

In general the decomposition of s is longer:

$$s = \sum_{k \geq -1} s_k$$ where $k = \text{antif}(s_k)$, i.e.: the variation of pure antifield no.

$\Rightarrow$ e.g.: YM theory $\longrightarrow$ $k_{max} = 0$ and $s = s_{-1} + s_0 = \delta + \gamma$

Since the field functionals $\Omega(\phi, c, \bar{c}, \dots)$ contain all ghost no., we can choose say $h = 0$ to be the physical fields:

$$\Omega_0(\phi,\dots) = \{\mathcal{V}[\phi,\dots] + s\Lambda[\phi,\dots]\} \in H^0(s,\Omega)$$

In the Green functions:

$$\langle (\mathcal{V}_1 + s\Lambda)\mathcal{V}_2\,\mathcal{V}_3\dots\rangle = \langle \mathcal{V}_1\,\mathcal{V}_2\,\mathcal{V}_3\dots\rangle + \langle s(\Lambda\,\mathcal{V}_2\,\mathcal{V}_3\dots)\rangle$$

$$\int \mathcal{D}\phi\dots\, e^{-S_{TOT}}\, s(\Lambda\,\mathcal{V}_2\,\mathcal{V}_3\dots) = 0$$

What happens to the other ghost no.? E.g.: $\Omega_1 = \{c^a f_a[\phi,\dots]\}$

$$s\Omega_1 = 0 \Leftrightarrow -i\frac{g}{2} f^a_{bc}\, c^b c^c f_a[\dots] - c^a\left[(D_\mu c)^b \frac{\delta f_a}{\delta A^b_\mu} + \partial_\nu (D_\mu c)^a \frac{\delta f_a}{\delta(\partial_\nu A_\mu)^a} + \dots\right] = 0$$

$\Rightarrow$ WZ consistency relation

We now have to understand how BRST symmetry (i.e. $s(S)=0$) acts on the Fock space of states. In general:

$$\phi' = \phi + \delta\phi \text{ and } S[\phi'] = S[\phi] \Rightarrow \exists J_\mu \mid \partial^\mu J_\mu = 0 \Rightarrow Q = \int d^{d-1}x\, J^0(x) \mid \dot{Q} = 0$$

then

$$s(S)=0 \Rightarrow \delta_\varepsilon S = \int d^d x\, J_\mu(x)\partial^\mu \varepsilon(x) \longrightarrow \boxed{Q_{BRST}} \text{ acting on states in } \mathcal{F}_v$$

We require

$$|\psi\rangle \in H(Q_{BRST}, \mathcal{F}_v) \Rightarrow Q_{BRST}|\psi\rangle = 0 \text{ and } |\psi\rangle \neq Q_{BRST}|\chi\rangle$$

$$\Downarrow$$

$$|\psi\rangle \sim |\psi\rangle + Q_{BRST}|\chi\rangle$$

Since $|\psi\rangle$ is created by operators $\mathcal{V}(\phi,\dots) \in V$ acting on vacuum:

$$\mathcal{V} \in H(Q_{BRST}, V) \Rightarrow [Q_{BRST}, \mathcal{V}] = 0 \text{ and } \mathcal{V} \neq [Q_{BRST}, \Omega]$$

or equivalently:

$$[Q_{BRST}, \mathcal{V}] = 0 \text{ and } \delta\mathcal{V} = [Q_{BRST}, \Omega] \text{ (gauge transf.)}$$

e.g.: (1) Free particle:

$$S = \int d\tau \left(P_m \dot{X}^m + \tfrac{1}{2} e P_m P^m\right) \longrightarrow S_{GF} = \int d\tau \left(-\tfrac{1}{2}\dot{X}_m \dot{X}^m + b\dot{c}\right)$$

$$\Rightarrow Q_{BRST} = c P^m P_m \text{ acting on } \mathcal{V}(X,c) = \varphi_0(x) + c\,\varphi_1(x) \quad \left(\text{NB: } \dots + \underbrace{\cancel{c^2\varphi_2}}_{=0} + \dots\right)$$

$\hookrightarrow Q_{BRST}^2 = c^2 P^m P_m P^r P_r = 0$ $\qquad$ $\hookrightarrow P, b$ are conjugate momenta

Then for a physical state:

i) $[Q_{BRST}, \mathcal{V}] = 0 \longrightarrow c\,\Box\varphi_0(X) = 0 \Rightarrow \Box\varphi_0(X) = 0$ (no constraints on $\varphi_1(x)$)

2) $\delta\mathcal{V} = [Q_{BRST}, \Omega]$ where $\Omega(X,c) = \omega_0(X) + c\,\omega_1(X) \longrightarrow \delta\varphi_0 + c\,\delta\varphi_1 = c\,\Box\omega_0 \Rightarrow \delta\varphi_1(X) = \Box\omega_0(X)$

$$\delta\varphi_0(X) = 0$$

Interpretation of physical states:

i) $\Box \varphi_0 = 0 \Rightarrow \varphi_0(x) = \varphi(X)$ IS A MASSLESS SCALAR (i.e.: a free massless particle)

ii) $\delta\varphi_1(X) = \Box\omega_0(X) \Rightarrow$ a) $\Box\omega_0 = 0 \Rightarrow$ CANNOT gauge away $\varphi_1(X)$

b) $\Box\omega_0 \neq 0 \Rightarrow \Box\varphi_1(X) = 0 \Rightarrow \varphi_1(X) = \varphi^*(X)$ IS A MASSLESS SCALAR

$\boxed{\text{can gauge } \varphi_1(X) \text{ away}}$

$\rightarrow \varphi(X)$ is the FIELD (ghost no. 0)

$\varphi^*(X)$ is the antifield (ghost no. 1)

↳ (or -1 dep. on conventions. Usually:

$c\,A(X) \longrightarrow$ ghost no. 1

$c\,\partial_c B(X) \longrightarrow$ ghost no. 2

etc.)

So $\mathcal{V}(X,c) \in H(\Omega_{BRST}, V)$ if $\Box\varphi(X) = \Box\varphi^*(X) = 0$.

e.g.: (2) String in LC gauge: [OPEN STRING] ($\hat{\ } :=$ 0-mode)

$$S = \int d\tau d\sigma \left(P_m \dot{X}^m + e\,(P_m + X'_m)(P_m + X'_m) \right) \longrightarrow S_{GF} = \int d\tau d\sigma \left(\dot{\hat{X}}^+ \hat{X}^- + \tfrac{1}{4}\partial X^i \partial X^i + \hat{b}\hat{c} \right)$$

$$\Rightarrow Q_{BRST} = \hat{c}\int d\sigma\,(P+X')^2 = \hat{c}\int d\sigma \left(M^2 + (P_1 + X'_1)(P_{-1} + X'_{-1}) + \dots \right)$$

Then consider $\mathcal{V} = \underbrace{\varphi(\hat{X}^+, \hat{X}^-, X^i)}_{\text{FIELD}} + \hat{c}\,\underbrace{\varphi^*(\hat{X}^+, \hat{X}^-, X^i)}_{\text{ANTIFIELD}}$

i) $[Q_{BRST}, \mathcal{V}] = 0 \Rightarrow [\hat{c}\int d\sigma\,(P+X')^2, \varphi(\dots)] = 0 \Rightarrow$ MASS SHELL FOR $\varphi(\hat{X}^+, \hat{X}^-, X^i)$

ii) $\delta\mathcal{V} = [Q_{BRST}, \Omega] \Rightarrow \delta\varphi^* = [\hat{c}\int d\sigma\,(P+X')^2, \Omega] \rightarrow$ gauge away φ^* unless ON SHELL

(i.e.: $M^2 = -(P_1 + X_1)(P_{-1} + X_{-1}) + \dots$)

↳ 0 eigenvalues of $[\hat{c}\int d\sigma\,(P+X')^2, \Omega]$

eg.: ③ String in CONFORMAL GAUGE [OPEN STRINGS]

$$S_{GF} = \int dz d\bar{z} \left(\partial X \cdot \bar{\partial} X + b \bar{\partial} c \right)$$

$$\rightarrow Q_{BRST} = \int d\sigma \left(c\, \partial X\, \partial X + b c\, \partial c \right)$$

$c(z) = c_{-1} + c_0 z + c_1 z^2 + \dots$

And consider $V(X, c, b) = \varphi_0(X) + c\,\varphi_1(X) + c\,\partial c\, \varphi_2(X) + c\,\partial c\, \partial^2 c\, \varphi_3(X)$
(no b's just for convenience)

$\Rightarrow$ as before $[Q_{BRST}, V] = 0$ is the MASS SHELL CONDITION. For example we can construct:

$k^2 = 1$: $c_{-1} e^{ik\cdot X}|0\rangle$ [all computed as $\lim_{z\to 0} V(z)|0\rangle$]

NB: $P^2 + M^2 = 0$
$\Downarrow$
$P^2 = -M^2$

$k^2 = 0$: $e^{ik\cdot X}|0\rangle$, $c_{-1}\alpha_1^m e^{ik\cdot X}|0\rangle$, $c_0 e^{ik\cdot X}|0\rangle$, $c_{-1}c_1 e^{ik\cdot X}|0\rangle, \dots$

$k^2 = -1$: $b_2 c_{-1} e^{ik\cdot X}|0\rangle$, $b_1 e^{ik\cdot X}|0\rangle, \dots$

and apply the cohomology directly on the states

$$Q_{BRST}|\psi\rangle = 0$$

E.g.: consider $k^2 = 0$: $|\psi\rangle = \Big(\underbrace{\xi_1 e^{ik\cdot X}|0\rangle}_{\text{gh. no. } 0} + \underbrace{c_{-1}\xi_2\cdot\alpha_1 e^{ik\cdot X} + c_0 \xi_3 e^{ik\cdot X}}_{\text{gh. no. } 1} + \underbrace{\dots}_{\text{other gh. no.}} \Big)|0\rangle$
and apply:

$$Q_{BRST} = \int d\sigma \left(c_{-1}(P+X')_1^2 + c_0 (P+X')_0^2 + \dots + b_{-1} c_0 c_{-1} + \dots \right)$$

such that (for gh. no. 1):

$$\left[c_0 (\partial X)_0^2 \left(c_{-1}\xi_2(X)\cdot\alpha_1 \right) + c_{-1} (\partial X)_1^2 \left(c_0 \xi_3(X) \right) + \dots \right] |\psi\rangle = 0$$
$$\hookrightarrow c_0 c_{-1} \left(\Box \xi_{2m}(X) - \partial_m \xi_3(X) \right) |\psi\rangle = 0 \Rightarrow \Box\xi_{2m}(X) - \partial_m \xi_3(X) = 0.$$

then:

- $\xi_1 = \Lambda = \text{const}$
- $\Box \xi_{2m}(X) = \partial_m \xi_3(X)$ and $\partial^m \xi_{2m}(X) = \xi_3(X)$

While $|\chi\rangle = Q_{BRST}|\Omega\rangle$:

- $\delta \xi_{2m} = \partial_m \Omega$ and $\delta \xi_3 = \Box \Omega$.