Non-Relativistic General Relativity

Ira Z. Rothstein[a1]

[1]*Carnegie Mellon University*

Dept. of Physics,

Pittsburgh PA 15213, USA

[a] Work supported by DOE contracts DOE-ER-40682-143 and DEAC02-6CH03000.

I. INTRODUCTION

In this chapter we will apply the ideas presented in the previous chapter to general relativity (GR)[1]. Some rudimentary understanding of GR will be assumed, although perhaps surprisingly, very little knowledge of the details of GR are required to generate interesting results relevant to gravitational wave astronomy. We will not go into any details regarding the phenomenology of this subject, but instead refer the reader to the literature [2]. The reader should be familiar with geometric quantities such as the Riemann tensor and its contractions, the scalar curvature, and the Ricci tensor. This chapter relies on the terminology and ideas introduced in the first few sections of the previous chapter. Some of the calculations, in particular those involving the calculation of multipole moments are standard and can be found in text books such as [2], but they have been included in order to keep the chapter more self contained.

We will be interested in calculating the gravity wave signal generated during the radiationally induced inspiral of two gravitationally bound compact objects. We will assume that initially the orbit is sufficiently large that the constitutents are moving at velocities small compared to the speed of light and, as such, the space is approximately flat (see the discussion in the next section). Towards the end of the inspiral, when the velocity grows, the curvature effects become so large that we have no hope of calculating analytically. So we will be restricting our analysis to the early stages of the inspiral where we can do perturbation theory around flat space. Our goal will be to calculate the gravitational power loss as well as the wave forms. To accomplish this we follow the line of reasoning of the previous chapter proceeding using the two stage approach. We start with two (or perhaps more) finite size objects with typical size of order R, separated by a distance of order r, where $r \gg R$. Next we integrate all of the physics distances shorter than R. This leads to an effective theory of point particles. This matching is done in isolation as each particle could have different properties leading to differing matching coefficients for the operators which reproduce the finite size effects. At this stage the only relevant power counting parameter is the ratio of the size of the objects to some long distance scale, typically the eventual wavelength of the radiation. Next we consider the interaction of the bodies, at which point a new power counting parameter will be determined once one specifies the nature of the two body system. In this chapter we will assume that the relative velocity of the two constituents is small compared to c. Another case of interest is the so-called extreme mass ratio scenario, where a small black hole orbits a large black hole. In this case one expands around the Schwarzchild solution and as such the results are not as analytically tractable. We will not be discussing this case here and refer the reader to [4] as an example the application of the EFT technique to this case.

In the second stage we consider what the system looks like at distances much longer than r since we will be observing the binaries asymptotically far away. At these very long distances the binary looks like single particle with time dependent multipole moments. This two stage approach is illustrated in figure (1).

II. THE SINGLE PARTICLE EFFECTIVE THEORY

Consider an isolated compact body, such as a black hole or neutron star. We wish to write down an effective theory which accounts for finite size effects that is consistent with all of the symmetries, which now include, general coordinate invariance and world line reparameterization invariance (RPI). The former implies that our action must be built out of scalar invariants. At the level of a point particle we parameterize the particles world line via $x^\mu(\lambda)$. The action for the point particle is then given by[1]

$$S^0 = -M \int d\tau - 2M_{pl}^2 \int d^4x \sqrt{g} R, \tag{1}$$

where

$$d\tau = d\lambda \sqrt{\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} g^{\mu\nu}(x(\lambda))}. \tag{2}$$

Varying this action leads to the geodesic equation for a point like particle.

Now we would like to include terms which account for the finite size of the object and generate non-geodesic motion. There are an infinite number of such terms in an expansion in the size of the object. The lowest dimensional terms that we can write down that are generally coordinate and RPI are [5]

$$S_2 = C_R \int d\lambda R(x(\lambda)) + C_{vvR} \int d\lambda v^\mu v^\nu R_{\mu\nu}(x(\lambda)). \tag{3}$$

[1] Note we are sticking with our choice of units where $\hbar = c = 1$. Thus even though we will be only interested in classical results, M_{pl} sits around during intermediate stages of the calculation. This is purely historical and a consequence of the author's educational upbringing. Changing back to more conventional units is hopefully not a great burden on the reader.

However these terms can be removed by a field redefinition (see section()), that is, they vanish by use of the vacuum field equations

$$R_{\mu\nu} = 0. \tag{4}$$

Note that we are allowed to use the vacuum equations because the sources all live on the world line. If we included these source terms the field redefinition will introduce singular delta functions ($\delta(0)$) on the world line which should be dropped[2].

The only possible terms that can not be eliminated must involve the Riemann tensor. This also implies that there are no terms linear in the metric tensor. This is consistent with Birkhoff's theorem, which states that the spherically symmetric solution to Einstein's equation is fixed by only one parameter, the mass[3]. Furthermore, on shell, we can write down all of the operators in terms of the trace free Weyl tensor defined by

$$C_{\rho\sigma\mu\nu} = R_{\rho\sigma\mu\nu} - (g_{\rho[\mu}R_{\nu]\sigma} - g_{\sigma[\mu}R_{\nu]\rho}) + \frac{1}{3}g_{\rho[\mu}g_{\nu]\sigma}R. \tag{5}$$

We can further decompose the Weyl tensor into pieces which transform covariantly under parity. Given a time like vector, which we will take to be the world line tangent vector (v^β) we define the electric and magnetic part of the Weyl tensor as

$$E_{\rho\mu} = C_{\rho\sigma\mu\nu}v^\sigma v^\nu \equiv R_{\rho\sigma\mu\nu}v^\sigma v^\nu \tag{6}$$

$$B_{\alpha\mu} = \frac{1}{2}\star C_{\rho\sigma\mu\nu}v^\sigma v^\nu \equiv \frac{1}{2}\star R_{\rho\sigma\mu\nu}v^\sigma v^\nu \tag{7}$$

where the Hodge dual is defined as

$$\star C_{\rho\sigma\mu\nu} = \epsilon_{\rho\sigma\beta\alpha}C^{\beta\alpha}{}_{\mu\nu}. \tag{8}$$

and the equivalences (6,7) hold only on shell, and are allowed at the level of the action (see above).

---

[2] This is discussed in the first problem at the end of this chapter.

[3] Any term in the action which has a linear piece in the metric will effect the one-point function, i.e. field value.

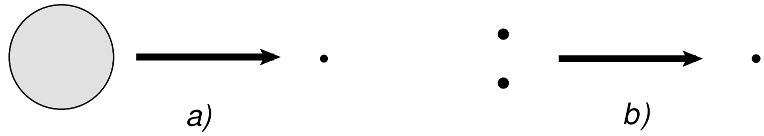

FIG. 1. The two stages of our effective field theory. In the first stage a) we consider each body in isolation and coarse grain to generate an effective field theory of point particles. In the second stage b) we consider the theory of two such point particles we generate an EFT of a single particle, representing the bound state, with time dependent multipole moments.

Symmetry allows us to include two terms that are bi-linear in the Weyl tensor

$$S_{E+B} = \int d\tau (C_E E_{\mu\nu} E^{\mu\nu} + C_B B_{\mu\nu} B^{\mu\nu}). \tag{9}$$

Note that a term a proportional to $E \cdot B$ violate parity and will have a vanishing coefficient for the cases of interest.

The action which includes (1) and (9) represents the first step in coarse graining as shown in the figure (1a). The corrections to this action are suppressed by powers of the radius (R). The scaling of the coefficients $C_{E,B}$ in R (not to be confused with the scalar curvature) is complicated for most compact objects and will depend upon the equation of state. However, for black holes we can take the size as the Schwarzchild radius which is related to the mass via

$$r_s = \frac{M}{16\pi M_{pl}^2}, \tag{10}$$

so that we can write down a precise scaling of $C_{E,B}$ in terms of r_s using dimensional analysis

$$C_{E,B} \sim r_s^5. \tag{11}$$

This world-line theory is valid for observables for which there are no relevant scales shorter than r_s (or whatever the radius is of the object(s) under consideration). The coefficients $C_{E,B}$ are analogous to the polarizabilities discussed in the case of electrodynamics in the previous chapter. In the present case, they represent how easily the object is tidally distorted. The coefficients (C_E, C_B) represent what are known as the leading order (electric, magnetic) "Love numbers". Terms with more derivatives would generate higher order multipole responses, while terms beyond quadratic in the Weyl tensor would generate non-linear responses. We will come back to the matching procedure later in the chapter.

A. Interacting World-Lines: The Post-Newtonian Expansion

The power of the world-line EFT manifests itself when considering interacting bodies. By matching for the Wilson coefficients in isolation we extract the relevant data needed for calculating the finite size corrections to the interactions in a much simpler setting. That is to say, we have "divided and conquered". By treating one scale at a time we avoid the complication involved in solving the whole problem at once. Furthermore, once we have these coefficients we can use them in many different setting/approximations. As discussed in the introduction, there are several approximations that can be considered. In the "extreme mass ratio" approximation, one considers a small black hole with Schwarzchild radius r_{s1} orbiting a large black hole with radius ($r_{s2} \gg r_{s1}$). One then approximates the small black hole dynamics to evolve according to (9). When the distance between the holes is of order the Schwarzchild radius of the large hole, we must work in the full Schwarzchild background. This case is calculationally nettlesome due to the fact that graviton propagator in a curved background is quite complicated, and one usually has to resort numerical methods.

Here will be concerned with the case where the space-time in which the black holes propagate is approximately flat. This criteria will be satisfied when the black holes [4] are sufficiently far apart, i.e. when the distance (r) between them obeys $r \gg r_1, r_2$. Expanding around flat space is called the post-Minkowskian approximation. This approximation need not correspond to a non-relativistic one, as it is an expansion in r_s/r. However, typically when two holes are interacting at long distances they will be bound, in which case the virial theorem implies

$$v^2 \sim \frac{M}{M_{pl}^2 r} \sim r_s/r, \tag{12}$$

and we see that approximate flatness is automatic if $v \ll 1$. This regime is called "Post-Newtonian" (PN)[3]. There exists a family of preferred frames for such system. Typically one chooses to work in the center of mass (COM) frame where the COM position is taken to be at the origin[5]. If we consider the evolution of a binary inspiral then we expect the PN approximation to be valid during the early stages only. At later stages of the inspiral, the holes enter the strong coupling regime and merge.

We will now utilize our world-line EFT to calculate gravity wave signals in the PN

[4] More generally we are also interested in neutron stars or other compact objects, but we will use black holes for all the examples we are considering, just for simplicity.

[5] Note that the center of mass position receives corrections from non-linearities. See the associated problem at the end of this chapter.

approximation. To do so we would need to generate a power counting scheme in v. The scheme we will use is derivative of NRQCD. Indeed, QCD and gravity (expanded around flat space) are both non-linear theories with quadratic propagator kernels. As in NRQCD we expect to have potential modes and radiation modes. Unlike NRQCD however, the soft modes are not relevant for classical gravity as they generate quantum corrections, as will be discussed in more detail in section (**??**). In principle we could include soft modes but they would generate corrections suppressed by $1/L$ where L is the angular momentum of the binary. This is easily uncovered once we repristinate the $\hbar$'s in our calculation and use dimensional analysis. Furthermore, we will be interested in classical sources, and as such, the origin of the relevant modes does not follow from the Landau criteria [6] but from the time dependence of the sources, as discussed in the last chapter.

To formally develop the power counting we first consider all of the relevant scales in the problem,

$$R, r, M, M_{pl}. \tag{13}$$

where M is the reduced mass, R is the radius of the compact body, and r is the radius of the orbit. The scale R will not play a role in the dynamics and can be ignored for the moment. With the remaining parameters we can form two independent dimensionless quantities which we will choose to be

$$\lambda = \frac{M}{r}\frac{1}{M_{pl}^2}, \tag{14}$$

For virialized objects we can relate the potential and kinetic energies leading to the relation

$$\lambda \sim v^2. \tag{15}$$

As the other dimensionless quantity we choose

$$L = \frac{M^2}{v M_{pl}^2}. \tag{16}$$

The inclusion of the dimensionless factor v will allow us to interpret L as the angular momentum below. Next we wish to write down an action with each term scaling homogeneously in each of these expansion parameters. We will work at leading order in $1/L$ since quantum

[6] This comment is relevant only to the readers of the earlier chapters on QCD.

corrections are more than negligible.

In the next step we take the action (1) for i bodies, ignoring finite size effects for the moment,

$$S_) = -2M_{pl}^2 \int d^4x \sqrt{g} R + \sum_i M_i \int d\tau_i \tag{17}$$

and expand it out so that each term in the action scales homogeneously in L and v. In the previous chapter we saw that the photon field itself does not scale homogeneously in v since it has support in both potential and radiation modes (see the discussion in the previous chapter), and the same reasoning applies here. It should not be surprising that the same modes will contribute despite the fact that we are now interested in classical sources, since the classical limit can be reached by ignoring recoil effects.

In analogy to what was done in electrodynamics in the previous chapter, we split the graviton into potential (H) and radiation ($\bar{h}$) modes such that

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{h_{\mu\nu}}{M_{pl}} \equiv \eta_{\mu\nu} + \frac{H_{\mu\nu}}{M_{pl}} + \frac{\bar{h}_{\mu\nu}}{M_{pl}}. \tag{18}$$

At first this decomposition might lead one to think there may be issues of double counting as well as problems with diffeomorphism invariance. However, this decomposition is indeed nothing more then a manifestation of the background field method originally developed by DeWitt [20]. The radiation mode should be thought of as a long wavelength background field which is frozen on the time scales of the potential modes.

Let us now expand the action beginning with the world-line term

$$S_M = -M \int d\tau, \tag{19}$$

we expand around Minkowski space and choose to parameterize the worldline by the global time coordinate

$$d\tau = dt \sqrt{\frac{dx^\mu}{dt}\frac{dx_\mu}{dt} + \frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\frac{h_{\mu\nu}(x(t))}{M_{pl}}}. \tag{20}$$

Then

$$d\tau \approx dt(1 - \frac{1}{2}\vec{v}^2 + \frac{1}{2}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\frac{h_{\mu\nu}(x(t))}{M_{pl}}), \tag{21}$$

so that the action is given by

$$S_m = -M \int dt(1 - \frac{1}{2}\vec{v}^2 + \frac{1}{2}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\frac{h_{\mu\nu}(x(t))}{M_{pl}} - \frac{1}{8}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\frac{h_{\mu\nu}(x(t))}{M_{pl}}\frac{dx^\rho}{dt}\frac{dx^\beta}{dt}\frac{h_{\rho\beta}(x(t))}{M_{pl}} + ...). \tag{22}$$

For the moment let us concentrate on calculating the potentials, so we will set aside the radiation graviton. Recall that the potential field is the instantaneous part of the graviton. If our effective field theory is to have homogenous scaling then we must enforce this instantaneity at the level of the action. To accomplish this we follow the strategy in the last chapter and pull out the large part of the potential's momentum by doing a partial Fourier transform [7] via

$$\mathbf{H}_{\mu\nu}(t, \vec{x}) = \int [d^3k] e^{i\vec{k}\cdot\vec{x}} H_{\vec{k}\ \mu\nu}(t) \equiv \int_{\vec{k}} e^{i\vec{k}\cdot\vec{x}} H_{\vec{k}\ \mu\nu}(t). \tag{23}$$

The potential modes have a typical wavelength of order r, which is to say that $\mathbf{k}$ scales as $1/r$. Now to calculate the propagator for the potential mode we need to gauge fix the action. It is convenient to choose the (harmonic) gauge fixing term as

$$S_{GF} = M_{Pl}^2 \int d^4x \sqrt{g}\Gamma_\mu \Gamma^\mu, \tag{24}$$

with $\Gamma_\mu = \partial_\alpha H^\alpha_\mu - \frac{1}{2}\partial_\mu H^\alpha_\alpha$. When we come back to the issue of radiation we will have to modify this gauge fixing term. In this gauge, the $\mathcal{O}(H^2)$ terms in the action are (after performing a rescaling of $H_{\mu\nu}$ to obtain a canonically normalized kinetic term)

$$\mathcal{L}_{H^2} = -\frac{1}{2}\int_{\vec{k}} \left[\vec{k}^2 H_{\vec{k}\mu\nu} H^{\mu\nu}_{-\vec{k}} - \frac{\vec{k}^2}{2} H_{\vec{k}} H_{-\vec{k}} - \partial_0 H_{\vec{k}\mu\nu}\partial_0 H^{\mu\nu}_{-\vec{k}} + \frac{1}{2}\partial_0 H_{\vec{k}}\partial_0 H_{-\vec{k}} \right], \tag{25}$$

where $H_{\vec{k}} = H^\mu_{\mu\vec{k}}$. As we will see the terms in the second line of this equation are suppressed relative to the first line by a power of v^2 and must be treated as perturbations which correspond to corrections to instantaneity. The propagator for $H_{\vec{k}\mu\nu}$ can be read off from the first term

$$\langle H_{\vec{k}\mu\nu}(x^0) H_{q\alpha\beta}(0)\rangle = -(2\pi)^3 \delta^3(\vec{k} + \vec{q})\frac{i}{\vec{k}^2}\delta(x_0) P_{\mu\nu;\alpha\beta}, \tag{26}$$

where $P_{\mu\nu;\alpha\beta} = \frac{1}{2}\left[\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \frac{2}{d-2}\eta_{\mu\nu}\eta_{\alpha\beta}\right]$ with d the spacetime dimension. Note the propagator is manifestly instantaneous (independent of energy). Assigning the usual non-

[7] In NRQCD we called this a label formalism.

relativistic scalings, $x^0 \sim r/v$, $\vec{k} \sim 1/r$, we learn from this that a potential graviton scales as $H_{\vec{k}\mu\nu} \sim v^{1/2}r^2$, while in coordinate space $H \sim v^{1/2}/r$. We will occasionally work in coordinate space when it proves convenient.

We now expand (22) to get a set of vertices (interactions) each of which has definite scaling in v (and L).

$$\begin{aligned}
L_{v^0} &= -\frac{M}{2M_{pl}} H_{\vec{k}00} \\
L_{v^1} &= -\frac{M}{2M_{pl}} v^i H_{\vec{k}0i} \\
L_{v^2} &= -\frac{M}{4M_{pl}} v^2 H_{\vec{k}00} - \frac{M}{2M_{pl}} v^i v^j H_{\vec{k}ij} - \frac{M}{8} \frac{H_{\vec{k}00} H_{-\vec{k}00}}{M_{pl}^2}
\end{aligned} \tag{27}$$

where there is an implied integral over momentum which has been suppressed for notational simplicity. Each of these terms will scale as $\sqrt{L}$.

With the results (26) and (27) in hand we are ready to begin integrating out the potential mode in order to generate potentials. To generate the leading order (Newtonian) potential we consider the coupling of the potential graviton between two world lines, which yields

$$\begin{aligned}
-iVT = (-i)^2 \int dt_1 dt_2 M_1 M_2 \frac{1}{4M_{pl}^2} \int [d^3k][d^3q] e^{i\vec{k}\cdot\vec{x}_1(t_1)} e^{i\vec{q}\cdot\vec{x}_2(t_2)} \langle 0 \mid T\left[H_{\vec{k}00}(t_1) H_{\vec{q}00}(t_2)\right] \mid 0 \rangle, \\
= \int dt M_1 M_2 \frac{1}{8M_{pl}^2} \int [d^3k] e^{-i\vec{k}\cdot(\vec{x}_1(t)-\vec{x}_2(t))} \frac{i}{\vec{k}^2}.
\end{aligned} \tag{28}$$

To perform this integral we will first evaluate the more general integral

$$\int \frac{d^dk}{(2\pi)^d} \frac{1}{(\vec{k}^2)^\alpha} e^{-i\vec{k}\cdot\vec{x}} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(d/2-\alpha)}{\Gamma(\alpha)} \left(\frac{\vec{x}^2}{4}\right)^{\alpha-d/2}. \tag{29}$$

which will be useful when we consider higher order corrections. Using this result we find

$$V_0(r) = -\frac{M_1 M_2}{32\pi M_{pl}^2 r}. \tag{30}$$

Notice that for the graviton correlator we wrote down a time ordered product, which in the end is extraneous since the correlator is instantaneous, as can be seen in (26), and we will drop the time ordering in potential correlators.

After re-deriving this monumental result let us pause to discuss the power counting in L.

Since the action has the units of angular momentum all the classical potential contribution to the action must scale like L, $\int dtV \sim L$. We can see that this is true using the fact that $dt \sim r/v$ and $M^2/M_{pl}^2 \sim vL$. However, we always wish to power count at the level of the action, including the graviton field, that way we can determine which diagrams (a combination of vertices) contribute at leading order in L, i.e. classically (as expected quantum contributions will involve closed graviton loops). We already determined that $H_k \sim r^2 v^{1/2}$, so that in the world line action we have $\int d^3k H_{\vec{k}} \sim v^{1/2}/r$. Then we can see that all of the world line terms in (27) scale as $\sqrt{L}$ once we include the time measure scaling $dt \sim r/v$. Thus connecting any two such vertices generates a classical contribution.

Let us now calculate the 1PN (order v^2) potential[8]. We begin by enumerating all of the possible contributions at order v^2 (not including diagrams where the masses are interchanged, which are trivially added when we are done)[9]. First we have all the possible single graviton diagrams which connect vertices whose net momentum scaling is v^2.

1. The simplest such example stems from the term $v^2 h_{00}$. By inserting this term we generate exactly the same expression as in (28) so that this contribution is

$$V_{v^2}^{(1)} = \frac{1}{2}(v_1^2 + v_2^2)V_0 \tag{31}$$

2. We can also have two insertions of the term

$$S_m = -M \int dt v^i e^{ik\cdot x} \frac{H_{ki0}}{M_{pl}} \tag{32}$$

one on each line. Expanding the exponential of L_{int} to second order in this perturbation leaves

$$\begin{aligned}
V_{v^2}^{(2)} &= -i \int dt_1 dt_2 M_1 M_2 \frac{1}{M_{pl}^2} v_1^i(t_1) v_2^j(t_2) (\langle H_{\vec{p}i0}(x_1(t_1)) H_{\vec{q}j0}(x_2(t_2))\rangle. \\
&= \int dt M_1 M_2 \frac{1}{M_{pl}^2} \vec{v}_1(t) \cdot \vec{v}_2(t) \frac{1}{2} \int [d^3p] e^{-i\vec{p}\cdot(\vec{x}_1(t) - \vec{x}_2(t))} \frac{1}{\vec{p}^2} \\
&= 4\vec{v}_1(t) \cdot \vec{v}_2(t) V_0.
\end{aligned} \tag{33}$$

[8] The nomenclature is such that that the N-th PN correction to the potential is order v^{2N}. The reason for this is that the conservative potential pieces must scale with even powers of the velocity by time reversal invariance.

[9] Using Feynman diagrams to calculate classical potentials goes back to the work [6], although in this reference the potential is extracted from S-matrix elements. The calculations in terms of classical sources can be found in [8].

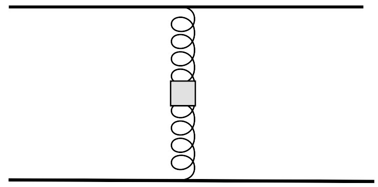

FIG. 2. Propagator correction which accounts for the leading deviation from instantaneity which is order v^2.

3. There is a first order correction to the propagator coming from the last line in (27)

$$V^{(3)}_{v^2} = -\int dt_1 dt_2 M_1 M_2 \frac{1}{8M_{pl}^2}\int d^4p e^{ip\cdot(x_1(t_1)-x_2(t_2))}\frac{p_0^2}{\vec{p}^4}. \tag{34}$$

Had we not expanded in v at the level of the action, so that it scaled homogeneously, then we would have seen this correction by expanding the graviton propagator.

$$\frac{1}{p_0^2-\vec{p}^2} \approx \frac{1}{\vec{p}^2} + \frac{p_0^2}{\vec{p}^4}. \tag{35}$$

In terms of Feynman diagrams such propagator corrections are drawn as in figure (2).

To perform this integral, it is simplest to integrate by parts, yielding

$$\begin{aligned} V^{(3)}_{v^2} &= -\int dt_1 dt_2 M_1 M_2 \frac{[d^4p]}{8M_{pl}^2}(\frac{\partial}{\partial t_1}\frac{\partial}{\partial t_2}e^{-ip_0(t_1-t_2)})e^{i\vec{p}\cdot(\vec{x}_1(t_1)-\vec{x}_2(t_2))}\frac{1}{\vec{p}^4} \\ V^{(3)}_{v^2} &= -\vec{v}_1\cdot\vec{v}_2\frac{V_0}{2} + (\vec{v}_1\cdot\vec{X})(\vec{v}_2\cdot\vec{X})\frac{V_0}{2X^2}, \end{aligned} \tag{36}$$

where $\vec{X} \equiv \vec{x}_1 - \vec{x}_2$. Notice that when we integrated by parts there exist an ambiguity. In particular, we could have chosen to write $\frac{\partial}{\partial t_1}\frac{\partial}{\partial t_1}$. However, this can not change the results for any physical observable since a total derivative can not affect the equations of motion.

4. We can also have one leading order insertion and one insertion of $v^i v^j h^{ij}$ (on either

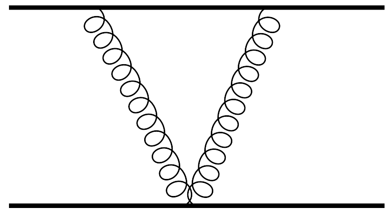

FIG. 3. Seagull contribution to the potential at order v^2.

world-line)

$$V^{(4)}_{v^2} = i\int dt_1 dt_2 M_1 M_2 \frac{(-i)^2}{4M_{pl}^2} \times$$
$$\Big(v_1^i(t_1) v_1^j(t_1) \langle T(H_{\vec{k}ij}(x_1(t_1) H_{\vec{q}00}(x_2(t_2))\rangle + v_2^i(t_2) v_2^j(t_2) \langle T(H_{\vec{k}00}(x_1(t_1) H_{\vec{q}ij}(x_2(t_2))\rangle \Big).$$
$$= i\int dt M_1 M_2 \frac{1}{8M_{pl}^2} (\vec{v}_1(t)^2 + \vec{v}_2(t)^2) \int [d^3p] e^{-i\vec{p}\cdot(\vec{x}_1(t)-\vec{x}_2(t))} \frac{i}{\vec{p}^2}, \tag{37}$$

so that

$$V^{(4)}_{(v^2)} = (\vec{v}_1^2 + \vec{v}_2^2) V_0. \tag{38}$$

5. The first non-linear contribution comes from the term quadratic in H in the expansion of the world-line action (22). This interaction generates so-called "sea-gull" diagrams as shown in figure (3). We consider a correlator of this operator with two insertions of the leading order interaction. We have

$$V^{(5)}_{(v^2)} = \int dt_1 dt_2 dt_2' M_1 M_2^2 \frac{1}{8\times 2\times 2} \langle \frac{H_{\vec{k}00}(x_1(t_1))}{M_{pl}} \frac{H_{\vec{q}00}(x_1(t_1))}{M_{pl}} \frac{H_{\vec{l}00}(x_2(t_2))}{M_{pl}} \frac{H_{\vec{p}00}(x_2(t_2'))}{M_{pl}} \rangle$$
$$+ (1 \leftrightarrow 2). \tag{39}$$

There are two possible contractions which give identical contributions, but we also pick up a factor of 1/2 from each propagator

$$V^{(5)}_{(v^2)} = \int dt_1 dt_2 dt_2' \frac{M_1 M_2^2}{M_{pl}^4} \frac{1}{8\times 2^4} \left(\int [d^3p] e^{-i\vec{p}\cdot(\vec{x}_2(t_2)-\vec{x}_1(t_1))} \frac{i}{\vec{p}^2} \times \int [d^3p'] e^{-i\vec{p}'\cdot(\vec{x}_2(t_2)-\vec{x}_1(t_1))} \frac{i}{\vec{p}'^2} \right)$$
$$+ (1 \leftrightarrow 2). \tag{40}$$

Let us pause to explain the symmetry factor. There is a 1/8 from the operator, and two factors of 1/2 from the leading order operator. There is a factor of 2 since there are two possible Wick contractions, but this is cancelled by a factor of 1/2 coming from the expansion of the exponential. Finally we have two additional factors of 1/2 coming from the propagators. Thus the net contribution from this "sea-gull" diagram is

$$V^{(5)}_{(v^2)} = -\int dt \frac{M_1 M_2^2}{128 M_{pl}^4}\frac{1}{(4\pi)^2}\frac{1}{r^2} \quad + (1 \leftrightarrow 2). \tag{41}$$

6. Finally we have the contribution from the three graviton vertex as shown in the left hand side of figure (II A). To see why this type of diagram shows up at 1PN we need to understand the scaling of the bulk non-linear vertices. Since the vertices arise as an expansion of the bulk curvature term. The generic form of the vertex involves two derivatives giving a term in the action which scales as

$$S_{3g} \sim \frac{1}{M_{pl}}\int d^4x h\partial h\partial h \sim \frac{1}{M_{pl}}\frac{r^4}{v}(\frac{\sqrt{v}}{r})^3\frac{1}{r^2} \sim \frac{M}{M_{pl}}\frac{v}{rMv}\sqrt{v} = \frac{v^2}{\sqrt{L}}. \tag{42}$$

Note that we have taken the measure to scale as r^4/v since we are dealing with potential gravitons (see previous chapter). This scaling will change when we discuss radiation gravitons. Given that each leading order world-line vertex scales as $\sqrt{L}$, we find that the diagram on the left hand side of figure (4) scales as Lv^2 as a 1PN potential should.

The Feynman rule for the three graviton vertex can be found in the appendix of reference [1]. We will not go through its derivation simply because as we shall see in the next section, one can make a simpler choice for the metric ansatz which will allow us to avoid the three graviton vertex at 1PN. But it is instructive to use the three graviton vertex since it will need to be included when working at higher orders. The calculation simplifies since the propagator which hooks up h_{00} gravitons obeys certain simple identities

$$P_{00:\alpha\beta}P^{00:\alpha\beta} = 1. \tag{43}$$

$$P_{00:\alpha\beta}\eta^{\alpha\beta} = -1. \tag{44}$$

$$P_{00:\alpha\beta}P^{00:\alpha\delta} = \frac{1}{4}\delta^\delta_\alpha. \tag{45}$$

$$k^\alpha P_{00:\alpha\beta} = -\frac{1}{2}k_\beta. \tag{46}$$

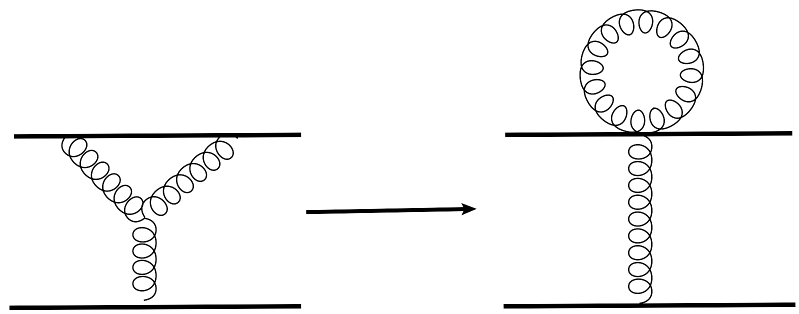

FIG. 4. The three graviton contribution to the EIH potential. Terms with momenta in the numerator eliminate one propagator and leads to a vanishing "tadpole" contribution as shown in the right hand side of the diagram.

Performing all of the requisite contractions [10], the net result is

$$V_{3g} = i\frac{M_1^2 M_2}{2!}(-i/(2M_{pl}))^3(-2i/M_{pl})(i)^3 \int dt \frac{[d^{d-1}k]}{-\vec{k}^2}\frac{[d^{d-1}p]}{-\vec{p}^2}\frac{[d^{d-1}r]}{-\vec{r}^2}\frac{-1}{8}(\vec{p}^2+\vec{k}^2+\vec{r}^2)\times \\ e^{i\vec{k}\cdot\vec{x}_1(t)}e^{i\vec{p}\cdot\vec{x}_1(t)}e^{i\vec{r}\cdot\vec{x}_2(t)}(2\pi)^3\delta^{(3)}(\vec{k}+\vec{p}+\vec{r}). \tag{47}$$

Note that first two of the three terms in this integral vanish by definition in dimensional regularization. To see this, consider the first term and perform the r integral leaving

$$V_{3g}^{div} \sim \iint dt \frac{[d^{d-1}k]}{-\vec{k}^2}[d^{d-1}p]\frac{1}{-(\vec{p}+\vec{k})^2}e^{i\vec{k}\cdot(\vec{x}_1(t)-\vec{x}_2(t)}e^{i\vec{p}\cdot(\vec{x}_1(t)-\vec{x}_2(t))}. \tag{48}$$

then shifting $p \to p-k$ leaves the k integral scaleless (see section **??** for details) It is important to understand that there is nothing magical about dimensional regularization. Had we used a different regulator, one in which this integral did not vanish, it still can not have any physical effect. To see this let us study the origin of this divergence. The $\vec{p}^2$ in the numerator cancels the propagator and forces the bulk vertex to live on the world line. The k integral becomes a loop which closes onto itself (this is sometimes called a tadpole) and corresponds to a mass renormalization as shown in second diagram in figure (4).

The result for the three graviton diagram is then given by

$$V_{3g} = \frac{i}{8}\frac{M_1^2 M_2}{2!}(-i/(2M_{pl}))^3(-2i/M_{pl})(i)^3 \int d\tau \frac{[d^{d-1}k]}{-\vec{k}^2}\frac{[d^{d-1}p]}{-\vec{p}^2}e^{i\vec{k}\cdot(\vec{x}_1-x_2)}e^{i\vec{p}\cdot(\vec{x}_1-x_2)}. \tag{49}$$

$$V_{(v^2)}^{(3g)} = \int dt \frac{M_1^2 M_2}{64 M_{pl}^4}\frac{1}{(4\pi)^2}\frac{1}{r^2} + (1 \leftrightarrow 2). \tag{50}$$

After symmetrizing in the masses, and adding all the contributions $(1-6)$ we find the

[10] When doing these contractions the author utilized FeynCalc [10].

(Einstein-Infeld-Hoffmann) potential, written in terms of $G_N = \frac{1}{32\pi M_{pl}^2}$ is

$$V_{EIH} = -\frac{G_N M_1 M_2}{2r}\left[3(v_1^2 + v_2^2) - 7v_1 \cdot v_2 - \frac{(v_1 \cdot r)(v_1 \cdot r)}{r^2}\right] + \frac{G_N^2 M_1 M_2 (M_1 + M_2)}{2r^2}. \quad (51)$$

It is again important to emphasize that this potential depends upon the choice of gauge fixing as well as the coordinate system.

B. A Simpler Way to Calculate

As mentioned in the previous section, the highly non-linear nature of GR makes higher order diagrams very cumbersome. At 1PN we run into the three graviton vertex which has 6 indices that leads to a plethora of contractions. In general, using the standard representation of the metric we will need the $n+2$ graviton vertex at nPN order. However, it was pointed out in [9] that there is a better way to parameterize the metric which pushes back the need for this vertex by one order. To accomplish this one utilizes the so-called Kaluza-Klein[11] variables where the metric is parameterized as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \equiv e^{2\phi}(dt - A_i dx^i)^2 - e^{-2\phi}\gamma_{ij} dx^i dx^j. \quad (52)$$

In terms of these variables the Einstein-Hilbert action is given by

$$S = -2M_{pl}^2 \int d^4x \sqrt{\gamma}\left(-R(\gamma) - \frac{1}{4}e^{4\phi}F_{ij}F^{ij} + 2\partial^i\phi\partial_i\phi + ...\right) \quad (53)$$

where the dots represent terms with time derivatives that are higher order in the PN expansion and $F_{ij} = \partial_i A_j - \partial_j A_i$. Now consider the 1-PN three-graviton contribution. At 1PN the world line couplings all involve $h_{00} \equiv \phi$ since any open vector indices must be contracted with velocities. Notice that there are no try-linear scalar (ϕ) couplings in this choice of variables, and thus the three-graviton coupling will not show up until 2PN[12]. Even with this simplification, as one goes to higher orders in the PN expansion one will need to calculate diagrams with even more vertices, and and some point the calculation will become

[11] Kaluza-Klein theories are extra dimensional attempts at unification. For instance, a five dimensional metric is split up into a 4-D metric and a vector fields which play the role of the electromagnetic gauge potential.

[12] The 2PN potential is calculated using these techniques in [12] and the 3PN was done in [13].

very computationally intensive. The 3PN calculation using the KK variables was done on a computer [13] and involved 20 differing topologies for diagrams.

Before closing this section we should note that it is possible to avoid the problem of higher order vertices altogether [11]. While the techniques for this go beyond the scope of this book, a few sentences can capture the basic idea of the methodology. The salient point is that in gauge theories, such as gravity, the action contains a large amount of redundancy. There are unphysical degrees of freedom that play no role in physical observables and lead to very complicated intermediate steps of calculations. Feynman diagrams include these unphysical states and if one looks at unphysical objects, such as individual diagrams, which are not gauge invariant, one often finds hopelessly complicated expressions. However, the sum of the diagrams usually is quite compact. It turns out that these higher order vertices are all part of the gauge redundancy in gravity and that one can calculate all S-matrix elements without reference to anything beyond the three point vertex. Since the potentials can be related to S-matrix elements through a matching procedure (see chapter on NRQCD) one can then calculate the potentials in a greatly simplified manner without summing Feynman diagrams with all their intrinsic redundancies.

C. The Quantum Calculation versus the Classical Calculation of the Potential

When considering diagrams such as those in figure (4) it is important not to interpret the source line as a propagator. In NRGR we are considering classical sources. We draw the (non-graviton) lines connecting the vertices to represent time propagation, though some may find this misleading. Instead one could draw the diagram corresponding to (4) by using classical source insertions as shown in figure (5). Drawing diagrams in this way has the advantage that it also makes clear what diagrams should *not* be included. For instance including the matter line propagators would lead to diagrams with the box topology, which one might naively include. However, if we instead draw the diagrams with source insertions it becomes clear that these are disconnected diagrams which should not be included when calculating the energy. Such diagrams would serve the purpose of exponentiating the energy (this is sometimes called the “linked cluster expansion”).

It is natural now to ask about the classical forces between quarks. One thing we can discern immediately is that all corrections to the Coulomb potential are quantum mechanical

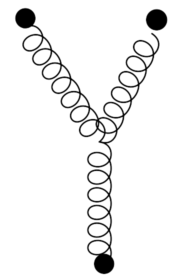

FIG. 5. An alternative way to draw first the diagram in figure (4) which makes it clear that the masses act as classical sources which do not fluctuate. The down side of this way of drawing things is that it does not make clear that one still integrates over the times of the source insertions.

if the quarks are static. This is obvious given the classical conformal symmetry of QCD. There is simply no potential that can we write down aside from $V \sim 1/r$. This becomes immediately clear in perturbation theory when the gluons hitting the external lines couple to static sources forcing them to be temporal. This is as opposed to gravity where even the static limit leads to corrections to Newtons law as can be seen in the last term of the EIH result (51). However, once we allow for non-static sources there will be a velocity dependent classical corrections to the Coulomb potential.

Let us further explore the differences between QCD and GR by considering the relative sizes of non-linear contributions to the bound orbit problem. In GR all quantum loops are small as long as the momentum (the inverse radius) are small compared to the Planck scale. Thus even when $v \sim 1$, i.e. when $r \sim M/M_{pl}^2$ quantum corrections are negligible as long as $M \gg M_{pl}$, corresponding to a large Schwarzchild radius with small scalar curvature at the horizon. On the otherhand the classical non-linearities become large when $v \sim 1$.

Whereas in QCD the size of the quantum mechanical non-linear effects is controlled by the dimensionless coupling g. So when the coupling gets large so do the quantum corrections. However, the size of the classical corrections also scale with the coupling. The size of the classical non-linearities will depend upon the size of the charge on the world lines which we will call Q. Thus if $Q \gg g$, then its possible for the classical contributions to dominante the quantum corrections. However, if $g \sim 1$, then the classical result will no longer be trustworthy. One possible scenario where the classical potential will dominate in QCD is to have large $Q \gg g$ and $g \ll 1$. Given that QCD is asymptotically free this scenario corresponds to a have quark bound state with very large color charge.

III. FINITE SIZE CORRECTIONS

Given that we have developed a systematic way to power count in the PN expansion we can ask the following question. At what order in the PN expansion do finite size effects show up? To answer this question all we have to do is to power count the finite size operators (9). For black holes this is straightforward since we know that the only relevant parameter is the Schwarzchild radius. Using the operator scaling in eq. (11) the action then scales as

$$S_{FS} \sim \frac{M^5}{M_{pl}^{10}} \int dt(\partial\partial h)^2 \sim \frac{M^5}{M_{pl}^{10}} \frac{r}{v} \frac{1}{r^4} (\frac{\sqrt{v}}{r})^2 \sim \frac{M^{10}}{M_{pl}^{10}} \frac{v^5}{(rMv)^5} \sim L^0 v^{10}. \tag{54}$$

This diminishment of finite size effects in general relativity is often termed "effacement" [16]. Note that the scaling in L is as it should be since we can generate a classical potential by drawing a sea-gull diagram, as in figure (3), where now the lower vertex is an insertion of the finite size operator. The top vertices are mass insertions which scale as $\sqrt{L}$ leaving a potential with the proper scaling. For other compact objects such as neutron stars, it is possible that this scaling is numerically enhanced since it will depend upon the equation of state.

Exercise 1.2 Consider the finite size world line action

$$S_2 = C_R \int d\lambda R(x(\lambda)) + C_{vvR} \int d\lambda v^\mu v^\nu R_{\mu\nu}(x(\lambda)). \tag{55}$$

Naively, on dimensional grounds, we might conclude that the couplings scales linearly with the size of the object. However, while this is correct in electrodynamics which is classically conformal, it is not necessarily correct in gravity. Let us consider the case of a black hole where the radius is $r_s \sim M/M_{pl}^2$. Since the coupling is now dimensionful we can make up the units with powers of M_{pl} as well. Write the scaling of the coupling as $C_R \sim (M/M_{pl})^n \frac{1}{M_{pl}}$ and determine n by making sure that the potential generated by this operator (which we know is unphysical since it can be removed via a field redefinition (see the discussion around eq. (4)) gives a sensible classical limit. i.e the potential scales as L.

There are additional finite size corrections which are not associated with the local terms (9) that arise from dissipative effects. Given that we're allowing our compact objects to deform, it is necessary to account for the fact that there is work done in this process. Some amount of energy will be absorbed by the compact object, i.e. it will heat up thus increasing its mass. How do we account for this effect? Since we are working in the point particle approximation we have already integrated out all of the degrees of freedom which would account for this heating process. Indeed, we really had no right to integrate out these modes, as they are gapless. i.e. they can be excited via an infinitesimal energy transfer. Thus we will need to add back in some degrees of freedom that live on the world-line if we are to have any hope of accounting for this dissipation.

Let us introduce a field $\phi(\tau)$ that lives on the world-line. It forms some representation of the (local) Lorentz group which we will fix in a moment. We want to couple this degree of freedom to the metric in a diffeomorphism invariant fashion. Moreover, these internal degrees of freedom should only be excited when they are placed in a background which has some tidal pull. There are no geometric invariants with only one derivative, so the field must couple to a two derivative object, i.e. the curvature. This tells us that the field ϕ must in fact be, at least, a two indexed tensor, which we will call $Q_{\mu\nu}(\tau)$. The object has an obvious physical interpretation as a dynamical quadrapole moment, in its rest frame. As before, to minimize redundancies we will couple $Q_{\mu\nu}(\tau)$ to the electric and magnetic pieces of the Weyl tensor. Then given the fact that E and B are of opposite parity we will define two distinct quadrapole degrees of freedom (Q, M) such that the action is given by

$$S_{dis} = \int d\tau (Q_{ab}E^{ab} + M_{ab}B^{ab}). \tag{56}$$

Here we have written the expression in terms of small Roman letters to denote local orthogonal coordinates, as opposed to the global coordinates denoted by greek letters. Working in the local frame will allow us to simplify the correlators of the $Q's$. The two coordinate systems are related by the vierbein which satisfies

$$e^\mu_a e^\nu_b \eta^{ab} = g^{\mu\nu} \qquad e^a_\mu e^b_\nu g^{\mu\nu} = \eta^{ab}. \tag{57}$$

The $Q's$ are symmetric and traceless. At lowest order the distinction between global and local frames is irrelevant.

Now we would like to calculate how these couplings affect the potential in the PN ex-

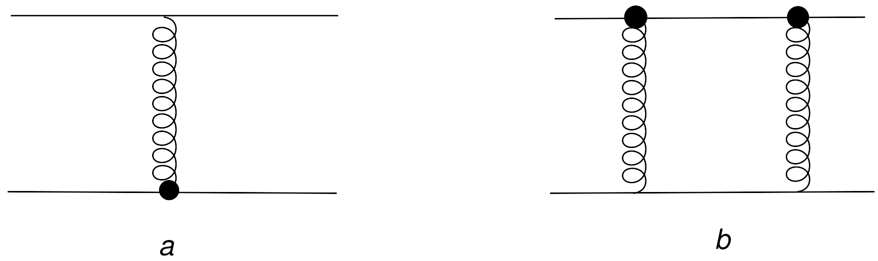

FIG. 6. Diagram (a) shows a possible leading order contribution due to the dynamical quadrapole moment which vanishes for spherically symmetric bodies. Diagram (b) shows the first non-vanishing contribution. The heavy dot denotes an insertions of the quadrapole operator while the other vertex is the lowest order mass insertion. Not shown are the symmetric diagrams which interchange (1) and (2).

pansion. To do so we need to power count the operators in (56), and thus we need to understand how the Q's scale. We will return to this issue once we understand how to make sense of them. We will take the Q and M to be operators in a Hilbert space though we know in the end that this is a classical problem and they really represent the (quasi)normal mode spectrum of the black holes. But as we have learned sometimes the quantum framework organizes our calculation in a clearer fashion [13]. Moreover, it is possible to think of the underlying degrees of freedom that cause dissipation as being dual to some quantum mechanical theory.

How can these terms effect the potentials? At lowest order we would start by considering one insertion of a quadrapole interaction with a leading order mass insertion as shown on the left-hand side of figure (6). For the moment let us concentrate on the electric piece of the interaction, in which case we have

$$-iVT = -i\frac{M_2}{2M_{pl}}\int d\tau_1 \langle 0 \mid Q_1^{ab}(\tau_1) \mid 0\rangle \langle E_{ab}(x_1) h_{00}(x_2)\rangle. \tag{58}$$

$$\tag{59}$$

Assuming that the hole has no permanent quadrapole moment, and that the black hole is not in an excited state, the matrix element vanishes[14] and we need to go to higher order to get a non-zero contribution to the potential. At next order in the interaction we have

[13] This is analogous to the intuition that we can gain about quantum field theory from string theory. Indeed it tempting to associate this description with the dimensional reduction of the CFT description of the near horizon physics [17]

[14] It is important to remember that, despite the appearance of the Feynman diagram, we are not calculating a scattering amplitude but a vacuum to vacuum persistence amplitude as emphasized in the last section.

$$-iVT=-\frac{M_2^2}{8M_{pl}^2}\int d\tau_1 d\tau_1' d\tau_2 d\tau_2' \langle 0 \mid T(Q_1^{ab}(\tau_1)Q_1^{cd}(\tau_1')) \mid 0\rangle \langle E_{ab}(\tau_1,x_1)h_{00}(\tau_2,x_2)\rangle$$
$$\times \langle E_{cd}(\tau_1',x_1)h_{00}(\tau_2',x_2)\rangle + (1\leftrightarrow 2). \tag{60}$$

At lowest order there is no distinction between the local and global coordinate, and thus all contractions can be performed using a flat space metric. Expanding the definition of the electric Weyl tensor to linear order and keeping only the leading order piece in the PN expansion we find[15]

$$E_{ij}=\frac{1}{2M_{pl}}\partial_i\partial_j h_{00} \tag{61}$$

in which case we have

$$\langle E_{ij}(\tau_1,x_1)h_{00}(\tau_2,x_2)\rangle = \frac{i}{16\pi M_{pl}}\delta(\tau_2-\tau_1)\partial_i\partial_j\frac{1}{\mid \vec{x}_1-\vec{x}_2\mid}, \tag{62}$$

and

$$-iVT=\frac{M_2^2}{1024\pi^2 M_{pl}^4}\int d\tau d\tau' \langle 0 \mid T(Q_1^{ij}(\tau)Q_1^{kl}(\tau')) \mid 0\rangle q_{ij}q_{kl} + (1\leftrightarrow 2), \tag{63}$$

where $q_{ij}(t) = \partial_i\partial_j\frac{1}{|\vec{x}_1-\vec{x}_2|}$. The correlator of the quadrapole operator encodes all of the physics of the underlying degrees of freedom on the world-line. We can determine this correlator via a matching procedure. However, note this is a very unusual matching in that we are matching for an *infra-red* quantity, whereas normally in an EFT we match to fix parameters arising from *ultra-violet* physics. So it is perhaps better to use the term "extracting" instead of matching.

Before doing the extraction, let us utilize the rotational covariance and time translation invariance to decompose the correlator. Given that the Q_{ij}'s are symmetric and traceless we may write

$$\langle 0 \mid T(Q_1^{ij}(\tau)Q_1^{kl}(\tau')) \mid 0\rangle = A(\tau-\tau')(\frac{2}{3}\delta^{ij}\delta^{kl}-\delta^{ik}\delta^{jl}-\delta^{il}\delta^{jk}). \tag{64}$$

$A(\tau-\tau')$ has the units of inverse mass squared. Inserting this result into (63) we have

$$VT=i\frac{M_2^2}{512\pi^2 M_{pl}^4}\int dtdt' A(t-t')(\frac{1}{3}q_{ii}(t)q_{ii}(t')-q_{ij}(t)q_{ij}(t'))+(1\leftrightarrow 2) \tag{65}$$

---

[15] Here the small Roman letters represent Euclidean 3-space.

where we have parameterized the world lines by time and dropped the accompanying velocity corrections. Note that the function A is in general complex, thus the potential will have real and imaginary parts. Here we will be interested in the complex piece as this is the part that leads to dissipation [16]. The physics behind this reasoning is that dissipation is a consequence of the "openness" of the system. Energy can flow into the degrees of freedom represented by the Q's and we are not tracking that energy flow. Thus from a quantum mechanical viewpoint the system is not unitary. Time evolution does not correspond to pure phase, i.e. the energy is complex. This demonstrates again how quantum mechanical intuition gives insight into a classical problem.

Now we would like to relate the dissipation to the power loss. Recall that for an unstable state the $Im(E) = \frac{i\Gamma}{2}$, where $1/\Gamma$ is the lifetime of the state. Furthermore, as we learned from the previous chapter, one can think of Γ as the average number of particles emitted per unit time. Thus if we write $\Gamma = \int d\omega \frac{d\Gamma}{d\omega}$ then the power follows as

$$P = \int \omega \frac{d\Gamma}{d\omega} d\omega. \tag{66}$$

The power loss due to the electric degrees of freedom is then given by

$$\begin{aligned} P &= -\frac{1}{T}\frac{M_2^2}{256\pi^2 M_{pl}^4}\int dt dt' Im\left[i\int_{-\infty}^{\infty}[d\omega]\omega\tilde{A}(\omega)e^{i\omega(t-t')}(q_{ij}(t)q_{ij}(t')) + (1\leftrightarrow 2)\right] \\ &= -\frac{1}{T}\frac{M_2^2}{256\pi^2 M_{pl}^4}\int_{-\infty}^{\infty}[d\omega]\omega Im(i\tilde{A}(\omega)) \mid q_{ij}(\omega)\mid^2 + (1\leftrightarrow 2), \qquad (67) \\ & \qquad (68) \end{aligned}$$

where we have used the fact that $q_{ii} = 0$.

To determine $\tilde{A}(\omega)$ we note that it can related to the absorptive scattering of gravitons off of a black hole via the optical theorem. Consider the scattering of a graviton off of our compact object. In particular we will be interested in the absorptive cross section which, at leading order in derivatives, will be generated by the diagram is shown in first diagram in figure (7). The absorptive cross section is then given by

$$\sigma_{abs} = \frac{1}{2\omega}\sum_f \sum_\lambda \mid \langle 0 \mid Q_{\rho\nu}(0)) \mid f\rangle W^{\rho\nu} \mid^2 (2\pi)\delta(P_i^0 + \omega - P_f^0) \tag{69}$$

[16] The real part would need to be subtracted out of the matching procedure for the E^2 and B^2 operators.

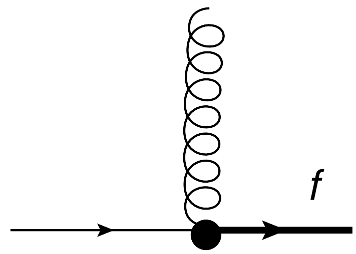

FIG. 7. Diagram showing the leading order contribution to graviton scattering off of a compact object. The heavy dot is an insertions of the $Q \cdot E$ operator.

where P_f is the final state momentum of the black hole, which is in the state f, λ are the graviton polarizations and

$$W_{\rho\nu} \equiv -\frac{1}{2M_{pl}}(k_\mu(k_\sigma \epsilon^{(\lambda)}_{\rho\nu} + k_\nu \epsilon^{(\lambda)}_{\rho\sigma} - k_\rho \epsilon^{(\lambda)}_{\nu\sigma}) - k_\nu(k_\sigma \epsilon^{(\lambda)}_{\rho\mu} + k_\mu \epsilon^{(\lambda)}_{\rho\sigma} - k_\rho \epsilon^{(\lambda)}_{\mu\sigma}))u^\sigma u^\nu). \tag{70}$$

Keep in mind that we are interested in the long wavelength limit, so we are expanding around flat space. At leading order in this expansion we may ignore the differences between the local and global coordinate systems. Furthermore, restricting ourselves to the physical polarization restricts the non-vanishing components of W to be spatial (working in the rest frame of the hole)

$$W_{ij} = \frac{1}{2M_{pl}}\omega^2 \epsilon_{ij}, \tag{71}$$

so that

$$\sigma = \sum_\lambda \frac{\omega^3}{8M_{pl}^2}\int dt e^{-i\omega t}\langle 0 \mid Q^E_{ij}(t) Q^E_{kl}(0) \mid 0\rangle \epsilon^\lambda_{ij}\epsilon^{\lambda\star}_{kl}, \tag{72}$$

where we have utilized the completeness of the black hole Hilbert space. We can relate the RHS of this equation to the time ordered product by considering the tensor

$$\begin{aligned} I_{ijkl}(\omega) &\equiv \int dt e^{-i\omega t} \mid \langle 0 \mid T(Q_{ij}(t)Q_{kl}(0))) \mid 0\rangle \equiv \tilde{A}(\omega)(\frac{2}{3}\delta^{ij}\delta^{kl} - \delta^{ik}\delta^{jl} - \delta^{il}\delta^{jk}) \\ &= \sum_P \frac{1}{2\pi i}\int \frac{d\tau}{\tau - i\epsilon} dt \mid \langle 0 \mid Q_{ij}(0) \mid P\rangle \mid^2 \left[e^{it(\tau+\omega-E_P+E_0)} + e^{it(-\tau+\omega+E_P-E_0)}\right] \\ &= \sum_P \frac{1}{i} \mid \langle 0 \mid Q_{ij}(0) \mid P\rangle \mid^2 \left[\frac{1}{(E_P - \omega - E_0 - i\epsilon)} + \frac{1}{(\omega + E_P - E_0 - i\epsilon)}\right] \end{aligned} \tag{73}$$

where E_0 is the ground state energy and use has been made of the integral representation of the step function. Then we note that

$$Im(iI_{ijkl}(\omega)) = \pi \sum_P \mid \langle 0 \mid Q_{ij}(0) \mid P \rangle \mid^2 [\delta(\omega - E_P + E_0) + \delta(\omega - E_P + E_0)] . \tag{74}$$

The second term corresponds to spontaneous emission (Hawking radiation), which is negligible for macroscopic black holes. We can then write (working in the rest frame of the black hole where E has only spatial indices)

$$\int e^{-i\omega t} \langle 0 \mid Q_{ij}(t) Q_{kl}(0) \mid 0 \rangle = \theta(\omega) \frac{1}{\pi} Im(iI_{ijkl}(\omega)), \tag{75}$$

so that

$$\sigma_{abs} = \frac{\omega^3}{8M_{pl}^2} \sum_\lambda \theta(\omega) Im(\frac{i}{\pi} \tilde{A}(\omega))(\frac{2}{3}\delta_{ij}\delta_{kl} - \delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl})\epsilon_{ij}\epsilon^\star_{kl}. \tag{76}$$

Summing over polarization and performing the requisite contraction gives

$$\sigma_{abs} = \frac{\omega^3}{2M_{pl}^2} Im(i\tilde{A}(\omega)). \tag{77}$$

At leading order in the derivative expansion the absorptive cross section for a graviton scattering on a black hole is given by [18]

$$\sigma_{abs} = \frac{4\pi r_s^6 \omega^4}{45}, \tag{78}$$

where $r_s = 2G_N M$ is the Schwarzchild radius. Now recall that this is the total cross section so to match we must include both the magnetic and electric parts of the world line interactions. The magnetic part of the calculation follows in the same way as the electric and is left as an exercise for the reader. We also use the fact that the magnetic analog of $\tilde{A}(\omega)$ is equal to $\tilde{A}(\omega)$ which can be seen by studying the full theory scattering (Teukolsky) equation [19]. Using this result we find

$$Im(i\tilde{A}(\omega)) = \frac{16}{45} m^6 G_N^5 \mid \omega \mid . \tag{79}$$

Plugging this result in (67) we get

$$P = -\frac{1}{T}\frac{M_2^2 M_1^6}{516\pi^2 M_{pl}^4}\int_{-\infty}^{\infty}[d\omega]\omega^2\frac{16}{45}G_N^5 \mid q_{ij}(\omega)\mid^2 + (1 \leftrightarrow 2), \tag{80}$$

$$\tag{81}$$

where the ω integral had been extended to $-\infty$ using the fact that integrand is even.

If we wish to consider other compact objects, such as neutron stars, we need only use the appropriate absorptive cross section in (77). Thus for a general compact object we may write the absorptive power loss as

$$\frac{dP}{d\omega} = -\frac{1}{T}\frac{G_N}{64\pi^2}\sum_{a\neq b}\frac{\sigma_{abs}^{(b)}}{\omega^2}M_{(a)}^2 \mid q_{ij}^{(a)}(\omega)\mid^2 . \tag{82}$$

Now we go back to coordinate space using the relation

$$q^{ij}(t) = \frac{\delta^{ij}}{r^3(t)} - \frac{3r_i r_j}{r^5}, \tag{83}$$

and after integrating by parts we find the power loss for a black hole

$$P = \frac{32}{5}G^7(M_1^6 M_2^2 + M_2^6 M_1^2)\left(2\frac{\dot{r}^2}{r^8} + \frac{\dot{\vec{r}}^{\,2}}{r^8}\right). \tag{84}$$

Now let us return to the question of the v scaling of the wordline line terms (101). We would like to determine the scaling of the absorptive potential (80) in both L and v. As previsouly emphasized we can read off the scaling by studying the scalings of the operators which make up the Feynman diagram (7b). The scaling of Q can be read off by looking at its correlator

$$Im\int dte^{i\omega t}\langle 0 \mid TQ_{ab}(t)Q_{cd}(0)\mid 0\rangle \propto \omega(GM)^6 M_{pl}^2. \tag{85}$$

In the context of of calculating the potential in the PN expansion ω scales as v/R and dt scales as R/v such that

$$Q(\tau)Q(0) \sim (v/R)^2 M^6/M_{pl}^{10} = v^4/L^2(M/M_{pl})^8 1/M_{pl}^2 = L^2 v^8/M_{pl}^2. \tag{86}$$

So that Q scales as Lv^4/M_{pl}. Now we may consider the scaling of action

$$S = \int dt Q_{\mu\nu} E^{\mu\nu} \sim \frac{r}{v} \times L v^4 M_{pl}^{-1} \times \frac{1}{M_{pl}} r^{-2} \times v^{1/2} r^{-1} \sim L \frac{v^{7/2}}{r^2 M_{pl}^2} \sim L \frac{v^{11/2}}{r^2 M^2 v^2} \times vL \sim v^{13/2}/L. \tag{87}$$

Two insertions of this operator with two mass insertions leads to an overall scaling of v^{13}. The lack of L suppression is consistent with a classical correction. A similar calculation shows that the magnetic contribution has the identical scaling. The power should then scale as v^{15}. The odd power of v is constitent with a non-conservative effect since it is not time reversal invariant.

Finally let us pause for a moment to try to gain some intuition for the degrees of freedom responsible for the dissipation. We chose to extract the time ordered product by matching to the full theory absorptive cross section. Suppose that instead of scattering off the black hole, we chose instead to place the black hole in a background field and measured the response. To properly perform the calculation we should work in the closed-time path in-in formalism, since we are interested in the time dependent response of a particular state. However, if we are only interested in the response to linear order in the external field we may use the well known result (see for instance [15])

$$\langle \delta O(\omega) \rangle = G_R(w) J(\omega) \tag{88}$$

where

$$G_R(\omega) = \int dt e^{i\omega t} \theta(t) \langle 0 \mid [O(t), O(0)] \mid 0 \rangle. \tag{89}$$

Using the spectral decomposition (73) and an analogous one for $G_R(\omega)$ one can see that $G_R(\omega) = G_F(\omega)$ for ($\omega > 0$, $\omega \in R$). Moreover, $Im(iG_R)[Re(iG_R))]$ is odd [even] under $\omega \to -\omega$ while $Im(iG_F)[Re(iG_F)]$ is even [odd]. These fact will be utilized below.

Now we have assumed that the system couples weakly to external perturbation in the low frequency limit. Indeed our power counting is predicated upon this assumption. The odd nature of $Im(iG_R)$ then implies that it starts at $O(\omega)$. On the other hand the real part could certainly have a constant piece. Furthermore our low energy weak coupling assumption implies that the correlator fall off faster then any power at large times. This does not preclude the existence of gapless excitations. Indeed we do not expect there to be a gap since dissipation should occur at all scales. However, the rapid fall odd does imply that the spectral density

$$\rho(\omega) = Im(iG_R(\omega)) \tag{90}$$

is regular near the origin. Note this implies there are no single particle massless excitation that would contribute to the spectral density as $\rho(\omega) \sim \delta(\omega)$ [22].

We may gather more intuition by writing the correlator as

$$G_R(\omega) = A_1 + i\omega B_1 + A_2\omega^2 + ... \tag{91}$$

A_i have a simple physical interpretation. A_1 is *related* to what is known as the "tidal Love number" and is a measure of the response to a static field, while A_2 is related to the first dynamic Love number. We emphasized the word "related" because the system responds to external fields via multiple operators in the effective theory. When we expand the gravitational field around a background probe field $g_{\mu\nu} = g^{bg}_{\mu\nu} + h_{|\mu\nu}$ we generate tadpole responses from both the coupling to the world line degrees of freedom via (56) as well as the contact interactions (22). The static Love number is then seen to be a linear combination of C_E and A_1. We might consider allowing the coupling (56) to subsume the local interactions. That is we could choose to set $C_E = 0$. But this would not be in the spirit of EFT as we should write down all allowable operators. Furthermore, we would expect that, in general, the existence of C_E will be necessary to absorb divergences that arise due to our use of the point particle approximation.

In [23] a matching calculation was performed and the result leads to a vanishing tidal Love number ($A_1 = 0$)! A_2 however, is finite. Moreover, there is an additional term of the form $\omega^2 Log(\omega)$ [24], which implies that the operator $\dot{E}^2$ runs! The vanishing of A_1 is a fascinating result in that at face value it is unexpected. In fact, it implies a fine tuning, as there are many possible diagrams (including power divergent ones) that should renormalize C_E. Note that the vanishing of the Love number does not mean that a black hole does not get distorted under the influence of static quadruple field. There will still be contributions in the effective theory to the distortion stemming from interactions with the mass. It's just that these distortions will be the same as those that would be generated by point particle of the same mass. The difference is that a generic a compact object, which is not a black hole, would have additional contributions to the distortion field due to the short distance physics which accounts for the finite size of the body. Black holes are obviously singular, in every sense of the word. It is important to note that the correlator as well as the E^2 operator are gauge independent. So the vanishing of the Love number (as defined above) is not a gauge artifact. However, the induced quadrupole moment is indeed gauge dependent [25].

IV. GRAVITATIONAL RADIATION

In the end, the relevant observables for binary inspirals are the phase and amplitude of the wave which is being detected. From a theoretical standpoint the phase is most easily calculated via the power loss. Taking for illustration a circular orbit in the extreme mass ratio limit (i.e. keeping only terms to leading order in the light mass) and using conservation of energy, $dE/dt = -P$, the gravitational wave phase seen by the detector is then given by

$$\Delta\phi(t) = \int_{t_0}^{t} d\tau \omega(\tau) = \frac{2}{G_N M} \int_{v(t)}^{v_0} dv' v'^3 \frac{dE/dv'}{P(v')}, \tag{92}$$

where $\omega(\tau)$ is the wave's frequency and M is the mass of the heavy partner. Calculating the amplitude at the detector itself corresponds to determining the field value far away from the source. In this section we will show how one calculates both the power and the amplitude.

We begin by integrating out the modes whose wavelength is of order the size of the binary. These are the potential modes responsible for binding the system. Once we have removed these modes we have coarse grained to the point where the details of the bound state are no longer accessible to us. As such, to be consistent (see the chapter on NRQCD) we must multipole expand the couplings of the radiation field to the sources, which is broken down into two steps. First the direct couplings of the radiation field to the world lines must be expanded, which is a straightforward exercise. In addition, we must expand the coupling of the radiation field to the potential, which involves more work. It is at this point that the power of the EFT becomes manifest. The net effect of these two steps is to generate an effective action for a *single* point particle with a set of multiple moments coupled to the metric in a way consistent with the symmetries. Once we have determined the multipole moments, via a matching procedure, we can calculate the power loss via the imaginary part of the the energy as in the last chapter.

When calculating the power loss, or amplitude, from the effective action, there are several other manifestations of the non-linearities of GR. For instance, when we calculate the power we must account for the fact that the radiation can scatter off the background (Schwarzchild) metric, leading to what is known as the "tail effect". Furthermore, it is also possible for the the radiation to scatter off itself, which is the so-called "memory effect". Both of these effects can be considered as manifestations of "radiative moments". We will return to these effects after we have calculated some of simple, non-radiative, moments.

When we integrate out the potential modes it desirable to maintain general coordinate invariance for the resulting action, which is a function of the radiation field $\bar{h}$. This is accomplished by working in the background field gauge, whose gauge fixing term is given by taking eq. (93) and replacing the derivatives with covariant derivatives whose connection is built from the radiation field

$$\Gamma_\mu = D_\alpha H^\alpha_\mu - \frac{1}{2} D_\mu H^\alpha_\alpha. \tag{93}$$

In this gauge we expect that, after integrating out the potential modes and performing the multipole expansion, the resulting action has the form

$$S_1 = \frac{1}{2}\int dt (Q^{ab} E_{ab} - \frac{4}{3} J^{ab} B_{ab} + \frac{1}{3} O^{abc} \nabla_c E_{ab} + ...), \tag{94}$$

where are the fields are evaluated at the origin . As in the case of finite size operators we express the gravitational field in terms of the electric and magnetic pieces of the Weyl tensor since they are irreducible under use of the equations of motion. The moments which couple to E and B are called "mass" and "current" multipole moments respectively.

The multipole moments should be thought of as matching coefficients. The numerical coefficients are chosen for later convenience. Each multipole moment transforms as an irreducible representation of the rotation group in the local rest frame (small Roman letters). The mass and current quadrapoles Q and J are symmetric and traceless, and form the $l = 2$ representation of $SO(3)$, while the octopole O is the completely symmetric and traceless $l = 3$ representation. Each of the multipole moments is orthogonal to each other in the sense that there will be no interference terms between them when they radiate gravitons. This is obvious from a group theory standpoint.

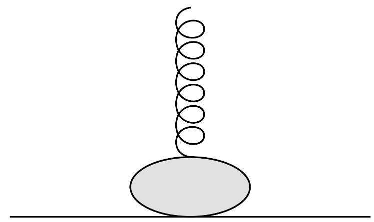

FIG. 8. Diagram showing the generic form of a tadpole diagram which generates a term of the form $h_{\mu\nu}T^{\mu\nu}$ in the effective action. The blob presents the coupling to gravitons through the non-linearities.

A. Calculating Multipole Moments

The graviton couples to the (pseudo[17]) stress energy tensor as follows

$$S = -\frac{1}{2M_{pl}} \int d^4x T_{\mu\nu}(x) h^{\mu\nu}(x), \tag{95}$$

Here h contains both the radiation field $\bar{h}$ as well as the potential generated by the *pair*. That is, even after integrating out the potentials whose typical wavelength of order the binary spacing, the pair itself still generates a potential which builds up the Schwarzchild solution of the effective one body. This potential will couple to conserved [18] quantities such as the mass an angular momentum (this is discussed in one of the problems at the end of this chapter).

The strategy will be to multipole expand (95) and manipulate it into the form given by (94). Recall that such an expansion is forced upon us to maintain manifest power counting at the level of the action, a tenent of EFT. Once we have done this, we will be able to read off an expression for the multipole moments in terms of derivatives of the stress energy tensor, which is generated both by the world-lines as well as the potential gravitons that bind the world-lines. The stress energy can be calculated via the expectation value of the radiation field $\langle \bar{h}_{\mu\nu} \rangle$, which is often called the "one-point" function and corresponds to Feynman diagrams with one open graviton line as illustrated in figure (8).

Before beginning our calculation, we must understand the v scaling of the radiation modes. Given that these modes are on shell, the energy and momentum must scale in the same way. Since the energy of the radiation scales with the frequency, $E \sim v/r$, we have

[17] This abusive term refers to the fact that the gravitational field itself contributes to the stress energy.
[18] These quantities are only exactly conserved when defined in such a way as to include the effects of radiation.

$k^\mu \sim (v/r, \vec{v}/r)$. Then using, what are hopefully familiar by now, arguments, we find that the coordinate space radiation field scales as v/r. In table (I) , we summarize the v power counting rules.

TABLE I. Scaling relations

M^2/M_{pl}^2	vL
$Md\tau$	$\frac{L}{v^2}$
h/M_{pl}	$\frac{v^{5/2}}{\sqrt{L}}$
H/M_{pl}	$\frac{v^2}{\sqrt{L}}$

To generate an action of the form (94) we start by multipole expanding the graviton field. Again, since we have maintained general coordinate invariance we know that the result of expanding must lead to a coupling to the curvature (or Weyl tensor). Let us consider expanding around the center of mass (taken to be the origin) of the binary to quadratic order

$$-\frac{1}{2M_{pl}}\int d^4x T^{\mu\nu}(x)h_{\mu\nu}(x) \approx -\frac{1}{2M_{pl}}\int d^4x T^{\mu\nu}(x)\left(h_{\mu\nu}(0) + x^\rho\partial_\rho h_{\mu\nu}(0) + \frac{1}{2}x^\rho x^\sigma \partial_\rho\partial_\sigma \bar{h}_{\mu\nu}(0)\right). \tag{96}$$

Notice that the first two terms in this series are absent in (94) since the Weyl tensor is quadratic in derivatives. This is easily understood once we interpret these first two terms in the non-relativistic limit. Noting that the stress energy tensors components have differing scalings, T_{00}, T_{0i} and T_{ij} start at orders v^0, v^1 and v^2 respectively, we find that the leading term in the Taylor expansion generates $v^{1/2}$ and $v^{3/2}$ pieces

$$S_{1/2} = -\frac{1}{2M_{pl}}\int dt \sum_a M_a \bar{h}_{00} \qquad S_{3/2} = -\frac{1}{M_{pl}}\int dt P_i h_{0i} = 0, \tag{97}$$

where *both* spatial and temporal derivatives hitting the radiation field scale as v. $P_i = 0$ is the center of mass momentum [19]. These coupling may seem troubling at first since they don't fit the form (94). However, under a small coordinate transformation

$$x^\mu \to x^\mu + \xi^\mu \qquad \bar{h}_{\mu\nu} \to \bar{h}_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu. \tag{98}$$

[19] At higher orders the effects of radiation must be taken into account if the charges are to be conserved.

these terms are invariant since M and P_i are conserved quantities. If we were to go to higher orders in v then there will be contributions to the mass coming from the kinetic energy and the potential between the world-lines. i.e. the mass would get get an additional contribution equal to $\sum_i \frac{1}{2}M_i v_i^2 - \frac{GM_1M_2}{r}$. Note that physical radiation only arises from the transverse components of h_{ij}. Thus h_{00} would seem to be irrelevant to the action. However, as previously mentioned, the total mass generates the Schwarzchild field of the effective one body, and couples to h_{00}. Radiation scatters off this background generating the aforementioned "tail effects" which we will discuss later in this chapter. In addition, we will have linear couplings to h_{0i} which couples to the conserved angular momentum. Generating this coupling will be left as an exercise for the reader.

At this point we could continue expanding in this way to generate other moments but this would not be the most efficient way to calculate. We need to take full advantage of the information at our disposal so that we don't end up reinventing the wheel. For starters, we know that the real radiation is transverse so we might was well start by considering the coupling to h_{ij}. Diffeomorphism invariance, i.e. the fact we know that the action has the form (94), will fix the dependence upon the other components of h [20]. This implies that the multipole moments will be functions of T_{ij}. Note that since T_{ij} starts at $O(v^2)$, this implies that the one point functions starts at $O(v^2)$. At this order the one point function will include the diagram where radiation is emitted off of a potential mode[21]. However, we can avoid this extra piece of work by utilizing moment relations between T_{ij} and T_{00}, T_{0i} that are easily derived using conservation of the stress energy (pseudo) tensor which includes the effects of the gravitational field and is divergence free. For instance, we have the relation

$$\int d^3x T_{ij} = \int d^3x x_i x_j \ddot{T}_{00}. \tag{99}$$

Note that in deriving these relations we did not have time derivatives hitting the x's. The stress energy tensor gains time dependence through proportionality to a delta function $\delta(x-$

[20] By calculating in the rest frame the moments will only have spatial components.
[21] Showing this is left as an exercise for the reader.

$y(t)$) which forces it on the world line. Using the linearized version of E and B

$$E_{ij} = \frac{1}{2M_{pl}}((h_{i0,0j} - h_{00,ij}) - (h_{ij,00} - h_{j0,i0}))$$
$$B_{ij} = \frac{\epsilon_{imn}}{2M_{pl}}(h_{n0,0m} + h_{jm,on}) \tag{100}$$

leads to the leading order quadrapole results

$$Q_{ij} = \int d^3x x_i x_j T_{00}. \tag{101}$$

This representation of the quadrapole does not transform irreducibly under rotations [22] since it is not trace free. Thus the quadrapole should formally have the trace subtracted, though practically it does not matter since Einstein's equations imply that

$$E_{ii} = B_{ii} = 0 = \partial_i E_{ij} = \partial_i B_{ij} = 0. \tag{102}$$

When one goes to higher orders insuring irreducibility will not be a simple formality. In any case, the result for the leading order mass quadrapole is the canonical form

$$Q^{(0)}_{ij} = \int d^3x(x_i x_j - \frac{1}{3}\delta_{ij})T^{(0)}_{00}, \tag{103}$$

and we have successfully derived the first term in (94). Where $T^{(0)}_{00}$ is the leading order contribution in v to the stress energy tensor. The (0) superscript in Q is there to remind the reader that this is only the leading order form of the mass quadrapole. The corrections to this result will be discussed below.

Let us next derive the current quadrapole and mass octopoles which arise from the first derivative term in (96). We decompose the tensor product $x_j T_{ik}$ into irreducible representations of $SO(3)$ via $1 \otimes 2 = 3 \oplus 2 \oplus 1$. The 1 wont couple since it corresponds to a three index object with two indices chewed by a delta function, which leads to a zero contraction due to (102). The 3 is the totally symmetric ocotopole while the 2 is the mixed symmetry magnetic (current) quadrapole. We can decompose the product as follows

$$x_j T_{ik} = \frac{1}{3}(x_j T_{ik} + x_i T_{jk} + x_k T_{ij}) + \frac{1}{3}(2x_j T_{ik} - x_i T_{jk} - x_k T_{ij}). \tag{104}$$

[22] It is assumed here that the reader has a rudimentary knowledge of the rotation group.

Using the moment relations

$$\begin{aligned}\frac{1}{2}\int d^3x x^i x^j x^k \ddot{T}_{00} &= \int d^3x (x_i T_{jk} + x_j T_{ik} + x_k T_{ij}) \\ \int d^3x (x_i x_j \dot{T}_{0k}) &= \int d^3x (x_i T_{kj} + x_j T_{ki})\end{aligned} \tag{105}$$

the first term, the octopole, can be reduced to

$$\int d^3x \frac{1}{3}(x_i T_{jk} + x_j T_{ik} + x_k T_{ij}) = \frac{1}{6}\int d^3x \left[x^i x^j x^k \ddot{T}_{00}\right]^{TF}, \tag{106}$$

where TF implies the trace has been removed, while the current quadrapole can be written as

$$\int d^3\mathbf{x}\frac{1}{3}\left[2x_j T_{ki} - (x_i T_{jk} + x_k T_{ij})\right]^{TF} = \int d^3\mathbf{x}\frac{1}{3}\left[\dot{T}_{0i}x_j x_k + \dot{T}_{0k}x_j x_i - 2\dot{T}_{0j}x_i x_k\right]^{TF}. \tag{107}$$

Plugging this result into (96), integrating by parts, and picking out the piece of the Weyl tensor proportional to derivatives of h_{ij} leaves

$$-\frac{1}{2M_{pl}}\frac{1}{2}\int d^3x (\partial_j h_{ik}) T_{ki} x_j = \frac{1}{6}O^{ijk}_{(0)}\partial^j E^{ik} - \frac{2}{3}B_{ij}J^{ij}_{(0)} \tag{108}$$

where

$$\begin{aligned}J^{ij}_{(0)} &= \int d^3x \left[\epsilon^{ilk}T^{0k}x^j x^l\right]_{STF} \\ O^{ijk}_{(0)} &= \int d^3x T^{00}\left[x^i x^j x^k\right]_{TF}\end{aligned} \tag{109}$$

where STF implies one must symmetrize in addition to removing the trace. The derivative acting in the octopole term may be automatically lifted to its covariant form since we have maintained diffeomorphism invariance by working in background field gauge.

If we wish to calculate the subleading contribution to the mass quadrapole, then we must consider two sources of corrections to the result (103). Higher order terms in the Taylor series expansion (96), that will change the form of the moment, and subleading corrections to T_{00}. The former corrections can be derived in a standard fashion (i.e. we would calculate these terms in the same way as is done more traditional gravity wave methodologies [2, 3]), however, these corrections are treated below to keep this discussion self-contained. On the other hand, the derivation of the latter corrections are native to the EFT approach.

The corrections to the form (101) arise from the trace pieces of higher order terms in the multipole expansion. The first of such corrections arises from third term in (96)

$$S = -\frac{1}{2M_{pl}} \int dt d^3x \frac{1}{2} x_k x_l (\partial_k \partial_l h_{ij}) T_{ij}. \tag{110}$$

We may decompose this term into irreducible representations of $SO(3)$ as follows

$$T^{ij} x^k x^l \sim \mathbf{2} \otimes (\mathbf{1} \otimes \mathbf{1})_S \sim \mathbf{2} \otimes (\mathbf{2} \oplus \mathbf{0})_S = \mathbf{4} \oplus \mathbf{3} \oplus \mathbf{2}_M \oplus \mathbf{2}_C \oplus \mathbf{1} \oplus 0, \tag{111}$$

consisting of a 16-pole mass moment ($\mathbf{4}$), a current octopole moment ($\mathbf{3}$), correction terms to mass and current quadrupoles ($\mathbf{2}_{M,C}$) as well as corrections to non-radiating moments transforming as $\mathbf{1}, \mathbf{0}$. The corrections to the quadrapoles arise from the traces which are subtracted out from the higher dimensional representations. As opposed to the leading order case, where the traces can not radiate, in this case the traces can contribute since there are a sufficient number of indices that the trace (i.e. δ_{ab}) need not lead to a contraction which vanishes by Einstein's equations (102). The form of the mass quadrapole correction follows by performing the appropriate decomposition of the tensor product and is given by

$$Q_{ij}^{(2)} = \int d^3\mathbf{x} \left[\frac{4}{21} T^{aa} x^i x^j - \frac{2}{7} \left(x^i x^a T^{ja} + x^j x^a T^{ia} \right) + \frac{11}{21} T^{ij} \mathbf{x}^2 \right]. \tag{112}$$

We will not go into the algebra needed to derive this result, and refer the reader to [27] where the form multipoles is given to all orders. Instead we will concentrate on the corrections to T_{00}.

T_{00} couples to h_{00}, as such, we may extract T_{00} by calculating the expectation value $\langle h_{00} \rangle$. Formally we should derive this by introducing the notion of the effective action, which is a idea familiar to readers who have taken an introductory quantum field theory course. However, we will side step this formalism, and instead will rely on some simple handwaving arguments. We begin by noting that if we were not to integrate out any modes whatsoever, then by definition we would have

$$T_{\mu\nu} = -2M_{pl}\frac{\delta S}{\delta h_{\mu\nu}}. \tag{113}$$

That is, if we linearize the action in the metric then we can directly read off the value of the stress energy tensor. Formally how do we relate this to the expectation value of $\langle h_{\mu\nu}\rangle$? We begin by noting that by definition

$$\langle h_{\mu\nu}(x)\rangle \equiv \int Dh_{\mu\nu} \ \left[e^{iS}h_{\mu\nu}(x)\right]. \tag{114}$$

Again assuming we have not integrated out any modes, we bring down one power of the lowest order interaction Lagrangian and use Wicks theorem [23]

$$\langle h_{\mu\nu}(x)\rangle = -\frac{i}{2M_{pl}}\int d^4y \ T^{\rho\sigma}(y)\langle h_{\mu\nu}(x)h_{\rho\sigma}(y)\rangle. \tag{115}$$

We see that to extract $T_{\mu\nu}$ all we need to do is calculate $\langle h_{\mu\nu}\rangle$, strip off the propagator and multiply by $2iM_{pl}$. As a trivial example let's use this technique to calculate the leading order contribution to T_{00} in the PN expansion. The linear term in the interacting part of the action is

$$S^{int}_{LO} = -\sum_a \frac{M_a}{2M_{pl}}\int dt h_{00} = -\sum_a \frac{M_a}{2M_{pl}}\int dt \int d^4y h_{00}(y)\delta^{(4)}(y-x(t)) \tag{116}$$

which leads to the elementary result

$$T_{00}(y) = \sum_a M_a \int dt\delta^{(4)}(y-x(t)). \tag{117}$$

We can see that for tadpole contributions, going through (115) is overly pedantic, and we can simply read off T from the Lagrangian. Subleading tad-pole contributions are easily read off from the action as well. For instance, the kinetic contribution to T_{00} will arise from the v^2 term in (101). However, in addition we have contributions to stress tensor that arise from interactions of the metric with the potential. Such a contribution is shown in

[23] Strictly speaking if we are interested in calculating field expectation values we would need to use the so-called "in-in" formalism [30] which involves doubling the time countour over which the path integral is performed. However, here since we are truncating the propagator this is unnecessary. We will return to this issue in the discussion of the wave form.

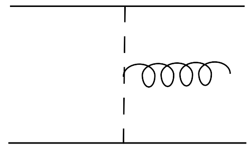

FIG. 9. A contribution to the one point function leading to a potential energy piece of the stress tensor.

figure (9). Before tackling such contributions to the stress tensor, note that what we are really interested in are the *moments* of the stress tensor, which can be determined easily by calculating in momentum space. In this way we may extract the moments by comparing the Taylor expansion of the one point function with

$$T^{\mu\nu}(x^0,\mathbf{k}) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \left[\int d^3\mathbf{x} T^{\mu\nu}(x^0,\mathbf{x}) x^{i_1} \cdots x^{i_n} \right] k_{i_1} \cdots k_{i_n}. \tag{118}$$

For tadpole diagrams this is trivial since the dependence on the external momentum (k) arises only from the wave function of the external state. That is, after truncating the propagator in momentum space [24] the exponential $e^{-i\vec{k}\cdot\vec{x}}$ remains. Let us be highly explicit by redoing our rudimentary example for the leading order T_{00} which arises from the tadpole.

$$\langle \tilde{h}_{\mu\nu}(y_0,\vec{k}) \rangle \equiv \int d^3y e^{-i\vec{k}\cdot\vec{y}} \langle h_{\mu\nu}(y_0,\vec{y}) \rangle = \sum_a \frac{-iM_a}{2M_{pl}} \int dt_1 \int d^3y e^{-i\vec{k}\cdot\vec{y}} \int [d^4p] \int dt \frac{iP_{\mu\nu,00}}{p^2} e^{-ip\cdot(x_a(t_1)-y)}. \tag{119}$$

Now to extract the stress energy we truncate and include a factor of $i2M_{pl}$ (see the definition (113)),

$$T_{00}(t,\vec{k}) = \sum_a M_a e^{-i\vec{k}\cdot\vec{x}_a}. \tag{120}$$

Then by comparing to (118) we find the trivial moments

$$\int d^3x T_{00} x^{i_1} x^{i_2} .. x^{i_n} = \sum_a M_a x_a^{i_1} x_a^{i_2} .. x_a^{i_n} \tag{121}$$

The $1PN$ corrections arising from the tapole follows immediately by making the replacement $M_a \to M_a + \frac{1}{2} M_a v_a^2$. Now we'd like to calculate the contribution to the moments of T_{00}

[24] For readers with less background in quantum field theory, this just means that we take the result for the diagram and multiply by k^2, where k the is four-momentum of the external graviton. In coordinate space this corresponds to the stripping of the propagator mentioned above (115). The exponential which remains after truncation is just the exponential weight of the Fourier transform.

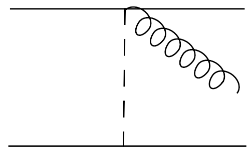

FIG. 10. A contribution to the one point function leading to a potential energy piece of the stress tensor. The dashed/curly line is a potential/radiation graviton.

stemming from the potential energy. We know that the result should be trivial, i.e. an additional contribution of the form $-\frac{1}{2}\sum_b \frac{G_N M_b}{|x_a - x_b|}$. However, doing the explicit calculation in a turn the crank fashion is an excellent exercise. The potential contributions to the mass-energy come the diagrams such as (9) and (10). Let us begin with the simpler of the two cases, namely figure (10). The mixed potential-radiation vertex can be read off from the quadratic part of (22), after making the substitution (18). The result for this Feynman diagram is then

$$\begin{aligned}\langle \tilde{h}_{00}(t,\vec{k})\rangle &= \sum_a \int d^3y e^{-i\vec{k}\cdot\vec{y}} \left(\frac{i2M_1}{8M_{pl}^2}\right)\left(\frac{-iM_2}{2M_{pl}}\right)\int dt_1 dt_2 \int [d^4q] \frac{ie^{iq\cdot(x_1(t_1)-x_2(t_2))}}{-2\vec{q}^2}\langle h_{00}(t_1,x_a)h_{00}(t,\vec{y})\rangle \\ &= -i\sum_a \int d^3y e^{-i\vec{k}\cdot\vec{y}}\left(\frac{M_1M_2}{8M_{pl}^3}\right)\int dt_1 dt_2 \int [d^4q]\frac{e^{iq\cdot(x_1(t_1)-x_2(t_2))}}{2\vec{q}^2}\int \frac{[d^4l]}{l^2}\frac{i}{2}e^{-il_0(t_1-t)+i\vec{l}\cdot(\vec{x}_a-\vec{y})}.\end{aligned} \tag{122}$$

Truncating the external graviton line leaves

$$\begin{aligned}T_{00}(t,\vec{k}) &= \sum_a e^{-i\vec{k}\cdot\vec{x}_a(t)}\left(\frac{M_1M_2}{8M_{pl}^2}\right)\int [d^3q]\frac{e^{i\vec{q}\cdot(\vec{x}_1(t)-\vec{x}_2(t))}}{\vec{q}^2} \\ &= -e^{-i\vec{k}\cdot\vec{x}_a(t)}V\end{aligned} \tag{123}$$

where

$$V = -\frac{M_1M_2}{32\pi M_{pl}^2}\frac{1}{r} = -\frac{GM_1M_2}{r} \tag{124}$$

and use of been made of the generalized integral

$$\int \frac{d^dk}{(2\pi)^d}\frac{1}{(\vec{k}^2)^\alpha}e^{-i\vec{k}\cdot\vec{x}} = \frac{1}{(4\pi)^{d/2}}\frac{\Gamma(d/2-\alpha)}{\Gamma(\alpha)}\left(\frac{\vec{x}^2}{4}\right)^{\alpha-d/2}. \tag{125}$$

Note there is no factor of two which would arise from putting the external line on either world line, since we are summing over worldlines. Also one must be careful in general to assign the correct coordinate to the radiation graviton wave function. For this particular diagram it is obvious that the graviton wave function should be evaluated at x_a, i.e. the line from which it was emitted.

Now let us consider the contribution from diagram (9). This diagram is more complicated due to the algebraic complexity of the three graviton vertex which is derived by expanding out the Einstein-Hilbert action, along with the background field gauge fixing term to third order in the fields and is a straightforward yet tedious exercise that will be bypassed here [25]. However, let us pause here to power count the three graviton vertex. Two of the gravitons are potential and one is radiation. In addition, since the vertex arises from the bulk curvature term the vertex involves two derivatives each of which scales as v/R and the vertex comes with a factor for $1/M_{pl}$. The bulk measure will scale, according to the rules developed in the previous chapters, as the inverse of $dk_0 d^3k$ [26] where dk_0 scales as v/r and $dk \sim 1/r$. Thus the vertex scales as

$$S_{3g} \sim \frac{1}{M_{pl}}(r^4/v)(v^2 M_{pl}/\sqrt{L})(v^{5/2} M_{pl}/\sqrt{L})(v/r)^2. \tag{126}$$

The result for this diagram is

$$T_{00}(t,k) = \sum_a \frac{M_a M_b}{2M_{Pl}^2} e^{-i\vec{k}\cdot\vec{x}_a} \int [d^3q] \frac{1}{\vec{q}^2} \frac{1}{(\vec{q}+\vec{k})^2} \left[-\frac{3}{8}\vec{k}\cdot\vec{q} - \frac{3}{8}\vec{q}^2 \right] e^{-i\vec{q}\cdot\vec{x}_{ab}}, \tag{127}$$

which can be re-written as

$$T_{00}(t,k) = \sum_a \frac{M_a M_b}{2M_{Pl}^2} e^{-i\vec{k}\cdot\vec{x}_a} \frac{3}{16} \int [d^3q] \left(\frac{1}{\vec{q}^2} - \frac{\vec{k}^2}{\vec{q}^2(\vec{q}+\vec{k})^2} + \frac{1}{(\vec{q}+\vec{k})^2} \right) e^{-i\vec{q}\cdot(\vec{x}_1-\vec{x}_2)}. \tag{128}$$

Given that the radiation graviton is emitted off the potential, it is not as obvious this time what the proper wave function, i.e. the argument of the exponential, should be. To clarify this issue we must go back to the Feynman rules for this diagram and consider all of the relevant phase factors. Taking the coordinate position of the bulk vertex to be z, and

[25] MATHEMATICA code for this vertex can be found on this books website.

[26] For those who have not read the previous chapters, this is equivalent to determining the scaling from the delta function which enforced by the spatial integral.

assigning momentum k, p to the potential lines and q, to the radiation line, the phase factor is

$$\int d^4p d^4q \int d^4z e^{-ik\cdot z} e^{-i(x_1-z)\cdot q} e^{-i(z-x_2)\cdot p} = (2\pi)^4 \int d^4q e^{-i(x_1-x_2)\cdot q} e^{ix_2\cdot k} \tag{129}$$

Note this result is symmetric in x_1 and x_2 allowing for shifts in the integration variables.

Now since for the moment we are only interested in the v^2 corrections to the mass quadrapole, equations (112) and (118) imply that we need keep only terms up to $O(k^2)$ this leaves

$$\begin{aligned} T_{00}(t,k) &= \sum_a \frac{M_a M_b}{2M_{Pl}^2} e^{-i\vec{k}\cdot\vec{x}_a} \frac{3}{16} \int [d^3q] \left(\frac{e^{-i\vec{q}\cdot\vec{x}_{ab}}}{q^2} - \frac{k^2 e^{-i\vec{q}\cdot\vec{x}_{ab}}}{q^4} + \frac{e^{-i(\vec{q}-\vec{k})\cdot\vec{x}_{ab}}}{q^2} \right) \\ &= \sum_a \frac{M_a M_b}{2M_{Pl}^2} \frac{3}{16} \int [d^3q] \left(2\frac{e^{-i\vec{k}\cdot\vec{x}_a} e^{-i\vec{q}\cdot\vec{x}_{ab}}}{q^2} - \frac{k^2 e^{-i\vec{q}\cdot(\vec{x}_a-\vec{x}_b)}}{q^4} \right) \end{aligned} \tag{130}$$

where $\vec{x}_{ab} = \vec{x}_a - \vec{x}_b$. We can drop the last term since we are only interested in the trace free part and when we differentiate that piece with respect to $k_i k_j$ this will yield a Kronicker delta function contribution. The first term is just $-3/2$ of figure 2.

$$T_{00}^3 = -\frac{3}{2} \sum_a V e^{-i\vec{k}\cdot\vec{x}_a}. \tag{131}$$

The sum of the non-trivial diagrams plus the mass couplings leaves

$$T_{00}(t,k) = e^{-i\vec{k}\cdot\vec{x}_a} \left(\sum_A m_A \frac{1}{2} v_A^2 - V \right). \tag{132}$$

The second moment then follows from the expanding the exponential and yields the expected correction to the mass from the kinetic and binding energies.

B. Calculating the Power Loss

As discussed in the previous chapter we may calculate the power loss by either calculating the imaginary part of the energy or, perhaps more simply, by directly calculating the semi-classical amplitude for graviton radiation. That is, we calculate the matrix element for the emission of one radiation graviton, and then square it to get an emission rate. In terms of Feynman diagrams this means we calculate all the diagrams with one external radiation graviton leg hanging off. We can save ourselves a lot of time by using the fact that the mass and angular momentum are conserved and thus do not radiate, so that the leading order PN contribution will stem from the emission of one graviton off of the quadrapole moment. Although this contribution will be leading order it will scale with a non-negative power of v, which is consistent with fact that there are no Newtonian gravitons.

The matrix element for one (on-shell $k_0^2 - \vec{k}^2 = 0$) graviton emission off of the quadrapole moment is given by

$$\begin{aligned} iA(k) &= \frac{i}{2}\langle 0 \mid \int dt Q_{ij}(t) E^{ij}(x(\tau)) \mid k\rangle \\ &= -\frac{i}{4M_{pl}^2} \int dt Q_{ij}(t) k_0^2 \epsilon_{ij}(k) e^{-ik\cdot x(t)} \\ &= -\frac{i}{4M_{pl}^2} \int dt \int d\omega \tilde{Q}_{ij}(\omega) e^{-i\omega t} k_0^2 \epsilon_{ij}(k) e^{-ik\cdot x(t)} \\ &= -\frac{i}{4M_{pl}^2} \tilde{Q}_{ij}(k_0) k_0^2 \epsilon_{ij}^h(k) e^{-ik\cdot x(t)} \end{aligned} \tag{133}$$

where we included only the traverse part of the metric in the definition of E since that is the only piece that radiates. The h superscript is the polarization index. The graviton emission rate is determined by squaring the amplitude and weighting it by the phase space factor

$$d\Gamma(k) = \frac{1}{T} \sum_{pol} \mid A(k) \mid^2 \frac{[d^3k]}{2k}, \tag{134}$$

where T is the emission time, which will drop out of our final result. The power is determined by weighting the graviton emission by the energy such that

$$P = \int k d\Gamma(k). \tag{135}$$

When evaluating this quantity we are free to use any gauge we wish since we started with

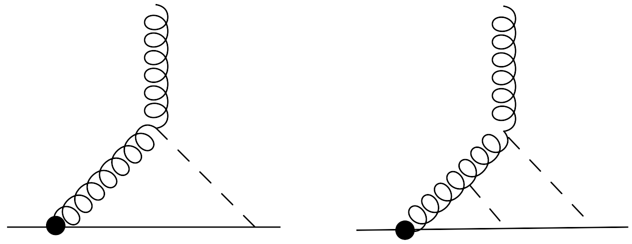

FIG. 11. The contributions to the tail. The heavy dot is an insertion of a quadrapole which radiates and scatters of a potential mode which is generated by a mass at the vertex. The dotted line corresponds to a static graviton which is part of the Schwarzchild background of the total binary system. There are an infinite number of diagrams of this form with more and more mass insertions. The sum of the divergence part generates the Coulomb phase.

a gauge invariant action. It is convenient to use the transverse traceless gauge where

$$\sum_h \epsilon^h_{ij}(k)\epsilon^{*h}_{rs}(k) \equiv P_{ij;rs}(\mathbf{k}) = \frac{1}{2}\left[\delta_{ir}\delta_{js} + \delta_{is}\delta_{jr} - \delta_{ij}\delta_{rs} + \frac{1}{\mathbf{k}^2}\left(\delta_{ij}k_rk_s + \delta_{rs}k_ik_j\right)\right.$$
$$\left. - \frac{1}{\mathbf{k}^2}\left(\delta_{ir}k_jk_s + \delta_{is}k_jk_r + \delta_{jr}k_ik_s + \delta_{js}k_ik_r\right) + \frac{1}{\mathbf{k}^4}k_ik_jk_rk_s\right], \quad (136)$$

Performing the cumbersome index contractions leads to the time honored result

$$\begin{aligned} P &= \frac{G_N}{5\pi T}\int_0^\infty dk k^6 |Q_{ij}(k)|^2 \\ &= \frac{G_N}{5}\langle \dddot{Q}_{ij}\dddot{Q}_{ij}\rangle, \end{aligned} \quad (137)$$

where the dots denote time derivatives, and the brackets denote a time average. Of course, this equation is just the usual quadrupole radiation formula. The power loss from one graviton emissiondue to other moments follows in a similar fashion.

1. *The Tail Effect*

One certainly does not need the EFT formalism to perform the simple calculations involved in the last section. Where the EFT formalism helps, is when we have processes involving both radiation and potential modes. The tail effect, is such a situation, where a radiation graviton scatters off of the background of the binary. The relevant Feynman diagram is shown in figure (11). Relative to the single quadrapole emission, this diagram includes one mass insertion and one three graviton insertion. Now it is important to recall here that the potential mode is not the same potential mode that was integrated out in our previous discussion. As such, the scaling of this potential differs from that listed in table (1). The potential mode again carries no energy, but its momentum no longer scales as $1/R$. Recall that this second EFT only knows about that scale $1/R$ via the matching coefficients (multipole moments). Given that the only energy or momentum scale in the problem is now the radiation energy-momentum $p \sim v/R$, the potential itself must carry momentumn of the same order, i.e. v/R and using our usual arguments we find that this potential field scales identically to the radiation, i.e. $H/M_{pl} \sim v^{5/2}/\sqrt{L}$.

Do determine the scaling of the tail effect we need to power count the three graviton vertex which now scales differently then in the case of the EIH calculation. This vertex scales as

$$S_{3g} \sim \int d^4x \frac{1}{M_{pl}} h(\partial h)^2, \tag{138}$$

thus this contribution is down, relative to the leading order quadrapole emission by v^3, and is thus a 1.5 PN correction to the leading order radiation result.

Evaluating this integral will be left as an exercise for the reader. In the infra-red region of the loop integral the integrand has the form

$$I = \frac{\int [d^3k]}{\vec{k}^2} \frac{1}{\vec{q} \cdot \vec{k}} \tag{139}$$

which is infra-red divergent. This should not be surprising as the Newtonian potential falls off as $1/r$ and thus one can scatter off the background at arbitrarily long distances. This is just the Coulomb phase, which drops out when we calculate the square of the amplitude. Indeed it is straightforward to resum the appropriate set of diagrams (the first two of which are shown in figure (11))that lead to the exponentiation of the phase (see [28, 29]).

2. Summing Logs in the Power: The Renormalization Group Trajectory

There is another very interesting piece of this particular calculation that deserves comment. Namely the second diagram in figure (11) while being IR divergent also generates a UV divergence which is not associated with the Coulomb phase. In addition, there are two other diagrams, without the tail topology, that have UV divergences as well, and the reader is encouraged to determine the relevant topologies as an exercise. The two loop diagrams are logarithmically divergent, leading to renormalization group running. We will not go into the details of the two loop calculations which can be found in the appendix of [29]. Here we will focus on the UV divergence since that is the piece which contains the information about the running. Normalizing to the leading order radiation[27] amplitude ($A_0 = \frac{\vec{k}^2}{4M_{pl}}\epsilon^\star_{ij}Q_{ij}$) the result for the amplitude is [28] of the form

$$\frac{A}{A_0} = (\omega G_N m)^2 \frac{107}{105}(\frac{1}{\epsilon} - Log(\vec{k}^2/\mu^2) + ...) \tag{140}$$

thus we can absorb the divergence into $Q(\omega,\mu)$. The reader should find this slightly discomforting since Q is a short distance matching coefficient whose physics is determined by wavelengths of order of the size of the system which is parametrically smaller than the inverse frequency by a factor of the velocity. Indeed it seems that perhaps we should really be absorbing the divergence into a counter-term for an operators of the form $Q\ddot{E}$, in which case the ω^2 contribution arises from the long distance contribution of the matrix element of the operator. The renormalization would then correspond to operator mixing, which is a subject that was discussed in the earlier chapters of this book. Thus to keep the classical discussion self contained we will stick to the technique whereby the matching coefficient depends upon ω. Needless to say, both methods yield the same result.

The μ independence of the (net) amplitude leads to the differential equation

$$\mu\frac{d}{d\mu}Q_{ij}(\mu,\omega) = -2\times(\omega G_N m)^2\frac{107}{105}Q_{ij}(\mu,\omega) \tag{141}$$

whose solution is given by

$$Q_{ij}(\mu,\omega) = (\mu/\mu_0)^{-\frac{214}{105}(\omega G_N m)^2}\,Q_{ij}(\mu_0,\omega). \tag{142}$$

μ_0 should be chosen to be the natural scale for the quadrapole. i.e. the size of the bound state (r) while μ should be chosen to minimize the logarithms in the amplitude (140), which is accomplished by taking $\mu \sim \omega$. We can expand the result to expose the complete series of

[27] Recall this is on-shell so $\omega^2 = \vec{k}^2$.

[28] This diagram also has an infrared divergence which is purely imaginary and will not contribute once we square the amplitude, as it represents an unphysical phase much, as in Coulomb scattering. We will discuss this in more details when we calculate the wave form.

logarithms that we have resummed. At the level of at the amplitude squared we have [29]

$$| \frac{A}{A_0} |^2 = 1 - \frac{428}{105}(G_n m\omega)^2 Log(\omega/\mu_0) + \frac{91592}{11025}(G_n m\omega)^4 Log^2(\omega/\mu_0) - \frac{32901376}{3472875}(G_n m\omega)^6 Log^3(\omega/\mu_0) + ... \tag{143}$$

The existence of the UV log divergence in the effective one body theory is telling us that had we calculated this diagrams in the "full" theory, we wold generate logs of the form $\log(\omega r)$. That is, the size of the state cuts off the apparent divergence in the full theory. Diagrammatically we can see how this arises by opening up and dissecting the quadrapole interaction in the effective theory. This is illustrated in figure (12) where now the single lines opens up into two lines corresponding to the constituents of the binary. By choosing $\mu_0 = 1/r$ we eliminate any large logs ($Log(v)$) from the quadruple moment.

It is important to realize that resumming the logs does not improve the accuracy of our prediction. This is as opposed to the case of the renormalization group in a quantum field theory. The reason is that each power of a log is accompanied by a factor of v^2, so we are not resumming a set of order one contributions. In the quantum field theory case we resummed the product of a large log and a coupling such that the product was order one. However, there is no limit in which $v^2 Log(v) \sim 1$. Furthermore, we can not be assured that at a given order in v we have captured *all* of the logs. In this case however, the terms in the series do indeed capture the leading order contribution at each order in $\log(v)$. In fact, the logs in (143) have been compared to the analytic solution derived by solving the wave equation in the extreme mass ratio limit (one body is taken as a test mass) and agreement has been found [32]. At each power of $\log(v)$ there are sub-leading corrections which arise which have additional powers of v in front. For instance, diagrams involving two quadrapole insertions renormalize the mass and lead to sub-leading log contributions [33] of the form $v^4(Gm\omega)^{2n+1}\log^n(v)$.

C. Radiation Reaction[29]

In this previous chapter we calculated the force on finite size spheres due to radiation back-reaction. The gravitational analogue of this force plays an important role in gravity wave phenomenology. Here we will discuss this force in the context of the PN expansion

[29] This section assumes the reader has read section (IIB) of the previous chapter.

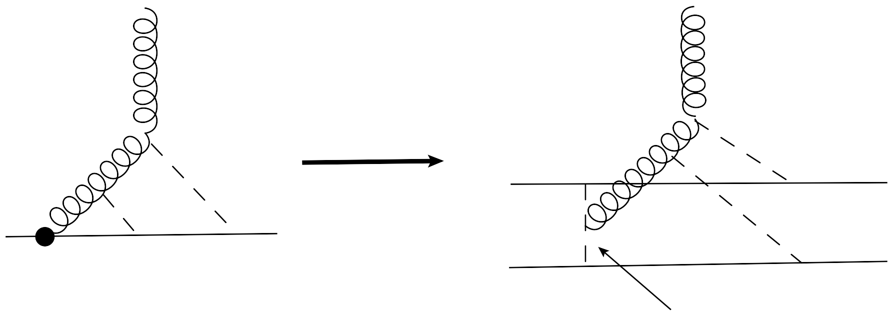

FIG. 12. This diagram shows how if one opens up the quadruple insertion (heavy dot) in the effective theory, into the full theory diagrams (on the RHS) the UV divergence is cut-off by the size of the binary. The arrow points to the propagator which softens the UV behavior of the integral.

[30]. The strategy is completely analogous to the electromagnetic case. Since we are working in the radiation theory, we start with the action (94) and calculate an effective action by integrating out (using retarded propagators) the radiation. The leading contribution, in the PN expansion, arises from the self energy diagram (see figure (7) of the previous chapter) where the vertices are now quadrapole insertions as dictated by the Feynman rules generated by (94). The amplitude for this diagram is given by

$$iM = -\frac{1}{2}\int d\tau d\tau' \langle E_{\mu\nu}(\tau)E_{\rho\sigma}(\tau')\rangle Q^{\mu\nu}(\tau)Q_{\rho\sigma}(\tau'). \tag{144}$$

Next we use the linearized form of Electric Weyl tensor (100), keeping in mind that in the radiation theory the time and space derivatives scale identically. The retarded (harmonic gauge) propagator is given by

$$D^R_{\mu\nu\alpha\beta} = \frac{P_{\mu\nu\alpha\beta}}{2\pi}\theta(x^0(\tau) - \theta(x^0(\tau'))\delta([x(\tau) - x(\tau')]^2), \tag{145}$$

where

$$P_{\mu\nu\alpha\beta} = \frac{1}{2}(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \eta_{\mu\nu}\eta_{\alpha\beta}). \tag{146}$$

By choosing the affine parameter to be $\sqrt{(y(\tau) - y(\tau'))^2}$, varying this effective action with respect to the latest time parameter, and using the leading order result for the quadruple

[30] For a discussion in the context of the extreme mass ratio limit see [4].

moment (107) leads to the the Burke-Thorn result [34]

$$\ddot{x}^i(t) = -\frac{2}{5} G_N \frac{d^5 Q^{ij}(t)}{dt^5} x^j. \tag{147}$$

There are two sources of higher order corrections: Non-linear interactions in the bulk as well as higher order multipole moments on the world-line. Note that non-linear terms on the world-line themselves will be quantum in nature and will thus be suppressed. To see this one notes that such diagrams will always lead to at least one closed graviton loop.

V. OTHER DEGREES OF FREEDOM: SPIN

We can generalize the theory further by allowing for the other degrees of freedom. The most obvious generalization would be the inclusion of spin which is highly relevant for the phenomenology of gravity waves. As in the case of absorption, we must allow for a new variable to live on the world line. The challenge is to introduce the spin vector $\vec{S}$ in a covariant way. We will follow the ideas presented in [35] but will not go into all of the technical details regarding the theory of constrained systems. We begin by studying the relativistic top in flat space [36]. One first introduces a field on the world-line, $\Lambda^\mu{}_\nu(\tau)$, which is an element of the Lorentz group $SO(3,1)$ which connects the inertial frame to the co-rotating frame. Λ obeys the constraint

$$\Lambda_\mu{}^\nu \Lambda_{\rho\nu} = \eta_{\mu\rho}. \tag{148}$$

The relevant velocities are then $\dot{x}^\mu$ and $\dot{\Lambda}^\mu{}_\nu$. We note that the quantity

$$\sigma^{\mu\nu} = \Lambda_\lambda{}^\mu \dot{\Lambda}^{\lambda\nu} = -\sigma^{\mu\nu}, \tag{149}$$

is the relativistic analog of the usual angular velocity ω^{ij} of a rotating frame

$$\frac{dx^i}{dt} = x^j \omega^{ji}. \tag{150}$$

The difference being of course that ω^{ij} has the correct number of degrees of freedom (3) to describe spin while $\sigma_{\mu\nu}$ has six. The spin $S_{\mu\nu}$ is defined as being the momentum conjugate to $\sigma_{\mu\nu}$.

Clearly we will have to eliminate three degrees of freedom using some constraint which is also called the Spin Supplementarity Conditions or SSC. The reason that spin is ambiguous is that one must recall that the body has finite size, and what we mean by its "position" must be defined in order to make sense of the concept of spin since an axis of rotation must be chosen. The classic work by Pryce [37] on this subject gives three useful definitions

- The rest frame center of momentum:

$$\vec{x}_{rf} = \frac{\sum_i E_0^i \vec{x}_i}{\sum_i E_0^i} \tag{151}$$

  where E_i is the energy of the ith component particle computed when the total momentum is zero. The corresponding constraint is covariant

$$S_{rf}^{\mu\nu} p_\nu = 0. \tag{152}$$

- Generalized center of momentum:

$$\vec{x}_g = \frac{\sum_i E^i \vec{x}_i}{\sum_i E^i} \tag{153}$$

  where E^i is the energy as measured by an arbitrary observer. The corresponding constraint is

$$S_g^{0\mu} = 0. \tag{154}$$

- The Newton-Wigner coordinate: This is a hybrid of the above two choices given by

$$\vec{x}_{NW} = \frac{M\vec{x}_{rf} + E\vec{x}_g}{M + E}. \tag{155}$$

  where M is the total invariant mass of the system. The corresponding constraint is

$$MS_{NW}^{0\mu} - S_{NW}^{\mu\nu} p_\nu = 0. \tag{156}$$

Each one of these definitions generates the necessary number (three) of constraints and leads to a different set of Poisson brackets (or quantum mechanically, commutation relations).

Indeed the Newton-Wigner coordinates are chosen such that the position operator commutes with itself, and is an attempt at making sense of the position as an operator in relativistic quantum mechanics. For the other choices of coordinates the algebra is non-canonical and will have to be treated with care. We will choose to work with the covariant constraint as it proves convenient when working in a curved space which we now discuss.

In a general space-time we can define a local orthogonal frame via the vierbein such that

$$g_{\mu\nu} = e_\mu^a e_\nu^b = \eta^{ab}. \tag{157}$$

Then we can relate the co-rotating frame labelled by capital Roman letters to the local orthogonal frame (lower case Roman letters) via the introduction of e_I^μ such that

$$e_a^\mu = \Lambda_a{}^I e_I^\mu. \tag{158}$$

Note that e_I^ν describes the state of the particle as opposed to e_a^ν which encodes information about the space-time and is not an explicit function of τ. To reiterate, we have established three frames: The global, sometimes called PN, frame labelled (Greek letters), the local orthogonal frame (small Roman letters), and the co-rotating frame (capital Roman letters). I the flat space limit, the PN and local orthogonal frame are identical, and the co-rotating frame is related to them by a generalized boost Λ.

The generalized coordinates and velocities which describe the particle are $(x^\mu, u^\mu, e_\mu^I, \dot{e}_\mu^I)$. The generalized angular velocity $(\Omega_{\mu\nu})$ may then be defined via

$$u \cdot \nabla e_I^\mu(\tau) = -\Omega^{\mu\nu} e_{I\nu}. \tag{159}$$

The indices of e_μ^I are raised and lowered by the space-time metric $g_{\mu\nu}$ and the local flat metric η_{IJ}. The spin is then defined as being conjugate to Ω

$$S_{\mu\nu} = -2\frac{\partial L}{\partial \Omega_{\mu\nu}} \tag{160}$$

where the -2 is chosen for later convenience.

Now we would like to write down an action for the spin that is both coordinate invariant as well as RPI. Given this ladder symmetry, this implies that the Hamiltonian should vanish and therefore

$$S \sim \int d\tau (p\dot{q}) \equiv \int d\tau S_{\mu\nu}(\tau)\Omega^{\mu\nu}(\tau). \tag{161}$$

where we note that $S_{\mu\nu}$ is RPI by dint of the fact that its conjugate to Ω. The action written in terms of the local orthogonal frame variables is then given by

$$S = -\frac{1}{2}\int d\tau S^{ab}(\Omega_{ab} + u_\alpha \omega^\alpha_{ab}) \tag{162}$$

where the normalization has been fixed in order to reproduce the correct non-relativisitic flat space limit. ω^α_{ab} is the spin connection defined via

$$\omega^\alpha_{ab} = (\nabla^\alpha e^\mu_a)e_{\mu b}. \tag{163}$$

Notice, that up to this point the choice of constraints has not played a role. We will return to this issue in a moment.

Given that S is a conjugate momentum we utilize a Routhian description of the system, whereby the Routhian R acts as a Hamiltonian for the spin degrees of freedom and a Lagrangian for the world line position

$$R = -\sum_i \left(M_i\sqrt{u_i^2} + \frac{1}{2}S_i^{ab}\omega_{ab\mu}u_i^\mu \right). \tag{164}$$

We have not included the term which is independent of the graviton $S_{ab}\Omega_{ab}$ as we will always be writing our final results in terms of S_{ab} and thus this term will not contribute to any potentials.

The equations of motion then follow from

$$\frac{\delta}{\delta x^\mu}\int R\, d\tau = 0, \quad \frac{dS^{ab}}{d\tau} = \{S^{ab}, R\}. \tag{165}$$

Obviously to get the equations of motion for the spin we will need to know the algebra for the phase space variable (x, p, S), where p is the canonical (i.e. not mechanical) momentum. The spin generates rotations in $SO(1,3)$ and thus obey

$$\{S_{ab}, S_{cd}\} = \eta_{ac}S_{bd} + \eta_{bd}S_{ac} - \eta_{ad}S_{bc} - \eta_{bc}S_{ad}. \tag{166}$$

These relations obviously include the redundant degrees of freedom which we could choose to eliminate, leading to the reduced phase space algebra [31]. However, for the sake of simplicity it is better not impose the SSC until the end of our calculation [38]. Note that the algebra applies to the spin in the local orthogonal frame.

If we are not going to impose the constraint at the level of the action/Routhian, then we must make sure that it is conserved by the dynamics. If we are to choose the covariant SSC then we have the evolution equation

$$u \cdot \nabla(p_\mu S^{\mu\nu}) = 0. \tag{167}$$

Without even knowing the form of the Routhian we have at our disposal the equations of motion

$$u \cdot \nabla(S_{\mu\sigma}) = S_{\rho\mu}\Omega^\rho{}_\sigma - S_{\mu\nu}\Omega^\nu{}_\sigma \tag{168}$$

$$u \cdot \nabla(p^\gamma) = -\frac{1}{2}R^\gamma_{\rho\alpha\beta}S^{\alpha\beta}u^\rho. \tag{169}$$

where at this point we have not specified the form of p^α. That is, these equations follow by varying the generalized action (see for instance [35])

$$S = \int d\tau(p \cdot u + \frac{1}{2}S \cdot \Omega). \tag{170}$$

Furthermore, by writing out the most general Lagrangian[32] in terms of $(\Omega, \dot{x})$ one can derive [36] the Mathisson-Papapetrou (MP equations)

$$u \cdot \nabla(S_{\mu\sigma}) = p^\mu u^\sigma - p^\sigma u^\mu. \tag{171}$$

We may use these equations to determine p, given that we wish to impose (167). A simple calculation yields

$$p^\alpha = \frac{1}{\sqrt{u^2}}\left(mu^\alpha + \frac{1}{2m}R_{\beta\nu\rho\sigma}S^{\alpha\beta}S^{\rho\sigma}u^\nu\right) + ... \tag{172}$$

Note that since we are utilizing the equations of motion (EOM) , we are implicitly doing a coordinate transformation. If we choose not to use the EOM then we will have acceleration

[31] It is in this reduced phase space that the coordinates no longer have a non-vanishing Poisson bracket with themselves.

[32] One writes down a irreducible set of Poincare invariant quantities and allows the Lagrangian to be arbitrary function of this set. The MP equations then follow by by varying the action.

dependent terms in the action. Of course all physical observables will not depend upon this choice (see [38]).

In (172) we have dropped terms which will be highly suppressed in the PN expansion. We will justify and quantify this approximation once we have set up the power counting for spin. Also in deriving the result (172) we used the approximate version of the SSC $v \cdot S = 0$. The extra term needed to preserve the SSC is quadratic in spin and will not play a role in inter-spin potentials proportional to $S_1 \cdot S_2$. However, before going on to calculate these potentials let us first pause to discuss the finite size spin effects.

Unlike the non-spinning case, we expect there to be world line operators linear in the metric which are induced by finite size effects. The spin generates a quadruple moment which then effects the one point function (i.e. the solution to Einstein's equation is sensitive to the spin) so as to generate the Kerr solution. Following our usual logic, the lowest dimensional operator that we can write down that this consistent with the symmetries is

$$L_{ES^2} = \frac{C_1}{2mM_{pl}} \frac{E_{ab}}{\sqrt{u^2}} S^a{}_c S^{cb}. \tag{173}$$

The velocity factor is there to preserve RPI. When calculating potentials proportional to S^2 one needs to include the effects of (173) as well as (172).

A. Power Counting and Feynman Rules for Spin

For a black hole or neutron star the natural length scale is $r_s \sim m/M_{pl}^2$ so the moment of inertia scales as $I \sim m^3/M_{pl}^4$. Thus the spin scales as

$$S = I\omega \sim \frac{m^3}{M_{pl}^4} \frac{v}{r_s}. \tag{174}$$

For generic black holes, the case of interest for phenomenology, the holes are believe to be near maximally rotating $v \sim 1$, thus

$$S \sim m^2/M_{pl}^2 \sim Lv. \tag{175}$$

Given this scaling along with the results tabulated in table 1, we are prepared to derive the scaling of vertices in the action. First we must expand out the terms in (164) in terms of the metric fluctuations.

We may re-write the spin dependent part of the Routhian (164) as

$$R = \frac{1}{2} S_{LM} e^L_\mu e^M_\nu \eta^{IJ} e^\nu_I (u^\alpha \nabla_\alpha e^\mu_J = u^\alpha(\partial_\alpha e^\mu_J + \Gamma^\mu_{\alpha\beta} e^\beta_J)) \tag{176}$$

Imposing the relation

$$\eta_{IJ} e^I_\mu e^J_\nu = \eta_{\mu\nu} + \frac{h_{\mu\nu}}{M_p}. \tag{177}$$

We find the relations

$$e^J_\mu = \Lambda^J_\mu + \frac{h^\nu_\mu \Lambda^J_\nu}{2M_{pl}} - \frac{h^\rho_\mu h_{\rho\sigma} \Lambda^{\sigma I}}{8M^2_{pl}}, \tag{178}$$

$$e^{J\nu} = \Lambda^{J\nu} - \frac{h^\nu_\rho \Lambda^{J\rho}}{2M_{pl}} + \frac{h^\rho_\sigma h^\nu_\rho \Lambda^{J\sigma}}{4M^2_{pl}}. \tag{179}$$

$$e^\nu_J = \Lambda^\nu_J - \frac{h^\nu_\rho \Lambda^\rho_J}{2M_{pl}} + \frac{3h^\rho_\sigma h^\nu_\rho \Lambda^\sigma_J}{8M^2_{pl}}. \tag{180}$$

Here we have included terms quadratic in the metric though we will only utilize the linear terms, since we will only explicitly calculate the leading order spin dependent potential.

Working at linear order we may use the relation

$$S_{LM} \Lambda^M_\mu = S_{L\mu}, \tag{181}$$

since the local orthogonal frame and the global frame are the same to leading order.

$$R = \frac{1}{2} S_\mu{}^J (-u^\alpha \delta^\rho_J \partial_\alpha \frac{h^\mu_\rho}{2M_{pl}} + u^\alpha \Gamma^\mu_{\alpha\beta} \delta^\beta_J)). \tag{182}$$

The first term vanishes due to anti-symmetry of the spin tensor. Now using the result

$$\Gamma^{(1)\mu}_{\alpha\beta} = \frac{1}{2M_{pl}} (h^\mu_{\alpha,\beta} + h^\mu_{\beta,\alpha} - h_{\alpha\beta}{}^{,\mu}). \tag{183}$$

We have

$$R = -\frac{1}{2M_{pl}} S_{\mu\beta} u^\alpha h_\alpha{}^{\beta,\mu}. \tag{184}$$

Note this is all in the global (PN) frame, and the constraint as well as the spin algebra are written in the local frame so one must be careful when working at higher orders.

FIG. 13. Diagrams generating the leading order spin dependent potentials. The square vertex represents a spin insertion. Diagram a) is the spin-orbit interaction which is order 1.5 PN while diagram b) is the spin1-spin2 potential which is order 2 PN.

We are now in position to calculate the leading order potential. However, before doing so let us power count this interaction. Expanding we find

$$R = -\frac{1}{2M_{pl}}(S_{0\beta}h_0{}^{\beta,0} + S_{i\beta}h_0{}^{\beta,i} + S_{0\beta}v_i h^{i\beta,0} + S_{j\beta}v_i h^{i\beta,j}). \tag{185}$$

Now given that we are interested in potentials, time derivatives scale as v^2 while spatial derivatives are linear in v. Furthermore, given our covariant constraints we have $S_{0i} \sim vS_{ij}$. Thus the leading order terms are

$$R = -\frac{1}{2M_{pl}}(S_{ij}H_0{}^{j,i} + S_{i0}H_0{}^{0,i}). \tag{186}$$

Where, following our notation, are now considering the potential graviton (H) since we're interested in calculating potentials. Using Table (1) we can read off the scaling of this interaction as $v^2\sqrt{L}$.

We are now prepared to calculate the spin dependent potentials. The leading order potential will come from interactions involving one insertion of the spin interaction connecting to a mass insertion on the other world line. Naiviely, we would think this potential is order v^2 (1PN), however, given that the interaction (186) is proportional to H_{0i}, the mass insertion on the other line must couple to the interaction in (27), because the graviton propagator (26) does not connect h_{00} to h_{0i}. The Feynman diagram responsible for this potential is shown in figure (13a). Performing the contraction between the vertices leads to the result

$$V = \frac{GM_2}{r^2}(S^{(1)}_{ij}n_i v_{1j} - 2S^{(1)}_{ij}n_i v_{2j} + S^{(1)}_{i0}n_i) + (1 \leftrightarrow 2). \tag{187}$$

The equations of motion are then given by [33]

$$\frac{dS_{ij}}{d\tau} = \{S_{ij}, -V\}, \tag{188}$$

from which is follows that

$$\frac{dS_{ab}}{d\tau} = -\frac{GM_2}{r^2}\left[(n_i v_{1j} - 2n_i v_{2j})\{S_{ab}, S_{ij}\}) + n_i\{S_{ab}, S_{i0}\}\right] \tag{189}$$

and

$$\frac{d\vec{s}_1}{dt} = 2\left(1 + \frac{M_2}{M_1}\right)\frac{\mu G_N}{r^2}(\vec{n} \times \vec{v}) \times \vec{s}_1 + \frac{M_2 G_N}{r^2}(\vec{s}_1 \times \vec{n}) \times \vec{v}_1 \tag{190}$$

where we have written the result in terms of the spin vector defined via

$$S_{ij} = \epsilon_{ijk} s_k, \tag{191}$$

and n is the normal vector in the direction of $\vec{r}$. This is not the complete result for this diagram. We must recall that the SSC is defined in the local frame, not in the PN frame in which the observer lives. We can relate the velocity in the local from (L) to the PN frame via the tetrad

$$\begin{aligned} v^j_L &\approx e^j_\mu v^\mu = v^\mu(\delta^j_\mu + \frac{1}{2M_{pl}} h_{\rho\mu}\eta^{j\rho}) = v^j + \frac{v^\mu}{2M_{pl}} h^j_\mu \\ v_{Lj} &= v_j + \frac{1}{2M_{pl}} h_{j0} \end{aligned} \tag{192}$$

The field value (one point function) generated by the second spin is easily calculated and is given by

$$\begin{aligned} \langle h_{j0}\rangle &= \frac{iS^{(2)}_{ab}}{2M_{pl}}\int d\lambda \langle h_{0j}(x) h_{0b,a}(y)\rangle. \\ &= \frac{iS^{(2)}_{ab}}{2M_{pl}}\frac{-in_a}{8\pi r^2}\delta_{bj} \end{aligned} \tag{193}$$

[33] The apparent extra minus sign is due to our choice of spin algebra (166) conventions.

Now let us reconsider the piece (189) proportional to S_{0i} in (189)

$$\begin{aligned}\frac{dS_{ab}}{dt} &= -\frac{GM_2}{r^2}n_i\{S_{ab}, S_{i0}\} \\ &= -\frac{GM_2}{r^2}n^i(-\delta_{ai}S_{b0} + \delta_{bi}S_{a0}) \\ &= \frac{G^2M_2}{r^4}(\vec{n} \times ((\vec{n} \times \vec{s}_2) \times \vec{s}_1)\end{aligned} \tag{194}$$

Note that this piece is bi-linear in spin, so one might not wish to consider this part of the spin orbit coupling, but this is just a matter of nomenclature. There are of course, other contribution to the spin1-spin2 potential coming from direct contribution. The leading order contribution is shown to in figure (13b), the resulting 2PN potential is given by

$$V^{ss}_{2PN} = -\frac{G_N}{r^3}(\vec{s}_1 \cdot \vec{s}_2 - 3\frac{\vec{r} \cdot \vec{s}_1 \vec{r} \cdot \vec{s}_2}{r^2}), \tag{195}$$

In addition at 2PN we have the contribution stemming from s_i^2 terms, which corresponds to a quadarpole-monopole potential. This contribution is generated by the contraction of a mass insertion on one side and a spin squared (finite size) insertion from (173) on the other. The spin squared term in the Routhian that is needed to preserve the SSC is higher order and not relevant at 2PN. The resulting potential is given by

$$V^{so}_{2PN} = -\frac{C_1 G_N M_2}{2M_1 r^3}\left(\vec{s}_1 \cdot \vec{s}_1 - 3(\vec{n} \cdot \vec{s}_1)^2\right) + \ 1 \leftrightarrow 2, \tag{196}$$

where C_1 is the matching coefficient for the finite size operator (173). This coefficient is easily matched by calculating the field value (one point function) due to the finite size operator and matching it onto the Kerr solution, with the result being $C_1 = 1$ [35].

VI. CALCULATING THE WAVE FORM

In addition to the power loss, which leads to the phase of the wave, the other relevant observable is the wave form itself, the calculation of which follows simply via the techniques we have developed so far. The important distinction being that we are now interested in real time expectation values of the field, $\langle h(x,t)\rangle$, and not in-out scattering matrix elements.

Thus the path integral technology that we have developed must be modified.

In quantum field theory the technology for calculating time sensitive quantities is called the in-in formalism [30, 31]. Going into this formalism would take us too far afield. Furthermore, the in-in formalism is a large hammer for our relatively small nail, since we are not interested in quantum effects. Indeed, for non-radiative moments calculating the field value at some a space-time point (x, t) simply follows by convolving the source multipole (Q) with a retarded Greens function (D_R) and has the generic form

$$h(x,t) \sim \int d^4y \partial..\partial D_R(x-y)Q(y). \tag{197}$$

The derivatives arise due to the form of the couplings of the E and B fields to the moments in (94). The Greens function (in our harmonic gauge) is given by (145

We project onto the the physical transverse traceless piece

$$h^{TT}_{ij} = \frac{1}{M_{pl}}\Lambda_{ij,kl}h_{kl}. \tag{198}$$

The M_{pl} is put there to make h dimensionless and is standard normalization. The projector is given by

$$\Lambda_{ij,kl} = (\delta_{ik} - n_i n_k)(\delta_{jl} - n_j n_l) - \frac{1}{2}(\delta_{ij} - n_i n_j)(\delta_{kl} - n_k n_l), \tag{199}$$

where n is the unit vector in the direction of $\vec{r}$, which connects the source to the observer.

The direct contribution of the multipoles to the wave form then follows from the coupling of the multipoles to E and B in (94)

$$h^{TT}_{ij}(t, \mathbf{x}) = -\frac{4G}{|\mathbf{x}|}\Lambda_{ij,kl}[\frac{1}{2}\frac{d^2Q^{kl}}{dt^2} - \epsilon^{ab(k}\frac{2}{3}\frac{d^2J^{l)b}}{dt^2}n_a + ...] \tag{200}$$

where we have kept only the leading order mass and current (electric and magnetic) contributions for the sake of illustration and the parenthesis around the indices indicate symmetrization. The procedure for calculating the multipole moments was discussed in section (IV A).

The radiative moments are formally more interesting and illustrate utility of the EFT approach. Once we have integrated out the potentials, we are working with a single particle endowed with dynamical multipole moments. The only scales around are the wavelength of the radiation and the distance to the detector, which is, for all intents and purposes, infinite.

Let us consider the tail effect previously discussed in the context of the power radiated. The leading order contribution to the wave form coming from scattering off of the effective Schwarzschild background of the one body is shown in figure (11). The in-in formalism implies that the only change needed, relative to our usual Feynman diagram calculation, is that we should use causal propagators. This change could have been guessed on physical grounds. For gravitons which couple to the time independent mass this is moot since they are instantaneous anyway, but for those which couple to time dependent moments, using the causal (retarded) propagator implies that we must change the $i\epsilon$ prescription as follows

$$\frac{1}{k_0^2 - \vec{k}^2 + i\epsilon} \to \frac{1}{(k_0 + i\epsilon)^2 - \vec{k}^2}. \tag{201}$$

In position space this prescription simply ensures that emission predates scattering.

The insuing integral for this diagram, which will be left as an exercise for the reader[34], is infrared divergent and is regulated by utilizing dimensional regularization,

$$iA_{Q^{ij}M}(\omega, \omega\mathbf{n}) = iA_{Q^{ij}} \times (iGM\omega)\left[-\frac{(\omega + i\epsilon)^2}{\pi\mu^2} e^{\gamma_E}\right]^{(d-4)/2} \times \left[\frac{2}{d-4} - \frac{11}{6}\right] \tag{202}$$

where $A_{I^{ij}} = \frac{i}{4m_p}\omega^2\epsilon^*_{ij}Q^{ij}(\omega)$ and ϵ_{ij} is the polarization tensor. After expanding d around $4 - 2\epsilon$ we can read off the contribution from the tail to the radiative quadrupole moment (we absorb the factor of π into μ from now on),

$$Q^{ij}_{\rm rad}(\omega) = Q^{ij}_0(\omega)\left\{1 + GM\omega\left(\mathrm{sign}(\omega)\pi + i\left[\frac{2}{\epsilon_{\rm IR}} + \log\frac{\omega^2}{\mu^2} + \gamma_E - \frac{11}{6}\right]\right)\right\}. \tag{203}$$

Furthermore, and crucially, one can show by summing the diagrams with further mass insertions [29] that the IR divergences exponentiate into a pure phase in the emission amplitude, i.e. $\mathcal{A} = \mathcal{A}_{\rm IR-finite}\, e^{2iGM\omega/\epsilon_{\rm IR}}$.

The divergence arises because we are scattering off of a $1/r$ potential. This may seem puzzling at first since we are interested in measuring this wave form and any measurable quantity must obviously be finite. But if we were to calculate in position space, then we would see that the phase is sensitive to arbitrary long times, since we are not solving the

[34] The calculation of this diagram is greatly simplified by the fact that we may take the external graviton to be on-shell, i.e. $k^2 = 0$. This is permissible because we are interested in the field at infinity, and off-shellness leads to a vanishing field at asymptotia.

initial value problem. That is, we can't expect to be able to predict the value of the phase since we have assumed that the source is eternal. However, we can predict differences in the phase such that once we measure the phase at some time, then we can predict it at a later time.

From the above expressions the contribution from the $Q^{ij}M$ coupling to the GW amplitude becomes

$$\left(h_{ij}^{TT}\right)_{I^{ij}+I^{ij}M}(\tilde{t},\mathbf{x}) = -\frac{2G}{|\mathbf{x}|}\Lambda_{ij,kl}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\left[-\omega^2 e^{i\omega\tilde{t}_{\rm ret}(\mu)}e^{i\theta_{\rm tail}(\omega,\mu)}\left(1+GM|\omega|\pi\right)\right]Q_0^{ij}(\omega), \tag{204}$$

with

$$\theta_{\rm tail}(\omega,\mu) \equiv GM\omega\left(\log\frac{\omega^2}{\mu^2}+\gamma_E-\frac{11}{6}\right), \tag{205}$$

and we have absorbed the $1/\epsilon_{\rm IR}$ into a time redefinition, namely

$$t_{\rm bare} \to \tilde{t} - 2GM/\epsilon_{\rm IR}. \tag{206}$$

(204) is precisely (up to an unphysical rescaling of μ) the result previously obtained in [39] using traditional methods.

$\tilde{t}$ represents the time variable used by the experimentalists. This is in accord with our previous discussion, in the sense that one must choose a time origin for the signal, which is of course arbitrary. Notice that even after this shift there will still be a dependence upon the arbitrary reference scale μ_0 whose variation, $\mu_0 \to \lambda\mu_0$ can be cancelled out by a simultaneous transformation $\tilde{t}(\lambda\mu_0) \to \tilde{t}(\mu_0) + 2GM\log\lambda$. Hence the GW amplitude is independent of μ.

Phases arising from other multipole-mass interactions will also appear and, while the logarithm (and Euler constant γ_E) is universal, the rational numbers which appear will not be and thus the difference between these numbers will be physical, see for instance [40].

VII. PROBLEMS

1. In the text we states that we could eliminate terms which vanished by Einsteins equations in the bulk. Determine what field re-definitions are necessary to eliminate the

following possible two terms which are allowed by the symmetries

$$L = c_1 \int d\tau R(x(\tau) + c_2 \int d\tau R_{\mu\nu}(x(\tau) R^{\mu\nu}(x(\tau)). \tag{207}$$

For the first operator it is suggest that one consider a conformal transformation of the metric. In both cases one should consider shifts in the metric proportional to the diffeomorphism invariant quantity,

$$\int d\tau \delta^{(4)}(x - x(\tau))/\sqrt{g}. \tag{208}$$

Choose the coefficient of this shift to cancel the operator of interest. Notice that one must also transform the world line operator as well. This leads to the product of delta functions on the world line should be taken to be zero.

2. Suppose we would like to calculate the classical force between static colored objects in QCD. First show that at order g^4 call the diagrams vanish. Then argue that to all orders in perturbation theory there are no corrections. Now consider allowing for a finite size charge distribution. Write down all of the allowed operators and then calculate the leading order finite size correction to the force law.

3. Consider a theory of scalar-tensor theory gravity. In such a theory there will be a new set of worldline couplings that one can write down that can not be eliminated by field redefinitions. Use PN power counting to determine the order of the first finite size operator on the worldline.

4. Using the boost invariance of the effective action for a binary system in the post-Newtonian limit show that Noethers' theorem imples that the center of mass position $X^i \equiv \int d^3x x^i T_{00}$ is a constant of the motion. Then calculate the 1PN correction to X^i. To accomplish this calculate the one point function of g_{00} and follow the reasoning sub-section A. If we did not choose the center of mass position to vanish, what would it couple to in eq. (94)? See problem (6) below.

5. Use the KK variables to derive the 1PN potential. To accomplish this begin by showing

that the propagators for the various pieces of the metric are given by

$$\begin{aligned}
\langle \phi_{\vec p}(t_a)\phi_{\vec q}(t_b)\rangle &= -\frac{1}{8}(2\pi)^3\delta^{(3)}(\vec p+\vec q)\frac{i}{\vec p^2}\delta(t_a-t_b)\\
\langle A^i_{\vec p}(t_a)A^j_{\vec q}(t_b)\rangle &= \frac{1}{2}(2\pi)^3\delta^{(3)}(\vec p+\vec q)\frac{i\delta_{ij}}{\vec p^2}\delta(t_a-t_b)\\
\langle \gamma^{ij}_{\vec p}(t_a)\gamma^{kl}_{\vec q}(t_b)\rangle &= -(2\pi)^3\delta^{(3)}(\vec p+\vec q)\frac{iP^{ij,kl}}{\vec p^2}\delta(t_a-t_b)
\end{aligned} \tag{209}$$

where $P^{ij,kl} = \frac{1}{2}(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk} - 2\delta^{ij}\delta^{kl})$.

6. Show that after multipole expanding the world line action, one generates a coupling of the angular momentum of the binary system to the spin connection ω^{μ}_{ab}. First show that at linear order $\omega^0_{ij} = \frac{1}{2}\left(\partial_j \bar h_{0i} - (i \leftrightarrow j)\right)$. To do this start with the relation

$$\omega^{\alpha}_{ab} = (\nabla^{\alpha} e^{\mu}_a)e_{\mu b} \tag{210}$$

and use the relations (57) to write e^{μ}_a in term of $h_{\mu\nu}$.

7. Prove the result (162). To do so utilize (159) to solve for $\Omega_{\mu\nu}$. Show that it gives the correct non-relativistic limit where the action is $\frac{1}{2}I\omega^2$, where I is the moment of inertia $\vec S = I\vec\omega$.

Exercise 2.7: This problem has to do with the *gravitational* Van Der Waals interaction between strings. First let us show that the classical force between such objects vanishes, when we take the tension (mass per unit length) of the objects to be small compared to the Planck scale M_{pl}, where $G_N = 1/(32\pi M_{pl}^2)$. This approximation allows us to expand the metric around flat space, such that

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{h_{\mu\nu}}{M_{pl}}. \tag{211}$$

The action for the strings are given by

$$S = -\sum_i \tau_i \int d^4x \sqrt{g^i}\,\delta^{(2)}(x - x_i) \tag{212}$$

where τ_i and g_i are the tension and induced metric of the i'th string. For simplicity we will take the string to lie along one coordinate axis and we will choose the local space-time coordinates on the string to coincide with the global coordinate system. In this way the induced metric is just the value of the bulk metric on the string restricted to the $0, 1$ coordinates.

We expand the action using

$$\sqrt{g^i} \approx 1 + \frac{h_a^b}{2M_{pl}} + \frac{h_a^a h_b^b}{8M_{pl}^2} - \frac{h_a^b h_a^b}{4M_{pl}^2} + ... \tag{213}$$

where the indices are now summed only over the two dimensional sub-space. The classical force will then arise from the one graviton exchange diagram. Using the background field method discussed in this chapter, where the graviton propagator takes the form

$$D_{\mu\nu,\rho\sigma}(q) = -i\frac{\eta_{\mu\sigma}\eta_{\nu\rho} + \eta_{\nu\sigma}\eta_{\nu\rho} - \frac{2}{d-1}\eta_{\mu\nu}\eta_{\rho\sigma}}{\vec{q}^2} \tag{214}$$

show that the classical force vanishes.

To prove that the classical force vanishes, to all orders in the tension, one must solve the full einstein equations to account for the curving of the space. However, as was shown in [**?**], co-dimension two objects leave space uncurved. The net effect of the tension is to remove a deficit angle in the space-time. That is, the space is conical. Thus there is no classical force to all orders in the tension.

To find the first correction due to the tension on the force, we consider the analog of figure (**??**). Using the action (**??**) show that this diagram generates the potential per unit length

$$V = -\frac{\tau_1 \tau_2}{64\pi^3 R^2 M_{pl}^4} \tag{215}$$

where R is the distance between the strings.

Note that in gravity we have additional contributions beyond those in diagram (**??**) arising from graviton self interactions. For the discussion of the complete force including these effects see [**?**].

[1] The theory discussed in this chapter is based on the approach in, W. D. Goldberger and I. Z. Rothstein, Phys. Rev. D **73**, 104029 (2006) [arXiv:hep-th/0409156].

[2] For example see, M. Maggiore, "Gravitational Waves. Vol. 1: Theory and Experiments," Oxford University Press, October 2007. 572p.

[3] Luc Blanchet, Living Rev. Rel. **9**, 4 (2006).

[4] C. R. Galley, Class. Quant. Grav. **29**, 015010 (2012) [arXiv:1012.4488 [gr-qc]].

[5] This operator was written down in the context of scalar-tensor gravity in, T. Damour and G. Esposito-Farese, Phys. Rev. D **58**, 042001 (1998) [gr-qc/9803031].

[6] Y. Iwasaki, Lett. Nuovo Cim. **1S2**, 783 (1971) [Lett. Nuovo Cim. **1**, 783 (1971)].

[7] I. J. Muzinich and S. Vokos, Phys. Rev. D **52**, 3472 (1995) [hep-th/9501083].

[8] M. J. Duff, Phys. Rev. D **7**, 2317 (1973).

[9] B. Kol and M. Smolkin, Phys. Rev. D **77**, 064033 (2008) [arXiv:0712.2822 [hep-th]].

[10] R. Mertig, M. Bohm and A. Denner, Comput. Phys. Commun, 64:345, 1991.

[11] D. Neill and I. Z. Rothstein, arXiv:1304.7263 [hep-th].

[12] J. B. Gilmore and A. Ross, Phys. Rev. D **78**, 124021 (2008) [arXiv:0810.1328 [gr-qc]].

[13] S. Foffa and R. Sturani, Phys. Rev. D **84**, 044031 (2011) [arXiv:1104.1122 [gr-qc]].

[14] From a path integral point of view this glibly follows by a re-defintion of the coordinates just as in the case of field theory. For a rigorous proof in particle dynamics see T. Damour and G. Schaefer, J. Math. Phys. **32**, 127 (1991).

[15] See for example, "Quantum Theory of Many-Particle Systems", A.L. Fetter and J.D. Walecka, *Dover Books on Physics*

[16] P. D. D'Eath, Phys. Rev. D **11**, 1387 (1975), R. Kates, Phys. Rev. D **22**, 1853 (1980), T. Damour in "Gravitational Radiation", eds. N. Dreulle and T. Piran, (Les Houches 1982), North-Holland.

[17] S. Carlip, Phys. Rev. Lett. **88**, 241301 (2002) [gr-qc/0203001].

[18] A. A. Starobinskii and S. M. Churilov, Zh. Eksp. Teor. Fiz. **65**, 3 (1973).

[19] For a review on black hole perturbation theory see, M. Sasaki and H. Tagoshi, Living Rev. Rel. **6**, 6 (2003) [gr-qc/0306120].

[20] B. S. DeWitt, Phys. Rev. **162**, 1195 (1967).

[21] L. F. Abbott, Acta Phys. Polon. B **13**, 33 (1982).

[22] S. Endlich, A. Nicolis, R. A. Porto and J. Wang, arXiv:1211.6461 [hep-th].

[23] T. Damour and A. Nagar, Phys. Rev. D **80**, 084035 (2009) [arXiv:0906.0096 [gr-qc]], T. Binnington and E. Poisson, Phys. Rev. D **80**, 084018 (2009) [arXiv:0906.1366 [gr-qc]]. B. Kol and M. Smolkin, JHEP **1202**, 010 (2012) [arXiv:1110.3764 [hep-th]].

[24] J. Hartung, J. Steinhoff and G. Schafer, arXiv:1303.0666 [gr-qc].

[25] H. Fang and G. Lovelace, Phys.Rev. **D72** (2005) 124016,

[26] R. P. Geroch, J. Math. Phys. **11**, 1955 (1970). , R. P. Geroch, J. Math. Phys. **11**, 2580 (1970).

[27] A. Ross, Phys. Rev. D **85**, 125033 (2012) [arXiv:1202.4750 [gr-qc]].

[28] S. Weinberg, Phys. Rev. **140** , B516, (1965).

[29] W. D. Goldberger and A. Ross, Phys. Rev. D **81**, 124015 (2010) [arXiv:0912.4254 [gr-qc]].

[30] J. S. Schwinger, J. Math. Phys. **2**, 407 (1961). K. T. Mahanthappa, Phys. Rev. **126**, 329 (1962). L. V. Keldysh, Zh. Eksp. Teor. Fiz. **47**, 1515 (1964) [Sov. Phys. JETP **20**, 1018 (1965)]. For an exposition in the spirit of this chapter see, R. D. Jordan, Phys. Rev. D **33**, 444 (1986).

[31] The in-in formalism in the context of EFT can be found in, C. R. Galley and M. Tiglio, Phys. Rev. D **79**, 124027 (2009) [arXiv:0903.1122 [gr-qc]].

[32] R. Fujita, Prog. Theor. Phys. **128**, 971 (2012) [arXiv:1211.5535 [gr-qc]].

[33] W. D. Goldberger, A. Ross and I. Z. Rothstein, arXiv:1211.6095 [hep-th].

[34] W.L. Burke and K.S. Thorne, in *Relativity*, edited by M. Carmelli, S.I Fickler and L. Witten, pp. 209-228, Plenum, New York, 1970.

[35] R. A. Porto, Phys. Rev. D **73**, 104031 (2006) [gr-qc/0511061].

[36] A. J. Hanson and T. Regge, Annals Phys. **87**, 498 (1974).

[37] M. H. L. Pryce, Proc. Roy. Soc. Lond. A **195**, 62 (1948).

[38] R. A. Porto and I. Z. Rothstein, Phys. Rev. D **78**, 044012 (2008) [Erratum-ibid. D **81**, 029904 (2010)] [arXiv:0802.0720 [gr-qc]].

[39] L. Blanchet and G. Schafer, Class. Quant. Grav. **10**, 2699 (1993).

[40] R. A. Porto, A. Ross and I. Z. Rothstein, JCAP **1209**, 028 (2012) [arXiv:1203.2962 [gr-qc]].