1. Kervaire-Milnor I: Construction of the h-cobordism Group.

We closely follow Kervaire and Milnor, "Groups of Homotopy Spheres: I" *Annals of Mathematics*, 1962. The following is nothing more than a "little bit of water added" to the original article. The main purpose of this exposition is to help me understand what is written in the original works rather than to produce anything new.

1.1. **Conventions.**

Convention 1.1. Unless stated otherwise, all manifolds are smooth, compact, and oriented. They may or may not have boundary, and the boundary will be denoted by ∂M. A manifold with reversed orientation is denoted by $-M$.

Notation 1.2. Disjoint sum is denoted by $\amalg$. Connected sum is denoted by $\#$.

1.2. **Homology Spheres and Homotopy Spheres.** I often get homology spheres and homotopy spheres mixed up (one is obviously a subset of the other), so we will discuss them here.

Definition 1.3. The manifold M is a *homotopy n-sphere* if M is closed and has the homotopy type of the sphere $\mathbb{S}^n$.

A trivial observation is that homotopy spheres are homology spheres, i.e.

$$H^k(M) = H^k(\mathbb{S}^n) = \begin{cases} \mathbb{Z} & k = 0, n \\ 0 & \text{otherwise} \end{cases} \tag{1.4}$$

and $\pi_k(M) = \pi_k(\mathbb{S}^n)$.

Theorem 1.5 (Topological Generalized Poincaré Conjecture.)**.** *An n-dimensional homotopy sphere is homeomorphic to $\mathbb{S}^n$.*

Remark 1.6 (Smooth Generalized Poincaré Conjecture). Milnor's construction of the exotic 7-sphere implies that the smooth generalized Poincaré conjecture is false. It is an open question (as of Feb 2019) whether there are non-trivial smooth homotopy spheres in dimension 4.

Homotopy spheres are the basic object of the Kervaire-Milnor paper.

We will say a few things about homology spheres as well.

Definition 1.7. The manifold M is a *homology n-sphere* if M has the same homology groups as $\mathbb{S}^n$.

Example 1.8. The Poincaré homology sphere is given by identifying opposite faces of a dodecahedron. See more about this on the blog post about Poincaré homology sphere.

Exercise 1.9. * Compute the simplicial homology of the Poincaré homology sphere using Mayer-Vietoris. Also compute π_1 using Seifert van Kampen.

Definition 1.10 (Homology Manifold.)**.** A locally compact topological space X is a *homology manifold* of dimension n if its local homology coincides with the local homology of an n-manifold, i.e.

$$H_k(X, X - x) = \begin{cases} \mathbb{Z} & k = n \\ 0 & \text{otherwise} \end{cases} \tag{1.11}$$

Exercise 1.12. * If A is a homology 3-sphere not homeomorphic to $\mathbb{S}^3$, then ΣA is a homology manifold that is not a topological manifold.

Exercise 1.13. ** With the same notation as in the previous exercise, prove that $\Sigma^2 A$ is homeomorphic to $\mathbb{S}^5$, but its triangulation is not a PL manifold.

1.3. ***h*-cobordism Group.**

Definition 1.14. Two closed n-manifolds M_1 and M_2 are *h-cobordant* if $M_1 + (-M_2) = \partial W$ for some manifold W and M_1 and $-M_2$ are deformation retracts of W.

Remark 1.15. The term "corbodism" refers to the first statement about $M_1 + (-M_2)$ being a manifold. The "homotopy" refers to the second statement about the deformation retraction.

h-cobordism is clearly an equivalence relation.

The following is the central object of the paper.

Theorem 1.16. *The h-corbodism classes of homotopy n-spheres form an abelian group under $\#$.*

Definition 1.17. The group in the previous theorem is the *nth homotopy sphere cobordism group* and is denoted by Θ_n.

Remark 1.18 (Interpretation of the Cobordism Group.). The Poincaré conjecture says that homotopy spheres are homeomorphic to $\mathbb{S}^n$. Smale proves that when $n \neq 3, 4$, two homotopy n-spheres are h-cobordant iff they are diffeomorphic. Thus, for $n \neq 3, 4$, the group Θ_n can be described as the set of all diffeomorphism classes of differentiable structures on topological n-spheres.

FIGURE 1. Table of $|\Theta_n|$ for first several n. (Image taken from the Kervaire-Milnor.) We know that $\Theta_3 = 0$, by Poincaré conjecture.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$[\Theta_n]$	1	1	?	1	1	1	28	2	8	6	992	1	3	2	16256	2	16	16.

Theorem 1.19 (Munkres and Whitehead.). *For $n \leq 3$, a topological n-manifold has differentiable structure which is unique up to diffeomorphism.*

Collary 1.20. $\Theta_n = 0$ *for* $n \leq 3$.

Proof. By Poincaré conjecture, homotopy spheres are homeomorphic to $\mathbb{S}^n$. But since there is a unique differentiable structure on $\mathbb{S}^n$ by the preceding theorem, it follows that all homotopy spheres for dimension at most 3 have unique differentiable structure. The result follows by Remark 1.18. □

1.4. **The h-cobordism group, Θ_n.** The goal of this section is to prove Theorem 1.16.

Unless stated otherwise, from here on, we assume that the dimension n is ≥ 4 since we have $\Theta_n = 0$ for $n \leq 3$ (Corollary 1.20).

Lemma 1.21. *The connected sum operation is well-defined, associative, and commutative up to orientation preserving diffeomorphism. The sphere $\mathbb{S}^n$ serves as the identity element.*

Exercise 1.22. Prove this.

Lemma 1.23. *The connected sum is a well-defined operation on a pair of closed, simply connected, h-cobordant manifolds.*

Proof. Let M_1 and M_1' be closed, simply connected, and h-cobordant so that $M_1 + (-M_1') = \partial W_1$ for some W_1. Take a smooth curve A from $p \in M_1$ to $p' \in -M_1'$ within W_1. Then by taking coordinate charts around each point of the curve, we can see that the tubular neighborhood of A is diffeomorphic to $\mathbb{R}^n \times [0, 1]$. This gives an embedding

$$i : \mathbb{R}^n \times [0, 1] \to W_1 \tag{1.24}$$

with $i(\mathbb{R}^n \times 0) \subseteq M_1,\ i(\mathbb{R}^n \times 1) \subseteq M_1'$, and $i(0 \times [0, 1]) = A$.

□

Exercise 1.25. **Eliminate the hypothesis for the simple connectivity of the two manifolds.

Lemma 1.26. *A simply connected manifold M is h-cobordant to the sphere $\mathbb{S}^n$ iff M bounds a contractible manifold.*

Proof. ($\implies$.) If $M + (-\mathbb{S}^n) = \partial W$, then gluing in a disk $\mathbb{D}^{n+1}$ inside W along $-\mathbb{S}^n$, we obtain a manifold W' such that $\partial W' = M$. If $\mathbb{S}^n$ is a deformation retract of W, then by extending the retraction map to W', we see that W' deformation retracts to $\mathbb{D}^{n+1}$, and so W' is contractible.

□

Exercise 1.27. * Fill in details for the converse ($\impliedby$.) in the above proof.

Lemma 1.28. *If M is a homotopy sphere, then $M\#(-M)$ bounds a contractible manifold.*

Proof. The idea is to use a clever surgery. Let $H^2 \subseteq \mathbb{D}^2$ denote the top half of a disc, i.e. the points $(t\sin\theta, t\cos\theta) \in \mathbb{D}^2$ for $t \in [0,1],\ \theta \in [0,\pi]$, and let $\frac{1}{2}\mathbb{D}^n \subseteq \mathbb{D}^n$ denote the disk of radius $\frac{1}{2}$.

Given an embedding $i : \mathbb{D}^n \to M$, form W from following identification: take the set

$$\left(M - i\left(\frac{1}{2}\mathbb{D}^n\right)\right) \times [0,\pi] \coprod \mathbb{S}^{n-1} \times H^2 \tag{1.29}$$

Then identify $i(tu) \times \theta$ with $u \times ((2t-1)\sin\theta, (2t-1)\cos\theta)$ for each $\frac{1}{2} < t \leq 1,\ 0 \leq \theta \leq \pi$. □

Exercise 1.30. * Verify that W is a differentiable manifold. (Hint: Totally disjoint action.)

Exercise 1.31. * Verify that $\partial W = M\#(-M)$.

Exercise 1.32. * Verify that W deformation retracts to $M - \text{Interior}\left(i\left(\frac{1}{2}\mathbb{D}^n\right)\right)$.

Exercise 1.33. * Give a concrete example that helps visualize the previous proof.

The proof of Theorem 1.16 follows immediately from the above four lemmas.

Proof of Theorem 1.16. By Lemma 1.21 and 1.23, the connected sum operation is a well-defined, associative, commutative operation on Θ_n with identity $\mathbb{S}^n$. By Lemma 1.28, for every $[M] \in \Theta_n$, there is $-M$ such that $M\#(-M)$ is contractible. But by Lemma 1.26, $M\#(-M)$ is a trivial element, and so every element in Θ_n has an inverse. Thus, $(\Theta_n, \#)$ is an abelian group. □

References

[1] Kervaire and Milnor, "Groups of Homotopy Spheres: I" *Annals of Mathematics.*
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[3] https://en.wikipedia.org/wiki/Homology_sphere
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