BOUNDARY STATE FOR MAGNETIZED D9 BRANES AND ONE-LOOP CALCULATION

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We construct the boundary state describing magnetized D9 branes in $R^{3,1} \times T^6$ and we use it to compute the annulus and Möbius amplitudes. We derive from them, by using open/closed string duality, the number of Landau levels on the torus T^d.

1 Introduction

String theories are perfectly consistent theories in ten non-compact dimensions. However, in order for them to be consistent with particle phenomenology, six of the ten dimensions have to be compactified. The compactification procedure produces a number of fields called moduli whose vacuum expectation value cannot be fixed in perturbation theory if supersymmetry is preserved. Therefore a major problem that one has to solve in string theory is to find ways to fix the vacuum expectation values of those moduli in order to be able to compare with particle phenomenology.

Recently, magnetized D9 branes have been used to produce semirealistic models where the moduli are stabilized and the tadpoles are canceled out [a]. This has been done by mostly using the Born-Infeld action that encodes their properties.

In this paper we give a string description of magnetized D9 branes by determining the corresponding boundary state in $R^{3,1} \times T^6$ following closely the procedure outlined in Ref. [3]. We use it to provide a straightforward derivation of the one-loop amplitudes corresponding to the annulus and Möbius diagrams and generalizing the results of Ref. [4] to the case of an arbitrary NS-NS B_2-field and arbitrary wrappings.

In the next section we will construct the boundary state corresponding to a number of magnetized D9 branes and in the third section we will use it for computing one-loop amplitudes.

2 The boundary state for magnetized D9 branes in $R^{3,1} \times T^6$

In order to determine the boundary state of magnetized D9 branes on $R^{3,1} \times T^6$ we start from the open string channel by considering the action which describes the interaction of a string with in general two arbitrary abelian gauge fields $A^{(0)}, A^{(\pi)}$ acting respectively at the endpoints of the string $\sigma = 0, \pi$. In the following we will consider only the six compact directions omitting the four non-compact ones because the boundary state corresponding to them has been already determined (see the two reviews on the boundary state [5,6]). We will also omit to discuss the part of the boundary state corresponding to the world-sheet fermion degrees of freedom that can be found in Ref.s [5,6].

[a] See for instance Ref.s [1,2] and Ref.s therein.

The action of an open string in a closed toroidal string background interacting with two arbitrary abelian gauge fields with constant field strength is given by[b]:

$$S = -\frac{1}{4\pi\alpha'}\int d\tau \int_0^\pi d\sigma \left[G_{ij}\partial_\alpha X^i \partial_\beta X^j \eta^{\alpha\beta} - B_{ij}\epsilon^{\alpha\beta}\partial_\alpha X^i \partial_\beta X^j\right] + S_{boundary} \quad (1)$$

where $S_{boundary}$ is equal to:

$$S_{boundary} = -q_0 \int d\tau A_i^{(0)} \partial_\tau X^i|_{\sigma=0} - q_\pi \int d\tau A_i^{(\pi)} \partial_\tau X^i|_{\sigma=\pi} =$$

$$= \frac{q_0}{2}\int d\tau F_{ij}^{(0)} X^j \dot{X}^i|_{\sigma=0} + \frac{q_\pi}{2}\int d\tau F_{ij}^{(\pi)} X^j \dot{X}^i|_{\sigma=\pi} \ \ ; \quad i,j = 1\ldots\hat{d} \qquad (2)$$

where q_0 and q_π are the charges located at the two end-points and, for the sake of generality, we keep i and j to vary between 1 and $\hat{d}$ (for a D9 brane $\hat{d} = 6$). We have used the expression $A_i = -\frac{1}{2}F_{ij}X^j$ with a constant field strength. From the previous action one can write the equation of motion in the bulk given by:

$$\partial_\alpha [G_{ij}\partial^\alpha X^j] = 0 \Rightarrow \partial_\alpha[\partial^\alpha X^i] = 0 \qquad (3)$$

and the two boundary conditions at $\sigma = 0, \pi$:

$$\left[G_{ij}\partial_\sigma X^j + (B_{ij} - 2\pi\alpha' q_0 F_{ij}^{(0)})\partial_\tau X^j\right]_{\sigma=0} = 0 \qquad (4)$$

and

$$\left[G_{ij}\partial_\sigma X^j + (B_{ij} + 2\pi\alpha' q_\pi F_{ij}^{(\pi)})\partial_\tau X^j\right]_{\sigma=\pi} = 0 \ \ . \qquad (5)$$

The most general solution of the bulk equation in Eq. (3) is given by:

$$X^i(\sigma,\tau) = F^i(\tau+\sigma) + G^i(\tau-\sigma) \qquad (6)$$

with $F^i(\tau+\sigma)$ and $G^i(\tau-\sigma)$ arbitrary functions. By inserting Eq. (6) in the boundary conditions in Eq.s (4) and (5) we get:

$$\partial_\tau G^i(\tau) = (R_0)^i{}_j \partial_\tau F^j(\tau) \ \ ; \ \ \partial_\tau F^i(\tau+\pi) = (R_\pi^{-1}R_0)^i{}_j \partial_\tau F^j(\tau-\pi) \qquad (7)$$

where

$$(R_0)^i{}_j = [(1-\mathcal{B}_0)^{-1}(1+\mathcal{B}_0)]^i{}_j \ \ ; \ \ (R_\pi)^i{}_j = [(1-\mathcal{B}_\pi)^{-1}(1+\mathcal{B}_\pi)]^i{}_j \qquad (8)$$

and

$$\mathcal{B}^i_{0\ j} = G^{ik}(B_{kj} - 2\pi\alpha' q_0 F_{kj}^{(0)}) \ \ ; \ \ \mathcal{B}^i_{\pi\ j} = G^{ik}(B_{kj} + 2\pi\alpha' q_\pi F_{kj}^{(\pi)}) \ \ . \qquad (9)$$

Notice that the matrices R_0 and R_π and therefore also the product $R \equiv R_\pi^{-1}R_0$ are orthogonal matrices, i.e. $R^T_{(0,\pi)} = R^{-1}_{(0,\pi)}$, $R^T = R^{-1}$. An orthogonal matrix can always be brought into a diagonal form with eigenvalues λ such that $|\lambda| = 1$. Moreover, being orthogonal, such a matrix is unitary and real and therefore in a space with an

[b]Hereafter we closely follow Ref.[7]

even number of dimensions ($\hat{d}$ is even) for each eigenvalue λ it exists also an eigenvalue λ^*. Therefore one can always bring the orthogonal matrix R in the following form:

$$R^i{}_j = e^{-2\pi i\nu_i}\delta^i{}_j \tag{10}$$

where $\nu_{2a} = -\nu_{2a-1}$ for $a = 1\ldots\frac{\hat{d}}{2}$. R is an orthogonal matrix of dimensionality $\hat{d}$ that can be diagonalized with eigenvectors and eigenvalues satisfying the following equations:

$$R^i{}_j C_a^j = e^{-2i\pi\nu_a} C_a^i \;\; ; \;\; R^i{}_j C_a^{*j} = e^{2i\pi\nu_a} C_a^{*i} \;\; ; \;\; a = 1\ldots\frac{\hat{d}}{2} \tag{11}$$

with [c] $0 \leq \nu_a \leq 1/2$. The eigenvectors C and C^* are orthonormal satisfying the conditions:

$$C_a^{*i} G_{ij} C_b^j = \delta_{ab} \;\; ; \;\; C_a^i G_{ij} C_b^j = C_a^{*i} G_{ij} C_b^{*j} = 0 \;\; . \tag{12}$$

The quantities ν_a may be zero. This happens when $\det(q_0 F^{(0)} + q_\pi F^{(\pi)})_{ij} = 0$ and in this case C and C^* may be taken real. In the following we will assume that the previous matrix has nonzero entries only along the directions $1\ldots d$, with $d \leq \hat{d}$ and even, and that the determinant of its not null submatrix is different from zero, i.e. $\det(q_0 F^{(0)} + q_\pi F^{(\pi)})_{AB} \neq 0$ for $0 \leq A, B \leq d$. All other entries are vanishing. This means that R has the form given in Eq. (10) for $i, j = 1\ldots d$, while the remaining diagonal elements are equal to 1 ($\nu_{2a} = \nu_{2a-1} = 0$ for $\frac{d}{2} < a \leq \frac{\hat{d}}{2}$).

The equation that the boundary state must satisfy can be derived from Eq. (4) with the substitution $\sigma \leftrightarrow \tau$. In so doing one gets:

$$\left[G_{ij}\partial_\tau X^j + (B_{ij} - 2\pi\alpha' q F_{ij})\partial_\sigma X^j\right]_{\tau=0} |B\rangle = 0 \;\; . \tag{13}$$

Inserting in the previous equation the mode expansion for a closed string:

$$\begin{aligned} X^i(\tau,\sigma) &= x^i + \sqrt{\alpha'}\left[2\hat{m}^i\sigma + 2G^{ij}\left(\hat{n}_j - B_{jk}\hat{m}^k\right)\tau\right] + \\ &\quad + i\frac{\sqrt{2\alpha'}}{2}\sum_{n\neq 0}\frac{1}{n}\left[\alpha_n^i e^{-2in(\tau-\sigma)} + \tilde{\alpha}_n^i e^{-2in(\tau+\sigma)}\right]. \end{aligned} \tag{14}$$

gives the following conditions:

$$(\hat{n}_i - 2\pi\alpha' q F_{ij}\hat{m}^j)|B\rangle = 0 \tag{15}$$

and

$$\left(\mathcal{E}_{ij}\alpha_n^j + \mathcal{E}_{ij}^T\tilde{\alpha}_{-n}^j\right)|B\rangle = 0 \;\; ; \;\; \mathcal{E}_{ij} = G_{ij} - B_{ij} + 2\pi\alpha' q F_{ij} \;\; , \tag{16}$$

being $qF = -q_\pi F_\pi$ on the boundary in $\sigma = \pi$ and $qF = q_0 F_0$ on the boundary in $\sigma = 0$. In Eq.s (14) and (15) we have inserted the hat to remember that $\hat{n}$ and $\hat{m}$ are operators.

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[c] In principle ν_a varies in the interval $0 \leq \nu_a < 1$ but we can restrict ourselves to the smaller interval $0 \leq \nu_a \leq 1/2$ because of the freedom that we have in defining C and C^*.

The boundary state satisfying Eq.s (15) and (16) is given by[3]:

$$|B\rangle_{0,\pi} = C_{0,\pi} N_{0,\pi} W_{0,\pi} \prod_{n=1}^{\infty} \left[e^{-\frac{1}{n}\alpha^i_{-n} G_{ik} (\mathcal{E}^{-1})^{kh} (\mathcal{E}^T)_{hj} \tilde{\alpha}^j_{-n}} \right] \times$$

$$\times \sum_{r_i, s^j \in \mathbb{Z}} \delta_{n_i \mp 2\pi\alpha' q_{0,\pi} F^{(0,\pi)}_{ij} m^j} |n_i = \frac{r_i}{W^i_{0,\pi}}\rangle |m^j = W^j_{0,\pi} s^j\rangle |0_{\alpha,\tilde{\alpha}}\rangle \tag{17}$$

where the plus sign refers to the boundary defined at $\sigma = \pi$ while the minus sign to the other one. W_0^i (W_π^i) is the wrapping number of the brane at $\sigma = 0$ $(\sigma = \pi)$ along the ith direction and $W_{0,\pi} = \prod_{i=1}^{\hat{d}} W^i_{0,\pi}$. Furthermore, $N_{0,\pi}$ is the number of D9 branes.

The gauge fields $q_{0,\pi} F^{(0,\pi)}_{ij}$ cannot be arbitrary because the following quantity corresponding to the first Chern class, given by:

$$(c_1)_{ij} = \frac{1}{2\pi} \int_{(i,j)} \frac{dx^i \wedge dx^j}{2} q_{0,\pi} F^{(0,\pi)}_{ij} = \frac{1}{2\pi} \cdot (2\pi\sqrt{\alpha'})^2 q_{0,\pi} F^{(0,\pi)}_{ij} W^i_{0,\pi} W^j_{0,\pi} \equiv f^{(0,\pi)}_{ij}, \tag{18}$$

has to be an integer. It is important to stress here that the coordinate of the brane along the ith compact direction is assumed to vary in the interval $(0, 2\pi\sqrt{\alpha'}W^i)$. $C_{0,\pi}$ is a normalization constant that in general can be determined by computing the annulus diagram both in the open and closed string channel and by comparing the two results. In some particular cases this constant is known and will allow us to compute the number of Landau levels by performing a modular transformation on the annulus amplitude computed in the closed string channel. The states corresponding to the zero modes are normalized[d] for any compact direction i as follows:

$$\langle n_i | (n')_i \rangle = (2\pi\sqrt{\alpha'})^{1/2} \delta_{n_i,(n')_i} \;\; ; \;\; \langle m^i | (m')^i \rangle = (2\pi\sqrt{\alpha'})^{1/2} \delta_{m^i,(m')^i} \tag{19}$$

3 One-loop amplitudes

In this section we use the previously constructed boundary state for computing the boundary-boundary interaction. We need the closed string propagator, taken to be equal to:

$$D = \frac{\alpha'\pi}{2} \delta_{L_0 - \tilde{L}_0, 0} \int_0^\infty dt \; e^{-\pi t (L_0 + \tilde{L}_0)} \tag{20}$$

where

$$L_0 + \tilde{L}_0 = N + \tilde{N} + \frac{1}{2} \left[G_{ij} \hat{m}^i \hat{m}^j + (\hat{n}_i - B_{ik}\hat{m}^k) G^{ij} (\hat{n}_j - B_{jh}\hat{m}^h) \right] \tag{21}$$

and

$$N = \sum_{n=1}^{\infty} G_{ij} \alpha^i_{-n} \alpha^j_n \;\; ; \;\; \tilde{N} = \sum_{n=1}^{\infty} G_{ij} \tilde{\alpha}^i_{-n} \tilde{\alpha}^j_n \;\; . \tag{22}$$

Here we have omitted to write the ghost contribution.

[d] The origin of the factor $(2\pi\sqrt{\alpha'})^{1/2}$ can be traced back to Eq. (2,19) of Ref.[8] and Eq. (6.28) of Ref.[6] with $\Phi = 2\pi\sqrt{\alpha'}$.

We are now ready to compute the annulus diagram that is given by $\langle B, f^{(0)}|D|B, f^{(\pi)}\rangle$. Taking into account the δ-function in Eq. (20) we can trade $\tilde{N}$ with N. The contribution of the zero modes can be easily computed and one gets:

$$\sum_{r_i, s^i, r'_i, s^{j'} \in \mathbb{Z}} \langle n_i = \frac{r_i}{W_0^i}|\langle m^j = W_0^j s^j|\delta_{n_i - 2\pi\alpha' q_0 F_{ij}^{(0)} m^j}\delta_{n'_i + 2\pi\alpha' q_\pi F_{ij}^{(\pi)} m^{j'}} \times$$

$$\times e^{-\frac{\pi}{2}t\left[G_{ij}\hat{m}^i\hat{m}^j + (\hat{n}_i - B_{ik}\hat{m}^k)G^{ij}(\hat{n}_j - B_{jh}\hat{m}^h)\right]}|(n')_i = \frac{r'_i}{W_\pi^i}\rangle|(m')^j = W_\pi^j s^{j'}\rangle$$

$$= \sum_{s^i, s^{j'} \in \mathbb{Z}} \langle \frac{f_{ij}^{(0)} s^j}{W_0^i}|\langle W_0^j s^j|e^{-\frac{\pi}{2}t\left[G_{ij}\hat{m}^i\hat{m}^j + (\hat{n}_i - B_{ik}\hat{m}^k)G^{ij}(\hat{n}_j - B_{jh}\hat{m}^h)\right]}|-\frac{f_{ij}^{(\pi)} s^{j'}}{W_\pi^i}\rangle|W_\pi^j s^{j'}\rangle$$

$$= (2\pi\sqrt{\alpha'})^{\hat{d}} \sum_{s^i, s^{j'} \in \mathbb{Z}} \delta_{W_0^j s^j - W_\pi^j (s')^j}\delta_{f_{ij}^{(0)} s^j / W_0^i + f_{ij}^{(\pi)} (s')^j / W_\pi^i} \times$$

$$\times\, e^{-\frac{\pi}{2}t\left[G_{ij}W_0^i s^i W_0^j s^j + (\frac{f_{ik}^{(0)} s^k}{W_0^i} - B_{ik}W_0^k s^k)G^{ij}(\frac{f_{jh}^{(0)} s^h}{W_0^j} - B_{jh}W_0^h s^h)\right]} \tag{23}$$

where $f_{ij}^{(0,\pi)}$ is defined in Eq. (18) and we have used Eq.s (19). In the case of the bosonic string (superstring) $\hat{d} = 22\ \ (6)$.

It is easy to see that the first δ-function can be satisfied only if

$$s^j = \frac{W_{lcm}^j}{W_0^j}u^j \ \ ; \ \ (s')^j = \frac{W_{lcm}^j}{W_\pi^j}u^j \tag{24}$$

where u^j is an arbitrary integer and W_{lcm}^i is the least common multiple of W_0^i and W_π^i. By inserting the previous values in the other δ-function one can write it as $\delta_{(q_0 F_{ij}^{(0)} + q_\pi F_{ij}^{(\pi)})W_{lcm}^j u^j}$ getting

$$(2\pi\sqrt{\alpha'})^{\hat{d}} \sum_{u^j \in \mathbb{Z}} \delta_{(q_0 F_{ij}^{(0)} + q_\pi F_{ij}^{(\pi)})W_{lcm}^j u^j}\, e^{-\frac{\pi}{2}t u^i W_{lcm}^i \mathcal{G}_{ij} u^j W_{lcm}^j} \tag{25}$$

in terms of the open string metric $\mathcal{G}_{ij}$ defined by:

$$\mathcal{G}_{ij} \equiv G_{ij} - \mathcal{B}_{ik}G^{kh}\mathcal{B}_{hj} = \mathcal{E}_{ik}^T G^{kh}\mathcal{E}_{hj} \ \ ; \ \ \mathcal{B}_{ij} \equiv B_{ij} - 2\pi\alpha' q_0 F_{ij}^{(0)} \ \ ; \ \ \mathcal{E}_{ij} \equiv G_{ij} - \mathcal{B}_{ij} \ . \tag{26}$$

The contribution of the non-zero modes is given by:

$$\prod_{n=1}^{\infty} \frac{1}{\det\left[\delta^i{}_j - \left(\mathcal{E}_0^{-1}\mathcal{E}_0^T\right)^i{}_h G^{hk}\left(\mathcal{E}_\pi \mathcal{E}_\pi^{-1T}\right)_{kj} e^{-2\pi n t}\right]} \tag{27}$$

where the following commutation relations have been used:

$$[\alpha_n^i, \alpha_m^j] = n\delta_{n+m;0}G^{ij} \ \ ; \ \ [\tilde{\alpha}_n^i, \tilde{\alpha}_m^j] = n\delta_{n+m;0}G^{ij} \ . \tag{28}$$

Putting all factors together and adding the contribution of the four non-compact directions and of the ghosts lead, in the case of the bosonic string, to the following expression[3] ($q \equiv e^{-\pi t}$):

$$\langle B, f^{(0)}|D|B, f^{(\pi)}\rangle = C_0 N_0 W_0 C_\pi N_\pi W_\pi (2\pi\sqrt{\alpha'})^{\hat{d}} \frac{\alpha' \pi}{2} V_4 \sum_{n^j \in \mathbb{Z}} \delta_{(q_0 F^{(0)}_{ij} + q_\pi F^{(\pi)}_{ij}) W^j_{lcm} u^j} \int_0^\infty dt \times$$

$$\times e^{-\frac{\pi}{2} t u^i W^i_{lcm} \mathcal{G}_{ij} u^j W^j_{lcm}} \times \frac{q^{-\hat{d}/12}}{(f_1(q))^2} \prod_{n=1}^\infty \frac{1}{\det\left[\delta^i{}_j - \left(\mathcal{E}_0^{-1}\mathcal{E}_0^T\right)^i{}_h G^{hk} \left(\mathcal{E}_\pi \mathcal{E}_\pi^{-1T}\right)_{kj} e^{-2\pi n t}\right]} \; . \quad (29)$$

Eq. (29) can be easily generalized to the case of superstring becoming:

$$\langle B, f^{(0)}|D|B, f^{(\pi)}\rangle = C_0 C_\pi (2\pi\sqrt{\alpha'})^6 \frac{\alpha' \pi}{2} V_4 \sum_{u^j \in \mathbb{Z}} \delta_{(q_0 F^{(0)}_{ij} + q_\pi F^{(\pi)}_{ij}) W^j_{lcm} u^j} \times$$

$$N_0 N_\pi W_0 W_\pi \int_0^\infty dt e^{-\frac{\pi}{2} t u^i W^i_{lcm} \mathcal{G}_{ij} u^j W^j_{lcm}} \times$$

$$\frac{1}{2}\left\{ \frac{1}{q} \left[\prod_{n=1}^\infty \frac{\det\left[\delta^i{}_j + \left(\mathcal{E}_0^{-1}\mathcal{E}_0^T\right)^i{}_h G^{hk} \left(\mathcal{E}_\pi \mathcal{E}_\pi^{-1T}\right)_{kj} q^{2n-1}\right] (1+q^{2n-1})^2}{\det\left[\delta^i{}_j - \left(\mathcal{E}_0^{-1}\mathcal{E}_0^T\right)^i{}_h G^{hk} \left(\mathcal{E}_\pi \mathcal{E}_\pi^{-1T}\right)_{kj} q^{2n}\right] (1-q^{2n})^2} + \right.\right.$$

$$\left. - \prod_{n=1}^\infty \frac{\det\left[\delta^i{}_j - \left(\mathcal{E}_0^{-1}\mathcal{E}_0^T\right)^i{}_h G^{hk} \left(\mathcal{E}_\pi \mathcal{E}_\pi^{-1T}\right)_{kj} q^{2n-1}\right] (1-q^{2n-1})^2}{\det\left[\delta^i{}_j - \left(\mathcal{E}_0^{-1}\mathcal{E}_0^T\right)^i{}_h G^{hk} \left(\mathcal{E}_\pi \mathcal{E}_\pi^{-1T}\right)_{kj} q^{2n}\right] (1-q^{2n})^2} \right] +$$

$$\left. - \left[2^{4-\hat{d}/2} \prod_{a=1}^{\hat{d}/2} (2\cos\pi\nu_a) \right] \prod_{n=1}^\infty \frac{\det\left[\delta^i{}_j + \left(\mathcal{E}_0^{-1}\mathcal{E}_0^T\right)^i{}_h G^{hk} \left(\mathcal{E}_\pi \mathcal{E}_\pi^{-1T}\right)_{kj} q^{2n}\right] (1+q^{2n})^2}{\det\left[\delta^i{}_j - \left(\mathcal{E}_0^{-1}\mathcal{E}_0^T\right)^i{}_h G^{hk} \left(\mathcal{E}_\pi \mathcal{E}_\pi^{-1T}\right)_{kj} q^{2n}\right] (1-q^{2n})^2} \right\} \; . \quad (30)$$

By using Eq. (8) together with the following equations:

$$(1 - \mathcal{B}_0)^i{}_j = G^{ik} \mathcal{E}^{(0)}_{kj} \equiv (\mathcal{E}_0)^i{}_j \;\; ; \;\; (1 - \mathcal{B}_\pi)^i{}_j = G^{ik} \mathcal{E}^{(\pi)}_{kj} \equiv (\mathcal{E}_\pi)^i{}_j \qquad (31)$$

and the fact that under a determinant we can change the order of the two matrices in the last line of Eq. (29), we can rewrite Eq. (30) as follows:

$$\langle B, f^{(0)}|D|B, f^{(\pi)}\rangle = C_0 C_\pi (2\pi\sqrt{\alpha'})^6 \frac{\alpha' \pi}{2} V_4 \sum_{u^j \in \mathbb{Z}} \delta_{(q_0 F^{(0)}_{ij} + q_\pi F^{(\pi)}_{ij}) W^j_{lcm} u^j} \times$$

$$\times N_0 N_\pi W_0 W_\pi \int_0^\infty dt \; e^{-\frac{\pi}{2} t u^i W^i_{lcm} \mathcal{G}_{ij} u^j W^j_{lcm}} \times$$

$$\frac{1}{2}\left\{ \frac{1}{q} \left[\prod_{n=1}^\infty \frac{\det\left[\delta^i{}_j + R^i{}_j q^{2n-1}\right] (1+q^{2n+1})^2}{\det\left[\delta^i{}_j - R^i{}_j q^{2n}\right] (1-q^{2n})^2} - \prod_{n=1}^\infty \frac{\det\left[\delta^i{}_j - R^i{}_j q^{2n-1}\right] (1-q^{2n+1})^2}{\det\left[\delta^i{}_j - R^i{}_j q^{2n}\right] (1-q^{2n})^2} \right] + \right.$$

$$- \left[2^{4-\hat{d}/2} \prod_{a=1}^{\hat{d}/2} (2\cos \pi\nu_a) \right] \prod_{n=1}^{\infty} \frac{\det \left[\delta^i{}_j + R^i{}_j q^{2n} \right] (1-q^{2n+1})^2}{\det \left[\delta^i{}_j - R^i{}_j q^{2n} \right] (1-q^{2n})^2} \Bigg\} \tag{32}$$

where $R = R_\pi^{-1} R_0$ and the matrices R_0 and R_π are the ones introduced in Eq. (8). The last two lines of the previous equation can be written as follows[e]:

$$\frac{1}{2} \left\{ \prod_{a=1}^{d/2} \left[\frac{\Theta_3(\nu_a|it)}{\Theta_1(\nu_a|it)} \right] \left(\frac{f_3(q)}{f_1(q)} \right)^{8-d} - \prod_{a=1}^{d/2} \left[\frac{\Theta_4(\nu_a|it)}{\Theta_1(\nu_a|it)} \right] \left(\frac{f_4(q)}{f_1(q)} \right)^{8-d} + \right.$$

$$\left. - \prod_{a=1}^{d/2} \left[\frac{\Theta_2(\nu_a|it)}{\Theta_1(\nu_a|it)} \right] \left(\frac{f_2(q)}{f_1(q)} \right)^{8-d} \right\} \prod_{a=1}^{d/2} (-2\sin \pi\nu_a) \ . \tag{33}$$

This implies that Eq. (32) becomes:

$$\langle B, f^{(0)}|D|B, f^{(\pi)}\rangle = C_0 C_\pi (2\pi\sqrt{\alpha'})^6 \frac{\alpha'\pi}{2} V_4 \sum_{u^j \in \mathbb{Z}} \delta_{(q_0 F^{(0)}_{ij} + q_\pi F^{(\pi)}_{ij}) W^j_{lcm} u^j} \times$$

$$\times N_0 N_\pi W_0 W_\pi \int_0^\infty dt \sum_{u^i, u^j} e^{-\frac{\pi}{2} t u^i W^i_{lcm} \mathcal{G}_{ij} u^j W^j_{lcm}} \prod_{a=1}^{d/2} (-2\sin \pi\nu_a) \times$$

$$\frac{1}{2} \left\{ \prod_{a=1}^{d/2} \left[\frac{\Theta_3(\nu_a|it)}{\Theta_1(\nu_a|it)} \right] \left(\frac{f_3(q)}{f_1(q)} \right)^{8-d} - \prod_{a=1}^{d/2} \left[\frac{\Theta_4(\nu_a|it)}{\Theta_1(\nu_a|it)} \right] \left(\frac{f_4(q)}{f_1(q)} \right)^{8-d} + \right.$$

$$\left. - \prod_{a=1}^{d/2} \left[\frac{\Theta_2(\nu_a|it)}{\Theta_1(\nu_a|it)} \right] \left(\frac{f_2(q)}{f_1(q)} \right)^{8-d} \right\} \ . \tag{34}$$

It is straightforward to compute Eq. (34) for $\hat{d} = d = 6$ and $d = 0$. In the first case we can use the following equation:

$$\begin{aligned} 1 - R \equiv 1 - R_\pi^{-1} R_0 = 1 - (1+\mathcal{B}_\pi)^{-1}(1-\mathcal{B}_\pi)(1-\mathcal{B}_0)^{-1}(1+\mathcal{B}_0) &= \\ = (1+\mathcal{B}_\pi)^{-1} \left[(1+\mathcal{B}_\pi)(1-\mathcal{B}_0) - (1-\mathcal{B}_\pi)(1+\mathcal{B}_0) \right] (1-\mathcal{B}_0)^{-1} &= \\ = (1+\mathcal{B}_\pi)^{-1} G^{-1} (4\pi\alpha')(q_\pi F_\pi + q_0 F_0)(1-\mathcal{B}_0)^{-1} & \end{aligned} \tag{35}$$

together with

$$\sqrt{\det(1 - R_\pi^{-1} R_0)} = \prod_{a=1}^{\hat{d}/2} (2\sin \pi\nu_a) \tag{36}$$

in order to rewrite ($0 \leq \nu_a \leq 1/2$)

$$\prod_{a=1}^{\hat{d}/2} (-2\sin \pi\nu_a) = \frac{(-1)^{\hat{d}/2} \sqrt{\det G_{ij}} \sqrt{\det[4\pi\alpha'(q_0 F^{(0)} + q_\pi F^{(\pi)})_{ij}]}}{\sqrt{\det(G_{ij} + B_{ij} + 2\pi\alpha' q_\pi F^{(\pi)}_{ij})} \sqrt{\det(G_{ij} + B_{ij} - 2\pi\alpha' q_0 F^{(0)}_{ij})}} \ . \tag{37}$$

[e] We use the definition of the Θ-functions given in App. A of Ref [9] where References to previous papers dealing with strings interacting with gauge fields with constant field strength, can also be found.

It is easy to convince oneself that, for $d = \hat{d} = 6$, one gets:

$$C_0 = \frac{T_9}{2}\sqrt{\det(G_{ij} + B_{ij} - 2\pi\alpha' q_0 F_{ij}^{(0)})}/(\det G_{ij})^{1/4} \;\; ; \;\; T_9 = \frac{\sqrt{\pi}}{(2\pi\sqrt{\alpha'})^6} \quad (38)$$

and

$$C_\pi = \frac{T_9}{2}\sqrt{\det(G_{ij} + B_{ij} + 2\pi\alpha' q_\pi F_{ij}^{(\pi)})}/(\det G_{ij})^{1/4} \;\; . \quad (39)$$

In fact the first two factors in the two previous equations are precisely those that one also gets in non-compact space [5,6]. The last factor in the denominator is instead peculiar of a compact space and is already present in the case of a single direction compactified on a circle of radius R as one can immediately check. After inserting Eq.s (38) and (39) in Eq. (34) we get ($\hat{d} = 6$):

$$\langle B, f^{(0)}|D|B, f^{(\pi)}\rangle^{d=6} = -\frac{V_4 N_0 N_\pi}{(8\pi^2\alpha')^2} N_{LL}\int_0^\infty dt \frac{1}{2}\left\{\prod_{a=1}^{3}\frac{\Theta_3(\nu_a|it)}{\Theta_1(\nu_a|it)}\left(\frac{f_3(q)}{f_1(q)}\right)^2 + \right.$$

$$\left. -\prod_{a=1}^{3}\left[\frac{\Theta_4(\nu_a|it)}{\Theta_1(\nu_a|it)}\right]\left(\frac{f_4(q)}{f_1(q)}\right)^2 - \prod_{a=1}^{3}\left[\frac{\Theta_2(\nu_a|it)}{\Theta_1(\nu_a|it)}\right]\left(\frac{f_2(q)}{f_1(q)}\right)^2\right\} \quad (40)$$

where N_{LL} is the number of Landau levels, given by [4]:

$$N_{LL} = (2\pi\sqrt{\alpha'})^6 W_0 W_\pi \sqrt{\det\left(\frac{q_\pi F^{(\pi)} + q_0 F^{(0)}}{2\pi}\right)} \quad (41)$$

and we have used the relation:

$$\left(\frac{T_9}{2}\right)^2 (2\pi\sqrt{\alpha'})^6 \frac{\alpha'\pi}{2} = \frac{1}{8(8\pi^2\alpha')^2} \;\; . \quad (42)$$

Eq. (40) becomes in the open string channel ($\tau = 1/t, k = e^{-\pi\tau}$):

$$\langle B, f^{(0)}|D|B, f^{(\pi)}\rangle^{d=6} = \frac{iV_4 N_0 N_\pi N_{LL}}{(8\pi^2\alpha')^2}\int_0^\infty \frac{d\tau}{\tau^3}\frac{1}{2}\left[\left(\frac{f_3(k)}{f_1(k)}\right)^2 \prod_{a=1}^{3}\frac{\Theta_3(i\nu_a\tau|i\tau)}{\Theta_1(i\nu_a\tau|i\tau)} + \right.$$

$$\left. -\left(\frac{f_4(k)}{f_1(k)}\right)^2 \prod_{a=1}^{3}\frac{\Theta_4(i\nu_a\tau|i\tau)}{\Theta_1(i\nu_a\tau|i\tau)} - \left(\frac{f_2(k)}{f_1(k)}\right)^2 \prod_{a=1}^{3}\frac{\Theta_2(i\nu_a\tau|i\tau)}{\Theta_1(i\nu_a\tau|i\tau)}\right] \;\; . \quad (43)$$

Also the case $d = 0$ is easy to treat. In fact in this case, by using Eq.s (38) and (39) (that are still valid) with $(q_0 F_{ij}^{(0)} + q_\pi F^{(\pi)})_{ij} = 0$, it is easy to show that Eq. (34) reads as:

$$\langle B, f^{(0)}|D|B, f^{(\pi)}\rangle^{d=0} = \frac{V_4 N_0 N_\pi W_0 W_\pi}{(8\pi^2\alpha')^2}\int_0^\infty dt \sum_{u^i, u^j} e^{-\frac{\pi}{2}t u^i W^i_{mcm}\mathcal{G}_{ij}u^j W^j_{mcm}} \times$$

$$\left[\det\left(\frac{\mathcal{G}_{ij}}{2}\right)\right]^{1/2}\frac{1}{2}\left[\left(\frac{f_3(q)}{f_1(q)}\right)^8-\left(\frac{f_4(q)}{f_1(q)}\right)^8-\left(\frac{f_2(q)}{f_1(q)}\right)^8\right] \tag{44}$$

where we have used the relation:

$$\left[\det\left(\frac{\mathcal{G}_{ij}}{2}\right)\right]^{1/2}=\frac{\det(G+B-2\pi\alpha' q_0 F^{(0)})_{ij}}{8\sqrt{\det G_{ij}}} \quad . \tag{45}$$

In the open string channel Eq. (44) becomes:

$$\langle B, f^{(0)}|D|B, f^{(\pi)}\rangle^{d=0}=\frac{V_4 N_0 N_\pi}{(8\pi^2\alpha')^2}W_{GCD}\int_0^\infty\frac{d\tau}{\tau^3}\sum_{u\in\mathbb{Z}^{6-d}}e^{-2\pi\tau\frac{u_i}{W^i_{lcm}}\mathcal{G}^{ij}\frac{u_j}{W^j_{lcm}}}\times$$

$$\times\frac{1}{2}\left[\left(\frac{f_3(k)}{f_1(k)}\right)^8-\left(\frac{f_4(k)}{f_1(k)}\right)^8-\left(\frac{f_2(k)}{f_1(k)}\right)^8\right] \tag{46}$$

where W^i_{GCD} is the greatest common divisor of W^i_0 and W^i_π[f].

The intermediate case is more difficult to treat and we will not derive it in detail, but we will only give the final expression:

$$\langle B, f^{(0)}|D|B, f^{(\pi)}\rangle=\frac{V_4 N_0 N_\pi}{(8\pi^2\alpha')^2}W^{>d}_{GCD}\int_0^\infty dt\, N^{(d)}_{LL}(-)^{d/2}\left[\det\left(\frac{W^i_{lcm}\mathcal{G}^{>d}_{ij}W^j_{lcm}}{2}\right)\right]^{1/2}\times$$

$$\sum_{u\in\mathbb{Z}^{6-d}}e^{-\frac{\pi t}{2}W^i_{lcm}u^i\mathcal{G}^{>d}_{ij}u^jW^j_{lcm}}\frac{1}{2}\left[\left(\frac{f_3(q)}{f_1(q)}\right)^{8-d}\prod_{a=1}^{d/2}\frac{\Theta_3(\nu_a|i\,t)}{\Theta_1(\nu_a|i\,t)}+\right.$$

$$\left.-\left(\frac{f_4(q)}{f_1(q)}\right)^{8-d}\prod_{a=1}^{d/2}\frac{\Theta_4(\nu_a|i\,t)}{\Theta_1(\nu_a|i\,t)}-\left(\frac{f_2(q)}{f_1(q)}\right)^{8-d}\prod_{a=1}^{d/2}\frac{\Theta_2(\nu_a|i\,t)}{\Theta_1(\nu_a|i\,t)}\right] \tag{47}$$

that becomes in the open string channel:

$$\begin{aligned}\langle B, f^{(0)}|D|B, f^{(\pi)}\rangle \;=\; &\frac{V_4 N_0 N_\pi}{(8\pi^2\alpha')^2}W^{>d}_{GCD}\int_0^\infty\frac{d\tau}{\tau^3}N^{(d)}_{LL}(-i)^{d/2}\sum_{u\in\mathbb{Z}^{6-d}}e^{-2\pi\tau\frac{u_i}{W^i_{lcm}}\mathcal{G}^{>d ij}\frac{u_j}{W^j_{lcm}}}\\ &\times\frac{1}{2}\left[\left(\frac{f_3(k)}{f_1(k)}\right)^{8-d}\prod_{a=1}^{d/2}\frac{\Theta_3(i\nu_a\tau|i\tau)}{\Theta_1(i\nu_a\tau|i\tau)}-\left(\frac{f_4(k)}{f_1(k)}\right)^{8-d}\prod_{a=1}^{d/2}\frac{\Theta_4(i\nu_a\tau|i\tau)}{\Theta_1(i\nu_a\tau|i\tau)}\right.\\ &-\left.\quad\left(\frac{f_2(k)}{f_1(k)}\right)^{8-d}\prod_{a=1}^{d/2}\frac{\Theta_2(i\nu_a\tau|i\tau)}{\Theta_1(i\nu_a\tau|i\tau)}\right]\end{aligned} \tag{48}$$

[f] Remember that $W_0 W_\pi = W_{lcm}W_{GCD}$.

where by the upper index $(> d)$ we mean that the indices i, j run in the interval $d < i, j \leq \hat{d}$ and now the number of Landau levels is given by:

$$N_{LL}^{(d)} = (2\pi\sqrt{\alpha'})^d W_0^d W_\pi^d \sqrt{\det\left(\frac{(q_\pi F^{(\pi)} + q_0 F^{(0)})_{ij}}{2\pi}\right)^{(d)}} \tag{49}$$

with $W_{0,\pi}^d \equiv \prod_{i=1}^d W_{0,\pi}^i$. The upper index d of the matrix in the last term indicates that we must compute the determinant of the $d \times d$ submatrix whose entries are non-vanishing. Eq. (49) generalizes to the torus T^d the result obtained in Ref. [10] for the torus $T2$.

In the last part of this paper, by comparing Eq.s (34) and (47), we are going to determine the normalization of the boundary state in the general case assuming, as we have already done, that the matrix $\det(q_0 F^{(0)} + q_\pi F^{(\pi)})_{AB} \neq 0$ only for $1 \leq A, B \leq d$. By comparing Eq.s (34) and (47) we get:

$$2^{d/2} N_{LL}^{(d)} \left(\det \mathcal{G}_{ij}^{(>d)}\right)^{1/2} = W_0^d W_\pi^d \hat{C}_0 \hat{C}_\pi \prod_{a=1}^{d/2} (2\sin\pi\nu_a) \; ; \; C_{0,\pi} = \frac{T_9}{2}\hat{C}_{0,\pi} \; . \tag{50}$$

In order to fix $\hat{C}_{0,\pi}$ we need to use again Eq. (35), but in this general case $\det(1-R) = 0$. In order to get a nonvanishing result we have to restrict ourselves to the determinant of the submatrix living in the subspace of the eigenvectors of $1 - R$ with nonvanishing eigenvalues. This can be done by grouping the eigenvectors with nonvanishing eigenvalues into a $\hat{d} \times d$-dimensional matrix $L^\dagger$ and its hermitian L:

$$L_\alpha{}^i = \begin{pmatrix} C_1^{Ti} \\ C_1^{\dagger i} \\ \dots \\ \dots \\ C_{d/2}^{Ti} \\ C_{d/2}^{\dagger i} \end{pmatrix} \; ; \; L^{\dagger i}{}_\alpha = \begin{pmatrix} C_1^{*i} & C_1^i & \dots & \dots & C_{d/2}^{*i} & C_{d/2}^i \end{pmatrix} \tag{51}$$

with $\alpha = 1...d$ and by computing the determinant of the following $d \times d$ matrix:

$$\sqrt{\det\left(L_\alpha{}^i G_{ik}(1-R)^k{}_j L^{\dagger j}{}_\beta\right)} = \prod_{a=1}^{d/2}(2\sin\pi\nu_a) \; . \tag{52}$$

On the other hand, by using Eq. (35) one can see that the previous determinant is also equal to:

$$\sqrt{\det\left(L_\alpha{}^i G_{ik}[(1+\mathcal{B}_\pi)^{-1}]^k{}_h G^{hA}\right)} \sqrt{\det\left[4\pi\alpha'(q_0 F^{(0)} + q_\pi F^{(\pi)})_{AB}\right]} \times$$

$$\times \sqrt{\det\left([(1-\mathcal{B}_0)^{-1}]^B{}_j L^{\dagger j}{}_\beta\right)} \; . \tag{53}$$

By inserting this equation in Eq. (50) and using Eq. (49), we get:

$$C_0 = \frac{T_9}{2} \frac{[\det \mathcal{G}_{ij}^{(>d)}]^{1/4}}{\sqrt{|\det\left([(1-\mathcal{B}_0)^{-1}]^B{}_j L^{\dagger j}{}_\beta\right)|}} \; ; \; C_\pi = \frac{T_9}{2} \frac{[\det \mathcal{G}_{ij}^{(>d)}]^{1/4}}{\sqrt{|\det(L_\alpha{}^i G_{ik}[(1+\mathcal{B}_\pi)^{-1}]^k{}_j G^{jA})|}} \tag{54}$$

where we have introduced the absolute value because Eq. (53) does not depend on the phases. For $d = 0$ the two denominators are absent and the two normalization constants reduce to Eq.s (38) and (39) that are also valid for $d = 0$. For $d = \hat{d} = 6$ the numerator is absent and in the denominator both the indices α and i run in the same interval. It follows:

$$C_0 = \frac{T_9}{2} \frac{\sqrt{\det(1-\mathcal{B}_0)}}{\sqrt{|\det L|}} = \frac{T_9}{2} \sqrt{\det(1-\mathcal{B}_0)}\, (\det G_{ij})^{1/4} = \frac{T_9}{2} \frac{\sqrt{\det(G-\mathcal{B}_0)_{ij}}}{(\det G_{ij})^{1/4}} \tag{55}$$

where we have used the relation ${L_\alpha}^i G_{ij} {L^{\dagger j}}_\beta = \delta_{\alpha\beta}$ that summarizes the ones in Eq.s (12). Furthermore Eq.s (38) and (55) are, as expected, in agreement.

We conclude by extending the previous calculation to Type I string theory. In this case all the amplitudes must be divided by a factor two due to the orientifold projection, and we have also to take into account the interaction boundary-crosscap given by:

$$\begin{aligned} \mathcal{M} &= -\frac{2^{5-d/2} V_4}{(8\pi^2\alpha')^2} (-1)^{d/2} W^{>d} N [\det \frac{\mathcal{G}^{>d}_{ij}}{2}]^{1/2} \int_0^\infty dt N^{(d)}_{LL} \sum_{u^j \in \mathbb{Z}} e^{-2\pi t u^i W^i \mathcal{G}^{>d}_{ij} u^j W^j} \\ &\times \frac{1}{2} \left[\prod_{a=1}^{d/2} \frac{\Theta_3(\hat{\nu}_a | it + \frac{1}{2})}{\Theta_1(\hat{\nu}_a | it + \frac{1}{2})} \left(\frac{f_3(iq)}{f_1(iq)} \right)^{8-d} - \prod_{a=1}^{d/2} \frac{\Theta_4(\hat{\nu}_a | it + \frac{1}{2})}{\Theta_1(\hat{\nu}_a | it + \frac{1}{2})} \left(\frac{f_4(iq)}{f_1(iq)} \right)^{8-d} \right. \\ & \left. - \prod_{a=1}^{d/2} \frac{\Theta_2(\hat{\nu}_a | it + \frac{1}{2})}{\Theta_1(\hat{\nu}_a | it + \frac{1}{2})} \left(\frac{f_2(iq)}{f_1(iq)} \right)^{8-d} \right] \end{aligned} \tag{56}$$

where

$$N^{(d)}_{LL} = W^d \, (2\pi\sqrt{\alpha'})^d \sqrt{\det\left(\frac{2q\, F}{2\pi}\right)_{ij}} \tag{57}$$

and $\hat{\nu}_a$ are the eigenvalues of the matrix R taken with $F_\pi = 0$.

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