Anomalies

Anomalies $\longleftrightarrow$ Symmetry breaking:

- * SOFT BREAKING : "small" term into $\mathcal{L}$ s.t. symm. is broken
- * SSB (Heisenberg '39 cond. mat., Goldstone '60s qft)
- * ANOMALOUS BREAKING $\Rightarrow$ chiral matter to gauge th. or gravity

Anom. break.:

Classical Symmetry $\longrightarrow$ Quantum Theory

$\Downarrow$

cannot realise symm. $\longrightarrow$ REGULARIZATION

Suppose symm:

$$\partial_\mu J^\mu_\alpha = 0 \quad \text{(classical)}$$

$\hookrightarrow$ quantum level : $\partial_\mu J^\mu_\alpha = \underline{\underline{a_\alpha(A)}}$

$\downarrow$

ALWAYS FINITE (connected to counterterms)

ANOMALIES :

$\rightarrow$ anom. for GLOBAL sym.

$\rightarrow$ anom. for LOCAL sym. $\rightarrow$ coupling to gauge field $\Rightarrow$ CONSISTENT ANOM. $\downarrow$ MUST VANISH!

$\rightarrow$ link to BRST symm.

Suppose interacting $D=2$ theory

$$\begin{cases} \gamma^0 = \sigma^1 \\ \gamma^1 = \sigma^2 \\ \gamma^3 = -i\sigma^3 \end{cases} \longrightarrow \begin{array}{l} J_\mu = \bar\psi \gamma_\mu \psi \\ J_{\mu 5} = i\bar\psi \gamma_\mu \gamma_5 \psi \end{array} \Rightarrow \begin{array}{l} \partial_\mu J^\mu = 0 \\ \partial_\mu J_5^\mu = 0 \end{array}$$

NB $\varepsilon^{\mu\nu}\gamma_\nu = i\gamma^\mu\gamma_5 \Rightarrow J_5^\mu = \varepsilon^{\mu\nu} J_\nu$ [axial current is dual to vector current]

$$\Rightarrow \langle J^\mu(x) J^\nu(y) \rangle = g^{\mu\nu}\Pi_1(x-y) - \frac{\partial^\mu\partial^\nu}{\Box}\Pi_2(x-y) + \left(\frac{\partial^\mu \epsilon^{\nu\alpha}\partial_\alpha}{\Box} + \frac{\partial^\nu \epsilon^{\mu\alpha}\partial_\alpha}{\Box}\right)\Pi_3(x-y)$$

$$\Rightarrow \langle J_5^\mu(x) J^\nu(y) \rangle = \epsilon^{\mu\nu}\Pi_1(x-y) - \epsilon^{\mu\alpha}\frac{\partial_\alpha\partial^\nu}{\Box}\Pi_2(x-y) - \left(g^{\mu\nu} - 2\frac{\partial^\mu\partial^\nu}{\Box}\right)\Pi_3(x-y)$$

$\rightarrow$ impose conservation:

VEC cons: $\Pi_1 = \Pi_2$, $\Pi_3 = 0$

AX cons: $\Pi_1 = 0$

} either the theory is trivial or there is an anomaly!

Now consider $D=4$:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^{\alpha} F^{\mu\nu}_{\alpha} + \mathcal{L}_{MATTER}(\bar\psi, \psi, D_\mu\bar\psi, D_\mu\psi)$$

ψ in R of group G

$$\Rightarrow [T_a, T_b] = i f_{ab}{}^c T_c$$

$$\delta A_\mu^R = \partial_\mu \epsilon^R - i[A_\mu^R, \epsilon^R], \qquad A_\mu^R = A_\mu^a T_a^{(R)}$$
$$\delta\psi^R = i\epsilon^R\psi \qquad\qquad \epsilon^R = \epsilon^a T_a^{(R)}$$

Then $(D_\mu F^{\mu\nu})^a = J^{a\nu}_{MATT.} \Rightarrow D_\mu J^{\mu a}_{MATT.} = 0$ [local gauge symmetry]

Abelian symmetry (global)

$$\mathcal{L}_{\text{MATT.}} = -\bar\psi(\not\partial - i\not A^a)\psi \qquad \Rightarrow \langle O_1(x_1)\dots O_n(x_n)\rangle = \int \mathcal{D}A_\mu \mathcal{D}\bar\psi \mathcal{D}\psi \, e^{iS[A,\bar\psi,\psi]}$$

$$Z[A] = \int \mathcal{D}\bar\psi \mathcal{D}\bar\psi \, e^{iS_{\text{MATT.}}} = \text{(after int. over } A) = e^{iW[A]}$$

$$\downarrow$$

$U = \exp(i\epsilon T\gamma_5)$ s.t. $T = T^\dagger$

$$[T, T^a] = 0$$
$$[T, \gamma_5] = 0$$

then $\psi \to U\psi$

$\bar\psi \to \bar\psi U$

$$\longrightarrow J_5^\mu = i\bar\psi\gamma^\mu\gamma_5 T\psi \Rightarrow \partial_\mu J_5^\mu = 0$$

Now we translate to quantum corrections:

$$\langle \partial_\mu J_5^\mu(x)\rangle_A = 0$$

suppose $\epsilon \to \epsilon(x)$

then $\begin{cases} \psi' = U(x)\psi \\ \bar\psi' = \bar\psi U(x) \end{cases} \longrightarrow$ NOT a symm., but a change of variables

$$\int \mathcal{D}\psi' \mathcal{D}\bar\psi' \, e^{iS_M[\bar\psi', \psi', A]}$$

$$\int \mathcal{D}\psi \mathcal{D}\bar\psi \, \underbrace{J(\epsilon(x))}_{\text{"JACOBIAN"}} \, e^{iS_M - \int d^4x \, \epsilon(x)\, \partial_\mu J_5^\mu(x)}$$

"JACOBIAN" $\longrightarrow$ leads to ANOMALIES!

$$\Rightarrow J(\epsilon(x)) = \exp\left(-i\int d^4x \, \epsilon(x)\, a(x)\right)$$

$\rightsquigarrow$ expand at 1st order in ϵ:

$$\partial_\mu \langle J_5^\mu(x)\rangle_A = \langle a(x)\rangle_A$$

background of gauge field ($\exists$ 1-pt function)

What happens to the vector symm?

$$U(x) = e^{i\epsilon T} \Rightarrow \begin{cases} \psi' = U\psi \\ \bar\psi' = \bar\psi U^\dagger \end{cases} \longrightarrow (\det U)(\det U^{-1}) = 1$$

Consider:

$$\langle x| U |y\rangle = \delta^4(x-y)\,\mathcal{U}(x)$$

$\Rightarrow$ since they're ferm. we have:

$$\left(\det \mathcal{U}\right)^2 = \exp\left(-2\,\mathrm{Tr}\ln \mathcal{U}\right) = \exp\left(i\int d^4x\, \epsilon^a\, a_a(x)\right)$$

$\boxed{\mathrm{tr}\,\gamma_5 = 0}$

$$\rightarrow \mathrm{Tr}\ln\mathcal{U} = \int d^4x\, \langle x|\,\mathrm{Tr}\ln\mathcal{U}\,|x\rangle = \int d^4x\, \delta^4(x-x)\,\mathrm{Tr}\ln\mathcal{U}(x) = \int d^4x\, \delta^4(x-x)\, i\epsilon^a\, \mathrm{Tr}[T\gamma_5]$$

$$= \underbrace{\delta^4(0)}_{\text{INFINITE }(\infty)} \int d^4x\, i\epsilon^a(x)\,\mathrm{Tr}(T\gamma_5) \longrightarrow \text{classical sum} = 0$$

INFINITE (∞) $\Rightarrow$ QUANTUM OBJECT (now we need to regularized)

$\Rightarrow$ REGULARIZATION

$\Lambda\to\infty \Rightarrow$ we recover the original determinant

$$-2\lim_{\Lambda\to\infty}\int d^4x\, \mathrm{Tr}\left[\langle x|\epsilon(x)\,\gamma_5\, T f\left(-\frac{\not{D}^2}{\Lambda^2}\right)|x\rangle\right]$$

$f(0)=1$

$f(\infty)=0$

$sf'(s)=0 \longrightarrow$ at $s=0,\infty$

$$\Rightarrow \langle x|\not{D}|x\rangle = \not{D}\langle x|x\rangle$$

$\rightarrow$ go to momentum space:

$$-2\lim_{\Lambda\to\infty}\int d^4x\,\frac{d^4q}{(2\pi)^4}\,\mathrm{Tr}\left[\langle x|q\rangle\langle q|\,\epsilon(x)\,\gamma_5\, T f\left(\frac{\not{D}^2}{\Lambda^2}\right)|x\rangle\right] =$$

$$= -2\lim_{\Lambda\to\infty}\left[\int d^4x\,\epsilon(x)\,\Lambda^4\int\frac{d^4q}{(2\pi)^4}\,\mathrm{Tr}\left(\gamma_5\, T f\left(-\left(-i\not{q}+\frac{\not{D}}{\Lambda}\right)^2\right)\right)\right.$$

$$\not{D}^2 = D_\mu D^\mu - \frac{1}{4}\left[\gamma^\mu,\gamma^\nu\right]F_{\mu\nu}$$

$\rightarrow$ expand $f(\dots)$:

Wick rotation

$$-2\lim_{\Lambda\to\infty}\left\{\int d^4x\,\epsilon(x)\int\frac{d^4q}{(2\pi)^4}\,\frac{1}{2}f''(q^2)\,\mathrm{Tr}\left[\gamma_5\, T\not{D}^4\right]\right\} \longrightarrow \int\frac{d^4q}{(2\pi)^4}\,\frac{1}{2}f''(q^2) = \frac{i}{32\pi^2}$$

$$\rightarrow -2\lim_{\Lambda\to\infty}\left\{\dots\ -i\epsilon^{\mu\nu\rho\lambda}\,\mathrm{Tr}_R\left[T F_{\mu\nu}F_{\rho\lambda}\right]\right\} \qquad \rightarrow \not{D}^2 = D_\mu D^\mu - \frac{i}{4}\left[\gamma^\mu,\gamma^\nu\right]F_{\mu\nu}$$

$$a(x) = -\frac{1}{16\pi^2}\,\epsilon^{\mu\nu\rho\lambda}\,\mathrm{Tr}_R\left(T F_{\mu\nu}F_{\rho\lambda}\right)$$

TRULY A QUANTUM EFFECT

Therefore:

$$\partial_\mu \langle J_5^\mu(x) \rangle_A = \frac{\hbar}{16\pi^2} \epsilon^{\mu\nu\rho\lambda} \langle Tr_R (T F_{\mu\nu} F_{\rho\lambda}) \rangle_A$$

trace over the rep. R of the group G.

NO LONGER NULL

($F_{\mu\nu}$ contains A_μ in a background made of As $\Rightarrow \langle\ \rangle_A$ are superflous.)

Take $T = \mathbb{I}$:

$$\int \mathcal{D}\psi \mathcal{D}\bar{\psi}\, e^{i\int d^4x\, \bar{\psi} \not{D} \psi} = \prod_n \lambda_n = \det \not{D}$$

$$\psi = \sum c_n \varphi_n(x)$$ Grassmann

$$\bar{\psi} = \sum \bar{c}_n \varphi_n^\dagger(x)$$

s.t.: $\not{D}\varphi_n(x) = \lambda_n \varphi_n(x)$

Now insert the regulator ($\prod \lambda_n$ is div):

$$\prod_n \lambda_n f\left(\frac{\lambda_n^2}{\Lambda^2}\right)$$

Euclidean space: (γ_0, too)

$$\begin{cases} i\not{D}_E \varphi_n = \lambda_n \varphi_n \\ T\varphi_k = t_k \varphi_k \end{cases} \longrightarrow \quad \mathbb{1} = \sum_n |\varphi_n\rangle\langle\varphi_n| \qquad Tr\, A = \sum_k \langle \varphi_k | A | \varphi_k \rangle$$

$$\{\gamma_5, \not{D}_E\} = 0$$

Take $\gamma_5\varphi_k \longrightarrow \not{D}_E \gamma_5 \varphi_k = -\gamma_5 \not{D}_E \varphi_k = -\gamma_5 \lambda_k \varphi_k = -\lambda_k (\gamma_5 \varphi_k)$

$$\varphi_k \longrightarrow \lambda_k$$
$$\gamma_5\varphi_k \longrightarrow -\lambda_k$$

Then $\varphi_{k^\pm} = \frac{1 \pm \gamma_5}{2} \varphi_k$ [no longer eigenstates of $\not{D}_E$]

however they're eigenvectors of $\not{D}_E^2$

$$f\left(\frac{\not{D}_E^2}{\Lambda^2}\right) \longrightarrow \not{D}_E^2 \varphi_{k^\pm} = \pm \lambda_k^2 \varphi_{k^\pm}$$

NB: the ker is trickier $\Rightarrow \not{D}_E \varphi_0 = 0$ does not say much...

φ_{0+}, φ_{0-} lead to the same result!

Take

φ_0^+ n_+ times

φ_0^- n_- times

and compute:

$$\lim_{\Lambda\to\infty} \mathrm{Tr}\left[\gamma_5 T f\left(-\frac{\not{D}_E^2}{\Lambda^2}\right)\right] = \lim_{\Lambda\to\infty} \sum_k \langle \varphi_k | \gamma_5 T f\left(-\frac{\not{D}_E^2}{\Lambda^2}\right) | \varphi_k \rangle =$$

$$= \lim_{\Lambda\to\infty} \sum_k f\left(-\frac{\lambda_k^2}{\Lambda^2}\right) t_k \langle \varphi_k | \gamma_5 | \varphi_k \rangle =$$

only non corresponding states survive!

$$\sum_{k=1}^{n_+} t_k - \sum_{k=1}^{n_-} t_k$$

$$\to T = \mathbb{I} \Rightarrow n_+ - n_- = \frac{1}{32\pi^2} \int d^4x\, \epsilon^{\mu\nu\rho\sigma}\, \mathrm{Tr}\left[F_{\mu\nu}, F_{\rho\lambda}\right]$$

(instanton density)

(II Chern class)

$$\int d^4x\, \varepsilon^{\mu\nu\rho\lambda} F^a_{\mu\nu} F_a^{\mu\nu} = 64\pi^2 \nu, \quad \nu \in \mathbb{Z}$$

$\Downarrow$

"measures" the sector of the theory

→ GENERAL (EVEN) DIMENSION $D = 2d$

$$\gamma_E = i^d \gamma^1 \gamma^2 \dots \gamma^{2d}$$

$$\lim_{\Lambda\to\infty} \mathrm{Tr}\left[\gamma_E f\left(-\frac{\not{D}_E^2}{\Lambda^2}\right)\right]$$

$$\int d^{2d}x\, \Lambda^{2d} \int \frac{d^{2d} q_E}{(2\pi)^{2d}} \mathrm{Tr}\left[\gamma_E f\left(-q_E^2 - 2i \frac{q_E^\mu D_\mu}{\Lambda^2} + \frac{\not{D}_E^2}{\Lambda^2}\right)\right]$$

Now the first non vanishing trace:

$$\mathrm{Tr}\left(\gamma_E \gamma^{\mu_1} \dots \gamma^{\mu_{2d}}\right) = (-1)^d\, 2^d\, \epsilon^{\mu_1 \dots \mu_{2d}}$$

$$\to \int \frac{d^{2d} q}{(2\pi)^{2d}} f^{(d)}(\dots) = \frac{1}{(4\pi)^d}$$

$$\Rightarrow \text{ind}(i\not{D}) = \frac{(-1)^d}{d!}\frac{1}{(4\pi)^d}\int d^{2d}x\, \varepsilon^{\mu_1 \dots \mu_{2d}}\, Tr\left[F_{\mu_1\mu_2}\dots F_{\mu_{2d-1}\mu_{2d}}\right]$$

COUPLING OF CURRENT WITH MATTER FIELDS

matter (chiral) $\longrightarrow \psi, \overline{\psi}$

define $\longrightarrow e^{iW[A]} = \int \mathcal{D}\psi\mathcal{D}\overline{\psi}\, e^{iS[\psi,\overline{\psi},A]}$

$$A_\mu \longrightarrow A'_\mu = A_\mu + D_\mu \epsilon$$

not A_a !

$$e^{iW[A']} = \int \mathcal{D}\psi\mathcal{D}\overline{\psi}\, e^{iS[\psi,\overline{\psi},A']} = \int \mathcal{D}\psi'\mathcal{D}\overline{\psi}'\, e^{iS[\psi',\overline{\psi}',A]} = \int \mathcal{D}\psi\mathcal{D}\overline{\psi}\, e^{i\int d^4x\, \varepsilon^a(x)\mathcal{A}_a(x) + S[\psi,\overline{\psi},A]}$$

$$\| $$

$$e^{i\int d^4x\, \varepsilon^a \mathcal{A}_a(x)}\, e^{iW[A]}$$

Take ε infinitesimal:

$$\delta_\epsilon W[A] = \int d^4x\, \varepsilon^a(x)\mathcal{A}^a(x) \longrightarrow \text{ANOMALY !}$$

$$= \int d^4x\, \delta A^a_\mu(x)\frac{\delta W}{\delta A^a_\mu(x)} = \int d^4x\, (D_\mu\epsilon)^a \frac{\delta W}{\delta A^a_\mu} = -\int d^4x\, \varepsilon^a(x) D_\mu\left(\frac{\delta W}{\delta A^a_\mu}\right) =$$

$$= -\int d^4x\, \varepsilon_a(x)\left(D_\mu \langle J^\mu(x)\rangle\right)^a$$

dynamical current coupled to gauge fields

$$\Rightarrow \left(D_\mu\langle J^\mu(x)\rangle\right)^a = -\mathcal{A}(x) \Longrightarrow$$ the <u>covariant</u> divergence is NOT conserved

<u>Feynman diagram</u>:

$$\ln \int \mathcal{D}\psi\mathcal{D}\overline{\psi}\, e^{\dots \int d^4x\, A^a_\mu(x)\, J^\mu_a(x)}$$

$$\frac{\delta}{\delta A^{a_1}_{\mu_1}(x_1)}\cdots\frac{\delta}{\delta A^{a_n}_{\mu_n}(x_n)}\left[W[A]\right]_{A_\mu=0} = \langle J^{\mu_1}_{a_1}(x_1)\dots J^{\mu_n}_{a_n}(x_n)\rangle$$

free theory ($A=0$)

$$\| $$

$$\Gamma^{\mu_1\dots\mu_n}_{a_1\dots a_n}(x_1\dots x_n)$$

(composite op. $\Rightarrow$ the only conn. graphs are loops)

$$\Rightarrow \frac{\delta}{\delta A_{\nu}(y)}\frac{\delta}{\delta A_{\rho}(z)}\langle \partial_\mu J^\mu_5(x)\rangle\Big|_{A=0} = -\frac{\delta}{\delta A_\nu(y)}\frac{\delta}{\delta A_\rho(z)}\, Q(x)\Big|_{A=0} \longrightarrow \partial^x_\mu\langle J^\mu_5(x)\, J^\nu(y)\, J^\rho(z)\rangle = -\frac{1}{2\pi^2}\varepsilon^{\mu\nu\rho\sigma}\left(\partial^y_\lambda\, \delta^4(y-x) - \partial^z_\sigma\, \delta^4(z-x)\right)$$

$Tr\, T^3 = \sum q^3$

$$\| $$

$$\Gamma^{\mu\nu\rho}_5(x,y,z)$$

→ MOMENTUM SPACE

Fourier transf.: $$\boxed{(p+q)_\mu \Gamma_5^{\mu\nu\rho}(-p-q,p,q) = \frac{1}{2\pi^2}\left(\sum_i q_i^3\right)\varepsilon^{\nu\rho\lambda\sigma} p_\lambda q_\sigma}$$

$$\rightarrow N \geq 4 \Rightarrow p_\mu^{(1)} T_5^{\mu\nu_2\dots\nu_N}(p^{(1)}, p^{(2)}, \dots, p^{(N)}) = 0$$

NON ABELIAN THEORY

$$\mathrm{Tr}[T T^a T^b]\, \varepsilon^{\mu\nu\rho\lambda} \mathrm{Tr}_R(F_{\mu\nu} F_{\rho\lambda})$$

$$\Rightarrow D_\mu \langle J^\mu(x)\rangle_a = -\mathcal{A}_a(x)$$

$$\hookrightarrow \mathcal{A}^{\nu\rho}_{abc} := \left.\frac{\delta^2 \mathcal{A}_a(x)}{\delta A^b_\nu(y)\, \delta A^c_\rho(z)}\right|_{A=0}$$

there are A_μ fields in D_μ

$$\Rightarrow \Gamma^{\mu\nu\rho}_{abc}(x,y,z) = -\langle J^\mu_a(x) J^\nu_b(y) J^\rho_c(z)\rangle$$

$$\Pi^{\rho\nu}_{eb}(x,y) = i\langle J^\rho_e(x) J^\nu_b(y)\rangle$$

$$\rightarrow \frac{\partial}{\partial x^\mu}\Gamma^{\mu\nu\rho}_{abc}(x,y,z) + f_{abc}\left[\delta^4(x-y)\, \Pi^{\nu\rho}_{dd}(y,z) - \delta^4(x-z)\, \Pi^{\nu\rho}_{ff}(y,z)\right] = -\mathcal{A}^{\nu\rho}_{abc}(x,y,z)$$

↳ NB: this part is natural

this is anomalous

Suppose $\mathcal{A}_a(x) = 4c\, \mathrm{Tr}_R[T^a T^b T^c]\, \varepsilon^{\mu\nu\rho\lambda} \partial_\mu A^c_\nu\, \partial_\rho A^c_\lambda + O(A^3)$

$\mathrm{Tr}[T^a T^b] = I_2(R)\,\delta^{ab}$

Dinkin index: $I_2(R) = C_2(R)\frac{d_R}{d_A}$ (d_R: dim of R, d_A: dim of adj)

$$\Downarrow$$

$$\mathcal{A}^{\nu\rho}_{abc}(-p-q,p,q) = 8c\, \varepsilon^{\nu\rho\lambda\sigma} p_\lambda q_\sigma D^R_{abc}$$

$\mathrm{Tr}[T_{(a} T_b T_{c)}]$

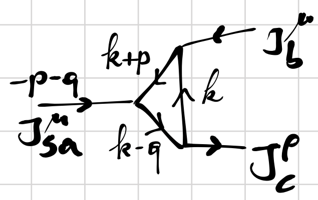
→ since there are NO ε tensors in 2pts func., in order to just take the anomaly, we use:

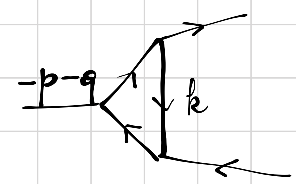
$$-i(p+q)_\mu \Gamma^{\mu\nu\rho}_{abc}(-p-q,p,q)\Big|_{\varepsilon\text{ part}} = 8c\, \varepsilon^{\nu\rho\lambda\sigma} p_\lambda q_\sigma D^R_{abc}$$

"some coefficient"

How to derive the axial anomaly from Feynman diagrams?

* Consider: $\begin{cases} J^\mu_{5a} = i\bar{\psi}\gamma^\mu\gamma_5 T_a \psi \\ J^\mu_a = i\bar{\psi}\gamma^\mu T_a \psi \end{cases} \longrightarrow \langle J^\mu_{5a} J^\nu_b J^\rho_c \rangle$

→ contractions: 

$+$ 

* axial vertices: $-\gamma^\mu\gamma_5 T_a$

* vector vertices: $-\gamma^\mu T_a$

$$\Rightarrow \Gamma^{\mu\nu\rho}_{5abc} = -i\int \frac{d^4k}{(2\pi)^4}\, \mathrm{Tr}\left[\gamma_5\gamma^\mu \frac{\not{k}+\not{p}}{(k+p)^2-i\varepsilon}\gamma^\nu \frac{\not{k}}{k^2-i\varepsilon}\gamma^\rho \frac{\not{k}-\not{q}}{(k-q)^2-i\varepsilon}\right]\mathrm{Tr}_R\left[T_a T_b T_c\right] + (p\to q,\ \nu\to\rho,\ a\to c)$$

⇒ linear divergent! ⟶ choose the regularization!

- dimens. reg ⇒ not the best choice
- Pauli-Villar ⇒ ✓

Define $\left(\Gamma^{\mu\nu\rho}_{5abc}\right)_{reg} = \Gamma^{\mu\nu\rho}_{5abc} - \Gamma^{\mu\nu\rho}_{5abc}(M)$ ⟶ $\dfrac{1}{\not{p}-M+i\varepsilon}$ ← i.e.: insert a massive ($M \gg 1$) part. with opposite statistic.

$$\Rightarrow -(p+q)_\mu\left(\Gamma^{\mu\nu\rho}_{5abc}\right)_{reg} = \int\frac{d^4k}{(2\pi)^4}(p+q)_\mu I^{\mu\nu\rho}_M(k,p,q) = \int\frac{d^4k}{(2\pi)^4}\frac{8i\varepsilon^{\nu\rho\lambda\sigma}p_\lambda q_\sigma M^2}{\left((k+p)^2+M^2-i\varepsilon\right)\left(k^2+M^2-i\varepsilon\right)\left((k+q)^2+M^2-i\varepsilon\right)}$$

→ We need to rescale everything by M:

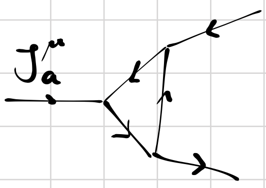
$$(k+p)^2+M^2-i\varepsilon \to (\tilde{k}M+p)^2+M^2-i\varepsilon = M^2\left[\left(\tilde{k}+\frac{p}{M}\right)^2+1-i\varepsilon\right]$$

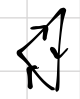
→ after $\int d^4k$ integration:

$$i(p+q)_\mu \Gamma^{\mu\nu\rho}_{5abc}(-p-q,p,q) = -\frac{1}{2\pi^2}\varepsilon^{\nu\rho\lambda\sigma}p_\lambda q_\sigma\, \mathrm{Tr}\left[T_a T_b T_c\right]$$

Now consider the non abelian anomaly:

→ chiral matter: $\psi_L = P_L\psi$ $\quad$ $\psi_R = P_R\psi$ $\qquad P_{L,R} = \frac{1\pm\gamma_5}{2}$ $\quad \to \bar\psi_L \not{D}\psi$ or $\bar\psi_R \not{D}\psi_R$

→ 

$+$  $\quad$ where $J^\mu_a = i\,\bar\psi_{L,R}\gamma^\mu T_a \psi_{L,R}$

→ prop: $\frac{\not{k}}{k^2 - i\varepsilon}$ $\qquad \Big\}\ \mathrm{Tr}\Big[i\gamma^\mu T_a \overbrace{P_L\ \ P_L}^{P_L} \frac{\not{k}}{k^2-i\varepsilon} \cdots\Big]$

vert $(i\gamma^\mu T_a)P_{L\,\mathrm{or}\,R}$

↓

same as always but with chiral vertex

$$\bar\psi(\not{\partial} - i\not{A}P_L)\psi$$

Go back to path integral approach: $\int \mathcal{D}\psi\mathcal{D}\bar\psi\, e^{i\int\bar\psi\not{D}\psi}\cdots$

$$\det(\underbrace{\not{\partial} - i\not{A}P_L}_{\not{D}_L}) \longrightarrow \text{does not transform covariantly}$$

$$\not{D}_L \to U_R^{-1}\not{D}_L U_L \qquad U_L = e^{iP_L\varepsilon_a T^a} \qquad U_R^{-1} = e^{-iP_R\varepsilon_a T^a}$$

→ At path integral level I use

$\not{D}_L = \not{D}P_L \Rightarrow$ the eigenvalue issue is valid here as well → it maps Hilbert spaces $\mathcal{H}_L \to \mathcal{H}_R$

then:

$$\not{D}_L^\dagger \not{D}_L \Rightarrow \mathcal{H}_L \to \mathcal{H}_L \Rightarrow \text{well defined operator, however what is it?}$$

$$\det(\not{D}^\dagger \not{D}_L) = |\det \not{D}_L|^2 \Rightarrow \text{this is GAUGE INVARIANT!}$$

→ the anomaly is in the IMAGINARY PART of the effective action!

Therefore $W[A] \to \det(\not{D}_L^{(\varepsilon)}) = -\exp^{W[A]}$

$$\hookrightarrow i(p+q)_\mu \Gamma^{\mu\nu\rho}_{abc}(-p-q,p,q) = -\frac{1}{12\pi^2} D^R_{abc}\,\varepsilon^{\nu\rho\lambda\sigma}p_\lambda q_\sigma, \qquad D^R_{abc} = \mathrm{Tr}_R[T_a T_b T_c]$$

In principle we need $O(A^3), O(A^4), \ldots$ terms in order to have the full anomaly!

Anom. are always a LOCAL (it depends on p, q) and FINITE result!

⇝ full form of the anomaly:

everything is Lie alg. valued!

$$\mathcal{A}_a^L(x) = -\frac{1}{24\pi^2}\varepsilon^{\mu\nu\rho\lambda}\,Tr\left[T_a\,\partial_\mu\left(A_\nu\partial_\rho A_\lambda - \frac{i}{4}A_\nu[A_\rho, A_\lambda]\right)\right]$$

Group theoretical factor is D^R_{abc} ⇒ exactly the same factor we found in the quadratic part (it partly generates the cubic part).

→ ANOMALY CANCELLATION

$$\mathcal{A}_a = \sum_i \mathcal{A}^L_{a,i} + \sum_j \mathcal{A}^R_{a,j} \quad (L\text{ ferm} + R\text{ ferm})$$

↳ group theoretical factor: $\sum_{i,L} D^{R,L}_{abc} - \sum D^{R,R}_{abc}$ ⇒ $=0$ in the SM!

In SM we have: ψ_L

ψ_R $\quad \psi_R^c = i\gamma^0 C\psi_R^*$

what's the representation?

$$\underbrace{(T_a^{R_R})^* = -(T_a^{R_R})^T}$$

we can use L particles and $\underline{\underline{L}}$ ANTIPARTICLES under conjugate representation!

→ $\sum_{L,R} D^{R_2}_{abc}$ ⇒ it all boils down to deciding whether this is 0 or not!

⇒ two rep are equiv if $\exists S \mid T_a^{R_1} = S T_a^{R_2} S^{-1}$

⇒ $(T_a^R)^T = -S T_a^R S^{-1}$ (R equiv to $\bar{R}$) ⇒ PSEUDOREAL repres.

($S = I$ ⇒ REAL representation)

Then $D^R_{abc} = 0$ if R is pseudoreal $\Rightarrow$ if the general gauge group has only pseudoreal repr. the anomaly always vanishes! $\longrightarrow$

$$\left.\begin{array}{l} SO(2n+1) \\ SO(4n) \\ Usp(2n) \\ G_2,\ F_4,\ E_7,\ E_8 \end{array}\right] \longrightarrow \text{pseudoreal}$$

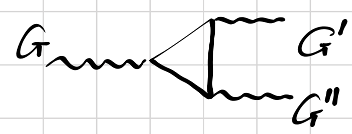
$SU(2) \longrightarrow$ real

$SO(4n+2)$ $n>1$, E_6 $\longrightarrow$ OK by chance

$\Rightarrow$ $SU(N)$ $N\geq 3$ or $U(1)$ $\Rightarrow$ NOT SAFE!

$\Rightarrow$ if one of $G_1 \otimes G_2 \otimes \dots$ is $U(1)$ or $SU(N)$ $\longrightarrow$ potentially anomalous!

e.g.: $G = G_1 \otimes G_2$



$\Rightarrow$ $U(1), U(1), U(1)$ $\rightarrow$ $D_{abc} \sim \sum_i q_i^3$

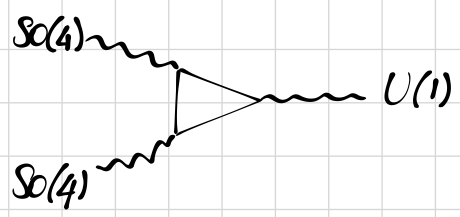
$\Rightarrow$ $U(1), G_s, G_s$ $\rightarrow$ $D_{abc} = 0$

$\rightarrow$ anomalies come from groups which are products of $U(1)$ and $SU(N)$ factors!

$\rightarrow$ Coupling to Gravity:

local $SO(1,3) \longrightarrow SO(4)$

stress-energy tensor $\Rightarrow$ gravity in SM



$\langle J^\mu T T \rangle$

$\longrightarrow$ "MAGIC" CANCELLATION!

$\rightarrow$ What happens with masses?

Dirac: $\bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L$ (need 2 types of fermions)

Majorana: $\bar{\psi}_L \psi_L^c + \bar{\psi}_L^c \psi_L$ (just 1 type of fermion) $\Rightarrow$ we can define it if and only if the rep is real

$\Rightarrow$ NO CONTRIBUTION TO ANOMALY

Now consider:

$$\delta_\varepsilon W[A] = \int d^4x \; \varepsilon^a(x) \mathcal{A}_a(x)$$

$$\delta_\varepsilon A_\mu^a = D_\mu \varepsilon^a$$

Define: $$-\underbrace{\left(D_\mu \frac{\delta}{\delta A_\mu}\right)^a}_{} W[A] = \mathcal{A}^a(x)$$

$$G_a(x) = -\frac{\partial}{\partial x^\mu}\frac{\delta}{\delta A_\mu^a(x)} - f_{abc} A_\mu^b \frac{\delta}{\delta A_\mu^c(x)} \Rightarrow [G_a(x), G_b(y)] = \delta^4(x-y) f_{ab}{}^c G_c(x)$$

$$\rightarrow G_a(x)\mathcal{A}_b(y) - G_b(y)\mathcal{A}_a(x) = f_{abc}\,\delta^4(x-y)\,\mathcal{A}^c(x) \Rightarrow \boxed{\text{WESS-ZUMINO CONSISTENCY RELATION}}$$

(constraint equation)

$\downarrow$

it can be translated into a BRS problem (with ghosts) $\rightarrow$

$$s\mathcal{A} = D\omega$$
$$s\omega = \dots$$