[05/03/17]

Conventions

$$X_{closed}(z,\bar{z}) = \frac{1}{2}\left(X_L(z) + \tilde{X}_R(\bar{z})\right)$$

$$\begin{cases} X_L(z) = x_L - 2\alpha' i p_L \ln z + i\sqrt{2\alpha'} \sum_{n\neq 0} \frac{\alpha_n}{n} z^{-n} \\ X_R(\bar{z}) = \bar{x}_R - 2\alpha' i p_R \ln \bar{z} + i\sqrt{2\alpha'} \sum_{n\neq 0} \frac{\tilde{\alpha}_n}{n} \bar{z}^{-n} \end{cases}$$

with $z = e^{2i(\tau+\sigma)} = e^{2\tau_E + 2i\sigma}$

- $[x^i, p^j] = i\, G^{ij}$ $\quad [\alpha^i_n, \alpha^j_m] = G^{ij}\, n\, \delta_{n+m,0}$

- $X(\sigma,\tau) = \frac{x_L + x_R}{2} + 2\alpha'(p_L + p_R)\tau -$

$$+ i\frac{\sqrt{2\alpha'}}{2} \sum{}' \left[\frac{\alpha_n}{n} e^{-2i(\tau+\sigma)n} + \frac{\tilde{\alpha}_n}{n} e^{-2in(\tau-\sigma)n} \right]$$

$$= \sum x_n e^{-2inσ} + 2RW\sigma$$

$\Rightarrow n \neq 0$ $\quad x_n = i\frac{\sqrt{2\alpha'}}{2n}\left[\alpha_n e^{-i 2n\tau} - \tilde{\alpha}_{-n} e^{i2n\tau}\right]$

$n = 0$ $\quad x_0 = \frac{1}{2}(x_L + x_R) + 2\alpha'(p_L + p_R)\tau$

- $\mathcal{P}_i(\sigma,\tau) = \frac{1}{2\pi\alpha'}\left(G_{ij}\,\dot{X}^j - B_{ij}\,X'^j\right)$

$$\Rightarrow \quad \frac{1}{2\pi\alpha'}\,\dot{X}_i$$

$2\pi\alpha'\mathcal{P}_i(\sigma,\tau)$

$$= 2\alpha'(p_L+p_R) + \sqrt{2\alpha'}\,{\sum_n}'\left[\alpha_n e^{-i2n(\tau+\sigma)} + \tilde{\alpha}_{-n}\, e^{-i2n(\tau-\sigma)}\right]$$

$$= \pi \sum p_n\, e^{-i2n\sigma}$$

$\Rightarrow \quad n \neq 0 \qquad \pi p_n = \sqrt{2\alpha'}\left[\alpha_n e^{-i2n\tau} + \tilde{\alpha}_{-n}\, e^{i2n\tau}\right]$

- $\left[X^i(\sigma), \mathcal{P}_j(\sigma')\right] = \pi \sum_{n,m} \left[x_n^i, p_{mj}\right] e^{-i2n\sigma - i2m\sigma'}$

$$= \pi \sum_n e^{-i2n(\sigma-\sigma')}\,\delta^i_j = \delta_P(\sigma-\sigma')\,\delta^i_j$$

$$\delta_P(\sigma+\pi) = \delta_P(\sigma) = \sum \frac{1}{\pi}\, e^{i2n\sigma}$$

$$\theta_P(\sigma+\pi) = \theta_P(\sigma) = \frac{1}{\pi}\left[\sigma + \frac{1}{2i}{\sum}' \frac{1}{n}\, e^{i2n\sigma}\right]$$

Chk

$$\theta_n = \int_0^\pi \frac{d\sigma}{\pi}\, e^{+i2n\sigma}\,\theta_P(\sigma_0-\sigma) = \int_0^{\sigma_0} \frac{d\sigma}{\pi}\, e^{i2n\sigma}$$

Toy model

- $(a-z)\,|z)=0$

solution $|z) = e^{z a^+} |0\rangle$

but $|z)) = \mathcal{N}\ \delta(a-z)\ |0\rangle$ is also.

but it is trivial since

$$|z)) = \mathcal{N}\ \delta(z)\ |0\rangle$$

- $(x - x_0)\ |x_0\rangle = 0$

Try $|x_0)) = \mathcal{N}\ \delta(x-x_0)\ |0\rangle_a$ since $x|0\rangle \neq 0$

$$|x_0\rangle = \mathcal{N} \int \frac{dl}{2\pi}\ e^{il(x-x_0)}\ |0_a\rangle =$$

$$= \mathcal{N} \int \frac{dl}{2\pi}\ e^{-ilx_0}\ e^{il\frac{a+a^+}{\sqrt{2}}}\ |0\rangle_a$$

BCH $[x,p] = \left[\frac{a+a^+}{\sqrt{2}}, i\frac{a^+-a}{\sqrt{2}}\right] = \frac{i}{2}\left(1-(-1)\right) = i$

$$= \mathcal{N} \int \frac{dl}{2\pi}\ e^{-ilx_0}\ e^{il\frac{a^+}{\sqrt{2}}}\ e^{-\frac{1}{2}\left(\frac{il}{\sqrt{2}}\right)^2 [a^+,a]}\ |0_a\rangle$$

$$= \mathcal{N} \int \qquad e^{-\frac{1}{4}l^2}\ |0_a\rangle$$

$$= \mathcal{N} \sqrt{\frac{\pi}{1/4}}\ e^{+\frac{1}{4}\frac{\left[i\left(a^+/\sqrt{2}-x_0\right)\right]^2}{1/4}}\ |0\rangle_a$$

$$= \mathcal{N} \sqrt{4\pi}\ e^{-\left(\frac{a^+}{\sqrt{2}}-x_0\right)^2}\ |0\rangle_a$$

Chk

$$x\,|x_0\rangle\rangle = \frac{a+a^+}{\sqrt{2}}\,|x_0\rangle\rangle = \frac{1}{\sqrt{2}}\left[a^+ - 2\frac{1}{\sqrt{2}}\left(\frac{a^+}{\sqrt{2}} - x_0\right)\right]\,|x_0\rangle\rangle$$

$$= \frac{1}{\sqrt{2}}\sqrt{2}\,x_0\,|x_0\rangle\rangle = x_0\,|x_0\rangle\rangle$$

Boundary with variable $F(\sigma)$

$$\left[\mathcal{P}_i(\sigma) - F_{ij}(\sigma) X'^j(\sigma) \right]\Big|_{\tau=0} |B[F]\rangle = 0 \qquad \sigma \neq \sigma_t$$

where we suppose $F(\sigma_t)$ is discontinuous

If $B_{ij}=0$ then $\mathcal{P}_i(\sigma) = \frac{1}{2\pi\alpha'} \dot{X}^j(\sigma)\, G_{ij}$

and we can write

$$\left[G_{ij} \dot{X}^j(\sigma) - 2\pi\alpha' F_{ij}(\sigma) X'^j(\sigma) \right] |B[F]\rangle = 0$$

Formally we can write the SOLUTION

$$|B[F]\rangle = \exp\left[i \int_0^{\pi} d\sigma' \frac{1}{2} F_{ij}(\sigma') X^i(\sigma') X'^j(\sigma') \right] |B\rangle$$

they commute

Chk

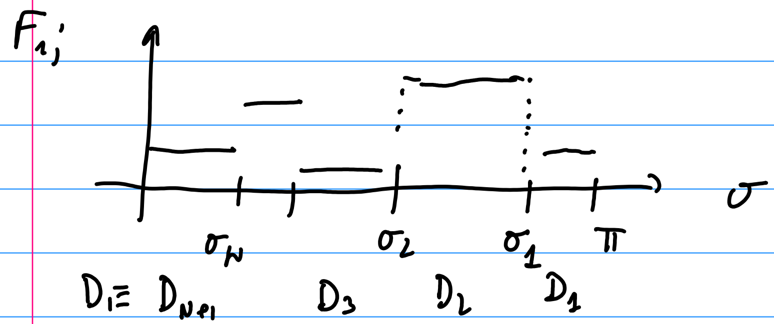
$$\left[\mathcal{P}_i(\sigma) , \int_0^{\pi} d\sigma' \frac{1}{2} F_{em} X^e X'^m \right]$$

$$= \frac{i}{2} \int_0^{\pi} d\sigma' \left[(-i) F_{im} \delta_P(\sigma-\sigma') X'^m - i F_{ei} X^e \partial_{\sigma'} \delta(\sigma-\sigma') \right]$$

$$= -i F_{im}(\sigma) X'^m(\sigma) + \frac{i}{2}(-1) \partial_\sigma F_{ie} X^e(\sigma)$$

If $\qquad F_{ij}(\sigma) = \sum_{t=1}^{N} F_{ij}^{(t)} \, \theta_p(\sigma_{t-1} > \sigma > \sigma_t)$

i.e.



then

$$\partial_\sigma F_{ij}(\sigma) = \sum \delta_p(\sigma - \sigma_t) \left[F_{ij}^{(t)} - F_{ij}^{(t+1)} \right]$$

since

$$F \supset F^{(t+1)} \theta(\sigma - \sigma_{t+1}) \, \theta(\sigma_t - \sigma) + F^{(t)} \theta(\sigma - \sigma_t) \theta(\sigma_{t-1} - \sigma)$$

implies

$$\partial_\sigma F \supset F^{(t+1)} \theta(\sigma - \sigma_{t+1}) (-) \delta(\sigma_t - \sigma) + F^{(t)} \delta(\sigma - \sigma_t) \, \theta(\sigma_{t-1} - \sigma)$$

$$= \left(F^{(t)} - F^{(t+1)} \right) \delta(\sigma - \sigma_t)$$

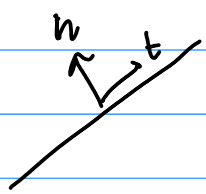
(·) Branes et angle

a) $D1 \subset \mathbb{R}^2$ $\qquad D1 = \{ x^2 = 0 \}$

$$\mathcal{P}_1(\sigma)\, |D1\rangle = \left(X^2(\sigma) - y \right) |D1\rangle = 0$$

b) $D1 \subset \mathbb{R}^2$ $\qquad D1 = \{ n \cdot x = y \}$

consider $\quad t \ / \ t^2 = 1 \qquad t \cdot n = 0 \qquad t$



(t, n) positive

$$t \cdot \mathcal{P}(\sigma)\, |D1\rangle = \left[n \cdot X(\sigma) - y \right] |D1\rangle = 0$$

c) More $D1 \subset \mathbb{R}^2$

$$t(\sigma) \cdot \mathcal{P}(\sigma)\, |D1\rangle = \left[n(\sigma) \cdot X(\sigma) - y(\sigma) \right] |D1\rangle = 0$$

SINCE

$$p^1(\sigma)\,|D2\rangle = p^2(\sigma)\,|D2\rangle = 0$$

The solutions are

a) $$|D1\rangle = \mathcal{N} \prod_\sigma \delta(X^2(\sigma) - y)\ |D2\rangle$$

not obvious how to fix since D2 does not reduce to D1 by a continuous transformation

b) $$|D1\rangle = \mathcal{N} \prod_\sigma (\ n\cdot X(\sigma) - g)\ \ |D2\rangle$$

c)

not necessary! $p^1(\sigma)|D2\rangle = p^2(\sigma)\,|D2\rangle = 0 \Rightarrow p^t|D2\rangle = p^n|D2\rangle = 0$

$$\prod_\sigma \left[\cancel{\delta(t(\sigma)\cdot p(\sigma))}\ \ \delta(n(\sigma)\cdot X(\sigma))\right] |D2\rangle = |D1\rangle$$

where there are not ordering problems since $[t\cdot p,\ n\cdot X] = t\cdot n\ \delta(\sigma - \sigma')$

NOTICE

1) We need use the discrete version from Fourier transform

2) The generator applied to $|D2\rangle$ is linear! while $|D2(F)\rangle$ is quadratic

a) $\left(X^2(\sigma) - y \right) |D1\rangle = 0$

$\Rightarrow \quad \left(x_n^2 - y\,\delta_{n,0} \right) |D1\rangle = 0$

$\Rightarrow \quad |D1\rangle = \mathcal{N} \prod_n \delta\left(x_n^2 - y\,\delta_{n,0} \right) |D2\rangle$

$= \mathcal{N} \prod{}' \delta(x_n^2) \quad \delta\left(x_0^2 - y\right) |D2\rangle$

Remember $\quad x_n = i\,\frac{N}{n}\left(\alpha_n - \tilde{\alpha}_{-n}\right) \qquad N = \frac{\sqrt{2\alpha'}}{2}$

1^{st} mode

$\prod{}' \delta(x_n^2)\, |D2\rangle = \prod{}' \int \frac{dl_n}{2\pi}\, e^{i l_n (x_n^2)} \quad |D2\rangle$

$= \int \prod{}' \frac{dl_n}{2\pi} \quad \exp\left[i\, l_n \; i\,\frac{N}{n}\left(\alpha_n - \tilde{\alpha}_{-n}\right)\right] \quad |D2\rangle$

$= \int \prod{}' \frac{dl_n}{2\pi} \quad e^{-\frac{N}{n} l_n \alpha_n} \quad e^{+\frac{N}{n} l_n \tilde{\alpha}_{-n}} \quad |D2\rangle$

We $\quad \left(\alpha_n + \tilde{\alpha}_{-n}\right) |D2\rangle = p_n \; |D2\rangle = 0$

$= \int \prod{}' \frac{dl_n}{2\pi} \quad e^{-\frac{N}{n} l_n \alpha_n} \quad e^{-\frac{N}{n} l_n \alpha_n} \quad |D2\rangle$

$= \int \prod{}' \frac{dl_n}{2\pi} \quad e^{-2\frac{N}{n} l_n \alpha_n} \quad |D2\rangle$

$= \int \prod_{n=1}^{\infty} \frac{dl_n}{2\pi}\, \frac{dl_{-n}}{2\pi} \quad e^{-2\frac{N}{n}\left(l_n \alpha_n + l_{-n}\alpha_{-n}\right)} \quad |D2\rangle$

Use $e^{A+B} = e^{A} e^{B} e^{-\frac{1}{2}[A,B]}$

$$= \int \prod_{n=1}^{\infty} \frac{d\ell_n \, d\ell_{-n}}{(2\pi)^2} \; e^{-2\frac{N}{n}\ell_{-n}\alpha_{-n}} \; e^{-2\frac{N}{n}\ell_n \alpha_n}$$

$$\exp\left[-\frac{1}{2}\left(-\frac{2N}{n}\right)^2 \ell_{-n}\ell_n(-n)\right] \quad |D2\rangle$$

$$= \prod_{n=1}^{\infty} \int \frac{d\ell_n \, d\ell_{-n}}{(2\pi)^2} \; e^{-2\ell_n \frac{N}{n}\alpha_{-n}} \; e^{+2\ell_n \frac{N}{n}\tilde{\alpha}_{-n}}$$

$$e^{\frac{2N^2}{n}\ell_{-n}\ell_n} \quad |D2\rangle$$

This expression is ∞!

Need to be regularized

Proceed naively

$$\frac{2N^2}{n}\ell_{-n}\ell_n - 2\frac{N}{n}\alpha_{-n}\ell_{-n} + 2\frac{N}{n}\tilde{\alpha}_{-n}\ell_n$$

$$= \frac{2N^2}{n}\left[\ell_{-n} + \frac{1}{N}\tilde{\alpha}_{-n}\right]\left[\ell_n - \frac{1}{N}\alpha_{-n}\right] + \frac{2}{n}\tilde{\alpha}_{-n}\alpha_{-n}$$

we get

$$|D1\rangle = M' \prod_{n=1}^{\infty} e^{\frac{2}{n}\alpha_{-n}^2\tilde{\alpha}_{-n}^2} \; \delta(x_0^2 - g) \quad |D2\rangle$$

now we have

$$|D2\rangle_{nzm} = e^{-\prod_{n=1}^{\infty}\frac{1}{n}\alpha_{-n}\cdot\tilde{\alpha}_{-n}} \quad |0_a\rangle$$

Chk

$n>0 \quad \alpha^i_n |D2\rangle_{n\neq m} = -\frac{1}{n} \cdot n \cdot \tilde{\alpha}^i_{-n} |D2\rangle$

$$\left(\alpha^i_n + \tilde{\alpha}^i_{-n}\right) |D2\rangle = 0$$

hence we find

$$|D1\rangle = M^1 \, e^{-\sum_{n=1}^{\infty} \frac{1}{n}\left(\alpha^1_{-n}\tilde{\alpha}^1_{-n} - \alpha^2_{-n}\tilde{\alpha}^2_{-n}\right)} |0\rangle_\alpha$$

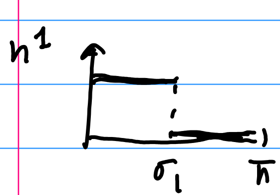
$$\delta(x^2_0 - g) \, |p^1 = p^2 = 0\rangle$$

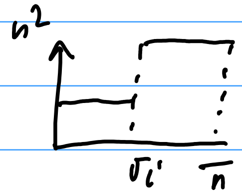
c)

$$\left[n(\sigma)\cdot X(\sigma) - g(\sigma)\right]_n = \int_0^{\pi} \frac{d\sigma}{\pi}\, e^{+i2n\sigma} \left(n(\sigma)\cdot X(\sigma) - g(\sigma) \right)$$

$$= \sum_{k} n_k \cdot x_{n-k} - g_n$$

Simplest case of $n(\sigma) = \begin{cases} \cos\alpha\, \vec{i} + \sin\alpha\, \vec{j} & 0<\sigma<\sigma_i \\ \vec{j} & \sigma_i < \sigma < \pi \end{cases}$





Then

$$n_k(\sigma) = \vec{i} \left[\cos\alpha \int_0^{\sigma_i} \frac{d\sigma}{\pi}\, e^{i2k\sigma} \right]$$

$$+ \vec{j} \left[\sin\alpha \int_0^{\sigma_i} \frac{d\sigma}{\pi}\, e^{i2k\sigma} + \int_{\sigma_i}^{\pi} \frac{d\sigma}{\pi}\, e^{i2k\sigma} \right]$$

In particular

$$n_0(\sigma) = \vec{i} \left[\cos\alpha\, \frac{\sigma_i}{\pi} \right] + \vec{j} \left[\sin\alpha\, \frac{\sigma_i}{\pi} + \frac{\pi - \sigma_i}{\pi} \right]$$

Let us try to compute ($n=0$ included!)

$$\prod_{n\in\mathbb{Z}} \delta\left(\sum n_{n-k}\cdot x_k - g_n\right) |D2\rangle$$

$$= \int \prod \frac{dl_n}{2\pi} \, e^{-i l_n g_n} \, e^{i l_n \sum_{k\neq 0} n_{n-k}\cdot \frac{i}{k} N(\alpha_k - \tilde{\alpha}_{-k})}$$

$$e^{i l_n n_n \cdot x_0} \qquad |D2\rangle$$

$$= \int \prod \frac{dl_n}{2\pi} \, e^{+i l_n (n_n\cdot x_0 - g_n)}$$

$$e^{-N l_n \sum_{k\neq 0} n_{n-k}\cdot \alpha_k \frac{1}{k}} \, e^{+N l_n \sum_{k\neq 0} n_{n-k}\cdot \tilde{\alpha}_{-k} \frac{1}{k}} |D2\rangle$$

Use $e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}$

$$e^{-N\sum_n l_n \sum_{k<0} \frac{1}{k} n_{n-k}\cdot\alpha_k - N\sum_m l_m \sum_{k>0}\frac{1}{k} n_{m-k}\cdot \alpha_k} =$$

$$= :\ :\ e^{-\frac{1}{2}(-N l_n)(-N l_m)\sum_{k<0} \frac{1}{k}\frac{1}{-k} \, k \, n_{n-k}\cdot n_{m+k}}$$

Check $\left[\sum_{k<0} \frac{1}{k} n_{n-k}\cdot\alpha_k , \sum_{l>0} \frac{1}{l} n_{m-l}\cdot\alpha_l\right]$

$$= \sum_{k<0}\sum_{l>0} \frac{1}{k}\frac{1}{l} \, n_{n-k}\cdot n_{m-l} \; k \, \delta_{k+l,0}$$

$$= :\ :\ \exp\left[-\frac{N^2}{2} l_n l_m \sum_{k>0} \frac{1}{k} n_{n-k}\cdot n_{m+k}\right]$$

Use $(\alpha_{+n} + \tilde{\alpha}_{-n})|D2\rangle = 0$

$$= \int \prod \frac{dl_n}{2\pi} e^{i l_n (n_1 \cdot x_0 - y_1)}$$

$$\times \exp\left[-\frac{N^2}{2} l_n l_m \sum_{k>0} \frac{1}{k} n_{n-k} \cdot n_{m+k}\right]$$

$$e^{-N l_n \sum_{k\leq 0} \frac{1}{k} n_{n-k} \cdot \alpha_k} e^{-N l_n \sum_{k>0} \frac{1}{k} n_{n-k} \cdot \alpha_k}$$

$$\times \exp\left[-\frac{N^2}{2} l_n l_m \sum_{k>0} \frac{1}{k} n_{n+k} \cdot n_{m-k}\right]$$

$$e^{-N l_n \sum_{k\leq 0} \frac{1}{k} n_{n+k} \cdot \tilde{\alpha}_k} e^{-N l_n \sum_{k>0} \frac{1}{k} n_{n+k} \cdot \tilde{\alpha}_k} |D2\rangle$$

$$= \int \prod \frac{dl_n}{2\pi} e^{i l_n (n_1 \cdot x_0 - y_1)}$$

$$\exp\left[-\frac{N^2}{2} l_n l_m \sum_{k>0} \frac{1}{k} \left(n_{n+k} \cdot n_{m-k} + n_{n-k} \cdot n_{m+k}\right)\right]$$

$$e^{-N l_n \sum_{k\leq 0} \frac{1}{k} n_{n-k} \cdot \alpha_k} e^{-N l_n \sum_{k\leq 0} \frac{1}{k} n_{n+k} \cdot \tilde{\alpha}_k}$$

$$e^{+N l_n \sum_{k>0} \frac{1}{k} n_{n-k} \cdot \tilde{\alpha}_{-k}} e^{+N l_n \sum_{k>0} \frac{1}{k} n_{n+k} \cdot \alpha_{-k}} |D2\rangle$$

$$= \int \prod \frac{d\ell_n}{2\pi} \, e^{i \ell_n (n_1 \cdot x_0 - y_n)}$$

$$\exp\left[-\frac{N^2}{2} \ell_n \ell_m \sum_{k>0} \frac{1}{k} \left(n_{n+k} \cdot n_{m-k} + n_{n-k} \cdot n_{m+k} \right) \right]$$

$$e^{-2N \ell_n \sum_{k \neq 0} \frac{1}{k} n_{n-k} \cdot \alpha_k} \quad e^{-2N \ell_n \sum_{k \neq 0} \frac{1}{k} n_{n+k} \cdot \tilde{\alpha}_k} \quad |D2\rangle$$

The key is the matrix

$$S_{nm} = S_{mn} = \sum_{k>0} \frac{1}{k} \left(n_{n+k} \cdot n_{m-k} + n_{n-k} \cdot n_{m+k} \right)$$

which corresponds to the bilocal operator

$$S(\sigma, \xi) = \sum_{n,m} S_{nm} \, e^{-i 2n\sigma} e^{-i 2m\xi}$$

$$= \sum_{n,m} e^{-i 2n\sigma - i 2m\xi} \sum_{k \neq 0}' \frac{1}{|k|} n_{n+k} \cdot n_{m-k}$$

$$= \sum_k' \left[\sum_n e^{-i 2(n+k)\sigma} n_{n+k} \cdot \sum_m e^{-i 2(m-k)\xi} n_{m-k} \, \frac{1}{|k|} e^{i 2k\sigma - i 2k\xi} \right]$$

$$= \sum_k' n(\sigma) \cdot n(\xi) \, \frac{e^{i 2k(\sigma - \xi)}}{|k|}$$

Now $f(\sigma) = \sum{}' \frac{e^{i2k\sigma}}{|k|} \Rightarrow f'(\sigma) = 2i \sum{}' e^{i2k\sigma} \mathrm{sgn}(k)$

$$= 2i \left[\sum_{k=1}^{\infty} e^{i2k\sigma} - \sum_{k=1}^{\infty} e^{-i2k\sigma} \right]$$

$$= 2i \left[\frac{e^{i2\sigma}}{1-e^{i2\sigma}} - \frac{e^{-i2\sigma}}{1-e^{-i2\sigma}} \right]$$

$$= 2i \left[\frac{e^{i2\sigma}}{1-e^{i2\sigma}} - \frac{1}{e^{i2\sigma}-1} \right]$$

$$= 2i \left[\frac{1+e^{i2\sigma}}{1-e^{i2\sigma}} \right] = \frac{\cos\sigma}{-\sin\sigma} = -\operatorname{ctg}\sigma$$

Then $f(\sigma) = f_0 - \int d\sigma \operatorname{ctg}\sigma = f_0 - \ln|\sin\sigma|$

$$S(\sigma,\zeta) = -n(\sigma)\cdot n(\zeta) \ln|\sin(\sigma-\zeta)|$$

$$\stackrel{!}{=} S(\zeta,\sigma)$$

Eigen

$$\int_0^{\pi} d\zeta \; S(\sigma,\zeta)\, v(\zeta) = \lambda_v \, v(\sigma)$$

Check $\quad n(\sigma) = n = \text{const} \quad \Rightarrow \quad n_K = n_0\, \delta_{K,0}$

$$S_{n,m} = \sum_{K>0} \frac{1}{K} \left(n_{n+K} \cdot n_{m-K} + n_{n-K} \cdot n_{m+K} \right)$$

$$= \sum_{K>0} \frac{1}{K} \left(\delta_{n+K}\, \delta_{m-K} + \delta_{n-K}\, \delta_{m+K} \right) n_0^2$$

$$\begin{cases} = \underbrace{\frac{1}{|n|}}_{K>0} \delta_{n+m,0}\, n_0^2 & n, m \neq 0 \\ = 0 & n = 0 \text{ or } m = 0 \end{cases}$$

hence

$$\langle D1 \rangle \propto \int \prod \frac{d\ell_n}{2\pi}\, e^{i \ell_n (n_n \cdot n_0 - g_n)}$$

$$\exp\left[-\frac{N^2}{2} \ell_n \ell_m \sum_{K>0} \frac{1}{K} \left(n_{n+K} \cdot n_{m-K} + n_{n-K} \cdot n_{m+K} \right) \right]$$

$$e^{-2N \ell_n \sum_{K<0} \frac{1}{K} n_{n-K} \cdot \alpha_K}\; e^{-2N \ell_n \sum_{K<0} \frac{1}{K} n_{n+K} \cdot \tilde{\alpha}_K} \quad |D2\rangle$$

$$= \int \dots e^{-\frac{N^2}{2|n|} \ell_n \ell_{-n}\, n_0^2} \dots \quad |D2\rangle$$

$$= \int \dots \underbrace{e^{-N^2 \sum_{n=1}^{\infty} \ell_n \ell_{-n}}}_{\text{factor } 2} \dots \quad |D2\rangle$$