# Quantum Field Theory in Curved Spacetime

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#### Abstract

We review the mathematically rigorous formulation of the quantum theory of a linear field propagating in a globally hyperbolic spacetime. This formulation is accomplished via the algebraic approach, which, in essence, simultaneously admits all states in all possible (unitarily inequivalent) Hilbert space constructions. The physically nonsingular states are restricted by the requirement that their two-point function satisfy the Hadamard condition, which insures that the ultra-violet behavior of the state be similar to that of the vacuum state in Minkowski spacetime, and that the expected stress-energy tensor in the state be finite. We briefly review the Unruh and Hawking effects from the perspective of the theoretical framework adopted here. A brief discussion also is given of several open issues and questions in quantum field theory in curved spacetime regarding the treatment of "back-reaction", the validity of some version of the "averaged null energy condition", and the formulation and properties of quantum field theory in causality violating spacetimes.

### 1 Introduction

The subject of quantum field theory in curved spacetime is the study of the behavior of quantum fields propagating in a classical gravitational field. It is used to analyze phenomena where the quantum nature of fields and the effects of gravitation are both important, but where the quantum nature of gravity itself is assumed not to play a crucial role, so that gravitation can be described by a classical, curved spacetime, as in the framework of general relativity.

The main initial development of the theory occurred in the late 1960's, driven primarily by the desire to analyze the phenomenon of particle creation occurring in the very early universe. By 1969, one can find the theory formulated in recognizably modern form and applied to cosmology in the paper of Parker [1]. In the early 1970's, the theory was applied to the study of particle creation near rotating and charged black holes, where the discovery of classical "superradiant scattering" (analogous to stimulated emission) strongly suggested that spontaneous particle creation should occur. This line of research culminated in the analysis by Hawking of particle creation resulting from the gravitational collapse of body to form a black hole [2]. It thereby was discovered that black holes radiate as perfect black bodies at temperature  $T = \kappa/2\pi$ , where  $\kappa$  denotes the surface gravity of the black hole. This result solidified an undoubtedly deep relationship between the laws of black hole physics and the laws of thermodynamics, the ramifications of which continue to be pondered today.

As a direct consequence of Hawking's remarkable discovery, there occurred in the mid-to-late 1970's a rapid and extensive development of the theory of quantum fields in curved spacetime and its applications to a variety of phenomena. A good summary of this body of work can be found in the monograph of Birrell and Davies [3]. Further important applications to cosmology were made in the early 1980's, as the methods and results of quantum field theory in curved spacetime were used to calculate the perturbations generated by quantum field fluctuations during inflation. Many of these lines of investigation begun in the late 1970's and early 1980's continue to be pursued today.

Although it would be more difficult to point to major historic landmarks, during the

past twenty years the theoretical framework of quantum field theory in curved spacetime has undergone significant development, mainly through the incorporation of key aspects of the algebraic approach to quantum field theory. As a result, the theory of a linear quantum field propagating in a globally hyperbolic spacetime can be formulated in an entirely mathematically rigorous manner insofar as the definition of the fundamental field observables is concerned.

My main goal here is to review the key developments leading to a mathematically rigorous formulation of quantum field theory in curved spacetime. (Much more detail can be found in my recent book on this subject [4], to which I refer the reader for a more comprehensive and pedagogically oriented discussion.) I will also briefly review some open issues and questions regarding the treatment of "back-reaction", positivity properties of the expected stress-energy tensor, and the formulation and properties of quantum field theory in causality violating spacetimes. Notational conventions follow those of [5].

# 2 The Formulation of Quantum Field Theory in Curved Spacetime

In the classical mechanics of a system with n degrees of freedom, the state of a system at any instant of time is described by a point in phase space,  $\mathcal{M}$ , which has the structure of a 2n-dimensional symplectic manifold, i.e., on  $\mathcal{M}$  is defined a non-degenerate, closed two-form  $\Omega_{ab}$ , referred to as a symplectic form. Most commonly,  $\mathcal{M}$  is obtained as the cotangent bundle of an n-dimensional configuration manifold  $\mathcal{Q}$ , in which case  $\Omega_{ab}$  is given by

$$\Omega_{ab} = \sum_{\mu=1}^{n} 2\nabla_{[a} p_{|\mu|} \nabla_{b]} q^{\mu} \tag{1}$$

where  $(q^{\mu}, p_{\mu})$  denote (local) canonical coordinates on the cotangent bundle. (The 2-form  $\Omega$  is then independent of the choice of these coordinates.) An observable in classical mechanics is simply a real-valued function on  $\mathcal{M}$ .

This basic structure of the classical description of a system stands in marked contrast

to the corresponding quantum description. In quantum mechanics, the state of a system at a given instant of time is described by a vector (or, more generally, a density matrix) in an infinite-dimensional, separable Hilbert space  $\mathcal{F}$ . An observable is a self-adjoint operator on  $\mathcal{F}$ . Since all infinite-dimensional, separable Hilbert spaces are isomorphic to each other, the content of a quantum theory corresponding to a given classical theory is completely specified by giving a map  $\hat{\ }$ :  $\mathcal{O}_c \to \mathcal{O}_q$ , where  $\mathcal{O}_c$  denotes the set of classical observables (i.e., real valued functions on  $\mathcal{M}$ ), and  $\mathcal{O}_q$  denotes the set of quantum observables (i.e., self-adjoint operators on  $\mathcal{F}$ ).

Since the structures of the classical and quantum theories are so different, it is far from obvious how the map  $\hat{}$  is to be determined. However, a key guiding principle arises from the comparison of the classical and quantum dynamics. In classical mechanics with Hamiltonian H, the rate of change of an observable, f, in the "Heisenberg representation" is given by

$$df/dt = \{f, H\} \tag{2}$$

where  $\{,\}$  denotes the Poisson bracket, defined by

$$\{f,g\} = \Omega^{ab} \nabla_a f \nabla_b g \tag{3}$$

where  $\Omega^{ab}$  is the inverse of  $\Omega_{ab}$ . On the other hand, in quantum mechanics with Hamiltonian  $\hat{H}$ , the rate of change of an observable,  $\hat{f}$ , in the Heisenberg representation is given by

$$d\hat{f}/dt = -i[\hat{f}, \hat{H}] \tag{4}$$

Consequently, there will be a close correspondence between classical and quantum dynamics (for any choice of Hamiltonian) if the map, ^, can be chosen so as to satisfy the "Poisson bracket goes to commutator" rule:

$$[\hat{f}, \hat{g}] = i\{\widehat{f, g}\}\tag{5}$$

In fact, it is well known that even in standard Schrodinger quantum mechanics, no map  $\hat{}$  exists which implements the relation (5) on all observables (see, e.g., [6]). However, the relation (5) can be implemented on a restricted class of observables. In particular, consider the case where  $\mathcal{M}$  has the structure of a symplectic vector space,

i.e.,  $\mathcal{M}$  is a vector space and the symplectic form  $\Omega_{ab}$  has constant components in a globally parallel basis, so that it may be viewed as an antisymmetric, bilinear map  $\Omega: \mathcal{M} \times \mathcal{M} \to \mathbb{R}$  rather than a tensor field on  $\mathcal{M}$ . (This situation arises whenever  $\mathcal{M}$  is the cotangent bundle of a configuration space  $\mathcal{Q}$  which has vector space structure and  $\Omega$  is defined by eq.(1).) In this case, we will refer to the 2n-dimensional vector space of linear functions on  $\mathcal{M}$  as the fundamental classical observables, since any classical observable (i.e., function on  $\mathcal{M}$ ) can be expressed as a function of the 2n elements of a basis for the linear observables. If we restrict attention to the fundamental observables, then the Poisson-bracket-commutator relationship (5) can be implemented. Moreover, in the sense explained below, it uniquely determines a map  $\hat{}$  on these observables.

Before stating this result more precisely, it is useful to note that since  $\Omega$  is non-degenerate, any fundamental observable (i.e., linear function) on  $\mathcal{M}$  can be written in the form  $\Omega(y,\cdot)$  for some  $y \in \mathcal{M}$ , where by  $\Omega(y,\cdot)$  we mean the function  $f: \mathcal{M} \to \mathbb{R}$  defined by  $f(z) = \Omega(y,z)$ . In this notation, the Poisson bracket of the fundamental observables is given by

$$\{\Omega(y_1,\cdot),\Omega(y_2,\cdot)\} = -\Omega(y_1,y_2)1$$
 (6)

where 1 denotes the function on  $\mathcal{M}$  which takes the value 1 at each point. Hence, the Poisson-bracket-commutator relationship (5) for the fundamental observables is simply

$$[\hat{\Omega}(y_1,\cdot),\hat{\Omega}(y_2,\cdot)] = -i\Omega(y_1,y_2)I \tag{7}$$

where I denotes the identity operator on the Hilbert space.

However, there are some potential technical difficulties with eq.(7) because  $\hat{\Omega}(y,\cdot)$  should be an unbounded operator and, hence, can be defined only on a dense domain, so compositions (and, hence, commutators) are not automatically well defined. These difficulties are most easily dealt with by working with the following exponentiated version of (7): For each y, define the classical observable W(y) by

$$W(y) = \exp[-i\Omega(y, \cdot)] \tag{8}$$

Then eq.(7) together with the self-adjointness of  $\hat{\Omega}(y,\cdot)$  is formally equivalent to the following Weyl relations

$$\hat{W}(y_1)\hat{W}(y_2) = \exp[i\Omega(y_1, y_2)/2]\hat{W}(y_1 + y_2)$$
(9)

$$\hat{W}^{\dagger}(y) = \hat{W}(-y) \tag{10}$$

We shall view the Weyl relations as providing a mathematically precise statement of the Poisson-bracket-commutator relationship, which avoids operator domain problems.

The key uniqueness result is provided by the Stone-von Neumann theorem: For a finite dimensional symplectic vector space  $(\mathcal{M}, \Omega)$ , any two strongly continuous (in y), irreducible representations of the Weyl relations (9), (10) are unitarily equivalent. Thus, for a classical linear system with finitely many degrees of freedom, the "Poisson bracket goes to commutator" rule determines in a natural, canonical way a corresponding quantum theory in so far as the fundamental observables are concerned. This provides the main justification for the standard choices of the Cartesian position and momentum operators for Schrodinger quantum mechanics, as found in standard texts. Note, however, that although all classical observables can be written as functions of the fundamental observables, "factor ordering" ambiguities generally arise when one attempts to express an arbitrary quantum observable as a function of the quantum representatives of the fundamental observables. Since the "Poisson bracket goes to commutator" rule cannot be implemented for all observables, there does not appear to be any natural way to resolve this factor ordering ambiguity. Thus, when a sufficiently general class of observables is considered, it appears that there are many quantum theories corresponding to a given classical theory. However, our primary interest here is in the fundamental observables, where the Stone-von Neumann theorem does provide the desired uniqueness result for a system with finitely many degrees of freedom.

All of the above basic structure present in a classical system with finitely many degrees of freedom also is present in the theory of a classical field propagating in a globally hyperbolic spacetime  $(M, g_{ab})$ . Consider, for definiteness, a linear, Klein-Gordon scalar field,  $\phi$ , satisfying

$$\nabla^a \nabla_a \phi - m^2 \phi = 0 \tag{11}$$

This equation has a well posed initial value formulation, with the initial data consisting of the pair of functions  $(\phi, \pi)$  on a Cauchy surface  $\Sigma$ , where  $\pi = n^a \nabla_a \phi$ , with  $n^a$  the unit normal to  $\Sigma$ . We have a choice of precisely what class of functions to allow in phase space  $\mathcal{M}$ , but a particularly convenient choice is to require  $\phi$  and  $\pi$  to be smooth and of

compact support, i.e., we define

$$\mathcal{M} = \{ (\phi, \pi) | \phi, \pi \in C_0^{\infty}(\Sigma) \}$$
(12)

The Klein-Gordon Lagrangian gives rise to a well defined, conserved symplectic structure  $\Omega: \mathcal{M} \times \mathcal{M} \to \mathbb{R}$  on  $\mathcal{M}$ , given by

$$\Omega[(\phi_1, \pi_1), (\phi_2, \pi_2)] = \int_{\Sigma} (\pi_1 \phi_2 - \pi_2 \phi_1)$$
(13)

We also have a choice as to precisely which linear observables on  $\mathcal{M}$  should be viewed as the "fundamental observables". A natural choice is to take the fundamental observables to consist of all linear maps from  $\mathcal{M}$  into  $\mathbb{R}$  which are of the form  $\Omega(y,\cdot)$  with  $y \in \mathcal{M}$ . On this class of observables, there is a well defined Poisson bracket, given again precisely by eq.(6).

It is worth pointing out that for any "test function", f, on spacetime (i.e., any  $f \in C_0^{\infty}(M)$ ) and any solution,  $\psi$ , of the Klein-Gordon equation with initial data of compact support on a Cauchy surface, we have the identity (see lemma 3.2.1 of [4])

$$\int_{M} f\psi = \Omega(Ef, \psi) \tag{14}$$

where Ef denotes the advanced minus retarded solution with source f, and the natural identification of phase space with the space of solutions to the Klein-Gordon equation is implicit on the right side of this equation. Consequently, the fundamental observable  $\Omega(y,\cdot)$  on phase space also may be viewed as the observable on solution space defined by the spacetime smearing of solutions with a particular test function. The corresponding quantum observable  $\hat{\Omega}(y,\cdot)$  will then have an alternative interpretation of being a spacetime smearing of the Heisenberg field operator. Our choice of fundamental observables thus corresponds to the collection of Heisenberg field operators smeared with arbitrary test functions f.

We have seen above that the theory of a linear Klein-Gordon field on a globally hyperbolic spacetime has a phase space structure and a choice of fundamental observables that parallels completely the case of a system with finitely many degrees of freedom. Thus, the only relevant difference which occurs when we attempt to construct a quantum

theory of a field arises from the fact that now  $\dim(\mathcal{M}) = \infty$ . However, this difference is crucial because the Stone-von Neumann theorem does not hold in infinite dimensions, and there are many inequivalent ways of implementing the "Poisson bracket goes to commutator" rule for the fundamental observables.

The class of possible quantum field theory constructions can be restricted in a natural way by exploiting the analogy of a quantum field with an infinite collection of harmonic oscillators, and performing a construction analogous to the standard construction of the quantum theory of a harmonic oscillator using annihilation and creation operators. A mathematically precise implementation of this idea can be achieved by introducing a real inner product  $2\mu$  on  $\mathcal{M}$  which makes  $\Omega$  be norm preserving, i.e., we introduce a symmetric, bilinear map  $\mu: \mathcal{M} \times \mathcal{M} \to \mathbb{R}$  such that

$$\mu(\psi_1, \psi_1) = \frac{1}{4} \sup_{\psi_2 \neq 0} \frac{|\Omega(\psi_1, \psi_2)|^2}{\mu(\psi_2, \psi_2)}$$
(15)

It can be shown (see [4]) that there always exist a wide class of  $\mu$ 's satisfying (15). Given a  $\mu$  satisfying (15), we complete  $\mathcal{M}$  in the inner product  $2\mu$ . We then use  $\Omega$  to define a complex structure, so as to convert the completion of  $\mathcal{M}$  into a complex Hilbert space  $\mathcal{H}$ . We construct the symmetric Fock space  $\mathcal{F}_S(\mathcal{H})$  based upon  $\mathcal{H}$  by

$$\mathcal{F}_S(\mathcal{H}) = C \oplus \mathcal{H} \oplus (\mathcal{H} \otimes_S \mathcal{H}) \oplus \dots \tag{16}$$

where C denotes the complex numbers and  $\mathcal{H} \otimes_S \mathcal{H}$  denotes the symmetrized tensor product of  $\mathcal{H}$  with itself. Finally, we define the fundamental quantum observables  $\hat{\Omega}(y,\cdot)$  in terms of annihilation and creation operators on  $\mathcal{F}_S(\mathcal{H})$ , in close analogy with the harmonic oscillator construction. Details of this construction can be found in [4].

However, if two maps  $\mu_1$  and  $\mu_2$  satisfying eq.(15) are sufficiently different in the precise sense stated in theorem 4.4.1 of [4], the quantum field constructions based upon them will be unitarily inequivalent. Thus, we must still face the key question: Which  $\mu$  should we choose to do the quantum field construction?

In Minkowski spacetime – and, more generally, in curved, stationary spacetimes – the presence of a time translation symmetry gives rise to the following natural, "preferred" choice of  $\mu$ , which is motivated by the choice of  $\mu$  which simplifies the description of

the time independent, quantum mechanical harmonic oscillator: For  $y_1, y_2 \in \mathcal{M}$ , we define  $\mu(y_1, y_2)$  to be the real part of the Klein-Gordon inner product of the positive frequency parts of the solutions corresponding to the initial data  $y_1$  and  $y_2$ . (A more mathematically precise and complete description of this construction can be found in section 4.3 of [4].) Furthermore, for this construction in stationary spacetimes, states in  $\mathcal{F}_S(\mathcal{H})$  then have a natural, physical interpretation in terms of "particles", with the one-dimensional subspace C corresponding to the "vacuum", with  $\mathcal{H}$  corresponding to the subspace of single particle states, with  $\mathcal{H} \otimes_S \mathcal{H}$  corresponding to the subspace of two-particle states, etc.

However, in a general, non-stationary spacetime, there does not appear to any mathematically preferred choice of  $\mu$  nor any physically preferred definition of "particles" – although the Hadamard condition (discussed below) does pick out a preferred unitary equivalence class of  $\mu$ 's for spacetimes with compact Cauchy surfaces [4]. So the question remains: Which  $\mu$  should one choose? Equivalently, how should one define the notion of "particles" in a general, curved spacetime?

My view is that this question has roughly the same status in quantum field theory as the following question in classical general relativity: Which coordinate system should one choose in a general, curved spacetime? This latter question is very natural one to ask (and often is asked by begining students) if one has learned special relativity via its formulation in terms of global inertial coordinate systems. It is "answered" by formulating general relativity in a geometrical, coordinate independent manner, so that the question becomes manifestly irrelevant. Similarly, the issue of how to define "particles" is a natural one to ask by those to whom quantum field theory was presented as though it were a theory of "particles". It also can be answered by reformulating the theory in a manner which makes the question manifestly irrelevant.

In order to do this, we need a framework for quantum field theory which (initially, at least) simultaneously admits all states occurring in all (unitarily inequivalent) Hilbert space constructions of the theory, so that no "preferred construction" need be specified in advance. The algebraic approach accomplishes this goal in a mathematically elegant and straightforward manner, and I shall now briefly describe some of the main elements

of the algebraic formulation.

The key idea of the algebraic approach is to reverse the logical order in which the notion of states and observables are specified. One first introduces a notion of observables and provides the set of observables with the mathematical structure of an abstract \*-algebra,  $\mathcal{A}$ . (For some purposes, it is required that  $\mathcal{A}$  have the additional structure of being a C\*-algebra.) On then defines a state to be a linear map  $\omega : \mathcal{A} \to C$  which satisfies the positivity condition

$$\omega(A^*A) \ge 0 \tag{17}$$

for all  $A \in \mathcal{A}$ , as well as the normalization condition  $\omega(I) = 1$ , where I denotes the identity element of the algebra. Although this notion of a state may appear to be drastically different from the usual notion of a state as a vector (or, more generally, a density matrix) in a Hilbert space, there is, in fact, a very close relationship between these notions: If  $\mathcal{F}$  is a Hilbert space and  $\mathcal{A}$  is a sub-algebra of the C\*-algebra,  $\mathcal{L}(\mathcal{F})$ , of bounded linear maps on  $\mathcal{F}$ , then any density matrix state  $\rho$  on  $\mathcal{F}$  gives rise to an algebraic state,  $\omega$ , on  $\mathcal{A}$  by the formula  $\omega(A) = \operatorname{tr}(\rho A)$  for all  $A \in \mathcal{A}$ . Conversely, if  $\omega$  is an algebraic state on the C\*-algebra,  $\mathcal{A}$ , then the GNS construction shows that there exists a Hilbert space  $\mathcal{F}$ , a representation  $\pi: \mathcal{A} \to \mathcal{L}(\mathcal{F})$ , and a cyclic vector  $\Psi \in \mathcal{F}$  such that for all  $A \in \mathcal{A}$  we have  $\omega(A) = \langle \Psi | \pi(A) | \Psi \rangle$ . Thus, every state in the algebraic sense corresponds to a state in the usual sense in some Hilbert space construction of the quantum field theory. The key advantage of the algebraic approach is that it allows one to consider, on an equal footing, all states arising in all unitarily inequivalent Hilbert space constructions of the theory.

To define the theory of a quantum field in a curved, globally hyperbolic spacetime via the algebraic approach, we must specify a C\*-algebra structure on the class of observables that we wish to consider. If one wishes – initially, at least – merely to have the fundamental observables defined in the theory, then a C\*-algebra structure can be specified as follows: We start with the classical phase space  $\mathcal{M}$ , eq.(12), with symplectic structure  $\Omega$  given by eq.(13). We then choose an inner product  $\mu$  on  $\mathcal{M}$  satisfying eq.(15), and we perform the Hilbert space quantum field theory construction outlined above to define the observables  $\hat{\Omega}(y,\cdot)$  as self-adjoint operators on a Fock space  $\mathcal{F}$ . Next, we define

corresponding unitary operators  $\hat{W}(y)$  by exponentiation (see eq.(8)). These operators satisfy the Weyl relations (9), (10). The finite linear combinations of the  $\hat{W}(y)$ 's then have the natural structure of a \*-algebra. Completion of this algebra in the norm provided by  $\mathcal{L}(\mathcal{F})$  defines the desired C\*-algebra,  $\mathcal{A}$ . In principle,  $\mathcal{A}$  could depend upon the choice of  $\mu$ . If it did, then we would be back in a situation very similar to the situation we faced when we had many unitarily inequivalent Hilbert space constructions of the theory; we now would have many different possible constructions of the algebra of fundamental observables and no obvious means of choosing a "preferred" one. Fortunately, this is not the case: Even when the inner products  $\mu_1$  and  $\mu_2$  yield unitarily inequivalent Hilbert space constructions of a quantum field theory, the algebras  $\mathcal{A}_1$ ,  $\mathcal{A}_2$  to which they give rise are isomorphic as abstract C\*-algebras. In other words, associated with the symplectic vector space  $(\mathcal{M}, \Omega)$  for a Klein-Gordon (or other linear, bosonic) field on a given globally hyperbolic spacetime is a unique, well defined C\*-algebra of fundamental observables – known as the Weyl algebra – constructed in the manner described above, using any choice of  $\mu$  satisfying (15).

The specification of the Weyl algebra,  $\mathcal{A}$ , as the C\*-algebra of observables completes the formulation of the quantum theory of a Klein-Gordon field in an arbitrary globally hyperbolic, curved spacetime insofar as the definition of fundamental observables is concerned. Note that the specification of a state on  $\mathcal{A}$  corresponds, roughly, to the specification of the complete list of all n-point distributions  $\langle \phi(x_1), ..., \phi(x_n) \rangle$ , subject to all of the conditions arising from the positivity requirement (17).

However, there are other observables besides the fundamental observables which one may wish to consider. Most prominent among these is the stress-energy tensor,  $T_{ab}$ , of the quantum field, since it is needed to describe "back-reaction" effects of the quantum field on the gravitational field. In order to extend the construction of quantum field theory in curved spacetime to encompass the stress-energy tensor, one would like to enlarge the Weyl algebra  $\mathcal{A}$  to a new algebra of observables which contains elements corresponding to the (presumably, spacetime smeared) stress-energy tensor  $T_{ab}$ . A first step in this regard would be to define  $\hat{T}_{ab}$  as an operator-valued-distribution in Hilbert space constructions of the theory. There are serious difficulties with doing this because, formally,  $\hat{T}_{ab}$  is

the product of two distributions at the same spacetime point, so some "regularization" is needed to give it a mathematically well defined meaning. Some progress towards defining  $\hat{T}_{ab}$  as an operator-valued-distribution has been reported recently [7], but most work to date has focused on the less ambitious goal of defining expectation values of the (unsmeared) stress-energy tensor. (In a Hilbert space construction, this corresponds to defining the stress-energy tensor as a quadratic form on the Hilbert space rather than as an operator-valued-distribution.) Note that for an algebraic state  $\omega$ , a knowledge of  $\langle T_{ab} \rangle_{\omega}$  is precisely what is needed in order to determine whether the semiclassical Einstein equation

$$G_{ab} = 8\pi \langle T_{ab} \rangle_{\omega} \tag{18}$$

is satisfied, so a knowledge of  $\langle T_{ab} \rangle$  is all that is needed to analyze back reaction within the context of the semiclassical approximation.

The main results of the analysis of  $\langle T_{ab} \rangle$  are the following (see [4] for more details): (i)  $\langle T_{ab} \rangle_{\omega}$  can be defined only for states,  $\omega$ , that satisfy the Hadamard condition, which, in essence, states that the "ultra-violet" behavior of the state – as measured by the short distance behavior of the two point distribution  $\langle \phi(x)\phi(x')\rangle_{\omega}$  – is similar in nature to the short distance behavior of the two-point distribution for the vacuum state in Minkowski spacetime. (A precise definition of the "global Hadamard condition" can be found in [8]; its equivalence to a "local Hadamard condition" was proven in [9].) States which fail to satisfy the Hadamard condition are to be viewed as "physically singular", in that their stress-energy is infinite (or otherwise ill defined). The Hadamard condition thus provides an important additional restriction on the class of states which otherwise would be admissible when only the fundamental observables are considered. (ii) The prescription for assigning an expected stress-energy,  $\langle T_{ab} \rangle_{\omega}$ , to all Hadamard states,  $\omega$ , is uniquely determined up to addition of (state independent) conserved local curvature terms, by a list of physical properties that  $\langle T_{ab} \rangle_{\omega}$  should satisfy. The "point-splitting" regularization prescription satisfies these properties and thus provides a completely satisfactory definition of  $\langle T_{ab} \rangle_{\omega}$ , up to the local curvature ambiguity.

Thus, the status of quantum field theory for a linear field in a globally hyperbolic spacetime,  $(M, g_{ab})$ , may be summarized as follows: From  $(M, g_{ab})$  and the classical

symplectic structure of the field, we can construct the Weyl algebra,  $\mathcal{A}$ , of fundamental observables (or a corresponding "anticommutator algebra" in the case of fermion fields). States are then defined in the algebraic sense with respect to  $\mathcal{A}$ . Nonsingular states must, in addition, satisfy the Hadamard condition. For Hadamard states,  $\langle T_{ab} \rangle$  is well defined up to the ambiguity of adding conserved local curvature terms.

We conclude this section by briefly describing the statements of the Unruh and Hawking effects within the framework developed above. (Again, much more detail can be found in [4].) In the Unruh effect, one considers the action of a one-parameter family of Lorentz boosts on Minkowski spacetime. For definiteness, we normalize the boost Killing field to have unit norm on an orbit of acceleration a. We note that the orbits of the boost isometries are timelike in the "right wedge" (as well as the "left wedge") region of Minkowski spacetime. Furthermore, this "right wedge" region – when viewed as a spacetime in its own right – is globally hyperbolic. Hence, there is a well defined Weyl algebra,  $A_R$ , of "right wedge" observables, which is naturally a subalgebra of the Weyl algebra,  $\mathcal{A}$ , for all of Minkowski spacetime. Hence, the restriction of the ordinary Minkowski vacuum state,  $\omega_0$ , to the subalgebra  $\mathcal{A}_R$  defines a state on the right wedge spacetime. The Unruh effect is the assertion that this state is a thermal state at temperature  $T = a/2\pi$ , in the precise sense that it satisfies the KMS condition with respect to the notion of "time translations" provided by the Lorentz boosts. In fact, this mathematical statement of the Unruh effect was proven by Bisognano and Wichmann [10], independently of (and simultaneously with) the paper of Unruh [11]. However, the physical interpretation of this fact – namely, that an observer with acceleration a will "feel himself" to be immersed in a thermal bath at temperature  $T = a/2\pi$  when the field is in the Minkowski vacuum state – is due to Unruh.

In the Hawking effect, one considers a spacetime describing the gravitational collapse of a body to form a black hole. One assumes that the black hole settles down to a stationary final state, with constant surface gravity  $\kappa$ . The Hawking effect [2], [12] is the assertion that any nonsingular (i.e., Hadamard) state asymptotically approaches, at late times, a thermal state at temperature  $T = \kappa/2\pi$  with respect to the subalgebra associated with solutions which "appear to emerge from the direction of the black hole".

In other words, a black hole "radiates" as a perfect black body; its physical temperature is  $\kappa/2\pi$ .

# 3 Some Open Issues

In this section, I will briefly discuss several open issues in quantum field theory in curved spacetime. My purpose in doing this is merely to give the reader a flavor of some topics of current research interest. No attempt will be made to provide a comprehensive account of all present research in quantum field theory in curved spacetime.

One key issue that remains unresolved concerns the treatment of back-reaction. There is general agreement that, in the context of the semiclassical approximation, back-reaction effects should be described by the semiclassical Einstein equation (18). However, the following two significant difficulties of principle arise when one attempts to solve this equation and extract physically relevant solutions: (i) As mentioned above, there is a local curvature ambiguity in the definition of  $\langle T_{ab} \rangle$ . By requiring that the ambiguous, conserved local curvature terms have the "correct dimension", this ambiguity for 4-dimensional spacetimes can be reduced to a two parameter family. However, as discussed in [4], at least one of these two parameters cannot, in principle, be determined by arguments involving only quantum field theory in curved spacetime (as opposed to quantum gravity). Thus, there is a fundamental ambiguity in eq. (18). (ii) Equation (18) is of a "higher derivative" character than the classical Einstein equation, and it admits many - presumably spurious - "run-away" solutions. However, it is not clear, in general, how to distinguish between the "physical" and the "unphysical, spurious" solutions. Some proposals in this regard have been given by Simon [13], and further discussion can be found in [14].

It is worth noting that the mathematical situation with regard to treating backreaction is considerably improved when one considers 2-dimensional "dilaton gravity" models [15], [16]. In particular, there now are no conserved local curvature terms of the correct dimension, so difficulty (i) does not arise. Difficulty (ii) also is considerably alleviated. In addition, in the conformal vacuum state  $\langle T_{ab} \rangle$  is given by a relatively simple, local expression in the conformal factor, so the back reaction equations are considerably more tractable than in the 4-dimensional case.

The second open issue I wish to mention concerns energy conditions which may hold for  $\langle T_{ab} \rangle$ . Even for a Klein-Gordon field in Minkowski spacetime, it is easy to find quantum states for which  $\langle T_{ab} \rangle$  violates any of the local positive energy conditions which hold for classical fields. However, it is possible that  $\langle T_{ab} \rangle$  may still satisfy some nontrivial global positive energy conditions. The most interesting of these conditions is the "averaged null energy condition" (ANEC), which states that for any quantum state and for any complete null geodesic, we have

$$\int \langle T_{ab} \rangle k^a k^b d\lambda \ge 0 \tag{19}$$

where  $\lambda$  denotes the affine parameter of the geodesic and  $k^a$  denotes its tangent. The validity of ANEC is sufficient for proofs of the positive energy theorem [17] and for "topological censorship" [18], so many of the key results of classical general relativity continue to hold when the pointwise energy conditions are replaced by ANEC. Although ANEC holds for Minkowski spacetime [19], [20], it is known to fail for arbitrary curved spacetimes (in 4-dimensions) [20]. Nevertheless, some recent research has indicated that (i) ANEC may come "close enough" to holding to exclude wormholes with curvature everywhere much smaller than the Planck scale [21] and (ii) when the semiclassical Einstein equation (18) is imposed, a version of ANEC may hold wherein one "transversely averages" (with a suitable smearing function) over null geodesics within roughly a Planck length of the given geodesic [14]. Thus, it is possible that violations of ANEC may be confined to regimes where the semiclassical approximation is not applicable.

The final topic of present research which I wish to mention concerns quantum field theory on spacetimes with closed causal curves. Consider, for definiteness, a causality violating spacetime which is "globally hyperbolic outside of a compact set", i.e., a spacetime obtained by modifying the metric of a globally hyperbolic spacetime in a compact region so as to produce closed timelike curves within that region. It seems plausible that – at least within a suitable subclass of such spacetimes – one will have a well defined, deterministic dynamics for classical fields [22]. Does there exist a similarly well defined

quantum field theory on such spacetimes?

For linear quantum fields, one needs only the symplectic vector space structure of the classical solution space in order to construct the Weyl algebra, so if the classical dynamics is well behaved there should be no difficulty defining fundamental observables and states. In particular, it should be possible to obtain a well defined, unitary S-matrix describing scattering processes. However, the association of a solution, Ef, to every test function f (see eq.(14)) uses global hyperbolicity, so it is not clear that the field observables can still be interpreted as distributions on spacetime. Even if they can, it can be shown that the condition of "F-compatibility" [23] must fail [24] at least at some points of the chronology horizon, so the local behavior of the field observables must be different from that occurring in globally hyperbolic spacetimes. Furthermore, the local Hadamard condition cannot be satisfied everywhere on the chronology horizon [24]. Thus, although in certain examples it is possible to choose states where the stress-energy tensor remains finite as one approaches the chronology horizon [25], the stress-tensor always must be singular (or ill defined) at least at some points of the chronology horizon. Thus, it seems far from clear that the quantum theory of linear fields can be sensibly defined even on the class of causality violating spacetimes which are "globally hyperbolic outside of a compact set".

The situation for nonlinear fields appears to be considerably worse in that conventional perturbation theory yields an S-matrix which is non-unitary (in the sense of not conserving probability) [26]. Alternative ideas for constructing quantum field theory on causality violating spacetimes are currently being actively pursued by a number of authors [27]-[30].

## 4 Summary

The theory of linear quantum fields propagating on a fixed, globally hyperbolic, curved spacetime is completely well posed mathematically insofar as the definition of the fundamental observables (or, equivalently, the smeared field operators) are concerned, and  $\langle T_{ab} \rangle$  is well defined (for Hadamard states) up to local curvature term ambiguities.

This provides the necessary tools to investigate many phenomena of interest involving quantum effects occurring in strong gravitational fields. Nevertheless, some interesting and important issues remain open, such as the ones mentioned in the previous section.

One may hope that some of the insights obtained from the study of quantum field theory in curved spacetime – particularly, the close relationship between Killing horizons and thermal states as seen in the Unruh and Hawking effects – will be helpful in guiding the development of a quantum theory of gravity.

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