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INTRODUCTION TO BRS SYMMETRY

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Abstract

This paper contains a revised version of the lecture notes of a short course on the quantization of gauge theories. Starting from a sketchy review of scattering theory, the paper describes the lines of the BRST-Faddeev-Popov quantization considering the problem of a non-perturbative extension of this method. The connection between the Slavnov-Taylor identity and the unitarity of the S-matrix is also discussed.

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1 Introduction

These lectures begin recalling some general results of scattering theory [1]. The reduction formulae for the S-matrix are given in terms of the Feynman functional in the case of a massive field theory. Then the analysis comes to gauge theories for which the concept of gauge orbit is introduced. The Faddeev-Popov definition [2] of a finite functional measure is given. BRST external differential operator along the orbits is introduced together with the full BRS operator [3]. The gauge algebra of the infinitesimal gauge transformations is briefly discussed. Assuming the existence of a global gauge fixing (no Gribov ambiguity [7]) the Slavnov-Taylor identity and the gauge fixing independence of the theory is deduced from the BRS invariance of the functional measure. The extension of the Faddeev-Popov formula to the case of Gribov ambiguities is briefly discussed together with that of the Slavnov-Taylor identity. The Slavnov-Taylor identity is then translated in terms of the proper functional (the effective action) and the extension of the method to the case of gauge algebras that are closed only modulo the field equations is discussed. Limiting for simplicity the study to the case of massive fields the Slavnov-Taylor identity is applied to the two-point functions, this leads to the introduction of the BRS symmetry for the asymptotic fields (in the form of Kugo and Ojima [14]) and to the proof the existence of a physical Hilbert space in which the S-matrix is unitary.

2 The S-matrix

In quantum field theory [1] the scattering amplitudes are computed by means of the reduction formula. This can be simply written using the Feynman functional generator of the theory that is defined according:

$$Z[j] \equiv e^{iZ_c} = <\Omega, T\left(e^{i\int d^4x\phi(x)J(x)}\right)\Omega > . \tag{1}$$

where ϕ and j in general label a set of quantized fields and corresponding sources. Z_c is the connected functional. The Feynman functional is computed by means of the formula:

$$Z[j] = \int d\mu \ e^{i \int d^4x \phi(x) J(x)} \ , \tag{2}$$

in terms of the functional measure $d\mu$ of the theory that is deduced from its bare action by the heuristic relation; $d\mu = \prod_x d\phi(x)e^{iS(\phi)}$.

Computing the two-point function according:

$$\frac{\delta^2}{\delta J_i(x)\delta J_i(0)} Z_c|_{J=0} \equiv \Delta^{ij}(x) , \qquad (3)$$

and excluding for simplicity the presence of massless fields, we can separate from Δ the asymptotic propagator Δ_{as} :

$$\Delta^{ij}(x) = \sum_{\lambda} \int \frac{dp}{(2\pi)^4} \frac{e^{ipx}}{m_{\lambda}^2 - p^2 - i0_+} \zeta_{\lambda}^{ij}(p) + R^{ij}(x) \equiv \Delta_{as}^{ij}(x) + R^{ij}(x) , \quad (4)$$

where the Fourier transform of R has no pole in p^2 . It is clear that the asymptotic propagator is by no means unique since ζ_{λ}^{ij} is defined up to a polynomial in p^2 vanishing at m_{λ}^2 ; however this lack of uniqueness does not affect the **S** matrix. Then one introduces the asymptotic free fields ϕ_{in} with the (anti-)commutation relations:

$$\left[\phi_{in}^{i(+)}(x), \phi_{in}^{j(-)}(0)\right]_{\pm} = \sum_{\lambda} \int \frac{dp}{(2\pi)^4} e^{ipx} \theta(p^0) \delta(p^2 - m_{\lambda}^2) \zeta_{\lambda}^{ij}(p) , \qquad (5)$$

and the asymptotic wave operator which satisfies:

$$K_{ij}(\partial)\phi_{in}^i(x) = 0 , \qquad (6)$$

and

$$K_{ik}(-ip)\tilde{\Delta}^{kj}(p)|_{p^2=m_\lambda^2} = \delta_i^j \tag{7}$$

for any λ . It should be clear that the asymptotic wave operator is by no means unique in much the same way as the matrix ζ .

Then one computes the S matrix of the theory according:

$$\mathbf{S} =: e^{\int d^4x \phi^i(x) K_{ij}(\partial) \frac{\delta}{\delta J_j(x)}} : Z|_{J=0} \equiv: e^{\Sigma} : Z|_{J=0} . \tag{8}$$

3 Gauge invariance and BRS symmetry

Now we come to the quantization of gauge theories. In this course we shall disregard the crucial problem of the explicit non-perturbative construction

of the theory, limiting our analysis to the formal and symmetry aspects that should simplify the construction and characterize the solution of the full quantum theory. We shall be often concerned with the functional measure of the theory; avoiding any consideration of its actual definition we shall indifferently pass from the Minkowskian form

$$d\mu = \prod_{x} d\phi(x)e^{iS(\phi)} \tag{9}$$

to the Euclidean one

$$d\mu = \prod_{x} d\phi(x)e^{-S(\phi)} . {10}$$

Furthermore, in order to simplify the notation we shall merge all the labels of the fields into a single index including the space-time variable. Assuming the usual convention of summation over repeated indices we shall also often omit the integration symbol. However one should keep firmly in mind that the fields are local variables and that locality is considered to play a crucial role in field theory. For many reasons we shall also avoid discussing many mathematical aspects that should bring our analysis too far from its purposes. Let us call \mathcal{F}_0 the field space, that is the configuration space upon which the gauge theory is constructed. In a gauge theory \mathcal{F}_0 is fibered by the gauge orbits \mathcal{O} that is the set of gauge transforms of a given configuration. Considering the infinitesimal transformations and translating everything in differential geometry terms (we are freely following e.g. [4]), we are given a system of partial differential operators $\{X\}$ on \mathcal{F}_0 , that we shall label with the index I, and that in any point of \mathcal{F}_0 define a system of tangent vectors to the corresponding orbit. Denoting the generic field coordinate in \mathcal{F}_0 by the ϕ^{α} , we can write these operators in the form

$$X_I = P_I^{\alpha}(\phi)\partial_{\phi^{\alpha}} . {11}$$

In the following we shall systematically use the symbol ϕ^{α} , or, more explicitly $\phi^{\alpha}(x)$, for the bosonic gauge and matter fields and we shall mention only occasionally the spinor fields. The bosonic gauge and matter field components are chosen real and hence they correspond to Hermitian operator-valued distributions.

The explicit form of Eq.(11) in a pure non abelian case is:

$$X_I(x) = \partial_\mu \frac{\delta}{\delta A_\mu^I(x)} - g f_{IJ}^K A_\mu^J(x) \frac{\delta}{\delta A_\mu^K(x)} . \tag{12}$$

The system $\{X\}$ is often called the differential system of the orbits, whose very existence implies:

$$[X_I, X_J] = C_{IJ}^K(\phi) X_K ,$$
 (13)

that is the complete integrability of $\{X\}$. In the standard situation the algebra (13) is a Lie algebra, the structure functions C_{IJ}^K are constants. As a matter of fact, having merged the space-time variables with the discrete indices, what we call a Lie algebra is an infinite Lie algebra. Putting the space-time variables into evidence, that is, replacing above I, J and K with the pairs (I, x), (J, y) and $(K, z), C_{IJ}^K$ decomposes into factors such as the constant f_{IJ}^K and the distribution $\delta(x - y)\delta(x - z)$. If the discrete indices run in the set $I = 1, \ldots, G$ we shall call G the dimension of the gauge Lie algebra.

We shall see in the following how (13) can be weakened restricting the integrability condition to the "mass shell", that is modulo the field equations. We also assume that a X-invariant measure is uniquely defined up to an orbit-independent normalization constant. This we shall call the vacuum invariant measure, the measure which is associated with the vacuum state of the theory. In general one is interested in the vacuum correlators of local observables; these correspond to the integrals of a different class of invariant measures that can be written as the product of the vacuum measure times gauge invariant functionals depending on the field variables corresponding to a suitably localized space-time domain. A generic invariant measure will be assumed to belong to this class.

The vacuum functional measure is constant over the orbits \mathcal{O} ; in general this makes the functional measure of \mathcal{F}_0 non integrable and the Feynman functional ill defined. This difficulty is cured by the Faddeev-Popov trick. In order to recall it conveniently let us assume that, even if this is in general not the case, that the original field variables trivialize the fibration; that is let us assume that the set of fields $\{\phi\}$ is decomposed according $\{\xi\}$ and $\{\eta\}$ where $\{\xi\}$ are constant along the orbits and $\{\eta\}$ are "vertical" coordinates along the orbits. Then it is natural to make the measure integrable by multiplying it by an integrable functional of η whose integral over \mathcal{O} corresponding to the above mentioned X-invariant measure shoould be independent of the orbit (of ξ). One often considers the invariant Dirac δ_{inv} $[\eta - \bar{\eta}]$:

$$\delta_{inv} \left[\eta - \bar{\eta} \right] \equiv \delta \left[\eta - \bar{\eta} \right] \det |X_I \, \eta^J| \, , \tag{14}$$

but more generally one can consider its convolution with a suitable $\bar{\eta}$ -functional.

Notice that the determinant appears in (14) since the action of a gauge transformation does not correspond to a Euclidean transformation on the η variables. The Faddeev-Popov measure [2] is obtained by the substitution:

$$d\mu \to d\mu \, \delta_{inv} \left[\eta - \bar{\eta} \right] \,, \tag{15}$$

The invariant Dirac measure can be easily written as a functional Fourier transform. Introducing two sets of Grassmann variables $\{\omega^I\}$ and $\{\bar{\omega}_J\}$ that can be simply identified with the generators of an exterior algebra and the corresponding derivatives that we label by $\{\partial_{\omega^I}\}$ and by $\{\partial_{\bar{\omega}_J}\}$, one introduces the Berezin integral:

$$\int d\omega^I \equiv \frac{1}{\sqrt{2\pi i}} \partial_{\omega^I} , \qquad (16)$$

for ω and an analogous definition for $\bar{\omega}$. Then using the so called Nakanishi-Lautrup multipliers $\{b_J\}$ one can reproduce the right-hand side of (14) in the form:

$$\int \prod db_I \prod d\omega^J \prod d\bar{\omega}_K \ e^{i\left[b_I\eta^I - \bar{\omega}_I\omega^J X_J\eta^I\right]} \ . \tag{17}$$

This formula can be interpreted as an enlargement of the field space \mathcal{F}_0 with the addition of a set of ordinary fields corresponding to the Nakanishi-Lautrup multipliers and of two sets of anticommuting fields corresponding to the exterior algebra generators. We call \mathcal{F}_C the new space. We also introduce the measure on \mathcal{F}_C :

$$d\mu_C \equiv d\mu \prod_I db_I \prod_J d\omega_J \prod_K d\bar{\omega}_K . \tag{18}$$

Looking now into the details of (17), we see that the differential operator $\omega^I X_I$ appearing in the exponent can be replaced by its minimal nilpotent extension:

$$d_V = \omega^I X_I - \frac{1}{2} \omega^I \omega^J C_{IJ}^K(\phi) \partial_{\omega_K} , \qquad (19)$$

This operator, that is often called the BRST operator [3], is nilpotent due to (13) and to the corresponding Jacobi identity. That is:

$$d_V^2 = 0 \leftrightarrow \begin{cases} \omega^I \omega^J \left(X_I X_J - \frac{1}{2} C_{IJ}^K(\phi) X_K \right) = 0; \\ \omega^I \omega^J \omega^K \left(C_{IJ}^M(\phi) C_{MK}^L(\phi) - X_I C_{JK}^L(\phi) \right) = 0. \end{cases}$$
(20)

Identifying the system $\{\omega\}$ with that of the X-left-invariant forms we can interpret the differential operator d_V as the vertical exterior differential operator on \mathcal{F}_0 , that is, with the operator on \mathcal{F}_0 that in any point is identified with the exterior differential operator on the orbit (see Appendix A).

Let us now come back to the trivializing coordinates, it is clear that these exist globally only in very special cases, in particular when the corresponding fibration is trivial. However in order to define a finite measure through Eq.(15) it is sufficient to identify a global section of \mathcal{F}_0 , if it exists, intersecting every orbit in a single point. This condition is equivalent to the existence of a system of local functionals $\{\Psi(\phi)\}$ that we shall continue to label with the index I, for which the Jacobian $det|X_I\Psi^J|$ does not vanish in the points where $\Psi = \bar{\Psi}$ for some $\bar{\Psi}$.

Assuming this condition, we shall replace in the exponent in (17) the coordinate η^I by a generic functional $\Psi^I(\phi)$ writing the exponent as:

$$iS_{GF} = i \left[b_I \left(\Psi^I - \bar{\Psi}^I \right) - \bar{\omega}_I d_V \Psi^I \right] , \qquad (21)$$

The above formula gives the definition of the gauge fixing action S_{GF} .

Eq.(21) can be translated into a simpler form introducing a new exterior derivative s acting on the algebra generated by ω and $\bar{\omega}$ whose action on ϕ and ω coincides with that of d_V and:

$$s\bar{\omega} = b$$
 , $sb = 0$, (22)

and hence

$$s = d_V + b_I \partial_{\bar{\omega}_I} . (23)$$

It it clear that s is nilpotent and that (21) is written:

$$S_{GF} = s \left[\bar{\omega}_I \left(\Psi^I - \bar{\Psi}^I \right) \right] , \qquad (24)$$

it is also obvious that s commutes with the physical functional measure $d\mu$. A further generalization of the measure, that includes also the convolutions of (24) with generic functionals of $\bar{\Psi}$, is obtained extending the choice of $\bar{\Psi}$ to b, ω and $\bar{\omega}$ -dependent local functionals. In the following we shall replace $\bar{\omega}_I \left(\Psi^I - \bar{\Psi}^I \right)$ with a generic functional Θ carrying the same quantum numbers. In the standard situation Θ is a strictly local quadratic functional of b, that is, the space-time integral of a second order polynomial in b, independent of its derivatives; therefore b is an auxiliary field. However there are models, in particular in supergravity, in which [5] b corresponds to propagating degrees of freedom that play the role of extra ghosts.

The most frequently met gauge choice in the case of renormalizable theories is the linear choice in which the gauge fixing function in Eq. (21) is a linear function. Showing explicitly the dependence on the space-time variables one sets:

$$\Psi^{I} = \int dx \left[V_{\alpha}^{I}(\partial)\phi^{\alpha} + \frac{\xi}{2}\delta^{IJ}b_{J} \right] , \qquad (25)$$

where δ is the Kronecker symbol and the matrix V is a real linear function of the space-time derivatives and has maximal rank, that is, rank equal to the number of independent components of the b fields. Furthermore $\bar{\Psi}=0$. It follows that:

$$S_{GF} = \int dx \left[b_I (V_\alpha^I(\partial)\phi^\alpha + \frac{\xi}{2}\delta^{IJ}b_J) - \bar{\omega}_I V_\alpha^I P_J^\alpha(\phi)\omega^J \right] . \tag{26}$$

Even within the linear gauge there are many possibilities among which one chooses depending on the calculation purposes. The most frequent choices are the covariant ones, typically Lorentz's gauge in which $V_{\alpha}^{I}(\partial)\phi^{\alpha}=k_{J}^{I}\partial^{\mu}A_{\mu}^{J}$ and 't Hooft gauges, $V_{\alpha}^{I}(\partial)\phi^{\alpha}=k_{J}^{I}\partial^{\mu}A_{\mu}^{J}+\rho_{i}^{I}\Phi_{i}$, where A_{μ}^{I} is a gauge vector field and Φ_{i} is a scalar field. Among the non-covariant choice one often meets the ligh-like axial gauge in which $V_{\alpha}^{I}\phi^{\alpha}=k_{J}^{I}n^{\mu}A_{\mu}^{J}$ where n is a light-like vector. However the covariant choices are the most convenient ones for a general discussion.

The above mentiones condition that $\det |X_I\Psi^J|$ does not vanish, that is, the condition for the gauge degrees of freedom to be completely fixed, implies that $\det |C_{0,J}^I|$, the determinant of the Fourier transform of the field-independent part of $X_JV_\alpha^I\phi^\alpha$, does not vanish for a generic choice of p. In the Lorentz gauge $C_{0,J}^I(p) = -k_J^Ip^2$ and in 't Hooft's one $C_{0,J}^I(p) = -k_J^Ip^2 + \rho_i^It_j^{ij}v_j$ where $t_J^{ij}v_j$ is the in-homogeneous part of the scalar field gauge transformation. Discussing the S-matrix unitarity we shall assume a covariant choice with the further condition that $C_{0,J}^I(p)$ is real and the ghost part of the Lagrangian is formally Hermitian together with the ghost field ω , while the anti-ghost is anti-Hermitian. Since the equation $\det |C_{0,J}^I| = 0$ is an algebraic equation of degree G in p^2 whose solutions in the semi-classical approximation are the ghost masses, we assume that all these solutions are real and positive and that the ghost particles are kinematically stable.

Thus under the standard assumption of a closed gauge algebra (13) S has the following structure:

$$S = S_{inv}(\phi) + S_{GF} + \int dx \left[\gamma_{\alpha} \omega^{I} P_{I}^{\alpha}(\phi) + \frac{1}{2} \zeta_{I} C_{JK}^{I} \omega^{J} \omega^{K} \right] . \tag{27}$$

In the case of renormalizable theories this structure is obliged by the condition that the dimension of the action should be limited by that of space-time.

A further comment on the auxiliary role of b and hence of $\bar{\omega}$ is here necessary. In this study we are tacitly considering the field space \mathcal{F}_C finite dimensional, strictly speaking this is, of course, not true since every field corresponds to an infinite number of variables; however one assumes that some mechanism, e.g. some kind of regularization, renders finite the number of effective degrees of freedom. With this proviso let us consider:

$$\int \prod_{I} db_{I} \prod_{J} d\bar{\omega}_{J} e^{is\Theta} = \lim_{\epsilon \to 0} \int \prod_{I} db_{I} \prod_{J} d\bar{\omega}_{J} e^{is\Theta - \epsilon \sum_{K} b_{k}^{2}}$$

$$= i \lim_{\epsilon \to 0} \int \prod_{I} db_{I} \prod_{J} d\bar{\omega}_{J} \int_{0}^{1} dt \ s(\Theta) e^{is\Theta t - \epsilon \sum_{K} b_{k}^{2}}$$

$$= d_{V} \left[i \lim_{\epsilon \to 0} \int \prod_{I} db_{I} \prod_{J} d\bar{\omega}_{J} \int_{0}^{1} dt \ \Theta e^{is\Theta t - \epsilon \sum_{K} b_{k}^{2}} \right] , \qquad (28)$$

where we have used the fact that the Berezin integral of a constant gives zero. Eq.(28) shows that the Faddev-Popov measure corresponds to the insertion of a d_V -exact factor into the functional measure and the fields b and $\bar{\omega}$ are auxiliary in the sense that they allow the explicit construction of this factor in local terms. Furthermore we see from (28) that the resulting measure on \mathcal{F}_0 that is:

$$d\mu \int \prod_{K} d\omega^{K} \prod_{I} db_{I} \prod_{J} d\bar{\omega}_{J} e^{is\Theta}$$
 (29)

is an exact top form. It follows that its integral over a compact cycle, such as a gauge group orbit of a lattice gauge theory, vanishes $[6]^2$. This is due to the fact that on a cycle the gauge fixing equation $\Psi = \bar{\Psi}$ has an even number of solutions whose contributions to the above measure cancel pairwise. However it should be clearly kept in mind that according to the Faddeev-Popov prescription the functional integral should not cover the whole orbits but only a compact subset of every orbit containing a single solution of the gauge fixing equation. To be explicit let us consider the extreme example in which \mathcal{F}_0 reduces to a circle, a single U(1) gauge orbit. Choosing $\Theta = \bar{\omega} \sin \varphi$, setting $s = i\omega \partial_{\varphi} + b\partial_{\bar{\omega}}$ and integrating over the whole space, one gets

$$\oint d\varphi \int db d\omega d\bar{\omega} e^{ibsin \varphi + \bar{\omega}\omega \cos \varphi} = -i \oint d\varphi \delta \left(\sin \varphi \right) \cos \varphi = 0 , \qquad (30)$$

²I thank M.Testa for calling my attention to this reference

while with the actual prescription, one has:

$$i\int_{-\epsilon}^{\epsilon} d\varphi \int db d\omega d\bar{\omega} e^{ibsin \varphi + \bar{\omega}\omega cos \varphi} = 1.$$
 (31)

To conclude this section let us remember [7] that in the case of covariant and local gauge choices the condition that $\Psi = \bar{\Psi}$ defines a global section of the orbit space does not hold true, the situation is less clear for the so called axial gauges, that however suffer even worst diseases [10]. We shall see how this difficulty can be overcome in the situation in which \mathcal{F}_0 can be divided into a system of cells U_a in which one can find for every cell a Ψ_a defining a section in the cell.

4 The Slavnov-Taylor identity

The particular structure of the functional measure allows an immediate proof, up to renormalization effects, of the Slavnov-Taylor (S-T) identity. That is: for any measurable functional Ξ :

$$\int d\mu_C \ e^{iS_{GF}} s \ \Xi = 0 \ . \tag{32}$$

Indeed, using the same arguments as for (28), we get:

$$\int d\mu_C \ e^{iS_{GF}} s \ \Xi = \int d\mu_C \ s \left[e^{iS_{GF}} \ \Xi \right]$$

$$= \int d\mu \prod_I d\omega^I dV \int \prod_J db_J \prod_K d\bar{\omega}_K \ e^{iS_{GF}} \ \Xi = 0 \ , \tag{33}$$

since the last expression apparently corresponds to an exact top form whose support, according to the general prescription, is contained in the integration domain, and hence it vanishes on the boundaries of this domain. Considering the extreme example given in the last section, one has for any continuous $A(\varphi)$:

$$i \int_{-\epsilon}^{\epsilon} d\varphi \int db \ d\omega d\bar{\omega} e^{ibsin \ \varphi + \bar{\omega}\omega\cos \ \varphi} s \left[\bar{\omega}A\left(\varphi\right)\right]$$
$$= i \int_{-\epsilon}^{\epsilon} d\varphi \int db \ d\omega d\bar{\omega} e^{ibsin \ \varphi + \bar{\omega}\omega\cos \ \varphi} \left[bA\left(\varphi\right) - i\bar{\omega}\omega A'\left(\varphi\right)\right]$$

$$= \int_{-\epsilon}^{\epsilon} \frac{d\varphi}{2\pi} \int db \ e^{ibsin \ \varphi + \bar{\omega}\omega\cos\varphi} \left[bA(\varphi)\cos\varphi - iA'(\varphi) \right]$$

$$= -i \int_{-\epsilon}^{\epsilon} \frac{d\varphi}{2\pi} \frac{d}{d\varphi} \int db \ e^{ibsin \ \varphi} A(\varphi) = 0 , \qquad (34)$$

since $\int db \ e^{ibsin \ \varphi} A(\varphi)$ vanishes at $\varphi = \pm \epsilon$.

The identity (32) can be interpreted saying that all correlators between elements of the image of s and s-invariants ones vanish. Indeed, according to the definition given in section 3, the s-invariant functionals can be considered to be generic local factors in the invariant measure $d\mu$. Considering the s operator as the natural extension of d_V , the exterior derivative corresponding to the gauge transformations, it is natural to assume as a basic principle of gauge theories the identification of observables with s-invariant functionals. Due to the nilpotency of s this set contains the image of s, whose elements, however, correspond to trivial observables according to (32). Therefore the non-trivial observables belong to the quotient space of the kernel of s versus its image, that its to the cohomology of s.

It remains to verify that the cohomology of s is equivalent to that of the vertical exterior differential operator d_V . Indeed consider the functional differential operator:

$$\Delta \equiv -\left(b_I \partial_{b_I} + \bar{\omega}_I \partial_{\bar{\omega}_I}\right) \ . \tag{35}$$

Let P be the projector on the kernel of Δ , that is on the $\bar{\omega}$ and b-independent functionals. It is apparent that Δ and hence P, commute with s. Therefore a generic functional X which is s-invariant, that is satisfying:

$$(d_V + b_I \partial_{\bar{\omega}_I}) X = 0 , \qquad (36)$$

is the sum of two terms: PX and (1 - P)X, satisfying:

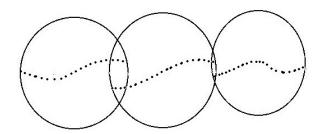
$$d_V P X = 0$$
 , $s(1-P) X = 0$. (37)

In much the same way, an element of the image of s: Z = sY, is decomposed according:

$$PZ = d_V PY$$
 , $(1 - P)Z = s(1 - P)Y$. (38)

Therefore the cohomology of s is the union of that of d_V in the kernel of Δ and that of s in the kernel of P. We want to verify that this second contribution is trivial. Indeed, consider the differential operator: $\bar{\omega}_I \partial_{b_I}$, satisfying:

$$\{\bar{\omega}_I \partial_{b_I}, s\} = -\Delta , \qquad (39)$$



for s-invariant X, this yields:

$$s\bar{\omega}_I\partial_{b_I}(1-P)X = -\Delta(1-P)X \to (1-P)X = -s\Delta^{-1}\bar{\omega}_I\partial_{b_I}(1-P)X$$
. (40)

Indeed, on account of the definition of P, $\bar{\omega}\partial_{\omega}(1-P)X$ belongs to the domain of Δ^{-1} . Thus (1-P)X belongs to the image of s.

A second consequence of (32) is the gauge fixing independence of the correlators of observables. Indeed let us compare the expectation values of the same s-invariant functional Ω computed with two different measures corresponding to choices of Θ differing by $\delta\Theta$. To first order in $\delta\Theta$ the difference of the expectation values is given by

$$i \int d\mu_C \ e^{iS_{GF}} s\left(\delta\Theta\right) \Omega = 0 \ . \tag{41}$$

Of course this implies the independence of the expectation values of the choice of Θ in a certain class of measurable functionals. Even in perturbation theory this is not enough to prove that the expectation values in a renormalizable gauge coincide with those in a non-renormalizable one.

We now come to the problem of extending the functional measure to the situation in which the gauge fixing is defined only locally. In general the orbit manifold has to be divided into cells, each corresponding to a different choice of Θ . Every cell in the orbit space corresponds to a cell U_a in \mathcal{F}_0 . Let $\chi_a(\phi)$ be a suitable smooth positive function with support in U_a and such that the set $\{\chi\}$ is a partition of unity on the union of the supports of the gauge-fixed measures $d\mu_C e^{is\Theta_a}$. This is shown in the figure where the dotted lines corresponds to the support of the measures and the circles to the cells. Explicitly a cell will be defined giving its center, that is a special configuration ϕ_a (background field), and defining the characteristic functions χ_a according:

$$\chi_a(\phi) \equiv \frac{\theta (R^2 - \|\phi - \phi_a\|^2) \theta^2 (R^2 - \inf_c \|\phi - \phi_c\|^2)}{\sum_b \theta (R^2 - \|\phi - \phi_b\|^2)}.$$
 (42)

where θ is a smoothed Heavyside function and $\|\phi - \phi_a\|$ is the L^2 norm of the difference $\phi - \phi_a$. Hints about the values of R_a can be found in [8].

The BRS invariant functional measure corresponding to this local gauge choice is given by [9]:

$$d\mu_C \left[\sum_a \chi_a e^{is\Theta_a} - i^n \sum_{n=1}^\infty \frac{(-1)^{\frac{n(n-1)}{2}}}{n+1} \left(s\chi_{a_1} ... s\chi_{a_n} \chi_{a_{n+1}} \right)_A \right]$$

$$\partial_{\Theta} \left(\Theta_{a_1} ... \Theta_{a_{n+1}} \right) e^{is\Theta_{a_1}, ..., a_{n+1}} , \qquad (43)$$

where we have used the definitions:

$$\partial_{\Theta} \left(\Theta_{a_1} ... \Theta_{a_n} \right) \equiv \sum_{l=1}^{n} (-1)^{l+1} \Theta_{a_1} ... \check{\Theta}_{a_l} ... \Theta_{a_n} . \tag{44}$$

and:

$$e^{is\Theta_{a_1,\dots,a_n}} \equiv \int_0^\infty \prod_{i=1}^n dt_i \,\,\delta\left(\sum_{j=1}^n t_j - 1\right) e^{is\sum_{k=1}^n t_k\Theta_{a_k}} \,\,. \tag{45}$$

In Appendix B [(137)] it is proven that under the hypothesis of a finite multiplicity of cell intersections the measure (43) satisfies the Slavnov-Taylor identity.

The lack of gauge invariance of the characteristic functions of the cells induces new contributions to the measure localized on the cell (regularized) boundaries. Of course the above measure could be ill defined if the cells would accumulate around some singularity of \mathcal{F}_0 . This could perhaps induce instabilities of BRS symmetry in the sense of [11].

Another version of the S-T identity concerns the Feynman functional involving the sources j_{α} of ϕ^{α} , J^{I} of b_{I} , $\bar{\sigma}_{I}$ of ω^{I} and σ^{I} of $\bar{\omega}_{I}$. The new functional is defined according:

$$Z[j, J, \bar{\sigma}, \sigma] \equiv \int d\mu_C \ e^{iS_{GF}} e^{i\int dx \left[j_{\alpha}\phi^{\alpha} + J^I b_I + \bar{\sigma}_I \omega^I + \sigma^I \bar{\omega}_I\right]}, \tag{46}$$

where we have explicitly shown the space-time integral symbol. This more explicit form will be essential for the discussion of unitarity.

The new form of the S-T identity is a particular version of Eq. (32), that is:

$$\int d\mu_C \ e^{iS_{GF}} s \ e^{i\int dx \left[j_\alpha \phi^\alpha + J^I b_I + \bar{\sigma}_I \omega^I + \sigma^I \bar{\omega}_I\right]} = 0 \ . \tag{47}$$

It is possible, exploiting the nilpotency of s, to translate (47) into a functional differential equation for Z; this requires the introduction of further sources

for the composite operators generated by the action of s on the fields. On these are the source has to introduce the source γ_{α} for $s\phi^{\alpha}$ and ζ_{I} for $s\omega^{I}$. These sources, which are often called anti-fields, appear in a further factor in the functional measure:

$$d\mu_C \ e^{iS_{GF}} \to d\mu_C \ e^{iS_{GF}} e^{i\int dx \left[\gamma_\alpha s\phi^\alpha + \zeta_I s\omega^I\right]} = d\mu_C \ e^{iS_{GF}} e^{-i\int dx \ s\left[\gamma_\alpha \phi^\alpha - \zeta_I \omega^I\right]}$$
(48)

which remains s-invariant due since s is nilpotent. Notice that the introduction of the sources for the fields and their variations has enlarged the functional exterior algebra upon which the Feynman functional is defined. In particular $\sigma, \bar{\sigma}$ and γ are odd elements of this algebra. In the following formulae many derivatives are in fact anticommuting derivatives, this induces some obvious changes of sign.

Now, inserting the new measure into (47) we get:

$$\int d\mu_C \ e^{iS_{GF}} e^{i\int dx \left[\gamma_\alpha s\phi^\alpha + \zeta_I s\omega^I\right]} s \ e^{i\int dx \left[j_\alpha \phi^\alpha + J^I b_I + \bar{\sigma}_I \omega^I + \sigma^I \bar{\omega}_I\right]} = 0 \ , \tag{49}$$

that is:

$$\int d\mu_C \ e^{iS_{GF}} e^{i\int dx \left[\gamma_\alpha s\phi^\alpha + \zeta_I s\omega^I\right]} \int dx \left[j_\beta s\phi^b - \bar{\sigma}_J s\omega^J - \sigma^K b_K\right] e^{i\int dx \left[j_\alpha \phi^\alpha + J^I b_I + \bar{\sigma}_I \omega^I + \sigma^I \bar{\omega}_I\right]} = 0 \ .$$
(50)

This is equivalent to the first order partial differential equation for the extended Feynman functional:

$$\int dx \left[j_{\beta} \frac{\delta}{\delta \gamma_{\beta}} - \bar{\sigma}_{J} \frac{\delta}{\delta \zeta_{J}} - \sigma^{K} \frac{\delta}{\delta J^{K}} \right] \int d\mu_{C} \ e^{iS_{GF}} e^{i \int dx \left[\gamma_{\alpha} s \phi^{\alpha} + \zeta_{I} s \omega^{I} \right]}
e^{i \int dx \left[j_{\alpha} \phi^{\alpha} + J^{I} b_{I} + \bar{\sigma}_{I} \omega^{I} + \sigma^{I} \bar{\omega}_{I} \right]}
\equiv \mathcal{S}Z = \int dx \left[j_{\beta} \frac{\delta}{\delta \gamma_{\beta}} - \bar{\sigma}_{J} \frac{\delta}{\delta \zeta_{J}} - \sigma^{K} \frac{\delta}{\delta J^{K}} \right] Z = 0 .$$
(51)

From now on, whenever we shall make explicit dependence on the space-time coordinates, we shall use for the functional derivative the notation used in Eq.(51). This equation translates the S-T identity in terms of the Green functions. The same equation holds true for the connected functional:

$$Z_{c}[j, J, \bar{\sigma}, \sigma, \gamma, \zeta] \equiv -i \log Z[j, J, \bar{\sigma}, \sigma, \gamma, \zeta] .$$
 (52)

It is very useful to translate (52) into a functional differential equation for the proper functional [1]. In perturbation theory the proper functional is the functional generator of the 1-particle-irreducible amplitudes and is generally defined as the Legendre transform of Z_c . It is often called the effective action, although this name is also shared by completely different objects. Introducing the collective symbol \mathcal{J} for the field sources $(j, J, \bar{\sigma}, \sigma)$, \mathcal{K} for the other sources (γ, ζ) and Φ for the fields $(\phi, b, \omega, \bar{\omega})$, one defines the field functional:

$$\Phi\left[\mathcal{J},\mathcal{K}\right] \equiv \frac{\delta}{\delta \mathcal{J}} Z_c \left[\mathcal{J},\mathcal{K}\right] - \frac{\delta}{\delta \mathcal{J}} Z_c \left[0,0\right] , \qquad (53)$$

then, assuming that the inverse functional $\mathcal{J}[\Phi,\mathcal{K}]$ is uniquely defined, one has the proper functional:

$$\Gamma\left[\Phi,\mathcal{K}\right] \equiv Z_c\left[\mathcal{J}\left[\Phi,\mathcal{K}\right],\mathcal{K}\right] - \int dx \mathcal{J}\left[\Phi,\mathcal{K}\right] \left(\Phi + \frac{\delta}{\delta \mathcal{J}} Z_c\left[0,0\right]\right) . \tag{54}$$

It is easy to verify that:

$$\frac{\delta}{\delta\Phi}\Gamma\left[\Phi\left[\mathcal{J},\mathcal{K}\right],\mathcal{K}\right] = \mp\mathcal{J},\qquad(55)$$

$$\frac{\delta}{\delta \mathcal{K}} \Gamma \left[\Phi, \mathcal{K} \right] |_{\Phi = \Phi[\mathcal{J}, \mathcal{K}]} = \frac{\delta}{\delta \mathcal{K}} Z_c \left[\mathcal{J}, \mathcal{K} \right] . \tag{56}$$

Therefore:

$$\frac{\delta}{\delta\Phi(x)}\frac{\delta}{\delta\Phi'(y)}\Gamma\left[\Phi,\mathcal{K}\right]|_{\Phi=\Phi[\mathcal{J},\mathcal{K}]} = \mp \left[\frac{\delta}{\delta\mathcal{J}(x)}\frac{\delta}{\delta\mathcal{J}'(y)}Z_{c}\left[\mathcal{J},\mathcal{K}\right]\right]^{-1}.$$
 (57)

That is: the second field-derivative of Γ gives the full wave operator. Notice that the minus sign in (55) and (57) refers to commuting fields while in the anti-commuting case one has the plus sign.

Using the above identities one can immediately write the S-T identity for the proper functional:

$$\int dx \left[\frac{\delta}{\delta \phi^{\alpha}} \Gamma \frac{\delta}{\delta \gamma_{\alpha}} \Gamma + \frac{\delta}{\delta \omega^{I}} \Gamma \frac{\delta}{\delta \zeta_{I}} \Gamma + b_{I} \frac{\delta}{\delta \bar{\omega}_{I}} \Gamma \right] = 0 .$$
 (58)

This identity is a crucial tool in many instances, we shall exploit it in the analysis of unitarity, even more important is however its role in renormalization.

It is apparent from Eq.(26) that the Lagrangian density depends on the field b only through linear and bilinear terms. Thus this field behaves like a free field and satisfies a linear equation of motion which is not affected by the perturbative corrections and in the functional language turns out to be:

$$J^{I} + \left[V_{\alpha}^{I}(\partial) \frac{\delta}{\delta j_{\alpha}} + \xi \delta^{IJ} \frac{\delta}{\delta J^{J}} \right] Z_{c} = 0 , \qquad (59)$$

and hence:

$$\frac{\delta}{\delta b^I} \Gamma = V_\alpha^I(\partial) \phi^\alpha + \xi \delta^{IJ} b_J \ . \tag{60}$$

Equation (59) combined with (51), which holds true also for Z_c , gives:

$$\sigma^{I} + V_{\alpha}^{I}(\partial) \frac{\delta}{\delta \gamma_{\alpha}} Z_{c} = 0 , \qquad (61)$$

that is:

$$\frac{\delta}{\delta \bar{\omega}_I} \Gamma + V_{\alpha}^I(\partial) \frac{\delta}{\delta \gamma_{\alpha}} \Gamma = 0 . \tag{62}$$

We shall use these equations in order to simplify the analysis of the S-matrix unitarity.

A third interesting application of Eq.(58) is the search for generalizations of the geometrical setting of gauge theories. This is based on the fact that (58) is verified by the classical action S of a gauge theory. Indeed the classical action is the first term in the loop-ordered perturbative expansion of Γ .

The search for generalizations is justified by the fact that the low energy effective actions of more general theories, such as e.g. supergravity, are free from dimensional constraints; this allows the introduction of terms of higher degree in the sources γ and ζ . Disregarding the gauge fixing, setting $\bar{\omega} = b = 0$, let us consider for example:

$$S = S_{inv} + \gamma_{\alpha}\omega^{I}P_{I}^{\alpha}(\phi) + \frac{1}{2}\omega^{I}\omega^{J}\gamma_{\alpha}\gamma_{\beta}R_{IJ}^{\alpha\beta} + \frac{1}{2}\zeta_{I}C_{JK}^{I}\omega^{J}\omega^{K} , \qquad (63)$$

that inserted into (58) gives:

$$\begin{split} &\omega^I P_I^\alpha \partial_\alpha S_{inv} = 0 \ , \\ &\gamma_\beta \omega^I \omega^J \left[P_J^\alpha \partial_\alpha P_J^\beta - \frac{1}{2} P_K^\beta C_{IJ}^K + R_{IJ}^{\alpha\beta} \partial_\alpha S_{inv} \right] = 0 \ , \\ &\zeta_K \omega^I \omega^J \omega^L \left[P_I^\alpha \partial_\alpha C_{JL}^K + C_{MI}^K C_{JL}^M \right] = 0 \ , \end{split}$$

$$\gamma_{\alpha}\gamma_{\beta}\omega^{I}\omega^{J}\omega^{L} \left[\partial\gamma P_{I}^{\beta}R_{JL}^{\gamma\alpha} + \frac{1}{2} \left(P_{I}^{\gamma}\partial_{\gamma}R_{JL}^{\alpha\beta} + R_{MI}^{\alpha\beta}C_{JL}^{M} \right) \right] = 0 ,
\gamma_{\alpha}\gamma_{\beta}\gamma_{\gamma}\omega^{I}\omega^{J}\omega^{K}\omega^{L} \left[R_{IJ}^{\delta\alpha} \partial_{\delta}R_{KL}^{\beta\gamma} \right] = 0
\gamma_{\alpha}\zeta_{K}\omega^{I}\omega^{J}\omega^{K}\omega^{L} \left[R_{IJ}^{\delta\alpha} \partial_{\delta}C_{KL}^{M} \right] = 0 .$$
(64)

In order to simplify the notation we have written ∂_{α} instead of $\partial_{\phi^{\alpha}}$. The first line above implies the invariance of S_{inv} under the action of the differential system X given in (11), the second one takes place of the first equation in (20). It is indeed clear that the first term in this line corresponds to the commutator of two X's, the second one prescribes the structure functions of the algebra while the last one is new; it defines the deviation from a closed algebra that, being proportional to the field derivative of the physical action, vanishes on the mass shell. The third equation in (64) prescribes the deformed structure of the Jacobi identity, that is, of the the second line of Eq.(20), while the remaining lines give constraints for $R_{IJ}^{\alpha\beta}$. These constraints depend on the particular choice of Eq.(27) which excludes terms of second order in ζ . Eq.(27) shows the simplest example of the extensions of our method to open algebras that have been introduced by Batalin and Vilkovisky [12].

A very simple example of a mass-shell closed gauge algebra can be found if one tries to use the BRS algorithm to compute a n-dimensional Gaussian integral in polar coordinates³. Let \vec{x} with components x_i , (i=1,..,n) be the variable and $S_{inv}=-\frac{x^2}{2}$ define the invariant measure under the action of the gauge group O(n) corresponding to the BRS transformations:

$$sx_i = \omega_{ij}x_j \tag{65}$$

where ω_{ij} is the O(n) ghost antisymmetric in its indices and, as usual, the sum over repeated indices is understood. The polar coordinate gauge choice corresponds to the vanishing of n-1 components of \vec{x} . This configuration has a residual O(n-1) invariance and therefore the Jacobian matrix in (14) is highly degenerate. To overcome this difficulty one has to enlarge the BRS structure adding ghosts for ghosts (γ_{ij}) ; and hence introducing the ghost transformations:

$$s\omega_{ij} = \gamma_{ij} - \omega_{ik} \ \omega_{jk}$$

$$s\gamma_{ij} = \gamma_{ik} \ \omega_{jk} - \omega_{ik} \ \gamma_{jk} \ . \tag{66}$$

³This exercise has been suggested by J.Fröhlich

With this choice s is not nilpotent; indeed $s^2xi = \gamma_{ij}x_j$. It is mass-shell nilpotent since the "field equations" are: $\partial_{x_i}S_{inv} = x_i = 0$. Disregarding the structure of the gauge fixing and introducing a suitable set of sources, we identify the form of the action (27) which is adapted to the present case according:

$$S = S_{inv} + \mu_i s x_i + \zeta_{ij} s \omega_{ij} + \eta_{ij} s \gamma_{ij} + \frac{1}{2} \mu_i \mu_j \gamma_{ij} . \tag{67}$$

Eq.(67) satisfies the Slavnov-Taylor identity (58) (for b = 0) The last term in Eq.(67) corresponds to the second order term in γ in Eq.(27).

5 Unitarity

The first step in the analysis of S unitarity is the study of the asymptotic propagators and wave operators of a gauge theory [13].

For simplicity we shall limit our study to the situations in which the whole gauge symmetry is spontaneously broken and hence no vector particle is left massless. We shall also assume that, contrary to the Electro-Weak model, all physical and unphysical particles are stable, and we choose a gauge fixing prescription of the 't Hooft kind in which no unphysical particle, in particular Faddeev-Popov ghost, is mass-less.

First of all let us now consider how one can extract information about asymptotic particle states from the second derivatives of the proper functional. We label by Φ_i the fields appearing in our theory, that is the variables upon which Γ depends. In our case the index $i = 1, \dots, N$ distinguishes vector field components from ghost and matter field ones. We choose the fields Φ Hermitian, with the exception of the anti-ghost which are chosen anti-Hermitian. Defining the Fourier transformed field:

$$\tilde{\Phi}_i(p) \equiv \int \frac{d^4x}{(2\pi)^4} e^{-ip\cdot x} \Phi_i(x) , \qquad (68)$$

and setting:

$$\partial_{\tilde{\Phi}_i(p)} \equiv \int d^4x \ e^{ip\cdot x} \frac{\delta}{\delta \Phi_i(x)} \quad , \quad \partial_{\Phi_i} \equiv \frac{\delta}{\delta \Phi_i(0)} \ ,$$
 (69)

we get the Fourier transformed wave matrix of the theory whose elements are the second derivatives:

$$\partial_{\tilde{\Phi}_i(p)} \partial_{\Phi_j} \Gamma|_{\Phi=0} \equiv \Gamma_{i,j}(p) \ .$$
 (70)

With the exception of the ghost field components this is in general an Hermitian matrix. Due to translation invariance one has $\Gamma_{i,j}(p) = \pm \Gamma_{j,i}(-p)$ where the upper and lower signs refer to commuting and anti-commuting fields respectively.

For simplicity we limit our discussion to the cases in which the determinant $\Delta(p^2) \equiv \det |\Gamma(p)|$ is an analytic function of p^2 for $p^2 < M^2$ and its zeros in the analyticity domain lie on the positive real axis.

Let $(\bar{p}_{\lambda})^2 = m_{\lambda}^2$ for $\lambda = 1, \dots, a$ be solutions of order $k_{\lambda} < N$ of the equation $\Delta(p^2) = 0$. We assume that the matrix equation $\Gamma(-\bar{p}_{\lambda})_{i,j}v_j(\bar{p}_{\lambda}) = 0$ has k_{λ} independent, non-trivial solutions. That is, the matrix $\Gamma(-\bar{p}_{\lambda})$ has k_{λ} independent null eigenvectors.

This is a simplifying hypothesis which excludes the presence of dipole, or even worst, singularities in the unphysical propagators. These singularities are often met in gauge theories with particular gauge choices, e.g. QED in the Landau gauge and in the example we shall present in the following. As a matter of fact it is shown in Appendix C that dipole singularities correspond to mass-degenerate asymptotic states with opposite norm, that is, they correspond to a pair of simple poles with opposite residue and degenerate in mass. Thus our analysis is easily extended to the dipole case, however here we prefer to begin considering the simplest case of simple poles.

Under the above assumptions near a zero \bar{p}_{λ} of the determinant one has:

$$\Gamma(p) = (p^2 - m_{\lambda}^2)\tilde{\zeta}_{\lambda}(p) + R_{\lambda}(p) , \qquad (71)$$

where the matrices $\tilde{\zeta}_{\lambda}$ and R_{λ} are Hermitian and by no means unique. As already noticed in section 2 this lack of uniqueness does not affect the scattering theory. Still, in a suitable neighborhood of \bar{p}_{λ} , we make the choice $\tilde{\zeta}_{\lambda}R_{\lambda}=R_{\lambda}\tilde{\zeta}_{\lambda}=0$.

With this choice $\tilde{\zeta}_{\lambda}(p)$ has rank k_{λ} and, if $P_{\lambda}(\bar{p}_l)$ is the projector upon the space spanned by the null eigenvectors of $\Gamma(-\bar{p}_{\lambda})$, one has:

$$P_{\lambda}(p)\tilde{\zeta}_{\lambda}(p) = \tilde{\zeta}_{\lambda}(p)P_{\lambda}(p) = \tilde{\zeta}_{\lambda}(p)$$
(72)

Furthermore another matrix $\zeta_{\lambda}(p)$ with rank k_{λ} exists such that $\tilde{\zeta}_{\lambda}(p)\zeta_{\lambda}(p) = \zeta_{\lambda}(p)\tilde{\zeta}_{\lambda}(p) = P_{\lambda}(p)$. Therefore one has:

$$\Gamma(p)P_{\lambda}(p) = (p^2 - m_{\lambda}^2)\tilde{\zeta}_{\lambda}(p) + O((p^2 - m_{\lambda}^2)^2),$$
 (73)

and hence:

$$P_{\lambda}(p) = (p^2 - m_{\lambda}^2)\Gamma^{-1}(p)\tilde{\zeta}_{\lambda}(p) + O((p^2 - m_{\lambda}^2)) , \qquad (74)$$

and

$$\Gamma^{-1}(p) = \frac{\zeta(p)}{p^2 - m_{\lambda}^2 - i0_{+}} + Q_{\lambda}(p) , \qquad (75)$$

where Q_{λ} is analytic in the mentioned neighborhood of \bar{p}_{λ} . This last equation leads to Eq.(4) and hence allows the construction of the scattering matrix with the asymptotic fields satisfying the (anti-)commutation relations (5).

In particular the expression for the scattering matrix, Eq.(8), can be written in the alternative form involving the Fourier transformed asymptotic fields $\tilde{\phi}_{in}$ and corresponding sources \tilde{J} :

$$\mathbf{S} =: \exp \int \frac{d^4 p}{(2\pi)^4} \tilde{\Phi}_{in}^i(p) \Gamma_{j,i}(-p) \frac{\delta}{\delta \tilde{J}_j(p)} : Z|_{j=0} , \qquad (76)$$

where, taking into account Eq. (5), we have replaced the asymptotic wave operator K with the proper two point function Γ .

Now we consider how the Slavnov-Taylor identity for the proper functional (58) and Eq.s (60) and Eq.s (62) constrain the wave matrix of our gauge theory. Setting:

$$\frac{\delta^2}{\delta\tilde{\phi}^{\alpha}(p)\delta\phi^{\beta}(0)}\Gamma|_{\Phi=0} \equiv \Gamma_{\alpha\beta}(p) , \qquad (77)$$

and

$$\frac{\delta^2}{\delta\omega^I(0)\delta\tilde{\gamma}_\beta(p)}\Gamma|_{\Phi=0} \equiv \Gamma_I^\beta(p) , \qquad (78)$$

and computing in the origin of the field manifold the second functional derivative of Eq.(58) with respect to $\tilde{\phi}^{\alpha}(p)$ and $\omega^{I}(0)$ we get:

$$\Gamma_{\alpha\beta}(p) \ \Gamma_I^{\beta}(p) = 0 \ .$$
 (79)

In QED this corresponds to the transversality condition for the vacuum polarization.

In general from Eq's (11) and (27) we see, up to quantum corrections, that Eq. (78) defines G vectors $\Gamma_I(p)$, where G is the dimension of the gauge Lie algebra mentioned in section 3. These vectors span the tangent space to

the orbits in the origin. Therefore they must be independent and they must remain such beyond the quantum corrections. Eq. (79) shows that, for a generic p the matrix Γ has G null eigenvectors and hence its determinant is identically zero. We shall call Γ a degenerate matrix. For a generic choice of the momentum Γ 's rank is equal to F-G where F is the number of its rows and columns .

From Eq.(60) one get:

$$\frac{\delta^2}{\delta\tilde{\phi}^{\alpha}(p)\delta b_I(0)}\Gamma|_{\Phi=0} = V_{\alpha}^I(ip) , \qquad (80)$$

thus setting, as above:

$$\frac{\delta}{\delta\omega^I(0)\delta\tilde{\omega}_J(p)}\Gamma|_{\Phi=0} \equiv C_I^J(p) \ . \tag{81}$$

and taking the second derivative of (58) with respect to $\tilde{b}_I(p)$ and ω^J , we get

$$\Gamma_J^{\alpha}(p)V_{\alpha}^I(-ip) = -C_J^I(p) . \tag{82}$$

We have furthermore from (60):

$$\frac{\delta^2}{\delta \tilde{b}_J(p)\delta b_I(0)} \Gamma|_{\Phi=0} \equiv \xi \delta^{IJ} \ . \tag{83}$$

The wave matrix of gauge and matter fields is then given by:

$$\begin{pmatrix} \Gamma_{\alpha\beta} & V_{\alpha}^{I} \\ V_{\beta}^{J*} & \xi \delta^{I,J} \end{pmatrix} (p) , \qquad (84)$$

where, taking account that $V(\partial)$ is a real linear matrix, we have set $V_{\alpha}^{I} \equiv V_{\alpha}^{I}(ip)$ and $V_{\alpha}^{I*} = V_{\alpha}^{I}(-ip)$.

The wave operator of the Faddeev-Popov ghosts has matrix elements $C_I^J(p)$.

Let us now look for a null eigenvector of the above gauge and matter field wave matrix. We have to solve the system $\Gamma_{i,j}(-p)\Phi^j(p)=0$, that is:

$$\Gamma_{\alpha\beta}(-p)\phi^{\beta}(p) + V_{\alpha}^{I}(-ip)b_{I}(p) = 0$$

$$V_{\beta}^{I}(ip)\phi^{\beta}(p) + \xi b^{I}(p) = 0$$
(85)

which implies:

$$\left[\Gamma_{\alpha\beta}(-p) - \frac{1}{\xi} V_{\alpha}^{I}(-ip) V_{\beta}^{I}(ip)\right] \phi^{\beta}(p) \equiv \bar{\Gamma}_{\alpha\beta}(-p) \phi^{\beta}(p) = 0 . \tag{86}$$

and

$$\xi b_{in}^{I}(p) = -V_{\alpha}^{I}(ip)\phi_{in}^{\alpha}(p) . \tag{87}$$

If the gauge fixing procedure is complete, the matrix $\bar{\Gamma}$ must be non-degenerate, and the solutions to the equation $\det |\bar{\Gamma}|(p^2) = 0$ correspond to the masses of the bosonic gauge and matter fields. Notice that the spinor degrees of freedom are not considered since they only carry physical degrees of freedom.

Now, taking into account Eq. (79), Eq. (82) and Eq. (86) we get:

$$\bar{\Gamma}_{\alpha\beta}(p)\Gamma_I^{\beta}(p) = \frac{V_{\alpha}^J(ip)C_I^J(p)}{\xi} . \tag{88}$$

This equation relates the masses in the ghost sector to those in the gauge matter sector. The matrix $\bar{\Gamma}$ is Hermitian and hence it has the structure appearing in Eq.(71).

If \bar{p}_g is a solution of the equation $\det |C| = 0$, the linear system

$$C_J^I(-\bar{p}_q)w^J(\bar{p}_q) = 0 (89)$$

has non trivial solutions. Let M_g be the number of the independent solutions of (89) that we label by $w_a^J(\bar{p}_g)$ for $a=1,\dots,M_g$. From (88) we have

$$\bar{\Gamma}_{\alpha\beta}(-\bar{p}_g)\Gamma_I^{\beta}(-\bar{p}_g)w_a^I(\bar{p}_g) \equiv \bar{\Gamma}_{\alpha\beta}(-\bar{p}_g)\phi_a^{\beta}(\bar{p}_g) = 0.$$
 (90)

The M_g vectors ϕ_a are linearly independent since the vectors Γ_I are linearly independent. In other words, if one had a non-trivial set of coefficients c^a such that $c^a\phi_a^\alpha = c^aw_a^I(\bar{p}_g)\Gamma_I^\alpha(-\bar{p}_g) = 0$, this would contradict, either the hypothesis of linear independence of the w_a 's, or that of independence of the Γ_I 's.

In the same situation also the linear system

$$\bar{w}_I(\bar{p}_g)C_J^I(\bar{p}_g) = 0 \tag{91}$$

has M_g linearly independent solutions that we label by $\bar{w}^a(\bar{p}_g)$. For every solution of det |C|(p) = 0 one has the same number of solutions of Eq.(89) and of Eq.(91).

Notice that, if there are solutions to Eq.(86) with $b^I \not\equiv 0$, they correspond to solutions of Eq.(91). Indeed under the above condition $V_{\alpha}^I(ip)\phi^{\alpha}(p) \not\equiv 0$, however one has:

$$\Gamma_I^{\alpha}(p)\bar{\Gamma}_{\alpha\beta}(-p)\phi^{\beta}(p) = -C_I^J(p)V_{\beta}^J(ip)\phi^{\beta}(p) = \xi C_I^J(p)b_J(p) = 0.$$
 (92)

These solutions with non-vanishing b do not necessarily exist, indeed, as it will be shown in a moment, they may be replaced with dipole singularities. In any case the number of solutions of Eq.(86) with independent non-vanishing b cannot exceed G. Let it be G-X. We claim that Eq.(86) has X independent dipole solutions.

In order to prove this claim we better specify our framework. We are considering models in which the bosonic fields are either gauge vector or scalar fields. Among the scalar fields there are G Goldstone bosons associated with the spontaneous breakdown of gauge symmetry and required by the Higgs mechanism, and the same number of vector fields. Indeed in this situation and with the 't Hooft gauge choice, one can exclude massive asymptotic fields. Furthermore the bosonic wave operator is the sum the space-like vector field operator, that of the Higgs scalar fields, and the gauge fixing wave operator for the mixed the Goldstone fields and scalar components of the vector fields, that is $\partial^{\mu}A_{\mu}$. We concentrate on this wave operator for the moment forgetting the rest. With our Hermitian choice of the field basis the Fourier transform of the gauge fixing term of the wave operator defined in Eq.(77) is a real p-dependent matrix which has the following 2×2 block structure:

$$\begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{12}^T & \Gamma_{22} \end{pmatrix} , (93)$$

where the suffix T means "transposed". The blocks are $G \times G$ matrices and in particular Γ_{11} is the restriction of the wave matrix to the $\partial^{\mu}A_{\mu}$ components and Γ_{22} that to the Goldstone bosons. According to Golstone's theorem $\Gamma_{22} = p^2 K$ and, in our framework, K, which is a real symmetric matrix, is also invertible, thus we can write:

$$\Gamma_{22} = O^T p^2 d O (94)$$

where d is a real, diagonal and strictly positive matrix and O an orthogonal one.

The matrix $\Gamma_I^{\beta}(p)$ defined in Eq.(78) has the corresponding block structure:

$$\begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \tag{95}$$

where, due to the assumption of complete symmetry breakdown, both γ_1 and γ_2 must be invertible matrices.

Eq. (79) is written in terms of the above block matrices in the form:

$$\Gamma_{11}\gamma_1 + \Gamma_{12}\gamma_2 = 0$$
 , $\Gamma_{12}^T\gamma_1 + \Gamma_{22}\gamma_2 = 0$. (96)

Changing the basis of the Goldstone bosons one can transform Γ_{22} into the unity matrix leaving Γ_{11} and γ_1 unchanged and transforming Γ_{12} into $\Gamma_{12}O^T\sqrt{1/(p^2d)}\equiv\Gamma'_{12}$ and γ_2 into $\sqrt{p^2d}O\gamma_2\equiv\gamma'_2$. After this transformation Eq.(96) becomes:

$$\Gamma_{11}\gamma_1 + \Gamma'_{12}\gamma'_2 = 0 \quad , \quad \Gamma'^T_{12}\gamma_1 + \gamma'_2 = 0 \quad ,$$
 (97)

which is solved by:

$$\Gamma'_{12} = -(\gamma_1^{-1})^T \gamma_2^{'T} \equiv R \quad , \quad \Gamma_{11} = RR^T .$$
 (98)

The gauge fixing wave matrix $\bar{\Gamma}$ defined in Eq.(86) can be written as the difference $\bar{\Gamma} = \Gamma - \Delta$ where Δ is deduced from Eq.(86). Here we generalize the gauge choice replacing the matrix $\xi \delta_{IJ}$ with the symmetric real and invertible matrix ξ^{IJ} getting the new matrix Δ in the block form:

$$\begin{pmatrix} \xi^{-1} & \xi^{-1}V^{T} \\ V\xi^{-1} & V\xi^{-1}V^{T} \end{pmatrix} . (99)$$

With the new choice of the Goldstone field basis Δ keeps the same form with V replaced with $\sqrt{1/(p^2d)}$ O $V \equiv W$.

The ghost wave matrix is computed from Eq.(82) getting:

$$C = -\gamma_1^T - \gamma_2^T V = -\gamma_1^T (I - RW) . {100}$$

Therefore, in order to study the masses of the asymptotic fields corresponding to the Goldstone bosons and scalar components of the gauge fields, we have to compute the determinant of the matrix whose block structure is:

$$\bar{\Gamma} \equiv \begin{pmatrix} RR^T - \xi^{-1} & R - \xi^{-1}W^T \\ R^T - W\xi^{-1} & I - W\xi^{-1}W^T \end{pmatrix} . \tag{101}$$

We shall use the following formula for a matrix in block form:

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A - BD^{-1}C) \det D , \qquad (102)$$

where, of course, we have assumed that $\det D$ does not vanish⁴. In order to exploit this formula let us consider the matrix:

$$RR^{T} - \xi^{-1} - (R - \xi^{-1}W^{T})(I - W\xi^{-1}W^{T})^{-1}(R^{T} - W\xi^{-1})$$

$$= RR^{T} - \xi^{-1} - (R - \xi^{-1}W^{T}) \sum_{n=0}^{\infty} (W\xi^{-1}W^{T})^{n}(R^{T} - W\xi^{-1})$$

$$= -(I - RW)\xi^{-1} \sum_{n=0}^{\infty} (W^{T}W\xi^{-1})^{n}(I - W^{T}R^{T})$$

$$= - - (I - RW)\xi^{-1}W^{T}(W^{-1})^{T}(I - W\xi^{-1}W^{T})^{-1}(I - W^{T}R^{T})$$
(103)

where we have assumed the convergence of the operator power series. This does not limit the generality of our analysis since the operator series certainly convergences for a suitable choice of the matrix ξ and, from Eq.(102) we get:

$$\det \bar{\Gamma} = (\det \xi)^{-1} (\det C)^2 / (\det \gamma_1)^2 \tag{104}$$

which is independent of $\det(I - W\xi^{-1}W^T) \equiv \det D$.

In order to simplify the discussion let us now assume that on a given ghost mass-shell, say $p^2 = \bar{m}_0^2$, there is a single ghost solution w_0 (Eq.(89)) and a single anti-ghost one \bar{w}_0 (Eq.(91)). This hypothesis does not limit the generality of our discussion since it can always be met through a suitable choice of the ghost fixing parameters. Our formula (104) shows that on the same mass-shell det $\bar{\Gamma}$ has a double zero Therefore, either Eq.(86) has two solutions, that is, $\hat{\phi}^{\alpha} = \Gamma_I^{\alpha}(-\bar{p}_0)w_0^I$ and a solution with $b_I = \bar{w}_{0,I}$, or the first solution is unique and the corresponding asymptotic field propagator has a dipole singularity. It is shown in Appendix C that in this case the positive frequency part of the asymptotic field creates two degenerate particle states with opposite norm.

The analysis of S-matrix unitarity can be pusher forward in both situations, it is however simpler in the case of two solutions [3]. Therefore, in the following, we shall consider this particular case.

⁴I thank C. Imbimbo for suggesting this particular equation among many equivalent forms

Let us now compute, in the framework of our gauge theory, the operator Σ appearing and defined in Eq.(8). Limiting the analysis to the gauge and bosonic matter fields and to the ghost fields, that is, disregarding the spinor fields, one has:

$$\Sigma_{B} = \int \frac{d^{4}p}{(2\pi)^{4}} \left[\phi_{in}^{\alpha}(p) \left[\Gamma_{\beta\alpha}(-p) \frac{\delta}{\delta \tilde{j}_{\beta}(p)} + V_{\alpha}^{I}(ip) \frac{\delta}{\delta \tilde{J}^{I}(p)} \right] \right]$$

$$+ b_{in}^{I}(p) \left[V_{\beta}^{I}(-ip) \frac{\delta}{\delta \tilde{j}_{\beta}(p)} + \xi \frac{\delta}{\delta \tilde{J}^{I}(p)} \right] - \omega_{in}^{I}(p) C_{I}^{J}(-p) \frac{\delta}{\delta \tilde{\sigma}^{J}(p)}$$

$$+ \bar{\omega}_{in,J}(p) C_{I}^{J}(p) \frac{\delta}{\delta \tilde{\sigma}_{I}(p)} \right] .$$

$$(105)$$

Using Eq.(87), we can transform this equation into the simpler form:

$$\Sigma_{B} = \int \frac{d^{4}p}{(2\pi)^{4}} \left[\phi_{in}^{\alpha}(p) \bar{\Gamma}_{\beta\alpha}(-p) \frac{\delta}{\delta \tilde{j}_{\beta}(p)} - \omega_{in}^{I}(p) C_{I}^{J}(-p) \frac{\delta}{\delta \tilde{\sigma}^{J}(p)} + \bar{\omega}_{in,J}(p) C_{I}^{J}(p) \frac{\delta}{\delta \tilde{\sigma}_{I}(p)} \right] . \tag{106}$$

It is clear that the operator Σ_B acts on the asymptotic states containing ghost and bosonic particles, since we are forgetting spinors. Among the bosonic particles there are those corresponding to the asymptotic fields $\phi_a^{\alpha}(p) = \Gamma_I^{\alpha}(-p)w_a^I(p)$ considered above in Eq.(90) and the asymptotic components of b. There also are the particles corresponding to the remaining independent solutions of Eq.(86) that we call *physical* together with the spinor particles.

Computing the commutator $[S, \Sigma_B]Z$ where;

$$S \equiv \int dx \left[j_{\beta} \frac{\delta}{\delta \gamma_{\beta}} - \bar{\sigma}_{J} \frac{\delta}{\delta \zeta_{J}} - \sigma^{K} \frac{\delta}{\delta J^{K}} \right] , \qquad (107)$$

is the functional differential operator appearing in Eq.(51) and Z is the Feynman functional, we get:

$$[S, \Sigma_B] Z = -\int \frac{d^4p}{(2\pi)^4} \left[\phi_{in}^{\alpha}(p) \bar{\Gamma}_{\beta\alpha}(-p) \frac{\delta}{\delta \tilde{\gamma}_{\beta}(p)} + \omega_{in}^I(p) C_I^J(-p) \frac{\delta}{\delta \tilde{J}^J(p)} \right] Z ,$$
(108)

where we have omitted the term $\int \frac{d^4p}{(2\pi)^4} \bar{\omega}_{in,J}(p) C_I^J(p) \frac{\delta}{\delta \tilde{\zeta}_I(p)} Z$ since $\frac{\delta}{\delta \tilde{\zeta}_I(p)} Z$ is a regular function of p^2 at $p^2 = m_g^2$ for any ghost mass m_g and the momentum integral in Eq.(108) is restricted to the union of ghost mass-shells where

a pole in $\frac{\delta}{\delta\tilde{\zeta}_I(p)}Z$ should be needed in order to compensate the zeros of the ghost wave operator C. For the same reason the functional derivative $\frac{\delta}{\delta\tilde{\gamma}_\beta(p)}Z$ appearing in the same integral should be replaced with its mass-shell singular part, which, using Eq.(78), is easily seen to be given by $\Gamma_I^\beta(p)\frac{\delta}{\delta\tilde{\sigma}_I(p)}Z$. Indeed one finds that the two-point function $\frac{\delta^2}{\delta\sigma^I(0)\delta\tilde{\gamma}_\alpha(p)}Z_c|_{\mathcal{J}=0}$ is equal to $\Gamma_J^\alpha(p)\frac{\delta^2}{\delta\sigma^I(0)\delta\tilde{\sigma}_J(p)}Z_c|_{\mathcal{J}=0}$ where the second factor is the ghost propagator.

Furthermore, from Eq.(59) one has that $\frac{\delta}{\delta \tilde{J}^I(p)}Z$ should be replaced with $-\frac{1}{\xi}V_I^{\alpha}(-ip)\frac{\delta}{\delta \tilde{J}^{\alpha}(p)}Z$. Hence, taking into account Eq.(88), Eq.(108) should be written:

$$[S, \Sigma_B] Z = -\int \frac{d^4 p}{(2\pi)^4} \left[\phi_{in}^{\alpha}(p) \Gamma_{\beta\alpha}(-p) \Gamma_I^{\beta}(p) \frac{\delta}{\delta \tilde{\sigma}_I(p)} \right]$$

$$-\frac{1}{\xi} \omega_{in}^I(p) C_I^J(-p) V_J^{\alpha}(-ip) \frac{\delta}{\delta \tilde{j}^{\alpha}(p)} Z$$

$$= \int \frac{d^4 p}{(2\pi)^4} \left[\omega_{in}^I(p) \Gamma_I^{\alpha}(-p) \bar{\Gamma}_{\beta\alpha}(-p) \frac{\delta}{\delta \tilde{j}_{\beta}(p)} \right]$$

$$-\frac{1}{\xi} \phi_{in}^{\alpha}(p) V_{\alpha}^J(ip) C_I^J(p) \frac{\delta}{\delta \tilde{\sigma}_I(p)} Z.$$

$$(109)$$

We have used the relation $\Gamma_{\alpha\beta}(p) = \Gamma_{\beta\alpha}(-p)$. This result is the kernel of the first rigorous unitarity proof given in [3]. Here we shall follow the simpler analysis given in [14].

Following Kugo-Ojima, we introduce an operator Q acting on the asymptotic state space and annihilating the vacuum state: $Q|0\rangle = 0$. The new operators satisfies the following relations

$$[Q, \phi_{in}^{\alpha}(p)] = -i \Gamma_I^{\alpha}(-p)\omega_{in}^I(p)$$

$$\{Q, \bar{\omega}_{in,I}(p)\} = i\frac{1}{\xi}V_{\alpha}^J(ip)\phi_{in}^{\alpha}(p) = -ib_{in}^I(p) . \tag{110}$$

We remind that with our conventions all the asymptotic fields are Hermitian except $\bar{\omega}_{in,I}$ which is anti-Hermitian, and the gauge fixing action is Hermitian. The operator Q generates a nilpotent transformation on the Fock space. Its kernel KerQ is the subspace generated by the action of the vacuum state of the positive frequency (creation) part of the physical fields and by $b_{in,I}^{(+)}$ and $\omega_{in}^{(+)I}$, while its image ImQ is the subspace generated by $b_{in,I}^{(+)}$ and $\omega_{in}^{(+)I}$.

Now it is interesting to define the cohomology of Q, that is, the quotient space KerQ/ImQ. This is the linear space of the Q-equivalence classes of the elements of KerQ. We consider Q-equivalent two elements of KerQ if their difference belongs to ImQ. It is easy to verify that the cohomology of Q coincides with Q-equivalence classes of the states of the subspace of the Fock space generated by the action on the vacuum state of the positive frequency part of the physical fields. Notice that the asymptotic properties of these field components are completely determined by the invariant part of Γ and hence it is expected that they generate a positive norm space. It is also a direct consequence of the nilpotency of Q that the states in the image of Q have vanishing scalar product with those in its kernel, they are in particular zero norm states.

Using (110) one has

$$i[Q, \Sigma_B] Z = [S, \Sigma_B] Z. \qquad (111)$$

It follows that:

$$[Q, \mathbf{S}] = [Q, : e^{\Sigma_B} :] Z|_{\mathcal{J}=0} =: [Q, e^{\Sigma_B}] : Z|_{\mathcal{J}=0} = -i[\mathcal{S}, : e^{\Sigma_B} :] Z|_{\mathcal{J}=0}$$

= $i : e^{\Sigma_B} : \mathcal{S}Z|_{\mathcal{J}=0} = 0$. (112)

Notice that Q acts separately on the positive and the negative frequency parts of the asymptotic fields and this justifies the second identity in Eq.(112) which shows that the commutator $[Q, \mathbf{S}]$ vanishes. Under the assumption that the measure corresponding to a Hermitian Lagrangian defines a "pseudo" unitary \mathbf{S} -matrix in the asymptotic Fock space:

$$\mathbf{S}\mathbf{S}^{\dagger} = \mathbf{S}^{\dagger}\mathbf{S} = I , \qquad (113)$$

and that the physical space is a positive metric space - both assumptions are true in perturbation theory - we conclude that, owing to (112) and to the above discussed properties of Q, the S-matrix is unitary in the physical space identified with the cohomology of Q. That is: if the initial state $|i\rangle$ is a physical state and hence it belongs to the kernel of Q and has positive norm, $S|i\rangle$ belongs to the kernel of Q and has the same norm as $|i\rangle$, therefore it defines a Q-equivalence class corresponding to a positive norm physical state.

We consider for example an SU(2) Higgs model in the tree approximation [3]. This model involves an iso-triplet of vector fields \vec{A}_{μ} , a triplet of Goldstone particles $\vec{\pi}$, and the Higgs field σ that appears in our calculations only through its vacuum expectation value V. Therefore now the symbol ϕ^{α} corresponds to \vec{A}_{μ} and $\vec{\pi}$. We add a further iso-triplet of Nakanishi-Lautrup multipliers \vec{b} . In the tree approximation the free Lagrangian density is given by:

$$\mathcal{L} = -\frac{\vec{F}_{\mu\nu} \cdot \vec{F}^{\mu\nu}}{4} + \frac{1}{2} \left(\partial \vec{\pi} - gV \vec{A} \right)^2 + \vec{b} \cdot \left(\partial \vec{A} + \rho \vec{\pi} \right) + \xi \frac{b^2}{2} - \vec{\omega} \cdot (\partial^2 + g\rho V) \vec{\omega} ,$$
(114)

From now on we shall disregard the isotopic indices since all the wave operators are diagonal in the isotopic space. The wave matrix Γ defined in (77) is given by:

$$\begin{pmatrix} p^{\mu}p^{\nu} - g^{\mu\nu}p^2 + g^2V^2g^{\mu\nu} & -igVp^{\mu} \\ igVp^{\nu} & p^2 \end{pmatrix}, \tag{115}$$

where the first row and line correspond to A and the second ones to π . The matrix V_I^{α} defining the gauge fixing in (25) corresponds to:

$$\begin{pmatrix} -ip^{\mu} \\ \rho \end{pmatrix}, \tag{116}$$

Notice that the isotopic indices are hidden. The ghost wave operator C(p) is given by $p^2 - g\rho V$.

The wave operator of the gauge and scalar fields $\bar{\Gamma}_{\alpha\beta}(-p)$ appearing in Eq. (86) is:

$$\begin{pmatrix}
p^{\mu}p^{\nu}(1-\frac{1}{\xi}) - g^{\mu\nu}p^{2} + g^{2}V^{2}g^{\mu\nu} & i(gV-\frac{\rho}{\xi})p^{\mu} \\
-i(gV-\frac{\rho}{\xi})p^{\nu} & p^{2}-\frac{\rho^{2}}{\xi}
\end{pmatrix}.$$
(117)

Forgetting the isospin degeneracy its determinant is given by:

$$\det |\bar{\Gamma}| = -\frac{1}{\xi} (p^2 - (gV)^2)^3 (p^2 - g\rho V)^2 , \qquad (118)$$

however Eq.(86) has only four independent solutions, three of them correspond to $p^2 = m_p^2 = (gV)^2$, the remaining one corresponding to $p^2 = m_g^2 = g\rho V$ where the above determinant has a double zero. Looking at the solutions of Eq.(86) and considering in particular the *longitudinal* ones in which $A_{\mu}^{(l)}(p) \sim p_{\mu}$, one sees that for a generic choice of ρ there is a single solution which, in contrast with the hypothesis made at the beginning of this chapter,

corresponds to a dipole singularity of the gauge and scalar field propagator which is degenerate with the simple pole of the ghost propagator. This is by no means surprising since the asymptotic longitudinal vector field satisfies the free field equations $(p^2 - \xi gV)A^{(l)}_{\mu}(p) = ip_{\mu}(\xi gV - \rho)\vec{\pi}$ which is analogous to the field equation in Landau's gauge QED. One can find some details on the structure of the asymptotic state space in Appendix C and in Landau's gauge QED in [15]. The dipole singularity disappears and one finds a fifth independent solution of Eq. (86) mass degenerate with the ghosts if $\rho = \xi gV$. This is called the *special* 't Hooft choice. It is easy to verify that Eq. (86) has solutions with $b \not\equiv 0$ only in this special case.

The matrix Γ_{α}^{I} defined in (78) in the tree approximation is:

$$\begin{pmatrix} ip^{\mu} \\ gV \end{pmatrix} . \tag{119}$$

It is apparent that its columns correspond to solutions of Eq. (86) on the ghost mass-shell in agreement with Eq. (90). The Q operator is defined by the conditions:

$$\begin{aligned}
&[Q, \vec{A}_{in}^{\mu}(p)] = -p^{\mu} \vec{\omega}_{in}(p) \quad , \quad [Q, \vec{\pi}_{in}(p)] = -igV \vec{\omega}_{in}(p) \\
&\{Q, \vec{\omega}_{in}(p)\} = \frac{1}{\xi} [p_{\mu} A_{in}^{\mu}(p) + i\rho \vec{\pi}_{in}(p)] \equiv -ib_{in}(p) .
\end{aligned} \tag{120}$$

This operator is apparently nilpotent since on the ghost mass-shell

$$\{Q, \left[Q, \vec{A}_{in}^{\mu}(p)\right]\} = 0$$
 (121)

In order to have a look at the physical content of the theory we introduce three space-like polarization vectors $\epsilon_a^{\mu}(p)$ orthogonal to the momentum p and such that $\epsilon_a^{\mu}(p)\epsilon_{b\nu}=-\delta_{ab}$. Due to Eq.(86) the asymptotic field $\vec{A}_{in}^{\mu}(p)=\sum_a \vec{\phi}_{a,in}(p)\epsilon_a^{\mu}(p)$ has support on the mass-shell $p^2=g^2V^2$ and its positive frequency part generates a positive norm subspace of the asymptotic space. Furthermore from the above relation it is apparent that $[Q,\phi_{a,in}(p)]=0$ and hence the Q-equivalence classes of these positive norm states identify the cohomology of Q, that is the physical state space.

Notice that the presence of dipole singularities remarked above, which, as shown in Appendix C, corresponds to ghost-degenerate unphysical states with opposite norm, does not affect the conclusions of our analysis [3][15].

A Appendix

The differential system $\{X\}$ being integrable one can choose local charts of coordinates trivializing the fibration; let us indicate by $\{\xi\}$ the gauge invariant coordinates, that are constant along the orbits, and by $\{\eta\}$ those parametrizing points on the orbits; the transition functions between two neighbouring charts are:

$$\xi_a = \xi_a \left(\xi_b \right) \quad , \quad \eta_a = \eta_a \left(\xi_b, \eta_b \right) .$$
 (122)

In a given chart the elements of the differential system are:

$$X_I \equiv X_I^J(\xi, \eta) \,\partial_{\eta^J} \,. \tag{123}$$

On every fibre we define the adjoint system of differential 1-forms according

$$\omega^I \equiv \omega_J^I(\xi, \eta) \, d\eta^J \,\,, \tag{124}$$

with:

$$X_I^L \omega_L^J = \delta_I^J \ . \tag{125}$$

Taking into account the commutation relation (13) one verifies directly that these 1-forms satisfy the Maurer-Cartan equation:

$$d_V \ \omega^I \equiv d\eta^L \partial_{\eta^L} \ \omega^I = -\frac{1}{2} C^I_{JK} \ \omega^J \omega^K \ . \tag{126}$$

The above equations define explicitly and, according to (122), globally on very orbit, the exterior algebra involved into the definition of the Faddeev Popov measure. However they cannot be directly translated into their field theory equivalent due to locality. Indeed, even if the differential system $\{X\}$ is given as a set of local functional differential operators in the gauge field variables, the trivializing coordinates are non-local with respect to the fields.

This difficulty is overcome replacing the system of generators of the vertical exterior algebra $\{d\eta\}$ with $\{\omega\}$ that in this way appear into the theory as new Grassmannian local field variables. This however requires that the vertical exterior derivarive d_V be written according (19):

$$d_V = \omega^I X_I - \frac{1}{2} C_{JK}^I \ \omega^J \omega^K \partial_{\omega^I} \ , \tag{127}$$

where the first term in the right-hand side accounts for the action of d_V on functions while the second term acts on the exterior algebra generators.

B Appendix

In this appendix we prove (43) exploiting (32). The first step will be the proof of two lemmas whose recursive use will lead to (43). Let us consider the set of cells $\{U_a\}$ and the corresponding partition of unity $\{\chi_a\}$ and gauge fixing functionals $\{\Theta_a\}$ We define:

$$\left(s\chi_{a_1}...s\chi_{a_{n-1}}\chi_{a_n}\right)_A \equiv \sum_{k=1}^n (-1)^{k-n}\chi_{a_k}s\chi_{a_1}...s\check{\chi}_{a_k}...s\chi_{a_n}, \qquad (128)$$

where the check mark above χ_{a_k} means that the corresponding term should be omitted. It is fairly evident that the functional (128) is antisymmetric with respect to permutations of the indices $(a_1, ..., a_n)$ and that its support is contained in the intersection of the corresponding cells.

It is apparent that:

$$s\left(s\chi_{a_1}...s\chi_{a_{n-1}}\chi_{a_n}\right)_A = (-1)^{n+1}n\ s\chi_{a_1}...s\chi_{a_n}\ . \tag{129}$$

Furthermore, taking into account that $\{\chi_a\}$ is a partition of unity on the support of the functional measure, one has:

$$\sum_{a_{n+1}} \left(s\chi_{a_1} ... s\chi_{a_n} \chi_{a_{n+1}} \right)_A = \sum_{a_{n+1}} \sum_{k=1}^{n+1} (-1)^{k-n-1} \chi_{a_k} s\chi_{a_1} ... s\check{\chi}_{a_k} ... s\chi_{a_{n+1}}$$

$$= s\chi_{a_1} ... s\chi_{a_n} , \qquad (130)$$

indeed only the term with k = n+1 contributes to the second member giving the right-hand side of this equation. Comparing (129) with (130), we get:

$$s\left(s\chi_{a_1}...s\chi_{a_{n-1}}\chi_{a_n}\right)_A = (-1)^{n+1}n \sum_{a_{n+1}} \left(s\chi_{a_1}...s\chi_{a_n}\chi_{a_{n+1}}\right)_A . \tag{131}$$

Let now $A_{a_1,...,a_{n+1}}$ be antisymmetric in its indices, one has:

$$\sum_{a_1,\dots,a_{n+1}} A_{a_1,\dots,a_{n+1}} \partial_{\Theta} (\Theta_{a_1} \dots \Theta_{a_n}) e^{is\Theta_{a_1,\dots,a_n}}$$

$$= i \frac{(-1)^n}{n+1} \sum_{a_1,\dots,a_{n+1}} A_{a_1,\dots,a_{n+1}} s \partial_{\Theta} (\Theta_{a_1} \dots \Theta_{a_{n+1}}) e^{is\Theta_{a_1,\dots,a_{n+1}}} . (132)$$

Indeed, using the identity:

$$\sum_{a_1,\dots,a_{n+1}} A_{a_1,\dots,a_{n+1}} V_{a_1,\dots,a_n} = \frac{(-1)^n}{n+1} \sum_{k=1}^{n+1} (-1)^{k+1} \sum_{a_1,\dots,a_{n+1}} A_{a_1,\dots,a_{n+1}} V_{a_1,\dots,\check{a}_k,\dots,a_{n+1}} ,$$
(133)

the left-hand side of (132) is written:

$$\frac{(-1)^n}{n+1} \sum_{a_1,\dots,a_{n+1}} A_{a_1,\dots,a_{n+1}} \sum_{k< l=1}^{n+1} (-1)^{k+l} \Theta_{a_1} ... \check{\Theta}_{a_k} ... \check{\Theta}_{a_l} ... \Theta_{a_{n+1}}
\left(e^{is\Theta_{a_1,\dots,\check{a}_k,\dots,a_{n+1}}} - e^{is\Theta_{a_1,\dots,\check{a}_l,\dots,a_{n+1}}} \right)
= i \frac{(-1)^n}{n+1} \sum_{a_1,\dots,a_{n+1}} A_{a_1,\dots,a_{n+1}} \sum_{k< l=1}^{n+1} (-1)^{k+l} \Theta_{a_1} ... \check{\Theta}_{a_k} ... \check{\Theta}_{a_l} ... \Theta_{a_{n+1}}
s (\Theta_{a_k} - \Theta_{a_l}) e^{is\Theta_{a_1,\dots,a_{n+1}}} ,$$
(134)

from which one reaches (131).

Let us now consider the extension of (33) to the case of a cell decomposition of \mathcal{F}_C , one has:

$$\int d\mu_{C} \sum_{a} \chi_{a} e^{is\Theta_{a}} sX = -\int d\mu_{C} \sum_{a_{1}} s\chi_{a_{1}} e^{is\Theta_{a_{1}}} X$$

$$= \int d\mu_{C} \sum_{a_{1},a_{2}} (s\chi_{a_{1}}\chi_{a_{2}})_{A} e^{is\Theta_{a_{1}}} X$$

$$= \frac{1}{2} \int d\mu_{C} \sum_{a_{1},a_{2}} (s\chi_{a_{1}}\chi_{a_{2}})_{A} \left(e^{is\Theta_{a_{1}}} - e^{is\Theta_{a_{2}}} \right) X$$

$$= -\frac{i}{2} \int d\mu_{C} \sum_{a_{1},a_{2}} (s\chi_{a_{1}}\chi_{a_{2}})_{A} s\partial_{\Theta} (\Theta_{a_{1}}\Theta_{a_{2}}) e^{is\Theta_{a_{1},a_{2}}} X . \tag{135}$$

In (135) we have used (130) and (132) with n = 1. From (135) we have:

$$\int d\mu_C \left[\sum_a \chi_a e^{is\Theta_a} + \frac{i}{2} \int d\mu_C \sum_{a_1, a_2} (s\chi_{a_1}\chi_{a_2})_A \,\partial_{\Theta} (\Theta_{a_1}\Theta_{a_2}) \, e^{is\Theta_{a_1, a_2}} \right] sX$$

$$= -\frac{i}{2} \int d\mu_C \sum_{a_1, a_2} s \left(s\chi_{a_1}\chi_{a_2} \right)_A \,\partial_{\Theta} (\Theta_{a_1}\Theta_{a_2}) \, e^{is\Theta_{a_1, a_2}} \, . \tag{136}$$

Using recursively the same equations we get:

$$d\mu_{C} \left[\sum_{a} \chi_{a} e^{is\Theta_{a}} - \sum_{n=1}^{m} i^{n} \frac{(-1)^{\frac{n(n-1)}{2}}}{n+1} \left(s\chi_{a_{1}} ... s\chi_{a_{n}} \chi_{a_{n+1}} \right)_{A} \right]$$

$$\partial_{\Theta} \left(\Theta_{a_{1}} ... \Theta_{a_{n+1}} \right) e^{is\Theta_{a_{1}} ..., a_{n+1}} sX$$

$$= i^{m} \frac{(-1)^{\frac{m(m-1)}{2}}}{m+1} \int d\mu_{C} s \left(s\chi_{a_{1}} ... s\chi_{a_{m}} \chi_{a_{m+1}} \right)_{A}$$

$$\partial_{\Theta} \left(\Theta_{a_{1}} ... \Theta_{a_{m+1}} \right) s e^{is\Theta_{a_{1}} ..., a_{m+1}} X . \tag{137}$$

It is clear that, if the maximum effective number of intersecting cells is N the right-hand side of (137) vanishes for $m \geq N$.

C Appendix

The aim of this Appendix is to clarify the structure of the Fock space associated with a generalized free Hermitian field ϕ whose propagator presents a dipole singularity, that is such that:

$$\langle 0|T(\phi(x)\phi(0))|0\rangle = i \int \frac{dp}{(2\pi)^4} \frac{e^{ipx}}{(m^2 - p^2 - i0 +)^2}$$
 (138)

Using the Lehmann spectral representation it follows that the Wightman function is

$$\langle 0|\phi(x)\phi(0)|0\rangle = \int \frac{dp}{(2\pi)^4} e^{ipx}\theta(p^0)\delta'(p^2 - m^2) ,$$
 (139)

and hence, if the field is Hermitian, the corresponding Fock space must be an indefinite metric space. Indeed $\delta'(x)$ is not a positive distribution. We shall call pseudo-Hermitian an Hermitian operator in an indefinite metric space and we shall label the pseudo-Hermitian conjugate by a dagger. It is clear that our field satisfies the linear equation:

$$(\partial^2 + m^2)^2 \phi(x) = 0 , \qquad (140)$$

whose general solution is

$$\phi(x) = \int \frac{dp}{(2\pi)^2} e^{ipx} [\alpha(p)\delta(p^2 - m^2) + \beta(p)\delta'(p^2 - m^2)] . \tag{141}$$

Here $\alpha(p)$ and $\beta(p)$ have analogous $p \equiv (p^0, \vec{p})$ dependence, in particular, $\alpha(p) = \alpha_+(\vec{p})\theta(p^0) + \alpha_-(\vec{p})\theta(-p^0)$. The stability condition for the vacuum state with our metric choice, $px = p^0x^0 - \vec{p} \cdot \vec{x}$, implies that $\alpha_-(\vec{p})|0\rangle = 0$ and $\beta_-(\vec{p})|0\rangle = 0$. Thus α_- and β_- are annihilation operators. An alternative expression for the field is given integrating over p^0 and taking into account that, if the function f(x) continuous with its derivative has n non-degenerate zeros x_i one has:

$$\delta(f(x)) = \sum_{i=1}^{n} \frac{\delta(x - x_i)}{|f'(x_i)|} \quad \text{and} \quad \delta'(f(x)) = \frac{1}{f'(x)} \sum_{i=1}^{n} \frac{\delta'(x - x_i)}{|f'(x_i)|} . \tag{142}$$

Setting $E(\vec{p}) = \sqrt{|\vec{p}|^2 + m^2}$ one gets:

$$\phi(x) = \int \frac{dp}{(2\pi)^2 2E(\vec{p})} e^{ipx} \left[\alpha_+(\vec{p})\delta(p^0 - E(\vec{p})) + \alpha_-(\vec{p})\delta(p^0 + E(\vec{p})) \right]$$

$$+ \frac{\beta_+(\vec{p})}{2p^0} \delta'(p^0 - E(\vec{p})) + \frac{\beta_-(\vec{p})}{2p^0} \delta'(p^0 + E(\vec{p})) \right]$$

$$= \int \frac{d\vec{p}}{(2\pi)^2 2E(\vec{p})} e^{ipx} |_{p^0 = E(\vec{p})} [\alpha_+(\vec{p}) + \frac{\beta_+(\vec{p})}{2E^2(\vec{p})} (1 - ix^0 E(\vec{p}))]$$

$$+ \int \frac{d\vec{p}}{(2\pi)^2 2E(\vec{p})} e^{ipx} |_{p^0 = -E(\vec{p})} [\alpha_-(\vec{p}) + \frac{\beta_-(\vec{p})}{2E^2(\vec{p})} (1 + ix^0 E(\vec{p}))] .$$
 (143)

Therefore we see that the field is pseudo-Hermitian if

$$\alpha_{+}^{\dagger}(\vec{p}) = \alpha_{-}(-\vec{p}) \quad , \quad \beta_{+}^{\dagger}(\vec{p}) = \beta_{-}(-\vec{p}) .$$
 (144)

Writing Eq(139) in terms of $\alpha(p)$ and $\beta(p)$ one gets from Eq.(141)

$$\langle 0|\phi(x)\phi(y)|0\rangle = \int \frac{d^4p}{(2\pi)^4 2E(\vec{p})} e^{ip(x-y)} \frac{\delta'(p^0 - E(\vec{p}))}{2p^0}$$

$$= \int \frac{d\vec{p}}{(2\pi)^4 2E^3(\vec{p})} e^{ipx}|_{p^0 = E(\vec{p})} (1 - i(x^0 - y^0)E(\vec{p}))$$

$$= \int \frac{d\vec{p} \ d\vec{q}}{(2\pi)^4 4E(\vec{p})E(\vec{q})} e^{i(px+qy)}|_{p^0 = E(\vec{p}), q^0 = -E(\vec{q})}$$

$$\langle 0|\left(\alpha_+(\vec{p}) + \frac{\beta_+(\vec{p})}{2E^2(\vec{p})} (1 - ix^0 E(\vec{p}))\right) \left(\alpha_-(\vec{q}) + \frac{\beta_-(\vec{q})}{2E^2(\vec{q})} (1 + iy^0 E(\vec{q}))\right) |0\rangle ,$$

from which one finds the commutation conditions:

$$[\alpha_{+}(\vec{p}), \beta_{-}(\vec{q})] = [\beta_{+}(\vec{p}), \alpha_{-}(\vec{q})] = 2E(\vec{p})\delta(\vec{p} + \vec{q})$$

$$[\alpha_{+}(\vec{p}), \alpha_{-}(\vec{q})] = -E^{-1}(\vec{p})\delta(\vec{p} + \vec{q}) \quad [\beta_{+}(\vec{p}), \beta_{-}(\vec{q})] = 0 . \quad (146)$$

Taking into account the pseudo-Hermiticity conditions, these commutation relations can be diagonalized introducing the annihilation and creation operators $A_{\sigma}(\vec{p})$ and $A_{\sigma}^{\dagger}(\vec{p})$ with $\sigma = \pm$ defined by:

$$A_{\sigma}(\vec{p}) \equiv \sqrt{\frac{2}{\sqrt{5}}} (E(\vec{p})\alpha_{-}(\vec{p}) + \sigma) \frac{\sqrt{5} + \sigma 1}{4E(\vec{p})} \beta_{-}(\vec{p}) , \qquad (147)$$

whose commutation rules are:

$$[A_{\sigma}(\vec{p}), A_{\sigma'}^{\dagger}(\vec{q})] = \delta_{\sigma,\sigma'}\sigma 2E(\vec{p})\delta(\vec{p} - \vec{q}) \quad [A_{\sigma}(\vec{p}), A_{\sigma'}(\vec{q})] = o . \tag{148}$$

It is apparent that the corresponding Fock space has indefinite metric corresponding to the operator $(-1)^{N_-}$ with $N_- = \int d\vec{p}/(2E(\vec{p}))A_-^{\dagger}(\vec{p})A_-(\vec{p})$.

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