Classical Effective Field Theories

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I. EFFECTIVE THEORY FOR CHARGED SPHERES

While effective field theories were first developed for quantum field theories, the basic ideas which underly them can be illustrated in a classical context. Moreover, even in classical settings EFT's provide a sharp calculational tool. In the next chapter we will introduce an EFT for gravitational interactions that has use in predicting gravity wave signatures for inspiralling binaries, but here we will start in the simpler setting of electrodynamics. An attempt has been made to make this chapter self contained. In addition, an effort has been made to elucidate some rudimentary points that may not be familiar to those with less exposure to quantum field theory. However, it will be assumed that the reader has some familiarity with the notion of the path integral as well as a basic understanding of the semi-classical treatment of radiation at the level of an advanced undergraduate text in quantum mechanics. For those who are not familiar with Feynman diagrams, an appendix at the end of the chapter has been included to provide the necessary background.

Let us consider the following problem. Suppose we have two spherical shells each of which carry a net charge. The charges on the sphere are dynamical, and thus the spheres have both electric and magnetic polarizabilities. We wish to calculate the positions of the spheres as a function of time given some initial data. This is a complicated problem to say the least. When the spheres are in motion, all sorts of effects need to be accounted for in order to track their phase space trajectories. First off, as they accelerate they will radiate, which will dampen the motion. Furthermore, the charges on the sphere will shift their orientation depending upon the position of the other sphere, which will change the nature of the interaction. In general there is no possible way to solve this problem analytically. One might hope to put it on a computer and solve the set of coupled differential equations, but this is not a task that one would ask any friend to tackle. However, we might hope to get an analytic handle on the problem in certain limits. In particular, when the spheres are far apart, compared to their radii, it makes sense to approximate each sphere as a point particle. That is we will integrate out the modes which are responsible for the internal dynamics of the sphere, in the spirit of the example in the introductory chapter. This approximation will clearly simplify the problem. However, we would like to be able to include finite size effects in a systematic fashion, and this is where the power of effective field theory becomes evident.

Before continuing, it is important to clarify some terminology. Sometimes in the literature one sees theories in which the point particle approximation has been used, referred to as effective field theories. This definition is much too broad to be useful, since by this account an EFT would amount to no more than the multipole expansion. The modern use of the term EFT refers to multi-scale theories in which the scales have been separated at the level of the action and each term in the action scales homogeneously in the expansion parameter of interest which is typically a ratio of scales.

In any case, for a classical EFT the starting point is the point particle approximation [1]. The action for a collection of point particles is given by

$$S = \sum_{i=1,2} \int d\lambda_i \left(m_i \sqrt{\frac{dx_i^\mu}{d\lambda_i} \frac{dx_{i\mu}}{d\lambda_i}} + e_i \frac{dx_i^\mu}{d\lambda_i} A_\mu(x_i(\lambda_i)) \right) - \int d^4x \frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \tag{1}$$

e_i is the net charge of the i'th particle and λ_i parameterize the world lines. We will separate the space into the *bulk* and the worldline so that the photons kinetic term lives in the bulk.

Now how do we account for the finite size of the spheres? We know were missing out on physics so we have to add *something* to this action to account for it. But that something, can't just be anything. It must respect the symmetries of the theory. In this case at hand those symmetries are gauge [1] and Lorentz invariance, as well as world-line reparameterization invariance (RPI)[2]. We know that in the bulk, away from the sources, we have nothing but Maxwells' electrodynamics[3] , so we can not mess with the bulk action. But on the worldline we are free to add operators [4]. Furthermore, it is clear that as the engineering dimensions of the new terms grow, their effect must diminish as powers of R_i/κ where κ is some external scale (say the distance between the spheres) and R_i is the radius of the i'th sphere.

We may form Lorentz invariants out of the vector potential A_μ, as well as the four velocity of the world lines, v^μ. We could also use higher time derivatives of the worldline x^μ. Of the

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[1] Terms need only be gauge invariant up to a total derivative, as is the case for the charge coupling to the gauge field.

[2] Meaning that the action should not depend upon how one wishes to parameterize the path.

[3] Readers who have gone through the earlier chapters will note that strictly speaking quantum effects will generate higher dimensional non-linear terms arising from integrating out heavy charged particles, but the effects of such terms will be irrelevant at long distances.

[4] The term "operators" is used despite the fact that we are considering a classical theory.

leading terms, those of lowest dimension will be [5]

$$S_{FS} = \int d\lambda \left(\frac{C_1}{\sqrt{v^2}} v^\mu F_{\mu\nu} v_\alpha F^{\alpha\nu} + C_2 \sqrt{v^2} F_{\mu\nu} F^{\mu\nu}\right) \tag{2}$$

C_1 and C_2 are coefficients which we will fix later in the chapter. The factors of $\sqrt{v^2}$ are forced upon us by RPI, where $v = dx/d\lambda$. On dimensional grounds, the coefficients of both of these operators must scale as R^3. The attentive reader may ask why the operator

$$O_a = \frac{v_\mu a_\nu}{v^2} F^{\mu\nu} \tag{3}$$

has not been included, since it would scale as R^2 and therefore would be the true leading order effect. This operator would indeed be there if the particles were being acted upon by an external force, or if the interactions between the spheres were sufficient to generate an appreciable acceleration. However, save for a short diversion, this term will not play a role in the remainder of this chapter[6].

Returning now to our leading order finite corrections (2), note that these coefficients encode "short distance" information, which are responsible for the polarizability. That is, when we put the spheres in an external field they will deform. The polarization could be due to structural changes, i.e. the sphere itself may deform if the charges are constrained, or due to the charge flow on the surfaces. The polarization can be a consequence of quantum or classical effects, it does not matter. In either case there will be a net polarization, the strength of which will be recorded in $C_{1,2}$. We will calculate these coefficients later in this chapter.

Note that in doing things this way we have assumed that the time scale for the deformation is short in comparison to the "external" time scales in the problem, such as the period of the orbits. In the absence of such a hierarchy, i.e. if there existed zero modes with vanishing frequency we would need to add additional degrees of freedom to the worldline and track their time dependence. We will address this issue of additional degrees of freedom below.

Now that we have the action in the point particle approximation, including the effects of finite size to leading order in the radii, we would like to use this action to calculate the

[5] This is perhaps the shortest path to the answer to the "why is the sky blue?" question. The symmetries of the theory imply the first interactions for neutral particles start at two derivatives, and thus the scattering cross section grows with frequency. Thus the sharpest answer to the childs question is: "the sky is blue because symmetries allow nothing else".

[6] One could also consider terms which do not involve the gauge field but will change the equations of motion of the spheres. See [7] and ([6]) for a discussion.

dynamics of the system. Notice that, at this order, the system is Gaussian, and thus completely solvable since the path integral can be done exactly (more on this below). However, there is really no point finding an exact solution because it would in no way improve our systematics. Such a solution would re-sum all the finite size effects due to the first correction in R, but given that we have truncated our Lagrangian, dropping higher order terms in R, we can not claim to be calculating at any accuracy beyond R^3.

We will now use the path integral formalism [7], developed for quantum field theories, to determine the dynamics of this classical system. This seems like using a rather large hammer for a very small nail. However, we will see that the formalism will pay dividends.

The first object we will calculate is the so-called "vacuum-persistence" amplitude which is defined by

$$Z[x,\dot{x}] = \frac{\langle 0, -\infty \mid 0, +\infty \rangle_{x,\dot{x}}}{\langle 0, -\infty \mid 0, +\infty \rangle} = \frac{\int DA_\mu e^{iS(A,x,\dot{x})}}{\int DA_\mu e^{iS(A)}} \tag{4}$$

where vacuum boundary conditions are applied to both ends of the path integral. Roughly speaking, Z is the probability amplitude to go from the vacuum at past infinity to the vacuum at future infinity in the presence of the sources x_i.

Let us now see what we can learn from this amplitude. We can write the amplitude as

$$Z[x,\dot{x}] = e^{-i\int dt E(x,\dot{x})} \tag{5}$$

where $E(x,\dot{x})$ is the energy in the presence of the source which need not be constant over the worldline (e.g. there can be acceleration). This result follows from the basic quantum mechanical fact that for a stationary state with energy E_0 (let's assume it's the ground state), the time evolution of the system is trivial. As time evolves the state just picks up a phase factor e^{-iE_0T} during a time interval T. Suppose we now perturb the system adiabatically with a source J (in our case the x_i). The state of the system will remain in the ground state, only now the energy level has been shifted. The net phase which is accrued over the path will be the integral over the energy. If we consider sources which are not static, then, in general, we should find that the probability to arrive back at the ground state at plus

[7] Readers not familiar with the path integral, or for those who are just rusty on the subject can consult [16].

infinity will no longer be one, since we expect there to be radiation.

We can understand this better by considering the example at hand, namely, sources coupled to the electromagnetic field whose action is given by

$$S = \int d^4x \left(-\frac{1}{4} F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\partial_\mu A^\mu)^2 + J_\mu A^\mu, \right) \tag{6}$$

where the second term in the action restricts the field to live in Feynman gauge[8]. This theory is Gaussian, and thus we can do the path integral exactly, leaving for the persistence amplitude

$$\boxed{Z[J] = e^{-\frac{i}{2}\int d^4x d^4y J_\mu(x) G_F^{\mu\nu}(x-y) J_\nu(y)} \equiv e^{-iW[J]}} \tag{7}$$

For a particle traversing a worldline $x^\mu(\tau)$ the form of the current is

$$J^\mu(x) = e \int d\tau v^\mu(\tau) \delta^{(4)}(x - x(\tau)). \tag{8}$$

G_F is the "Feynman" propagator which will be defined below. Intuitively we can see that the exponent looks like an energy. The convolution of the Greens function with one of the sources gives the value for the field at the position of the other source. However, this is not quite correct, because the Greens function is the acausal[9], complex, Feynman prescription Greens function, not the causal retarded Greens function we would need to calculate the value of the field away from the sources. Notice that we had no choice in our prescription for the Greens function. The path integral tells us we must use Feynman. A simple way of reaching this conclusion is to first note that the Minkowskian path integral does not converge, as it is highly oscillatory. By discretization, the path integral can be written as a product of Gaussians (see Peskin), each of which is oscillatory (recall that in the amplitude the action gets multiplied by a factor of i). We may restore convergence by adding a damping term to the Lagrangian [10].

$$\Delta L = -\frac{i}{2}\epsilon A^2 \quad \epsilon > 0. \tag{9}$$

[8] We will restrict ourselves to this gauge unless otherwise stated

[9] The Feynman prescription Greens function has support outside the light-cone and is inherently quantum mechanical.

[10] Note the sign is due to our choice of metric convention $(+,-,-,-)$. The time like photon polarization would appear to have the wrong sign for the convergence factor. However, since they decouple as a consequence of current conservation, they contribute an overall factor which cancels with the denominator in (4).

In momentum space we may write the action as

$$S = \frac{i}{2}\int [d^4k]\left(\tilde{A}_\mu(k)(k^2+i\epsilon)\tilde{A}^\mu(-k) + J_\mu(-k)\tilde{A}^\mu(k)\right) \tag{10}$$

The propagator may be read off by inverting the kernel of the quadratic term in the gauge field which automatically gives rise to the Feynman prescription (Feynman gauge) result

$$\tilde{G}_F^{\mu\nu}(k) = \frac{-g^{\mu\nu}}{k^2+i\epsilon}. \tag{11}$$

In coordinate space we have [11]

$$G_F^{\mu\nu}(x) = \frac{g_{\mu\nu}}{4\pi^2}\frac{-i}{x^2-i\epsilon}. \tag{12}$$

which has support outside the light-cone and is complex. On the other hand, the "retarded" propagator

$$G_R^{\mu\nu}(x) = -g^{\mu\nu}\int \frac{[d^4k]e^{-ik\cdot x}}{((k_0+i\epsilon)^2-\vec{k}^2)} = \frac{g^{\mu\nu}}{2\pi}\theta(x_0)\delta(x^2) \tag{13}$$

is causal having support only on the light cone.

Thus given that we must use the Feynman prescription, we can expect that our energy functional will be complex. But for accelerating sources that is exactly what we would expect since the system wants to "decay" by radiating photons[12]. We can see this more clearly by writing $W[J]$ in momentum space

$$W[J] = -\frac{1}{2}\int \frac{[d^4k]}{k^2+i\epsilon}\tilde{J}(-k)\cdot\tilde{J}(k). \tag{14}$$

We can see that the only imaginary part comes from the pole and is proptional to $\delta(k^2)$, which corresponds to a real propagating photon, i.e. radiation. Moreover, given the signs, we can see that that radiation will only decrease the vacuum persistence amplitude. If the sources are static then $\tilde{J}(k) \sim \delta(k_0)$ and the pole is not supported. In this case $W[J]$ is

[11] There is a distinction between the Greens function and the two point function, $\langle T(\phi(x)\phi(0))\rangle = iG_F$. The two point function arises when performing Wick contractions and thus is what is relevant for Feynman rules.

[12] Readers who have considered the chapter on SCET, can appreciate that the kink in the wilson lines means that the Sudakov form factor will not be a pure phase.

purely real, and corresponds to the Coulomb energy. More generally, we have

$$ImW[J] = \frac{1}{2}\int \frac{[d^3k]}{2E}\tilde{J}(-k)\cdot\tilde{J}(k), \tag{15}$$

where $k_0 =\mid \vec{k} \mid$ is implied. Given current conservation $k_\mu J^\mu(k) = 0$ we can re-write the result as

$$ImW[J] = -\frac{1}{2}\int \frac{[d^3k]}{2E}(\mid \tilde{J}^T(k) \mid^2) \tag{16}$$

where we have decomposed the current into transverse and longitudinal pieces $\tilde{J}_\mu = \tilde{J}^T_\mu + k_\mu \tilde{J}^L$.

The probability of producing no photons is given by the modulus squared of the vacuum to vacuum transition amplitude (7) is given by

$$P_0 = e^{-\int \frac{d^3k}{2E}(|\tilde{J}^T(k)|^2)}. \tag{17}$$

What is the proper way to interpret the exponent? It is simple to show that it gives the average number of photons emitted, i.e. To see this consider the probability to emit n photons [13], which follows from squaring the transition amplitude and integrating over phase space of the final state photons,

$$P_n = \frac{1}{n!}\Pi^n_{i=1}\int \frac{[d^3k_i]}{2E_i} \mid \langle k_1...k_n \mid e^{-i\int d^4x A_\mu(x)J^\mu(x)} \mid 0\rangle \mid^2 . \tag{18}$$

The factor of $n!$ accounts for the fact that we have identical bosons in the final state. Now decompose A_μ into its creation (A^+) and annihilation (A^-) pieces as is done in standard semi-classical calculations. Then using the result

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]} \tag{19}$$

which holds when $[A,B]$ is a c-number, we can write

$$e^{-i\int d^4x A_\mu(x)J^\mu(x)} = e^{-i\int d^4x A^+_\mu(x)J^\mu(x)}e^{-i\int d^4x A^-_\mu(x)J^\mu(x)}e^{\frac{1}{2}\int \frac{[d^3k]}{2E}\tilde{J}(k)\cdot\tilde{J}(-k)} \tag{20}$$

[13] We will sum over polarizations. This sum is implied in all the following formulae.

where use was made of the result [14]

$$[A_\mu^+(x), A_\nu^-(y)] = g_{\mu\nu} \int [d^4k] e^{-ik\cdot(x-y)} \theta(k_0)\delta(k^2). \tag{21}$$

We can see immediately that our result for the vacuum persistence amplitude is consistent with P_0 since $A^- \mid 0\rangle = \langle 0 \mid A^+ = 0$ by defintion.

Now let's consider the amplitude

$$A_n = \langle k_1 ... k_n \mid e^{-i\int d^4x A_\mu(x) J^\mu(x)} \mid 0\rangle. \tag{22}$$

For A_n we may ignore A^- since the exponent will vanish when operating on the vacuum. Leaving

$$\int d^4x A^+ \cdot J = \int \frac{[d^3k]}{2E} a_k^\dagger \epsilon^\mu(k) \tilde{J}_\mu(k) \tag{23}$$

where ϵ is the photon polarization. Note that the conserved current picks out the physical transverse polarizations. Expanding the exponential we find

$$A_n = (-i)^n \epsilon_1 \cdot \tilde{J}(k_1) .... \epsilon_n \cdot \tilde{J}(k_n) \tag{24}$$

where the $1/n!$ cancelled since there are $n!$ identical terms.

Squaring and summing over polarizations, gives

$$P_n = \frac{1}{n!} \left(\int \frac{[d^3k]}{2E} \tilde{J}(-k) \cdot \tilde{J}(k) \right)^n e^{-\int \frac{[d^3k]}{2E} \tilde{J}(k)\cdot\tilde{J}(-k)}. \tag{25}$$

We see that we may write the probability as a Poisson distribution with

$$P_n = e^{-\langle n\rangle} \frac{\langle n\rangle^n}{n!}, \tag{26}$$

if we interpret (17) as $\langle n\rangle$. With this knowledge in hand, we are prepared to calculate the power loss in the system. We first note that we may make the following identification

$$Re \ln(Z[J]) = -\frac{\Gamma T}{2} \tag{27}$$

[14] This result follows from using the field normalization $A_\mu(x) = \int \frac{[d^3k]}{2E_k} (\epsilon_\mu e^{-ik\cdot x} a_k + \epsilon_\mu^\star e^{ik\cdot x} a_k^\dagger)$ and the canonical commutation relations.

where Γ is the width of the state, so that the decay rate is Γ^{-1} and T is the observation time. Heuristically we might have guessed this result just from our intuition for unstable states in quantum mechanics. Thus to get the power loss all we have to do is to weight the phase space measure with the energy

$$P = \frac{1}{2T}\int [d^3k] \mid \tilde{J}(k) \mid^2 . \tag{28}$$

It is important to note that in general if we wish to measure time dependent quantities, then the traditional path integral formulation where we are only concerned about the states at plus and minus temporal infinity will not be applicable. As, will be discussed later in this chapter, when we wish to measure quantities that are local in time we will need to so-called "in-in" formalism [8](as opposed to the "in-out" formalism utilized above). However, for time averaged quantities such as the power loss we see that the "in-out" formalism is sufficient.

A. Some Simple Examples

So far this discussion has been formal, so let us now consider some explicit examples. Take two static sources, with charges (e_1, e_2) separated by $\vec{R}$. Their world lines are parameterized as

$$x_1^\mu = (t, \vec{0}) \quad ; \quad x_2^\mu = (t', \vec{R}) \tag{29}$$

so that

$$\tilde{J}_\mu(k) = (2\pi)e_1 v_{1\mu}\delta(k_0) + (2\pi)e_2 v_{2\mu}\delta(k_0)e^{-i\vec{k}\cdot\vec{R}}. \tag{30}$$

Then we have

$$\begin{aligned} W[J] &= \frac{1}{2}\int [d^4k]\tilde{J}_\mu(k)G_F^{\mu\nu}(k)\tilde{J}_\nu(-k) \\ &= \frac{(2\pi)\delta(0)}{2}\int \frac{[d^3k]}{\vec{k}^2}\left(e_1 v_1^\mu + e_2 v_2^\mu e^{-i\vec{k}\cdot\vec{R}}\right)\left(e_1 v_{1\mu} + e_2 v_{2\mu} e^{i\vec{k}\cdot\vec{R}}\right). \end{aligned} \tag{31}$$

The terms proportional to e_i^2 are self energy contributions, independent of R, which contribute an overall unphysical phase to Z. This divergent contribution to the energy can be

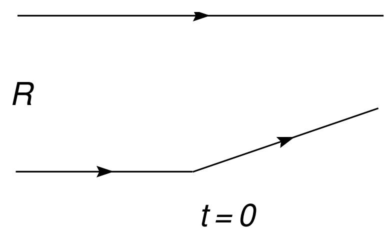

FIG. 1. Two charged particle world lines, separated by R until the time $t = 0$ at which time the second particle instantaneously accelerates to a fixed velocity $v^{2\prime}_{\mu}$.

thought of as renormalizing the mass of the sources. The physical piece comes from the cross terms and for static sources yields

$$W[J] \equiv TV = (2\pi)\delta(0)\frac{e_1 e_2}{4\pi \mid \vec{R} \mid}. \tag{32}$$

Identifying

$$\int dt = 2\pi\delta(0), \tag{33}$$

we can extract the usual Coulomb energy of the static sources. We can see that in this static case the Greens function prescription was irrelevant as the pole was not an issue.

In the case where we have accelerating charges leading to real ("on-shell") radiation the $(i\epsilon)$ pole prescription is what will lead to an apparent lack of unitarity, in that the phase will no longer have unit modulus. This fact was made clear in the previous section. Consider two charged particles, one, with charge and four velocity $(e_1, v_{1\mu})$, which is fixed at the origin, and the other, labelled by $(e_2, v_{2\mu})$, is initially fixed a distance $\delta\vec{x} = \vec{R}$ away. At a time $t = 0$ this second particle instantaneously accelerates to a velocity $v'_{2\mu}$ in some direction. This set up is depicted in figure (1).

Thus the current takes the form

$$\begin{aligned}\tilde{J}_\mu(k) &= e_1 \int_{-\infty}^{\infty} dt v_{1\mu}(t) e^{it(k\cdot v_1 + i\epsilon)} + e_2 \int_0^{\infty} dt v'_{2\mu}(t) e^{i(k\cdot v'_2 + i\epsilon)t - ik\cdot\vec{R}} + e_2 \int_{-\infty}^{0} dt v_{2\mu}(t) e^{i(k.v_2 - i\epsilon)t - ik\cdot\vec{R}} \\ &= e_1 v_{1\mu}(2\pi)\delta(k\cdot v_1) + ie_2\left(\frac{v'_{2\mu}e^{-i\vec{k}\cdot\vec{R}}}{k\cdot v'_2 + i\epsilon} - \frac{v_{2\mu}e^{-i\vec{k}\cdot\vec{R}}}{k\cdot v_2 - i\epsilon}\right),\end{aligned} \tag{34}$$

where convergence factors have been added to make the time integrals well defined. Let us consider the exponent $W[J]$. As before, the static self interaction of particle one will lead to an overall phase renormalizationand can be ignored. Now consider the cross (X) terms

$$W_X[J] = -\frac{i}{2}e_1 e_2 \int \frac{[d^4k](2\pi)\delta(k\cdot v_1)}{k^2+i\epsilon}\left(\frac{v_1\cdot v_2' e^{-i\vec{k}\cdot\vec{R}}}{k\cdot v_2'+i\epsilon}+\frac{v_1\cdot v_2' e^{i\vec{k}\cdot\vec{R}}}{k\cdot v_2'-i\epsilon}-\frac{v_1\cdot v_2 e^{i\vec{k}\cdot\vec{R}}}{k\cdot v_2+i\epsilon}-\frac{v_1\cdot v_2 e^{-i\vec{k}\cdot\vec{R}}}{k\cdot v_2-i\epsilon}\right). \tag{35}$$

The terms involving v_2 will give one half [15] of the net (time integrated) Coulomb energy of the two static particle case, while the terms involving v_2' will give the time integrated Coloumb energy between the static particle and particle two after it has been accelerated. Finally, we have the contribution coming from the e_2^2 terms. As we will now see, these terms are responsible for the radiation and the diminishment in the amplitude. Furthermore, this suppression will arise only because we are working with the Feynman prescription for the Greens function. The relevant terms for radiation (R) are

$$\begin{aligned} W_R[J] = -\frac{1}{2}e_2^2 \int \frac{[d^4k]}{k^2+i\epsilon}\bigg(&\frac{1}{(k\cdot v_2'-i\epsilon)(k\cdot v_2'+i\epsilon)}+\frac{1}{(k\cdot v_2+i\epsilon)(k\cdot v_2-i\epsilon)} \\ &-\frac{v_2\cdot v_2'}{(k\cdot v_2'-i\epsilon)(-k\cdot v_2+i\epsilon)}-\frac{v_2\cdot v_2'}{(k\cdot v_2'+i\epsilon)(-k\cdot v_2-i\epsilon)}\bigg) \end{aligned} \tag{36}$$

The term in the brackets is purely real by construction and thus the only contribution to the imaginary part will come from the pole in the photon propagator at $k_0 =\mid \vec{k}\mid$. These poles correspond to real ("on-shell", $k^2=0$) radiation. Notice that if we used the retarded propagator, where the poles are on the same side of the axis, we could deform the contour so as to completely avoid the poles. Also note that if we were to calculate the power loss by weighting the integral by the energy we would find a divergent answer. The reason is that we have assumed an instantaneous acceleration which puts a kink in the worldline[16]. A physical path would have to resolve this singularity and lead to a finite result.

[15] To get the factor of one half correct one must be careful to drop the principle part of the linear denominators.

[16] Readers who have had some experience with HQET (or SCET) will recognize this divergence as responsible for the "cusp-anomalous dimension".

B. The non-Relativistic Approximation

In anticipation of applying the ideas developed in this chapter to the non-linear theory of General Relativity (GR), let us consider working in the limit where the spheres are moving slowly compared to the speed of light. In this limit we wish to develop a power counting scheme[17] in v/c and concentrate on the case of a captured orbit, as this is a case of significant phenomenological relevance in GR. We will calculate the potentials, which can be used to determine the equations of motion, as well as the power loss. The calculation will be performed in two ways. First, we will expand our exact result (7) in powers of the velocity, then we will show how to reproduce this result using an effective field theory action, by developing a power counting scheme at the level of the Lagrangian. We will only work to order v^2, but the proper way to include higher order corrections will hopefully be clear to the reader.

1. *Expanding the Exact Solutions*

The two worldlines will be labelled by $x_1^\mu(t)$ and $x_2^\mu(t')$, respectively. As before parameterize the lines using the time coordinate

$$\tilde{J}_\mu(k) = e_1 \int_{-\infty}^{\infty} dt \frac{dx_1^\mu}{dt} e^{ik\cdot x_1} + e_2 \int_{-\infty}^{\infty} dt \frac{dx_2^\mu}{dt} e^{ik\cdot x_2}. \tag{37}$$

Thus

$$W = -\frac{1}{2}\int dt dt' \int \frac{[d^4k]}{k^2+i\epsilon} \Big(e_1^2(1-\vec{v}_1(t)\cdot\vec{v}_1(t'))e^{ik\cdot(x_1(t)-x_1(t'))} + e_2^2(1-\vec{v}_2(t)\cdot\vec{v}_2(t'))e^{ik\cdot(x_2(t)-x_2(t'))} \\ + e_1e_2(1-\vec{v}_1(t)\cdot\vec{v}_2(t'))e^{ik\cdot(x_1(t)-x_2(t'))} + e_1e_2(1-\vec{v}_1(t')\cdot\vec{v}_2(t))e^{-ik\cdot(x_1(t')-x_2(t))}\Big) \tag{38}$$

Let us first consider the real part, which is the piece independent of the $(i\epsilon)$ pole prescription. We note that in the static limit, after making a change of variables, we can perform the time integration, which leads to k_0 vanishing. Thus, $k_0 \sim v^n$, where n will be fixed to one momentarily. Then we can expand the integrand in powers of k_0/k, given that k will not vanish in the static limit and therefore on dimensional grounds $k \sim 1/r$, where r is the radius of the orbit. Furthermore, in this limit the interactions are instantaneous, i.e.

[17] Recall we will work in units where $c = 1$.

they are potentials. The real part of the e_1^2 and e_2^2 terms are unphysical self energies which contribute an overall irrelevent phase that can be renormalized away. At leading order we find (recalling (32)) for the crossed terms,

$$V_C(R) = \frac{e_1 e_2}{4\pi \mid \vec{x}_1 - \vec{x}_2 \mid} \tag{39}$$

which follows after dropping k_0 in the denominator. While at order v^2 we have

$$Re[W]_{v^2} = -\int dt V_C(x_1 - x_2)\vec{v}_1 \cdot \vec{v}_2 - \frac{e_1 e_2}{2} \int dt dt' \frac{k_0^2 [d^4k]}{\vec{k}^4} (e^{ik_0(t-t') - i\vec{k}\cdot(\vec{x}_1(t) - \vec{x}_2(t'))} + c.c), \tag{40}$$

We can see that after integration by parts, each factor of k_0 will bring down one power of v so that $k_0 / \mid \vec{k} \mid \sim v$. The resulting integrals leads to the net potential

$$\boxed{V = V_C - \frac{e_1 e_2}{8\pi r}\left(\vec{v}_1 \cdot \vec{v}_2 + \vec{v}_1 \cdot \frac{\vec{x_1} - \vec{x_1}}{\mid \vec{x}_1 - \vec{x}_2 \mid}\vec{v}_2 \cdot \frac{\vec{x}_1 - \vec{x}_2}{\mid \vec{x}_1 - \vec{x}_2 \mid}\right).} \tag{41}$$

Note that this potential is expressed in terms of the canonical coordinates and the velocities. So it should be interpreted as the potential part of the Lagrangian.

Now let us consider the imaginary part of W which is non-vanishing only when the pole prescription is relevant, i.e. in the on-shell region $k_0 \sim k$. We will consider the case where the system is bound, as this will be the configuration of relevant interest when we discuss gravitational waves. In this case $\lambda \sim 1/\omega \sim r/v$. This v suppression is expressing the fact that the wavelength of the radiation is parametrically large compared to the radius of the orbit in the non-relativistic limit. First, it proves convenient to re-write (38) as

$$W = -\frac{1}{2}\int \frac{[d^4k]}{k^2 + i\epsilon} \int dt dt' \left(\sum_{i=1,2} e_i \frac{dx_i^\mu(t)}{dt} e^{ik\cdot x_i(t)}\right)\left(\sum_{i=1,2} e_i \frac{dx_{i\mu}(t')}{dt'} e^{-ik\cdot x_i(t')}\right) \tag{42}$$

We may then perform a multipole expansion, which is an expansion in $R/\lambda \sim v$. The leading order term is given by

$$W = -\frac{1}{2}\int \frac{[d^4k]}{k^2 + i\epsilon} \int dt dt' \left(e_1 e^{ik_0 t} + e_2 e^{ik_0 t}\right)\left(e_1 e^{-ik_0 t'} + e_2 e^{-ik_0 t'}\right) \tag{43}$$

which we see fails to have an imaginary part. This is just the statement that the monopole moment (the total charge) does not radiate since it is a conserved quantity. It should be clear that to get a non-vanishing contribution there must be non-trivial dependence on both t and t'. At next order we have

$$\frac{dx_i^\mu(t)}{dt}e^{-i\vec{k}\cdot\vec{x}_i(t)} \sim (\eta^{0\mu}(1 - i\vec{k}\cdot\vec{x}_i(t)) + v_i^a(t)\delta_a^\mu), \tag{44}$$

where v^a is the three velocity. Using $Im\frac{1}{k^2+i\epsilon} = -\pi\delta(k^2)$, we find to $O(v^2)$

$$ImW = -\frac{1}{2}\int dtdt' \int \frac{[d^3k]}{2k}\sum_{i=1,2} e^{ik(t-t')}(e_i(\vec{k}\cdot\vec{x}_i(t)\vec{k}\cdot\vec{x}_i(t')) - e_i\vec{k}^2\vec{x}_i(t)\cdot\vec{x}_i(t')) \tag{45}$$

where in the second term we have integrated by parts. We then get Γ by first multiplying by a factor of two (recall (27)), and the elementary power loss follows[18] after weighting the integral by the energy

$$\boxed{\langle P\rangle = \frac{1}{6\pi T}\int dt\ddot{\vec{p}}(t)\cdot\ddot{\vec{p}}(t)} \tag{46}$$

where $\vec{p}$ is the dipole moment

$$\vec{p}_i = \sum_i e_i\vec{x}_i. \tag{47}$$

2. Building up the Solution from an Effective Theory

In non-Gaussian theories we do not have the luxury of expanding the exact solution. Instead we must formulate our non-relativistic approximation at the level of the action within the context of an effective field theory. The idea is to separate the scales in such a way that we can construct a set of Feynman rules where each vertex or propagator scales homogeneously [19] . In this way we can construct a set of diagrams which encode all of the contribution of a given order in the small v expansion.

To construct an effective theory we must first get a handle on the scales involved. Our theory of interacting spheres has three relevant scales: R the size of the spheres (we'll assume that all the sphere have radii of the same order), r the distance between the spheres, and λ

[18] Note our units of charge differ from the choice in Landau and Lifshitz.

[19] For those who are not familiar with Feynman rules there is an appendix at the end of this chapter which gives a rapid fire review.

the wavelength of the radiation. We have already eliminated[20] the scale R, by working in the point particle approximation. For the moment we will ignore the finite size corrections and will return to them later.

When we calculate Feynman diagrams we typically work in momentum space where the scales are easily manifested. In the last section we saw that radiation had both energy and momenta that scale as v/r whereas in the potentials the momenta scale as $1/r$ and the energy scales as v/r. When we construct effective field theories we eliminate the scales one at a time from short to long distances. So we will first address the potentials which get contributions from scales of order $1/r$. Once we've eliminated this scale, the only relevant degree of freedom will be a composite formed from the spheres since no knowledge of the scale $1/r$ will remain. The only remnants of this scale will be numbers, or coefficients of operators in the Lagrangian, but the fields themselves will have no support on scales of order r.

The way we accomplish this separation of scales is to write the gauge field in our path integral as

$$A_\mu(x) = \mathbf{A}_\mu + \bar{A}_\mu. \tag{48}$$

where the barred (bold) field is the radiation (potential) field. Naively this would seem to lead to double counting, so we have to make sure we peform this separation in a way such that $\mathbf{A}_\mu(\bar{A}_\mu)$ captures only the physics responsible for potentials (radiation). This occurs naturally[21] by making sure that each type of mode has the proper pole structure built into its propagator, which as we shall see below, follows automatically when we enforce homogeneity in velocity scaling.

Technically the separation (48) falls under the rubric of the "background field method" developed by DeWitt [2] and Abbott [3] and is very reminiscent of the Born-Oppenheimer approximation for systems with fast and slow variables. The underlying idea is that since the potential part of the field is "off-shell", i.e. $k_\mu k^\mu \sim 1/r^2 > 0$ its virtual life is fleeting compared to the long lived on-shell radiation field. One thinks of the potential field as a fluctuation on a (relatively) static background radiation field. In this way of doing things we may "integrate out" the potential field, leaving an effective action for the background

[20] We have not really eliminated R, the coefficients of higher dimensional operators will scale with R. But this R dependence is trivial.

[21] For more complicated effective theories it is not so natural, and one has to worry about so-called "zero-bins", see chapter () for details. In this chapter such subtlties will not arise.

radiation field. The term "integrate out" stems from the fact that formally one does the path integral over this field as discussed in the introduction. The classical limit of integrating out a field corresponds to performing the saddle point approximation, which effectively means solving for the field value and plugging it back into the action. Formally we may write

$$Z[J] = \int D\bar{A} D\mathbf{A} e^{iS(\bar{A},\mathbf{A},J)} = \int D\bar{A} e^{iS_{eff}(\bar{A},J)} \tag{49}$$

where S_{eff} is the resulting action for the effective theory. Typicallly, i.e. for non-Gaussian theories, one can not do this integral exactly (even in the classical approximation), since it's not possible to solve for the exact field value in a non-linear theory. Instead one does a matching procedure in which one chooses S_{eff} to reproduce the physics of the full theory at some fixed order of the relevant expansion parameter (in the present case v) to the order of interest. This matching procedure is at the heart of effective field theories.

At this point we should discuss the issue of gauge invariance. In the background field formalism [3], one may choose distinct gauges for $\bar{A}$ and $\mathbf{A}$. There is a preferred gauge called the "background field gauge" in which the act of integrating out $\mathbf{A}$ leaves a gauge invariant action (S_{eff}) for $\bar{A}$. The gauge fixing term for $\mathbf{A}$ is fixed by covariantizing the typical gauge fixing term with respect to the background field. For a linear theory where the gauge field does not transform under the gauge symmetry the covariantization has no action, but for a non-linear theory such as General Relativity (GR), this gauge fixing term will shift the action, as we shall be made clear in the next chapter. In the linear theory in which we are presently interested, we may gauge fix $\mathbf{A}$ and $\bar{A}$ independently without concern of generating a non-gauge invariant S_{eff}.

Let us now see how this matching ("integrating out") procedure is done in our toy model. We begin with the complete (or "full theory") action ,

$$L = -\sum_i \int m_i d\tau_i + \sum_i \int e_i v_\mu(x_i) A^\mu(x_i) d\tau_i - \frac{1}{4}\int d^4x F^2 - \frac{1}{2}(\partial_\mu A^\mu)^2, \tag{50}$$

where for the moment we have neglected the finite size effects. So the first, trivial, step in the effective field theory construction has been utilized, i.e. we have removed the scale R (size of the objects) by working in the point particle approximation. All vestiges of this scale would be in coefficients of the higher dimensional terms (2) which we are neglecting for the moment. Now we perform our mode decomposition (48) and furthermore perform a partial Fourier transform[22]

$$\mathbf{A}_\mu(t,\vec{x}) = \int [d^3k] e^{i\vec{k}\cdot\vec{x}} \mathbf{A}_{\vec{k}\ \mu}(t) \equiv \int_{\vec{k}} e^{i\vec{k}\cdot\vec{x}} \mathbf{A}_{\vec{k}\ \mu}(t). \tag{51}$$

The reason for performing this transformation is that it effectively "removes" the large scale $1/r$ from the field fluctuations and makes any dependence on the scale $1/r$ manifest, as will be seen clearly below.

Plugging our decomposition into the action (50) leads to

$$L = \sum_i \int e_i \frac{dx_i^\mu}{dt_i}(t_i)\left(\bar{A}_\mu(x_i) + \int_{\vec{k}} e^{i\vec{k}\cdot\vec{x}_i(t_i)} \mathbf{A}_{\vec{k}\mu}(t_i)\right) dt_i + \frac{1}{2}\int dt \int_{\vec{k}} (\vec{k}^2 \mathbf{A}^\mu_{\vec{k}} \mathbf{A}_{-\vec{k}\mu} + \partial_0 \mathbf{A}_{\vec{k}\mu} \partial_0 \mathbf{A}^\mu_{-\vec{k}})$$
$$+ \frac{1}{2}\int d^4x (\bar{A}_\mu \Box \bar{A}^{\ \mu}), \tag{52}$$

where the kinetic term for the point particle has been dropped as it wont play a role here, and the affine parameter has been chosen to be the time coordinate [23]. The terms linear in the fluctuating field $\mathbf{A}_{\vec{k}\mu}(t_i)$ are dropped.

Now each term in the action should have definite scalings in v. To see how this comes about we must first fix the scalings of the fields. This may be done by noting that the kinetic term for the photon must be leading order to have a sensible perturbative expansion. Let us first consider the radiation field. To fix its scaling we use the fact that $\Box \sim v^2/r^2$. We also need to determine how the measure scales. Perhaps the simplest method is to note that the spatial integration generates a momentum conserving delta function, and since all the momenta scale as v/r the measure scales as r^4/v^4. Imposing the fact that kinetic term is leading order yields

$$\boxed{\bar{A}_\mu \sim v/r.} \tag{53}$$

Alternatively one can determine the field scaling by considering the propagator (see chapter). For the potential field, we have $\partial_0 \sim v/r$ and $k \sim 1/r$. Thus we *must* treat the term with temporal derivatives as a perturbation. To determine the scaling of the potential field we

[22] Readers who have familiarized themselves with NRQCD and SCET may wonder why this is an integral and not a sum. The answer is that in the classical theories there is no residual momentum for the external sources so all of the associated subtleties are absent.

[23] At higher orders in v one must be careful not to drop contributions arising from this choice of parameterization.

note $dt \sim r/v$ thus the field scales

$$\boxed{\mathbf{A}_{\vec{k}\mu}(t) \sim v^{1/2} r^2.} \tag{54}$$

The propagator for the potential photon may be found by inverting the associated quadratic in the action to give

$$\langle \mathbf{A}_{\vec{k}\mu}(t_1)\mathbf{A}_{\vec{q}\nu}(t_2)\rangle = (2\pi)^3\delta(t_1 - t_2)\delta^3(\vec{k} + \vec{q})\frac{ig_{\mu\nu}}{\vec{k}^2}. \tag{55}$$

We see immediately that since the potential propagator is independent of the energy, its Fourier transform is proportional to $\delta(t)$, i.e. instantaneous. Notice that there is no energy pole in this propagator, thus eliminating the possibility of double counting the radiation field which gets its support from the region $k_0 \sim \mid \vec{k} \mid$. The temporal derivative terms correct for this instantaneity, and are down by v^2. These corrections correspond to the k_0 corrections in (52). The dominant interaction will involve the temporal potential photon. This interaction will clearly generate the leading order Coulomb potential. The couplings to non-temporal photons are suppressed by v.

We now have a well defined power counting in the action and are prepared for the next step in the EFT process, namely integrating out the potential modes. In so doing we will have eliminated the scale $1/r$ from the theory, and the only remaining scale will be v/r. In this next theory, since we have coarse grained beyond the scale $1/r$ we can not distinguish between the constituents of the binary. Thus the resulting theory will be a one body theory which couples only to radiation. Let us now generate the one body theory for our toy model. Since the theory is Gaussian, we could, of course, solve it exactly as we did in the last section, but instead we will build up an approximate solution by considering the relevant Feynman diagrams order by order. To calculate the diagram we will need a set of Feynman rules. The potential photon propagator represented by dashed lines was given in (55). There are two types of vertices which follow from expanding out the interaction

$$\sum_i \int e_i \frac{dx_i^\mu}{dt_i}(t_i)\left(\int_{\vec{k}} e^{i\vec{k}\cdot\vec{x}_i(t_i)}\mathbf{A}_{\vec{k}\mu}(t_i)\right)dt_i \approx \sum_i e_i\left(\int_{\vec{k}} e^{i\vec{k}\cdot\vec{x}_i(t_i)}(\mathbf{A}_{\vec{k}0}(t_i) - v_a\mathbf{A}_{\vec{k}a}(t_i)\right)dt_i, \tag{56}$$

a leading order coupling to the temporal photon, and an order v vertex coupling to the

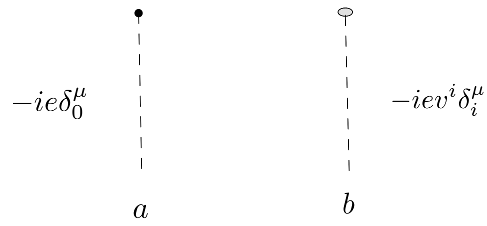

FIG. 2. Feynman rules for the interaction of the source with a photon, a) leading order in velocity, b) $O(v)$ interaction.

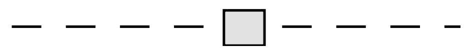

FIG. 3. The box corresponds to the first correction to instantaneity, arising from the temporal derivative term in (52). One can think of this as the corrected propagator. Higher order corrections to instantaneity, say with n insertions of the square go as $\sim \frac{1}{(k_i^2)^n}(\frac{d}{dt_1}\frac{d}{dt_2})^n\delta(t_1-t_2)$.

spatial photon.

The corresponding Feynman rules are shown in figure (2). Not shown are the explicit factors of $\int_{\vec{k}} e^{i\vec{k}\cdot x_i(t)}$ which are associated with the potential $A_{\vec{k}}$ at the vertex coupling to worldlines $x_i(t)$. The corrections to instantaneity are accounted for by correcting the propagator which arise as a consequence of the last term in the first line of (52). The leading order corrected propagator is depicted in figure (3) and is given by

$$\langle \mathbf{A}_{\vec{k}\mu}(t_1)\mathbf{A}_{\vec{k}'\nu}(t_2)\rangle = i(2\pi)^3 g_{\mu\nu}\delta^3(\vec{k}-\vec{k}')\frac{d}{dt_1}\frac{d}{dt_2}\delta(t_1-t_2)\frac{1}{\vec{k}^4}. \tag{57}$$

where the subscript on the expectation value reminds us that this includes the first order correction to instantaneity. Those are all of the rules we need in this simple Gaussian theory since the radiation photon does not couple to the potential photon. Thus it is simple for us to integrate out the potential photon. Up to order v^2 there are only three diagrams, which are depicted in figure (4).

Let us start with the leading order contribution figure (4a). This diagram arises from the Wick contraction

$$iM_{LO} = -\int dt_1 dt_2 \int [d^3k][d^3q] e_1 e_2 \langle \mathbf{A}^0_{\vec{k}}(t_1)\mathbf{A}^0_{\vec{q}}(t_2)\rangle e^{i\vec{k}\cdot\vec{x}_1} e^{i\vec{q}\cdot\vec{x}_2}. \tag{58}$$

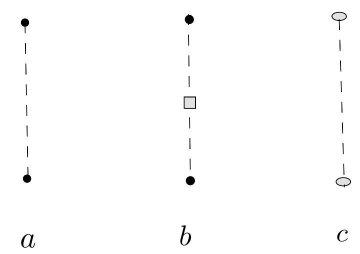

FIG. 4. The Feynman diagrams which generate potentials up to order v^2. Diagram (a) is the leading order Coulomb potential, while (b) corresponds to a velocity dependent vertex correction. This diagram has an associated figure where dot is inserted on the other worldline. Diagrams (c) corresponds to an instantaneity correction.

Power counting this diagram we find that it scales as $e_1 e_2/v$. Naively this seems to not have a well defined $v \to 0$ limit, until we recall that for a Coulombic bound state $e_1 e_2 \sim v$. We might also worry that since this exchange is leading order, that multiple exchanges of this type would also be leading order. This is indeed true, but we must recall that we are interested in calculating the potential which lives in the exponent. So we will need an infinite number of diagrams to sum into this exponential. The resummation into an exponential follows since the nth order diagram factorizes into a product of independent propagators[24] . The exponential follows by carefully keeping track of symmetry factors as discussed in the appendix. The reader is encouraged to complete the following exercise.

Exercise 2.1: Consider the vacuum energy for two static sources. Show that the infinite sum of diagrams exponentiates to form the Coulomb potential in the exponent by working out all of the permutations for the contraction of photon lines.

To complete the leading order calculation we use the result for the potential mode propagator (54) to find

$$iM_{LO} = -ie_1 e_2 \int dt \frac{[d^3k]}{\vec{k}^2} e^{i\vec{k}\cdot(\vec{x}_1(t) - x_2(t))}. \tag{59}$$

To extract the potentials from the Feynman diagrams (which are given by iM, M being the amplitude) we use the relation

$$iM = -iVT. \tag{60}$$

This follow the fact that diagrams arise from expanding out the matrix element in (4) and

[24] Notice in a non-Abelian theory care must be taken to account for the fact that the vertices don't commute.

using Wick's theorem. Using (60) and performing the integral gives the Coulomb potential which we previously derived by expanding the the full theory (39).

To calculate the first relativistic correction we have to include the two diagrams (4b,4c). By inspection, we can see that these are the only diagrams which contribute at order v^2 relative to the leading order contribution. Let us consider (4b). According to our Feynmen rules this diagrams is given by

$$fig\ (4b) = -ie_1e_2 \int dt_1dt_2 \int \frac{[d^3k]}{(\vec{k}^2)^2} e^{i\vec{k}\cdot(\vec{x}_1(t)-\vec{x}_2(t))} \frac{d}{dt_1}\frac{d}{dt_2}\delta(t_1-t_2). \tag{61}$$

After an integration by parts the calculation of this integral is a simple extension of the integral (59). Performing this integral and then including the result for diagram (4c) reproduces the result (41).

Exercise 2.2: Show that the result for diagram (4b) is given by

$$V_b^{(2)} = \vec{v}_1\cdot\vec{v}_2\frac{V_0}{2} - (\vec{v}_1\cdot\vec{X})(\vec{v}_2\cdot\vec{X})\frac{V_0}{2X^2} \tag{62}$$

Once we have integrated out the potentials, we have an effective Lagrangian which is of the form

$$L = \sum_i V_i + L_{rad}(\bar{A}(x)). \tag{63}$$

However, our goal of eliminating the scale $1/r$ has yet to be achieved, as the radiation field still know about this scale as a consequence of its coupling to the worldlines. That is, the potential still couples distinctly to the individual worldlines. We have yet to coarse grain the system sufficiently. To remove this scale we must multipole expand the fields to generate an effective one body action. Note that this is not a matter of choice, to have a well defined effective theory it is necessary that all terms in the action scale homogeneously, and if we were not to multipole expand this criteria would not be satisfied [4].

Consider the coupling of the radiation photon. We must expand

$$\begin{aligned}\sum_i e_i v^i_\nu A^\nu(x_i) &= \sum_i e_i(v^i_\nu A^\nu(t,0) - v^i_\nu(\vec{x}^i\cdot\vec{\partial})A^\nu(t,0)+...)\\ &= \sum_i e_i(A^0(t,0) - (\vec{x}^i\cdot\vec{\partial})A^0(t,0) + x^i\dot{A}^i(t,0)...)\\ &= QA_0(t,0) + \vec{p}\cdot\vec{E}(t,0)+... \end{aligned} \tag{64}$$

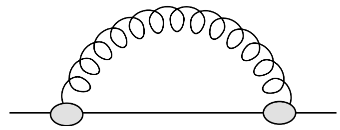

FIG. 5. The Feynman diagram responsible for the leading order power loss.

where we have gone to the COM frame of the system. Q and $\vec{p}$ are nothing but the total charge and net dipole moment respectively. This is the standard dipole coupling. Keeping higher orders in v would generate the magnetic dipole and the electric quadrapole etc. The dipole moment is the moment of the composite object, and there is no longer any reference to the scale of the orbit itself. Notice that the orbital scale $r \sim x_i$ now shows up *explicitly* as part of an operator coefficient. Just as the object sizes themselves show for the finite size operators. The dependence on the orbit size is manifest, which is what insures that we can power count in a systematic fashion.

Now let us calculate the power loss. At leading order there is only one relevant diagram which is shown in figure (5). As an exercise, let us calculate this diagram directly using Wicks theorem which is the long way of doing things. Usually we use a short cut where we derive a set of Feynman rules which allow us to construct diagrams directly, but for readers who have little to no quantum field theory experience this exercise will hopefully be instructive.

Expanding our L_{int} to second order [25] in the dipole interaction we see that there is only one possible Wick contraction, and it is given by

$$fig(5) = -\frac{1}{2}\int dt_1 dt_2 p_i(t_1) p_j(t_2) T\langle(\partial_i A_0(t_1,0)\partial_j A_0(t_2,0) + \partial_0 A_i(t_1,0)\partial_0 A_j(t_2,0))\rangle. \tag{65}$$

Using the radiation photon propagator (as usual in Feynman gauge)

$$\int [d^4x] e^{ik\cdot x}\langle T(A_\mu(x)A_\nu(0))\rangle = \frac{-ig_{\mu\nu}}{k^2+i\epsilon}. \tag{66}$$

[25] There is no contribution to the vacuum persistence amplitude at leading order in the interaction since all of the photon legs must be contracted.

such that

$$ImM = \frac{1}{6}\int dt_1 dt_2 \int [d^3k]\ |\vec{k}|\ \vec{p}(t_1)\cdot\vec{p}(t_2)e^{-i|\vec{k}|(t_1-t_2)}. \tag{67}$$

After weighting the integral by the energy, and using (27) we regain the standard dipole radiation formula (46).

Exercise 2.3: Complete the calculation of the dipole formula (46) by performing the necessary integrals in (65).

We have successfully reproduced the full theory results to order v^2 in the non-relativistic expansion. It is hopefully clear to the reader how to carry out the expansion to higher orders by including higher order multipole moments. As stated above, in this trivial, solvable, case there was no need to go through the whole exercise of separating out the scales, as the power and potentials are calculated in many elementary texts using canonical methods. This section hopefully served as a warm up for the payoff which will come when we consider non-linear theories.

II. INCLUDING FINITE SIZE EFFECTS

The power of effective field theory really resides in its systematics. That is, it is a "turn the crank method" of calculating results to any order desired. So far we have only calculated in the zero radius limit. But we are set up to calculate to any order in R, as long as we are willing to put in the work. The first step in calculating finite size effects is to fix C_E and C_B.

A. Interpreting the Finite Size Coefficients

To determine the coefficients $C_{1,2}$, we perform a matching calculation. We first choose some object to calculate in both the full and effective theories. Whereby "full" we mean the complete theory, i.e. no approximations, the real microscopic theory. We then choose our matching coefficients $C_{1,2}$ so that the full theory reproduces the effective theory to the appropriate order in R. The immediate question then arises: "If I knew how to calculate in the full theory, then why would I bother with the effective theory in the first place?". The answer is that we are free to choose *any* object to calculate when we do the matching procedure, as long as the calculation is sensitive to the relevant operator. If we're smart, then we'll match using some very simple configuration that will make the full theory side doable. Its worth expounding upon this point. We could consider the interaction between any number of differing shaped objects. But the only full theory calculation that we need to do involves only one object at a time, as the matching coefficients of the finite size operators only depend upon the details of the individual objects in isolation. Once we've done this calculation to fix $C_{1,2}$, then we can use the effective theory to calculate something more complicated. This point will become more clear when we do an explicit calculation below.

Let's see how this works in our example. First sine well be interested in non-relativistic physics we will make our lives easier by working the in frame were the velocities are small and we will keep only the leading piece in velocity. So we will redefine our couplings so that

$$S_{fs} = \int dt (C_E \vec{E}^2 + C_B \vec{B}^2) \tag{68}$$

The couplings are related by

$$C_E = -2(C_1 + C_2) \qquad C_B = 2C_2. \tag{69}$$

Next to matching let us choose to calculate the induced field generated by a sphere in the presence of some external field $\mathbf{A}_\mu$. We begin performing the decomposition

$$A_\mu(x) = \mathbf{A}_\mu + \mathsf{A}_\mu, \tag{70}$$

where A_μ is the induced field and $\mathbf{A}_\mu$ is the background field which we will take to be static. When we expand the field in this way in the effective theory,

$$\begin{aligned}
S_{FS} &= \int dt \left(-2C_E(\mathbf{E}^i(\partial^i \mathsf{A}^0(t,x))) + 2C_B(\mathbf{B}^i \epsilon^{ijk} \partial^j \mathsf{A}^k(t,x)\right) \\
&= \int d^3y dt \left(-2C_E(\mathbf{E}^i(y)(\partial^i \mathsf{A}^0(y)) + 2C_B(\mathbf{B}^i(y)\epsilon^{ijk}\partial^j \mathsf{A}^k(y))\right) \delta^{(3)}(x(t)-y) \\
&= \int d^3y dt \left[2C_E \partial^i(\delta^{(3)}(x(t)-y)\mathbf{E}^i(y))\mathsf{A}^0 - 2C_B \partial^j(\delta^{(3)}(x(t)-y)\mathbf{B}^i(y)\epsilon^{ijk})\mathsf{A}^k\right] \\
&\equiv \int d^4y J^{eff}_\nu(y) \mathsf{A}^\nu(y),
\end{aligned} \tag{71}$$

and

$$J_\nu^{eff}(y) = \left[2C_E\partial^i(\delta^{(3)}(x(t)-y)\mathbf{E}^i(y))\delta_\nu^0 - 2C_B\partial^j(\delta^{(3)}(x(t)-y)\mathbf{B}^i(y)\epsilon^{ijk})\delta_\nu^k\right] \tag{72}$$

where we have kept only the terms relevant for the linear response. For simplicity we have taken the external field to be static. Recall, we are free to match *any* observable as long as the resulting solution to the matching equation is unique and non-zero. We see that the finite size terms look like sources when we expand around the external field. We can calculate the induced field by convolving the effective source with the retarded Green's function

$$D^r_{\mu\nu}(z-x) = g_{\mu\nu}\frac{\theta(z_0-x_0)}{4\pi\mid\vec{z}-\vec{x}\mid}\delta(z_0-x_0-\mid\vec{z}-\vec{x}\mid). \tag{73}$$

Placing the particle at the origin $\vec{y}=0$, leaves

$$\mathsf{A}_\mu(z) = -\frac{C_B}{2\pi}\frac{(z\times\mathbf{B})^i}{\mid\vec{z}\mid^3}\delta^i_\mu - \frac{C_E}{2\pi}\delta^0_\mu\frac{z^i}{\mid\vec{z}\mid^3}\mathbf{E}^i \tag{74}$$

We immediately see that these new terms in the action are responsible for generating a polarizability. Given that a dipole $(\vec{P},\vec{M})$ generates a potential $(A_0 = \frac{\vec{P}\cdot\vec{r}}{4\pi r^3}, A_i = \frac{\vec{m}\times\vec{r}}{4\pi r^3})$, we can then read off the induced dipole moments which are given by

$$\vec{P} = -2C_E\vec{E} \quad : \quad \vec{M} = -2C_B\vec{B} \tag{75}$$

and thus the Wilson coefficients are related to the electric and magnetic susceptibilities defined via $p = 4\pi\alpha_E E$ and $m = 4\pi\alpha_B B$

$$C_E = -2\pi\alpha_E \quad : \; C_B = -2\pi\alpha_B. \tag{76}$$

To calculate the corresponding field in the full theory we must make some choice for the properties of the shell. For simplicity, suppose that the spheres are perfect conductors which is a standard problem in electrostatics. The vanishing of field inside the conductor can be accomplished by placing image dipoles at the origin with magnitudes

$$\vec{P} = R^3\vec{E} \qquad \vec{M} = \frac{R^3}{2}\vec{B}. \tag{77}$$

Comparing this result with (75) gives the matching result for the sphere

$$C_B = -\frac{R^3}{4}, \quad C_E = -\frac{R^3}{2}. \tag{78}$$

B. Radiation Reaction

The question of the classical motion of a charged body turns out to be a rather vexing subject. As the body accelerates under the influence of an external force it will radiate. The radiation in turn backreacts on the particle which affects its equations of motion. The idea is to try to find a differential equation which describes the motion of the body. While the solution for the case of a point particle was solved by Abraham Dirac and Lorentz (ADL), for the case of a finite sized object solving the problem exactly, at least analytically, seems to be out of the question. However, if we assume that the radiation wavelength is much larger then the object size, then we may hope that the point particle approximation is a good starting point.

The radiation reaction equation of motion for a point particle is given by the celebrated ADL equation. You can find numerous derivations in textbooks [10]. The result is

$$m\ddot{x}^\beta = \frac{e^2}{4\pi}\frac{2}{3}(\dddot{x}^\beta - \dot{x}^\beta(\ddot{x}\cdot\ddot{x})). \tag{79}$$

We should be able to re-derive this result using our effective field theory developed in this chapter. This is accomplished by applying the Euler-Lagrange to $W[J]$. However, we need to modify our formalism slightly in order to study these equations of motion. In the end the needed change will be only slight and intuitively obvious.

We begin by noticing that in calculating conservative equations of motion, i.e. those due to potentials, the $i\epsilon$ prescription is irrelevant since potential propagators are off-shell. However, as is hopefully clear to the reader by now, when we are interested in the effects of radiation, the pole structure is crucial. So far when it comes to radiation all we've been concerned about is the power output. That is, we've asked the question, given some initial state with zero radiation, what is the probability of finding a final state with some fixed final state? Now however, we are no longer interested in calculating transitions from in to out states. We are interested in calculating instantaneous quantities.

To treat this subject properly one needs to use the so-called "in-in" formalism [**?**].

However, it is not difficult to intuit the proper method of calculating. First off we should replace the Feynman propagator by the retarded propagator. This seems obvious since the Feynman propagator has support outside the light-cone which would lead to acausal equations of motion. The retarded propagator is given by

$$D^R_{\mu\nu}(x) \equiv \langle A^\mu(x(\tau))A^\nu(x(\tau'))\rangle_R = \frac{1}{2\pi}\delta((x(\tau)-x(\tau'))^2)\theta(x^0(\tau)-x^0(\tau')) \tag{80}$$

Furthermore, when we apply the equations of motions we should vary only the vertex which is at the latest time. The leading diagram is shown in figure (6). The amplitude is given by

$$iM = -\frac{e^2}{2}\int d\tau d\tau' v_\mu(\tau)v_\nu(\tau')\langle A^\mu(\tau)A^\nu(\tau')\rangle_R \tag{81}$$

Varying this action with respect to $y^\beta(\sigma)$ (with the respect to latest σ) leads to the Euler-Lagrange equation

$$m\ddot{y}^\beta(\sigma) = \frac{e^2}{2}\frac{d}{d\sigma}\int d\tau v_\mu(\tau)\langle A^\mu(\tau)A_\beta(\sigma)\rangle_R - \frac{e^2}{2}\int d\tau v_\mu(\tau)v_\nu(\sigma)\langle A^\mu(\tau)\frac{d}{dy^\beta(\sigma)}A^\nu(\sigma)\rangle_R. \tag{82}$$

To evaluate this expression we choose the affine parameter to be z such that $z^2 = (y(\tau)-y(\sigma))^2$. This will allow us to evaluate the integral. First we replace $\frac{d}{dy(\sigma)}$by $-\frac{d}{dy(z)}$ and then use

$$\frac{d}{dy^\beta(z)} = \frac{y^\beta(z)-y^\beta(\sigma)}{z}\frac{d}{dz}. \tag{83}$$

Since the propagator is proportional to $\delta(z)$ we may expand the integrand around $z=0$ and integrate by parts. The resulting equations of motion are

$$m\ddot{y}^\beta = \frac{e^2}{4\pi}\left(\frac{2}{3}(\dddot{y}^\beta - \dot{y}^\beta\ddot{y}\cdot\ddot{y}) - \int dz\delta(z)\frac{\ddot{y}^\beta}{2z}\right). \tag{84}$$

Note that for the first two term the derivatives have been re-expressed in terms of the proper time. When the proper time is the affine parameter the expression simplifies due to the many identities which follow from the constraint $\dot{y}^2 = 1$.

This equation does not include the external force necessary for the initial acceleration. We see that the last term is divergent, but it can be absorbed into the mass. This *had* to be the case since all short divergences must be local and must correspond to some term in the action, since the action includes all possible terms consistent with the symmetry. Note that if we regulate the integral in a way which breaks one of our symmetries then this statement no longer holds, and it one might need to include terms in the action which are not invariant under the broken symmetry group to absorb the divergence. We see that after this renormalization this reproduces (79).

Now consider the power corrections corresponding to finite size effects. We have already considered the first such operators and they are down by R^3. However, as previously discussed, if we have external forces, as we must here, we should include the operator (3). In addition, we must consider operators which are independent of the gauge field that are allowed by the symmetries such as

$$O = a^2. \tag{85}$$

The matching coefficient for this operator is $-\frac{2}{9}e^2 R$ [7].

This operator accounts for the conservative part of the finite size effects and it coefficient scales as R. However this operator can be eliminated using a shift in the world line [6] . To see this note that using the equations of motion we can re-write this operator as

$$O_{a^2} = \frac{e}{m} v_\mu F^{\mu\nu} a_\nu \tag{86}$$

Thus the effects of the a^2 operator scale as$\sim Re^2/m$, assuming the sphere has a relatively uniform density. Thus the leading order contribution to the finite size effects will come from the operators

$$O_a = \frac{1}{v^2} v^\mu a^\nu F_{\mu\nu}. \tag{87}$$

If we compare the size of the coeffcients

$$C_{a^2}/C_a \sim \frac{e^2/R}{m} \tag{88}$$

which is the ratio of the binding energy to the rest mass is vanishingly small for a macroscopic object. Physically this operator represents the acceleration induced dipole moment due to the finite size of the sphere.

We may now use this to calculate the first correction to the ADL equation. This correction

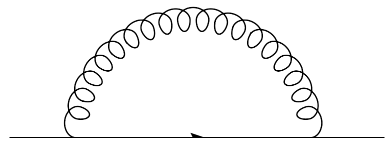

FIG. 6. The Feynman diagram responsible for the leading order radiation reaction force. The vertices are time ordered with time increasing to the right.

FIG. 7. The Feynman diagrams responsible for the first finite size correction. The heavy dot is an insertion of O_a, and time runs from left to right. The vertex furthest to the right is the one which is varied when calculating the equations of motion.

follows from applying the Euler-Lagrange equation to the diagrams show in figure (7). In analyzing this diagram, we must keep track of the time ordering of the vertices, since only the later vertex should be varied, given that the earlier vertex generates the radiation which induces the force [26] A rather tedious, yet straightforward calculation leads to the result [5]

$$m\ddot{y}^\beta = \frac{4eC_a}{3}(y^\beta_{(5)} - y^\nu_{(5)}\dot{y}_\nu\dot{y}^\beta + 2\dot{y}\cdot y_{(3)}(y^\beta_{(3)} - y^\nu_{(3)}\dot{y}_\nu\dot{y}^\beta) - 2\ddot{y}^\beta y^\nu_{(4)}\dot{y}_\nu). \tag{89}$$

where $y_{(n)}$ denotes the n'th derivative. The matching procedure fixes $C_a = e/6$ for a spherical shell [5].

C. The Force Between Neutral Bodies

We may utilize our effective theory to calculate the interaction between neutral bodies which have no permanent mulitpole moments. Of course classically there will be no forces between such objects, and since we are presently discussing classical effective theories this subject is slightly off topic. Furthermore, since we will be considering loops, i.e. quantum corrections, our analysis of the mode decomposition will have to be extended. That is we will need another mode, which is not potential but radiation, that is needed to reproduce the proper full theory loop corrections.

[26] To see a proof of this seemingly ad-hoc, yet physical, procedure use must be made of the "in-in formalism" see [?].

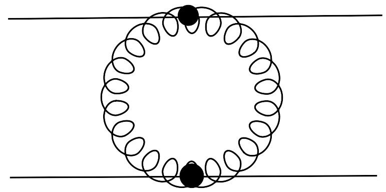

FIG. 8. The Feynman diagram responsible for the leading order force between neutral objects. The dot represents an insertion of either one of the finite size operators that arises at leading order.

We will be interested in static source, so we will ignore any relativistic effects. Since we are interested in neutral objects, the action will be given by [27]

$$S = \sum_{i=1,2} \int dt_i(m_i + C_E^i \vec{E}^2 + C_B^i \vec{B}^2) - \int d^4x \frac{1}{4} F_{\mu\nu}F^{\mu\nu}. \tag{90}$$

The dominant force, which arises at $O(R^6)$, will come from the diagram shown in figure (8). On dimensional grounds we can see that this diagram leads to a potential which scales like

$$V \sim \frac{R^6}{r^7} \tag{91}$$

since it involves one insertion of a finite size operator, scaling as R^3, on each line. Notice that this diagram has a closed photon loop in it. This is a clear sign that the result is actually quantum mechanical. To understand this let us recall the origin of the effect. The force arises as a consequence of the fact that microscopic [28] objects with no multipole moments can fluctuate quantum mechanically into states with non-vanishing moments, resulting in a force between neutral objects. Alternatively, we may think of the effect as arising from a vacuum fluctuations in the electric field inducing a virtual dipole [12] in the atoms.

We should expect the scaling (91) to hold as long as the distance between the particles is large compared to $1/\Delta E$, where ΔE is the energy of the first excited state of the sphere(s). When we constructed the effective theory we assumed that the only relevant scale was the size of the particle. We have integrated out all the excitations assuming that we wont be probing the system at distances short enough for any excited states to be relevant. For many situations of physical interest, this is not the case, so we will have to return to this issue in the next section.

[27] Again the terms which are independent of the gauge field will be suppressed in the non-relativistic limit.

[28] This force can also manifest itself on larger scales as well and is responsible for the Geckos remarkable climbing abilities.

Before moving on with the calculation, the discussion in the last paragraph brings up an important point. Unless we know all of the scales in the full theory there will always be some uncertainty in the scale at which it breaks down. In our example of the spheres, it is fair to assume that the first excited state will have a gap of order $\Delta E \sim 1/R$, but this need not be the case. In fact, the sphere could be a molecule, or a macroscopic object. The theory should apply equally well in both case. But it is clear that the hierarchy of scales in these cases will be quite disparate. Suppose we were considering Hydrogen. In this case the breakdown scales will be the energy splittings ($\Delta E \sim \alpha^2 m_e$). Thus the error in any such EFT calculation will in general involve two expansion parameters $1/(\lambda \Delta E)$ as well as (R/λ) where λ is the typical scale for the observable of interest. In our case here $\lambda = r$, the distance between the spheres. Another way to think about this mismatch is to note that when we naively assume that higher dimensional operators are down by powers of R, what were really assuming is that the gap in the energy spectrum is set by the size, which would be the case for a simple system like a particle in a box.

For now let us assume that we are free to consider distances long enough that our EFT is valid. We will begin by calculating the result for the diagram in figure (8). Before doing so we will take advantage of the fact that we're not interested in relativistic effects. Lets re-write the finite size action as

$$S_{fs} = \int dt (C_E \vec{E}^2 + C_B \vec{B}^2) \tag{92}$$

For the moment we will take $C_B = 0$, which we will justify later when we do the matching to a full theory model. Let us also assume that the two particles are identical. The amplitude is given by

$$iM = 2(i^2) \times C_E^2 \int dt_1 dt_2 \langle E_i(t_1) E_a(t_2) \rangle \langle E_i(t_1) E_a(t_2) \rangle. \tag{93}$$

It behooves us to calculate in coordinate space where

$$\langle T(A^\mu(x) A^\nu(0)) \rangle = \frac{g_{\mu\nu}}{4\pi^2} \frac{-i}{x^2 - i\epsilon} \tag{94}$$

Then

$$Diag = iM = i\alpha_E^2 \frac{23}{4\pi r^7} T = -iVT \tag{95}$$

where the time interval $T \equiv \int dt$ and $\mid \vec{x} - \vec{y} \mid \equiv r$. So that

$$\boxed{V = -\frac{23\alpha_E^2}{4\pi r^7}.} \tag{96}$$

This force is usually referred to as the Casimir-Polder effect.

Notice that the photons will not have zero energy and are thus are not instantaneous. This occurs even though there is no radiation, as the sources can be completely static. In the literature one often sees this referred to as a "retardation effect". In momentum space we would find that the effect is coming from the purely quantum mechanical region of momentum space where $k_0 \sim k \sim 1/r$. This is called the "soft" region and will be discussed in the chapter on non-relativistic gauge theories.

Exercise 2.5: List the set of operators which correspond to the leading corrections to (2). Then draw the Feynman diagrams which will generate the first corrections to the $1/r^7$ potential when $1/r \ll \Delta E$. On dimensional grounds determine how these corrections scale in powers of $1/r$.

Exercise 2.6: Suppose we allow our neutral particles to be asymmetric. Furthermore suppose that the charge distribution is such that these particles have no dipole moments. The asymmetry of the particle breaks the rotational invariance of the world line physics. To account for this we introduce a tensor κ_{ij}^a associated with particle a which as we shall see will be related to the tensor susceptibility of the particles. We would like to determine the force between two such asymmetric particles. Working in the non-relativistic approximation we may write down the wordline coupling

$$S = \int \sum_a d\lambda_a (\kappa_{ij}^a E_i E_j). \tag{97}$$

Use this interaction to calculate the force between two static particles. Then use the matching procedure to show relate κ_{ij}^a to the tensor susceptibility χ_{ij} defined via the relation

$$P_i = \chi_{ij} E_i^B. \tag{98}$$

Show that your answer for the energy reduces to (96) in the limit where $\chi_{ij} = \delta_{ij}$.

D. The Van Der Vaals Force

So far we have considered two scales in this problem, r, R. However, the internal dynamics of the sphere may generate other scales. We might assume that the modes of the sphere have typical energy of order $1/R$, but this is not necessarily the case. Let us call the energy gap of the degrees of freedom on the sphere to be ΔE. In this case instead of an expansion in $R/r \sim kR$, there are corrections which scale as $k/(\Delta E)$. This does not mean that there are no kR corrections as well. If we start to probe the system at distances small enough that we excite the internal dynamics above the ground state the expansion parameter $k/(\Delta E)$ becomes of order one. We will need to "integrate back in" these excited states by introducing new degrees of freedom on the world-line. However, we need to do so in a systematic fashion. That is, we need to make sure that whatever we add to the action, we know its scaling in the relevant hierarchy and we furthermore know the size of the next correction.

We begin by introducing a new dynamical variable $O(t)$ that lives on the world line that couples in a way which is consistent with the relevant symmetries which are: global translations and rotations, and gauge invariance. If we wished we could covariantize the action to make it consistent with relativity.

How do we interpret the operator $O(t)$? This operator, acting on the Hilbert space of the particle, should excite the lowest lying excitations. In our case we are interested in allowing for a fluctuating dipole moment, so the relevant operator will transform as a vector under the rotation group and is odd under parity, thus we label it as $\vec{p}(t)$. The operator $\vec{p}(t)$ will act on a Hilbert space carried by the worldline, and its coupling will be given by the interaction

$$S = \int dt \vec{p} \cdot \vec{E}. \tag{99}$$

The effective theory knows nothing about the dynamics of the electric dipole $\vec{p}(t)$. We will have to input that information at some future point.

Since we'll be working in the non-relativistic limit, the electric field will be decomposed into radiation and potential pieces, as discussed in the previous section. Thus this interaction will lead to new power loss mechanisms via radiation. Here we will only be interested in the static force between neutrals, but the inclusion of radiation effects would follow in a manner similar to the cases discussed in the previous section. Note that there is also a magnetic dipole term which arises at the same order in the multipole expansion but is suppressed since the magnetic field couples to angular momentum and we assume that the constituents of the sphere are non-relativistic [29].

What diagrams contribute to the potential? The fact that the worldline now carries degrees of freedom implies that box topologies are no longer considered iterations of the leading order exchange. The leading order potential will arise from two insertions of dipole operators on each line. $\langle p_i(t)\rangle$ vanishes under our assumption of spherical symmetry so one insertion of the interaction will not suffice. The box diagram (see (**??**)) gives

$$Box = -\frac{1}{4}\int dt_1 dt_2 dt_1' dt_2' \langle T(p_a(t_1)p_b(t_2))\rangle\langle T(\mathbf{p}_c(t_1')\mathbf{p}_b(t_2'))\rangle\langle T(E_a(t_1)E_c(t_1'))\rangle\langle T(E_b(t_2)E_d(t_2'))\rangle \tag{100}$$

The 1/4 comes from the fact that we have two identical vertices for both sides so we have to expand out the time evolution operator to quadratic order in both interactions. Bold letters have been used to distinguish between worldlines, as the particles need not have the same properties.

Write the time ordered product for the dipoles as

$$\begin{aligned}\langle T(p_a(t_1)p_b(t_2))\rangle &= \frac{1}{2\pi i}\int d\tau \frac{1}{\tau - i\epsilon}\sum_n \langle 0 \mid p_a(0) \mid n\rangle\langle n \mid p_b(0) \mid 0\rangle e^{-i(E_0-E_n+\tau)(t_1-t_2)} \\ &+ \frac{1}{2\pi i}\int d\tau \frac{1}{\tau - i\epsilon}\sum_n e^{-i(E_0-E_n+\tau)(t_2-t_1)}\langle 0 \mid p_b(0) \mid n\rangle\langle n \mid p_a(0) \mid 0\rangle\end{aligned} \tag{101}$$

where we have used the integral representation for the theta function. The sum over intermediate energy eigenstates includes degenerate states. The box diagram is then given as

$$\begin{aligned}&Box = \frac{1}{16\pi^2}\int dt_1 dt_2 dt_1' dt_2' \int \frac{[d^4k]}{k_0^2 - \vec{k}^2 + i\epsilon}\frac{[d^4p]}{p_0^2 - \vec{p}^2 + i\epsilon} \\ &N_{ijkl}(k,p)e^{-ik_0(t_1-t_1')+i\vec{k}\cdot(\vec{x}_1(t_1)-\vec{x}_2(t_1'))}e^{-ip_0(t_2-t_2')+i\vec{p}\cdot(\vec{x}_1(t_2)-\vec{x}_2(t_2'))} \\ &\sum_{n,m}\int d\tau \frac{1}{\tau - i\epsilon}\langle 0 \mid p_a(0) \mid n\rangle\langle n \mid p_b(0) \mid 0\rangle(\delta_{ia}\delta_{jb}e^{-i(E_0-E_n+\tau)(t_1-t_2)} + \delta_{ib}\delta_{ja}e^{-i(E_0-E_n+\tau)(t_2-t_1)}) \\ &\times \int d\tau' \frac{1}{\tau' - i\epsilon}\langle 0 \mid \mathbf{p}_c(0) \mid m\rangle\langle m \mid \mathbf{p}_d(0) \mid 0\rangle(e^{-i(E_0-E_m+\tau')(t_1'-t_2')}\delta_{ck}\delta_{dl} + \delta_{cl}\delta_{dk}e^{-i(E_0-E_m+\tau)(t_2'-t_1')})\end{aligned} \tag{102}$$

[29] The coupling to intrinsic spin has no explicit powers of the velocity, but is still suppressed as can be seen by the explicit powers of $1/c$ in the magnetic moment.

N_{ijkl} is the tensor that arises from

$$\langle T(E_i E_k)\rangle\langle T(E_j E_l)\rangle = \langle T((\partial_i A_0 - \partial_0 A_i)(\partial_k A_0 - \partial_0 A_k))\rangle\langle T((\partial_j A_0 - \partial_0 A_j)(\partial_l A_0 - \partial_0 A_l))\rangle \tag{103}$$

which we will return to in a moment. Next we perform the time integrals ,

$$\begin{aligned} Box = \quad & \frac{(2\pi)^3}{16\pi^2}\delta(0)\int\frac{[d^4k]}{k_0^2-\vec{k}^2+i\epsilon}\frac{[d^3p]}{k_0^2-\vec{p}^2+i\epsilon}N_{ijkl}(k,p)e^{i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)}e^{i\vec{p}\cdot(\vec{x}_1-\vec{x}_2)} \\ & \sum_{n,m}\langle 0 \mid (p_a(0)\mid n\rangle\langle n\mid p_b(0))\mid 0\rangle(\delta_{ia}\delta_{jb}\frac{1}{E_n-E_0-k_0-i\epsilon}+\delta_{ib}\delta_{ja}\frac{1}{E_n-E_0+k_0-i\epsilon}) \\ \times & \langle 0\mid(\mathbf{p}_c(0)\mid m\rangle\langle m\mid \mathbf{p}_d(0))\mid 0\rangle(\frac{1}{E_m-E_0+k_0-i\epsilon}\delta_{ck}\delta_{dl}+\delta_{cl}\delta_{dk}\frac{1}{E_m-E_0-k_0-i\epsilon}) \end{aligned} \tag{104}$$

Following an analogous calculation it is straightforward to show that the cross-box diagram gives the same contribution as a consequence of the fact that the initial and final states are identical on both sides of the diagram.

Let us pause at this point to discuss the power counting. This integral has two relevant external scales: $k \sim 1/r$ and ΔE, the energy splittings. By studying the pole structure we can see that the k_0 integral will get contributions from both of these scales and on dimensional grounds $|\vec{k}|\sim 1/r$. Now since we are interested in the regime where $\Delta E \ll k$ the integral will be dominated by the region where $k_0 \sim \Delta E$, so to leading order we may drop k_0 in the photon propagator. Thus corrections to our approximation will be suppressed by $\Delta E/k$, which will act as the expansion parameter. We may also conclude that the photon is a potential mode and instantaneous. We will return to the systematics of this approximation in the next section. If we only keep the leading order terms in k_0, we have

$$N_{ijkl} = k_i p_j k_k p_l. \tag{105}$$

Dropping terms which have poles on the same side of the k_0 real axis we find

$$Box = i\frac{2}{4\times 9}\frac{(2\pi)^2 T}{4\pi^2}\int \frac{[d^3k]}{\vec{k}^2}\frac{[d^3p]}{\vec{p}^2} N_{iikk}(k,p) e^{i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)} e^{i\vec{p}\cdot(\vec{x}_1-\vec{x}_2)}$$
$$\sum_{n,m}\langle 0 \mid (p_A(0)\mid n\rangle\langle n \mid p_A(0)) \mid 0\rangle \times \frac{1}{-2E_0+E_n+E_m}\langle 0 \mid (p_B(0)\mid m\rangle\langle m \mid p_B(0)) \mid 0\rangle). \tag{106}$$

Evaluating the integral we find

$$-iV_{box}(r) = 2\times i\frac{6}{4}\frac{1}{(4\pi)^2 r^6}\sum_{n,m}\frac{\mid\langle 0 \mid p_z \mid n\rangle\mid^2\mid\langle 0 \mid \mathbf{p}_z \mid m\rangle\mid^2}{-2E_0+E_n+E_m-i\epsilon}, \tag{107}$$

where we have used spherical symmetry to reduce the matrix elements and then re-express them in terms of the dipole moment along the z axis to get the final result into the text book form

Adding in the crossed box gives the final result

$$\boxed{V_{VdV}(r) = -\frac{3}{8\pi^2 r^6}\sum_{n,m}\frac{\mid\langle 0 \mid p_z \mid n\rangle\mid^2\mid\langle 0 \mid \mathbf{p}_z \mid m\rangle\mid^2}{-2E_0+E_n+E_m}} \tag{108}$$

Which is the standard result that one could get, using time independent quantum mechanical perturbation theory. Notice that we can drop the $i\epsilon$ since the pole is never reached.

Notice that our method of calculation violated a basic precept of effective field theory in that the integrand did not scale homogeneously in the power counting parameter. That is because when we started the calculation we were working in some hybrid theory where we had degrees of freedom living on the worldline and we had yet to sort out the power counting. Now that we know that the photon exchange is instantaneous (which we might have guessed with a little thought) we could go back and work at the level of the action and separate the soft radiation and potential modes, to generate interactions which scale homogeneously.

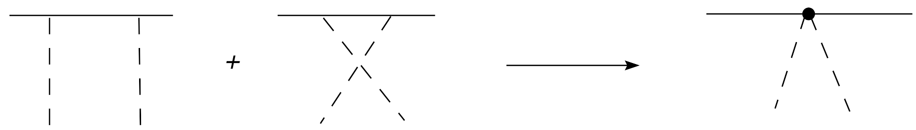

FIG. 9. The Feynman diagram on the left are the full theory diagrams. The intermediate line is off shell and can be shrunk to a point and generates the F^2 operator.

III. MATCHING THE COEFFICIENTS C_E AND C_B

When we work with the theory (90), valid for $k \ll \Delta E$, we have two unknown coefficients when we include the leading order finite size effects, C_E and C_B. If we did not know the underlying theory, then this is about all we could say. We would have to fit the coefficients of the finite size operators using the data. This is not to say we would not have predictive power, since there are numerous observables but only a fixed number of coefficients at any finite order. However, we would not really know when the theory broke down, we could only make an educated guess that it would be near $r \sim R$, since a naive guess would be $\Delta E \sim 1/R$. The only way we would know for sure would be to compare our predictions with experiment for smaller and smaller values of R, until our predictions fail by factors of order unity.

However, we *do* know the the underlying theory, it is given by the action (99). Thus we can perform a matching procedure to fix the coefficients C_E, C_B as well as the coefficients of all other higher dimensional operators in the action. The matching is performed by calculating some quantities in the full and effective theories. We then expand the full theory result to the order in the power expansion of interest and choose the matching coefficients to reproduce the full theory result.

For the problem at hand, we will choose the scattering of low energy (i.e. $k \sim 1/r$) photons off of the worldline. An incoming photon [30] will polarize the particle prior to it re-radiating. The relevant diagrams are shown in figure (9). The result of these two diagrams will fix the matching coefficients. The sum of these two diagrams is given by

$$iM = 2 \times \int dt dt' \frac{(i)^2}{2} \langle T \left(\vec{\mathbf{p}}(t) \cdot (\vec{\epsilon} k_0 - \vec{k} \epsilon^0) \vec{\mathbf{p}}(t') \cdot (\vec{\epsilon}^\star p_0 - \vec{p} \epsilon^{\star 0}) \right) \rangle e^{-ip\cdot x(t')} e^{ik \cdot x(t)}, \tag{109}$$

[30] We can choose to match with physical radiation photons or off shell potential photons since gauge invariance implies that there is only one matching coefficient.

The two follows from the fact that the diagrams give the same result.

Expanding out the time ordered product as in the previous calculation we find

$$\begin{aligned} iM &= -(4\pi i)\delta(k_0-p_0)\sum_n \frac{\langle 0 \mid \mathbf{p_i}(0) \mid n\rangle\langle n \mid \mathbf{p_j}(0) \mid 0\rangle}{E_0-E_n-k_0-i\epsilon}(\epsilon_i k^0 - k_i\epsilon^0)(\epsilon_j^\star p^0 - p_j\epsilon^{\star 0})e^{i\vec{p}\cdot\vec{x}}e^{-i\vec{k}\cdot\vec{x}} \\ &\approx -(4\pi i)\delta(k_0-p_0)\sum_n \frac{\langle 0 \mid \mathbf{p_i}(0) \mid n\rangle\langle n \mid \mathbf{p_j}(0) \mid 0\rangle}{E_0-E_n}(\epsilon_i k^0 - k_i\epsilon^0)(\epsilon_j^\star p^0 - p_j\epsilon^{\star 0})e^{i\vec{p}\cdot\vec{x}}e^{-i\vec{k}\cdot\vec{x}} + ... \end{aligned} \tag{110}$$

Where we have expanded in small k_0. If we were interested in matching to operators with more then two derivatives we would have to keep higher orders in k_0.

The goal now is to find operators in the effective theory which will reproduce this result to the order we are interested. Let us consider the contributions in the effective theory from the operators E^2 and (B^2). For incoming momentum polarization $(k,\epsilon^\rho(k))$ and outgoing $(p,\epsilon^\sigma(p))$

$$iM_{E^2} = 2\times iC_E(2\pi)\delta(k_0-p_0)e^{-i\vec{k}\cdot\vec{x}}e^{i\vec{p}\cdot\vec{x}}(k^0\epsilon^i - k^i\epsilon^0)(p^0\epsilon^{i\star} - p^i\epsilon^{0\star}) \tag{111}$$

$$iM_{B^2} = 2\times iC_B(2\pi)\delta(k_0-p_0)e^{-i\vec{k}\cdot\vec{x}}e^{i\vec{p}\cdot\vec{x}}(k^i\epsilon^j)(p^a\epsilon^{\star b})\epsilon^{ijd}\epsilon^{abd} \tag{112}$$

The overall factors of two coming from the fact that we can contract the external photons in two ways, both of which give the same result. To solve for the two unknowns we may choose two different final state polarization configurations. Let us choose two (unphysical) timelike polarizations $\epsilon^0\epsilon^0$ for which only the electric operator contributes

$$iM_{E^2} = 2\times iC_E(2\pi)\delta(k_0-p_0)e^{-i\vec{k}\cdot\vec{x}}e^{i\vec{p}\cdot\vec{x}}\epsilon^0\epsilon^{\star 0}(\vec{p}\cdot\vec{k})) \tag{113}$$

Comparing to the full theory side (110), and using our assumption of spherical symemtry we can exract

$$\alpha_E = \frac{1}{2\pi}\sum_n \frac{\langle 0 \mid \mathbf{p_z}(0) \mid n\rangle\langle n \mid \mathbf{p_z}(0) \mid 0\rangle}{E_n - E_0} \tag{114}$$

which agrees with the standard text book result for the static polarizability.

IV. POWER CORRECTIONS

A. Corrections to the Casimir Polder Result

Our leading order finite size operators start at $O(R^3)$[31] , and the first corrections to our calculation will come from the next set of allowable higher dimensional terms. These terms are restricted by gauge and Lorentz invariance. In addition, when going to higher orders we also have to consider charge conjugation invariance which reduces to the statement that the action should be invariant under

$$A_\mu \to -A_\mu \qquad J_\mu \to -J_\mu. \tag{115}$$

Finally, we have reparameterization invariance which can necessitate the inclusion of factors of $\sqrt{v^2}$ in the operator. We have one other constraint at our disposal. We may eliminate operators which vanish by use of the equations of motion. So for instance, any operator which contains $\partial_\mu F^{\mu\nu}$ may be set to zero. To illustrate this point in a simple example let's consider a scalar field instead of the photon. Furthermore, let us suppose we added a term to the free worldline action such that

$$S = -\int d^4y \ \frac{1}{2}\phi\partial^2\phi + \int d\tau C R^n F(\phi)\partial^2\phi, \tag{116}$$

where C is dimensionless and n is fixed by the units of F . We can see that we can eliminate this term via the field redefinition

$$\phi(y) \to \phi(y) + CR^n F(\phi)\delta^4(y^\mu - x^\mu(\tau)). \tag{117}$$

Note that, in general, this transformation will generate new operators on the world line. For instance plugging in the shift into the worldline operator itself will generated a, divergent, order $O(R^{2n})$ contribution. But since we have to write down all possible operators to begin with, that operator was already there anyway. This shift will renormalize the (unknown until fixed) coefficient of that operator. One can think of this as a change of basis. There is some freedom in choice of world line operators, and the coefficients will depend upon that

[31] Again ignoring terms which vanish in the static limit.

choice of basis. Physical results are independent of this choice, as is proved in section ().

It is interesting to ask whether or not we are free to eliminate operators which vanish by use of the leading order equations of motion for the worldline itself? That is, suppose we had a term in the worldline action of the form

$$S = \int d\tau a^\mu F^\mu(x, v) \tag{118}$$

where a^μ is the four acceleration. Can we drop this term if there are no external forces? The answer is yes, but [14] the appropriate coordinate transformation will lead to a change into a non-inertial frame. In general relativity this is of course a non-issue [15]. Note however, that at the level of the equations of motion, we may indeed use the lower order equations of motion in higher order corrections. This is sometimes called "order reduction", and is really just implementing perturbation theory. In fact, it is crucial that when solving the equations of motion, that all finite size effects be treated as perturbations. Typically the finite size effects lead to higher time derivatives of the worldline coordinate, and if we were to treat these as leading order contributions we would run into all sorts of pathologies.

We are now in a position to write down the first corrections the action. For our neutral particle interaction example, we may ignore acceleration dependent terms. In which case the first terms which have been left off in the action (2) will contain four photon fields. The enumeration of the these terms will be left as an exercise for the reader.

B. Corrections to the Van der Waals Result

There are multiple sources of corrections to the classic Van der Waals result. We have restricted ourselves to one particular expansion parameter, $\Delta E/k$, ignoring corrections from the magnetic dipole moment which we took to be suppressed by an additional parameter v, the velocity of the internal constituents of the composite particle. If we wished we could include those effects with relative ease by including the coupling to the magnetic field. Note that these velocity dependent corrections should be distinguished from the velocity dependent corrections which would arise from the motion of the composites themselves. These corrections could also be easily included by covariantizing the theory. We will discuss such a covariant theory in the next chapter.

If we continue to restrict ourselves to only $\Delta E/r$ corrections, the next order multipole moment would be the electric quadrapole Q_{ij} which couples via the interaction

$$S = \int dt Q_{ij} \partial_i E_j. \tag{119}$$

On dimensional grounds, since $Q_{ij} \sim x^2$, this operator will be suppressed by one power of R/r relative to the dipole. The inclusion of such power correction effects is left as an exercise.

Exercise 2.8: Introduce an electric quadrapole $Q_{ij}(t)$ degrees of freedom onto the world-line and use it to calculate the quadrapole-dipole force analog of the the Van der Waals force. You may compare your answer to the result in [13].

The existence of additional sources of power corrections follows from our investigation of the box and crossed box diagrams. We noticed previously that the result in (104) does not scale homogeneously in our expansion parameter. These diagrams get contributions from two regions of the momentum integral

$$\begin{aligned} k_0 \sim\mid \vec{k} \mid\sim 1/r, \\ k_0 \sim \Delta E, \mid k \mid\sim 1/r. \end{aligned} \tag{120}$$

The reason we are still getting this mixing of scales is that we have one more step to perform, before reaching our final low energy theory which only contains the lowest energy scale $\Delta E \ll 1/r$. We must integrate out the scale $1/r$. To accomplish this goal we now match onto an effective theory, which in this case is nothing but a set of potentials. In a sense the lowest energy theory is rather simple since there are no photons left. The only dynamical degrees of freedom left are the word-line excitations over which we will sum.

In this case the matching procedure is rather trivial because the effective theory has no diagrams to calculate. So we all we have to do is to calculate in the full theory and expand in powers of $\delta \equiv \Delta E r$. This will generate a set of potentials. Fortunately, we have already done all the work. The relevant integral is given by (104). To match we calculate this integral and then expand it in δ. Or we can asymptotically expand the integrand to pick out the relevant pieces as we shall now do.

Consider matching at next order. As previously discussed there are two regions of k_0

space which dominate the integral. The leading piece came from the region where $k_0 \sim \Delta E$ and $| \vec{k} | \sim 1/r$ and gave us the Van der Walls result. Going to next order in $k_0/ | \vec{k} |$ (staying in this region) we have two contributions. One from the time derivative terms ($O(k_0^2)$) we drop in (103), and the other from expanding the denominator. The result of keeping the $O(k_0^2)$ pieces of the numerator is to use

$$N_{ijkl} = -((E_0 - E_n)^2 \delta_{ik} k_j k_l + (E_0 - E_m)^2 \delta_{jl} p_j p_l) \tag{121}$$

in equation (106). However this leads to a contribution proportional to $\delta^3(\vec{x}_1 - \vec{x}_2)$ and thus vanishes.

So the first power correction comes from expanding the denominator.

$$\begin{aligned} iM_{full} \approx \quad & 2 \times \frac{(2\pi)^3}{16\pi^2} \delta(0) \int \frac{[d^4k] k_0^2}{\vec{k}^4} \frac{[d^3p]}{\vec{p}^2} N_{ijkl}(k,p) e^{i\vec{k}\cdot(\vec{x}_1 - \vec{x}_2)} e^{i\vec{p}\cdot(\vec{x}_1 - \vec{x}_2)} \\ & \times \sum_{n,m} \langle 0 \mid (p_a(0) \mid n\rangle\langle n \mid p_b(0)) \mid 0\rangle (\delta_{ia}\delta_{jb} \frac{1}{E_n - E_0 - k_0 - i\epsilon} + \delta_{ib}\delta_{ja} \frac{1}{E_n - E_0 + k_0 - i\epsilon}) \\ & \times \langle 0 \mid (\mathbf{p}_c(0) \mid m\rangle\langle m \mid \mathbf{p}_d(0)) \mid 0\rangle (\frac{1}{E_m - E_0 + k_0 - i\epsilon} \delta_{ck}\delta_{dl} + \delta_{cl}\delta_{dk} \frac{1}{E_m - E_0 - k_0 - i\epsilon}), \end{aligned} \tag{122}$$

where the factor of two comes from the fact that we can expand either denominator. This integral is down by $(\Delta E r)^2$ relative to the leading order result. Performing the energy integral by contours contracting all the indices and then performing the tensor integral over the spatial momenta, the correction of order $(\Delta E r)^2$ stemming from dipole fluctuations is given by

$$V_{VdW}^{ret} = -\frac{3}{16\pi^2} \frac{1}{r^4} \sum_{n,m} \mid \langle 0 \mid (p_a(0) \mid n\rangle \mid^2 \mid \langle 0 \mid (\mathbf{p}_c(0) \mid m\rangle \mid^2 \left[(\frac{(E_n - E_0)^2 + (E_m - E_0)^2}{E_n + E_m - 2E_0}). \right]$$

Physically we may interpret this correction as being a retardation effect.

The attentive reader should be puzzled by this result. What seems odd is that we have a sum over exicted states that is not restricted. It is natural to ask, in what sense is this correction suppressed? We claimed it was down by $(\Delta E r)^2$, but what precisely do we mean by ΔE. We assumed that we are probing the system at short enough wavelengths that some set of excited states act as long (relative to the time scale $1/r$) lived intermediate states. But surely there will be some subset of the excitations which will not satisfy this criteria. Those states should not be included in the sums. We should truncate the sum, and not include any state E_n for which $E_n - E_0 > 1/r$. On the other hand these additional states will have some effect, albeit suppressed, on the force law. How do they show up? The answer is that they get integrated out, and generate the operators E^2 and B^2. So if we restrict ourselves to the electric dipole sector of the theory, then we have the following result for the corrections to force law. Assuming, again, for the sake of simplicity that the two particles have the same excitation spectrum

$$V = -\frac{23\alpha_E(E_{c+1})^2}{4\pi r^7} + \bar{V}_{VdW} - \frac{3}{8\pi^2}\frac{1}{r^4}\overline{\sum}_{n,m} \mid \langle 0 \mid p_a(0) \mid n \rangle \mid^2 \mid \langle 0 \mid \mathbf{p}_c(0) \mid m \rangle \mid^2 \left[\left(\frac{(E_n - E_0)^2}{E_n + E_m - 2E_0} \right). \right] \tag{123}$$

where the barred sum is over all states whose energy is less then $(E_c - E_0) > 1/r$. Here the susceptability has been generalized, and depends upon E_c. It the susceptability of a imagined particle for which the states below E_c are ignored. The most energetic of such states is labelled by E_c. The Wilson coefficient $\alpha(E_{c+1})$ for the Casimir-Polder term is fixed by the matching calculation (114) but now the sum only include states whose energy is above E_c. The final result is independent of the choice of E_c, as long at it is larger than $1/r$. This result is valid over the entire range as long as the multipole expansion is valid $(R \ll r)$. When $k < E_1 - E_0$, $E_c = E_0$ and there is no sum to perform. As $1/r$ increases the sums turn on, and the effects of the Casmir-Polder term decreases.

Finally we may ask which operator in the effective theory generates this power correction? We have seen that higher multipole generate corrections in R/r where quantum mechanically $R \sim \langle x \rangle$. So the corrections of the form $\Delta E r$ come from operators of the form $\ddot{\vec{p}} \cdot \vec{E}$, where single time derivatives are excluded by time reversal invariance.

V. APPENDIX: CALCULATING THE VACUUM PERSISTENCE AMPLITUDE USING FEYNMAN DIAGRAMS

This appendix is meant to allow those who have little or no experience using Feynman diagrams to calculate the potentials discussed in this chapter. Readers interested in more details are urged to consult [16] or some equivalent text. We start with the definition of the vacuum persistence amplitude[32] (partition function) for a photon (A_μ) coupled to a source J and split the Lagrangian into a leading order (kinetic piece) L_0 and an interaction Lagrangian L_I.

$$Z[J] = \int DA_\mu e^{i\int d^4x(L_0(A,\partial A)+\lambda L_I(A_\mu)+J_\mu A^\mu)} \tag{124}$$

where we include the gauge fixing term in the leading order action

$$L_0(A,\partial A) = -\frac{1}{4}F^2 - \frac{1}{2}(\partial\cdot A)^2. \tag{125}$$

The interaction will be treated in an expansion in λ and L_I is a polynomial in A_μ. Thus we may write

$$Z[J] = e^{i\int d^4x L_I(\frac{\delta}{\delta J})}\int DA_\mu e^{i\int d^4x(L_0(A,\partial A)+J_\mu A^\mu)} \tag{126}$$

Since L_0 is quadratic in the fields, we may perform the Gaussian integral over the fields leaving

$$Z[J] = e^{i\int d^4x L_I(\frac{\delta}{\delta J})}e^{-\frac{i}{2}\int d^4x d^4y\left(J_\mu(x)G_F^{\mu\nu}(x-y)J_\nu(y)\right)}. \tag{127}$$

This is the master formula from which all diagrams will follow. For now we will leave the currents arbitrary, and at the end we will set

$$J_\mu(x) = \sum_i \int d\tau_i e_i v_i^\mu(\tau_i)\delta^{(4)}(x^\mu - y_i^\mu(\tau_i)), \tag{128}$$

in accordance with the theory introduced in this chapter.

The finite size effects are incorporated into L_I. Let us consider the case where

$$L_I = \sum_i \int d\tau_i C_1^i F^2(x(\tau_i)) = -2C_1^i \sum_i \int d\tau_i A_\mu(x(\tau_i))(\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu)A_\nu(x(\tau_i)) \tag{129}$$

Instead of working out a general set of Feynman rules we will consider one simple example. It will hopefully be clear how the rules follow from a general interaction.

Let us now see how we can reproduce the result for the Casimir-Polder potential (96). Since we are interested in the potential between the worldlines, the leading order contribution

[32] We will ignore the denominator in the definition since it is independent of the sources and only serves to change an irrelevant overall factor.

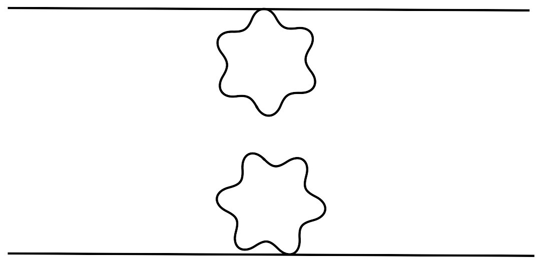

FIG. 10. The Feynman diagram corresponding to the contribution from (132).

will come from expanding L_I to linear order in each of the C^i_E. To this order we have

$$Z[J]\mid_{C_E^2} = 4(i)^2 \int d\tau_1 d\tau_2 C_1^1 C_1^2 (\partial_{[\mu} \frac{\delta}{\delta J_{\nu]}(x(\tau_1))})(\partial^{[\mu} \frac{\delta}{\delta J^{\nu]}(x(\tau_1))})(\partial_{[\rho} \frac{\delta}{\delta J_{\sigma]}(x(\tau_2))})(\partial^{[\rho} \frac{\delta}{\delta J^{\sigma]}(x(\tau_2))}) \\ e^{-\frac{i}{2}\int d^4x d^4y \left(J_\mu(x) G_F^{\mu\nu}(x-y) J_\nu(y)\right)} \tag{130}$$

There are various ways in which the functional derivatives can hit the exponential, and each possibility corresponds to a particular Feynman diagram, some of which give identical contributions, and others of which will give irrelevant (divergent) constants which are independent of the spatial separations of the wordlines. For instance, when two derivatives acting with the same wordline argument hit the same quadratic, i.e.

$$\frac{\delta}{\delta J_{\nu]}(x(\tau_i))} \frac{\delta}{\delta J_{\nu]}(x(\tau_i))} \int d^4x d^4y \left(J_\mu(x) G_F^{\mu\nu}(x-y) J_\nu(y)\right), \tag{131}$$

we get a contribution of the form

$$(4i^2)(-i)^2 \int d\tau_1 d\tau_2 C_1^1 C_1^2 (\partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu) G^F_{\mu\nu}(0)(\partial^2 g^{\rho\sigma} - \partial^\rho \partial^\sigma) G^F_{\rho\sigma}(0). \tag{132}$$

Now we can draw the corresponding Feynman diagram. For each photonic Greens function $-iG_F(x-y)$ we associate a curly line which extends from x to y. We may write the two worldlines as solid lines which traverse the top and the bottom of the diagram. For the contribution (132) we have $G_F(0)$ which arises from $G_F(x(\tau_i) - x(\tau_i))$. Thus the corresponding diagram is show in figure (10). These diagrams are divergent and, as should be clear, independent of the distance to the worldline, they contribute an overall constant that plays no

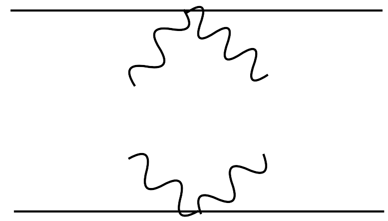

FIG. 11. Two F^2 vertex insertions which generate all possible diagrams at this order.

physical role[33].

The first relevant contribution comes from the diagram (8). This diagram arises from the contribution where one functional derivative from each worldline hits the same quadratic. A little algebra shows that there are two ways in which this can occur and the resulting contribution is given by

$$2 \times 4 \times (4i^2)(-i)^2 \int d\tau_1 d\tau_2 C_1^1 C_1^2 (\partial_\mu^{x_1} \partial_{[\rho}^{x_2} G_{\nu\sigma]}(x_1 - x_2) + \partial_\nu^{x_1} \partial_{[\sigma}^{x_2} G^F_{\mu\rho]}(x_1 - x_2)) \partial^\mu_{x_1} \partial^\rho_{x_2} G^F_{\nu\sigma}(x_1 - x_2). \tag{133}$$

where the factor of two comes from the ways in which we can get this contribution after differentiating, and the four comes from the anti-symmetry property of the field strength.

In retrospect we can see that we can generate all the possible contributions by drawing two vertices each with two photon lines coming out of them as shown in (11), and then contracting the lines in all possible ways. The possibilities corresponds to the ways in which the functional derivatives can hit the exponential.

Doing things in this way is rather cumbersome. There is a simple mnemonic device which greatly simplifies matters. If were interested in a contribution (diagram) coming from two insertions of the F^2 operator, then after expanding the interaction Lagrangian piece of the exponential, we may write the contribution as an expectation value namely

$$(i^2) \int d\tau_1 d\tau_2 \langle F_{\mu\nu} F^{\mu\nu}(x_1) F_{\rho\sigma} F^{\rho\sigma}(x_2) \rangle \tag{134}$$

Then we associate pairs of A_μ with a factor of $-iG_F$ which we write as

$$-iG_F^{\mu\nu} \equiv \langle A^\mu A^\nu \rangle, \tag{135}$$

[33] The divergent constant can be absorbed into the masses.

and contract the pairs in all possible ways. This is called Wicks' theorem and follows from expanding the expression (127).

The contribution, corresponding to figure (10) gives the product of $G_F(0)^2$ which we have already discussed while the other is given by

$$2 \times (i^2) \int d\tau_1 d\tau_2 \langle F_{\mu\nu}(x_1) F^{\mu\nu}(x_2)\rangle \langle F_{\rho\sigma}(x_1) F^{\rho\sigma}(x_2)\rangle. \tag{136}$$

where the two follows from the fact that we have two ways of contracting which are identical. Factors such as this are usually called "symmetry factors". This final result reproduces the result (93).

When doing simple diagrams such as this one it is easy to simply expand out the exponential and use Wicks' theorem to contract the fields. However, when we consider more complicated diagrams it is simpler to first write down, all the possible diagrams and then use a set of Feynman rules to write down the corresponding integral expression.

Let us consider working in the effective non-relativistic theory to see how we can reproduce to the electromagnetic potential up to $O(v^2)$. The leading order Lagrangian is given by

$$L_0 = \int dt \int_{\vec{k}} \frac{1}{2} \vec{k}^2 \mathbf{A}^{\mu}_{\vec{k}} \mathbf{A}_{-\vec{k}\mu}(t) \tag{137}$$

and the interaction, or perturbing, Lagrangian is given by

$$L_I = \sum_i \int dt_i e_i \int_{\vec{k}} e^{i\vec{k}\cdot\vec{x}_i(t_i)} \mathbf{A}_{\vec{k}0}(t_i) - \sum_i e_i \int_{\vec{k}} e^{i\vec{k}\cdot\vec{x}_i(t_i)} v_i^a \mathbf{A}^a_{\vec{k}}(t_i) dt_i + \frac{1}{2}\int dt \int_{\vec{k}} \partial_0 \mathbf{A}_{\vec{k}\mu} \partial_0 \mathbf{A}^{\mu}_{-\vec{k}} \tag{138}$$

Notice that the first term, a propagator correction, is order v^2 while the second a vertex correction, is only down by v. However, in the Feynman gauge, we will need two vertices will be needed to get a non-vanishing result.

Let's start with the leading order, Coulomb potential. Using the standard notation that the value diagram is equal to iM we have

$$iM = (i)^2 \langle \int dt_1 dt_2 e_1 e_2 \int_{\vec{k}} e^{i\vec{k}\cdot\vec{x}_1(t_1)} \mathbf{A}_{\vec{k}0}(t_1) \int_{\vec{k}'} e^{i\vec{k}'\cdot\vec{x}_2(t_2)} \mathbf{A}_{\vec{k}'0}(t_2) \rangle \tag{139}$$

there is only one possible contraction, corresponding to single photon exchange. Then using the result (55)

$$\langle \mathbf{A}_{\vec{k}\mu}(t_1)\mathbf{A}_{\vec{q}\nu}(t_2)\rangle \equiv G_{\mu\nu}(\vec{k}) = (2\pi)^3\delta(t_1-t_2)\delta^3(\vec{k}+\vec{q})\frac{ig_{\mu\nu}}{\vec{k}^2}, \tag{140}$$

we find

$$iM = -ie_2e_2\int\frac{[d^3k]}{\vec{k}^2}e^{i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)} = -i\int dt\frac{e_1e_2}{4\pi\mid\vec{x}_1-\vec{x}_2\mid} \tag{141}$$

It is hopefully clear to the read that we could have avoided using Wicks theorem by just drawing the diagram involving the leading order vertex in figure (**??**), and the propagator connecting them.

The relevant pieces on the Lagrangian at $O(v^2)$ are the last two terms in (138). The first of these terms generates a correction to the propagator . To see how this comes about we can consider Wicks theorem. When expanding the exponential we will get a piece proportional to

$$iM = \frac{(i)^3}{2}\langle\int dt_1dt_2e_1e_2\int_{\vec{k}}e^{i\vec{k}\cdot\vec{x}_1(t_1)}\mathbf{A}_{\vec{k}0}(t_1)\int_{\vec{k}'}e^{i\vec{k}'\cdot\vec{x}_2(t_2)}\mathbf{A}_{\vec{k}'0}(t_2)\int dt\int_{\vec{k}}\partial_0\mathbf{A}_{\vec{k}\mu}\partial_0\mathbf{A}^{\mu}_{-\vec{k}}\rangle \tag{142}$$

There are two possible ways of contracting the fields, which will kill the overall factor of 1/2. The contractions give two propagators except now we have two derivatives hitting the delta functions.

VI. PROBLEMS

1. List the set of operators which correspond to the leading corrections to (2). Then draw the Feynman diagrams which will generate the first corrections to the $1/r^7$ potential when $1/r \ll \Delta E$. On dimensional grounds determine how these corrections scale in powers of $1/r$.

2. This problem[34] has to do with the *gravitational* Van Der Waals interaction. That is, we will consider a set of cases where force between two massive objects is purely quantum mechanical, since the loop expansion is also an expansion in $\hbar$. This scenario arises for co-dimension two [35] objects.

[34] This problem is for those who have some training in general relativity and is highly instructive. Those who do not have the background may wish to first read the next chapter and return to this problem.

[35] This means that the spatial dimensions of the objects equals $d-2$, where d is the number of spatial dimensions.

First let us show that the classical force between such objects vanishes, when we take the tension (mass per unit length) of the objects to be small compared to the Planck scale M_{pl}, where $G_N = 1/(32\pi M_{pl}^2)$. This approximation allows us to expand the metric around flat space, such that

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{h_{\mu\nu}}{M_{pl}}. \tag{143}$$

The action for the strings are given by

$$S = -\sum_i \tau_i \int d^4x \sqrt{g^i} \delta^{(2)}(x - x_i) \tag{144}$$

where τ_i and g_i are the tension and induced metric of the i'th string. For simplicity we will take the string to lie along one coordinate axis and we will choose the local space-time coordinates on the string to coincide with the global coordinate system. In this way the induced metric is just the value of the bulk metric on the string restricted to the $0, 1$ coordinates.

We expand the action using

$$\sqrt{g^i} \approx 1 + \frac{h_a^b}{2M_{pl}} + \frac{h_a^a h_b^b}{8M_{pl}^2} - \frac{h_a^b h_a^b}{4M_{pl}^2} + ... \tag{145}$$

where the indices are now summed only over the two dimensional sub-space. The classical force will then arise from the one graviton exchange diagram. Using the background field method discussed in this chapter, where the graviton propagator takes the form

$$D_{\mu\nu,\rho\sigma}(q) = -i\frac{\eta_{\mu\sigma}\eta_{\nu\rho} + \eta_{\nu\sigma}\eta_{\nu\rho} - \frac{2}{d-1}\eta_{\mu\nu}\eta_{\rho\sigma}}{\vec{q}^2} \tag{146}$$

show that the classical force vanishes.

To prove that the classical force vanishes, to all orders in the tension, one must solve the full einstein equations to account for the curving of the space. However, as was shown in [17], co-dimension two objects leave space uncurved. The net effect of the tension is to remove a deficit angle in the space-time. That is, the space is conical. Thus there is no classical force to all orders in the tension.

To find the first correction due to the tension on the force, we consider the analog of figure (8). Using the action (52) show that this diagram generates the potential per unit length

$$V = -\frac{\tau_1 \tau_2}{64\pi^3 R^2 M_{pl}^4} \tag{147}$$

where R is the distance between the strings.

Note that in gravity we have additional contributions beyond those in diagram (8) arising from graviton self interactions. For the discussion of the complete force including these effects see [18].

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