Local BRST cohomology in gauge theories

Glenn Barnich [a,b], Friedemann Brandt [c], Marc Henneaux [a,d]

[a] *Physique Théorique et Mathématique, Université Libre de Bruxelles, Boulevard du Triomphe, Campus Plaine C.P. 231, B-1050 Bruxelles, Belgium*

[b] *Departament de Física Teòrica, Universitat de València, C. Dr. Moliner 50, E-46100 Burjassot (València), Spain*

[c] *Institut für Theoretische Physik, Universität Hannover, Appelstr. 2, D-30167 Hannover, Germany*

[d] *Centro de Estudios Científicos, Casilla 1469, Valdivia, Chile*

E-mail: gbarnich@ulb.ac.be, brandt@itp.uni-hannover.de, henneaux@ulb.ac.be

Abstract

The general solution of the anomaly consistency condition (Wess-Zumino equation) has been found recently for Yang-Mills gauge theory. The general form of the counterterms arising in the renormalization of gauge invariant operators (Kluberg-Stern and Zuber conjecture) and in gauge theories of the Yang-Mills type with non power counting renormalizable couplings has also been worked out in any number of spacetime dimensions. This Physics Report is devoted to reviewing in a self-contained manner these results and their proofs. This involves computing cohomology groups of the differential introduced by Becchi, Rouet, Stora and Tyutin, with the sources of the BRST variations of the fields ("antifields") included in the problem. Applications of this computation to other physical questions (classical deformations of the action, conservation laws) are also considered. The general algebraic techniques developed in the Report can be applied to other gauge theories, for which relevant references are given.

Contents

1 Conventions and notation

Spacetime. Greek indices from the middle of the alphabet generally are spacetime indices, $\mu = 0, 1, \ldots, n-1$. We work in n-dimensional Minkowskian space with metric $\eta_{\mu\nu} = \mathrm{diag}(-1, +1, ..., +1)$. The spacetime coordinates are denoted by x^μ. The Levi-Civita tensor $\epsilon^{\mu_1 \ldots \mu_n}$ is completely antisymmetric in all its indices with $\epsilon^{01 \ldots n-1} = 1$. Its indices are lowered with the metric $\eta_{\mu\nu}$. The differentials dx^μ anticommute, the volume element is denoted by $d^n x$,

$$dx^\mu dx^\nu = dx^\mu \wedge dx^\nu = -dx^\nu dx^\mu, \quad d^n x = dx^0 \wedge \ldots \wedge dx^{n-1}.$$

Gauge theories of the Yang-Mills type. The gauge group is denoted by G, its Lie algebra by $\mathcal{G}$. Capital Latin indices from the middle of the alphabet $I, J, \ldots$ generally refer to a basis for $\mathcal{G}$. The structure constants of the Lie algebra in that basis are denoted by $f_{IJ}{}^K$.

The gauge coupling constant(s) are denoted by e and are explicitly displayed in the formulae. However, in most formulae we use a collective notation which does not distinguish the different gauge coupling constants when the gauge group is the direct product of several (abelian or simple) factors.

The covariant derivative is defined by $D_\mu = \partial_\mu - e A_\mu^I \rho(e_I)$ where $\{e_I\}$ is a basis of $\mathcal{G}$ and ρ a representation of $\mathcal{G}$. The field strength is defined by $F_{\mu\nu}^I = \partial_\mu A_\nu^I - \partial_\nu A_\mu^I + e f_{JK}{}^I A_\mu^J A_\nu^K$, the corresponding 2-form by $F^I = \frac{1}{2} F_{\mu\nu}^I dx^\mu dx^\nu$.

General. The Einstein summation convention over repeated upper or lower indices generally applies.

Complete symmetrization is denoted by ordinary brackets $(\cdots)$, complete antisymmetrization by square brackets $[\cdots]$ including the normalization factor:

$$M_{(\mu_1 \ldots \mu_k)} = \frac{1}{k!} \sum_{\sigma \in \mathcal{S}^k} M_{\mu_{\sigma(1)} \ldots \mu_{\sigma(k)}}, \quad M_{[\mu_1 \ldots \mu_k]} = \frac{1}{k!} \sum_{\sigma \in \mathcal{S}^k} (-)^\sigma M_{\mu_{\sigma(1)} \ldots \mu_{\sigma(k)}},$$

where the sums run over all elements σ of the permutation group $\mathcal{S}^k$ of k objects and $(-)^\sigma$ is 1 for an even and -1 for an odd permutation.

Dependence of a function f on a set of fields ϕ^i and a finite number of their derivatives is collectively denoted by $f([\phi])$,

$$f([\phi]) \equiv f(\phi^i, \partial_\mu \phi^i, \ldots, \partial_{(\mu_1 \ldots \mu_r)} \phi^i).$$

All our derivatives are left derivatives.

The antifield of a field ϕ^i is denoted by ϕ_i^*; for instance the antifield corresponding to the Yang-Mills gauge potential A_μ^I is $A_I^{*\mu}$.

The Hodge dual of a p-form ω is denoted by $\star\omega$,

$$\omega = \frac{1}{p!} dx^{\mu_1} \ldots dx^{\mu_p} \omega_{\mu_1 \ldots \mu_p}, \quad \star\omega = \frac{1}{p!(n-p)!} dx^{\mu_1} \ldots dx^{\mu_{n-p}} \epsilon_{\mu_1 \ldots \mu_n} \omega^{\mu_{n-p+1} \ldots \mu_n}.$$

2   Introduction

2.1   Purpose of report

Gauge symmetries underlie all known fundamental interactions. While the existence of the gravitational force can be viewed as a consequence of the invariance of the laws of physics under arbitrary spacetime diffeomorphisms, the non-gravitational interactions are dictated by the invariance under an internal non-Abelian gauge symmetry.

It has been appreciated in the last twenty years or so that many physical questions concerning local gauge theories can be powerfully reformulated in terms of local BRST cohomology. The BRST differential was initially introduced in the context of perturbative quantum Yang-Mills theory in four dimensions [45, 46, 47, 48, 209]. One of the aims was to relate the Slavnov-Taylor identities [188, 200] underlying the proof of power-counting renormalizability [145, 146, 147, 148, 168, 169, 170, 171] to an invariance of the gauge-fixed action (for a recent historical account, see [233]). However, it was quickly realized that the scope of BRST theory is much wider. It can not only be formulated for any theory with a gauge freedom, but also it is quite useful at a purely classical level.

The purpose of this report is to discuss in detail the local BRST cohomology for gauge theories of the Yang-Mills type. We do so by emphasizing whenever possible the general properties of the BRST cohomology that remain valid in other contexts. At the end of the report, we give some references to papers where the local BRST cohomology is computed for other gauge theories by means of similar techniques.

2.2   Gauge theories of the Yang-Mills type

We first define the BRST differential in the Yang-Mills context. The Yang-Mills gauge potential is a one-form, which we denote by $A^I = dx^\mu A^I_\mu$. The gauge group can be any finite dimensional group of the form $G = G_0 \times G_1$, where G_0 is Abelian and G_1 is semi-simple. In practice, G is compact so that G_0 is a product of $U(1)$ factors while G_1 is compact and semi-simple. This makes the standard Yang-Mills kinetic term definite positive. However, it will not be necessary to make this assumption for the cohomological calculation. This is important as non-compact gauge groups arise in gravity or supergravity. The Lie algebra of the gauge group is denoted by $\mathcal{G}$.

The matter fields are denoted by ψ^i and can be bosonic or fermionic. They are assumed to transform linearly under the gauge group according to a completely reducible representation. The corresponding representation matrices of $\mathcal{G}$ are denoted by T_I and the structure constants of $\mathcal{G}$ in that basis are written $f_{IJ}{}^K$,

$$[T_I, T_J] = f_{IJ}{}^K T_K \ . \tag{2.1}$$

The field strengths and the covariant derivatives of the matter fields are denoted by $F^I_{\mu\nu}$ and $D_\mu \psi^i$ respectively,

$$
\begin{aligned}
F^I_{\mu\nu} &= \partial_\mu A^I_\nu - \partial_\nu A^I_\mu + e\, f_{JK}{}^I A^J_\mu A^K_\nu \ , \tag{2.2}\\
D_\mu \psi^i &= \partial_\mu \psi^i + e\, A^I_\mu T^i_{Ij} \psi^j. \tag{2.3}
\end{aligned}
$$

5

Here e denotes the gauge coupling constant(s) (one for each simple or abelian factor).

The (infinitesimal) gauge transformations read

$$\delta_\epsilon A_\mu^I = D_\mu \epsilon^I, \quad \delta_\epsilon \psi^i = -e \, \epsilon^I T_{Ij}^i \psi^j, \tag{2.4}$$

where the ϵ^I are the "gauge parameters" and

$$D_\mu \epsilon^I = \partial_\mu \epsilon^I + e \, f_{JK}{}^I A_\mu^J \epsilon^K. \tag{2.5}$$

In the BRST formalism, the gauge parameters are replaced by anticommuting fields C^I; these are the "ghost fields" of [104, 84, 102].

The Lagrangian L is a function of the fields and their derivatives up to a finite order ("local function"),

$$L = L([A_\mu^I], [\psi^i]) \tag{2.6}$$

(see section 1 for our notation and conventions). It is invariant under the gauge transformations (2.4) up to a total derivative, $\delta_\epsilon L = \partial_\mu k^\mu$ for some k^μ that may be zero. The detailed form of the Lagrangian is left open at this stage except that we assume that the matter sector does not carry a gauge-invariance of its own, so that (2.4) are the only gauge symmetries. This requirement is made for definiteness. The matter fields could carry further gauge symmetries (e.g., p-form gauge symmetries) which would bring in further ghosts; these could be discussed along the same lines but for definiteness and simplicity, we exclude this possibility.

A theory with the above field content and gauge symmetries is said to be of the "Yang-Mills type". Specific forms of the Lagrangian are $L = (-1/4)\delta_{IJ}F_{\mu\nu}^I F^{J\mu\nu}$, for which $\delta_\epsilon L = 0$ (Yang-Mills original theory [229]) or $d^3x \, L = \text{Tr}(AdA + (e/3)A^3)$ which is invariant only up to a non-vanishing surface term, $\delta_\epsilon L = \partial_\mu k^\mu$ with $k^\mu \neq 0$ (Chern-Simons theory in 3 dimensions [80]).

We shall also consider "effective Yang-Mills theories" for which the Lagrangian contains all possible terms compatible with gauge invariance [119, 224] and thus involves derivatives of arbitrarily high order.

In physical applications one usually assumes, of course, that L is in addition Lorentz or Poincaré invariant. The cohomological considerations actually go through without this assumption.

The BRST differential acts in an enlarged space that contains not only the original fields and the ghosts, but also sources for the BRST variations of the fields and the ghosts. These sources are denoted by $A_I^{*\mu}$, ψ_i^* and C_I^* respectively, and have Grassmann parity opposite to the one of the corresponding fields. They have been introduced in order to control how the BRST symmetry gets renormalized [45, 46, 47, 231, 232]. They play a crucial role in the BV construction [28, 29, 30] where they are known as the antifields; for this reason, they will be indifferently called antifields or BRST sources here.

The BRST-differential decomposes into the sum of two differentials

$$s = \delta + \gamma \tag{2.7}$$

with δ and γ acting as

Z	δZ	γZ
A_μ^I	0	$D_\mu C^I$
ψ^i	0	$-e\,C^I T_{Ij}^i \psi^j$
C^I	0	$\frac{1}{2}\,e\,f_{KJ}{}^I\,C^J C^K$
C_I^*	$-D_\mu A_I^{*\mu} - e\,\psi_i^* T_{Ij}^i \psi^j$	$e\,f_{JI}{}^K\,C^J C_K^*$
$A_I^{*\mu}$	L_I^μ	$e\,f_{JI}{}^K\,C^J A_K^{*\mu}$
ψ_i^*	L_i	$e\,C^I \psi_j^* T_{Ii}^j$

(2.8)

where

$$D_\mu C^I = \partial_\mu C^I + e\,f_{JK}{}^I A_\mu^J C^K \tag{2.9}$$

$$D_\mu A_I^{*\mu} = \partial_\mu A_I^{*\mu} - e\,f_{JI}{}^K A_\mu^J A_K^{*\mu} \tag{2.10}$$

$$L_I^\mu = \frac{\delta L}{\delta A_\mu^I}\,, \quad L_i = (-)^{\epsilon_i}\frac{\delta L}{\delta \psi^i}\,. \tag{2.11}$$

Here $\delta L/\delta\phi$ denotes the Euler-Lagrange derivative of L with respect to ϕ (taken from the left), and ϵ_i is the Grassmann parity of ψ^i ($\epsilon_i = 0$ for a boson, $\epsilon_i = 1$ for a fermion). The parity dependent sign in L_i originates from the convention that δ is defined through Euler-Lagrange right-derivatives of L.

δ, γ (and thus s) are extended to the derivatives of the variables by the rules $\delta\partial_\mu = \partial_\mu\delta$, $\gamma\partial_\mu = \partial_\mu\gamma$. These rules imply $\delta d + d\delta = \gamma d + d\gamma = sd + ds = 0$, where d is the exterior spacetime derivative $d = dx^\mu \partial_\mu$, because dx^μ is odd. Furthermore, δ, γ, s and d act as left (anti-)derivations, e.g., $\delta(ab) = (\delta a)b + (-1)^{\epsilon_a} a\delta b$, where ϵ_a is the Grassmann parity of a.

The BRST differential is usually not presented in the above manner in the literature; we shall make contact with the more familiar formulation in the appendix 2.A below. However, we point out already that on the fields A_μ^I, ψ^i and the ghosts C^I, s reduces to γ and the BRST transformation takes the familiar form of a "gauge transformation in which the gauge parameters are replaced by the ghosts" (for the fields A_μ^I and ψ^i), combined with $sC^I = \frac{1}{2}ef_{KJ}{}^I C^J C^K$ in order to achieve nilpotency. The differential δ, known as the Koszul-Tate differential, acts non-trivially only on the antifields. It turns out to play an equally important rôle in the formalism.

The decomposition (2.7) is related to the various gradings that one introduces in the algebra generated by the fields, ghosts and antifields. The gradings are the pure ghost number *puregh*, the antifield number *antifd* and the (total) ghost number *gh*. They are not independent but related through

$$gh = puregh - antifd. \tag{2.12}$$

The antifield number is also known as "antighost number". The gradings of the basic

variables are given by

$$
\begin{array}{c|c|c|c}
Z & puregh(Z) & antifd(Z) & gh(Z) \\
\hline
A_\mu^I & 0 & 0 & 0 \\
\psi^i & 0 & 0 & 0 \\
C^I & 1 & 0 & 1 \\
C_I^* & 0 & 2 & -2 \\
A_I^{*\mu} & 0 & 1 & -1 \\
\psi_i^* & 0 & 1 & -1
\end{array}
\tag{2.13}
$$

The BRST differential and the differentials δ and γ all increase the ghost number by one unit. The differential δ does so by decreasing the antifield number by one unit while leaving the pure ghost number unchanged, while γ does so by increasing the pure ghost number by one unit while leaving the antifield number unchanged. Thus, one has $antifd(\delta) = -1$ and $antifd(\gamma) = 0$. The decomposition (2.7) corresponds to an expansion of s according to the antifield number, $s = \sum_{k \geq -1} s_k$, with $antifd(s_k) = k$, $s_{-1} = \delta$ and $s_0 = \gamma$. For Yang-Mills gauge models, the expansion stops at γ. For generic gauge theories, in particular theories with open algebras, there are higher-order terms s_k, $k \geq 1$ (when the gauge algebra closes off-shell, $s_1, s_2, \ldots$ vanish on the fields, but not necessarily on the antifields).

2.3   Relevance of BRST cohomology

As stated above, the BRST symmetry provides an extremely efficient tool for investigating many aspects of a gauge theory. We review in this subsection the contexts in which it is useful. We shall only list the physical questions connected with BRST cohomological groups without making explicitly the connection here. This connection can be found in the references listed in the course of the discussion. [Some comments on the link between the BRST symmetry and perturbative renormalization are surveyed – very briefly – in appendix 2.A below to indicate the connection with the gauge-fixed formulation in which these questions arose first.]

A crucial feature of the BRST differential s is that it is a differential, i.e., it squares to zero,

$$
s^2 = 0. \tag{2.14}
$$

In Yang-Mills type theories (and in other gauge theories for which $s = \delta + \gamma$) this is equivalent to the fact that δ and γ are differentials that anticommute,

$$
\delta^2 = 0, \ \delta\gamma + \gamma\delta = 0, \ \gamma^2 = 0. \tag{2.15}
$$

The BRST cohomology is defined as follows. First, the BRST cocycles A are objects that are "BRST -closed", i.e., in the kernel of s,

$$
sA = 0. \tag{2.16}
$$

8

Because s squares to zero, the BRST-coboundaries, i.e., the objects that are BRST-exact (in the image of s) are automatically closed,

$$A = sB \quad \Rightarrow \quad sA = 0. \tag{2.17}$$

The BRST-cohomology is defined as the quotient space $\mathrm{Ker}\, s / \mathrm{Im}\, s$,

$$H(s) = \frac{\mathrm{Ker}\, s}{\mathrm{Im}\, s}. \tag{2.18}$$

An element in $H(s)$ is an equivalence class of BRST-cocycles, where two BRST-cocycles are identified if they differ by a BRST-coboundary. One defines similarly $H(\delta)$ and $H(\gamma)$.

One can consider the BRST cohomology in the space of local functions or in the space of local functionals. In the first instance, the "cochains" are local functions, i.e., are functions of the fields, the ghosts, the antifields and a finite number of their derivatives. In the second instance, the cochains are local functionals i.e., integrals of local volume-forms, $A = \int a$, $a = f\, d^n x$, where f is a local function in the above sense (n denotes the spacetime dimension). When rewritten in terms of the integrands (see subsection 4.4), the BRST cocycle and coboundary conditions $s \int a = 0$ and $\int a = s \int b$ become respectively

$$sa + dm = 0 \qquad \text{(cocycle condition)}, \tag{2.19}$$
$$a = sb + dn \qquad \text{(coboundary condition)}, \tag{2.20}$$

for some local $(n-1)$-forms m and n. For this reason, we shall reserve the notation $H(s)$ for the BRST cohomology in the space of local functions and denote by $H(s|d)$ the BRST cohomology in the space of local functionals. So $H(s|d)$ is defined by (2.19) and (2.20), while $H(s)$ is defined by

$$sa = 0 \qquad \text{(cocycle condition)}, \tag{2.21}$$
$$a = sb \qquad \text{(coboundary condition)}. \tag{2.22}$$

In both cases, a and b (and m, n) are local forms.

One may of course fix the ghost number and consider cochains with definite ghost number. At ghost number j, the corresponding groups are denoted $H^j(s)$ and $H^{j,n}(s|d)$ respectively, where in the latter instance the cochains have form-degree equal to the spacetime dimension n. The calculation of $H^{j,n}(s|d)$ is more complicated than that of $H^j(s)$ because one must allow for the possibility to integrate by parts. It is useful to consider the problem defined by (2.19), (2.20) for other values p of the form-degree. The corresponding cohomology groups are denoted by $H^{j,p}(s|d)$.

Both cohomologies $H(s)$ and $H(s|d)$ capture important physical information about the system. The reason that the BRST symmetry is important at the quantum level is that the standard method for quantizing a gauge theory begins with fixing the gauge. The BRST symmetry and its cohomology become then substitutes for gauge invariance, which would be otherwise obscure. The realization that BRST theory is also useful at the classical level is more recent. We start by listing the quantum questions for which the local BRST cohomology is relevant (points 1 though 4 below). We then list the classical ones (points 5 and 6).

1. Gauge anomalies: $H^{1,n}(s|d)$.

   Before even considering whether a quantum gauge field theory is renormalizable, one must check whether the gauge symmetry is anomaly-free. Indeed, quantum violations of the classical gauge symmetry presumably spoil unitarity and probably render the quantum theory inconsistent. As shown in [45, 46, 47], gauge anomalies $\mathcal{A} = \int a$ are ghost number one local functionals constrained by the cocycle condition $s\mathcal{A} = 0$, i.e.,

   $$sa + dm = 0, \qquad (2.23)$$

   which is the BRST generalization of the Wess-Zumino consistency condition [227]. Furthermore, trivial solutions can be removed by adding local counterterms to the action. Thus, the cohomology group $H^{1,n}(s|d)$ characterizes completely the form of the non trivial gauge anomalies. Once $H^{1,n}(s|d)$ is known, only the coefficients of the candidate anomalies need to be determined.

   The computation of $H^{1,n}(s|d)$ was started in [89, 192, 55, 193, 234, 235, 40, 201, 41, 174, 96, 97, 13, 14] and completed in the antifield-independent case in [61, 63, 62, 92, 101]. Some aspects of solutions with antifields included have been discussed in [47, 88, 11, 12, 65] ; the general solution was worked out more recently in [21, 23, 24].

2. Renormalization of Yang-Mills gauge models: $H^{0,n}(s|d)$.

   Yang-Mills theory in four dimensions is power-counting renormalizable [145, 146, 147, 148]. However, if one includes interactions of higher mass dimensions, or considers the Yang-Mills Lagrangian in higher spacetime dimensions, one loses power-counting renormalizability. To consistently deal with these theories, the effective theory viewpoint is necessary [222] (for recent reviews, see [184, 115, 155, 175, 180]). The question arises then as to whether these theories are renormalizable in the "modern sense", i.e., whether the gauge symmetry constrains the divergences sufficiently, so that these can be absorbed by gauge-invariant counterterms at each order of perturbation theory [119, 223, 224, 225]. This is necessary for dealing meaningfully with loops – and not just tree diagrams. This question can again be formulated in terms of BRST cohomology. Indeed, the divergences and counterterms are constrained by the cocycle condition (2.23) but this time at ghost number zero; and trivial solutions can be absorbed through field redefinitions [231, 45, 46, 47, 87, 210, 217, 218, 219, 167, 7, 8, 224].

   Thus it is $H^{0,n}(s|d)$ that controls the counterterms. The question raised above can be translated, in cohomological terms, as to whether the most general solution of the consistency condition $sa + dm = 0$ at ghost number zero can be written as $a = h \, d^n x + sb + dn$ where h is a local function which is off-shell gauge invariant up to a total derivative. The complete answer to this question, which is affirmative when the gauge-group is semi-simple, has been worked out recently [21, 23, 24].

3. Renormalization of composite, gauge-invariant operators: $H^0(s)$ and $H^{0,n}(s|d)$.

   The renormalization of composite gauge-invariant operators arises in the analysis of the operator product expansion, relevant, in particular, to deep inelastic scattering. In the mid-seventies, it was conjectured that gauge-invariant operators can only mix, upon renormalization, with gauge-invariant operators or with gauge-variant operators which vanish in physical matrix elements [156, 157, 158] (see also [86, 90]).

   As established also there, the conjecture is equivalent to proving that in each BRST cohomological class at ghost number zero, one can choose a representative that is strictly gauge-invariant. For local operators involving the variables and their derivatives (at a given, unspecified, point) up to some finite order, the relevant cohomology is $H^0(s)$. The problem is to show that the most general solution of $sa = 0$ at ghost number zero is of the form $a = I + sb$, where I is an invariant function of the curvatures $F^I_{\mu\nu}$, the fields ψ^i and a finite number of their covariant derivatives. For operators of low mass dimension, the problem involves a small number of possible composite fields and can be analyzed rather directly. However, power counting becomes a less constraining tool for operators of high mass dimension, since the number of composite fields with the appropriate dimensions proliferates as the dimension increases. One must use cohomological techniques that do not rely on power counting. The conjecture turns out to be correct and was proved first (in four spacetime dimensions) in [152]. The proof was streamlined in [135] using the above crucial decomposition of s into $\delta + \gamma$. Further information on this topic may be found in [77]. Recent applications are given in [78, 129].

   In the same way, the cohomological group $H^{0,n}(s|d)$ controls the renormalization of integrated gauge-invariant operators $\int d^n x \, O(x)$, or, as one also says, operators at zero momentum. As mentioned above, its complete resolution has only been given recently [21, 23, 24].

4. Anomalies for gauge invariant operators: $H^1(s)$ and $H^{1,n}(s|d)$.

   The question of mixing of gauge invariant operators as discussed in the previous paragraph may be obstructed by anomalies if a non gauge invariant regularization and renormalization scheme is or has to be used [91]. To lowest order, these anomalies are ghost number 1 local functions a and have to satisfy the consistency condition $sa = 0$. BRST-exact anomalies are trivial because they can be absorbed through the addition of non BRST invariant operators, which compensate for the non invariance of the scheme. This means that the cohomology constraining these obstructions is $H^1(s)$. Similarly, the group $H^{1,n}(s|d)$ controls the anomalies in the renormalization of integrated gauge invariant composite operators [91].

5. Generalized conservation laws - "Characteristic cohomology" and BRST cohomology in negative ghost number

The previous considerations were quantum. The BRST cohomology captures also important classical information about the system. For instance, it has been proved in [23] that the BRST cohomology at negative ghost number is isomorphic to the so-called "characteristic cohomology" [214, 215, 72], which generalizes the familiar notion of (non-trivial) conserved currents. A conserved current can be defined (in Hodge dual terms) as an $(n-1)$-form that is d-closed modulo the equations of motion,

$$dj \approx 0 \qquad (2.24)$$

where $\approx$ means "equal when the equations of motion hold". One says that a conserved current is (mathematically) trivial when it is on-shell equal to an exact form (see e.g. [178]),

$$j \approx dm, \qquad (2.25)$$

where m is a *local* form. The characteristic cohomology in form degree $n-1$ is by definition the quotient space of conserved currents by trivial ones. The characteristic cohomology in arbitrary form degree is defined by the same equations, taken at the relevant form degrees. The characteristic cohomology in form-degree $n-2$ plays a rôle in the concept of "charge without charge" [177, 211].

The exact correspondence between the BRST cohomology and the characteristic cohomology is as follows [23]: the characteristic cohomology H_{char}^{n-k} in form-degree $n-k$ is isomorphic to the BRST cohomology $H^{-k,n}(s|d)$ in ghost number $-k$ and maximum form-degree n.[1] Cocycles of the cohomology $H^{-k,n}(s|d)$ involve necessarily the antifields since these are the only variables with negative ghost number.

The reformulation of the characteristic cohomology in terms of BRST cohomology is particularly useful in form degree $< n-1$, where it has yielded new results leading to a complete calculation of H_{char}^{n-k} for $k > 1$. It is also useful in form-degree $n-1$, where it enables one to work out the explicit form of all the conserved currents that are not gauge-invariant (and cannot be invariantized by adding trivial terms) [22]. The calculation of the gauge-invariant currents is more complicated and depends on the specific choice of the Lagrangian, contrary to the characteristic cohomology in lower form-degree.

6. Consistent interactions: $H^{0,n}(s_0|d)$, $H^{1,n}(s_0|d)$ - Uniqueness of Yang-Mills cubic vertex

Is is generally believed that the only way to make a set of massless vector fields consistently interact is through the Yang-Mills construction - apart from interactions that do not deform the abelian gauge transformations and involve only the abelian curvatures or abelian Chern-Simons terms. Partial proofs of this result exist [9, 220, 20] but these always make implicitly some restrictive assumptions

---

[1]The isomorphism assumes the De Rham cohomology of the spacetime manifold to be trivial. Otherwise, the statement needs to be refined along the lines of [100]. A proof of the isomorphism is given in sections 6 and 7.

on the number of derivatives involved in the coupling or the polynomial degree of the interaction. In fact, a counterexample exists in three spacetime dimensions, which generalizes the Freedman-Townsend model for two-forms in four dimensions [111, 3, 4].

The problem of constructing consistent (local) interactions for a gauge field theory has been formulated in general terms in [50]. It turns out that this formulation has in fact a natural interpretation in terms of deformation theory and involves the computation of the free BRST cohomologies $H^{0,n}(s_0|d)$ and $H^{1,n}(s_0|d)$[19] (see also [118, 190, 137]), where s_0 is the free BRST differential. The BRST point of view systematizes the search for consistent interactions and the demonstration of their uniqueness up to field redefinitions.

We present in this report complete results on $H^k(s)$. We also provide, for a very general class of Lagrangians, a complete description of the cohomological groups $H(s|d)$ in terms of the non trivial conserved currents that the model may have. So, all these cohomology groups are known once all the global symmetries of the theory have been determined – a problem that depends on the Lagrangian. Furthermore, we show that the cocycles in the cohomological groups $H^{0,n}(s|d)$ (counterterms) and $H^{1,n}(s|d)$ (anomalies) may be chosen not to involve the conserved currents when the Yang-Mills gauge group has no abelian factor (in contrast to the groups $H^{g,n}(s|d)$ for other values of g). So, in this case, we can also work out completely $H^{0,n}(s|d)$ and $H^{1,n}(s|d)$, without specifying L. We also present complete results for $H^{-k,n}(s|d)$ (with $k > 1$) as well as partial results for $H^{-1,n}(s|d)$. Finally, we establish the uniqueness of the Yang-Mills cubic vertex in four spacetime dimensions, using a result of Torre [203].

2.4 Cohomology and antifields

The cohomological investigation of the BRST symmetry was initiated as early as in the seminal papers [45, 46, 47], which gave birth to the modern algebraic approach to the renormalization of gauge theories (for a recent monograph on the subject, see [182]; see also [181, 54, 103, 120]). Many results on the antifield-independent cohomology were established in the following fifteen years. However, the antifield-dependent case remained largely unsolved and almost not treated at all although a complete answer to the physical questions listed above requires one to tackle the BRST cohomology without a priori restrictions on the antifield dependence.

In order to deal efficiently with the antifields in the BRST cohomology, a new qualitative ingredient is necessary. This new ingredient is the understanding that the antifields are algebraically associated with the equations of motion in a well-defined fashion, which is in fact quite standard in cohomology theory. With this novel interpretation of the antifields, new progress could be made and previous open conjectures could be proved.

Thus, while the original point of view on the antifields (sources coupled to the BRST variations of the fields [45, 46, 47, 231, 232]) is useful for the purposes of renormalization theory, the complementary interpretation in terms of equations of

motion is quite crucial for cohomological calculations. Because this interpretation of the antifields, related to the so-called "Koszul-Tate complex", plays a central rôle in our approach, we shall devote two entire sections to explaining it (sections 5 and 6).

The relevant interpretation of the antifields originates from work on the Hamiltonian formulation of the BRST symmetry, developed by the Fradkin school [107, 108, 109, 27, 110, 31], where a similar interpretation can be given for the momenta conjugate to the ghosts [130, 99, 176, 71, 131, 105, 189]. The reference [105] deals in particular with the case of reducible constraints, which is the closest to the Lagrangian case from the algebraic point of view. The extension of the work on the Hamiltonian Koszul-Tate complex to the antifield formalism was carried out in [106, 132]. Locality was analyzed in [133].

What enabled one to identify the Koszul-Tate differential as a key building block was the attempt to generalize the BRST construction to more general settings, in which the algebra of the gauge transformations closes only "on-shell". In that case, it is crucial to introduce the Koszul-Tate differential from the outset; the BRST differential is then given by $s = \delta + \gamma +$ "more", where "more" involves derivations of higher antifield number. Such a generalization was first uncovered in the case of supergravity [154, 191]. The construction was then systematized in [83, 212] and given its present form in [28, 30, 33]. However, even when the algebra closes off-shell, the Koszul-Tate differential is a key ingredient for cohomological purposes.

2.5 Further comments

The antifield formalism is extremely rich and we shall exclusively be concerned here with its cohomological aspects in the context of local gauge theories. So, many of its properties (meaning of antibracket and geometric interpretation of the master equation [228], anti-BRST symmetry with antifields [34], quantum master equation and its regularization [32, 204, 149, 202]) will not be addressed. A review with the emphasis on the algebraic interpretation of the antifields used here is [134]. Other reviews are [205, 206, 117].

It would be impossible - and out of place - to list here all references dealing with one aspect or the other of the antifield formalism; we shall thus quote only some papers from the last years which appear to be representative of the general trends. These are [35, 159, 1, 122, 123] (geometric aspects of the antifield formalism), [199, 116, 37] ($Sp(2)$-formalism), [36] (quantum antibrackets), [51] (higher antibrackets). We are fully aware that this list is incomplete but we hope that the interested reader can work her/his way through the literature from these references.

Finally, a monograph dealing with aspects of anomalies complementary to those discussed here is [52].

2.6 Appendix 2.A: Gauge-fixing and antighosts

The BRST differential as we have introduced it is manifestly gauge-independent since nowhere in the definitions did we ever fix the gauge. That this is the relevant differential for the classical questions described above (classical deformations of the

action, conservation laws) has been established in [19, 23]. Perhaps less obvious is the fact that this is also the relevant differential for the quantum questions. Indeed, the BRST differential is usually introduced in the quantum context only after the gauge has been fixed and one may wonder what is the connection between the above definitions and the more usual ones.

A related aspect is that we have not included the antighosts. There is in fact a good reason for this, because these do not enter the cohomology: they only occur through "trivial" terms because, as one says, they are in the contractible part of the algebra [194].

To explain both issues, we first recall the usual derivation of the BRST symmetry. First, one fixes the gauge through gauge conditions

$$\mathcal{F}^I = 0, \tag{2.26}$$

where the $\mathcal{F}^I$ involve the gauge potential, the matter fields and their derivatives. For instance, one may take $\mathcal{F}^I = \partial^\mu A^I_\mu$ (Lorentz gauge). Next, one writes down the gauge-fixed action

$$S_F = S^{inv} + S^{gf} + S^{ghost} \tag{2.27}$$

where $S^{inv} = \int d^n x \, L$ is the original gauge-invariant action, S^{gf} is the gauge-fixing term

$$S^{gf} = \int d^n x \, b_I(\mathcal{F}^I + \frac{1}{2}b^I) \tag{2.28}$$

and S^{ghost} is the ghost action,

$$S^{ghost} = -\mathrm{i} \int d^n x \left[D_\mu C^J \frac{\delta(\bar{C}_I \mathcal{F}^I)}{\delta A^J_\mu} - e \, C^J T^i_{Jj} \psi^j \frac{\delta(\bar{C}_I \mathcal{F}^I)}{\delta \psi^i} \right]. \tag{2.29}$$

The b^I's are known as the auxiliary fields, while the $\bar{C}_I$ are the antighosts. We take both the ghosts and the antighosts to be real.

The gauge-fixed action is invariant under the BRST symmetry $\sigma A^I_\mu = D_\mu C^I$, $\sigma \psi^i = -eC^I T^i_{Ij} \psi^j$, $\sigma C^I = (1/2)ef_{KJ}{}^I C^J C^K$, $\sigma \bar{C}_I = \mathrm{i} b_I$ and $\sigma b_I = 0$. This follows from the gauge-invariance of the original action as well as from $\sigma^2 = 0$. σ coincides with s on A^I_μ, ψ^i and C^I but we use a different letter to avoid confusion. One can view the auxiliary fields b^I as the ghosts for the (abelian) gauge shift symmetry $\bar{C}_I \to \bar{C}_I + \epsilon_I$ under which the original Lagrangian is of course invariant since it does not depend on the antighosts.

To derive the Ward-Slavnov-Taylor identities associated with the original gauge symmetry and this additional shift symmetry, one introduces sources for the BRST variations of all the variables, including the antighosts and the auxiliary b^I-fields. This yields

$$S^{total} = S^{inv} + S^{gf} + S^{ghost} + S^{sources} \tag{2.30}$$

with

$$S^{sources} = -\int d^n x \, (\sigma A^I_\mu K^\mu_I + \sigma \psi^i K_i + \sigma C^I L_I + \mathrm{i} b_I M^I) \tag{2.31}$$

where K_I^μ, K_i, L_I and M_I are respectively the sources for the BRST variations of A_μ^I, ψ_i, C^I and $\bar{C}_I$. We shall also denote by N^I the sources associated with b_I. The antighosts $\bar{C}_I$, the auxiliary fields b_I and their sources define the "non-minimal sector".

The action S^{total} fulfills the identity

$$(S^{total}, S^{total}) = 0 \tag{2.32}$$

where the "antibracket" $(,)$ is defined by declaring the sources to be conjugate to the corresponding fields, i.e.,

$$(A_\mu^I(x), K_J^\nu(y)) = \delta_J^I \delta_\mu^\nu \delta^n(x-y), \quad (C^I(x), L_J(y)) = \delta_J^I \delta^n(x-y) \quad \text{etc.} \tag{2.33}$$

The generating functional Γ of the one-particle irreducible proper vertices coincides with S^{total} to zeroth order in $\hbar$,

$$\Gamma = S^{total} + \hbar \Gamma^{(1)} + O(\hbar^2). \tag{2.34}$$

The perturbative quantum problem is to prove that the renormalized finite Γ, obtained through the addition of counterterms of higher orders in $\hbar$ to S^{total}, obeys the same identity,

$$(\Gamma, \Gamma) = 0 \tag{2.35}$$

These are the Ward-Slavnov-Taylor identities (written in Zinn-Justin form) associated with the original gauge symmetry and the antighost shift symmetry. The problem involves two aspects: anomalies and stability.

General theorems [172, 163, 164] guarantee that to lowest order in $\hbar$, the breaking Δ_k of the Ward identity,

$$(\Gamma, \Gamma) = \hbar^k \Delta_k + O(\hbar^{k+1}), \tag{2.36}$$

is a local functional. The identity $(\Gamma, (\Gamma, \Gamma)) = 0$, then gives the lowest order consistency condition

$$\mathcal{S}\Delta_k = 0, \tag{2.37}$$

where $\mathcal{S}$ is the so-called (linearized) Slavnov operator, $\mathcal{S} \cdot = (S^{total}, \cdot)$ which fulfills $\mathcal{S}^2 = 0$ because of (2.32). Trivial anomalies of the form $\mathcal{S}\Sigma_k$ can be absorbed through the addition of finite counterterms, so that, in the absence of non trivial anomalies, (2.35) can be fulfilled to that order. Hence, non trivial anomalies are constrained by the cohomology of $\mathcal{S}$ in ghost number 1.

The remaining counterterms S_k of that order must satisfy

$$\mathcal{S}S_k = 0, \tag{2.38}$$

in order to preserve the Ward identity to that order. Solutions of the form $\mathcal{S}(\text{something})$ can be removed through field redefinitions or a change of the gauge conditions. The question of stability in the minimal, physical, sector is the question whether any non trivial solution of this equation can be brought back to the form S^{inv} by redefinitions of the coupling constants and field redefinitions. Thus, it is the cohomology of $\mathcal{S}$ in ghost number 0 which is relevant for the analysis of stability.

As it has been defined, $\mathcal{S}$ acts also on the sources and depends on the gauge-fixing. However, the gauge conditions can be completely absorbed through a redefinition of the antifields

$$A_I^{*\mu} = K_I^\mu + \frac{\delta\Psi}{\delta A_\mu^I}, \tag{2.39}$$

$$\psi_i^* = K_i + \frac{\delta\Psi}{\delta\psi^i}, \tag{2.40}$$

$$\bar{C}^{*I} = M^I + \frac{\delta\Psi}{\delta\bar{C}_I}, \tag{2.41}$$

$$C_I^* = L_I + \frac{\delta\Psi}{\delta C^I}, \tag{2.42}$$

$$b^{*I} = N^I + \frac{\delta\Psi}{\delta b_I}, \tag{2.43}$$

where, in our case, Ψ is given by

$$\Psi = \mathrm{i}\,\bar{C}_I(\mathcal{F}^I + \frac{1}{2}b^I). \tag{2.44}$$

This change of variables does not affect the antibracket ("canonical transformation generated by Ψ"). In terms of the new variables, S^{total} becomes

$$S^{total} = S^{inv} - \int d^n x\left[(sA_\mu^I)A_I^{*\mu} + (s\psi^i)\psi_i^* + (sC^I)C_I^*\right] + S^{NonMin} \tag{2.45}$$

where the non-minimal part is just

$$S^{NonMin} = -\mathrm{i}\int d^n x\; b_I\,\bar{C}^{*I} \tag{2.46}$$

The Slavnov operator becomes then

$$\mathcal{S} = s + \mathrm{i}\int d^n x\left[b_I(x)\frac{\delta}{\delta\bar{C}_I(x)} - \bar{C}^{*I}(x)\frac{\delta}{\delta b^{*I}(x)}\right] \tag{2.47}$$

where here, s acts on the variables A_μ^I, ψ^i, C^I of the "minimal sector" and on their antifields $A_I^{*\mu}$, ψ_i^*, C_I^* and takes exactly the form given in (2.8), while the remaining piece is contractible and does not contribute to the cohomology (see appendix B). Thus, $H(\mathcal{S})$ and $H(s)$ are isomorphic, as are $H(\mathcal{S}|d)$ and $H(s|d)$: any cocycle of $\mathcal{S}$ may be assumed not to depend on the variables $\bar{C}_I$, $\bar{C}^{*I}$, b^{*I} and b_I of the "non-minimal sector" and is then a cocycle of s. Furthermore, for chains depending on the variables of the minimal sector only, s-coboundaries and $\mathcal{S}$-coboundaries coincide. Hence, the cohomological problems indeed reduce to computing $H(s)$ and $H(s|d)$.

Remarks:

(i) Whereas the choice made above to introduce sources also for the antighosts $\bar{C}_I$ and the auxiliary b^I fields is motivated by the desire to have a symmetrical description of fields and sources with respect to the antibracket (2.33) and a BRST transformation

that is canonically generated on all the variables, other authors prefer to introduce sources for the BRST variations of the variables A^I_μ, ψ_i and C^I only. In that approach, the final BRST differential in the physical sector is the same as above, but the non minimal sector is smaller and consists only of $\bar{C}_I$ and b^I.

(ii) In the case of standard Yang-Mills theories in four dimensions, one may wonder whether one has stability of the complete action (2.30) not only in the relevant, physical sector but also in the gauge-fixing sector, i.e., whether linear gauges are stable. Stability of linear gauges can be established by imposing legitimate auxiliary conditions. In the formalism where the antighosts and auxiliary fields have no antifields, these auxiliary conditions are the gauge condition and the ghost equation, fixing the dependence of Γ on b^I and $\bar{C}_I$ respectively. The same result can be recovered in the approach with antifields for the antighosts and the auxiliary fields. The dependence of Γ on the variables of the non minimal sector is now fixed by the same ghost equation as before, while the gauge condition is modified through the additional term $-\mathrm{i}\bar{C}^*_I b^I$. In addition, one imposes: $\delta\Gamma/\delta\bar{C}^*_I = -\mathrm{i}b^I$ and $\delta\Gamma/\delta b^*_I = 0$.

(iii) One considers sometimes a different cohomology, the so-called gauge-fixed BRST cohomology, in which there is no antifield and the equations of motion of the gauge-fixed theory are freely used. This cohomology is particularly relevant to the "quantum Noether method" for gauge-theories [150, 151]. The connection between the BRST cohomology discussed here and the gauge-fixed cohomology is studied in [136, 26], where it is shown that they are isomorphic under appropriate conditions which are explicitly stated.

2.7   Appendix 2.B: Contractible pairs

We show here that the antighosts and the auxiliary fields do not contribute to the cohomology of $\mathcal{S}$. This is because $\bar{C}_I$ and b_I form "contractible pairs",

$$\mathcal{S}\bar{C}_I = \mathrm{i}\, b_I, \ \mathcal{S}b_I = 0, \tag{2.48}$$

and furthermore, the $\mathcal{S}$-transformations of the other variables do not involve $\bar{C}_I$ or b_I. We shall repeatedly meet the concept of "contractible pairs" in this work.

Let N be the operator counting $\bar{C}_I$ and b_I and their derivatives,

$$N = \bar{C}_I\frac{\partial}{\partial\bar{C}_I} + b_I\frac{\partial}{\partial b_I} + \sum_{t>0}\left[\partial_{\mu_1\ldots\mu_t}\bar{C}_I\frac{\partial}{\partial(\partial_{\mu_1\ldots\mu_t}\bar{C}_I)} + \partial_{\mu_1\ldots\mu_t}b_I\frac{\partial}{\partial(\partial_{\mu_1\ldots\mu_t}b_I)}\right]. \tag{2.49}$$

One has $[N,\mathcal{S}] = 0$ and in fact $N = \mathcal{S}\varrho + \varrho\mathcal{S}$ with

$$\varrho = -\mathrm{i}\,\bar{C}_I\frac{\partial}{\partial b_I} - \mathrm{i}\sum_{t>0}\partial_{\mu_1\ldots\mu_t}\bar{C}_I\frac{\partial}{\partial(\partial_{\mu_1\ldots\mu_t}b_I)}. \tag{2.50}$$

ϱ is called a contracting homotopy for N with respect to $\mathcal{S}$. Now, let a be $\mathcal{S}$-closed. One can expand a according to the N-degree, $a = \sum_{k\geq0}a_k$ with $Na_k = ka_k$. One has $\mathcal{S}a_k = 0$ since $[N,\mathcal{S}] = 0$. It is easy to show that the components of a with $k > 0$ are all $\mathcal{S}$-exact. Indeed, for $k > 0$ one has $a_k = (1/k)Na_k = (1/k)(\mathcal{S}\varrho + \varrho\mathcal{S})a_k$

and thus $a_k = \mathcal{S}b_k$ with $b_k = (1/k)\varrho a_k$. Accordingly, $a = a_0 + \mathcal{S}(\sum_{k>0} b_k)$. Hence, the "non-minimal part" of an s-cocycle is always trivial. Furthermore, by analogous arguments, an s-cocycle a is trivial if and only if its "minimal part" a_0 is trivial in the minimal sector.

Similarly, if a is a solution of the consistency condition, $\mathcal{S}a + dm = 0$, one has $\mathcal{S}a_k + dm_k = 0$ and one gets, for $k \neq 0$, $a_k = \mathcal{S}b_k + dn_k$ with $b_k = (1/k)\varrho a_k$ and $n_k = (1/k)\varrho m_k$ (thanks to $[N, d] = 0$, $d\varrho + \varrho d = 0$). Thus, the cohomology of $\mathcal{S}$ can be non-trivial only in the space of function(al)s not involving the antighosts and the auxiliary fields b_I.

The same reasoning applies to the antifields $\bar{C}^{*I}$ and b^{*I}, which form also contractible pairs since

$$\mathcal{S}\bar{C}^{*I} = 0, \ \mathcal{S}b^{*I} = -\mathrm{i}\,\bar{C}^{*I}. \tag{2.51}$$

The argument is actually quite general and constitutes one of our primary tools for computing cohomologies. We shall make a frequent use of it in the report, whenever we have a pair of independent variables (x, y) and a differential Δ such that $\Delta x = y$ and $\Delta y = 0$, and the action of Δ on the remaining variables does not involve x or y.

3 Outline of report

The calculation of the BRST cohomology is based on the decomposition of s into $\delta + \gamma$, on the computation of the individual cohomologies of δ and γ and on the descent equations. Our approach follows [21, 23, 24], but we somewhat streamline and systematize the developments of these papers by starting the calculation *ab initio*. This enables one to make some shortcuts in the derivation of the results.

We start by recalling some useful properties of the exterior derivative d in the algebra of local forms (section 4). In particular, we establish the important "algebraic Poincaré lemma" (theorem 4.2), which is also a tool repeatedly used in the whole report.

We compute then $H(\delta)$ and $H(\delta|d)$ (sections 5 and 6, respectively). One can establish general properties of $H(\delta)$ and $H(\delta|d)$, independently of the model and valid for other gauge theories such as gravity or supergravity. In particular, the relationship between $H(\delta|d)$ and the (generalized) conservation laws is quite general, although the detailed form of the conservation laws does depend of course on the model. We have written section 6 with the desire to make these general features explicit. We specialize then the analysis to gauge theories of the Yang-Mills type. Within this set of theories (and on natural regularity and normality conditions on the Lagrangian), the groups $H(\delta|d)$ are completely calculated, except $H_1^n(\delta|d)$, which is related to the global symmetries of the model and can be fully determined only when the Lagrangian is specified.

In section 7, we establish the general link between $H(s)$, $H(\delta)$ and $H(\gamma)$ (respectively, $H(s|d)$, $H(\delta|d)$ and $H(\gamma|d)$). The connection follows the line of "homological perturbation theory" and applies also to generic field theories with a gauge freedom.

We compute next $H(\gamma)$ (section 8). The calculation is tied to theories of the Yang-Mills type but within this class of models, it does not depend on the form of the Lagrangian since γ involves only the gauge transformations and not the detailed dynamics. We first show that the calculation of $H(\gamma)$ reduces to the problem of computing the Lie algebra cohomology of the gauge group formulated in terms of the curvature components, the matter fields, the antifields, and their covariant derivatives, and the undifferentiated ghosts. This is a well-known mathematical problem whose general solution has been worked out long ago (for reductive Lie algebras). Knowing the connection between $H(\delta)$, $H(\gamma)$ and $H(s)$ makes it easy to compute $H(s)$ from $H(\delta)$ and $H(\gamma)$.

We then turn to the computation of $H(s|d)$. The relevant mathematical tool is that of the descent equations, which we first review (section 9). Equipped with this tool, we calculate $H(s|d)$ in all form and ghost degrees in a smaller algebra involving only the forms C^I, dC^I, A^I, dA^I and their exterior products (section 10). Although this problem is a sub-problem of the general calculation of $H(s|d)$, it turns out to be crucial for investigating solutions of the consistency condition $sa + dm = 0$ that "descend non trivially". The general case (in the algebra of all local forms not necessarily expressible as exterior products of C^I, dC^I, A^I and dA^I) is treated next (section 11), paying due attention to the antifield dependence.

Because the developments in section 11 are rather involved, we discuss their phys-

ical implications in a separate section (section 12). The reader who is not interested in the proofs but only in the results may skip section 11 and go directly to section 12. We explain in particular there why the antifields can be removed from the general solution of the consistency condition at ghost numbers zero (counterterms) and one (anomalies) when the gauge group is semisimple. We also explain why the anomalies can be expressed solely in terms of C^I, dC^I, A^I and dA^I in the semisimple case [61, 62, 101]. These features do not hold, however, when there are abelian factors or for different values of the ghost number.

The case of a system of free abelian gauge fields, relevant to the construction of consistent couplings among massless vector particles, has special features and is therefore treated separately in section 13. We also illustrate the general results in the case of pure Chern-Simons theory in three dimensions, where many solutions disappear because the Yang-Mills curvature vanishes when the equations of motion hold (section 14).

Finally, the last section reviews the literature on the calculation of the local BRST cohomology $H(s|d)$ for other local field theories with a gauge freedom.

4 Locality - Algebraic Poincaré lemma: $H(d)$

Since locality is a fundamental ingredient in our approach, we introduce in this subsection the basic algebraic tools that allow one to deal with locality. The central idea is to consider the "fields" ϕ^i and their partial derivatives of first and higher order as independent coordinates of so-called jet-spaces. The approach is familiar from the variational calculus where the fields and their partial derivatives are indeed treated as independent variables when computing the derivatives $\partial L/\partial \phi^i$ or $\partial L/\partial(\partial_\mu \phi^i)$ of the Lagrangian L. Fields, in the usual sense of "functions of the space-time coordinates", emerge in this approach as sections of the corresponding jet-bundles.

What are the "fields" will depend on the context. In the case of gauge theories of the Yang-Mills type, the "fields" ϕ^i may be the original classical fields A^I_μ and ψ^i, or may be these fields plus the ghosts. In some other instances, they could be the original fields plus the antifields, or the complete set of all variables introduced in the introduction. The considerations of this section are quite general and do not depend on any specific model or field content. So, we shall develop the argument without committing ourselves to a definite set of variabes.

4.1 Local functions and jet-spaces

A local function f is a smooth function of the spacetime coordinates, the field variables ϕ^i and a finite number of their derivatives, $f = f(x, [\phi])$, where the notation $[\phi]$ means dependence on $\phi^i, \phi^i_\mu, \ldots, \phi^i_{(\mu_1\ldots\mu_k)}$ for some finite k (with $\phi^i_\mu \equiv \partial_\mu \phi^i$, $\phi^i_{(\mu_1\ldots\mu_k)} \equiv \partial_{(\mu_1} \ldots \partial_{\mu_k)} \phi^i$). A local function is thus a function on the "jet space of order k" $J^k(E) = M \times V^k$ (for some k), where M is Minkowski (or Euclidean) spacetime and where V^k is the space with coordinates given by $\phi^i, \phi^i_\mu, \ldots, \phi^i_{(\mu_1\ldots\mu_k)}$ – some of which may be Grassmannian. The fields and their various derivatives are considered as completely independent in $J^k(E)$ except that the various derivatives commute, so that only the completely symmetric combinations are independent coordinates.

In particular, the jet space of order zero $J^0(E) \equiv E$ is coordinatized by x^μ and ϕ^i. A field history is a section $s : M \longrightarrow E$, given in coordinates by $x \longmapsto (x, \phi(x))$. A section of E induces a section of $J^k(E)$, with $\phi^i_{(\mu_1\ldots\mu_k)}|_s = \partial^k \phi^i(x)/\partial x^{\mu_1} \ldots \partial x^{\mu_k}$. Evaluation of a local function at a section yields a spacetime function. The independence of the derivatives reflects the fact that the only local function $f(x, [\phi])$ which is zero on all sections is the function $f \equiv 0$.

Because the order in the derivatives of the relevant functions is not always known a priori, it is useful to introduce the infinite jet-bundle $\pi^\infty : J^\infty(E) = M \times V^\infty \to M$, where coordinates on V^∞ are given by $\phi^i, \phi^i_\mu, \phi^i_{(\mu_1\mu_2)}, \ldots$.

In our case where spacetime is Minkowskian (or Euclidean) and the field manifold is homeomorphic to $\mathbb{R}^m$ (with m the number of independent real fields ϕ^i, $i = 1, \ldots, m$), the jet-bundles are trivial and the use of bundle terminology may appear a bit pedantic. However, this approach is crucial for dealing with more complicated situations in which the spacetime manifold and/or the field manifold is topologically non-trivial. Global aspects related to these features will not be discussed

here. They come over and above the local cohomologies analyzed in this report, which must in any case be understood[2]. We refer to [174, 101, 25] for an analysis of BRST cohomology taking into account some of these extra global features.

In the case of field theory, the local functions are usually polynomial in the derivatives. This restriction will turn out to be important in some of the next sections. However, for the present purposes, it is not necessary. The theorems of this section and the next are valid both in the space of polynomial local functions and in the space of arbitrary smooth local functions. For this reason, we shall not restrict the functional space of local functions at this stage.

4.2 Local functionals - Local p-forms

An important class of objects are local functionals. They are given by the integrals over space-time of local n-forms evaluated at a section. An example is of course the classical action.

Local p-forms are by definition exterior forms with coefficients that are local functions,

$$\omega = \frac{1}{p!} \, dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p} \omega_{\mu_1 \ldots \mu_p}(x, [\phi]). \tag{4.1}$$

We shall drop the exterior product symbol $\wedge$ in the sequel since no confusion can arise.

Thus, local functionals evaluated at the section s take the form $\mathcal{F}(f, s) = \int_M \omega|_s$ with ω the n-form $\omega = f d^n x$, where $d^n x = dx^0 \cdots dx^{n-1}$. In the case of the action, f is the Lagrangian density.

It is customary to identify local functionals with the formal expression $\int_M \omega$ prior to evaluation.

4.3 Total and Euler-Lagrange derivatives

The total derivative ∂_μ is the vector field defined on local functions by

$$\partial_\mu = \frac{\partial}{\partial x^\mu} + \phi^i_\mu \frac{\partial}{\partial \phi} + \phi^i_{(\mu\nu)} \frac{\partial}{\partial \phi^i_\nu} + \ldots = \frac{\partial}{\partial x^\mu} + \sum_{k \geq 0} \phi^i_{(\mu\nu_1 \ldots \nu_k)} \frac{\partial}{\partial \phi^i_{(\nu_1 \ldots \nu_k)}}, \tag{4.2}$$

where for convenience, we define the index $(\nu_1 \ldots \nu_k)$ to be absent for $k = 0$. These vector fields commute, $[\partial_\mu, \partial_\nu] = 0$. Note that $\partial_\mu \phi = \phi_\mu$, $\partial_\mu \partial_\nu \phi = \phi_{\mu\nu}$ etc... Furthermore, evaluation at a section and differentiation commute as well:

$$(\partial_\mu f)|_s = \frac{\partial}{\partial x^\mu}(f|_s). \tag{4.3}$$

The Euler-Lagrange derivative $\frac{\delta}{\delta \phi^i}$ is defined on a local function f by

$$\frac{\delta f}{\delta \phi^i} = \frac{\partial f}{\partial \phi^i} - \partial_\mu \frac{\partial f}{\partial \phi^i_\mu} + \ldots = \sum_{k \geq 0} (-)^k \partial_{(\mu_1 \ldots \mu_k)} \frac{\partial f}{\partial \phi^i_{(\mu_1 \ldots \mu_k)}}, \tag{4.4}$$

---

[2]Our considerations are furthermore sufficient for the purposes of perturbative quantum field theory.

where $\partial_{(\mu_1...\mu_k)} = \partial_{\mu_1} \ldots \partial_{\mu_k}$.

4.4 Relation between local functionals and local functions

A familiar property of total derivatives $\partial_\mu j^\mu$ is that they have vanishing Euler-Lagrange derivatives. The converse is also true. In fact, one has

Theorem 4.1 *(i) A local function is a total derivative iff it has vanishing Euler-Lagrange derivatives with respect to all fields,*

$$f = \partial_\mu j^\mu \quad \Leftrightarrow \quad \frac{\delta f}{\delta \phi^i} = 0 \quad \forall \, \phi^i \ . \tag{4.5}$$

(ii) Two local functionals $\mathcal{F}, \mathcal{G}$ agree for all sections s, $\mathcal{F}(f, s) = \mathcal{G}(g, s)$ iff their integrands differ by a total derivative, $f = g + \partial_\mu j^\mu$, for some local functions j^μ, whose boundary integral vanishes, $\oint_{\partial M} j = 0$.

Proof: (i) Let

$$N = \sum_{k \geq 0} \phi^i_{(\mu_1...\mu_k)} \frac{\partial}{\partial \phi^i_{(\mu_1...\mu_k)}} \ .$$

We start from the identity

$$f(x, [\phi]) - f(x, 0) = \int_0^1 d\lambda \frac{d}{d\lambda} f(x, [\lambda\phi]) = \int_0^1 \frac{d\lambda}{\lambda} [Nf](x, [\lambda\phi]). \tag{4.6}$$

Using integrations by parts and $f(x, 0) = \partial_\mu k^\mu(x)$ (which holds as a consequence of the standard Poincaré lemma for $\mathbb{R}^n$), one gets

$$f(x, [\phi]) = \partial_\mu j^\mu + \int_0^1 \frac{d\lambda}{\lambda} [\phi^i \frac{\delta f}{\delta \phi^i}](x, [\lambda\phi]). \tag{4.7}$$

for some local functions j^μ. Thus, $\delta f/\delta \phi^i = 0 \; \forall i$ implies $f = \partial_\mu j^\mu$. Evidently, j^μ is polynomial whenever f is.

Conversely, we have

$$[\frac{\partial}{\partial \phi^i_{(\mu_1...\mu_k)}}, \partial_\nu] = \delta^{\mu_1}_{(\nu} \cdots \delta^{\mu_k}_{\lambda_{k-1})} \frac{\partial}{\partial \phi^i_{(\lambda_1...\lambda_{k-1})}}. \tag{4.8}$$

[For $k = 0$: $[\partial/\partial\phi^i, \partial_\nu] = 0$.] This gives

$$\sum_{k \geq 0} (-)^k \partial_{(\mu_1...\mu_k)} \frac{\partial(\partial_\nu j^\nu)}{\partial \phi^i_{(\mu_1...\mu_k)}} = \sum_{k \geq 0} (-)^k \partial_{(\mu_1...\mu_k \nu)} \frac{\partial j^\nu}{\partial \phi^i_{(\mu_1...\mu_k)}}$$

$$+ \sum_{k \geq 1} (-)^k \partial_{(\nu\lambda_1...\lambda_{k-1})} \frac{\partial j^\nu}{\partial \phi^i_{(\lambda_1...\lambda_{k-1})}} = 0. \tag{4.9}$$

24

(ii) That two local functionals whose integrands differ by a total derivative with vanishing boundary integral agree for all sections s follows from (4.3) and Stokes theorem. Conversely, $\mathcal{F}(f, s) = \mathcal{G}(g, s)$ for all s, implies $I = \int_M d^n x \, (f - g)|_{\phi^i(x) + \varepsilon \eta^i(x)} = 0$ for all ε. Thus, for sections $\eta^i(x)$ which vanish with a sufficient number of their derivatives at ∂M, using integrations by parts and (4.3), one gets $0 = \frac{d}{d\varepsilon}|_{\epsilon=0} I = \int_M d^n x \, \eta^i(x) \frac{\delta(f-g)}{\delta \phi^i}|_{\phi^i(x)}$, which implies $\frac{\delta(f-g)}{\delta \phi^i} = 0$ for all ϕ^i and concludes the proof by using (i). $\square$

Remarks:

(i) The first part of this theorem can be reformulated as the statement that "terms in a classical Lagrangian give no contributions to the classical equations of motion iff they are total derivatives". It is crucial to work in jet space for this statement to be true since the Poincaré lemma for the standard De Rham cohomology in $\mathbb{R}^n$ implies that any function of x can be written as a total derivative. Thus, if we were to consider naïvely the Lagrangian L as a function of the spacetime coordinates ($L = L(x)$), we would get an apparent contradiction, since on the one hand the Euler-Lagrange equations are in general not empty while on the other hand L can be written as a total derivative in x-space. The point is of course that L would not be given in general by the total derivative of a *local* vector density.

(ii) The theorem leads to the following view on local functionals, put forward in explicit terms in [114]: local functionals can be identified with equivalence classes of local functions modulo total derivatives. This is legitimate whenever the surface terms can be neglected or are not under focus. This is the case in the physical situations described in the introduction (e.g. in renormalization theory, the classical fields in the effective action Γ are in fact sources yielding 1-particle irreducible Green functions. They can be assumed to be of compact support in that context). In the remainder of the report, we always use this identification.

4.5 Algebraic Poincaré lemma - $H(d)$

Let Ω be the algebra of local forms. The exterior (also called horizontal) differential in Ω is defined by $d = dx^\mu \partial_\mu$ with ∂_μ given by formula (4.2).

The derivative d satisfies $d^2 = 0$ because the dx^μ anticommute while the ∂_μ commute. The complex (Ω, d) is called the horizontal complex. Its elements, the local forms, are also called the horizontal forms, or just the "forms". The p-forms ω^p satisfying $d\omega^p = 0$ are called closed, or cocycles (of d), the ones given by $\omega^p = d\omega^{p-1}$, which are necessarily closed, are called exact or coboundaries (of d). The cohomology group $H(d, \Omega)$ is defined to be the space of equivalence classes of cocycles modulo coboundaries, $[\omega^p] \in H^p(d, \Omega)$ if $d\omega^p = 0$ with $\omega^p \sim \omega^p + d\omega^{p-1}$.

Integrands of local functionals are local n-forms. A local n-form $\omega^n = d^n x \, f$ is exact, $\omega^n = d\omega^{n-1}$ (with ω^{n-1} a local $(n-1)$-form), if and only if f is a total derivative; one has

$$d^n x f = d\omega^{n-1} \quad \Leftrightarrow \quad f = \partial_\mu j^\mu \quad \Leftrightarrow \quad \frac{\delta f}{\delta \phi^i} = 0 \quad \forall \, \phi^i \qquad (4.10)$$

where the first equivalence holds with $\omega^{n-1} = -\frac{1}{(n-1)!}\epsilon_{\mu\nu_1...\nu_{n-1}}dx^{\nu_1}\ldots dx^{\nu_{n-1}}j^\mu$ and the second one holds by theorem 4.1. Due to $d\omega^n = 0$, theorem 4.1 can thus be reformulated as the statement that local functionals are described by $H^n(d, \Omega)$, and that $H^n(d, \Omega)$ is given by the equivalence classes $[\omega^n]$ having identical Euler-Lagrange derivatives, $\omega^n = fd^nx \sim \omega'^n = f'd^nx$ if $\frac{\delta}{\delta\phi^i}(f - f') = 0$. An important result is the following on the cohomology of d in lower form degrees.

Theorem 4.2 *The cohomology of d in form degree strictly smaller than n is exhausted by the constants in form degree 0,*

$$0 < p < n : \quad d\omega^p = 0 \quad \Leftrightarrow \quad \omega^p = d\omega^{p-1} ;$$
$$p = 0 : \quad d\omega^0 = 0 \quad \Leftrightarrow \quad \omega^0 = constant . \qquad (4.11)$$

Theorem 4.1 (part (i)) and theorem 4.2, which give the cohomology of d in the algebra of local forms, are usually collectively referred to as the "algebraic Poincaré lemma". The proof below applies to the case of a polynomial dependence on derivatives. Proofs covering the smooth case can be found in the literature cited at the end of this section.

Proof: As in (4.6), we have the decomposition $\omega(x, dx, [\phi]) = \omega_0 + \tilde{\omega}$, where $\omega_0 = \omega(x, dx, 0)$ and $\tilde{\omega}(x, dx, [\phi]) = \int_0^1 \frac{d\lambda}{\lambda}[N\omega](x, dx, [\lambda\phi])$.

The condition $d\omega = 0$ then implies separately $dx^\mu\frac{\partial}{\partial x^\mu}\omega_0 = 0$ and $d\tilde{\omega}(x, dx, [\phi]) = 0$ because d is homogeneous of degree zero in λ and commutes with N.

Defining $\rho' = x^\nu\frac{\partial}{\partial dx^\nu}$, we have $\{dx^\mu\frac{\partial}{\partial x^\mu}, \rho'\} = x^\mu\frac{\partial}{\partial x^\mu} + dx^\nu\frac{\partial}{\partial dx^\nu}$. Using a homotopy formula analogous to (4.6) for the variables x^μ, dx^μ, we get $\omega_0(x, dx, 0) = \omega(0, 0, 0) + d\int_0^1 \frac{d\lambda}{\lambda}[\rho'\omega_0](\lambda x, \lambda dx, 0)$, which is the standard Poincaré lemma.

Let $t^\nu = \sum_{k\geq 0} k\delta^\nu_{(\mu_1}\delta^{\lambda_1}_{\mu_2}\ldots\delta^{\lambda_{k-1}}_{\mu_k)}\phi^i_{(\lambda_1...\lambda_{k-1})}\frac{\partial}{\partial\phi^i_{(\mu_1...\mu_k)}}$. Then

$$[t^\nu, \partial_\mu] = \delta^\nu_\mu N. \qquad (4.12)$$

If one defines $D^{+\nu}\tilde{\omega} = \int_0^1 \frac{d\lambda}{\lambda}[t^\nu\tilde{\omega}](x, dx, [\lambda\phi])$, one gets

$$[D^{+\nu}, \partial_\mu]\tilde{\omega} = \delta^\nu_\mu\tilde{\omega}, \qquad (4.13)$$

because ∂_μ is homogeneous of degree 0 in λ. With $\rho = D^{+\nu}\frac{\partial}{\partial dx^\nu}$, one has

$$\{d, \rho\}\tilde{\omega} = [D^{+\nu}\partial_\nu - dx^\mu\frac{\partial}{\partial dx^\mu}]\tilde{\omega} = [\partial_\nu D^{+\nu} + (n - dx^\mu\frac{\partial}{\partial dx^\mu})]\tilde{\omega}. \qquad (4.14)$$

Let $\alpha = n - p$, for $p < n$. Apply the previous relation to a d-closed p-form $\tilde{\omega}^p$ to get

$$d\tilde{\omega}^p = 0 \quad \Rightarrow \quad \tilde{\omega}^p = d\frac{\rho}{\alpha}\tilde{\omega}^p - \frac{1}{\alpha}\partial_\nu D^{+\nu}\tilde{\omega}^p. \qquad (4.15)$$

We want to use this formula recursively. In order to do so, we need some relations for the operators $P_m = \partial_{\nu_1}\ldots\partial_{\nu_m}D^{+\nu_1}\ldots D^{+\nu_m}$ where, by definition $P_0 = 1$. (4.13) implies $[P_1, d]\tilde{\omega} = d\tilde{\omega}$ and $P_1 P_m\tilde{\omega} = [P_{m+1} + mP_m]\tilde{\omega}$. The latter allows one to express

26

P_m in terms of P_1: $P_m\tilde{\omega} = \Pi_{l=0}^{m-1}(P_1 - l)\tilde{\omega}$. If $\tilde{\omega}$ is closed, it follows that $dP_m\tilde{\omega} = 0$. Together with (4.14) (applied to a p-form $P_m\tilde{\omega}^p$), this yields

$$d\tilde{\omega}^p = 0 \quad \Rightarrow \quad (\alpha + m)P_m\tilde{\omega}^p = d\rho P_m\tilde{\omega}^p - P_{m+1}\tilde{\omega}^p. \tag{4.16}$$

Injecting this relation for $m = 1$ into (4.15), we find

$$\tilde{\omega}^p = d[\frac{\rho}{\alpha} - \frac{\rho}{\alpha(\alpha + 1)}P_1]\tilde{\omega}^p + \frac{1}{\alpha(\alpha + 1)}P_2\tilde{\omega}^p.$$

Going on recursively, the procedure will stop because $\tilde{\omega}^p$ is polynomial in derivatives: the total number of derivatives contained $\tilde{\omega}^p$ is bounded by some $\bar{m}$, so that $P_{\bar{m}+1}\tilde{\omega}^p = 0$. Hence, the final result is $d\tilde{\omega}^p = 0 \Rightarrow \tilde{\omega}^p = d\eta^{p-1}$, with

$$\eta^{p-1} = \sum_{m=0}^{\bar{m}} \frac{(-)^m \rho\, P_m\, \tilde{\omega}^p}{\alpha(\alpha + 1)\cdots(\alpha + m)} \,, \quad \alpha = n - p.$$

This proves the theorem, by noting that $\omega(0,0,0)$ can never be d exact. $\square$

Note that the above construction preserves polynomiality: η^{p-1} is polynomial in the fields and their derivatives if ω^p is.

4.6 Cohomology of d in the complex of x-independent local forms

The previous theorem holds in the space of forms that are allowed to have an explicit x-dependence. It is sometimes necessary to restrict the analysis to translation-invariant local forms, which have no explicit x-dependence. In that case, the cohomology is bigger, because the constant forms (polynomials in the dx^μ's with constant coefficients) are closed but not exact in the algebra of forms without explicit x-dependence ($dx^\mu = d(x^\mu)$, but x^μ is not in the algebra). In fact, as an adaptation of the proof of the previous theorem easily shows, this is the only additional cohomology.

Theorem 4.3 (algebraic Poincaré lemma in the x-independent case): In the algebra of local forms without explicit x-dependence, the cohomology of d in form degree strictly less than n is exhausted by the constant forms:

$$p < n: \quad d\omega^p = 0 \quad \Leftrightarrow \quad \omega^p = d\omega^{p-1} + c_{\mu_1\ldots\mu_p}dx^{\mu_1}\ldots dx^{\mu_p} \tag{4.17}$$

where the $c_{\mu_1\ldots\mu_p}$ are constants.

If one imposes Lorentz invariance, the cohomology in form-degree $0 < p < n$ disappears since there is no Lorentz-invariant constant form except for $p = 0$ or $p = n$ (see Sections 11.1 and 11.2 for a discussion of Lorentz invariance).

4.7 Effective field theories

The Lagrangian of effective Yang-Mills theory contains all terms compatible with gauge invariance [119, 224]. Consequently, it is not a local function since it contains an infinite number of derivatives.

However, the above considerations are relevant to the study of effective theories. Indeed, in that case the Lagrangian L is in fact a formal power series in some free parameters (including the gauge coupling constants). The coefficients of the independent powers of the parameters in this expansion are local functions in the above sense and are thus defined in a jet-space of finite order. Equality of two formal power series in the parameters always means equality of all the coefficients. For this reason, formal power series in parameters with coefficients that are local function(al)s – in fact polynomials in derivatives by dimensional analysis – can be investigated by means of the tools introduced for local function(al)s. These are the objects that we shall manipulate in the context of effective field theories.

4.8 A guide to the literature

Useful references on jet-spaces are [178, 186, 6]. The algebraic Poincaré lemma has been proved by many authors and repeatedly rediscovered. Besides the references just quoted, one can list (without pretendence of being exhaustive!) [214, 195, 208, 5, 82, 207, 62, 100, 85, 221]. We have followed the proof given in [94].

The very suggestive terminology "algebraic Poincaré lemma" appears to be due to Stora [192].

5 Equations of motion and Koszul-Tate differential: $H(\delta)$

The purpose of this section is to compute the homology[3] of the Koszul-Tate differential δ in the algebra of local forms Ω_F on the jet-space $J^\infty(F)$ of the original fields A_μ^I and ψ, the ghosts C^I, the antifields, and their derivatives. To that end, it is necessary to make precise a few properties of the equations of motion.

The action of the Koszul-Tate differential δ for gauge theories of the Yang-Mills type has been defined in the introduction on the basic variables A_μ^I, ψ, C^I, A_μ^{*I}, ψ_i^* and C_I^*, through formula (2.8). The differential δ is then extended to the jet space $J^\infty(F)$ by requiring that δ be a derivation (of odd degree) that commutes with ∂_μ. This yields explicitly

$$
\begin{aligned}
\delta \; = \; & \sum_{l \geq 0} \partial_{(\mu_1 \ldots \mu_l)} L_I^\mu \frac{\partial}{\partial A_{I|(\mu_1 \ldots \mu_l)}^{*\mu}} + \sum_{l \geq 0} \partial_{(\mu_1 \ldots \mu_l)} L_i \frac{\partial}{\partial \psi_{i|(\mu_1 \ldots \mu_l)}^*} \\
& + \sum_{m \geq 0} \partial_{(\nu_1 \ldots \nu_m)} [- D_\mu A_I^{*\mu} - e \psi_i^* T_{Ij}^i \psi^j] \frac{\partial}{\partial C_{I|(\nu_1 \ldots \nu_m)}^*} \; ,
\end{aligned}
$$

an expression that makes it obvious that δ is a derivation. By setting $\delta(dx^\mu) = 0$, one extends δ trivially to the algebra Ω_F of local forms.

5.1 Regularity conditions

5.1.1 Stationary surface

The Euler-Lagrange equations of motion and their derivatives define surfaces in the jet-spaces $J^r(E)$, $J^\infty(E)$ of the original fields A_μ^I, ψ^i, the ghosts C^I and their derivatives.

Consider the collection R^∞ of equations $L_I^\mu = 0$, $L_i = 0$, $\partial_\nu L_I^\mu = 0$ $\partial_\nu L_i = 0$, $\partial_{(\nu_1 \nu_2)} L_I^\mu = 0$, $\partial_{(\nu_1 \nu_2)} L_i = 0, \ldots$ defined on $J^\infty(E)$. They define the so-called stationary surface Σ^∞ on $J^\infty(E)$. In a given jet-space $J^r(E)$ of finite order r, the stationary surface Σ^r is the surface defined by the subset $R^r \subset R^\infty$ of the above collection of equations which is relevant in $J^r(E)$.

Note that the equations of motion involve only the original classical fields A_μ^I and ψ^i and their derivatives. They do not constrain the ghosts because one is dealing with the original gauge-covariant equations and not those of the gauge-fixed theory. This fact will turn out to be quite important later on.

5.1.2 Noether identities

Because of gauge invariance, the left hand sides of the equations of motion are not all independent functions on $J^\infty(E)$, but they satisfy some relations, called Noether

---

[3]One speaks of "homology", rather than cohomology, because δ acts like a boundary (rather than coboundary) operator: it decreases the degree (antifield number) of the objects on which it acts.

identities. These read

$$\partial_{(\nu_1 \ldots \nu_k)}[D_\mu L_I^\mu + eL_i T_{Ij}^i \psi^j] = 0, \tag{5.1}$$

for all $k = 0, 1, \ldots$ and $I = 1, \ldots, dim(\mathcal{G})$.

5.1.3 Statement of regularity conditions

The Yang-Mills Lagrangian $L = (-1/4)\delta_{IJ}F_{\mu\nu}^I F^{J\mu\nu}$ fulfills important regularity conditions which we now spell out in detail. For all r, the collection R^r of equations of motion can be split into two groups, the "independent equations" L_a and the "dependent equations" L_Δ.[4] The independent equations are by definition such that they can be taken to be some of the coordinates of a new coordinate system on V^r, while the dependent equations hold as consequences of the independent ones: $L_a = 0$ implies $L_\Delta = 0$. Furthermore, there is one and only one dependent equation for each Noether identity: these identities are the only relations among the equations and they are not redundant. To be precise, when viewed as equations for the L_a and L_Δ, the Noether identities $\partial_{(\nu_1 \ldots \nu_k)}D_\mu L_I^\mu = 0$ are strictly equivalent to

$$L_\Delta - L_a\, k_\Delta^a = 0, \tag{5.2}$$

for some local functions k_Δ^a of the fields and their derivatives (a complete set $\{L_\Delta, L_a\}$ is given explicitly below). Thus, the left hand sides of the Noether identities $\partial_{(\nu_1 \ldots \nu_k)}D_\mu L_I^\mu = 0$, viewed as equations for the L_a and L_Δ, are of the form

$$\partial_{(\nu_1 \ldots \nu_k)}D_\mu L_I^\mu = (L_\Delta - L_a\, k_\Delta^a)\mathcal{M}_{(\nu_1 \ldots \nu_k)I}^\Delta, \tag{5.3}$$

for some invertible matrix $\mathcal{M}_{(\nu_1 \ldots \nu_k)I}^\Delta$ that may depend on the dynamical variables (the range of $(\nu_1 \ldots \nu_k)I$ is equal to the range of Δ).

Similarly, one can split the gauge fields and their derivatives into "independent coordinates" y_A, which are not constrained by the equations of motion in the jet-spaces, and "dependent coordinates" z_a, which can be expressed in terms of the y_A on the stationary surface(s). For fixed y_A, the change of variables from L_a to z_a is smooth and invertible (a complete set $\{y_A, z_a\}$ is given explicitly below).

The same regularity properties hold for the Chern-Simons action, or if one minimally couples scalar or spinor fields to the Yang-Mills potential as in the standard model.

The importance of the regularity conditions is that they will enable us to compute completely the homology of δ by identifying appropriate contractible pairs. We shall thus verify them explicitly.

We successively list the L_a, L_Δ, y_A and z_a for the massless free Dirac field, for the massless free Klein-Gordon field, for pure Yang-Mills theory and for the Chern-Simons theory in three dimensions.

---

[4]By an abuse of terminology, we use "equations of motions" both for the actual equations and for their left hand sides.

Dirac field: We start with the simplest case, that of the free (real) Dirac field, with equations of motion $\mathcal{L} \equiv \gamma^\mu \partial_\mu \Psi = 0$. These equations are clearly independent (no Noether identity) and imply no restriction on the undifferentiated field components. So, the stationary surface in $J^0(E)$ is empty. The equations of motion start "being felt" in $J^1(E)$ since they are of the first order. One may rewrite them as $\partial_0 \Psi = \gamma^0 \gamma^k \partial_k \Psi$, so the derivatives $\partial_k \Psi$ may be regarded as independent, while $\partial_0 \Psi$ is dependent. Similarly, the successive derivatives of $\partial_0 \Psi$ may be expressed in terms of the spatial derivatives of Ψ by differentiating the equations of motion, so one gets the following decompositions:

$$\{L_a\} \equiv \{\mathcal{L}, \partial_\mu \mathcal{L}, \partial_{(\mu_1 \mu_2)} \mathcal{L}, \ldots\}, \quad \{L_\Delta\} \text{ is empty}, \tag{5.4}$$

$$\{y_A\} \equiv \{\Psi, \Psi_s, \Psi_{(s_1 s_2)}, \ldots, \Psi_{(s_1 \ldots s_m)}, \ldots\}, \tag{5.5}$$

$$\{z_a\} \equiv \{\Psi_0, \Psi_{(\rho_1 0)}, \ldots, \Psi_{(\rho_1 \ldots \rho_m 0)}, \ldots\}. \tag{5.6}$$

The subset of equations R^r relevant in $J^r(E)$ is given by the Dirac equations and their derivatives up to order $r - 1$.

Klein-Gordon field: Again, the equation of motion $\mathcal{L} \equiv \partial_\mu \partial^\mu \phi = 0$ and all its differential consequences are independent. Furthermore, they can clearly be used to express any derivative of ϕ involving at least two temporal derivatives in terms of the other derivatives.

$$\{L_a\} \equiv \{\mathcal{L}, \partial_\mu \mathcal{L}, \partial_{(\mu_1 \mu_2)} \mathcal{L}, \ldots\}, \quad \{L_\Delta\} \text{ is empty}, \tag{5.7}$$

$$\{y_A\} \equiv \{\phi, \phi_\rho, \phi_{(s_1 \rho)}, \ldots, \phi_{(s_1 \ldots s_m \rho)}, \ldots\}, \tag{5.8}$$

$$\{z_a\} \equiv \{\phi_{(00)}, \phi_{(\rho_1 00)}, \ldots, \phi_{(\rho_1 \ldots \rho_m 00)}, \ldots\}. \tag{5.9}$$

Pure Yang-Mills field: The equations of motion $L_I^\mu \equiv D_\nu F_I^{\nu\mu} = 0$ are defined in $J^2(E)$, where they are independent. The Noether identities involve the spacetime derivatives of the Euler-Lagrange derivatives of the pure Yang-Mills Lagrangian and so start playing a rôle in $J^3(E)$. They can be used to express ∂_0 of the field equation for A_0^I in terms of the other equations. Thus, in $J^3(E)$, $L_I^\mu = 0$, $\partial_\rho L_I^m = 0$ and $\partial_k L_I^0 = 0$ are independent equations, while $\partial_0 L_I^0 = 0$ are dependent equations following from the others. One can solve the equations $L_I^m = 0$ and $L_I^0 = 0$ for $A_{m|(00)}^I$ and $A_{0|(11)}^I$ in terms of the other second order derivatives of the fields. Similarly, the derivatives of the equations $L_I^m = 0$ can be solved for $A_{m|(\rho_1 \ldots \rho_s 00)}^I$ while the independent derivatives of $L_I^0 = 0$ can be solved for $A_{0|(s_1 \ldots s_n 11)}^I$. A possible split of the equations and the variables fulfilling the above requirements is therefore given by

$$\{L_a\} \equiv \{L_I^\mu, \partial_\rho L_I^m, \ldots, \partial_{(\rho_1 \ldots \rho_s)} L_I^m, \ldots,$$

$$\partial_k L_I^0, \ldots, \partial_{(k_1 \ldots k_2)} L_I^0, \ldots\}, \tag{5.10}$$

$$\{L_\Delta\} \equiv \{\partial_0 L_I^0, \partial_\rho \partial_0 L_I^0, \ldots, \partial_{(\rho_1 \ldots \rho_s)} \partial_0 L_I^0, \ldots\}, \tag{5.11}$$

$$\{y_A\} \equiv \{A_\mu^I, A_{\mu|\rho}^I, A_{m|(s_1 \rho)}^I, \ldots, A_{m|(s_1 \ldots s_k \rho)}^I, \ldots,$$

$$A_{0|(\lambda 0)}^I, \ldots, A_{0|(\lambda_1 \ldots \lambda_k 0)}^I, \ldots,$$

$$A^I_{0|(\bar{l}m)}, \ldots, A^I_{0|(\bar{l}_1\ldots\bar{l}_k m)}, \ldots \} \quad (\bar{l}, \bar{l}_i > 1), \qquad (5.12)$$

$$\{z_a\} \equiv \{A^I_{m|(00)}, A^I_{m|(\rho 00)}, \ldots, A^I_{m|(\rho_1\ldots\rho_s 00)}, \ldots,$$

$$A^I_{0|(11)}, \ldots, A^I_{0|(s11)}, \ldots, A^I_{0|(s_1\ldots s_n 11)}, \ldots \}. \qquad (5.13)$$

Finally, the matrix $\mathcal{M}^{\Delta}_{(\nu_1\ldots\nu_k)I}$ in Eq. (5.3) associated with this split of the equations of motion is easily constructed: it is a triangular matrix with entries 1 on the diagonal and thus it is manifestly invertible. Indeed, one has, for the undifferentiated Noether identities, $D_\mu L^\mu_I = \delta^J_I \partial_0 L^0_J +$ "more", where "more" denotes terms involving only the independent equuations. By differentiating these relations, one gets $\partial_{(\nu_1\ldots\nu_k)} D_\mu L^\mu_I = \delta^J_I \partial_{(\nu_1\ldots\nu_k)} \partial_0 L^0_J +$ "lower" + "more", where "lower" denotes terms involving the previous dependent equations.

Chern-Simons theory in three dimensions: The equations are this time $L^\mu_I \equiv \varepsilon^{\mu\rho\sigma} F_{I\rho\sigma} = 0$. They can be split as above since the Noether identities take the same form. But there are less independent field components since the equations of motion are of the first order and thus start being relevant already in J^1. From the equations $L^i_I = 0$ and their derivatives, one can express the derivatives of the spatial components A^I_i of the vector potential with at least one ∂_0 in terms of the derivatives of A^I_0, which are unconstrained. Similarly, from the equations $L^0_I = 0$ and their spatial derivatives, which are independent, one can express all spatial derivatives of A^I_2 with at least one ∂_1 in terms of the spatial derivatives of A^I_1. Thus, we have L_a and L_Δ as in (5.10) and (5.11), but the y_A and z_a are now given by

$$\{y_A\} \equiv \{A^I_\mu, A^I_{0|\rho}, \ldots, A^I_{0|(\rho_1\ldots\rho_k)}, \ldots,$$

$$A^I_{1|s}, \ldots, A^I_{1|(s_1\ldots s_k)}, \ldots,$$

$$A^I_{2|2}, \ldots, A^I_{2|2\ldots 2}, \ldots \}, \qquad (5.14)$$

$$\{z_a\} \equiv \{A^I_{m|0}, A^I_{m|(0\rho)}, \ldots, A^I_{m|(0\rho_1\ldots\rho_k)}, \ldots,$$

$$A^I_{2|1}, A^I_{2|(1s)}, \ldots, A^I_{2|(1s_1\ldots s_k)}, \ldots \}. \qquad (5.15)$$

We have systematically used $\Psi_{(s\rho)} = \partial_{(s\rho)}\Psi$ etc and $A^I_{0|(s11)} = \partial_{(s11)} A^I_0$ etc (with $s = 1, \ldots, n-1$ and $\rho = 0, \ldots, n-1$). Note that the above splits are not unique. Furthermore, they are not covariant. We will in practice not use any of these splits. The only thing that is needed is the fact that such splits exist.

It is clear that the regularity conditions continue to hold if one minimally couples the Klein-Gordon or Dirac fields to the Yang-Mills potential since the coupling terms involve terms with fewer derivatives. Therefore, the regularity conditions hold in particular for the Lagrangian of the standard model.

For more general local Lagrangians of the Yang-Mills type, the regularity conditions are not automatic. For instance, they are not fulfilled in pure Chern-Simons theory in five dimensions because the equations of motion of that theory have no part linear in the fields and can therefore not be used as new admissible jet coordinates. The results on $H(\delta)$ derived in this section are valid only for Lagrangians fulfilling the regularity conditions.

For non-local Lagrangians of the type appearing in the discussion of effective field theories, the question of whether the regularity conditions are fulfilled does not arise since the equations of motion imply no restriction in the jet-spaces $J^r(E)$ of finite order. In some definite sense to be made precise below, one can say, however, that these theories also fulfill the regularity conditions.

5.1.4   Weakly vanishing forms

An antifield independent local form vanishing when the equations of motion hold is said to be weakly vanishing. This is denoted by $\omega \approx 0$. An immediate consequence of the regularity conditions is

Lemma 5.1 *If an antifield independent local form $\omega \in \Omega_E$ is weakly vanishing, $\omega \approx 0$, it can be written as a linear combination of equations of motion with coefficients which are local forms, and is thus δ-exact in the space Ω_F of local forms,*

$$\omega \approx 0, \quad antifd(\omega) = 0 \quad \Leftrightarrow \quad \omega = \delta\eta, \quad antifd(\eta) = 1.$$

Proof   In the coordinate system (x, dx, L_a, y_A), ω satisfies $\omega(x, dx, 0, y_A) = 0$. Using a homotopy formula like in (4.6), one gets $\omega = L_a \int_0^1 d\lambda [\frac{\partial}{\partial L_a}\omega](x, dx, \lambda L_b, y_A)$. Since the L_a are equations of motion, there are antifield variables ϕ_a^* such that $\delta\phi_a^* = L_a$. This gives $\omega = \delta\eta$ where $\eta = \phi_a^* \int_0^1 d\lambda [\frac{\partial}{\partial L_a}\omega](x, dx, \lambda L_b, y_A)$. Going back to the original coordinate system proves the lemma. $\square$

If both the equations of motion and the form ω are polynomial, η is also polynomial.

5.2   Koszul-Tate resolution

Forms defined on the stationary surface can be viewed as equivalence classes of forms defined on the whole of jet-space modulo forms that vanish when the equations of motion hold. It turns out that the homology of δ is precisely given, in degree zero, by this quotient space. Furthermore, its homology in all other degrees is trivial. This is why one says that the Koszul-Tate differential implements the equations of motion.

More precisely, one has

Theorem 5.1 *(Homology of δ in the algebra Ω_F of local forms involving the original fields, the ghosts and the antifields)*

The homology of δ in antifield number 0 is given by the equivalence classes of local forms $(\in \Omega_E)$ modulo weakly vanishing ones, $H_0(\delta, \Omega_F) = \{[\omega_0]\}$, with $\omega_0 \sim \omega_0'$ if $\omega_0 - \omega_0' \approx 0$.

The homology of δ in strictly positive antifield number is trivial, $H_m(\delta, \Omega_F) = 0$ for $m > 0$.

In mathematical terminology, one says that the Koszul-Tate complex provides a "resolution" of the algebra of local forms defined on the stationary surface.

Proof The idea is to exhibit appropriate contractible pairs using the regularity conditions. First, one can replace the jet-space coordinates A_μ^I, ψ^i and their derivatives by y_A and L_a. As we have seen, this change of variables is smooth and invertible.

In the notation (a, Δ), the antifields A_μ^{*I}, ψ_i^* and their derivatives are $(\phi_a^*, \phi_\Delta^*)$ with $\delta\phi_a^* = L_a$ and $\delta\phi_\Delta^* = L_\Delta$. The second step is to redefine the antifields ϕ_Δ^* using the matrix $\mathcal{M}_{(\nu_1...\nu_k)I}^\Delta$ for the Noether identities (5.1) analogous to the matrix in (5.3) for the pure Yang-Mills case. One defines $\tilde\phi_{(\nu_1...\nu_k)I}^* := (\phi_\Delta^* - \phi_a^* k_\Delta^a)\mathcal{M}_{(\nu_1...\nu_k)I}^\Delta$. This definition makes the action of the Koszul-Tate differential particularly simple since one has $\delta\tilde\phi_{(\nu_1...\nu_k)I}^* = 0$. Indeed $(L_\Delta - L_a k_\Delta^a)\mathcal{M}_{(\nu_1...\nu_k)I}^\Delta$ identically vanishes by the Noether identity. In fact, one has $\tilde\phi_{(\nu_1...\nu_k)I}^* = \delta C_{I|(\nu_1...\nu_k)}^*$.

In terms of the new variables, the Koszul-Tate differential reads

$$\delta = L_a\frac{\partial}{\partial\phi_a^*} + \sum_{k\geq 0}\tilde\phi_{(\nu_1...\nu_k)I}^*\frac{\partial}{\partial C_{I|(\nu_1...\nu_k)}^*},$$

which makes it clear that $L_a, \phi_a^*, \tilde\phi_{(\nu_1...\nu_k)I}^*$ and $C_{I|(\nu_1...\nu_k)}^*$ form contractible pairs dropping from the homology. This leaves only the variables y_A, as well as the ghosts C^I and their derivatives, as generators of the homology of δ. In particular, the antifields disappear from the homology and there is thus no homology in strictly positive antifield number. $\square$

A crucial ingredient of the proof is the fact that the Noether identities are independent and exhaust all the independent Noether identities. This is what guaranteed the change of variables used in the proof of the theorem to be invertible. It allows one to generalize the theorem to theories fulfilling regularity conditions analogous to those of the Yang-Mills case. For theories with "dependent" Noether identities ("reducible case"), one must add further antifields at higher antifield number. With these additional variables, the theorem still holds. The homological rationale for the antifield spectrum is explained in [105, 106].

Remarks: (i) We stress that the equations of motion that appear in the theorem are the gauge-covariant equations of motion derived from the gauge-invariant Lagrangian L (and not any gauge-fixed form of these equations).

(ii) When the Lagrangian is Lorentz-invariant, it is natural to regard the antifields $A_I^{*\mu}$, ψ_i^* and C_I^* as transforming in the representation of the Lorentz group contragredient to the representation of A_μ^I, ψ^i and C^I, respectively. Thus, the $A_I^{*\mu}$ are Lorentz vectors while the C_I^* are Lorentz scalars. Because $\delta A_I^{*\mu}$, $\delta\psi_i^*$ and δC_I^* have the same transformation properties as $A_I^{*\mu}$, ψ_i^* and C_I^*, δ commutes with the action of the Lorentz group. One can consider the homology of δ in the algebra of Lorentz-invariant local forms. Using that the Lorentz group is semi-simple, one checks that this homology is trivial in strictly positive antifield number, and given by the equivalence classes of Lorentz-invariant local forms modulo weakly vanishing ones in antifield number zero (alternatively one may verify this directly be means of the properties of the contracting homotopy used in the proof). Similar considerations apply to other linearly realized global symmetries of the Lagrangian.

(iii) Again, if the equations of motion are polynomial, theorem 5.1 holds in the algebra of local, polynomial forms.

5.3 Effective field theories

The results for the homology of δ in the Yang-Mills case extends to the analysis of effective field theories. The problem is to compute the homology of δ in the space of formal power series in the free parameters, generically denoted by g_α, with coefficients that are local forms. We normalize the fields so that they have canonical dimensions. This means, in particular, that the Lagrangian takes the form

$$L = L_0 + O(g_\alpha) \tag{5.16}$$

where the zeroth order Lagrangian L_0 is the free Lagrangian and is the sum of the standard kinetic term for free massless vector fields and of the free Klein-Gordon or Dirac Lagrangians.

Corresponding to this decomposition of L, there is a decomposition of δ,

$$\delta = \delta_0 + O(g_\alpha), \tag{5.17}$$

where δ_0 is the Koszul-Tate differential of the free theory. Now, δ_0 is acyclic (no homology) in positive antifield number. The point is that this property passes on to the complete δ. Indeed, let a be a form which is δ-closed, $\delta a = 0$, and has positive antifield number. Expand a according to the degree in the parameters, $a = a_i + a_{i+1} + \ldots$. Since there are many parameters, a_j is in fact the sum of independent monomials of degree j in the g_α's. The terms a_i of lowest order in the parameters must be δ_0-closed, $\delta_0 a_i = 0$. But then, they are δ_0-exact, $a_i = \delta_0 b_i$, where b_i is a local form (since a_i has positive antifield number by assumption). This implies that $a - \delta b_i$ starts at some higher order i' ($i' > i$) if it does not vanish. By repeating the reasoning at order i', and then successively at the higher orders, one sees that a is indeed equal to the δ of a (in general infinite, formal) power series in the parameters, where each coefficient is a local form.

Similarly, at antifield number zero, a formal power series is δ-exact if and only if it is a combination of the Euler-Lagrange derivatives of L (such forms may be called "weakly vanishing formal power series"). Thus Theorem 5.1 holds also in the algebra of formal power series in the parameters with coefficients that are local forms, relevant to effective field theories. In that sense, effective theories fulfill the regularity conditions because the leading term L_0 does.

6 Conservation laws and symmetries: $H(\delta|d)$

In this chapter, we relate $H(\delta|d)$ to the characteristic cohomology of the theory. The argument is quite general and not restricted to gauge theories of the Yang-Mills type. It only relies on the fact that the Koszul-Tate complex provides a resolution of the algebra of local p-forms defined on the stationary surface, so that lemma 5.1 and theorem 5.1 hold for the gauge theory under study.

We then compute this cohomology for irreducible gauge theories in antifield number higher than 2 on various assumptions and specialize the results to the Yang-Mills case.

6.1 Cohomological version of Noether's first theorem

In this section, the fields ϕ^i are the original classical fields and $\mathcal{L}_i$ the Euler-Lagrange derivatives of L with respect to ϕ^i. The corresponding jet-spaces are denoted by $J^r(D)$, $J^\infty(D)$. In order to avoid cluttered formulas, we shall assume for simplicity that the fields are all bosonic. The inclusion of fermionic fields leads only to extra sign factors in the formulas below.

An infinitesimal field transformation is characterized by local functions $\delta_Q \phi^i = Q^i(x, [\phi])$, to which one associates the vector field

$$\vec{Q} = \sum_{l \geq 0} \partial_{(\mu_1 \ldots \mu_l)} Q^i \frac{\partial}{\partial \phi^i_{(\mu_1 \ldots \mu_l)}}$$

on the jet-space $J^\infty(D)$. It commutes with the total derivative,

$$[\partial_\mu, \vec{Q}] = 0. \tag{6.1}$$

The Q^i are called the "characteristics" of the field transformation.

A symmetry of the theory is an infinitesimal field transformation leaving the Lagrangian invariant up to a total derivative:

$$\vec{Q}L \equiv \delta_Q L = \partial_\mu k^\mu, \tag{6.2}$$

for some local functions k^μ. One can rewrite this equation using integrations by parts as

$$Q^i \mathcal{L}_i + \partial_\mu j^\mu = 0, \tag{6.3}$$

for some local vector density j^μ. This equation can also be read as

$$\partial_\mu j^\mu \approx 0, \tag{6.4}$$

which means that j^μ is a conserved current. This is just Noether's result that to every symmetry there corresponds a conserved current. Note that this current could be zero in the case where $Q^i = M^{[ij]} \mathcal{L}_j$ (such Q^i are examples of trivial symmetries, see below), which means that the correspondence is not one to one. On the other hand, one can associate to a given symmetry the family of currents $j^\mu + \partial_\nu k^{[\nu\mu]}$, which means

that the correspondence is not onto either. As we now show, one obtains bijectivity by passing to appropriate quotient spaces.

Defining $\omega_0^{n-1} = \frac{1}{(n-1)!} dx^{\mu_1} \ldots dx^{\mu_{n-1}} \epsilon_{\mu_1\ldots\mu_n} j^{\mu_n}$, we can rewrite Eq. (6.4) in terms of the antifields as

$$d\omega_0^{n-1} + \delta\omega_1^n = 0. \tag{6.5}$$

This follows from lemma 5.1 and theorem 5.1, the superscript denoting the form degree and the subscript the antifield number (we do not write the pure ghost number because the ghosts do not enter at this stage; the pure ghost number is always zero in this and the next subsection). Because δ and d anticommute, a whole class of solutions to this equation is provided by

$$\omega_0^{n-1} = d\eta_0^{n-2} + \delta\eta_1^{n-1}, \tag{6.6}$$

with $\omega_1^n = d\eta_1^{n-1}$. This suggests to define equivalent conserved currents $\omega_0^{n-1} \sim \omega_0'^{n-1}$ as conserved currents that differ by terms of the form given in the right-hand side of (6.6). In other words, equivalence classes of conserved currents are just the elements of the cohomology group $H_0^{n-1}(d|\delta)$ (defined through the cocycle condition (6.5) and the coboundary condition (6.6)). Expliciting the coboundary condition in dual notation, one thus identifies conserved currents which differ by identically conserved currents of the form $\partial_\nu k^{[\nu\mu]}$ modulo weakly vanishing currents,

$$j^\mu \sim j^\mu + \partial_\nu k^{[\nu\mu]} + t^\mu, \ t^\mu \approx 0. \tag{6.7}$$

Equations (6.4) and (6.7) define the characteristic cohomology H_{char}^{n-1} in form degree $n-1$, which can thus be identified with $H_0^{n-1}(d|\delta)$.

Let us now turn to symmetries of the theory. In a gauge theory, gauge transformations do not change the physics. It is therefore natural to identify two symmetries that differ by a gauge transformation. A general gauge transformation involves not only standard gauge transformations, but also, "trivial gauge transformations" that vanish on-shell [153, 83, 212] (for a recent discussion, see e.g. [134]).

A trivial, local, gauge symmetry reads

$$\delta_M\phi^i = \sum_{m,k\geq 0} (-)^k \partial_{(\mu_1\ldots\mu_k)}[M^{j(\nu_1\ldots\nu_m)i(\mu_1\ldots\mu_k)}\partial_{(\nu_1\ldots\nu_m)}\mathcal{L}_j], \tag{6.8}$$

where the functions $M^{j(\nu_1\ldots\nu_m)i(\mu_1\ldots\mu_k)}$ are arbitrary local functions antisymmetric for the exchange of the indices

$$M^{j(\nu_1\ldots\nu_m)i(\mu_1\ldots\mu_k)} = -M^{i(\mu_1\ldots\mu_k)j(\nu_1\ldots\nu_m)}. \tag{6.9}$$

It is direct to verify that trivial gauge symmetries leave the Lagrangian invariant up to a total derivative.

If

$$\delta_f\phi^i = \sum_{l=0}^{\bar{l}_\alpha} R_\alpha^{i(\mu_1\ldots\mu_l)}\partial_{(\mu_1\ldots\mu_l)}f^\alpha, \tag{6.10}$$

where the f^α are arbitrary local functions, provides a complete set of nontrivial gauge symmetries in the sense of [134][5], then, the most general gauge symmetry is given by the sum of a transformation (6.10) and a trivial gauge transformation (6.8)

$$\delta_{f,M}\phi^i = \delta_f\phi^i + \delta_M\phi^i. \tag{6.11}$$

We thus define equivalent global symmetries as symmetries of the theory that differ by a gauge transformation of the form (6.11) with definite choices of the local functions f^α and $M^{j(\nu_1...\nu_m)i(\mu_1...\mu_k)}$. The resulting quotient space is called the space of non trivial global symmetries.

The Koszul-Tate differential is defined through

$$\delta\phi_i^* = \mathcal{L}_i, \tag{6.12}$$

$$\delta C_\alpha^* = \sum_{l=0}^{\bar{l}_\alpha} R_\alpha^{+i(\mu_1...\mu_l)}\partial_{(\mu_1...\mu_l)}\phi_i^*, \tag{6.13}$$

where the $R_\alpha^{+i(\mu_1...\mu_l)}$ are defined in terms of the $R_\alpha^{i(\mu_1...\mu_l)}$ through

$$\sum_{l=0}^{\bar{l}_\alpha}(-)^l\partial_{(\mu_1...\mu_l)}[R_\alpha^{+i(\mu_1...\mu_l)}f^\alpha] = \sum_{l=0}^{\bar{l}_\alpha}R_\alpha^{i(\mu_1...\mu_l)}\partial_{(\mu_1...\mu_l)}f^\alpha \tag{6.14}$$

and where the C_α^* are the antifields conjugate to the ghosts. For instance, for pure Yang-Mills theory, one has $R_{\mu J}^I = ef_{KJ}{}^I A_\mu^K = R_{\mu J}^{+I}$, $R_{\mu J}^{I\nu} = \delta_\mu^\nu\delta_J^I = -R_{\mu J}^{+I\nu}$ and one recovers from (6.13) the formula (2.8) for δC_I^*.

To any infinitesimal field transformation Q^i, one can associate a local n-form linear in the antifields ϕ_i^* through the formula $\omega_1^n = d^n x\, a_1 = d^n x\, Q^i\phi_i^*$. Conversely, given an arbitrary local n-form of antifield number one, $\omega_1^n = d^n x\, a_1 = d^n x\sum_{l\geq 0} a^{i(\mu_1...\mu_l)}\phi_{i|(\mu_1...\mu_l)}^*$, one can add to it a d-exact term in order to remove the derivatives of ϕ_i^*. The coefficient of ϕ_i^* in the resulting expression defines an infinitesimal transformation. Explicitly, $Q^i = \frac{\delta a_1}{\delta\phi_i^*} = \sum_{l\geq 0}(-)^l\partial_{(\mu_1...\mu_l)}a^{i(\mu_1...\mu_l)}$. There is thus a bijective correspondence between infinitesimal field transformations and equivalence classes of local n-forms of antifield number one (not involving the ghosts), where one identifies two such local n-forms that differ by a d-exact term.

Now, it is clear that Q^i defines a symmetry of the theory if and only if the corresponding ω_1^n's are δ-cocycles modulo d. In fact, one has

Lemma 6.1 *Equivalence classes of global symmetries are in bijective correspondence with the elements of $H_1^n(\delta|d)$.*

Proof: The proof simply follows by expanding the most general local n-form a_2 of antifield number 2, computing δa_2 and making integrations by parts. It is left to the reader. We only remark that the antisymmetry in (6.9) follows from the fact that the antifields are Grassmann odd.

This cohomological set-up allows to prove Noether's first theorem in the case of (irreducible) gauge theories in a straightforward way, using theorems 4.2 and 5.1.

---

[5]For more information on complete sets of gauge transformations, see appendix A.

Theorem 6.1 *The cohomology groups $H_1^n(\delta|d)$ and $H_0^{n-1}(d|\delta)$ are isomorphic in spacetime dimensions $n > 1$. In classical mechanics (n=1), they are isomorphic up to the constants, $H_1^1(\delta|d) \simeq H_0^0(d|\delta)/\mathbb{R}$.[6] In other words, there is an isomorphism between equivalence classes of global symmetries and equivalence classes of conserved currents (modulo constant currents when $n = 1$).*

Proof: The proof relies on the triviality of the (co)homologies of δ and d in appropriate degrees and follows a standard pattern. We define a mapping from $H_1^n(\delta|d)$ to $H_0^{n-1}(d|\delta)/\delta_0^{n-1}\mathbb{R}$ as follows. Let ω_1^n be a δ-cocycle modulo d in form-degree n and antifield number 1,

$$\delta\omega_1^n + d\omega_0^{n-1} = 0 \qquad\qquad (6.15)$$

for some ω_0^{n-1} of form-degree $(n-1)$ and antifield number 0. Note that ω_0^{n-1} is a d-cocycle modulo δ. Furthermore, given ω_1^n, ω_0^{n-1} is defined up to a d-closed term, i.e., up to a d-exact term $(n > 1)$ or a constant $(n = 1)$ (algebraic Poincaré lemma). If one changes ω_1^n by a term which is δ-exact modulo d, ω_0^{n-1} is changed by a term which is d-closed modulo δ. Formula (6.15) defines therefore a mapping from $H_1^n(\delta|d)$ to $H_0^{n-1}(d|\delta)/\delta_0^{n-1}\mathbb{R}$. This mapping is surjective because (6.15) is the cocycle condition both for $H_1^n(\delta|d)$ and for $H_0^{n-1}(d|\delta)$. It is also injective because $H_1^n(\delta) = 0$. $\square$

6.2 Characteristic cohomology and $H(\delta|d)$

We now consider the cohomology groups $H_0^{n-p}(d|\delta)$ for all values $p = 1, \ldots, n$, and not just for $p = 1$. Using again lemma 5.1 and theorem 5.1, these groups can be described independently of the antifields. In that context, they are known as the "characteristic cohomology groups" H_{char}^{n-p} of the stationary surface Σ^∞ and define the "higher order conservation laws". For instance, for $p = 2$, they are given, in dual notation, by "supercurrents" $k^{[\mu\nu]}$ such that $\partial_\mu k^{[\mu\nu]} \approx 0$, where two such supercurrents are identified if they differ on shell by an identically conserved supercurrent: $k^{[\mu\nu]} \sim k^{[\mu\nu]} + \partial_\lambda l^{[\lambda\mu\nu]} + t^{[\mu\nu]}$, where $t^{[\mu\nu]} \approx 0$.

In the same way, one generalizes non-trivial global symmetries by considering the cohomology groups $H_k^n(\delta|d)$, for $k = 1, 2, \ldots$.. These groups are referred to as "higher order (non-trivial) global symmetries".

The definitions are such that the isomorphism between higher order conserved currents and higher order symmetries still holds:

Theorem 6.2 *One has the following isomorphisms*

$$k < n: \quad H_k^n(\delta|d) \simeq H_{k-1}^{n-1}(\delta|d) \simeq \ldots \simeq H_1^{n-k+1}(\delta|d) \simeq H_0^{n-k}(d|\delta); \qquad (6.16)$$

$$H_n^n(\delta|d) \simeq H_{n-1}^{n-1}(\delta|d) \simeq \ldots \simeq H_1^1(\delta|d) \simeq H_0^0(d|\delta)/\mathbb{R}; \qquad (6.17)$$

$$k > n: \quad H_k^n(\delta|d) \simeq H_{k-1}^{n-1}(\delta|d) \simeq \ldots \simeq H_{k-n+1}^1(\delta|d) \simeq H_{k-n}^0(\delta) = 0. \qquad (6.18)$$

---

[6]We derive and write cohomological results systematically for the case that the cohomology under study is computed over $\mathbb{R}$. They hold analogously over $\mathbb{C}$.

and

$$H_k^p(\delta|d) \simeq H_{k-1}^{p-1}(d|\delta) \quad \text{for } k > 1 \text{ and } 0 < p \le n. \tag{6.19}$$

In particular, the cohomology groups $H_k^n(\delta|d)$ ($1 \le k \le n$) are isomorphic to the characteristic cohomology groups $H_0^{n-k}(d|\delta)$ (modulo the constants for $k = n$).

Proof: One proves equation (6.19) and the last isomorphisms in (6.16), (6.17) as Theorem 6.1. The last equality in (6.18) holds because of the acyclicity of δ in all positive antifield numbers (see Section 5). The proof of the remaining isomorphisms illustrates the general technique of the descent equations of which we shall make ample use in the sequel. Let a_j^i be a δ-cocycle modulo d, $\delta a_j^i + da_{j-1}^{i-1} = 0$, $i > 1$, $j > 1$. Then, $d\delta a_{j-1}^{i-1} = 0$, from which one infers, using the triviality of d in form-degree $i - 1$ ($0 < i - 1 < n$) that $\delta a_{j-1}^{i-1} + da_{j-2}^{i-2} = 0$ for some a_{j-2}^{i-2}. Thus, a_{j-1}^{i-1} is also a δ-cocycle modulo d. If a_j^i is modified by trivial terms ($a_j^i \to a_j^i + \delta b_{j+1}^i + db_j^{i-1}$), then, a_{j-1}^{i-1} is also modified by trivial terms. This follows again from $H^{i-1}(d) = 0$. Thus the "descent" $[a_j^i] \to [a_{j-1}^{i-1}]$ from the class of a_j^i in $H_j^i(\delta|d)$ to the class of a_{j-1}^{i-1} in $H_{j-1}^{i-1}(\delta|d)$ defines a well-defined application from $H_j^i(\delta|d)$ to $H_{j-1}^{i-1}(\delta|d)$. This application is both injective (because $H_j(\delta) = 0$) and surjective (because $H_{j-1}(\delta) = 0$). Hence, the groups $H_j^i(\delta|d)$ and $H_{j-1}^{i-1}(\delta|d)$ are isomorphic. $\square$

Remark: The isomorphism $H_1^{n-k+1}(\delta|d) \simeq H_0^{n-k}(d|\delta)$ ($n > k$) uses $H^{n-k}(d) = 0$, which is true only in the space of forms with an explicit x-dependence. If one does not allow for an explicit x-dependence, $H^{n-k}(d)$ is isomorphic to the space $\wedge^{n-k}\mathbb{R}$ of constant forms. The last equality in (6.16) reads then $H_1^{n-k+1}(\delta|d) \simeq H_0^{n-k}(d|\delta)/\wedge^{n-k}\mathbb{R}$.

The results on the groups $H_k^p(\delta|d)$ are summarized in the table below. The first row contains the characteristic cohomology groups H_{char}^{n-p} while $\mathcal{F}(\Sigma)$ corresponds to the local functionals defined on the stationary surface, i.e., the equivalence classes of n-forms depending on the original fields alone, where two such forms are identified if they differ, on the stationary surface, by a d-exact n-form, $\omega_0^n \sim \omega_0^n + d\eta_0^{n-1} + \delta\eta_1^n$. The characteristic cohomology group H_{char}^0, in particular, contains the functions that are constant when the equations of motion hold. All the cohomology groups $H_i^i(\delta|d)$ along the principal diagonal are isomorphic to $H_{\text{char}}^0/\mathbb{R}$; those along the parallel diagonals are isomorphic among themselves. The unwritten groups $H_k^p(\delta|d)$ with $k > n$ all vanish.

$k \backslash p$	0	1	2	...	...	$n-2$	$n-1$	n
0	$H^0_{\text{char}}/\mathbb{R}$	H^1_{char}	H^2_{char}			H^{n-2}_{char}	H^{n-1}_{char}	$\mathcal{F}(\Sigma)$
1	0	$H^0_{\text{char}}/\mathbb{R}$	H^2_1			H^{n-2}_1	H^{n-1}_1	H^n_1
2	0	0	$H^0_{\text{char}}/\mathbb{R}$			H^{n-2}_2	H^{n-1}_2	H^n_2
$\vdots$								
$\vdots$								
$n-2$	0	0	0			$H^0_{\text{char}}/\mathbb{R}$	H^{n-1}_{n-2}	H^n_{n-2}
$n-1$	0	0	0			0	$H^0_{\text{char}}/\mathbb{R}$	H^n_{n-1}
n	0	0	0			0	0	$H^0_{\text{char}}/\mathbb{R}$

6.3 Ghosts and $H(\delta|d)$

So far, we have not taken the ghosts into account in the calculation of the homology of δ modulo d (the ghosts are denoted by C^α; in Yang-Mills theories one has $C^\alpha \equiv C^I$). These are easy to treat since they are not constrained by the equations of motion. As we have seen, δ acts trivially on them, $\delta C^\alpha = 0$, and they do not occur in the δ-transformation of any antifield. Let N_C be the counting operator for the ghosts and their derivatives, $N_C = \sum_{n \geq 0} C^\alpha_{(\mu_1 \ldots \mu_n)} \partial/\partial C^\alpha_{(\mu_1 \ldots \mu_n)}$. This counting operator is of course the pure ghost number. The vector space of local forms is the direct sum of the vector space of forms with zero pure ghost number (= forms which do not depend on the $C^\alpha_{(\mu_1 \ldots \mu_n)}$) and the vector space of forms that vanish when one puts the ghosts and their derivatives equal to zero, $\Omega^* = \Omega^*_{N_C=0} \oplus \Omega^*_{N_C>0}$. In fact, $\Omega^*_{N_C>0}$ is itself the direct sum of the vector spaces of forms with pure ghost number one, two, etc. Since $[N_C, \delta] = 0$, $\delta(\Omega^*_{N_C=0})$ is included in $\Omega^*_{N_C=0}$ and $\delta(\Omega^*_{N_C>0})$ is included in $\Omega^*_{N_C>0}$.

Theorem 6.3 *The cohomology of δ modulo d in form degree n and positive antifield number vanishes for forms in $\Omega^*_{N_C>0}$, $H^n_k(\delta|d, \Omega^*_{N_C>0}) = 0$ for $k \geq 1$.*

Proof: Let $\omega_k = d^n x\, a_k$, with $k \geq 1$, be a cycle of $H^n_k(\delta|d, \Omega^*_{N_C>0})$, $\delta a_k + \partial_\mu k^\mu = 0$. Because of theorem 4.1, $\frac{\delta}{\delta C^\alpha} \delta a_k = 0$. Since δ does not involve the ghosts, this implies $\delta(\frac{\delta}{\delta C^\alpha} a_k) = 0$. Theorem 5.1 then yields

$$\frac{\delta a_k}{\delta C^\alpha} = \delta b_{\alpha\,k+1}. \tag{6.20}$$

Now, by an argument similar to the one that leads to the homotopy formula (4.7), one finds that a_k satisfies

$$a_k(\xi, [C]) - a_k(\xi, 0) = \partial_\mu k^\mu + \int_0^1 \frac{d\lambda}{\lambda} [C^\alpha \frac{\delta a_k}{\delta C^\alpha}](\xi, [\lambda C^\beta]) \tag{6.21}$$

where ξ stands for x, the antifields, and all the fields but the ghosts. The second term in the left-hand side of (6.21) is zero when a_k belongs to $\Omega^*_{N_C>0}$. Using this

information and (6.20) in (6.21), together with $[\delta, C^\alpha] = 0$, one finally gets that $a_k = \delta b_{k+1} + \partial_\mu k^\mu$ for some b_{k+1}, k^μ. $\square$

Using that the isomorphisms of theorem 6.2 remain valid when the ghosts are included in $\Omega^*_{N_C>0}$ (because they are only based on the algebraic Poincaré lemma and on the vanishing homology of δ in all positive antifield numbers), we have

Corollary 6.1 *The cohomology groups $H^p_k(\delta|d, \Omega^*_{N_C>0})$ and $H^{n-k}_0(d|\delta, \Omega^*_{N_C>0})$ vanish for all $k \geq 1$.*

Note that the constant forms do not appear in $H^{n-k}_0(d|\delta, \Omega^*_{N_C>0})$, even if one considers forms with no explicit x-dependence. This is because the constant forms do not belong to $\Omega^*_{N_C>0}$.

To summarize, any mod-d δ-closed form can be decomposed as a sum of terms of definite pure ghost number, $\omega = \sum_l \omega^l$, where ω^l has pure ghost number l. Each component ω^l is δ-closed modulo d. According to the above discussion, it is then necessarily δ-exact modulo d, unless $l = 0$.

6.4 General results on $H(\delta|d)$

6.4.1 Cauchy order

In order to get additional vanishing theorems on $H^n_k(\delta|d)$, we need more information on the detailed structure of the theory.

An inspection of the split of the field variables in Eqs. (5.4) through (5.15) shows that for the Dirac and Klein-Gordon field, the set of independent variables $\{y_A\}$ is closed under spatial differentiation: $\partial_\mu y_A \subset \{y_B\}$ for $\mu = 1, \ldots n-1$, while there are y_A such that $\partial_0 y_A$ involves z_a. For the standard Yang-Mills and Chern-Simons theories, we have $\partial_\mu y_A \subset \{y_B\}$ for $\mu = 2, \ldots n-1$, while there are y_A such that $\partial_0 y_A$ or $\partial_1 y_A$ involves z_a.

We define the Cauchy order of a theory to be the minimum value of q such that the space of local functions $f(y)$ is stable under ∂_μ for $\mu = q, q+1, \ldots, n-1$ (or, equivalently, $\partial_\mu y_A = f_{\mu A}(y)$ for all A and all $\mu = q, \ldots, n-1$ where $f_{\mu A}(y)$ are local functions which can be expressed solely in terms of the y_A). The minimum is taken over all sets of space-time coordinates and all choices of $\{y_A\}$. The Dirac and Klein-Gordon theories are of Cauchy order one, while Chern-Simons and Yang-Mills theories are of Cauchy order two. The Lagrangian of the standard model defines therefore a theory of Cauchy order two.

The usefulness of the concept of Cauchy order lies in the following theorem.

Theorem 6.4 *For theories of Cauchy order q, the characteristic cohomology is trivial for all form-degrees $p = 1, \ldots, n-q-1$:*

$$H^p_0(d|\delta) = \delta^p_0 \mathbb{R} \quad for \quad p < n-q.$$

Equivalently, among all cohomological groups $H^n_k(\delta|d)$ only those with $k \leq q$ may possibly be nontrivial.

The proof of the theorem is given in the appendix 6.B.

In particular, for Klein-Gordon or Dirac theory, only $H_0^{n-1}(d|\delta) \simeq H_1^n(\delta|d)$ may be nonvanishing (standard conserved currents), while for Yang-Mills or Chern-Simons theory, there can be in addition a nonvanishing $H_0^{n-2}(d|\delta) \simeq H_2^n(\delta|d)$. We shall strengthen this result by showing that this latter group is in fact zero unless there are free abelian factors.

Remark: The results on $H_k^n(\delta|d)$ in theorem 6.4 hold in the space of forms with or without an explicit coordinate dependence. By contrast, the results on the characteristic cohomology hold only in the space of x-dependent forms. If one restricts the forms to have no explicit x-dependence, there is additional cohomology: the constant forms encountered above are nontrivial even if one uses the field equations.

6.4.2   Linearizable theories

Let $N = N_\phi + N_{\phi^*} + N_{C^*}$, where $N_\phi = \sum_{l \geq 0} \phi^i_{(\lambda_1 \dots \lambda_l)} \frac{\partial}{\partial \phi^i_{(\lambda_1 \dots \lambda_l)}}$ (and similarly for the other fields), i.e., N is the counting operator for the fields, the antifields and their derivatives. Decompose the Lagrangian L and the reducibility functions $R_\alpha^{+i(\mu_1 \dots \mu_l)}$ according to the N-degree, $L = \sum_{n \geq 2} L^{(n)}$, $R_\alpha^{+i(\mu_1 \dots \mu_l)} = \sum_{n \geq 0} (R_\alpha^{+i(\mu_1 \dots \mu_l)})^{(n)}$, $\delta = \sum_{n \geq 0} \delta^{(n)}$. So, $L^{(2)}$ is quadratic in the fields and their derivatives, $L^{(3)}$ is cubic etc, while $\delta^{(0)}$ preserves the polynomial degree, $\delta^{(1)}$ increases it by one unit, etc. We say that a gauge theory can be linearized if the cohomology of $\delta^{(0)}$ (i) is trivial for all positive antifield numbers and (ii) is in antifield number 0 given by the equivalence classes of local forms modulo forms vanishing on the surface in the jet space defined by the linearized equations of motion $\partial_{\mu_1 \dots \mu_k} \frac{\delta L^{(2)}}{\delta \phi^j} = 0$. This just means that the field independent $(R_\alpha^{+i(\mu_1 \dots \mu_l)})^{(0)}$ provide an irreducible generating set of Noether identities for the linearized theory.

The Lagrangian of the standard model is clearly linearizable since its quadratic piece is the sum of the Lagrangians for free Klein-Gordon, Dirac and $U(1)$ gauge fields. Pure Chern-Simons theory in three dimensions is linearizable too, and so are effective field theories sketched in Section 5.3.

One may view the condition of linearizability as a regularity condition on the Lagrangian, which is not necessarily fulfilled by all conceivable Lagrangians of the Yang-Mills type, although it is fulfilled in the cases met in practice in the usual physical models. An example of a non-linearizable theory is pure Chern-Simons theory in $(2k+1)$ dimensions with $k > 1$. The lowest order piece of the Lagrangian is of order $(k+1)$ and so $L^{(2)} = 0$ when $k > 1$. The zero Lagrangian has a much bigger gauge symmetry than the Yang-Mills gauge symmetry. The non-linearizability of pure Chern-Simons theory in $(2k+1)$ dimensions $(k > 1)$ explains some of its pathologies. [By changing the "background" from zero to a non-vanishing one, one may try to improve on this, but the issue will not be addressed here].

Theorem 6.5 *For irreducible linear gauge theories, (i)* $H_k^n(\delta|d) = 0$ *for* $k \geq 3$, *(ii) if* $N_{C^*}\omega_2^n = 0$, *then* $\delta\omega_2^n + d\omega_1^{n-1} = 0$ *implies* $\omega_2^n = \delta\eta_3^n + d\eta_2^{n-1}$ *and (iii) if* $\omega_1^n \approx 0$, *then* $\delta\omega_1^n + d\omega_0^{n-1} = 0$ *implies* $\omega_1^n = \delta\eta_2^n + d\eta_1^{n-1}$.

For irreducible linearizable gauge theories, the above results hold in the space of forms with coefficients that are formal power series in the fields, the antifields and their derivatives.

The proof is given in the appendix 6.B.

The theorem settles the case of effective field theories where the natural setting is the space of formal power series. In order to go beyond this and to make sure that the power series stop and are thus in fact local forms in the case of theories with a local Lagrangian, an additional condition is needed.

6.4.3 Control of locality. Normal theories

For theorem 6.5 to be valid in the space of local forms, we need more information on how the derivatives appear in the Lagrangian. Let N_∂ be the counting operator of the derivatives of the fields and antifields, $N_\partial = N_{\partial\phi} + N_{\partial\phi^*} + N_{\partial C^*}$, where $N_{\partial\phi} = \sum_k k\phi_{(\lambda_1...\lambda_k)}^i \frac{\partial}{\partial\phi_{(\lambda_1...\lambda_k)}^i}$ and similarly for the antifields. The equations of motions $\mathcal{L}_i = 0$ are partial differential equations of order r_i and gauge transformations involve a maximum of $\bar{l}_\alpha$ derivatives ($\bar{l}_\alpha = 1$ for theories of the Yang-Mills type). We define

$$A = \sum_{k\geq 0} \left[r_i \, \phi_{i(\lambda_1...\lambda_k)}^* \, \frac{\partial}{\partial\phi_{i(\lambda_1...\lambda_k)}^*} + m_\alpha C_{\alpha(\lambda_1...\lambda_k)}^* \, \frac{\partial}{\partial C_{\alpha(\lambda_1...\lambda_k)}^*} \right]$$

where $m_\alpha = \bar{l}_\alpha + \max_{i,l,(\nu_1...\nu_l)}\{r_i + n_{\partial\phi}(R_\alpha^{+i(\nu_1...\nu_l)})\}$ with $n_{\partial\phi}(R_\alpha^{+i(\nu_1...\nu_l)})$ the largest eigenvalue of $N_{\partial\phi}$ contained in $R_\alpha^{+i(\nu_1...\nu_l)}$. In standard, pure Yang-Mills theory, A reads explicitly

$$A = \sum_{k\geq 0} \left[2A_{I(\lambda_1...\lambda_k)}^{*\mu} \frac{\partial}{\partial A_{I(\lambda_1...\lambda_k)}^{*\mu}} + 3C_{I(\lambda_1...\lambda_k)}^* \frac{\partial}{\partial C_{I(\lambda_1...\lambda_k)}^*} \right], \qquad (6.22)$$

since the equations of motion are of second order. For pure Chern-Simons theory in three dimensions, A is

$$A = \sum_{k\geq 0} \left[A_{I(\lambda_1...\lambda_k)}^{*\mu} \frac{\partial}{\partial A_{I(\lambda_1...\lambda_k)}^{*\mu}} + 2C_{I(\lambda_1...\lambda_k)}^* \frac{\partial}{\partial C_{I(\lambda_1...\lambda_k)}^*} \right], \qquad (6.23)$$

since the equations of motion are now of first order.

The degree $K = N_\partial + A$ is such that $[K, \partial_\mu] = [N_\partial, \partial_\mu] = \partial_\mu$ and

$$[K, \delta] = \sum_{k\geq 0} \left[\partial_{(\mu_1...\mu_k)}[(N_{\partial\phi} - r_i)\mathcal{L}_i] \frac{\partial}{\partial\phi_{i(\mu_1...\mu_k)}^*} \right.$$

$$\left. + \partial_{(\mu_1...\mu_k)}[\sum_{l\geq 0}^{\bar{l}_\alpha}(N_{\partial\phi} + r_i + l - m_\alpha)R_\alpha^{+i(\nu_1...\nu_l)}\phi_{i(\nu_1...\nu_l)}^*] \frac{\partial}{\partial C_{\alpha(\mu_1...\mu_k)}^*} \right]. \qquad (6.24)$$

It follows that $\delta = \sum_t \delta^t$, $[K, \delta^t] = t\delta^t$ with $t \leq 0$. For a linearizable theory, we have now two degrees: the degree of homogeneity in the fields, antifields and their derivatives, for which δ has only nonnegative eigenvalues and the K degree, for which δ has only nonpositive eigenvalues.

A linearizable theory is called a normal theory if the homology of $\delta^{(0),0}$ is trivial in positive antifield number. Let us define furthermore $\delta^{\text{int},t} := \sum_{n\geq1} \delta^{(n),t}$. Examples of normal theories are (i) pure Chern-Simons theory in three dimensions, (ii) pure Yang-Mills theory; (iii) standard model. For instance, in the first case, $\delta^{(0),0}$ reduces to the Koszul-Tate differential of the $U(1)^{dim(G)}$ Chern-Simons theory, while in the second case, it reduces to the Koszul-Tate differential for a set of free Maxwell fields. For these free theories, we have seen that theorem 5.1 holds, and thus, indeed, we have "normality" of the full theory.

Theorem 6.6 *For normal theories, the results of theorem 6.5 extend to the space of forms with coefficients that are polynomials in the differentiated fields, the antifields and their derivatives and power series in the undifferentiated fields. Furthermore, if $\delta^{\text{int},0} = 0$, they extend to the space of polynomials in the fields, the antifields and their derivatives.*

The proof is given in the appendix 6.B.

The condition in the last part of this theorem is fulfilled by the Lagrangian of pure Chern-Simons theory, pure Yang-Mills theory or the standard model, because the interaction terms in the Lagrangian of those theories contain less derivatives than the quadratic terms. Thus theorem 6.6 holds in full in these cases. The condition would not be fulfilled if the theory contained for instance the local function $\exp(\partial_\mu \partial^\mu \phi / k)$.

We thus see that normal, local theories and effective theories have the same properties from the point of view of the cohomology groups $H(\delta|d)$. For this reason, the terminology "normal theories" will cover both cases in the sequel.

Remark. Part (iii) of theorems 6.5, 6.6 means that global symmetries with on-shell vanishing characteristics are necessarily trivial global symmetries in the sense of lemma 6.1. In particular, in the absence of non trivial Noether identities, weakly vanishing global symmetries are necessarily related to antisymmetric combinations of the equations of motions through integrations by parts.

6.4.4 Global reducibility identities and $H_2^n(\delta|d)$

We define a "global reducibility identity" by a collection of local functions f^α such that they give a gauge transformation $\delta_f \phi^i$ as in Eq. (6.10) which is at the same time an on-shell trivial gauge symmetry $\delta_M \phi^i$ as in Eqs. (6.8). Explicitly a global reducibility identity requires thus

$$\sum_{l=0}^{\bar{l}_\alpha} R_\alpha^{i(\mu_1...\mu_l)} \partial_{(\mu_1...\mu_l)} f^\alpha = -\sum_{k,m\geq0} (-)^k \partial_{(\mu_1...\mu_k)} [M^{j(\nu_1...\nu_m)i(\mu_1...\mu_k)} \partial_{(\nu_1...\nu_m)} \mathcal{L}_j] \quad (6.25)$$

for some local functions $M^{j(\nu_1...\nu_m)i(\mu_1...\mu_k)}$ with the antisymmetry property (6.9). Note that this is a stronger condition than just requiring that the transformations $\delta_f \phi^i$ vanish on-shell.

A global reducibility identity is defined to be trivial if all f^α vanish on-shell, $f^\alpha \approx 0$, because $f^\alpha \approx 0$ implies $\delta_f \phi^i = -\delta_M \phi^i$ for some $\delta_M \phi^i$. This is seen as follows: $f^\alpha \approx 0$ means $f^\alpha = \delta g_1^\alpha$ for some g_1^α and implies $\phi_i^* \delta_f \phi^i = \delta(C_\alpha^* \delta g_1^\alpha) + \partial_\mu(\ )^\mu = -\delta[(\delta C_\alpha^*)g_1^\alpha] + \partial_\mu(\ )^\mu$; taking now the Euler-Lagrange derivative with respect to ϕ_i^* yields indeed $\delta_f \phi^i = -\delta_M \phi^i$ because $(\delta C_\alpha^*)g_1^\alpha$ is quadratic in the $\phi_{i(\mu_1...\mu_m)}^*$.

The space of nontrivial global reducibility identities is defined to be the space of equivalence classes of global reducibility identities modulo trivial ones.

Theorem 6.7 *In normal theories, $H_2^n(\delta|d)$ is isomorphic to the space of non trivial global reducibility identities.*

Proof: Every cycle of $H_2^n(\delta|d)$ can be assumed to be of the form $d^n x\, a_2$ with $a_2 = C_\alpha^* f^\alpha + M_2 + \partial_\mu(\ )^\mu$, where $M_2 = \frac{1}{2}\sum_{n,m\geq 0}\phi_{j(\nu_1...\nu_n)}^* \phi_{i(\mu_1...\mu_m)}^* M^{j(\nu_1...\nu_n)i(\mu_1...\mu_m)}$ such that (6.9) holds (indeed, all derivatives can be removed from C_α^* by subtracting a total derivative from a_2; the antisymmetry of the M's follows from the odd grading of the ϕ_i^*). Taking the Euler-Lagrange derivative of the cycle condition with respect to ϕ_i^* gives (6.25). Conversely, multiplying (6.25) by ϕ_i^* and integrating by parts an appropriate number of times yield $\delta a_2 + \partial_\mu(\ )^\mu = 0$. Hence, cycles of $H_2^n(\delta|d)$ correspond to global reducibility identities and vice versa.

We still have to show that an element of $H_2^n(\delta|d)$ is trivial iff the corresponding global reducibility identity is trivial. The term b_3 in the coboundary condition $a_2 = \delta b_3 + \partial_\mu(\ )^\mu$ contains terms with one C^* and one ϕ^* and terms trilinear in ϕ^*'s. Taking the Euler-Lagrange derivative with respect to C_α^* of the coboundary condition implies $f^\alpha \approx 0$, or $f^\alpha = \delta g_1^\alpha$. Conversely, $f^\alpha = \delta g_1^\alpha$ implies that $a_2 - \delta(C_\alpha^* g_1^\alpha)$ is a δ-cycle modulo a total derivative in antifield number 2, which does not depend on C_α^*. Part (ii) of theorems 6.5 or 6.6 then implies that $a_2 - \delta(C_\alpha^* g_1^\alpha)$ is a δ-boundary modulo a total derivative, and thus that $d^n x\, a_2$ is trivial in $H_2^n(\delta|d)$. $\square$

The same result applies to effective field theories since one can then use theorem 6.5.

6.4.5 Results for Yang-Mills gauge models

For irreducible normal gauge theories, we have entirely reduced the computation of the higher order characteristic cohomology groups to properties of the gauge transformations.

We now perform explicitly the calculation of the global reducibility identities in the case of gauge theories of the Yang-Mills type, which are irreducible. We start with free electromagnetism, which has a non vanishing $H_2^n(\delta|d)$. The Koszul-Tate differential is defined on the generators by $\delta A_\mu = 0$, $\delta A^{*\mu} = \partial_\nu F^{\nu\mu}$, $\delta C^* = -\partial_\mu A^{*\mu}$.

Theorem 6.8 *For a free abelian gauge field with Lagrangian $L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ in dimensions $n > 2$, $H_2^n(\delta|d)$ is represented by $d^n x\, C^*$. A corresponding representative of the characteristic cohomology H_{char}^{n-2} is $\star F = \frac{1}{(n-2)!2}dx^{\mu_1}\ldots dx^{\mu_{n-2}}\epsilon_{\mu_1...\mu_n}F^{\mu_{n-1}\mu_n}$.*

Proof: According to theorem 6.7, $H_2^n(\delta|d)$ is determined by the nontrivial global reducibility conditions. A necessary condition for the existence of a global reducibility condition is that a gauge transformation $\delta_f A_\mu = \partial_\mu f$ vanishes weakly, i.e. $df \approx 0$. Hence, f is a cocycle of $H_0^0(d|\delta)$. By the isomorphisms (6.17) we have $H_0^0(d|\delta) \simeq H_n^n(\delta|d) + \mathbb{R}$. $H_n^n(\delta|d)$ vanishes for $n > 2$ according to part (i) of theorem 6.5. Hence, if $n > 2$, we conclude $f \approx constant$, i.e. the nontrivial global reducibility conditions are exhausted by constant f. The nontrivial representatives of $H_2^n(\delta|d)$ can thus be taken proportional to $d^n x\, C^*$ if $n > 2$ which proves the first part of the theorem. The second part of the theorem follows from the chain of equations $\delta C^* + \partial_\mu A^{*\mu} = 0$, $\delta A^{*\mu} - \partial_\nu F^{\nu\mu} = 0$ by the isomorphisms (6.16) (see also proof of these isomorphisms). $\square$

The reason that there is a non-trivial group $H_2^n(\delta|d)$ for free electromagnetism is that there is in that case a global reducibility identity associated with gauge transformations with constant gauge parameter. As the proof of the theorem shows, this property remains true if one includes gauge-invariant self-couplings of the Born-Infeld or Euler-Heisenberg type. The corresponding representatives of H_{char}^{n-2} are obtained through the descent equations. Furthermore the result extends straightforwardly to models with a set of abelian gauge fields A_μ^I, $I = 1, 2, \ldots$: then $H_2^n(\delta|d)$ is represented by $d^n x\, C_I^*$, $I = 1, 2, \ldots$

However, if one turns on self-couplings of the Yang-Mills type, which are not invariant under the abelian gauge symmetries, or if one includes minimal couplings to charged matter fields, the situation changes: there is no non-trivial reducibility identity any more. Indeed, gauge transformations leaving the Yang-Mills field A_μ^I and the matter fields ψ^i invariant on-shell fulfill

$$D_\mu f^I \approx 0, \quad f^I T_{Ij}^i \psi^j \approx 0 \tag{6.26}$$

whose only solution $f^I([A], [\psi])$ is $f^I \approx 0$. By theorem 6.7, $H_2^n(\delta|d)$ vanishes in those cases. To summarize, we get the following result.

Corollary 6.2 *For normal theories of the Yang-Mills type in dimensions $n > 2$, the cohomology groups $H_k^n(\delta|d)$ vanish for $k > 2$. The group $H_2^n(\delta|d)$ also vanishes, unless there are abelian gauge symmetries under which all matter fields are uncharged, in which case $H_2^n(\delta|d)$ is represented by those $d^n x\, C_I^*$ which correspond to these abelian gauge symmetries.*

This theorem covers pure Chern-Simons theory in three dimensions, pure Yang-Mills theory, the standard model as well as effective theories of the Yang-Mills type (this list is not exhaustive).

Finally, the group $H_1^n(\delta|d)$ is related to the standard conserved currents through theorem 6.1. Its dimension depends on the specific form of the Lagrangian, which may or may not have non trivial global symmetries. The complete calculation of $H_1^n(\delta|d)$ is a question that must be investigated on a case by case basis. For free theories, there is an infinite number of conserved currents. At the other extreme, for effective theories, which include all possible terms compatible with gauge symmetry and a definite set of global symmetries (such as Lorentz invariance), the only global symmetries and conservation laws should be the prescribed ones.

6.5 Comments

The characteristic cohomology associated with a system of partial differential equations has been investigated in the mathematical literature for some time [215, 207, 216, 72]. The connection with the Koszul-Tate differential is more recent [23]. This new point of view has even enabled one to strengthen and generalize some results on the characteristic cohomology, such as the result on $H_0^{n-2}(d|\delta)$ (isomorphic to $H_2^n(\delta|d)$). The connection with the reducibility properties of the gauge transformations was also worked out in this more recent work and turns out to be quite important for p-form gauge theories, where higher order homology groups $H_k^n(\delta|d)$ are non-zero [138, 213].

The relation to the characteristic cohomology provides a physical interpretation of the nontrivial homology groups $H(\delta|d)$ in terms of conservation laws. In particular it establishes a useful cohomological formulation of Noether's first theorem and a direct interpretation of the (nontrivial) homology groups $H_1^n(\delta|d)$ in terms of the (nontrivial) global symmetries. Technically, the use of the antifields allows one, among other things, to deal with trivial symmetries in a very efficient way. For instance the rather cumbersome antisymmetry property (6.9) of on-shell trivial symmetries is automatically reproduced through the coboundary condition in $H_1^n(\delta|d)$ thanks to the odd Grassmann parity of the antifields.

6.6 Appendix 6.A: Noether's second theorem

We discuss in this appendix the general relationship between Noether identities, gauge symmetries and "dependent" field equations.

In order to do so, it is convenient to extend the jet-spaces by introducing a new field ϵ. A gauge symmetry on the enlarged jet-space is defined to be an infinitesimal field transformation $\vec{Q}(\epsilon)$ leaving the Lagrangian invariant up to a total derivative,

$$\vec{Q}(\epsilon)L_0 + \partial_\mu j'^\mu(\epsilon) = 0. \tag{6.27}$$

The characteristic $Q^i(\epsilon) = \sum_{l\geq 0} Q^{i(\mu_1...\mu_l)}\epsilon_{(\mu_1...\mu_l)}$ depends linearly and homogeneously on ϵ and its derivatives $\epsilon_{(\mu_1...\mu_l)}$ up to some finite order.

A "Noether operator" is a differential operator $N^{i(\mu_1...\mu_l)}\partial_{(\mu_1...\mu_l)}$ that yields an identity between the equations of motion,

$$\sum_{l\geq 0} N^{i(\mu_1...\mu_l)}\partial_{(\mu_1...\mu_l)}\mathcal{L}_i = 0. \tag{6.28}$$

We consider theories described by a Lagrangian that fulfills regularity conditions as described in section 5.1.3 ("irreducible gauge theories"). Namely, the original equations of motion are equivalent to a set of independent equations $\{L_a\}$ (which can be taken as coordinates in a new coordinate system on the jet-space) and to a set of dependent equations $\{L_\Delta\}$ (which hold as a consequence of the independent ones). Explicitly, $\partial_{(\mu_1...\mu_l)}\mathcal{L}_i = L_\Delta\mathcal{N}^\Delta_{(\mu_1...\mu_l)i} + L_a\mathcal{N}^a_{(\mu_1...\mu_l)i}$, where the matrix $\mathcal{N}^M_{(\mu_1...\mu_l)i}$, with $M = \{a, \Delta\}$ is invertible.

Furthermore, we assume that the dependent equations are generated by a finite set $\{L_\alpha\}$ of equations (living on finite dimensional jet-spaces) through repeated differentiations, $\{L_\Delta\} \equiv \{L_\alpha, \partial_\rho L_\alpha, \partial_{(\rho_1 \rho_2)} L_\alpha, \ldots\}$, and that these successive derivatives are independent among themselves. For instance, the split $\{L_\Delta, L_a\}$ made in section 5.1.3 in the pure Yang-Mills case corresponds to $\{L_\alpha\} \equiv \{\partial_0 L_I^0\}$.

Lemma 6.2 *Associated to the dependent equations of motions, there exists a set of Noether operators $\{\sum_{l \geq 0}^{\bar{l}_\alpha} R_\alpha^{+i(\mu_1 \ldots \mu_l)} \partial_{(\mu_1 \ldots \mu_l)}\}$, which are non trivial, in the sense that they do not vanish weakly, and which are irreducible, in the sense that if $\sum_{m \geq 0} Z^{+\alpha(\nu_1 \ldots \nu_m)} \partial_{(\nu_1 \ldots \nu_m)} \circ [\sum_{l \geq 0}^{\bar{l}_\alpha} R_\alpha^{+i(\mu_1 \ldots \mu_l)} \partial_{(\mu_1 \ldots \mu_l)}] \approx 0$ (as an operator identity) then $\sum_{m \geq 0} Z^{+\alpha(\nu_1 \ldots \nu_m)} \partial_{(\nu_1 \ldots \nu_m)} \approx 0$ (i.e. all Z's vanish weakly).*

Proof: Applying the equivalent of lemma 5.1 to the equations L_α, we get $L_\alpha = L_{\bar{a}} k_\alpha^{\bar{a}}$ where $\{L_{\bar{a}}\}$ is a finite subset of $\{L_a\}$. These are Noether identities whose left hand sides can be written in terms of the original equations of motion,

$$\sum_{l \geq 0}^{\bar{l}_\alpha} R_\alpha^{+i(\mu_1 \ldots \mu_l)} \partial_{(\mu_1 \ldots \mu_l)} \mathcal{L}_i = L_\alpha - L_{\bar{a}} k_\alpha^{\bar{a}}, \tag{6.29}$$

for some $R_\alpha^{+i(\mu_1 \ldots \mu_l)}$. Note that the expression on the right hand side takes the form

$$L_\alpha - L_{\bar{a}} k_\alpha^{\bar{a}} = R_\alpha^{+\beta} L_\beta + R_\alpha^{+\bar{a}} L_{\bar{a}}, \tag{6.30}$$

where $R_\alpha^{+\beta} = \delta_\alpha^\beta$ and $R_\alpha^{+\bar{a}} = -k_\alpha^{\bar{a}}$, i.e., $R_\alpha^{+i(\mu_1 \ldots \mu_l)} \equiv (R_\alpha^{+\beta}, R_\alpha^{+\bar{a}})$. The presence of δ_α^β then implies the first part of the lemma.

Taking derivatives $\partial_{(\rho_1 \ldots \rho_m)}$, $m = 0, 1, \ldots$, of (6.29) we get the identities

$$\partial_{(\rho_1 \ldots \rho_m)} [R_\alpha^{+i(\mu_1 \ldots \mu_l)} \partial_{(\mu_1 \ldots \mu_l)} \mathcal{L}_i] = [L_\Delta - L_a k_\Delta^a] \mathcal{M}_{(\rho_1 \ldots \rho_m)\alpha}^\Delta, \tag{6.31}$$

for some functions k_Δ^a and for an invertible matrix $\mathcal{M}_{(\rho_1 \ldots \rho_m)\alpha}^\Delta$ analogous to the one in (5.3). The Noether identities $L_\Delta - L_a k_\Delta^a = 0$ are equivalent to $R_\Delta^{+\Gamma} L_\Gamma + R_\Delta^{+a} L_a = 0$, where $R_\Delta^{+\Gamma} = \delta_\Delta^\Gamma$ and $R_\Delta^{+a} = -k_\Delta^a$.

Thus, because of δ_Δ^Γ, if $Z^{+\Delta}(R_\Delta^{+\Gamma}, R_\Delta^{+a}) \approx 0$ then $Z^{+\Delta} \approx 0$. The lemma follows from the (6.31), the fact that $Z^{+\Delta}$ is related to $Z^{+\alpha(\rho_1 \ldots \rho_m)}$, $m = 0, 1, \ldots$ through the invertible matrix $\mathcal{M}_{(\rho_1 \ldots \rho_m)\alpha}^\Delta$ and the fact that the (L_Δ, L_a) are related to $\partial_{(\mu_1 \ldots \mu_l)} \mathcal{L}_i$ through the invertible matrix $\mathcal{N}_{(\mu_1 \ldots \mu_l)i}^M$. $\square$

In terms of the equations $L_\Delta = 0$ and $L_a = 0$, the acyclicity of the Koszul-Tate operator $\delta C_\Delta^* = \phi_\Delta^* - \phi_a^* k_\Delta^a$, $\delta \phi_\Delta^* = L_\Delta$, $\delta \phi_a^* = L_a$ follows directly by introducing new generators $\tilde{\phi}_\Delta^* = \phi_\Delta^* - \phi_a^* k_\Delta^a$. Using both matrices $\mathcal{N}_{(\mu_1 \ldots \mu_l)i}^M$ and $\mathcal{M}_{(\rho_1 \ldots \rho_m)\alpha}^\Delta$, one can verify that the Koszul-Tate operator is given in terms of the original equations by (6.13).

49

Lemma 6.3 *The irreducible set of Noether operators associated to the dependent equations of motion is a generating set of non trivial Noether identities in the following sense: every Noether operator $\sum_{m \geq 0} N^{i(\mu_1 \ldots \mu_m)} \partial_{(\mu_1 \ldots \mu_m)}$ can be decomposed into the direct sum of*

$$\sum_{n \geq 0} Z^{+\alpha(\rho_1 \ldots \rho_n)} \partial_{(\rho_1 \ldots \rho_n)} \circ \left[\sum_{l \geq 0}^{\bar{l}} R_\alpha^{+i(\lambda_1 \ldots \lambda_l)} \partial_{(\lambda_1 \ldots \lambda_l)} \right] \tag{6.32}$$

with $Z^{+\alpha(\rho_1 \ldots \rho_n)} \not\approx 0$, $n = 0, 1, \ldots$ and the weakly vanishing piece

$$\sum_{m,n \geq 0} M^{j(\nu_1 \ldots \nu_n) i(\mu_1 \ldots \mu_m)} [\partial_{(\nu_1 \ldots \nu_n)} \mathcal{L}_j] \partial_{(\mu_1 \ldots \mu_m)}, \tag{6.33}$$

where $M^{j(\nu_1 \ldots \nu_n) i(\mu_1 \ldots \mu_m)}$ is antisymmetric in the exchange of $j(\nu_1 \ldots \nu_n)$ and $i(\mu_1 \ldots \mu_m)$.

Proof: Every Noether identity can be written as a δ cycle in antifield number 1, $\delta(\sum_m N^{i(\mu_1 \ldots \mu_m)} \phi^*_{i|(\mu_1 \ldots \mu_m)}) = 0$, which implies because of theorem 5.1 that $\sum_m N^{i(\mu_1 \ldots \mu_m)} \phi^*_{i|(\mu_1 \ldots \mu_m)} = \delta b_2$ with

$$b_2 = \sum_{n \geq 0} C^*_{\alpha|(\rho_1 \ldots \rho_n)} Z^{+\alpha(\rho_1 \ldots \rho_n)} + \frac{1}{2} \sum_{n,m \geq 0} \phi^*_{j(\nu_1 \ldots \nu_n)} \phi^*_{i(\mu_1 \ldots \mu_m)} M^{j(\nu_1 \ldots \nu_n) i(\mu_1 \ldots \mu_m)}, \tag{6.34}$$

or explicitly

$$\sum_{m \geq 0} N^{i(\mu_1 \ldots \mu_m)} \phi^*_{i|(\mu_1 \ldots \mu_m)} = \sum_{n \geq 0} Z^{+\alpha(\rho_1 \ldots \rho_n)} \partial_{(\rho_1 \ldots \rho_n)} \left[\sum_{l \geq 0}^{\bar{l}_\alpha} R_\alpha^{+i(\lambda_1 \ldots \lambda_l)} \phi^*_{i(\lambda_1 \ldots \lambda_l)} \right]$$
$$+ \sum_{m,n \geq 0} M^{j(\nu_1 \ldots \nu_n) i(\mu_1 \ldots \mu_m)} [\partial_{(\nu_1 \ldots \nu_n)} \mathcal{L}_j] \phi^*_{i(\mu_1 \ldots \mu_m)}. \tag{6.35}$$

Identification of the coefficients of the independent $\phi^*_{i|(\mu_1 \ldots \mu_m)}$ gives the result that every Noether operator can be written as the sum of (6.32) and (6.33). In order to prove that the decomposition is direct for non weakly vanishing $Z^{+\alpha(\rho_1 \ldots \rho_n)}$, we have to show that every weakly vanishing Noether identity can be written as in (6.33) (and in particular we have to show this for a Noether identity of the form (6.32), without sum over n and $Z^{+\alpha(\rho_1 \ldots \rho_n)} \approx 0$ for this n). Using the set of indices (a, Δ), a weakly vanishing Noether identity is defined by $N^a L_a + N^\Delta L_\Delta = 0$ with $N^a \approx 0$ and $N^\Delta \approx 0$. The last equation implies as in the proof of lemma 5.1, that $N^\Delta = l^{\Delta a} L_a$ so that the Noether identity becomes $(N^a + L_\Delta l^{\Delta a}) L_a = 0$. In terms of the new generators $\tilde{\phi}^*_\Delta = \phi^*_\Delta - k^a_\Delta \phi^*_a$, $\tilde{\phi}^*_a = \phi^*_a$ the Koszul-Tate differential $\delta = L_a \frac{\partial}{\partial \tilde{\phi}^*_a} + \tilde{\phi}^*_\Delta \frac{\partial}{\partial C^*_\Delta}$ involves only the contractible pairs. The Noether identity $\delta[(N^a + L_\Delta l^{\Delta a}) \tilde{\phi}^*_a] = 0$ then implies $(N^a + L_\Delta l^{\Delta a}) \tilde{\phi}^*_a = \delta[\frac{1}{2} \tilde{\phi}^*_b \tilde{\phi}^*_a \mu^{[ba]}]$. This proves the corollary, because we get $N^a = \mu^{[ba]} L_b - L_\Delta l^{\Delta a}$ and $N^\Delta = l^{\Delta a} L_a$. $\square$

Theorem 6.9 *(Noether's second theorem)* *To every Noether operator* $\sum_{l\geq 0} N^{i(\mu_1...\mu_l)}\partial_{(\mu_1...\mu_l)}$ *there corresponds a gauge symmetry* $\vec{Q}(\epsilon)$ *given by* $Q^i(\epsilon) = \sum_{l\geq 0}(-)^l\partial_{(\mu_1...\mu_l)}[N^{i(\mu_1...\mu_l)}\epsilon]$ *and, vice versa, to every gauge symmetry* $Q^i(\epsilon)$, *there corresponds the Noether operator defined by* $\sum_{l\geq 0} Q^{+i(\nu_1...\nu_l)}\partial_{(\nu_1...\nu_l)} \equiv \sum_{l\geq 0}(-)^l\partial_{(\mu_1...\mu_l)}[Q^{i(\mu_1...\mu_l)} \cdot]$.

Proof: The first part follows by multiplying the Noether identity

$$\sum_{l\geq 0} N^{i(\mu_1...\mu_l)}\partial_{(\mu_1...\mu_l)}\mathcal{L}_i \tag{6.36}$$

by ϵ and then removing the derivatives from the equations of motion by integrations by parts to get $\sum_{l\geq 0}(-)^l\partial_{(\mu_1...\mu_l)}[\epsilon N^{i(\mu_1...\mu_l)}]\mathcal{L}_i + \partial_\mu j^\mu = 0$, which can be transformed to (6.27).

The second part follows by starting from (6.27) and doing the reverse integrations by parts to get $\epsilon(\sum_{l\geq 0}(-)^l\partial_{(\mu_1...\mu_l)}[Q^{i(\mu_1...\mu_l)}\mathcal{L}_i]) + \partial_\mu j''^\mu = 0$. Taking the Euler-Lagrange derivatives with respect to ϵ, which annihilates the total derivative according to theorem 4.1, proves the theorem. $\square$

The gauge transformations associated with a generating (or "complete") set of Noether identities are said to form a generating (or complete) set of gauge symmetries.

Trivial gauge symmetries are defined as those that correspond to weakly vanishing Noether operators:

$$Q_T^i(\epsilon) = \sum_{m,k\geq 0}(-)^k\partial_{(\mu_1...\mu_k)}[\partial_{(\nu_1...\nu_m)}\mathcal{L}_j M^{j(\nu_1...\nu_m)i(\mu_1...\mu_k)}\epsilon]$$

$$= \sum_{m,n\geq 0} M^{+j(\nu_1...\nu_m)i(\lambda_1...\lambda_n)}\partial_{(\nu_1...\nu_m)}\mathcal{L}_j\epsilon_{(\lambda_1...\lambda_n)}, \tag{6.37}$$

where the last equation serves as the definition of the functions $M^{+j(\nu_1...\nu_m)i(\lambda_1...\lambda_n)}$.

Note that trivial gauge symmetries do not only vanish weakly, they are moreover related to antisymmetric combinations of equations of motions through integrations by parts.

Non trivial gauge transformations are defined as gauge transformations corresponding to non weakly vanishing Noether identities. In particular, the gauge transformations corresponding to the generating set constructed above are given by

$$R_\alpha^i(\epsilon) = \sum_{l\geq 0}^{\bar{l}_\alpha}(-)^l\partial_{(\mu_1...\mu_l)}[R_\alpha^{+i(\mu_1...\mu_l)}\epsilon] = \sum_{l\geq 0}^{\bar{l}_\alpha} R_\alpha^{i(\mu_1...\mu_l)}\partial_{(\mu_1...\mu_l)}\epsilon . \tag{6.38}$$

The operator $Z^{+\alpha} \equiv \sum_{m\geq 0} Z^{+\alpha(\rho_1...\rho_m)}\partial_{(\rho_1...\rho_m)} \approx 0$ iff the operator $Z^\alpha \equiv \sum_{m\geq 0}(-)^m\partial_{(\rho_1...\rho_m)}\circ Z^{+\alpha(\rho_1...\rho_m)} \approx 0$. A direct consequence of theorem 6.9 and lemma 6.3 is then

Corollary 6.3 *Every gauge symmetry* $Q^i(\epsilon)$ *can be decomposed into the direct sum* $Q^i(\epsilon) = R_\alpha^i(Z^\alpha(\epsilon)) + Q_T^i(\epsilon)$, *where the operator* Z^α *is not weakly vanishing, while* $Q_T^i(\epsilon)$ *is weakly vanishing. Furthermore,* $Q_T^i(\epsilon)$ *is related to an antisymmetric combination of equations of motion through integrations by parts.*

It is in that sense that a complete set of gauge transformations generates all gauge symmetries.

6.7   Appendix 6.B: Proofs of theorems 6.4, 6.5 and 6.6

Proof of theorem 6.4:   We decompose the spacetime indices into two subsets, $\{\mu\} = \{a, \ell\}$ where $a = 0, \ldots, q-1$ and $\ell = q, \ldots, n-1$. The cocycle condition $da + \delta b = 0$ decomposes into

$$d^1 a^M + \delta b^{M+1} = 0, \quad d^0 a^M + d^1 a^{M-1} + \delta b^M = 0, \quad \ldots \qquad (6.39)$$

where the superscript is the degree in the dx^ℓ and M is the highest degree in the decomposition of a,

$$a = \sum_{m \leq M} a^m, \quad d^1 = dx^\ell \partial_\ell, \quad d^0 = dx^a \partial_a .$$

Note that M cannot exceed $n - q$ because the dx^ℓ anticommute. Without loss of generality we can assume that a depends only on the y_A, dx^μ, x^μ because this can be always achieved by adding a δ-exact piece to a if necessary. In particular, a^M can thus be assumed to be of the form

$$a^M = dx^{\ell_1} \ldots dx^{\ell_M} f_{\ell_1 \ldots \ell_M}(dx^a, x^\mu, y_A).$$

Since we assume that the theory has Cauchy order q, $d^1 a^M$ depends also only on the y_A, dx^μ, x^μ and therefore vanishes on-shell only if it vanishes even off-shell. The first equation in (6.39) implies thus

$$d^1 a^M = 0. \qquad (6.40)$$

To exploit this equation, we need the cohomology of d^1. It is given by a variant of the algebraic Poincaré lemma in section 4.5 and can be derived by adapting the derivation of that lemma as follows. Since d^1 contains only the subset $\{\partial_\ell\}$ of $\{\partial_\mu\}$, the jet coordinates $\partial_{(a_1 \ldots a_k)} \phi^i$ play now the same rôle as the ϕ^i in the derivation of the algebraic Poincaré lemma (it does not matter that the set of all $\partial_{(a_1 \ldots a_k)} \phi^i$ is infinite because a local form contains only finitely many jet coordinates). The dx^a and x^a are inert to d^1 and play the rôle of constants. Forms of degree $n - q$ in the dx^ℓ take the rôle of the volume forms. One concludes that the d^1-cohomology is trivial in all dx^ℓ-degrees $1, \ldots, n-q-1$ and in degree 0 represented by functions $f(dx^a, x^a)$. Eq. (6.40) implies thus:

$$0 < M < n - q: \quad a^M = d^1 \eta^{M-1} ;$$
$$M = 0: \quad a^0 = f(dx^a, x^a). \qquad (6.41)$$

In the case $0 < M < n - q$ we introduce $a' := a - d\eta^{M-1}$ which is equivalent to a. Since a' contains only pieces with dx^ℓ-degrees strictly smaller than M, one can repeat the arguments until the dx^ℓ-degree drops to zero. In the case $M = 0$, one has $a = a^0 = f(dx^a, x^a)$ and the cocycle condition imposes $d^0 f(dx^a, x^a) \approx 0$. This

requires $d^0 f(dx^a, x^a) = 0$ which implies $f(dx^a, x^a) = constant + dg(dx^a, x^a)$ by the ordinary Poincaré lemma in $\mathbb{R}^q$. Hence, up to a constant, a is trivial in $H(d|\delta)$ whenever $M < n - q$. This proves the theorem because one has $M < n - q$ whenever the form-degree of a is smaller than $n - q$ since M cannot exceed the form-degree. $\square$

Proof of theorem 6.5: Let us first establish two additional properties satisfied by Euler-Lagrange derivatives. These are

$$\sum_{k\geq 0}(-)^k \partial_{(\lambda_1...\lambda_k)}\left[\frac{\partial(\partial_\mu f)}{\partial \phi_{(\lambda_1...\lambda_k)}}\, g\right] = -\sum_{k\geq 0}(-)^k \partial_{(\lambda_1...\lambda_k)}\left[\frac{\partial f}{\partial \phi_{(\lambda_1...\lambda_k)}}\, \partial_\mu g\right], \qquad (6.42)$$

for any local functions f, g and

$$\vec{Q}\frac{\delta f}{\delta \phi^j} = (-)^{\epsilon_j \epsilon_Q}\frac{\delta}{\delta \phi^j}\left[Q^i \frac{\delta f}{\delta \phi^i}\right] - (-)^{\epsilon_j \epsilon_Q}\sum_{k\geq 0}(-)^k \partial_{(\lambda_1...\lambda_k)}\left[\frac{\partial Q^i}{\partial \phi^j_{(\lambda_1...\lambda_k)}}\frac{\delta f}{\delta \phi^i}\right] \qquad (6.43)$$

for local functions f and Q^i (ϵ_i and ϵ_Q denote the Grassmann parities of ϕ^i and $\vec{Q}$ respectively). Indeed, using (4.8), the left hand side of (6.42) is

$$\sum_{k\geq 0}(-)^k \partial_{(\lambda_1...\lambda_k)}\left[\partial_\mu(\frac{\partial f}{\partial \phi_{(\lambda_1...\lambda_k)}})\, g\right] + \sum_{k\geq 0}(-)^k \partial_{(\nu_1...\nu_{k-1}\mu)}\left[\frac{\partial f}{\partial \phi_{(\nu_1...\nu_{k-1})}}\, g\right].$$

Integrating by parts the ∂_μ in the first term and using the same cancellation as before in (4.9) gives (6.42). Similarly, because of (6.1), the left hand side of (6.43) is $\sum_{k\geq 0}(-)^k \partial_{(\lambda_1...\lambda_k)}[\vec{Q}(\frac{\partial f}{\partial \phi^j_{(\lambda_1...\lambda_k)}})]$. Commuting $\vec{Q}$ with $\frac{\partial}{\partial \phi^j_{(\lambda_1...\lambda_k)}}$ gives (6.43): the terms with $\vec{Q}$ and $\frac{\partial}{\partial \phi^j_{(\lambda_1...\lambda_k)}}$ in reverse order reproduce the first term on the right hand side of (6.43) due to theorem 4.1, while the commutator terms yield the second term upon repreated use of Eq. (6.42).

Let us now turn to the proof of the theorem. Using $\omega_k = d^n x\, a_k$, the cocycle condition reads $\delta a_k + \partial_\mu j^\mu = 0$. Using theorem 4.1, the Euler-Lagrange derivatives of this condition with respect to a field and or antifield Z gives

$$\frac{\delta}{\delta Z}\sum_{\Phi^*=C^*_\alpha, \phi^*_i}(\delta\Phi^*)\frac{\delta a_k}{\delta \Phi^*} = 0. \qquad (6.44)$$

Using Eq. (6.43) (for $\vec{Q} \equiv \delta$), the previous formula is now exploited for $Z \equiv C^*_\alpha$, $Z \equiv \phi^*_i$ and $Z \equiv \phi^i$. For $Z \equiv C^*_\alpha$ it gives

$$\delta\,\frac{\delta a_k}{\delta C^*_\alpha} = 0.$$

When $k \geq 3$, $\delta a_k/\delta C^*_\alpha$ has positive antifield number. Due to the acyclicity of δ in positive antifield number, the previous equation gives

$$\frac{\delta a_k}{\delta C^*_\alpha} = \delta\sigma^\alpha_{k-1}. \qquad (6.45)$$

53

For the proof of part (ii) of the theorem, we note that this relation holds trivially with $\sigma_{k-1}^{\alpha} = 0$. For $Z \equiv \phi_i^*$, Eq. (6.44) gives

$$\delta \, \frac{\delta a_k}{\delta \phi_i^*} = R_{\alpha}^i \left(\frac{\delta a_k}{\delta C_{\alpha}^*} \right)$$

where we used the same notation as in Eq. (6.38). Using (6.45) in the previous equation, and once again the acyclicity of δ in positive antifield number gives

$$\frac{\delta a_k}{\delta \phi_i^*} = R_{\alpha}^i (\sigma_{k-1}^{\alpha}) + \delta \sigma_k^i. \tag{6.46}$$

For the proof of part (iii) of the theorem, we note that this relation holds with $\sigma_{k-1}^{\alpha} = 0$ because by assumption $\delta a_1 / \delta \phi_i^*$ is weakly vanishing, and so it is δ-exact. Finally Eq. (6.44) gives for $Z \equiv \phi^i$, using (6.43) and (6.42) repeatedly,

$$
\begin{aligned}
\delta \, \frac{\delta a_k}{\delta \phi^i} &= -\sum_{k \geq 0} (-)^k \partial_{(\lambda_1 \ldots \lambda_k)} \left[\frac{\partial \mathcal{L}_j}{\partial \phi_{(\lambda_1 \ldots \lambda_k)}^i} \frac{\delta a_k}{\delta \phi_j^*} \right. \\
&\quad \left. + \phi_j^* \sum_{l \geq 0} \frac{\partial R_{\alpha}^{j(\mu_1 \ldots \mu_l)}}{\partial \phi_{(\lambda_1 \ldots \lambda_k)}^i} \partial_{(\mu_1 \ldots \mu_l)} \frac{\delta a_k}{\delta C_{\alpha}^*} \right].
\end{aligned}
$$

One now inserts (6.45), (6.46) in the previous equation and uses

$$\sum_{k \geq 0} \sum_{l \geq 0} (-)^k \partial_{(\lambda_1 \ldots \lambda_k)} \left[\frac{\partial (R_{\alpha}^{j(\mu_1 \ldots \mu_l)} \mathcal{L}_j)}{\partial \phi_{(\lambda_1 \ldots \lambda_k)}^i} \partial_{(\mu_1 \ldots \mu_l)} \sigma_{k-1}^{\alpha} \right] = 0,$$

which follows from repeated application of (6.42) and the fact that $R_{\alpha}^{+i(\mu_1 \ldots \mu_l)}$ defines a Noether identity. This gives an expression $\delta(\ldots) = 0$. Using then acyclicity of δ in positive antifield number, one gets $(\ldots) = \delta \sigma_{ik+1}$:

$$
\begin{aligned}
\frac{\delta a_k}{\delta \phi^i} &= -\sum_{k \geq 0} (-)^k \partial_{(\lambda_1 \ldots \lambda_k)} \left[\frac{\partial \mathcal{L}_j}{\partial \phi_{(\lambda_1 \ldots \lambda_k)}^i} \sigma_k^j \right. \\
&\quad \left. - \sum_{l \geq 0} \frac{\partial (R_{\alpha}^{+j(\mu_1 \ldots \mu_l)} \phi_{j(\mu_1 \ldots \mu_l)}^*)}{\partial \phi_{(\lambda_1 \ldots \lambda_k)}^i} \sigma_{k-1}^{\alpha} \right] + \delta \sigma_{ik+1} \, . \tag{6.47}
\end{aligned}
$$

On the other hand, we have

$$N a_k = \phi^i \frac{\delta a_k}{\delta \phi^i} + \phi_i^* \frac{\delta a_k}{\delta \phi_i^*} + C_{\alpha}^* \frac{\delta a_k}{\delta C_{\alpha}^*} + \partial_{\mu} j^{\mu}. \tag{6.48}$$

Using (6.45)-(6.47), integrations by parts and (6.43), we get

$$
\begin{aligned}
N a_k &= \delta(\phi^i \sigma_{ik+1} - \phi_i^* \sigma_k^i + C_{\alpha}^* \sigma_{k-1}^{\alpha}) + \partial_{\mu} j'^{\mu} \\
&\quad + \left[2\mathcal{L}_j - \frac{\delta(N_{\phi} L)}{\delta \phi^j} \right] \sigma_k^j + \sum_{l \geq 0} (N_{\phi} R_{\alpha}^{+i(\mu_1 \ldots \mu_l)}) \phi_{i(\mu_1 \ldots \mu_l)}^* \sigma_{k-1}^{\alpha}. \tag{6.49}
\end{aligned}
$$

If the theory is linear, the two terms in the last line vanish. We can then use this result in the homotopy formula $a_k = \int_0^1 \frac{d\lambda}{\lambda}[Na_k](x, \lambda\phi, \lambda\phi^*, \lambda C^*)$ and the fact that $\delta = \delta^{(0)}$ and ∂_μ are homogeneous of degree 0 in λ to conclude that $a_k = \delta(\ ) + \partial_\mu(\ )^\mu$. This ends the proof in the case of irreducible linear gauge theories.

If a theory is linearizable, we decompose a_k, with $k \geq 1$ into pieces of definite homogeneity n in all the fields, antifields and their derivatives, $a_k = \sum_{n \geq l} a_k^{(n)}$ where $l \geq 2$ due to the assumptions of the theorem. We then use the acyclicity of $\delta^{(0)}$ to show that if $c_k = \delta e_{k+1}$ with the expansion of c starting at homogeneity $l \geq 1$, then the expansion of e can be taken to start also at homogeneity l. Indeed, we have $\delta^{(0)} e_{k+1}^{(1)} = 0$, $\delta^{(0)} e_{k+1}^{(2)} + \delta^{(1)} e_{k+1}^{(1)} = 0, \ldots$, $\delta^{(0)} e_{k+1}^{(l-1)} + \ldots + \delta^{(l-2)} e_{k+1}^{(1)} = 0$. The first equation implies $e_{k+1}^{(1)} = \delta^{(0)} f_{k+2}^{(1)}$, so that the redefinition $e_{k+1} - \delta f_{k+2}^{(1)}$, which does not modify c_k allows to absorb $e_{k+1}^{(1)}$. This process can be continued until $e_{k+1}^{(l-1)}$ has been absorbed.

Hence, we can choose $\sigma_{ik+1}, \sigma_k^i, \sigma_{k-1}^\alpha$ to start at homogeneity $l - 1$. This implies that the two last terms in (6.49) are of homogeneity $\geq l + 1$. Due to $a_k = \frac{1}{l} Na_k + \sum_{n > l} a'^{(n)}$, Eq. (6.49) yields $a_k = \delta(\ ) + \partial_\mu(\ )^\mu + a_k''$, where a_k'' starts at homogeneity $l + 1$ (unless it vanishes). Going on recursively proves the theorem. $\square$

Proof of theorem 6.6: In the space of forms which are polynomials in the derivatives of the fields, the antifields and their derivatives with coefficients that are power series in the fields, the K degree is bounded. It is of course also bounded in the space of forms that are polynomials in the undifferentiated fields as well. We can use the acyclicity of $\delta^{(0),0}$ to prove the acyclicity of δ^0 in the respective spaces. Indeed, suppose that c is of strictly positive antifield number, its polynomial expansion starts with l and its K bound is M. From $\delta^0 c_M = 0$ it follows that $\delta^{(0),0} c_{l,M} = 0$ and then that $c_{l,M} = \delta^{(0),0} e_{l,M}$. This means that $c - \delta^0 e_{l,M}$ starts at homogeneity $l + 1$. Going on in this way allows to absorb all of c_M. Note that if c is a polynomial in the undifferentiated fields and $\delta^{\text{int},0} = 0$, the procedure stops after a finite number of steps because the terms modifying the terms in c_M of homogeneity higher than l in the absorption of $c_{l,M}$ have a K degree which is strictly smaller than M. One can then go on recursively to remove $c_{M-1}, \ldots$. Because δ^0 is acyclic, we can assume that in the equation $c_k = \delta e_{k+1}$, the K degrees of e_{k+1} is bounded by the same M bounding the K degree of c_K. Indeed, if the K degree of e_{k+1} were bounded by $N > M$, we have $\delta^0 e_{k+1,N} = 0$. Acyclicity of δ^0 then implies $e_{k+1,N} = \delta^0 f_{k+1,N}$ and, the redefinition $e - \delta f_{k+1,N}$, which does not affect c_k allows to absorb $e_{k+1,N}$.

If the K degree of a_k is bounded by M, the K degree of $\frac{\delta a_k}{\delta\phi^i}, \frac{\delta a_k}{\delta\phi_i^*}, \frac{\delta a_k}{\delta C_\alpha^*}$ is bounded respectively by M, $M - r_i$, $M - m_\alpha$, because of $[K, \partial_\mu] = \partial_\mu$. It follows from the definitions of r_i and m_α that the K degree of $\sigma_{ik+1}, \sigma_k^i, \sigma_{k-1}^\alpha$ is also bounded respectively by $M, M - r_i, M - m_\alpha$ and that the K degree of the second line of equation (6.49), modifying the terms of higher homogeneity in the fields in the absorption of the term of order l in the proof of theorem 6.5, is also bounded by M. This proves the theorem, by noticing as before that in the space of polynomials in all the variables, with $\delta^{\text{int},0} = 0$, the procedure stops after a finite number of steps. $\square$

7 Homological perturbation theory

7.1 The longitudinal differential γ

In the introduction, we have defined the γ-differential for Yang-Mills gauge models in terms of generators. Contrary to the Koszul-Tate differential, γ does not depend on the Lagrangian but only on the gauge symmetries. Thus, it takes the same form for all gauge theories of the Yang-Mills type. One has explicitly

$$
\begin{aligned}
\gamma &= \sum_m [\partial_{\mu_1\ldots\mu_m}(D_\mu C^I)\frac{\partial}{\partial(\partial_{\mu_1\ldots\mu_m}A_\mu^I)} + \partial_{\mu_1\ldots\mu_m}(-e\,C^I T_{Ij}^i\psi^j)\frac{\partial}{\partial(\partial_{\mu_1\ldots\mu_m}\psi^i)}] \\
&+ \sum_m \partial_{\mu_1\ldots\mu_m}(\tfrac{1}{2}\,e\,f_{KJ}{}^I\,C^J C^K)\frac{\partial}{\partial(\partial_{\mu_1\ldots\mu_m}C^I)} \\
&+ \sum_m [\partial_{\mu_1\ldots\mu_m}(e\,f_{JI}{}^K\,C^J A_K^{*\mu})\frac{\partial}{\partial(\partial_{\mu_1\ldots\mu_m}A_I^{*\mu})} + \partial_{\mu_1\ldots\mu_m}(e\,C^I\psi_j^* T_{Ii}^j)\frac{\partial}{\partial(\partial_{\mu_1\ldots\mu_m}\psi_i^*)}] \\
&+ \sum_m \partial_{\mu_1\ldots\mu_m}(e\,f_{JI}{}^K\,C^J C_K^*)\frac{\partial}{\partial(\partial_{\mu_1\ldots\mu_m}C_I^*)}.
\end{aligned}
\tag{7.1}
$$

Clearly, $\gamma^2 = 0$. The differential γ increases the pure ghost number by one unit, $[N_C, \gamma] = \gamma$.

One may consider the restriction γ_R of γ to the algebra generated by the original fields and the ghosts, without the antifields,

$$
\begin{aligned}
\gamma_R &= \sum_m [\partial_{\mu_1\ldots\mu_m}(D_\mu C^I)\frac{\partial}{\partial(\partial_{\mu_1\ldots\mu_m}A_\mu^I)} + \partial_{\mu_1\ldots\mu_m}(-e\,C^I T_{Ij}^i\psi^j)\frac{\partial}{\partial(\partial_{\mu_1\ldots\mu_m}\psi^i)}] \\
&+ \sum_m \partial_{\mu_1\ldots\mu_m}(\tfrac{1}{2}\,e\,f_{KJ}{}^I\,C^J C^K)\frac{\partial}{\partial(\partial_{\mu_1\ldots\mu_m}C^I)}.
\end{aligned}
\tag{7.2}
$$

One has also $\gamma_R^2 = 0$, i.e., the antifields are not necessary for nilpotency of γ. It is sometimes this differential which is called the BRST differential.

However, the fact that this restricted differential – or even (7.1) – is nilpotent is an accident of gauge theories of the Yang-Mills type. For more general gauge theories with so-called "open algebras", γ_R (known as the "longitudinal exterior differential along the gauge orbits") is nilpotent only on-shell, $\gamma_R^2 \approx 0$. Accordingly, it is a differential only on the stationary surface. Alternatively, when the antifields are included, γ fulfills $\gamma^2 = -(\delta s_1 + s_1\delta)$ and is a differential only in the homology of δ. Thus, one can define, in general, only the cohomological groups $H(\gamma, H(\delta))$. [For Yang-Mills theories, however, $H(\gamma)$ makes sense even in the full algebra since γ is strictly nilpotent on all fields and antifields. The cohomology $H(\gamma)$ turns out to be important and will be computed below.]

In the general case the BRST differential s is not simply given by $s = \delta + \gamma$, but contains higher order terms

$$
s = \delta + \gamma + s_1 + \text{``higher order terms''},
\tag{7.3}
$$

where the higher order terms have higher antifield number, and s_1 and possibly higher order terms are necessary for s to be nilpotent, $s^2 = 0$. This can even happen for a "closed algebra". Indeed, in the case of non constant structure functions, γ^2 does not necessarily vanish on the antifields and a non vanishing s_1 may be needed.

The construction of s from δ and γ follows a recursive pattern known as "homological perturbation theory". We shall not explain it here since this machinery is not needed in the Yang-Mills context where s is simply given by $s = \delta + \gamma$. However, even though the ideas of homological perturbation theory are not necessary for constructing s in the Yang-Mills case, they are crucial in elucidating some aspects of the BRST cohomology and in relating it to the cohomologies $H(\delta)$ and $H(\gamma, H(\delta))$. In particular, they show the importance of the antifield number as auxiliary degree useful to split the BRST differential. They also put into light the importance of the Koszul-Tate differential in the BRST construction[7]. It is this step that has enabled one, for instance, to solve long-standing conjectures regarding the BRST cohomology.

7.2   Decomposition of BRST cohomology

The BRST cohomology groups are entirely determined by cohomology groups involving the first two terms δ and γ in the decomposition $s = \delta + \gamma + s_1 + \dots$ This result is quite general, so we shall state and demonstrate it without sticking to theories of the Yang-Mills type. In fact, it is based solely on the acyclicity of δ in positive antifield number which is crucial for the whole BRST construction,

$$H_k(\delta) = 0 \quad \text{for} \quad k > 0. \tag{7.4}$$

Theorem 7.1 *In the space of local forms, one has the following isomorphisms:*

$$H^g(s) \quad \simeq \quad H_0^g(\gamma, H(\delta)), \tag{7.5}$$

$$H^{g,p}(s|d) \quad \simeq \quad \begin{cases} H_0^{g,p}(\gamma, H(\delta|d)) & \text{if} \quad g \geq 0, \\ H_{-g}^p(\delta|d) & \text{if} \quad g < 0, \end{cases} \tag{7.6}$$

where the superscripts g and p indicate the (total) ghost number and the form-degree respectively and the subscript indicates the antifield number.

Explanation and proof. Both isomorphisms are based upon the expansion in the antifield number and state that solutions a to $sa = 0$ or $sa + dm = 0$ can be fully characterized in the cohomological sense (i.e., up to respective trivial solutions) through properties of the lowest term in their expansion. Let g denote the ghost number of a. When g is nonnegative, a may contain a piece that does not involve

---

[7]In fact, the explicit decomposition $s = \delta + \gamma$ appeared in print relatively recently, even though it is of course rather direct.

an antifield at all; in contrast, when g is negative, the lowest possible term in the expansion of a has antifield number $-g$,

$$a = a_{\underline{k}} + a_{\underline{k}+1} + a_{\underline{k}+2} + \dots \ , \quad antifd(a_k) = k, \ \underline{k} \geq \begin{cases} 0 \text{ if } g \geq 0, \\ -g \text{ if } g < 0, \end{cases} \quad (7.7)$$

because there are no fields of negative ghost number.

Now, (7.5) expresses on the one hand that every nontrivial solution a to $sa = 0$ has an antifield independent piece a_0 which fulfills

$$\gamma a_0 + \delta a_1 = 0, \quad a_0 \neq \gamma b_0 + \delta b_1 \quad (7.8)$$

and is thus a nontrivial element of $H_0^g(\gamma, H(\delta))$ since $\gamma a_0 + \delta a_1 = 0$ and $a_0 = \gamma b_0 + \delta b_1$ are the cocycle and coboundary condition in that cohomology respectively (a cocycle of $H_0^g(\gamma, H(\delta))$ is γ-closed up to a δ-exact form since δ-exact forms vanish in $H(\delta)$). Note that (7.8) means $\gamma a_0 \approx 0$ and $a_0 \not\approx \gamma b_0$. In particular, $H(s)$ thus vanishes at all negative ghost numbers because then a has no antifield independent piece a_0. Furthermore (7.5) expresses that each solution a_0 to (7.8) can be completed to a nontrivial s-cocycle $a = a_0 + a_1 + \dots$ and that this correspondence between a and a_0 is unique up to terms which are trivial in $H^g(s)$ and $H_0^g(\gamma, H(\delta))$ respectively.

To prove these statements, we show first that $sa = 0$ implies $a = sb$ whenever $\underline{k} > 0$, for some b whose expansion starts at antifield number $\underline{k} + 1$,

$$sa = 0, \quad \underline{k} > 0 \quad \Rightarrow \quad a = s(b_{\underline{k}+1} + \dots). \quad (7.9)$$

This is seen as follows. $sa = 0$ contains the equation $\delta a_{\underline{k}} = 0$ which implies $a_{\underline{k}} = \delta b_{\underline{k}+1}$ when $\underline{k} > 0$, thanks to (7.4). Consider now $a' := a - sb_{\underline{k}+1}$. If a' vanishes we get $a = sb_{\underline{k}+1}$ and thus that a is trivial. If a' does not vanish, its expansion in the antifield number reads $a' = a'_{\underline{k}'} + \dots$ where $\underline{k}' > \underline{k}$ because of $a'_{\underline{k}} = a_{\underline{k}} - \delta b_{\underline{k}+1} = 0$. Furthermore we have $sa' = sa - s^2 b_{\underline{k}+1} = 0$. Applying the same arguments to a' as before to a, we conclude $a'_{\underline{k}'} = \delta b'_{\underline{k}'+1}$ for some $b'_{\underline{k}'+1}$. We now consider $a'' = a' - sb'_{\underline{k}'+1} = a - s(b_{\underline{k}+1} + b'_{\underline{k}'+1})$. If a'' vanishes we get $a = s(b_{\underline{k}+1} + b'_{\underline{k}'+1})$ and stop. If a'' does not vanish we continue until we finally get $a = sb$ for some $b = b_{\underline{k}+1} + b'_{\underline{k}'+1} + b''_{\underline{k}''+1} + \dots$ (possibly after infinitely many steps).

(7.9) shows that every s-cocycle with $\underline{k} > 0$ is trivial. When $\underline{k} = 0$, a_0 satisfies automatically $\delta a_0 = 0$ since it contains no antifield. The first nontrivial equation in the expansion of $sa = 0$ is then $\gamma a_0 + \delta a_1 = 0$, while $a = sb$ contains $a_0 = \gamma b_0 + \delta b_1$. Hence, every nontrivial s-cocycle contains indeed a solution to (7.8).

To show that every solution to (7.8) can be completed to a nontrivial s-cocycle, we consider the cocycle condition in $H_0^g(\gamma, H(\delta))$, $\gamma a_0 + \delta a_1 = 0$, and define $X := s(a_0 + a_1)$. When X vanishes we have $sa = 0$ with $a = a_0 + a_1$ and thus that a is an s-cocycle. When X does not vanish, its expansion starts at some antifield number ≥ 1 due to $X_0 = \gamma a_0 + \delta a_1 = 0$. Furthermore we have $sX = s^2(a_0 + a_1) = 0$. Applying (7.9) to X yields thus $X = -sY$ for some $Y = a_k + \dots$ where $k \geq 2$. Hence we get $X = s(a_0 + a_1) = -s(a_k + \dots)$ and thus $sa = 0$ where $a = a_0 + a_1 + a_k + \dots$ with $k \geq 2$. So, each solution of $\gamma a_0 + \delta a_1 = 0$ can indeed be completed to an s-cocycle

58

a. Furthermore a is trivial if $a_0 = \gamma b_0 + \delta b_1$ and nontrivial otherwise. Indeed, $a_0 = \gamma b_0 + \delta b_1$ implies that $Z := a - s(b_0 + b_1)$ fulfills $sZ = 0$ and $Z_0 = a_0 - (\gamma b_0 + \delta b_1) = 0$, and thus, by arguments used before, either $Z = 0$ or $Z = -s(b_k + \ldots)$, $k \geq 2$ which both give $a = sb$. $a_0 \neq \gamma b_0 + \delta b_1$ guarantees $a \neq sb$ because $a = sb$ would imply $a_0 = \gamma b_0 + \delta b_1$. We have thus seen that (non)trivial elements of $H(s)$ correspond to (non)trivial elements of $H_0^g(\gamma, H(\delta))$ and vice versa which establishes (7.5).

(7.6) expresses that every nontrivial solution a to $sa + dm = 0$ has a piece a_0 if $g \geq 0$, or a_{-g} if $g < 0$, fulfilling

$$g \geq 0: \quad \gamma a_0 + \delta a_1 + dm_0 = 0, \quad a_0 \neq \gamma b_0 + \delta b_1 + dn_0, \quad (7.10)$$

$$g < 0: \quad \delta a_{-g} + dm_{-g-1} = 0, \quad a_{-g} \neq \delta b_{-g+1} + dn_{-g}. \quad (7.11)$$

(7.10) states that a_0 is a nontrivial cocycle of $H_0^{g,*}(\gamma, H(\delta|d))$ because $\gamma a_0 + \delta a_1 + dm_0 = 0$ and $a_0 = \gamma b_0 + \delta b_1 + dn_0$ are the cocycle and coboundary condition in that cohomology respectively. (7.11) states that a_{-g} is a nontrivial cocycle of $H_{-g}^p(\delta|d)$. Furthermore (7.6) expresses that every solution to (7.10) or (7.11) can be completed to a nontrivial solution $a = a_0 + a_1 + \ldots$ or $a = a_{-g} + a_{-g+1} + \ldots$ of $sa + db = 0$. Note that a_0 contains no antifield, while a_{-g} contains no ghost due to $gh(a) = g$. Hence, (7.10) means $\gamma a_0 + dm_0 \approx 0$ and $a_0 \not\approx \gamma b_0 + dn_0$, while (7.11) means that a_{-g} is related to a nontrivial element of the characteristic cohomology as explained in detail in Section 6.

These statements can be proved along lines whose logic is very similar to the derivation of (7.5) given above. Therefore we shall only sketch the proof, leaving the details to the reader. The derivation is based on corollary 6.1 which itself is a direct consequence of (7.4) as the proof of that theorem shows. The rôle of Eq. (7.9) is now taken by the following result:

$$sa + dm = 0, \; \underline{k} > \begin{cases} 0 \text{ if } g \geq 0 \\ -g \text{ if } g < 0 \end{cases} \Rightarrow a = s(b_{\underline{k}+1} + \ldots) + d(n_{\underline{k}} + \ldots). \quad (7.12)$$

This is proved as follows. $sa + dm = 0$ contains the equation $\delta a_{\underline{k}} + dm_{\underline{k}-1} = 0$. When $\underline{k} > 0$ and $g \geq 0$, or when $\underline{k} > -g$ and $g < 0$, $a_{\underline{k}}$ has both positive antifield number and positive pureghost number (due to $gh = antifd + puregh$). Using theorem 6.3, we then conclude $a_{\underline{k}} = \delta b_{\underline{k}+1} + dn_{\underline{k}}$ for some $b_{\underline{k}+1}$ and $n_{\underline{k}}$. One now considers $a' := a - sb_{\underline{k}+1} - dn_{\underline{k}}$ which fulfills $sa' + dm' = 0$ $(m' = m - sn_{\underline{k}})$ and derives (7.12) using recursive arguments analogous to those in the derivation of (7.9). The only values of $\underline{k}$ which are not covered in (7.12) are $\underline{k} = 0$ if $g \geq 0$, and $\underline{k} = -g$ if $g < 0$. In these cases, $sa + dm = 0$ contains the equation $\gamma a_0 + \delta a_1 + dm_0 = 0$ if $g \geq 0$, or $\delta a_{-g} + dm_{-g-1} = 0$ if $g < 0$. These are just the first equations in (7.10) and (7.11) respectively. To finish the proof one finally shows that a is trivial $(a = sb + dn)$ if and only if $a_0 = \gamma b_0 + \delta b_1 + dn_0$ for $g \geq 0$, or $a_{-g} = \delta b_{-g+1} + dn_{-g}$ for $g < 0$ by arguments which are again analogous to those used in the derivation of (7.5). $\square$

7.3   Bounded antifield number

As follows from the proof, the isomorphisms in theorem 7.1 hold under the assumption that the local forms in the theory may contain terms of arbitrarily high antifield

number. That is, if one expands the BRST cocycle a associated with a given element a_0 of $H_0^g(\gamma, H(\delta))$ or $H_0^{g,p}(\gamma, H(\delta|d))$ according to the antifield number as in Eq. (7.7), there is no guarantee, in the general case, that the expansion stops even if a_0 is a local form. So, although each term in the expansion would be a local form in this case, a may contain arbitrarily high derivatives if the number of derivatives in a_k grows with k. This is not a problem for effective field theories, but is in conflict with locality otherwise.

In the case of normal theories with a local Lagrangian, which include, as we have seen, the original Yang-Mills theory, the standard model as well as pure Chern-Simons theory in 3 dimensions (among others), one can easily refine the theorems and show that the expansion (7.7) stops, so that a is a local form. This is done by introducing a degree that appropriately controls the antifield number as well as the number of derivatives.

To convey the idea, we illustrate the procedure in the simplest case of pure electromagnetism. We leave it to the reader to extend the argument to the general case. The degree in question – call it D – may then be taken to be the sum of the degree counting the number of derivatives plus the degree assigning weight one to the antifields $A^{*\mu}$ and C^*. Our assumption of locality and polynomiality in the derivatives for a_0 implies that it has bounded degree D. In fact, since the differentials δ, γ and d all increase this degree by one unit, one can assume that a_0 is homogeneous of definite (finite) D-degree k. The recursive equations in $sa + dm = 0$ determining a_{i+1} from a_i read in this case $\delta a_{i+1} + \gamma a_i + dm_i = 0$ (thanks to $s = \delta + \gamma$), and so, one can assume that a_{i+1} has also D-degree k. Thus, all terms in the expansion (7.7) have same D-degree equal to k. This means that as one goes from one term a_i to the next a_{i+1} in (7.7), the antifield number increases (by definition) while the number of derivatives decreases until one reaches $a_m = a_{m+1} = \cdots = 0$ after a finite number of (at most $2k$) steps.

The fact that the expansion (7.7) stops is particularly convenient because it enables one to analyse the BRST cohomology starting from the last term in (7.7) (which exists). Although this is not always necessary, this turns out to be often a convenient procedure.

7.4 Comments

The ideas of homological perturbation theory appeared in the mathematical literature in [144, 124, 125, 126, 165, 127, 128]. They have been applied in the context of the antifield formalism in [106] (with locality analyzed in [133, 23]) and are reviewed in [134], chapters 8 and 17.

8 Lie algebra cohomology: $H(s)$ and $H(\gamma)$ in Yang-Mills type theories

8.1 Eliminating the derivatives of the ghosts

Our first task in the computation of $H(s)$ for gauge theories of the Yang-Mills type is to get rid of the derivatives of the ghosts. This can be achieved for every Lagrangian L fulfilling the conditions of the introduction; it is performed by making a change of jet-space coordinates adapted to the problem at hand (see [62, 101]).

We consider subsets W^k ($k = -1, 0, 1, \ldots$) of jet coordinates where W^{-1} contains only the undifferentiated ghosts C^I, while W^k for $k \geq 0$ contains

$$A^I_{\mu|(\nu_1\ldots\nu_l)}, \ \psi^i_{(\nu_1\ldots\nu_l)}, \ C^I_{(\nu_1\ldots\nu_{l+1})},$$
$$A^{*\mu}_{I(\nu_1\ldots\nu_{l-2})}, \ \psi^*_{i(\nu_1\ldots\nu_{l-m})}, \ C^*_{I(\nu_1\ldots\nu_{l-3})} \tag{8.1}$$

for $l = 0, \ldots, k$ and $m = 1, 2$, for matter fields with first and second order field equations respectively. [These definitions are in fact taylored to the standard model; if the gauge fields obey equations of motion of order k, ν_{l-2} should be replaced by ν_{l-k} and ν_{l-3} by ν_{l-k-1} in (8.1); similarly, m is generally the derivative order of the matter field equations.]

We can take as new coordinates on W^k the following functions of the old ones:

$$\partial_{(\nu_1\ldots\nu_l}A^I_{\mu)}, \ \partial_{(\nu_1\ldots\nu_l}D_{\mu)}C^I, \tag{8.2}$$

$$C^I, \tag{8.3}$$

$$D_{(\nu_1}\ldots D_{\nu_{l-1}}F^I_{\nu_l)\mu}, \ D_{(\nu_1}\ldots D_{\nu_l)}\psi^i, \tag{8.4}$$

$$D_{(\nu_1}\ldots D_{\nu_{l-2})}A^{*\mu}_I, \ D_{(\nu_1}\ldots D_{\nu_{l-3})}C^*_I, \ D_{(\nu_1}\ldots D_{\nu_{l-m})}\psi^*_i, \tag{8.5}$$

for $l = 0, \ldots, k$ and $m = 1, 2$. This change of coordinates is invertible because $\partial_{\nu_1\ldots\nu_l}A^I_\mu = \partial_{(\nu_1\ldots\nu_l}A^I_{\mu)} + \frac{l}{l+1}D_{(\nu_1}\ldots D_{\nu_{l-1}}F^I_{\nu_l)\mu} + O(l-1)$ where $O(l-1)$ collects terms with less than l derivatives. There are no algebraic relations between the $\partial_{(\nu_1\ldots\nu_l}A^a_{\mu)}$ and the $D_{(\nu_1}\ldots D_{\nu_{l-1}}F^a_{\nu_l)\mu}$, which correspond to the independent irreducible components of $\partial_{\nu_1\ldots\nu_l}A^I_\mu$. Similarly, one has $D_{(\nu_1}\ldots D_{\nu_l)}\psi^i = \partial_{\nu_1\ldots\nu_l}\psi^i + O(l-1)$.

The new coordinates can be grouped into two sets : contractible pairs (8.2) on the one hand, and gauge covariant coordinates (8.4), (8.5) plus undifferentiated ghosts (8.3) on the other hand. These sets transform among themselves under s. Indeed we have $s\partial_{(\nu_1\ldots\nu_l}A^I_{\mu)} = \partial_{(\nu_1\ldots\nu_l}D_{\mu)}C^I$ and $s\partial_{(\nu_1\ldots\nu_l}D_{\mu)}C^I = 0$. Similarly, if we collectively denote by χ^u_Δ the coordinates (8.4) and (8.5), we have $\gamma\chi^u_\Delta = -eC^I T^u_{Iv}\chi^v_\Delta$ where the T^u_{Iv} are the entries of representation matrices of $\mathcal{G}$ and Δ labels the various multiplets of $\mathcal{G}$ formed by the χ's. For instance, for every fixed set of spacetime indices, the $D_{(\nu_1}\ldots D_{\nu_{l-1}}F^K_{\nu_l)\mu}$ ($K = 1, \ldots, dim(\mathcal{G})$) form a multiplet χ^K_Δ (with fixed Δ) of the coadjoint representation with T^u_{Iv} given by $f_{IJ}{}^K$.[8]

---

[8] Our notation is slightly sloppy because the index u (and in particular its range) really depends on the given multiplet and thus should carry a subindex Δ. The substitution $u \longrightarrow u_\Delta$ should thus be understood in the formulas below.

Finally, sC^I is a function of the ghosts alone, and δ on the antifields (8.5) only involves the coordinates (8.5) and (8.4). The latter statement about the δ-transformations is equivalent to the gauge covariance of the equations of motion. It can be inferred from $\gamma\delta + \delta\gamma = 0$, without referring to a particular Lagrangian. Namely $(\gamma\delta + \delta\gamma)A_I^{*\mu} = 0$ gives $\gamma L_I^\mu = eC^J f_{JI}{}^K L_K^\mu$ while $(\gamma\delta + \delta\gamma)\psi_i^* = 0$ gives $\gamma L_i = eC^I T_{Ii}^j L_j$, see Eq. (2.8). The absence of derivatives of the ghosts in γL_I^μ and γL_i implies that L_I^μ and L_i can be expressed solely in terms of the variables (8.4) (and the x^μ when the Lagrangian involves x^μ explicitly) because γ is stable in the subspaces of local functions with definite degree in the variables (8.2).

The coordinates (8.2) form thus indeed contractible pairs and do not contribute to the cohomology of s according to a reasoning analogous to the one followed in section 2.7.

Note that the removal of the vector potential, its symmetrized derivatives, and the derivatives of the ghosts, works both for $H(s)$ and $H(\gamma)$ since s and γ coincide in this sector.

8.2 Lie algebra cohomology with coefficients in a representation

One of the interests of the elimination of the derivatives of the ghosts is that the connection between $H(\gamma)$, $H(s)$ and ordinary Lie algebra cohomology becomes now rather direct.

We start with $H(\gamma)$, for which matters are straightforward. We have reduced the computation of the cohomology of γ in the algebra of all local forms to the calculation of the cohomology of γ in the algebra $\mathcal{K}$ of local forms depending on the covariant objects χ_Δ^u and the undifferentiated ghosts C^I. More precisely, the relevant algebra is now

$$\mathcal{K} = \Omega(\mathbb{R}^n) \otimes \mathcal{F} \otimes \Lambda(C) \tag{8.6}$$

where $\Omega(\mathbb{R}^n)$ is the algebra of exterior forms on $\mathbb{R}^n$, $\mathcal{F}$ the algebra of functions of the covariant objects χ_Δ^u, and $\Lambda(C)$ the algebra of polynomials in the ghosts C^I (which is just the antisymmetric algebra with $dim(\mathcal{G})$ generators).

The subalgebra $\Omega(\mathbb{R}^n) \otimes \mathcal{F}$ provides a representation of the Lie algebra $\mathcal{G}$, the factor $\Omega(\mathbb{R}^n)$ being trivial since it does not transform under $\mathcal{G}$. We call this representation ρ. The differential γ can be written as

$$\gamma = eC^I \rho(e_I) + \frac{e}{2} C^J C^I f_{IJ}{}^K \frac{\partial}{\partial C^K} \tag{8.7}$$

where the e_I form a basis for the Lie algebra $\mathcal{G}$ and the $\rho(e_I)$ are the corresponding "infinitesimal generators" in the representation,

$$\rho(e_I) = -T_{Iv}^u \, \chi_\Delta^v \, \frac{\partial}{\partial \chi_\Delta^u} \; . \tag{8.8}$$

The identification of the polynomials in C^I with the cochains on $\mathcal{G}$ then allows to identify the differential γ with the standard Chevalley-Eilenberg differential [76] for Lie algebra cochains with values in the representation space $\Omega(\mathbb{R}^n) \otimes C^\infty(\chi_\Delta^u)$.

Thus, we see that the cohomology of γ is just standard Lie algebra cohomology with coefficients in the representation ρ of functions in the covariant objects χ^u_Δ (times the spacetime exterior forms).

The space of smooth functions in the variables χ^u_Δ is evidently infinite-dimensional. In order to be able to apply theorems on Lie algebra cohomology, it is necessary to make some restrictions on the allowed functions so as to effectively deal with finite dimensional representations. This condition will be met, for instance, if one considers polynomial local functions in the χ^u_Δ with coefficients that can possibly be smooth functions of invariants (e.g. $\exp\phi$ can occur if ϕ does not transform under $\mathcal{G}$). This space is still infinite-dimensional, but splits as the direct sum of finite-dimensional representation spaces of $\mathcal{G}$. Indeed, because $\rho(e_I)$ is homogeneous of degree 0 in the χ^u_Δ, we can consider separately polynomials of a given homogeneity in the χ^u_Δ, which form finite dimensional representation spaces. Thus, the problem of computing the Lie algebra cohomology of $\mathcal{G}$ with coefficients in the representation ρ is effectively reduced to the problem of computing the Lie algebra cohomology of $\mathcal{G}$ with coefficients in a finite-dimensional representation. The same argument applies, of course, to effective field theories. From now on, it will be understood that such restrictions are made on $\mathcal{F}$.

8.3 $H(s)$ versus $H(\gamma)$

The previous section shows that the computation of $H(\gamma)$ boils down to a standard problem of Lie algebra cohomology with coefficients in a definite representation. This is also true for $H(s)$, but the representation space is now different.

Indeed, we have seen that $H(s) \simeq H(\gamma, H(\delta))$. This result was established in the algebra of all local forms depending also on the differentiated ghosts and symmetrized derivatives of the vector potential, but also holds in the algebra $\mathcal{K}$. Moreover, the cohomology of the Koszul-Tate differential in $\mathcal{K}$ can be computed in the same manner as above. The antifields drop out with the variables constrained to vanish with the equations of motion. More precisely, among the field strength components, the matter field components, and their covariant derivatives, some can be viewed as constrained by the equations of motion and the others can be viewed as independent. Let X^u_A be the independent ones and $\mathcal{F}^R$ be the algebra of smooth functions in X^u_A (with restrictions analogous to those made in the previous subsection). Because the equations of motion are gauge covariant (see above), one can take the X^u_A to transform in a linear representation of $\mathcal{G}$, which we denote by ρ^R. Again, since one can work order by order in the derivatives, the representation ρ^R effectively splits as a direct sum of finite-dimensional representations. [See below for an explicit construction of the X^u_A in the standard model.]

By our general discussion of section 7, it follows that

Lemma 8.1 *The cohomology of s is isomorphic to the cohomology of γ in the space of local forms depending only on the undifferentiated ghosts C^I and the X^u_A.*

In the case of the standard model, the X^u_A may be constructed as follows. First, the $D_{(\nu_1}\ldots D_{\nu_{l-1}}F^I_{\nu_l)\mu}$ are split into the algebraically independent completely trace-

less combinations $(D_{(\nu_1} \dots D_{\nu_{l-1}} F^I_{\nu_l)\mu})_{\text{tracefree}}$ (i.e., $\eta^{\nu_1\nu_2}(D_{(\nu_1} \dots D_{\nu_{l-1}} F^I_{\nu_l)\mu})_{\text{tracefree}} = 0$) and the traces $D_{(\nu_1} \dots D_{\nu_{l-2}} D_{\lambda)} F^{I\lambda}_\mu$ [203]. Second the covariant derivatives $D_{(s_1} \dots D_{s_l)} \psi^i$ of the matter fields are replaced by $D_{(s_1} \dots D_{s_l)} \hat{\psi}^i$ and $D_{(\nu_1} \dots D_{\nu_{l-1})} \hat{\mathcal{L}}_i$ for matter fields $\hat{\psi}^i$ with first order equations and by $D_{(s_1} \dots D_{s_l)} \tilde{\psi}^i$, $D_{(s_1} \dots D_{s_{l-1}} D_{0)} \tilde{\psi}^i$, $D_{(\nu_1} \dots D_{\nu_{l-2})} \tilde{\mathcal{L}}_i$ for matter fields $\tilde{\psi}^i$ with second order field equations. The X^u_A are then the coordinates $(D_{(\nu_1} \dots D_{\nu_{l-1}} F^I_{\nu_l)\mu})_{\text{tracefree}}$, $D_{(s_1} \dots D_{s_l)} \hat{\psi}^i$, $D_{(s_1} \dots D_{s_l)} \tilde{\psi}^i$, $D_{(s_1} \dots D_{s_{l-1}} D_{0)} \tilde{\psi}^i$. Other splits are of course available.

In the algebra

$$\mathcal{K}^R = \Omega(\mathbb{R}^n) \otimes \mathcal{F}^R \otimes \Lambda(C) \tag{8.9}$$

the differential γ reads

$$\gamma = eC^I \rho^R(e_I) + \frac{e}{2} C^J C^I f_{IJ}{}^K \frac{\partial}{\partial C^K} \tag{8.10}$$

where the $\rho^R(e_I)$ are the infinitesimal generators in the representation ρ^R,

$$\rho^R(e_I) = -T^u_{Iv} X^v_A \frac{\partial}{\partial X^u_A} \ . \tag{8.11}$$

The representations of $\mathcal{G}$ which occur in $\mathcal{F}$ and $\mathcal{F}^R$ are the same ones, but with a smaller multiplicity in $\mathcal{F}^R$. One can thus identify the cohomology of s with the Lie-algebra cohomology of $\mathcal{G}$, with coefficients in the representation ρ^R. The difference between $H(s)$ and $H(\gamma)$ lies only in the space of coefficients.

8.4 Whitehead's theorem

In order to proceed, we shall now assume that the gauge group G is the direct product of a compact abelian group G_0 times a semi-simple group G_1. Thus, $G = G_0 \times G_1$, with $G_0 = (U(1))^q$. This assumption on G was not necessary for the previous analysis, but is used for the subsequent developments since in this case one has complete results on the Lie algebra cohomology. On these conditions, it can then be shown that any finite-dimensional representation of $\mathcal{G}$ is completely reducible [60].

We can now follow the standard literature (e.g. [121]). For any representation space V with representation ρ, let $V_{\rho=0}$ be the invariant subspace of V carrying the trivial representation, which may occur several times ($v \in V_{\rho=0} \Rightarrow \rho(e_I)v = 0 \ \forall I$). Note that the space $\Lambda(C)$ of ghost polynomials is a representation of $\mathcal{G}$ for

$$\rho^C(e_I) = -C^J f_{IJ}{}^K \frac{\partial}{\partial C^K} \ .$$

(With the above interpretation of the C^I, it is the extension of the coadjoint representation to $\Lambda\mathcal{G}^*$.) The total representation on $V \otimes \Lambda(C)$ is $\rho^T(e_I) = \rho(e_I) + \rho^C(e_I)$. It satisfies

$$\left\{ \gamma, \frac{\partial}{\partial C^I} \right\} = e\rho^T(e_I), \quad [\gamma, \rho^T(e_I)] = 0, \tag{8.12}$$

the first relation following by direct computation, the second one from the first and $\gamma^2 = 0$. Hence the cohomology $H(\gamma, (V \otimes \Lambda(C))_{\rho^T = 0})$ is well defined. We can write

$$\gamma = eC^I \rho(e_I) + \frac{e}{2} C^I \rho^C(e_I) = eC^I \rho^T(e_I) + \hat{\gamma} \ ,$$

where $\hat{\gamma}$ is the restriction of γ to $\Lambda(C)$, up to the sign:

$$\hat{\gamma} = \frac{e}{2} C^I C^J f_{IJ}{}^K \frac{\partial}{\partial C^K} \ .$$

It follows that

$$H(\gamma, (V \otimes \Lambda(C))_{\rho^T = 0}) \simeq H(\hat{\gamma}, (V \otimes \Lambda(C))_{\rho^T = 0}). \tag{8.13}$$

Note that we also have,

$$\left\{ \hat{\gamma}, \frac{\partial}{\partial C^I} \right\} = -e\rho^C(e_I), \quad [\hat{\gamma}, \rho^C(e_I)] = 0. \tag{8.14}$$

The first mathematical result that we shall need reduces the problem of computing the Lie algebra cohomology of $\mathcal{G}$ with coefficients in V to that of finding the invariant subspace $V_{\rho=0}$ and computing the Lie algebra cohomology of $\mathcal{G}$ with coefficients in the trivial, one-dimensional, representation.

Theorem 8.1 (i) $H(\gamma, V \otimes \Lambda(C))$ is isomorphic to $H(\gamma, (V \otimes \Lambda(C))_{\rho^T = 0})$. In particular, $H(\hat{\gamma}, \Lambda(C))$ is isomorphic to $\Lambda(C)_{\rho^C = 0}$.
(ii) $H(\gamma, (V \otimes \Lambda(C))_{\rho^T = 0})$ is isomorphic to $V_{\rho=0} \otimes \Lambda(C)_{\rho^C = 0}$.

The proof of this theorem is given in the appendix 8.A. The result $H(\gamma, (V \otimes \Lambda(C)) \simeq V_{\rho=0} \otimes H(\hat{\gamma}, \Lambda(C))$ is known as Whitehead's theorem.

To determine the cohomology of s, we thus need to determine on the one hand the invariant monomials in the X_A^u, which depends on the precise form of the matter field representations and which will not be discussed here; and on the other hand, $H(\hat{\gamma}, \Lambda(C)) \simeq \Lambda(C)_{\rho^C = 0}$. This latter cohomology is known as the Lie algebra cohomology of $\mathcal{G}$ and is discussed in the next section.

8.5   Lie algebra cohomology - Primitive elements

We shall only give the results (in ghost notations), without proof. We refer to the mathematical literature for the details [162, 121].

The cohomology $H^g(\hat{\gamma}, \Lambda(C))$ can be described in terms of particular ghost polynomials $\theta_r(C)$ representing the so-called primitive elements. These are in bijective correspondence with the independent Casimir operators $\mathcal{O}_r$,

$$\mathcal{O}_r = d^{I_1 \ldots I_{m(r)}} \delta_{I_1} \ldots \delta_{I_{m(r)}}, \quad r = 1, \ldots, rank(\mathcal{G}). \tag{8.15}$$

The $d^{I_1 \ldots I_{m(r)}}$ are symmetric invariant tensors,

$$\sum_{i=1}^{m(r)} f_{JL}{}^K d^{I_1 \ldots I_{i-1} L I_{i+1} \ldots I_{m(r)}} = 0, \tag{8.16}$$

while $\delta_I = \rho(e_I)$ for some representation $\rho(e_I)$ of $\mathcal{G}$. The ghost polynomial $\theta_r(C)$ corresponding to $\mathcal{O}_r$ is homogeneous of degree $2m(r) - 1$, and given by

$$\theta_r(C) = (-e)^{m(r)-1} \frac{m(r)!(m(r)-1)!}{2^{m(r)-1}(2m(r)-1)!} f_{I_1\ldots I_{2m(r)-1}} C^{I_1} \ldots C^{I_{2m(r)-1}} ,$$

$$f_{I_1 J_1 \ldots I_{m(r)-1} J_{m(r)-1} K_{m(r)}} = d_{K_1 \ldots K_{m(r)-1}[K_{m(r)}} f_{I_1 J_1}{}^{K_1} \ldots f_{I_{m(r)-1} J_{m(r)-1}]}{}^{K_{m(r)-1}} \quad (8.17)$$

This definition of $\theta_r(C)$ involves, for later purpose, a normalization factor containing both the gauge coupling constant and the order $m(r)$ of $\mathcal{O}_r$. The $d_{K_1\ldots K_{m(r)}}$ arise from the invariant symmetric tensors in (8.15) by lowering the indices with the invertible metric g_{IJ} obtained by adding the Killing metrics for each simple factor, trivially extended to the whole of $\mathcal{G}$, and with the identity for an abelian δ_I. Using an appropriate (possibly complex) matrix representation $\{T_I\}$ of $\mathcal{G}$ (cf. example below), $\theta_r(C)$ can also be written as

$$\theta_r(C) = (-e)^{m(r)-1} \frac{m(r)!(m(r)-1)!}{(2m(r)-1)!} \mathrm{Tr}(C^{2m(r)-1}), \quad C = C^I T_I . \quad (8.18)$$

Indeed, using that the C^I anticommute and that $\{T_I\}$ represents $\mathcal{G}$ ($[T_I, T_J] = f_{IJ}{}^K T_K$), one easily verifies that (8.18) agrees with (8.17) for

$$d_{I_1 \ldots I_{m(r)}} = \mathrm{Tr}[T_{(I_1} \ldots T_{I_{m(r)})}] .$$

Those $\theta_r(C)$ with degree 1 coincide with the abelian ghost fields (if any), in accordance with the above definitions: the abelian elements of $\mathcal{G}$ count among the Casimir operators as they commute with all the other elements of $\mathcal{G}$. We thus set

$$\{\theta_r(C) : m(r) = 1\} = \{\text{abelian ghosts}\}. \quad (8.19)$$

Note that this is consistent with (8.18), as it corresponds to the choice $\{T_I\} = \{0, \ldots, 0, 1, 0, \ldots, 0\}$, where one of the abelian elements of $\mathcal{G}$ is represented by the number 1, while all the other elements of $\mathcal{G}$ are represented by 0.

Each $\theta_r(C)$ is $\hat{\gamma}$ closed, as is easily verified using the matrix notation (8.18),

$$\hat{\gamma} \mathrm{Tr}(C^{2m-1}) = e \, \mathrm{Tr}(C^{2m}) = 0,$$

where the first equality holds due to $\hat{\gamma}C = eC^2$ and the second equality holds because the trace of any even power of a Grassmann odd matrix vanishes. The cohomology of $\hat{\gamma}$ is generated precisely by the $\theta_r(C)$, i.e., the corresponding cohomology classes are represented by polynomials in the $\theta_r(C)$, and no nonvanishing polynomial of the $\theta_r(C)$ is cohomologically trivial,

$$\hat{\gamma}h(C) = 0 \quad \Leftrightarrow \quad h(C) = P(\theta(C)) + \hat{\gamma}g(C) ;$$
$$P(\theta(C)) = \hat{\gamma}g(C) \quad \Leftrightarrow \quad P = 0. \quad (8.20)$$

Note that the $\theta_r(C)$ anticommute because they are homogeneous polynomials of odd degree in the ghost fields. Therefore the dimension of the cohomology of $\hat{\gamma}$ is

$2^{rank(\mathcal{G})}$. Note also that the highest nontrivial cohomology class (i.e., the one with highest ghost number) is represented by the product of all the $\theta_r(C)$. This product is always proportional to the product of all the ghost fields (see, e.g., [63]), and has thus ghost number equal to the dimension of $\mathcal{G}$,

$$\prod_{r=1}^{rank(\mathcal{G})} \theta_r(C) \propto \prod_{I=1}^{dim(\mathcal{G})} C^I. \tag{8.21}$$

Example 1. Let us spell out the result for the gauge group of the standard model, $G = U(1) \times SU(2) \times SU(3)$. The $U(1)$-ghost is a θ by itself, see (8.19). We set

$$\theta_1(C) = \text{U}(1)\text{-ghost}.$$

$SU(2)$ has only one Casimir operator which has order 2. The corresponding θ has thus degree 3 and is given by

$$\theta_2(C) = -\frac{e_{\text{su}(2)}}{3} \text{Tr}_{\text{su}(2)}(C^3),$$

with $C = C^I T_I$, $\{T_I\} = \{0, i\sigma_\alpha, 0, \ldots, 0\}$ where the zeros represent $u(1)$ and $su(3)$, and σ_α are the Pauli matrices.

$SU(3)$ has two independent Casimir operators, with degree 2 and 3 respectively. This gives two additional θ's of degree 3 and 5 respectively,

$$\theta_3(C) = -\frac{e_{\text{su}(3)}}{3} \text{Tr}_{\text{su}(3)}(C^3), \quad \theta_4(C) = \frac{e_{\text{su}(3)}^2}{10} \text{Tr}_{\text{su}(3)}(C^5),$$

with $\{T_I\} = \{0, 0, 0, 0, i\lambda_a\}$ where λ_a are the Gell-Mann matrices.

A complete list of inequivalent representatives of $H(\hat{\gamma}, \Lambda(C))$ is:

ghost number	representatives of $H(\hat{\gamma}, \Lambda(C))$
0	1
1	θ_1
3	θ_2 , θ_3
4	$\theta_1\theta_2$, $\theta_1\theta_3$
5	θ_4
6	$\theta_2\theta_3$, $\theta_1\theta_4$
7	$\theta_1\theta_2\theta_3$
8	$\theta_2\theta_4$, $\theta_3\theta_4$
9	$\theta_1\theta_2\theta_4$, $\theta_1\theta_3\theta_4$
11	$\theta_2\theta_3\theta_4$
12	$\theta_1\theta_2\theta_3\theta_4$

Example 2. Let us denote by C^{I_a} the ghosts of the abelian factors, $I_a = 1, \ldots, l$. A basis of the first cohomology groups $H^g(\hat{\gamma}, \Lambda(C))$ ($g = 0, 1, 2$) is given by (i) 1 for $g = 0$; (ii) C^{I_a}, with $I_a = 1, \ldots, l$, for $g = 1$; and (iii) $C^{I_a} C^{J_a}$, with $I_a < J_a$, for $g = 2$. In particular, $H^1(\hat{\gamma}, \Lambda(C))$ and $H^2(\hat{\gamma}, \Lambda(C))$ are trivial if there is no abelian factor. For a compact group, a basis for $H^3(\hat{\gamma}, \Lambda(C))$ is given by $C^{I_a} C^{J_a} C^{K_a}$ ($I_a < J_a < K_a$) and $f_{I_s J_s K_s} C^{I_s} C^{J_s} C^{K_s} \equiv \mathrm{Tr}_{\mathcal{G}_s} C^3$, where s runs over the simple factors G_s in G.

8.6 Implications for the renormalization of local gauge invariant operators

Class I local operators are defined as local, non integrated, gauge invariant operators (built out of the gauge potentials and the matter fields) that are linearly independent, even when the gauge covariant equations of motions are used [158, 152, 90]. In the absence of anomalies, it can be shown that these operators only mix, under renormalization, with BRST closed local operators (built out of all the fields and antifields). BRST exact operators are called class II operators. They can be shown not to contribute to the physical S matrix and to renormalize only among themselves. The question is then whether class I operators can only mix with class I operators and class II operators under renormalization.

That the answer is affirmative follows from lemma 8.1 in the case of ghost number 0. Indeed, the γ cohomology in the space of forms in the X_A^u, (i.e., combinations of the covariant derivatives of the field strength components and the matter field components not constrained by the equations of motions) reduces to the gauge invariants in these variables. This is precisely the required statement that class I operators give a basis of the BRST cohomology $H^{0,*}(s, \Omega)$ in ghost number 0.

The statement was first proved in [152]. A different proof has been given in [135].

8.7 Appendix 8.A: Proof of theorem 8.1

Our proof of theorem 8.1 uses ghost notations.

(i) Let us first of all prove general relations for a completely reducible representation commuting with a differential, i.e., take into account only the second relation of (8.12). This relation implies that the representation $(\rho^T)^\#(e_I)$ induced in cohomology by $(\rho^T)^\#(e_I)[a] = [\rho^T(e_I)a]$ is well defined. The induced representation is completely reducible: since the space of γ cocycles Z is stable under $\rho^T(e_I)$, there exists a stable subspace $E \subset V \otimes \Lambda(C)$, such that $V \otimes \Lambda(C) = Z \oplus E$. Similarly, because the space of γ coboundaries B is stable under $\rho^T(e_I)$, there exists a stable subspace $F \subset Z$ such that $Z = F \oplus B$. It follows that $H(\gamma, V \otimes \Lambda(C))$ is isomorphic to F. Since F is completely reducible for $\rho^T(e_I)$, so is $H(\gamma, V \otimes \Lambda(C))$ for $(\rho^T)^\#(e_I)$. Complete reducibility also implies that $Z(\rho^T(V \otimes \Lambda(C))) = Z \cap \rho^T(V \otimes \Lambda(C)) = \rho^T Z$. This means that $H(\gamma, \rho^T(V \otimes \Lambda(C))) \subset (\rho^T)^\# H(\gamma, V \otimes \Lambda(C))$. In the same way, $H(\gamma, (V \otimes \Lambda(C))_{\rho^T=0}) \subset H(\gamma, V \otimes \Lambda(C))_{(\rho^T)^\#=0}$. Complete reducibility of ρ^T then implies $H(\gamma, V \otimes \Lambda(C)) = H(\gamma, (V \otimes \Lambda(C))_{\rho^T=0}) \oplus H(\gamma, \rho^T(V \otimes \Lambda(C)))$, while complete reducibility of $(\rho^T)^\#$ implies $H(\gamma, V \otimes \Lambda(C)) = H(\gamma, V \otimes \Lambda(C))_{(\rho^T)^\#=0} \oplus$

$(\rho^T)^\# H(\gamma, V \otimes \Lambda(C))$. It follows that $H(\gamma, (V \otimes \Lambda(C))_{\rho^T=0}) = H(\gamma, V \otimes \Lambda(C))_{(\rho^T)^\#=0}$ and $H(\gamma, \rho^T(V \otimes \Lambda(C))) = (\rho^T)^\# H(\gamma, V \otimes \Lambda(C))$. Using now the first relation of (8.12), it follows that $(\rho^T)^\# = 0$ so that $H(\gamma, \rho^T(V \otimes \Lambda(C))) = 0$ and $H(\gamma, V \otimes \Lambda(C)) = H(\gamma, (V \otimes \Lambda(C))_{\rho^C=0})$. This proves the first part of (i).

For the second part, we note that in $\Lambda(C)$, the representation $\rho^C(e_I)$ reduces to the representation of the semi-simple factor. Let us denote the generators of this factor by e_a and its representation on $\Lambda(C)$ by $\rho^C(e_a)$. This representation is completely reducible, because defining properties of semi-simple Lia algebras are (a) the Killing metric $g_{ab} = f_{ac}{}^d f_{bd}{}^c$ is invertible; (b) it is the direct sum of simple ideals, (a Lie algebra being simple if it is non abelian and contains no proper non trivial ideals); (c) all its representations in finite dimensional vector spaces are completely reducible [60]. This proves then the second part by the same reasoning as above with $V = \rho = 0$, respectively (8.14) instead of (8.12).

(ii) $\Lambda(C) = \Lambda(C)_{\rho^C=0} \oplus \rho^C \Lambda(C)$ implies $(V \otimes \Lambda(C))_{\rho^T=0} = V_{\rho=0} \otimes \Lambda(C)_{\rho^C=0} \oplus (V \otimes \rho^C \Lambda(C))_{\rho^T=0}$. Because it follows from $[\hat{\gamma}, \rho^T] = 0 = [\hat{\gamma}, \rho^C] = [\hat{\gamma}, \rho]$ that all the spaces are stable under $\hat{\gamma}$, the Künneth formula gives $H(\hat{\gamma}, (V \otimes \Lambda(C))_{\rho^T=0}) = V_{\rho=0} \otimes H(\hat{\gamma}, \Lambda(C)_{\rho^C=0}) \oplus H(\hat{\gamma}, (V \otimes \rho^C \Lambda(C))_{\rho^T=0})$. The result (ii) then follows from (8.13) and $\hat{\gamma} = -\frac{e}{2} C^I \rho^C(e_I)$, if we can show that $H(\hat{\gamma}, (V \otimes \rho^C \Lambda(C))_{\rho^T=0}) = 0$.

The contracting homotopy that allows to prove $H(\hat{\gamma}, \rho^C \Lambda(C)) = 0$ can be constructed explicitly as follows. Define the Casimir operator $\Gamma = \frac{1}{2} g^{ab} \rho^C(e_a) \rho^C(e_b)$, where g^{ab} is the inverse of the Killing metric associated to the semi-simple Lie subalgebra of $\mathcal{G}$. From the complete skew symmetry of the structure constants lowered or raised through the Killing metric or its inverse (this being a consequence of the Jacobi identity), it follows that this operator commutes with all the operators of the representation, $[\Gamma, \rho^C(e_a)] = 0$, while the first relation of (8.14) implies that $[\hat{\gamma}, \Gamma] = 0$.

A property of semi-simple Lie algebras is that the Casimir operator Γ is invertible on $\rho^C \Lambda(C)$. Obviously, in this case, $[\Gamma^{-1}, \rho^C(e_a)] = 0$ and $[\hat{\gamma}, \Gamma^{-1}] = 0$. Take $a \in \rho^C \Lambda(C)$, with $\hat{\gamma} a = 0$. We have $a = \Gamma \Gamma^{-1} a = -\frac{1}{2} g^{ab} \{\hat{\gamma}, \frac{\partial}{\partial C^a}\} \rho^C(e_b) \Gamma^{-1} a = -\hat{\gamma} \rho^C(e_b) \frac{1}{2} g^{ab} \frac{\partial}{\partial C^a} \Gamma^{-1} a$, where we have used in addition $g^{ab} [\frac{\partial}{\partial C^a}, \rho^C(e_b)] = 0$, as follows from the first relation of (8.14) and the graded Jacobi identity for the graded commutator of operators. Hence, $H(\hat{\gamma}, \rho^C \Lambda(C)) = 0$.

Similarly, let $a = v \otimes b$, where $b \in \rho^C \Lambda(C)$, with $\gamma a = 0 = e C^I \rho(e_I) v \otimes b - e(-)^v v \otimes \hat{\gamma} b$ and $\rho^T(e_I) a = 0 = \rho(e_I) v \otimes b + v \otimes \rho(e_I) b$. We have $a = -v \otimes (\hat{\gamma} \rho^C(e_b) \frac{1}{2} g^{ab} \frac{\partial}{\partial C^a} \Gamma^{-1} b + \rho^C(e_b) \frac{1}{2} g^{ab} \frac{\partial}{\partial C^a} \Gamma^{-1} \hat{\gamma} b) = \gamma \rho^C(e_b) \frac{1}{2} g^{ab} \frac{\partial}{\partial C^a} \Gamma^{-1} (v \otimes b)$. Furthermore, direct computation using skew symmetry of structure constants with lifted indices shows that $[\rho^T(e_I), \rho^C(e_b) \frac{1}{2} g^{ab} \frac{\partial}{\partial C^a} \Gamma^{-1}] = 0$, which proves that $H(\hat{\gamma}, (V \otimes \rho^C \Lambda(C))_{\rho^T=0} = 0$ and thus (ii). $\square$

9 Descent equations: $H(s|d)$

9.1 Introduction

The descent equation technique is a powerful tool to calculate $H(s|d)$ which we shall use below. Its usefulness rests on the fact that it relates $H(s|d)$ to $H(s)$ which is often much simpler than $H(s|d)$ - and which we have determined.

In subsections 9.2, 9.3 and 9.4, we shall review general properties of the descent equations and work out the relation between $H(s|d)$ and $H(s)$ in detail. Our only assumption for doing so will be that in the space of local forms under study, the cohomology of d is trivial at all form-degrees $p = 1, \ldots, n-1$ and is represented at $p = 0$ by pure numbers,

$$H^p(d) = \delta_0^p \mathbb{R} \quad \text{for} \quad p < n. \tag{9.1}$$

Since this is the only assumption being made at this stage, the considerations in subsections 9.2 through 9.4 are not restricted to gauge theories of the Yang-Mills type but apply whenever (9.1) is fulfilled.

In the case of theories of the Yang-Mills type (in $\mathbb{R}^n$), the considerations apply in the space of local smooth forms or in the space of local polynomial forms, provided one allows for an explicit x-dependence. Indeed, the algebraic Poincaré lemma guarantees then that 9.1 holds (theorem 4.2). Although our ultimate goal is to cover the polynomial (or formal power series) case, such a restriction is not necessary in this section. The considerations are also valid in subalgebras of the algebra of local forms for which 9.1 remains true.

However the considerations of this section do not immediately apply, for instance, if no explicit spacetime coordinate dependence is allowed. In this case, the cohomology of d is non-trivial in degrees $\neq 0$ and contains the constant forms (see theorem 4.3). It turns out, however, that the constant forms cannot come in the way, so that the same descent equation techniques in fact apply. This is explained in subsection 9.5.

Finally, we carry out the explicit derivation of the descent equations in the case of the differential s, but a similar discussion applies to γ or δ (or, for that matter, any differential D such that $Dd + dD = 0$). In fact, this tool has already been used in section 6 for the mod d cohomology of δ, to prove the isomorphism $H_k^p(\delta|d) \simeq H_{k-1}^{p-1}(\delta|d)$ for $p > 1, k > 1$. The same techniques can be followed for $H(s|d)$ (or $H(\gamma|d)$), with, however, one complication. While one had $H_k(\delta) = 0$ ($k > 0$) for δ, the cohomology of s is non trivial. As a result, while it is easy to "go down" the descent (because this uses the triviality of d – see below), it is more intricate to "go up".

9.2 General properties of the descent equations

We shall now review the derivation and some basic properties of the descent equations, assuming that (9.1) holds.

Derivation of the descent equations. Let ω^m be a cocycle of $H^{*,m}(s|d)$,

$$s\omega^m + d\omega^{m-1} = 0. \tag{9.2}$$

Two cocycles are equivalent in $H^{*,m}(s|d)$ when they differ by a trivial solution of the consistency condition,

$$\omega^m \sim \omega'^m \quad \Leftrightarrow \quad \omega^m - \omega'^m = s\eta^m + d\eta^{m-1}. \tag{9.3}$$

Applying s to Eq. (9.2) yields $d(s\omega^{m-1}) = 0$ (due to $s^2 = 0$ and $sd + ds = 0$), i.e., $s\omega^{m-1}$ is a d-closed $(m-1)$-form. Let us assume that $m - 1 > 0$ (the case $m - 1 = 0$ is treated below). Using (9.1), we conclude that $s\omega^{m-1}$ is d-exact, i.e., there is an $(m-2)$-form such that $s\omega^{m-1} + d\omega^{m-2} = 0$. Hence, ω^{m-1} is a cocycle of $H^{*,m-1}(s|d)$. Moreover, due to the ambiguity (9.3) in ω^m, ω^{m-1} is also determined only up to a coboundary of $H^{*,m-1}(s|d)$. Indeed, when ω^m solves Eq. (9.2), then $\omega'^m = \omega^m + s\eta^m + d\eta^{m-1}$ fulfills $s\omega'^m + d\omega'^{m-1} = 0$ with $\omega'^{m-1} = \omega^{m-1} + s\eta^{m-1} + d\eta^{m-2}$. Now, two things can happen:

(a) either ω^{m-1} is trivial in $H^{*,m-1}(s|d)$, $\omega^{m-1} = s\eta^{m-1} + d\eta^{m-2}$; then we can substitute $\omega'^{m-1} = \omega^{m-1} - s\eta^{m-1} - d\eta^{m-2} = 0$ and $\omega'^m = \omega^m - d\eta^{m-1}$ for ω^{m-1} and ω^m respectively and obtain $s\omega'^m = 0$; we say that ω^m has a trivial descent;

(b) or ω^{m-1} is nontrivial in $H^{*,m-1}(s|d)$; then there is no way to make ω^m s-invariant by adding a trivial solution to it; we say that ω^m has a nontrivial descent.

In case (b), we treat $s\omega^{m-1} + d\omega^{m-2} = 0$ as Eq. (9.2) before: acting with s on it gives $d(s\omega^{m-2}) = 0$; if $m - 2 \neq 0$, this implies $s\omega^{m-2} + d\omega^{m-3} = 0$ for some $(m-3)$-form thanks to (9.1). Again there are two possibilities: either ω^{m-2} is trivial and can be removed through suitable redefinitions such that $s\omega'^{m-1} = 0$; or it is nontrivial. In the latter case one continues the procedure until one arrives at $s\omega^{\underline{m}} = 0$ at some nonvanishing form-degree $\underline{m}$ (possibly after suitable redefinitions), or until the form-degree drops to zero and one gets the equation $d(s\omega^0) = 0$.

From the equation $d(s\omega^0) = 0$, one derives, using once again (9.1), $s\omega^0 = \alpha$ for some constant $\alpha \in \mathbb{R}$. If one assumes that the equations of motion are consistent - which one better does! - , then α must vanish and the conclusion is the same as in the previous case. This is seen by decomposing $s\omega^0 = \alpha$ into pieces with definite antifield number and pure ghost number. Since α is a pure number and has thus vanishing antifield number and pure ghost number, the decomposition yields in particular the equation $\delta a = \alpha$ where a is the piece contained in ω^0 which has antifield number 1 and pure ghost number 0. Due to $\delta a \approx 0$, this makes only sense if $\alpha = 0$ because otherwise the equations of motion would be inconsistent (as one could have, e.g., $0 = 1$ on-shell[9]).

---

[9]Such an inconsistency would arise, for instance, if one had a neutral scalar field Φ with Lagrangian $L = \Phi$. This Lagrangian yields the equation of motion $1 = 0$ and must be excluded. Having $1 = s\omega$ would of course completely kill the cohomology of s. We have not investigated whether inconsistent Lagrangians (in the above sense) are eliminated by the general conditions imposed on L in the introduction, and so, we make the assumption separately. Note that a similar difficulty does not arise in the descent associated with γ; an equation like $1 = \gamma\omega$ is simply impossible, independently of the Lagrangian, because 1 has vanishing pure ghost number while $\gamma\omega$ has pure ghost number equal to or greater than 1. For s, the relevant grading is the total ghost number and can be negative.

We conclude: when (9.1) holds, Eq. (9.2) implies in physically meaningful theories that there are forms ω^p, $p = \underline{m}, \ldots, m$ fulfilling

$$s\omega^p + d\omega^{p-1} = 0 \quad \text{for} \quad p = \underline{m} + 1, \ldots, m \ , \quad s\omega^{\underline{m}} = 0 \tag{9.4}$$

with $\underline{m} \in \{0, \ldots, m\}$. Eqs. (9.4) are called the descent equations. We call the forms ω^p, $p = \underline{m}, \ldots, m - 1$ descendants of ω^m, and $\omega^{\underline{m}}$ the bottom form of the descent equations.

Furthermore we have seen that ω^m and its descendants are determined only up to coboundaries of $H(s|d)$. In fact, for given cohomology class $H^{*,m}(s|d)$ represented by ω^m, this is the only ambiguity in the solution of the descent equations, modulo constant forms at form-degree 0. This is so because a trivial solution of the consistency condition can only have trivial descendants, except that ω^0 can contain a constant. Indeed, assume that ω^p is trivial, $\omega^p = s\eta^p + d\eta^{p-1}$. Inserting this in $s\omega^p + d\omega^{p-1} = 0$ gives $d(\omega^{p-1} - s\eta^{p-1}) = 0$ and thus, by (9.1), $\omega^{p-1} = s\eta^{p-1} + d\eta^{p-2} + \delta_0^{p-1}\alpha$ where $\alpha \in \mathbb{R}$ can occur only when $p - 1 = 0$. Hence, when ω^p is trivial, its first descendant ω^{p-1} is necessarily trivial too, except for a possible pure number when $p = 1$. By induction this applies to all further descendants too.

Shortest descents. The ambiguity in the solution of the descent equations implies in particular that all nonvanishing forms which appear in the descent equations can be chosen such that none of them is trivial in $H(s|d)$ because otherwise we can "shorten" the descent equations. In particular, there is thus a "shortest descent" (i.e., a maximal value of $\underline{m}$) for every nontrivial cohomology class $H^{*,m}(s|d)$. A shortest descent is realized precisely when all the forms in the descent equations are nontrivial. An equivalent characterization of a shortest descent is that the bottom form $\omega^{\underline{m}}$ is nontrivial in $H^{*,\underline{m}}(s|d)$ if $\underline{m} > 0$, respectively that it is nontrivial in $H^{*,0}(s|d)$ even up to a constant if $\underline{m} = 0$ (i.e., that $\omega^{\underline{m}} \neq s\eta^{\underline{m}} + d\eta^{\underline{m}-1} + \delta_0^{\underline{m}-1}\alpha$, $\alpha \in \mathbb{R}$). The latter statement holds because the triviality of any nonvanishing form in the descent equations implies necessarily that all its descendants, and thus in particular $\omega^{\underline{m}}$, are trivial too except for a number that can contribute to ω^0. Of course, the shortest descent is not unique since one may still make trivial redefinitions which do not change the length of a descent.

9.3 Lifts and obstructions

We have seen that the bottoms $\omega^{\underline{m}}$ of the descent equations associated with solutions ω^m of the consistency conditions $s\omega^m + d\omega^{m-1} = 0$ are cocycles of s, $s\omega^{\underline{m}} = 0$, which are non trivial in $H^{*,0}(s|d)$ (even up to a constant if $\underline{m} = 0$). In particular, they are non trivial in $H(s)$.

One can conversely ask the following questions. Given a non trivial cocycle of $H(s)$: (i) Is it trivial in $H(s|d)$?; (ii) Can it be viewed as bottom of a non trivial descent? These questions were raised for the first time in [96, 97] and turn out to contain the key to the calculation of $H(s|d)$ in theories of the Yang-Mills type.

We say that an s-cocycle ω^p can be "lifted" k times if there are forms $\omega^{p+1}, \ldots, \omega^{p+k}$ such that $d\omega^p + s\omega^{p+1} = 0, \ldots, d\omega^{p+k-1} + s\omega^{p+k} = 0$. Contrary to the descent, which is never obstructed, the lift of an element of $H(s)$ can be obstructed because the cohomology of s is non trivial. Let a be an s-cocycle and let us try to construct an element "above it". To that end, one must compute da and see whether it is s-exact. It is clear that da is s-closed; the obstructions to it being s-exact are thus in $H(s)$. Two things can happen. Either da is not s-exact,

$$da = m \tag{9.5}$$

with m a non trivial cocycle of $H(s)$. Or da is s-exact, in which case one has

$$da + sb = 0 \tag{9.6}$$

for some b. Of course, b is defined up to a cocycle of s.

In the first case, it is clear that "the obstruction" m to lifting a, although non trivial in $H(s)$ is trivial in $H(s|d)$. Furthermore a itself cannot be trivial in $H(s|d)$ since trivial elements $a = su + dv$ can always be lifted $(da = s(-du))$.

In the second case, one may try to lift a once more. Thus one computes db. Again, it is easy to verify that db is an s-cocycle. Therefore, either db is not s-exact,

$$db + sc = n \qquad \text{"Case A"} \tag{9.7}$$

for some non trivial cocycle n of $H(s)$ (we allow here for the presence of the exact term sc - which can be absorbed in n - because usually, one has natural representatives of the classes of $H(s)$, and db may differ from such a representative n by an s-exact term). Or db is s-exact,

$$db + sc = 0 \qquad \text{"Case B"} \tag{9.8}$$

for some c.

Note that in case A, b is defined up to the addition of an s-cocycle, so the "obstruction" n to lifting a a second time is really present only if n cannot be written as $dt + sq$ where t is an s-cocycle, i.e., if n is not in fact the obstruction to the first lift of some s-cocycle. The obstructions to second lifts are therefore in the space $H(s)/\operatorname{Im} d$ of the cohomology of s quotientized by the space of obstructions to first lifts. If the obstruction to lifting a a second time is really present, then a is clearly non trivial in $H(s|d)$. And in any case, n is trivial in $H(s|d)$.

In case B, one can continue and try to lift a a third time. This means computing dc. The analysis proceeds as above and is covered by the results of the next subsection.

9.4 Length of chains and structure of $H(s|d)$

By following the above procedure, one can construct a basis of $H(s)$ which displays explicitly the lift structure and the obstructions.

Theorem 9.1 *If $H^p(d, \Omega) = \delta_0^p \mathbb{R}$ for $p = 0, \ldots, n-1$ and the equations of motions are consistent, there exists a basis*

$$\{[1], [h_{i_r}^0], [\hat{h}_{i_r}], [e_{\alpha_s}^0]\} \tag{9.9}$$

of $H(s)$ such that the representatives fulfill

$$\begin{aligned} sh_{i_r}^{r+1} + dh_{i_r}^r &= \hat{h}_{i_r}, \\ sh_{i_r}^r + dh_{i_r}^{r-1} &= 0, \\ &\vdots \\ sh_{i_r}^1 + dh_{i_r}^0 &= 0, \\ sh_{i_r}^0 &= 0 \end{aligned} \tag{9.10}$$

and

$$\begin{aligned} \text{form degree } e_{\alpha_s}^s &= n, \\ se_{\alpha_s}^s + de_{\alpha_s}^{s-1} &= 0, \\ &\vdots \\ se_{\alpha_s}^1 + de_{\alpha_s}^0 &= 0, \\ se_{\alpha_s}^0 &= 0, \end{aligned} \tag{9.11}$$

for some forms $h_{i_r}^q$, $q = 1, \ldots, r+1$ and $e_{\alpha_s}^p$, $p = 1, \ldots, s$. Here, $[a]$ denotes the class of the s-cocycle a in $H(s)$.

We recall that a set $\{f_A\}$ of s-cocycles is such that the set $\{[f_A]\}$ forms a basis of $H(s)$ if and only if the following two properties hold: (i) any s-cocycle is a linear combination of the f_A's, up to an s-exact term; and (ii) if $\lambda^A f_A = sg$, then the coefficients λ^A all vanish.

The elements of the basis (9.9) have the following properties: The $h_{i_r}^0$ can be lifted r times, until one hits an obstruction given by $\hat{h}_{i_r}$. By contrast, the $e_{\alpha_s}^0$ can be lifted up to maximum degree without meeting any obstruction. We stress that the superscripts of $h_{i_r}^q$ and $e_{\alpha_s}^p$ in the above theorem do *not* indicate the form-degree but the increase of the form-degree relative to $h_{i_r}^0$ and $e_{\alpha_s}^0$ respectively. The form-degree of $h_{i_r}^0$ is not determined by the above formulae except that it is smaller than $n - r$. $e_{\alpha_s}^0$ has form-degree $n - s$.

We shall directly construct bases with such properties in the Yang-Mills setting, for various (sub)algebras fulfilling (9.1), thereby proving explicitly their existence in the concrete cases relevant for our purposes. The proof of the theorem in the general case is given in appendix 9.A following [142]. We refer the interested reader to the pioneering work of [96, 97, 98] for a proof involving more powerful homological tools ("exact couples").

For a basis of $H(s)$ with the above properties, the eqs (9.10) provide optimum lifts of the $h_{i_r}^0$. The $\hat{h}_{i_r}$ represent true obstructions; by using the ambiguities in the successive lifts of $h_{i_r}^0$, one cannot lift $h_{i_r}^0$ more than r times. This is seen by using

74

a recursive argument. It is clear that the $h^0_{i_0}$ cannot be lifted at all since the $\hat{h}_{i_0}$ are independent in $H(s)$. Consider next $h^0_{i_1}$ and the corresponding chain, $sh^0_{i_1} = 0$, $dh^0_{i_1} + sh^1_{i_1} = 0$, $dh^1_{i_1} + sh^2_{i_1} = \hat{h}_{i_1}$. Suppose that the linear combination $\alpha^{i_1} h^0_{i_1}$ could be lifted more than once, which would occur if and only if $\alpha^{i_1} \hat{h}_{i_1}$ was the obstruction to the single lift of an s-cocycle, i.e., $\alpha^{i_1} \hat{h}_{i_1} = da + sb$, $sa = 0$. Since (9.9) provides a basis of $H(s)$, one could expand a in terms of $\{1, h^0_{i_r}, \hat{h}_{i_r}, e^0_{\alpha_s}\}$ (up to an s-exact term), $a = \alpha^{i_0} h^0_{i_0} + \cdots$. Computing da using (9.10) and (9.11), and inserting the resulting expression into $\hat{h}_{i_1} = da + sb$, one gets $\hat{h}_{i_1} = \alpha^{i_0} \hat{h}_{i_0} + s(\cdot)$, leading to a contradiction since the $\hat{h}_{i_1}$ and $\hat{h}_{i_0}$ are independent in cohomology. The argument can be repeated in the same way for bottoms leading to longer lifts and is left to the reader.

It follows from this analysis that the $h^0_{i_r}$ are s-cocycles that are non-trivial in $H(s|d)$ - while, of course, the $\hat{h}_{i_r}$ are trivial. In fact, the advantage of a basis of the type of (9.9) for $H(s)$ is that it gives immediately the cohomology of $H(s|d)$.

Theorem 9.2 *If $\{[1], [h^0_{i_r}], [\hat{h}_{i_r}], [e^0_{\alpha_s}]\}$ is a basis of $H(s)$ with the properties of theorem 9.1, then an associated basis of $H(s|d)$ is given by*

$$\{[1], [h^q_{i_r}], [e^p_{\alpha_s}] : q = 0, \dots, r, \ p = 0, \dots, s\} \tag{9.12}$$

where in this last list, $[\ ]$ denotes the class in $H(s|d)$.

Proof: The proof is given in the appendix 9.B.

The theorem shows in particular that the $e^0_{\alpha_s}$, just like the $h^0_{i_r}$, are non trivial s-cocycles that remain non trivial in $H(s|d)$. This property holds even though they can be lifted all the way to maximum form-degree n (while the lifts of the $h^0_{i_r}$ are obstructed before). The basis of $H(s|d)$ is given by the non trivial bottoms $h^0_{i_r}$, $e^0_{\alpha_s}$ and all the terms in the descent above them (up to the obstructions in the case of $h^0_{i_r}$).

9.5   Descent equations with weaker assumptions on $H(d)$

So far we have assumed that (9.1) holds. We shall now briefly discuss the modifications when (9.1) is replaced by an appropriate weaker prerequisite. Applications of these modifications are described below.

Let $\{\alpha^p_{i_p}\}$ be a set of p-forms representing $H^p(d)$ (hence, the superscript of the α's indicates the form-degree, the subscript i_p labels the inequivalent α's for fixed p). That is, any d-closed p-form is a linear combination of the $\alpha^p_{i_p}$ with constant coefficients λ^{i_p}, modulo a d-exact form,

$$d\omega^p = 0 \quad \Leftrightarrow \quad \omega^p = \lambda^{i_p} \alpha^p_{i_p} + d\eta^{p-1} , \quad d\alpha^p_{i_p} = 0, \tag{9.13}$$

and no nonvanishing linear combination of the $\alpha^p_{i_p}$ is d-exact.

Now, in order to derive the descent equations, it is quite crucial that the equality $d(s\omega^p) = 0$ implies $s\omega^p + d\omega^{p-1} = 0$. However, if $H^p(d)$ is non trivial, we must allow for a combination of the $\alpha^p_{i_p}$ on the right hand side, and this spoils the descent. This

phenomenon cannot occur if no $\alpha_{i_p}^p$ is s-exact modulo d. Thus, in order to be able to use the tools provided by the descent equations, we shall assume that the forms non trivial in $H^p(d)$ remain non trivial in $H(s|d)$ for $p < n$. More precisely, it is assumed that the $\alpha_{i_p}^p$ with $p < n$ have the property that no nonvanishing linear combination of them is trivial in $H(s|d)$,

$$ p < n : \quad \lambda^{i_p} \alpha_{i_p}^p = s\eta^p + d\eta^{p-1} \quad \Rightarrow \quad \lambda^{i_p} = 0 \quad \forall\, i_p . \tag{9.14} $$

This clearly implies the central property of the descent,

$$ p < n : \quad d(s\omega^p) = 0 \quad \Leftrightarrow \quad \exists\, \omega^{p-1} : \quad s\omega^p + d\omega^{p-1} = 0. \tag{9.15} $$

Indeed, by (9.13), $d(s\omega^p) = 0$ implies $s\omega^p + d\omega^{p-1} = \lambda^{i_p}\alpha_{i_p}^p$ for some ω^{p-1} and λ^{i_p}. (9.14) implies now $\lambda^{i_p} = 0$ whenever $p < n$.

When (9.14) holds, the discussion of the descent equations proceeds as before. The only new feature is the fact that the $\alpha_{i_p}^p$ yield additional nontrivial classes of $H(s|d)$ and $H(s)$. Indeed, $d\alpha_{i_p}^p = 0$ implies $d(s\alpha_{i_p}^p) = 0$ and thus, due to (9.15), $s\alpha_{i_p}^p + d\alpha_{i_p}^{p-1} = 0$ for some $\alpha_{i_p}^{p-1}$ (which may vanish). Hence, the $\alpha_{i_p}^p$ are cocycles of $H(s|d)$ and they are nontrivial by (9.14). In particular, some of the $\alpha_{i_p}^p$ may have a nontrivial descent. Theorems (9.1) and (9.2) get modified because the $\alpha_{i_p}^p$ and their nontrivial descendants (if any) represent classes of $H(s|d)$ in addition to the $[h_{i_r}^q]$ and $[e_{\alpha_s}^p]$ ($q = 0, \ldots, r$, $p = 0, \ldots, s$), while $H(s)$ receives additional classes represented by nontrivial bottom forms corresponding to the $\alpha_{i_p}^p$.

Applications. 1. The above discussion is important to cover the space of local forms which are not allowed to depend explicitly on the x^μ. Indeed, in that space $H^p(d)$ is represented for $p < n$ by the constant forms $c_{\mu_1 \ldots \mu_p} dx^{\mu_1} \ldots dx^{\mu_p}$ (theorem 4.3). Now, the equations of motion may be such that some of the constant forms become trivial in $H(s|d)$. We know that this cannot happen in form-degree zero, but nothing prevents it from happening in higher form-degrees. For instance, for a single abelian gauge field with Lagrangian $L = (-1/4)F^{\mu\nu}F_{\mu\nu} + k^\mu A_\mu$ with constant k^μ, the equations of motion read $\partial_\nu F^{\nu\mu} + k^\mu = 0$ and imply that the constant $(n-1)$-form $\star k = \frac{1}{(n-1)!} dx^{\mu_1} \ldots dx^{\mu_{n-1}} \epsilon_{\mu_1 \ldots \mu_n} k^{\mu_n}$ is trivial in $H(s|d)$, $s \star A^* + d \star F + (-)^n \star k = 0$. Hence, for this Lagrangian (9.14) is not fulfilled. Of course, the example is academic and the Lagrangian is not Lorentz-invariant. The triviality of $\star k$ in $H(s|d)$ is a consequence of the linear term in the Lagrangian.

(9.14) is fulfilled in the space of x-independent local forms for Lagrangians having no linear part in the fields, for which the equations of motion reduce identically to $0 = 0$ when the fields are set to zero. It is also fulfilled if one restricts one's attention to the space of Poincaré invariant local forms. [And it is also trivially fulfilled in the space of all local forms with a possible explicit x-dependence, as we have seen]. For this reason, (9.14) does not appear to be a drastic restriction in the space of x-independent local forms.

Note that the classification of the elements (and the number of these elements) in a basis of $H(s)$ having the properties of theorem 9.1 depends on the context.

76

For instance, for a single abelian gauge field with ghost C, $dx^\mu C$ is non trivial in the algebra of local forms with no explicit x-dependence, and so can be taken as a $h^0_{i_1}$; but it becomes trivial if one allows for an explicit x-dependence, $dx^\mu C = d(x^\mu C) + s(x^\mu A)$, and so can be regarded in that case as a $\hat h_{i_0}$.

2. Another instance where the descent equations with the above assumption on $H(d)$ play a rôle is the cohomology $H(\delta|d, \mathcal{I}_\chi)$, where $\mathcal{I}_\chi$ is the space of $\mathcal{G}$-invariant local forms depending only on the variables χ^u_Δ defined in section 8.1 and on the x^μ and dx^μ. $H^p(d, \mathcal{I}_\chi)$ is represented for $p < n$ by the "characteristic classes", i.e., by $\mathcal{G}$-invariant polynomials $P(F)$ in the curvature 2-forms F^I [62, 101] (these polynomials are d-exact in the space of all local forms, but they are not d-exact in $\mathcal{I}_\chi$). Using this result on $H^p(d, \mathcal{I}_\chi)$, one proves straightforwardly by means of the descent equations that $H(\delta|d, \mathcal{I}_\chi)$ is isomorphic to the characteristic cohomology in the space of gauge invariant local forms ("equivariant characteristic cohomology") which will play an important rôle in the analysis of the consistency condition performed in section 11.

3. Finally we mention that the above discussion was used within the computation of the local BRST cohomology in Einstein-Yang-Mills theory [25]. In that case $H^p(d)$ is nontrivial in certain form-degrees $p < n$ due to the nontrivial De Rham cohomology of the manifold in which the vielbein fields take their values. The corresponding $\alpha^p_{i_p}$ fulfill (9.14) and have a nontrivial descent (in contrast, the constant forms and the characteristic classes met in the two instances discussed before do not descend).

9.6 Cohomology of $s + d$

The descent equations establish a useful relation between $H^{*,n}(s|d)$, $H(d)$ and the cohomology of the differential $\tilde s$ that combines s and d,

$$\tilde s = s + d. \tag{9.16}$$

Note that $\tilde s$ squares to zero thanks to $s^2 = sd + ds = d^2 = 0$ and defines thus a cohomology $H(\tilde s)$ in the space of formal sums of forms with various degrees,

$$\tilde\omega = \sum_p \omega^p \ .$$

We call such sums total forms. Cocycles of $H(\tilde s)$ are defined through

$$\tilde s \tilde\omega = 0, \tag{9.17}$$

while coboundaries take the form $\tilde\omega = \tilde s \tilde\eta$. Consider now a cocycle $\tilde\omega = \sum_{p=\underline{m}}^m \omega^p$ of $H(\tilde s)$. The cocycle condition (9.17) decomposes into

$$d\omega^m = 0, \quad s\omega^m + d\omega^{m-1} = 0, \quad \dots \quad , \quad s\omega^{\underline{m}} = 0.$$

These are the descent equations with top-form ω^m and the supplementary condition $d\omega^m = 0$. The extra condition is of course automatically fulfilled if $m = n$. Hence, assuming that Eq. (9.14) holds, every cohomology class of $H^{*,n}(s|d)$ gives rise to a cohomology class of $H(\tilde s)$. Evidently, this cohomology class of $H(\tilde s)$ is nontrivial if

its counterpart in $H^{*,n}(s|d)$ is nontrivial (since $\tilde{\omega} = \tilde{s}\tilde{\eta}$ implies $\omega^n = s\eta^n + d\eta^{n-1}$). All additional cohomology classes of $H(\tilde{s})$ (i.e. those without counterpart in $H^{*,n}(s|d)$) correspond precisely to the cohomology classes of $H(d)$ in form degrees $< n$ and thus to the $\alpha_{i_p}^p$ with $p < n$ in the notation used above. Indeed, $d\omega^m = 0$ implies $\omega^m = \lambda^{i_m}\alpha_{i_m}^m + d\eta^{m-1}$ by (9.13). Furthermore, as shown above, each $\alpha_{i_m}^m$ gives rise to a solution of the descent equations and thus to a representative of $H(\tilde{s})$. These representatives are nontrivial and inequivalent because of (9.14). This yields the following isomorphism whenever (9.14) holds:

$$H(\tilde{s}) \simeq H^{*,n}(s|d) \oplus H^{n-1}(d) \oplus \cdots \oplus H^0(d). \tag{9.18}$$

This isomorphism can be used, in particular, to determine $H^{*,n}(s|d)$ by computing $H(\tilde{s})$. The cohomology of s modulo d in lower form degrees is however not given by $H(\tilde{s})$.

9.7 Appendix 9.A: Proof of theorem 9.1

In order to prove the existence of (9.9), we note first that the space Ω of local forms admits the decomposition $\Omega = E_0 \oplus G \oplus sG$ for some space $E_0 \simeq H(s, \Omega)$ and some space G.

Following [97, 197, 142], we define recursively differentials $d_r : H(d_{r-1}) \longrightarrow H(d_{r-1})$ for $r = 0, \ldots, n$, with $d_{-1} \equiv s$ as follows: the space $H(d_{r-1})$ is given by equivalence classes $[X]_{r-1}$ of elements $X \in \Omega$ such that there exist $c_1, \ldots, c_r$ satisfying $sX = 0, dX + sc_1 = 0, \ldots, dc_{r-1} + sc_r = 0$ (i.e., X can be lifted at least r times). We define $c_0 \equiv X$. The equivalence relation $[\;]_{r-1}$ is defined by $X \sim_{r-1} Y$ iff $X - Y = sZ + d(v_0^0 + \ldots + v_{r-1}^{r-1})$ where $sv_j^j + dv_j^{j-1} = 0, \ldots, sv_j^0 = 0$. d_r is defined by $d_r[X]_{r-1} = [dc_r]_{r-1}$. Let us check that this definition makes sense. Applying d to $dc_{r-1} + sc_r = 0$ gives $sdc_r = 0$, so that $[dc_r]_{r-1} \in H(d_{r-1})$ (one can simply choose the required $c_1', \ldots, c_r'$ to be zero because of $d(dc_r) = 0$). Furthermore, $[dc_r]_{r-1}$ does not depend on the ambiguity in the definition of $X, c_1, \ldots, c_r$. Indeed, the ambiguity in the definition of X is $sZ + d(v_0^0 + \ldots + v_{r-1}^{r-1})$, the ambiguity in the definition of c_1 is $dZ + m_{r-1}^0$, where $sm_{r-1}^0 = 0, dm_{r-1}^0 + sm_{r-1}^1 = 0, \ldots, dm_{r-1}^{r-2} + sm_{r-1}^{r-1} = 0$. Similarly, the ambiguity in c_2 is $m_{r-1}^1 + m_{r-2}^0$, where $sm_{r-2}^0 = 0, dm_{r-2}^0 + sm_{r-2}^1 = 0, \ldots, dm_{r-2}^{r-3} + sm_{r-2}^{r-2} = 0$. Going on in the same way, one finds that the ambiguity in c_r is $m_{r-1}^{r-1} + \ldots + m_0^0$, where $sm_j^j + dm_j^{j-1} = 0$. This means that the ambiguity in dc_r is $d(m_{r-1}^{r-1} + \ldots + m_0^0)$, which is zero in $H(d_{r-1})$. (For $r = 0$, the ambiguity in $dc_0 \equiv dX$ is $-sdZ$, which is zero in $H(s)$.) Finally, $d_r^2 = 0$ follows from $d^2 = 0$.

The cocycle condition $d_r[X]_{r-1} = 0$ reads explicitly $dc_r = sW + d(w_0^0 + \ldots + w_{r-1}^{r-1})$, where $sw_j^j + dw_j^{j-1} = 0, \ldots, sw_j^0 = 0$. This means that $sX = 0, dX + s(c_1 - w_{r-1}^0) = 0$, $d(c_1 - w_{r-1}^0) + s(c_2 - w_{r-1}^1 - w_{r-2}^0) = 0, \ldots, d(c_r - w_{r-1}^{r-1} - \ldots - w_0^0) + s(-W) = 0$. The coboundary condition $[X]_{r-1} = d_r[Y]_{r-1}$ gives $X = db_r + sZ + d(v_0^0 + \ldots + v_{r-1}^{r-1})$, where $sY = 0, dY + sb_1 = 0, \ldots, db_{r-1} + sb_r = 0$. The choices $c_1' = c_1 - w_{r-1}^0, \ldots, c_r' = c_r - w_{r-1}^{r-1} - \ldots - w_0^0, c_{r+1}' = -W$, respectively $v_j'^j = v_j^j$ for $j = 0, \ldots, r-1$ and $v_r^r = b_r$, $v_r^{r-1} = b_{r-1}, \ldots, v_r^1 = b_1, v_r^0 = Y$, show that $H(d_r)$ is defined by the same equations as $H(d_{r-1})$ with $r - 1$ replaced by r, as it should.

78

Because the maximum form degree is n, one has $d_n \equiv 0$ and the construction stops.

It is now possible to define spaces $E_r \subset E_{r-1} \subset E_0 \subset \Omega$ and spaces $F_{r-1} \subset E_{r-1} \subset \Omega$ for $r = 1, \ldots, n$ such that $E_{r-1} = E_r \oplus d_{r-1}F_{r-1} \oplus F_{r-1}$ with $E_r \simeq H(d_{r-1})$. This leads to the decomposition

$$E_0 = E_n \oplus (d_{n-1}F_{n-1} \oplus F_{n-1}) \oplus (d_{n-1}F_{n-1} \oplus F_{n-1}) \oplus \ldots \oplus (d_0 F_{n-1} \oplus F_0). \quad (9.19)$$

Note that this decomposition involves choices of representatives (E_i) and of supplementary subspaces (F_i) and is not "canonical". But it does exist.

The $e^0_{\alpha_s}$ are elements of a basis of E_n. They can be lifted s times to form degree n, i.e., they are of form degree $n - s$. The element 1 also belongs to E_n. The $\hat{h}_{i_r}$ and $h^0_{i_r}$ are elements of a basis of $d_r F_r$ and F_r respectively. $\square$

9.8   Appendix 9.B: Proof of theorem 9.2

We verify here that if a basis (9.9) with properties (9.10) and (9.11) exists, then a basis of $H(s|d)$ is given by (9.12).

The set (9.12) is complete.   Suppose that $s\omega^l + d\omega^{l-1} = 0$, $s\omega^{l-1} + d\omega^{l-2} = 0, \ldots,$ $s\omega^0 = 0$ where the superscript indicates the length of the descent (number of lifts) rather than the form-degree. We shall prove that then

$$\omega^l = C + \sum_{0 \le q \le l} \sum_{r \ge l-q} \lambda_q^{i_r} h_{i_r}^{l-q} + \sum_{0 \le p \le l} \sum_{s \ge l-p} \mu_p^{\alpha_s} e_{\alpha_s}^{l-p} + s\eta^l + d[\eta^{l-1} + \sum_{r \ge 0} \nu^{(l)i_r} h_{i_r}^r], \quad (9.20)$$

with C a constant and $\eta^{-1} = 0$.

To prove this, we proceed recursively in the length l of the descent. For $l = 0$, we have $s\omega^0 = 0$. Using the assumption that the h^0's, $\hat{h}$'s, e^0's and the number 1 provide a basis of $H(s)$, this gives $\omega^0 = C + \sum_{r \ge 0} \lambda_0^{i_r} h_{i_r}^0 + \sum_{s \ge 0} \mu_0^{\alpha_s} e_{\alpha_s}^0 + \sum_{r \ge 0} \nu^{(0)i_r} \hat{h}_{i_r} + s\tilde{\eta}^0 = \sum_{r \ge 0} \lambda_0^{i_r} h_{i_r}^0 + \sum_{s \ge 0} \mu_0^{\alpha_s} e_{\alpha_s}^0 + s\eta^0 + d(\sum_{r \ge 0} \nu^{(0)i_r} h_{i_r}^r)$, where $\eta^0 = \tilde{\eta}^0 + \sum_{r \ge 0} \nu^{(0)i_r} h_{i_r}^{r+1}$. This is (9.20) for $l = 0$.

We assume now that (9.20) holds for $l = L$. Then we have for $l = L + 1$, that in $s\omega^{L+1} + d\omega^L = 0$, ω^L is given by (9.20) with l replaced by L. Using (9.10) and (9.11), one gets $d\omega_L = s[-\sum_{0 \le q \le L} \sum_{r \ge L-q+1} \lambda_q^{i_r} h_{i_r}^{L-q+1} - \sum_{0 \le p \le l} \sum_{s \ge L-p+1} \mu_p^{\alpha_s} e_{\alpha_s}^{L-p+1} - d\eta^L - \sum_{0 \le q \le L} \lambda_q^{i_{L-q}} h_{i_{L-q}}^{L-q+1}] + \sum_{0 \le q \le L} \lambda_q^{i_{L-q}} \hat{h}_{i_{L-q}}$. Injecting this into $s\omega^{L+1} + d\omega^L = 0$, we find, using the properties of the basis and the first relation of (9.10), on the one hand that $\lambda_q^{i_{L-q}} = 0$, and on the other hand that $\omega^{L+1} = C + \sum_{0 \le q \le L} \sum_{r \ge L-q+1} \lambda_q^{i_r} h_{i_r}^{L-q+1} + \sum_{0 \le p \le l} \sum_{s \ge L-p+1} \mu_p^{\alpha_s} e_{\alpha_s}^{L-p+1} + \sum_{r \ge 0} \lambda_{L+1}^{i_r} h_{i_r}^0 + \sum_{s \ge 0} \mu_{L+1}^{\alpha_s} e_{\alpha_s}^0 + s\eta^{L+1} + d[\eta^L + \sum_{r \ge 0} \nu^{(L+1)i_r} h_{i_r}^r]$, which is precisely (9.20) for $l = L + 1$.

The elements of (9.12) are cohomologically independent.   Suppose now that

$$C + \sum_{0 \le q \le l} \sum_{r \ge l-q} \lambda_q^{i_r} h_{i_r}^{l-q} + \sum_{0 \le p \le l} \sum_{s \ge l-p} \mu_p^{\alpha_s} e_{\alpha_s}^{l-p} = s\tilde{\eta}^{(l)} + d\eta^{(l)}. \quad (9.21)$$

We have to show that this implies that $\lambda_q^{i_r} = 0$, $\mu_p^{\alpha_s} = 0$ and $C = 0$.

Again, we proceed recursively on the length of the descent. For $l = 0$, we have $C + \sum_{r \geq 0} \lambda_0^{i_r} h_{i_r}^0 + \sum_{s \geq 0} \mu_0^{\alpha_s} e_{\alpha_s}^0 = s\tilde{\eta}^{(0)} + d\eta^{(0)}$. Applying s and using the triviality of the cohomology of d, we get that $\eta^{(0)}$ is an s modulo d cocycle, $s\eta^{(0)} + d(\cdot) = 0$ (without constant since the equations of motion are consistent). Suppose that the descent of $\eta^{(0)}$ stops after l' steps with $0 \leq l' \leq n - 1$, i.e., $\eta^{(0)} \equiv \eta^{(0)l'}$ with $s\eta^{(0)l'} + d\eta^{(0)l'-1} = 0, \ldots, s\eta^{(0)0} = 0$ (again no constant here). It follows that $\eta^{(0)l'}$ is given by (9.20) with l replaced by l'. Evaluating then $d\eta^{(0)l'}$ in the equation for $l = 0$ and using the properties of the basis implies that $\lambda_0^{i_r} = \mu_0^{\alpha_s} = C = 0$.

Suppose now that the result holds for $l = L$. If we apply s to (9.21) at $l = L + 1$ and use the triviality of the cohomology of d we get $\sum_{0 \leq q \leq L} \sum_{r \geq L+1-q} \lambda_q^{i_r} h_{i_r}^{L-q} + \sum_{0 \leq p \leq L} \sum_{s \geq L+1-p} \mu_p^{\alpha_s} e_{\alpha_s}^{L-p} = s\eta^{(L+1)} + d(\ )$. The induction hypothesis implies then that $\lambda_q^{i_r} = 0 = \mu_p^{\alpha_s}$ for $0 \leq q \leq L$ and $0 \leq p \leq L$. This implies that the relation at $l = L+1$ reduces to $\sum_{r \geq 0} \lambda_{L+1}^{i_r} h_{i_r}^0 + \sum_{s \geq 0} \mu_{L+1}^{\alpha_s} e_{\alpha_s}^0 = s\tilde{\eta}^{(l)} + d\eta^{(l)}$. As we have already shown, this implies that $\lambda_{L+1}^{i_r} = 0 = \mu_{L+1}^{\alpha_s}$. $\square$

10 Cohomology in the small algebra

10.1 Definition of small algebra

The "small algebra" $\mathcal{B}$ is by definition the algebra of polynomials in the undifferentiated ghosts C^I, the gauge field 1-forms A^I and their exterior derivatives dC^I and dA^I.

$$\mathcal{B} = \{\text{polynomials in } C^I,\ A^I,\ dC^I,\ dA^I\}.$$

It is stable under d and s ($b \in \mathcal{B} \Rightarrow db \in \mathcal{B},\ sb \in \mathcal{B}$). This is obvious for d and holds for s thanks to

$$s\,C^I = \tfrac{1}{2}\,e\,f_{JK}{}^I C^K C^J, \quad s\,A^I = -dC^I - e\,f_{JK}{}^I C^J A^K,$$

$$s\,dC^I = -e\,f_{JK}{}^I C^J dC^K, \quad s\,dA^I = e\,f_{JK}{}^I(C^K dA^J - A^J dC^K). \tag{10.1}$$

Accordingly, the cohomological groups $H(s|d, \mathcal{B})$ of s modulo d in $\mathcal{B}$ are well defined[10].

The small algebra $\mathcal{B}$ is only a very small subspace of the complete space of all local forms (in fact $\mathcal{B}$ is finite dimensional whereas the space of all local forms is infinite dimensional). Nevertheless it provides a good deal of the BRST cohomology in Yang-Mills theories, in that it contains all the antifield-independent solutions of the consistency condition $sa + db = 0$ that descend non-trivially, in a sense to be made precise in section 11. Furthermore, it will also be proved there that the representatives of $H(s|d, \mathcal{B})$ remain nontrivial in the full cohomology, with only very few possible exceptions.

For this reason, the calculation of $H(s|d, \mathcal{B})$ is an essential part of the calculation of $H(s|d)$ in the full algebra. This calculation was done first in [96, 97] (in fact in the universal algebra defined below).

Let us briefly outline the construction of $H(s|d, \mathcal{B})$ before going into the details. It is based on an analysis of the descent equations described in Section 9. The central task in this approach is the explicit construction of a particular basis of $H(s, \mathcal{B})$ as in theorem 9.1 which provides $H(s|d, \mathcal{B})$ by theorem 9.2. Technically it is of great help that the essential steps of this construction can be carried out in a free differential algebra associated with $\mathcal{B}$. This is shown first.

10.2 Universal algebra

The free differential algebra associated with $\mathcal{B}$ is denoted by $\mathcal{A}$. It was called the "universal algebra" in [96, 97]. It has the same set of generators as $\mathcal{B}$, but these are not constrained by the condition coming from the spacetime dimension that there is no exterior form of form-degree higher than n.

Explicitly, $\mathcal{A}$ is generated by anticommuting variables $C_\mathcal{A}^I$, $A_\mathcal{A}^I$ and commuting variables $(dC)_\mathcal{A}^I$, $(dA)_\mathcal{A}^I$ which correspond to C^I, A^I, dC^I and dA^I respectively, $\mathcal{A}$ is the space of polynomials in these variables,

$$\mathcal{A} = \{\text{polynomials in } C_\mathcal{A}^I,\ A_\mathcal{A}^I,\ (dC)_\mathcal{A}^I,\ (dA)_\mathcal{A}^I\}.$$

---

[10]Note that s and γ coincide in the small algebra, which contains no antifield.

These variables are subject only to the commutation/anticommutation relations $C_\mathcal{A}^I C_\mathcal{A}^J = -C_\mathcal{A}^J C_\mathcal{A}^I$ etc but are not constrained by the further condition that the forms are zero whenever their form-degree exceeds n. Thus, the free differential algebra $\mathcal{A}$ is independent of the spacetime dimension; hence its name "universal".

The fundamental difference between $\mathcal{A}$ and $\mathcal{B}$ is that the $A_\mathcal{A}^I$, $(dC)_\mathcal{A}^I$ and $(dA)_\mathcal{A}^I$ are variables by themselves, whereas their counterparts A^I, dC^I and dA^I are composite objects containing the differentials dx^μ and jet space variables (fields and their derivatives), $A^I = dx^\mu A_\mu^I$, $dC^I = dx^\mu \partial_\mu C^I$, $dA^I = dx^\mu dx^\nu \partial_\mu A_\nu^I$.

The universal algebra $\mathcal{A}$ is infinite dimensional, whereas $\mathcal{B}$ is finite dimensional since the spacetime dimension bounds the form-degree of elements in $\mathcal{B}$. By contrast, $\mathcal{A}$ contains elements with arbitrarily high degree in the $(dC)_\mathcal{A}^I$ and $(dA)_\mathcal{A}^I$.

The usefulness of $\mathcal{A}$ for the computations in the small algebra rests on the fact that $\mathcal{A}$ and $\mathcal{B}$ are isomorphic at all form-degrees smaller than or equal to the spacetime dimension n. To make this statement precise, we first define a "p-degree" which is the form-degree in $\mathcal{A}$,

$$
p = A_\mathcal{A}^I \frac{\partial}{\partial A_\mathcal{A}^I} + (dC)_\mathcal{A}^I \frac{\partial}{\partial (dC)_\mathcal{A}^I} + 2(dA)_\mathcal{A}^I \frac{\partial}{\partial (dA)_\mathcal{A}^I} \ ,
$$

and clearly coincides with the form-degree in $\mathcal{B}$ for the corresponding objects. The p-degree is indicated by a superscript. $\mathcal{A}$ and $\mathcal{B}$ decompose into subspaces with definite p-degrees,

$$
\mathcal{B} = \bigoplus_{p=0}^{n} \mathcal{B}^p \quad , \quad \mathcal{A} = \bigoplus_{p=0}^{\infty} \mathcal{A}^p \ .
$$

We then introduce the natural mappings π^p from $\mathcal{A}^p$ to $\mathcal{B}^p$ which just replace each generator $C_\mathcal{A}^I$, $A_\mathcal{A}^I$, $(dC)_\mathcal{A}^I$, $(dA)_\mathcal{A}^I$ by its counterpart in $\mathcal{B}$,

$$
\pi^p : \begin{cases} \mathcal{A}^p \longrightarrow \mathcal{B}^p \\ a(C_\mathcal{A}, A_\mathcal{A}, (dC)_\mathcal{A}, (dA)_\mathcal{A}) \longmapsto a(C, A, dC, dA) \ . \end{cases} \tag{10.2}
$$

These mappings are defined for all p, with $\mathcal{B}^p \equiv 0$ for $p > n$. Evidently they are surjective (each element of $\mathcal{B}^p$ is in the image of π^p). The point is that they are also injective for all $p \le n$. Indeed, the kernel of π^p is trivial for $p \le n$ by the following lemma:

Lemma 10.1 *For all $p \le n$, the image of $a^p \in \mathcal{A}^p$ under π^p vanishes if and only if a^p itself vanishes,*

$$
\forall\, p \le n : \quad \pi^p(a^p) = 0 \quad \Leftrightarrow \quad a^p = 0. \tag{10.3}
$$

Lemma 10.1 holds due to the fact that the components A_μ^I, $\partial_\mu C^I$, $\partial_{[\mu} A_{\nu]}^I$ of A^I, dC^I, dA^I are algebraically independent variables in the jet space (see section 4) and occur in elements of $\mathcal{B}$ always together with the corresponding differential(s). For instance, consider $a^p = k_{I_1 \dots I_p} A_\mathcal{A}^{I_1} \dots A_\mathcal{A}^{I_p}$ where the $k_{I_1 \dots I_p}$ are constant coefficients. Without loss of generality the $k_{I_1 \dots I_p}$ are antisymmetric since the $A_\mathcal{A}^I$ anticommute. Hence, one has $a^p = p! \sum_{I_i < I_{i+1}} k_{I_1 \dots I_p} A_\mathcal{A}^{I_1} \dots A_\mathcal{A}^{I_p}$ and $\pi^p(a^p) =$

$p! \sum_{I_i < I_{i+1}} k_{I_1 \ldots I_p} A^{I_1} \ldots A^{I_p}$. Vanishing of $\pi^p(a^p)$ in the jet space requires in particular that the coefficients of $(dx^1 A_1^{I_1}) \ldots (dx^p A_p^{I_p})$ vanish, for all sets $\{I_1, \ldots, I_p : I_i < I_{i+1}\}$. This requires vanishing of all coefficients $k_{I_1 \ldots I_p}$, and thus $a^p = 0$. The general case is a straightforward extension of this example.

We can thus conclude:

Corollary 10.1 *The mappings π^p are bijective for all $p \leq n$ and establish thus isomorphisms between $\mathcal{A}^p$ and $\mathcal{B}^p$ for all $p \leq n$.*

In order to use these isomorphisms, $\mathcal{A}$ is equipped with differentials $s_{\mathcal{A}}$ and $d_{\mathcal{A}}$ which are the counterparts of s and d. Accordingly, $s_{\mathcal{A}}$ and $d_{\mathcal{A}}$ are defined on the generators of $\mathcal{A}$ as follows:

$$
\begin{array}{c||c|c}
Z_{\mathcal{A}} & s_{\mathcal{A}} Z_{\mathcal{A}} & d_{\mathcal{A}} Z_{\mathcal{A}} \\
\hline
C_{\mathcal{A}}^I & \frac{1}{2} e\, f_{JK}{}^I C_{\mathcal{A}}^K C_{\mathcal{A}}^J & (dC)_{\mathcal{A}}^I \\
A_{\mathcal{A}}^I & -(dC)_{\mathcal{A}}^I - e\, f_{JK}{}^I C_{\mathcal{A}}^J A_{\mathcal{A}}^K & (dA)_{\mathcal{A}}^I \\
(dC)_{\mathcal{A}}^I & -e\, f_{JK}{}^I C_{\mathcal{A}}^J (dC)_{\mathcal{A}}^K & 0 \\
(dA)_{\mathcal{A}}^I & e\, f_{JK}{}^I (C_{\mathcal{A}}^K (dA)_{\mathcal{A}}^J - A_{\mathcal{A}}^J (dC)_{\mathcal{A}}^K) & 0
\end{array}
\qquad (10.4)
$$

The definition of $s_{\mathcal{A}}$ is extended to all polynomials in the generators by the graded Leibniz rule, $s_{\mathcal{A}}(ab) = (s_{\mathcal{A}} a)b + (-)^{\epsilon_a} a(s_{\mathcal{A}} b)$, and the definition of $d_{\mathcal{A}}$ is analogously extended. With these definitions, $s_{\mathcal{A}}$ and $d_{\mathcal{A}}$ are anticommuting differentials in $\mathcal{A}$,

$$s_{\mathcal{A}}^2 = d_{\mathcal{A}}^2 = s_{\mathcal{A}} d_{\mathcal{A}} + d_{\mathcal{A}} s_{\mathcal{A}} = 0.$$

One can therefore define $H(d_{\mathcal{A}}, \mathcal{A})$ and $H(s_{\mathcal{A}}, \mathcal{A})$ (and $H(s_{\mathcal{A}}|d_{\mathcal{A}}, \mathcal{A})$ as well).

By construction one has

$$\forall p: \quad \pi^p \circ s_{\mathcal{A}} = s \circ \pi^p, \quad \pi^{p+1} \circ d_{\mathcal{A}} = d \circ \pi^p \quad \text{(on } \mathcal{A}^p\text{)}. \qquad (10.5)$$

This just means that π^p and π^{p+1} map $s_{\mathcal{A}} a^p$ and $d_{\mathcal{A}} a^p$ to $s\pi^p(a^p)$ and $d\pi^p(a^p)$ respectively, for every $a^p \in \mathcal{A}^p$ (note that $s\pi^p(a^p)$ and $d\pi^p(a^p)$ vanish for $p > n$ and $p \geq n$ respectively).

Now, π^p can be inverted for $p \leq n$ since it is bijective (see corollary 10.1). Hence, (10.5) gives

$$
\begin{aligned}
\forall p \leq n: \quad s &= \pi^p \circ s_{\mathcal{A}} \circ (\pi^p)^{-1} && \text{(on } \mathcal{B}^p\text{)}, \\
d &= \pi^{p+1} \circ d_{\mathcal{A}} \circ (\pi^p)^{-1} && \text{(on } \mathcal{B}^p\text{)}, \\
s_{\mathcal{A}} &= (\pi^p)^{-1} \circ s \circ \pi^p && \text{(on } \mathcal{A}^p\text{)}, \\
\forall p < n: \quad d_{\mathcal{A}} &= (\pi^{p+1})^{-1} \circ d \circ \pi^p && \text{(on } \mathcal{A}^p\text{)}
\end{aligned}
$$

where the last relation holds only for $p < n$ (but not for $p = n$) since π^{n+1} has no inverse due to $\pi^{n+1}(\mathcal{A}^{n+1}) = 0$. We conclude:

Corollary 10.2 *$H(s, \mathcal{B}^p)$ is isomorphic to $H(s_{\mathcal{A}}, \mathcal{A}^p)$ for all $p \leq n$, and $H(d, \mathcal{B}^p)$ is isomorphic to $H(d_{\mathcal{A}}, \mathcal{A}^p)$ for all $p < n$. At the level of the representatives $[b^p] \in \mathcal{B}^p$*

and $[a^p] \in \mathcal{A}^p$ of these cohomologies, the isomorphisms are given by the mappings (10.2),

$$\forall\, p \leq n: \quad H(s, \mathcal{B}^p) \simeq H(s_\mathcal{A}, \mathcal{A}^p), \quad [b^p] = \pi^p([a^p]) \ ;$$
$$\forall\, p < n: \quad H(d, \mathcal{B}^p) \simeq H(d_\mathcal{A}, \mathcal{A}^p), \quad [b^p] = \pi^p([a^p]) \ . \tag{10.6}$$

In the following this corollary will be used to deduce the cohomologies of s and d in $\mathcal{B}$ (except for $H(d, \mathcal{B}^n)$) from their counterparts in $\mathcal{A}$.

10.3 Cohomology of d in the small algebra

It is very easy to compute $H(d_\mathcal{A}, \mathcal{A})$ since all generators of $\mathcal{A}$ group in contractible pairs for $d_\mathcal{A}$ given by $(C_\mathcal{A}^I, (dC)_\mathcal{A}^I)$ and $(A_\mathcal{A}^I, (dA)_\mathcal{A}^I)$, see (10.4). One concludes by means of a contracting homotopy (cf. Section 2.7):

Lemma 10.2 *$H(d_\mathcal{A}, \mathcal{A}^p)$ vanishes for all $p > 0$ and $H(d_\mathcal{A}, \mathcal{A}^0)$ is represented by the constants (pure numbers),*
$$H(d_\mathcal{A}, \mathcal{A}^p) = \delta_0^p\, \mathbb{R} \ . \tag{10.7}$$

By corollary 10.2, this implies

Corollary 10.3 *$H(d, \mathcal{B}^p)$ vanishes for $0 < p < n$ and $H(d, \mathcal{B}^0)$ is represented by the constants,*
$$H(d, \mathcal{B}^p) = \delta_0^p\, \mathbb{R} \quad \textit{for} \quad p < n. \tag{10.8}$$

Corollary 10.3 guarantees that we can apply theorems 9.1 and 9.2 of Section 9 to compute $H(s|d, \mathcal{B})$ since (9.1) holds.

10.4 Cohomology of $s_\mathcal{A}$

The cohomology of $s_\mathcal{A}$ can be derived using the techniques of Section 8. One first gets rid of the exterior derivatives of the ghosts and of the gauge potentials by introducing a new basis of generators of $\mathcal{A}$, which are denoted by $\{u^I, v^I, w^i\}$ with

$$u^I = A_\mathcal{A}^I \ , \quad v^I = -(dC)_\mathcal{A}^I - e\, f_{JK}{}^I C_\mathcal{A}^J A_\mathcal{A}^K \ ,$$
$$\{w^i\} = \{C_\mathcal{A}^I, F_\mathcal{A}^I\} \ , \quad F_\mathcal{A}^I = (dA)_\mathcal{A}^I + \tfrac{1}{2} e\, f_{JK}{}^I A_\mathcal{A}^J A_\mathcal{A}^K \ . \tag{10.9}$$

Note that v^I and $F_\mathcal{A}^I$ replace the former generators $(dC)_\mathcal{A}^I$ and $(dA)_\mathcal{A}^I$ respectively and that $F_\mathcal{A}^I$ corresponds of course to the field strength 2-forms $F^I = \tfrac{1}{2} dx^\mu dx^\nu F_{\mu\nu}^I = \pi^2(F_\mathcal{A}^I)$. Note also that the change of basis preserves the polynomial structure: $\mathcal{A}$ is the space of polynomials in the new generators.

Using (10.4), one easily verifies that $s_\mathcal{A}$ acts on the new generators according to

$$s_\mathcal{A}\, u^I = v^I \ , \quad s_\mathcal{A}\, v^I = 0 \ ,$$
$$s_\mathcal{A}\, C_\mathcal{A}^I = \tfrac{1}{2} e\, f_{JK}{}^I C_\mathcal{A}^K C_\mathcal{A}^J \ , \quad s_\mathcal{A}\, F_\mathcal{A}^I = -e\, f_{JK}{}^I C_\mathcal{A}^J F_\mathcal{A}^K \ . \tag{10.10}$$

The u^I and v^I form thus contractible pairs for $s_\mathcal{A}$ and drop therefore from $H(s_\mathcal{A}, \mathcal{A})$. Hence, $H(s_\mathcal{A}, \mathcal{A})$ reduces to $H(s_\mathcal{A}, \mathcal{A}_w)$ where $\mathcal{A}_w$ is the space of polynomials in the $C_\mathcal{A}^I$ and $F_\mathcal{A}^I$.

From (10.10) and our discussion in section 8, it follows that $H(s_\mathcal{A}, \mathcal{A}_w)$ is nothing but the Lie algebra cohomology of $\mathcal{G}$ in the representation space of the polynomials in the $F_\mathcal{A}^I$, tansforming under the extension of the coadjoint representation. As shown in section 8, it is generated by the ghost polynomials $\theta_r(C_\mathcal{A})$ and by $\mathcal{G}$-invariant polynomials in the $F_\mathcal{A}^I$.

We can make the description of $H(s_\mathcal{A}, \mathcal{A})$ completely precise here, because the space of invariant polynomials in the $F_\mathcal{A}^I$ is completely known. Indeed, the space of $\mathcal{G}$-invariant polynomials in the $F_\mathcal{A}^I$ is generated by a finite set of such polynomials given by

$$f_r(F_\mathcal{A}) = \mathrm{Tr}(F_\mathcal{A}^{m(r)}) , \quad F_\mathcal{A} = F_\mathcal{A}^I T_I , \quad r = 1, \ldots, R , \quad R = rank(\mathcal{G}) \qquad (10.11)$$

where we follow the notations of subsection 8.5: r labels the independent Casimir operators of $\mathcal{G}$, $m(r)$ is the order of the rth Casimir operator, and $\{T_I\}$ is the same matrix representation of $\mathcal{G}$ used for constructing $\theta_r(C)$ in Eq. (8.18). More precisely one has (see e.g. [121, 161, 230]):

(i) Every $\mathcal{G}$-invariant polynomial in the $F_\mathcal{A}^I$ is a polynomial $P(f_1(F_\mathcal{A}), \ldots, f_R(F_\mathcal{A}))$ in the $f_r(F_\mathcal{A})$.

(ii) A polynomial $P(f_1(F_\mathcal{A}), \ldots, f_R(F_\mathcal{A}))$ in the $f_r(F_\mathcal{A})$ vanishes as a function of the $F_\mathcal{A}^I$ if and only if $P(f_1, \ldots, f_R)$ vanishes as a function of commuting independent variables f_r (i.e., in the free differential algebra of variables $f_1, \ldots, f_R$).

Note that (i) states the completeness of $\{f_r(F_\mathcal{A})\}$ while (ii) states that the $f_r(F_\mathcal{A})$ are algebraically independent. Analogous properties hold for the $\theta_r(C_\mathcal{A})$ (they generate the space of $\mathcal{G}$-invariant polynomials in the anticommuting variables $C_\mathcal{A}^I$). In particular, the algebraic independence of the $\theta_r(C_\mathcal{A})$ and $f_r(F_\mathcal{A})$ implies that a polynomial in the $\theta_r(C_\mathcal{A})$ and $f_r(F_\mathcal{A})$ vanishes in $\mathcal{A}$ if and only if it vanishes already as a polynomial in the free differential algebra of anticommuting variables θ_r and commuting variables f_r. Summarizing, we have:

Lemma 10.3 $H(s_\mathcal{A}, \mathcal{A})$ *is freely generated by the* $\theta_r(C_\mathcal{A})$ *and* $f_r(F_\mathcal{A})$:
(i) Every $s_\mathcal{A}$-*closed element of* $\mathcal{A}$ *is a polynomial in the* $\theta_r(C_\mathcal{A})$ *and* $f_r(F_\mathcal{A})$ *up to an* $s_\mathcal{A}$-*exact element of* $\mathcal{A}$,

$$s_\mathcal{A}\, a = 0, \quad a \in \mathcal{A} \quad \Leftrightarrow$$
$$a = P(\theta_1(C_\mathcal{A}), \ldots, \theta_R(C_\mathcal{A}), f_1(F_\mathcal{A}), \ldots, f_R(F_\mathcal{A})) + s_\mathcal{A}\, a', \quad a' \in \mathcal{A} . \quad (10.12)$$

(ii) No nonvanishing polynomial $P(\theta_1, \ldots, \theta_R, f_1, \ldots, f_R)$ *gives rise to an* $s_\mathcal{A}$-*exact polynomial in* $\mathcal{A}$,

$$P(\theta_1(C_\mathcal{A}), \ldots, \theta_R(C_\mathcal{A}), f_1(F_\mathcal{A}), \ldots, f_R(F_\mathcal{A})) = s_\mathcal{A}\, a, \quad a \in \mathcal{A} \quad \Rightarrow$$
$$P(\theta_1, \ldots, \theta_R, f_1, \ldots, f_R) = 0 . \quad (10.13)$$

By lemma 10.3, a basis of $H(s_\mathcal{A}, \mathcal{A})$ is obtained from a basis of all polynomials in anticommuting variables θ_r and commuting variables f_r, $r = 1, \ldots, R$. Such a basis is simply given by all monomials of the following form:

$$\theta_{r_1} \cdots \theta_{r_K} f_{s_1} \cdots f_{s_N} : \quad K, N \geq 0 , \quad r_i < r_{i+1} , \quad s_i \leq s_{i+1} . \tag{10.14}$$

Here it is understood that $\theta_{r_1} \cdots \theta_{r_0} \equiv 1$ if $K = 0$, and $f_{s_1} \cdots f_{s_0} \equiv 1$ if $N = 0$. The requirements $r_i < r_{i+1}$ and $s_i \leq s_{i+1}$ take the commutation relations (Grassmann parities) of the variables into account.

10.5   Cohomology of s in the small algebra

(10.14) induces a basis of $H(s, \mathcal{B})$ thanks to corollary 10.2. However, this basis is not best suited for our ultimate goal, the determination of $H(s|d, \mathcal{B})$, because it is not split into what we called $h^0_{i_r}$, $e^0_{\alpha_s}$ and $\hat{h}_{i_r}$ in theorem 9.1. Therefore we will now construct a better suited basis, using (10.14) as a starting point. For this purpose we order the Casimirs according to their degrees in the F's, namely, we assume that the Casimir labels $r = 1, \ldots, R$ are such that, for any two such labels r and r',

$$r < r' \quad \Rightarrow \quad m(r) \leq m(r') . \tag{10.15}$$

Note that the ordering (10.15) is ambiguous if two or more Casimir operators have the same order. This ambiguity will not matter, i.e., any ordering that satisfies (10.15) is suitable for our purposes.

The set of all monomials (10.14) is now split into three subsets. The first subset is just given by $\{1\}$. The second subset contains all those monomials which have the property that the lowest appearing Casimir label is carried by a θ. These monomials are denoted by $M_{r_1 \ldots r_K | s_1 \ldots s_N}(\theta, f)$,

$$M_{r_1 \ldots r_K | s_1 \ldots s_N}(\theta, f) = \theta_{r_1} \cdots \theta_{r_K} f_{s_1} \cdots f_{s_N} ,$$
$$K \geq 1 , \ N \geq 0 , \ r_i < r_{i+1} , \ s_i \leq s_{i+1} , \ r_1 = \min\{r_i, s_i\} . \tag{10.16}$$

Note that $r_1 = \min\{r_i, s_i\}$ requires $r_1 \leq s_1$ if $N \neq 0$ while it does not impose an extra condition if $N = 0$ (it is already implied by $r_i < r_{i+1}$ if $N = 0$).

The third subset contains all remaining monomials. In these monomials at least one of the f's has a lower Casimir label than all the θ's. Denoting the lowest occurring label again by r_1, the monomials of the third set can thus be written as

$$\bar{N}_{r_1 \ldots r_K s_1 \ldots s_N}(\theta, f) = f_{r_1} \theta_{r_2} \cdots \theta_{r_K} f_{s_1} \cdots f_{s_N} ,$$
$$K \geq 1 , \ N \geq 0 , \ r_i < r_{i+1} , \ s_i \leq s_{i+1} , \ r_1 = \min\{r_i, s_i\} . \tag{10.17}$$

Thus, for instance, in the case of $G = U(1) \times SU(2)$, there are two f's and two θ's, namely, $f_1 = F^{u(1)}$, $f_2 = \text{Tr}_{su(2)} F^2$, $\theta_1 = C^{u(1)}$ and $\theta_2 = \text{Tr}_{su(2)} C^3$. The monomial $\theta_1 f_2$ belongs to the second subset, i.e., it is an "M", because the label on θ is clearly smaller than that on f. The monomial $\theta_2 f_1$, by contrast, belongs to the third set. We shall see that $\theta_1 f_2$ is non trivial in $H(s|d)$ – and can be regarded as an $h^0_{i_r}$ –, while $\theta_2 f_1$ is s-exact modulo d and is a $\hat{h}_{i_r}$ (it arises as an obstruction in the lift of $\theta_1 \theta_2$).

We now define the following polynomials:

$$N_{r_1 \ldots r_K s_1 \ldots s_N}(\theta, f) = \sum_{r:\, m(r)=m(r_1)} f_r \, \frac{\partial M_{r_1 \ldots r_K | s_1 \ldots s_N}(\theta, f)}{\partial \theta_r} \,. \tag{10.18}$$

$N_{r_1 \ldots r_K s_1 \ldots s_N}$ is the sum of $\bar{N}_{r_1 \ldots r_K s_1 \ldots s_N}$ and a linear combination of M's,

$$N_{r_1 \ldots r_K s_1 \ldots s_N}(\theta, f) = \bar{N}_{r_1 \ldots r_K s_1 \ldots s_N}(\theta, f) - \sum_{\substack{i:\, i \geq 2, \\ m(r_i)=m(r_1)}} (-)^i M_{r_1 \ldots \hat{r}_i \ldots r_K | r_i s_1 \ldots s_N}(\theta, f)$$

where $\hat{r}_i$ means omission of r_i. This shows that the set of all M's and N's, supplemented by the number 1, is a basis of polynomials in the θ_r and f_r (as the same holds for the M's and $\bar{N}$'s). Due to lemma 10.3 this provides also a basis of $H(s_{\mathcal{A}}, \mathcal{A})$ after substituting the $\theta_r(C_{\mathcal{A}})$ and $f_r(F_{\mathcal{A}})$ for the θ_r and f_r:

Corollary 10.4 *A basis of $H(s_{\mathcal{A}}, \mathcal{A})$ is given by*

$$\{B_\alpha\} = \{1\,,\, M_{r_1 \ldots r_K | s_1 \ldots s_N}(\theta(C_{\mathcal{A}}), f(F_{\mathcal{A}}))\,,\, N_{r_1 \ldots r_K s_1 \ldots s_N}(\theta(C_{\mathcal{A}}), f(F_{\mathcal{A}}))\}. \tag{10.19}$$

Again, "basis" is meant here in the cohomological sense: (i) every $s_{\mathcal{A}}$-closed element of $\mathcal{A}$ is a linear combination of the B_α up to an $s_{\mathcal{A}}$-exact element ($s_{\mathcal{A}} a = 0 \Leftrightarrow a = \lambda^\alpha B_\alpha + s_{\mathcal{A}} a'$); (ii) no nonvanishing linear combination of the B_α is $s_{\mathcal{A}}$-exact ($\lambda^\alpha B_\alpha = s_{\mathcal{A}} a \Rightarrow \lambda^\alpha = 0 \,\forall \alpha$).

Note that each B_α has a definite p-degree p_α which equals twice its degree in the $F_{\mathcal{A}}^I$. Hence, by corollary 10.2, those B_α with $p_\alpha \leq n$ provide a basis of $H(s, \mathcal{B})$. The requirement $p_\alpha \leq n$ selects those $M_{r_1 \ldots r_K | s_1 \ldots s_N}$ with $\Sigma_{i=1}^N 2m(s_i) \leq n$ and those $N_{r_1 \ldots r_K s_1 \ldots s_N}$ with $2m(r_1) + \Sigma_{i=1}^N 2m(s_i) \leq n$. We conclude:

Corollary 10.5 *A basis of $H(s, \mathcal{B})$ is given by $\{1\,,\, M_a\,,\, N_i\}$ where*

$$\begin{aligned}
\{M_a\} &\equiv \{M_{r_1 \ldots r_K | s_1 \ldots s_N}(\theta(C), f(F)) : \Sigma_{i=1}^N 2m(s_i) \leq n\}, \\
\{N_i\} &\equiv \{N_{r_1 \ldots r_K s_1 \ldots s_N}(\theta(C), f(F)) : 2m(r_1) + \Sigma_{i=1}^N 2m(s_i) \leq n\}. \tag{10.20}
\end{aligned}$$

10.6   Transgression formulae

To derive $H(s|d, \mathcal{B})$ from corollary 10.5, we need to construct the lifts of the elements M of the previous basis. The most expedient way to achieve this task is to use the celebrated "transgression formula" (also called Russian formula) [192, 193, 234]

$$(s + d)\, q_r(C + A, F) = \mathrm{Tr}\left(F^{m(r)}\right) = f_r(F), \tag{10.21}$$

where $C = C^I T_I$, $A = A^I T_I$, $F = F^I T_I$, and

$$q_r(C + A, F) = m(r) \int_0^1 dt\, \mathrm{Tr}\left((C + A)\left[tF + e(t^2 - t)(C + A)^2\right]^{m(r)-1}\right). \tag{10.22}$$

Here t is just an integration variable and should not be confused with the spacetime coordinate x^0. A derivation of Eqs. (10.21) and (10.22) is given at the end of this

subsection. $q_r(C+A,F)$ is nothing but the Chern-Simons polynomial $q_r(A,F)$ with $C+A$ substituting for A. It fulfills $dq_r(A,F) = f_r(F)$, as can be seen from (10.21) in ghost number 0.

The usefulness of (10.21) for the determination of $H(s|d,\mathcal{B})$ rests on the fact that it relates $\theta_r(C)$ and $f_r(F)$ via a set of equations obtained by decomposing (10.21) into parts with definite form-degrees,

$$d[\theta_r]^{2m(r)-1} = f_r(F),$$
$$s[\theta_r]^p + d[\theta_r]^{p-1} = 0 \quad \text{for } 0 < p < 2m(r),$$
$$s[\theta_r]^0 = 0 \tag{10.23}$$

where $[\theta_r]^p$ is the p-form contained in $q_r(C+A,F)$,

$$q_r(C+A,F) = \sum_{p=0}^{2m(r)-1} [\theta_r]^p . \tag{10.24}$$

The 0-form contained in $q_r(C+A,F)$ is nothing but $\theta_r(C)$,

$$
\begin{aligned}
[\theta_r]^0 &= m(r)\,\mathrm{Tr}(C^{2m(r)-1})e^{m(r)-1}\int_0^1 dt\,(t^2-t)^{m(r)-1} \\
&= (-e)^{m(r)-1}\frac{m(r)!(m(r)-1)!}{(2m(r)-1)!}\,\mathrm{Tr}(C^{2m(r)-1}) = \theta_r(C). \tag{10.25}
\end{aligned}
$$

Note that $f_r(F)$ and some of the $[\theta_r]^p$ vanish in sufficiently low spacetime dimension (when $n < 2m(r)$) but that $q_r(C+A,F)$ never vanishes completely since it contains $\theta_r(C)$. The same formulae hold in the universal algebra $\mathcal{A}$, but there, of course, none of the f_r vanishes.

Consider now the polynomials $M_{r_1...r_K|s_1...s_N}(q(C+A,F),f(F))$ arising from the $M_{r_1...r_K|s_1...s_N}(\theta,f)$ in Eq. (10.16) by substituting the $q_r(C+A,F)$ and $f_r(F)$ for the corresponding θ_r and f_r. Analogously to (10.24), these polynomials decompose into pieces of various form-degrees,

$$M_{r_1...r_K|s_1...s_N}(q(C+A,F),f(F)) = \sum_{p=\underline{s}}^{\bar{p}}[M_{r_1...r_K|s_1...s_N}]^p ,$$
$$\underline{s} = \Sigma_{i=1}^K 2m(s_i), \quad \bar{p} = \underline{s} + \Sigma_{i=1}^K(2m(r_i)-1) \tag{10.26}$$

where some or all $[M_{r_1...r_K|s_1...s_N}]^p$ may vanish in sufficiently low spacetime dimension. Due to (10.25), one has

$$[M_{r_1...r_K|s_1...s_N}]^{\underline{s}} = M_{r_1...r_K|s_1...s_N}(\theta(C),f(F)). \tag{10.27}$$

The polynomials (10.26) give rise to transgression equations that generalize Eqs. (10.23). These equations are obtained by evaluating $(s+d)M_{r_1...r_K|s_1...s_N}(q(C+A,F),f(F))$: one gets a sum of terms in which one of the $q_r(C+A,F)$ in M is replaced by the corresponding $f_r(F)$ as a consequence of (10.21) (note that one has

$(s+d)f_r(F) = (s+d)^2 q_r(C+A,F) = 0$ due to $(s+d)^2 = 0$, i.e., $(s+d)$ acts non-trivially only on the q's contained in M). Hence, $(s+d)M$ is obtained by applying the operation $\sum_r f_r \partial/\partial q_r$ to M. This makes it easy to identify the part of lowest form-degree contained in the resulting expression: it is $N_{r_1\ldots r_K s_1\ldots s_N}(\theta(C), f(F))$ given in Eq. (10.18) thanks to the ordering (10.15) of the Casimir labels. One thus gets generalized transgression equations

$$s[M_{r_1\ldots r_K|s_1\ldots s_N}]^{\underline{s}+2m(r_1)} + d[M_{r_1\ldots r_K|s_1\ldots s_N}]^{\underline{s}+2m(r_1)-1} = N_{r_1\ldots r_K s_1\ldots s_N}(\theta(C), f(F)),$$

$$s[M_{r_1\ldots r_K|s_1\ldots s_N}]^{\underline{s}+q} + d[M_{r_1\ldots r_K|s_1\ldots s_N}]^{\underline{s}+q-1} = 0 \quad \text{for} \quad 0 < q < 2m(r_1),$$

$$s[M_{r_1\ldots r_K|s_1\ldots s_N}]^{\underline{s}} = 0. \tag{10.28}$$

Note that (10.23) is just a special case of (10.28), arising for $M_{r_1\ldots r_K|s_1\ldots s_N}(\theta, f) \equiv \theta_r$.

Derivation of Eqs. (10.21) and (10.22). The derivation is performed in the free differential algebra $\mathcal{A}$.

As shown in subsection 10.3, the cohomology of $d_\mathcal{A}$ is trivial in the algebra $\mathcal{A}$ of polynomials in $A^I_\mathcal{A}, (dA)^I_\mathcal{A}, C^I_\mathcal{A}, (dC)^I_\mathcal{A}$. The contracting homotopy is explicitly given by $\rho = A^I_\mathcal{A}\frac{\partial}{\partial(dA)^I_\mathcal{A}} + C^I_\mathcal{A}\frac{\partial}{\partial(dC)^I_\mathcal{A}}$. For a $d_\mathcal{A}$-cocycle $f(A^I_\mathcal{A}, (dA)^I_\mathcal{A}, C^I_\mathcal{A}, (dC)^I_\mathcal{A})$ with $f(0,0,0,0) = 0$, we get

$$f(A^I_\mathcal{A}, (dA)^I_\mathcal{A}, C^I_\mathcal{A}, (dC)^I_\mathcal{A}) = d_\mathcal{A} \int_0^1 \frac{dt}{t} [\rho f](tA^I_\mathcal{A}, t(dA)^I_\mathcal{A}, tC^I_\mathcal{A}, t(dC)^I_\mathcal{A}). \tag{10.29}$$

We then consider the change of generators $A^I_\mathcal{A}, (dA)^I_\mathcal{A} \longrightarrow A^I_\mathcal{A}, F^I_\mathcal{A}$ in $\mathcal{A}$ (while $C^I_\mathcal{A}, (dC)^I_\mathcal{A}$ remain unchanged). The differential $d_\mathcal{A}$ acts on $A^I_\mathcal{A}$ and $F^I_\mathcal{A}$ according to

$$\begin{aligned}
d_\mathcal{A} A^I_\mathcal{A} &= F^I_\mathcal{A} - \frac{1}{2} e\, f_{JK}{}^I A^J_\mathcal{A} A^K_\mathcal{A}, \\
d_\mathcal{A} F^I_\mathcal{A} &= -e\, f_{JK}{}^I A^J_\mathcal{A} F^K_\mathcal{A}.
\end{aligned} \tag{10.30}$$

For a $d_\mathcal{A}$-cocycle $f(A^I_\mathcal{A}, F^I_\mathcal{A})$ with $f(0,0) = 0$, the homotopy formula (10.29) gives

$$f(A^I_\mathcal{A}, F^I_\mathcal{A}) = d_\mathcal{A}\left[A^L_\mathcal{A} \int_0^1 dt\, [\frac{\partial f(A^I_\mathcal{A}, F^I_\mathcal{A})}{\partial F^L_\mathcal{A}}](tA^I_\mathcal{A}, tF^I_\mathcal{A} + \tfrac{1}{2} e\,(t^2-t) f_{JK}{}^I A^J_\mathcal{A} A^K_\mathcal{A})\right]. \tag{10.31}$$

This formula generalizes straightforwardly to an analogous one for the differential

$$\tilde{s}_\mathcal{A} = s_\mathcal{A} + d_\mathcal{A}. \tag{10.32}$$

Indeed, with $\tilde{C}^I_\mathcal{A} = C^I_\mathcal{A} + A^I_\mathcal{A}$, one gets

$$\begin{aligned}
\tilde{s}_\mathcal{A} \tilde{C}^I_\mathcal{A} &= \frac{1}{2} e\, f_{JK}{}^I \tilde{C}^K_\mathcal{A} \tilde{C}^J_\mathcal{A} + F^I_\mathcal{A}, \\
\tilde{s}_\mathcal{A} F^I_\mathcal{A} &= -e\, f_{JK}{}^I \tilde{C}^J_\mathcal{A} F^K_\mathcal{A}.
\end{aligned} \tag{10.33}$$

Comparing (10.30) and (10.33), we see that the differential algebras $(d_{\mathcal{A}}, \Lambda(A^I_{\mathcal{A}}, F^I_{\mathcal{A}}))$ and $(\tilde{s}_{\mathcal{A}}, \Lambda(\tilde{C}^I_{\mathcal{A}}, F^I_{\mathcal{A}}))$ are isomorphic. It follows that

$$f(\tilde{C}^I_{\mathcal{A}}, F^I_{\mathcal{A}}) = \tilde{s}_{\mathcal{A}} \left[\tilde{C}^L_{\mathcal{A}} \int_0^1 dt \left[\frac{\partial f(\tilde{C}^I_{\mathcal{A}}, F^I_{\mathcal{A}})}{\partial F^L_{\mathcal{A}}} \right] (t\tilde{C}^I_{\mathcal{A}}, tF^I_{\mathcal{A}} + \tfrac{1}{2} e \, (t^2 - t) f_{JK}{}^I \tilde{C}^J_{\mathcal{A}} \tilde{C}^K_{\mathcal{A}}) \right], \quad (10.34)$$

for an $\tilde{s}_{\mathcal{A}}$-cocycle $f(\tilde{C}^I_{\mathcal{A}}, F^I_{\mathcal{A}})$ with $f(0,0) = 0$. Taking $f = \mathrm{Tr} F^{m(r)}_{\mathcal{A}}$, this yields (10.21) and (10.22) through the mappings (10.2).

10.7 $H(s|d)$ in the small algebra

We are now in the position to determine $H(s|d, \mathcal{B})$. Namely, Eqs. (10.28) imply that the basis of $H(s, \mathcal{B})$ given in corollary 10.5 has the properties described in theorems 9.1 and 9.2 of section 9. To verify this, consider Eqs. (10.28) first in the cases $\underline{s} + 2m(r_1) \leq n$. In these cases, Eqs. (10.28) reproduce Eqs. (9.10) in theorem 9.1, with the identifications

$$\underline{s} + 2m(r_1) \leq n :$$
$$\hat{h}_{i_r} \equiv N_{r_1 \ldots r_K s_1 \ldots s_N}(\theta(C), f(F))$$
$$h^q_{i_r} \equiv [M_{r_1 \ldots r_K | s_1 \ldots s_N}]^{\underline{s}+q}, \quad q = 0, \ldots, r$$
$$r = 2m(r_1) - 1.$$

In particular this yields $h^0_{i_r} \equiv M_{r_1 \ldots r_K | s_1 \ldots s_N}(\theta(C), f(F))$ for $\underline{s} + 2m(r_1) \leq n$ by Eq. (10.27).

Next consider Eqs. (10.28) in the cases $n - 2m(r_1) < \underline{s} \leq n$. This reproduces Eqs. (9.11) in theorem 9.1, with the identifications

$$n - 2m(r_1) < \underline{s} \leq n :$$
$$e^q_{\alpha_s} \equiv [M_{r_1 \ldots r_K | s_1 \ldots s_N}]^{\underline{s}+q}, \quad q = 0, \ldots, s$$
$$s = n - \underline{s}.$$

In particular this yields $e^0_{\alpha_s} \equiv M_{r_1 \ldots r_K | s_1 \ldots s_N}(\theta(C), f(F))$ for $n - 2m(r_1) < \underline{s} \leq n$.

Hence, the basis of $H(s, \mathcal{B})$ given in corollary 10.5 has indeed the properties described in theorem 9.1. By theorem 9.2, a basis of $H(s|d, \mathcal{B})$ is thus given by the $[M_{r_1 \ldots r_K | s_1 \ldots s_N}]^p$ specified in the above equations. The whole set of these representatives can be described more compactly through $p = \underline{s}, \ldots, \overline{m}$ where $\overline{m} = \min\{\underline{s} + 2m(r_1) - 1, n\}$.

We can summarize the result in the form of a receipe. Given the gauge group G and the spacetime dimension n, one obtains $H(s|d, \mathcal{B})$ as follows:

1. Specify the independent Casimir operators of G and label them such that

$$r < r' \quad \Rightarrow \quad m(r) \leq m(r') \qquad (10.35)$$

where $m(r)$ is the order of the rth Casimir operator.

2. Specify the following monomials:

$$M_{r_1...r_K|s_1...s_N}(\theta, f) = \theta_{r_1} \cdots \theta_{r_K} f_{s_1} \cdots f_{s_N} \ :$$
$$K \geq 1 \ , \ N \geq 0 \ , \ r_i < r_{i+1} \ , \ s_i \leq s_{i+1} \ ,$$
$$r_1 = \min\{r_i, s_i\} \ , \ \sum_{i=1}^{N} 2m(s_i) \leq n \ . \tag{10.36}$$

3. Replace in (10.36) the θ_r and f_r by the corresponding $q_r(C + A, F)$ and $f_r(F)$ given in (10.22) and (10.21) and decompose the resulting polynomials in the $q_r(C + A, F)$ and $f_r(F)$ into pieces of definite form-degree,

$$M_{r_1...r_K|s_1...s_N}(q(C + A, F), f(F)) = \sum_{p} [M_{r_1...r_K|s_1...s_N}]^p \ . \tag{10.37}$$

4. A basis of $H(s|d, \mathcal{B})$ is then given by the number 1 and the following $[M_{r_1...r_K|s_1...s_N}]^p$:

$$[M_{r_1...r_K|s_1...s_N}]^p \ : \quad p = \underline{s}, \ldots, \overline{m} \ ,$$
$$\underline{s} = \Sigma_{i=1}^{N} 2m(s_i),$$
$$\overline{m} = \min\{\underline{s} + 2m(r_1) - 1, n\}. \tag{10.38}$$

A similar results hold in the universal algebra $\mathcal{A}$, but in this case, there is no $e^0_{\alpha_s}$ but only $h^0_{i_r}$: all lifts are obstructed at some point. This implies, in particular, that any solution of the consistency condition in $\mathcal{A}$ can be seen as coming from an obstruction living above. For instance, the Adler-Bardeen-Bell-Jackiw anomaly in four dimensions, which is a four-form, comes from the six-form $\mathrm{Tr}F_{\mathcal{A}}^3$ through the Russian formula. This makes sense only in the universal algebra, although the anomaly itself is meaningful both in $\mathcal{A}$ and $\mathcal{B}$.

10.8 $H^{0,n}(s|d)$ and $H^{1,n}(s|d)$ in the small algebra

Physically important representatives of $H(s|d, \mathcal{B})$ are those with form-degree n and ghost numbers 0 or 1 as they provide possible counterterms and gauge anomalies respectively. To extract these representatives from Eqs. (10.35) through (10.38), one uses that $q_r(C + A, F)$ and $f_r(F)$ have total degree (= form-degree + ghost number) $2m(r) - 1$ and $2m(r)$ respectively. The total degree of $[M_{r_1...r_K|s_1...s_N}]^p$ is thus $\sum_{i=1}^{K}(2m(r_i) - 1) + \sum_{i=1}^{N} 2m(s_i) = \underline{s} + \sum_{i=1}^{K}(2m(r_i) - 1)$. Hence, representatives with form-degree n and ghost number g fulfill

$$n + g = \underline{s} + \sum_{i=1}^{K}(2m(r_i) - 1).$$

Furthermore, representatives with form-degree n fulfill

$$n \leq \underline{s} + 2m(r_1) - 1$$

because of the requirement $\overline{m} = \min\{\underline{s} + 2m(r_1) - 1, n\}$ in Eq. (10.38). Combining these two conditions, one gets

$$\sum_{i=2}^{K} (2m(r_i) - 1) \leq g .\tag{10.39}$$

Note that here the sum runs from 2 to K, and that we have $K \geq 1$ by (10.36). Hence, for $g = 0$, (10.39) selects the value $K = 1$. The representatives of $H^{0,n}(s|d, \mathcal{B})$ arise thus from (10.37) by setting $K = 1$ and selecting the ghost number 0 part. These representatives are

$$[M_{r|s_1...s_N}]^{\underline{s}+2m(r)-1} = [\theta_r]^{2m(r)-1} f_{s_1}(F) \cdots f_{s_N}(F) .\tag{10.40}$$

Note that (10.36) imposes $r \leq s_1 \leq s_2 \leq \ldots \leq s_N$ if $N \neq 0$. $[\theta_r]^{2m(r)-1}$ is nothing but the Chern-Simons form corresponding to $f_r(F)$, see equation (10.23). (10.40) is thus a Chern-Simons form too, corresponding to $f_r(F)f_{s_1}(F) \cdots f_{s_N}(F)$. All representatives (10.40) have odd form-degree and occur thus only in odd spacetime dimensions.

For $g = 1$, (10.39) leaves two possibilities: $K = 1$, or $K = 2$ where the latter case requires in addition $m(r_2) = 1$. For $K = 1$, this yields the following representatives of $H^{1,n}(s|d, \mathcal{B})$,

$$[M_{r|s_1...s_N}]^{\underline{s}+2m(r)-2} = [\theta_r]^{2m(r)-2} f_{s_1}(F) \cdots f_{s_N}(F) .\tag{10.41}$$

Again, (10.36) imposes $r \leq s_1 \leq s_2 \leq \ldots \leq s_N$ if $N \neq 0$. By (10.22) one has

$$[\theta_r]^{2m(r)-2} = \text{Tr}(CF^{m(r)-1} + \ldots).$$

All representatives (10.41) have even form-degree. They represent the consistent chiral gauge anomalies in even spacetime dimensions.

The remaining representatives of $H^{1,n}(s|d, \mathcal{B})$ have $K = 2$ and $m(r_2) = 1$. $m(r_2) = 1$ requires $m(r_1) = 1$ by (10.35) and (10.36). The Casimir operators of order 1 are the abelian generators (see Section 8.5). The corresponding $q_r(C + A, F)$ coincide with the abelian $C + A$,

$$\{q_r(C + A, F) : m(r) = 1\} = \{\text{abelian } C^I + A^I\}.\tag{10.42}$$

The representatives of $H^{1,n}(s|d, \mathcal{B})$ with $K = 2$ read

$$[M_{IJ|s_1...s_N}]^{\underline{s}+1} = (C^I A^J - C^J A^I) f_{s_1}(F) \cdots f_{s_N}(F) \quad (\text{abelian } I, J).\tag{10.43}$$

(10.36) imposes $I < J$, $s_i \leq s_{i+1}$ and that $f_{s_1}(F)$ is not an abelian F^K with $K < I$. Note that the representatives (10.43) have odd form-degree and are only present if the gauge group contains at least two abelian factors. They yield candidate gauge anomalies in odd spacetime dimensions.

10.9   Examples

To illustrate the results, we shall now spell out $H^{0,n}(s|d,\mathcal{B})$ and $H^{1,n}(s|d,\mathcal{B})$ for specific choices of n and G. We list those $\theta_r(C)$ (up to the normalization factor) and $f_r(F)$ needed to construct $H^{0,n}(s|d,\mathcal{B})$ and $H^{1,n}(s|d,\mathcal{B})$ and give a complete set of the inequivalent representatives ("Reps.") of these cohomological groups and the corresponding obstructions ("Obs.") in the universal algebra $\mathcal{A}$ (except that in the last example we leave it to the reader to spell out the obstructions as it is similar to the second example). The inclusion of $SO(1,9)$ and $SO(1,10)$ in the last two examples is relevant in the gravitational context because the Lorentz group plays a rôle similar to the Yang-Mills gauge group when one includes gravity in the analysis.

$\underline{n = 4,\ G = U(1) \times SU(2) \times SU(3)}$

r	1	2	3	4
$m(r)$	1	2	2	3
$\theta_r(C)$	$C^{\mathrm{u}(1)}$	$\mathrm{Tr}_{\mathrm{su}(2)}C^3$	$\mathrm{Tr}_{\mathrm{su}(3)}C^3$	$\mathrm{Tr}_{\mathrm{su}(3)}C^5$
$f_r(F)$	$F^{\mathrm{u}(1)}$	$\mathrm{Tr}_{\mathrm{su}(2)}F^2$	$\mathrm{Tr}_{\mathrm{su}(3)}F^2$	0

$H^{0,4}(s|d,\mathcal{B})$:   empty

$H^{1,4}(s|d,\mathcal{B})$:

Reps.	$C^{\mathrm{u}(1)}(F^{\mathrm{u}(1)})^2$	$C^{\mathrm{u}(1)}f_2(F)$	$C^{\mathrm{u}(1)}f_3(F)$	$[\theta_3]^4$
Obs.	$(F_{\mathcal{A}}^{\mathrm{u}(1)})^3$	$F_{\mathcal{A}}^{\mathrm{u}(1)}f_2(F_{\mathcal{A}})$	$F_{\mathcal{A}}^{\mathrm{u}(1)}f_3(F_{\mathcal{A}})$	$f_4(F_{\mathcal{A}})$

where $[\theta_3]^4 = \mathrm{Tr}_{\mathrm{su}(3)}[Cd(AdA + \frac{1}{2}e_{\mathrm{su}(3)}A^3)],\ f_4(F_{\mathcal{A}}) = \mathrm{Tr}_{\mathrm{su}(3)}(F_{\mathcal{A}})^3$

$\underline{n = 10,\ G = SO(32)}$

r	1	2	3
$m(r)$	2	4	6
$\theta_r(C)$	$\mathrm{Tr}C^3$	$\mathrm{Tr}C^7$	$\mathrm{Tr}C^{11}$
$f_r(F)$	$\mathrm{Tr}F^2$	$\mathrm{Tr}F^4$	0

$H^{0,10}(s|d,\mathcal{B})$:   empty

$H^{1,10}(s|d,\mathcal{B})$:

Reps.	$[\theta_1]^2(f_1(F))^2$	$[\theta_1]^2 f_2(F)$	$[\theta_3]^{10}$
Obs.	$(f_1(F_{\mathcal{A}}))^3$	$f_1(F_{\mathcal{A}})f_2(F_{\mathcal{A}})$	$f_3(F_{\mathcal{A}})$

where $[\theta_1]^2 = \mathrm{Tr}(CdA),\ [\theta_3]^{10} = \mathrm{Tr}[C(dA)^5 + \dots],$
$f_3(F_{\mathcal{A}}) = \mathrm{Tr}(F_{\mathcal{A}})^6$

$\underline{n = 11, \ G = SO(1,10)}$

r	1	2	3
$m(r)$	2	4	6
$\theta_r(C)$	$\mathrm{Tr}C^3$	$\mathrm{Tr}C^7$	$\mathrm{Tr}C^{11}$
$f_r(F)$	$\mathrm{Tr}F^2$	$\mathrm{Tr}F^4$	0

$H^{0,11}(s|d,\mathcal{B}):$

Reps.	$[\theta_1]^3(f_1(F))^2$	$[\theta_1]^3 f_2(F)$	$[\theta_3]^{11}$
Obs.	$(f_1(F_\mathcal{A}))^3$	$f_1(F_\mathcal{A})f_2(F_\mathcal{A})$	$f_3(F_\mathcal{A})$

where $\quad [\theta_1]^3 = \mathrm{Tr}(AdA + \frac{1}{3}eA^3), \ [\theta_3]^{11} = \mathrm{Tr}[A(dA)^5 + \ldots],$
$\qquad\quad f_3(F_\mathcal{A}) = \mathrm{Tr}(F_\mathcal{A})^6$

$H^{1,11}(s|d,\mathcal{B}): \quad$ empty

$\underline{n = 10, \ G = SO(1,9) \times SO(32)}$

r	1	2	3	4	6	7
$m(r)$	2	2	4	4	6	6
$\theta_r(C)$	$\mathrm{Tr}_{\mathrm{so}(1,9)}C^3$	$\mathrm{Tr}_{\mathrm{so}(32)}C^3$	$\mathrm{Tr}_{\mathrm{so}(1,9)}C^7$	$\mathrm{Tr}_{\mathrm{so}(32)}C^7$	$\mathrm{Tr}_{\mathrm{so}(1,9)}C^{11}$	$\mathrm{Tr}_{\mathrm{so}(32)}C^{11}$
$f_r(F)$	$\mathrm{Tr}_{\mathrm{so}(1,9)}F^2$	$\mathrm{Tr}_{\mathrm{so}(32)}F^2$	$\mathrm{Tr}_{\mathrm{so}(1,9)}F^4$	$\mathrm{Tr}_{\mathrm{so}(32)}F^4$	0	0

$H^{0,10}(s|d,\mathcal{B}): \quad$ empty

$H^{1,10}(s|d,\mathcal{B})$ (Reps.) : $\quad [\theta_1]^2(f_1(F))^2, \ [\theta_1]^2 f_1(F)f_2(F), \ [\theta_1]^2(f_2(F))^2,$
$\qquad\qquad\qquad\qquad\qquad\quad [\theta_1]^2 f_3(F), \ [\theta_1]^2 f_4(F),$
$\qquad\qquad\qquad\qquad\qquad\quad [\theta_2]^2(f_2(F))^2, \ [\theta_2]^2 f_3(F), \ [\theta_2]^2 f_4(F),$
$\qquad\qquad\qquad\qquad\qquad\quad [\theta_6]^{10}, \ [\theta_7]^{10}$

Remark: the Pfaffian of $SO(1,9)$ yields $f_5(F)$ with $m(5) = 5$; however it does not contribute to $H^{1,10}(s|d,\mathcal{B})$ in this case (it would contribute through $C^{\mathrm{u}(1)}f_5(F)$ if G contained in addition a $U(1)$).

11 General solution of the consistency condition in Yang-Mills type theories

11.1 Assumptions

We shall now put the pieces together and determine $H(s|d, \Omega)$ completely in Yang-Mills type theories for two cases:

$$
\begin{array}{lll}
\text{Case I:} & \Omega = \{\text{all local forms}\}, & \\
\text{Case II:} & \Omega = \{\text{Poincaré-invariant local forms}\}, & (11.1)
\end{array}
$$

on the following assumptions:

(a) the gauge group does not contain abelian gauge symmetries under which all matter fields are uncharged – we shall call such special abelian symmetries "free abelian gauge symmetries" in the following;

(b) the spacetime dimension (denoted by n) is larger than 2;

(c) the theory is normal and the regularity conditions hold;

(d) in case II it is assumed that the Lagrangian itself is Poincaré-invariant (in case I it need not be Poincaré-invariant).

Assumption (c) is a technical one and has been explained in Sections 5 and 6. Assumptions (a) and (b) reflect special properties of free abelian gauge symmetries and 2-dimensional (pure) Yang-Mills theory which complicate somewhat the general analysis. These special properties of free abelian gauge symmetries are illustrated and dealt with in Section 13 where we compute the cohomology for a set of free abelian gauge fields. Two-dimensional pure Yang-Mills theory is treated separately in the appendix to this Section. Note that assumption (a) does not exclude abelian gauge symmetries. It excludes only the presence of free abelian gauge fields, or of abelian gauge fields that couple exclusively non-minimally to matter or gauge fields (i.e., through the field strengths and their derivatives only).

The space of all local forms is the direct product $\mathcal{P} \otimes \Omega(\mathbb{R}^n)$ where $\mathcal{P}$ is the space of local functions of the fields, antifields, and all their derivatives, while $\Omega(\mathbb{R}^n)$ is the space of ordinary differential forms $\omega(x, dx)$ in $\mathbb{R}^n$. Depending on the context and Lagrangian, $\mathcal{P}$ can be, for instance, the space of polynomials in the fields, antifields, and all their derivatives (when the Lagrangian is polynomial too), or it can be the space of local forms that depend polynomially on the derivatives of the fields and antifields but may depend smoothly on (some of) the undifferentiated fields (when the Lagrangian has the same property). The latter would be the case for instance for Yang-Mills theory coupled to a dilaton. It can also be the space of formal power series in the fields, antifields, and all their derivatives, with coefficients that depend on the coupling constants (in the case of effective theories). More generally speaking, the results and their derivation apply whenever the various cohomological results derived and discussed in the previous sections (especially those on $H(s)$ and $H_{\mathrm{char}}(d)$) hold, since these will be used within the computation below.

The space of Poincaré-invariant local forms is a subspace of $\mathcal{P} \otimes \Omega(\mathbb{R}^n)$. It contains only those local forms which do not depend explicitly on the spacetime coordinates x^μ

and are Lorentz-invariant. Lorentz-invariance requires here simply that all Lorentz indices (including the indices of derivatives and differentials, and the spinor indices of spacetime fermions) are contracted in an $SO(1, n-1)$-invariant manner.

Two central ingredients in our computation of $H(s|d)$ are the results on the characteristic cohomology of d in Section 6 and on the cohomology of s in Section 8. We shall repeat them here, for case I and case II, and reformulate the result on $H(s)$ since we shall frequently use it in that formulation. For the characteristic cohomology of d we have:

Corollary 11.1 $H^p_{\text{char}}(d, \Omega)$ *vanishes at all form-degrees $0 < p < n-1$ and is at form-degree 0 represented by the constants,*

$$
\begin{aligned}
0 < p < n-1 : && d\omega^p \approx 0, \ \omega^p \in \Omega && \Leftrightarrow && \omega^p \approx d\omega^{p-1}, \ \omega^{p-1} \in \Omega \ ; \\
p = 0 : && d\omega^0 \approx 0, \ \omega^0 \in \Omega && \Leftrightarrow && \omega^0 \approx constant.
\end{aligned}
\tag{11.2}
$$

In case I this follows directly from the results of Section 6 thanks to assumptions (a), (b) and (c), where assumptions (a) and (b) are only needed for the vanishing of $H^{n-2}_{\text{char}}(d, \Omega)$ ($H^{n-2}_{\text{char}}(d, \Omega)$ does not vanish when free abelian gauge symmetries are present, and is not exhausted by the constants in 2-dimensional pure Yang-Mills theory; this makes these two cases special). The analysis and results of Section 6 extend to case II because both d and δ are Lorentz invariant according to the definition of Lorentz invariance used here (i.e., d and δ commute in Ω with $SO(1, n-1)$-rotations of all Lorentz indices). This is obvious for d and holds for δ thanks to assumption (d) because that assumption guarantees the Lorentz-covariance of the equations of motion.

The results on $H(s)$ in section 8 can be reformulated as follows:

Corollary 11.2 *Description of $H(s, \Omega)$:*

$$
s\omega = 0, \ \omega \in \Omega \quad \Leftrightarrow \quad \omega = I^\alpha \Theta_\alpha + s\eta, \ I^\alpha \in \mathcal{I}, \ \eta \in \Omega \ ;
\tag{11.3}
$$

$$
I^\alpha \Theta_\alpha = s\omega, \ I^\alpha \in \mathcal{I}, \ \omega \in \Omega \quad \Leftrightarrow \quad I^\alpha \approx 0 \ \forall \alpha.
\tag{11.4}
$$

Here, $\{\Theta_\alpha\}$ is a basis of all polynomials in the $\theta_r(C)$,

$$
\{\Theta_\alpha\} = \{1, \prod_{\substack{i=1,\ldots,K \\ r_i < r_{i+1}}} \theta_{r_i}(C) : K = 1, \ldots, rank(\mathcal{G})\},
\tag{11.5}
$$

and $\mathcal{I}$ is the antifield independent gauge invariant subspace of Ω given in the two cases under study respectively by

Case I: $\mathcal{I} = \{\mathcal{G}\text{-invariant local functions of } F^I_{\mu\nu}, \psi^i, D_\rho F^I_{\mu\nu}, D_\rho \psi^i, \ldots\} \otimes \Omega(\mathbb{R}^n)$

Case II: $\mathcal{I} = \{\mathcal{G}\text{-invariant and Lorentz-invariant local functions of}$

$$
dx^\mu, F^I_{\mu\nu}, \psi^i, D_\rho F^I_{\mu\nu}, D_\rho \psi^i, \ldots\}.
\tag{11.6}
$$

Again, the precise definition of "local functions" depends on the context and Lagrangian.

Let us now briefly explain how and why corollary 11.2 reformulates the results in Section 8. We first treat case I. The results in Section 8 give (using a notation as in that section) $H(s,\Omega) = \Omega(\mathbb{R}^n) \otimes V^X_{\rho=0} \otimes \Lambda(C)_{\rho^C=0}$ where $V^X_{\rho=0}$ is the space of $\mathcal{G}$-invariant local functions of the X^u_A specified in Section 8.3, and $\Lambda(C)_{\rho^C=0}$ is the space of polynomials in the $\theta_r(C)$. Now, one has $\Omega(\mathbb{R}^n) \otimes V^X_{\rho=0} \subset \mathcal{I}$ and thus $\Omega(\mathbb{R}^n) \otimes V^X_{\rho=0} \otimes \Lambda(C)_{\rho^C=0} \subset \mathcal{I} \otimes \Lambda(C)_{\rho^C=0}$. This yields the implication $\Rightarrow$ in (11.3). The implication $\Leftarrow$ holds because all elements of $\mathcal{I} \otimes \Lambda(C)_{\rho^C=0}$ are s-closed. Furthermore, since $\Omega(\mathbb{R}^n) \otimes V^X_{\rho=0}$ is only a subspace of $\mathcal{I}$, one has $\mathcal{I} = (\Omega(\mathbb{R}^n) \otimes V^X_{\rho=0}) \oplus (\Omega(\mathbb{R}^n) \otimes V^X_{\rho=0})^\top$. By the results in Section 8, all nonvanishing elements of $\Omega(\mathbb{R}^n) \otimes V^X_{\rho=0} \otimes \Lambda(C)_{\rho^C=0}$ are nontrivial in $H(s,\Omega)$ whereas all elements of $(\Omega(\mathbb{R}^n) \otimes V^X_{\rho=0})^\top \otimes \Lambda(C)_{\rho^C=0}$ are trivial in $H(s,\Omega)$ and vanish on-shell. This gives (11.4).

We now turn to case II. Thanks to assumption (d), both δ and γ commute with Lorentz transformations. Therefore the results of Section 8 hold analogously in the Lorentz-invariant subspace of $\mathcal{P} \otimes \Omega(\mathbb{R}^n)$. This gives in case II $H(s,\Omega) = V^{X,dx}_{\rho=0} \otimes \Lambda(C)_{\rho^C=0}$ where $V^{X,dx}_{\rho=0}$ is the space of Lorentz-invariant and $\mathcal{G}$-invariant local functions of the X^u_A and dx^μ. Corollary 11.2 holds now by reasons analogous to case I.

Finally we shall often use the following immediate consequence of the isomorphism (7.5) (cf. proof of that isomorphism):

Corollary 11.3 *An s-cocycle with nonnegative ghost number is s-exact whenever its antifield independent part vanishes on-shell,*

$$s\omega = 0, \ \omega \in \Omega, \ gh(\omega) \geq 0, \ \omega_0 \approx 0 \quad \Rightarrow \quad \omega = sK, \ K \in \Omega$$

where ω_0 is the antifield independent part of ω.

11.2 Outline of the derivation and result

We shall determine the general solution of the consistency condition

$$s\omega^p + d\omega^{p-1} = 0, \quad \omega^p, \omega^{p-1} \in \Omega \tag{11.7}$$

for all values of the form-degree p and of the ghost number (the ghost number will not be made explicit throughout this section, contrary to the form-degree). Since the precise formulation and derivation of the result are involved, we shall first outline the crucial steps of the computation and describe the various nontrivial solutions. The precise formulation of the result and its proof will be given in the following Section 11.3.

The computation relies on the descent equation technique described in Section 9, which can be used because d has trivial cohomology at all form-degrees different from 0 and n,

$$H^p(d,\Omega) = \delta^p_0 \mathbb{R} \quad \text{for} \quad p < n. \tag{11.8}$$

This holds in the space of all local forms (case I) by the algebraic Poincaré lemma (theorem 4.2). It also holds in the space of Poincaré invariant local forms because d is Lorentz-invariant (cf. text after corollary 11.1). In fact, one may even deduce this

directly from the proof of the algebraic Poincaré lemma given in Section 4 because the operators ρ and P_m used there are manifestly Lorentz-invariant, and because there are no Lorentz-invariant constant forms $c_{\mu_1\ldots\mu_p}dx^{\mu_1}\ldots dx^{\mu_p}$ with form degree $0 < p < n$. In fact, that proof of the algebraic Poincaré lemma is not just an existence proof but provides an explicit construction of ω^{p-1} for given d-closed ω^p such that $\omega^p = d\omega^{p-1}$, both in case I and case II.

One distinguishes between solutions with a trivial descent and solutions with a nontrivial descent.

11.2.1 Solutions with a trivial descent.

These are solutions to (11.7) that can be redefined by the addition of trivial solutions such that they solve $s\omega^p = 0$. The result on $H(s)$ (corollary 11.2) implies then that these solutions have the form

$$\omega^p = I^{p\,\alpha}\Theta_\alpha \ , \quad I^{p\,\alpha} \in \mathcal{I} \tag{11.9}$$

modulo trivial solutions. We note that (11.9) can be trivial in $H(s|d)$ even if $I^{p\,\alpha} \not\approx 0$.

11.2.2 Lifts and equivariant characteristic cohomology

The solutions with a non trivial descent are those for which it is impossible to make ω^{p-1} vanish in (11.7) through the addition of trivial solutions. Their determination is more difficult. In particular it calls for the solution of a cohomological problem that we have not discussed so far. Namely, one has to determine the characteristic cohomology in the space $\mathcal{I}$ defined in (11.6). This cohomology is well defined because $d\mathcal{I} \subset \mathcal{I}$. Indeed, for $I \in \mathcal{I}$, one has $dI = DI$, where $D = dx^\mu D_\mu$, with D_μ the covariant derivative on the fields and $D_\mu x^\nu = \delta_\mu^\nu$. Furthermore DI is $\mathcal{G}$-invariant. We call this cohomology the "equivariant characteristic cohomology" and denote it by $H_{\text{char}}(d,\mathcal{I})$. It is related to, but different from the ordinary characteristic cohomology discussed in Section 6.

To understand the difference, assume that $I \in \mathcal{I}$ is weakly d-exact in Ω, i.e., $I \approx d\omega$ for some $\omega \in \Omega$. The equivariant characteristic cohomology poses the following question: is it possible to choose $\omega \in \mathcal{I}$? We shall answer this question in the affirmative with the exception when I contains a "characteristic class". Abusing slightly standard terminology, a characteristic class is in this context a $\mathcal{G}$-invariant polynomial in the curvature 2-forms F^I and is thus, in particular, an element of the small algebra.[11] Furthermore, we shall show that no characteristic class with form-degree $< n$ is trivial in $H_{\text{char}}(d,\mathcal{I})$ (at form-degree n there may be exceptions).

Hence, $H_{\text{char}}(d,\mathcal{I})$ is the sum of a subspace of $H_{\text{char}}(d,\Omega)$ (given by $H_{\text{char}}(d,\Omega)\cap\mathcal{I}$) and of the space of characteristic classes. Using corollary 11.1, one thus gets that

---

[11]It is a semantic coincidence that the word "characteristic" is used in the literature both in the context of the polynomials $P(F)$ and to term the weak cohomology of d. The invariant cohomology of d without antifields and use of the equations of motion has been investigated in [61, 62, 101]. The fact that it contains only the characteristic classes has been called the "covariant Poincaré lemma" in [61, 62].

$H_{\text{char}}(d, \mathcal{I})$ is at all form-degrees $< n - 1$ solely represented by characteristic classes (at form-degree $n - 2$ this is due to assumptions (a) and (b)). In contrast, there are in general additional nontrivial representatives of $H_{\text{char}}(d, \mathcal{I})$ at form-degrees n and $n - 1$.[12] At form-degree n they are present because an n-form is automatically d-closed but not necessarily weakly d-exact. The additional representatives at form-degree $n - 1$ are gauge-invariant nontrivial Noether currents written as $(n-1)$-forms.

The result on $H_{\text{char}}(d, \mathcal{I})$ is interesting in itself and a cornerstone of the local BRST cohomology in Yang-Mills type theories. The technical assumption of "normality" assumed throughout the calculation is made in order to be able to characterize completely $H_{\text{char}}(d, \mathcal{I})$. The properties of $H_{\text{char}}(d, \mathcal{I})$ are at the origin of the importance of the small algebra for the cohomology, as we now explain.

The equivariant characteristic cohomology arises as follows when discussing the descent equations. One has

$$s\omega^p + d\omega^{p-1} = 0 \ , \ \ s\omega^{p-1} + d\omega^{p-2} = 0 \ , \ \ \ldots \ , \ \ s\omega^{\underline{m}} = 0.$$

Without loss of generality, one can assume that the bottom form $\omega^{\underline{m}}$ is a nontrivial solution of the consistency condition. It is thus a solution with a trivial descent and can be taken of the form

$$\omega^{\underline{m}} = I^{\underline{m}\alpha}\Theta_\alpha \ , \quad I^{\underline{m}\alpha} \in \mathcal{I} \ . \tag{11.10}$$

In the case of a nontrivial descent, $\omega^{\underline{m}}$ satisfies additionally

$$s\omega^{\underline{m}+1} + d\omega^{\underline{m}} = 0 \tag{11.11}$$

which is the last but one descent equation. It turns out that this equation is a very restrictive condition on the bottom $\omega^{\underline{m}}$. Few bottoms can be lifted at least once. In order to be "liftable", all $I^{\underline{m}\alpha}$ must be representatives of the equivariant characteristic cohomology.

Indeed, one has

$$d(I^{\underline{m}\alpha}\Theta_\alpha) = (dI^{\underline{m}\alpha})\Theta_\alpha - s(I^{\underline{m}\alpha}[\Theta_\alpha]^1) \tag{11.12}$$

where we used $sI^{\underline{m}\alpha} = 0$ (which holds due to $I^{\underline{m}\alpha} \in \mathcal{I}$) and

$$d\Theta_\alpha + s[\Theta_\alpha]^1 = 0, \quad [\Theta_\alpha]^1 = A^I \frac{\partial\Theta_\alpha}{\partial C^I} \ . \tag{11.13}$$

(11.13) is nothing but the equation with $q = 1$ contained in Eqs. (10.28), for the particular case $M_{r_1\ldots r_K|s_1\ldots s_N} \equiv M_{r_1\ldots r_K}$.

Now, (11.10) through (11.12) imply $(dI^{\underline{m}\alpha})\Theta_\alpha = s(\ldots)$. By corollary 11.2 this implies that all $dI^{\underline{m}\alpha}$ vanish weakly,

$$\forall\, \alpha : \quad dI^{\underline{m}\alpha} \approx 0. \tag{11.14}$$

---

[12]An exception is 3-dimensional pure Chern-Simons theory (with semisimple gauge group) where $H_{\text{char}}(d, \mathcal{I})$ vanishes even in form-degrees $n = 3$ and $n - 1 = 2$, see Section 14.

Furthermore, if $I^{\underline{m}\,\alpha} \approx dI^{\underline{m}-1}$ for some $I^{\underline{m}-1} \in \mathcal{I}$ and some α, the piece $I^{\underline{m}\,\alpha}\Theta_\alpha$ (no sum over α here) can be removed by subtracting a trivial term from $\omega^{\underline{m}}$. Indeed, $I^{\underline{m}\,\alpha} \approx dI^{\underline{m}-1}$ implies that $I^{\underline{m}\,\alpha} = dI^{\underline{m}-1} + sK^{\underline{m}}$ for some $K^{\underline{m}}$ by corollary 11.3 and thus that $I^{\underline{m}\,\alpha}\Theta_\alpha$ is trivial, $I^{\underline{m}\,\alpha}\Theta_\alpha = d(I^{\underline{m}-1}\Theta_\alpha) + s(I^{\underline{m}-1}[\Theta_\alpha]^1 + K^{\underline{m}}\Theta_\alpha)$ (no sum over α). Hence, without loss of generality one can assume that

$$\forall\,\alpha: \quad I^{\underline{m}\,\alpha} \not\approx dI^{\underline{m}-1\,\alpha}, \quad I^{\underline{m}-1\,\alpha} \in \mathcal{I} \; . \tag{11.15}$$

By Eqs. (11.14) and (11.15), every $I^{\underline{m}\,\alpha}$ is a nontrivial representative of the equivariant characteristic cohomology. This cohomology qualifies thus to some extent the bottom forms that appear in nontrivial descents: all those $I^{\underline{m}\,\alpha}$ with $\underline{m} < n - 1$ can be assumed to be characteristic classes $P(F)$, while those with $\underline{m} = n - 1$ can additionally contain nontrivial gauge-invariant Noether currents (in the special case $n = 2$ or when free gauge symmetries are present, there can be bottom forms of yet another type with form-degree $n - 2$).

11.2.3 Solutions with a nontrivial descent

Of course, the previous discussion gives not yet a complete characterization of the bottom forms because $I^{\underline{m}\,\alpha}\Theta_\alpha$ may be trivial in $H(s|d,\Omega)$ even when all $I^{\underline{m}\,\alpha}$ represent nontrivial classes of the equivariant characteristic cohomology (the nontriviality of $I^{\underline{m}\,\alpha}$ in $H_{\mathrm{char}}(d,\mathcal{I})$ is necessary but not sufficient for the nontriviality of $I^{\underline{m}\,\alpha}\Theta_\alpha$). Furthermore one still has to investigate how far the nontrivial bottom forms can be maximally lifted (so far we have only discussed lifting bottom forms once). Nevertheless the above discussion gives already an idea of the result. Namely, one finds ultimately that the consistency condition has at most three types of solutions with a nontrivial descent:

1. Solutions which lie in the small algebra $\mathcal{B}$. These solutions are linear combinations of those $[M_{r_1\ldots r_K|s_1\ldots s_N}]^p$ in (10.38) with $\sum_{i=1}^N 2m(s_i) < p$ (the solutions with $\sum_{i=1}^N 2m(s_i) = p$ are s-closed and have thus a trivial descent). Here and in the following, such linear combinations are denoted by B^p,

$$B^p = \sum_{\underline{s}=p-2m(r_1)+1}^{p-1} \lambda^{r_1\ldots r_K|s_1\ldots s_N}[M_{r_1\ldots r_K|s_1\ldots s_N}]^p, \quad \underline{s} = \sum_{i=1}^N 2m(s_i) \tag{11.16}$$

where the $\lambda^{r_1\ldots r_K|s_1\ldots s_N}$ are constant coefficients. We shall prove that no non-vanishing B^p is trivial in $H(s|d,\Omega)$. In other words: B^p is trivial only if all coefficients $\lambda^{r_1\ldots r_K|s_1\ldots s_N}$ vanish (because the $[M_{r_1\ldots r_K|s_1\ldots s_N}]^p$ are linearly independent, see Section 10). The solutions B^p descend to bottom forms involving characteristic classes $P(F)$.

2. Antifield dependent solutions which involve nontrivial global symmetries corresponding to gauge invariant nontrivial Noether currents. These solutions cannot be given explicitly in a model independent manner because the set of

global symmetries is model dependent. To describe them we introduce the notation $\{j_\Delta^\mu\}$ for a basis of those Noether currents which can be brought to a form such that the corresponding $(n-1)$-forms are elements of $\mathcal{I}$ (possibly by the addition of trivial currents, see Section 6),

$$j_\Delta = \frac{1}{(n-1)!} dx^{\mu_1} \ldots dx^{\mu_{n-1}} \epsilon_{\mu_1 \ldots \mu_n} j_\Delta^{\mu_n} \in \mathcal{I} , \quad dj_\Delta \approx 0. \qquad (11.17)$$

"Basis" means here that (a) every Noether current which has a representative in $\mathcal{I}$ is a linear combination of the j_Δ^μ up to a current which is trivial in Ω, and (b) no nonvanishing linear combination of the j_Δ is trivial in $H_{\mathrm{char}}(d, \Omega)$,

$$dI^{n-1} \approx 0, \quad I^{n-1} \in \mathcal{I} \quad \Rightarrow \quad I^{n-1} \approx \lambda^\Delta j_\Delta + d\omega^{n-2} ; \qquad (11.18)$$
$$\lambda^\Delta j_\Delta \approx d\omega^{n-2} \quad \Rightarrow \quad \lambda^\Delta = 0 \quad \forall \Delta . \qquad (11.19)$$

Note that, in general, $\{j_\Delta\}$ does not represent a complete basis of $H_{\mathrm{char}}^{n-1}(d, \mathcal{I})$ because our definition does not use the coboundary condition in $H_{\mathrm{char}}^{n-1}(d, \mathcal{I})$ ($\{j_\Delta\}$ does not contain characteristic classes $P(F)$ while $H_{\mathrm{char}}^{n-1}(d, \mathcal{I})$ may contain characteristic classes when n is odd). Note also that, in general, $\{j_\Delta^\mu\}$ differs in case I and II. For instance, the various components $T_0{}^\mu$, ..., $T_{n-1}{}^\mu$ of the energy momentum tensor $T_\nu{}^\mu$ can normally be redefined such that they are gauge invariant[13] and provide then n elements of $\{j_\Delta\}$ in case I; however, they are not contravariant Lorentz-vectors and therefore they do not provide elements of $\{j_\Delta\}$ in case II. Similarly, in globally supersymmetric Yang-Mills models, the supersymmetry currents provide normally elements of $\{j_\Delta\}$ in case I, but not in case II.

Since j_Δ is gauge invariant, one has $dj_\Delta = Dj_\Delta$ and therefore dj_Δ is gauge invariant and s-invariant too. By corollary 11.3, $dj_\Delta \approx 0$ implies thus the existence of a volume form K_Δ such that

$$sK_\Delta + dj_\Delta = 0. \qquad (11.20)$$

K_Δ encodes the global symmetry corresponding to j_Δ (see Section 6 and also Section 7 for the connection between s and δ). We now define the n-forms

$$V_{\Delta\alpha} = K_\Delta \Theta_\alpha + j_\Delta [\Theta_\alpha]^1 \qquad (11.21)$$

with $[\Theta_\alpha]^1$ as in (11.13). These forms solve the consistency condition. Indeed, Eqs. (11.13) and (11.20) give immediately

$$sV_{\Delta\alpha} + d\,[j_\Delta \Theta_\alpha] = 0. \qquad (11.22)$$

This also shows that $j_\Delta \Theta_\alpha$ is a bottom form corresponding to $V_{\Delta\alpha}$ (as one has $s(j_\Delta \Theta_\alpha) = 0$ due to $j_\Delta \in \mathcal{I}$).

---

[13] An example where $T_0{}^\mu$ cannot be made gauge invariant is example 3 in Section 12.1.3.

3. In special cases (i.e., for special Lagrangians), there are additional nontrivial antifield dependent solutions. They emerge from nontrivial conserved currents which can *not* be made gauge invariant by the addition of trivial currents. These "accidental" solutions complicate the derivation of the general solution of the consistency condition but, even though exceptional, they must be covered since we do not make restrictions on the Lagrangian besides the technical ones explained above.

We shall prove that, if $n > 2$ and free abelian gauge symmetries are absent, such currents exist if and only if characteristic classes with maximal form-degree n are trivial in $H_{\mathrm{char}}(d, \mathcal{I})$ (examples are given in Section 12). Assume that $\{P_A(F)\}$ is a basis for characteristic classes of this type, i.e., assume that every characteristic class with form-degree n which is trivial in $H_{\mathrm{char}}(d, \mathcal{I})$ is a linear combination of the $P_A(F)$ and the $P_A(F)$ are linearly independent,

$$
\begin{align}
P_A(F) &\approx dI_A^{n-1}, \quad I_A^{n-1} \in \mathcal{I} ; & (11.23) \\
P^n(F) &\approx dI^{n-1}, \quad I^{n-1} \in \mathcal{I} \quad \Rightarrow \quad P^n = \lambda^A P_A(F) ; & (11.24) \\
\lambda^A P_A(F) &= 0 \quad \Rightarrow \quad \lambda^A = 0 \quad \forall A . & (11.25)
\end{align}
$$

Note that $P_A(F) \approx dI_A^{n-1}$ implies $d(I_A^{n-1} - q_A^{n-1}) \approx 0$ where q_A^{n-1} is a Chern-Simons $(n-1)$-form fulfilling $P_A(F) = dq_A^{n-1}$. The $I_A^{n-1} - q_A^{n-1}$ are thus conserved $(n-1)$-forms; they are the afore-mentioned Noether currents that cannot be made gauge invariant. $P_A(F) - dI_A^{n-1}$ is s-invariant (as it is in $\mathcal{I}$) and vanishes weakly; hence, corollary 11.3 guarantees the existence of a volume form K_A such that

$$
P_A(F) = dI_A^{n-1} + sK_A . \tag{11.26}
$$

This gives $sK_A + d(I_A^{n-1} - q_A^{n-1}) = 0$, i.e. K_A is a cocycle of $H^{-1,n}(s|d)$ (it contains the global symmetry corresponding to the Noether current $I_A^{n-1} - q_A^{n-1}$).

Every linear combination of n-forms $P_A(F) \Theta_\alpha$ belongs to the cohomology of s in the small algebra and can therefore be expanded in the N_i and M_a in (10.20). We now consider only those linear combinations which can be written solely in terms of N_i (which must have form degree n). A basis for these linear combinations is denoted by $\{N_\Gamma\}$:

$$
\begin{align}
N_\Gamma &= k_\Gamma^i N_i = k_\Gamma^{A\alpha} P_A(F) \Theta_\alpha ; & (11.27) \\
\lambda^i N_i &= \lambda^{A\alpha} P_A(F) \Theta_\alpha \quad \Leftrightarrow \quad \lambda^i N_i = \lambda^\Gamma N_\Gamma ; & (11.28) \\
\lambda^\Gamma N_\Gamma &= 0 \quad \Rightarrow \quad \lambda^\Gamma = 0 \quad \forall \Gamma . & (11.29)
\end{align}
$$

In particular one can choose the basis $\{N_\Gamma\}$ such that it contains $\{P_A(F)\}$ (since $\{N_i\}$ contains a basis of all $P(F)$).

Now, on the one hand, one gets

$$
\begin{align}
N_\Gamma &= k_\Gamma^{A\alpha} (dI_A^{n-1} + sK_A) \Theta_\alpha \\
&= k_\Gamma^{A\alpha} \left[d(I_A^{n-1} \Theta_\alpha) + s(I_A^{n-1} [\Theta_\alpha]^1 + K_A \Theta_\alpha) \right] & (11.30)
\end{align}
$$

where we used (11.26) and (11.13). On the other hand, N_Γ is a linear combination of those N_i with form-degree n and thus of the form

$$N_\Gamma = s b_\Gamma^n + d B_\Gamma^{n-1} \ , \quad b_\Gamma^n , \ B_\Gamma^{n-1} \in \mathcal{B} \tag{11.31}$$

by the first equation in (10.28) [b_Γ^n is a linear combination of the $[M_{r_1 \dots r_K | s_1 \dots s_N}]^n$ with $n = 2m(r_1) + \sum_{i=1}^N 2m(s_i)$ and is therefore not of the form (11.16), in contrast to B_Γ^{n-1}]. Subtracting (11.30) from (11.31), one gets

$$s W_\Gamma + d[B_\Gamma^{n-1} - k_\Gamma^{A\alpha} I_A^{n-1} \Theta_\alpha] = 0 \tag{11.32}$$

where

$$W_\Gamma = b_\Gamma^n - k_\Gamma^{A\alpha} \left(K_A \Theta_\alpha + I_A^{n-1} [\Theta_\alpha]^1 \right) . \tag{11.33}$$

W_Γ descends to a bottom form in the small algebra which involves characteristic classes $P(F)$, namely to the same bottom form to which B_Γ^{n-1} descends. From the point of view of the descent equations this means the following. In the small algebra, the bottom-form corresponding to B_Γ^{n-1} can only be lifted to form-degree $(n-1)$; there is no way to lift it to an n-form in the small algebra because this lift is obstructed by N_Γ. However, there is no such obstruction in the full algebra because the characteristic classes contained in N_Γ are trivial in $H_{\mathrm{char}}^n(d, \mathcal{I})$ (W_Γ is not entirely in the small algebra: in particular it contains antifields through the K_A).

11.3   Main result and its proof

According to the discussion in Section 11.2 it may appear natural to determine first $H_{\mathrm{char}}(d, \mathcal{I})$ and afterwards $H(s|d, \Omega)$. In fact that strategy was followed in previous computations [23, 24]. However, it is more efficient to determine $H_{\mathrm{char}}(d, \mathcal{I})$ and $H(s|d, \Omega)$ at a stroke. The reason is that these two cohomologies are strongly interweaved. In fact, not only does one need $H_{\mathrm{char}}(d, \mathcal{I})$ to compute $H(s|d, \Omega)$; but also, $H_{\mathrm{char}}(d, \mathcal{I})$ at form-degree p can be computed by means of $H(s|d, \Omega)$ at lower former degrees using descent equation techniques. That makes it possible to determine both cohomologies simultaneously in a recursive manner, starting at form-degree 0 where $H(s|d, \Omega)$ reduces to $H(s, \Omega)$, and then proceeding successively to higher form-degrees. This strategy streamlines the derivation as compared to previously used approaches (but reaches of course identical conclusions!), and is reflected in the formulation and proof of the theorem given below. The theorem is formulated such that it applies both to case I and to case II; however, the different meaning of Ω, $\mathcal{I}$ (and thus also j_Δ and $V_{\Delta\alpha}$) in these cases should be kept in mind.

To formulate and prove the theorem, we use the same notation as in Section 11.2. In addition we introduce, similarly to (11.16), the notation M^p, N^p and b^p for linear combinations of the $[M_{r_1 \dots r_K | s_1 \dots s_N}]^{\underline{s}}$, $N_{r_1 \dots r_K s_1 \dots s_N}(\theta(C), f(F))$, and $[M_{r_1 \dots r_K | s_1 \dots s_N}]^{\underline{s} + 2m(r_1)}$ with form-degree p respectively,

$$M^p \ \equiv \ \sum_{\underline{s} = p} \lambda^{r_1 \dots r_K | s_1 \dots s_N} M_{r_1 \dots r_K | s_1 \dots s_N}(\theta(C), f(F)) \ , \tag{11.34}$$

$$N^p \equiv \sum_{\underline{s}+2m(r_1)=p} \lambda^{r_1\dots r_K s_1\dots s_N} N_{r_1\dots r_K s_1\dots s_N}(\theta(C), f(F)) \tag{11.35}$$

$$b^p \equiv \sum_{\underline{s}+2m(r_1)=p} \lambda^{r_1\dots r_K|s_1\dots s_N} [M_{r_1\dots r_K|s_1\dots s_N}]^{\underline{s}+2m(r_1)} \tag{11.36}$$

where we used once again the notation $\underline{s} = \sum_{i=1}^{N} 2m(s_i)$. Note that we have, for every B^p as in (11.16),

$$sB^p = -d(B^{p-1} + M^{p-1}), \quad dB^p = -s(B^{p+1} + b^{p+1}) + N^{p+1} \tag{11.37}$$

for some B^{p-1}, M^{p-1}, B^{p+1}, b^{p+1} and N^{p+1} by Eqs. (10.28).

We can now formulate the result as follows.

Theorem 11.1 *Let $\omega^p \in \Omega$ with Ω as in (11.1), $I^p, I^{p\alpha} \in \mathcal{I}$ with $\mathcal{I}$ as in (11.6). Let $P^p(F)$ denote characteristic classes (p indicating the form-degree respectively), and $\sim$ denoting equivalence in $H(s|d, \Omega)$ (i.e. $\omega'^p \sim \omega^p$ means $\omega'^p = \omega^p + s\eta^p + d\eta^{p-1}$ for some $\eta^p, \eta^{p-1} \in \Omega$). For Yang-Mills type theories without free abelian gauge symmetries, the following statements hold in all spacetime dimensions $n > 2$:*

(i) At all form-degrees $p < n$, the general solution of the consistency condition is given, up to trivial solutions, by the sum of a term $I^\alpha \Theta_\alpha$ and a solution in the small algebra as in Eq. (11.16); at form-degree $p = n$ it contains in addition a linear combination of the $V_{\Delta\alpha}$ and W_Γ given in (11.21) and (11.33) respectively,

$$s\omega^p + d\omega^{p-1} = 0 \iff \omega^p \sim I^{p\alpha}\Theta_\alpha + B^p + \delta_n^p(\lambda^{\Delta\alpha}V_{\Delta\alpha} + \lambda^\Gamma W_\Gamma). \tag{11.38}$$

(ii) $I^{p\alpha}\Theta_\alpha + B^p + \delta_n^p(\lambda^{\Delta\alpha}V_{\Delta\alpha} + \lambda^\Gamma W_\Gamma)$ is trivial in $H(s|d, \Omega)$ if and only if B^p and all coefficients $\lambda^{\Delta\alpha}$, λ^Γ vanish and $I^{p\alpha}\Theta_\alpha$ is weakly equal to $N^p + (dI^{p-1\alpha})\Theta_\alpha$ for some N^p and $I^{p-1\alpha}$,[14]

$$I^{p\alpha}\Theta_\alpha + B^p + \delta_n^p(\lambda^{\Delta\alpha}V_{\Delta\alpha} + \lambda^\Gamma W_\Gamma) \sim 0$$

$$\iff B^p = 0, \; \lambda^{\Delta\alpha} = 0 \; \forall (\Delta, \alpha), \; \lambda^\Gamma = 0 \; \forall \Gamma, \; I^{p\alpha}\Theta_\alpha \approx N^p + (dI^{p-1\alpha})\Theta_\alpha \tag{11.39}$$

(iii) If $I^p \in \mathcal{I}$ ($p > 0$) is trivial in $H^p_{\mathrm{char}}(d, \Omega)$ then it is the sum of a characteristic class and a piece which is trivial in $H^p_{\mathrm{char}}(d, \mathcal{I})$,

$$p > 0: \quad I^p \approx d\omega^{p-1} \iff I^p \approx P^p(F) + dI^{p-1}. \tag{11.40}$$

(iv) No nonvanishing characteristic class with form-degree $p < n$ is trivial in $H_{\mathrm{char}}(d, \mathcal{I})$,

$$p < n: \quad P^p(F) \approx dI^{p-1} \implies P^p(F) = 0. \tag{11.41}$$

Proof.

Step 1. To prove the theorem, we first verify that (i), (ii) and (iv) hold at form-degree 0. There are no B^0 or N^0 since the ranges of values for $\underline{s}$ in the sums in Eqs. (11.16) and (11.35) are empty for $p = 0$. Hence, for $p = 0$, (i) and (ii) reduce to

---

[14]Note that this requires $I^{p\alpha} \approx dI^{p-1\alpha} + P^{p\alpha}(F)$ with $P^{p\alpha}(F)$ such that $P^{p\alpha}(F)\Theta_\alpha = N^p$.

$s\omega^0 = 0 \Leftrightarrow \omega^0 = I^{0\alpha}\Theta_\alpha + s\eta^0$ and $I^{0\alpha}\Theta_\alpha = s\eta^0 \Leftrightarrow I^{0\alpha} \approx 0$ respectively and hold by the results on $H(s)$ (corollary 11.2). (iv) reduces for $p = 0$ to $constant \approx 0 \Leftrightarrow constant = 0$ which holds for every meaningful Lagrangian (if it would not hold then the equations of motion were inconsistent, see Section 9).

Step 2. In the second (and final) step we show that (i) through (iv) hold for $p = m$ if they hold for $p = m - 1$, excluding $m = n$ in the case (iv).

(iv) By corollary 11.3, $P^m(F) \approx dI^{m-1}$ implies $P^m(F) = dI^{m-1} + sK^m$ for some local K^m. On the other hand one has $P^m(F) = dq^{m-1}$ for some Chern-Simons $(m-1)$-form q^{m-1} which we choose to be the B^{m-1} corresponding to $P^m(F)$.[15] This gives $sK^m + d(I^{m-1} - q^{m-1}) = 0$, i.e., K^m is a cocycle of $H^{-1,m}(s|d, \Omega)$. One has $H^{-1,m}(s|d, \Omega) \simeq H_1^m(\delta|d, \Omega) \simeq H_0^{m-1}(d|\delta, \Omega) = H_{\text{char}}^{m-1}(d, \Omega)$ for $m > 1$ and analogously $H^{-1,1}(s|d, \Omega) \simeq H_{\text{char}}^0(d, \Omega)/\mathbb{R}$ by theorems 7.1 and 6.2. By corollary 11.1 this gives $H^{-1,m}(s|d, \Omega) = 0$ (since we are assuming $0 < m < n$) and thus $K^m \sim 0$. This implies $I^{m-1} - q^{m-1} \sim 0$ by the standard properties of the descent equations[16]. Now, since we assume that (ii) holds for $p = m - 1$, we conclude from $I^{m-1} - q^{m-1} \sim 0$ in particular that $q^{m-1} = 0$ (since q^{m-1} is a B^{m-1}) and thus that $P^m(F) = dq^{m-1} = 0$ which is (iv) for $p = m < n$.

(i) $s\omega^m + d\omega^{m-1} = 0$ implies descent equations (Section 9). In particular there is some ω^{m-2} such that $s\omega^{m-1} + d\omega^{m-2} = 0$. Since we assume that (i) holds for $p = m - 1$, we conclude

$$\omega^{m-1} = I^{m-1\,\alpha}\Theta_\alpha + B^{m-1} \tag{11.42}$$

for some $I^{m-1\,\alpha}$ and B^{m-1} (without loss of generality, since trivial contributions to any form in the descent equations can be neglected, see Section 9). By (11.37) we have

$$dB^{m-1} = -s(\hat{B}^m + b^m) + N^m \tag{11.43}$$

for some $\hat{B}^m$, b^m, N^m. Using in addition (11.13), we get

$$d\omega^{m-1} = (dI^{m-1\,\alpha})\Theta_\alpha - s(I^{m-1\,\alpha}[\Theta_\alpha]^1 + \hat{B}^m + b^m) + N^m.$$

Inserting this in $s\omega^m + d\omega^{m-1} = 0$, we obtain

$$s(\omega^m - I^{m-1\,\alpha}[\Theta_\alpha]^1 - \hat{B}^m - b^m) + [dI^{m-1\,\alpha} + P^{m\,\alpha}(F)]\Theta_\alpha = 0 \tag{11.44}$$

where we used that

$$N^m = P^{m\,\alpha}(F)\Theta_\alpha \tag{11.45}$$

---

[15] Note that there can be an ambiguity in the choice of Chern-Simons forms. Indeed, consider $P^m(F) = f_1(F)f_2(F)$. One has $P^m(F) = d[\alpha q_1(A, F)f_2(F) + (1 - \alpha)q_2(A, F)f_1(F)]$ where α is an arbitrary number. Our prescription in Section 10 selects $B^{m-1} = q_1(A, F)f_2(F)$ $(\alpha = 1)$. This extends to all characteristic classes $P^m(F)$: our prescription selects precisely one B^{m-1} among all Chern-Simons forms corresponding to $P^m(F)$.

[16] If one of the forms in the descent equations is trivial, then all its descendants are trivial too, see Section 9.

for some $P^{m\alpha}(F)$. Using corollary 11.2, we conclude from (11.44) that

$$dI^{m-1\alpha} + P^{m\alpha}(F) \approx 0 \quad \forall \alpha. \tag{11.46}$$

To go on, we must distinguish the cases $m < n$ and $m = n$.

$\underline{m < n}$. We have just proved that (iv) holds for $p = m$ if $m < n$. Using this, we conclude from (11.46) that

$$m < n: \quad P^{m\alpha}(F) = 0 \quad \forall \alpha . \tag{11.47}$$

Hence, N^m vanishes, see (11.45). Therefore b^m vanishes as well because b^m is present in Eq. (11.43) only if N^m is present too [using Eqs. (10.28), one verifies this by making the linear combinations of the $[M_{r_1...r_K|s_1...s_N}]^p$ and $N_{r_1...r_K s_1...s_N}$ explicit that enter in (11.37)]. Hence, we have

$$m < n: \quad N^m = b^m = 0, \quad dB^{m-1} = -s\hat{B}^m. \tag{11.48}$$

Moreover, using (11.47) in (11.46), we get $dI^{m-1\alpha} \approx 0$. Hence, $I^{m-1\alpha}$ is weakly d-closed and has form-degree $< n-1$ (since we are discussing the cases $m < n$). We conclude, using corollary 11.1, that $I^{m-1\alpha} \approx d\omega^{m-2\alpha}$ for some $\omega^{m-2\alpha}$ if $m-1 > 0$, or $I^{0\alpha} = \lambda^\alpha$ for some constants $\lambda^\alpha \in \mathbb{R}$ if $m-1 = 0$. Since we assume that (iii) holds for $p = m-1$, we get

$$m < n: \quad I^{m-1\alpha} \approx dI^{m-2\alpha} + P^{m-1\alpha}(F) \quad \forall \alpha \tag{11.49}$$

(with $P^{0\alpha}(F) \equiv \lambda^\alpha$ if $m-1 = 0$). Using corollary 11.3 we conclude from (11.49) that $I^{m-1\alpha} - dI^{m-2\alpha} - P^{m-1\alpha}(F)$ is s-exact,

$$m < n: \quad I^{m-1\alpha} = sK^{m-1\alpha} + dI^{m-2\alpha} + P^{m-1\alpha}(F) \quad \forall \alpha . \tag{11.50}$$

Using (11.50) in (11.42), we get

$$m < n: \quad \omega^{m-1} = [P^{m-1\alpha}(F) + sK^{m-1\alpha} + dI^{m-2\alpha}]\Theta_\alpha + B^{m-1}. \tag{11.51}$$

To deal with the first term on the right hand side of (11.51), we use that every $P^{m-1\alpha}(F)\Theta_\alpha$ can be written as $P^{m-1\alpha}(F)\Theta_\alpha = N^{m-1} + M^{m-1} + \delta_0^{m-1}\hat{\lambda}$ for some N^{m-1} and M^{m-1} and some constant $\hat{\lambda}$ which can only contribute if $m - 1 = 0$. This is guaranteed because $\{1, M_{r_1...r_K|s_1...s_N}(\theta(C), f(F)), N_{r_1...r_K s_1...s_N}(\theta(C), f(F))\}$ is a basis of all $P^\alpha(F)\Theta_\alpha$, see Section 10 (note that this is the place where we use the completeness property of this basis).

Now, N^{m-1} is trivial in $H(s|d, \Omega)$ since each $N_{r_1...r_K s_1...s_N}(\theta(C), f(F))$ is trivial, see Eqs. (10.28). Furthermore, $(sK^{m-1\alpha} + dI^{m-2\alpha})\Theta_\alpha$ is trivial too, due to

$$(sK^{m-1\alpha} + dI^{m-2\alpha})\Theta_\alpha = s(K^{m-1\alpha}\Theta_\alpha + I^{m-2\alpha}[\Theta_\alpha]^1) + d(I^{m-2\alpha}\Theta_\alpha)$$

where we used once again (11.13). Since trivial contributions to ω^{m-1} can be neglected (see above), we can thus assume, without loss of generality,

$$m < n: \quad \omega^{m-1} = M^{m-1} + B^{m-1} + \delta_0^{m-1}\,\hat{\lambda}. \tag{11.52}$$

For every M^{m-1} there is a $\tilde{B}^m$ such that $dM^{m-1} = -s\tilde{B}^m$. This holds by Eqs. (10.28) (more precisely: the equation with $q = 1$ there). Using in addition Eq. (11.48), we get

$$m < n: \quad d\omega^{m-1} = -sB^m, \quad B^m = \tilde{B}^m + \hat{B}^m.$$

Using this in $s\omega^m + d\omega^{m-1} = 0$, we get

$$m < n: \quad s(\omega^m - B^m) = 0.$$

From this we conclude, using corollary 11.2,

$$m < n: \quad \omega^m \sim B^m + I^{m\alpha}\Theta_\alpha\,. \tag{11.53}$$

This proves (11.38) for $p = m$ if $m < n$.

$\underline{m = n}$. In this case we conclude from (11.46), using (11.23) and (11.24),

$$P^{n\alpha} = \lambda^{A\alpha}P_A\,, \quad d(I^{n-1\alpha} + \lambda^{A\alpha}I_A^{n-1}) \approx 0 \tag{11.54}$$

for some constant coefficents $\lambda^{A\alpha}$. Using (11.18), we conclude from (11.54) that $I^{n-1\alpha} + \lambda^{A\alpha}I_A^{n-1} \approx \lambda^{\Delta\alpha}j_\Delta + d\omega^{n-2}$ for some constant coefficients $\lambda^{\Delta\alpha}$ and some ω^{n-2}. Hence, $I^{n-1\alpha} + \lambda^{A\alpha}I_A^{n-1} - \lambda^{\Delta\alpha}j_\Delta \in \mathcal{I}$ is weakly d-exact. Using (iii) for $p = m-1 = n-1$ and then once again corollary 11.3, we conclude from (11.54)

$$I^{n-1\alpha} = -\lambda^{A\alpha}I_A^{n-1} + \lambda^{\Delta\alpha}j_\Delta + P^{n-1\alpha}(F) + dI^{n-2\alpha} + sK^{n-1\alpha} \tag{11.55}$$

for some $K^{n-1\alpha}$. Furthermore, because of (11.28) and (11.54), we have $N^n = \lambda^\Gamma N_\Gamma$ in Eq. (11.45), for some λ^Γ such that $\lambda^{A\alpha} = \lambda^\Gamma k_\Gamma^{A\alpha}$. Now consider

$$\begin{aligned}\hat{\omega}^n &:= \omega^n - \lambda^{\Delta\alpha}V_{\Delta\alpha} - \lambda^\Gamma W_\Gamma \\ \hat{\omega}^{n-1} &:= \omega^{n-1} - \lambda^{\Delta\alpha}j_\Delta\Theta_\alpha - \lambda^\Gamma[B_\Gamma^{n-1} - k_\Gamma^{A\alpha}\,I_A^{n-1}\Theta_\alpha] \\ &= [P^{n-1\alpha}(F) + dI^{n-2\alpha} + sK^{n-1\alpha}]\Theta_\alpha + \hat{B}^{n-1} \end{aligned} \tag{11.56}$$

where $\hat{B}^{n-1} = B^{n-1} - \lambda^\Gamma B_\Gamma^{n-1}$. One has $s\hat{\omega}^n + d\hat{\omega}^{n-1} = 0$, due to (11.22) and (11.32) (and $s\omega^n + d\omega^{n-1} = 0$). The last line in (11.56) is analogous to (11.51). By the same arguments that have led from (11.51) to (11.53), we conclude that

$$\hat{\omega}^n \sim B^n + I^{n\alpha}\Theta_\alpha\,.$$

This yields (11.38) for $p = m = n$ due to $\omega^n = \hat{\omega}^n + \lambda^{\Delta\alpha}V_{\Delta\alpha} + \lambda^\Gamma W_\Gamma$.

(ii) We shall treat the case $m = n$; the proof for $m < n$ is simpler and obtained from the one for $m = n$ by setting $\lambda^{\Delta\alpha} = \lambda^\Gamma = 0$ and substituting m for n in the following formulae.

Consider the n-form $\omega^n = I^{n\,\alpha}\Theta_\alpha + B^n + \lambda^{\Delta\alpha}V_{\Delta\alpha} + \lambda^\Gamma W_\Gamma$. Due to (11.37), (11.22) and (11.32), one has

$$s(I^{n\,\alpha}\Theta_\alpha) = 0 \tag{11.57}$$
$$sB^n = -d(B^{n-1} + M^{n-1}) \tag{11.58}$$
$$sV_{\Delta\alpha} = -d\,[j_\Delta\Theta_\alpha] \tag{11.59}$$
$$sW_\Gamma = -d[B_\Gamma^{n-1} - k_\Gamma^{A\alpha}\,I_A^{n-1}\Theta_\alpha] \tag{11.60}$$

for some B^{n-1} and M^{n-1}. Hence, one has $s\omega^n + d\omega^{n-1} = 0$ where $\omega^{n-1} = \hat{B}^{n-1} + \hat{I}^{n-1\,\alpha}\Theta_\alpha$ with

$$\hat{B}^{n-1} = B^{n-1} + \lambda^\Gamma B_\Gamma^{n-1} \tag{11.61}$$
$$\hat{I}^{n-1\,\alpha}\Theta_\alpha = M^{n-1} + \lambda^{\Delta\alpha}j_\Delta\Theta_\alpha - \lambda^\Gamma k_\Gamma^{A\alpha}\,I_A^{n-1}\Theta_\alpha \tag{11.62}$$

We assume now that ω^n is trivial,

$$I^{n\,\alpha}\Theta_\alpha + B^n + \lambda^{\Delta\alpha}V_{\Delta\alpha} + \lambda^\Gamma W_\Gamma \sim 0. \tag{11.63}$$

Then ω^{n-1} is trivial too (see footnote 16),

$$\hat{B}^{n-1} + \hat{I}^{n-1\,\alpha}\Theta_\alpha \sim 0. \tag{11.64}$$

Since we assume that (ii) holds for $p = n - 1$, we conclude from Eq. (11.64)

$$B^{n-1} = -\lambda^\Gamma B_\Gamma^{n-1} \quad (\Leftrightarrow\ \hat{B}^{n-1} = 0) \tag{11.65}$$
$$\hat{I}^{n-1\,\alpha}\Theta_\alpha \approx N^{n-1} + (dI^{n-2\,\alpha})\Theta_\alpha\ . \tag{11.66}$$

(11.65) implies that both $\lambda^\Gamma B_\Gamma^{n-1}$ and B^{n-1} vanish. This is trivial if n is odd because then no N_Γ is present (recall that N_Γ is a polynomial in the C^I and F^I and has thus even form-degree). If n is even, then no M^{n-1} can be present in Eq. (11.58) (as M^{n-1} is a polynomial in the C^I and F^I too). By (11.31), we have $d(\lambda^\Gamma B_\Gamma^{n-1}) = -s(\lambda^\Gamma b_\Gamma^n) + \lambda^\Gamma N_\Gamma$. Using this and (11.65) in Eq. (11.58), for n even, one gets $s(B^n + \lambda^\Gamma b_\Gamma^n) = \lambda^\Gamma N_\Gamma$, i.e., $\lambda^\Gamma N_\Gamma$ is s-exact in the small algebra. This implies $\lambda^\Gamma N_\Gamma = 0$ because $\lambda^\Gamma N_\Gamma$ is a linear combination of nontrivial representatives of $H(s, \mathcal{B})$ by construction (recall that it is a linear combination of the N_i in corollary 10.5) and is thus s-exact in $\mathcal{B}$ only if it vanishes. $\lambda^\Gamma N_\Gamma = 0$ implies that all coefficients λ^Γ vanish because the N_Γ are linearly independent by assumption, see Eq. (11.29). Hence, we get indeed

$$\lambda^\Gamma = 0 \quad \forall\Gamma \tag{11.67}$$

and thus also, by Eq. (11.65),

$$B^{n-1} = 0. \tag{11.68}$$

Using $\lambda^\Gamma = 0$, Eqs. (11.62) and (11.66) give

$$N^{n-1} - M^{n-1} \approx \lambda^{\Delta\alpha} j_\Delta \Theta_\alpha - (dI^{n-2\,\alpha})\Theta_\alpha \ . \qquad (11.69)$$

We have

$$N^{n-1} - M^{n-1} = P^{n-1\,\alpha}(F)\Theta_\alpha(C) \qquad (11.70)$$

for some $P^{n-1\,\alpha}(F)$. By assumption no nonvanishing linear combination of the j_Δ is weakly d-exact, see Eq. (11.19). Since each $P^{n-1\,\alpha}(F)$ is d-exact, (11.69) implies

$$\lambda^{\Delta\alpha} = 0 \quad \forall\,(\Delta,\alpha). \qquad (11.71)$$

Since we assume that (iv) holds for $p = n-1$, we conclude from (11.69) through (11.71) also that all $P^{n-1\,\alpha}(F)$ vanish and thus that $N^{n-1} - M^{n-1} = 0$. The latter implies that N^{n-1} and M^{n-1} vanish separately because they contain independent representatives of $H(s,\mathcal{B})$,

$$N^{n-1} = 0, \quad M^{n-1} = 0.$$

Using this and Eq. (11.68) in (11.58), the latter turns into $sB^n = 0$. By the very definition (11.16), B^n is a linear combination of terms with nonvanishing and linearly independent s-transformations. Hence, $sB^n = 0$ holds if and only if B^n itself vanishes. We conclude

$$B^n = 0. \qquad (11.72)$$

(11.67), (11.71) and (11.72) provide already the assertions for λ^Γ, $\lambda^{\Delta\alpha}$ and B^n in part (ii) of the theorem. We still have to prove those for $I^{n\,\alpha}\Theta_\alpha$. Using $\lambda^\Gamma = \lambda^{\Delta\alpha} = B^n = 0$, (11.63) reads

$$I^{n\,\alpha}\Theta_\alpha = s\eta^n + d\eta^{n-1} \qquad (11.73)$$

where we made the trivial terms explicit. Acting with s on this equation gives $d(s\eta^{n-1}) = 0$ and thus

$$s\eta^{n-1} + d\eta^{n-2} = 0 \qquad (11.74)$$

for some η^{n-2}, thanks to the algebraic Poincaré lemma. Since we assume that (i) holds for $p = n-1$, we conclude from (11.74) that

$$\eta^{n-1} = \tilde{B}^{n-1} + \tilde{I}^{n-1\,\alpha}\Theta_\alpha + s\tilde{\eta}^{n-1} + d\tilde{\eta}^{n-2}. \qquad (11.75)$$

As above, we have

$$d\tilde{B}^{n-1} = -s(\tilde{B}^n + \tilde{b}^n) + \tilde{N}^n$$
$$d(\tilde{I}^{n-1\,\alpha}\Theta_\alpha) = (d\tilde{I}^{n-1\,\alpha})\Theta_\alpha - s(\tilde{I}^{n-1\,\alpha}[\Theta_\alpha]^1).$$

Using this in (11.73), we get

$$[I^{n\,\alpha} - d\tilde{I}^{n-1\,\alpha} - \tilde{P}^{n\,\alpha}]\Theta_\alpha = s(\eta^n - \tilde{B}^n - \tilde{b}^n - d\tilde{\eta}^{n-1} - \tilde{I}^{n-1\,\alpha}[\Theta_\alpha]^1), \qquad (11.76)$$

109

where

$$\tilde{P}^{n\alpha}\Theta_\alpha = \tilde{N}^n . \tag{11.77}$$

Using corollary 11.2 we conclude from (11.76) that

$$I^{n\alpha} - d\tilde{I}^{n-1\alpha} - \tilde{P}^{n\alpha} \approx 0. \tag{11.78}$$

(11.77) and (11.78) complete the demonstration of (ii).

(iii) $I^m \approx d\omega^{m-1}$ implies $I^m \sim 0$, i.e. I^m is trivial in $H(s|d, \Omega)$. Indeed, $I^m \approx d\omega^{m-1}$ means that $I^m = \delta\omega^m + d\omega^{m-1}$ for some ω^m with antifield number 1. Hence, I^m is a cocycle of $H(s|d, \Omega)$ (since it is s-closed due to $I^m \in \mathcal{I}$) and trivial in $H(\delta|d, \Omega)$. It is therefore also trivial in $H(s|d, \Omega)$ by theorem 7.1 (cf. proof of (7.6)). Now, $I^m \sim 0$ is just a special case of $I^{p\alpha}\Theta_\alpha \sim 0$ (due to $1 \in \{\Theta_\alpha\}$). Hence, using (ii) for $p = m$ (which we have already proved), we conclude $I^m \approx dI^{m-1} + P^m(F)$ for some $I^{m-1} \in \mathcal{I}$ and some $P^m(F)$. Conversely, if $m > 0$, we have $P^m(F) = dq^{m-1}$ for some Chern-Simons form q^{m-1} and thus $I^m \approx dI^{m-1} + P^m(F)$ implies $I^m \approx d\omega^{n-1}$ with $\omega^{n-1} = I^{m-1} + q^{m-1}$. $\square$

11.4 Appendix 11.A: 2-dimensional pure Yang-Mills theory

Pure 2-dimensional Yang-Mills theory needs a special treatment because $H^{n-2}_{char}(d, \Omega) \equiv H^0_{char}(d, \Omega)$ is not given by the global reducibility identities associated with the abelian gauge symmetries (theorem 6.8), but is much bigger and in fact infinite-dimensional (see explicit description at the end of the appendix). This feature disappears if one couples coloured matter fields. We discuss the pure Yang-Mills case for the sake of completeness contenting ourselves with case I, i.e., with the solution of the consistency condition $s\omega^p + d\omega^{p-1} = 0$ in the space of all local forms.

We consider the standard Lagrangian

$$L = -\tfrac{1}{4}\, F^I_{\mu\nu} F^{\mu\nu}_I$$

where $F^{\mu\nu}_I = g_{IJ}F^{\mu\nu J}$ involves an invertible $\mathcal{G}$-invariant symmetric tensor g_{IJ}. The gauge group may contain abelian factors.

Due to $n = 2$, we have $F^I_{01} = -F^I_{10} = (1/2)\epsilon^{\mu\nu}F^I_{\mu\nu} = \star F^I$ and the equations of motion set all covariant derivatives of F_{01I} to zero. The result for $H(s)$ (corollary 11.2) implies thus immediately that

$$\omega^0 = I^\alpha(x, F_{01})\Theta_\alpha + s\eta^0 \tag{11.79}$$

where the $I^\alpha(x, F_{01})$ are arbitrary $\mathcal{G}$-invariant local functions of the F_{01I} and the x^μ (the latter can occur because we are discussing case I). (11.79) is therefore the general solution of the consistency condition for $p = 0$.

To find the general solutions with $p = 1$ and $p = 2$, we use the descent equations and examine whether an ω^0 as in Eq. (11.79) can be lifted to solutions of the consistency condition with form-degree 1 or 2. In order to lift ω^0 to form-degree 1, it

is necessary and sufficient that the $dI^\alpha(x, F_{01})$ vanish weakly, for all α (see Section 11.2). Since the $I^\alpha(x, F_{01})$ are $\mathcal{G}$-invariant, we have

$$dI^\alpha(x, F_{01}) = dx^\mu \left[\frac{\partial I^\alpha(x, F_{01})}{\partial x^\mu} + (D_\mu F_{01I}) \frac{\partial I^\alpha(x, F_{01})}{\partial F_{01I}} \right] \approx dx^\mu \frac{\partial I^\alpha(x, F_{01})}{\partial x^\mu}$$

where we have used $D_\mu F_{01I} \approx 0$. No nonvanishing function of the undifferentiated F_{01I} is weakly zero since the equations of motion contain derivatives of F_{01I}. Hence, ω^0 can be lifted to form-degree 1 if and only if $\partial I^\alpha / \partial x^\mu = 0$, i.e., the I^α must not depend explicitly on the x^μ. It turns out that this also suffices to lift ω^0 to form-degree 2. To show this, we introduce

$$\star\tilde{C}_I^* := \star C_I^* + \star A_I^* + \star F_I$$

where $\star C_I^* = d^2x\, C_I^*$, $\star A_I^* = dx^\mu \epsilon_{\mu\nu} A_I^{*\nu}$ and $\star F_I = \frac{1}{2}\epsilon_{\mu\nu} F_I^{\mu\nu}$. One has

$$(s+d) \star \tilde{C}_I^* = (C^J + A^J) e f_{JI}{}^K \star \tilde{C}_K^* \ ,$$

i.e., the $\star\tilde{C}_I^*$ transform under $(s+d)$ according to the adjoint representation of $\mathcal{G}$ with "$(s+d)$-ghosts $(C^I + A^I)$". $\mathcal{G}$-invariant functions of the $\star\tilde{C}_I^*$ are thus $(s+d)$-closed,

$$(s+d)I^\alpha(\star\tilde{C}^*) = 0.$$

Recall that the Θ_α are polynomials in the $\theta_r(C)$ and that the latter are related to $\mathcal{G}$-invariant polynomials $f_r(F)$ via the transgression formula (10.21) which decomposes into Eqs. (10.23). In two dimensions, all $f_r(F)$ with degree $m(r) > 1$ in the F^I vanish. The transgression formula gives thus

$$m(r) > 1: \quad (s+d)q_r(C+A, F) = 0, \quad q_r(C+A, F) = [\theta_r]^0 + [\theta_r]^1 + [\theta_r]^2.$$

For $m(r) = 1$ one gets $(s+d)(C^I + A^I) = F^I$ where C^I, A^I and F^I are abelian. One has

$$\begin{aligned}
\text{abelian } F^I: \quad F^I &= \tfrac{1}{2} dx^\mu dx^\nu F^I_{\mu\nu} \\
&= d(\tfrac{1}{2} x^\mu dx^\nu F^I_{\mu\nu}) + \tfrac{1}{2} x^\mu dx^\nu dx^\rho \partial_\rho F^I_{\mu\nu} \\
&= d(\tfrac{1}{2} x^\mu dx^\nu F^I_{\mu\nu}) - s(\tfrac{1}{2} d^2x\, x^\mu \epsilon_{\mu\nu} A^{*\nu I}) \\
&= (s+d)(\tfrac{1}{2} x^\mu dx^\nu F^I_{\mu\nu} - \tfrac{1}{2} d^2x\, x^\mu \epsilon_{\mu\nu} A^{*\nu I})
\end{aligned}$$

where we have used that one has $\partial_\mu F^I_{01} = s(\epsilon_{\mu\nu} A^{*\nu I})$ for abelian F^I. Hence we have two different quantities whose $(s+d)$-transformation equals F^I in the abelian case ($C^I + A^I$ and the quantity in the previous equation). The difference of these quantities is thus an $(s+d)$-closed extension of the abelian ghosts. Hence, we can complete every $\theta_r(C)$, whether nonabelian or abelian, to an $(s+d)$-invariant quantity $\tilde{q}_r$,

$$\begin{aligned}
m(r) > 1: \quad &\tilde{q}_r = q_r(C+A, F) = [\theta_r]^0 + [\theta_r]^1 + [\theta_r]^2 \\
m(r) = 1: \quad &\tilde{q}_r = C^I + A^I - \tfrac{1}{2}(x^\mu dx^\nu F^I_{\mu\nu} - d^2x\, x^\mu \epsilon_{\mu\nu} A^{*\nu I}) \quad \text{(abelian } I\text{)}.
\end{aligned}$$

Due to $(s+d)\tilde{q}_r = 0$ and $(s+d)I^\alpha(\star\tilde{C}^*) = 0$, we have

$$(s+d)\left[I^\alpha(\star\tilde{C}^*)\Theta_\alpha(\tilde{q})\right] = 0, \tag{11.80}$$

where $\Theta_\alpha(\tilde{q})$ arises from Θ_α by substituting the $\tilde{q}_r$ for the $\theta_r(C)$. The decomposition of (11.80) into pieces with definite form-degree reads

$$s[I^\alpha\Theta_\alpha]^2 + d[I^\alpha\Theta_\alpha]^1 = 0,$$
$$s[I^\alpha\Theta_\alpha]^1 + d[I^\alpha\Theta_\alpha]^0 = 0,$$
$$s[I^\alpha\Theta_\alpha]^0 = 0,$$

where $[I^\alpha\Theta_\alpha]^p$ is the p-form contained in $I^\alpha(\star\tilde{C}^*)\Theta_\alpha(\tilde{q})$,

$$I^\alpha(\star\tilde{C}^*)\Theta_\alpha(\tilde{q}) = \sum_{p=0}^{2}[I^\alpha\Theta_\alpha]^p.$$

Every $I^\alpha(F_{01})\Theta_\alpha = [I^\alpha\Theta_\alpha]^0$ can thus indeed be lifted to solutions of the consistency condition with form-degrees 1 and 2.

It is now easy to complete the analysis. $s\omega^1 + d\omega^0 = 0$ yields $s(\omega^1 - [I^\alpha\Theta_\alpha]^1 - d\eta^0) = 0$. By the result on $H(s)$, the general solution of $s\omega^1 + d\omega^0 = 0$ is accordingly

$$\omega^1 = [I^\alpha\Theta_\alpha]^1 + dx^\mu I^\alpha_\mu(x, F_{01})\Theta_\alpha + s\eta^1 + d\eta^0 \tag{11.81}$$

where the $I^\alpha_\mu(x, F_{01})$ are arbitrary $\mathcal{G}$-invariant local functions of the F_{01I} and the x^μ.

We know already that every $[I^\alpha\Theta_\alpha]^1$ can be lifted to $[I^\alpha\Theta_\alpha]^2$. In order to lift an ω^1 as in Eq. (11.81), it is therefore necessary that the piece $\hat{\omega}^1 := dx^\mu I^\alpha_\mu(x, F_{01})\Theta_\alpha$ can be lifted too. By arguments analogous to those used above, this requires $d_x\hat{\omega}^1 = 0$ where $d_x = dx^\mu\partial/\partial x^\mu$. Since $H^1(d_x)$ is trivial (ordinary Poincaré lemma in $\mathbb{R}^2$), this gives $\hat{\omega}^1 = d_x J^\alpha(x, F_{01})\Theta_\alpha$ for some $\mathcal{G}$-invariant local functions $J^\alpha(x, F_{01})$. This implies that $\hat{\omega}^1$ is trivial in $H(s|d)$. Indeed, using $D_\mu F_{01I} = \delta(\epsilon_{\mu\nu}A_I^{*\nu})$, the $\mathcal{G}$-invariance of I^α and J^α, and Eq. (11.13), one obtains

$$d_x J^\alpha(x, F_{01})\Theta_\alpha = d[J^\alpha(x, F_{01})\Theta_\alpha] - (dx^\mu D_\mu F_{01I})\frac{\partial J^\alpha(x, F_{01})}{\partial F_{01I}}\Theta_\alpha - J^\alpha(x, F_{01})d\Theta_\alpha$$

$$= d[J^\alpha(x, F_{01})\Theta_\alpha] + s\left[\star A_I^*\frac{\partial J^\alpha(x, F_{01})}{\partial F_{01I}}\Theta_\alpha + J^\alpha(x, F_{01})A^I\frac{\partial\Theta_\alpha}{\partial C^I}\right].$$

Hence, those solutions ω^1 which can be lifted are of the form $[I^\alpha\Theta_\alpha]^1 + s\eta'^1 + d\eta'^0$. Inserting this in $s\omega^2 + d\omega^1 = 0$ yields $s(\omega^2 - [I^\alpha\Theta_\alpha]^2 - d\eta'^1) = 0$. Every element of $H(s)$ with form-degree 2 is d_x-closed and thus d_x-exact, due to $H^2(d_x) = 0$. Using arguments as before, one concludes that the general solution of $s\omega^2 + d\omega^1 = 0$ is

$$\omega^2 = [I^\alpha\Theta_\alpha]^2 + s\eta''^2 + d\eta''^1. \tag{11.82}$$

Remark. Using the isomorphism $H^0_{\text{char}}(d, \Omega) \simeq H^2_2(\delta|d, \Omega) \oplus \mathbb{R} \simeq H^{-2,2}(s|d, \Omega) \oplus \mathbb{R}$ (see theorems 6.2 and 7.1), one deduces from the above result that $H^0_{\text{char}}(d, \Omega)$ and $H^0_{\text{char}}(d, \mathcal{I})$ are represented by arbitrary $\mathcal{G}$-invariant polynomials in the F_{01I}. These cohomological groups are thus infinite dimensional. This explains the different results as compared to higher dimensions where the nontrivial representatives of $H^{n-2}_{\text{char}}(d, \Omega)$ correspond one-to-one to the free abelian gauge symmetries.

12 Discussion of the results for Yang-Mills type theories

Theorem 11.1 gives the general solution of the consistency condition $sa + db = 0$ at all form-degrees and ghost numbers for theories of the Yang-Mills type without free abelian gauge symmetries (in the sense of subsection 11.1) and in spacetime dimensions greater than 2. The case of free abelian symmetries is treated in the next section. In this section, we spell out the physical implications of the theorem by expliciting the results in the relevant ghost numbers.

To that end we shall use the notation $f([F, \psi]_D)$ for functions that depend only on the Yang-Mills field strengths, the matter fields and their covariant derivatives,

$$f([F, \psi]_D) \equiv f(F_{\mu\nu}^I, D_\rho F_{\mu\nu}^I, D_\rho D_\sigma F_{\mu\nu}^I, \ldots, \psi^i, D_\mu \psi^i, D_\mu D_\nu \psi^i, \ldots).$$

We recall that the results are valid for general Lagrangians of the Yang-Mills type, provided these fulfill the technicality assumptions of "regularity" and "normality" explained above. The results cover in particular the standard model and effective gauge theories.

12.1 $H^{-1,n}(s|d)$: Global symmetries and Noether currents

12.1.1 Solutions of the consistency condition at negative ghost number

We start with the discussion of the results at negative ghost number. First, we recall that the groups $H^{-q,n}(s|d)$ are trivial for $q > 1$. This implies that there is no characteristic cohomology in form degree $< n - 1$, i.e., no non trivial higher order conservation law. Any conserved local antisymmetric tensor $A^{\mu_1 \cdots \mu_q}$ $(q > 1)$ is trivial, i.e., equal on-shell to the divergence of a *local* antisymmetric tensor with one more index,

$$q > 1: \qquad \partial_{\mu_1} A^{\mu_1 \cdots \mu_q} \approx 0, \ A^{\mu_1 \cdots \mu_q} = A^{[\mu_1 \cdots \mu_q]}$$
$$\Rightarrow \ A^{\mu_1 \cdots \mu_q} \approx \partial_{\mu_0} B^{\mu_0 \mu_1 \cdots \mu_q}, \ B^{\mu_0 \mu_1 \cdots \mu_q} = B^{[\mu_0 \mu_1 \cdots \mu_q]}.$$

We stress again that the important point in this statement is that the $B^{\mu_0 \mu_1 \cdots \mu_q}$ are local functions; the statement would otherwise be somewhat empty due to the ordinary Poincaré lemma for $\mathbb{R}^n$.

The only non-vanishising group at negative ghost number is $H^{-1,n}(s|d, \Omega)$. The nontrivial representatives of $H^{-1,n}(s|d, \Omega)$ are the generators of the nontrivial global symmetries, denoted by K_Δ and K_A in the previous section. Indeed, one must set $\Theta_\alpha = 1$ in order that Eqs. (11.21) and (11.33) yield solutions with ghost number -1. The general solution of the consistency condition with ghost number -1 is thus

$$\omega^{-1,n} \sim \lambda^\Delta K_\Delta + \lambda^A K_A,$$

in form-degree n, where K_Δ and K_A are related to the gauge invariant conserved currents j_Δ and to the characteristic classes $P_A(F)$ respectively, through

$$sK_\Delta + dj_\Delta = 0, \quad j_\Delta \in \mathcal{I} \ ,$$
$$sK_A + dI_A^{n-1} = P_A(F), \quad I_A^{n-1} \in \mathcal{I} \ .$$

That is, the coefficients of the antifields $A_I^{*\mu}$ and ψ_i^* in the K's determine the transformations of the corresponding field in the global symmetry associated with the conserved Noether currents.

12.1.2   Structure of global symmetries and conserved currents.

A determination of a complete set of gauge invariant nontrivial conserved currents j_Δ depends on the specific model under study. It also depends on the detailed form of the Lagrangian whether or not invariants I_A^{n-1} exist which are related to characteristic classes by Eq. (11.23). However, we can make the description of the K_Δ and K_A a little more precise without specifying L. Namely, as we prove in the appendix to this section, one can always choose the j_Δ and I_A^{n-1} such that all K_Δ and K_A take the form

$$K_\Delta = d^n x \left[A_I^{*\mu} Q_{\Delta\mu}^I(x, [F, \psi]_D) + \psi_i^* Q_\Delta^i(x, [F, \psi]_D) \right] \tag{12.1}$$

$$K_A = d^n x \left[A_I^{*\mu} Q_{A\mu}^I(x, [F, \psi]_D) + \psi_i^* Q_A^i(x, [F, \psi]_D) \right] \tag{12.2}$$

where the $Q_{\Delta\mu}^I(x, [F, \psi]_D)$ transform under $\mathcal{G}$ according to the coadjoint representation and the $Q_\Delta^i(x, [F, \psi]_D)$ according to the same representation as the ψ^i. We assume here that we work in the space of all local forms (case I). An analogous statement holds in the space of Poincaré invariant local forms (case II) where (12.1) and (12.2) hold with Poincaré invariant K's.

This result, and the relationship between the K's and the conserved currents, enables us to draw the following conclusions (in Yang-Mills type theories without free abelian gauge symmetries, when the spacetime dimension exceeds 2):

1. In odd dimensional spacetime, every nontrivial conserved current is equivalent to a gauge invariant conserved current,

$$n = 2k + 1: \quad \partial_\mu j^\mu \approx 0 \; \Rightarrow \; j^\mu \sim j_{\text{inv}}^\mu(x, [F, \psi]_D)$$

where $j_{\text{inv}}^\mu(x, [F, \psi]_D)$ is $\mathcal{G}$-invariant and $\sim$ means "equal modulo trivial conserved currents",

$$j^\mu \sim h^\mu \; :\Leftrightarrow \; j^\mu \approx h^\mu + \partial_\nu m^{[\nu\mu]}.$$

2. In even dimensional spacetime a nontrivial conserved current is either equivalent to a completely gauge invariant current or to a current that is gauge invariant except for a Chern-Simons term,

$$n = 2k: \quad \partial_\mu j^\mu \approx 0 \; \Rightarrow \; j^\mu \sim \begin{cases} j_{\text{inv}}^\mu(x, [F, \psi]_D) \\ \text{or} \\ I_{\text{inv}}^\mu(x, [F, \psi]_D) + q_{\text{CS}}^\mu(A, \partial A) \end{cases}$$

where $j_{\text{inv}}^\mu(x, [F, \psi]_D)$ and $I_{\text{inv}}^\mu(x, [F, \psi]_D)$ are $\mathcal{G}$-invariant, and $q_{\text{CS}}^\mu(A, \partial A)$ is dual to a Chern-Simons $(n-1)$-form, i.e.,

$$q_{\text{CS}}^\mu(A, \partial A) = \epsilon^{\mu\nu_1\mu_2\nu_2\dots\mu_k\nu_k} d_{I_1\dots I_k} A_{\nu_1}^{I_1} \partial_{\mu_2} A_{\nu_2}^{I_2} \dots \partial_{\mu_k} A_{\nu_k}^{I_k} + \dots$$

One can choose the basis of inequivalent conserved currents such that those currents which contain Chern-Simons terms correspond one-to-one to the characteristic classes $P_A(F)$ which are trivial in the equivariant characteristic cohomology. In particular, all conserved currents can be made strictly gauge invariant when no characteristic class is trivial in the equivariant characteristic cohomology.[17]

3. Every nontrivial global symmetry can be brought to a gauge covariant form. More precisely, let δ_Q be the generator of a global symmetry whose characteristics $\delta_Q A_\mu^I = Q_\mu^I$ and $\delta_Q \psi^i = Q^i$ are local functions of the fields (Q_μ^I or Q^i may depend explicitly on the x^μ). Then one can bring δ_Q to a form (by subtracting trivial symmetries if necessary) such that the characteristics depend only on the x^μ, the Yang-Mills field strengths and their covariant derivatives, and the matter fields and their covariant derivatives,

$$\delta_Q A_\mu^I = Q_\mu^I(x, [F, \psi]_D), \quad \delta_Q \psi^i = Q^i(x, [F, \psi]_D),$$

where the Q_μ^I transform under the coadjoint representation of $\mathcal{G}$ and the Q^i transform under the same representation of $\mathcal{G}$ as the ψ^i.

Note that the gauge covariant global symmetries commute with the gauge transformations (2.4): for instance, one has

$$
\begin{aligned}
[\delta_Q, \delta_\epsilon] A_\mu^I &= \delta_Q(\partial_\mu \epsilon^I + e\, f_{JK}{}^I A_\mu^J \epsilon^K) - \delta_\epsilon Q_\mu^I \\
&= e\, f_{JK}{}^I Q_\mu^J \epsilon^K - (-e\, \epsilon^K f_{KJ}{}^I Q_\mu^J) = 0.
\end{aligned}
$$

Here we used $\delta_Q \epsilon^I = 0$ where ϵ^I are arbitrary fields. Of course, in general δ_Q would not commute with a special gauge transformation obtained by substituting functions of the A_μ^I, ψ^i and their derivatives for ϵ^I.

12.1.3   Examples.

1. It should be noted that the gauge covariant form of a global symmetry is not always its most familiar version. In order to make a global symmetry gauge covariant, it may be necessary to add a trivial symmetry to it. We illustrate this feature now for conformal transformations. Consider 4-dimensional massless scalar electrodynamics,

$$n = 4, \quad L = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} - \tfrac{1}{2} (D_\mu \varphi) D^\mu \bar{\varphi}$$

---

[17]To our knowledge, it is still an open problem whether in standard Yang-Mills theory characteristic classes $P(F)$ with form-degree n can be trivial in $H_{\mathrm{char}}(d, \mathcal{I})$. (The problem occurs only in the space of forms with explict x^μ-dependence.) In Section 13 of [23], we have claimed that the answer is negative for a polynomial dependence on x^μ. However, the proof of the assertion given there is incorrect because the $sl(n)$-decomposition of the equations of motion used there does not yield pieces which are all weakly zero separately. If the answer were positive (contrary to our expectations), it would mean that non covariantizable currents could occur in standard Yang-Mills theory, contrary to the claim in theorem 2 in [22].

where φ is a complex scalar field, $\bar\varphi$ is the complex conjugate of φ, and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ , \ \ D_\mu \varphi = \partial_\mu \varphi + \mathrm{i}e A_\mu \varphi \ , \ \ D_\mu \bar\varphi = \partial_\mu \bar\varphi - \mathrm{i}e A_\mu \bar\varphi \ .$$

The action is invariant under the following infinitesimal conformal transformations

$$\begin{aligned}
\delta_{\mathrm{conf}} A_\mu &= \xi^\nu \partial_\nu A_\mu + (\partial_\mu \xi^\nu) A_\nu \ , \\
\delta_{\mathrm{conf}} \varphi &= \xi^\nu \partial_\nu \varphi + \tfrac{1}{4}(\partial_\nu \xi^\nu)\varphi \ , \\
\xi^\mu &= a^\mu + \omega^{[\mu\nu]} x_\nu + \lambda x^\mu + b^\mu x_\nu x^\nu - 2x^\mu b^\nu x_\nu
\end{aligned}$$

where the a^μ, $\omega^{[\mu\nu]}$, λ and b^μ are constant parameters of conformal transformations and $x_\mu = \eta_{\mu\nu} x^\nu$. δ_{conf} is not gauge covariant. To make it gauge covariant we add a trivial global symmetry to it, namely a special gauge transformation with "gauge parameter" $\epsilon = -\xi^\nu A_\nu$. This special gauge transformation is $\delta_{\mathrm{trivial}} A_\mu = \partial_\mu(-\xi^\nu A_\nu)$, $\delta_{\mathrm{trivial}} \varphi = \mathrm{i}e\xi^\nu A_\nu \varphi$. $\hat\delta_{\mathrm{conf}} = \delta_{\mathrm{conf}} + \delta_{\mathrm{trivial}}$ is gauge covariant,

$$\hat\delta_{\mathrm{conf}} A_\mu = \xi^\nu F_{\nu\mu} \ , \ \ \hat\delta_{\mathrm{conf}} \varphi = \xi^\nu D_\nu \varphi + \tfrac{1}{4}(\partial_\nu \xi^\nu)\varphi \ .$$

Since $\hat\delta_{\mathrm{conf}}$ and δ_{conf} differ only by a special gauge transformation, they are equivalent and yield the same variation of the Lagrangian,

$$\hat\delta_{\mathrm{conf}} L = \delta_{\mathrm{conf}} L = \partial_\mu(\xi^\mu L - b^\mu \varphi \bar\varphi).$$

The Noether current corresponding to $\hat\delta_{\mathrm{conf}}$ is gauge invariant,

$$j^\mu_{\mathrm{inv}}(x, [F, \varphi, \bar\varphi]_D) = \sum_{\Phi = A_\nu, \varphi, \bar\varphi} (\hat\delta_{\mathrm{conf}} \Phi) \frac{\partial L}{\partial(\partial_\mu \Phi)} - \xi^\mu L + b^\mu \varphi \bar\varphi \ .$$

2. We shall now illustrate the unusual situation in which a nontrivial Noether current contains a Chern-Simons term. As a first example we consider 4-dimensional Yang-Mills theory with gauge group $SU(2)$ coupled nonminimally to a real $SU(2)$-singlet scalar field ϕ via the following Lagrangian,

$$n = 4, \quad L = -\tfrac{1}{4} F^I_{\mu\nu} F^{\mu\nu J} \delta_{IJ} - \tfrac{1}{2}(\partial_\mu \phi)\partial^\mu \phi + \tfrac{1}{4} \phi \epsilon^{\mu\nu\rho\sigma} F^I_{\mu\nu} F^J_{\rho\sigma} \delta_{IJ}$$

where

$$F^I_{\mu\nu} = \partial_\mu A^I_\nu - \partial_\nu A^I_\mu + e\,\epsilon_{IJK} A^J_\mu A^K_\nu \ .$$

The action is invariant under constant shifts of ϕ,

$$\begin{aligned}
\delta_{\mathrm{shift}} \phi = -1 \quad &, \quad \delta_{\mathrm{shift}} A^I_\mu = 0 \quad \Rightarrow \quad \delta_{\mathrm{shift}} L = -\partial_\mu q^\mu_{\mathrm{CS}}(A, \partial A), \\
& q^\mu_{\mathrm{CS}}(A, \partial A) = \epsilon^{\mu\nu\rho\sigma}(\delta_{IJ} A^I_\nu \partial_\rho A^J_\sigma + \tfrac{1}{3} e\,\epsilon_{IJK} A^I_\nu A^J_\rho A^K_\sigma).
\end{aligned}$$

δ_{shift} is obviously nontrivial and gauge covariant. The corresponding Noether current contains the Chern-Simons term $q^\mu_{\mathrm{CS}}(A, \partial A)$ and is otherwise gauge invariant,

$$j^\mu = \partial^\mu \phi + q^\mu_{\mathrm{CS}}(A, \partial A) \ .$$

3. A variant of the previous example arises when one replaces the scalar field by the time coordinate x^0,

$$n = 4, \quad L = -\tfrac{1}{4} F_{\mu\nu}^I F^{\mu\nu J} \delta_{IJ} + \tfrac{1}{4} x^0 \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^I F_{\rho\sigma}^J \delta_{IJ}$$

with $F_{\mu\nu}^I$ as in the previous example. This example breaks of course Lorentz invariance and is given only for illustrative purposes. One can get rid of x^0 by integrating by parts the last term, at the price of introducing an undifferentiated A_μ. The action is therefore invariant under temporal translations,

$$\delta_{\text{time}} A_\mu^I = F_{0\mu}^I \quad \Rightarrow \quad \delta_{\text{time}} L = \partial_0 L - \partial_\mu q_{\text{CS}}^\mu (A, \partial A)$$

with $q_{\text{CS}}^\mu (A, \partial A)$ as in the previous example. $\delta_{\text{time}} A_\mu^I = F_{0\mu}^I$ is already the gauge covariant version of temporal translations (one has $\delta_{\text{time}} A_\mu^I = \partial_0 A_\mu^I + \delta_{\text{trivial}} A_\mu^I$ where $\delta_{\text{trivial}} A_\mu^I$ is a special gauge transformation with $\epsilon^I = -A_0^I$). Note that we used $\delta_{\text{time}} x^0 = 0$, i.e., we transformed only the fields. The conserved Noether current corresponding to δ_{time} is the component $\nu = 0$ of the energy momentum tensor $T_\nu{}^\mu$. In the present case, it cannot be made fully gauge invariant but contains the Chern-Simons term $q_{\text{CS}}^\mu (A, \partial A)$,

$$T_0{}^\mu = F_{0\nu}^I \frac{\partial L}{\partial(\partial_\mu A_\nu^I)} - \delta_0^\mu L + q_{\text{CS}}^\mu (A, \partial A).$$

12.2 $H^{0,n}(s|d)$: Deformations and BRST-invariant counterterms

We now turn to the local BRST cohomology at ghost number zero. This case covers deformations of the action and controls therefore the stability of the theory. We first make the results of theorem 11.1 more explicit for the particular value 0 of the ghost number; we then discuss the implications.

The most general solution of the consistency condition with ghost number 0 and form-degree n is

$$\omega^{0,n} \sim I^n + B^{0,n} + V^{0,n} + W^{0,n}$$

where:

1. $I^n \in \mathcal{I}$, i.e., I^n is a strictly gauge invariant n-form,

$$\text{Case I:} \quad I^n = d^n x\, I_{\text{inv}}(x, [F, \psi]_D)$$
$$\text{Case II:} \quad I^n = d^n x\, I_{\text{inv}}([F, \psi]_D).$$

(we recall that in case I, one computes the cohomology in the algebra of forms having a possible explicit x-dependence; while the forms in case II have no explicit x-dependence and are Lorentz-invariant)

2. $B^{0,n}$ is a linear combination of the independent Chern-Simons n-forms, see Eq. (10.40). Solutions $B^{0,n}$ can thus only exist in odd spacetime dimensions.

3. $V^{0,n}$ are linear combinations of the solutions $V_{\Delta\alpha}$ (11.21) related to global symmetries. In order that $V_{\Delta\alpha}$ has ghost number 0, the Θ_α which appears in it must have ghost number 1. There are such Θ_α only when the gauge group has abelian factors, in which case the Θ_α are the abelian ghosts. In the absence of abelian factors, there are thus no $V_{\Delta\alpha}$ at ghost number zero. Using Eq. (12.1), one gets explicitly

$$
\begin{aligned}
V^{0,n} &= \sum_{I:\text{abelian}} \lambda_I^\Delta \left(K_\Delta C^I + j_\Delta A^I \right) \\
&= d^n x \sum_{I:\text{abelian}} \lambda_I^\Delta \left[A_J^{*\mu} Q_{\Delta\mu}^J C^I + \psi_i^* Q_\Delta^i C^I - (-)^{\epsilon_\Delta} j_\Delta^\mu A_\mu^I \right] \quad (12.3)
\end{aligned}
$$

where the j_Δ^μ are the nontrivial gauge invariant Noether currents, $Q_{\Delta\mu}^J$ and Q_Δ^i are the corresponding gauge covariant symmetries, and ϵ_Δ is the parity of j_Δ^μ (e.g., $\epsilon_\Delta = 1$ when j_Δ^μ is the conserved current of a global supersymmetry). We stress again that the sets of Q's and j's are different in case I and case II, see text after Eq. (11.19).

4. $W^{0,n}$ are linear combinations of solutions W_Γ (11.33) with ghost number 0; such solutions exist only for peculiar choices of Lagrangians discussed below - and again only when there are abelian factors.

Nontriviality of the solutions. A solution $I^n + B^{0,n} + V^{0,n} + W^{0,n}$ is only trivial when $B^{0,n}$, $V^{0,n}$ and $W^{0,n}$ all vanish and I^n is weakly d-exact,

$$
I^n + B^{0,n} + V^{0,n} + W^{0,n} \sim 0 \iff B^{0,n} = V^{0,n} = W^{0,n} = 0, \ I^n \approx d\omega^{n-1}.
$$

$I^n \approx d\omega^{n-1}$ is equivalent to $I^n \approx dI^{n-1} + P(F)$ for some $I^{n-1} \in \mathcal{I}$ and some characteristic class $P(F)$.

Semisimple gauge group. When the gauge group G is semisimple there are no solutions $V^{0,n}$ or $W^{0,n}$ at all because all these solutions require the presence of abelian gauge symmetries. Hence, when G is semisimple, all representatives of $H^{0,n}(s|d)$ can be taken to be strictly gauge invariant except for the Chern-Simons forms in odd spacetime dimensions. In particular, the antifields can then be removed from all BRST-invariant counterterms and integrated composite operators by adding cohomologically trivial terms, and the gauge transformations are stable, i.e., they cannot be deformed in a continuous and nontrivial manner. This result implies, in particular, the structural stability of effective Yang-Mills theories in the sense of [117].

Comment. We add a comment on Chern-Simons forms which should also elucidate a bit the distinction between case I and case II. Chern-Simons forms $B^{0,n}$ are Lorentz-invariant in n-dimensional spacetime and occur thus among the solutions both in case I and in case II. However, these are not the only solutions constructible out of Chern-Simons forms. For instance, in 4 dimensions there is the solution $\omega^{0,4} = B^{0,3} dx^0$ where

$B^{0,3}$ is a Chern-Simons 3-form. This solution is not Lorentz-invariant and is thus present only in case I but not in case II. Have we missed this solution? The answer is "no" because it is equivalent to the solution $I^4 = x^0 P(F)$ where $P(F) = dB^{0,3}$. Namely we have $B^{0,3}dx^0 = d(-x^0 B^{0,3}) + x^0 dB^{0,3}$ and thus indeed $B^{0,3}dx^0 \sim x^0 P(F)$. Note that in order to establish this equivalence it is essential that we work in the space of local forms that may depend explicitly on the x^μ.

The exceptional solutions $W^{0,n}$. The existence of a solution $W^{0,n}$ requires a relation $k^i N_i = k^{A\alpha} P_A(F) \Theta_\alpha$ at ghost number 1, cf. Eq. (11.27). The N_i with ghost number number 1 are linear combinations of terms $(C^I F^J - C^J F^I) P(F)$ where C^I, C^J, F^I, F^J are abelian and $P(F)$ is some characteristic class. The Θ_α with ghost number 1 are the abelian ghosts, and the $P_A(F)$ are characteristic classes which are trivial in the equivariant characteristic cohomology $H^n_{\text{char}}(d, \mathcal{I})$. Hence, in order that a solution $W^{0,n}$ exists, a nonvanishing linear combination of terms $(C^I F^J - C^J F^I) P(F)$ must be equal to a linear combination of the $P_A(F) C^I$, where C^I, C^J, F^I, F^J are abelian. The gauge group must therefore contain at least two abelian factors and, additionally, there must be at least two different $P_A(F)$ containing abelian field strengths. This is really a very special situation not met in practice (to our knowledge), which must be included in the discussion because we allow here for general Lagrangians. We illustrate it with a simple example:

$$n = 4, \quad L = \sum_{I=1}^{2} \left[-\tfrac{1}{4} F^I_{\mu\nu} F^{\mu\nu I} - \tfrac{1}{2} (\partial_\mu \phi^I) \partial^\mu \phi^I + \tfrac{1}{4} \phi^I \epsilon^{\mu\nu\rho\sigma} F^I_{\mu\nu} F^2_{\rho\sigma} \right]$$

where $F^I_{\mu\nu} = \partial_\mu A^I_\nu - \partial_\nu A^I_\mu$ are abelian field strengths and ϕ^1 and ϕ^2 are real scalar fields. In this case we have $\{P_A(F)\} = \{F^1 F^2, F^2 F^2\}$ with corresponding $\{-I^3_A\} = \{\star d\phi^1, \star d\phi^2\}$ and $\{K_A\} = \{\star \phi^*_1, \star \phi^*_2\}$. Furthermore we have one N_Γ with ghost number 1 given by $\epsilon_{JI} C^I F^J F^2$ and corresponding b^4_Γ given by $(1/2)\epsilon_{IJ} A^I A^J F^2$ ($\epsilon_{IJ} = -\epsilon_{JI}$). (11.33) gives now the following solution:

$$W^{0,4} = d^4 x \sum_{I,J=1}^{2} \epsilon_{IJ} [\tfrac{1}{4} \epsilon^{\mu\nu\rho\sigma} A^I_\mu A^J_\nu F^2_{\rho\sigma} + A^I_\mu \partial^\mu \phi^J - \phi^*_I C^J]. \tag{12.4}$$

12.3 $H^{1,n}(s|d)$: Anomalies

We know turn to $H^{1,n}(s|d)$, i.e., to anomalies. The most general solution of the consistency condition with ghost number 1 and form-degree n is

$$\omega^{1,n} \sim \sum_{I:\text{abelian}} C^I I^n_I + B^{1,n} + V^{1,n} + W^{1,n}$$

where:

1. $I^n_I \in \mathcal{I}$, i.e., $\{I^n_I\}$ is a set of strictly gauge invariant n-forms.

2. When n is even, $B^{1,n}$ is a linear combination of the celebrated chiral anomalies listed in Eq. (10.41), except for those which contain abelian ghosts (the chiral anomalies with abelian ghosts are already included in $\sum_{I:\text{abelian}} C^I I_I^n$). When n is odd, $B^{1,n}$ is a linear combination of the solutions (10.43) which exist only when the gauge group contains at least two abelian factors.

3. $V^{1,n}$ are linear combinations of the solutions $V_{\Delta\alpha}$ (11.21) with ghost number 1 related to global symmetries; in order that $V_{\Delta\alpha}$ has ghost number 1, the Θ_α which appears in it must have ghost number 2 and must thus be a product of two abelian ghosts. Hence, solutions $V^{1,n}$ exist only if the gauge group contains at least two abelian factors. They are given by

$$V^{1,n} = \sum_{I,J:\text{abelian}} \lambda_{IJ}^\Delta \left[K_\Delta C^I C^J + j_\Delta (A^I C^J - A^J C^I) \right]. \qquad (12.5)$$

Using (12.1), the antifield dependence of $V^{1,n}$ can be made explicit, analogously to (12.3).

4. A discussion of Eq. (11.27) similar to the one performed for $W^{0,n}$ shows that the solutions $W^{1,n}$ are even more exceptional than their counterparts in ghost number zero; they exist only in the following situation: the gauge group must contain at least three abelian factors and, additionally, there must be at least three different $P_A(F)$ containing abelian field strengths. An example is the following:

$$n = 4, \quad L = \sum_{I=1}^{3} \left[-\tfrac{1}{4} F_{\mu\nu}^I F^{\mu\nu I} - \tfrac{1}{2} (\partial_\mu \phi^I) \partial^\mu \phi^I + \tfrac{1}{4} \phi^I \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^I F_{\rho\sigma}^3 \right]$$

where $F_{\mu\nu}^I = \partial_\mu A_\nu^I - \partial_\nu A_\mu^I$ are abelian field strengths, and ϕ^I are real scalar fields. A solution $W^{1,4}$ is

$$W^{1,4} = d^4 x \sum_{I,J,K=1}^{3} \epsilon_{IJK} \left[\tfrac{1}{4} \epsilon^{\mu\nu\rho\sigma} C^I A_\mu^J A_\nu^K F_{\rho\sigma}^3 + C^I A_\mu^J \partial^\mu \phi^K - \tfrac{1}{2} C^I C^J \phi_K^* \right]. \quad (12.6)$$

Nontriviality of the solutions. A solution $\sum_{I:\text{abelian}} C^I I_I^n + B^{1,n} + V^{1,n} + W^{1,n}$ is only trivial when $B^{1,n}$, $V^{1,n}$ and $W^{1,n}$ all vanish and, additionally,

$$\sum_{I:\text{abelian}} C^I I_I^n \approx \sum_{I:\text{abelian}} C^I dI_I^{n-1} + \sum_{I,J:\text{abelian}} (C^I F^J - C^J F^I) P_{IJ}(F)$$

for some $I_I^{n-1} \in \mathcal{I}$ and some characteristic classes $P_{IJ}(F)$.

Semisimple gauge group. Note that all nontrivial solutions $\omega^{1,n}$ involve abelian ghosts, except for the solutions $B^{1,n}$ in even dimensions. Hence, when the gauge group is semisimple, the candidate gauge anomalies are exhausted by the well-known nonabelian chiral anomalies in even dimensions. These live in the small algebra

and can be obtained from the characteristic classes living in two dimensions higher through the Russian formula of section 10.6. Furthermore, these anomalies are in finite number (independently of power counting arguments) and do not depend on the specific form of the Lagrangian. For some groups, there may be none ("anomaly-safe groups"), in which case the consistency condition implies absence of anomalies, for any Lagrangian.

In 4 dimensions, $B^{1,4}$ is the non abelian gauge anomaly [17]:

$$B^{1,4} = d_{IJK} C^I d[A^J dA^K + \tfrac{1}{4} e f_{LM}{}^J A^K A^L A^M],
\qquad (12.7)$$

where d_{IJK} is the general symmetric $\mathcal{G}$-invariant tensor. Hence, the gauge group is anomaly safe for any Lagrangian in 4 dimensions if there is no d_{IJK}-tensor.

12.4 The cohomological groups $H^{g,n}(s|d)$ with $g > 1$

The results for ghost numbers $g > 1$ are similiar to those for $g = 0$ and $g = 1$; one gets

$$\omega^{g,n} \sim I^{n \, \alpha_g} \Theta_{\alpha_g} + B^{g,n} + V^{g,n} + W^{g,n}$$

where $\{\Theta_{\alpha_g}\}$ is the subset of $\{\Theta_\alpha\}$ containing those Θ's with ghost number g. Of course, this subset depends on the gauge group G. It depends both on G and on the spacetime dimension which solutions are present for given g. For instance, for $G = SU(2)$ and $n = 4$, one has $\omega^{2,4} \sim V^{2,4}$. The highest ghost number for which nontrivial solutions exist is $g = \dim(G)$ because this is the ghost number of the product of all $\theta_r(C)$. Antifield dependent solutions $V^{g,n}$ exist up to ghost number $g = \dim(G) - 1$. As we have mentioned already several times, solutions $W^{g,n}$ exist only for exceptional Lagrangians.

12.5 Appendix 12.A: Gauge covariance of global symmetries

(12.1) and (12.2) can be achieved because the equations of motion are gauge covariant. This is seen as follows. Consider an n-form $I \in \mathcal{I}$ which vanishes weakly, $I \approx 0$. This is equivalent to $I = \delta \hat{K}$ for some n-form $\hat{K}$. The equations of motion are gauge covariant in the sense that one has $\delta A_I^{*\mu} = L_I^\mu(x, [F, \psi]_D)$ and $\delta \psi_i^* = L_i(x, [F, \psi]_D)$ where the $L_I^\mu(x, [F, \psi]_D)$ are in the adjoint representation of $\mathcal{G}$ and the $L_i(x, [F, \psi]_D)$ in the representation dual to the representation of the ψ^i (cf. section 8.1). In particular, δ is stable in the space of $\mathcal{G}$-invariant functions $f_{\text{inv}}(x, [F, \psi, A^*, \psi^*, C^*]_D)$, i.e., it maps this space into itself. We can thus choose

$$\hat{K} \;=\; d^n x \left[A_I^{*\mu} \hat{Q}_\mu^I(x, [F, \psi]_D) + \psi_i^* \hat{Q}^i(x, [F, \psi]_D) \right.$$
$$\left. + (D_\nu A_I^{*\mu}) \hat{Q}_\mu^{I\nu}(x, [F, \psi]_D) + (D_\nu \psi_i^*) \hat{Q}^{i\nu}(x, [F, \psi]_D) + \dots \right]$$

where $\hat{K}$ is $\mathcal{G}$-invariant. Note that $\hat{K}$ contains in general covariant derivatives of antifields. To deal with these terms, we write

$$\hat{K} \;=\; d^n x \left[A_I^{*\mu} Q_\mu^I + \psi_i^* Q^i + D_\nu R^\nu \right],$$

$$
\begin{aligned}
Q_\mu^I &= \hat{Q}_\mu^I - D_\nu \hat{Q}_\mu^{I\nu} + \dots , \\
Q^i &= \hat{Q}^i - D_\nu \hat{Q}^{i\nu} + \dots , \\
R^\nu &= A_I^{*\mu} \hat{Q}_\mu^{I\nu} + \psi_i^* \hat{Q}^{i\nu} + \dots .
\end{aligned}
$$

Since R^ν is $\mathcal{G}$-invariant, we have $D_\nu R^\nu = \partial_\nu R^\nu$ and thus

$$
\begin{aligned}
\hat{K} &= K + dR , \\
K &= d^n x \left[A_I^{*\mu} Q_\mu^I + \psi_i^* Q^i \right] , \\
R &= (-)^n \tfrac{1}{(n-1)!} \, dx^{\mu_1} \dots dx^{\mu_{n-1}} \epsilon_{\mu_1 \dots \mu_n} R^{\mu_n} .
\end{aligned}
$$

Using this in $I = \delta \hat{K}$, we get

$$
I + d\delta R = sK
$$

because the $\mathcal{G}$-invariance of K implies $\delta K = sK$.

(12.1) follows by setting $I = -dj$ where $j \in \mathcal{I}$ is a conserved current (we have $dj = Dj \in \mathcal{I}$). Namely the above formula gives in this case $d(j - \delta R) = -sK$. Note that $j - \delta R$ is equivalent to j and gauge invariant (due to $\delta R \approx 0$ and $\delta R \in \mathcal{I}$). Hence we can indeed choose the basis $\{j_\Delta\}$ of the inequivalent gauge invariant currents such that $dj_\Delta = -sK_\Delta$ with K_Δ as in (12.1).

Now consider the equation $P(F) \approx dI^{n-1}$ with $I^{n-1} \in \mathcal{I}$. Setting $I = P(F) - dI^{n-1}$, the above formula gives $P(F) - d(I^{n-1} - \delta R) = sK$. Hence, we can choose all I_A^{n-1} such that $P_A(F) = sK_A + dI_A^{n-1}$ with K_A as in (12.2).

13 Free abelian gauge fields

13.1 Pecularities of free abelian gauge fields

We now compute the local BRST cohomology for a set of R abelian gauge fields with a free Lagrangian of the Maxwell type,

$$L = -\frac{1}{4} \sum_{I=1}^{R} F_{\mu\nu}^I F^{\mu\nu I}, \quad F_{\mu\nu}^I = \partial_\mu A_\nu^I - \partial_\nu A_\mu^I . \tag{13.1}$$

This question is relevant for determining the possible consistent interactions that can be defined among massless vector particles, where both groups $H^{0,n}$ and $H^{1,n}$ play a rôle, as we shall discuss in subsection 13.3 below.

As we have already mentioned, theorem 11.1 does not hold for (13.1). The reason is that the characteristic cohomology group $H_{\text{char}}^{n-2}(d,\Omega)$ does not vanish in the free model. Rather, by theorem 6.8, this cohomological group is represented in all spacetime dimensions $n > 2$ by the Hodge-duals of the abelian curvature 2-forms $F^I = dA^I$,

$$\star F^I = \frac{1}{(n-2)!\, 2} \, dx^{\mu_1} \dots dx^{\mu_{n-2}} \epsilon_{\mu_1 \dots \mu_n} F^{\mu_{n-1}\mu_n I} . \tag{13.2}$$

This modifies the results for the form-degrees $p = n - 1$ and $p = n$ as compared to theorem 11.1, by allowing solutions of a new type. These solutions are precisely those that appear in the non Abelian deformation of (13.1).

In contrast, the results for lower form-degrees remain valid as an inspection of the proof of the theorem shows since one still has $H_{\text{char}}^p(d,\Omega) = \delta_0^p \mathbb{R}$ for $p < n - 2$.

The discussion of this section applies also to abelian gauge fields with self-couplings involving the curvature only (like in the Born-Infeld Lagrangian), or in the case of non-minimal interactions with matter through terms involving only the field strength (e.g., $F_{\mu\nu} \bar{\psi}\gamma^{[\mu}\gamma^{\nu]}\psi$, where ψ is a Dirac spinor). In that case, the matter fields do not transform under the abelian gauge symmetry so that the global reducibility identities behind theorem 6.8 are still present.

13.2 Results

We shall now work out the modifications for form-degrees $p = n - 1$ and $p = n$, assuming the spacetime dimension n to be greater than 2.

13.2.1 Results in form-degree $p = n - 1$

Let ω^{n-1} be a cocycle of $H^{*,n-1}(s|d,\Omega)$,

$$s\omega^{n-1} + d\omega^{n-2} = 0. \tag{13.3}$$

The same arguments as in the proof of theorem 11.1 until Eq. (11.48) included yield $\omega^{n-2} = I^{n-2\,\alpha}\Theta_\alpha$ where (i) $I^{n-2\,\alpha}$ is gauge-invariant and fulfills $dI^{n-2\,\alpha} \approx 0$; and (ii) the Θ_α form a basis of polynomials in the undifferentiated ghosts (in the purely

abelian case each ghost polynomial is invariant; furthermore, Eq. (11.48) has no solution B^{n-2} in the purely abelian case because the B's cannot be lifted, see below). The condition $dI^{n-2\,\alpha} \approx 0$ gives now $I^{n-2\,\alpha} \approx \lambda_I^\alpha \star F^I + P^{n-2\,\alpha}(F) + dI^{n-3\,\alpha}$ where the linear combination $\lambda_I^\alpha \star F^I$ of the $\star F^I$ comes from $H_{\mathrm{char}}^{n-2}(d, \Omega)$. It is here that the extra characteristic cohomology enters and gives extra terms in $I^{n-2\,\alpha}$ compared with Eq. (11.49). These extra terms fulfill

$$d \star F^I = -s \star A^{*I} \tag{13.4}$$

where $\star A^{*I}$ is the antifield dependent $(n-1)$-form

$$\star A^{*I} = \frac{1}{(n-1)!}\, dx^{\mu_1} \dots dx^{\mu_{n-1}} \epsilon_{\mu_1\dots\mu_n} A_J^{*\mu_n} \delta^{JI}. \tag{13.5}$$

It is then straightforward to adapt Eqs. (11.50) through (11.53). This gives, instead of Eq. (11.53),

$$\omega^{n-1} \sim \lambda_I^\alpha(\star A^{*I}\Theta_\alpha + (\star F^I)[\Theta_\alpha]^1) + B^{n-1} + I^{n-1\,\alpha}\Theta_\alpha \;.$$

Since we are dealing with a purely abelian case, the Θ_α are just products of the undifferentiated ghosts. We can therefore write the result, up to trivial solutions, as

$$\omega^{n-1} \sim \star A^{*I} P_I(C) + (\star F^I) A^J \partial_J P_I(C) + B^{n-1} + I^{n-1\,\alpha} P_\alpha(C) \tag{13.6}$$

where $P_I(C)$ and $P_\alpha(C)$ are arbitrary polynomials in the undifferentiated ghosts, and

$$\partial_I \equiv \frac{\partial}{\partial C^I} \;.$$

Furthermore, the descent is particularly simple in the small algebra because a non trivial bottom can be lifted only once; at the next step, one meets an obstruction. This implies that the solutions B^{n-1} can occur in (13.6) only when the spacetime dimension n is even: Eq. (11.16) gives in the purely abelian case only solutions with odd form-degrees, which are linear in the one-forms A^I,

$$\begin{aligned} B^{2N+1} &= \sum_{i=1}^K \lambda_{I_1\dots I_K J_1\dots J_N} C^{I_1} \cdots C^{I_{i-1}} A^{I_i} C^{I_{i+1}} \cdots C^{I_K} F^{J_1} \cdots F^{J_N} \\ B^{2k} &= 0 \end{aligned} \tag{13.7}$$

where $I_i < I_{i+1}$, $J_i \leq J_{i+1}$, and (if $N > 0$) $I_1 \leq J_1$. These solutions descend on the gauge-invariant term $\lambda_{I_1\dots I_K J_1\dots J_N} C^{I_1} \cdots C^{I_K} F^{J_1} \cdots F^{J_N}$.

Eq. (13.6) gives the general solution of (13.3), both in the space of all local forms (case I) and in the space of Poincaré invariant local forms (case II), with $I^{n-1\,\alpha} \in \mathcal{I}$ where $\mathcal{I}$ is the respective gauge invariant subspace of local forms,

Case I: $\quad \mathcal{I} = \{\text{local functions of } F_{\mu\nu}^I, \partial_\rho F_{\mu\nu}^I, \dots\} \otimes \Omega(\mathbb{R}^n)$ (13.8)

Case II: $\quad \mathcal{I} = \{\text{Lorentz-invariant local functions of } dx^\mu, F_{\mu\nu}^I, \partial_\rho F_{\mu\nu}^I, \dots\}$.(13.9)

[As before, $\Omega(\mathbb{R}^n)$ denotes the space of ordinary differential forms in $\mathbb{R}^n$.]

13.2.2 Results in form-degree $p = n$

Let ω^n be a cocycle of $H^{*,n}(s|d, \Omega)$,

$$s\omega^n + d\omega^{n-1} = 0. \tag{13.10}$$

By the standard arguments of the descent equation technique, ω^{n-1} is a cocycle of $H^{*,n-1}(s|d, \Omega)$ and trivial contributions to ω^{n-1} can be neglected without loss of generality. Hence, ω^{n-1} can be assumed to be of the form (13.6) and we have to analyse the restrictions imposed on it by the fact that it can be lifted once to give ω^n through (13.10). To this end we compute $d\omega^{n-1}$.

To deal with the first two terms in (13.6), we use once again (13.4) as well as

$$d \star A^{*I} = -s \star C^{*I} \tag{13.11}$$

where

$$\star C^{*I} = d^n x\, C_J^* \,\delta^{JI}. \tag{13.12}$$

This yields[18]

$$d[\star A^{*I} P_I(C) + (\star F^I) A^J \partial_J P_I(C)] = (-)^n (\star F^I) F^J \partial_J P_I(C)$$
$$-s[\star C^{*I} P_I(C) + (\star A^{*I}) A^J \partial_J P_I(C) + \tfrac{1}{2}(\star F^I) A^J A^K \partial_K \partial_J P_I(C)]. \tag{13.13}$$

The remaining terms in (13.6) are dealt with as in the proof of part (i) of theorem 11.1. One gets

$$d[B^{n-1} + I^{n-1\,\alpha} P_\alpha(C)] = -s[b^n + I^{n-1\,\alpha} A^I \partial_I P_\alpha(C)] + N^n + (dI^{n-1\,\alpha}) P_\alpha(C) \tag{13.14}$$

where N^n is in the small algebra (it is an obstruction to a lift in the small algebra, see corollary 10.4 and theorem 11.1). Using Eqs. (13.13) and (13.14) in Eq. (13.10), one obtains

$$s(\omega^n - \ldots) = (-)^n F^J \star F^I \partial_J P_I(C) + N^n + (dI^{n-1\,\alpha}) P_\alpha(C). \tag{13.15}$$

The right hand side of (13.15) has zero antighost number and does not contain the derivatives of the ghosts. Due to $\gamma A_\mu^I = \partial_\mu C^I$ and $\gamma C^I = \gamma A_I^{*\mu} = \gamma C_I^* = 0$, $\gamma(\omega^n - \ldots)$ is a sum of field monomials each of which contains derivatives of the ghosts (unless it vanishes). Hence, (13.15) implies that the term on the right hand side is δ-exact, i.e., that it vanishes weakly,

$$(-)^n (\star F^I) F^J \partial_J P_I(C) + N^n + (dI^{n-1\,\alpha}) P_\alpha(C) \approx 0. \tag{13.16}$$

To analyse this condition, we must distinguish cases I and II.

We treat first case II, i.e. the space of Poincaré-invariant local forms, for which the analysis can be pushed to the end. In this case we have $I^{n-1\,\alpha} \in \mathcal{I}$ with $\mathcal{I}$ as in (13.9).

---

[18]Eq. (13.13) can be elegantly derived using the quantities $\star \tilde{C}^{*I} = \star C^{*I} + \star A^{*I} + \star F^I$ and $\tilde{C}^I = C^I + A^I$. One has $(s + d) \star \tilde{C}^{*I} = 0$ and $(s + d)\tilde{C}^I = F^I$. This implies $(s + d)[\star \tilde{C}^{*I} P_I(\tilde{C})] = (-)^n (\star \tilde{C}^{*I}) F^J \partial_J P_I(C)$ whose n-form part is Eq. (13.13).

Hence, $dI^{n-1\,\alpha}$ is a sum of field monomials each of which contains a first or higher order derivative of at least one of the $F^I_{\mu\nu}$. Furthermore the equations of motion contain at least first order derivatives of the $F^I_{\mu\nu}$. Hence, in case II, the part of Eq. (13.16) which contains only undifferentiated $F^I_{\mu\nu}$ reads $(-)^n F^J \star F^I \partial_J P_I(C) + N^n = 0$.[19] This implies that both $F^J \star F^I \partial_J P_I(C)$ and N^n vanish since N^n contains the $F^I_{\mu\nu}$ only via wedge products of the F^I (all wedge products of the F^I are total derivatives while no $F^J \star F^I$ is a total derivative). Eq. (13.16) yields thus

$$\text{Case II:} \quad F^J \star F^I \partial_J P_I(C) = 0, \quad N^n = 0, \quad (dI^{n-1\,\alpha})P_\alpha(C) \approx 0. \qquad (13.17)$$

Since $F^J \star F^I = -\frac{1}{2} d^n x F^J_{\mu\nu} F^{\mu\nu I}$ is symmetric in I and J, the first condition in (13.17) gives

$$\text{Case II:} \quad \partial_I P_J(C) + \partial_J P_I(C) = 0. \qquad (13.18)$$

The general solution of Eq. (13.18) is obtained from the cohomology $H(D,\mathcal{C})$ of the differential $D = \xi^I \partial_I$ in the space $\mathcal{C}$ of polynomials in commuting extra variables ξ^I and anticommuting variables C^I. Indeed, by contracting Eq. (13.18) with $\xi^I \xi^J$, it reads $Da = 0$ where $a = \xi^I P_I(C)$. Using the contracting homotopy $\varrho = C^I \partial/\partial \xi^I$ (see appendix 2.7), one easily proves that $H(D,\mathcal{C})$ is represented solely by pure numbers ("Poincaré lemma for D"; note that D is similar to $dx^\mu \partial/\partial x^\mu$ except that the "differentials" ξ^I commute while the "coordinates" C^I anticommute). In particular this implies that $Da = 0 \Rightarrow a = DP(C)$ for $a = \xi^I P_I(C)$, i.e.,

$$\text{Case II:} \quad P_I(C) = \partial_I P(C) \qquad (13.19)$$

for some polynomial $P(C)$ in the C^I. The analysis of Eq. (13.10) can now be finished along the lines of the proof of theorem 11.1. One obtains in case II that the general solution of Eq. (13.10) is, up to trivial solutions, given by

$$\text{Case II:} \quad \omega^n = [\star C^{*I} \partial_I + (\star A^{*I}) A^J \partial_J \partial_I + \tfrac{1}{2}(\star F^I) A^J A^K \partial_K \partial_J \partial_I]P(C)$$
$$+ B^n + I^{n\,\alpha} P_\alpha(C) + [K_\Delta + j_\Delta A^I \partial_I]P^\Delta(C) \qquad (13.20)$$

where $P(C)$, $P_\alpha(C)$ and $P^\Delta(C)$ are arbitrary polynomials in the undifferentiated ghosts, B^n occurs only in odd dimensional spacetime due to (13.7), $j_\Delta \in \mathcal{I}$ are gauge-invariant and Poincaré-invariant conserved $(n-1)$-forms, see Eqs. (11.17) through (11.19), and K_Δ contains the global symmetry corresponding to j_Δ and satisfies $sK_\Delta + dj_\Delta = 0$. There are no solutions W_Γ because all characteristic classes $P(F)$, including those with form-degree n, are nontrivial in the equivariant characteristic cohomology $H_{\text{char}}(d,\mathcal{I})$ with $\mathcal{I}$ as in (13.9) (cf. footnote 19).

Let us finally discuss case I, i.e., the space of all local forms (with a possible, explicit x-dependence). In this case Eq. (13.16) holds for $I^{n-1\,\alpha} \in \mathcal{I}$ with $\mathcal{I}$ as in (13.8). In contrast to case II, $dI^{n-1\,\alpha}$ may thus contain field monomials which involve only undifferentiated $F^I_{\mu\nu}$ because $I^{n-1\,\alpha}$ may depend explicitly on the spacetime coordinates x^μ. Therefore the arguments that have led us to Eq. (13.19) do not apply in case I. In fact one finds in all spacetime dimensions $n \neq 4$ that Eq. (13.19)

---

[19] The same argument yields $P(F) \approx dI, I \in \mathcal{I} \Rightarrow P(F) = 0$ in case II.

need not hold in case I. Rather, if $n \neq 4$, (13.10) does not impose any restriction on the $P_I(C)$ at all, i.e., the terms in ω^{n-1} related to the $P_I(C)$ can be lifted to a solution ω^n for any set $\{P_I(C)\}$. This solution is $d^n x \, a$ where

$$
\begin{aligned}
a &= (C^{*I} + A^{*\mu I} A^J_\mu \partial_J - \tfrac{1}{2} F^{\mu\nu I} A^J_\mu A^K_\nu \partial_K \partial_J) P_I(C) \\
&\quad + \tfrac{2}{n-4} F^I_{\mu\nu} (x^\mu A^{*\nu J} + x^\mu F^{\rho\nu J} A^K_\rho \partial_K + \tfrac{1}{4} F^{\mu\nu J} x^\rho A^K_\rho \partial_K) \partial_{(I} P_{J)}(C). \quad (13.21)
\end{aligned}
$$

$d^n x \, a$ fulfills (13.19), i.e., $s(d^n x \, a) + da^{n-1} = 0$ where a^{n-1} is indeed of the form (13.6),

$$
\begin{aligned}
a^{n-1} &= [\star A^{*I} + (\star F^I) A^J \partial_J] P_I(C) + \tfrac{1}{(n-1)!} dx^{\mu_1} \dots dx^{\mu_{n-1}} \epsilon_{\mu_1 \dots \mu_n} I^{\mu_n}, \\
I^\mu &= \tfrac{2}{n-4} \left(\tfrac{1}{4} x^\mu F^I_{\nu\rho} F^{\nu\rho J} + F^{\mu\nu I} F^J_{\nu\rho} x^\rho \right) \partial_{(I} P_{J)}(C).
\end{aligned}
$$

Note that both a and a^{n-1} depend explicitly on x^μ and are therefore present only in case I but not in case II, except when $\partial_{(I} P_{J)}(C) = 0$ (then $d^n x \, a$ reproduces the first line of (13.20). When one multiplies a by $n-4$, one gets solutions for all n. For $n = 4$, they become solutions of the form $[K_\Delta + j_\Delta A^I \partial_I] P^\Delta(C)$ with gauge invariant Noether currents $j_\Delta \in \mathcal{I}$ involving explicitly the x^μ. One may now proceed along the previous lines. However, two questions remain open in case I: Does (13.10) impose restrictions on the $P_I(C)$ when $n = 4$? Are there characteristic classes $P(F)$ with form-degree n which are trivial in the equivariant characteristic cohomology $H_{\mathrm{char}}(d, \mathcal{I})$ with $\mathcal{I}$ as in (13.8)? (See also footnote 17.)

13.3 Uniqueness of Yang-Mills cubic vertex

We now use the above results to discuss the consistent deformations of the action (13.1). Requiring that the interactions be Poincaré invariant, the relevant results are those of case II.

As shown in [19], the consistent deformations of an action are given, to first order in the deformation parameter, by the elements of $H^{0,n}(s|d)$, i.e., here, from (13.20),

$$
\begin{aligned}
\omega^{0,n} &= [\star C^{*I} \partial_I + (\star A^{*I}) A^J \partial_J \partial_I + \tfrac{1}{2} (\star F^I) A^J A^K \partial_K \partial_J \partial_I] P(C) \\
&\quad + B^n + I^n + [K_\Delta + j_\Delta A^I \partial_I] P^\Delta(C) \quad (13.22)
\end{aligned}
$$

where $P(C)$ has ghost number 3 and $P^\Delta(C)$ ghost number one ($P_\alpha(C)$ in (13.20) has ghost number zero and thus is a constant; this has been taken into account in (13.22)).

The term B^n is the familiar Chern-Simons term [80], and exists only in odd dimensions. It belongs to the small algebra and is of the form $AF \dots F$. The term I^n is strictly gauge-invariant and thus involves the abelian field strengths and their derivatives. Born-Infeld or Euler-Heisenberg deformations are of this type. Since these terms are well understood and do not affect the gauge symmetry, we shall drop them from now on and focus on the other two terms, which are,

$$
[\star C^{*I} \partial_I + (\star A^{*I}) A^J \partial_J \partial_I + \tfrac{1}{2} (\star F^I) A^J A^K \partial_K \partial_J \partial_I] P(C) \quad (13.23)
$$

and

$$
[K_\Delta + j_\Delta A^I \partial_I] P^\Delta(C) \quad (13.24)
$$

127

Expression (13.23) involves the antifields conjugate to the ghosts, while (13.24) involves only the antifields conjugate to A^I.

Now, it has also been shown in [19] (see [137] for further details) that deformations involving nontrivially the antifields do deform the gauge symmetries. Those that involve the antifields conjugate to the ghosts deform not only the gauge transformations but also their algebra; while those that involve only A_I^* modify the gauge transformations but leave the gauge algebra unchanged (at least to first order in the deformation parameter).

Writing $P(C) = (1/3!)f_{IJK}C^I C^J C^K$ (with f_{IJK} completely antisymmetric), one gets from (13.23) that the deformations of the theory that deform the gauge algebra are given by

$$\tfrac{1}{2} \star C^{*I} f_{IJK} C^J C^K + (\star A^{*I}) A^J f_{IJK} C^K + \tfrac{1}{2} (\star F^I) A^J A^K f_{IJK}$$
$$= -d^n x \left(\tfrac{1}{2} f_{IJK} C^K C^J C^{*I} + f_{IJK} A_\mu^J C^K A^{*\mu I} + \tfrac{1}{2} f_{IJK} F^{\mu\nu I} A_\mu^J A_\nu^K \right). \quad (13.25)$$

The term independent of the antifields is the first order deformation of the action, and one recognizes the standard Yang-Mills cubic vertex – except that the f_{IJK} are not subject to the Jacobi identity at this stage. This condition arises, however, when one investigates consistency of the deformation to second order: the deformation is obstructed at second order by a non trivial element of $H^{1,n}(s|d)$ unless $\sum_K (f_{IJK} f_{KLM} + f_{JLK} f_{KIM} + f_{LIK} f_{KJM}) = 0$ [20]. The obstruction is precisely of the type (13.23), with $P(C) = -(1/36) \sum_K f_{IJK} f_{KLM} C^I C^J C^L C^M$. The field monomials in front of the antifields in (13.25) give the deformations of the BRST transformations of the ghosts and gauge fields respectively (up to a minus sign) and provide thus the nonabelian extension of the abelian gauge transformations and their algebra. Thus, one recovers the known fact that the Yang-Mills construction provides the only deformation of the action for a set of free abelian gauge fields that deforms the algebra of the gauge transformations at first order in the deformation parameter. Any consistent interaction which deforms nontrivially the gauge algebra at first order contains therefore the Yang-Mills vertex. Furthermore, this deformation automatically incorporates the Lie algebra structure underlying the Yang-Mills theory, without having to postulate it a priori. This result has been derived recently in [220], along different lines und under stronger assumptions on the form of the new gauge symmetries. These extra assumptions are in fact not necessary as the cohomological derivation shows.

Having dealt with the deformations (13.23), we can turn to the deformations (13.24), which do not deform the (abelian) gauge algebra at first order although they do deform the gauge transformations. These involve Lorentz covariant and gauge invariant conserved currents j_Δ^μ. An example of a deformation of this type is given by the Freedman-Townsend vertex in three dimensions [111, 3]. In four dimensions, however, the results of [203] indicate that there is no (non trivial) candidate for j_Δ^μ. There is an infinite number of conservation laws because the theory is free, but these do not involve gauge invariant Lorentz vectors. Thus, there is no Poincaré invariant deformation of the type (13.24) in four dimensions. This strengthens the above result on the uniqueness of the Yang-Mills cubic vertex, which is the only

vertex deforming the gauge transformations in four dimensions. Accordingly, in four dimensions, the most general deformation of the action for a set of free abelian gauge fields is given, at first order, by the Yang-Mills cubic vertex and by strictly gauge invariant deformations. We do not know whether the results of [203] generalize to higher dimensions, leaving the (unlikely in our opinion) possibility of the existence of interactions of the type (13.24) in $n > 4$ dimensions, which would deform the gauge transformations without modifying the gauge algebra at first order in the deformation parameter.

14 Three-dimensional Chern-Simons theory

14.1 Introduction − $H(s)$

We shall now describe the local BRST cohomology in 3-dimensional pure Chern-Simons theory with general gauge group G, i.e., G may be abelian, semisimple, or the direct product of an abelian and a semisimple part. Since pure Chern-Simons theory is of the Yang-Mills type, theorem 11.1 applies to it when the gauge group is semisimple. So, the Chern-Simons case is not really special from this point of view. However, the results are particularly simple in this case because the theory is topological. As we shall make it explicit below, there is no non-trivial local, gauge-invariant function and the BRST cohomology reduces to the Lie algebra cohomology with coefficients in the trivial representation.

It is because of this, and because of the physical interest of the Chern-Simons theory, that we devote a special section to it.

The Chern-Simons action is [80]

$$S_{\text{CS}} = \int g_{IJ}[\tfrac{1}{2} A^I dA^J + \tfrac{1}{6} e f_{KL}{}^I A^J A^K A^L]. \tag{14.1}$$

where $g_{IJ} = \delta_{IJ}$ for the abelian part of G and $g_{IJ} = f_{IK}{}^L f_{JL}{}^K$ for the nonabelian part. This yields explicitly

$$\begin{aligned}
s\, A_I^{\mu*} &= \tfrac{1}{2} g_{IJ} \epsilon^{\mu\nu\rho} F_{\nu\rho}^J + e\, C^J f_{JI}{}^K A_K^{\mu*}, \\
s\, C_I^* &= -D_\mu A_I^{\mu*} + e\, C^J f_{JI}{}^K C_K^*.
\end{aligned} \tag{14.2}$$

Again we shall determine $H(s,\Omega)$ both in the space of all local forms (case I) and in the space of Poincaré-invariant local forms (case II).

We first specify $H(s,\Omega)$ in these cases, using the results of Section 8. Since in pure Chern-Simons theory the field strengths vanish weakly and do not contribute to $H(s,\Omega)$ at all, we find from this section that in case I $H(s,\Omega)$ is represented by polynomials in the $\theta_r(C)$ which can also depend explicitly on the spacetime coordinates x^μ and the differentials dx^μ,

$$\text{Case I:} \quad s\omega = 0 \iff \omega = P(\theta(C), x, dx) + s\eta. \tag{14.3}$$

In case II, a similar result holds, but now no x^μ can occur and Lorentz invariance enforces that the differentials can contribute nontrivially only via the volume form d^3x,

$$\text{Case II:} \quad s\omega = 0 \iff \omega = Q(\theta(C)) + d^3x P(\theta(C)) + s\eta. \tag{14.4}$$

(14.3) and (14.4) provide the solutions of the consistency condition with a trivial descent. They also yield the bottom forms which can appear in nontrivial descents. To find all solutions with a nontrivial descent, we investigate how far these bottom forms can be lifted to solutions with higher form-degree.

14.2   BRST cohomology in the case of x-dependent forms

In order to lift a bottom form $P(\theta(C), x, dx)$ once, it is necessary and sufficient that $d_x P(\theta(C), x, dx) = 0$ where $d_x = dx^\mu \partial/\partial x^\mu$ (this is nothing but Eq. (11.14), specified to $P(\theta(C), x, dx) = I^\alpha(x, dx)\Theta_\alpha$). $d_x P(\theta(C), x, dx) = 0$ implies $P(\theta(C), x, dx) = Q(\theta(C)) + d_x P'(\theta(C), x, dx)$ by the ordinary Poincaré lemma, and thus $P(\theta(C), x, dx) = Q(\theta(C)) + dP'(\theta(C), x, dx) + s[A^I \partial_I P'(\theta(C), x, dx)]$ where we used once again Eq. (11.13) and the notation

$$\partial_I = \frac{\partial}{\partial C^I} \ .$$

The pieces $dP'(\ldots) + s[A^I \partial_I P'(\ldots)]$ are trivial and can thus be neglected without loss of generality. Hence, all bottom forms which can be lifted once can be assumed to be of the form $Q(\theta(C))$. Furthermore there are no obstructions to lift these bottom forms to higher form-degrees. An elegant way to see this and to construct the corresponding solutions at higher form-degree is the following. We introduce

$$\mathcal{C}^I = C^I + A^I + \star A^{*I} + \star C^{*I} \ , \tag{14.5}$$

where

$$\star A^{*I} = \frac{1}{2} dx^\mu dx^\nu \epsilon_{\mu\nu\rho} g^{IJ} A_J^{*\rho} \quad , \quad \star C^{*I} = d^3 x g^{IJ} C_J^* \ ,$$

(see also for instance [75]). Using (14.2), one verifies

$$(s + d)\, \mathcal{C}^I = \frac{1}{2}\, e\, f_{JK}{}^I \mathcal{C}^K \mathcal{C}^J. \tag{14.6}$$

Hence, $(s+d)$ acts on the $\mathcal{C}^I$ exactly as s acts on the C^I. This implies $(s+d)\theta_r(\mathcal{C}) = 0$ (which is analogous to $s\theta_r(C) = 0$) and thus

$$(s + d)\, Q(\theta(\mathcal{C})) = 0. \tag{14.7}$$

This equation decomposes into the descent equations $s[Q]^3 + d[Q]^2 = 0$, $s[Q]^2 + d[Q]^1 = 0$, $s[Q]^1 + d[Q]^0 = 0$, $s[Q]^0 = 0$ where $[Q]^p$ is the p-form contained in $Q(\theta(\mathcal{C}))$,

$$
\begin{aligned}
Q(\theta(\mathcal{C})) &= \sum_{p=0}^{3} [Q]^p \\
[Q]^0 &= Q(\theta(C)) \\
[Q]^1 &= A^I \partial_I\, Q(\theta(C)) \\
[Q]^2 &= [\tfrac{1}{2} A^I A^J \partial_J \partial_I + \star A^{*I} \partial_I]\, Q(\theta(C)) \\
[Q]^3 &= [\tfrac{1}{6} A^I A^J A^K \partial_K \partial_J \partial_I + A^I \star A^{*J} \partial_J \partial_I + \star C^{*I} \partial_I]\, Q(\theta(C)). \quad (14.8)
\end{aligned}
$$

We conclude that the general solution of the consistency condition is at the various form-degrees given by

$$
\begin{aligned}
\text{Case I:} \quad H^{*,0}(s|d,\Omega): \quad & \omega^0 \sim P(\theta(C), x) \\
H^{*,1}(s|d,\Omega): \quad & \omega^1 \sim [Q]^1 + dx^\mu P_\mu(\theta(C), x) \\
H^{*,2}(s|d,\Omega): \quad & \omega^2 \sim [Q]^2 + dx^\mu dx^\nu P_{\mu\nu}(\theta(C), x) \\
H^{*,3}(s|d,\Omega): \quad & \omega^3 \sim [Q]^3 \quad\quad\quad\quad\quad\quad\quad\quad\quad (14.9)
\end{aligned}
$$

where one can assume that $dx^\mu P_\mu(\theta(C), x)$ and $dx^\mu dx^\nu P_{\mu\nu}(\theta(C), x)$ are not d_x-closed because otherwise they are trivial by the arguments given above. For the same reason every contribution $d^3x P(\theta(C), x)$ to ω^3 is trivial (it is a volume form and thus automatically d_x-closed) and has therefore not been written in (14.9).

14.3 BRST cohomology in the case of Poincaré invariant forms

This case is easy. By Eq. (14.4), all nontrivial bottom forms that can appear in case II are either 0-forms $Q(\theta(C))$ or volume-forms $d^3x P(\theta(C))$. We know already that the former can be lifted to the above Poincaré-invariant solutions $[Q]^1$, $[Q]^2$ and $[Q]^3$. Hence, we only need to discuss the volume-forms $d^3x P(\theta(C))$. They are nontrivial in case II, in contrast to case I, except for the banal case that $P(\theta)$ vanishes identically. Indeed, assume that $d^3x P(\theta(C))$ is trivial, i.e., that $d^3x P(\theta(C)) = s\eta_3 + d\eta_2$ for some Poincaré-invariant local forms η_3 and η_2. The latter equation has to hold identically in all the fields, antifields and their derivatives. In particular, it must therefore be fulfilled when we set all fields, antifields and their derivatives equal to zero except for the undifferentiated ghosts. This yields an equation $P(\theta(C)) = sh(C)$ since η_2 does not involve x^μ in case II (in contrast to case I). By the Lie algebra cohomology, $P(\theta(C)) = sh(C)$ implies $P(\theta(C)) = 0$, see Section 8. Hence, in the space of Poincaré-invariant local forms, no nonvanishing volume-form $d^3x P(\theta(C))$ is trivial and the general solution of the consistency condition reads

$$\text{Case II:} \quad \begin{aligned} H^{*,0}(s|d, \Omega): & \quad \omega^0 \sim [Q]^0 \\ H^{*,1}(s|d, \Omega): & \quad \omega^1 \sim [Q]^1 \\ H^{*,2}(s|d, \Omega): & \quad \omega^2 \sim [Q]^2 \\ H^{*,3}(s|d, \Omega): & \quad \omega^3 \sim [Q]^3 + d^3x P(\theta(C)). \end{aligned} \quad (14.10)$$

Antifield dependence. $[Q]^2$ and $[Q]^3$ contain antifields. This antifield dependence can actually be removed by the addition of trivial solutions, except when $Q(\theta(C))$ contains abelian ghosts. For instance, consider a $\theta_r(C) = -\frac{1}{3}e\,\mathrm{Tr}(C^3)$ with $m(r) = 2$. We know that this $\theta_r(C)$ can be completed to $q_r(\tilde{C}, F) = \mathrm{Tr}[\tilde{C}F - \frac{1}{3}e\tilde{C}^3]$, $\tilde{C} = C + A$, which satisfies $(s+d)q_r(\tilde{C}, F) = 0$ in 3 dimensions, see subsection 10.6. The solutions arising from $q_r(\tilde{C}, F)$ do not involve antifields and are indeed equivalent to those obtained from $\theta_r(\mathcal{C}) = -\frac{1}{3}e\,\mathrm{Tr}(\mathcal{C}^3)$. Namely one has

$$\mathrm{Tr}(\tilde{C}F - \tfrac{1}{3}\,e\,\tilde{C}^3) = -\tfrac{1}{3}\,e\,\mathrm{Tr}(\mathcal{C}^3) + (s+d)\mathrm{Tr}(\tilde{C} \star A^* + \tilde{C} \star C^*)$$

where $\star A^* = \star A^{*I} T_I$, $\star C^* = \star C^{*I} T_I$. Analogous statements apply to all $\theta_r(C)$ with $m(r) > 1$ and thus to all polynomials thereof. In particular, when the gauge group is semisimple, all $[Q]^p$ can be replaced by antifield independent representatives arising from polynomials $Q(q(\tilde{C}, F))$. It is then obvious that (14.9) and (14.10) reproduce theorem 11.1 when the gauge group is semisimple: in the notation of theorem 11.1, one gets solutions B^1, B^2 and B^3 given by the 1-, 2- and 3-forms in $Q(q(\tilde{C}, F))$, and

solutions $I^\alpha(x,dx)\Theta_\alpha$ given by $P(\theta(C),x)$, $dx^\mu P_\mu(\theta(C),x)$ and $dx^\mu dx^\nu P_{\mu\nu}(\theta(C),x)$ in case I, and by $[Q]^0$ and $d^3xP(\theta(C))$ in case II respectively. In particular there are no solutions $V_{\Delta\alpha}$ when the gauge group is semisimple case because there are no nontrivial Noether currents at all in that case. The latter statement follows directly from the results, because $H^{-1,3}(s|d,\Omega)$ is empty when the gauge group is semisimple.

In contrast, the antifield dependence cannot be completely removed when $Q(\theta_r(C))$ contains abelian ghosts (recall that the abelian ghosts coincide with those $\theta_r(C)$ which $m(r)=1$). For instance, the abelian $\mathcal{C}^I$ satisfy $(s+d)\mathcal{C}^I=0$ and provide thus solutions to the descent equations by themselves. The 3-form solution in an abelian $\mathcal{C}^I$ is $d^3x\,C_I^*$. This is a nontrivial solution because it is also a nontrivial representative of $H_2^3(\delta|d,\Omega)$, see Section 6. Since it is a nontrivial solution with negative ghost number, it is impossible to make it antifield independent.

14.4 Examples

Let us finally spell out the results for $H^{g,3}(s|d,\Omega)$, $g\leq 1$, when the gauge group is either simple and compact, or purely abelian. The generalization to a general gauge group (product of abelian factors times a semi-simple group) is straightforward.

Simple compact gauge group. In this case $H^{g,3}(s|d,\Omega)$ vanishes for $g<0$ and $g=1$, while $H^{0,3}(s|d,\Omega)$ is one-dimensional and represented by the 3-form $\omega^{0,3}$ contained in $-\frac{1}{3}e\mathrm{Tr}(\mathcal{C}^3)$ (both in case I and II),

$$H^{g,3}(s|d,\Omega)=0 \quad \text{for } g<0 \text{ and } g=1, \tag{14.11}$$

$$\omega^{0,3}=-e\mathrm{Tr}[\tfrac{1}{3}A^3+(CA+AC)\star A^*+C^2\star C^*]. \tag{14.12}$$

Purely abelian gauge group. In this case we have $Q(\theta(C))\equiv f_{I_1\ldots I_k}C^{I_1}\cdots C^{I_k}$ where $f_{I_1\ldots I_k}$ are arbitrary constant antisymmetric coefficients. In case I, this yields the following nontrivial representatives of $H^{g,3}(s|d,\Omega)$ for $g=-2,\ldots,1$:

$$
\begin{aligned}
\omega^{-2,3} &= f_I\star C^{*I}\\
\omega^{-1,3} &= 2f_{IJ}(A^I\star A^J+C^I\star C^{*J})\\
\omega^{0,3} &= f_{IJK}(A^IA^JA^K+6C^IA^J\star A^{*K}+3C^IC^J\star C^{*K})\\
\omega^{1,3} &= f_{IJKL}C^I(4A^JA^KA^L+12C^JA^K\star A^{*L}+4C^JC^K\star C^{*L}). \quad (14.13)
\end{aligned}
$$

In case II one gets in addition representatives of $H^{0,3}(s|d,\Omega)$ and $H^{1,3}(s|d,\Omega)$ given by the volume element d^3x and by $d^3x\,a_IC^I$ respectively (with a_I arbitrary constant coefficients).

15   References for other gauge theories

In this section, we give references to works where the previous algebraic techniques have been used to find the general solution of the consistency condition with antifields included (and for the BRST differential associated with gauge symmetries) in other field theoretical contexts.

Some aspects of local BRST cohomology for the Stueckelberg model are investigated in [95]. The general solution of the Wess-Zumino consistency condition for gauged non-linear σ-models is discussed in [141, 143].

Cohomological techniques (Poincaré lemma, BRST cohomology) have been recently analyzed on the lattice in [173, 112, 196].

Algebraic aspects of gravitational anomalies [2] are discussed in [18, 38, 166, 57, 187, 93]. The general solution of the consistency condition without antifields is given in [64]; this work is extended to include the antifields in spacetime dimensions strictly greater than two in [25], where again, the cohomology of the Koszul-Tate differential is found to play a crucial rôle.

In 2 spacetime dimensions, these groups have been analyzed in the context of the (bosonic) string world sheet action coupled to backgrounds in [42, 44, 49, 226, 15, 16, 73, 198, 53]. Complete results, with the antifields included, are derived in [67, 68, 70].

Algebraic results on the Weyl anomaly [74, 79, 81] may be found in [56, 58].

Algebraic aspects of the consistency condition for $N = 1$ supergravity in 4 dimensions have been discussed in [59, 43, 66, 183]. The complete treatment, with antifields included, is given in [69].

For p-form gauge theories, the local BRST cohomology groups without antifields have been investigated in [39] and more recently, with antifields, in [139, 138, 213, 140, 113, 142, 160].

Acknowledgements

References

[1] M. Alexandrov, M. Kontsevich, A. Schwartz and O. Zaboronsky, "The Geometry of the Master equation and Topological Quantum Field Theory," Int. J. Mod. Phys. **A12** (1997) 1405 hep-th/9502010.

[2] L. Alvarez-Gaumé and E. Witten, "Gravitational Anomalies," Nucl. Phys. **B234** (1984) 269.

[3] S.C. Anco, "New spin-one gauge theory in three dimensions," J. Math. Phys. **36** (1995) 6553.

[4] S.C. Anco, "Novel generalization of three-dimensional Yang-Mills theory," J. Math. Phys. **38** (1997) 3399.

[5] I.M. Anderson and T. Duchamp, "On the existence of global variational principles," Amer. J. Math. **102** (1980) 781.

[6] I.M. Anderson, "Introduction to the Variational Bicomplex," Cont. Math. **132** (1992) 51.

[7] D. Anselmi, "Removal of divergences with the Batalin-Vilkovisky formalism," Class. Quant. Grav. **11** (1994) 2181.

[8] D. Anselmi, "More on the subtraction algorithm," Class. Quant. Grav. **12** (1995) 319 hep-th/9407023.

[9] R. Arnowitt and S. Deser, "Interaction between Gauge Vector Fields," Nucl. Phys. **49** (1963) 133.

[10] J.A. de Azcárraga and J.M. Izquierdo, "Lie groups, Lie algebras, cohomology and applications in physics," Cambridge Monographs on Mathematical Physics (Cambridge University Press 1995).

[11] G. Bandelloni, A. Blasi, C. Becchi and R. Collina, "Nonsemisimple Gauge Models: 1. Classical Theory And The Properties Of Ghost States," Ann. Inst. Henri Poincaré Phys. Theor. **28** (1978) 225.

[12] G. Bandelloni, A. Blasi, C. Becchi and R. Collina, "Nonsemisimple Gauge Models: 2. Renormalization," Ann. Inst. Henri Poincaré Phys. Theor. **28** (1978) 255.

[13] G. Bandelloni, "Yang-Mills Cohomology In Four-Dimensions," J. Math. Phys. **27** (1986) 2551.

[14] G. Bandelloni, "Nonpolynomial Yang-Mills Local Cohomology," J. Math. Phys. **28** (1987) 2775.

[15] G. Bandelloni and S. Lazzarini, "Diffeomorphism cohomology in Beltrami parametrization," J. Math. Phys. **34** (1993) 5413.

[16] G. Bandelloni and S. Lazzarini, "Diffeomorphism cohomology in Beltrami parametrization. 2: The 1 forms," J. Math. Phys. **36** (1995) 1 [hep-th/9410190].

[17] W.A. Bardeen, "Anomalous Ward Identities In Spinor Field Theories," Phys. Rev. **184** (1969) 1848.

[18] W. A. Bardeen and B. Zumino, "Consistent And Covariant Anomalies In Gauge And Gravitational Theories," Nucl. Phys. **B244** (1984) 421.

[19] G. Barnich and M. Henneaux, "Consistent couplings between fields with a gauge freedom and deformations of the master equation," Phys. Lett. **B311** (1993) 123 [hep-th/9304057].

[20] G. Barnich, M. Henneaux and R. Tatar, "Consistent interactions between gauge fields and local BRST cohomology: The Example of Yang-Mills models," Int. J. Mod. Phys. **D3** (1994) 139 [hep-th/9307155].

[21] G. Barnich and M. Henneaux, "Renormalization of gauge invariant operators and anomalies in Yang-Mills theory," Phys. Rev. Lett. **72** (1994) 1588 [hep-th/9312206].

[22] G. Barnich, F. Brandt and M. Henneaux, "Conserved currents and gauge invariance in Yang-Mills theory," Phys. Lett. **B346** (1995) 81 hep-th/9411202.

[23] G. Barnich, F. Brandt and M. Henneaux, "Local BRST cohomology in the antifield formalism. I. General theorems," Commun. Math. Phys. **174** (1995) 57 [hep-th/9405109].

[24] G. Barnich, F. Brandt and M. Henneaux, "Local BRST cohomology in the antifield formalism. II. Application to Yang-Mills theory," Commun. Math. Phys. **174** (1995) 93 [hep-th/9405194].

[25] G. Barnich, F. Brandt and M. Henneaux, "Local BRST cohomology in Einstein Yang-Mills theory," Nucl. Phys. **B455** (1995) 357 [hep-th/9505173].

[26] G. Barnich, M. Henneaux, T. Hurth and K. Skenderis, "Cohomological analysis of gauged-fixed gauge theories," hep-th/9910201, to appear in Phys. Lett. **B**.

[27] I.A. Batalin and G.A. Vilkovisky, "Relativistic S Matrix Of Dynamical Systems With Boson And Fermion Constraints," Phys. Lett. **69B** (1977) 309.

[28] I.A. Batalin and G.A. Vilkovisky, "Gauge Algebra And Quantization," Phys. Lett. **102B** (1981) 27.

[29] I.A. Batalin and G. A. Vilkovisky, "Feynman Rules For Reducible Gauge Theories," Phys. Lett. **B120** (1983) 166.

[30] I.A. Batalin and G.A. Vilkovisky, "Quantization Of Gauge Theories With Linearly Dependent Generators," Phys. Rev. **D28** (1983) 2567.

[31] I.A. Batalin and E.S. Fradkin, "A Generalized Canonical Formalism And Quantization Of Reducible Gauge Theories," Phys. Lett. **122B** (1983) 157.

[32] I.A. Batalin and G.A. Vilkovisky, "Closure Of The Gauge Algebra, Generalized Lie Equations And Feynman Rules," Nucl. Phys. **B234** (1984) 106.

[33] I.A. Batalin and G.A. Vilkovisky, "Existence Theorem For Gauge Algebra," J. Math. Phys. **26** (1985) 172.

[34] I.A. Batalin, P.M. Lavrov and I.V. Tyutin, "Remarks on the Sp(2) covariant Lagrangian quantization of gauge theories," J. Math. Phys. **32** (1991) 2513.

[35] I.A. Batalin, K. Bering and P.H. Damgaard, "Gauge independence of the Lagrangian path integral in a higher-order formalism," Phys. Lett. **B389** (1996) 673 [hep-th/9609037].

[36] I. Batalin and R. Marnelius, "General quantum antibrackets," hep-th/9905083.

[37] I. Batalin and R. Marnelius, "Open group transformations within the Sp(2)-formalism," hep-th/9909223.

[38] L. Baulieu and J. Thierry-Mieg, "Algebraic Structure Of Quantum Gravity And The Classification Of The Gravitational Anomalies," Phys. Lett. **B145** (1984) 53.

[39] L. Baulieu, "Algebraic Construction Of Gauge Invariant Theories," PAR-LPTHE 84/4 *Based on lectures given at Cargèse Summer School: Particles and Fields, Cargèse, France, Jul 6-22, 1983.*

[40] L. Baulieu, "Anomalies And Gauge Symmetry," Nucl. Phys. **B241** (1984) 557.

[41] L. Baulieu, "Perturbative Gauge Theories," Phys. Rept. **129** (1985) 1.

[42] L. Baulieu, C. Becchi and R. Stora, "On The Covariant Quantization Of The Free Bosonic String," Phys. Lett. **B180** (1986) 55.

[43] L. Baulieu, M. Bellon and R. Grimm, "BRS Symmetry Of Supergravity In Superspace And Its Projection To Component Formalism," Nucl. Phys. **B294** (1987) 279.

[44] L. Baulieu and M. Bellon, "Beltrami Parametrization And String Theory," Phys. Lett. **B196** (1987) 142.

[45] C. Becchi, A. Rouet and R. Stora, "The Abelian Higgs-Kibble Model. Unitarity Of The S Operator," Phys. Lett. **B52** (1974) 344.

[46] C. Becchi, A. Rouet and R. Stora, "Renormalization Of The Abelian Higgs-Kibble Model," Commun. Math. Phys. **42** (1975) 127.

[47] C. Becchi, A. Rouet and R. Stora, "Renormalization Of Gauge Theories," Annals Phys. **98** (1976) 287.

[48] C. Becchi, A. Rouet and R. Stora, "Renormalizable Theories With Symmetry Breaking", in "Field Theory Quantization and Statistical Physics", E. Tirapegui ed., Reidel, Dordrecht 1981 (preprint dated 1975).

[49] C. Becchi, "On The Covariant Quantization Of The Free String: The Conformal Structure," Nucl. Phys. **B304** (1988) 513.

[50] F.A. Berends, G.J. Burgers and H. van Dam, "On The Theoretical Problems In Constructing Interactions Involving Higher Spin Massless Particles," Nucl. Phys. **B260** (1985) 295.

[51] K. Bering, P. H. Damgaard and J. Alfaro, "Algebra of Higher Antibrackets," Nucl. Phys. **B478** (1996) 459 [hep-th/9604027].

[52] R.A. Bertlmann, "Anomalies in quantum field theory," *Oxford, UK: Clarendon (1996) 566 p. (International series of monographs on physics: 91).*

[53] P. A. Blaga, L. Tătaru and I. V. Vancea, "BRST cohomology for 2D gravity," Rom. J. Phys. **40** (1995) 773 [hep-th/9504037].

[54] G. Bonneau, "Some Fundamental But Elementary Facts On Renormalization And Regularization: A Critical Review Of The Eighties," Int. J. Mod. Phys. **A5** (1990) 3831.

[55] L. Bonora and P. Cotta-Ramusino, "Some Remarks On BRS Transformations, Anomalies And The Cohomology Of The Lie Algebra Of The Group Of Gauge Transformations," Commun. Math. Phys. **87** (1983) 589.

[56] L. Bonora, P. Cotta-Ramusino and C. Reina, "Conformal Anomaly And Cohomology," Phys. Lett. **B126** (1983) 305.

[57] L. Bonora, P. Pasti and M. Tonin, "The Anomaly Structure Of Theories With External Gravity," J. Math. Phys. **27** (1986) 2259.

[58] L. Bonora, P. Pasti and M. Bregola, "Weyl Cocycles," Class. Quant. Grav. **3** (1986) 635.

[59] L. Bonora, P. Pasti and M. Tonin, "The Chiral Anomaly In Supersymmetric Gauge Theories Coupled To Supergravity," Phys. Lett. **B167** (1986) 191.

[60] N. Bourbaki, "Groupes et algèbres de Lie I," (Hermann, Paris 1960).

[61] F. Brandt, N. Dragon and M. Kreuzer, "All Consistent Yang-Mills Anomalies," Phys. Lett. **B231** (1989) 263.

[62] F. Brandt, N. Dragon and M. Kreuzer, "Completeness And Nontriviality Of The Solutions Of The Consistency Conditions," Nucl. Phys. **B332** (1990) 224.

[63] F. Brandt, N. Dragon and M. Kreuzer, "Lie Algebra Cohomology," Nucl. Phys. **B332** (1990) 250.

[64] F. Brandt, N. Dragon and M. Kreuzer, "The Gravitational Anomalies," Nucl. Phys. **B340** (1990) 187.

[65] F. Brandt, "Antifield dependence of anomalies," Phys. Lett. **B320** (1994) 57 [hep-th/9310080].

[66] F. Brandt, "Anomaly candidates and invariants of D = 4, N=1 supergravity theories," Class. Quant. Grav. **11** (1994) 849 [hep-th/9306054].

[67] F. Brandt, W. Troost and A. Van Proeyen, "The BRST–antibracket cohomology of 2d gravity," Nucl. Phys. **B464** (1996) 353 [hep-th/9509035].

[68] F. Brandt, W. Troost and A. Van Proeyen, "Background charges and consistent continuous deformations of 2d gravity theories," Phys. Lett. **B374** (1996) 31 [hep-th/9510195].

[69] F. Brandt, "Local BRST cohomology in minimal D = 4, N = 1 supergravity," Annals Phys. **259** (1997) 253 [hep-th/9609192].

[70] F. Brandt, J. Gomis and J. Simón, "Cohomological analysis of bosonic D-strings and 2d sigma models coupled to abelian gauge fields," Nucl. Phys. **B523** (1998) 623 [hep-th/9712125].

[71] A.D. Browning and D. McMullan, "The Batalin, Fradkin, And Vilkovisky Formalism For Higher Order Theories," J. Math. Phys. **28** (1987) 438.

[72] R.L. Bryant and P.A. Griffiths, "Characteristic Cohomology Of Differential Systems I: General Theory," J. Am. Math. Soc. **8** (1995) 507.

[73] I. L. Buchbinder, B. R. Mistchuk and V. D. Pershin, "BRST - BFV analysis of anomalies in bosonic string theory interacting with background gravitational field," Phys. Lett. **B353** (1995) 457 [hep-th/9502087].

[74] D. M. Capper and M. J. Duff, "Trace Anomalies In Dimensional Regularization," Nuovo Cim. **23A** (1974) 173.

[75] M. Carvalho, L. C. Vilar, C. A. Sasaki and S. P. Sorella, "BRS Cohomology of Zero Curvature Systems I. The Complete Ladder Case," J. Math. Phys. **37** (1996) 5310 [hep-th/9509047].

[76] C. Chevalley and S. Eilenberg, "Cohomology theory of Lie groups and Lie algebras," Trans. Amer. Math. Soc. **63** (1948) 85.

[77] J.C. Collins, "Renormalization. An Introduction To Renormalization, The Renormalization Group, And The Operator Product Expansion," *Cambridge, Uk: Univ. Pr. (1984) 380p*, chapter 12.6.

[78] J.C. Collins and R.J. Scalise, "The Renormalization of Composite Operators in Yang-Mills Theories using General Covariant Gauge," Phys. Rev. **D50** (1994) 4117 [hep-ph/9403231].

[79] S. Deser, M. J. Duff and C. J. Isham, "Nonlocal Conformal Anomalies," Nucl. Phys. **B111** (1976) 45.

[80] S. Deser, R. Jackiw and S. Templeton, "Three-Dimensional Massive Gauge Theories," Phys. Rev. Lett. **48** (1982) 975.

[81] S. Deser and A. Schwimmer, "Geometric classification of conformal anomalies in arbitrary dimensions," Phys. Lett. **B309** (1993) 279 [hep-th/9302047].

[82] M. De Wilde, "On the Local Chevalley Cohomology of the Dynamical Lie algebra of a Symplectic Manifold," Lett. Math. Phys. **5** (1981) 351.

[83] B. de Wit and J.W. van Holten, "Covariant Quantization Of Gauge Theories With Open Gauge Algebra," Phys. Lett. **79B** (1978) 389.

[84] B.S. DeWitt, "Quantum Theory of Gravity. II. The Manifestly Covariant Theory," Phys. Rev. **162** (1967) 1195.

[85] L.A. Dickey, "On Exactness of the Variational Bicomplex", Cont. Math. **132** (1992) 307.

[86] J.A. Dixon and J.C. Taylor, "Renormalization of Wilson Operators in Gauge Theories", Nucl. Phys. **B78** (1974) 552.

[87] J.A. Dixon, "Field Redefinition And Renormalization In Gauge Theories," Nucl. Phys. **B99** (1975) 420.

[88] J.A. Dixon, "Cohomology And Renormalization Of Gauge Theories. 1," unpublished preprint.

[89] J.A. Dixon, "Cohomology And Renormalization Of Gauge Theories. 2," HUTMP 78/B64.

[90] W.S. Deans and J.A. Dixon, "Theory Of Gauge Invariant Operators: Their Renormalization And S Matrix Elements," Phys. Rev. **D18** (1978) 1113.

[91] J. Dixon and M. Ramon Medrano, "Anomalies of higher-dimension composite fields," Phys. Rev. **D22** (1980) 429.

[92] J.A. Dixon, "Calculation of BRS cohomology with spectral sequences," Commun. Math. Phys. **139** (1991) 495.

[93] N. Dragon, "Regular gravitational Lagrangians," Phys. Lett. **B276** (1992) 31.

[94] N. Dragon, "BRS Symmetry and Cohomology," Lectures given at First National Summer School for Graduate Students, Saalburg 1995, hep-th/9602163.

[95] N. Dragon, T. Hurth and P. van Nieuwenhuizen, "Polynomial form of the Stueckelberg model," Nucl. Phys. Proc. Suppl. **56B** (1997) 318 [hep-th/9703017].

[96] M. Dubois-Violette, M. Talon and C.M. Viallet, "New Results On BRS Cohomology In Gauge Theory," Phys. Lett. **158B** (1985) 231.

[97] M. Dubois-Violette, M. Talon and C.M. Viallet, "BRS Algebras: Analysis Of The Consistency Equations In Gauge Theory," Commun. Math. Phys. **102** (1985) 105.

[98] M. Dubois-Violette, M. Talon and C.M. Viallet, "Anomalous Terms In Gauge Theory: Relevance Of The Structure Group," Ann. Inst. Henri Poincaré Phys. Theor. **44** (1986) 103.

[99] M. Dubois-Violette, "Systèmes dynamiques contraints: l'approche homologique," Ann. Inst. Fourier **37** (1987) 45.

[100] M. Dubois-Violette, M. Henneaux, M. Talon and C. Viallet, "Some results on local cohomologies in field theory," Phys. Lett. **B267** (1991) 81.

[101] M. Dubois-Violette, M. Henneaux, M. Talon and C. Viallet, "General solution of the consistency equation," Phys. Lett. **B289** (1992) 361 [hep-th/9206106].

[102] L.D. Faddeev and V.N. Popov, "Feynman Diagrams For The Yang-Mills Field," Phys. Lett. **B25** (1967) 29.

[103] R. Ferrari and P.A. Grassi, "Constructive algebraic renormalization of the Abelian Higgs-Kibble model," Phys. Rev. **D60** (1999) 065010 [hep-th/9807191].

[104] R.P. Feynman, "Quantum Theory Of Gravitation," Acta Phys. Polon. **24** (1963) 697.

[105] J. Fisch, M. Henneaux, J. Stasheff and C. Teitelboim, "Existence, Uniqueness And Cohomology Of The Classical BRST Charge With Ghosts Of Ghosts," Commun. Math. Phys. **120** (1989) 379.

[106] J.M.L. Fisch and M. Henneaux, "Homological Perturbation Theory And The Algebraic Structure Of The Antifield - Antibracket Formalism For Gauge Theories," Commun. Math. Phys. **128** (1990) 627.

[107] E.S. Fradkin and G.A. Vilkovisky, "Quantization Of Relativistic Systems With Constraints," Phys. Lett. **B55** (1975) 224.

[108] E. S. Fradkin and M. A. Vasilev, "Hamiltonian Formalism, Quantization And S Matrix For Supergravity," Phys. Lett. **B72** (1977) 70.

[109] E.S. Fradkin and G.A. Vilkovisky, "Quantization Of Relativistic Systems With Constraints: Equivalence Of Canonical And Covariant Formalisms In Quantum Theory Of Gravitational Field," CERN-TH-2332.

[110] E.S. Fradkin and T.E. Fradkina, "Quantization Of Relativistic Systems With Boson And Fermion First And Second Class Constraints," Phys. Lett. **72B** (1978) 343.

[111] D.Z. Freedman and P.K. Townsend, "Antisymmetric Tensor Gauge Theories And Nonlinear Sigma Models," Nucl. Phys. **B177** (1981) 282.

[112] T. Fujiwara, H. Suzuki and K. Wu, "Non-commutative differential calculus and the axial anomaly in Abelian lattice gauge theories," hep-lat/9906015.

[113] J.A. Garcia and B. Knaepen, "Couplings between generalized gauge fields," Phys. Lett. **B441** (1998) 198 [hep-th/9807016].

[114] I.M. Gel'fand and I.Ya. Dorfman, "Hamiltonian operators and associated algebraic structures," Funct. Anal. Appl. **13** (1979) 174.

[115] H. Georgi, "Thoughts on effective field theory," Nucl. Phys. Proc. Suppl. **29BC** (1992) 1.

[116] B. Geyer, P. M. Lavrov and D. Mülsch, "OSp(1,2)-covariant Lagrangian quantization of reducible massive gauge theories," J. Math. Phys. **40** (1999) 6189 [hep-th/9806117].

[117] J. Gomis, J. París and S. Samuel, "Antibracket, antifields and gauge theory quantization," Phys. Rept. **259** (1995) 1 [hep-th/9412228].

[118] J. Gomis and J. París, "Perturbation theory and locality in the field - antifield formalism," J. Math. Phys. **34** (1993) 2132.

[119] J. Gomis and S. Weinberg, "Are Nonrenormalizable Gauge Theories Renormalizable?," Nucl. Phys. **B469** (1996) 473 [hep-th/9510087].

[120] P.A. Grassi, T. Hurth and M. Steinhauser, "Practical algebraic renormalization," hep-ph/9907426.

[121] W. Greub, S. Halperin and R. Vanstone, "Connections, Curvature and Cohomology. Volume III: Cohomology of Principal Bundles and Homogeneous Spaces", Pure and Applied Mathematics. A Series of Monographs and Textbooks, vol. 47, eds. S. Eilenberg and H. Bass, (Academic Press, New York: 1976).

[122] M.A. Grigorev and P.H. Damgaard, "Superfield BRST charge and the master action," hep-th/9911092.

[123] M.A. Grigorev, A. M. Semikhatov and I. Y. Tipunin, "BRST formalism and zero locus reduction," hep-th/0001081.

[124] V.K.A.M. Gugenheim and J.P. May, "On the theory and applications of torsion products," Mem. AMS **142** (1974).

[125] V.K.A.M. Gugenheim, "On a perturbation theory for the homology of a loop space," J. Pure Appl. Alg. **25** (1982) 197.

[126] V.K.A.M. Gugenheim and J.D. Stasheff, "On perturbations and A_∞-structures," Bull. Soc. Math. Belgique **38** (1986) 237.

[127] V.K.A.M. Gugenheim and L. Lambe, "Applications of perturbation theory to differential homological algebra," Ill. J. Math. **33** (1989) 556.

[128] V.K.A.M. Gugenheim, L. Lambe and J.D. Stasheff, "Applications of perturbation theory to differential homological algebra II," Ill. J. Math. **34** (1990) 485.

[129] B.W. Harris and J. Smith, "Anomalous dimension of the gluon operator in pure Yang-Mills theory," Phys. Rev. **D51** (1995) 4550 [hep-ph/9409405].

[130] M. Henneaux, "Hamiltonian Form Of The Path Integral For Theories With A Gauge Freedom," Phys. Rept. **126** (1985) 1.

[131] M. Henneaux and C. Teitelboim, "BRST Cohomology In Classical Mechanics," Commun. Math. Phys. **115** (1988) 213.

[132] M. Henneaux, "Lectures On The Antifield - BRST Formalism For Gauge Theories," *Lectures given at 20th GIFT Int. Seminar on Theoretical Physics, Jaca, Spain, Jun 5-9, 1989, and at CECS, Santiago, Chile, June/July 1989*, Nucl. Phys. B (Proc. Suppl.) **A18** (1990) 47.

[133] M. Henneaux, "Space-Time Locality of the BRST Formalism," Commun. Math. Phys. **140** (1991) 1.

[134] M. Henneaux and C. Teitelboim, "Quantization of gauge systems," *Princeton, USA: Univ. Pr. (1992) 520 p.*

[135] M. Henneaux, "Remarks on the renormalization of gauge invariant operators in Yang-Mills theory," Phys. Lett. **B313** (1993) 35 [hep-th/9306101].

[136] M. Henneaux, "On The Gauge-Fixed BRST Cohomology," Phys. Lett. **B367** (1996) 163 [hep-th/9510116].

[137] M. Henneaux, "Consistent interactions between gauge fields: The cohomological approach," *Proceedings of a Conference on Secondary Calculus and Cohomological Physics, Aug 24-31, 1997, Moscow, Russia,* Cont. Math. **219** (1998) 93 [hep-th/9712226].

[138] M. Henneaux, B. Knaepen and C. Schomblond, "Characteristic cohomology of p-form gauge theories," Commun. Math. Phys. **186** (1997) 137 [hep-th/9606181].

[139] M. Henneaux, B. Knaepen and C. Schomblond, "BRST cohomology of the Chapline-Manton model," Lett. Math. Phys. **42** (1997) 337 [hep-th/9702042].

[140] M. Henneaux and B. Knaepen, "All consistent interactions for exterior form gauge fields," Phys. Rev. **D56** (1997) 6076 [hep-th/9706119].

[141] M. Henneaux and A. Wilch, "Local BRST cohomology of the gauged principal non-linear sigma model," Phys. Rev. **D58** (1998) 025017 [hep-th/9802118].

[142] M. Henneaux and B. Knaepen, "The Wess-Zumino consistency condition for p-form gauge theories," Nucl. Phys. **B548** (1999) 491 [hep-th/9812140].

[143] M. Henneaux and A. Wilch, "Semi-invariant terms for gauged non-linear sigma-models," Phys. Lett. **B471** (2000) 373 [hep-th/9906121].

[144] G. Hirsch, "Sur les groups d'homologie des espaces fibrés," Bull. Soc. Math. Belgique **6** (1953) 79.

[145] G. 't Hooft, "Renormalization Of Massless Yang-Mills Fields," Nucl. Phys. **B33** (1971) 173.

[146] G. 't Hooft, "Renormalizable Lagrangians For Massive Yang-Mills Fields," Nucl. Phys. **B35** (1971) 167.

[147] G. 't Hooft and M. Veltman, "Regularization And Renormalization Of Gauge Fields," Nucl. Phys. **B44** (1972) 189.

[148] G. 't Hooft and M. Veltman, "Combinatorics Of Gauge Fields," Nucl. Phys. **B50** (1972) 318.

[149] P.S. Howe, U. Lindström and P. White, "Anomalies And Renormalization In The BRST-BV Framework," Phys. Lett. **B246** (1990) 430.

[150] T. Hurth and K. Skenderis, "Quantum Noether method," Nucl. Phys. **B541** (1999) 566 hep-th/9803030.

[151] T. Hurth and K. Skenderis, "The quantum Noether condition in terms of interacting fields," [hep-th/9811231].

[152] S.D. Joglekar and B.W. Lee, "General Theory Of Renormalization Of Gauge Invariant Operators," Ann. Phys. **97** (1976) 160.

[153] R.E. Kallosh, "Gauge Invariance In Supergravitation. (In Russian)," Pisma Zh. Eksp. Teor. Fiz. **26** (1977) 575.

[154] R.E. Kallosh, "Modified Feynman Rules In Supergravity," Nucl. Phys. **B141** (1978) 141.

[155] D.B. Kaplan, "Effective field theories," *Lectures at the Seventh Summer School in Nuclear Physics: Symmetries,* (Seattle, 1995), nucl-th/9506035.

[156] H. Kluberg-Stern and J.B. Zuber, "Ward Identities And Some Clues To The Renormalization Of Gauge Invariant Operators," Phys. Rev. **D12** (1975) 467.

[157] H. Kluberg-Stern and J.B. Zuber, "Renormalization Of Nonabelian Gauge Theories In A Background Field Gauge: 1. Green Functions," Phys. Rev. **D12** (1975) 482.

[158] H. Kluberg-Stern and J.B. Zuber, "Renormalization Of Nonabelian Gauge Theories In A Background Field Gauge. 2. Gauge Invariant Operators," Phys. Rev. **D12** (1975) 3159.

[159] O.M. Khudaverdian and A.P. Nersessian, "On the geometry of the Batalin-Vilkovsky formalism," Mod. Phys. Lett. **A8** (1993) 2377 [hep-th/9303136].

[160] B. Knaepen, "Local BRST cohomology for p-form gauge theories," hep-th/9912021.

[161] S. Kobayashi and K. Nomizu, "Foundations of Differential Geometry. Volume II," John Wiley and Sons (1996), section 12.

[162] J.-L. Koszul, "Homologie et cohomologie des algèbres de Lie," Bull. Soc. Math. France **78** (1950) 65-127.

[163] Y.P. Lam, "Perturbation Lagrangian Theory For Scalar Fields: Ward-Takahasi Identity And Current Algebra," Phys. Rev. **D6** (1972) 2145.

[164] Y.P. Lam, "Equivalence Theorem On Bogoliubov-Parasiuk-Hepp-Zimmermann Renormalized Lagrangian Field Theories," Phys. Rev. **D7** (1973) 2943.

[165] L. Lambe and J.D. Stasheff, "Applications of perturbation theory to iterated fibrations," Manus. Math. **58** (1987) 363.

[166] F. Langouche, T. Schücker and R. Stora, "Gravitational Anomalies Of The Adler-Bardeen Type," Phys. Lett. **B145** (1984) 342.

[167] P.M. Lavrov and I. V. Tyutin, "Effective Action In General Gauge Theories," Sov. J. Nucl. Phys. **41** (1985) 1049.

[168] B.W. Lee and J. Zinn-Justin, "Spontaneously Broken Gauge Symmetries. I. Preliminaries," Phys. Rev. **D5** (1972) 3121.

[169] B.W. Lee and J. Zinn-Justin, "Spontaneously Broken Gauge Symmetries. II. Perturbation Theory And Renormalization," Phys. Rev. **D5** (1972) 3137.

[170] B.W. Lee and J. Zinn-Justin, "Spontaneously Broken Gauge Symmetries. III. Equivalence," Phys. Rev. **D5** (1972) 3155.

[171] B.W. Lee and J. Zinn-Justin, "Spontaneously Broken Gauge Symmetries. IV. General Gauge Formulation," Phys. Rev. **D7** (1973) 1049.

[172] J.H. Lowenstein, "Differential Vertex Operations In Lagrangian Field Theory," Commun. Math. Phys. **24** (1971) 1.

[173] M. Lüscher, "Topology and the axial anomaly in abelian lattice gauge theories," Nucl. Phys. **B538** (1999) 515.

[174] J. Mañes, R. Stora and B. Zumino, "Algebraic Study Of Chiral Anomalies," Commun. Math. Phys. **102** (1985) 157.

[175] A.V. Manohar, "Effective field theories," *Lectures at the 1996 Schladming Winter School,* hep-ph/9606222.

[176] D. McMullan, "Yang-Mills Theory And The Batalin-Fradkin-Vilkovisky Formalism," J. Math. Phys. **28** (1987) 428.

[177] C.W. Misner and J.A. Wheeler, "Classical physics as geometry: gravitation, electromagnetism, unquantized charge and mass as properties of curved empty space," Ann. Phys. (N.Y.) **2** (1957) 525.

[178] P.J. Olver, Applications of Lie Groups to Differential Equations", Springer, New York: 1986.

[179] L. O'Raifeartaigh, "Group structure of gauge theories," (Cambridge University Press 1986).

[180] A. Pich, "Effective field theory: Course," *Talk given at Les Houches Summer School in Theoretical Physics, Session 68: Probing the Standard Model of Particle Interactions, Les Houches, France, 28 Jul - 5 Sep 1997*, hep-ph/9806303.

[181] O. Piguet and A. Rouet, "Symmetries In Perturbative Quantum Field Theory," Phys. Rept. **76** (1981) 1.

[182] O. Piguet and S.P. Sorella, "Algebraic renormalization: Perturbative renormalization, symmetries and anomalies," *Berlin, Germany: Springer (1995) 134 p. (Lecture notes in physics: m28).*

[183] O. Piguet and S. Wolf, "The supercurrent trace identities of the N = 1, D = 4 super-Yang-Mills theory in the Wess-Zumino gauge," JHEP **9804** (1998) 001 [hep-th/9802027].

[184] J. Polchinski, "Effective field theory and the Fermi surface," *Lectures presented at TASI 92, Boulder, CO, Jun 3-28, 1992.* Published in Boulder TASI 92:0235-276 [hep-th/9210046].

[185] M. Postnikov, "Leçons de géométrie: Groupes et algèbres de Lie", (traduction française: Editions Mir, 1985).

[186] D.J. Saunders, "The Geometry of Jet bundles", Cambridge University Press (Cambridge, 1989).

[187] T. Schücker, "The Cohomological Proof Of Stora's Solutions," Commun. Math. Phys. **109** (1987) 167.

[188] A.A. Slavnov, "Ward Identities In Gauge Theories," Theor. Math. Phys. **10** (1972) 99.

[189] J. Stasheff, "Homological (ghost) approach to constrained Hamiltonian systems," hep-th/9112002.

[190] J. Stasheff, "Deformation theory and the Batalin-Vilkovisky master equation," q-alg/9702012.

[191] G. Sterman, P.K. Townsend and P. van Nieuwenhuizen, "Unitarity, Ward Identities, And New Quantization Rules Of Supergravity," Phys. Rev. **D17** (1978) 1501.

[192] R. Stora, "Continuum Gauge Theories," *Lectures given at Summer Inst. for Theoretical Physics, Cargèse, France, Jul 12-31, 1976*, published in *New Developments in Quantum Field Theory and Statistical Physics,* eds. M. Lévy and P. Mitter NATO ASI Series B26 (Plenum, 1977).

[193] R. Stora, "Algebraic Structure And Topological Origin Of Anomalies," *Seminar given at Cargèse Summer Inst. Sep 1-15, 1983*, published in *Progress in Gauge Field Theory,* eds. 't Hooft et al. (Plenum Press, 1984)

[194] D. Sullivan, "Infinitesimal computations in topology," Pub. Math. IHES **47** (1977) 269.

[195] F. Takens, "A global version of the inverse problem to the calculus of variations," J. Diff. Geom. **14** (1979) 543.

[196] H. Suzuki, "Anomaly cancellation condition in lattice gauge theory," hep-lat/0002009.

[197] M. Talon, "Algebra Of Anomalies," PAR-LPTHE-85/37, *Presented at Cargèse Summer School, Cargèse, France, Jul 15-31, 1985.*

[198] L. Tătaru and I. V. Vancea, "BRST cohomology in Beltrami parametrization," Int. J. Mod. Phys. **A11** (1996) 375 [hep-th/9504036].

[199] L. Tătaru and R. Tătar, "Koszul-Tate cohomology for an Sp(2)-covariant quantization of gauge theories with linearly dependent generators," Int. J. Mod. Phys. **A13** (1998) 1981 [hep-th/9708159].

[200] J.C. Taylor, "Ward Identities And Charge Renormalization Of The Yang-Mills Field," Nucl. Phys. **B33** (1971) 436.

[201] J. Thierry-Mieg, "Classification Of The Yang-Mills Anomalies In Even And Odd Dimension," Phys. Lett. **B147** (1984) 430.

[202] M. Tonin, "Dimensional regularization and anomalies in chiral gauge theories," Nucl. Phys. Proc. Suppl. **29BC** (1992) 137.

[203] C.G. Torre, "Natural symmetries of the Yang-Mills equations," J. Math. Phys. **36** (1995) 2113 [hep-th/9407129].

[204] W. Troost, P. van Nieuwenhuizen and A. Van Proeyen, "Anomalies And The Batalin-Vilkovisky Lagrangian Formalism," Nucl. Phys. **B333** (1990) 727.

[205] W. Troost and A. Van Proeyen, "Regularization, the BV method, and the antibracket cohomology," hep-th/9410162.

[206] W. Troost and A. Van Proeyen, "Regularization and the BV formalism," hep-th/9307126.

[207] T. Tsujishita, "On variational bicomplexes associated to differential equations," Osaka J. Math. **19** (1982) 311.

[208] W.M. Tulczyjew, "The Euler-Lagrange resolution," in Lecture Notes in Mathematics, No. 836 (1980) 22.

[209] I.V. Tyutin, "Gauge Invariance In Field Theory And Statistical Physics In Operator Formalism," LEBEDEV-75-39.

[210] I.V. Tyutin and B.L. Voronov, "Renormalization Of General Gauge Theories," *In *Moscow 1981, Proceedings, Quantum Gravity*, 481-501.*

[211] W.G. Unruh, "Excluded Possibilities of Geometrodynamical Analog to Electric Charge," Gen. Relativ. Gravit. **2** (1971) 27.

[212] J.W. van Holten, "On the construction of supergravity theories," Ph.D. thesis, University of Leiden (1980).

[213] A. Verbovetsky, "Notes on the horizontal cohomology," in *Proceedings of a Conference on Secondary Calculus and Cohomological Physics, Aug 24-31, 1997, Moscow, Russia,* Cont. Math. **219** (1998) 211 [math.dg/9803115].

[214] A.M. Vinogradov, "On the algebra-geometric foundations of Lagrangian field theory," Sov. Math. Dokl. **18** (1977) 1200.

[215] A.M. Vinogradov, "A spectral sequence associated with a nonlinear differential equation and algebra-geometric foundations of Lagrangian field theory with constraints," Sov. Math. Dokl. **19** (1978) 144.

[216] A.M. Vinogradov, "The $\mathcal{C}$-spectral sequence, Lagrangian formalism, and conservation laws. I. The linear theory. II. The non linear theory," J. Math. Anal. Appl. **100** (1984) 1.

[217] B.L. Voronov and I.V. Tyutin, "Formulation Of Gauge Theories Of General Form. I," Theor. Math. Phys. **50** (1982) 218.

[218] B.l. Voronov and I.V. Tyutin, "Formulation Of Gauge Theories Of General Form. II. Gauge Invariant Renormalizability And Renormalization Structure," Theor. Math. Phys. **52** (1982) 628.

[219] B. L. Voronov, P. M. Lavrov and I. V. Tyutin, "Canonical Transformations And The Gauge Dependence In General Gauge Theories ," Sov. J. Nucl. Phys. **36** (1982) 292.

[220] R.M. Wald, "Spin-two Fields And General Covariance," Phys. Rev. **D33** (1986) 3613.

[221] R.M. Wald, "On identically closed forms locally constructed from a field," J. Math. Phys. **31** (1990) 2378.

[222] S. Weinberg, "Phenomenological Lagrangians," Physica **96A** (1979) 327.

[223] S. Weinberg, "The Quantum theory of fields. Vol. 1: Foundations," *Cambridge, UK: Univ. Pr. (1995) 609 p.*

[224] S. Weinberg, "The quantum theory of fields. Vol. 2: Modern applications," *Cambridge, UK: Univ. Pr. (1996) 489 p.*

[225] S. Weinberg, "What is quantum field theory, and what did we think it was?," hep-th/9702027.

[226] M. Werneck de Oliveira, M. Schweda and S. P. Sorella, "Supersymmetric structure of the bosonic string theory in the Beltrami parametrization," Phys. Lett. **B315** (1993) 93 [hep-th/9305148].

[227] J. Wess and B. Zumino, "Consequences Of Anomalous Ward Identities," Phys. Lett. **B37** (1971) 95.

[228] E. Witten, "A Note On The Antibracket Formalism," Mod. Phys. Lett. **A5** (1990) 487.

[229] C.N. Yang and R.L. Mills, "Conservation of isotopic spin and isotopic gauge invariance," Phys. Rev. **96** (1954) 191.

[230] D.P. Želobenko, "Compact Lie Groups and their Representations," (Translations of Mathematical Monographs vol. 40) American Mathematical Society (Providence, Rhode Island 1973).

[231] J. Zinn-Justin, "Renormalization Of Gauge Theories," *Lectures given at Int. Summer Inst. for Theoretical Physics, Jul 29 - Aug 9, 1974, Bonn, West Germany* published in *Trends in Elementary Particle Physics,* Lectures Notes in Physics 37, H. Rollnik and K. Dietz eds. (Springer Verlag, Berlin 1975).

[232] J. Zinn-Justin, "Quantum Field Theory And Critical Phenomena," *Oxford, UK: Clarendon (1989), 914 p. (International series of monographs on physics, 77).* Third ed. 1996, chapter 21.

[233] J. Zinn-Justin, "Renormalization of gauge theories and master equation," Mod. Phys. Lett. **A14** (1999) 1227 hep-th/9906115.

[234] B. Zumino, "Chiral Anomalies And Differential Geometry" *Lectures given at Les Houches Summer School on Theoretical Physics, Les Houches, France, Aug 8 - Sep 2, 1983,* published in *Relativity, Groups and Topology II,* eds. B.S. De-Witt and R. Stora (North Holland, Amsterdam 1984).

[235] B. Zumino, Y. Wu and A. Zee, "Chiral Anomalies, Higher Dimensions, And Differential Geometry," Nucl. Phys. **B239** (1984) 477.