Recheck

02/02/18

- Solution of e.o.m.

$$X^I(u,\bar{u}) = X^I_L(u) + X^I_R(\bar{u})$$

- b.c. $X_L(x+i0^+) = U_t X_R(x-i0^-)$ $x \in (x_t, x_{t-1})$

$\Rightarrow X_L(x+i0^+) = U_t X_R(x-i0^+) + \Delta_t \leftarrow$ constant

- Doubling

$$\mathcal{X}^I_{(\bar{t})}(z) = \begin{cases} X^I_L(u) & z=u \\ U_{(\bar{t})}{}^I{}_J \; X^J_R(\bar{u}) & z=\bar{u} \end{cases}$$

This means

$$\mathcal{X}_{(\bar{t})}(x+i0^+) = X_L(x+i0^+) = U_{(t)} X_R(x-i0^+) + \Delta_{(t)}$$

$$= U_{(t)}\left(U_{\bar{t}}^{-1} \mathcal{X}_{(\bar{t})}(x-i0^+) - \Delta_{(\bar{t})}\right) + \Delta_{(t)}$$

$$= U_{(t)} U_{(\bar{t})}^{-1} \mathcal{X}_{(\bar{t})}(x-i0^+) + \Delta_{(t)} - U_{(t)} \Delta_{(\bar{t})}$$

- If we glue on $(x_{\bar{t}}, x_{\bar{t}-1})$ we can set $\Delta_{\bar{t}} = 0$

- In particular

$$X^I(u,\bar{u}) = \mathcal{X}^I_{(\bar{t})}(u) + \left(U^{-1}_{(\bar{t})}\right)^I{}_J \; \mathcal{X}^J_{(\bar{t})}(\bar{w})$$

In particular in $D=2$ we need $\mathcal{X}^z(z)$ and $\mathcal{X}^{\bar{z}}(z)$

[23/01/18] b.c.

We require

$$X^I(x_t, \bar{x}_t) = X^I(x_t + i0^+, x_t - i0^+) = 0 \qquad 2 \leq t \leq N-1$$

$$\Rightarrow \quad \mathcal{H}_{(\bar{t})}(x_t + i0^+) + U^{-1}_{(\bar{t})} \mathcal{H}_{(\bar{t})}(x_t - i0^+) = 0$$

We glue along $\bar{t} = N+1$, i.e. for $x<0$

Q: Does this mean that $\Delta_{(t)}[X] = 0$?

Since

$$X(x_t, \bar{x}_t) = (\mathbb{1} + U_{(t)}) X_R(x_t^+) + \Delta_{(t)}[X]$$
$$= (\mathbb{1} + U_{(t+1)}) X_R(x_t^-) + \Delta_{(t+1)}[X]$$

we get only $(\mathbb{1} - U_{(t)}) \Delta_{(t)}[X] = 0 \quad \forall t$

- Metric

We define for any solutions F, G:

$$\langle F, G\rangle = \oint \frac{dz}{i\pi}\, \partial_z \mathcal{F}^T g\, G = \oint \frac{dz}{i\pi}\, \partial_z \mathcal{F}^I G^J g_{IJ}$$

whenever $\Delta_t[G]=0 \quad \forall t \in \{1, .. N, N+1\}$

and $\Delta_t(F) \in \mathbb{C}$, i.e. $\Delta_t[F]$ may be not zero

but $(\mathbb{1} - \mathcal{U}_{(t)})\, \Delta_{(t)}[F]=0$

$$\langle F, G\rangle = -\frac{1}{2\pi}(F, G) = -\frac{1}{2\pi}\left[\oint_{|u|=r_0} * F^T g\, \overleftarrow{d} G + \mathcal{B}_+(r_0) + \mathcal{B}_-(-r_0)\right]$$

$$= -\frac{1}{2\pi}\left[\int_0^{\pi} d\vartheta\, r_0 \left(F^T g\, \partial_r G - \partial_r F^T g\, G\right)\right]$$

whenever $\Delta_t[F]=\Delta_t[G]=0 \quad \forall t \in \{1, .. N, N+1\}$

Let us see how to get the previous result

The current

$$\hat{J} = F^T g \overleftrightarrow{d} G$$

is conserved because eom

$$d{*}\hat{J} = dF^T g * dG + F^T g\, d * dG$$
$$- d * dF^T g\, G + * dF^T g\, dG$$

$$= dF^T g * dG + * dF^T g\, dG$$
$$= 0$$

since $* dx = dy$, $* dy = -dx$

$$\omega \wedge * \chi = \omega_x \chi_x\, dx\, dy + \omega_y \chi_y\, dy\, (-dx)$$
$$= (\omega_x \chi_x + \omega_y \chi_y)\, dx\, dy$$

$$* \omega \wedge \chi = \omega_y \chi_y (-dx)\, dy + \omega_x \chi_x\, dy\, dx$$
$$= (-\omega_x \chi_x - \omega_y \chi_y)\, dx\, dy$$

Notice that in a more pedestrian way we have

$$\hat{j}_\alpha = F \cdot \overleftrightarrow{\partial_\alpha} G \qquad \leftarrow \qquad \hat{J} = F \cdot \overleftarrow{d} G$$

$$\Rightarrow \quad \partial_\alpha \hat{j}_\alpha = \partial_\alpha F \cdot \partial_\alpha G + F \cdot \Box G - \Box F \cdot G - \partial_\alpha F \cdot \partial_\alpha G \quad \Leftrightarrow$$

Then we can compute

$$\oint d\star\hat{j} = 0 = \left[\int_{r_1} - \int_{r_0} + \int_{r_0}^{r_1} + \int_{-r_1}^{-r_0} \right] \star\hat{j}$$

so we can define

$$(F, G) = \int_{r_0} \star\hat{j} + \hat{\mathcal{B}}_+(r_0) + \hat{\mathcal{B}}_-(r_0)$$

with

$$\int_{r_0}^{r_1} \star\hat{j} = \hat{\mathcal{B}}_+(r_1) - \hat{\mathcal{B}}(r_0)$$

$$\int_{-r_1}^{-r_0} \star\hat{j} = -\hat{\mathcal{B}}_-(-r_0) + \hat{\mathcal{B}}_-(-r_1)$$

Now

$$-\int_{r_0}^{r_1} \star\hat{j} = +\int_{r_0}^{r_1} dx \quad \hat{j}_y \Big|_{y=0}$$

$$= \int_{r_0}^{r_1} dx \quad (F \cdot \partial_y G - \partial_y F \cdot G) \Big|_{y=0}$$

$$= \int_{r_0}^{r_1} dx \Big[(F_L(x) + F_R(x)) \cdot (i G_L'(x) - i G_R'(x))$$
$$- i(F_L'(x) - F_R'(x)) \cdot (G_L(x) + G_R(x)) \Big]$$

$$= i \int_{r_0}^{r_1} dx \Big[F_L \cdot G_L' - F_L' \cdot G_L - F_R \cdot G_R' + F_R' \cdot G_R$$
$$- F_L \cdot G_R' + F_R \cdot G_L' - F_L' \cdot G_R + F_R' \cdot G_L \Big]$$

If $(r_0, r_1) \subset (x_t, x_{t-1})$

$$\Rightarrow \; i' \int_{(r_0,r_1)\in J_t}^{r_1} dx \Big\{ (F_R^T U_t^T + \Delta_t^T[F])\, g\, U_t G_R' - F_R^T g\, G_R'$$

$$- F_R'^{\,T} U_t^T g (U_t G_R + \Delta_t[G]) + F_R'^{\,T} g\, G_R$$

$$- (F_R^T U_t^T + \Delta_t^T[F])\, g\, G_R' - F_R^{T\prime} g\, U_t G_R$$

$$+ F_R^T g\, U_t G_R' + F_R'^{\,T} g (U_t G_R + \Delta_t[G]) \Big\}$$

$$= i' \int_{(r_0,r_1)\subset J_t} dx \Big\{ \Delta_t^T[F]\, g\, U_t G_R' - \Delta_t^T(G)\, g\, U_t F_R'$$

$$- (F_R^T g\, U_t G_R)' - \Delta_t^T[F]\, g\, G_R'$$

$$+ (F_R^T g\, U_t G_R)' + F_R^{T\prime} g\, \Delta_t[G] \Big\}$$

$$= i' \Big[-\Delta_t^T(F)\, g\, (1 - U_t) G_R + \Delta_t^T[G]\, g\, (1 - U_t) F_R \Big]\Big|_{r_0}^{r_1}$$

$$= - \int_{+r_0}^{+r_1} x\, \hat{j} = \hat{\beta}_t(r_0) - \hat{\beta}_t(r_1)$$

where we used $U_t^T g\, U_t = g$

Check for $(r_0, r_1) \subset (x_t, x_{t-1})$ in a better way

$$- \int_{r_0}^{r_1} \star \hat{j} = i \int_{r_0}^{r_1} dx \Big\{ \left[(\mu + V_t) F_n(x) + \Delta_t [F] \right]^T g \left[-(\mu - V_t) G_n' \right]$$

$$- \left[-(\mu - V_t) F_n' \right]^T g \left[(\mu + V_t) G_n + \Delta_t [G] \right] \Big\}$$

$$= i \int_{r_1}^{r_1} dx \quad -\Delta_t^T [F] g (\mu - V_t) G_n' + F_n'^T (\mu - V_t)^T g \, \Delta_t [G]$$

$$= i \left[- \Delta_t^T [F] g (\mu - V_t) G_n + F_n^T (\mu - V_t)^T g \, \Delta_t [G] \right] \Big|_{r_0}^{r_1}$$

$$= \hat{\beta}_+ (r_0) - \hat{\beta}_+ (r_1)$$

hence

$$\int_{(r_0, r_1)} \star j = i \int_{(r_0, r_1)} \star \hat{j} = \left[- \Delta_t^T [F] g (\mu - V_t) G_n \ldots \right] \Big|_{r_0}^{r_1}$$

Hence we need the continuity at the boundary x_t

$x_t^+ \in D_t$ $\qquad$ $x_t^- \in D_{t+1}$

i.e.

IMPORTANT

$$\left[-\Delta_t^T(F)\, g\, (\mathbb{1}-U_t)\, G_n + \Delta_t^T[G]\, g\, (\mathbb{1}-U_t) F_n \right](x_t^+)$$

$$= \left[-\Delta_{t+1}^T(F)\, g\, (\mathbb{1}-U_{t+1})\, G_n + \Delta_{t+1}^T[G]\, g\, (\mathbb{1}-U_{t+1}) F_n \right](x_t^-)$$

to write

$$\hat{\mathcal{B}}_t(r_0) =$$

$$-r' \left[-\Delta_t^T(F)\, g\, (\mathbb{1}-U_t)\, G_n + \Delta_t^T[G]\, g\, (\mathbb{1}-U_t) F_n \right]\Big|_{r_0}$$

Notice we can satisfy the constraints in many ways

1) $\Delta_t[F] = \Delta_t[G] = 0$ $\Rightarrow$ constraints are ok

2) $\frac{1}{2}(\mathbb{1}-U_t)\, G(x_t^+) = f_t^{\perp t}[G] = 0$ and $f_t^{\perp t}[F] = 0$

$\Rightarrow$ constraints are ok

[21/01/18] In particular we may want

$$X \equiv F$$

since after we use $\partial \mathcal{F}$ and we express $X \sim \mathcal{J}$

Then we have

$$X(x_t, \bar{x}_t) = 0 \qquad t = 2, \dots N-1$$

This means $X(x_t^+, \bar{x}_t^+) = (\mathbb{1} + U_t) X_R(\bar{x}_t^+) + \Delta_t[X]$

so

$$(\mathbb{1} - U_t) X(x_t, \bar{x}_t) = (\mathbb{1} - U_t) \Delta_t [X] = 0$$

and $\qquad X(x_t^-, \bar{x}_t^-) = (\mathbb{1} + U_{t+1}) X_R(\bar{x}_t^-) + \Delta_{t+1}[X]$

$$\Rightarrow \qquad (\mathbb{1} - U_{t+1}) \Delta_{t+1} [X] = 0$$

Then we are left with

$$\Big[- \cancel{\Delta_t^T(F) \, g \, (\mathbb{1} - U_t) G_R}^{\to 0} + \Delta_t^T[G] \, g \, (\mathbb{1} - U_t) F_R \Big](x_t^+)$$

$$= \Big[- \cancel{\Delta_{t+1}^T(F) \, g \, (\mathbb{1} - U_{t+1}) G_R}_{\to 0} + \Delta_{t+1}^T[G] \, g \, (\mathbb{1} - U_{t+1}) F_R \Big](x_t^-)$$

Then either $\quad F_R(\bar{x}_t) = 0 \quad$ or $\quad (\mathbb{1} - U_t) \Delta_t [G] = 0 \ \forall t$

The continuity constraint in $D=2$

$$\to U(t) = \begin{array}{c|cc} & z & \bar{z} \\ z & \\ \bar{z} & \end{array} \begin{pmatrix} 0 & e^{i2\alpha_t\pi} \\ e^{-i2\alpha_t\pi} & 0 \end{pmatrix} = U_{(t)}^{-1} = U_{(t)}^{\dagger}$$

$$\to (\mathbb{1} - U) F = \begin{pmatrix} 1 & -e^{i2\alpha\pi} \\ -e^{-i2\alpha\pi} & 1 \end{pmatrix} \begin{pmatrix} F^z \\ F^{\bar{z}} \end{pmatrix}$$

$$= \begin{pmatrix} F^z - e^{i2\alpha\pi} F^{\bar{z}} \\ -e^{-i2\alpha\pi} F^z + F^{\bar{z}} \end{pmatrix} = \begin{pmatrix} e^{i\alpha\pi}\, 2i\, \mathrm{Im}(e^{-i\alpha\pi} F^z) \\ -e^{-i\alpha\pi}\, 2i\, \mathrm{Im}(e^{-i\alpha\pi} F^z) \end{pmatrix}$$

since F is real so $F^{\bar{z}} = (F^z)^*$

$$= 2i\, \mathrm{Im}(e^{-i\alpha} F^z) \begin{pmatrix} e^{i\alpha} \\ -e^{-i\alpha} \end{pmatrix}$$

$$\to \Delta^T g (\mathbb{1} - U) F = 2i\, \mathrm{Im}(e^{-i\alpha\pi} F^z) \begin{pmatrix} \Delta^z & \Delta^{\bar{z}} \end{pmatrix} \begin{pmatrix} & \frac{1}{2} \\ \frac{1}{2} & \end{pmatrix} \begin{pmatrix} e^{i\alpha} \\ -e^{-i\alpha} \end{pmatrix}$$

$$= i\, \mathrm{Im}(e^{-i\alpha\pi} F^z) \left(-\Delta^z e^{-i\alpha\pi} + \Delta^{\bar{z}} e^{i\alpha\pi} \right)$$

$$= 2\, \mathrm{Im}(e^{-i\alpha\pi} F^z)\, \mathrm{Im}(e^{-i\alpha\pi} \Delta^z)$$

$\to$ Suppose F_n, G_n are continuous then we need

$$\begin{aligned} &-2\, \mathrm{Im}(e^{-i\alpha_t} \Delta^z_t[F])\, \mathrm{Im}(e^{-i\alpha_t} G^z_n(x_t)) \\ &+ 2\, \mathrm{Im}(e^{-i\alpha_t} \Delta^z_t[G])\, \mathrm{Im}(e^{-i\alpha_t} F^z_n(x_t)) \\ &= -2\, \mathrm{Im}(e^{-i\alpha_{t+1}} \Delta^z_{t+1}[F])\, \mathrm{Im}(e^{-i\alpha_{t+1}} G^z_n(x_t)) \\ &\quad + 2\, \mathrm{Im}(e^{-i\alpha_{t+1}} \Delta^z_{t+1}[G])\, \mathrm{Im}(e^{-i\alpha_{t+1}} F^z_n(x_t)) \end{aligned}$$

[23/01/18]

IF $F \doteq X$ and $f_{t=2,..N-1} = 0$

$$(M - U)\Delta[F] = 0 \Rightarrow \mathrm{Im}\,(e^{-i\alpha\pi} F^{z}) = 0$$

so we get

$$\mathrm{Im}\,(e^{-i\pi\alpha_t}\Delta^{z}_{t}[F])\ \mathrm{Im}\,(e^{-i\pi\alpha_t} G^{z}_{R}/x_t)) =$$

$$\mathrm{Im}\,(e^{-i\pi\alpha_{t+1}}\Delta^{z}_{(t+1)}[F])\ \mathrm{Im}\,(e^{-i\pi\alpha_{t+1}} G^{z}_{R}/x_t))$$

Consider the other piece $\int_{r_0} * \hat{J}$

Use

$$* dr^2 = 2r * dr = *(2x\,dx + 2y\,dy) = 2(x\,dy - y\,dx)$$
$$= 2r^2(d\vartheta)$$
$$\Rightarrow \quad * dr = r\,d\vartheta$$

to get

$$\int_{r_0} * J = \int_0^{\pi} d\theta \; r_0 \; j_r$$

$$= r_0 \int_0^{\pi} d\vartheta \left[F \cdot \partial_r G - \partial_r F \cdot G \right] \Big|_{r=r_0}$$

Now

$$\partial_r G = \partial_r G_L(re^{i\vartheta}) + \partial_r G_R(re^{-i\vartheta})$$
$$= G_L'(u)\, e^{i\vartheta} + G_R'(\bar{u})\, e^{-i\vartheta}$$
$$= \frac{-i}{r} \partial_\vartheta G_L + \frac{i}{r} \partial_\vartheta G_R$$

check $\quad G_L \to re^{i\vartheta} \qquad \partial_r G_L \to e^{i\vartheta} \qquad \partial_\vartheta G_L \to ire^{i\vartheta}$

hence

$$= r_0 \int_0^{\pi} d\vartheta \Big[\left(F_L(u) + F_R(\bar{u}) \right) \cdot \left(\frac{-i}{r_0} \partial_\vartheta G_L + \frac{i}{r_0} \partial_\vartheta G_R \right)$$
$$- \left(\frac{-i}{r_0} \partial_\vartheta F_L + \frac{i}{r_0} \partial_\vartheta F_R \right) \cdot \left(G_L + G_R \right) \Big] \Big|_{r=r_0}$$

$$= -i \int_0^{\pi} d\vartheta \Big[F_L \cdot \partial_\vartheta G_L - \partial_\vartheta F_L \cdot G_L$$
$$- F_R \cdot \partial_\vartheta G_R + \partial_\vartheta F_R\, G_R$$
$$- F_L\, \partial_\vartheta G_R - \partial_\vartheta F_L \cdot G_R$$
$$+ F_R \cdot \partial_\vartheta G_L + \partial_\vartheta F_R\, G_L \Big]$$

$$= i\left(F_L\cdot G_R - F_R\cdot G_L\right)\Big|_{r_0}^{-r_0}$$

$$-i\left(F_L\cdot G_L - F_R\cdot G_R\right)\Big|_{r_0}^{-r_0}$$

$$+ i\int_0^{\pi} d\vartheta\ \left[\partial_\vartheta F_L^T g\, G_L - \partial_\vartheta F_R^T g\, G_R\right]$$

$$= +i\left(F_L + F_R\right)^T g\,\left(G_L - G_R\right)\Big|_{-r_0}^{+r_0}$$

$$+ i\int_0^{\pi} d\vartheta\ \left[\partial_\vartheta F_L^T g\, G_L - \partial_\vartheta F_R^T g\, G_R\right]$$

$$= i\left[\left((\mathbb{1}+U_t)F_R + \Delta_t[F]\right)^T g\left[-(\mathbb{1}-U_t)G_R + \Delta_t[G]\right]\right]\Big|_{-r_0}^{+r_0}$$

$$+ i \oint du\ \partial_u F_L^T g\, G_L - i\oint d\bar{u}\ \partial_{\bar{u}} F_R^T g\, G_R$$

since $\displaystyle\int_0^{\pi} d\vartheta\ \partial_\vartheta\left(e^{-in\vartheta}\right) e^{-im\vartheta} = \oint_{\bar{u}\in H^-} d\bar{u}\ \bar{\partial}\left(\bar{u}^n\right)\bar{u}^m$

and $G_L(x) = U_t\, G_R(x) + \Delta_t[G] \quad x\in(x_t, x_{t-1})$

and $G_R(\bar{u}) = U_{\bar{t}}^{-1}\, g_{(\bar{t})}(\bar{u})$

$$= i\left[F_R(\mathbb{1}+U_t)^T g\,\Delta_t[G] - \Delta_t[F]^T g\,(\mathbb{1}-U_t)G_R + \Delta_t[F]^T g\,\Delta_t[G]\right]\Big|_{r_0}$$

$$- i\left[\qquad\qquad\right]\Big|_{-r_0}$$

$$+ i\oint_{z\in H} dz\ \partial_z \mathcal{F}_{(\bar{t})}^T g\, \mathcal{G}_{(\bar{t})} + i\oint_{z\in H^-} dz\ \partial_z \mathcal{F}_{(\bar{t})}^T g\, \mathcal{G}_{(\bar{t})}$$

Putting all together

$$(F, G) = \oint_{r_0} *j + \hat{\mathcal{B}}_+(r_0) + \hat{\mathcal{B}}_-(-r_0)$$

$$= 2i \oint dz\, \partial_z \mathcal{F}^T(z)\, g\, \mathcal{G}(z)$$

$$+ i\,[\; F_R\,(1+U_t)^T g\, \Delta_t[G] - \Delta_t[F]^T g\,(1-U_t) G_R + \Delta_t[F]^T g\, \Delta_t[G]\;]\Big|_{r_0}$$

$$- i\,[\qquad\qquad\qquad\qquad]\Big|_{-r_0,\; t=N+1}$$

$$+ i\left[-\Delta_t^T(F)\, g\, (1-U_t)\, G_R + \Delta_t^T[G]\, g\, (1-U_t) F_R \right]\Big|_{r_0}$$

$$- i\left[-\Delta_{N+1}^T(F)\, g\, (1-U_{N+1})\, G_R + \Delta_{N+1}^T[G]\, g\, (1-U_{N+1}) F_R \right]\Big|_{-r_0}$$

Then simplify the boundary contribution at r_0

$$+i\Big[\; F_R\,(\mu+V_t)^T g\, \Delta_t[G] - \Delta_t[F]^T g\,(\mu-V_t)G_R$$

$$+\;\Delta_t[F]^T g\;\Delta_t[G]\Big]\Big|_{r_0}$$

$$-i\;\Delta_t^T(F)\; g\,(\mu-V_t)\,G_R + i\;\Delta_t^T[G]\; g\,(\mu-V_t)\,F_R\Big\}\Big|_{\bar r_0} =$$

$$= \; i\;\Delta_t^T[F]\,g\;\Delta_t[G] + 2i\;\Delta_t^T[G]\;g\;F_R(\bar r_0)$$

$$-\;i\;\Delta_t^T[F]\;g\;(\mu-V_t)\;G_R(\bar r_0)$$

true? NO — since F_R is not local w.r.t $F(u,\bar u)$ we need

$$\Delta_t[G]=0$$

Finally

$$(F,G) = 2i \oint_{\tilde{S}} dz\, \partial_z \mathcal{F}^T(z)\, g\, \mathcal{G}(z)$$

$$+ i\, \Delta_t^T[G]\, g \left(\Delta_t[F] + 2\, F_R(\bar{r}_0)\right)$$

$$- 2i\, \Delta_t^T[F]\, g\, (M - U_t)\, G_R(\bar{r}_0)$$

$$- i\, \Delta_{N+1}^T[G]\, g \left(\Delta_{N+1}[F] + 2\, F_R(-\bar{r}_0)\right)$$

$$- 2i\, \Delta_{N+1}^T[F]\, g\, (M - U_{N+1})\, G_R(-\bar{r}_0)$$

true! with $\Delta_t[G] = 0$ since $F_R(\bar{u})$ is not local wrt $F(u,\bar{u})$

NO and $\Delta_t[F] = 0$

Now we want $F \equiv X$ and $(M - U_{(t)})\Delta_{(t)}[F] = 0$

so we are left with

$$= 2i \oint_{\tilde{S}} dz\, \partial_z \mathcal{F}^T(z)\, g\, \mathcal{G}(z)$$

$$+ i\, \Delta_t^T[G]\, g \left(\Delta_t[F] + 2\, F_R(\bar{r}_0)\right)$$

$$- i\, \Delta_{N+1}^T[G]\, g \left(\Delta_{N+1}[F] + 2\, F_R(-\bar{r}_0)\right)$$

if $\Delta_{(t)}[G] = 0$

$$= 2i \oint dz\, \partial_z \mathcal{F}^T g\, \mathcal{G}$$

$$= -2\pi \oint \frac{dz}{i\pi}\, \partial_z \mathcal{F}^T g\, \mathcal{G} = -2\pi \langle F, G \rangle$$

Notice that

$$(F, G) = -(G, F) =$$

$$= -i \oint dz\, \partial_z \mathcal{G}^T y \mathcal{F}$$

$$= i \oint dz\, \partial_z \mathcal{F}^T y \mathcal{G} \quad - \quad i \quad \mathcal{G}^T(z) y \mathcal{F}(z) \Big|_{z = r_0 e^{i\varepsilon}}^{z = -r_0 e^{i\varepsilon}}$$

$$- i \quad \mathcal{G}^T y \mathcal{F} \Big|_{z = -r_0 e^{-i\varepsilon}}^{z = r_0 e^{-i\varepsilon}}$$

but $\quad \mathcal{G}^T y \mathcal{F} \Big|_{r_0 e^{i(\pi-\varepsilon)}} = G_L^T(r_0 e^{i(\pi-\varepsilon)})\, y\, F_L(r_0 e^{i(\pi-\varepsilon)})$

$$\mathcal{G}^T y \mathcal{F} \Big|_{r_0 e^{-i(\pi-\varepsilon)}} = G_R^T(r_0 e^{-i(\pi-\varepsilon)})\, U_t^T y U_t\, F_R(r_0 e^{\ldots})$$

then

$$\mathcal{G}^T y \mathcal{F} \Big|_{r_0}^{-r_0} + \mathcal{G}^T y \mathcal{F} \Big|_{-\bar{r}_0}^{\bar{r}_0} = G_L^T(-r_0)\, y\, F_L(-r_0) - G_L^T(r_0)\, y\, F_L(r_0)$$
$$- G_R^T(-\bar{r}_0)\, y\, F_R(-\bar{r}_0) + G_R(\bar{r}_0)\, y\, F_R(\bar{r}_0)$$

$$= \left[U_{N+1} G_R(-\bar{r}_0) + \Delta_{N+1}[G] \right]^T y \left[U_{N+1} F_R(-\bar{r}_0) + \Delta_{N+1}[F] \right]$$
$$- G_R^T(-\bar{r}_0)\, y\, F_R(-\bar{r}_0)$$

$$- \left(-r_0 \to +r_0 \right)$$

$$= G_R^T(-\bar{r}_0)\, U_{N+1}^T y\, \Delta_{N+1}[F] + \Delta_{N+1}^T[G]\, y\, U_{N+1} F_R(-\bar{r}_0) + \Delta_{N+1}^T[G]\, y\, \Delta_{N+1}^T[F]$$
$$- \left(-r_0 \to +r_0 \right)$$

then $\quad \Delta_t[F] = \Delta_t[G] = 0 \implies$ (previous expression) $= 0$

so we integrate by parts and get

$$= -i \oint dz\, \mathcal{F}^T y\, \partial_z \mathcal{G}$$

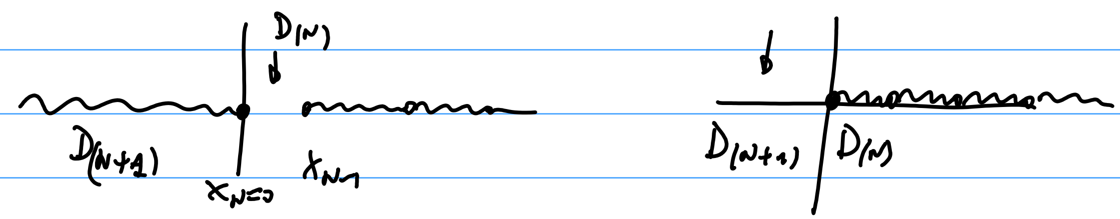
Boris (TOO RESTRICTIVE)

We write y^{z}_{n} in place of $y^{z}_{(\bar{t})N}$

$$y^{z}_{N}(z) = e^{i\pi\alpha_N} z^{-n+\varepsilon_N} \prod_{t=2}^{N-1} \left(1-\frac{z}{x_t}\right)^{\varepsilon_t} \qquad y^{\bar{z}}_{n} = 0$$

$$\bar{y}^{\bar{z}}_{n}(z) = e^{-i\pi\alpha_N} z^{-n+\bar{\varepsilon}_N} \prod_{t=2}^{N-1} \left(1-\frac{z}{x_t}\right)^{\bar{\varepsilon}_t} \qquad \bar{y}^{z}_{n} = 0$$

We identify $y \to y_{(N)}$ or with $y_{(N+1)}$ we put the cuts



This basis has

1) the expected number of elements
$\to$ they are 1-1 with $\mathcal{X}^{z}_{(2,1)n} = z^{-n+\varepsilon_N}$ $\quad \bar{\mathcal{X}}^{\bar{z}}_{(2,1)n} = z^{-n+\bar{\varepsilon}_N}$
the $N=2$ $M=1$ in-states

2) The proper behaviour at x_t, $t=2 \dots N-1$
$\to$ $y(x_t)=0$ $\quad t=2, \dots N-1$

3) The proper discontinuities $\Delta_t[y^{z}]=0$. $\quad t=1 \dots N+1$

⊙ Consider the case $y_{(N)}$

$$y^{z}_{Ln}(u) = y_{(N)n}(u) \qquad y^{\bar z}_{Rn}(\bar u) = \left(U^{-1}_{(N)}\right)^{\bar z}_{z} \, y^{z}_{n}(\bar u)$$
$$= e^{-i\, 2\pi \alpha_N} \, y^{\bar z}_{n}(\bar u)$$

• $0 < x < x_{N-1}$ $\quad x \in D_{(N)}$ $\qquad$ right direction for $D_{(N)}$

$$y_{(N)}(x + i0^+) = |y_{(N)}| \, e^{i \pi \alpha_N} = y_{(N)}(x - i0^+)$$

$$\Rightarrow \Delta^{z}[Y_n] = Y^{z}_{L}(x+i0) - U_{(N)}{}^{z}{}_{\bar z} \, Y^{\bar z}_{2}(x - i0) = 0$$

$x_{N-1} < x < x_{N-2}$ $\quad x \in D_{(N-1)}$

$$y_{(N)n}(x + i0^+) = e^{i\pi\alpha_N} \, e^{-i\pi(\varepsilon_{N-1} + n)} \, |y_{(N)n}(x)|$$

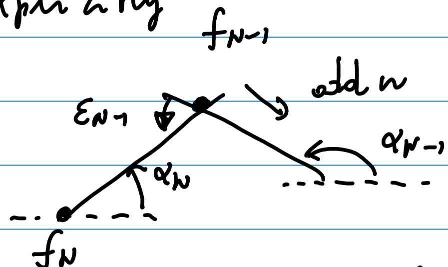
where $\qquad \varepsilon_{N-1} = \alpha_N - \alpha_{N-1} + \theta(\alpha_{N-1} - \alpha_N)$

$$y_{(N)n}(x + i0^+) = e^{i\pi\alpha_{N-1}} \, e^{i\pi[\theta(\alpha_{N-1} - \alpha_N) + n]} \, |y_{(N)n}(x)|$$

right direction for D_{N-1}

but sign depends on $n + \theta(\alpha_{N-1} - \alpha_N)$

Explicitly



$$y = e^{i\pi\alpha_{N-1}} \, e^{i\pi(n+1)} \, |y|$$

Now

$$y_{(N)n}(x-i0^+) = e^{i\pi\alpha_N}\, e^{i\pi\varepsilon_{N-1}}\, |y_{(N)n}(x)|$$

$$= e^{i\pi(2\alpha_N - \alpha_{N-1} + \theta(\alpha_{N-1}-\alpha_N))}\, |y_{(N)n}(x)|$$

$\uparrow$ wrong direction

Then $\quad y^{z}_{Ln}(x+i0^+) = y^{z}_{(N)n}(x+i0^+) =$

$$e^{i\pi\alpha_N}\, e^{-i\pi(\varepsilon_{N-1}+n)}\, |y_{(N)n}(x)|$$

and

$$y^{\bar z}_{Rn}(x-i0^+) = U_{(N)}{}^{\bar z}{}_{z} \cdot y^{z}_{(N)n}(x-i0^+)$$

$$= e^{-i2\pi\alpha_N} \cdot e^{i\pi\alpha_N}\, e^{i\pi\varepsilon_{N-1}}\, |y_{(N)n}(x)|$$

Now we can compute

$$\Delta^{z}_{(N-1)}[y_n] = e^{i\pi(\alpha_N - \varepsilon_{N-1})}|y|$$

$$- \underbrace{e^{+i2\pi\alpha_{N-1}}}_{U_{(N-1)}{}^{z}{}_{\bar z}} \left[e^{-i2\pi\alpha_N} \cdot e^{i\pi\alpha_N + i\pi\varepsilon_{N-1}} |y| \right]$$

$$= \left[e^{i\pi(\alpha_N - \varepsilon_{N-1})} - e^{-i2\pi\varepsilon_{N-1}} \cdot e^{i\pi(\alpha_N+\varepsilon_{N-1})} \right] |y| = 0$$

• $x<0$ $\quad x \in D_{(N+1)}$

$$y_n^z(x+i0^+) = e^{i\pi\alpha_N} e^{i\pi(-n+\varepsilon_N)} |\;|$$
$$= e^{i\pi\alpha_{N+1}} e^{i\pi(\theta(\alpha_N-\alpha_{N+1})-n)} ||$$

right direction

and
$$y_n^z(x-i0^+) = e^{i\pi\alpha_N} e^{-i\pi(-n+\varepsilon_N)} |\;|$$

Then $\quad y_{Ln}^z(x) = y_n^z(x+i0^+) = e^{i\pi(\alpha_N+\varepsilon_N-n)} |\;|$

and
$$y_{Rn}^{\bar{z}}(\bar{x}) = U_{(N)\,z}^{\bar{z}}\, y_n^z(x-i0^+)$$
$$= e^{-i\pi\alpha_N} e^{-i\pi(-n+\varepsilon_N)} |\;|$$

It follows

$$\Delta_{(N+1)}^z[Y_n] = e^{i\pi(\alpha_N+\varepsilon_N-n)} || - e^{i2\pi\alpha_{N+1}} e^{-i\pi\alpha_N} \cdot$$
$$\cdot\, e^{i\pi(\alpha_N-\varepsilon_N+n)} ||$$

$$= \left[e^{i\pi(\alpha_N+\varepsilon_N-n)} - e^{i\pi(2\varepsilon_N+\alpha_N-\varepsilon_N+n)}\right] ||$$

$$=0$$

• If $y = y_{(N+1)}$

$x<0 \quad x \in D_{(N+1)}$

$$y^{z}(x \pm i0^{+}) = e^{+i\pi \alpha_N} e^{i\pi \varepsilon_N} |y^{z}|$$

$$= e^{i\pi(\alpha_N + \alpha_{N+1} - \alpha_N + \vartheta(\alpha_N - \alpha_{N+1}))} |y^{z}|$$

$$= e^{i\pi(\alpha_{N+1} + \vartheta(\alpha_N - \alpha_{N+1}))} |y^{z}|$$

$\uparrow$

right $D_{N+1} \equiv D_1$ direction

• $0 < x < x_{N-1} \quad x \in D_{(N)}$

$$y^{z}(x+i0^{+}) = e^{i\pi \alpha_N} |y^{z}(x)|$$

$$y^{z}(x-i0^{+}) = e^{i\pi \alpha_N} e^{-i2\pi \varepsilon_N} |y^{z}|$$

so

$$y^{z}_{L}(x) = e^{i\pi \alpha_N} |y^{z}(x)|$$

$$y^{\bar{z}}_{R}(\bar{x}) = e^{-i2\pi \alpha_{N+1}} \left(e^{i\pi \alpha_N} e^{-i2\pi \varepsilon_N} |y^{z}| \right)$$

then

$$\Delta^{z}_{(N)}[y] = y^{z}_{L}(x) - U^{z}_{(N)\bar{z}} \, y^{\bar{z}}_{R}(\bar{x})$$

$$= e^{i\pi \alpha_N} | 1 - e^{i2\pi \alpha_N} \cdot [e^{-i2\pi \alpha_N}$$

⊙ Therefore we could NAIVELY write

$$\begin{cases} \mathcal{H}^{z}_{(1)}(z) = \sum_n y_n \, y^z_n(z) \\ \mathcal{H}^{\bar z}_{(1)}(z) = \sum_m \bar{y}_m \, \bar{y}^{\bar z}_m(z) \end{cases}$$

so that

$$\begin{pmatrix} X^{z}(u,\bar u) \\ X^{\bar z}(u,\bar u) \end{pmatrix} = \sum_n \left[y_n \begin{pmatrix} y^z_n(u) \\ U^{\bar z}_{(t)z} \, y^z_n(\bar u) \end{pmatrix} + \bar{y}_n \begin{pmatrix} U^{z}_{(t)\bar z} \, \bar{y}^{\bar z}_n(\bar u) \\ \bar{y}^{\bar z}_n(u) \end{pmatrix} \right]$$

or

$$X^z(u,\bar u) = \sum_n \left[y_n \, y^z_n(u) + \bar{y}_n \, U^{z}_{(t)\bar z} \, \bar{y}^{\bar z}_n(\bar u) \right]$$

Notice that the $y_n, \bar{y}_n$ are independent.

Q: Why too RESTRICTIVE?

Because if we write

$$X(u,\bar u) = \sum \left[x_n \, \mathcal{J}_n(u) + \bar{x}_n \, U_{(\bar t)} \bar{\mathcal{J}}_n(\bar u) \right]$$

and imposes

$$X(x_t, \bar{x}_t) = 0$$

we get something more general

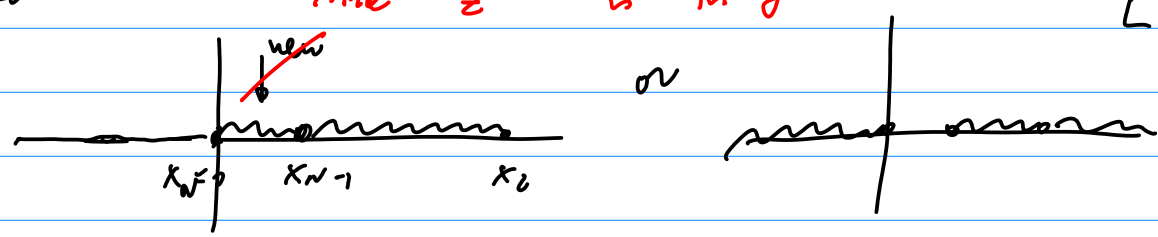
Basis (TOO LARGE)

We now introduce the basis for the dual space

$$\mathcal{J}^{z}_{n}(z) = e^{i\pi\alpha_N} \int_{x_N}^{z} dw \; w^{-n+\varepsilon_N-1} \prod_{t=2}^{N-1} \left(1-\frac{w}{x_t}\right)^{\varepsilon_t-1} \qquad \mathcal{J}^{\bar z}_{n} = 0$$

$$\bar{\mathcal{J}}^{z}_{n} = 0 \qquad \bar{\mathcal{J}}^{\bar z}_{m} = e^{-i\pi\alpha_N} \int_{x_N}^{\bar z} dw \; w^{-n+\bar\varepsilon_N-1} \prod_{t=2}^{N-1} \left(1-\frac{w}{x_t}\right)^{\bar\varepsilon_t-1}$$

Notice that for all $\mathcal{J}$ s.t. $\partial\mathcal{J}$ has a pole at $z=0$ or $z=\infty$ we need not to introduce another cut since $z^{-n+\varepsilon_N}$ is in $\mathcal{Y}$



Notice

1) $\mathrm{span}(\mathcal{J}) \supset \mathrm{span}(\mathcal{Y})$

In particular

$$\mathcal{J}(x_{t+1}, 0^+) \neq \mathcal{J}(x_t - 1, 0^+)$$

Let us define (when exists)

$$I_{(t+1)n} = \int_{x_{t+1}}^{x_t} dx \; x^{-n+\varepsilon_{N-1}} \prod_{t=2}^{N-2} \left|1 - \frac{x}{x_t}\right|^{\varepsilon_t - 1} \qquad t = N-1 \dots 1$$

since $x \in D_{t+1}$

and

$$I_{(t+1)n}(x) = \int_{x_t}^{x} dy \; y^{-n+\varepsilon_{N-1}} \prod \left|1 - \frac{y}{x_t}\right|^{\varepsilon_t - 1}$$

so $I_{(t+1)n}(x_t) = I_{(t+1)n}$

For $\mathcal{J} = \mathcal{J}_{(N)}$

we have

• $0 < x < \alpha_{N-1}$

$$\mathcal{J}^{z}_{(N)n}(x+i0^+) = \mathcal{J}^{z}_{(N)n}(x-i0^+) = I_{(N)n}(x)$$

Then
$$I^{z}_{Ln}(x+i0^+) = I_{(N)n}(x)$$
$$I^{\bar z}_{Rn}(x-i0^+) = U_{(N)}{}^{\bar z}{}_{z}\, I^{z}_{(N)n}(x-i0^+)$$
$$= e^{-i2\pi\alpha_N}\, I^{z}_{(N)n}(x-i0^+)$$

Therefore
$$\Delta^{z}_{(N)}[I_n] = I_{(N)n} - \underbrace{e^{i2\pi\alpha_N}}_{U_{(N)}{}^{z}{}_{\bar z}} \left(e^{-i2\pi\alpha_N}\, I^{z}_{(N)n}(x-i0^+)\right) \to 0$$

• $x < 0$
$$\mathcal{J}^{z}_{(N)n}(x+i0^+) = e^{i\alpha_N\cdot\pi}\, e^{i\pi(-n-\bar\varepsilon_N)} \,|\quad|$$
$$\mathcal{J}^{z}_{(N)n}(x-i0^+) = e^{i\pi\alpha_N}\, e^{-i\pi(-n-\bar\varepsilon_N)} \,|\quad|$$

then
$$I^{z}_{Ln}(x) = e^{i\pi(\alpha_N+\varepsilon_N-1-n)} \,|\quad|$$
$$I^{\bar z}_{Rn}(\bar x) = \underbrace{e^{-i2\pi\alpha_N}}_{U_{(N)}{}^{\bar z}{}_{z}} \cdot e^{i\pi(\alpha_N+n-\varepsilon_N+1)} \,|\quad|$$

so we get

$$\Delta^{z}_{(N-1)}[I_n] = e^{i\pi(\alpha_N+\varepsilon_N+1+n)}\,|\,|$$

$$- e^{i2\pi\alpha_{N-1}}\left[e^{-i2\pi\alpha_N}\cdot e^{i\pi(\alpha_N+n-\varepsilon_N+1)}\,|\,|\right]$$

$$= e^{i\pi(\alpha_N+\varepsilon_N+1+n)}\left[1-e^{-i\pi\varepsilon_N}\cdot e^{i2\pi\varepsilon_N}\cdot e^{-i\pi\varepsilon_N}\right]|\,| = 0$$

- $x_{N-1} < x < x_{N-2}$ $\qquad x \in D_{N-1}$

$$\prod^{z}_{n(N)}(x+i0^+) = \left(e^{-i\pi}\right)^{(-\bar\varepsilon_{N-1})}|\,|\cdot e^{i\pi\alpha_N}$$

$$\prod^{z}_{n(N)}(x-i0^+) = \left(e^{+i\pi}\right)^{(-\bar\varepsilon_{N-1})}|\,|\cdot e^{i\pi\alpha_N}$$

then $\qquad I^{z}_{Ln}(x+i0^+) = e^{i\pi(\alpha_N+\bar\varepsilon_{N-1})}\,|\,|$

$$I^{\bar z}_{Rn}(x-i0^+) = e^{-i2\pi\alpha_N}\cdot e^{i\pi(\alpha_N-\bar\varepsilon_{N-1})}\,|\,|$$

therefore

$$\Delta^{z}_{(N-1)}(I_n) = \left[e^{i\pi(\alpha_N+\bar\varepsilon_{N-1})} - e^{i2\pi\alpha_{N-1}}\cdot e^{i\pi(-2\alpha_N+\alpha_N-\bar\varepsilon_{N-1})}\right] * |\,|$$

$$= e^{i\pi(\alpha_N+\bar\varepsilon_{N-1})}\left[1-e^{i\pi(2\alpha_{N-1}-2\alpha_N-2\bar\varepsilon_{N-1})}\right]|\,|$$

since $\quad -2\bar\varepsilon_{N-1} = -2+2\varepsilon_{N-1}$

$\qquad = -2+2\alpha_N-2\alpha_{N-1}\pm 2$

$=0$

- $x_{N-2} < x < x_{N-3}$

$$\mathcal{J}^{z}_{(N)n}(x+i0^+) = e^{i\pi\alpha_N}\Big[e^{+i\pi\bar{\varepsilon}_{N-1}} I_{(N-1)n} + e^{i\pi(\bar{\varepsilon}_{N-1}+\bar{\varepsilon}_{N-2})} I_{(N-2)n}(x)\Big]$$

$$\mathcal{J}^{z}_{(N)n}(x-i0^+) = e^{i\pi\alpha_N}\Big[e^{-i\pi\bar{\varepsilon}_{N-1}} I_{(N-1)n} + e^{-i\pi(\bar{\varepsilon}_{N-1}+\bar{\varepsilon}_{N-2})} I_{(N-2)n}(x)\Big]$$

then

$$I^{z}_{Ln}(x+i0^+) = \mathcal{J}^{z}_{(D)n}(x+i0^+)$$

$$I^{\bar{z}}_{Rn}(x-i0^+) = e^{-i2\pi\alpha_N}\,\mathcal{J}^{z}_{(D)n}(x-i0^+)$$

Therefore

$$\Delta^{z}_{(N-2)}[I_n] = e^{i\pi\alpha_N}\Big\{ e^{i\pi\bar{\varepsilon}_{N-1}}\Big[1 - e^{-i2\pi\alpha_{N-2}}\cdot e^{-i2\pi(\alpha_N+\bar{\varepsilon}_{N-1})}\Big) I_{(N-1)n}$$

$$+ e^{i\pi(\bar{\varepsilon}_{N-1}+\bar{\varepsilon}_{N-2})}\Big[1 - e^{+i2\pi\alpha_{N-2}}\cdot e^{-i2\pi\alpha_N}\, e^{-i2\pi(\bar{\varepsilon}_{N-1}+\bar{\varepsilon}_{N-2})}\Big] I_{(N-2)n}(x)\Big\}$$

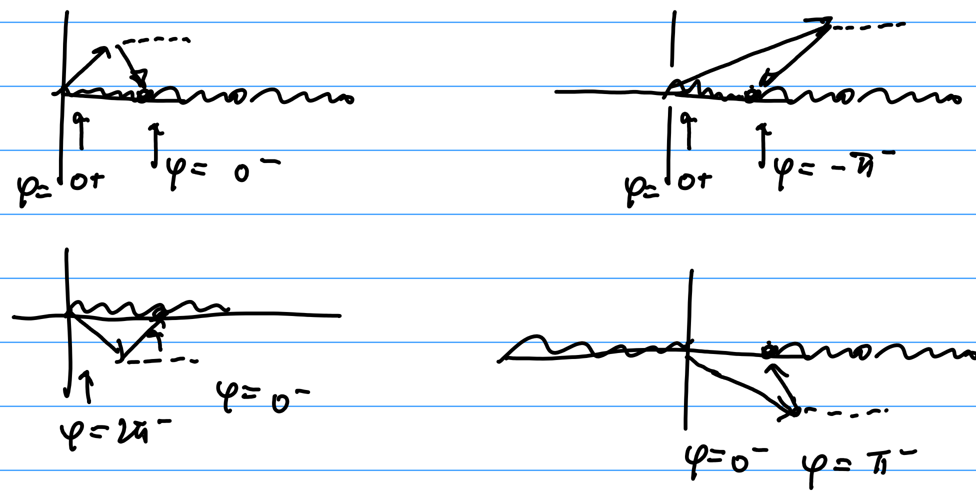
$$\downarrow$$

$$\underset{\text{mod } 2}{\equiv} -2(-\alpha_{N-2}+\alpha_N+1-\alpha_N+\cancel{\alpha_{N-1}}+1-\cancel{\alpha_{N-1}}+\alpha_{N-2}) = 0$$

hence $\Delta^{z}_{(N-2)}[I_n] = 0$

For $\mathcal{J} = \mathcal{J}_{(N+1)}$

We have after choosing the simplest cuts for this computation



$$\mathcal{J}^z_n(0+i0^+) = \int_{x_{N-1}}^{0} dx \; x^{-n-\bar{\varepsilon}_N} \prod | \quad |^{-\bar{\varepsilon}_k} = -I_{(N\cancel{-1})n}$$

$$\mathcal{J}^z_n(0-i0^+) = \left(e^{+i2\pi}\right)^{(-\bar{\varepsilon}_N)} \int_{x_{N-1}}^{0} = - e^{-i2\pi\bar{\varepsilon}_N} I_{(N\cancel{-1})n}$$

$$\mathcal{J}^z_n(x_{N-2}+i0^+) = \left(e^{-i\pi}\right)^{(-\bar{\varepsilon}_{N-1})} I_{(N-1)n} = e^{i\pi\bar{\varepsilon}_{N-1}} I_{(N-1)n}$$

$$\mathcal{J}^z_n(x_{N-2}-i0^+) = \left(e^{i2\pi}\right)^{-\bar{\varepsilon}_N} \left(e^{+i\pi}\right)^{(-\bar{\varepsilon}_{N-1})} I_{(N-1)n}$$

$$= e^{-i2\pi\bar{\varepsilon}_N - i\pi\varepsilon_{N-1}} I_{(N-1)n}$$

$$\mathcal{J}^{z}_{(N+1)n}(x_{N-3}+i0^{+}) - \mathcal{J}^{z}_{(N+1)n}(x_{N-2}+i0^{+}) =$$

$$= \left(e^{-i\hbar}\right)^{(-\bar{\varepsilon}_{N-1}-\bar{\varepsilon}_{N-2})} I_{(N-2)n} = e^{i\hbar\sum_{t=N-2}^{N-1}\bar{\varepsilon}_{t}} I_{(N-2)n}$$

$$\mathcal{J}^{z}_{n}(x_{N-3}-i0^{+}) - \mathcal{J}^{z}_{n}(x_{N-2}+i0^{+}) =$$

$$= e^{-i\hbar\bar{\varepsilon}_{N}} \cdot e^{-i\hbar\sum_{t=N-2}^{N-1}\bar{\varepsilon}_{t}} I_{(N-2)n}$$

The general expression is

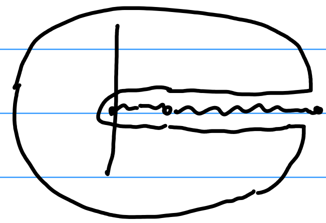
$$\mathcal{J}^{z}_{n}(x_{t}+i0^{+}) - \mathcal{J}^{z}_{n}(x_{t+1}+i0^{+}) =$$

$$= e^{+i\hbar\sum_{u=t+2}^{N-2}\bar{\varepsilon}_{u}} I_{(t+1)n} \qquad t \leq N-2$$

$$\mathcal{J}^{z}_{n}(x_{t}-i0^{+}) - \mathcal{J}^{z}_{n}(x_{t+1}-i0^{+}) =$$

$$= e^{-i\hbar\bar{\varepsilon}_{N}} e^{-i\hbar\sum_{u=t+2}^{N-2}\bar{\varepsilon}_{u}} I_{(t+1)n} \qquad t \leq N-2$$

In particular we have



$= 0$

but

$$\left|\oint dw\, w^{-n-\bar{\varepsilon}_N-\bar{\varepsilon}_{N-1}-\bar{\varepsilon}_2}\left(1+O\left(\frac{1}{w}\right)\right)\right| \le 2\pi R\, R^{-n-(\bar{M}-\bar{\varepsilon}_1)} \xrightarrow[R\to\infty]{} 0$$

when $1-n-(\bar{M}-\bar{\varepsilon}_1) < 0 \Rightarrow n > 1-\bar{M}+\bar{\varepsilon}_1$

$n \geq 2-\bar{M}$

and

$$\left|\oint dw\, w^{-n-\bar{\varepsilon}_N}(1+O(w))\right| \le 2\pi\varepsilon\, \varepsilon^{-n-\bar{\varepsilon}_N} \xrightarrow[\varepsilon\to 0]{} 0$$

when $1-n-\bar{\varepsilon}_N > 0 \Rightarrow n < 1-\bar{\varepsilon}_N$

$n \le 0$

Hence for $2-\bar{M} \le n \le 0$

$$0 = \oint =$$

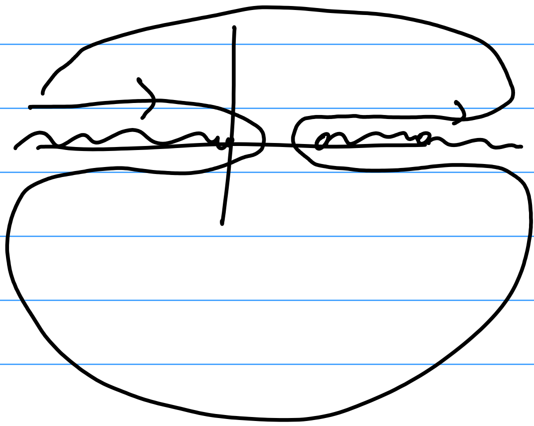
i.e.

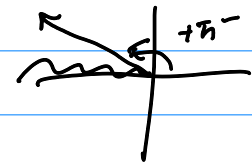
$$0 = -\left(1-e^{-i2\pi\bar{\varepsilon}_N}\right) I_{(N)} + \left(e^{i\pi\bar{\varepsilon}_{N-1}} - e^{-i2\pi\bar{\varepsilon}_N - i\pi\bar{\varepsilon}_N}\right) I_{(N-1)} + \dots$$

$$= 2i\, e^{-i\pi\bar{\varepsilon}_N}\Big[-\sin(\pi\bar{\varepsilon}_N)\, I_{(N)} + \sin(\pi\bar{\varepsilon}_N+\pi\bar{\varepsilon}_{N-1})\, I_{(N-1)} + \dots + \sin(\pi\bar{\varepsilon}_N + \quad + \pi\bar{\varepsilon}_2)\, I_{(2)}\Big]$$

$$\Rightarrow I_{(N)\,n} = \frac{\sin\pi(\bar{\varepsilon}_N+\bar{\varepsilon}_{N-1})\, I_{(N-1)}}{\sin(\pi\bar{\varepsilon}_N)} + \dots + \frac{\sin\pi(\bar{\varepsilon}_N+\dots\pi\bar{\varepsilon}_2)\, I_{(2)}}{\sin(\pi\bar{\varepsilon}_N)}$$

Would we have chosen the cuts





Then

$$\int = \int_{-\infty}^{0} dw\, w^{-n-\bar{\varepsilon}_N} \prod{}' \left(1-\frac{w}{x_t}\right)^{-\bar{\varepsilon}_t} = -e^{-i\pi\bar{\varepsilon}_N}\int_0^{\infty} dx\, x^{-n-\bar{\varepsilon}_N} \prod{}' \left|\,\right|^{-\varepsilon_t}$$

$$= -\left[e^{-i\pi\bar{\varepsilon}_N}\left(I_{(N)} + \hat{I}_{(N-1)} + \quad + \hat{I}_{(2)}\right)\right]$$

and

$$\int = + e^{i\pi\bar{\varepsilon}_N}\left(I_{(N)} + \quad + \hat{I}_{(2)}\right)\Big]$$

where $\hat{I}_{(t+1)} := \int_{x_{t+1}}^{x_t} dx\, x^{-n-\bar{\varepsilon}_N} \prod_{t=2}^{N-1}\left(1+\frac{x}{x_t}\right)^{-\varepsilon_t}$

so that for $\qquad 2-\bar{M} \leq n \leq 0$

$$0 = -2i \sin(\pi\bar{\varepsilon}_N)\left[\hat{I}_{(N-\cancel{1})} + \qquad + I_{(2)}\right]$$

$$+ 2i \sin(\pi\bar{\varepsilon}_{N-1}) I_{(N-1)} \qquad + \cdots$$

$$+ 2i \sin(\pi\bar{\varepsilon}_{N-2} + \pi\bar{\varepsilon}_2) I_{(2)}$$

$$\Rightarrow \quad I_{(N-\cancel{1})} \sin(\pi\bar{\varepsilon}_N) = \left[\sin(\pi\bar{\varepsilon}_{N-1}) I_{(N-1)} - \sin(\pi\bar{\varepsilon}_N)\hat{I}_{(N-1)}\right]$$

$$+$$

$$+ \left[\sin(\pi\bar{\varepsilon}_{N-1} + \pi\bar{\varepsilon}_2) I_{(2)} - \sin(\pi\varepsilon_N)\hat{I}_{(2)}\right]$$

which is not very useful!

⊙ We can then expand on $\mathbb{C} - (-\infty,0) - (x_{N-1}, X_F = \infty)$

$$y_n^z(z) = \sum_{m=n-(N-2)}^{N} c_{nm} \, \mathcal{J}_m^z(z) + c_n$$

$$\bar{y}_n^{\bar{z}}(z) = \sum \bar{c}_{nm} \, \bar{\mathcal{J}}_m^{\bar{z}}(z) + \bar{c}_n$$

with $c_n = \bar{c}_n = 0$ since $y^z(x_{N-1}) = \mathcal{J}^z(x_{N-1}) = 0$

Moreover the c.s are constrained as

$$y_n^z(x_t) = 0 = \sum c_{nm} \, \mathcal{J}_m^z(x_t \pm i0^+) \qquad t = 2, \dots N-2$$

$$\bar{y}_n^{\bar{z}}(x_t) = 0 = \sum \bar{c}_{nm} \, \bar{\mathcal{J}}_m^{\bar{z}}(x_t \pm i0^+)$$

This can formulated by saying that the matrices c, $\bar{c}$ have zero eigenvectors

$$c \, I_{(t)} = 0 \qquad t = 1, \dots N-2$$

$$\bar{c} \, \bar{I}_{(t)} = 0$$

$$\# t = N-3$$

Since $y(x_t) - y(x_{t+1}) = 0$ and $\mathcal{J}(x_t + i0^+) - \mathcal{J}(x_{t+1} + i0) = e^{\pm i\pi \sum \varepsilon_u} I_{(t+1)}$

Actually it may happen that there are other zero eigenvectors:

$$\begin{cases} y_n^z(0) = 0 & n \leq 0 \\ y_n^z(\infty) = 0 & n \geq M \end{cases}$$

check $\quad y_n^z \underset{z\to\infty}{\sim} z^{-n+\varepsilon_N+\sum_2^{N-1}\varepsilon_n} = z^{-n+M-\varepsilon_1} \underset{n\geq M}{\longrightarrow} 0$

GAP for $M \geq 2$: $1 \leq n \leq M-1$! $\Rightarrow$ $M=1$

Let us see what happens for $I(*)$

$$\begin{cases} y_n^z(0) = \sum_{m=n-\bar{N}}^{n} C_{nm}\, J_m^z(0) \ \& \ n \leq 0 \Rightarrow m \leq 0 \\ y_n^z(\infty) = \sum_{m=n-\bar{N}}^{n} C_{nm}\, J_m^z(\infty) \ \& \ n \geq M \Rightarrow m \geq M-\bar{N} . \end{cases}$$

NAIVELY

so we need $\quad M-\bar{N} \geq 1 \Rightarrow M \geq \bar{N}+1 = N-1$

for avoid overlap in m !

This means $\quad M = N-1 \ (\bar{M}=1)$ since $\quad M \leq N-1$

TRUELY we need $M=1$

Then there is an overlap $-(N-3)_2 < m \leq 0$

For $N=3$ only $m=0$ overlaps thus allows to write

$$I_{(x)m} = \begin{cases} \mathcal{J}^z_m(0) & m \leq 0 \\ \mathcal{J}^z_m(\infty)\,\dfrac{\mathcal{J}^z_0(0)}{\mathcal{J}^z_0(\infty)} & m \geq 0 \end{cases}$$

or better:

$$I_{(x)m} = \begin{cases} \mathcal{J}^z_m(0) & m \leq 0 \\ \left(\mathcal{J}^z_m(\infty) - \mathcal{J}^z_m(x_2)\right)\dfrac{\mathcal{J}^z_0(0)}{\mathcal{J}^z_0(\infty) - \mathcal{J}^z_0(x_2)} & m \geq 0 \end{cases}$$

The same is true for $\bar{c}$ with $M = N-1$ $(\bar{M}=1)$

I.e. $N=3$ $M=1$ $(\bar{M}=2)$ c $I_{(x)}=0$

$N=3$ $M=2$ $(\bar{M}=1)$ $\bar{c}$ $\bar{I}_{(x)}=0$

Q.: What about f.x. $N=4$? $(M=1, \bar{M}=3)$

Overlap $m=0,-1$

so we need
$$\frac{\mathcal{J}^{z}_{0}(0)}{\mathcal{J}^{z}_{0}(\infty)} = \frac{\mathcal{J}^{z}_{-1}(0)}{\mathcal{J}^{z}_{-1}(\infty)}$$

From the previous relations for $2-\bar{M}=-1 \leq m \leq 0$

$$\sin(\pi\bar{\varepsilon}_4)\, I_{(3)} = \sin(\pi\bar{\varepsilon}_4 + \pi\bar{\varepsilon}_3)\, I_{(2)} + \sin(\pi\bar{\varepsilon}_4 + \pi\bar{\varepsilon}_3 + \pi\bar{\varepsilon}_2)\, I_{(1)}$$

then
$$0 = y^{z}_{n}(\infty) - y^{z}_{n}(x_2) = \sum C_{nm}\, e^{i\pi(\bar{\varepsilon}_2+\bar{\varepsilon}_3)}\, I_{(2)m} \qquad n \geq M$$
$$0 = y^{z}_{n}(x_2) - y^{z}_{n}(x_1) = \sum C_{nm}\, e^{i\pi\bar{\varepsilon}_3}\, I_{(1)m} \qquad \forall n$$

so we can combine them and get
$$0 = \sum C_{nm} \left[\sin(\pi\bar{\varepsilon}_4 + \pi\bar{\varepsilon}_3)\, I_{(2)m} + \sin(\pi\bar{\varepsilon}_4 + \pi\bar{\varepsilon}_3 + \pi\bar{\varepsilon}_2)\, I_{(1)m}\right)$$
with $n \geq M$

then we can define
$$I_{(*)m} = \begin{cases} \sin(\pi\bar{\varepsilon}_4)\, I_{(1)m} & m \leq 0 \\ \sin(\pi\bar{\varepsilon}_4 + \pi\bar{\varepsilon}_3)\, I_{(2)m} + \sin(\pi\bar{\varepsilon}_4 + \pi\bar{\varepsilon}_3 + \pi\bar{\varepsilon}_2)\, I_{(1)m} & m \geq -1 \end{cases}$$

Are $I_{(x)}$ well defined?

$$m \leq 0: \; I_m(x) = \int_{x_{N-1}}^{0} dw \; w^{-m+\varepsilon_N - 1} \prod_{t=2}^{N-1} \left(1 - \frac{w}{x_t}\right)^{\varepsilon_t - 1}$$

$$\sim \int_{x_{N-1}}^{0} dw \; w^{|m| + \varepsilon_N - 1} \left(1 + O(w)\right) \qquad \text{ok}$$

$m \geq M + 2 - N$
$= 2 - \bar{M}$

$$I_m(x) \sim \int_{x_{N-1}}^{\infty} dw \; w^{-m + \varepsilon_N - 1 + \sum_2^{N-1} (\varepsilon_t - 1)} \left(1 + O\left(\tfrac{1}{w}\right)\right)$$

$$= \int_{x_{N-1}}^{\infty} dw \; w^{-m + M - \varepsilon_1 - (N-1)} \left(1 + O\left(\tfrac{1}{w}\right)\right)$$

$$= \int^{\infty} dw \; w^{-m - (\bar{M} - \bar{\varepsilon}_1)} \left(1 + O\left(\tfrac{1}{w}\right)\right)$$

ok if $\quad 1 - m - \bar{M} + \bar{\varepsilon}_1 < 0 \quad \Rightarrow \quad m > 1 - \bar{M} + \bar{\varepsilon}_1$

$m \geq 2 - \bar{M} \qquad$ ok!

Summary of constraints

$(N,M) = (3,1)$ $\quad c\, I_{(*)} = 0$

$(3,2)$ $\quad \bar{c}\, \bar{I}_{(*)} = 0$

$(4,1)$ $\quad c\, I_{(2)} = \bar{c}\, \bar{I}_{(2)} = c\, I_{(*)}$

$(4,2)$ $\quad c\, I_{(2)} = \bar{c}\, \bar{I}_{(2)} = 0$

$(4,3)$ $\quad c\, I_{(2)} = \bar{c}\, \bar{I}_{(2)} = 0 = \bar{c}\, \bar{I}_{(*)}$

$(5,1)$ $\quad c\, I_{(2)} = c\, I_{(3)} = \bar{c}\, \bar{I}_{(2)} = \bar{c}\, \bar{I}_{(3)} = \cancel{c\, I_{(*)}} = 0$

$(5,2)$ $\quad c\, I_{(2)} = c\, I_{(3)} = \bar{c}\, \bar{I}_{(2)} = \bar{c}\, \bar{I}_{(3)} = 0$

$(5,3)$ $\quad c\, I_{(2)} = c\, I_{(3)} = \bar{c}\, \bar{I}_{(2)} = \bar{c}\, \bar{I}_{(3)} = 0$

$(5,4)$ $\quad c\, I_{(2)} = c\, I_{(3)} = \bar{c}\, \bar{I}_{(2)} = \bar{c}\, \bar{I}_{(3)} = 0 = \cancel{\bar{c}\, \bar{I}_{(*)}}$

Since $\bar{c}_{m,1-n} = - c_{n,1-m}$

i.e. $\bar{c}\, R = -(c\, R)^T$, $\quad \bar{c} = -R c^T R$

with $R_{m,n} = R_{n,m} = \delta_{m+n,1} = (R^{-1})_{m,n}$

chk $(\bar{c} R)_{mn} = \bar{c}_{m,1-n}$

$[(c\, R)^T]_{mn} = (c R)_{nm} = c_{n,1-m}$

Constraints can be formulated using c only as

$M = 1$ & $N = 3$ $\quad c^T\, I_{(*)} = 0$

$2 \le M \le N-2$ $\quad c\, I_{(k)} = c^T R\, \bar{I}_{(k)} = 0$

$M = N-1$ & $N=3$ $\quad \bar{c}\, \bar{I}_{(*)} = 0$

Let us compute the coeff.s $c_{n,m}$

$$\partial y^z_m = \sum c_{nm} \, \partial_m y^z_m$$

$$\Rightarrow z^{-n+\varepsilon_N} \prod_{t=2}^{N-1} \left(1-\frac{z}{x_t}\right)^{\varepsilon_t} \left[\frac{-n+\varepsilon_N}{z} + \sum_{k=2}^{N-1} \frac{\varepsilon_k}{z-x_k}\right]$$

$$= \sum_m c_{nm} \, z^{-m+\varepsilon_N-1} \prod \left(1-\frac{z}{x_t}\right)^{\varepsilon_t-1}$$

$$\Rightarrow z \cdot \prod_{t=2}^{N-1} \left(1-\frac{z}{x_t}\right) \left[\frac{-n+\varepsilon_N}{z} + \sum_{k=2}^{N-1} \frac{\varepsilon_k}{z-x_k}\right] = Pol_{N-2}(z)$$

$$= \sum_m c_{nm} \, z^{n-m}$$

$$\Rightarrow \quad 0 \le n-m \le N-2 \quad \Leftrightarrow \quad n-(N-2) \le m \le n$$

More explicitly

$$\sum_{m=n-\bar{N}}^{n} c_{nm} \, z^{n-m} = (-n+\varepsilon_N)\cdot z^0$$

$$+ \left[(-n+\varepsilon_N) \sum_{t=2}^{N-2} \frac{-1}{x_t} + 1 \cdot \sum_{u=2}^{N-2} \left(- \frac{\varepsilon_u}{x_u} \right) \right] z^1$$

$$+ \dots$$

$$+ \left[(-n+\varepsilon_N) \prod_{t=2}^{N-2} \frac{-1}{x_t} + \sum_{u=2}^{N-2} \frac{-\varepsilon_u}{x_u} \prod_{t \neq 1,u,N-1,N} \frac{-1}{x_t} \right] z^{N-2}$$

$$= (-n+\varepsilon_N)$$

$$+ z \sum_{t=2}^{N-2} \frac{-\varepsilon_t - \varepsilon_N + n}{x_t} + \dots$$

$$+ z^{N-2} \prod_{t=2}^{N-2} \frac{-1}{x_t} \left[(-n+\varepsilon_N) + \sum_{u=2}^{N-2} \varepsilon_u \right]$$

$$\Rightarrow \quad c_{n,n} = -n+\varepsilon_N$$

$$c_{n,n-1} = \sum_{t=2}^{N-2} \frac{-\varepsilon_t - \varepsilon_N + n}{x_t}$$

$$\vdots$$

$$c_{n,n-\bar{N}} = \sum_{t=2}^{\bar{N}} \frac{-1}{x_t} \left[+ \sum_{u=2}^{\bar{N}} \varepsilon_u + \varepsilon_N - n \right]$$

Simplest case $n=3$

$$\mathrm{Pol}_{n-2}(z) = \mathrm{Pol}_1(z) = z\left(1-\frac{z}{x_2}\right)\left[\frac{-n+\varepsilon_3}{z} + \frac{\varepsilon_2}{z-x_2}\right]$$

$$= \left(1-\frac{z}{x_2}\right)(-n+\varepsilon_3) + z\,\frac{\varepsilon_2}{-x_2}$$

$$= (-n+\varepsilon_3) + z\left[-\frac{-n+\varepsilon_3}{x_2} - \frac{\varepsilon_2}{x_2}\right]$$

$$= (-n+\varepsilon_3) + z\,\frac{n-(n-\varepsilon_1)}{x_2}$$

Now we can compute

$$\langle Y_n, Y_m\rangle = \oint \frac{dz}{2\pi i}\, \partial y_n^z \cdot 0 \cdot y_m^z = 0$$

$$\langle Y_n, \bar{Y}_m\rangle = \oint \frac{dz}{2\pi i}\, \partial y_n^z \, \frac{1}{2}\, \bar{y}_m^{\bar{z}}$$

$$= \oint \frac{dz}{2\pi i} \sum_{k=n-\bar{N}}^{n} c_{nk}\, \partial y_k^z \, \bar{y}_m^{\bar{z}}$$

$$= \sum_{k=n-\bar{N}}^{n} c_{nk} \oint \frac{dz}{2\pi i}\, z^{-k+\varepsilon_N-1} \prod_{t=2}^{N-1}\left(1-\frac{z}{x_t}\right)^{\varepsilon_t-1}$$

$$\cdot\, z^{-m+\bar{\varepsilon}_N} \prod_{t=2}^{N-1}\left(1-\frac{z}{x_t}\right)^{\bar{\varepsilon}_t}$$

$$= \sum_{k=n-\bar{N}}^{n} c_{nk} \oint \frac{dz}{2\pi i}\, \underbrace{z^{-k-m}}_{\delta_{k+m,1}}$$

$$= c_{n,1-m}$$

Similarly

$$\langle \bar{Y}_m, Y_n\rangle = \bar{c}_{m,1-n}$$

but

$$\langle \bar{Y}_m, Y_n\rangle = -\langle Y_n, \bar{Y}_m\rangle$$

so

$$\bar{c}_{m,1-n} = -c_{n,1-m}$$

or $$\bar{c}_{m,n} = -c_{1-n,1-m}$$

let us perform some checks

- $C_{n,1-m} \neq 0 \iff n-\bar{N} \leq 1-m \leq n \iff 1-n \leq m \leq 1+\bar{N}-n$

$\bar{C}_{m,1-n} \neq 0 \iff m-\bar{N} \leq 1-n \leq m \iff 1-n \leq m \leq \bar{N}+1-n$

ok

- $C_{n,1} = -n + \varepsilon_N$

$\| ?$

$-\bar{C}_{1-n,1-n} = -\left[-(1-n) + \bar{\varepsilon}_N\right] = -n + 1 - \bar{\varepsilon}_N = -n + \varepsilon_N$ ok

- $$C_{n,n-1} = \sum_{t=2}^{N-2} \frac{-\varepsilon_t - \varepsilon_N + n}{x_t}$$

$\| ?$

$$-\bar{C}_{2-n,1-n} = -\sum_{t=2}^{N-2} \frac{-\bar{\varepsilon}_t - \bar{\varepsilon}_N + (2-n)}{x_t}$$

$$= -\sum_{t=2}^{N-2} \frac{\varepsilon_t + \varepsilon_N - n}{x_t}$$ ok

[21/01/18] ⊙ True (?) expansion

Let us write

$$X(u,\bar{u}) = \sum_n \left[x_n\, \mathcal{J}_n(u) + \bar{x}_n\, U(\bar{t})\, \bar{\mathcal{J}}_n(\bar{u}) \right]$$

or with the indexes

$$X^z(u,\bar{u}) = \sum_n \left[x_n\, \mathcal{J}^z_{\ n}(u) + \bar{x}_n\, U(\bar{t})^z_{\ \bar{z}}\, \bar{\mathcal{J}}^{\bar{z}}_{\ n}(\bar{u}) \right]$$

Then we impose

$$0 = X^z(x_t,\bar{x}_t) = \sum_n \left[x_n\, \mathcal{J}^z_{\ n}(x_t + i0^+) + \bar{x}_n\, U(\bar{t})^z_{\ \bar{z}}\, \bar{\mathcal{J}}^{\bar{z}}_{\ n}(x_t - i0^+) \right]$$

$t = 2, \dots N-2$

Eventually we also have

$$X^{z(+)}(x_N, \bar{x}_N) = X^{z(+)}(i0^+, -i0^+) = 0$$

and

$$X^{z(-)}(x_1, \bar{x}_1) = X^{z(-)}(+\infty + i0^+, +\infty - i0^+) = 0$$

$\Rightarrow$ If we adopt this point of view $x_n, \bar{x}_n$ are the fundamental operators

and $y, \bar{y}$ are the "dual bases"

⊙ Let us consider the $\mathcal{H}^z, \mathcal{H}^{\bar z}$ expansion $(\bar N = N-2)$

WRONG!

$$\mathcal{H}^z = \sum_n y_n \, y_n^z(z)$$

$$= \sum_n y_n \sum_{m=n-\bar N}^{n} c_{nm} \, \mathcal{J}_m^z$$

$$= \sum_m \left(\sum_{n=m}^{m+\bar N} y_n c_{nm} \right) \mathcal{J}_m^z$$

since $n-\bar N \le m \le n \iff m \le n \le m+\bar N$

$$= \sum_m x_m \, \mathcal{J}_m^z$$

with

$$x_m = \sum_{n=m}^{m+\bar N} y_n c_{nm} \quad , \quad x = c^T y$$

WRONG!!

Now the x_m are not independent since

$$\mathcal{H}^z(n_t) = 0 = \sum x_m I_m(t) = X^T I(t) \quad t = 2, \dots, N-1$$

with $I_m(t) = \mathcal{J}_m^z(x_t)$

Similarly we get $\bar X^T \bar I(t) = 0$ (classically quantum?)

Q: Why is it wrong?

Because X should not be expanded on y but on $\mathcal{J}$ plus constraints from $X(n_t, \bar n_t) = 0$!

[21/01/18] From what written above these constraints are too restrictive!

It is enough to request

$$\sum_n \Big[\kappa_n \, e^{+i\pi \sum_{k=t+2}^{N-2} \bar{\varepsilon}_k} \, I_{n\,(t)}$$

$$+ \bar{\kappa}_n \, U_{(N+1)}{}^{z} \, \bar{z} \, e^{-i\,l\pi\,\varepsilon_N} \, e^{-i\pi \sum_{k=t+1}^{N-2} \varepsilon_k} \, \bar{I}_{n\,(t)} \Big] = 0$$

$t = 2, \ldots, N-3$

or

$$\sum \Big[\kappa_n \, I_{n\,(t)} + \bar{\kappa}_n \, U_{(N+1)}{}^{z} \, \bar{t} \, e^{-i\,l\pi\,\varepsilon_N} \, (-)^{(N-t)} \, \bar{I}_{n\,(t)} \Big] = 0$$

⊙ Now we can compute

$$\langle X, \bar{Y}_n \rangle = \oint \frac{dz}{2\pi} \partial x^z \frac{1}{2} y_n^{\bar{z}}$$

$$= \sum_m x_m \langle I_m, \bar{Y}_n \rangle$$

$$= x_{1-n}$$

$$\text{ok} = \sum y_k \langle Y_k, \bar{Y}_n \rangle = \sum y_k \, c_{k,1-n} \overset{\text{df}}{\equiv} x_{1-n}$$

and

$$\langle X, Y_n \rangle = \bar{x}_{1-n}$$

① Radial canonical formalism

$$S_E' = T \int_{r_0}^{r_1} dr\, r \int_0^{\pi} d\theta\, \frac{1}{2}\left[\partial_r X^T g\, \partial_r X + \frac{1}{r^2} \partial_\theta X^T g\, \partial_\theta X \right]$$

$$\rightarrow P_I(\sigma) = \frac{\delta S_E'}{\delta \partial_r X^I(\sigma)} = T \cdot r \cdot g_{IJ}\, \partial_r X^J$$

$$\partial_r X^I = \frac{1}{T \cdot r} g^{IJ} P_J(\sigma)$$

$$\rightarrow \quad H_E = \int_0^{\pi} d\theta\, \frac{1}{2r}\left[\frac{1}{T} P^T g^{-1} P - T\, \partial_\theta X^T g\, \partial_\theta X\right]$$

$$\rightarrow \{ X^I(\sigma), P_J(\sigma') \} = \delta^I_J\, \delta_{b.c.}(\sigma - \sigma')$$

$\rightarrow$ e.o.m

$$\begin{cases} \partial_r X^I(\sigma) = \{X^I(\sigma), H_E\} = \frac{1}{r}\frac{1}{T} g^{IJ} P_J(\sigma) \\ \partial_r P_I(\sigma) = \{P_I(\sigma), H_E\} = -\frac{T}{r} \int_0^{\pi} d\theta\, (-)\partial_\theta \delta(\sigma - \theta)\, g_{IJ}\, \partial_\theta X^J \end{cases}$$

$$= -\frac{T}{r} g_{IJ}\, \partial_\theta^2 X^J$$

then

$$\partial_r P_I = \partial_r\left(T r\, g_{IJ}\, \partial_r X^J \right) = T g_{IJ}\, \partial_r (r\, \partial_r X^J)$$

$$= -\frac{T}{r} g_{IJ}\, \partial_\theta^2 X^J$$

$$\Rightarrow \quad \partial_r (r \partial_r X^I) + \frac{1}{r} \partial_\theta^2 X^I = 0$$

$$\frac{1}{r} \partial_r (r \partial_r X^I) + \frac{1}{r^2} \partial_\theta^2 X^I = \Box X^I = 0$$

⊙ Commutation relations

$$[x_n, x_m] = [\langle X, \bar{Y}_{1-n}\rangle, \langle X, \bar{Y}_{1-m}\rangle]$$

$$= [\langle \bar{Y}_{1-n}, X\rangle, \langle \bar{Y}_{1-m}, X\rangle]$$

$$= M \langle \bar{Y}_{1-n}, \bar{Y}_{1-m}\rangle = 0$$

$$[x_n, \bar{x}_m] = M \langle \bar{Y}_{1-n}, Y_{1-m}\rangle = M \bar{c}_{1-n,m} = -M c_{1-m,n}$$

$$[\bar{x}_m, x_n] = M \langle Y_{1-m}, \bar{Y}_{1-n}\rangle = M c_{1-m,n} = -M \bar{c}_{1-n,m}$$

$$[\bar{x}_n, \bar{x}_m] = 0$$

Constraints (which come from $c I_{(t)} = \bar{c} \bar{I}_{(t)} = 0$)

$M = 1$: $[\bar{x}_m, x^T I_{(x)}] = M (C I_{(x)})_{1-m} = M (R C I_{(x)})_m = 0$

$$\left.\begin{aligned} &[x_n, \bar{x}^T \bar{I}_{(t)}] = M (\bar{c} \bar{I}_{(t)})_{1-n} = 0 \\ &[\bar{x}_m, x^T I_{(t)}] = M (c I_{(t)})_{1-m} = 0 \end{aligned}\right\} 2 \le M \le N-2$$

$M = N-1$: $[x_n, \bar{x}^T \bar{I}_{(x)}] = M (R \bar{c} \bar{I}_{(x)})_n = 0$

⊙ Now we can try to compute (NECESSARY!)

$[y_k, \bar{x}_m]$ $\qquad$ $[x_n, \bar{y}_k]$

We have

$$[x_n, \bar{x}_m] = \sum_{k=n}^{n+\bar{N}} c_{kn} [y_k, \bar{x}_m] = M c_{1-m,n} = -M \bar{c}_{1-n,m}$$

or $$[x, \bar{x}^T] = c^T [y, \bar{x}^T] = M (Rc)^T = M c^T R = -M R \bar{c}$$

$$[\bar{x}, x^T] = [\bar{x}, y^T] c = -M R c$$

with $$R_{mk} = \delta_{m+k,1} = R_{km} \Leftrightarrow R_{m,1-m} = 1$$

Check $$[(Rc)^T]_{nm} = (Rc)_{mn} = c_{1-m,n}$$

Then $$[y, \bar{x}^T] = M R + K$$

with $$c^T K = 0$$

Remember that for $2 \leq M \leq N-2$

$$c\, I_{(t)} = c^T R \bar{I}_{(t)} = 0$$

For $M=1$ $\quad I_{(t)} \to I_{(\hat{t})}$, $\bar{I}_{(t)} \to \bar{I}_{(t)}$ $\quad \hat{t} = t, *$

$M = N-1$ $\quad I_{(t)} \to I_{(t)}$ $\quad \bar{I}_{(t)} \to \bar{I}_{(\hat{t})}$ $\quad \hat{t} = t, *$

then we can write

$$K = M \sum_t R \bar{I}_{(t)} J_{(t)}^T$$

and $$[y, \bar{x}^T] = M R + M R \sum_t \bar{I}_{(t)} J_{(t)}^T$$
$$= M R \left[M + \sum_{t=2}^{N-2} \bar{I}_{(t)} J_{(t)}^T \right]$$

What about $J(t)$?

We have also $[x, \bar{x}^T \bar{I}(t)] = 0$

We could ask $[y, \bar{x}^T \bar{I}(t)] = 0$

i.e. $$\bar{I}(t) + \sum_{u=2}^{N-2} \bar{I}(u) \, J^T_{(u)} \, \bar{I}(t) = 0$$

or $$J^T_{(u)} \, \bar{I}(t) = -\delta_{u,t}$$

But is this necessary? Yes since $\bar{x}\,\bar{I}(t) = 0 = x^{\bar{z}}(x_t)$

For $N=1$ $$\bar{I}(x) + \sum_{u=2}^{N-2} \bar{I}_{(u)} \, \bar{J}^T_{(u)} \, \bar{I}(x) + \bar{I}_{(x)} \, \bar{J}^T_{(x)} \, \bar{I}_{(x)} = 0$$

$$\bar{I}(t) + \sum_{u=2}^{N-2} \bar{I}(u) \, J^T_{(u)} \, \bar{I}(t) + \bar{I}(x) \, \bar{J}(x)^T \, \bar{I}(t) = 0$$

$\Rightarrow$ Notice there are many solutions for $J_{(u)}$.
The necessity of representing this algebra in a Hilbert space will fix J

Moreover $\quad [x_1, x^T] = 0 \quad = c^T [y, x^T] = 0$

$$\Rightarrow \quad [y, x^T] = K_{(1)} = M R \sum_{t=2}^{N-2} \bar{I}_{(t)} \bar{J}_{1(t)}^T$$

Constraints from Jacobi identity ? no

$$[[y_n, \underset{\overset{\|}{0}}{\bar{x}_m}], x_k] + [\underset{\overset{\|}{0}}{[x_k, y_n]}, \bar{x}_m] + [\underset{\overset{\|}{0}}{[\bar{x}_m, x_k]}, y_n] = 0$$

We need also to use $x^T I_{(t)} = 0$

to get
$$\sum_{u=2}^{N-2} \bar{I}_{(u)} \bar{J}_{1(u)}^T I_{(t)} = 0$$

If $\bar{I}_{(u)}$ are independent then

$$\bar{J}_{1(u)}^T I_{(t)} = 0 \qquad \forall u, t \in \{2, \dots N-2\}$$

Representation of the algebra

Since we can easily compute x_n we try to give to them a representation even if they are constrained.

$$\mathcal{J}^z_n \underset{z\to 0}{\sim} \int_{x_{N-1}}^{z} dw\, w^{-n+\varepsilon_N-1}\left(1+O(w)\right) \sim \text{const}_n + \frac{z^{-n+\varepsilon_N}}{-n+\varepsilon_N}$$

$$\Rightarrow \quad \mathcal{J}^z_n \underset{z\to 0}{\sim} \infty \qquad n \geq 1$$

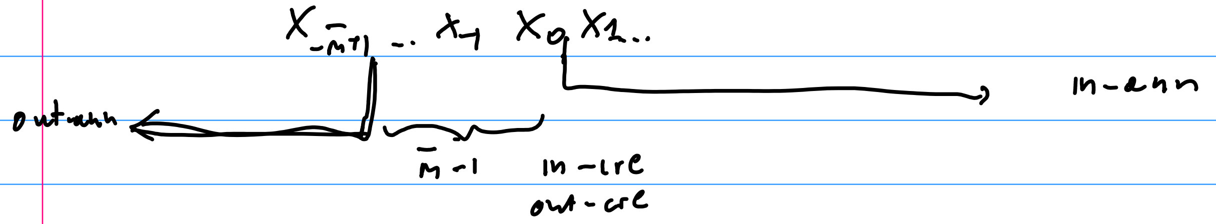
$$\underset{z\to 0}{\longrightarrow} \text{ finite} \qquad n \leq 0$$

$$\mathcal{J}^z_n \underset{z\to\infty}{\sim} \int_{x_2}^{z} dw\, w^{-n-(\bar{M}-\varepsilon_1)}\left(1+O\left(\tfrac{1}{w}\right)\right) \sim \text{const}'_n + \frac{z^{1+\varepsilon_1-n-\bar{M}}}{1+\varepsilon_1-n-\bar{M}}$$

$$\Rightarrow \quad \mathcal{J}^z_n \underset{z\to\infty}{\sim} \infty \qquad n \leq -\bar{M}+1$$

$$\underset{z\to\infty}{\longrightarrow} \text{ finite} \qquad n \geq -\bar{M}+2$$

So we can picture



Since $X_n = \sum_{k=n}^{n+\bar N} y_k \, c_{kn}$ $\qquad (\bar N = N-2)$

$X_{n\geq 1}$ in-ann $\Rightarrow$ $y_{k\geq 1}$ in-ann

$X_{n\leq -\bar M+1}$ out-ann $\Rightarrow$ $y_k \leq -\bar M + 1 + \bar N = +M-1$ out-ann

So we have

$$y_{-1} \quad y_0, \quad y_1 \dots \quad y_{M-1} \quad y_M$$

in-ann

out-ann

The only way to have $y_{1\leq k\leq M-1}$ in-ann and out-ann is to require

$$[\, y_k, \bar x_m] = 0 \qquad 1 \leq k \leq M-1 \qquad \forall m$$

Since

$$[y_n, \bar X_m] = M \quad [M + \sum_{t=2}^{N-2} \bar I_{(t)} \, J^T_{(t)}]_{1-n,m}$$

(for $2 \leq M \leq N-2$,

otherwise for $M=1$ $\quad I_{(t)} \to I_{(\hat t)}$, $\bar I_{(t)} \to \bar I_{(t)}$ $\quad \hat t = t, *$

$M = N-1$ $\quad I_{(t)} \to I_{(t)}$, $\bar I_{(t)} \to \bar I_{(\hat t)}$ $\quad \hat t = t, *$)

we get

$$\sum_{t=2}^{N-2} \bar I_{(t)\,1-k} \; \bar J_{(t)\,m} = - \delta_{1-k,m} \qquad k=1,\dots M-1$$

This must be added to $\quad \sum_n \bar J_{(t)n} \, \bar I_{(u)n} = - \delta_{u,t}$

Take $M=2$ $N\geq 4 \Rightarrow K=1$

$$\begin{cases} \sum_{t=2}^{N-2} \bar{I}_{(t)0} \, \bar{J}_{(t)m} = -\delta_{m,0} \\ \sum_{n} \bar{J}_{(t)n} \, \bar{I}_{(u)n} = -\delta_{u,t} \end{cases}$$

- $N=4$

$$\begin{cases} \bar{I}_{(2)0} \, \bar{J}_{(2)m} = -\delta_{m,0} \\ \sum_{n} \bar{J}_{(2)n} \, \bar{I}_{(2)n} = -1 \end{cases}$$

then $1^{st} \Rightarrow$ $\bar{J}_{(2)m\neq 0} = 0$ $\qquad \bar{J}_{(2)0} = -\frac{1}{\bar{I}_{(2)0}}$

so $2^{nd} \Rightarrow$ $\sum_{n} \bar{J}\,\bar{I} = \bar{J}_{(2)0}\,\bar{I}_{(2)0} = -1$ ok!

- $N=5$

$$\begin{cases} \bar{I}_{(2)0}\,\bar{J}_{(2)m} + \bar{I}_{(3)0}\,\bar{J}_{(3)m} = -\delta_{m,0} \\ \sum_n \bar{J}_{(t)n}\,\bar{I}_{(u)n} = -\delta_{u,t} \end{cases}$$

Try $\bar{J}_{(2)m} = \bar{J}_{(3)m} = 0 \qquad m \neq 0$

then

$$\begin{cases} \bar{J}_{(2)0}\,\bar{I}_{(2)0} = -1 \\ \bar{J}_{(2)0}\,\bar{I}_{(3)0} = 0 \end{cases} \Rightarrow \text{incompatible!}$$

$$\begin{cases} \bar{J}_{(3)0}\,\bar{I}_{(3)0} = -1 \\ \bar{J}_{(3)0}\,\bar{I}_{(2)0} = 0 \end{cases}$$

Try $\bar{J}_{(t)m} = 0 \qquad m \neq 0, 1 \qquad \# \bar{J} = 2 \cdot 2$

$$\begin{cases} \bar{J}_{(2)0}\,\bar{I}_{(2)0} + \bar{J}_{(2)1}\,\bar{I}_{(2)1} = -1 \\ J_{(2)0}\,\bar{I}_{(3)0} + \bar{J}_{(2)1}\,\bar{I}_{(3)1} = 0 \end{cases} \to \text{Fix } J_{(2)0,1}$$

$$\begin{cases} \bar{J}_{(3)0}\,\bar{I}_{(2)0} + \bar{J}_{(3)1}\,\bar{I}_{(2)1} = 0 \\ \qquad\qquad = -1 \end{cases} \to \text{Fix } J_{(3)0,1}$$

Compatible with

$$\begin{cases} \bar{J}_{(2)0}\,\bar{I}_{(2)0} + \bar{J}_{(3)0}\,\bar{I}_{(3)0} = -1 \\ \bar{J}_{(2)0}\,\bar{I}_{(2)1} + \bar{J}_{(3)0}\,\bar{I}_{(3)1} = 0 \end{cases} \quad ?$$

4 unknowns : $\bar{J}_{(2)0,1} \quad \bar{J}_{(3)0,1}$

6 eq.s

$$
\begin{pmatrix}
\bar{I}_{(2)0} & \bar{I}_{(2)1} & & \\
\bar{I}_{(3)0} & \bar{I}_{(3)1} & & \\
 & & \bar{I}_{(2)0} & \bar{I}_{(2)1} \\
 & & \bar{I}_{(3)0} & \bar{I}_{(3)1} \\
\bar{I}_{(2)0} & & \bar{I}_{(3)0} & \\
 & \bar{I}_{(2)1} & & \bar{I}_{(3)1}
\end{pmatrix}
\begin{pmatrix}
\bar{J}_{(2)0} \\
\bar{J}_{(2)1} \\
\bar{J}_{(3)0} \\
\bar{J}_{(3)1}
\end{pmatrix}
=
\begin{pmatrix}
-1 \\ 0 \\ 0 \\ -1 \\ -1 \\ 0
\end{pmatrix}
$$

Consider $\bar{J}_{(t)n} = 0 \qquad n \neq 0, 1, \dots F-1$

unknowns $\# \bar{J}_{(t)n} = \# t \cdot \# K = 2 \cdot F$

then we get

$$\begin{cases} \bar{J}_{(t)} \bar{I}_{(u)} = -\delta_{u,t} & 4 \text{ eq.s} \\ \sum_t \bar{J}_{(t)0} \bar{I}_{(t)K} = -\delta_{K,0} & F\text{-eq.s} \end{cases}$$

so we need $4 + F \leq 2 \cdot F \Rightarrow F \geq 4$

① Green functions

We write f.x

$$\partial_z X^z = \sum \Big[x_{cc}\, \partial Y^z_{cc} + x_{ca}\, \partial Y^z_{ca} + x_{ac}\, \partial Y^z_{ac} \Big]$$

out cc / in-cc out cc / in-ann out ann / in cc

We can compute

$$\langle \partial_z X^z(z)\, \partial_w \bar{X}^{\bar z}(w) \rangle =$$

$$= \sum \Big[\partial_z Y^z_{cc}(z)\, \partial_w \bar{Y}^{\bar z}_{\bar c \bar c}(w) \langle x_{cc}\, \bar{x}_{\bar c \bar c} \rangle$$

$$+ \partial_z Y^z_{cc}(z)\, \partial_w \bar{Y}^{\bar z}_{\bar a \bar c}(w) \langle x_{cc}\, x_{\bar a \bar c} \rangle$$

$$+ \partial_z Y^z_{ca}(z)\, \partial_w \bar{Y}^{\bar z}_{\bar c \bar c}(w) \langle x_{ca}\, \bar{x}_{\bar c \bar c} \rangle$$

$$+ \partial_z Y^z_{ca}(z)\, \partial_w \bar{Y}^{\bar z}_{\bar a \bar c}(w) \langle x_{ca}\, x_{\bar a \bar c} \rangle \Big]$$

Q: How can we fix $\langle x_{cc}\, \bar{x}_{\bar c \bar c} \rangle$, $\langle x_{cc}\, x_{cc_1} \rangle$ $\langle \bar{x}_{\bar c \bar c},\, \bar{x}_{\bar c \bar c_1} \rangle$?

Using $\langle x_{cc}\, X(x_t, \bar{x}_t) \rangle = 0$
exactly as done with the usual approach!