LACES 2017

Introduction to SUGRA

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SUSY (rigid) + GR $\longrightarrow$ local supersimmetry $\to$ SUGRA

* The passage is $(D=4)$: $\left[x^\mu = (ct, x^i)\,,\ \eta_{\mu\nu} = \begin{pmatrix} - & & & \\ & + & & \\ & & + & \\ & & & + \end{pmatrix}\right]$

Lorentz $\to$ Poincaré $\to$ Conformal group $\to$ SuperPoincaré

$\to$ LORENTZ GROUP:

- finite: $\eta_{\mu\nu} x^\mu y^\nu = \eta_{\mu\nu} x'^\mu y'^\nu \Rightarrow x' = \Lambda x \to \Lambda^T \eta \Lambda = \eta$
- infinitesimal: $\Lambda \sim \mathbb{I} + \omega \Rightarrow \omega^T \eta + \eta \omega = 0 \to \omega_{\mu\nu} = -\omega_{\nu\mu}$

$$\omega_{\mu\nu} \text{ antisymm} \Rightarrow \frac{4\cdot 3}{2} = 6 \text{ d/fs} \begin{cases} 3 \text{ boosts} \\ 3 \text{ rotations} \end{cases}$$

Inside the Lorentz group we find:

- "usual" Lorentz transformations
- P, T transformations.

$\det \Lambda = \pm 1 : L_\pm$

$\Lambda^0{}_0 \gtrless 0 : L^{\uparrow\downarrow}$

e.g.: $\Lambda^0{}_0 > 0$ and $\det\Lambda = +1 \Rightarrow$ restricted Lorentz group $[L^\uparrow_+]$ (continuosly connected to the id.)

There are connections with other subgroups:

$$\begin{array}{cc} L^\uparrow_+ & L^\uparrow_- \\ L^\downarrow_- & L^\downarrow_+ \end{array}$$

$L^\uparrow_+$ is the most relevant in physics

→ POINCARÉ GROUP

$$* \quad (x-y)^2 = (x'-y')^2 \longrightarrow x' = \Lambda x + a$$

$$(\Lambda_1, a_1) \cdot (\Lambda_2, a_2) = (\Lambda_1 \Lambda_2, a_1 + \Lambda_1 a_2)$$

Now notice that $L_+^\uparrow$ is <u>not</u> simply connected because it is isomorphic to a simply connected group:

$$SO(3,1) \longrightarrow SL(2,\mathbb{C})$$

Call ξ_α $(\alpha = 1,2)$ an object that transforms under $SL(2,\mathbb{C})$:

$$\xi'^\alpha = A^\alpha{}_\beta \, \xi^\beta \quad \text{and} \quad \det A = 1$$

We can use $\epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha} = -\epsilon_{\alpha\beta}$ to lower or raise the indices:

$$A^\alpha_\beta A^\delta{}_\gamma \, \epsilon^{\beta\gamma} \epsilon_{\alpha\delta}$$

A general $SL(2,\mathbb{C})$ tensor ($\dot\alpha, \dot\beta, \ldots$ are the conj. rep.):

$$\psi_{\alpha_1 \ldots \alpha_{2J_1}, \dot\alpha_1 \ldots \dot\alpha_{2J_2}}$$

is an irrep $\Leftrightarrow$ dotted and undotted indices are separately symmetric:

(J_1, J_2) identifies the representation

$$J_1 + J_2 \in \mathbb{Z} + \tfrac{1}{2} \qquad\qquad J_1 + J_2 \in \mathbb{Z}$$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

ψ is a Fermi field $\qquad\qquad$ ψ is a Bose field

* Notice that: $A \in SL(2,\mathbb{C})$ as well as $-A \in SL(2,\mathbb{C})$

→ let $\psi'_{\alpha_1 \ldots \alpha_{2J_1}, \dot\alpha_1 \ldots \dot\alpha_{2J_2}} = A_{\alpha_1}{}^{\beta_1} \ldots A_{\alpha_{2J_1}}{}^{\beta_{2J_1}} \bar A_{\dot\alpha_1}{}^{\dot\beta_1} \ldots \bar A_{\dot\alpha_{2J_2}}{}^{\dot\beta_{2J_2}} \psi_{\beta_1 \ldots \beta_{2J_1}, \dot\beta_1 \ldots \dot\beta_{2J_2}}$

and let $A \mapsto -A$ then:

$$\underline{\psi' \longmapsto (-)^{2J_1 + 2J_2} \psi'} \Rightarrow \text{Spin-stat. theorem}$$

We can classify the representations thanks to the fact that

$$SL(2,\mathbb{C}) \longrightarrow L_+^{\uparrow}$$

is a 2:1 homomorphism: $(A,-A) \mapsto \Lambda$. Then

$$L_+^{\uparrow} \sim SL(2,\mathbb{C})/\mathbb{Z}_2$$

We usually refer to representations thanks to the fact that:

(locally, i.e.: Lie alg.) $\quad sl(2,\mathbb{C}) \sim su(2) \oplus su(2)$

then

$$\psi_\alpha : \left(\tfrac{1}{2},0\right) \qquad A_\mu : \left(\tfrac{1}{2},\tfrac{1}{2}\right)$$
$$\psi_{\dot\alpha} : \left(0,\tfrac{1}{2}\right) \qquad F_{\mu\nu} : (1,0)\ [F^+] \,/\, F_{\mu\nu} : (0,1)\ [F^-]$$

In $D=4$:

$$\left(\tfrac{1}{2},0\right)^* \longmapsto \left(0,\tfrac{1}{2}\right)$$

and in general:

$$(J_1,J_2)^* \longmapsto (J_2,J_1)$$

i.e.: if $J_1 = J_2 \Rightarrow$ the rep is REAL!

Moreover we can build Dirac spinors: $\left(\tfrac{1}{2},0;0,\tfrac{1}{2}\right)$ (and add Majorana condition if needed):

$$\psi_D = \begin{pmatrix} \psi_\alpha \\ \chi^{\dot\alpha} \end{pmatrix}$$

Majorana: self conjugated form of $\psi_D \longrightarrow \psi_D^C = \gamma^0 \bar\psi_D^T = \psi_D$

$$\Rightarrow \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\begin{pmatrix} \psi^{\dot\alpha} \\ \chi_\alpha \end{pmatrix} = \begin{pmatrix} \chi_\alpha \\ \psi^{\dot\alpha} \end{pmatrix} = \begin{pmatrix} \psi_\alpha \\ \chi^{\dot\alpha} \end{pmatrix} \Rightarrow \boxed{\psi^{\dot\alpha} = \chi^{\dot\alpha}} \Rightarrow \psi_M = \begin{pmatrix} \psi_\alpha \\ \psi^{\dot\alpha} \end{pmatrix}$$

In Majorana rep. everything is real, therefore

$$\psi_L = \frac{1+i\gamma_5}{2}\psi_M \text{ and } \psi_R = \psi_L^*.$$

Noether Currents

In general QFT:

ϵ_A continuous symmetry $\longrightarrow$ Noether Currents $\partial^\mu J_\mu^A = 0$
(Lie algebra with parameter ϵ_A) (μ: index on spacetime
A: index on the param. space)

s.t. $Q^A = \int d^3x\, J_0^A$ and $[Q^A, Q^B] = f^{AB}{}_C Q^C$.

How to write canonical currents?

$$\delta S = \int d^4x\, \delta\mathcal{L} = \int d^4x \left\{ \left(\frac{\delta \mathcal{L}}{\delta\varphi} - \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\varphi)}\right)\right)\delta\varphi + \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\varphi)}\delta\varphi\right)\right\} = \int d^4x\, \partial^\mu K_\mu$$

(internal symm)

$\hookrightarrow$ $J^\mu = \dfrac{\delta\mathcal{L}}{\delta(\partial_\mu\varphi)}\delta\varphi - K^\mu$ CANONICAL Noether current.

We can actually extend the concept. E.g.: the canonical $T_{\mu\nu}$ is not generally manifestly symm $\longrightarrow$ we can add a term

$$T_{\mu\nu} + [\partial_\mu\partial_\nu - g_{\mu\nu}\Box]\phi^2 + \epsilon_{\mu\nu\rho\lambda}\partial^\lambda(\bar{\psi}\gamma^\rho\gamma^5\psi)$$

to make it symm. The important thing is to add a term s.t.

$\partial^\mu I_\mu \equiv 0$ (the change must be identically 0 not to change the underlying physics).

e.g.: $x^\mu \mapsto x^\mu + \epsilon^\mu \Rightarrow T^\mu{}_\nu = \dfrac{\delta\mathcal{L}}{\delta(\partial_\mu\varphi)}\delta(\partial_\nu\varphi) - \delta^\mu_\nu\mathcal{L}$ is not symm, but with the improvement term might become periodic.

e.g.: the same goes for internal symm. $\Rightarrow$ $J^\mu_A = \dfrac{\delta\mathcal{L}}{\delta(\partial_\mu\varphi^i)}\left(\theta^i_A - \partial_\sigma\varphi^i\Delta^\sigma_A\right) + \Delta^\mu_A\mathcal{L}$

New look at the conformal group.

* Generators : $\{ \underbrace{P_\mu, M_{\mu\nu}}_{\text{Poincaré}}, \underset{\text{dilations}}{D}, K_\lambda \}$ $K_\lambda \to$ special conformal transf.

we can compute the Noether current:

$$J_\mu \sim \xi^\rho \theta_{\rho\mu} \quad \text{where } \xi^\rho(x) = \lambda x^\rho + \omega^\rho{}_\mu x^\mu + a^\rho + (c^\rho x^2 - 2c^{\rho\nu} x_\mu x_\nu)$$

$$\hookrightarrow \partial_\mu \xi_\nu + \partial_\nu \xi_\mu = \frac{2}{D} \eta_{\mu\nu} \partial^\lambda \xi_\lambda$$

then $\quad Q = \int d^2x \; \xi^\rho \theta_{\rho 0} \quad (D=2)$

$$\partial^\mu (\xi^\rho \theta_{\rho\mu}) = 0$$

For later use, we define the Casimir operators:

LORENTZ:
$$J_1 = M_{\mu\nu} M^{\mu\nu}$$
$$J_2 = \epsilon_{\mu\nu\rho\lambda} M^{\mu\nu} M^{\rho\lambda}$$

$$\Rightarrow 2 J_1 (J_1 + 1) \pm 2 J_2 (J_2 + 1)$$

POINCARÉ:
$$W_1 = P_\mu P^\mu$$
$$W_2 = W_\mu W^\mu, \quad \text{where } W_\mu = \epsilon_{\mu\nu\rho\lambda} M^{\nu\rho} P^\lambda \quad \text{(Pauli-Lubanski)}$$

CONFORMAL GROUP:

$$J_{AB} J^{AB} = C_1$$
$$\epsilon_{ABCDEF} J^{AB} J^{CD} J^{EF} = C_2$$
$$J^A{}_B J^B{}_C J^C{}_D J^D{}_A = C_3$$

SUSY algebra

Before we start we must define a GRADED ALGEBRA

$$[A_m, A_n] = f_{mn}{}^{l} A_l$$

$$[A_m, Q_\alpha] = f_{m\alpha}{}^{\beta} Q_\beta$$

$$\{Q_\alpha, Q_\beta\} = F_{\alpha\beta}{}^{n} A_n$$

can be recasted into:

$$[A, B\} = AB - (-)^{n_A n_B} BA$$

where n_A, n_B are the gradings:

$$n_Q = \begin{cases} 0 & \text{if } Q \text{ is a BOSONIC generator} \\ 1 & \text{if } Q \text{ is a FERMIONIC generator} \end{cases}$$

Moreover a generalized Jacobi identity holds:

$$\sum_{i,j,k} [[A_i, A_j\}, A_k\} = 0.$$

→ Lie algebra classification:

$$\left.\begin{array}{lll} A_{n-1} & : & sl(n) \\ B_n & : & so(2n+1) \\ C_n & : & sp(2n) \\ D_n & : & so(2n) \end{array}\right\} \longrightarrow \text{classical Lie alg.}$$

together with additional exceptional algebras.

	G_2	F_4	E_6	E_7	E_8
dimens.	14	52	78	133	248
adj rep.	7	26	27	56	248

→ adj and fund. rep are the same

This applies to the space where fields ϕ^i live, that is:

$$\mathcal{L}_\phi = -\sqrt{-g}\, g^{ab}(\phi)\, \partial^\mu \phi_a \partial^\nu \phi_b\, g_{\mu\nu}$$

$\downarrow$

$g_{ab}(\phi)$ is the pullback metric

→ Riem. manifolds: $Hol(g) \subset SO(n)$

→ SUSY / SUGRA ⟶ holonomy is <u>restricted</u> or exceptional

* local SUSY (SUGRA): special geometry $(n_q \leq 8)$ and G/H $(n_q > 8)$
  where $G \longrightarrow$ NON COMPACT
  $H \longrightarrow$ MAXIMAL COMPACT SUBGROUP

* rigid SUSY: KÄHLER SPACES ($U(n)$ holonomy)

SUPERGRAVITY

SUGRA ~ $U(1)$ bundle

* rigid SUSY : $\partial_i W(\phi)$
* local SUSY: $\underbrace{D_i W(\phi)}_{\text{defined on Hodge-Kähler manifold}} = (\partial_i + K_i) W(\phi)$

e.g.: $\underline{D=4 \quad N=8}$

↳ vector multiplet (VM) : A_μ, λ_L, ϕ

hypermultiplet (HM) : ϕ^I_A, χ^I_L

↳ defined over a quaternionic space

Holonomy: $SU(2) \times USp(2N) \subset SO(4)$

$\downarrow$ R-sym. index A $\quad$ $\downarrow$ counting index I

[Quaternionic spaces are Einstein spaces : $Ric_{IJ} = \lambda g_{IJ}$]

In the presence of more VM :

$$(A^i_\mu, \lambda^i_L, \phi^i) \quad i = 1, \dots, m$$

we need to define more structures, e.g.:

$D=4$ <
- $N=1 \Rightarrow$ Hodge Kähler geometry
- $N=2 \Rightarrow$ HK geom. + $\underline{\text{SPECIAL GEOMETRY}}$ (more VM)

$D=5 \longrightarrow$ real special geometry

VM : A_μ, χ, ϕ (NB: real scalar)

* the geom of ϕ-space is defined through a cubic relation of dim +1 w.r.t. the no. of fields:

$$C_{IJK} L^I(\phi) L^J(\phi) L^K(\phi) = 1$$

→ it's a hypersurface defining the relation between fields ⇒ C_{IJK} defines the fields as a Chern-Simons (CS) term in D = 5 (and they are connected to Jordan alg.)

$$C_{IJK} \int d^5x \, A^I \wedge F^J \wedge F^K .$$

There may be other geometries and possibilities:

(1,0) D = 6 SUGRA:

* Hypermultiplet (HM) → QUATERNIONIC SPACE
* VM → NO SCALAR
* Tensor multiplet (TM) → SYMMETRIC SPACE

e.g.: n_T TMs $(B^I_{\mu\nu}, \chi^I_R, \phi^I)$ need $SO(1,n_T)/SO(n_T)$

We therefore have a separation in the space:

$$\mathcal{M}_{\substack{D=4\\D=5}} = \mathcal{M}_{VEC} \times \mathcal{M}_{HM}$$

$$\mathcal{M}_{D=6} = \mathcal{M}_{TM} \times \mathcal{M}_{HM}$$

Then how to go from D=6 to D=5?

D=6		D=5	
Tensor	→	Vector	} they give restrictions on C_{IJK}
Vector		Vector	

⇒ pure D = 5 ⇒ $C_{IJK} L^I L^J L^K = 1$

reduction from D=6 to D=5 ⇒ <u>restrictions</u> on C_{IJK}

What about $n_q > 8$ and COSET SPACES?

* $D=4$ $N=4$ and n VM

  ↳ Coulomb branch : (A_μ, λ_A, ϕ)

  YM : $(A_\mu^{\wedge}, \lambda_A^{\wedge}, \phi_{[AB]})$

the content completely fixes:

* Coulomb : $\frac{SL(2,\mathbb{R})}{SO(2)} \Rightarrow \phi$ is the parameter
* YM : $\frac{SL(2,\mathbb{R})}{SO(2)} \times \frac{SO(6,n_V)}{\underbrace{SO(6)}_{} \times SO(n_V)} \Rightarrow 6$ param of $\phi_{[AB]}$

  ↓

  R-symmetry group

  $SO(6) \sim SU(4)$

  $\Rightarrow \phi_{[AB]}$ is 6 of $SU(4)$

  ⇓

  REAL REP.

{ usually it's the H in G/H without matter, but in the presence of matter there can be other terms:

no matter : $G/H \rightarrow H = H_R$

matter : $\frac{G}{H} \rightarrow H = H_R \times H_M$ }

* $D=4$ $N=3$: ϕ_i is in 3 of $SU(3) \Rightarrow$ COMPLEX REP.

  ↓

$$\frac{SU(3,n)}{\underbrace{SU(3) \times U(1)}_{H_R = U(3)\ :\ R\text{-symmetry}} \times SU(n)}$$

$\Rightarrow$ MATTER FIXES THE DYNAMICS OF SUGRA

Therefore: $D=4$

$n_q \leq 8 \Rightarrow$ "some freedom" $N=1,2$

$8 < n_q \leq 16 \Rightarrow$ SUGRA + matter (which fixes dynamics) $N=3,4$

$n_q > 16 \Rightarrow$ Pure SUGRA $N=5,6,7,8$

Since we know what the dimension of G/H (→ no. of scalars) should be as well as the R-symmetry group, then the choice of G is restricted:

$D = 4$

$N = 8$: $\dim H = 63$, $\dim G/H = 70$ $\Rightarrow \dim G = 133 \Rightarrow G/H = E_{7(7)}/SU(8)$

$N = 6$: $G/H = SO^*(12)/U(6)$ → non compact form of $SO(12)$

$N = 5$: $G/H = SU(5,1)/U(5)$

NOTE:

$SU^*(2n) \subset SL(2n, \mathbb{C}) \longrightarrow$ commutes with MM^*

$SO^*(2n) \subset SO(2n, \mathbb{C}) \longrightarrow$ commutes with $\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$

and

COMPACT	NON COMPACT FORM (and maximal comp. sub)
$SU(n)$	$SL(n, \mathbb{R}) \supset SO(n)$
$SU(p,q)$ $\cup$ $SU(p) \times SU(q) \times U(1)$	$SU^*(2(p+q)) \supset USp(2n)$
$SO(p,q) \supset SO(p) \times SO(q)$	
$SO^*(2n) \supset U(n)$	
$Sp(2n, \mathbb{R}) \supset U(n)$	
G_2	$G_{2(2)} \supset SU(2) \times SU(2)$
F_4	$F_{4(-20)} \supset SO(8)$ $F_{4(4)} \supset SU(2) \times USp(6)$
E_7	$E_{7(7)} \supset SL(2,\mathbb{R})^7$ $\supset SU(8)$ $\supset SO(1,1)^7$ $E_{7(-5)}$ $F \; (-25$

Orthogonal groups are of particular interest since they are associated to Clifford algebra $\mathcal{C}$:

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu} \Rightarrow \dim \mathcal{C} = \begin{cases} 2^D & \text{if } D \text{ even} \\ 2^{D-1} & \text{if } D\text{-}1 \text{ odd} \end{cases}$$

$\hookrightarrow$ spinor representation $\rightarrow \begin{cases} 2^{D/2} & \text{if } D \text{ even} \\ 2^{\frac{D-1}{2}} & \text{if } D \text{ odd} \end{cases}$

* CLIFFORD ALGEBRAS:

$\mathcal{C}^{\mathbb{K}}_{t,s} = \{e_M\} \mid \{e_M, e_N\} = 2\eta_{MN}$ over a field $\mathbb{K}$ where $M,N = 1,\dots,d = t+s$

$$\eta_{MN} = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & -1 & \\ & & & & \ddots \\ & & & & & -1 \end{pmatrix}$$

$\rightarrow \dim \mathcal{C}^{\mathbb{K}}_{t,s} = 2^d \Rightarrow$ basis $\mathcal{B}_{\mathcal{C}^{\mathbb{K}}} = \{\mathbb{I}, e_{\mu_1}, e_{\mu_1\mu_2}, e_{\mu_1\mu_2\mu_3}, \dots, e_{\mu_1\dots\mu_{d-1}}, e_{1\dots d}\}$

$$1 < \mu_1 < \mu_2 < \dots < \mu_{d-1} < d$$

{or $T(E) = \mathbb{K} \oplus E \oplus (E \times E) \oplus \dots$ tensor alg. over E; $\mathcal{J}(Q)$ two sided ideal $x \otimes x - Q(x)$, $x \in E$ in $T(E)$. Then $T(E)/\mathcal{J}(Q) = \mathcal{C}^{\mathbb{K}}_{t,s}$ }

$\hookrightarrow$ quadratic form on E (signature (t,s))

$\rightarrow$ matrix algebra:

- Real alg: $\mathcal{C}_{0,0} = \mathbb{R}$; $\mathcal{C}_{1,0} = \mathbb{R} \oplus \mathbb{R}$; $\mathcal{C}_{2,0} = \mathrm{Mat}_2(\mathbb{R})$;
  $\mathcal{C}_{1,1} = \mathrm{Mat}_2(\mathbb{R})$; $\mathcal{C}_{0,1} = \mathbb{C}$; $\mathcal{C}_{0,2} = \mathbb{H}$

- Complex alg: $\bar{\mathcal{C}}_d = \mathcal{C}^{\mathbb{C}}_{t,s} \sim \mathcal{C}_{t,s} \otimes_{\mathbb{R}} \mathbb{C} \Rightarrow$ complexification of $\mathcal{C}_{t,s}$

$$\bar{\mathcal{C}}_d = \begin{cases} M_{2^{\lfloor d/2 \rfloor}}(\mathbb{C}) & \text{if } d \text{ even (simple alg)} \\ M_{2^{\lfloor d/2 \rfloor}}(\mathbb{C}) \oplus M_{2^{\lfloor d/2 \rfloor}}(\mathbb{C}) & \text{if } d \text{ odd} \end{cases}$$

$\Rightarrow$ Dirac Γ matrices:

$$\gamma : \bar{\mathcal{C}}_d \longrightarrow \mathrm{Mat}_{2^k}(\mathbb{C})$$
$$e_M \longmapsto \gamma_M \quad \text{where } d = \begin{cases} 2k \\ 2k+1 \end{cases}$$

$\longrightarrow$ Clifford vacuum: $|\Omega\rangle = |-\frac{1}{2}, \dots, -\frac{1}{2}\rangle$ wrt Cartan sub of $so(2n)$

Spin representations:

$$SO(2,1) \longrightarrow SL(2,\mathbb{R})$$
$$SO(4,1) \longrightarrow USp(2,2)$$
$$SO(4) \longrightarrow Usp(2) \times Usp(2)$$
$$SO(5,1) \longrightarrow SU^*(4)$$
$$SO(5) \longrightarrow Usp(4)$$

They all enter GRADED LIE ALGEBRAS:

GLA $\rightarrow$ 2 Lie groups to form appropriate bos. rep

* $SO(N) \times Sp(2M) \sim OSp(N|M)$
"Orthosymplectic GLA"
* $SU(N) \times SU(M) \times U(1) \sim PSU(N|M)$
"Unitary GLA"

$\Rightarrow$ SUSY ALGEBRA

$D=4 \quad N=1$ (Wess-Zumino)

$$\{Q_\alpha, Q_{\dot\alpha}\} = (\sigma_\mu)_{\alpha\dot\alpha} P^\mu$$

$$[Q_\alpha, M_{\mu\nu}] = \frac{1}{2}(\sigma_{\mu\nu})_\alpha{}^\beta Q_\beta$$

$$[Q_\alpha, P_\mu] = 0$$

$\rightarrow \{Q_\alpha, P_\mu\} \subset$ SUSY : supertransl.

$$[P_\mu, P_\nu] = 0$$

$$\{Q^I_\alpha, Q_{\dot\beta J}\} = (\sigma_\mu)_{\alpha\dot\beta}\, \delta^I_J P^\mu$$

$\rightarrow$ CENTRAL EXTENSIONS:

$D=2$
$$\{Q^A_\alpha, Q^B_\beta\} = \epsilon_{\alpha\beta} Z^{[AB]} + \sigma^{\mu\nu}_{\alpha\beta} Z^{(AB)}_{\mu\nu}$$

$D=4$
$$\{Q^A_\alpha, Q^B_\beta\} = \epsilon_{\alpha\beta} Z^{[AB]} + \gamma^{\mu\nu}_{\alpha\beta} Z^{(AB)}_{\mu\nu} + \gamma^{\mu\nu\rho\sigma\tau}_{\alpha\beta} Z^{AB}_{\mu\nu\rho\sigma\tau}$$

The central ext account for extended obj couplings: membranes!

$\rightsquigarrow$ Infinitesimal transf.: $e^{\epsilon^A Q_A} = \mathbb{I} + \epsilon^A Q_A + O(\epsilon^2)$

$\hookrightarrow$ Noether current: $\bar{\epsilon}^\alpha J_{\mu\alpha} \Rightarrow Q_\alpha = \int d^{D-1}x \, J_{0\alpha}(x,t)$

VECTOR-SPINOR CURRENT

Improved supercurrent:

$$\partial^\mu J_\mu{}^\alpha = 0$$

$$J_\mu{}^\alpha \gamma^\mu_{\alpha\beta} = 0$$

then $\exists I_\mu = x_\nu \gamma^\nu_{\alpha\beta} J_\mu{}^\alpha \rightarrow \partial^\mu I_\mu = \gamma^\mu J_\mu = 0$

$\hookrightarrow$ Superconformal invariance

Why extended objects?

EM : $\partial^\mu F_{\mu\nu} = J_\nu \Rightarrow Q \sim J_0$ (locally) $\Rightarrow$ NO INDICES

GENERAL: $\partial^{\mu_1} H_{\mu_1 \dots \mu_{p+2}} = J_{\mu_2 \dots \mu_{p+2}} \Rightarrow Q_{\mu_3 \dots \mu_{p+2}} \sim J_{0\mu_3 \dots \mu_{p+2}} \Rightarrow p$ INDICES

$$\downarrow$$

$$Z_{\mu_1 \dots \mu_p}$$

$\hookrightarrow Z^{AB} = \epsilon^{AB} Z \Rightarrow |Z| = M^E_{BH}$ (extremal BH in SUGRA $\rightarrow$ BPS)

Connection? $\Rightarrow$ DYONS (dual magn obj) with charge (e, p)

$p = \frac{D}{2} - 2$ (p = dim of ext. obj.)

(e_Λ, m^Λ) $\cdot (e'_\Lambda, m'^\Lambda)$ $\Rightarrow e_\Lambda m'^\Lambda - e'_\Lambda m^\Lambda \propto 2\pi n \longrightarrow Sp(2N, \mathbb{Z})$ for $p = 1, 3$

$\Lambda = 1, \dots, N$

$e_\Lambda m'^\Lambda + e'_\Lambda m^\Lambda \propto 2\pi n \longrightarrow SO(N,N)$ for $p = 0, 2$

* DIMENSIONAL REDUCTION arguments

⇒ definitions of SUGRA in D dim. must be recovered from KK reduction:

$$\mathcal{N}=1 \quad D=11 \Rightarrow 2^{\frac{11-1}{2}} = 2^5 = 32 \text{ supersymm. charges}$$

$\downarrow$ KK red.

$$\mathcal{N}=8 \quad D=4 \Rightarrow 8 \cdot 2^{\frac{4}{2}} = 8 \cdot 4 = 32 \text{ supersymm. charges}$$

the no. of B dofs and F dofs must be preserved:

$D=11$

$$\left.\begin{array}{ll} \psi_{\mu\alpha} \Rightarrow 2^{\frac{11-3}{2}} \cdot (11-3) = 128 \text{ F} & \\ g_{\mu\nu} \Rightarrow \frac{(11-2)(11-1)}{2} - 1 = 44 \text{ B} & \\ A_{\mu\nu\rho} \Rightarrow \frac{(11-2)(11-3)(11-4)}{3!} = 84 \text{ B} & \end{array}\right\} 128 \text{ B}$$

$D=4$

$$\psi_{\hat{\mu}\alpha}, \ \chi_{I\alpha} \quad (\hat{\mu}=0,\dots,3;\ I=4,\to 10)$$

$$\begin{array}{ccc} \downarrow & & \downarrow \\ 2\times 8 & & 2\times(7\cdot 8) \\ \| & & \| \\ 16 \text{ F} & + & 112 \text{ F} = 128 \text{ F} \end{array}$$

$$g_{\hat{\mu}\hat{\nu}}, \ g_{\hat{\mu}I}, \ g_{IJ}$$

$$2\text{B} + 28\text{B} + 14\text{B} = 44\text{B}$$

$$A_{\hat{\mu}\hat{\nu}\hat{\rho}}, \ A_{\hat{\mu}\hat{\nu}I}, \ A_{\hat{\mu}IJ}, \ A_{IJK}$$

$$1\text{B} + 7\text{B} + 21\text{B} + 35\text{B} = 84\text{B}$$

(non dyn) (dual to scal)

⇒ 28 vectors $(g_{\hat{\mu}I}, A_{\hat{\mu}IJ}) \rightarrow SO(8)$

70 scalars $(g_{IJ}, A_{\mu IJ}, \widetilde{A_{\mu\nu I}})$

⇒ $1(2)$ $8(\frac{3}{2})$ $28(1)$ $56(\frac{1}{2})$ $70(0)$ ⇒ $SU(8)$ <u>e.o.m. only</u> invar.

* DIMENSIONAL REDUCTION on CY_3 MANIFOLD

Consider (1,1) and (2,0) D=10 SUGRA to D=4 CY_3

→ CY manifold ⇒ Hodge no.: $h_{(p,q)}$ → $h_{(3,0)} = h_{(0,3)} = 1$

$h_{(1,1)} = h_{(2,2)}$

$h_{(1,2)} = h_{(2,1)}$

$h_{(0,0)} = h_{(3,3)} = 1$

Surviving massless modes depend on $h_{(p,q)}$

$(1,1)$: $g_{\mu\nu}$, $B_{\mu\nu}$, ϕ, A_μ, $A_{\mu\nu\rho}$ (type IIA)

$(2,0)$: " " ", $C_{\mu\nu}$, $C^{SD}_{\mu\nu\rho\lambda}$, C (type IIB)

- Type IIA:

	MODE	ASSOCIATED $h_{(p,q)}$	In the spectrum?
	g_{ij}	$h_{(1,1)}$	✓
$g_j^{\bar k \bar l} =$	$\Omega^{\bar k \bar l \bar i} g_{ij}$	$h_{(2,2)}$	✓
	A_i	$h_{(1,0)}$	×
	A_μ		✓
	$A_{\mu\nu i}$	$h_{(2,0)}$	×
	$A_{\mu i \bar j}$	$h_{(1,1)}$	✓
	$B_{i\bar j}$	$h_{(1,1)}$	✓
	$B_{\mu\nu}$	dual to scalar	✓
	$A_{i\ell\bar k}$	$h_{(2,1)}$	✓
	A_{ijk}	$h_{(3,0)}$	✓
	$A_{\bar i \bar j \bar k}$	$h_{(0,3)}$	✓

+ the same for type IIB. Therefore:

IIA: $h_{11} \times VM\,(A_{\mu i\bar j},\ g_{i\bar j},\ b_{i\bar j})$

$h_{12}+1 \times HM\,(A_{ik\bar l},\ g_{ij}) + (A_{ijk},\ B_{\mu\nu},\ \phi)$ "universal hypermult."

IIB: $h_{12} \times VM$

$h_{11}+1 \times HM$

$\Rightarrow \quad h_{11} \longleftrightarrow h_{12} \Rightarrow$ IIA $\longleftrightarrow$ IIB : MIRROR SYMMETRY!

i.e.: IIA on $CY_3(h_{11}, h_{12})$ = IIB on $CY_3'(h_{12}, h_{11})$

NB: many field forms can be dualized to scalars
$\longrightarrow$ QUATERNIONIC MANIFOLDS

* LAGRANGIANS :

$$\mathcal{L} = -\frac{\sqrt{-g}}{2} R - \sqrt{-g}\, g_{xy}\, \partial_\mu \varphi^x \partial_\nu \varphi^y g^{\mu\nu} + g_{\Lambda\Sigma} H^\Lambda \wedge * H^\Sigma + \underbrace{\theta_{\Lambda\Sigma} H^\Lambda \wedge H^\Sigma}_{\text{if } \exists \text{ dyons}}$$

$$- \frac{1}{2}\sqrt{-g}\, \psi_\mu \gamma^{\mu\nu\rho} D_\nu \psi_\rho - g_{AB} \bar{\chi}^A \gamma^\mu \partial_\mu \chi^B \sqrt{-g} \quad (\text{+ CS term in odd dim})$$

NB: $\gamma^\mu = e^{\mu a} \gamma_a$ (vielbein)

$\Rightarrow$ Define $\mathcal{N}_{\Lambda\Sigma} = \theta_{\Lambda\Sigma} + i\, g_{\Lambda\Sigma}$ ($\mathrm{Im}\, \mathcal{N} < 0$) transf. $Sp(2N, \mathbb{R}) / U(N)$

$$\Rightarrow \mathcal{N}' = (A + B\mathcal{N})^{-1}(C + D\mathcal{N})$$

s.t.: $A^T C$, $B^T D$ symm and $A^T D - C^T B = \mathbb{1}$

NB: $N = 8$ SUGRA $\longrightarrow E_{7(-7)}$ in $56 \Rightarrow \mathcal{N} \to E_{7(7)}/SU(8)$

$$\mathcal{N}' \sim \underset{\text{transl.}}{c} - \underbrace{a^T \mathcal{N} - \mathcal{N} a}_{\text{rotat.}} - \mathcal{N} \underset{\text{bilinear}}{b} \mathcal{N}$$

- $b = 0 \Rightarrow$ lower triang. subgr. ($\mathcal{L}$ inv. up to a total derivative)
  $\hookrightarrow$ ELECTRIC SUBGROUP $G_e \subset G$

SUSY BREAKING

$W_i = \partial_i W$

* $Q_\alpha |0\rangle \neq 0$, $V = W_i \bar{W}_{\bar{j}} \delta^{i\bar{j}} > 0$

$\rightarrow$ if $\exists i \mid \partial_i W \neq 0 \Rightarrow$ VEV $\Rightarrow$ SUSY BREAKING

$\Rightarrow \partial_j V = W_{ij} \bar{W}_{\bar{k}} \delta^{i\bar{k}} = 0$

if $\exists W_i \neq 0 \Rightarrow \det[W_{ij}] \neq 0 \Rightarrow \exists$ GOLDSTINO!