Conformal Weight of a Vertex Operator

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Consider:

$$
\begin{aligned}
T(z) &= \frac{1}{2}\eta_{\alpha\beta} : \partial_z X^\alpha(z) X^\beta(z) : , \\
V^{\alpha\beta}(k,w) &= : \partial_w X^\alpha(w) \partial_w X^\beta(w) : e^{ik\cdot X(w)}.
\end{aligned}
$$

Then for $z \to w$:

$$
\begin{aligned}
T(z) V^{\mu\nu}(k,w) &= \frac{1}{2}\eta_{\alpha\beta} : \partial_z X^\alpha(z) \partial_z X^\beta(z) : : \partial_w X^\mu(w) \partial_w X^\nu(w) e^{ik\cdot X(w)} : = \\
&= \frac{1}{2}\eta_{\alpha\beta} \Big(\langle \partial_z X^\alpha(z) \partial_w X^\mu(w) \rangle \partial_z X^\beta(z) \partial_w X^\nu(w) e^{ik\cdot X(w)} + \\
&+ \langle \partial_z X^\alpha(z) \partial_w X^\nu(w) \rangle \partial_z X^\beta(z) \partial_w X^\mu(w) e^{ik\cdot X(w)} + \\
&+ \left\langle \partial_z X^\alpha(z) e^{ik\cdot X(w)} \right\rangle \partial_z X^\beta(z) \partial_w X^\mu(w) \partial_w X^\nu(w) \\
&+ \left\langle \partial_z X^\beta(z) \partial_w X^\mu(w) \right\rangle \partial_z X^\alpha(z) \partial_w X^\nu(w) e^{ik\cdot X(w)} + \\
&+ \left\langle \partial_z X^\beta(z) \partial_w X^\nu(w) \right\rangle \partial_z X^\alpha(z) \partial_w X^\mu(w) e^{ik\cdot X(w)} + \\
&+ \left\langle \partial_z X^\beta(z) e^{ik\cdot X(w)} \right\rangle \partial_z X^\alpha(z) \partial_w X^\mu(w) \partial_w X^\nu(w) + \\
&+ \langle \partial_z X^\alpha(z) \partial_w X^\mu(w) \rangle \left\langle \partial_z X^\beta(z) \partial_w X^\nu(w) \right\rangle e^{ik\cdot X(w)} \\
&+ \langle \partial_z X^\alpha(z) \partial_w X^\nu(w) \rangle \left\langle \partial_z X^\beta(z) \partial_w X^\mu(w) \right\rangle e^{ik\cdot X(w)} \\
&+ \langle \partial_z X^\alpha(z) \partial_w X^\mu(w) \rangle \left\langle \partial_z X^\beta(z) e^{ik\cdot X(w)} \right\rangle \partial_w X^\nu(w) + \\
&+ \langle \partial_z X^\alpha(z) \partial_w X^\nu(w) \rangle \left\langle \partial_z X^\beta(z) e^{ik\cdot X(w)} \right\rangle \partial_w X^\mu(w) + \\
&+ \left\langle \partial_z X^\alpha(z) e^{ik\cdot X(w)} \right\rangle \left\langle \partial_z X^\beta(z) e^{ik\cdot X(w)} \right\rangle \partial_w X^\mu(w) \partial_w X^\nu(w) \Big) = \\
&= \frac{1}{(z-w)^4} \eta^{\mu\nu} e^{ik\cdot X(w)} + \frac{i}{(z-w)^3} k^{(\mu} \partial_w X^{\nu)} e^{ik\cdot X(w)} + \\
&+ \frac{1}{(z-w)^2} \left(\frac{1}{2}k^2 + 1 \right) \partial_w X^\mu(w) \partial_w X^\nu(w) + \\
&+ \frac{1}{z-w} \partial_w \left(\partial_w X^\mu(w) \partial_w X^\nu(w) e^{ik\cdot X(w)} \right) + \mathrm{O}(1)
\end{aligned}
$$

A similar treatment of $T(z) V^\mu(k,w)$, where $V^\mu(k,w) = \partial_w^2 X^\mu(w) e^{ik\cdot X(w)}$, leads to the conditions for

$$
\xi_{\mu\nu} V^{\mu\nu}(k,w) + \xi_\mu V^\mu(k,w)
$$

to be a physical vertex operator (e.g.: $k^2 = -2$).