• Introduction to AdS / CFT correspondence [D. Martelli]

1997 Maldacena: $\boxed{\text{GRAVITY}} \longleftrightarrow \boxed{\text{QFT}}$

(d+1)- dimen. d- dimen.

$\Downarrow$

AdS
BULK
(d+1)-dim.

CFT
BOUNDARY
d- dim

If SUSY is present $\Rightarrow$ SCFT on boundary $\longleftrightarrow$ SUGRA in bulk

$\downarrow$

as a limit of string theory

GRAVITY

$$S_{EH} = \frac{1}{2K_D^2}\int d^D x \underbrace{\sqrt{-g}}_{e = \det(e_a{}^\mu)} R \qquad \text{where } R = g^{\mu\nu}R_{\mu\nu}\,, \quad R_{\mu\nu} = R_\mu{}^\lambda{}_{\nu\lambda}$$

$$2K_D^2 = 16\pi G_D$$

* FERMIONS $\longrightarrow$ Orthonormal frame:

$$g_{\mu\nu} = e^a{}_\mu e^b{}_\nu \eta_{ab} \qquad [e^a_\mu := \text{"vielbein"}]$$

$\Rightarrow$ "spin connection": $\omega_\mu{}^{ab} = e_\nu{}^a \Gamma^\nu_{\sigma\mu} e^{\sigma b} + e_\nu{}^a \partial_\mu e^{\nu b}$

"Curvature 2-form": $R_{\mu\nu}{}^{ab} = \partial_\mu \omega_\nu{}^{ab} - \partial_\nu \omega_\mu{}^{ab} + \omega_\mu{}^{ac}\omega_\nu{}^b{}_c - \omega_\nu{}^{ac}\omega_\mu{}^b{}_c$

$\downarrow$

$$R = e^\mu{}_a e^\nu_b R_{\mu\nu}{}^{ab}$$

$\hookrightarrow$ RARITA-SCHWINGER ACTION: $S_{RS} = \frac{1}{2K_D^2}\int d^D x \; e\, \bar\psi_\mu \gamma^{\mu\nu\rho} D_\nu \psi_\rho$

where ψ_μ : gravitino;

$\gamma^{\mu\nu\rho} = \gamma^{[\mu}\gamma^\nu\gamma^{\rho]}$;

$D_\nu \psi_\rho = \partial_\nu \psi_\rho + \frac{1}{4}\omega_\nu{}^{ab}\gamma_{ab}\psi_\rho$.

$\Rightarrow S = S_{EH} + S_{RS} \longrightarrow$ invariant under $\underset{\downarrow}{\text{local}}$ SUSY transf.

$$\varepsilon \to \varepsilon(x)$$

s.t.

$$\delta e_\mu{}^a = \frac{1}{2}\bar{\varepsilon}\,\gamma^a\,\psi_\mu$$

$$\delta\psi_\mu = D_\mu\varepsilon = \partial_\mu\varepsilon + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\,\varepsilon$$

$$\Rightarrow \boxed{\delta S = 0}$$

$\to S_{EH} = \frac{1}{2K_D^2}\int d^Dx\; e\, e^{a\mu} e^{b\nu} R_{\mu\nu ab} \Rightarrow \delta S_{EH} = \frac{1}{2K_D^2}\int d^Dx\left[-\bar{\varepsilon}\,\gamma^\mu\psi^a e^{b\nu} R_{\mu\nu ab} + e\frac{1}{2}\bar{\varepsilon}\gamma^\rho\psi_\rho R\right]$

$$= \frac{1}{2K_D^2}\int d^Dx\; e\left[\frac{1}{2}R\,g_{\mu\nu} - R_{\mu\nu}\right]\bar{\varepsilon}\,\gamma^\mu\psi^\nu.$$

$\to \delta S_{RS} = \frac{1}{2K_D^2}\cdot\frac{1}{2}\int d^Dx\; e\,\bar{\psi}_\mu\gamma^{\mu\nu\rho}R_{\nu\rho}{}^{ab}\gamma_{ab}\,\varepsilon = \ldots = -\delta S_{EH}$

$$R_{\nu\rho}{}^{ab} \to \left(D_{[\nu}D_{\rho]}\right)^{ab}$$

$$\delta(D_\nu\psi_\rho) = D_\nu\delta\psi_\rho + \cancel{\text{cubic in } \psi_\mu}$$

$\Rightarrow$ if we keep higher orders we have to study case by case!
↳ add more fields!

$\to$ EOM: $\begin{cases} R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = T_{\mu\nu}(\psi_\mu) \\ \gamma^{\mu\nu\rho}D_\nu\psi_\rho = 0 \end{cases}$

$\Rightarrow$ set fermions $= 0$: $\psi_\mu = 0 \Rightarrow R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0 \Leftrightarrow R_{\mu\nu} = 0$ "RICCI-FLAT"

These sol. are connected to SUSY solut:

$$\delta e_\mu{}^a = 0 \;\checkmark$$

$$\delta\psi_\mu = \boxed{D_\mu\varepsilon = 0} \;\checkmark \qquad \left(D_\mu = \partial_\mu + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\right)$$

Killing Spinors / Kill spin. eqn.

non trivial eqn for: 1) $g_{\mu\nu}$ 2) ε

Solutions: $\varepsilon = \text{const}$ (s.t. $\partial_\mu \varepsilon = 0$)

$g_{\mu\nu} = \eta_{\mu\nu} \Rightarrow \omega_\mu{}^{ab} = 0 \Rightarrow D_\mu \varepsilon = 0$

NB $R_{\mu\nu} = 0$; $D_\mu \varepsilon = 0 \Rightarrow$ "SPECIAL HOLONOMY MANIFOLDS"

$\hookrightarrow$ SUGRA/SUSY $\longleftrightarrow$ GEOMETRY

Now add a COSMOLOGICAL CONST.:

$$S = \frac{1}{2K_D^2} \int d^D x \; e \left[R - \bar{\psi}_\mu \gamma^{\mu\nu\rho} D_\nu \psi_\rho + \frac{(D-1)(D-2)}{L^2} - \frac{(D-2)}{2L} \bar{\psi}_\mu \gamma^{\mu\nu} \psi_\nu \right]$$

$\hookrightarrow$ $\delta e_\mu{}^a$ = UNCHANGED

cannot be absorbed in ω_μ (only one γ_μ)

$$\delta \psi_\mu = \hat{D}_\mu \varepsilon = D_\mu \varepsilon - \frac{1}{2L} \gamma_\mu \varepsilon$$

$\Rightarrow$ e.o.m.: $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{1}{2L^2} (D-1)(D-2) g_{\mu\nu}$

where we took a background s.t. $\psi_\mu \equiv 0$.

Now take the trace and simplify: $R_{\mu\nu} = - \frac{D-1}{L^2} g_{\mu\nu}$

$\Rightarrow$ "EINSTEIN SPACES"

Further trace: $R = - \frac{D(D-1)}{L^2} < 0 \Rightarrow$ NEGATIVE CURVATURE!

We now have $D_\mu \varepsilon = \frac{1}{2L} \gamma_\mu \varepsilon$ (which $\Rightarrow R_{\mu\nu} = - \frac{(D-1)}{L^2} g_{\mu\nu}$)

ANTI de SITTER (AdS_D)

AdS_D is a solution to (maximally (super) symm.)

$$\begin{cases} R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{1}{2L^2}(D-1)(D-2) g_{\mu\nu} \\ D_\mu \varepsilon = \frac{1}{2L} \gamma_\mu \varepsilon \end{cases}$$

$$\Rightarrow R_{\mu\nu\rho\sigma} = -\frac{1}{L^2}\left(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}\right) \qquad \begin{pmatrix}\text{MAX} \\ \text{SYMM}\end{pmatrix}$$

	$\Lambda < 0$	$\Lambda > 0$
Lorentzian	AdS_D	dS_D
Euclidean	H_D	S^D

$\hookrightarrow$ "Hyperbolic spaces"

Take $\mathbb{R}^{2,D-1} \longrightarrow X_0, X_D, X_i \qquad i = 1, \dots, D-1$

$$\Rightarrow X_0^2 + X_D^2 - \sum_{i=1}^{D-1} X_i^2 = L^2$$

L → RADIUS OF AdS (even though it's not compact)

$$\hookrightarrow ds^2\left(\mathbb{R}^{2,D-1}\right) = -dX_0^2 - dX_D^2 + \sum_{i=1}^{D-1} dX_i^2$$

Isometry group $O(2, D-1)$.

Introduce

"Global AdS":

$$X_0 = L \cosh\rho \cos\tau \qquad \rho \in \mathbb{R}^+$$
$$\tau \in [0, 2\pi]$$
$$X_D = L \cosh\rho \sin\tau$$
$$X_i = L \sinh\rho \, \omega_i \qquad \text{s.t. } \sum_i \omega_i^2 = 1$$

ω_i: coord on S^{D-2}

$$\Rightarrow ds^2(AdS_D) = L^2\left(-\cosh^2\rho \, d\tau^2 + d\rho^2 + \sinh^2\rho \, d\Omega_{D-2}^2\right)$$

$d\tau^2$ ↓ "universal cover": $\tau \in [0, 2\pi] \longrightarrow \mathbb{R}$.

NB: $\rho \to 0 \Rightarrow \mathbb{R}_\tau \times \mathbb{R}^{D-1}$

$\rho \to \infty \Rightarrow$ "CONFORMAL BOUNDARY"

New introduce Poincaré coord:

$$X_0 = \frac{1}{2r}\left(1 + r^2(L^2 + \vec{x}^2 - t^2)\right)$$
$$X_D = L r t$$
$$X_i = L r x_i \qquad i = 1, \dots, D-2$$
$$X_{D-1} = \frac{1}{2r}\left(1 - r^2(L^2 - \vec{x}^2 + t^2)\right)$$

$$\hookrightarrow ds^2 = L^2\left(\frac{dr^2}{r^2} + r^2 \underbrace{\left(-dt^2 + d\vec{x}^2\right)}_{dx^\mu dx_\mu}\right)$$

Minkowski

$\Rightarrow r \to \infty$: "CONFORMAL BOUNDARY" $\left(ds^2 \sim L^2 r^2 dx^\mu dx_\mu\right)$

$r \to 0$: HORIZON

Other coords: $\left(r = \frac{1}{z}\right)$

$$ds^2 = L^2 \, \frac{dz^2 + dx^\mu dx_\mu}{z^2} \Rightarrow z \to 0 : \text{"CONF. BOUND"}$$

and $(r = e^y)$

$$ds^2 = L^2\left(dy^2 + e^{2y} dx^\mu dx_\mu\right)$$

MINIMAL GAUGED SUGRA IN D = 4

⇒ FIELD CONTENT:

Majorana $\quad \psi_\mu = \psi^1_\mu + i\psi^2_\mu$ (Dirac)

$$e_\mu^a \text{ (graviton)} + \psi_\mu^1 \text{ (gravitino)} \quad + \quad A_\mu, \; \psi_\mu^2$$

⇒ A_μ gauging on $SO(2) \sim U(1)$ R-symmetry

Change notation: $D_\mu = \nabla_\mu - \frac{i}{L} A_\mu = \partial_\mu + \frac{1}{4}\omega_\mu^{ab}\gamma_{ab} - \frac{i}{L}A_\mu$

$$S = \frac{1}{2K_D^2}\int d^4x \, e\Big[R - 2\bar{\psi}_\mu \gamma^{\mu\nu\rho} D_\nu \psi_\rho + \frac{2}{L}\bar{\psi}_\mu \gamma^{\mu\nu}\psi_\nu + \frac{6}{L^2} -$$

"Einstein-Maxwell" $(+\,\Lambda)$

$$- F_{\mu\nu}F^{\mu\nu} + \frac{i}{4}F^{\mu\nu}\bar{\psi}_\rho \gamma_{[\mu}\gamma^{\rho\sigma}\gamma_{\nu]}\psi_\sigma\Big]$$

$$\Rightarrow \delta e_\mu^a = \mathrm{Re}\left[\bar{\varepsilon}\gamma^a\psi_\mu\right]$$

$$\delta\psi_\mu = \hat{D}_\mu \varepsilon \qquad \left(\hat{D}_\mu = \nabla_\mu + \frac{1}{2L}\gamma_\mu + \frac{i}{4}F_{\nu\rho}\gamma^{\nu\rho}\gamma_\mu\right)$$

$$\delta A_\mu = \mathrm{Im}\left[\bar{\varepsilon}\psi_\mu\right]$$

⇒ KSE: $\hat{D}_\mu \varepsilon = 0 \Rightarrow$ eqn for $g_{\mu\nu}$, A_μ and ε

e.o.m.: Maxwell: $d * F = 0$

Einstein: $R_{\mu\nu} + \frac{3}{L^2}g_{\mu\nu} = 2\left(F_\mu{}^\rho F_{\nu\rho} - \frac{1}{4}F^2 g_{\mu\nu}\right)$

$(F_{\mu\nu} = 0 \longrightarrow AdS_4$ is the solution$)$

We are interested in solutions that <u>at least</u> approach AdS ASYMPTOTICALLY.

$$\hat{D}_\mu \varepsilon = 0 \longrightarrow$$

Killing vec.

e.g.: $V_\mu = i\bar{\varepsilon}\gamma_\mu\varepsilon \longrightarrow$ real vector s.t. 1) $\nabla_{(\mu}V_{\nu)} = 0 = \mathcal{L}_V g_{\mu\nu}$

2) $\mathcal{L}_V F_{\mu\nu} = 0$

Simple ex of SUSY sol:

* "SELF-DUAL" sol.: $*F = -F \Rightarrow R_{\mu\nu} = -\frac{3}{L^2} g_{\mu\nu}$

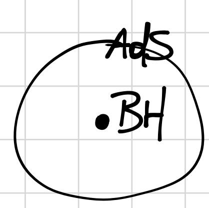
$(d*F = 0)$

Weyl tensor

$\hookrightarrow$ plug into KSE: $C_{\mu\nu\rho\sigma} = \frac{1}{2}\varepsilon_{\mu\nu\tau\lambda}C^{\tau\lambda}{}_{\rho\sigma}$

(QUATERNIONIC KÄHLER)

* AdS_4 BH



$$ds^2 = -f(r)\,dt^2 + \frac{1}{f(r)}dr^2 + r^2 \underbrace{ds^2(H^2)}_{\text{metric on hyperb.}}$$

(as $ds^2(S^2)$)

$$f(r) = \left(\frac{r}{L} - \frac{L}{2r}\right)^2 \Rightarrow \text{SUSY PRESERVED}$$

Compare REISSNER-NORDSTRÖM BH: $f_{RN}(r) = 1 - \frac{2MG}{r} + \frac{q^2 G}{4\pi r^2}$

$\hookrightarrow$ · NO SUSY (RN) vs SUSY (AdS_4)

· $r \to \infty$: $f(r) \sim \frac{r^2}{L^2}$: asymptotically AdS_4

$$ds^2 \overset{r\to\infty}{\sim} \frac{L^2}{r^2}dr^2 + r^2\left(-\frac{dt^2}{L^2} + ds^2(H^2)\right)$$

$\hookrightarrow$ not exactly S^2

$f_{RN}(r) \sim 0$: asymptotically FLAT

$\hookrightarrow$ for $\frac{q^2}{4\pi G} = M^2$ we have BPS BH (SUSY)

The fact that $f(r)$ is SUSY $\Rightarrow$ sth similar to BPS!

NB: $D=4$ gauged SUGRA can be embedded in $D=11$ SUGRA

D = 11 SUGRA

Field content: $e_\mu^{\ a}$ $\quad \psi_\mu$ (Majorana) $\quad$ (graviton + gravitino)

$$C_{\mu\nu\rho} \longrightarrow G = dC \quad (\text{"flux"})$$

$$\Rightarrow S_{11} = \frac{1}{2K_{11}^2}\int d^{11}x \left[\sqrt{-g}\left(R - \frac{1}{2}G^2\right) - \frac{1}{8}\, C\wedge G\wedge G\right] + \cancel{\text{fermions}}$$

$\hookrightarrow$ KSE: $\quad 0 = \delta\psi_\mu = \hat{D}_\mu \varepsilon \quad$ where $\hat{D}_\mu = \nabla_\mu + \frac{1}{288}\,\times$

$$\times\left(\gamma_\mu^{\ \nu\rho\sigma\tau} - 8\,\delta_\mu^{\ \nu}\gamma^{\rho\sigma\tau}\right)G_{\nu\rho\sigma\tau}$$

e.o.m.: $\quad R_{\mu\nu} = \dots$

8-form $\longleftarrow$ $\underbrace{d \underbrace{* G}_{11-4=7\text{ form}}}_{7+1=8\text{ form}} + \frac{1}{2}\underbrace{G\wedge G}_{4+4=8\text{ form}} = 0$ $\quad$ "Maxwell eqn."

$\underline{D=11 \text{ SUGRA}}$ is usual referred to as low-energy limit of M-theory

$\hookrightarrow$ it contains all the others through dim. reduction (KK)

$$\begin{array}{ccc}
D=11 \text{ SUGRA} & \xleftarrow{\text{low en.}} & \text{M-theory} \\
\downarrow \text{reduction} & & \downarrow \text{reduction} \\
D=10 \text{ IIA SUGRA} & \xleftarrow{\text{low en.}} & \text{Type IIA superstring th.} \\
\updownarrow \text{T-duality} & & \updownarrow \text{T-duality} \\
\text{IIB SUGRA} & \longleftarrow & \text{Type IIB} \quad ''
\end{array}$$

Two well known solutions:

* $AdS_4 \times S^7$ with $G \propto \mathrm{vol}(AdS_4)$
* $AdS_7 \times S^4$ $\quad ''\quad$ $G \propto \mathrm{vol}(S^4)$

They both preserve 32 SUSY charges $\longrightarrow$ maximally SUSY

$\Rightarrow$ Type IIA [& Type IIB] SUGRA

Reduction: $\mu = 0, \dots, 10$

$$\psi_\mu < \begin{array}{ll} \psi_i^+ , \ \psi_i^- & \text{(Majorana Weyl GRAVITINOs)} \\ \lambda^+ , \ \lambda^- & \text{(Majorana-Weyl DILATINOs)} \end{array}$$

$$C_{\mu\nu\rho} < \begin{array}{llll} C_{ijk} \longrightarrow A^{(3)}_{ijk} & \text{RR field} & \Longrightarrow & \text{D2 (el), D4 (mag)} \\ C_{11\,ij} \longrightarrow B_{ij} & \text{NSNS field} & & \end{array}$$

F1 (el) (string), NS5 (mag)

$$g_{\mu\nu} < \begin{array}{llll} g_{ij} & & & \\ g_{11,11} \longrightarrow e^\phi & \text{NSNS dilaton} & & \\ g_{11\,i} \longrightarrow A^{(1)}_i & \text{RR field} & \Longrightarrow & \text{couples to D0 (elect), D6 (magn)} \end{array}$$

$\Downarrow$

Type IIA SUGRA (= massless Type IIA Superstr th.)

NB: $C^{(p+1)} \Longrightarrow * dC^{(p+1)} = d\tilde{C}^{(\tilde{p}+1)} \Rightarrow \tilde{p} = 6 - p$

In SUGRA D-branes are sol. to e.o.m.

e.g. D2-brane solution $\longrightarrow A^{(3)}_{ijk} \quad g_{ij} \quad e^\phi$

What about Type IIB spectrum?

NSNS spectrum is universal $\Rightarrow B_{ij} \quad g_{ij} \quad e^\phi$ ✓ $\longrightarrow$ F1, NS5

$$\begin{array}{llll} \text{RR} \Rightarrow & A^{(0)} & A^{(2)} & A^{(4)} \\ & \downarrow & \downarrow & \downarrow \\ & D(-1) & D1 & D3 \\ & D7 & D5 & D3 \end{array}$$

$F^{(5)} = * F^{(5)}$ (SELF-DUAL)

$F^2 = 0$

Now focus on D-brane solutions.

Set to 0 all unnecessary fields. From string th.:

IIA, IIB

string frame

$$S_{sf} = \frac{1}{(2\pi)^7 \ell_s^8} \int d^{10}x \sqrt{-g} \left\{ e^{-2\phi}\left(R + 4\,\partial_\mu\phi\,\partial^\mu\phi\right) - \frac{2}{(8-p)!} F_{p+2}^2 \right\}$$

$F_{p+2} = dA_{p+1}$

⇓ to Einstein frame

$$g^{Ef}_{\mu\nu} = e^{-\phi/2}\, g^{sf}_{\mu\nu}$$

$$\ell_s^2 = \alpha'$$

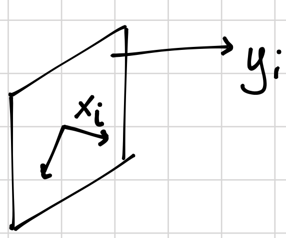
coupling const: $g_s = \langle e^{\phi} \rangle$

depends on p

$$\hookrightarrow S_{Ef} = \frac{1}{(2\pi)^7 \ell_s^8} \int d^{10}x \sqrt{-g^E}\left(R_E - \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{2}{(8-p)!}\, e^{a_p \phi} F_{p+2}^2\right)$$

Derive e.o.m. and consider the Ansatz:

$$ds^2 = H_p^{-1/2}(r)\left(-dt^2 + dx_1^2 + \ldots + dx_p^2\right) + H_p^{1/2}(r)\left(dy_1^2 + \ldots + dy_{9-p}^2\right)$$



x_i: on the brane

y_i: orth. dir.

$$\Rightarrow r^2 = y_1^2 + \ldots + y_{9-p}^2$$

$\Rightarrow$ Potential: $A^{(p+1)}_{t x_1 \ldots x_p} = -\frac{1}{2}\left(H_p(r)\right)^{-1}$ and $e^{\phi} = g_s \left(H_p(r)\right)^{\frac{3-p}{4}}$

s.t. $r \to \infty$ → recover Minkowski

→ plug into e.o.m. $\Rightarrow$ $H_p(r) = 1 + \dfrac{L^{7-p}}{r^{7-p}}$ ⟶ Harmonic in transverse space

constant + quant. cond.:

$$\frac{1}{(2\pi \ell_s)^{p+1} g_s} \int_{\underbrace{S^{8-p}}_{\text{cycle wrapping the brane at } \infty}} * F_{p+2} = N \in \mathbb{N} \Rightarrow L^{7-p} = (2\pi)^{p-2} d_p\, \ell_s^{7-p}\, g_s N$$

no. of p-branes

... other factors of 2π ...

Remarks:

$\tfrac{1}{2}$ SUSY

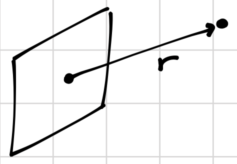
* these are SUSY solutions (they solve the KSE)
* $\exists$ non-SUSY versions called "black p-branes"
* for general p they are singular as $r \to 0$ (except for $p = 3$)

Choose $p = 3$:

$$e^{\phi} = g_s$$

$$H_3(r) = 1 + \frac{L^4}{r^4}, \qquad L^4 = 4\pi g_s (\alpha')^2 N = (2\pi)^4 \frac{g_s (\alpha')^2 N}{4 \underbrace{\mathrm{Vol}(S^5)}_{\pi^3}}$$

i.e. "near brane"



Consider $p = 3$. The limit $r \to 0$ ("near horizon" limit)

Let

$$\alpha' \to 0 \quad \text{with} \quad u = \frac{r}{\alpha'} \quad \text{FIXED} \quad (\text{energy scale})$$

$$G_{10} = 2^3 \pi^6 (\alpha')^4 g_s^2 \longrightarrow 0 \qquad \text{keep } g_s \text{ fixed}$$

$\Downarrow$

brane theory DECOUPLES FROM THE BULK

and α' corrections in S_{SUGRA} are suppressed.

Then we find:

flat Minkowski

$$H_3 = H \xrightarrow{\alpha' \to 0} \frac{4\pi g_s N}{u^4 (\alpha')^2} \Rightarrow ds^2 = \alpha' \left[(4\pi g_s N)^{1/2} \left(\frac{du^2}{u^2} + ds^2(S^5) \right) + \frac{u^2}{(4\pi g_s N)^{1/2}} dx^\mu dx_\mu \right]$$

NB: the N branes are all on top of each other

in spher. coord.

$\Rightarrow$ Near brane geometry looks like $AdS_5 \times S^5$ $\quad$ [L = radius of AdS_5 = $\quad$ = " S^5]

D3 preserves SUSY (all 32 actually...)

The "curvature" $\sim$ set by L

$\hookrightarrow$ Small curvature $\Rightarrow \frac{L^4}{\ell_s^4} \gg 1 \Rightarrow g_s N \gg 1$

Moreover to string loop corrections $\quad N \gg 1$ (fixed g_s)

$\downarrow$

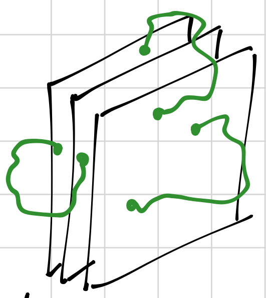
view SUGRA from ST low en. - limit $\Rightarrow \alpha' \to 0$ suppresses stringy corrections.

$\Rightarrow N \gg 1 \quad$ " the loop contrib.

D-BRANE SYM THEORY

1) D-brane $\rightarrow$ SUGRA SOL. of CLOSED STRING

$\searrow$ EXTENDED OBJ. with (open) STRINGS ATTACHED



Minkowski: $\mathbb{R}^{1,p}$

N Dp-branes

$\longrightarrow$ low energy effective th. on N coincident Dp-branes

$\Downarrow$

$(p+1)$-dim. SYM

$G = U(N)$

- FIELDS:

  * non Abelian A_μ
  * $9-p$ scalars in ADJ. REP of $U(N)$
  * gauginos λ

$\Downarrow$

SUSY fixes uniquely the action

$\mathcal{N}=1$ Dp-brane : $S_{DBI} = -T_p \int d^{p+1}x \, e^{-\phi} \sqrt{-\det(g + 2\pi\alpha' F)}$

$$T_p = \frac{1}{(2\pi)^p (\alpha')^{\frac{p+1}{2}} g_s}$$

Now take $\alpha' \to 0$ with $g_s \alpha'^{\left(\frac{p-3}{2}\right)}$ fixed $\Rightarrow S \sim \frac{1}{g_{YM}^2} \int d^{p+1}x \, e^{-\phi} F^2 \Rightarrow g_{YM}^2 = g_s (2\pi)^{p-2} (\alpha')^{\frac{p-3}{2}}$

For $p=3 \longrightarrow 4\pi g_s = g^2_{YM}$:

- A_μ in $1+3$ dim
- 6 scalar
- λ
- 16 SUSY $\longrightarrow \boxed{\mathcal{N}=4 \text{ SYM}}$

$\Rightarrow \mathcal{L}$ has manifest SU(4) GLOBAL SYMM: $so(4) \sim so(6)$

$\Downarrow$

R-SYMMETRY

6 scalars: Φ^I in $\underline{6}$ of $SU(4)$ $\quad I=1,..,6$

4 fermions: λ_a in $\underline{4}$ of $SU(4)$ $\quad a=1,\dots,4$

$\hookrightarrow$ supercharges: Q_a in $\underline{4}$ of $SU(4)$

$\longleftrightarrow \mathcal{N}=4$

$\Rightarrow$ FULL SYMM. of the $\mathcal{L}$ $\Rightarrow$ Poincaré in $p+1$ dimensions $\times SO(6)$

$\Downarrow$

4 P_μ

6 $J_{\mu\nu}$

$\mathcal{N}=4$ SYM is a conformal theory $\left\{ \begin{array}{ll} 1 & D \\ 4 & K_\mu \end{array} \right.$

15 generators $\Longrightarrow SO(2,4)$

$\rightsquigarrow$ "actual" full symmetry: $SO(2,4) \times SO(6)$ (bosonic)

$\Rightarrow$ MALDACENA 1997:

The following are equivalent:

① Type IIB ST in background of $AdS_5 \times S^5$ w/ N units of RR flux and = radius of AdS_5 and S^5

$$L = \left(4\pi g_s (\alpha')^2 N\right)^{1/4}$$

② $\mathcal{N}=4$ SYM in $d=1+3$ w/ $G=SU(N)$ s.t.

$$g_{YM}^2 = 4\pi g_s$$

This referred to as "STRONG FORM" of the correspondence.

In ST side we have

$\alpha' \to 0$

$g_s N \gg 1$ for the validity of SUGRA.

$N \gg 1$

Define 't Hooft coupling:

$$\lambda := g_{YM}^2 N$$

where • $N \to \infty$ at fixed $\lambda \Rightarrow$ 't Hooft / "large N" limit

• curvature $\sim \frac{1}{L^2} \sim \frac{1}{\lambda^{1/2}\alpha'}$

$$\hookrightarrow \mathcal{L}_{eff} = \frac{\alpha'}{L^2}\mathcal{L}_1 + \left(\frac{\alpha'}{L^2}\right)^2 \mathcal{L}_2 + \ldots =$$

$$= \lambda^{-\frac{1}{2}}\mathcal{L}_1 + \lambda^{-1}\mathcal{L}_2 + \ldots \quad \Rightarrow \quad \alpha' \to 0 \sim \lambda \to \infty .$$

LARGE N LIMIT

Gravity $\Rightarrow$ large no. of branes

QFT side $\Rightarrow$ consider th. containing ADJ (of $U(N)$ in part.)

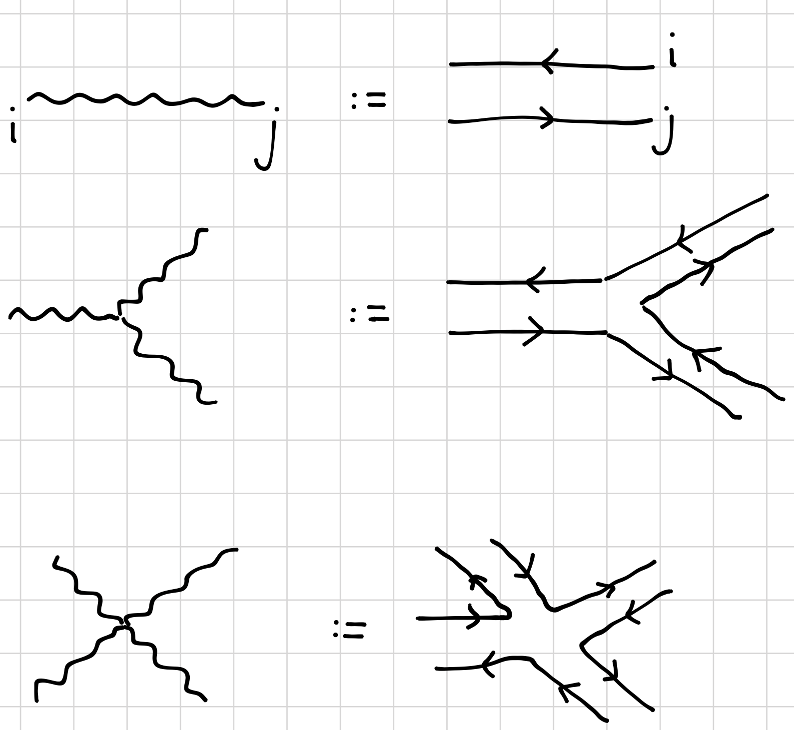
$$\boxed{\ln Z = \sum_{g=0}^{\infty} N^{2-2g} f_g(\lambda)}$$

$\Rightarrow$ for large N $\rightarrow$ $g=0$ dominates: $\ln Z \overset{N\to\infty}{\sim} N^2 f_0(\lambda) =$

$$= N^2\left(c_0 + c_1\lambda + c_2\lambda^2 + \dots\right)$$

$\Rightarrow$ OK if $\lambda \ll 1$ but this is <u>not</u> the case.

Use the "DOUBLE LINE NOTATION":

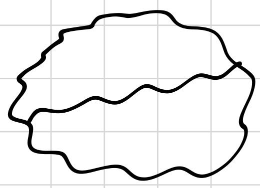
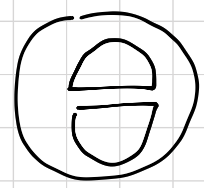


$$\Rightarrow \mathcal{L}_{YM} \sim (DA)^2 \sim \left(dA + g_{YM} A^2\right)^2 = dA^2 + g_{YM} A^2 dA + g_{YM}^2 A^4$$

Then introduce $A \rightarrow \frac{1}{g_{YM}} A \Rightarrow \mathcal{L}_{YM} = \frac{N}{\lambda}\left(dA^2 + A^3 + A^4\right)$

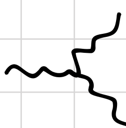
$\hookrightarrow$ in double line notation, for any loop $\Rightarrow$ factor N

Consider for example:

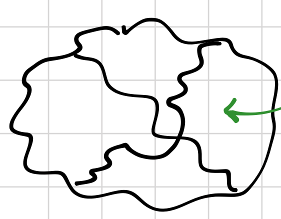
 = 

$\rightarrow$ 3 loops $\rightarrow N^3$

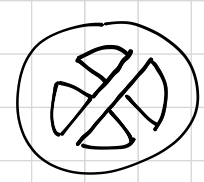
$\rightarrow$ 3 propagators $\rightarrow \left(\frac{\lambda}{N}\right)^3$

$\rightarrow$ 2  $\rightarrow \left(\frac{N}{\lambda}\right)^2$

$$\Rightarrow \mathcal{A} \sim N^2 \lambda = N^3 g_{YM}^2$$



not a vertex

= 

$\rightarrow$ 2 loops $\rightarrow N^2$

$\rightarrow$ 6 prop $\rightarrow \left(\frac{\lambda}{N}\right)^6$

$\rightarrow$ 4 vert. $\rightarrow \left(\frac{N}{\lambda}\right)^4$

$$\Rightarrow \mathcal{A} \sim N^0 \lambda^2$$

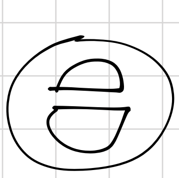
Therefore:

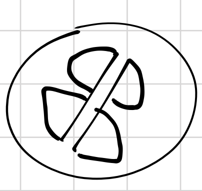
no. of loops, no. of prop, no. of vert

$$N^{L} \left(\frac{\lambda}{N}\right)^{E} \left(\frac{N}{\lambda}\right)^{V} = N^{L+V-E} \lambda^{-V+E} = N^{\chi} \lambda^{E-V}$$

Euler charact.:

CLOSED SURF. : $\chi = 2 - 2g$ ← genus

N.B:  : $g = 0 \longrightarrow$ we can draw it on a sphere
(PLANAR DIAGRAM)

 $\longrightarrow g = 1 \Rightarrow$ TORUS $(\chi = 0)$

$\Rightarrow$ LARGE $N \longrightarrow$ PLANAR LIMIT (strong 't Hooft coupling)

→ MATCHING OF SYMMETRIES

QUANTUM

• $\mathcal{N}=4$ SYM

• $AdS_5 \times S^5$

CLASSIC

$$\underbrace{\underbrace{SO(2,4)}_{\text{conformal group}} \times \overbrace{SU(4)}^{\text{R-symm.}} + 32 \text{ supercharges}}_{SU(2,2|4) \text{ SUPERALGEBRA}}$$

isometry of S^5

isometry of AdS_5

REMARKS:

* keep AdS_5
replace $S^5 \longrightarrow M_5$: we can break (some) SUSY w/ diff. symm. on QFT side

manifold (e.g.: Sasaki-Einstein)

⇒ "PHYSICS IN THE BULK" = "PHYSICS ON THE BOUNDARY"

Therefore:

AdS_5 :

$g_{\mu\nu}$
scalars
etc.
} w/ prescribed b.c.

CFT (e.g.: $\mathcal{N}=4$ SYM)

A_μ, ϕ^I, λ^a in ADJ of $SU(N)$

NOT gauge inv.
(not physical obs.)

I CAN'T SIMPLY COMPARE THEM!
(on AdS_5 there are no gauge indices)

We must assemble fields in gauge inv. operators. E.g.:

$$\mathrm{Tr}\left(F_{\mu\nu}F^{\mu\nu}\right)$$

$$\mathrm{Tr}\left(\phi^I\phi^J\phi^K_{\dots}\right)$$

$$\mathrm{Tr}\left(F_{\mu\nu}F^{\mu\nu}\phi^I\right)$$

We need to derive from ST a 5D theory $\Rightarrow$ KK REDUCTION

$\Rightarrow$ KK SPECTRUM of TYPE IIB on $AdS_5 \times S^5$

x = coordinates on AdS_5

y = " S^5

Generic field:

$$\phi(x,y) = \sum_n \phi_n(x)\, Y_n(y)$$ spherical armonics on S_5

The KK spectrum (in 5 steps):

1) Write every field of Type IIB SUGRA:

$$\phi = \mathring{\phi} + \delta\phi$$

$\delta\phi$: FLUCTUATIONS

$\mathring{\phi}$: BACKROUND $AdS_5\times S^5$ (it solves e.o.m.)

2) Plug e.o.m. and linearize around $\mathring{\phi} \Rightarrow$ linear eq. for $\delta\phi$

3) $\delta\phi = \Sigma\,(\text{harmonics})$

4) Coupled eq.'s $\longrightarrow$ DIAGONALIZE the linear system

5) COMPUTE EIGENVALUES

The symm. is $SU(2,2|4) \rightarrow$ eigenv. will be labelled by "QUANTUM No.s"

↳ these modes are the "fields in AdS_5"

Back in $\mathcal{N}=4$ SYM:

- focus on $SO(2,4) \supset SO(1,1) \times SO(1,3)$

$$\underset{\substack{\text{DILATATION} \\ (D)}}{SO(1,1)} \qquad \underset{\substack{\text{LORENTZ} \\ (J_{\mu\nu})}}{SO(1,3)}$$

Under D:

$$x^\mu \rightarrow \omega x^\mu$$

$$\mathcal{O}(x) \rightarrow \omega^{\Delta} \mathcal{O}(\omega x)$$

scaling dim.

↳ $$\left.\begin{array}{l} [D, P_\mu] = -i P_\mu \\ [D, K_\mu] = i K_\mu \end{array}\right\} \text{classical} \rightarrow \hat{D}\,\mathcal{O} = -i\Delta\,\mathcal{O}$$

$$\Downarrow$$

$$D(P_\mu \mathcal{O}) = -i(\Delta+1) P_\mu \mathcal{O}$$
$$D(K_\mu \mathcal{O}) = -i(\Delta-1) K_\mu \mathcal{O}$$

Unitarity requires a finite spectrum s.t. $K_\mu \mathcal{O}_{Low} = 0$.

$$\Downarrow$$

$$K_\mu \mathcal{O}_{lowest} = 0 \qquad (\text{analogous to } \alpha_n|0\rangle,\ n>0)$$

$$\Downarrow$$

"PRIMARY OP."

(acting w/ P_μ I find "descendents")

Use SUSY $\Rightarrow$ we have Q's:

$$[D, Q] = -\frac{i}{2} Q \quad \longrightarrow \quad \Delta(Q\mathcal{O}) = \Delta + \frac{1}{2}$$

in $\mathcal{N}=4$ there are 16 Q's

$\Rightarrow$ "LONG REP" of superalg.

↳ they all form the same rep

$\rightarrow$ in this case "Δ is not protected" $(\Delta = \Delta(N, \lambda))$

↳ NB: SUGRA is a specific limit for $\lambda \Rightarrow$ op. in CFT must not depend on λ

$\Rightarrow$ if (comb. of Q's) $\mathcal{O} = 0 \Rightarrow \mathcal{O}$ is called CHIRAL PRIMARY

↳ "SHORT REP."

↳ for this rep, Δ is UNIQUELY FIXED IN TERMS OF THE R-SYMMETRY $(SU(4))$

$\Rightarrow$ R-charges ~ integers

(for $\mathcal{N}=4$: (q_1, q_2, q_3) Casimirs of $SO(2)^3 \subset SO(6)$)

TOTALLY SYMM., TRACELESS OF $SO(6)$

e.g.:

$$\mathcal{O}_n = \mathrm{Tr}\left(\phi^{(I_1} \dots \phi^{I_n)}\right)$$

↳ $\Delta = n, \quad n \geq 2$

Then $$Q^4 \mathcal{O}_{n+2} = \mathrm{Tr}\left(F_{\mu\nu} F^{\mu\nu} \phi^{(I_1} \dots \phi^{I_n)}\right) \longrightarrow \Delta = n+4 \quad n \geq 0$$

↳ on the AdS_5 part those corresp. to scalars:

$$(\Box_{AdS_5} - m^2)\phi = 0$$

m^2 comes from KK-red $\Rightarrow$ Casimirs of $SO(6)$

$$\Rightarrow \Delta = 2 + \sqrt{4 + m^2 L^2} \quad \text{on } AdS_5$$

$$\left[\Delta = \frac{1}{2}\left(d + \underbrace{\sqrt{d^2 + 4m^2L^2}}\right) \quad \text{on } AdS_{d+1}\right]$$

$$d^2 + 4m^2L^2 \geq 0 \iff \underbrace{m^2 \geq -\frac{d^2}{4L^2}}_{\text{even the tachyon has bounds}}$$

$\Rightarrow$ BF BOUND

THE AdS/CFT "MASTER FORMULA"

⇒ must answer two questions:

1) What is the physical motivation to associate a field ϕ in the bulk to an operator $\mathcal{O}$ on the boundary?
2) In a CFT the observables are

$$\langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle$$

How do I relate them to the gravity side?

$$\Downarrow$$

$$\boxed{\begin{gathered} Z_{QFT}[J] = Z_{string}[\phi_0] \\ \downarrow \\ J = \phi_0 \end{gathered}}$$

where

$$Z_{QFT}[J] = \int \mathcal{D}[\underbrace{\text{dynam. fields}}_{\mathcal{N}=4\ SYM:\ A_\mu,\ \phi^I,\ \lambda^a}]\ e^{-S_{QFT} + \int d^4x\, \mathcal{O}(x) J(x)}$$

composite op.s sources

$$\Rightarrow \langle \mathcal{O}(x_1) \dots \mathcal{O}(x_L) \rangle = \frac{\delta^L}{\delta J(x_1) \dots \delta J(x_L)} Z_{QFT}[J] \Big|_{J=0}$$

On the gravity side:

$$Z_{string}[\phi_0] = \int \mathcal{D}[\text{dyn. fields}]\ e^{-S_{string}}$$

$$\phi_0 = \phi \big|_{\partial AdS_5}$$

Then $\quad Z_{QFT}[J] = Z_{string}[\phi_0] \longrightarrow J = \phi_0$

We use the "WEAK FORM" (computable):

$$\begin{array}{l} N \to \infty \\ \lambda \sim \dfrac{1}{(\alpha')^2} \to \infty \end{array} \Rightarrow Z_{string}[\phi_0] = \underbrace{e^{-N^2 S_{SUGRA}[\phi]}}_{} + O(\alpha') \times \begin{pmatrix}\text{stringy} \\ \text{correct.}\end{pmatrix}$$

$$\Downarrow$$

$$\boxed{Z_{QFT}[J] = e^{-N^2 S_{SUGRA}[\phi]}}$$

We then need to compute $S_{SUGRA}[\phi_0]$:

1) solve e.o.m. for ϕ w/ prescribed $\phi|_{\partial AdS} = \phi_0$

2) plug ϕ back into $S_{SUGRA} \longrightarrow S[\phi_0]$

Remarks:

a) S is typically divergent $\longrightarrow$ HOLOGRAPHIC RENORMALIZATION

b) in practice S_{SUGRA} is an effective action $(d=5)$ from KK reduction from Type IIB ST $(d=10)$.

In $d=5$: $\quad g_{\mu\nu} = \bar{g}_{\mu\nu}(AdS_5) + \delta g_{\mu\nu}$

$$e^{-S_{SUGRA}[\delta g_{\mu\nu}]} = \left\langle e^{\int d^4x \sqrt{g}\, g_{\mu\nu} T^{\mu\nu}} \right\rangle$$

energy-mom. tensor of the DUAL FIELD THEORY

$\Downarrow$

QUANTUM OPERATOR

$$e^{-S_{SUGRA}[\delta A_i]} = \left\langle e^{\int d^4x \sqrt{g}\, A_i J^i} \right\rangle$$

A_μ: gauge field in the bulk with $A_\mu|_{\partial AdS} = A_i$

J_i: conserved current

That is $\quad A^\mu J_\mu \longrightarrow A_\mu \to A_\mu + \partial_\mu \Lambda \Rightarrow \partial^i J_i = 0$

Consider now:

- fixed $g_{\mu\nu}$ (AdS_5)
- fluctuations of ϕ on top of $g_{\mu\nu}$

$$\Rightarrow \quad \left(\Box_{AdS} - m^2\right)\phi = 0$$

Choose $\quad ds^2(AdS_5) = \frac{L^2}{z^2}\left(dz^2 + dx^i dx_i\right)$ (coord. x^i, z, $i = 1, \dots, 4$)

$z = \varepsilon \to 0$ boundary

↳ use $t \to -i\bar{t}$ to have $d\vec{x}^2$

$$\Box = \frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}\, g^{\mu\nu}\partial_\nu\right) = \frac{1}{L^2} z^5 \partial_z\left(z^{-3}\partial_z\right) + \frac{z^2}{L^2}\Box_{4d}$$

↳ $\sqrt{g} = \left(\frac{L}{z}\right)^5$

Consider: $\quad \phi(z, x^i) = \int \frac{d^4p}{(2\pi)^4}\phi_p(z)\, e^{ip\cdot x}$

$$\Rightarrow \quad z^5 \partial_z\left(z^{-3}\partial_z\right)\phi_p(z) - p^2 z^2 \phi_p(z) - m^2 L^2 \phi_p(z) = 0$$

↳ $\phi(z) = \phi_0(x)\, z^\alpha$ ansatz

$$z^\alpha \phi_0\left[\alpha(\alpha-4) - m^2L^2\right] - z^{\alpha+2} p^2 \phi_0 = 0$$

as $z \to 0$, I must impose

$$\alpha(\alpha-4) - m^2L^2 = 0 \Rightarrow \alpha_\pm = 2 \pm \sqrt{4 + m^2L^2}$$

Then for $z \to 0$:

$$\phi(x,z) \sim \phi_0(x) z^{\alpha_-} + \phi_1(x) z^{\alpha_+} + \dots$$

NB $\quad \alpha_- - \alpha_+ < 0$

Then z^{α_-} is dominant. $\Rightarrow$ $\underbrace{\phi_0}$ is the b.c. of $\phi(x,z)$.

$\phi_1, \phi_2, \dots$ are specified from ϕ_0

$\rightarrow$ α_- is related to Δ :

$$\left.\begin{array}{l} x^\mu \rightarrow \omega x^\mu \\ z \rightarrow \omega z \end{array}\right\} \Rightarrow \phi \text{ is inv.}$$

$$\hookrightarrow \phi_0 \rightarrow \phi_0 \omega^{-\alpha_-} \Rightarrow \int d^4x\, \phi_0\, \mathcal{O}_\Delta \Rightarrow \mathcal{O}_\Delta \sim \omega^{4-\alpha_-} \mathcal{O}_\Delta$$

$$\boxed{\Delta = 4 - \alpha_- = \alpha_+}$$

$$\Rightarrow \Delta = 2 + \sqrt{4 + m^2 L^2}$$

Going back to the KK spectrum $AdS_5 \times S^5$:

$$m^2 L^2 = k(k-4) \longrightarrow k \geq 2$$

comes from mixing modes π, b

$$\Rightarrow \Delta = 2 + \sqrt{4 + k^2 - 4k} = k \Rightarrow \text{the modes have increasing } \Delta\,!$$

HOLOGRAPHIC (SCALAR) TWO-POINT FUNCTION

$$\langle \mathcal{O}_\Delta(x)\, \mathcal{O}_\Delta(y) \rangle = ?$$

$\Rightarrow Z \sim e^{-S[\phi_0]}$ $\rightarrow$ I need S up to ϕ_0^2 since I'm interested in two point functions.

Consider :

$$\left(\Box_{AdS} - m^2\right)\phi = 0$$

Take $u = pz \longrightarrow u^5 \frac{d}{du}\left(u^{-3}\frac{d}{du}\right) f(u) - u^2 f(u) - m^2 L^2 f(u) = 0$

where $f(u) \overset{u=pz}{=} \phi(z)$; then $f(u) = u^2 F(u)$:

$$u^5 \frac{d}{du}\left(u^{-3}\left(2uF + u^2F^2\right)\right) - u^4 F - m^2L^2 u^2 F = 0$$

$\Leftrightarrow$ two solutions: $\left.\begin{matrix} I_{\Delta-2}(u) \\ K_{\Delta-2}(u) \end{matrix}\right\}$ $F(u) = A\, I_{\Delta-2}(u) + B\, K_{\Delta-2}(u)$

$\hookrightarrow$ $u \to \infty$ ($z \to \infty$: Poincaré) : $I_{\Delta-2}(u) \sim \frac{e^u}{\sqrt{u}}$

$\Downarrow$ $K_{\Delta-2}(u) \sim \frac{e^{-u}}{\sqrt{u}}$

$A = 0$

$u \to 0$ ($z \to 0$: boundary) : $K_{\Delta-2}(u) \sim \frac{\Gamma(\Delta-2)}{2}\left(\frac{2}{u}\right)^{\Delta-2}$

$$\Rightarrow f(u) \sim u^{4-\Delta} \quad \checkmark$$

α_-

So far : $\phi(x,z) = B \cdot (pz)^2 K_{\Delta-2}(pz)\, e^{ip\cdot x}$

$\hookrightarrow$ $z = \epsilon \Rightarrow \phi(x,z) = e^{ip\cdot x}$

$$\Rightarrow \phi(x,z) = \frac{(pz)^2\, K_{\Delta-2}(pz)}{(p\epsilon)^2\, K_{\Delta-2}(p\epsilon)}\, e^{ip\cdot x}$$

NB : the final result has $\varepsilon \to 0 \Rightarrow$ it must not depend on ε !!!

Plug back the phase

$$S = C\int d^5x\,\sqrt{g}\left((\partial\phi)^2 + m^2\phi^2\right) \overset{IBP}{=}$$

normal to the boundary

$$= \underbrace{C\int d^5x\,\sqrt{g}\left(-\phi\,\Box\phi + m^2\phi^2\right)}_{=\,0} + C\int_{\text{boundary}} d^4x\,\sqrt{h}\,\phi\, n^\mu \nabla_\mu \phi =$$

$n^\mu = k\left(\frac{\partial}{\partial z}\right)^\mu$

s.t. $n^\mu n^\nu g_{\mu\nu} = 1$

$k = \frac{z}{L}$

$$= C L^3 \int_{z=\epsilon} d^4x\,\frac{1}{z^3}\,\phi\,\partial_z\phi =$$

$$= C L^3 (2\pi)^4\,\delta^4(p+q)\,\overbrace{\left[\frac{1}{z^3}\,\phi(z)\,\partial_z\phi(z)\right]_{z=\epsilon\to 0}}^{F(\epsilon,p) := \text{"FLUX FACTOR"}}$$

Fourier expansion

$\Rightarrow$ Taylor expand $F(\epsilon,p)$ for $\epsilon \to 0$: there are divergent terms $\frac{1}{\epsilon^\#}$

$$\underbrace{\frac{1}{\epsilon^\#} + \ldots}_{} + \text{finite} + O(\epsilon)$$

can we remove these? (Renorm.)

Now focus on the finite part. We have

$$F(\epsilon,p)\Big|_{\text{finite}} \sim p^{2\Delta-4} \;\longrightarrow\; \langle \hat{\mathcal{O}}(p)\tilde{\mathcal{O}}(q)\rangle \sim \delta^4(p+q)\,p^{2\Delta-4}$$

$$\Rightarrow \langle \mathcal{O}(x)\mathcal{O}(y)\rangle \sim \frac{1}{|x-y|^{2\Delta}} \qquad \text{(anti-Fourier transf.)}$$

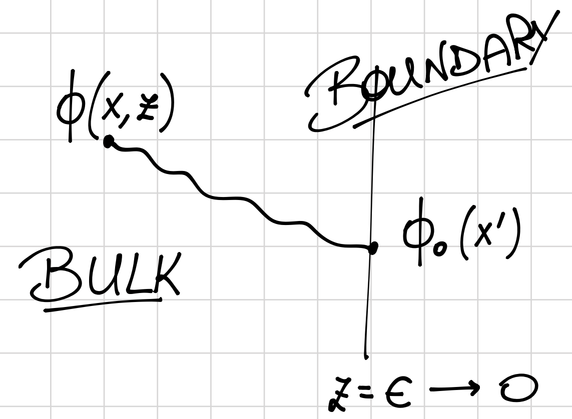
CORRELATION FUNCTIONS IN X-SPACE ⇒ WITTEN DIAGRAMS

- propagators: bulk-to-boundary propagator
  ↳ Witten's diagrams ($\neq$ Feynman because NOT quantum)

⇒ Higher n-point functions, other than scalars, too!
↳ AIM: compute $S = C \int d^5x \sqrt{g}\left((\partial\phi)^2 + m^2\phi^2\right)$

↳ start from $\left(\Box_{AdS} - m^2\right)\phi^2 = 0$

Define



$\to \quad \Box_{x,z} K(z,x,x') = \delta^4(x-x')$

$\Rightarrow \phi(x,z) = \int d^4x' \, K(z,x,x')\, \phi(x',z)$

⇒ SOLUTION: $K(z,x,x') = C \dfrac{z^\Delta}{\left(z^2 + |x-x'|^2\right)^\Delta} \quad \Rightarrow C = \dfrac{\Gamma(\Delta)}{\pi^2\,\Gamma(\Delta-2)}$

↳ $K \to 0$ if $x - x' \neq 0$ as $z \to 0$

$K \sim z^{-\Delta} \longrightarrow \infty$ if $x = x'$ as $z \to 0$

After IBP:

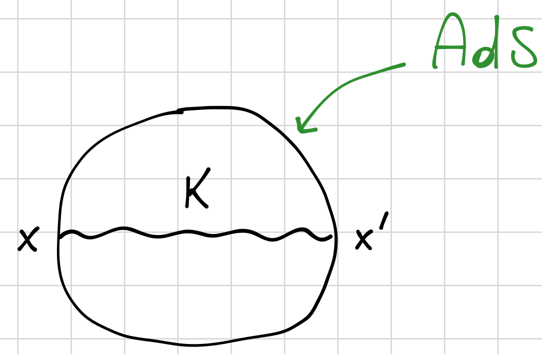
$$S = C \int_{z=\epsilon} d^4x \, L^3 \frac{1}{z^3} \phi(x,z)\, \partial_z \phi(x,z) \qquad \text{NB: } \phi(x,z) \overset{z\to 0}{\sim} z^{4-\Delta}\phi_0(x)$$

Insert $\phi = \int K \phi_0$:

$$S \sim \int_{z=\epsilon} d^4x \, \frac{1}{z^3} \, z^{4-\Delta}\phi_0(x)\, z^{\Delta-1} \int d^4x' \frac{\phi_0(x')}{|x-x'|^{2\Delta}} =$$

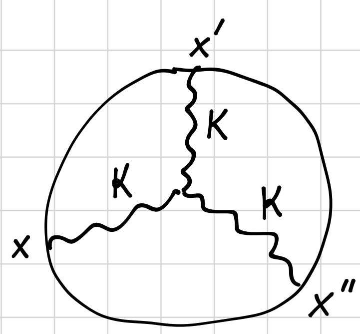
$$= \int d^4x \, d^4x' \, \frac{\phi_0(x)\,\phi_0(x')}{|x-x'|^{2\Delta}}$$

This is related to:



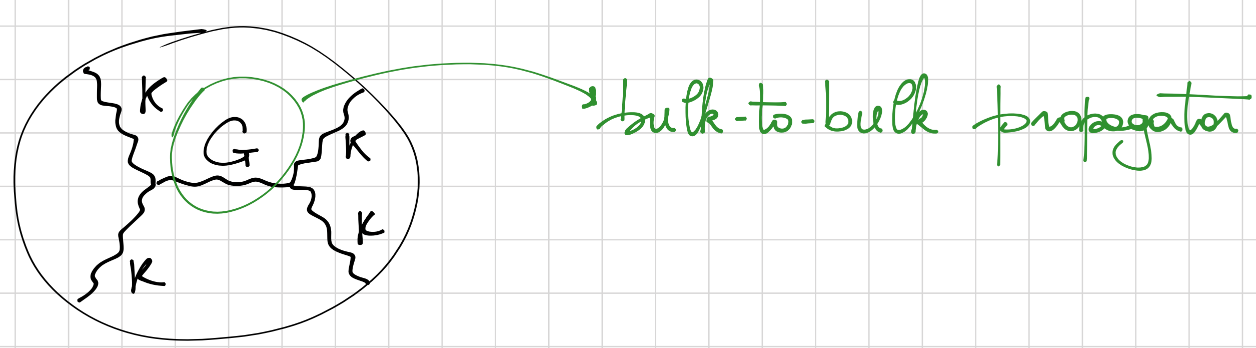
once we take the right no. of funct. derivatives.

For the 3 point function:



$\Rightarrow$ we need $S \sim \int \phi^3$ to have it!

For 4 point functions:



$$\Rightarrow S_{bulk} = S_{kin} + \int d^5x \sqrt{g}\, \frac{1}{3} \lambda \phi^3$$

$$\hookrightarrow \left(-\Box + m^2\right)\phi = \lambda \phi^2$$

$\Rightarrow$ PERTURBATION THEORY

$$\phi = \phi^{[0]} + \phi^{[1]} + \phi^{[2]} + \dots$$

where $\phi^{[0]}(x,z) = \int d^4x' \, K(z,x,x')\, \phi_0(x')$. Then:

$$\phi^{[1]} = \lambda \int d^4x' dz' \, G(z,z',x,x') \left(\phi^{[0]}(x',z')\right)^2$$

$$\Downarrow$$

$$\left(-\Box + m^2\right) G = \frac{1}{\sqrt{g}}\, \delta(z - z')\, \delta^4(x - x')$$

HOLOGRAPHIC RENORMALIZATION

$z = \epsilon \to 0 \Rightarrow \exists$ DIVERGENCES !!! e.g.: in $\langle \mathcal{O}(p)\, \mathcal{O}(q) \rangle$

What are the symm. we want to preserve ?

* SYMMETRIES OF AdS !

$$\to S_{HE} \sim \int d^D x \sqrt{g} \left(R + \frac{(D-1)(D-2)}{L^2} \right)$$

for AdS: $R = -\frac{D(D-1)}{L^2}$

$$\Downarrow$$

$$S_{HE}\Big|_{on\text{-}shell} \sim \int d^{D-1}x \int_0^{+\infty} \frac{dz}{z^D} \longrightarrow \infty$$

THIS IS WHERE THE PROBLEM LIES

$\Rightarrow$ This is a general feature: $g_{\mu\nu} \neq AdS_{d+1}$

$$\Downarrow$$

$g_{\mu\nu}$: asymptotically (locally) AdS

I can choose

$$ds^2 = \frac{1}{z^2}\left(dz^2 + g_{ij}\, dx^i dx^j \right)$$

they dep. on $g_{(0)}$ — ONLY IF d EVEN — free

$$\text{where} \quad g_{ij}(x,z) = g_{(0)ij}(x) + \ldots + z^d \left(g_{(d)ij}(x) + h_{(d)ij}(x) \ln z^2 \right) + \ldots$$

$\hookrightarrow$ FEFFERMAN-GRAHAM

$\Rightarrow$ corresponds to the expansion of ϕ w/ $\Delta = d \Rightarrow$ compatible with $m^2 = 0$

Then we can plug it into the action:

$$S_{reg}\Big|_{on\text{-}shell} = \int d^dx \sqrt{g}\left[\epsilon^{-2\Delta+d} a_{(0)} + \epsilon^{-2\Delta+d+2} a_{(2)} + \ldots + \ln\epsilon\, a_{(d)} + O(\varepsilon^0)\right]_{z=\epsilon}$$

⇒ Define counterterms:

$$S_{ct} = -\left(\text{divergent part of } S_{reg}\big|_{on\text{-}shell}\right)$$

↳ $S_{sub} = S_{reg} + S_{ct} = \int d^dx \sqrt{g}\ (\text{FINITE})$

→ compute, then $\epsilon \to 0$

$$\Rightarrow S_{ren}[\phi_0] = \lim_{\varepsilon\to 0} S_{sub}[\phi_0, \varepsilon]$$

e.g.:

Go back to KG eqn:

* SCALAR in fixed metric on AdS_{d+1}:

$$\left(\Box_{AdS} - m^2\right)\Phi(x,z) = 0$$

$$\Rightarrow \Phi(x,z) = z^{d-\Delta}\phi(x,z)$$

$$↳ \left(\Delta(\Delta-d) - m^2L^2\right)\phi(x,z) + (1+d-2\Delta)\, z\,\partial_z\phi(x,z) + $$

$$+ z^2\left(\partial_z^2\phi(x,z) + \Box_d\phi(x,z)\right) = 0$$

⇒ Ansatz: $\phi(x,z) = \phi_0(x) + z^2\phi_{(2)}(x) + \ldots$ (odd powers are set to 0)

↓

at 1st NON TRIV. ord: $(1+d-2\Delta)\phi_{(2)}(x) + \phi_{(2)}(x) + \frac{1}{2}\Box\phi_{(0)}(x) = 0$

$$\Rightarrow \phi_{(2)}(x) = \frac{-\Box\phi_{(0)}}{2(d+2-2\Delta)}$$

Then $\phi_{(4)} = \dfrac{1}{4(2\Delta - d - 4)} \Box \phi_{(2)}$

$\vdots$

$$\phi_{(2n)} = \frac{1}{2n\underbrace{(2\Delta - d - 2n)}} \Box \phi_{(2n-2)}$$

AT SOME THIS CAN VANISH

if at some point $\exists k \mid 2\Delta - d - 2k = 0$, then add

$$\phi(x,z) = \phi_{(0)}(x) + \dots + z^{2\Delta - d}\left(\phi_{(2k)}(x) + \ln z^2 \tilde{\phi}_{(2k)}(x)\right) + \dots$$

$\big\downarrow$ w/ this

$$\tilde{\phi}_{(2k)} = -\frac{1}{2} \Box \phi_{(2k-2)}$$

$$\Rightarrow S = C \int d^{d+1}x \sqrt{g} \left((\partial \Phi)^2 + m^2 \Phi^2 \right) =$$

$$= C \int_{z=\epsilon} d^d x \frac{1}{\epsilon^{2\Delta - d}} \left((d - \Delta)\phi^2 + \phi \epsilon \partial_\epsilon \phi \right)$$

$$\Rightarrow \epsilon \partial_\epsilon \phi = 2\epsilon^2 \phi_{(2)} + 4 \epsilon^4 \phi_{(4)} + \dots$$

$$\hookrightarrow S = - C L^{d-1} \int_{z=\epsilon} d^d x \, \epsilon^{d - 2\Delta} \left[(d - \Delta)\phi_{(0)}^2 + (2d - 2\Delta + 2)\phi_{(0)}\phi_{(2)} \epsilon^2 + \dots \right]$$

Referring to the previous expansion:

$$a_{(0)} = - C L^{d-1} (d - \Delta)$$

$$a_{(2)} = C L^{d-1} \frac{d - \Delta + 1}{d - 2\Delta + 2} \phi_{(0)} \Box \phi_{(0)}$$

Now determine

$\phi_0(x)$ as a function of $\Phi(x,z)$:

$$\phi_{(0)} = \epsilon^{-d+\Delta}\left(\Phi(x,\epsilon) - \frac{1}{2(2\Delta-d-2)}\Box_\gamma \Phi(x,\epsilon)\right)$$

$$\gamma_{ij} = \frac{1}{\epsilon^2}\delta_{ij}$$

$$\phi_{(2)} = \epsilon^{-d+\Delta-2}\frac{1}{2(2\Delta-d-2)}\Box_\gamma \Phi(x,\epsilon)$$

there are ϵ terms in this but we hide them to show the full AdS symmetry

Therefore:

$$S_{ct}[\Phi,\epsilon] = CL^{d-1}\int_{z=\epsilon} d^dx\sqrt{\gamma}\left[(d-\Delta)\Phi^2 + \frac{1}{2\Delta-d-2}\Phi\Box_\gamma\Phi\right] + \dots$$

Sufficient to renorm. up to $\Delta < \frac{d}{2} + 2$

NB: $\Delta = \frac{d}{2} + 1 \longrightarrow$ add ln term:

$$\frac{1}{2\Delta-d+2}\Phi\Box_\gamma\Phi \Longrightarrow \overbrace{-\frac{1}{2}\ln\epsilon\;\Phi\Box_\gamma\Phi}^{\text{ANOMALY}}$$

explicitly insert ϵ!

HOLOGRAPHIC WEYL ANOMALY

→ focus on metric using FG expansion: det. $g_{(2n)ij}(x)$.

$$\longrightarrow S = \frac{1}{16\pi G}\int d^{d+1}x\sqrt{g}\left(R + \frac{d(d-1)}{L^2}\right) - \frac{1}{8\pi G}\int_\partial d^dx\sqrt{\gamma}\,K$$

induced metric on boundary

$$K = \gamma^{\mu\nu}\nabla_\mu n_\nu$$

The additional term will cancel some of the div $\Rightarrow$ NOT A CT !

(NB the boundary term DOES NOT affect the e.o.m.)

$$\Rightarrow ds^2 = \frac{L^2}{z^2}\left(dz^2 + g_{ij}\, dx^i dx^j\right) \quad \text{where} \quad g_{ij} = g_{(0)ij} + z^2 g_{(2)ij} + \dots$$

$$\left(\text{NB: } g_{(2)ij} = \frac{L^2}{d-2}\left(R^{(0)}_{ij} - \frac{1}{2(d-1)} R^{(0)} g_{(0)ij}\right)\right)$$

Up to $g_{(d-2)ij}$ everything is det., but we know few things on $g_{(d-2)ij}$.

e.g.:

$$\mathrm{Tr}\, g_{(4)} = \frac{1}{4}\, \mathrm{Tr}\, g_{(2)}^2 \quad \text{w/ } g_{(0)ij}$$

$$\rightarrow S_{ct} = \frac{1}{8\pi G}\int_{z=\epsilon} d^d x \sqrt{\gamma}\left[\frac{d-1}{L} + \frac{L}{2(d-2)} \overset{\text{computed w/ full } \gamma_{ij}}{R} + \underbrace{\frac{L^3}{2(d-4)(d-2)^2}\left(R_{ij}R^{ij} - \frac{d}{4(d-1)}R^2\right)}\right.$$
$$\left. + \dots\right]$$

$$\text{e.g.: } d=4 \Rightarrow -\frac{L^3}{2(d-2)^2} \ln\epsilon \left(R_{ij}R^{ij} - \frac{1}{12}R^2\right)$$

In CFT $\rightarrow$ D invariance leads to

$$\langle g^{ij} T_{ij} \rangle = \mathcal{A} \neq 0 \rightarrow \text{ANOMALY}$$

at quantum level.

$$S_{ct} = \frac{1}{8\pi G}\int d^4x \sqrt{g}\left[\frac{3}{L} + \frac{L}{4}R - \frac{L^3}{8}\left(R^{ij}R_{ij} - \frac{1}{3}R^2\right)\ln\epsilon\right]$$

$\hookrightarrow \langle T \rangle = \mathcal{A} \neq 0 \Rightarrow$ WEYL ANOMALY

$$ds^2 = L^2\left(\frac{d\rho^2}{4\rho^2} + \frac{1}{\rho}g_{ij}(\rho,x)\,dx^i dx_i\right) \qquad \rho = z^2$$

$\Rightarrow$ PBH diffeom in the bulk ($=$ Weyl var. at ∂):

$$\begin{cases} \rho = \rho'(1-2\sigma(x')) \\ x^i = x^{i'} + a^i(x^i,\rho') \end{cases}$$

where $a^i(x,\rho) = \frac{L^2}{2}\int_0^\rho d\hat\rho\, g^{ij}(x,\hat\rho)\,\partial_j\sigma(x)$

$$\Rightarrow g_{ij} \longrightarrow g_{ij} + 2\sigma\left(1-\rho\frac{\partial}{\partial\rho}\right)g_{ij} + 2\nabla_{(i}a_{j)}$$

$\hookrightarrow$ for $\rho\to 0 \to a_i \to 0$

$$g_{ij} \xrightarrow[\text{at boundary}]{\rho\to 0} g_{ij}(1+2\sigma) \simeq e^{2\sigma}g_{ij}$$

Then $\delta_{PBH}(S + S_{ct}) = \qquad \delta_{PBH}S_{ct} = \int_{\rho=\epsilon} d^4x\left(\underset{-2\sigma\rho}{\delta\rho}\frac{\delta}{\delta\rho} + \underset{2\sigma g_{ij}}{\delta g_{ij}}\frac{\delta}{\delta g_{ij}}\right)S_{ct} =$

picks up the $\ln\epsilon$ term in S_{ct}

$$\Rightarrow \mathcal{A} = \frac{1}{16\pi G}\,2L^3\left(\frac{1}{8}R^{ij}R_{ij} - \frac{1}{24}R^2\right)$$

$\Rightarrow$ Field theory (general CFT):

$$\langle T^i{}_i\rangle = \frac{c}{16\pi^2}I_4 - \frac{a}{16\pi^2}E_4$$

$$\Rightarrow I_4 = R_{ijkl}R^{ijkl} - 2R_{ij}R^{ij} + \frac{1}{3}R^2$$

$$E_4 = R^{ijkl}R_{ijkl} - 4R^{ij}R_{ij} + R^2$$

$\Rightarrow$ Then

1) $a = c$

2) $a = \dfrac{\pi L^3}{8 G_5}$ $\qquad G_5 = \dfrac{G_{10}}{L^5 \, \mathrm{vol}(S^5)}$ $\qquad 16\pi G_{10} = (2\pi)^7 \ell_s^8 g_s^2$

$$L^4 = \frac{(2\pi)^4 g_s \ell_s^4 N}{4 \, \mathrm{vol}(S^5)}$$

$$\Rightarrow a = \frac{\pi^3 N^2}{4 \, \mathrm{vol}(S^5)} = \frac{N^2}{4}$$

Now cfr. with $\mathcal{N}=4$ SYM:

$$a = c = \frac{N^2}{4} - \frac{1}{4}$$

$\rightarrow$ The Weyl anomaly DOES NOT depend on the coupling.

NB: the comp. goes on even w/o S^5: $\quad a = \dfrac{\pi^3 N^2}{4 \, \mathrm{vol}(Y_5)}$.

5D manifold

DEFORMATIONS of AdS_5:

$$AdS_5 \times S^5$$

deform AdS_5
keep S^5

deform S^5
keep AdS_5

a linear comb.

SUPERSYMMETRY BREAKING

→ Introduce finite T:

1) go to Euclidean

2) compactify time $t_E \sim t_E + \beta$

PARTITION F.

$$Z_E[\beta] = \int_{\substack{\phi(\vec{x},t_E) = \phi(\vec{x},t_E+\beta) \\ \psi(\vec{x},t_E) = -\psi(\vec{x},t_E+\beta)}} \mathcal{D}\phi \; e^{-S_E[\phi]} = \mathrm{Tr}\, e^{-\beta H}$$

SUSY break.

What about GRAVITY side?

⇒ Do the same!

↳ e.g. SCHWARZSCHILD BH (not AdS, but good to start with)

$$d=4 \Rightarrow ds^2 = + \left(1 - \frac{2MG}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2MG}{r}} + r^2 ds^2(S^2)$$

EUCL.

⇒ Horizon at $r_* = 2MG$ ⇒ IN EUCL. THERE IS NO HORIZON

↳ is it regular? YES but we check for possible sing. which we can pick up somewhere else

NB: $\tau \sim \tau + \beta$ → ~ polar coord → in (r, τ) the space looks like $\mathbb{R}^2$

↳ change coord: $\rho^2 = r - 2MG$

Near $\rho \to 0$:

$$ds^2 \simeq \frac{1}{r_*} \rho^2 d\tau^2 + 4 r_* d\rho^2 + r_*^2 ds^2(S^2) =$$

$$= 4 r_* \left(\rho^2 \left(\frac{d\tau}{2r_*}\right)^2 + d\rho^2 \right) + r_*^2 ds^2(S^2)$$

$\Rightarrow \quad \frac{\tau}{2r_*} \in [0, 2\pi] \Rightarrow \frac{1}{T} = 8\pi M G$ (temperature)

$\hookrightarrow \quad dU = T\, dS \Rightarrow S = \int dM \frac{1}{T(M)} = 4\pi G M^2 = \frac{\text{Area}}{4G}$

$\quad U \sim M$

Now consider BLACK HOLES in AdS:

1) Poincaré patch
2) Global coord.

$\Rightarrow$ POINCARÉ COORD.

Go back to D3-brane solution.

Deform:

$$ds^2 = (H(r))^{-1/2} \left(-f(r)\, dt^2 + d\vec{x}^2\right) + (H(r))^{1/2} \left(\frac{dr^2}{f(r)} + r^2 ds^2(S^5) \right)$$

$*$ $H(r) = 1 - \frac{L^4}{r^4}$

$*$ $\underbrace{f(r)}_{\text{"BLACKENING FACTOR"}} = 1 - \underbrace{\left(\frac{r_*}{r}\right)^4}$ $\to$ deform. $\Rightarrow$ not SUSY!

Now take

1) near-horizon

2) $t \to -i\tau$

3) $z = \frac{L^2}{r}$

THIS IS ALSO SUSY SOL.

$$\Downarrow$$

$$ds^2 = \frac{L^2}{z^2}\left(\left(1-\frac{z^4}{z_*^4}\right)d\tau^2 + d\vec{x}^2 + \frac{dz^2}{1-\frac{z^4}{z_*^4}}\right) + L^2 ds^2(S^5)$$

with $\tau \sim \tau + \beta$.

$\boxed{AdS_5}$ in Poincaré patch.

Now define ρ : $\frac{z}{z_*} = 1 - \frac{\rho^2}{L^2}$

$$\Rightarrow z \to z_* \leftrightarrow \rho \to 0 \longrightarrow ds^2 \overset{\rho\to 0}{\simeq} d\rho^2 + \frac{4}{z_*^2}\rho^2 d\tau^2 + \frac{L^2}{z_*^2}d\vec{x}^2$$

$$\Rightarrow \frac{1}{\beta} = T = \frac{1}{\pi z_*}$$

$$\hookrightarrow S = \frac{A}{4G_5} \Longrightarrow A = \left(\frac{L}{z_*}\right)^3 \mathrm{vol}(\mathbb{R}^3)$$

$$\frac{S}{\mathrm{vol}(\mathbb{R}^3)} = \frac{L^3}{z_*^3}\cdot\frac{L^5\,\mathrm{vol}(S^5)}{G_{10}\cdot 4} = \frac{N^2}{\mathrm{vol}(S^5)}\cdot\frac{\pi^5}{2}T^3 =$$

$$= \frac{1}{2}\pi^2 T^3 N^2$$

$\Rightarrow$ this is hol. dual of $\mathcal{N}=4$ SYM @ temp $= T$ $(N\to\infty,\ \lambda \gg 1)$

$\hookrightarrow$ SUSY broken + strong coupl. $\Rightarrow$ plasma?

The comparison can be made in free theory: $s = \dfrac{S}{\mathrm{vol}(\mathbb{R}^3)}$ in free $\mathcal{N}=4$ SYM @ T

$$s_{free} = \frac{2}{3}\pi^2 N^2 T^3$$

while $s = \frac{1}{2}\pi^2 N^2 T^3 \Rightarrow \frac{4}{3}$ mismatch.

$\Rightarrow \dfrac{s_{\lambda=\infty}}{s_{\lambda=0}} = \dfrac{3}{4} \simeq 75\%$ while in lQCD $\dfrac{s_{\lambda=\infty}}{s_{\lambda=0}} \simeq 80\%$!

Now let's look at:

* finite charge density J^0 $(\rightarrow \partial_i J^i = 0)$ $\Rightarrow$ chemical potential a_0

$$\Rightarrow \int d^d x \, J^i a_i$$

$\hookrightarrow$ bulk: A_μ (gauge field)

$\Rightarrow AdS_5 \times S^5 \longrightarrow$ symm.: $SO(6) \supset U(1)^3$ (Cartan subalg.)

Charge dens. at bound. corresponds to ELECTRICALLY CHARGED BULK SOL./BH:

$$A \rightarrow a_0 dt \text{ for } z \rightarrow 0$$

→ EFFECTIVE SUGRA (Einstein-Maxwell + Λ)

$$ds^2 = \frac{L^2}{z^2}\left(-f(z)\,dt^2 + d\vec{x}^2 + \frac{dz^2}{f(z)}\right) \qquad f(z) = 1 - \left(\frac{z}{z_*}\right)^4 (1+c\mu^2) + c\mu^2\left(\frac{z}{z_*}\right)^6$$

⇒ 2 param: $z_* \sim \frac{1}{T}$, μ

$$A = \mu\left(1 - \left(\frac{z}{z_*}\right)^2\right)dt \overset{z\to 0}{\sim} \mu\,dt \qquad (a_0 = \mu)$$

$$\rightarrow \quad T = \frac{1}{\pi z_*}\left(1 - \frac{c\mu^2}{2}\right)$$

A for $z \to \hat{z} \mid f(\hat{z}) = 0$ (near-horizon) ⇒ $\boxed{A \to 0}$

THERMODYNAMICS

↳ Grand-canonical ensemble:

$$\Omega(\mu, T, V) = U - TS - \mu N$$

We have

$$Z_{QFT} = e^{-\beta\Omega} = e^{-S_{SUGRA}}$$

$$\Omega = T\, S_{SUGRA}(T, \mu, V)$$

Now consider AdS_5 in global coord. ($\mathcal{N}=4$ SYM $\mathbb{R}\times S^3$)
+ finite T

$\Rightarrow$ Euclidean $S^1_\beta \times S^3$

$$\Rightarrow \quad Z_{QFT}\left[S^1_\beta \times S^3\right] = \sum_i Z_{SUGRA}[X_i]$$

possible inner manifolds (all of them)
(SUSY $\rightarrow$ unique)
(~~SUSY~~ $\rightarrow$ not unique)

$$\Rightarrow X_1 : \text{"thermal } AdS_5\text{"} \rightarrow ds^2 = \left(1+\frac{r^2}{L^2}\right)d\tau^2 + \frac{dr^2}{1+\frac{r^2}{L^2}} + r^2 ds^2(S^3)$$

$\hookrightarrow$ $r\rightarrow 0$: $S^1 \times \mathbb{R}^4$

$$\Rightarrow X_2 : \quad ds^2 = f(r)\, d\tau^2 + \frac{1}{f(r)}\, dr^2 + r^2 ds^2(S^3)$$

$$f(r) = 1 - \frac{\mu}{r} + \frac{r^2}{L^2}$$

$$\left(\text{horizon: } f(r_h) = 0\right)$$

$$\Rightarrow \quad T = \frac{1}{2\pi}\frac{|f'(r_h)|}{2} = \frac{1}{4\pi r_h L^2}\left(2L^2 + 4r_h^2\right) \qquad f(r_h) = 0$$

$$\hookrightarrow T_{min} = \frac{1}{2\pi L}\sqrt{d(d-2)} \qquad (d=4)$$

$\rightarrow$ Sum over two sol:

on-shell act.

$$Z_{QFT} = e^{-S[X_1]} + e^{-S[X_2]} =$$

$$= e^{-S[X_1]}\left(1 + e^{-(S[X_2]-S[X_1])}\right)$$

$$\Rightarrow S[X_2] - S[X_1] = \frac{1}{T}\frac{r_h^{d-2}}{2\kappa^2}\,\mathrm{vol}(S^{d-1})\left(1 - \frac{r_h^2}{L^2}\right)$$

$r_h = L$ is a phase transition at charact. $T \Rightarrow$ CONF./DECONF. PHASE TRANS.

$$\Rightarrow \quad \left(\mathcal{O}(x)\right)^{\dagger} = x^{-2\Delta_{\mathcal{O}}} \mathcal{O}\left(\frac{x^\mu}{x^2}\right)$$

$$\left(\mathcal{O}(x)|0\rangle\right)^{\dagger} = x^{-2\Delta_{\mathcal{O}}} \langle 0| \mathcal{O}\left(\frac{x^\mu}{x^2}\right)$$

Now take the usual limit: $\left(y^\mu = \frac{x^\mu}{x^2}\right)$:

$$\langle \mathcal{O}| = \lim_{y\to\infty} y^{2\Delta_{\mathcal{O}}} \langle 0|\mathcal{O}(y)$$

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle = \langle 0| \left(x_1^{-2\Delta_{\mathcal{O}}} \mathcal{O}(y_1)\right)^{\dagger} \mathcal{O}(x_2)|0\rangle$$

$$= x_1^{-2\Delta_{\mathcal{O}}} \langle 0| \left(e^{y_1 P}\mathcal{O}(0)e^{-y_1 P}\right)^{\dagger} e^{x_2 P}\mathcal{O}(0)e^{-x_2 P}|0\rangle$$

$$= x_1^{-2\Delta_{\mathcal{O}}} \langle 0| \mathcal{O}^{(\dagger)}(0)\, e^{y_1 K}\, e^{x_2 P}\, \mathcal{O}(0)|0\rangle =$$

$$= x_1^{-2\Delta_{\mathcal{O}}} \langle \mathcal{O}| e^{y_1 K} e^{x_2 P} |\mathcal{O}\rangle =$$

$|\mathcal{O}\rangle$ is scalar

$$\sim x_1^{-2\Delta_{\mathcal{O}}} \left(\langle \mathcal{O}|\mathcal{O}\rangle + \frac{x_1^\mu x_2^\nu}{|x_1|^2} \langle \mathcal{O}| K_\mu P_\nu |\mathcal{O}\rangle\right)$$

$$= x_1^{-2\Delta_{\mathcal{O}}} \left(\langle \mathcal{O}|\mathcal{O}\rangle + \frac{x_1^\mu x_2^\nu}{|x_1|^2} \langle \mathcal{O}| [K_\mu, P_\nu] |\mathcal{O}\rangle\right) =$$

$$= x_1^{-2\Delta_{\mathcal{O}}} \left(\langle \mathcal{O}|\mathcal{O}\rangle + \frac{x_1^\mu x_2^\nu}{|x_1|^2} \langle \mathcal{O}| (2\Delta_{\mathcal{O}}\eta_{\mu\nu} - \cancelto{0}{2M_{\mu\nu}}) |\mathcal{O}\rangle\right)$$

$$= x_1^{-2\Delta_{\mathcal{O}}} \left(1 + 2\Delta_{\mathcal{O}} \frac{x_1 \cdot x_2}{|x_1|^2}\right) \underbrace{\langle \mathcal{O}|\mathcal{O}\rangle}_{=1}.$$

NB we know $\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle = \frac{1}{|x_{12}|^{2\Delta_{\mathcal{O}}}}$ expansion at $x_2 \to 0$

NB • Reflection positivity in EQFT:

$$\langle \psi | \psi \rangle \geq 0$$

Define $|\mathcal{O}\rangle$ s.t.:

$$\langle \mathcal{O} | \mathcal{O} \rangle = 1.$$

Do the descendants have positive norm?

$$\left| P_0 |\mathcal{O}\rangle \right|^2 = \langle \mathcal{O} | K_0 P_0 | \mathcal{O} \rangle = 2\Delta_{\mathcal{O}} \langle \mathcal{O} | \mathcal{O} \rangle = 2\Delta_{\mathcal{O}}$$

$\Rightarrow$ it must be $\Delta_{\mathcal{O}} \geq 0$ (for SCALARS!)

In general

collective index

$$\langle \mathcal{O}^I | \mathcal{O}_J \rangle = \delta^I{}_J \Rightarrow \left(P_\mu | \mathcal{O}^I \rangle \right)^\dagger \left(P_\nu | \mathcal{O}_J \rangle \right) = \langle \mathcal{O}^I | K_\mu P_\nu | \mathcal{O}_J \rangle =$$

$$= \langle \mathcal{O}^I | [K_\mu, P_\nu] | \mathcal{O}_J \rangle =$$

irrep. of $M_{\mu\nu}$

$$= \langle \mathcal{O}^I | 2\Delta_{\mathcal{O}} \eta_{\mu\nu} - 2R(M_{\mu\nu})_I{}^J | \mathcal{O}_J \rangle =$$

$$= 2\Delta_{\mathcal{O}} \eta_{\mu\nu} \delta^I{}_J - 2 \langle \mathcal{O}^I | R(M_{\mu\nu})_I{}^J | \mathcal{O}_J \rangle =$$

$$= 2\Delta_{\mathcal{O}} \eta_{\mu\nu} \delta^I{}_J - 2 R(M_{\mu\nu})^I{}_J$$

$\Rightarrow \Delta_{\mathcal{O}} \geq$ MAX EIGENVALUE OF $R(M_{\mu\nu})^I{}_J$

$$\Rightarrow \quad R(M_{\mu\nu})^I{}_J = \frac{1}{2}\underset{\substack{\uparrow \\ (L^{ab})_{\mu\nu} = \delta^a_\mu \delta^b_\nu - \delta^a_\nu \delta^b_\mu}}{(L^{ab})_{\mu\nu}} \cdot (S_{ab})^I{}_J$$

$$\Rightarrow \quad L^{ab} S_{ab} = \frac{1}{2}\Big[(L+S)^2 - \underbrace{L^2 - S^2}_{\text{Casimir oper.}}\Big] =$$

$$= \frac{1}{2}\Big[\mathrm{Cas}(V_1) + \mathrm{Cas}(r) - \mathrm{Cas}(r \otimes V_1)\Big]$$

Suppose r is traceless-symm. w/ ℓ indices:

$$\mathrm{Cas}(\ell) = \ell(\ell + d - 2)$$

$$\Rightarrow \Delta \geq \frac{1}{2}\Big[\mathrm{Cas}(\ell) + \mathrm{Cas}(\ell = 1) - \underbrace{\mathrm{Cas}(V_\ell \otimes V_1)}\Big]$$

$$V_1 \otimes V_\ell = \underline{\underline{V_{\ell-1}}} \oplus \dots$$

Therefore $\qquad \Delta \geq \ell + d - 2$

$\hookrightarrow$ the larger the spin the larger the dim.

$$\Rightarrow \quad |P_\mu P^\mu |\phi\rangle| \geq 0 \Rightarrow \Delta_\theta \geq \frac{d-2}{2} \quad (\ell = 0)$$

NB : $\quad \mathbb{I} \longrightarrow \Delta = 0, \ \ell = 0$

These are called UNITARITY BOUNDS.

When one of them is saturated, then one of the desc. has zero norm. E.g.:

$$J_\mu \quad \Rightarrow \Delta_J \geq d-1$$

When $\Delta_J = d-1 \Rightarrow \left| P_\mu | J^\mu \rangle \right|^2 = 0$

↳ "$\partial_\mu J^\mu = 0$" as a quantum op.

("SHORT MULTIPLET")

If $\ell=2$:

$$T^{\mu\nu} \longrightarrow \Delta_T = d \Longrightarrow \left| P_\mu | T^{\mu\nu} \rangle \right|^2 = 0 \Rightarrow \partial_\mu T^{\mu\nu} = 0 \ \checkmark$$

OPERATOR PRODUCT EXPANSION

$$\bullet O_1(x) \ \bullet O_2(x_2) \ |\psi\rangle = \sum_{\Delta,\ell} C_{\Delta\ell} \, | O_{\Delta,\ell} \rangle$$

$\Rightarrow \langle \underbrace{O_1 \, O_2}_{} \dots O_n \rangle$

take two of them an replace $|\psi\rangle$ $\bullet O_1 \ \bullet O_2$ by the state containing the irreps of them.

$$\Rightarrow O_1(x_1) O_2(x_2) = \sum_k C_{12k}(x_{12}, \partial_2) \, O_k(x_2)$$

$$= \sum_k \tilde{C}_{12k}(x_{12}, x_{23}, \partial_3) \, O_k(x_3)$$

↳ valid INSIDE CORR. FUNCT.

"OPE"

e.g.: $\langle \underbrace{O_1(x_1) O_2(x_2)}_{} O_3(x_3) \rangle = \sum_k C_{12k}(x_{12}, \partial_2) \langle O_k(x_2) O_3(x_3) \rangle$

$$= \sum_k C_{12k}(x_{12}, \partial_2) \frac{\delta_{k,3}}{x_{23}^{2\Delta_0}} =$$

$$= \underbrace{C_{123}(x_{12}, \partial_2)}_{\text{DETERMINE } C_{123}} \frac{1}{x_{23}^{2\Delta_3}}$$

$$\| \qquad \frac{f_{123}}{x_{12}^{\Delta_1+\Delta-\Delta_3} \, x_{13}^{\Delta_1-\Delta_2+\Delta_3} \, x_{23}^{-\Delta_1+\Delta_2+\Delta_3}}$$

$\Delta_1 = \Delta_2 = \Delta$, Δ_3 arb.

$$\Rightarrow C_{123}(x, \partial) \overset{\downarrow}{=} f_{123} \, x^{\Delta_3 - 2\Delta} \left(1 + \frac{1}{2} x \cdot \partial + \alpha \, x^\mu x^\nu \partial_\mu \partial_\nu + \dots \right)$$

where $\alpha = \frac{\Delta_3 + 2}{8(\Delta_3+1)}$.

SASAKI - EINSTEIN / $\mathcal{N}=1$ SCFT DUALITY

$\frac{1}{4}$ SUSY $\longleftarrow$ $AdS_5 \times S^5$ $\hookrightarrow$ replace w/ Y_5

Gravity side $\Rightarrow$ Type IIB :

<u>Ansatz</u>
$$\begin{cases} ds^2_{10} = ds^2(AdS_5) + ds^2(Y_5) \\ F_5 = \frac{4}{L}\left(\mathrm{vol}(AdS_5) + \mathrm{vol}(Y_5)\right) \end{cases}$$

$\Rightarrow R_{ij} = -\frac{4}{L^2} g_{ij}$ "EINSTEIN METRIC"

$\rightarrow$ 10d : $\epsilon = \underbrace{\chi}_{AdS_5} \otimes \underbrace{\xi}_{Y_5}$

$$\text{KSE} \Rightarrow \nabla_\mu \chi = \frac{1}{2L}\gamma_\mu \chi \Rightarrow \nabla_i \xi + \frac{i}{2L}\gamma_i \xi = 0$$

(check that Y_5 has to be Einstein)

REMARKS :

1) generically $\exists$ a ξ solution with ξ Dirac

2) any SE manifold have CANONICAL KILLING VECTOR

$$\bar{\xi}\gamma^\mu \xi = V^\mu = \left(\frac{\partial}{\partial\psi}\right)^\mu$$

$$\hookrightarrow \frac{1}{L^2}\, ds^2(Y_5) = \underbrace{ds^2(M_4)}_{R^{(4)}_{ij} = 6\, g^{(4)}_{ij}} + (d\psi + a)^2$$

(Kähler)

SYMMETRY IS $SO(2,4) \times U(1)$:

* $\mathcal{N}=1$ (bosonic) superconformal alg.

  $\hookrightarrow$ dual 4d is SCFT ($\mathcal{N} \geq 1$)

Strategy :

i) look at $\mathcal{N}=1$ Lagrangians

ii) require conf. invariance (CFT)

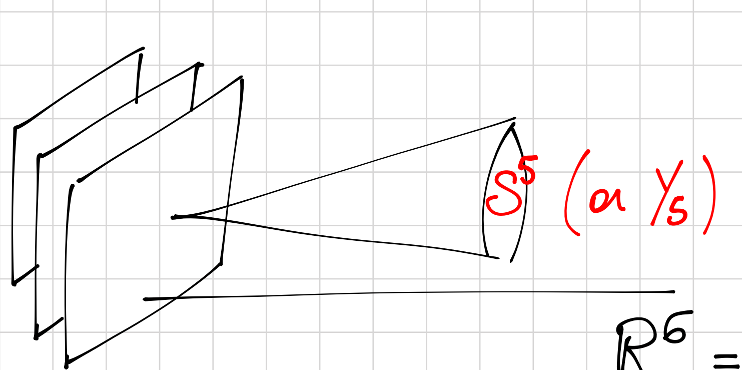
Once we have a candidate for $\mathcal{N}=1$ SCFT :

$\Rightarrow$ CHECKLIST :

1) Symm.
2) KK spectrum
3) anomalies $\longrightarrow$ Weyl

Take D3-brane picture :

$\hookrightarrow$ F_5 must come from this !



N D3-branes

$\mathbb{R}^6 = \underbrace{C}_{\text{"cone"}}(S^5)$ $\qquad ds^2 = dr^2 + r^2 ds^2(S^5)$

$$\Rightarrow \quad ds^2_{10} = \left(H(r)\right)^{-1/2} ds^2\left(\mathbb{R}^{1,3}\right) + \left(H(r)\right)^{1/2} \underbrace{\left(dr^2 + r^2 ds^2(S^5)\right)}_{\text{Non Compact variety}}$$

Calabi-Yau man. $\leftarrow$ $\nabla_i \epsilon = 0$ (Special Holonomy)

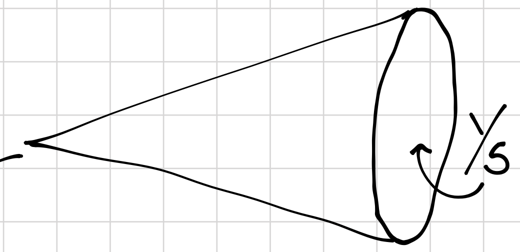
$\Rightarrow$ instead of $C(S^5) \longrightarrow C(Y_5)$

$\Downarrow$

CALABI-YAU CONE from SE manifold

$\Rightarrow$ $H(r) = 1 + \frac{L^4}{r^4}$ is unchanged.

Take near horizon limit:



SINGULAR

(but not visible to curv. invariants) $\rightarrow$ we cannot cut it out as usual

↳ CALABI-YAU SINGULARITIES

↳ ALGEBRAIC GEOMETRY (study of manifolds in $\mathbb{C}^N$)

$\downarrow$

the field th. comes from this

KEY CONSTRAINTS on our $\mathcal{N}=1$ SCFT:

* it comes from N D3-branes

$\Rightarrow$ SYM theory $\rightarrow$ it cannot be pure SYM (not conf.) nor $\mathcal{N}=4$ (it's $\mathcal{N}=1$)

$\downarrow$

we must add MATTER in rep. of G

* $\mathcal{N}=1$ SUSY $\Rightarrow$ multiplets and Lagrangians have det. structure

* it has AdS_5 dual $\Rightarrow$ CFT

The mult:

- vector: $\begin{cases} A_\mu^a \\ \lambda^a \\ D^a \end{cases}$ in ADJ of G

- chiral: $\begin{cases} \phi_i \\ \psi_i \\ F_i \end{cases}$ in R of G

$$\Rightarrow \mathcal{L} \supset -\frac{1}{4} Tr(F_{\mu\nu} F^{\mu\nu}) + (D_\mu \phi)^\dagger (D^\mu \phi) + |F|^2 + $$
$$+ D^a D_a + (\text{fermions})$$

Integrate out F_i, D^a:

$$F_i = \frac{\partial W}{\partial \phi^i} \qquad \text{"F-terms"}$$

$$\Rightarrow |F|^2 + D^a D_a \sim V(\phi)$$

$$D^a = \phi^i T^a_{ij} \phi^j \qquad \text{"D-terms"}$$

$\Rightarrow$ Vacuum Moduli space: CONST SOLUTION:

$$V = 0 \Longrightarrow D^a = 0 \qquad \left(\text{enforced by } \delta\psi = 0,\ \delta\lambda = 0\right)$$
$$F_i = 0$$

SUSY Vacuum mod. sp. $\Longrightarrow$ KÄHLER VARIETY

In some cases it can be CY (any dim. because it's not the phys. spacetime)

$\Rightarrow$ SE $\longrightarrow C(Y_5)$: CY $\longrightarrow$ SUSY $\longrightarrow$ vacuum moduli space

$\Rightarrow$ G is gauge group $G = SU(N)^{\chi}$ with different g^2_{YM}

$\Rightarrow$ $W = \sum_{k=1}^{M} h_k \prod_j \phi_j$ $\qquad \Rightarrow$ UV $\to \mathcal{L} \to$ IR $\rightarrow$ must flow to conf. point

additional couplings

$\Rightarrow$ R is usually $(N, \underline{N})$: "bi-fundamental"

We use quartic superpot.

Impose Conf. inv. $\Rightarrow$ $\beta_i \equiv 0$

$\hookrightarrow \lambda \gg 1 \quad N \to \infty \Rightarrow$ How can we compute β?

For $\mathcal{N}=1$ th. the β is exact ("Russian β-function")
NSVZ

We have:

$$g_{YM},\ h_k \Rightarrow \beta(g_{YM})=0,\ \beta(h_k)=0$$

$$\Downarrow$$

$$\beta(g_{YM})=0 \Leftrightarrow 0 = 1 - \sum_I (1-R_I)$$

chiral fields charged under gauge fields

$$\beta(h_k)=0 \Leftrightarrow \sum_I R_I - 2 = 0$$

sum over the chirals

R-charges

$$\Rightarrow R = \frac{2}{3}\Delta \quad \text{for chiral fields}$$

R-charge of ϕ

$\to$ we need to fix the R-charges $\Rightarrow$ a-MAXIMIZATION

KLEBANOV-WITTEN DUALITY

$AdS_5 \times SE_5 \longleftrightarrow \mathcal{N}=1$ SYM + matter flowing to a SCFT

$SO(4) \simeq SU_1(2) \times SU_2(2)$

$\Rightarrow$ <u>metric</u>: $ds^2(T^{1,1}) = \frac{1}{6}\left(d\theta_1^2 + \sin^2\theta_1\, d\phi_1^2 + 1 \leftrightarrow 2\right) +$

$$+\frac{1}{9}\left(d\psi - \cos\theta_1\, d\phi_1 - \cos\theta_2\, d\phi_2\right)^2$$

$\phi_1, \phi_2 \in [0, 2\pi]$

$\psi \in [0, 4\pi]$

$\theta_1, \theta_2 \in [0, \pi]$

$$\Rightarrow \mathrm{vol}(T^{1,1}) = \frac{1}{6^2}\cdot\frac{1}{3}\left(\int_0^{2\pi} d\phi_1\right)^2\left(\int_0^{\pi} d\theta_1 \sin\theta_1\right)\int_0^{4\pi} d\psi = \frac{16}{27}\pi^3 = \frac{16}{27}\mathrm{vol}(S^5)$$

$C(T^{1,1}) :=$ "CONIFOLD"

$\Rightarrow$ 1 eq. in $\mathbb{C}^4$: $z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0$

$\Rightarrow SO(4)$ SYMM.

We can also use: $\tilde{z}_1 \tilde{z}_2 = \tilde{z}_3 \tilde{z}_4 \qquad \tilde{z}_i \in \mathbb{C}^4$

$\tilde{z}_1 = A_1 B_1 \qquad \tilde{z}_3 = A_1 B_2$

$\tilde{z} = A_2 B_2 \qquad \tilde{z}_4 = A_2 B_1$

$\Rightarrow A_i \sim \lambda A_i \qquad B_i \sim \lambda^{-1} B_i \qquad \lambda \in \mathbb{C}^*$

$\lambda = s e^{i\alpha}$

$\Rightarrow |A_1|^2 + |A_2|^2 - |B_1|^2 - |B_2|^2 = 0$

Impose $A_i \sim e^{i\alpha}$ $B_i \sim e^{-i\alpha} B_i$ (complex ident.)

$\Rightarrow$ Kähler quotient: $\mathbb{C}^4/U(1)$

$\hookrightarrow$ charges: $Q = (1,1,-1,-1)$

$\hookrightarrow$ we have then $A_i \in \underline{2}$ of $SU(2)_1$
$B_i \in \underline{2}$ of $SU(2)_2$

We can look at this as vac. mod. sp. of $\mathcal{N}=1$ SCFT:

$\rightarrow$ gauge th. is $U(1)_1 \times U(1)_2$

$$Q = (1,1,-1,-1) \Longrightarrow \begin{tikzcd} U(1)_1 \arrow[r, bend right, "B_i"'] & U(1)_2 \arrow[l, bend right, "A_i"'] \end{tikzcd} \qquad \text{"QUIVER"}$$

add $W(\phi) = \lambda \epsilon^{ij}\epsilon^{kl}\,\mathrm{Tr}[A_i B_k A_j B_l]$ to promote to $SU(2)_1 \times SU(2)_2$

Effectively makes A_i, B_i abelian on the VMS = Sym N (conifold) :

$$\begin{tikzcd} SU(N)_1 \arrow[r, bend right, "B_i"'] & SU(N)_2 \arrow[l, bend right, "A_i"'] \end{tikzcd}$$

$$+ \; W = \lambda \epsilon^{ij}\epsilon^{kl}\,\mathrm{Tr}\left(A_i B_k A_j B_l\right)$$

is then our QFT $\Longrightarrow$ CHECKLIST:

* couplings: g_1, g_2, λ

$$\beta_1 = 0 \Leftrightarrow 1 - \frac{1}{2}\left(1 - R[A_1] + 1 - R[A_2] + 1 - R[B_1] + 1 - R[B_2]\right) = 0$$

assume $R[A_1] = R[A_2] = R[A]$ (same for B):

$$R[A] + R[B] = 1$$

Now assume a $\mathbb{Z}_2$ symm. $(A \leftrightarrow B)$:

$$R[A] = R[B] = \frac{1}{2} \quad (\text{below unitarity bound})$$

$\Rightarrow$ KK spec. works

$\Rightarrow$ Weyl anomaly:

$$a = \frac{3}{32}\left(3 \, \mathrm{Tr}\, R^3 - \mathrm{Tr}\, R\right)$$

$$c = \frac{1}{32}\left(9 \, \mathrm{Tr}\, R^3 - 5 \, \mathrm{Tr}\, R\right)$$

NB: $\mathrm{Tr}\, R^\alpha = \sum_{ferm} \underbrace{(\underbrace{R_i}_{scal} - 1)^\alpha}_{ferm}$

$$\longrightarrow \mathrm{Tr}(R) = N^2\Big[\underbrace{\left(\tfrac{1}{2} - 1\right) \cdot 4}_{4} + \underbrace{2}_{\lambda\ (R[\lambda]=1)}\Big] = 0$$

$$\mathrm{Tr}(R^3) = N^2\left(\left(\tfrac{1}{2} - 1\right)^3 \cdot 4 + 2\right) = N^2 \cdot \frac{3}{2}$$

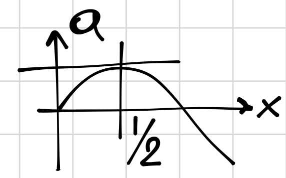
$$\Rightarrow a = c = \frac{27}{64} N^2 = \frac{\pi^3 N^2}{4 \, \mathrm{vol}(T^{1,1})}$$

We can forget $\mathbb{Z}_2$ symm. $\Rightarrow$ 1 eq for 3 coupl $\Rightarrow \exists$ 2d manifold

$$\longrightarrow \underbrace{R[A]}_{x} + R[B] = 1$$

$\hookrightarrow \mathrm{Tr}\, R = 0 \quad \forall x \, ; \quad \mathrm{Tr}\, R^3 = N^2\left(2(x-1)^3 + 2(-x)^3 + 2\right)$

$$\Rightarrow \quad a = \frac{27}{16} N^2 x(1-x) \quad \Rightarrow$$



$\exists\, a_{max}$? $\Rightarrow a\left(x_{max} = \frac{1}{2}\right) = \frac{27}{64} N^2$

a - MAXIM.: CFT $\Rightarrow$ a is max in R-symm. space

$\Rightarrow$ the geom. dual is:

SASAKI-EINSTEIN / $\mathcal{N}=1$ SCFT

(in $d=2$ it's c-maximization).