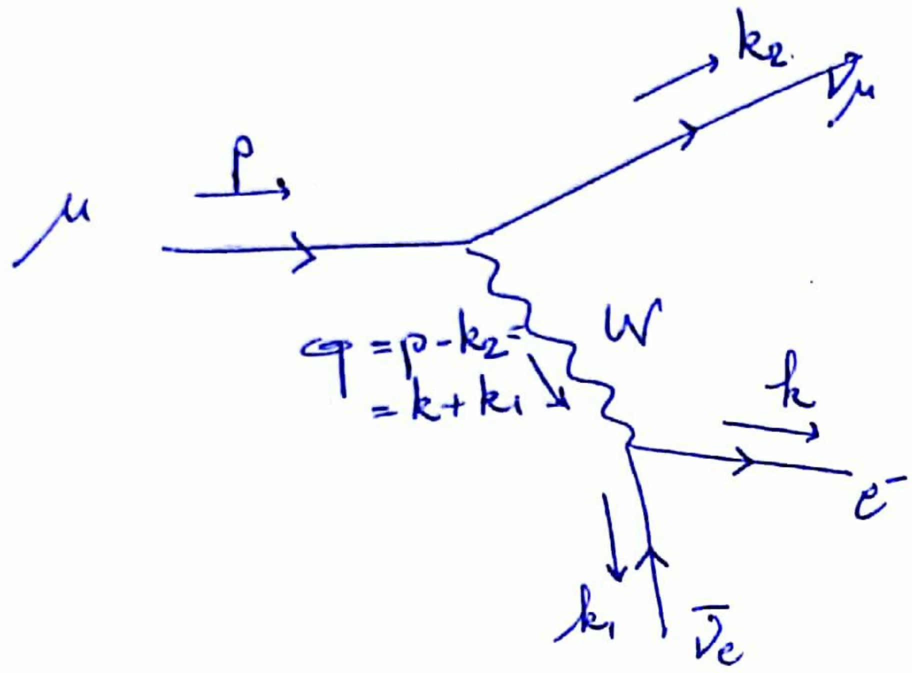
# MUON DECAY

$$\mu^-(p) \longrightarrow e^-(k) + \bar{\nu}_e(k_1) + \nu_\mu(k_2)$$

$$\mathcal{L}_c = \frac{g}{\sqrt{2}}\, \bar{\nu}_{eL}\, \gamma_\mu\, e_L\, W^{+\mu} + h.c.$$




$$iM = \frac{ig}{2\sqrt{2}}\, \bar{u}(k_2)\,\gamma^\mu(1-\gamma_5)\,u(p) \cdot (-i)\, \frac{g_{\mu\nu} - \frac{q_\mu q_\nu}{m_W^2}}{q^2 - m_W^2} \cdot \frac{ig}{2\sqrt{2}}\, \bar{u}(k)\,\gamma^\nu(1-\gamma_5)\,v(k_1)$$

i) $q^2 = (p-k_2)^2 = m_\mu^2 - 2m_\mu|\vec{k}_2| \leq m_\mu^2 \ll m_W^2$

$\hookrightarrow$ rest frame of the decaying muon

$\Rightarrow$ we can neglect $q^2$ in the denominator

ii) $\underset{(k+k_1)_\nu}{q_\nu}\, \bar{u}(k)\,\gamma^\nu(1-\gamma_5)\,v(k_1) = \overbrace{\bar{u}(k)\,\not{k}}^{m_e \bar{u}(k)}\,(1-\gamma_5)\,v(k_1) +$

$+\, \bar{u}(k)(1+\gamma_5)\underbrace{\not{k}_1 v(k_1)}_{= -m_{\nu_e} v(k_1)} \simeq 0 \longrightarrow$ neglect $m_e$ and $m_{\nu_e}$

$$\Rightarrow M = \underbrace{\frac{g^2}{8m_W^2}}_{=\frac{G_F}{\sqrt{2}}}\, \bar{u}(k_2)\,\gamma^\mu(1-\gamma_5)\,u(p) \cdot \bar{u}(k)\,\gamma_\mu(1-\gamma_5)\,v(k_1)$$

$$\Rightarrow \sum_{pol} |M|^2 = \frac{G_F^2}{2} \sum_{pol} \bar{u}(k_2)\,\gamma^\mu(1-\gamma_5)\,u(p)\,\bar{u}(k)\,\gamma_\mu(1-\gamma_5)\,v(k_1) \cdot \bar{u}(p)\,\gamma^\nu(1-\gamma_5)\,u(k_2)\,\bar{v}(k_1)\,\gamma_\nu(1-\gamma_5)\,u(k)$$

Now use $\sum_\lambda u(p,\lambda)\,\bar{u}(p,\lambda) = \not{p} + m$, etc.

Then

$$\sum_{pol} |M|^2 = \frac{G_F^2}{2} \sum_{pol} \left[ \bar{u}(k_2)\gamma^\mu(1-\gamma_5)u(p)\bar{u}(p)\gamma^\nu_\mu(1-\gamma_5)u(k_2) \right] \cdot$$

$$\cdot \left[ \text{same for the others} \right] =$$

never contributes...

$$= \frac{G_F^2}{2} \mathrm{Tr}\left[ \gamma^\mu(1-\gamma_5)(\not{p} + m_\mu)\gamma^\nu_\mu(1-\gamma_5)\not{k}_2 \right] \cdot$$

$$\cdot \mathrm{Tr}\left[ \gamma_\mu(1-\gamma_5)\not{k}_1 \gamma_\nu (1-\gamma_5)\not{k} \right] =$$

$$= \frac{G_F^2}{2} 2\mathrm{Tr}\left[ \gamma^\mu \cancel{(1-\gamma_5)} \not{p} \gamma^\nu_\mu \not{k}_2 (1+\gamma_5) \right] \cdot 2\mathrm{Tr}\left[ \gamma_\mu \cancel{(1-\gamma_5)} \not{k}_1 \gamma_\nu \not{k} (1+\gamma_5) \right]$$

NB: $\mathrm{Tr}\left[ \gamma^\mu \not{p} \gamma^\nu_\mu \not{k}_2 (1-\gamma_5) \right] = 4\left( p^\mu k_2^\nu - g^{\mu\nu} p\cdot k_2 + p^\nu k_2^\mu + i\varepsilon^{\mu\alpha\nu\beta} p_\alpha k_{2\beta} \right)$

$$\rightarrow \sum_{pol} |M|^2 = 128\, G_F^2 (p\cdot k_1)(k_2 \cdot k)$$

unpolarized initial muon

$$\hookrightarrow d\Gamma = \frac{1}{2m_\mu} \sum_{pol} |M|^2 \cdot \frac{1}{2} \underbrace{d\phi_3(p; k_1, k_2, k)}_{\text{3-body phase space}}$$

$$\rightarrow d\phi_3 = \frac{d^3k}{(2\pi)^3 2k^0} \frac{d^3k_1}{(2\pi)^3 2k_1^0} \frac{d^3k_2}{(2\pi)^3 2k_2^0} (2\pi)^4 \delta^4(p-k-k_1-k_2)$$

→ we want the diff. rate as a function of the energy of the electron (i.e. $\frac{d\Gamma}{dk}$)

$$I^{\alpha\beta}(K) = \int \frac{d^3k_1}{k_1^0} \frac{d^3k_2}{k_2^0} \underbrace{k_1^\alpha k_2^\beta}_{\text{because of } \sum_{pol}|M|^2} \delta^4(K-k_1-k_2) = A K^\alpha K^\beta + B g^{\alpha\beta} K^2$$

- $k_1 \cdot k_2 = \frac{1}{2}(k_1+k_2)^2 = \frac{K^2}{2}$
- $(k\cdot k_1)(k\cdot k_2) = \frac{K^4}{4}$

i) $g_{\alpha\beta} I^{\alpha\beta} = \int \frac{d^3k_1}{k_1^0}\frac{d^3k_2}{k_2^0} \underbrace{k_1\cdot k_2}_{\cancel{k^2}/2}\, \delta^4(k-k_1-k_2) =$

$= A\cancel{k^2} + 4B\cancel{k^2}$

$\Rightarrow \quad I(k) = \int \frac{d^3k_1}{k_1^0}\frac{d^3k_2}{k_2^0}\, \delta^4(k-k_1-k_2)$

$\hookrightarrow \frac{I}{2} = A + 4B$

ii) $k_\alpha k_\beta I^{\alpha\beta} = \int \frac{d^3k_1}{k_1^0}\frac{d^3k_2}{k_2^0} \underbrace{(k\cdot k_1)(k\cdot k_2)}_{\cancel{k^4}/4}\, \delta^4(k-k_1-k_2) =$

$= A\cancel{k^4} + B\cancel{k^4}$

$\Rightarrow \quad \frac{I}{4} = A + B$

$\rightarrow \begin{cases} A = \frac{I}{6} \\ B = \frac{I}{4} \end{cases}$

Now consider back-to-back scott. : $\boxed{\vec{k}_1 + \vec{k}_2 = 0}$

$I(k) = \int \frac{d^3k_1}{k_1^0 k_2^0}\, \delta(k^0 - k_1^0 - k_2^0) \int d^3k_2\, \delta^3(\cancel{\vec{k}} - \vec{k}_1 - \vec{k}_2) =$

$\overset{|\vec{k}_1| = |\vec{k}_2|}{=} \int \frac{4\pi |\vec{k}_1|^2\, d|\vec{k}_1|}{\cancel{k_1}\, \cancel{k_2}}\, \delta(k^0 - 2|\vec{k}_1|) = 4\pi \cdot \frac{1}{2} = 2\pi$

$\Rightarrow \begin{cases} A = \frac{\pi}{3} \\ B = \frac{\pi}{6} \end{cases}$

$\rightarrow \Gamma = \frac{1}{4m_\mu}\, \underset{128}{2^7}\, G_F^2 \frac{1}{(2\pi)^9}\, \frac{1}{2^3}\, (2\pi)^4 \int \frac{d^3k}{k^0}\, p_\alpha k_\alpha I^{\alpha\beta}(k) =$

$= \frac{G_F^2}{8\pi^5}\, \frac{1}{m_\mu}\, \frac{\pi}{3} \int \frac{d^3k}{k^0}\left[(p\cdot \underbrace{(p-k)}_{k})(p\cdot k - k^2) + \frac{1}{2}(p^2 + k^2 - 2p\cdot k)\right] =$

$$= \frac{G_F^2}{24\pi^4}\frac{1}{m_\mu}\int\frac{d^3k}{k^0}\left(\frac{3}{2}m_\mu^3 k_0 - 2m_\mu^2 k_0^2\right) =$$

$$= \frac{G_F^2}{\cancel{12}4\pi^3}\frac{1}{m_\mu}\int_0^{?} dE\, E^2\left(3m_\mu^2 - 4m_\mu E\right)$$

$$\begin{cases} m_\mu = |\vec{k}| + |\vec{k}_1| + |\vec{k}_2| \\ 0 = \vec{k} + \vec{k}_1 + \vec{k}_2 \end{cases} \rightarrow |\vec{k}|^2 = |\vec{k}_1|^2 + |\vec{k}_2|^2 + 2|k_1||k_2|\cos\vartheta$$

$$\Downarrow$$

max energy:

$$m_\mu = 2|\vec{k}|_{max} = 2E_{max}$$

$$= \frac{G_F^2}{\cancel{12}4\pi^2}\cancel{\frac{1}{m}}\int_0^{m_\mu/2} dE\, E^2\left(3m_\mu^2 - 4m_\mu E\right) =$$

$$= \frac{G_F^2 m_\mu^5}{192\pi^3} \simeq \left(2.19\times 10^{-6}\,s\right)^{-1}$$

$$\rightarrow \tau_\mu^{(exp)} = (2.1969811 \pm 0.0000022)\,s$$

(include $O\left(\left(\frac{m_e}{m_\mu}\right)^2\right)$ to get better results)

# PERTURBATIVE UNITARITY (Fermi Theory)

$$S^\dagger S = \mathbb{I} \,, \qquad S = \mathbb{I} + iT$$

$$\rightarrow \boxed{T^\dagger T = -i\left(T - T^\dagger\right)}$$

$$\langle a | T^\dagger T | b \rangle = -i\left(\langle a|T|b\rangle - \langle a|T^\dagger|b\rangle\right)$$

we define

$$\langle f | T | a \rangle = M_{af} \, (2\pi)^4 \, \delta^{(4)}(p_a - p_f)$$

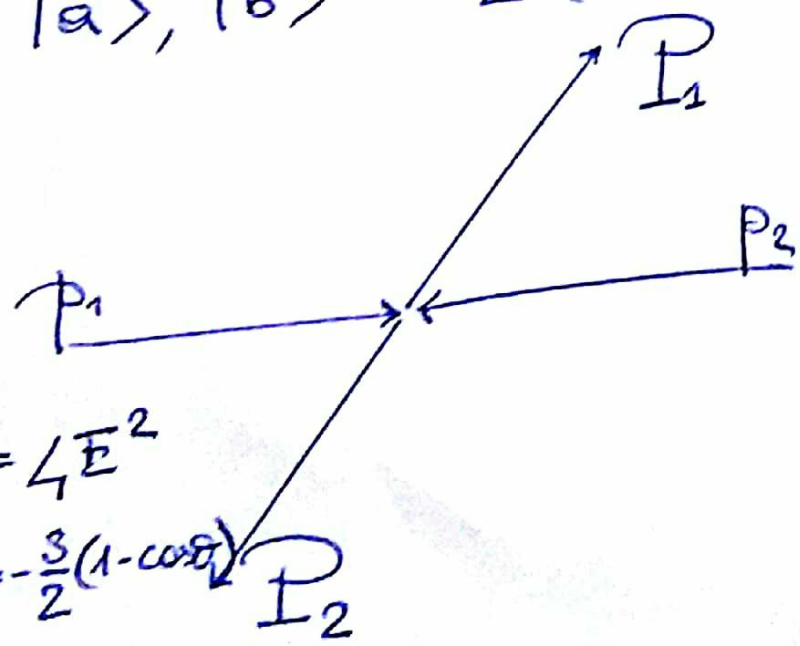
$$\mathbb{I} = \sum_f \prod_i \int \frac{d^3 p_i^f}{(2\pi)^3 2E_i^f} |f\rangle\langle f|$$

$$\Rightarrow \sum_f \prod_i \int \frac{d^3 p_i^f}{(2\pi)^3 2E_i^f} M_{af}^* (2\pi)^4 \delta^{(4)}\Big(p_a - \sum_i p_i^f\Big) M_{bf} (2\pi)^4 \delta^{(4)}\Big(p_b - \sum_i p_i^f\Big)$$

$$= -i\left(M_{ba} - M_{ab}^*\right)(2\pi)^4 \delta^{(4)}(p_a - p_b)$$

$$\Rightarrow \sum_f \prod_i \left[\int \frac{d^3 p_i^f}{(2\pi)^3 2E_i^f}\right] (2\pi)^4 \delta^{(4)}\Big(p_a - \sum_i p_i^f\Big) |M_{af}|^2 = 2\,\mathrm{Im}\, M_{aa}$$

Suppose $|a\rangle, |b\rangle$ 2-(massless) particle state




$$p_1 = (E, 0, 0, E)$$
$$p_2 = (E, 0, 0, -E)$$
$$P_1 = (E, E\sin\theta, 0, E\cos\theta)$$
$$P_2 = (E, -E\sin\theta, 0, -E\cos\theta)$$

$$s = (p_1 + p_2)^2 = 4E^2$$
$$t = (p_1 - P_1)^2 = -\frac{s}{2}(1 - \cos\theta)$$

For a fixed value of $s$, $M(s,t)$ of such scattering is just a function of $\underline{\underline{\cos\theta}}$ $\to$ partial wave expansion: ② EW

$$M(s,t) = 16\pi \sum_J (2J+1)\, a_J(s)\, P_J(\cos\theta)$$

$$\begin{cases} \int_{-1}^{+1} dz\, P_J(z) P_K(z) = \frac{2}{2J+1}\, \delta_{JK} \\ P_J(1) = 1 \end{cases}$$

$$\to a_J(s) = \frac{1}{32\pi} \int_{-1}^{+1} d\cos\theta\, P_J(\cos\theta)\, M(s,t)$$

OPTICAL THEOREM:

$$\text{LHS}: \int \frac{d^3 p_1'}{(2\pi)^3 2E_1}\, \frac{d^3 p_2'}{(2\pi)^3 2E_2}\, (2\pi)^4 \delta^4(p_1+p_2-p_1'-p_2')\, |M|^2 =$$

$$= \frac{1}{16\pi} \int_{-1}^{+1} d\cos\theta\, |M(s,t)|^2 =$$

$$= \frac{1}{16\pi} \int_{-1}^{+1} d\cos\theta \left[16\pi \sum_J (2J+1)\, a_J(s)\, P_J(\cos\theta)\right]\left[16\pi \sum_K (2K+1)\, a_K(s) P_K(\cos\theta)\right]^*$$

$$= 32\pi \sum_J (2J+1)\, |a_J(s)|^2 \leq$$

$$\leq 2\, \text{Im}\, M(s, \underset{\text{fwd scatt.}}{0}) = 32\pi \sum_J (2J+1)\, \text{Im}\, a_J(s)\, P_J(1)$$

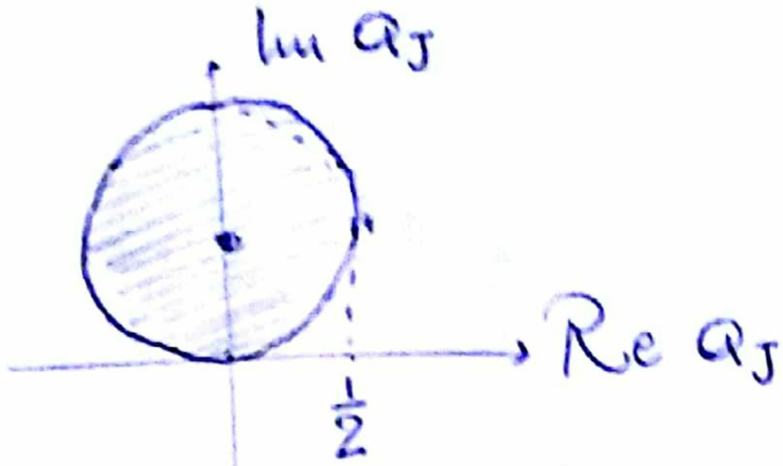
$\hookrightarrow$ only 2 particle, elastic scatt. states

$$\Rightarrow \quad \sum_J (2J+1)\, |a_J(s)|^2 \leq \sum_K (2K+1)\, \text{Im}\, a_K(s)$$

$\rightarrow \underline{|a_J|^2 \leq \mathrm{Im}\, a_J}, \quad \forall J$

$\hookrightarrow (\mathrm{Re}\, a_J)^2 + (\mathrm{Im}\, a_J)^2 \leq \mathrm{Im}\, a_J$

$\rightarrow (\mathrm{Re}\, a_J)^2 + \left(\mathrm{Im}\, a_J - \frac{1}{2}\right)^2 \leq \frac{1}{4}$



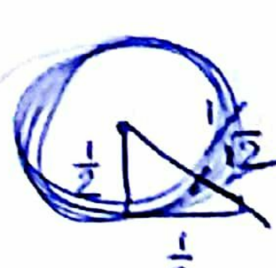

FULL THEORY $\rightarrow$ boundary
PERT. THEORY $\rightarrow$ unitarity bound

$$\downarrow$$

$$\boxed{|\mathrm{Re}\, a_J| < \frac{1}{2}}$$

$\rightarrow$ compute the amount of correction we need not to break pert. th.

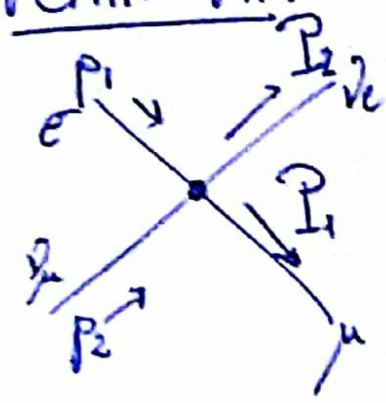
e.g. suppose $\mathrm{Re}\, a_J = \frac{1}{2}$




$\Delta = \frac{1}{\sqrt{2}} - \frac{1}{2} = \frac{1}{2}(\sqrt{2} - 1)$

~40% correction

eg $\underline{\text{Fermi th:}}$




$$\sum_{pol} |M|^2 = 32\, G_F^2\, s^2$$

$$M(s,t) = 16\pi\left(a_0(s) + 3a_1(s)\cos\theta + \dots\right)$$

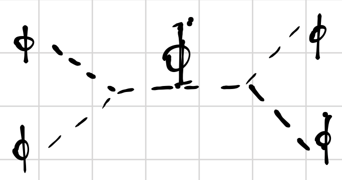
$$\rightarrow \cancel{a_0(s) =} \sum_{pol} |M|^2 = (16\pi)^2 |a_0|^2$$

$$\rightarrow a_0 = \frac{1}{16\pi}\sqrt{32}\, G_F\, s = \frac{G_F s}{2\sqrt{2}\pi} = |\mathrm{Re}\, a_0| < \frac{1}{2} \rightarrow \sqrt{s} < \sqrt{\frac{\sqrt{2}\pi}{G_F}}$$

$\sim 620$G

What happens to the unitarity bound at a resonance?

Suppose a trilinear interaction: $(m_\phi \ll m_\Phi)$



$$\mathcal{M} \overset{\sqrt{s}\gg m_\phi}{\sim} A^2 \frac{1}{s-m_\Phi^2} \Rightarrow a_0 \sim \frac{A^2}{16\pi}\frac{1}{s-m_\Phi^2}$$

$|\mathrm{Re}\, a_0| < \frac{1}{2} \Rightarrow (\sqrt{s}=m_\Phi)\ A=0$ → we can evaluate how far from the pole we should be in order not to come across this problem!

Consider the Dyson resummed prop.:

$$\frac{i}{s-M_\Phi^2}\left[1-(i\Pi(s))\frac{i}{s-M_\Phi^2}+\dots\right] = \frac{i}{s-M_\Phi^2-\Pi(s)}$$

$$\hookrightarrow \Pi(s=M_\Phi^2) = \underbrace{\mathrm{Re}\,\Pi(M_\Phi^2)}_{=0} + i\,\mathrm{Im}\,\Pi(M_\Phi^2) = -i\, M_\Phi \Gamma_\Phi$$

optical th. for decaying particle

$$\rightsquigarrow \mathcal{M} \sim A^2 \frac{1}{s-M_\Phi^2+iM_\Phi\Gamma_\Phi}$$

take relative error:

$$\frac{\left|\frac{i}{s-M_\Phi^2}\right|^2 - \left|\frac{i}{s-M_\Phi^2+iM_\Phi\Gamma_\Phi}\right|^2}{\left|\frac{i}{s-M_\Phi^2}\right|^2} < \Delta \quad (\text{e.g.: } \Delta = 10\%)$$

$$\Rightarrow |s-M_\Phi^2| > \sqrt{\frac{1}{\Delta}-1}\ \Gamma_\phi M_\phi \rightsquigarrow \text{"stay away from the critical region"}$$