Algebra Basic Definitions

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- **Semigroup** $(G, \cdot) \longrightarrow$ e.g.: $(\mathbb{N} \setminus \{0\}, +)$, $(\mathbb{Z}, \min)$:

  i) G is a *set*,

  ii) $\cdot : G \times G \to G$, $(g, h) \mapsto l \mid (g \cdot h) \cdot l = g \cdot (h \cdot l) \quad \forall g, h, l \in G$.

- **Monoid** $(M, \cdot) \longrightarrow$ e.g.: $(\mathbb{N}, +), (\mathbb{N} \setminus \{0\}, \cdot)$:

  i) M is a *semigroup*,

  ii) $\exists \mathbb{1}_M \mid m \cdot \mathbb{1}_M = \mathbb{1}_M \cdot m = m \quad \forall m \in M$.

- **Group** $(\mathbb{G}, \cdot) \longrightarrow$ e.g.: $(\mathbb{Z}_n, \cdot)$, $(S_n, \cdot)$, $(SO(n), \cdot)$:

  i) $\mathbb{G}$ is a *monoid*,

  ii) $\forall g \in \mathbb{G} \, \exists g^{-1} \in \mathbb{G} \mid g \cdot g^{-1} = g^{-1} \cdot g = \mathbb{1}_{\mathbb{G}}$.

- **Ring** $(\mathbb{R}, +, \cdot) \longrightarrow$ e.g.: $(\mathbb{Z}, +, \cdot)$, $\text{End}(\mathbb{G})$ where $\mathbb{G}$ is an *abelian group*:

  i) $(\mathbb{R}, +)$ is an *abelian group* (i.e. $g + h = h + g \quad \forall g, h \in \mathbb{R}$),

  ii) $(\mathbb{R}, \cdot)$ is a *monoid*,

  iii) $r \cdot (s + t) = r \cdot s + r \cdot t \quad \forall r, s, t \in \mathbb{R}$,

  iv) $(r + s) \cdot t = r \cdot t + s \cdot t \quad \forall r, s, t \in \mathbb{R}$.

- **Division ring** $(R, +, \cdot) \longrightarrow$ e.g.: $(\mathbb{H}, +, \cdot)$:

  i) $(R, +, \cdot)$ is a *ring*,

  ii) $\forall r \in R \, \exists r^{-1} \in R \mid r \cdot r^{-1} = r^{-1} \cdot r = \mathbb{1}_R$.

- **Field** $(\mathbb{F}, +, \cdot) \longrightarrow$ e.g.: $(\mathbb{C}, +, \cdot)$:

  i) $(\mathbb{F}, +, \cdot)$ is a *commutative ring* (i.e. $f \cdot g = g \cdot f \quad \forall f, g \in \mathbb{F}$),

  ii) $\exists \mathbb{1}_{\mathbb{F}} \in \mathbb{F} \mid \mathbb{1}_{\mathbb{F}} \cdot f = f \cdot \mathbb{1}_{\mathbb{F}} = f \quad \forall f \in \mathbb{F}$.

- **R-module** M $\longrightarrow$ e.g.: $C^\infty(M)$-module of $X \in \Gamma(M, TM)$:

  i) $(M, +)$ is an *abelian group*,

  ii) $(R, +, \cdot)$ is a *ring*,

  iii) $\cdot : R \times M \to M$, $(r, m) \mapsto r \cdot m \mid r \cdot (m + n) = r \cdot m + r \cdot n \quad \forall r \in R, \forall m, n \in M$,

  iv) $\cdot : R \times M \to M$, $(r, m) \mapsto r \cdot m \mid (r + s) \cdot m = r \cdot m + s \cdot m \quad \forall r, s \in R, \forall m \in M$,

  v) $\cdot : R \times M \to M$, $(r, m) \mapsto r \cdot m \mid (r \cdot s) \cdot m = r \cdot (s \cdot m) \quad \forall r, s \in R, \forall m \in M$,

  vi) $\cdot : R \times M \to M$, $(r, m) \mapsto r \cdot m \mid \mathbb{1}_R \cdot m = m \cdot \mathbb{1}_R = m \, \forall m \in M$.

- **$\mathbb{F}$-vector space** V $\longrightarrow$ e.g.: $(\mathbb{R}^n, +, \cdot)$

  i) V is a *$\mathbb{F}$-module*,

  ii) $\mathbb{F}$ is a *field*.