1. Kervaire-Milnor III: .

1.1. The Group of Manifolds Bounding Parallelizable Ones. The central object of this section is the following:

Notation 1.1. Let ∂P_{n+1} be the subgroup of Θ_n of homotopy n-spheres M which are boundaries of parallelizable manifolds.

There are two things to justify:

(1) the condition on M depends only on the
(2) ∂P_{n+1} is a group

Moreover, here is the main result of this section:

Theorem 1.2. *The quotient group $\Theta_n/\partial P_{n+1}$ is finite.*

Remark 1.3. We will prove later than ∂P_{n+1} is zero for n even (i.e. if W is a parallelizable manifold with boundary, then ∂W is h-cobordant to $\mathbb{S}^n$). For n odd, it is finite cyclic. (Again, the $n = 3$ case follows from the 3-dimensional Poincaré conjecture.)

Figure 1. Table of $|\partial P_{n+1}|$ for small n.

n	1	3	5	7	9	11	13	15	17	19
order of bP_{n+1}	1	?	1	28	2	992	1	8128	2	130,816.

Moreover, $|\partial P_{n+1}| = 1$ or 2 for $n \equiv 1 \mod 4$, but it has exponential growth for $n \equiv 3 \mod 4$.

1.2. Thom-Pontryagin Construction. We give a brief detour to descibe the Thom-Pontryagin construction. See Chapter 7 of [2] for details on the Pontryagin construction, or [3] for a more general discussion.

We start with the concrete version:

Definition 1.4. A *framing* of the submanifold $N \subseteq M$ is a smooth section $\mathfrak{v}$ of the normal bundle $TN^\perp \subseteq TM$. The pair $(N, \mathfrak{v})$ is a *framed submanifold* of M. Two framed submanifolds $(N, \mathfrak{v})$ and $(N', \mathfrak{w})$ are *framed cobordant* if there exists a cobordism $X \subseteq M \times [0, 1]$ between N and N' and a framing $\mathfrak{u}$ of X so that

$$u^i(x, t) = (v^i(x), 0) \quad (x, t) \in N \times [0, \epsilon) \tag{1.5}$$

$$u^i(x, t) = (w^i(x), 0) \quad (x, t) \in N \times (1 - \epsilon, 1] \tag{1.6}$$

where $u^i(x, t)$ is the ith component of $\mathfrak{u}(x, t) \in T_x X$.

Next, we give a way of associating a frame to a manifold $f^{-1}(y)$ obtained by regular value theorem.

Fix a positively oriented basis $\mathfrak{v} = (v^1, ..., v^p) \in T_y\mathbb{S}^p$, and take a smooth map $f : M \to \mathbb{S}^p$ and a regular value $y \in \mathbb{S}^p$. f induces a basis $f^*\mathfrak{v}$ as follows. For each $x \in f^{-1}(y)$, $\ker df_x = T_x f^{-1}(y)$ and

$$df_x|_{T_x f^{-1}(y)^\perp} : T_x f^{-1}(y)^\perp \to df_x(T_x f^{-1}(y)^\perp) \tag{1.7}$$

is an isomorphism onto its image. Hence there is a unique vector $(df_x)^{-1}v^i \in T_x f^{-1}(y)^\perp$ that maps into v^i under df_x.

Definition 1.8. Let $f^*\mathfrak{v}$ be the frame on $f^{-1}(y)$ be given by

$$f^*\mathfrak{v}(x) := ((df_x)^{-1}v^1, ..., (df_x)^{-1}v^p) \tag{1.9}$$

The framed manifold $(f^{-1}(y), f^*\mathfrak{v})$ is the *Pontryagin manifold* associated with f.

Here are the main results. Let notation be as above.

Theorem 1.10. *The Pontryagin manifold associated with f is unique up to framed cobordism, i.e. if y' is another regular value of f, and $\mathfrak{v}'$ is the associated positively oriented vasis for $T_{y'}\mathbb{S}^p$, then the Pontryagin manifolds $(f^{-1}(y), f^*\mathfrak{v})$ and $(f^{-1}(y'), f^*\mathfrak{v}')$ are framed cobordant.*

Theorem 1.11. *Two mappings from M to $\mathbb{S}^p$ are smoothly homotopic iff the associated Pontryagin manifolds are framed cobordant.*

Theorem 1.12. *Any compact framed submanifold $(N, \mathfrak{w})$ of codimension p in M occurs as Pontryagin manifold for some smooth mapping $f : M \to \mathbb{S}^p$.*

From the above three theorems, we have the following:

Theorem 1.13 (Pontryagin Construction.)**.** *Let M be a smooth, closed, connected manifold. Then there is a bijection between the set of framed cobordisms Ω^n_{framed} and the smooth homotopy classes of maps $[M, \mathbb{S}^p]$.*

We can further place a group structure on Ω^n_{framed} to make the above a group isomorphism.

A consequence of this is Hopf degree theorem.

Theorem 1.14 (Hopf Degree Theorem.)**.** *For a closed, oriented, connected manifold M, two maps $f : M \to \mathbb{S}^p$ are smoothly homotopic iff they have the same degree.*

Remark 1.15. The Pontryagin construction should be thought of as a generalization of the notion of a degree of a map in the following sense: a degree is an invariant (a number, i.e. a 0-submanifold of $\mathbb{R}$) associated to the regular value y of a map f when the domain and codomain have the same dimension, and this number does not depend on the point y. Further, Hopf degree theorem allows one to classify homotopy types of maps using the degree.

Similarly, the Pontryagin manifold is an invariant (a framed manifold) associated to the regular value y of a map f in the general case when the domain and codomain have different dimensions, and again, it does not depend on the point y. The theorems above says we can further classify homotopy types of maps from M to $\mathbb{S}^p$ using the framed cobordism classes in M.

The Pontryagin-Thom construction is a generalization of the Pontrjagin construction. See [5] for a preliminary discussion on Pontryagin-Thom construction, and [3] for details.

To state the Pontryagin-Thom construction, we introduce more ideas.

Recall from the Freudenthal suspension theorem that the suspension map $\pi_i(X) \to \pi_{i+1}(SX)$ is an isomorphism for $i < n+1$ when X is an n-connected CW complex. Therefore, in the sequence of iterated suspensions

$$\pi_i(X) \to \pi_{i+1}(SX) \to \pi_{i+2}(S^2X) \to ... \tag{1.16}$$

all maps are eventually isomorphisms.

Definition 1.17. The colimit $\lim_{k\to\infty} \pi_{n+k}(S^kX)$ is the **stable homotopy group** denoted $\pi^s_n(X)$.

In particular, $\pi^s_n(X) = \pi_{n+k}(S^kX)$ for large k. See Chapter 4 of [4] for more about the stable homotopy group.

Theorem 1.18 (Pontrjagin-Thom Construction.)**.** *For a smooth, closed, connected manifold M, there exists a group isomorphism*

$$\pi^s_n(M) \cong \Omega^{(B,f)}_n(M) \tag{1.19}$$

1.3. **Characterization of the Boundary Group ∂P_{n+1} in Terms of the Pontrjagin-Thom Construction.** We proceed to the proof of Theorem 1.2.

The fundamental object is the subset $p(M)$ of Π_n given by the Pontrjagin-Thom construction. We establish some results concerning this subset, and the proof of the theorem will follow.

Given an s-parallelizable closed manifold M^n, choose an embedding $i : M \to \mathbb{S}^{n+k}$ with $k > n+1$. (Such a map can be obtained using Whitney embedding theorem to embed M into $\mathbb{R}^{n+k}$ and taking the one-point compactification.)

Exercise 1.20. * Prove that the embedding i of an s-parallelizable closed manifold M^n into $\mathbb{S}^{n+k}$ with $k > n+1$ is unique up to differentiable isotopy.

Since M is s-parallelizable, the normal bundle of M in $\mathbb{S}^{n+k}$ is trivial. Choose a normal k-frame φ, and make $(M, \varphi) \in \Omega_n^{\text{framed}}$. Then by passing through Pontrjagin-Thom map, we obtain an element $p(M, \varphi) \in \Pi_n := \pi_{n+k}(\mathbb{S}^k) = [\mathbb{S}^{n+k}, \mathbb{S}^k]$Varying φ, we obtain a family of maps $p(M) = \{p(M, \varphi)\} \subseteq \Pi_n$.

The following are the significances of the set $p(M)$:

Lemma 1.21 (Characterization of Manifolds Bounding Parallelizable Ones.)**.** *The subset $p(M) \subseteq \Pi_n$ contains the zero element iff M bounds a parallelizable manifold.*

Lemma 1.22 (p is a map on h-cobordism classes.)**.** *If M_0 and M_1 are h-cobordant, then $p(M_0) = p(M_1)$.*

Lemma 1.23 (Semiadditivity.)**.** *If M, M' are s-parallelizable then*

$$p(M) + p(M') \subseteq p(M \# M') \subseteq \Pi_n \tag{1.24}$$

Lemma 1.25. *The set $p(\mathbb{S}^n)$ is a subgroup of the stable homotopy group Π_n. For any homotopy sphere Σ, the set $p(\Sigma)$ is a coset of this subgroup $p(\mathbb{S}^n)$. This defines a correspondence*

$$p' : \Theta_n \to \Pi_n / p(\mathbb{S}^n) \tag{1.26}$$

$$\Sigma \mapsto p(\Sigma) \tag{1.27}$$

1.4. **Proof of Main Result.** We provide proofs to the above results.

Proof of Lemma 1.21. ($\implies$.) Let M bound the parallelizable manifold W.

Exercise 1.28. *If M bounds a parallelizable manifold W, then an embedding $i : M \to \mathbb{S}^{n+k}$ can be extended to an embedding $\tilde{i} : W \to \mathbb{D}^{n+k+1}$.

Choose a normal k-frame ψ over W, and let φ be the restriction to M.

Exercise 1.29. ** The Pontrjagin-Thom map $P(M, \varphi) : \mathbb{S}^{n+k} \to \mathbb{S}^k$ extends to a map on $\mathbb{D}^{n+k+1}$.

Since the disc is contractible, the map is null-homotopic.

($\impliedby$.) Suppose $p(M, \varphi)$ is null homotopic.

Exercise 1.30. * If $p(M, \varphi)$ is null homotopic, then M bounds a manifold W contained in $\mathbb{D}^{n+k+1}$ where φ extends to a field ψ on W.

Exercise 1.31. * Prove that the normal bundle of M is trivial.

But a submanifold of a sphere with large dimension is s-parallelizable iff its normal bundle is trivial. Also, s-parallelizable manifolds are parallelizable for connected manifolds without boundary. This proves the claim. □

Proof of Lemma 1.22. If $M_0 \amalg (-M_1) = \partial W$, we choose an embedding of W in $\mathbb{S}^{n+k} \times [0,1]$ so that $M_q \to \mathbb{S}^{n+k} \times \{q\}$ for $q = 0, 1$. Let φ_0 be a normal frame on M_0. Choose any normal frame ψ on W which restricts φ_0 on M_0.

Exercise 1.32. * Show that ψ gives rise to a homotopy between $p(M_0, \varphi_0)$ and $p(M_1, \varphi_1)$.

□

Proof of Lemma 1.23. The proof is similar to the proof of Lemma 1.22.

Let W_1, W_2 be $(n+1)$-manifolds with connected boundary. We define the *connected sum along the boundary* of W_1, W_2 denoted $(W_1, \partial W_1) \# (W_2, \partial W_2)$ as follows.

Let H^{n+1} be the closed half-disk given by

$$H^{n+1} := \{x := (x_0, ..., x_n) \in \mathbb{R}^{n+1} : |x| \leq 1,\ x_0 \geq 1\} \tag{1.33}$$

and let $\mathbb{D}^n = \{x \in H^{n+1} : x_0 = 0\}$. Take the embedding $i_q : (H^{n+1}, \mathbb{D}^n) \hookrightarrow (W_q, \partial W_q)$, $q = 1, 2$ so that $i_2 \circ i_1^{-1}$ reverses orientation. Let W be the space given by identifying the points $i_1(tu)$ and $i_2((1-t)u)$ for $t \in (0,1)$, $u \in \mathbb{S}^n \cap H^{n+1}$ in the space $(W_1 - i_1(0)) \amalg (W_2 - i_2(0))$.

Exercise 1.34. * Prove that W is a differentiable manifold with boundary $\partial W_1 \# \partial W_2$, and that it has the homotopy type of $W_1 \wedge W_2$.

Exercise 1.35. * Draw a picture of this construction for the case when 1.) W_1 is the solid torus and W_2 is the 3-ball, and 2.) when W_1 and W_2 are both solid tori.

Let $W_1 := M \times [0,1]$ and $W_2 := M' \times [0,1]$, and let $W := (W_1, \partial W_1) \# (W_2, \partial W_2)$. Then

$$\partial W (M \# M') \amalg (-M) \amalg (-M') \tag{1.36}$$

Choose an embedding of W in $\mathbb{S}^{n+k}$ so that $(-M)$ and $(-M')$ go into separated submanifolds of $\mathbb{S}^{n+k} \times 0$ and so that $M \# M'$ goes into $\mathbb{S}^{n+k} \times 1$. Given normal k-fields φ, φ' on $-M$ and $-M'$, take any extension defined on W, and take the restriction ψ on $M \# M'$.

Exercise 1.37. * Show that this induces a homotopy of $p(M, \varphi) + p(M', \varphi')$ with $p(M \# M', \psi)$.

□

Proof of Lemma 1.25. Using Lemma 1.23 with $\mathbb{S}^n \# \mathbb{S}^n = \mathbb{S}^n$, we have

$$p(\mathbb{S}^n) + p(\mathbb{S}^n) \subseteq p(\mathbb{S}^n) \tag{1.38}$$

and so, $p(\mathbb{S}^n)$ is closed under addition. Since $\mathbb{S}^n$ bounds a parallelizable manifold (i.e. the disc), by Lemma 1.21, it contains the zero element. Thus, $p(\mathbb{S}^n)$ is a subgroup.

Second part follows from a similar argument: using Lemma 1.23 with $\mathbb{S}^n \# \Sigma = \Sigma$, we have

$$p(\mathbb{S}^n) + p(\Sigma) \subseteq p(\Sigma) \tag{1.39}$$

Thus, $p(\Sigma)$ is a union of cosets of $p(\mathbb{S}^n)$. Moreover, from the identity $\Sigma \# (-\Sigma)$ is h-cobordant to $\mathbb{S}^n$, by Lemma 1.22 and Lemma 1.23,

$$p(\Sigma) + p(-\Sigma) \subseteq p(\mathbb{S}^n) \tag{1.40}$$

Thus, $p(\mathbb{S}^n)$ must be a single coset. This proves the second part.

For the third part, the first two parts together shows that p' is a well-defined map from Θ_n to $\Pi_n / p(\mathbb{S}^n)$. The second part together with Lemma 1.23 shows that p' is a homomorphism of abelian groups. □

Proof of Theorem 1.2. By Lemma 1.21, the kerp' is the set of h-cobordism classes of homotopy n-spheres which bound parallelizable manifolds, and thus coincides with the set ∂P_{n+1}. By the First Isomorphism Theorem, $\Theta_n / \partial P_{n+1}$ is isomorphic to the subgroup of $\Pi_n / p(\mathbb{S}^n)$. But Π_n is finite by Serre's result, so $\Theta_n / \partial P_{n+1}$ is finite, as claimed. □

References

[1] Kervaire, "Smooth Homology Spheres and their Fundamental Groups"
[2] Milnor, *Topology from a Differentiable Viewpoint.*
[3] Stong, *Notes on Cobordism Theory.* https://www.jstor.org/stable/j.ctt183pnqj
[4] Hatcher, *Algebraic Topology.*
[5] MathOverflow: Explanation for the Thom-Pontryagin construction (and its generalisations), https://mathoverflow.net/questions/7375/explanation-for-the-thom-pontryagin-construction-and-its-generalisations