Boundary States for Strings in Non-constant EM Background

Consider the action:

$$S = -\frac{1}{4\pi\alpha'}\int d\tau d\sigma \left[-\partial_\tau X^\mu \partial_\tau X_\mu + \partial_\sigma X^\mu \partial_\sigma X_\mu\right]$$

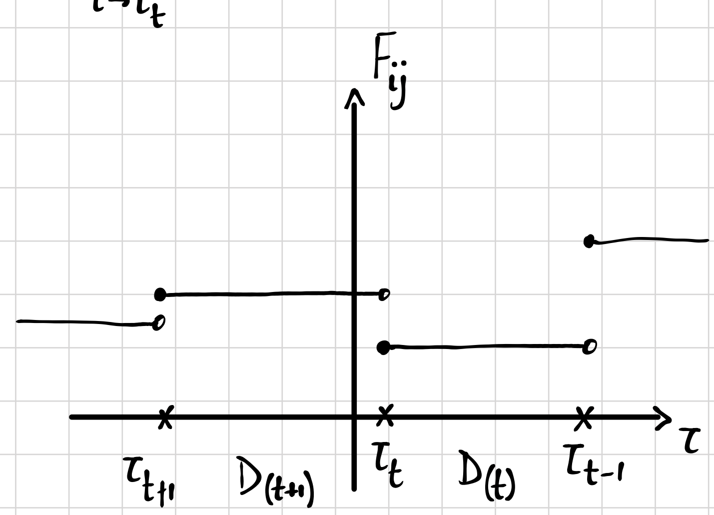
and add EM potential to OPEN STRING ENDPOINTS (suppose space-filling D9 or D25):

$$S = -\frac{1}{4\pi\alpha'}\int d\tau d\sigma \left(-\partial_\tau X^\mu \partial_\tau X_\mu + \partial_\sigma X^\mu \cdot \partial_\sigma X_\mu\right) + \int d\tau\, A_\mu \partial_\tau X^\mu \Big|_{\sigma=\pi} - \int d\tau\, A_\mu \partial_\tau X^\mu \Big|_{\sigma=0}$$

Now consider N D-branes, each with a different value of the potential A_μ. Let $F_{\mu\nu}$ be its field strength such that:

$$\partial_\tau F_{\mu\nu}(\tau) = 0 \quad \text{if} \quad \tau_t < \tau < \tau_{t-1} \; (\text{i.e.: when the string is on the brane } D_{(t)})$$

and $\lim_{\tau\to\tau_t^+} F_{\mu\nu}(\tau) \neq \lim_{\tau\to\tau_t^-} F_{\mu\nu}(\tau)$ (i.e.: $F_{\mu\nu}$ has discontinuities):



NB: t defined mod N

$$F_{\mu\nu}(\tau) = \sum_{t=1}^{N} F_{\mu\nu(t)}(\tau)\, \theta(\tau-\tau_t)\,\theta(\tau_{t-1}-\tau)$$

$$\Rightarrow \quad A_\mu(\tau,\sigma) = F_{\mu\nu(t)}(\tau)\, X^\nu(\tau,\sigma) \quad \text{for} \quad \tau_t < \tau < \tau_{t-1}.$$

Define the potential on the brane $D_{(t)}$:

$$A_{\mu(t)} = F_{\mu\nu(t)} X^\nu_{(t)}$$

where

$$X^\mu_{(t)} = X^\mu\, \theta(\tau-\tau_t)\,\theta(\tau_{t-1}-\tau).$$

Now we take $\tau = i\tau_E$:

$$\delta S = \frac{1}{2\pi\alpha'}\int d\tau_E d\sigma \left(\partial_{\tau_E} X^\mu \cdot \partial_{\tau_E}\delta X_\mu + \partial_\sigma X^\mu \cdot \partial_\sigma \delta X_\mu \right) - \sum_{t=1}^{\infty}\int_{\tau_t}^{\tau_{t-1}} d\tau_E \Big(F_{\mu\nu(t)}\delta X^\nu_{(t)}\, \partial_{\tau_E} X^\mu_{(t)}\Big|_{\sigma=\pi} + F_{\mu\nu(t)} X^\nu_{(t)}\, \partial_{\tau_E}\delta X^\mu_{(t)}\Big|_{\sigma=\pi} -$$

$$\Big\downarrow \text{ call } \tau \equiv \tau_E \text{ for simplicity} \qquad\qquad - \ldots \Big|_{\sigma=0} \Big) =$$

$$= \frac{1}{2\pi\alpha'}\int d\tau d\sigma \left(\partial_\tau\partial_\tau X^\mu + \partial_\sigma\partial_\sigma X^\mu \right)\delta X_\mu + \frac{1}{2\pi\alpha'}\int d\tau\, \partial_\sigma X^\mu \cdot \delta X_\mu \Big|_{\sigma=0,\pi} + \sum_{t=1}^{\infty}\int d\tau \Big(F_{\mu\nu(t)}\,\partial_\tau X^\mu_{(t)}\Big|_\sigma -$$

$$- F_{\mu\nu(t)}\,\partial_\tau X^\mu \Big|_{\sigma=0} \Big)\delta X^\nu_{(t)} =$$

$\Rightarrow$ EOM: $\Box X^\mu = 0$

BC: $\partial_\sigma X_{\mu(t)} - 2\pi\alpha' F_{\mu\nu(t)}\,\partial_\tau X^\nu_{(t)} \Big|_{\sigma=0} = 0.$

Now look at closed strings: $X^\mu(\tau,\sigma) = X^\mu_L(\tau + i\sigma) + X^\mu_R(\tau - i\sigma)$:

$$X^\mu_L(\tau+i\sigma) = \frac{1}{2}\tilde{x}^\mu + \ell^2 p^\mu(\tau + i\sigma) + i\ell\sum_{n\neq 0}\frac{\alpha^\mu_n}{n} e^{-in(\tau+i\sigma)}$$

$$X^\mu_R(\tau-i\sigma) = \frac{1}{2}\tilde{x}^\mu_0 + \tilde{\ell}^2 \tilde{p}^\mu(\tau - i\sigma) + i\tilde{\ell}\sum_{n\neq 0}\frac{\tilde{\alpha}^\mu_n}{n} e^{-in(\tau-i\sigma)}$$

Then we impose $X^\mu(\tau,\sigma+2\pi) = X^\mu(\tau,\sigma)$:

$$\ell^2 p^\mu \cdot 2i\pi - \tilde{\ell}^2 \tilde{p}^\mu \cdot 2i\pi = 0 \quad \text{and } n \in \mathbb{Z}$$

Then: $\ell^2 = \tilde{\ell}^2$, $p^\mu = \tilde{p}^\mu$ and $n\in\mathbb{Z}$:

$$X^\mu_{(c)}(\tau,\sigma) = \frac{1}{2}\left(x^\mu_0 + \tilde{x}^\mu_0\right) + 2\ell^2 p^\mu \tau + i\ell\sum_{n\neq 0}\frac{1}{n}\left(\alpha_n e^{-in(\tau+i\sigma)} + \tilde{\alpha}_n e^{-in(\tau-i\sigma)}\right).$$

Then we look at the centre-of-mass momentum:

$$\frac{1}{2\pi}\int_0^{2\pi} d\sigma\, \mathcal{P}^\mu = \frac{1}{2\pi}\left(\frac{1}{2\alpha'}\right)\int_0^{2\pi} d\sigma\, \partial_\tau X^\mu = \frac{1}{2\alpha'}\cdot 2\ell^2 p^\mu = \Pi^\mu_{CM} \leftrightarrow \ell^2 = \alpha',\ p^\mu = \Pi^\mu_{CM}$$

$$\Rightarrow X^\mu_{(c)}(\tau,\sigma) = \frac{1}{2}\left(x^\mu_0 + \tilde{x}^\mu_0\right) + 2\alpha'\Pi^\mu_{CM}\tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\left(\alpha^\mu_n e^{-in(\tau+i\sigma)} + \tilde{\alpha}^\mu_n e^{-in(\tau-i\sigma)}\right).$$

We try now to impose the Neumann conditions in the closed string channel:

$$\partial_\tau X_{\mu(t)} - 2\pi\alpha' F_{\mu\nu(t)}\, \partial_\sigma X^{\nu}_{(t)} \Big|_{\tau=0} |B_{(t)}\rangle = 0 .$$

where:

$$\partial_\tau X_{\mu(t)}\Big|_{\tau=0} = 2\alpha'\pi^{\mu}_{cm} + i\sqrt{2\alpha'}\sum_{n\neq 0}\left(\alpha^{\nu}_{n} e^{-in\sigma} + \tilde{\alpha}^{\nu}_{n} e^{in\sigma}\right)\eta_{\mu\nu} \qquad \left(\eta_{\mu\nu} = \mathrm{diag}\,(-,+,+,\dots)\right)$$

$$\partial_\sigma X^{\nu}_{(t)}\Big|_{\tau=0} = i\sqrt{2\alpha'}\sum_{n\neq 0}\left(-i\alpha^{\nu}_{n} e^{-in\sigma} + i\tilde{\alpha}^{\nu}_{n} e^{in\sigma}\right)$$

$$\hookrightarrow 2\alpha'\pi_{\mu,cm} + i\sqrt{2\alpha'}\sum_{n\neq 0}\left[\left(\alpha^{\nu}_{n}\eta_{\mu\nu} - i\,2\pi\alpha' F_{\mu\nu(t)}\alpha^{\nu}_{n}\right)e^{-in\sigma} + \left(\tilde{\alpha}^{\nu}_{n}\eta_{\mu\nu} + i\,2\pi\alpha' F_{\mu\nu(t)}\tilde{\alpha}^{\nu}_{n}\right)e^{in\sigma}\right]|B_{(t)}\rangle = 0$$

$$\Rightarrow 2\alpha'\pi_{\mu,cm} + i\sqrt{2\alpha'}\sum_{n\neq 0}\Big(\alpha^{\nu}_{n}(\eta_{\mu\nu} \underbrace{-2\pi i\alpha' F_{\mu\nu(t)}}_{\mathcal{F}_{\mu\nu(t)}})e^{-in\sigma} + \tilde{\alpha}^{\nu}_{n}(\eta_{\mu\nu} \underbrace{+2\pi i\alpha' F_{\mu\nu(t)}}_{\mathcal{F}_{\mu\nu(t)}})e^{in\sigma}\Big)|B_{(t)}\rangle = 0$$

$$\Rightarrow \quad \pi^{\mu}_{cm}|B_{(t)}\rangle = 0 \quad \text{(state is localized on the brane)}$$

$$\sum_{n\neq 0}\left(\alpha^{\nu}_{n}\left(\eta_{\mu\nu} - \mathcal{F}_{\mu\nu(t)}\right) + \tilde{\alpha}^{\nu}_{-n}\left(\eta_{\mu\nu} + \mathcal{F}_{\mu\nu(t)}\right)\right)|B_{(t)}\rangle = 0$$

As in the usual case

$$\sum_{n\neq 0}\left(j_n + \tilde{j}_{-n}\right)|B_0\rangle = 0 \;\rightarrow\; |B_0\rangle \propto \exp\left(-\sum_{n=1}^{\infty}\frac{1}{n} j^{\mu}_{-n}\tilde{j}^{\nu}_{-n}\eta_{\mu\nu}\right),$$

then, in this case:

$$|B_{(t)}\rangle = \mathcal{N}\,\delta^{9}(y - x_0)\exp\left(-\sum_{n=1}^{\infty}\frac{1}{n}\alpha^{\mu}_{-n}\left(\eta_{\mu\nu} - \mathcal{F}_{\mu\nu(t)}\right)^{\dagger}\left(\eta_{\rho\sigma} + \mathcal{F}_{\rho\sigma(t)}\right)\alpha^{\rho}_{-n}\eta_{\mu\rho}\right).$$