Advanced QFT - Anomalies

[A. Bilal]

$$\begin{array}{ccccc} \text{Classical th} & \longrightarrow & \text{Noether current} & \longrightarrow & \text{conservation of current} \\ \downarrow & & \downarrow & & \downarrow \\ \text{Quantum th} & \longrightarrow & \text{Ward/Slavnov-Taylor} & \longrightarrow & \text{conserv. of expectation values of the current} \end{array}$$

local (gauge) symm. $\Downarrow$ fundamental for unitarity, Lorentz invariance,...

global symm. $\Downarrow$ e.g.: $\pi^0 \to \gamma\gamma$ suppression by anomalous term

$\Rightarrow$ Non Abelian gauge th.:

generators: $t_\alpha \longrightarrow [t_\alpha, t_\beta] = i\, C_{\alpha\beta}{}^{\gamma}\, t_\gamma \quad , \quad \alpha,\beta,\gamma = 1,\dots,\dim G, \quad C_{\alpha\beta}{}^{\gamma} \in \mathbb{R}.$

$\downarrow$ Jacobi satisfied

matter field: $\psi_K \longrightarrow \psi'_L = \left(e^{i\theta^\alpha t_\alpha}\right)_L{}^K \psi_K \qquad L,K = 1,\dots,\dim R$ $\longrightarrow$ representation

ψ in adj $\Rightarrow \delta\psi^\gamma = \left(\varepsilon^\alpha t_\alpha^{ADJ}\right)^\gamma{}_\beta\, \psi^\beta = i\varepsilon^\alpha C^\gamma{}_{\alpha\beta}\, \psi^\beta$

e.g.: adj $\Rightarrow (t_\alpha)^\beta{}_\gamma = C^\beta{}_{\alpha\gamma}$
$(\dim R = \dim G)$

ψ in rep R: $\quad \psi^\gamma t_\gamma^R = \psi^R \Rightarrow \delta\psi^R = [\varepsilon^R, \psi^R]\ , \quad \varepsilon^R = \varepsilon^\alpha t_\alpha^{ADJ}$

When local:

$$\delta\psi^\ell(x) = i\, \varepsilon^\alpha(x) (t_\alpha^R)^\ell{}_k\, \psi^k(x) = i(\varepsilon^R(x))^\ell{}_k\, \psi^k(x).$$

$\to$ covariant derivative: $\quad D_\mu\psi = \partial_\mu\psi - iA_\mu\psi\ , \qquad A_\mu = A_\mu^\alpha t_\alpha^R$

$\to \delta(D_\mu\psi) = i\varepsilon D_\mu\psi \Rightarrow \delta A_\mu = \partial_\mu\varepsilon - i[A_\mu, \varepsilon]$ Lie Alg.-valued

$\downarrow$ in components

$$\delta A_\mu^\alpha = \partial_\mu\varepsilon^\alpha + C^\alpha{}_{\beta\gamma} A_\mu^\beta \varepsilon^\gamma.$$

What about $D_\nu D_\mu \psi$?

$$D_\nu D_\mu \psi^R - D_\mu D_\nu \psi^R = -i F^R_{\mu\nu} \psi^R \Rightarrow F^R_{\mu\nu} = F^\alpha_{\mu\nu} t^R_\alpha$$

$$\searrow \quad \delta F^R_{\mu\nu} = i[\varepsilon^R, F^R_{\mu\nu}] \Rightarrow \delta F^\alpha_{\mu\nu} = C^\alpha_{\beta\gamma} F^\beta_{\mu\nu} \varepsilon^\gamma.$$

Then $F^R_{\mu\nu} = \partial_\mu A^R_\nu - \partial_\nu A^R_\mu - i[A^R_\mu, A^R_\nu] \Longleftrightarrow F^\alpha_{\mu\nu} = \partial_\mu A^\alpha_\nu - \partial_\nu A^\alpha_\mu + C^\alpha_{\beta\gamma} A^\beta_\mu A^\gamma_\nu.$

$F_{\mu\nu}$ satisfies BIANCHI IDENTITY: $D_{[\mu} F_{\nu\rho]} = 0$.

coupling const.

For finite rep. and compact Lie algebras I can choose $(t^R_\alpha)^\dagger = t^R_\alpha \Rightarrow tr_R[t_\alpha t_\beta] = tr[t^R_\alpha t^R_\beta] = g^2 C_R \delta_{\alpha\beta}$

Now look at the action:

$$S = \int d^4x \, \mathcal{L} \longrightarrow \mathcal{L} = tr F_{\mu\nu} F^{\mu\nu} \Rightarrow \delta tr F_{\mu\nu} F^{\mu\nu} = tr[\varepsilon, F_{\mu\nu} F^{\mu\nu}] = tr \varepsilon F^2 - tr F^2 \varepsilon = tr \varepsilon F^2 - tr \varepsilon F^2 = 0$$

Normalization? $tr F_{\mu\nu} F^{\mu\nu} = F^\alpha_{\mu\nu} F^{\mu\nu\beta} tr(t_\alpha t_\beta \delta^{\alpha\beta}) \Rightarrow$ choose to have $-\frac{1}{4}$ in front!

$$\Rightarrow S = \int d^4x \left[-\frac{1}{4} F^\alpha_{\mu\nu} F^{\mu\nu}_\alpha + \mathcal{L}_{MAT}(\psi, D_\mu \psi) \right]$$

= 0 if Abelian!

EOM: $$\partial_\mu \underbrace{\frac{\partial \mathcal{L}}{\partial \partial_\mu A^\alpha_\nu}}_{-F^{\alpha,\mu\nu}} = \frac{\partial \mathcal{L}}{\partial A^\alpha_\nu} = \underbrace{\frac{\partial \mathcal{L}_{MAT}}{\partial A^\alpha_\nu}}_{J^{\alpha\nu}_{MAT}} + \underbrace{\frac{\partial \mathcal{L}_{YM}}{\partial A^\alpha_\nu}}_{C_{\gamma\alpha\beta} A^\beta_\mu F^{\gamma,\mu\nu}}$$

$$\Leftrightarrow \partial_\mu F^{\mu\nu,\alpha} + C^{\alpha\beta\gamma} A_{\mu\beta} F_\gamma{}^{\mu\nu} = -J^{\alpha\nu}_{MAT} \Leftrightarrow (D_\mu F^{\mu\nu})^\alpha = -J^{\alpha\nu}_{MAT}$$

Then

$$(D_\nu J^\nu)^\alpha = -(D_\nu D_\mu F^{\mu\nu})^\alpha = 0 \Rightarrow \text{Statement of gauge symmetry}$$

What about gauge symm.? $\Rightarrow \langle D_\nu J^{\nu\alpha} \rangle \stackrel{?}{=} 0$

$\Rightarrow$ QUANTIZATION OF NON ABELIAN GAUGE TH.

classical fields

quantum fields (operator)

Path integral: $\int \mathcal{D}A_\mu^a \mathcal{D}\psi \mathcal{D}\bar\psi \; \psi(x)\bar\psi(y)\, e^{iS[A_\mu,\psi,\bar\psi]} = \mathcal{N}\langle VAC|\, T[\psi(x)\bar\psi(y)]\, |VAC\rangle$

$$\Rightarrow \frac{\int \mathcal{D}A_\mu^\alpha \mathcal{D}\psi\mathcal{D}\bar\psi\; \psi\bar\psi e^{iS}}{\int \mathcal{D}A_\mu^\alpha \mathcal{D}\psi\mathcal{D}\bar\psi\; e^{iS}} = \frac{\langle VAC|\,T[\psi\bar\psi]\,|VAC\rangle}{\langle VAC|VAC\rangle}$$

What about gauge theories where some compon. are not physical?

$$\int \mathcal{D}A_\mu^\alpha \mathcal{D}\psi \mathcal{D}\bar\psi \;\; \mathrm{Det}(\qquad)\; \psi(x)\bar\psi(y)\, e^{i(S+S_{GF}}$$

ADJ rep.

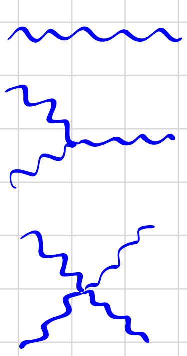
$$\underbrace{\int \mathcal{D}\omega_\alpha^* \mathcal{D}\omega_\beta}\, e^{i\int \omega_\alpha^* D_{\alpha\beta}\omega_\beta} = \mathrm{Det}\, D_{\alpha\beta}$$

anticommuting scalars] GHOSTS

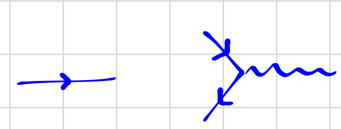
$\rightarrow$ I can rewrite: $\int \mathcal{D}A_\mu^\alpha \mathcal{D}\psi\mathcal{D}\bar\psi \mathcal{D}\omega_\alpha^*\mathcal{D}\omega_\beta \;\psi(x)\bar\psi(y)\, e^{i(S+S_{GF}+S_{gh})}$

e.g.: $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^\alpha F_\alpha^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^{\mu\alpha})^2 + \partial_\mu\omega_\alpha^*\,(D^\mu\omega)^\beta$

$\hookrightarrow \partial^\mu\omega^\beta + C^{\beta\gamma\delta}A_\gamma^\mu\omega_\delta$



$\hookrightarrow + \mathcal{L}_{MAT} \Rightarrow$



What about anomalies?

$$\int \mathcal{D}A_\mu^\alpha \mathcal{D}\omega^\alpha \mathcal{D}\omega^{*\alpha} \mathcal{D}\psi\mathcal{D}\bar\psi \;\; \mathcal{O}_1\ldots\mathcal{O}_N \; e^{i\int \mathcal{L}_{YM}+\mathcal{L}_{GF}+\mathcal{L}_{gh}+\mathcal{L}_{MAT}} = \mathcal{N}\langle VAC|\mathcal{O}_1\ldots\mathcal{O}_N|VAC\rangle$$

If $\mathcal{O}_1,\ldots,\mathcal{O}_N$ do not dep on $\psi,\bar\psi$:

$$\int \mathcal{D}A_\mu^\alpha \mathcal{D}\omega^\alpha\mathcal{D}\omega^{*\alpha}\; \mathcal{O}_1\ldots\mathcal{O}_N\; e^{i\int \mathcal{L}_{YM}+\mathcal{L}_{GF}+\mathcal{L}_{gh}} \underbrace{\int \mathcal{D}\psi\mathcal{D}\bar\psi\; e^{i\int \mathcal{L}_{MAT}[A_\mu,\psi,\bar\psi]}}_{= e^{i\hat W[A_\mu]}}$$

focus on this to deal with the issue! $\longrightarrow$

→ FUNCTIONAL MEASURE

$$e^{i\hat{W}[A_\mu]} = \int \mathcal{D}\psi \mathcal{D}\bar{\psi}\, e^{i\int \mathcal{L}_{MAT}[A_\mu, \psi, D_\mu\psi, \bar{\psi}, D_\mu\bar{\psi}]}$$

→ what transf on $\mathcal{D}\psi\mathcal{D}\bar{\psi}$? [think of $\mathcal{L}_{MAT} = -\bar{\psi}(\gamma^\mu D_\mu + m)\psi$ with $\eta^{\mu\nu} = \mathrm{diag}(-,+,+,+)$

$\Rightarrow \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{I}$; $\{\gamma^5, \gamma^\mu\} = 0, (\gamma^5)^2 = \mathbb{I}$, $\gamma_5^* = \gamma_5$]

DEF: $\mathcal{D}\psi = ?$

$\downarrow$

$\psi(x) = \sum_n c_n \psi_n(x)$ s.t. $(i\not{D} + m)\psi_n = \lambda_n \psi_n$, $\lambda_n \in \mathbb{R}$

$$\Rightarrow \mathcal{D}\psi = \prod_n c_n$$
$$\mathcal{D}\bar{\psi} = \prod_n \bar{c}_n$$

Suppose generally $\psi(x) \longrightarrow U(x)\psi(x)$ (general LOCAL transf.: can be gauge)

$\bar{\psi}(x) \longrightarrow \bar{\psi}(x)\bar{U}(x)$

where $\bar{\psi} = \psi^\dagger(i\gamma_0)$ and $\bar{U} = i\gamma_0 U^\dagger i\gamma_0$

we can think

$$\psi(x) = \int d^4y\, \mathcal{U}(x,y)\psi(y) \rightarrow \mathcal{U}(x,y) = \delta^4(x-y) \text{ INTEGRAL KERNEL}$$

$$\Rightarrow \mathcal{D}\tilde{\psi} = (\mathrm{Det}\,\mathcal{U})^{-1}\mathcal{D}\psi$$
$$\mathcal{D}\tilde{\bar{\psi}} = (\mathrm{Det}\,\bar{\mathcal{U}})^{-1}\mathcal{D}\bar{\psi}$$

$$\Rightarrow \mathcal{D}\psi\mathcal{D}\bar{\psi} \rightarrow \mathcal{D}\tilde{\psi}\mathcal{D}\tilde{\bar{\psi}} = (\mathrm{Det}\,\mathcal{U}\,\mathrm{Det}\,\bar{\mathcal{U}})^{-1}\mathcal{D}\psi\mathcal{D}\bar{\psi}$$

→ Q: is $(\mathrm{Det}\,\mathcal{U}\,\mathrm{Det}\,\bar{\mathcal{U}})^{-1} = 1$?

a) unitary, non chiral (no γ_5) transf.

b) 〃 chiral 〃

NON CHIRAL a) $U(x) = e^{i\varepsilon^\alpha(x) t_\alpha}$, $t_\alpha^\dagger = t_\alpha$, $[t_\alpha, \gamma^\mu] = 0$ (e.g.: gauge transf.)

$\hookrightarrow \bar{U}(x) = i\gamma^0 e^{-i\varepsilon^\alpha(x) t_\alpha} i\gamma_0 = -e^{-i\varepsilon^\alpha t_\alpha} = U^{-1}(x)$

Then what is $\mathcal{U}(x,y)$ w.r.t. $U(x)$?

$$\langle x | \mathcal{U}(x,y) | y \rangle = U(x)\, \delta^4(x-y).$$

$$\Rightarrow \det M = \prod_n \lambda_n = e^{\sum_n \ln \lambda_n} = e^{\operatorname{tr} \ln M} \Rightarrow \operatorname{Det} \mathcal{U} = e^{\operatorname{tr} \ln \mathcal{U}}$$

$$\Rightarrow \langle x | \mathcal{U}^2 | y \rangle = \int d^4z \, \langle x | \mathcal{U} | z \rangle \langle z | \mathcal{U} | y \rangle = (U(x))^2 \delta^4(x-y)$$

$$\langle x | f(\mathcal{U}) | y \rangle = f(U)\, \delta^4(x-y) \qquad [\mathcal{U} : L^2(\mathbb{R}^4) \to L^2(\mathbb{R}^4)]$$

$$\hookrightarrow \operatorname{Det} \mathcal{U} = e^{\operatorname{Tr} \ln \mathcal{U}} = e^{\int d^4x \langle x | \operatorname{tr} \ln \mathcal{U} | x \rangle} =$$

$$= e^{\int d^4x \operatorname{tr} \ln U(x) \langle x | x \rangle} =$$

$$= e^{\int d^4x \operatorname{tr}(i\varepsilon^\alpha(x) t_\alpha)\, \delta^4(0)} =$$

NB: $\delta^4(x-y) = \int \frac{d^4p}{(2\pi)^4} e^{ip\cdot(x-y)} \Big|_{x=y} = \int \frac{d^4p}{(2\pi)^4} = \Lambda^4$ CUT OFF

$$= e^{\int d^4x \, i\varepsilon^\alpha \operatorname{tr} t_\alpha \Lambda^4}$$

$$\operatorname{Det} \bar{\mathcal{U}} = e^{-\int d^4x \, i\varepsilon^\alpha \operatorname{tr} t_\alpha \Lambda^4}$$

$$\Rightarrow \mathcal{D}\psi \mathcal{D}\bar{\psi} = \mathcal{D}\tilde{\psi} \mathcal{D}\tilde{\bar{\psi}} \quad \text{i.e.: } [\operatorname{Det} \mathcal{U} \cdot \operatorname{Det} \bar{\mathcal{U}}] = 1.$$

CHIRAL b) $U(x) = e^{i\varepsilon^\alpha(x)\, t_\alpha \gamma_5}$ $\qquad t_\alpha^\dagger = t_\alpha \qquad [t_\alpha, \gamma^\mu] = 0$

$$\bar U(x) = i\gamma_0\, e^{-i\varepsilon^\alpha t_\alpha \gamma_5}\, i\gamma_0 = (i\gamma_0)^2 e^{i\varepsilon^\alpha t_\alpha \gamma_5} = U(x)$$

Then $\quad \mathrm{Det}\, \mathcal{U} = e^{\int d^4x\, i\varepsilon^\alpha(x)\, \mathrm{tr}(t_\alpha\gamma_5)\Lambda^4}$

$\quad \mathrm{Det}\, \bar{\mathcal{U}} = \mathrm{Det}\, \bar{\mathcal{U}}$

Dirac indeces

$$\Rightarrow \left(\mathrm{Det}\, \mathcal{U}\, \mathrm{Det}\, \bar{\mathcal{U}}\right)^{-1} = e^{-2\int d^4x\, i\varepsilon^\alpha(x)\, \mathrm{tr}_R t_\alpha \cdot \cancel{\mathrm{tr}_D \gamma_5}^{\to 0}\, \Lambda^4}$$

$\to \infty$

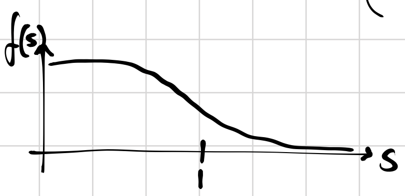
$$\sim e^{-2(0 \times \infty)} \qquad ???$$

We have to regularize better [consider $m = 0$]

$$e^{-2\,\mathrm{Tr}\ln \mathcal{U}} = \left(\mathrm{Det}\, \mathcal{U}\, \mathrm{Det}\, \bar{\mathcal{U}}\right)^{-1} = \left(\mathrm{Det}\, \mathcal{U}\right)^{-2} = e^{i\int d^4x\, \varepsilon^\alpha(x)\, a_\alpha(x)}$$

$$\Rightarrow \mathrm{Tr}\ln \mathcal{U} = i\,\mathrm{Tr}\left(\varepsilon^\alpha t_\alpha \gamma_5\right) \xrightarrow{REG} i\,\mathrm{Tr}\left(f\left(\left(\tfrac{i\not{D}}{\Lambda}\right)^2\right) \varepsilon^\alpha t_\alpha \gamma_5\right)$$

where $f(s) \sim e^{-s}$



then $\mathrm{Tr}\ln \mathcal{U}\big|_{REG} = i\int d^4x\, \mathrm{tr}_{R,D} \langle x|\, f\left(\left(\tfrac{i\hat{\not{D}}}{\Lambda}\right)^2\right) \varepsilon^\alpha(\hat{x})\, t_\alpha \gamma_5 \,|x\rangle =$

$$= i\int d^4x\, \varepsilon^\alpha(x) \int d^4p\, \underbrace{\langle x|\, f\left(\left(\tfrac{i\hat{\not{D}}}{\Lambda}\right)^2\right) |p\rangle}_{= f\left(\left(\frac{i\not{D}}{\Lambda}\right)^2\right) \underbrace{\langle x|p\rangle}_{\frac{e^{ip\cdot x}}{(2\pi)^2}}}\, t_\alpha \gamma_5 \langle p|x\rangle$$

$\frac{e^{-ip\cdot x}}{(2\pi)^2}$

NB: $(i\not{D})^2 = -\gamma^\mu\gamma^\nu D_\mu D_\nu = -\frac{1}{2}\cdot 2\eta^{\mu\nu} D_\mu D_\nu - \frac{1}{2}[\gamma^\mu, \gamma^\nu] D_\mu D_\nu =$

$$= -D_\mu D^\mu - \frac{1}{4}[\gamma^\mu, \gamma^\nu][D_\mu, D_\nu] =$$

$$= -D_\mu D^\mu + \frac{i}{2}\gamma^{\mu\nu} F_{\mu\nu}$$

$$\Rightarrow f\left(\left(\frac{i\not{D}}{\Lambda}\right)^2\right) e^{ip\cdot x} = e^{ip\cdot x} f\left(\frac{-(ip_\mu + D_\mu)(ip^\mu + D^\mu) + \frac{i}{2}\gamma^{\mu\nu}F_{\mu\nu}}{\Lambda^2}\right)$$

$$\Rightarrow \operatorname{Tr}\ln \mathcal{U}(x) = i\int d^4x\, \varepsilon^\alpha(x)\, \operatorname{tr}_{R,D} \int \frac{d^4p}{(2\pi)^4}\, f\left(\frac{-(ip_\mu + D_\mu)(ip^\mu + D^\mu) + \frac{i}{2}\gamma^{\mu\nu}F_{\mu\nu}}{\Lambda^2}\right) t_\alpha \gamma_5 =$$

Take $p = \Lambda q \Rightarrow f\left(q^2 + \frac{q\cdot D}{\Lambda} + \frac{D\cdot D}{\Lambda^2} + \frac{i}{2}\gamma^{\mu\nu}F_{\mu\nu}\frac{1}{\Lambda^2}\right)$

NB: $\operatorname{tr}_D \gamma_5 = 0$

$\operatorname{tr}_D \gamma^{\mu_1}\ldots\gamma^{\mu_n}\gamma_5 \neq 0$ only if $n > 4$

Then I need exactly the quad. term in $f(\ )$ expansion:

$$i\int d^4x\, \varepsilon^\alpha(x) \underbrace{\int d^4q}_{\frac{1}{4}2\pi^2\int_0^\infty q^3 dq\ldots} \frac{1}{2} f''(q^2)\, \operatorname{tr}\left(\frac{i}{2}\gamma^{\mu\nu}F_{\mu\nu}\right)^2 t_\alpha \gamma_5 =$$

$$-\frac{1}{4}\operatorname{tr}_R F_{\mu\nu}(x)F_{\rho\sigma}(x)\, t_\alpha \cdot \operatorname{tr}_D \gamma^{\mu\nu}\gamma^{\rho\sigma}\gamma_5$$

$$= i\,\frac{i}{32\pi^2}\operatorname{tr}_R\left(F_{\mu\nu}(x)F_{\rho\sigma}(x)\, t_\alpha\right)\varepsilon^{\mu\nu\rho\sigma}$$

Then $Q_\alpha(x) = -\frac{1}{16\pi^2}\varepsilon^{\mu\nu\rho\sigma}\operatorname{tr}_R\left(F_{\mu\nu}(x)F_{\rho\sigma}(x)\, t_\alpha\right)$ "ANOMALY FUNCTION"

$$\Rightarrow \mathcal{D}\psi\,\mathcal{D}\bar\psi \xrightarrow{U = e^{i\varepsilon^\alpha t_\alpha}} \mathcal{D}\psi\,\mathcal{D}\bar\psi\, e^{i\int d^4x\, \varepsilon^\alpha(x)\, Q_\alpha(x)}$$

Is it a symm. of sth?

$S = \int d^4x\, \bar\psi(\gamma^\mu D_\mu + m)\psi$ is invariant for chiral transf. if

$m = 0$

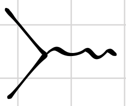
and

$\varepsilon^\alpha = \text{CONST.}$

$\Rightarrow$ symm. of class. action but quantum anomaly!

Let's look at

$$\mathcal{A}_\alpha(x) = -\frac{1}{16\pi^2}\,\varepsilon^{\mu\nu\rho\sigma}\,\underbrace{tr_R(t_\alpha t_\beta t_\gamma)}_{= D^R_{\alpha\beta\gamma} = \; Str_R(t_\alpha t_\beta t_\gamma) \text{ "symm. trace"}}\, F^\beta_{\mu\nu} F^\gamma_{\rho\sigma} \sim O(g^3)$$

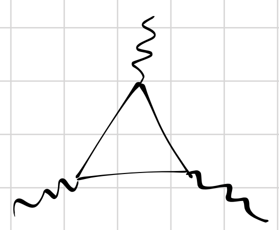
NB: $\bar\psi \gamma^\mu A^\alpha_\mu t_\alpha \psi \rightarrow$  $\propto g \Rightarrow \mathcal{A}_\alpha$ is higher order in P.T.

Introduce loop counting parameter: $e^{iS} \rightarrow e^{i\frac{S}{\lambda}} \Rightarrow$ prop $\sim \lambda$, vertex $\sim \lambda^{-1}$

$$\Rightarrow \mathcal{A} \sim \lambda^{I-V} = \lambda^{L-1}$$

I: int. lines
V: vertices
L: loops

Since what we have in Det cannot be $\propto \lambda \Rightarrow L = 1 \Rightarrow \mathcal{A}_\alpha$ is a 1L eff.



Why did we choose $f\left(\left(\frac{i\not D}{\Lambda}\right)^2\right)$ instead of $f\left(\left(\frac{i\not\partial}{\Lambda}\right)^2\right)$?

$\Rightarrow$ it's a choice we do once and for all and it should better be gauge covariant

Suppose now:

$$G \longrightarrow t_\alpha \in G$$
$$U(1) \longrightarrow t \in U(1) \;(\text{charge } q)$$

$$\Rightarrow U = e^{i\varepsilon t \gamma_5} \Longrightarrow \mathcal{A}(x) = -\frac{1}{16\pi^2}\,\varepsilon^{\mu\nu\rho\sigma}\, tr\left(t\, t_\beta t_\gamma\right) F^\beta_{\mu\nu} F^\gamma_{\rho\sigma} =$$

"ABELIAN ANOMALY" $\quad = -\frac{1}{16\pi^2}\,\varepsilon^{\mu\nu\rho\sigma}\, t\, C_R\, F^\alpha_{\mu\nu} F_{\rho\sigma,\alpha}$

$(\pi^0 \longrightarrow \gamma\gamma)$

How is this related to current NON-conservation:

$$S = -\int d^4x\, \bar{\psi}\, \gamma^\mu D_\mu \psi$$

$$U = e^{i\varepsilon\gamma_5} \quad \text{symmetry if } \varepsilon = \text{const.}$$

$\Rightarrow$ if $\delta S = 0$ for const $\varepsilon \Rightarrow \exists j^\mu \mid \partial_\mu j^\mu \big|_{EOM} = 0$

$$\hookrightarrow \delta S = -\int d^4x\, J^\mu \partial_\mu \varepsilon(x) = \int d^4x\, \partial_\mu J^\mu\, \varepsilon(x)$$

with $\quad \delta\psi = i\varepsilon\gamma_5\psi$

i.e.: $\delta S = \int d^4x\, (-)\, \bar{\psi}\gamma^\mu \partial_\mu \varepsilon\, i\gamma_5\psi \Rightarrow J_5^\mu = i\bar{\psi}\gamma^\mu\gamma_5\psi$

$$\Rightarrow \partial_\mu \langle J_5^\mu \rangle \overset{?}{=} 0 \qquad U = e^{i\varepsilon(x)t\gamma_5}$$

$$\rightarrow \langle J_5^\mu \rangle = \int \mathcal{D}\psi' \mathcal{D}\bar{\psi}'\, J_5^\mu(x)\, e^{i\int \mathcal{L}_{MAT}[A_\mu, \psi', \bar{\psi}']} = \quad (\text{let } \psi' = U\psi,\ \bar{\psi}' = \bar{\psi}\bar{U})$$

$$= \int \mathcal{D}\psi \mathcal{D}\bar{\psi}\, e^{i\int d^4x\, \varepsilon(x)\mathcal{a}(x)}\, e^{iS_{MAT} + i\int d^4x\, \partial_\mu J_5^\mu(x)\varepsilon(x)} =$$

$$\sim \int \mathcal{D}\psi \mathcal{D}\bar{\psi}\, e^{iS_{MAT}[A_\mu, \psi, \bar{\psi}]}\left(1 + i\int d^4x\, \varepsilon(x)\left(\mathcal{a}(x) + \partial_\mu J_5^\mu(x)\right)\right)$$

$$\Leftrightarrow \int d^4x\, \varepsilon(x)\left[\mathcal{a}(x) + \frac{\partial}{\partial x^\mu}\langle J_5^\mu(x)\rangle_A\right] = 0$$

$$\Leftrightarrow \frac{\partial}{\partial x^\mu}\langle J_5^\mu(x)\rangle_A = -\mathcal{a}(x) = \frac{1}{16\pi^2}\varepsilon^{\mu\nu\rho\sigma}\, \mathrm{tr}\left(F_{\mu\nu}(x) F_{\rho\sigma}(x)\right).$$

If $\varepsilon = \text{const}$:

$$\varepsilon\int d^4x\, \mathcal{a}(x) = -\frac{\varepsilon}{16\pi^2}\int d^4x\, \underbrace{\varepsilon^{\mu\nu\rho\sigma}\, \mathrm{tr}\left(F_{\mu\nu}F_{\rho\sigma}\right)}$$

$\partial_\mu(\ldots A \ldots)^\mu \Rightarrow$ at boundary it's pure gauge: $F(x) \underset{|x|\to\infty}{\sim} 0$

$$\downarrow$$

$\frac{64\pi^2}{g^2}\, \nu$ INSTANTON NO.

$$\Rightarrow \varepsilon\int d^4x\, \mathcal{a}(x) = -4C_R\, \nu\, \varepsilon.$$

Consider another way:

$\Rightarrow$ Euclidean signature: $i\gamma^\mu D_\mu := i\not{D}$ s.t. $\not{D}\gamma_5 = -\gamma_5\not{D}$

$$\to i\not{D}\varphi_k = \lambda_k \varphi_k \Rightarrow i\not{D}(\gamma_5\varphi_k) = -\lambda_k(\gamma_5\varphi_k)$$

$$\Rightarrow (i\not{D})^2 \begin{Bmatrix}\varphi_k \\ \gamma_5\varphi_k\end{Bmatrix} = \lambda_k^2 \begin{Bmatrix}\varphi_k \\ \gamma_5\varphi_k\end{Bmatrix}, \quad \lambda_k \neq 0 \Rightarrow \varphi_k \perp \gamma_5\varphi_k$$

Then I can take $\mathcal{L}(\varphi_k, \gamma_5\varphi_k) \Rightarrow \varphi_{k,\pm} = \frac{1\pm\gamma_5}{2}\varphi_k \Rightarrow \gamma_5\varphi_{k,\pm} = \pm\varphi_{k,\pm}$

$$(i\not{D})^2\varphi_{k,\pm} = \lambda_k^2\varphi_{k,\pm}$$

If $\lambda_k = 0$: $\gamma_5\varphi_u = \varphi_u$ and $\gamma_5\varphi_v = -\varphi_v$ where $u = 1,\dots,n_+$; $v = 1,\dots,n_-$

then:

$$\mathrm{Tr}\,\gamma_5 \overset{t=1}{f}\left(\left(\frac{i\not{D}}{\Lambda}\right)^2\right) = \sum_{k>0}\mathrm{tr}\langle\varphi_{k,+}|\gamma_5 f\left(\left(\frac{i\not{D}}{\Lambda}\right)^2\right)|\varphi_{k,+}\rangle + \sum_{k>0}\mathrm{tr}\langle\varphi_{k,-}|\gamma_5 f\left(\left(\frac{i\not{D}}{\Lambda}\right)^2\right)|\varphi_{k,-}\rangle$$

zero modes $\longrightarrow$ $+\sum_{u=1}^{n_+}\dots + \sum_{v=1}^{n_-}\dots =$

$$= n_+ - n_- = \mathrm{index}(i\not{D}_E)$$

Therefore:

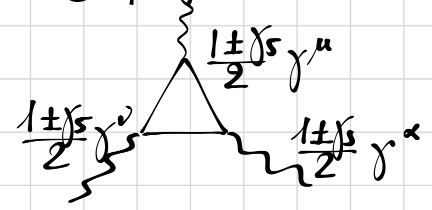
$$\int d^4x\, a(x) = -2\,\mathrm{index}(i\not{D}_E) = -\frac{1}{16\pi^2}\int d^4x\,\varepsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}\left(F_{\mu\nu}(x)F_{\rho\sigma}(x)\right)$$

$\Rightarrow$ ANOMALIES OF GAUGE TRANSFORMATIONS

* Abelian anomaly with $\mathcal{L}_{MAT} = -\bar\psi\gamma^\mu D_\mu\psi$ and $U = e^{i\varepsilon\gamma_5}$
* Gauge anomaly:

$$U(x) = e^{i\varepsilon^\alpha(x)t_\alpha}$$

$$\mathcal{L}_{MAT} = -\bar\psi\gamma^\mu D_\mu\left(\frac{1\pm\gamma_5}{2}\right)\psi$$ $\to$ chirality projector in $\mathcal{L}_{MAT}$



Consider:
$$e^{i\tilde{W}[A_\mu]} = \int \mathcal{D}\psi\, \mathcal{D}\bar{\psi}\; e^{i\int \mathcal{L}_{MAT}[A_\mu, \psi, \bar{\psi}]}$$

and Slavnov-Taylor id.

Take
$$\phi^r(x) \to \phi'^r(x) = \phi^r(x) + \underbrace{\delta\phi^r(x)}_{= \varepsilon F^r(x, \phi(x))}$$

and
$$S[\phi + \delta\phi] = S[\phi] \quad \text{with however} \quad \mathcal{D}\phi' = \mathcal{D}\phi\, e^{i\varepsilon\int d^4x\, \mathcal{A}(x)}$$

$$\begin{aligned} e^{iW[J]} &= \int \mathcal{D}\phi\; e^{iS[\phi] + i\int J_r \phi^r} = \\ &= \int \mathcal{D}\phi'\; e^{iS[\phi'] + i\int J_r \phi^{r\prime}} = \\ &= \int \mathcal{D}\phi\; e^{i\varepsilon\int \mathcal{A}(x) + iS[\phi] + i\int J_r \phi^r + i\varepsilon\int J_r F^r} = \\ &\sim \int \mathcal{D}\phi\; e^{iS[\phi] + i\int J_r \phi^r} \left(1 + i\varepsilon\left[\int \mathcal{A}(x) + \int J_r F^r(x)\right]\right) \end{aligned}$$

$$\Leftrightarrow \quad 0 = \int \mathcal{D}\phi\; e^{iS[\phi] + i\int J_r \phi^r} \left[\int \mathcal{A}(x) + \int J_r(x) F^r(x)\right]$$

$$\hookrightarrow \quad \frac{\int \mathcal{D}\phi\; e^{iS[\phi] + i\int J_r \phi^r} \left[\int \mathcal{A}(x) + \int J_r(x) F^r(x)\right]}{\int \mathcal{D}\phi\; e^{iS[\phi] + i\int J_r \phi^r}} = 0$$

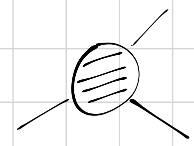
$$\Rightarrow \int d^4x \Big(\underbrace{\langle \mathcal{A}(x) \rangle_J}_{\substack{\text{if does not} \\ \text{dep. on } \phi \\ \Downarrow \\ = \mathcal{A}(x)}} + J_r(x) \langle F^r(x, \phi) \rangle_J \Big) = 0$$

Legendre transform.: $\langle \phi^r(x) \rangle_J = \varphi^r(x) = \dfrac{\delta W[J]}{\delta J_r(x)}$

1PI generator

$$\Rightarrow \Gamma[\varphi] = W[J_r] - \int d^4x\, \varphi^r(x) J_r(x) \longrightarrow \text{QUANTUM EFF. ACTION}$$

$$\Rightarrow \Gamma[\varphi] = \dots \sum_n \frac{1}{n!} \Gamma^{(n)}_{r_1\dots r_n}(x_1,\dots,x_n)\, \varphi^{r_1}(x_1)\dots\varphi^{r_n}(x_n)$$

$\hookrightarrow$ $n \geq 3$: 

all loop contrib!

$n=2$: $\Gamma^{(2)} = \left(G^{(2)}\right)^{-1} \Rightarrow$ prop.

$\hookrightarrow$ at LO, it is the action

Then

$$J_r = -\frac{\delta\Gamma}{\delta\varphi^r} \quad \text{and (if linear)} \quad \langle F^r \rangle_J = \delta\varphi^r$$

$$\hookrightarrow \quad \delta\Gamma = \int d^4x\, \delta\varphi^r \frac{\delta\Gamma}{\delta\varphi^r} = -\int \mathcal{A}(x)$$

Once we do GF, then BRST is left:

$$\delta_{BRST} A_\mu = D_\mu \omega \longrightarrow \text{NB: } \partial_\mu\omega - i[A_\mu,\omega] \Rightarrow \text{NON LINEAR !}$$

but necessary only when I want to compute $\int \mathcal{D}A_\mu$

$$\delta_{BRST}\psi = i\omega\psi$$

$$\delta_{BRST}\omega = i\omega\omega$$

$$\delta_{BRST}\omega^* = -h$$

$$\delta_{BRST} h = 0$$

Let's consider

$$e^{i\tilde{W}[A]} = \int \mathcal{D}\psi\mathcal{D}\bar\psi\, e^{i\int \mathcal{L}_{MAT}[A,\psi,\bar\psi]} \rightarrow \text{is it gauge inv ?}$$

$$\hookrightarrow e^{i\tilde{W}[A']} = \int \mathcal{D}\psi'\mathcal{D}\bar\psi'\, e^{i\int \mathcal{L}_{MAT}[A',\psi',\bar\psi']} = \int \mathcal{D}\psi\mathcal{D}\bar\psi\, e^{i\int \varepsilon^\alpha Q_\alpha(x)}\, e^{i\int \mathcal{L}_{MAT}[A,\psi,\bar\psi]}$$

$$= e^{i\int \varepsilon^\alpha Q_\alpha}\, e^{i\tilde{W}[A]}$$

$$\Rightarrow \tilde{W}[A'] - \tilde{W}[A] = \delta_\varepsilon \tilde{W}[A] = \int \varepsilon^\alpha Q_\alpha(x)$$

Now consider any $\mathcal{W}$:

$$\delta_\varepsilon \mathcal{W}[A] = \mathcal{W}[A+\delta A] - \mathcal{W}[A] = \int d^4x \underbrace{\delta A^\alpha_\mu(x)}_{(D_\mu \varepsilon^\alpha(x))} \frac{\delta \mathcal{W}[A]}{\delta A^\alpha_\mu(x)}$$

$$\Rightarrow \delta_\varepsilon \mathcal{W}[A] = -\int d^4x\, \varepsilon^\alpha(x)\, D_\mu \left(\frac{\delta \mathcal{W}[A]}{\delta A_\mu}\right)_\alpha$$

In our case:

$$-\int d^4x\, \varepsilon^\alpha(x)\, D_\mu\left(\frac{\delta\tilde{W}[A]}{\delta A_\mu}\right)_\alpha = \int d^4x\, \varepsilon^\alpha(x)\, \mathcal{Q}_\alpha(x)$$

$$\Rightarrow \left(D_\mu \frac{\delta \tilde{W}[A]}{\delta A_\mu}\right)^\alpha = -\mathcal{Q}^\alpha(x) \Leftrightarrow D_\mu \langle J^\mu_\alpha(x)\rangle = -\mathcal{Q}_\alpha(x).$$

We have $J^\mu_\alpha(x) = i\bar{\psi}\gamma^\mu t_\alpha \psi$.

ANOMALOUS WARD IDENTITIES:

$$\frac{\delta \tilde{W}[A]}{\delta A^\alpha_\mu(x)} = \langle J^\mu_\alpha(x)\rangle$$

Then

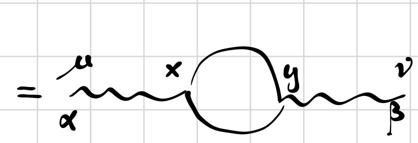
$$\frac{\delta}{\delta A^\beta_\nu(y)}\frac{\delta}{\delta A^\alpha_\mu(x)}\tilde{W}[A] = i\langle T[J^\nu_\beta(y) J^\mu_\alpha(x)]\rangle_A$$

$$\vdots$$

$$\frac{\delta}{\delta A^{\alpha_1}_{\mu_1}(x_1)}\cdots\frac{\delta}{\delta A^{\alpha_n}_{\mu_n}(x_n)}\tilde{W}[A]\Big|_{A=0} = i^{n-1}\langle T[J^{\mu_1}_{\alpha_1}(x_1)\ldots J^{\mu_n}_{\alpha_n}(x_n)\rangle_{A=0}$$

e.g.:

$$\langle T[J^\mu_\alpha(x) J^\nu_\beta(y)]\rangle = \langle T[\bar{\psi}\gamma^\mu t_\alpha \psi(x)\bar{\psi}\gamma^\nu t_\beta\psi(y)]\rangle =$$



$$\Rightarrow \langle T[J^{\mu_1}_{\alpha_1}(x_1)\ldots J^{\mu_n}_{\alpha_n}(x_n)]\rangle = \Gamma^{\mu_1\ldots\mu_n}_{\alpha_1\ldots\alpha_n}(x_1,\ldots,x_n)\Big|_{1L}$$

ABELIAN ANOMALY $\Rightarrow \partial_\mu \langle J_5^\mu(x)\rangle_A = -\mathcal{A}_\alpha(x) = \frac{1}{16\pi^2}\varepsilon^{\mu\nu\rho\sigma}\, tr\left(t_\alpha F_{\mu\nu}(x) F_{\rho\sigma}(x)\right)$

$$\hookrightarrow \frac{\delta}{\delta A^\nu_\beta(y)}\frac{\delta}{\delta A^\rho_\alpha(z)}\frac{\partial}{\partial x^\mu}\langle J_5^\mu(x) J^\nu_\beta(y) J^\rho_\alpha(z)\rangle\Big|_{A=0} = \frac{1}{2\pi^2}\, tr\Big(t_\alpha t_\beta t_\gamma\, \varepsilon^{\mu\nu\rho\sigma}\frac{\partial}{\partial y^\sigma}\delta^4(x-y) \times \frac{\partial}{\partial z^\rho}\delta^4(x-z)\Big)$$

$$\Rightarrow (p+q)_\mu\, \Gamma_5^{\mu\nu\rho}(-p-q,p,q) = -\frac{1}{2\pi^2}\left(\sum q^3\right)\varepsilon^{\nu\rho\lambda\sigma}p_\lambda q_\sigma$$

GAUGE ANOMALY $\longrightarrow$ we still don't know the RHS but we compute it with Feynman diag.

$$\Rightarrow D_\mu \langle J^\mu(x)\rangle^\alpha_A = -a^\alpha(x)$$

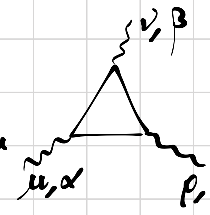
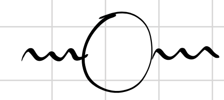
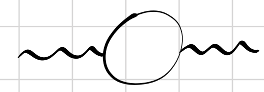
$$\hookrightarrow \partial_\mu\langle J^\mu_\alpha(x)\rangle_A + C_{\alpha\beta\gamma}A^\beta_\mu(x)\langle J^{\mu\gamma}(x)\rangle_A = -a_\alpha(x)$$

$$\hookrightarrow \frac{\delta}{\delta A^\beta_\nu(y)}(\,LHS\,) = i\,\partial_\mu\langle J^\nu_\beta(y) J^\mu_\alpha(x)\rangle_A + C_{\alpha\beta\gamma}\,\delta^4(y-x)\langle J^{\nu\gamma}(x)\rangle_A + C_{\alpha\beta\gamma}A^\beta_\mu(x)\langle J^\nu_\beta J^\mu_\gamma\rangle_A$$

$$\hookrightarrow \frac{\delta}{\delta A^\rho_\gamma(z)}(\ldots) = \frac{\partial}{\partial x^\mu}\langle T[J^\rho_\gamma(z) J^\nu_\beta(y) J^\mu_\alpha(x)]\rangle_{A=0} + C_{\alpha\beta\zeta}\,\delta^4(y-x)\langle T[J^\rho_\gamma(z) J^\nu_\zeta(x)]\rangle + C_{\alpha\delta\gamma}\,\delta^4(z-x)\langle T[J^\rho_\delta(z) J^\nu_\gamma(y)]\rangle = \frac{\delta^2 a_\alpha}{\delta A^\beta_\nu\,\delta A^\gamma_\rho}$$

$$\Leftrightarrow \frac{\partial}{\partial x^\mu}\Gamma^{\mu\nu\rho}_{\alpha\beta\gamma}(x,y,z) + C_{\alpha\beta\zeta}\,\Gamma^{\rho\nu}_{\beta\zeta}(y,x) + C_{\alpha\beta\gamma}\,\Gamma^{\rho\nu}_{\beta\gamma}(z,x) = \frac{\delta^2 a_\alpha(x)}{\delta A^\beta_\nu\,\delta A^\gamma_\rho}$$

Suppose $a_\alpha(x) = 4c\,\varepsilon^{\mu\nu\rho\sigma}\, tr\left(t_\alpha\, \partial_\mu A_\nu\, \partial_\rho A_\sigma\right)$, then:

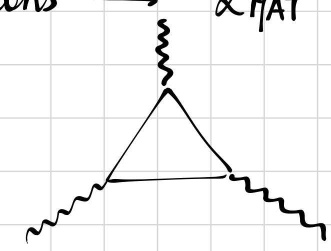
$(p+q)_\mu\, \Gamma^{\mu\nu\rho}_{\alpha\beta\gamma}(x,y,z) \sim (p+q)_\mu$  $=$  $+$  $+ \frac{\delta^2 a}{\delta A\,\delta A}\Big|_{A=0}$

$\Rightarrow$ from the coup. we can read c in $a_\alpha(x)$.

NB: the Ward id. are techically ∞ just by taking more and more derivatives.

Let's sketch the computation:

* ψ Dirac fermions $\longrightarrow$ $\mathcal{L}_{MAT} = -\bar{\psi}(\partial\!\!\!/ - iA\!\!\!/)\psi$

$\longrightarrow$ no anomaly  Linearly divergent $\Rightarrow$ DIMENSIONAL REG.

$\downarrow$

$\mathcal{A}_\alpha = 0$

* CHIRAL FERMIONS: $P_L = \frac{1+\gamma_5}{2}$, $P_R = \frac{1-\gamma_5}{2}$

$$\psi = \psi_L + \psi_R \quad \text{where} \quad \psi_{L,R} = P_{L,R}\,\psi$$

$$\Rightarrow \mathcal{L}_{MAT} = -\bar{\psi}_L(\partial\!\!\!/ - iA\!\!\!/)\psi_L = -\bar{\psi}(\partial\!\!\!/ - iA\!\!\!/)P_L\psi$$

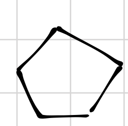
NB: $m\bar{\psi}_L\psi_L = m\bar{\psi}P_R P_L\psi = 0$

$\Rightarrow$ no mass term for chiral fermions

$\Downarrow$

I can forget one of the P_L

prop: $\cancel{P_L}\,\frac{1}{ip\!\!\!/}$

$\Rightarrow$  $\sim (\gamma^\mu t_\alpha \overbrace{P_L) P_L}^{P_L} \frac{1}{ip\!\!\!/}(\dots)$

vertices: $\gamma^\mu t_\alpha P_L$

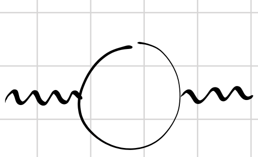
Then:

$$\mathcal{L}_{MAT} = -\bar{\psi}\partial\!\!\!/\psi - \bar{\psi}(-iA\!\!\!/)P_L\psi = -\bar{\psi}_R\partial\!\!\!/\psi_R - \bar{\psi}_L D\!\!\!\!/\,\psi_L$$

$\downarrow$

not interacting at tree level but present at 1L

In this case DIM. REG. is <u>not</u> a good idea. We use PAULI-VILLARS regularization:

 $$\sim \int d^4k \left[\frac{1}{ik\!\!\!/}\cdot\frac{1}{i(p\!\!\!/-k\!\!\!/)} - \underbrace{\frac{1}{ik\!\!\!/+M}\cdot\frac{1}{i(p\!\!\!/-k\!\!\!/)+M}}_{\text{regulariz. with MASSIVE, OPPOSITE STAT. fields.}}\right]$$

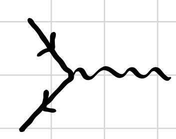
Now look at:

$$-i(p+q)_\mu \, \Gamma^{\mu\nu\rho}_{\alpha\beta\gamma}(-p-q,p,q) = -C_{\alpha\beta\gamma}\left(\pi^{\nu\rho}(p,-q) - \pi^{\nu\rho}(-p,q)\right) + \mathcal{A}^{\nu\rho}_{\alpha\beta\gamma}(-p-q,p,q)$$

If $\mathcal{A}_\alpha(x) = 4c\,\varepsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}\left(t_\alpha\, \partial_\mu A_\nu(x)\, \partial_\rho A_\sigma(x)\right) + O(A^3)$, then

$$\Rightarrow -i(p+q)_\mu \Gamma^{\mu\nu\rho}_{\alpha\beta\gamma}(-p-q,p,q)\Big|_{\varepsilon\text{-term}} = 8c\,\varepsilon^{\nu\rho\lambda\sigma}\, p_\lambda q_\sigma\, D^R_{\alpha\beta\gamma}$$

$\hookrightarrow$ $\mathcal{L}_{MAT} = -\bar\psi_L \not D \psi_L = \bar\psi \not D P_L \psi$ s.t. $\longrightarrow \; = -\frac{i}{(2\pi)^4}\cancel{P_L}\,\frac{1}{i\not k}$ forget this in loops

 $= (2\pi)^4\,\delta^4(\dots)\, t_\alpha\, \gamma^\mu P_L$

or equivalently: $\widetilde{\mathcal{L}_{MAT}} = -\bar\psi_R \not\partial \psi_R - \bar\psi_L \not D \psi_L$

Then $\int \mathcal{D}\psi_L \mathcal{D}\bar\psi_L \; e^{-i\int d^4x\, \psi_L \not D \bar\psi_L} \sim \mathrm{Det}(\not D P_L)$

NB: $\not D P_L : \mathcal{H}_L \longrightarrow \mathcal{H}_L \Rightarrow$ cannot def $\mathrm{Det}(\not D P_L)$

$A : \mathcal{H} \longrightarrow \mathcal{H}$ s.t. $\mathrm{Im}(A) \subset \mathrm{Det}(A)$ ✓

Use $(i\not D P_L)^\dagger = P_L(i\not D) = i\not D P_R$ and define

$$\mathrm{Det}\begin{pmatrix} 0 & (\not D P_L)^\dagger \\ \not D P_L & 0 \end{pmatrix}\begin{matrix}\mathcal{H}_L \\ \mathcal{H}_R\end{matrix} = \mathrm{Det}(\not D P_L)\,\mathrm{Det}(\not D P_L)^\dagger = |\mathrm{Det}(\not D P_L)|^2 \rightarrow \text{defined up to a phase.}$$

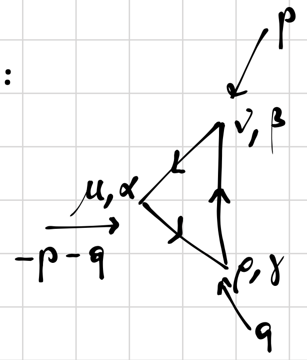
Then

$$e^{i\widetilde W_E[A]} = \int \mathcal{D}\psi \mathcal{D}\bar\psi \; e^{-i\int d^4x\, \bar\psi_L \not D_L \psi_L} \Rightarrow$$

$\mathrm{Re}\,\widetilde W_E[A]$ is well defined

$\mathrm{Im}\,\widetilde W_E[A]$ contains ε-terms.

Now let's compute:



$+ \ (\nu, \beta, p) \longleftrightarrow (\rho, \gamma, q)$

opposite direction of the arrow

$$\Rightarrow \ i(2\pi)^4 \, \Gamma_{L\,\alpha\beta\gamma}^{\mu\nu\rho}(-p-q, p, q) = -\int d^4k \, \mathrm{tr}_{R,D}\Big(-(2\pi)^4 t_\alpha \gamma^\mu P_L\Big)\left(-\frac{i}{(2\pi)^4}\cdot\frac{-i(\not{k}+\not{p})}{(k+p)^2 - i\varepsilon}\right)\Big(-(2\pi)^4 t_\beta \gamma^\nu P_L\Big) \times$$

$$\times \left(-\frac{i}{(2\pi)^4}\cdot\frac{-i\not{k}}{k^2 - i\varepsilon}\right)\Big(-t_\gamma \gamma^\rho P_L (2\pi)^4\Big)\left(-\frac{i}{(2\pi)^4}\cdot\frac{-i(\not{k}-\not{q})}{(k-q)^2 - i\varepsilon}\right) \ + \text{perm.}$$

$$\Leftrightarrow \ \Gamma_{L\,\alpha\beta\gamma}^{\mu\nu\rho}(-p-q, p, q) = i\int \frac{d^4k}{(2\pi)^4} \, \mathrm{tr}_D\Big(\gamma^\mu P_L (\not{k}+\not{p})\gamma^\nu P_L \not{k} \gamma^\rho P_L (\not{k}-\not{q})\Big)\frac{\mathrm{tr}_R(t_\alpha t_\beta t_\gamma)}{((k+p)^2 - i\varepsilon)(k^2 - i\varepsilon)((k-q)^2 - i\varepsilon)}$$

$+$ perm.

$\Rightarrow$ LINEARLY DIVERGENT

$\downarrow$

Pauli-Villars regularization: $(\eta_0 = 1, M_0 = 0), (\eta_1 = -1, M_1), (\eta_2 = +1, M_2), (\eta_3 = -1, M_3)$

$$\Gamma_{L\,\alpha\beta\gamma}^{\mu\nu\rho}(-p-q, p, q)\Big|_{REG} = i\sum_s \eta_s \int \frac{d^4k}{(2\pi)^4} \, \mathrm{tr}_D\Big(\gamma^\mu P_L[(\not{k}+\not{p})+iM_s]\gamma^\nu P_L[\not{k}+iM_s]\gamma^\rho P_L[(\not{k}-\not{q})+iM_s]\Big)\frac{\mathrm{tr}_R(t_\alpha t_\beta t_\gamma)}{((k+p)^2+M_s^2 - i\varepsilon)(k^2+M_s^2 - i\varepsilon)((k-q)^2+M_s^2 - i\varepsilon)}$$

$+$ perm.

$$= i\int \frac{d^4k}{(2\pi)^4} \sum_{s=0}^{3} \eta_s \, J_{M_s}^{\mu\nu\rho}(k,p,q) \, \mathrm{tr}_R(t_\alpha t_\beta t_\gamma)$$

$$\hookrightarrow \ J_{M_s}^{\mu\nu\rho}(k,p,q) = \frac{\mathrm{tr}_D\Big(\gamma^\mu P_L[(\not{k}+\not{p})+iM_s]\gamma^\nu P_L[\not{k}+iM_s]\gamma^\rho P_L[(\not{k}-\not{q})+iM_s]\Big)}{((k+p)^2+M_s^2 - i\varepsilon)(k^2+M_s^2 - i\varepsilon)((k-q)^2+M_s^2 - i\varepsilon)} =$$

$$= \frac{\mathrm{tr}_D\Big((\not{k}-\not{q})\gamma^\mu \ (\not{k}+\not{p})\gamma^\nu \ \not{k}\gamma^\rho P_L\Big)}{((k+p)^2+M_s^2 - i\varepsilon)(k^2+M_s^2 - i\varepsilon)((k-q)^2+M_s^2 - i\varepsilon)} =$$

$$= \frac{\mathrm{tr}_D\Big((\not{k}-\not{q})\gamma^\mu P_L (\not{k}+\not{p})\gamma^\nu P_L \not{k}\gamma^\rho P_L\Big)}{((k+p)^2+M_s^2 - i\varepsilon)(k^2+M_s^2 - i\varepsilon)((k-q)^2+M_s^2 - i\varepsilon)} =$$

$= \ldots$

$$\Rightarrow (p+q)_\mu \, J^{\mu\nu\rho}_{M_s}(k,p,q) = \frac{\mathrm{tr}_D\left((\not k-\not q)\gamma^\nu \not k \gamma^\rho P_L\right)}{(k^2+M_s^2-i\varepsilon)((k-q)^2+M_s^2-i\varepsilon)} - \frac{\mathrm{tr}_D\left((\not k+\not p)\gamma^\nu \not k \gamma^\rho P_L\right)}{((k+p)^2+M_s^2-i\varepsilon)(k^2+M_s^2-i\varepsilon)} +$$

$$+ M_s^2 \frac{\mathrm{tr}_D\left((\not p+\not q)\gamma^\nu \not k \gamma^\rho P_L\right)}{[(k+p)^2+M_s^2-i\varepsilon][k^2+M_s^2-i\varepsilon][(k-q)^2+M_s^2-i\varepsilon]}$$

no contrib. to ε tensor

no contrib. to ε tensor

$$\Rightarrow (p+q)_\mu \, \Gamma^{\mu\nu\rho}_{L\,\alpha\beta\gamma}\Big|_{\varepsilon\text{-tensor}} = i\int\frac{d^4k}{(2\pi)^4}\sum_{s=0}^{3}\eta_s M_s^2 \frac{\mathrm{tr}_D\left((\not p+\not q)\gamma^\nu \not k \gamma^\rho \gamma_5\right)}{[(k+p)^2+M_s^2-i\varepsilon][k^2+M_s^2-i\varepsilon][(k-q)^2+M_s^2-i\varepsilon]}$$

Use

$$\frac{1}{D_1D_2D_3} = 2\int_0^1 dx\int_0^{1-x} dy \frac{1}{(xD_1+yD_2+(1-x-y)D_3)^3}$$

$$(xD_1+yD_2+(1-x-y)D_3) \longrightarrow \Big[(\underbrace{k+xp-yq}_{k'})^2 + M_s^2 + r^2(p,q,x,y)\Big]^3$$

$$\Rightarrow (p+q)_\mu \, \Gamma^{\mu\nu\rho}_{L\,\alpha\beta\gamma}\Big|_{\varepsilon\text{-tensor}} = \int dxdy \; i\int\frac{d^4k'}{(2\pi)^4}\sum_s \eta_s M_s^2 \frac{\mathrm{tr}\left((\not p+\not q)\gamma^\nu(\cancelto{0}{\not k'} - x\not p + y\not q)\gamma^\rho\gamma_5\right)}{(k'^2+M_s^2+r^2(\dots)-i\varepsilon)^3} =$$

$$= \int_0^1 dx\int_0^{1-x} dy \; i \; \mathrm{tr}\left((\not p+\not q)\gamma^\nu(y\not q - x\not p)\gamma^\rho\gamma_5\right)\cdot I$$

$$I = \sum_s \eta_s \, i\int\frac{d^4k_E}{(2\pi)^4} M_s^2 \frac{1}{(k_E^2+M_s^2)^3} = \quad k = M_s \ell$$

$$= \frac{i}{(2\pi)^4}\sum_s \eta_s \int\frac{d^4\ell_E}{(\ell_E^2+1)^3} = \dots = \frac{i}{32\pi^2}\sum_s \eta_s$$

$$\Rightarrow (p+q)_\mu \, \Gamma^{\mu\nu\rho}_{L\,\alpha\beta\gamma}\Big|_{\varepsilon\text{-tensor}} = -\frac{1}{32\pi^2}\cdot\frac{1}{6}\,\mathrm{tr}_D\left((\not p+\not q)\gamma^\nu(\not q-\not p)\gamma^\rho\gamma_5\right)\cdot\sum_s \eta_s =$$

$$= -\frac{1}{32\pi^2}\frac{4i}{6}\varepsilon^{\lambda\nu\gamma\rho}\underbrace{(p+q)_\lambda(q-p)_\gamma}_{p_\lambda q_\gamma - q_\lambda p_\gamma = 2p_{[\lambda}q_{\sigma]}}\cdot\sum_s\eta_s =$$

$$= -\frac{i}{24\pi^2}\varepsilon^{\lambda\nu\sigma\rho}p_\lambda q_\sigma\sum_s\eta_s =$$

the term. gets the same sign and must be added!

$$= \frac{-i}{24\pi^2}\varepsilon^{\nu\rho\lambda\sigma}p_\lambda q_\sigma$$

$$\Rightarrow -i(p+q)_\mu \, \Gamma^{\mu\nu\rho}_{L\,\alpha\beta\gamma}\Big|_{\varepsilon} = -\frac{1}{12\pi^2}\varepsilon^{\nu\rho\lambda\sigma}p_\lambda q_\sigma \, D^R_{\alpha\beta\gamma} \qquad \mathrm{tr}_R(t_\alpha t_\beta t_\gamma)$$

More completely:

$$-i(p+q)_\mu \Gamma^{\mu\nu\rho}_{L\,\alpha\beta\gamma}(-p-q,p,q) = -\frac{1}{12\pi^2}\varepsilon^{\nu\rho\lambda\sigma} p_\lambda q_\sigma D^R_{\alpha\beta\gamma} - C_{\alpha\beta\gamma}\left[\Pi^{\nu\rho}_{\Pi}(p) - \Pi^{\nu\rho}_{\Pi}(q)\right]$$

FINITE and polynomial in p^μ (local)

$\downarrow$

THIS IS A GENERAL FEATURE

vacuum polarization: $(p^\nu p^\rho - \eta^{\nu\rho}p^2)\Pi_\Pi(p^2)$ — $\ln p^2$

$\Downarrow$

add counterterm

NB: gauge anomalies := "chiral anomaly" and chiral ferm $\exists$ in $d=2n$ since

$\gamma^{d+1} = i\gamma^0\dots\gamma^{d-1} \propto \mathbb{I}$ if $d = 2n+1$

$\Rightarrow$ "$\nexists\gamma_5$"

NB: a convergent Feynman diagram CANNOT BE ANOMALOUS:

in $d=2r$ $\Rightarrow$ int. lines $> 2r$ $\Rightarrow$ No anomalies ($\pentagon$ in $d=4$ IS NOT anomalous)

Take now n-point, 1L: $\Gamma^{(n)}(p_1,\dots,p_n)$

$$\mathrm{tr}\left[\frac{1}{\not{k}+\not{p}_1}\gamma^{\mu_1}P_L\frac{1}{\not{k}+\not{p}_2}\gamma^{\mu_2}P_L\dots\right] \longrightarrow \sim \frac{d^dk}{k^n} \rightarrow D = 2r-n$$

then take

$$\frac{\partial}{\partial p_1^i}\left[\mathrm{tr}(\cdots)\right] \Rightarrow \frac{\partial}{\partial p_i^i}\Gamma^{(n)}(\dots) \Rightarrow D = 2r-n-1$$

in general $\left(\frac{\partial}{\partial p_i}\right)^s\left((p+q)\Gamma^{(n)}\right) \Rightarrow D = 2r-n-s \Rightarrow$ if $s = 2r-n+2$ then it is convergent $\Rightarrow$ NO REG NEEDED

$\rightarrow$ since acting with s deriv. we end up with no anom $\Rightarrow$ the anomaly MUST BE POLYNOMIAL of degree $2r-n+1$

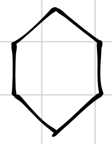
e.g.: $d=2r=4$, $n=3$ $\Rightarrow \mathcal{A} \sim \varepsilon^{\mu\nu\rho\sigma}p_\rho q_\sigma \rightarrow$ degree $4-3+1=2$.

Consider now the ε dependence:

$$\left(\sum_{i=1}^{n-1} p_i\right)_\mu \Gamma_{(n)}^{\mu\nu_1\ldots\nu_{n-1}}\left(-\sum_{i=1}^{n-1} p_i, p_1, \ldots, p_{n-1}\right)\Big|_{\text{anomalous}} = \varepsilon^{\overbrace{\cdots}^{r=n-1}\overbrace{\cdots}^{r=n-1}} p_1 \ldots p_{n-1}$$

$\Rightarrow$ the smallest diagram is s.t. $r = n-1 \longrightarrow n = \frac{d}{2}+1 = r+1$

d	n
4	3
6	4
8	5
10	6

$\longrightarrow$ Superstring th: 

$\Rightarrow$ by dimens. analysis: $(p)_\mu \Gamma^{\mu\nu_1\ldots\nu_r} \sim \int \frac{d^{2r}k}{(k\ldots)^{r+1}} \sim \Lambda^r \qquad [\Lambda] = M$

$$\Rightarrow (\ldots p\ldots)_\mu \Gamma^{\mu\nu_1\ldots\nu_r}\Big|_\varepsilon = \# \, \varepsilon^{\cdots} \, p_1\ldots p_r$$

$p_1\ldots p_r \rightarrow \Lambda^r$

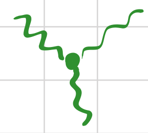
$\#$: CAN ONLY BE NUMERICAL!

Is the anomaly RELEVANT or IRREL.?

i.e.: Can we find a COUNTERTERM to cancel the anomaly?

a) YES $\Rightarrow S_{YM} \to S_{YM} + \Delta S_{CT} \Rightarrow$ cancel the anomaly! [IRREL.]

b) NO $\Rightarrow$ anomalous th. [REL.]

$\mathcal{A} \sim D^R_{\alpha\beta\gamma} \Rightarrow$  $\Rightarrow \mathcal{O}(g^3_{YM})$

$\Rightarrow \Delta S_{CT}$ should be LOCAL and $\sim A^3$

$$\longrightarrow \Delta\mathcal{L}_{CT} \sim p^\sigma_{(1,2,3)} A^\mu_\alpha(k) A^\nu_\beta(p) A^\rho_\gamma(q) \, \varepsilon_{\mu\nu\rho\sigma} \, \mathrm{tr}(t_\alpha t_\beta t_\gamma)$$

$$\Rightarrow \Delta\Gamma^{\mu\nu\rho}_{\alpha\beta\gamma}(k,p,q)\Big|_\varepsilon = \varepsilon^{\mu\nu\rho\sigma}(a k_\sigma + b p_\sigma + c q_\sigma) \, D^R_{\alpha\beta\gamma}$$

to have the correct symm: $\begin{cases} a = -b \\ b = -c \\ a = -c \end{cases} \Rightarrow a = b = c = 0$

$\Rightarrow$ The anomaly is relevant.

We found:

by symm.

$$\mathcal{A}^L_\alpha(x) = -\frac{1}{24\pi^2}\varepsilon^{\mu\nu\rho\sigma}\partial_\mu A^\beta_\nu(x)\,\partial_\rho A^\gamma_\sigma(x)\,D^R_{\alpha\beta\gamma} = -\frac{1}{24\pi^2}\varepsilon^{\mu\nu\rho\sigma}\partial_\mu\left[A^\beta_\nu\partial_\rho A^\gamma_\sigma - \frac{i}{4}A^\beta_\nu[A_\rho,A_\sigma]^\gamma\right]D^R_{\alpha\beta\gamma}$$

We considered

$$e^{i\tilde{W}[A]} = \int \mathcal{D}\psi_L\mathcal{D}\bar{\psi}_L\, e^{-i\int d^4x\,\bar{\psi}_L \not{D}\psi_L}$$

What about

$$\int \mathcal{D}A_\mu\,\mathcal{D}\omega\,\mathcal{D}\omega^* \ldots\, e^{i\int d^4x\,\left(-\frac{1}{4}\right)\mathrm{tr}\,F_{\mu\nu}F^{\mu\nu} + S_{GF} + S_{gh} + \tilde{W}[A]}$$

$\Rightarrow$ higher loops can always be regularized and are not anomalous

$\Rightarrow$ the chiral th. should not be anomalous!

We hope that anomalies cancel:

$$e^{i\tilde{W}[A]} = \int \underbrace{\mathcal{D}\psi_1\mathcal{D}\bar{\psi}_1}_{e^{i\int a_1}}\,\underbrace{\mathcal{D}\psi_2\mathcal{D}\bar{\psi}_2}_{e^{i\int a_2}}\,\underbrace{\mathcal{D}\psi_3\mathcal{D}\bar{\psi}_3}_{e^{i\int a_3}}\ldots \longrightarrow a_1+a_2+a_3=0$$

NB: $P_L \to P_R \Rightarrow \mathcal{A}^R_\alpha(x) = -\mathcal{A}^L_\alpha(x)$

Therefore I should consider:

$$\sum_{i\in L}\mathcal{A}^{L,i}_\alpha(x) + \sum_{i\in R}\mathcal{A}^{R,i}_\alpha(x) \longrightarrow \sum_{i\in L} D^{R,i}_{\alpha\beta\gamma} - \sum_{j\in L} D^{R,j}_{\alpha\beta\gamma}$$

$$\Rightarrow \mathrm{tr}_{R=\oplus_j R_j}\, t_\alpha t_\beta t_j = \sum_j \mathrm{tr}_{R_j}\, t_\alpha t_\beta t_\gamma$$

e.g.: neutrino $\to$ L $\longrightarrow$ ψ_L in rep. R^L

antineutrino $\to$ R $\longrightarrow$ $(\psi_L)^c = i\gamma^0\mathcal{C}(\psi_L)^*$

$\Downarrow$

since $\mathcal{C}(\gamma^\mu)^T = -\gamma^\mu\mathcal{C}$, $\mathcal{C}\gamma_5^T = \gamma_5\mathcal{C}$

then $\gamma_5\psi^c = i\gamma_5\gamma_0\mathcal{C}\psi^* = -i\gamma^0\mathcal{C}\gamma_5^T\psi^* = -i\gamma^0\mathcal{C}(\underbrace{\gamma_5\psi}_{=\pm\psi})^* =$

$= \mp i\gamma^0\mathcal{C}\psi^* = \mp\psi^c$

What rep ?

$$\psi \to e^{i\theta^\alpha t_\alpha^R}\psi$$

$$\psi^* \to e^{-i\theta^\alpha (t_\alpha^R)^*}\psi^* = e^{i\theta^\alpha(-t_\alpha^R)^T}\psi^*$$

The anomaly ?

$$-D_{\alpha\beta\gamma}^{R^*} = -\mathrm{tr}\left(t_\alpha^{R^*} t_\beta^{R^*} t_\gamma^{R^*}\right) = -(-)^3\,\mathrm{tr}\left((t_\alpha^R)^T (t_\beta^R)^T (t_\gamma^R)^T\right) =$$

$$= \mathrm{tr}\left(t_\gamma^R t_\beta^R t_\alpha^R\right)^T = D_{\alpha\beta\gamma}^R$$

$\Rightarrow$ L-handed part. and R-handed antiparticles contribute the same in 4d.

$\hookrightarrow$ in general $\mathcal{C}\gamma_5^T = (\;)\gamma_5\mathcal{C}$

$\hookrightarrow$ + in 4 mod 4 dimensions $\Rightarrow \psi, \psi^c$ opp. chir.

– in 2 mod 4 dimensions $\Rightarrow \psi, \psi^c$ same "

However the no. of (–) signs is s.t. there is no difference and $-D_{\alpha\beta\gamma}^{R^*} = D_{\alpha\beta\gamma}^R$

In 2 mod 8 dim we have Majorana-Weyl.

Consider $d=4$:

$\Rightarrow D_{\alpha\beta\gamma}^{R^*} = -D_{\alpha\beta\gamma}^R$

$\Rightarrow R$ is real or pseudoreal if $\exists S$ s.t. $t_\alpha^{R^*} = (-t_\alpha^R)^T = S t_\alpha^R S^{-1}$

$$\Downarrow$$

$$D_{\alpha\beta\gamma}^{R^*} = D_{\alpha\beta\gamma}^R = -D_{\alpha\beta\gamma}^R \Leftrightarrow \boxed{D_{\alpha\beta\gamma}^R = 0} \text{ for pseudoreal}$$

$$\downarrow$$

$$G = \prod_s G_s \times U(1)^n$$

$\hookrightarrow$ G_s : $SU(n)$, $SO(4n)$, $SO(4n+2)$, $SO(2n+1)$, $Sp(2n)$, G_2, F_4, E_6, E_7, E_8

$\Rightarrow$ they all have pseudoreal rep. apart from $SU(n\geq 3)$ and $U(1)$

In particular:

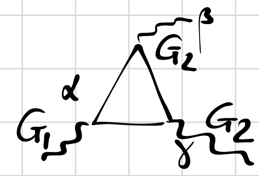
can be non irreps.

$$U(1) \to t \Rightarrow tr_R(ttt) = \sum_i q_i^3$$

$$SU(n) \to t_\alpha \Rightarrow D^R_{\alpha\beta\gamma}$$

Suppose, for instance:

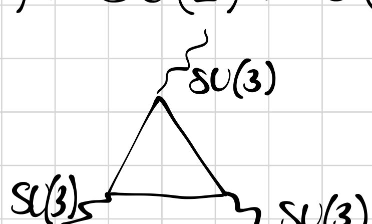
$$G_1 \times G_2 \times G_2 \longrightarrow$$



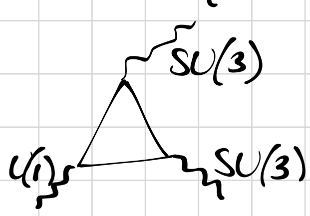
$$\Rightarrow tr_R(t_\alpha t_\beta t_\gamma) = \underbrace{tr_{R_1}(t_\alpha)}_{\neq 0 \Leftrightarrow G_1 = U(1)} \underbrace{tr_{R_2}(t_\beta t_\gamma)}_{\delta_{\beta\gamma} C_R}$$

i.e.: if one of the groups is $\neq$, it can only be $U(1)$.

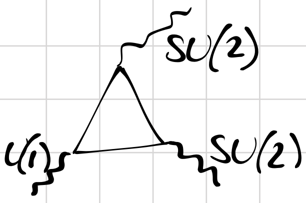
$\Rightarrow G = SU(3) \times SU(2) \times U(1)$:



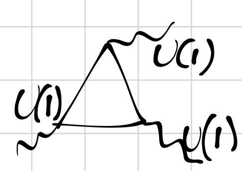
$$\longrightarrow D^{R, SU(3)}_{\alpha\beta\gamma}$$



$$\longrightarrow \sum_i q_i \; \delta_{\beta\gamma} C_R^{SU(3)}$$



$$\longrightarrow \sum_i q_i \; \delta_{\beta\gamma} C_R^{SU(2)}$$



$$\longrightarrow (\sum q_i)^3$$

Then consider: $\underbrace{e, \nu_e}_{SU(2)}$; $\underbrace{u, d}_{SU(3)}$ $\Rightarrow$

	SU(3)	SU(2)	U(1)
$\begin{pmatrix} e \\ \nu_e \end{pmatrix}_L$	1	2	$\frac{1}{2}$
$(e_R)^c$	1	1	-1
$\begin{pmatrix} u \\ d \end{pmatrix}_L$	3	2	$-\frac{1}{2}$
$(u_R)^c$	$\bar{3}$	1	$\frac{2}{3}$
$(d_R)^c$	$\bar{3}$	1	$-\frac{1}{3}$

Therefore the possible anomalies are

$$(SU(3))^3 \longrightarrow 3 + 3 + \bar{3} + \bar{3} = 0$$

$$U(1) \times (SU(2))^2$$

$$U(1) \times (SU(3))^2$$

$$(U(1))^3$$

Gravitational and Gauge anomalies

- Diff. form:

equivalence class of $\zeta_{(p)} = 0 \longrightarrow \zeta_{(p)} \sim \zeta_{(p)} + d\chi_{(p-1)}$

$\Downarrow$

p-th De Rham cohomol. $H^{(p)} \Rightarrow \dim H^{(p)} = b_p$ "BETTI NO."

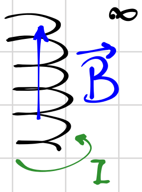
$\downarrow$

sometimes also referred to the manifold.

e.g.: $A = A_\mu dx^\mu \Rightarrow dA = \frac{1}{2} F_{\mu\nu} dx^\mu dx^\nu$

if $A = d\phi$ (pure gauge) $\Rightarrow F = dA = d^2\phi = 0$

if $F = dA = 0 \Rightarrow A$ closed $\rightarrow$ is A pure gauge ?



$\Rightarrow M = \mathbb{R}^3 \setminus$ solenoid

$\Downarrow$

$dA = 0 \rightarrow A \overset{?}{=} d\phi$

However we can compute A and it is NOT pure gauge!

$$A = \frac{B}{2\pi} d\varphi = d\left(\frac{B}{2\pi}\varphi\right)$$

$\Uparrow$

NOT WELL DEF ON M !

$\Rightarrow \boxed{b_1 = 1}$

e.g. $M = S^2$

$\Omega = \sin\theta \, d\theta \, d\varphi \rightarrow d\Omega = 0 \Rightarrow \int \Omega = 4\pi \Rightarrow \Omega$ IS NOT EXACT

it looks like $\Omega = d(-\cos\theta \, d\varphi)$ but φ is not well def. at the pole.

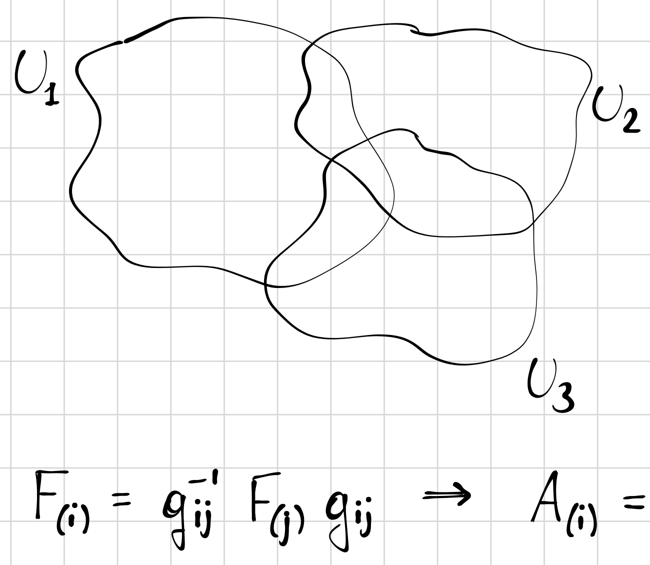
Then the expression for the anomaly:

$$\mathrm{tr}\left(\varepsilon^\alpha t_\alpha dA\, dA\right) = \mathrm{tr}\left(\varepsilon^\alpha t_\alpha\, \partial_\mu A_\nu \partial_\rho A_\sigma\, dx^\mu dx^\nu dx^\rho dx^\sigma\right) =$$
$$= \mathrm{tr}\left(\varepsilon^\alpha t_\alpha\, \partial_\mu A_\nu \partial_\rho A_\sigma\, \varepsilon^{\mu\nu\rho\sigma} \sqrt{-g}\, d^4x\right)$$
$$\hookrightarrow \int \varepsilon^\alpha\, \mathrm{tr}\left(t_\alpha\, dA\, dA\right) \quad \checkmark$$

If non abelian:

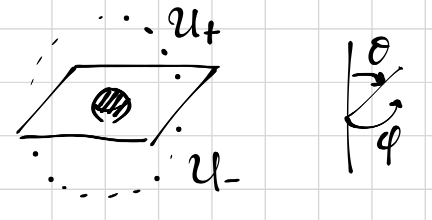
- $A = A_\mu^\alpha t_\alpha dx^\mu \Rightarrow A \wedge A = A_\mu^\alpha t_\alpha A_\nu^\beta t_\beta\, dx^\mu dx^\nu =$
  $= \frac{1}{2}\left[A_\mu, A_\nu\right] dx^\mu dx^\nu := A^2$
  $\hookrightarrow F = dA - i A^2 = \frac{1}{2}\underbrace{\left(\partial_\mu A_\nu - \partial_\nu A_\mu - i\left[A_\mu, A_\nu\right]\right)}_{F_{\mu\nu}} dx^\mu dx^\nu$
- $\delta A = d\varepsilon - i\left[A, \varepsilon\right]$ and def $D = dx^\mu D_\mu = d - iA$

Redef $A \to -iA$ to get rid of the i's. How do we def. transition functions on patches for gauge fields?



$$U_i \cap U_j \to F_{(i)} = g_{ij}^{-1} F_{(j)} g_{ij} \Rightarrow A_{(i)} = \underbrace{g_{ij}^{-1}\left(A_{(j)} + d\right) g_{ij}}_{\text{i.e.: gauge transf.}}$$

$\Rightarrow$ GAUGE BUNDLE

Now consider $M = \mathbb{R}^3 \setminus \{r \le r_0\} \Rightarrow$ 

- on $\mathcal{U}_+$: $A_+ = \gamma(1-\cos\theta)\, d\varphi \Rightarrow$ well def at $\theta = 0$
- on $\mathcal{U}_-$: $A_- = \gamma(-1-\cos\theta)\, d\varphi \Rightarrow$ well def at $\theta = \pi$

$\Rightarrow \; F_+ = F_- = \gamma \sin\theta \, d\theta \, d\varphi \;\; \checkmark \;\; \Rightarrow A_+ - A_- = i g_{+-}^{-1} d g_{+-} = 2\gamma \, d\varphi \;\Rightarrow$ well def at overlap

$\Downarrow$

$g_{+-} = e^{-2i\gamma\varphi}$

which must be single valued

for $\varphi \to \varphi + 2\pi$:

$\gamma \in \frac{\mathbb{Z}}{2}$

Now consider

$$\int_{S^2} F = \int_{S^2_+} F_+ + \int_{S^2_-} F_- = \gamma \, 4\pi = \underbrace{2\pi k}, \quad k \in \mathbb{Z}$$

MAGNETIC FLUX

(QUANTIZED) $\rightarrow$ PURELY TOPOLOGICAL

$\Downarrow$

magnetic monopole

AT THE ORIGIN!

$$\int_{S^2_+} dA_+ + \int_{S^2_-} dA_- =$$

$$= \int_{\partial S^2_+} A_+ + \int_{\partial S^2_-} A_- =$$

$$= \int_{S^1} (A_+ - A_-) = \int_{S^1} (i g_{+-}^{-1} d g_{+-})$$

the equator

$\Downarrow$

$\int_M F :=$ "CHARACTERISTIC CLASS"

(depends only on transition function)

$\downarrow$

TOPOLOGICAL

INVARIANTS

$$\int_M \mathrm{tr} \underbrace{F \wedge \ldots \wedge F}_{p \text{ times}}$$

⇒ CHARACTERISTIC CLASS

P : a local form on $\mathcal{M} \to P(F)$

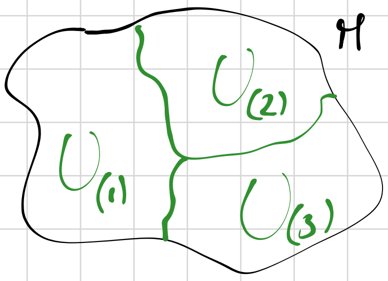
↳ $dP = 0$ but $P \neq dQ$ _globally_

↳ $P_{(i)} = dQ_{(i)}$ however on local patch $U_{(i)}$

$$\Rightarrow \int_{U_{(i)}} P_{(i)} = \int_{\partial U_{(i)}} Q_{(i)}$$

Therefore:

$$\int P = \sum_i \int_{U_{(i)}} P_{(i)} = \sum_{(i,j)} \int_{U_{(i,j)}} \left(Q_{(i)} - Q_{(j)} \right)$$



Consider:

$$P_m(F) = \mathrm{tr}\, F^m = \mathrm{tr}\, \underbrace{F \wedge \ldots \wedge F}_{m \text{ times}} \quad \text{where} \quad F \to gFg^{-1} \Rightarrow P_m \text{ is gauge invariant}$$

$$\Rightarrow dF = FA - AF \longrightarrow dP_m = m\, \mathrm{tr}\left(dF \wedge F^{m-1} \right) = m\, \mathrm{tr}\left[(FA - AF) \wedge F^{m-1} \right] = 0$$

Now take:

$$\left. \begin{aligned} A_1^{(i)} &= g_{ij} \left(A_1^{(j)} + d \right) g_{ij}^{-1} \\ A_0^{(i)} &= g_{ij} \left(A_0^{(j)} + d \right) g_{ij}^{-1} \end{aligned} \right\} \quad A_1^{(i)} - A_0^{(i)} = g_{ij} \left(A_1^{(j)} - A_0^{(j)} \right) g_{ij}^{-1}$$

$$\Downarrow$$

$$P_m(F_1) - P_m(F_0) = dR \quad [\text{only this is glob. def.}]$$

$$\Rightarrow \int \left(P_m(F_1) - P_m(F_0) \right) = \int dR = 0$$

Now define:

$$A_t = A_0 + t(A_1 - A_0), \quad t \in [0,1]$$

$$\Rightarrow F_t = dA_t + A_t^2 \Rightarrow \frac{\partial}{\partial t} F_t = D_t(A_1 - A_0)$$

$$\Rightarrow P_m(F_t) = \mathrm{tr}\, F_t^m \Rightarrow \frac{\partial}{\partial t} P_m(F_t) = m\, \mathrm{tr}\left(\frac{\partial}{\partial t} F_t \wedge F_t^{m-1} \right) = m\, \mathrm{tr}\left(D_t(A_1 - A_0) \wedge F_t^{m-1} \right)$$

since $D_t F_t = 0$ by Bianchi id.

$$= m D_t \left[\mathrm{tr}\left((A_1 - A_0) \wedge F_m^{t-1} \right) \right] = m\, d\left(\mathrm{tr}\left((A_1 - A_0) \wedge F_t \wedge \ldots \wedge F_t \right) \right).$$

↳ all the trans. funct. disappear from the trace!

Then:
$$P_m(F_1) - P_m(F_0) = d \underbrace{\int_0^1 dt\, m\, \mathrm{tr}\left[(A_1 - A_0)\wedge F_t \wedge \ldots \wedge F_t\right]}_{R} = dR$$

On $U_{(i)} \to P_m^{(i)} = d Q_{2m-1}^{(i)}$. Choose $\begin{cases} A_0 = 0 \\ A_1 = A \end{cases} U_{(i)}$

$$\Downarrow$$

$$Q_{2m-1} = m \int_0^1 dt\, \mathrm{tr}\left(A F_t^{m-1}\right)$$

and $F_t = t(dA + tA^2)$

$$Q_{2m-1} = m \int_0^1 dt\, t^{m-1}\, \mathrm{tr}\left[A\,(dA + tA^2)^{m-1}\right] \quad : \text{"CHERN-SIMONS FORMS"}$$

e.g.: $m = 2$: $\mathrm{tr}\, F^2 = dQ_3 \Rightarrow Q_3 = \mathrm{tr}\left(A\, dA + \frac{2}{3} A \wedge A \wedge A\right)$

e.g.: $Q_5 = \mathrm{tr}\left(A\, dA\, dA + \frac{3}{2} A^3 dA + \frac{3}{5} A^5\right)$

Now consider:

$$P_m = \mathrm{tr}\, F^m \to dP_m = 0 \text{ and } \delta_g P_m = 0$$

$\to$ locally $dQ_{2m-1} = P_m$

What about the gauge variation?

contains the gauge param

$$d\, \delta Q_{2m-1} = \delta(dQ_{2m-1}) = 0 \longrightarrow \text{locally } \delta Q_{2m-1} = dQ^1_{2m-2}$$

e.g.: $Q_3 = \mathrm{tr}\left(A\, dA + \frac{2}{3} A^3\right) = \mathrm{tr}\left(AF - \frac{1}{3} A^3\right)$ $\quad (-i\varepsilon = v)$ $\to$ gauge var. param.

$$\hookrightarrow \delta Q_3 = \mathrm{tr}\, dv\, dA = d(\mathrm{tr}\, v\, dA) = -d(\mathrm{tr}\, dv\, A) = dQ_2^1$$

$$\to Q_2^1 \simeq Q_2^1 + d\alpha_1^1$$

e.g.: $\Rightarrow \delta Q_5 = dQ_4^1$

$$Q_4^1 = \mathrm{tr}\left(v\, d\left(A dA + \tfrac{1}{2} A^3\right)\right) \simeq \mathrm{tr}\left[\varepsilon\, d\left(A dA + \tfrac{1}{2} A_3\right)\right] = \varepsilon^{\mu\nu\rho\sigma} \varepsilon^\alpha\, \mathrm{tr}\left(t_\alpha t_\beta t_\gamma\, \partial_\mu A_\nu^\beta\, \partial_\rho A_\sigma^\gamma\right) + \mathcal{O}(A^3)$$

THE ANOMALY

Therefore:

$$\left.\begin{aligned} &P_m = \mathrm{tr}\, F^m ; \qquad dP_m = 0 \text{ and } \delta P_m = 0 \\ &dQ_{2m-1} = P_m \\ &\delta Q_{2m-1} = dQ'_{2m-2} \end{aligned}\right\} \rightarrow \text{"DESCENT EQUATIONS"}$$

NB: $Q_{2m-1} \simeq Q_{2m-1} + d\gamma_{2m-2}$

$Q'_{2m-2} \simeq Q'_{2m-2} + d\beta'_{2m-3} + \delta\gamma_{2m-2}$ → equivalence relations!

↳ we can always try to add sth like this to cancel the anom.

⇒ WESS-ZUMINO CONDITIONS

Gauge variation: $\varepsilon^\alpha(x)$ parameter ⟶ assume $\varepsilon^\alpha(x) \neq 0$ only on a single coord. patch

$$\Rightarrow \delta_\varepsilon \Gamma[A] = \int d^4x\, \varepsilon^\alpha(x) \mathcal{A}_\alpha(x) = -\int d^4x\, \varepsilon^\alpha(x) \left(D_\mu \frac{\delta\Gamma}{\delta A_\mu} \right)_\alpha$$

$$\Leftrightarrow \mathcal{A}_\alpha(x) = -\left(D_\mu \frac{\delta\Gamma}{\delta A_\mu}\right)_\alpha$$

→ D_μ of the quantum current

→ Define $G_\alpha(x) = -\left(D_\mu \frac{\delta}{\delta A_\mu}\right)_\alpha$ s.t. $\mathcal{A}_\alpha(x) = G_\alpha(x)\Gamma[A]$

↳ $[iG_\alpha(x), iG_\beta(y)] = \delta^4(x-y)\, iC_{\alpha\beta\gamma}\, [iG^\gamma(x)]$ "local version of Lie alg." (affine Lie alg.?)

$$\Rightarrow \boxed{G_\alpha(x)\mathcal{A}_\beta(x) - G_\beta(x)\mathcal{A}_\alpha(x) = \delta^4(x-y)\, C_{\alpha\beta\gamma}\, \mathcal{A}^\gamma(x)}$$

↳ WZ CONSIST. CONDITION ⇒ we can det. the form of $\mathcal{A}_\alpha(x)$ [not the coeff.]

BRST TRANSFORMATION

Define s s.t.

$$\delta A^\alpha_\mu = \partial_\mu \varepsilon^\alpha + C^{\alpha\beta\gamma} A_{\mu\beta}\varepsilon_\gamma$$

GHOSTS $\qquad \omega = \omega^\alpha t_\alpha$

$$s A^\alpha_\mu = \partial_\mu \omega^\alpha + C^{\alpha\beta\gamma} A_{\mu\beta}\omega_\gamma \Rightarrow sA_\mu = \partial_\mu \omega - i\left[A_\mu, \omega\right]$$

$$\Rightarrow sA = d\omega - i\left\{A, \omega\right\}$$

$$\omega\, dx^\mu = -dx^\mu\, \omega$$

Then:

$$s\psi = i\omega\psi$$

$$s\psi^\dagger = \pm i\psi^\dagger\omega$$

NOT ω^* !

$$s\omega = i\omega\omega = i\,\omega^\alpha t_\alpha \omega^\beta t_\beta = \omega^\alpha\omega^\beta \frac{i}{2}\left[t_\alpha, t_\beta\right] = -C_{\alpha\beta}{}^{\gamma}\omega^\alpha\omega^\beta t_\gamma$$

$$s\omega^* = -h$$

$$sh = 0$$

$\boxed{\text{NB: } s^2 = 0}$

Then:

$$\delta_\varepsilon\Gamma[A] = \int \varepsilon^\alpha \mathcal{A}_\alpha \longrightarrow s\Gamma[A] = \int \omega^\alpha \mathcal{A}_\alpha = \mathcal{A}[\omega, A]$$

$$\Rightarrow s\mathcal{A}[\omega, A] = 0 \Rightarrow \text{WZ CONSISTENCY CONDITION}$$

$$\Rightarrow s\mathcal{A}[\omega, A] = s\int \omega^\alpha \mathcal{A}_\alpha = \int \underbrace{(s\omega^\alpha)}_{\frac{1}{2}C^{\alpha\beta\gamma}\omega_\beta\omega_\gamma} \mathcal{A}_\alpha - \int \omega^\alpha \underbrace{(s\mathcal{A}_\alpha)}_{\omega^\beta G_\beta \mathcal{A}_\alpha} = \frac{1}{2}\int \omega_\alpha\omega_\beta\left[C^{\alpha\beta\gamma}\mathcal{A}_\gamma + G^\beta \mathcal{A}^\alpha - G^\alpha \mathcal{A}^\beta\right]$$

Since $s^2 = 0 \longrightarrow$ BRST COHOMOLOGY:

$sF = 0 \longrightarrow F$ is BRST closed

$F = sH \longrightarrow$ " exact

$$\Rightarrow sF = 0 \Rightarrow F \simeq F + sH \Rightarrow \mathcal{A}[\omega, A] \rightarrow \mathcal{A}[\omega, A] + sF$$

Relevant anomaly is given by BRST cohomology class AT GHOST NO. 1.

all the possible 4-forms

We already had:

$$\mathcal{A}[\omega, A] = \frac{i}{24\pi^2} \int \mathrm{tr}\left[\omega\left(dA\,dA + b_1 A^2 dA + b_2\, A\,dA\,A + b_3\, dA\,A^2 + c A^4\right)\right]$$

$\Rightarrow s\left[\mathrm{tr}\, \omega A^4\right] = \mathrm{tr}\left[\omega^2 A^4\right] + O(\omega d\omega) \Rightarrow \boxed{c = 0}$ in order to have $s\mathcal{A} = 0$.

$\Rightarrow$ similarly:

$$b_1 = -b_2 = b_3 = \frac{1}{2}$$

$$\Rightarrow \mathcal{A}[\omega, A] = \frac{i}{24\pi^2} \int \underbrace{\mathrm{tr}\left\{\omega\left(d(A\,dA) + \frac{1}{2} A^3\right)\right\}}_{Q_4^{1}}$$

$Q_4^{1} \longrightarrow$ GHOST NO. 1

$$\Rightarrow dQ_4^1 = \delta Q_5 \quad \text{and} \quad dQ_5 = \mathrm{tr}(F \wedge F \wedge F)$$

This is a formal derivation of the 4-form in 4-dim. spacetime which comes from a 6-dim. auxiliary spacetime and its 6-form dQ_5 [e.g. consider $\mathcal{M}_4 \times S^2 \simeq \mathcal{M}_6$].

$\Rightarrow$ in general $\quad \mathcal{M}_{2r} \times S^2 \simeq \mathcal{M}_{2r+2}$

$$P_{r+1} = \mathrm{tr}\left(F^{r+1}\right) \longrightarrow Q_{2r+1} \longrightarrow Q_{2r}^1$$

$$\Downarrow$$

$$s\int_{\mathcal{M}_{2r}} Q_{2r}^1 = 0 \quad \text{and} \quad \int_{\mathcal{M}_{2r}} Q_{2r}^1 \text{ IS NOT exact!}$$

Consider

$$0 = s\left(s Q_{2r+1}\right) = s\left(dQ_{2r}^1\right) = -d\underbrace{\left(s Q_{2r}^1\right)}_{\text{CLOSED}}$$

$$\Downarrow$$

on the patch it's exact: $sQ_{2r}^1 = d\alpha_{2r-1}^2$

$$\Rightarrow s\int Q_{2r}^1 = \int d\alpha_{2r-1}^2 = 0$$

But also $\int Q_{2r}^1 \neq s(\ )$.

Remarks:

$$\text{anomaly} \to \underbrace{c}_{\text{this has to be computed}} \int Q^1_{2r}$$

For a set of fields $S = \{\psi_i\} \to c_i$ for each of them:

$$\sum_{\psi_i \in S} c_i \int Q^1_{2r} \longleftrightarrow \underbrace{\sum_{\psi_i \in S} c_i P_{r+1}}$$

$$\downarrow$$

$$I^{\text{total}}_{2r+2} = \begin{cases} 0 \\ \text{non-zero} \end{cases}$$

- abelian anomaly: $\int \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} = \int \mathrm{tr}\, F \wedge F \longleftrightarrow \text{index } \not{D}_{M_4}$
- gauge anomaly: $\int \mathrm{tr}\, F \wedge F \wedge F \longrightarrow d+2$ form (e.g.: $4+2$ in $d=4$)

$\hookrightarrow$ index $\not{D}_{M_{d+2}}$? Yes, but must define $\not{D}_{M_{d+2}}$ first!

$$\Rightarrow \text{index } \not{D}_{M_{d+2}} = C_r \,\mathrm{tr}\, F^{r+1}$$

$$\Downarrow$$

we get even the normal. !!!

EUCLIDEAN SPACETIME

$$\boxed{x^0_M = -i x^0_E} \quad \boxed{S_M = i S_E}$$

$$\Rightarrow S_M = \int d^d x_M \sqrt{-g} \left(-\frac{\alpha}{2p!} \zeta^{\mu_1 \dots \mu_p} \zeta_{\mu_1 \dots \mu_p} \right.$$

$$S_E = \int d^d x_E \sqrt{-g_E} \left(\frac{\alpha}{2p!} \zeta_E^{\mu_1 \dots \mu_p} \zeta_{E\, \mu_1 \dots \mu_p} \right.$$

$$\to \zeta^M_{2r} = \zeta^E_{2r} \text{ after Wick rot.}$$

NB $\zeta_{2r} = \frac{1}{2r!} \zeta_{\mu_1 \dots \mu_{2r}} \underbrace{dx^{\mu_1} \dots dx^{\mu_{2r}}}_{\varepsilon^{\mu_1 \dots \mu_{2r}} \sqrt{-g}\, d^{2r}x}$

We then need to def: $\left| \mathrm{Det}\, \not{D} P_L \right| = \sqrt{\underbrace{(\mathrm{Det}(\not{D} P_L)^\dagger)(\mathrm{Det}(\not{D} P_L))}_{\geq 0 \text{ if no } 0\text{-modes}}} = e^{-\Gamma[A]}$

$$\Rightarrow \delta\Gamma_E = -i\int_{M_E^{2r}} I_{2r}^1 \longleftrightarrow \delta\Gamma_M = \int_{M_M^{2r}} I_{2r}^1$$

Define $\hat{D} = \not{D}_+ + \not{\partial}_-$ as before $\Rightarrow \mathrm{Det}(i\hat{D}) = e^{-\tilde{W}[A]} = e^{-\Gamma[A]}$

$$\Rightarrow i\hat{D} = \begin{pmatrix} & i\not{D}_+ \\ i\not{\partial}_- & \end{pmatrix} \longrightarrow (i\hat{D})^\dagger (i\hat{D}) = \begin{pmatrix} i\not{\partial}_+ i\not{\partial}_- & \\ & i\not{D}_- i\not{D}_+ \end{pmatrix}$$

$$\Rightarrow \mathrm{Det}\,(i\hat{D})^\dagger (i\hat{D}) = \mathrm{const} \times \mathrm{Det}\, i\not{D}_- i\not{D}_+ = \mathrm{const} \times \begin{pmatrix} & i\not{D}_+ \\ i\not{D}_- & \end{pmatrix} = \mathrm{const} \times \mathrm{Det}(i\not{D})$$

Therefore: $e^{-\Gamma[A]} = (\mathrm{Det}\,(i\not{D}))^{\frac{1}{2}}\, e^{i\phi[A]}$.

Now suppose to have a gauge transform: $g(x,\theta)$, $\theta \in [0, 2\pi]$

$$\Rightarrow A^\theta(x) := A(x,\theta) = g(x,\theta)^{-1}(d + A(x))\,g(x,\theta) \text{ s.t. } g(x,2\pi) = g(x,0) = 1.$$

$$\hookrightarrow A(x,2\pi) = A(x,0) = A(x).$$

$$\Rightarrow \phi[A,2\pi] - \phi[A,0] = 2\pi m, \quad m \in \mathbb{Z}$$

$$\hookrightarrow \int_0^{2\pi} d\theta\, \frac{\partial\phi[A,\theta]}{\partial\theta} = 2\pi m \text{ by def.!}$$

THE ANOMALY!

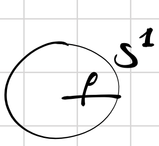
$$\Longrightarrow i\frac{\partial\phi}{\partial\theta} = -\frac{\delta\Gamma[A^\theta]}{\delta\theta} = -\int d^dx\,\sqrt{-g}\,\underbrace{\frac{\partial A^\theta_{\mu\alpha}(x)}{\partial\theta}}_{= D^\theta v}\,\frac{\delta\Gamma[A^\theta]}{\delta A^\theta_{\mu\alpha}(x)} \overset{\text{IBP}}{=} \int d^dx\,\sqrt{-g}\; v\Big(D^\theta \frac{\delta\Gamma}{\delta A^\theta}\Big)$$

$$\longrightarrow v = g^{-1}\frac{\partial}{\partial\theta}g.$$

We relate $m = \frac{1}{2\pi}\int_0^{2\pi} d\theta\,\frac{\partial\phi}{\partial\theta} = \frac{1}{2\pi i}\int_0^{2\pi} d\theta \int d^dx\,\sqrt{-g}\; v^{\theta\alpha}(x)\Big(D^\theta \frac{\delta\Gamma[A^\theta]}{\delta A^\theta_{\mu\alpha}(x)}\Big)$.

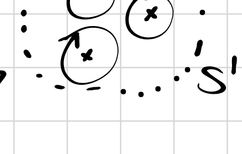
$\Rightarrow$ What's m?

$\theta \in S^1 \Rightarrow$ we are considering $M_d \times S^1$



$\Downarrow$

m is $\sim$ to a winding no. around this S^1.



$\Rightarrow$ we deform S^1 to circle around only the 0-modes $\Rightarrow$ m counts + right-chirality 0-modes $-$ left-chir. 0-modes $\Longrightarrow$ $m \longleftrightarrow \mathrm{index}\,(i\not{D}_{M_d \times S'})$

Then $\text{index}\left(i\not{D}_{2r+2}(A)\right) = \frac{(-i)^{r+1}}{(r+1)!(2\pi)^{r+1}} \int_{M_{2r}\times S^2} \operatorname{tr} F^{r+1} \xleftrightarrow{\text{compare}} \# \int_0^{2\pi} d\theta \int_{M_{2r}} Q^1_{2r}(v^\theta, A, F)$

$\Rightarrow$ the anomaly is related to $\text{index}(i\not{D})$!

In detail: $\delta\Gamma_E[A] = \frac{-i(-)^{r+1}}{(r+1)!(2\pi)^r} \int_{M_{2r}} Q^1_{2r}(\mathcal{E}, A, F) \Rightarrow$ we get even the coefficient !

Now we can go back to Minkowski space:

$$\delta\Gamma_m[A] = \frac{(-)^{r+1}}{(r+1)!(2\pi)^r} \int_{M_{2r}} Q^1_{2r}(\mathcal{E}, A, F) = \int_{M_{2r}} \hat{I}^1_{2r}$$

$$\rightarrow d\hat{I}^1_{2r} = \delta\hat{I}_{2r+1} \longrightarrow d\hat{I}_{2r+1} = \hat{I}_{2r+2} = \frac{(-i)^{r+1}}{(r+1)!(2\pi)^r} \operatorname{tr} F^{r+1} = 2\pi \times \text{indexdensity}$$

→ GRAVITATIONAL ANOMALIES

→ introduce $g_{\mu\nu}(x) = e^a_\mu(x) e^b_\nu(x) \eta_{ab} \Rightarrow$ transf. tensors $\Sigma^{ab}{}_c = e^a_\mu e^b_\nu E^\rho{}_c \Sigma^{\mu\nu}{}_\rho$

$$e^a_\sigma E^\sigma_b = \delta^a_b$$
$$e^a_\sigma E^\mu{}_a = \delta_\sigma^\mu$$

→ Define: $D = d + [\omega, \cdot\,] \Rightarrow (D\Sigma)^{ab}{}_c = d\Sigma^{ab}{}_c + \omega^a{}_{a'}\Sigma^{a'b}{}_c + \omega^b{}_{b'}\Sigma^{ab'}{}_c - (-)^p \Sigma^{ab}{}_{c'}\omega^{c'}{}_c$

SPIN CONNECTION

$$\nabla = d + [\Gamma, \cdot\,] \Rightarrow (\nabla\Sigma)^{\mu\nu}{}_\rho = d\Sigma^{\mu\nu}{}_\rho + \Gamma^\mu{}_{\mu'}\Sigma^{\mu'\nu}{}_\rho + \Gamma^\nu{}_{\nu'}\Sigma^{\mu\nu'}{}_\rho - (-)^p\Sigma^{\mu\nu}{}_{\rho'}\Gamma^{\rho'}{}_\rho$$

COMPATIBILITY: $de^a_\mu + \omega^a{}_b e^b_\mu - e^a_\rho \Gamma^\rho{}_\mu = 0 \Rightarrow de^a + \omega^a{}_b e^b = 0$

NB: at every point we can perform local Lorentz transf.: e