Boundary CFT and D-branes

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* OUTLINE
  1. Define BCFT and open strings
  2. Define BS
  3. Define T-duality and branes with EM fields
  4. Define BS for branes with EM fields

→ BOUNDARY CFT

* Free boson and T-duality

$$\boxed{S = -\frac{1}{4\pi\alpha'}\int d\tau\, d\sigma \left(\partial_\tau X\cdot\partial_\tau X + \partial_\sigma X\cdot\partial_\sigma X\right)}$$

⇒ EOM: $\frac{\delta S}{\delta X} = 0 \longrightarrow \delta S = \frac{-1}{2\pi\alpha'}\int_\Sigma d\tau\, d\sigma \left[-(\partial_\tau^2+\partial_\sigma^2)X\cdot\delta X + \partial_\sigma\left(\partial_\sigma X\cdot\delta X\right)\right]$

$$\Rightarrow \Box X^\mu(\tau,\sigma) = \partial_w\partial_{\bar w}\left(X_L^\mu(w) + X_R^\mu(\bar w)\right) = 0$$

where $w = \tau - i\sigma$, $\bar w = \tau + i\sigma = w^*$.

$$\Downarrow$$

$$X^\mu(w,\bar w) = x_0^\mu + \frac{a^2}{\pi}p^\mu w + \frac{b^2}{\pi}q^\mu \bar w + \frac{a}{\sqrt{2}\pi}\sum_{n\neq 0}\frac{\alpha_n^\mu}{n}e^{-nw} + \frac{b}{\sqrt{\pi}2}\sum_{n\neq 0}\frac{\tilde\alpha_n^\mu}{n}e^{-n\bar w}$$

⇒ Moreover we have to consider:

$$\Downarrow$$

$$\Sigma = (-\infty,+\infty)\times[0,\pi] \Rightarrow \partial_\sigma X^\mu\big|_{\sigma=0,\pi} = 0.$$

What about $\delta X\big|_{\sigma=0,\pi} = 0$?

→ Suppose to have closed strings s.t.: $X^\mu(w-i\pi, \bar w + i\pi) = X^\mu(w,\bar w)$. Then

$$X^\mu(w,\bar w) = x_0^\mu + \frac{a^2}{\pi}p^\mu(w+\bar w) + \frac{a}{2\pi}\sum_{n\neq 0}\frac{\alpha_n^\mu}{2n}e^{-2nw} + \frac{a}{2\pi}\sum_{n\neq 0}\frac{\tilde\alpha_n^\mu}{2n}e^{-2n\bar w}$$

where $\pi_{cm}^\mu = p^\mu = \frac{1}{\pi}\int_0^\pi d\sigma\,\frac{\delta S}{\delta\dot X_\mu} = \frac{1}{\pi}\cdot\frac{2}{4\pi\alpha'}2a^2p^\mu\cdot\pi = \frac{a^2}{\pi\alpha'}p^\mu \Rightarrow a^2 = \pi\alpha'$

$$\Rightarrow X^\mu(w,\bar w) = x_0^\mu + \alpha' p^\mu(w+\bar w) + \text{osc.}$$

Suppose now to have a compact dimension:

$$X^i(w-i\pi, \bar{w}+i\pi) = X^i(w,\bar{w}) + 2\pi R$$

then we find:

$$p^i + q^i = \frac{2n}{R}\sqrt{\frac{\alpha'}{2}}$$

$$p^i - q^i = \sqrt{\frac{2}{\alpha'}}\, w R$$

$$\Rightarrow M^2 = -(p+q)^\mu (p+q)_\mu = \frac{2}{\alpha'}(p^i)^2 + \frac{4}{\alpha'}(N-1)$$

$$= \frac{2}{\alpha'}(q^i)^2 + \frac{4}{\alpha'}(\tilde{N}-1)$$

where:

$$N = \sum_{k=1}^{\infty} \alpha^\mu_{-k}\alpha_{k,\mu},$$

$$p^i = \left(\frac{n}{R} + \frac{wR}{\alpha'}\right)\sqrt{\frac{\alpha'}{2}},$$

$$q^i = \left(\frac{n}{R} - \frac{wR}{\alpha'}\right)\sqrt{\frac{\alpha'}{2}}.$$

$\Rightarrow$ In the $R \to 0$ limit the compact dimension does not disappear!

Then

$$R \mapsto \frac{\alpha'}{R} \Rightarrow M^2 \text{ unchanged} \rightarrow \boxed{\text{"T-DUALITY"}}$$

$\Rightarrow$ What is the action of T-duality on the oscillators?

$$p^i \xmapsto{T} p^i \qquad \qquad \Rightarrow X^i_L(w) + X^i_R(\bar{w}) \xmapsto{T} X^i_L(w) - X^i_R(\bar{w})$$

$$q^i \xmapsto{T} -q^i$$

$\rightarrow$ T-duality: "spacetime parity on right modes".

⇒ What is the action on OPEN STRINGS?

→ Open strings do not wrap around the compact dimension:

↓

The compact dimension disappears

⇒ from a D dimensional theory we get a D-1 theory of open strings. → Its endpoints are constrained on a (D-1)-dimensional plane:

$$\boxed{\partial_\sigma X^i \xmapsto{T} \partial_\tau X^i = 0}$$

⇒ T-duality exchanges the boundary conditions of the endpoints:

$$\begin{cases} \partial_\sigma X^\mu \big|_{\sigma=0,\pi} = 0 & \leftarrow \text{Neumann conditions} \\ \partial_\tau X^\mu \big|_{\sigma=0,\pi} = 0 & \leftarrow \text{Dirichlet conditions } [\text{equivalent to } \delta X^\mu = 0] \end{cases}$$

(or in general: $X^\mu\big|_{\sigma=0,\pi} = x_0^\mu$)

If we apply T-duality to p+1 dimensions, we constrain the theory on a (p+1)-dimensional surface:

⇓

$$\boxed{\text{Dp-brane} := \text{(p+1)-dimensional hypersurface where string endpoints live.}}$$

How do we implement the b.c. on the string? $\left[X(\tau,\sigma) = x_0 + \alpha' p\tau + \alpha' q\sigma + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\alpha_n}{n}e^{-n(\tau-i\sigma)} + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\tilde{\alpha}_n}{n}e^{-n(\tau+i\sigma)}\right]$

1. NEUMANN: $\partial_\sigma X\big|_{\sigma=0} = 0 \Rightarrow \begin{cases} q = 0 \\ \alpha_n - \tilde{\alpha}_n = 0 \end{cases}$

2. DIRICHLET: $\partial_\tau X\big|_{\sigma=0} = 0 \Rightarrow \begin{cases} p = 0 \longrightarrow \text{CM momentum} \\ \alpha_n + \tilde{\alpha}_n = 0 \end{cases}$

Combining the two endpoints:

1. NN ⇒ $\alpha_n - \tilde{\alpha}_n = 0$ and $n \in \mathbb{Z}$,
2. DD ⇒ $\alpha_n + \tilde{\alpha}_n = 0$ and $n \in \mathbb{Z}' + \frac{1}{2}$,
3. ND ⇒ $\alpha_n - \tilde{\alpha}_n = 0$ and $n \in \mathbb{Z} + \frac{1}{2}$,
4. DN ⇒ $\alpha_n + \tilde{\alpha}_n = 0$ and $n \in \mathbb{Z}$.

That is:

1. NN $\Rightarrow X(\tau,\sigma) = x_0 + \alpha' p\tau + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{\alpha_n}{n} e^{-n\tau} \cos(n\sigma)$

2. DD $\Rightarrow X(\tau,\sigma) = x_0 + \alpha' q\sigma + \sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{\alpha_n}{n} e^{-n\tau} \sin(n\sigma)$

3. ND $\Rightarrow X(\tau,\sigma) = x_0 + i\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}+\frac{1}{2}} \frac{\alpha_n}{n} e^{-n\tau} \cos(n\sigma)$

4. DN $\Rightarrow X(\tau,\sigma) = x_0 + \sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}+\frac{1}{2}} \frac{\alpha_n}{n} e^{-n\tau} \sin(n\sigma)$

Define:

$$z = e^{w} \quad ; \qquad \bar{z} = e^{\bar{w}}.$$

The canonical commutation relations are:

$$[\alpha_n, \alpha_m] = i n \delta_{n+m,0}$$

which shows that:

$$T(z) = \frac{1}{2} : \partial_z X_L(z) \partial_z X_L(z) : = \frac{1}{2} \sum_{n\in\mathbb{Z}} L_n z^{-n-2},$$

$$\bar{T}(\bar{z}) = \frac{1}{2} : \partial_{\bar{z}} X_R(\bar{z}) \partial_{\bar{z}} X_R(\bar{z}) : = \frac{1}{2} \sum_{n\in\mathbb{Z}} \bar{L}_n \bar{z}^{-n-2}.$$

We need to impose, by both N and D conditions:

$$T(z) = \bar{T}(\bar{z}) \Rightarrow L_n = \bar{L}_n \quad [c = \bar{c}] \Rightarrow \text{There exists } \underline{\text{only one}} \text{ Virasoro Algebra}$$

$$\Downarrow$$

$$\mathcal{V}_{T\bar{T}} = V_T \oplus V_{\bar{T}}$$

We can then do the same computations for the superstring. Consider:

$$S = -\frac{1}{4\pi\alpha'}\int d\tau\, d\sigma \left(-i\,\overline{\psi}^{\mu}\rho^{\alpha}\partial_{\alpha}\psi_{\mu}\right)$$

where

$$\rho^{0} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \rho^{1} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \overline{\psi} = \psi^{\dagger}\rho^{0}, \quad \psi = \begin{pmatrix} \psi \\ \tilde{\psi} \end{pmatrix},$$

then:

$$S = \frac{1}{4\pi\alpha'}\int dw\, d\bar{w}\left(\psi^{\mu}\partial_{w}\psi_{\mu} + \tilde{\psi}^{\mu}\partial_{\bar{w}}\tilde{\psi}_{\mu}\right),$$

s.t.:

CLOSED:

$$\psi^{\mu}(w+2\pi i) = e^{2\pi i\nu}\psi^{\mu}(w)$$
$$\tilde{\psi}^{\mu}(\bar{w}+2\pi i) = e^{-2\pi i\tilde{\nu}}\tilde{\psi}^{\mu}(\bar{w})$$

where $\nu, \tilde{\nu} = \begin{cases} 0 \longrightarrow \text{RAMOND} \\ \frac{1}{2} \longrightarrow \text{NEVEU-SCHWARZ} \end{cases}$

$\Downarrow$

4 different L-R sectors!

$$\Rightarrow \psi^{\mu}(z) = \sum_{r\in\mathbb{Z}+\nu}\psi^{\mu}_{r}\, z^{-r-\frac{1}{2}};$$

$$\tilde{\psi}^{\mu}(\bar{z}) = \sum_{r\in\mathbb{Z}+\tilde{\nu}}\tilde{\psi}^{\mu}_{r}\, \bar{z}^{-r-\frac{1}{2}}, \quad \text{where } z = e^{-w}, \; \bar{z} = e^{-\bar{w}}.$$

$\Downarrow$

NB: branch cut in R sector!

OPEN:

$$\psi^{\mu}(\tau,0) = e^{2\pi i\nu}\tilde{\psi}^{\mu}(\tau,0)$$
$$\psi^{\mu}(\tau,\pi) = \tilde{\psi}^{\mu}(\tau,\pi)$$

$\Rightarrow$ only 2 sectors!

NB.

NS: $\psi^{\mu}_{r}|0\rangle_{NS} = 0 \;\; \forall r > 0$

R: $\psi^{\mu}_{r}|0\rangle_{R} = 0 \;\; \forall r > 0 \Rightarrow$ degeneracy for $\{\psi^{\mu}_{0}, \psi^{\nu}_{0}\} = \eta^{\mu\nu} \Rightarrow$ CLIFFORD ALGEBRA

$$|0\rangle_{R} = |s_1, s_2, s_3, s_4, s_5\rangle_{R} = |\vec{s}\rangle_{R}$$

$\Downarrow$

$$Spin(10) \Rightarrow 32 = 16 \oplus 16'$$

NB: we use 5 bosons $H^{a}(z)$ to express $S_{\alpha} = e^{i s_a H^{a}(z)}$ which is the SPIN FIELD which creates $|\vec{s}\rangle_{R}$ from $|0\rangle_{NS}$.

Because of the branch cut, the spacetime field theory is NOT LOCAL. We implement the GSO projection by means of $\Gamma_{11}(-1)^{\sum \psi_{-m}\cdot\psi_m}$:

$$\boxed{\begin{array}{lll} \text{Type IIA}: & \text{- bosons: } G_{\mu\nu},\ B_{\mu\nu},\ \phi & (35 \oplus 28 \oplus 1) \\ & \text{- forms: } C_\mu,\ C_{\mu\nu\rho} & (8 \oplus 56) \\ & \text{- fermions: } \psi^\mu_\alpha,\ \tilde{\psi}^\mu_{\dot\alpha},\ \lambda_\alpha,\ \tilde{\lambda}_{\dot\alpha} & (56 \oplus \overline{56} \oplus 8 \oplus \overline{8}) \\ \text{Type IIB}: & \text{- bosons: } G_{\mu\nu},\ B_{\mu\nu},\ \phi & (35 \oplus 28 \oplus 1) \\ & \text{- forms: } C,\ C_{\mu\nu},\ C^+_{\mu\nu\rho\lambda} & (1 \oplus 28 \oplus 35) \\ & \text{- fermions: } \psi^\mu_\alpha,\ \chi^\mu_\alpha,\ \lambda_\alpha, \sigma_\alpha & (56 \oplus 56 \oplus 8 \oplus 8) \end{array}}$$

We know that from the open strings, the boundary conditions exchange L and R sectors:

SUSY current: $j(z) + \tilde{j}(\bar{z}) \longrightarrow$ only $Q_\alpha + \tilde{Q}_\alpha$ can be conserved

$\Downarrow$

branes are BPS

Under T-duality R modes change the sign:

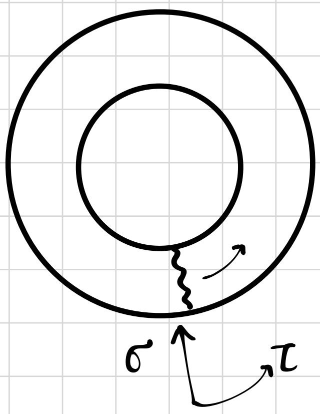
$$Q_\alpha \xmapsto{T} Q_\alpha$$

$$\tilde{Q}_\alpha \xmapsto{T} \beta^i_{\alpha\beta} Q_\beta \qquad \text{where } i \text{ is the T-dual coordinate.}$$

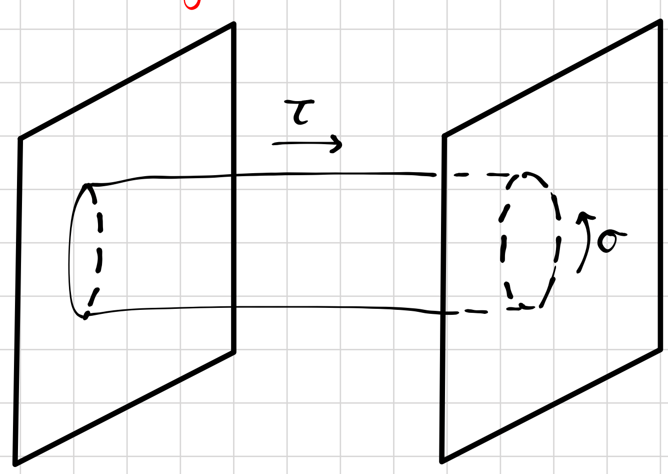
$\Rightarrow$ The unbroken SUSY is $Q_\alpha + \beta^\perp \tilde{Q}_\alpha$, where $\beta^\perp = \prod_{\{T\text{-dual}\}} \beta^i$.

* Boundary states and partition functions

Consider the vacuum 1-loop amplitude for the BOSONIC OPEN STRING:



$\longrightarrow$ under the exchange $(\tau,\sigma) \leftrightarrow (\sigma,\tau)$ we can interpret the amplitude AS A CLOSED STRING being emitted and absorbed by boundaries (D-branes)



$\Downarrow$

OPEN STRING CHANNEL		CLOSED STRING CHANNEL
N: $\partial_\sigma X\big|_{\sigma=0} = 0$	$\longrightarrow$	$\partial_\tau X\big|_{\tau=0} \lvert B_N\rangle = 0 \Rightarrow (\alpha_n + \tilde{\alpha}_{-n})\lvert B_N\rangle = 0$ and $p\lvert B_N\rangle = 0.$
D: $\partial_\tau X\big|_{\sigma=0} = 0$	$\longrightarrow$	$\partial_\sigma X\big|_{\tau=0} \lvert B_D\rangle = 0 \Rightarrow (\alpha_n - \tilde{\alpha}_{-n})\lvert B_D\rangle = 0.$ and s.t: $X\big|_{\tau=0}\lvert B_D\rangle = x_0\lvert B_D\rangle$

Then we have for the bosonic string:

$$\lvert B\rangle = \mathcal{N}\,\delta^{\perp}(y - x_0)\exp\left[-\sum_{n=1}^{\infty}\frac{1}{n}\alpha^{\mu}_{-n}S_{\mu\nu}\tilde{\alpha}^{\nu}_{-n}\right]\lvert 0; k=0, y\rangle$$

where:

$$S_{\mu\nu} = \begin{pmatrix} \eta^{\alpha\beta} & \\ & -\delta_{ij} \end{pmatrix} \longrightarrow \begin{array}{l} \alpha,\beta \in \{\parallel\} \\ i,j \in \{\perp\} \quad \text{(applied T-duality).} \end{array}$$

How do we fix $\mathcal{N}$?

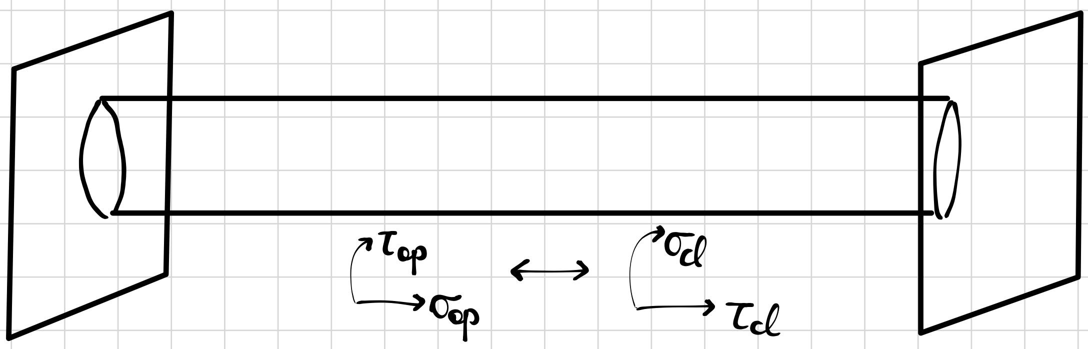
We have to define the D-brane action, which is the Dirac-Born-Infeld action. Let ξ^a be the coordinates on the brane, then:

$$S_{DBI} = -T_p \int d^{p+1}\xi \, e^{-\phi} \left[-\det\left(\partial_a X^\mu \partial_b X^\nu G_{\mu\nu} + \partial_a X^\mu \partial_b X^\nu B_{\mu\nu} + 2\pi\alpha' F_{ab} \right) \right]^{1/2}.$$

The tension of the Dp-brane is then

$$\tau_p = T_p \, e^{-\langle \phi \rangle}$$

We can then consider the exchange of closed strings between the branes:



Then we find (we assume the superstring framework):

$$\mathcal{A} = \underbrace{(1-1)}_{\text{NSNS} - \text{RR} \Rightarrow \text{supersymmetry}} V_{p+1} \frac{1}{2} \int \frac{dt}{t} (2\pi t)^{-(p+1)/2} \left(\frac{t}{2\pi\alpha'} \right)^4 e^{-\frac{t y^2}{8\pi\alpha'^2}}$$

where Y is the separation between the branes. We are now interested in the coupling of massless CLOSED STRINGS $\Rightarrow t \to 0$ in NS-NS sector:

$$\mathcal{A}_{NS\text{-}NS} = V_{p+1} \, 2\pi \left(4\pi^2 \alpha' \right)^{3-p} G_{9-p}(Y^2)$$

where

$$G_{9-p}(Y) = \frac{1}{4} \pi^{\frac{p-9}{2}} \Gamma\left(\frac{7-p}{2} \right) Y^{p-7} \quad \left[\text{NB: } \int dt \, d^p y \, d^{d-p-1} x \, e^{i k_\perp \cdot x} G(x) = \frac{V_{p+1}}{k_\perp^2} \right]$$

is the Green function of the massless string modes. This should then be compared with field theory computation of graviton-dilaton theory in 10 dimensions:

$$S = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-\tilde{G}} \left(\tilde{R} - \frac{1}{6} \nabla_\mu \tilde{\phi} \nabla^\mu \tilde{\phi} \right)$$ (then expand to first order $\tilde{G}$ and second order $\tilde{\phi}$)

$$\Rightarrow \mathcal{F}\left[\langle 0 | T(\tilde{\phi}(y) \tilde{\phi}(0)) | 0 \rangle \right](k) = -\frac{2 i \kappa^2}{k^2}$$

$$\mathcal{F}\left[\langle 0 | T(h_{\mu\nu}(y) h_{\rho\sigma}(0)) | 0 \rangle \right](k) = -\frac{2 i \kappa^2}{k^2} \left(\eta_{\mu\sigma}\eta_{\nu\rho} + \eta_{\mu\rho}\eta_{\nu\sigma} - \frac{1}{4} \eta_{\mu\nu}\eta_{\rho\sigma} \right)$$

Then the D-brane action becomes:

$$S_p = -\tau_p \int d^{p+1}\xi \left(\frac{p-11}{12} \tilde{\phi} - \frac{1}{2} h_{aa} \right)$$

$$\Rightarrow \mathcal{A} = \frac{2 i \kappa^2 \tau_p^2}{k_\perp^2} V_{p+1}$$

Therefore:

$$\tau_p^2 = \frac{\pi}{\kappa^2} (4\pi^2 \alpha')^{3-p} .$$

The same can be applied to the RR contribution (coupling Dp-brane to a (p+1)-form potential):

$$S = -\frac{1}{4\kappa^2} \int d^{10}x \sqrt{-G} \, | F_{p+2} |^2 + \mu_p \int C_{p+1}$$

then the amplitude for the exchange of a (p+1)-form is:

$$\mathcal{A} = -2 i \kappa^2 \mu_p^2 G_{9-p}(Y)$$

$$\Rightarrow \mu_p^2 = \frac{\pi}{\kappa^2} (4\pi^2 \alpha')^{3-p} = e^{2\phi} \tau_p^2 = T_p^2$$

The same result can be computed from the BS formalism introducing the propagator of the closed string:

$$D_a = \frac{\alpha'}{4\pi} \int_{|z| \leq 1} \frac{d^2 z}{|z|^2} z^{L_0 - a} \bar{z}^{\tilde{L}_0 - a}$$

and the amplitude

$$\langle B | D_1 | B \rangle = \mathcal{A}$$

Then:

$$|B\rangle = \frac{\sqrt{\pi}}{2} \delta^{\perp}(y - x_0) (4\pi^2 \alpha')^{3-p/2} \exp\left(-\sum_{k=1}^{\infty} \frac{1}{n} \alpha_{-n}^{\mu} S_{\mu\nu} \alpha_{-n}^{\nu} \right) | 0; k=0, y \rangle$$

We can then extend what we did to the SUPERSTRING framework:

$$|B_\psi\rangle \sim \exp\left(i\sum_{n>0}\psi^\mu_{-n}\, S_{\mu\nu}\,\tilde{\psi}^\nu_{-n}\right)|0\rangle$$

$$|B_{gh}\rangle \sim \exp\left(\sum_{n>0}\left(c_{-n}\tilde{b}_{-n} + \tilde{c}_{-n}\, b_{-n} + i\left(\gamma_{-n}\tilde{\beta}_{-n} + \tilde{\gamma}_{-n}\, b_{-n}\right)\right)\right)\frac{1}{2}\left(c_0+\tilde{c}_0\right)|\downarrow\downarrow\rangle$$

in the closed string scenario. After GSO projection:

$$|B_{NS}\rangle = \frac{\sqrt{\pi}}{2}\left(4\pi^2\alpha'\right)^{3-p/2}\delta^{\perp}\left(y-x_0\right)\, e^{-\sum_{n>0}\frac{1}{n}\alpha^\mu_{-n}S_{\mu\nu}\alpha^\nu_{-n}}\,\sin\left(\sum_{r=1/2}^{\infty} b^\mu_{-r}\, S_{\mu\nu}\, b^\nu_{-r}\right)|\Omega_{NS};\, k=0,\, y\rangle$$

where :

$$|\Omega_{NS}\rangle = \lim_{z\to 0} c(z)\, e^{-\phi(z)}\,|0\rangle$$

fixing the picture to $(-1,-1)$. We then consider the massless state:

$$V_{NSNS} = \varepsilon_{\mu\nu}\, V^\mu_{-1}\left(\frac{k}{2},z\right)\tilde{V}^\nu_{-1}\left(\frac{k}{2},\bar{z}\right) =$$

$$= \varepsilon_{\mu\nu}\; e^{-\phi(z)}\, c(z)\,\psi^\mu(z)\, e^{-i\frac{k}{2}\cdot X(z)}\;\; e^{-\tilde{\phi}(\bar{z})}\,\tilde{c}(\bar{z})\,\tilde{\psi}^\nu(\bar{z})\, e^{-i\frac{k}{2}\cdot X(\bar{z})}$$

and

w/o ghosts.

$$\langle 0_{gh}|\, V_{NSNS}\, D_0\,|B_{NS}\rangle = \lim_{z,\bar{z}\to 0}\varepsilon_{\mu\nu}\,\langle 0|\, b^\mu_{\frac{1}{2}}\,\tilde{b}^\nu_{\frac{1}{2}}\, D_{\frac{1}{8}}\,|\hat{B}_{NS}\rangle =$$

$$= -\frac{T_p}{2}\,\frac{V_{p+1}}{k_\perp^2}\, S^{\mu\nu}\varepsilon_{\mu\nu}$$

$\Rightarrow$ IRREPS :

$$h_{\mu\nu} = 2\, T_p\,\frac{V_{p+1}}{k_\perp^2}\left(-\frac{d-p-3}{2(d-2)}\,\eta_{\alpha\beta}\qquad\frac{p+1}{2(d-2)}\,\delta_{ij}\right)$$

$$\varphi = T_p\,\frac{V_{p+1}}{k_\perp^2}\,\frac{d-2p-4}{2\sqrt{d-2}}$$

$$B_{\mu\nu} = 0.$$

$\Rightarrow$ They represent the same behaviour and coupling of a solution to the DBI action for a D-brane.

In the R-R sector we consider the vertex operator:

TYPE IIA] $W_{RR} = \mathcal{A}_{\dot\alpha\dot\beta}\, \mathcal{V}^{\dot\alpha}_{-3/2}\, \tilde{\mathcal{V}}^{\dot\beta}_{-\frac{1}{2}} + i\,4\sqrt{2}\, \mathcal{F}_{\dot\alpha\beta}\, \mathcal{V}^{\dot\alpha}_{-\frac{3}{2}}\, \tilde{\mathcal{V}}^{\beta}_{-3/2}\, \bar\partial \tilde{c}\, \tilde{\xi}$

and

$$\mathcal{V}^{\alpha}_{-\frac{1}{2}}(k,z) = e^{-\frac{1}{2}\phi(z)}\, c(z)\, S^{\alpha}(z)\, e^{ik\cdot X(z)},$$

$[C :=$ charge conj.$]$

$$\mathcal{A}_{\dot\alpha\dot\beta} = \frac{1}{4\sqrt{2}\,n!}\left(C\Gamma^{\mu_1\ldots\mu_n}\right)_{\dot\alpha\dot\beta} A_{\mu_1\ldots\mu_n}, \qquad n \text{ odd}$$

$$\mathcal{F}_{\dot\alpha\beta} = \frac{i}{16(n+1)!}\left(C\Gamma^{\mu_1\ldots\mu_{n+1}}\right)_{\dot\alpha\beta} F_{\mu_1\ldots\mu_{n+1}}$$

$$\Rightarrow [Q_{BRST}, W_{RR}] = 0 \Rightarrow F = dA, \quad d{*}A = 0.$$

Type IIB] similar, but:

$$\mathcal{A}_{\dot\alpha\dot\beta} = \#\left(C\Gamma^{\mu_1\ldots\mu_n}\right)_{\dot\alpha\dot\beta} A_{\mu_1\ldots\mu_n}, \qquad n \text{ even}$$

Then

$$|B_R\rangle = \pm\frac{T_p}{2}\,\delta^{\perp}(y-x_0)\, e^{-\sum_{n>0}\frac{1}{n}\alpha^{\mu}_{-n}S_{\mu\nu}\alpha^{\nu}_{-n}}\left[\cos\Big(\sum_{n>0}\psi^{\mu}_{-n}S_{\mu\nu}\tilde\psi^{\nu}_{-n}\Big)|\Omega^{(1)}\rangle + \sin\Big(\sum_{n>0}\psi^{\mu}_{-n}S_{\mu\nu}\tilde\psi^{\nu}_{-n}\Big)|\Omega^{(2)}\rangle\right]$$

where:

$$|\Omega^{(1)}\rangle = \begin{cases} (C\Gamma^{0\ldots p})_{\alpha\beta}\, S^{\alpha}\tilde S^{\beta}\, e^{-\frac{1}{2}\phi}\, e^{-\frac{3}{2}\tilde\phi}|0\rangle & \text{if type IIA} \quad (p \text{ even}) \\ (C\Gamma^{0\ldots p})_{\dot\alpha\beta}\, S^{\dot\alpha}\tilde S^{\beta}\, e^{-\phi/2}\, e^{-\frac{3}{2}\tilde\phi}|0\rangle & \text{if type IIB} \quad (p \text{ odd}) \end{cases}$$

$$|\Omega^{(2)}\rangle = \begin{cases} (C\Gamma^{0\ldots p})_{\dot\alpha\dot\beta}\, S^{\dot\alpha}\tilde S^{\dot\beta}\, e^{-\frac{1}{2}\phi}\, e^{-\frac{3}{2}\tilde\phi}|0\rangle & \text{if type IIA} \quad (p \text{ even}) \\ (C\Gamma^{0\ldots p})_{\alpha\dot\beta}\, S^{\alpha}\tilde S^{\dot\beta}\, e^{-\phi/2}\, e^{-\frac{3}{2}\tilde\phi}|0\rangle & \text{if type IIB} \quad (p \text{ odd}). \end{cases}$$

Now we have for instance (type IIA)

$$\langle 0|\, \mathcal{V}^{\dot\alpha}_{-3/2}\, \tilde{\mathcal{V}}^{\dot\beta}_{-\frac{1}{2}}\, D_0\, |B_R\rangle = \mp\frac{T_p}{2}\,\frac{V_{p+1}}{k_\perp^2}\left(C^{-1}\Gamma^{0\ldots p}C\right)^{\dot\alpha\dot\beta}.$$

$\Rightarrow$ We know that $A_{0\ldots p}$ is the Γ-trace of the vertex op:

$$A_{\mu_1\ldots\mu_{p+1}} = \mp 2\sqrt{2}\, T_p\, \frac{V_{p+1}}{k_\perp^2}\, \varepsilon_{\mu_1\ldots\mu_{p+1}} \neq 0 \Rightarrow \text{since } B_{\mu\nu} = 0 \rightarrow \text{the D-brane is charged only under R-R.}$$

* Branes with EM fields

Consider the coupling of the string endpoints to a potential:

$$S_{EM} = \int d\tau \, A_m \dot{X}^m \big|_{\sigma=\pi} - \int d\tau \, A_m \dot{X}^m \big|_{\sigma=0} .$$

Suppose $F_{mn} = \text{const} \Rightarrow A_n = \frac{1}{2} F_{mn} X^m$. Then:

$$S = -\frac{1}{4\pi\alpha'} \int d\tau d\sigma \left(\dot{X}_\mu \dot{X}^\mu + X'_\mu X'^\mu \right) + \frac{1}{2} \int d\tau \, F_{mn} \left(X^m \dot{X}^n \big|_{\sigma=\pi} - X^m \dot{X}^n \big|_{\sigma=0} \right)$$

Focus only on the brane-world indices (m,n) at $\sigma = \pi$.

EOM: $\quad \partial_\sigma X^m - 2\pi\alpha' F_{mn} \partial_\tau X^n = 0$

→ ELECTRIC FIELD:

Suppose that X^9 is wrapped on a circle and:

$$F_{9,0} = E_9 = E \quad \text{and} \quad F_{\mu\nu} = 0 \text{ everywhere else.}$$

Then

$$\partial_\sigma X^0 - \overbrace{2\pi\alpha' E}^{\mathcal{E}} \partial_\tau X^9 = 0$$
$$-2\pi\alpha' E \, \partial_\tau X^0 + \partial_\sigma X^9 = 0$$

Choose an appropriate basis of coordinates:

$$\Rightarrow \quad \partial_+ \begin{pmatrix} X^0 \\ X^9 \end{pmatrix} = \begin{pmatrix} \frac{1+\mathcal{E}^2}{1-\mathcal{E}^2} & \frac{2\mathcal{E}}{1-\mathcal{E}^2} \\ \frac{2\mathcal{E}}{1-\mathcal{E}^2} & \frac{1+\mathcal{E}^2}{1-\mathcal{E}^2} \end{pmatrix} \partial_- \begin{pmatrix} X^0 \\ X^9 \end{pmatrix}$$

Now consider a T-duality transformation on X^9 (coord on the brane):

$$\partial_+ \begin{pmatrix} X^0 \\ X^9 \end{pmatrix} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \partial_- \begin{pmatrix} X^0 \\ X^9 \end{pmatrix} \xrightarrow{\;T\;} \partial_+ \begin{pmatrix} X^0 \\ \tilde{X}^9 \end{pmatrix} = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \partial_- \begin{pmatrix} X^0 \\ \tilde{X}^9 \end{pmatrix}$$

(i.e.: from N to D cond. on X^9).

Now suppose to boost the brane along $\tilde{X}^9$:

$$\begin{pmatrix} X^{0\prime} \\ \tilde{X}^{9\prime} \end{pmatrix} = M \begin{pmatrix} X^0 \\ \tilde{X}^9 \end{pmatrix} \quad \text{where} \quad M = \gamma \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix}$$

Then we have

$$\partial_+ \begin{pmatrix} X^0 \\ \tilde{X}^9 \end{pmatrix} = M^{-1} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} M \, \partial_- \begin{pmatrix} X^0 \\ \tilde{X}^9 \end{pmatrix}$$

and:

$$\partial_+ \begin{pmatrix} X^0 \\ X^9 \end{pmatrix} = \underbrace{M^{-1} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} M \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}} \partial_- \begin{pmatrix} X^0 \\ X^9 \end{pmatrix}$$

$$\downarrow$$

$$= \begin{pmatrix} \frac{1+\beta^2}{1-\beta^2} & \frac{2\beta}{1-\beta^2} \\ \frac{2\beta}{1-\beta^2} & \frac{1+\beta^2}{1-\beta^2} \end{pmatrix}$$

Therefore:

$$2\pi\alpha' E = \beta \Rightarrow |\beta| < 1 \Rightarrow |E| < \frac{1}{2\pi\alpha'} = E_{crit}$$

$$\Downarrow$$

A Dp-brane wrapped on a circle and carrying an electric field is T-dual to a D(p-1)-brane moving on the dual circle.

→ MAGNETIC FIELD

Now consider (X^2, X^3) and compactify X^3 on a circle. Then let

$$F_{23} = B_{23} = B$$

and $2\pi\alpha' B = \mathcal{B}$. Then:

$$\partial_\sigma X^2 - \mathcal{B}\, \partial_\tau X^3 = 0$$
$$\partial_\sigma X^3 + \mathcal{B}\, \partial_\tau X^2 = 0$$

$$\Downarrow$$

$$\partial_+ \begin{pmatrix} X^2 \\ \tilde{X}^3 \end{pmatrix} = \begin{pmatrix} \frac{1-\mathcal{B}^2}{1+\mathcal{B}^2} & \frac{2\mathcal{B}}{1+\mathcal{B}^2} \\ -\frac{2\mathcal{B}}{1+\mathcal{B}^2} & \frac{1-\mathcal{B}^2}{1+\mathcal{B}^2} \end{pmatrix} \begin{pmatrix} X^2 \\ X^3 \end{pmatrix}$$

Now consider T-duality on a Dp-brane (it becomes D(p-1) after that):

$$\partial_+ \begin{pmatrix} X'^2 \\ X'^3 \end{pmatrix} = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \partial_- \begin{pmatrix} X'^2 \\ X'^3 \end{pmatrix}.$$

Then perform a rotation $R = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$:

$$\partial_+ \begin{pmatrix} X^2 \\ X^3 \end{pmatrix} = R^{-1} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} R \, \partial_- \begin{pmatrix} X^2 \\ X^3 \end{pmatrix}$$

$$\partial_+ \begin{pmatrix} X^2 \\ \tilde{X}^3 \end{pmatrix} = \underbrace{R^{-1} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} R \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}}_{\begin{pmatrix} \cos 2\alpha & -\sin 2\alpha \\ \sin 2\alpha & \cos 2\alpha \end{pmatrix}} \partial_- \begin{pmatrix} X^2 \\ \tilde{X}^3 \end{pmatrix}$$

$$\Rightarrow \quad 2\pi\alpha' B = \tan\alpha$$

$$\Downarrow$$

The tilted D(p-1)-brane is T-dual to a Dp-brane in a const. magnetic field.

→ Flux compactification:

Let $F_{23} = B$ and

$$A_2 = 0 \quad \text{and} \quad A_3 = B X^2.$$

Now, we compactify X^3:

$$A_3 \sim A_3 + \frac{n}{R_3}, \quad n \in \mathbb{Z}.$$

Then:

$$\Delta A_3 = B \, \Delta X^2 = \frac{n}{R_3}.$$

Now consider the T-dual coord. and compactify X^2:

$$\Delta X^2 = \frac{2\pi n R_3}{2\pi\alpha' B} = \frac{2\pi n R_3}{\tan\alpha}$$

$$\Delta X^2 = 2\pi R_2$$

Then $\tan\alpha = n \frac{R_3}{R_2}$ → n is the no. of times the tilted brane wraps the T^2 identified by $X^2 \sim X^2 + 2\pi n R_2$ and $X^3 \sim X^3 + 2\pi m R_3$.

In the T-dual world with the magnetic flux we can compute the total magnetic flux on the torus:

$$\phi = B \cdot \text{Area} = B \cdot 2\pi \tilde{R}_3 \cdot 2\pi R_2$$

then we have:

$$\phi = -2\pi \frac{R_2}{R_3} \tan\alpha = -2\pi n \Rightarrow \text{QUANTIZED!}$$

(the D-brane can be tilted only at specific angles)

What does it become in the closed string formalism?

* <u>BS in non flat background</u>

Suppose now to have a string embedded in a non trivial background:

$$S = -\frac{1}{4\pi\alpha'}\int d\tau d\sigma \left[G_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu \eta^{\alpha\beta} - B_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu \varepsilon^{\alpha\beta}\right] - \frac{q_0}{2}\int d\tau\, F_{ij} X^j \dot{X}^i \Big|_{\sigma=0} + \frac{q_\pi}{2}\int d\tau\, F_{ij} X^j \dot{X}^i \Big|_{\sigma=0}$$

Then:

$$\text{BCFT} \longrightarrow G_{ij}\,\partial_\sigma X^j + \left(B_{ij} - 2\pi\alpha' q_0 F_{ij}\right)\dot{X}^j \Big|_{\sigma=0} = 0$$

(same for $\sigma=\pi$)

$$\text{EOM} \longrightarrow \partial_\tau X_R^i = (R_0)^i{}_j\, \partial_\tau X_L^j \quad \text{and} \quad \partial_\tau X_L^i = (R_\pi^{-1} R_0)^i{}_j\, \partial_\tau X_R^j$$

where: $R_0{}^i{}_j = \left((1-B_0)^{-1}(1+B_0)\right)^i{}_j$; $R_\pi{}^i{}_j = \left((1-B_\pi)^{-1}(1+B_\pi)\right)^i{}_j$

and $B_0{}^i{}_j = G^{ik}\left(B_{kj} - 2\pi\alpha' q_0 F_{kj}\right)$; $B_\pi{}^i{}_j = G^{ik}\left(B_{kj} + 2\pi\alpha' q_\pi F_{kj}\right)$.

$$\Rightarrow X^i(\tau,\sigma) = x_0^i + \sqrt{\alpha'}\left[2\hat{m}^i\sigma + 2G^{ij}\left(\hat{n}_j - B_{jk}\hat{m}^k\right)\tau\right] + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\left[\alpha_n^i e^{-2in(\tau-\sigma)} + \tilde{\alpha}_n^i e^{-2in(\tau+\sigma)}\right]$$

We then ask for

$$\left[G_{ij}\dot{X}^j + \left(B_{ij} - 2\pi\alpha' q F_{ij}\right)X'^j\right]_{\tau=0}|B\rangle = 0$$

This translates into:

$$\left(\hat{n}_i - 2\pi\alpha' q F_{ij}\hat{m}^j\right)|B\rangle = 0$$

$$\left(\mathcal{E}_{ij}\alpha^j_n + \mathcal{E}^T_{ij}\tilde{\alpha}^j_{-n}\right)|B\rangle = 0$$

$\Rightarrow$ n_i, m_j are the wrappings of the D-branes.

where $\mathcal{E}_{ij} = G_{ij} - B_{ij} + 2\pi\alpha' F_{ij}$.

Therefore:

$$|B\rangle = \mathcal{N} \underset{\text{no. of D9}}{N} \cdot W \sum_{r_i, s_j \in \mathbb{Z}} \delta_{n_i \mp 2\pi\alpha' q F_{ij} m^j, 0} \prod_{n=1}^{\infty} \exp\left(e^{-\frac{1}{n}\alpha^i_{-n} \overbrace{G_{ik}(\mathcal{E}^{-1})^k{}_h(\mathcal{E})^h{}_j}^{S_{ij}} \alpha^j_{-n}}\right)|n^i, m^j; 0, k=0\rangle$$

$W = \prod_i W_i$: wrappings in the i-th direction.

or for the superstring:

$$|B\rangle_\nu \sim \exp\left[i\eta \sum_{t \in \mathbb{Z}+\nu} \psi^i_{-t} S_{ij} \tilde{\psi}^j_{-t}\right] \rightarrow \nu = \begin{cases} 0 & \text{if R} \\ 1/2 & \text{if NS} \end{cases}.$$