1. Kervaire-Milnor II: s-parallelizability of Homotopy Spheres.

We continue our exposition of Kervaire and Milnor, "Groups of Homotopy Spheres: I" *Annals of Mathematics*, 1962.

1.1.

Notation 1.1. Let M be a manifold with tangent bundle TM, and let ε^r denote the rank r trivial bundle over M.

Definition 1.2. The bundle $TM \oplus \varepsilon^1$ is the *stable tangent bundle* of M

Definition 1.3. A manifold M is *s-parallelizable* if the stable tangent bundle $TM \oplus \varepsilon^1$ is a trivial bundle.

Exercise 1.4. Show directly (using differential topology) that $\mathbb{S}^n$ is s-parallelizable.

Here is the main claim.

Theorem 1.5. *Every homotopy sphere is s-parallelizable.*

Remark 1.6. In fact, Theorem 1.5 is true for homology spheres as well. See Kervaire "Smooth Homology Spheres and their Fundamental Groups" and MathOverflow: Are homology spheres stably parallelisable?

Before we give a proof of Theorem 1.5, we first give a discussion on the significance of s-parallelizability.

Lemma 1.7 (J.H.C. Whitehead)**.** *Let M be an n-dimensional submanifold of $\mathbb{S}^{n+k}$, $n < k$. Then M is s-parallelizable iff its normal bundle is trivial.*

Lemma 1.8. *A connected manifold M with non-vacuous boundary is s-parallelizable iff it is parallelizable.*

The second lemma shows that the lack of boundary (e.g. as in $\mathbb{S}^n$) is the only thing that can go wrong for an s-parallelizable manifold to be parallelizable.

We need the following lemma to prove the above two results.

Lemma 1.9. *Let ξ be a k-dimensional vector bundle over an n-dimensional complex, $k > n$. If $\xi \oplus \varepsilon^r$ is trivial, then ξ itself is trivial.*

Proof. Suppose we assume the result for when ϵ is a trivial line bundle. Then to prove the general claim for $r > 1$, we can apply the result for trivial line bundle r times. Thus, we can assume wlog that $r = 1$.

We can also assume that ξ is oriented. Let γ^k be the bundle of oriented k-planes in $(k+1)$-space. γ^k is a line bundle over $\mathbb{S}^k$ by looking at the normal vectors of the k-planes. The vector bundle isomorphism $\xi \oplus \varepsilon^1 \cong \varepsilon^{k+1}$ gives rise to the bundle map $f : \xi \to \gamma^k$. Since the base space of ξ has dimension n, and since the base space of γ^k is the sphere $\mathbb{S}^k$ with $k > n$, the map f is null-homotopic. Thus, ξ is trivial. □

Exercise 1.10. * In the above proof, justify why ξ can be assumed to be oriented.

Exercise 1.11. * Work out the details for the assertion made above that γ^k is a line bundle over $\mathbb{S}^k$.

Proof of Lemma 1.7. ($\Longleftarrow$.) Let ν be the normal bundle of M in $\mathbb{S}^{n+k}$. The bundle $TM \oplus \nu$ is trivial. So in particular, $(TM \oplus \epsilon^1) \oplus \nu$ is trivial, and by Lemma 1.9, $TM \oplus \epsilon^1$ is trivial, as required.

□

Exercise 1.12. * Show that $TM \oplus \nu$ is trivial (by explicitly constructing a vector bundle isomorphism). In particular, where are we using s-parallelisability and the hypothesis about $\mathbb{S}^{n+k}$?

Exercise 1.13. * Prove the forward direction.

Proof of Lemma 1.8. The converse is immediate. The forward direction follows by a similar proof as Lemma 1.7. □

Exercise 1.14. * Prove that any map from M^n into $\mathbb{S}^n$ is null-homotopic. Deduce the forward direction of the above proof as a consequence.

We now provide a proof of the main claim.

Proof of 1.5. □

1.2. **Which homotopy spheres bound parallelizable manifolds?**

References

[1] Kervaire, "Smooth Homology Spheres and their Fundamental Groups"
[2] https://mathoverflow.net/questions/242412/are-homology-spheres-stably-parallelisable