Stress Energy Tensor - OPE

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October 25, 2017

Consider the Virasoro algebra:

$$[L_m,\, L_n] = (m-n)\, L_{m+n} + \frac{c}{12} m \left(m^2 - 1\right) \delta_{m+n,\,0},$$

where $\delta_{i,j}$ is the usual Kronecker delta. This is encoded in

$$T(z)\, T(w) = \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2} T(w) + \frac{1}{z-w} \partial_w T(w)\,.$$

As a matter of fact, we have:

$$\begin{aligned}
[L_m,\, L_n] = &\oint_{0,\,|z|>|w|} \frac{\mathrm{d}z}{2\pi i} \oint_0 \frac{\mathrm{d}w}{2\pi i} z^{m+1} w^{n+1} T(z)\, T(w) - \\
&- \frac{\mathrm{d}z}{2\pi i} \oint_0 \frac{\mathrm{d}w}{2\pi i} \oint_{0,\,|z|<|w|} z^{m+1} w^{n+1} T(w)\, T(z) = \\
&= \oint_0 \frac{\mathrm{d}w}{2\pi i} w^{n+1} \oint_w \frac{\mathrm{d}z}{2\pi i} z^{m+1} \mathrm{R}\left(T(w)\, T(z)\right) = \\
&= \oint_0 \frac{\mathrm{d}w}{2\pi i} w^{n+1} \oint_w \frac{\mathrm{d}z}{2\pi i} z^{m+1} \left(\frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2} T(w) + \frac{1}{z-w} \partial_w T(w) \right) = \\
&= \oint_0 \frac{\mathrm{d}w}{2\pi i} w^{n+1} \left(\frac{1}{3!} \frac{c}{2} (m+1)\, m\, (m-1)\, w^{m-2} + 2T(w)\, (m+1)\, w^m + w^{m+1} \partial_w T(w) \right) = \\
&= \oint_0 \frac{\mathrm{d}w}{2\pi i} \left(\frac{c}{12} m \left(m^2-1\right) w^{n+m-1} + 2\,(m+1)\, T(w)\, w^{n+m+1} + w^{n+m+2} \partial_w T(w) \right) = \\
&= \frac{c}{12} m \left(m^2-1\right) \delta_{m+n,\,0} + 2\,(m+1)\, L_{m+n} - \oint_0 \frac{\mathrm{d}w}{2\pi i} (n+m+2)\, w^{n+m+1} T(w) = \\
&= \frac{c}{12} m \left(m^2-1\right) \delta_{m+n,\,0} + 2\,(m+1)\, L_{m+n} - (n+m+2)\, L_{m+n} = \\
&= (m-n)\, L_{m+n} + \frac{c}{12} m \left(m^2-1\right) \delta_{m+n,\,0}.
\end{aligned}$$