Cooley & Hansen (1989) Replication
The inflation tax in a real business cycle model

Jonas Bruno, Marcos Constantinou

5th January 2026

University of Lausanne – Faculty of Business and Economics
MScE Dynamic Macroeconomic Models

Contents

List of Tables

List of Figures

1 Introduction

Does money matter? In the typical Real Business Cycles (RBC) model, the answer is no. This is from the fact that the model is in "real terms", thus leading to the result of money neutrality as none of the main variables related to output, consumption, and labour are affected by nominal variables. Cooley & Hansen (1989) explore whether the result holds up in the case of an RBC model with money through a Cash-In-Advance (CIA) constraint. Specifically, they explore the role of inflation on real variables as well as the associated welfare losses. The authors mention that the structure of their model gives no role to unanticipated inflation as they do not include sticky price mechanisms, and thus explain that their results are looking at the impact of anticipated inflation. Their decision to implement a strictly real model could be explained either by them being neo-classical theorists, or by choice in order to strictly only study the effects of anticipated inflation, or by both.

Cooley & Hansen describe their model as a one-sector stochastic optimal growth model featuring indivisible labour with an employment lottery in a real economy. Stochasticity is implemented through exogenous technology (ϵ_t), and money growth (ξ_t) shocks. Money is introduced using the CIA constraint, applying only to the consumption good. This means that agents within the model must hold cash in order to consume goods, and otherwise would not like to hold any as it offers no returns compared to other asset classes (capital). This also shows the mechanism through which the inflation tax operates: if agents want to reduce cash holdings in response to higher inflation, they can only do so through a reduction in consumption of goods.

The authors compare this model to the framework described in Hansen (1985), which is identical in every respect except for the inclusion of the CIA constraint. Consequently, whenever this constraint is not binding, the simulated economy behaves exactly like the original Hansen model. Specifically, the CIA constraint remains non-binding provided that money is supplied optimally, which is when the gross growth rate of money, g, equals the household's inter-temporal discount factor, β. This distinction allows the comparison between the model under optimal circumstances (non-binding, where $g = \beta$) and sub-optimal circumstances (binding, where $g > \beta$). This comparison is the basis of the welfare analysis of the inflation tax.

In this paper, we attempt to replicate their analysis of the impact of the growth rate of money on the business cycle, the inflation tax, and its effects on welfare. We do so by deriving the model manually in section 2, in which we also include log-linearisation of certain equations, and the values used for calibration. We then use Dynare in order to solve and simulate the model in order to replicate the two main tables present in the original paper, which we show in section 3. We additionally include Impulse Response Functions (IRFs) to the two main shocks on key variables, as well as an explanation of the economic intuition behind them. Finally, in section 4, we describe our experience related to the replication.

2 Model

2.1 Setup

As described earlier, the model featured in Cooley & Hansen (1989) is an RBC model with production, capital, indivisible labour, money (CIA constraint), and stochasticity through exogenous technology & money growth shocks.

2.1.1 Households

The households' utility function is described in equation (1), where the representative agent derives utility from both consumption, c_t, and leisure, l_t. They are endowed with one unit of time each period which is split between a fixed amount of work hours (since labour is indivisible), h_0, and leisure. They do not decide whether they are employed, however, as employment is distributed via the labour lottery equation (2) which determines which households work or not. The households themselves only choose π_t, the probability of working in period t. Essentially, rather than choosing exact work hours, households decide their expected amount of hours worked h_t which is computed in (2). With this, we combine equations (1) and (2) in (3) in order to specify utility as a function of labour instead of leisure, which we then rewrite to obtain equation (4) that will be used in the optimisation problem. Households are subject to two constraints: the budget constraint (10) and the CIA constraint (8), in which money growth dynamics are described in (5) & (6) and capital dynamics in (7). Note that uppercase variables are per capita aggregates while lowercase are individual. This is visible in the budget constraint (8), where agents enter period t with nominal money balances equal to m_{t-1} in addition to a lump-sum transfer equal to $(g_t - 1)M_{t-1}$, a fraction of the per capita money supply. We do some algebra in order to include money and capital dynamics in the two constraints to obtain the final forms (9) & (11).

Preferences:

$$U(c_t, l_t) = \log(c_t) + A\log(l_t) \tag{1}$$

Labour lottery:

$$h_t = \pi_t h_0 \tag{2}$$

Period Utility:

$$\begin{aligned} U(c_t, h_t) &= \log(c_t) + \pi_t A \log(1 - h_0) + (1 - \pi_t) A \log(1) \qquad (3) \\ &= \log(c_t) + h_t A \frac{\log(1 - h_0)}{h_0} \\ &= \log(c_t) - B h_t \qquad (4) \\ \text{where } B &\equiv -A\frac{\log(1 - h_0)}{h_0} \end{aligned}$$

Money Growth:

$$M_t = g_t M_{t-1} \tag{5}$$

$$\log(g_{t+1}) = \alpha \log(g_t) + (1-\alpha)\log(\bar{g}) + \xi_{t+1} \tag{6}$$

$$\text{where } \xi_t \overset{\text{iid}}{\sim} N(0, \sigma_\xi^2)$$

Capital Stock Dynamics:

$$k_{t+1} = (1-\delta)k_t + x_t, \quad \text{where } 0 \leq \delta \leq 1 \tag{7}$$

CIA Constraint:

$$p_t c_t \leq m_{t-1} + (g_t - 1)M_{t-1} \tag{8}$$

$$\frac{p_t c_t}{M_t} \leq \frac{m_{t-1} + (g_t - 1)M_{t-1}}{M_t}$$

$$\hat{p}_t c_t \leq \frac{m_{t-1} + (g_t - 1)M_{t-1}}{g_t M_{t-1}}$$

$$\hat{p}_t c_t \leq \frac{\hat{m}_{t-1} + (g_t - 1)}{g_t} \tag{9}$$

$$\text{where } \hat{p}_t = \frac{p_t}{M_t}, \quad \& \quad \hat{m}_t = \frac{m_t}{M_t}$$

Budget Constraint:

$$c_t + x_t + \frac{m_t}{p_t} \leq w_t h_t + r_t k_t + \frac{m_{t-1} + (g_t - 1)M_{t-1}}{p_t} \tag{10}$$

$$c_t + k_{t+1} + \frac{\hat{m}_t}{\hat{p}_t} \leq w_t h_t + r_t k_t + (1-\delta)k_t + \frac{\hat{m}_{t-1} + (g_t - 1)}{\hat{p}_t g_t}$$

$$k_{t+1} + \frac{\hat{m}_t}{\hat{p}_t} \leq w_t h_t + r_t k_t + (1-\delta)k_t \tag{11}$$

2.1.2 Firms

Firms in this model behave in a very standard way, with a Cobb-Douglas production function (12) using capital, K_t, and labour, H_t as inputs. These are magnified by the exponential of technology, z_t, following an AR(1) process described by (13). We now have the two exogenous shocks, ϵ_t & ξ_t, that we will use to perturbate our economy in section 3.

Production:

$$Y_t = \hat{z}_t K_t^{\theta} H_t^{1-\theta} \tag{12}$$
$$\text{where } 0 \leq \theta \leq 1 \quad \& \quad \hat{z}_t = \exp(z_t)$$

Technology:

$$z_{t+1} = \gamma z_t + \epsilon_{t+1} \tag{13}$$
$$\text{where } 0 \leq \gamma \leq 1 \quad \& \quad \epsilon_t \overset{\text{iid}}{\sim} N(0, \sigma_\epsilon^2)$$

2.2 Optimisation

2.2.1 Households

Households maximise the expected path of their inter-temporal utility subject to the budget and CIA constraints, as shown in the Lagrangian (14). They must choose consumption, c_t, labour, h_t, cash holdings, $\hat{m}_t$, and investment, x_t. The FOCs are described in equations (15), (16), (17), and (18). We combine the second and fourth conditions to obtain the Euler equation (19), and the first three conditions to obtain the Labour Supply equation (20).

$$\begin{aligned}\mathcal{L} = \max_{\{c_t, h_t, \hat{m}_t, k_{t+1}\}} \mathbb{E}_0 \Bigg\{ \sum_{t=0}^{+\infty} \beta^t (\log c_t - B h_t) \\ + \lambda_{t,1} \left[\frac{\hat{m}_{t-1} + (g_t - 1)}{g_t} - \hat{p}_t c_t \right] \\ + \lambda_{t,2} \left[r_t k_t + (1-\delta) k_t + w_t h_t - k_{t+1} - \frac{\hat{m}_t}{p_t} \right] \Bigg\}\end{aligned} \tag{14}$$

FOCs:

$$0 \overset{!}{=} \frac{\partial \mathcal{L}}{\partial c_t} = \beta^t \frac{1}{c_t} - \lambda_{t,1} \hat{p}_t \tag{15}$$

$$0 \overset{!}{=} \frac{\partial \mathcal{L}}{\partial h_t} = -\beta^t B + \lambda_{t,2} w_t \tag{16}$$

$$0 \overset{!}{=} \frac{\partial \mathcal{L}}{\partial \hat{m}_t} = -\frac{\lambda_{t,2}}{\hat{p}_t} + \mathbb{E}_t \left[\frac{\lambda_{t+1,1}}{g_{t+1}} \right] \tag{17}$$

$$0 \overset{!}{=} \frac{\partial \mathcal{L}}{\partial k_{t+1}} = -\lambda_{t,2} + \mathbb{E}_t [\lambda_{t+1,2}(1 + r_{t+1} - \delta)] \tag{18}$$

Euler:

$$\mathbb{E}_t\left[\beta\frac{w_t}{w_{t+1}}(1+r_{t+1}-\delta)\right]=1 \tag{19}$$

Labour Supply:

$$\mathbb{E}_t\left[\frac{\beta}{g_{t+1}c_{t+1}\hat{p}_{t+1}}\right]=\frac{B}{w_t\hat{p}_t} \tag{20}$$

2.2.2 Firms

Firms maximise their profits as described in (21) with respect to both labour and capital. The FOCs are described in equations (22) & (23), which we rewrite to obtain the Labour Demand equation (24) and the Capital Demand equation (25).

$$\max_{\{K_t,H_t\}}\Pi_t=\hat{z}_tK_t^{\theta}H_t^{1-\theta}-w_tH_t-r_tK_t \tag{21}$$

FOCs:

$$0\overset{!}{=}\frac{\partial\Pi_t}{\partial H_t}=-w_t+(1-\theta)\hat{z}_tK_t^{\theta}H_t^{-\theta} \tag{22}$$

$$0\overset{!}{=}\frac{\partial\Pi_t}{\partial K_t}=-r_t+\theta\hat{z}_tK_t^{\theta-1}H_t^{1-\theta} \tag{23}$$

Labour Demand:

$$w_t=(1-\theta)\hat{z}_t\left(\frac{K_t}{H_t}\right)^{\theta} \tag{24}$$

Capital Demand:

$$r_t=\theta\hat{z}_t\left(\frac{K_t}{H_t}\right)^{\theta-1} \tag{25}$$

2.2.3 Market Clearing

Before establishing the equilibrium of this economy, we must impose the market clearing equation (26). In addition to this, since we are in the representative agent case, all per capita aggregate variables are equal to their individual levels, as described in equations (27), (28), (29), and (30).

Goods:

$$y_t=c_t+x_t \tag{26}$$

Aggregates:

$$K_t = k_t \tag{27}$$

$$H_t = h_t \tag{28}$$

$$C_t = c_t \tag{29}$$

$$M_t = m_t \tag{30}$$

$$\implies \hat{m}_t = 1 \ \forall t$$

2.3 Equilibrium

2.3.1 System

We can now describe the equilibrium conditions that we then input into Dynare to solve. We have a system of 10 equations and 10 unknowns.

10 Unknowns:

$$\begin{cases} \text{3 state variable} : k_t, z_t, g_t \\ \text{7 control variables} : c_t, h_t, \hat{p}_t, w_t, r_t, y_t, x_t \end{cases}$$

10 Equations:

$$\text{1)} \quad \text{Technology:} \quad z_{t+1} = \gamma z_t + \epsilon_{t+1} \tag{31}$$

$$\text{2)} \quad \text{Money Growth:} \quad \log(g_{t+1}) = \alpha \log(g_t) + (1-\alpha)\log(\bar{g}) + \xi_{t+1} \tag{32}$$

$$\text{3)} \quad \text{Euler:} \quad \mathbb{E}_t \left[\beta \frac{w_t}{w_{t+1}} (1 + r_{t+1} - \delta) \right] = 1 \tag{33}$$

$$\text{4)} \quad \text{Labour Supply:} \quad \mathbb{E}_t \left[\frac{\beta}{g_{t+1} c_{t+1} \hat{p}_{t+1}} \right] = \frac{B}{w_t \hat{p}_t} \tag{34}$$

$$\text{5)} \quad \text{Budget Constraint:} \quad k_{t+1} + \frac{1}{\hat{p}_t} = r_t k_t + (1-\delta) k_t + w_t h_t \tag{35}$$

$$\text{6)} \quad \text{CIA Constraint:} \quad \hat{p}_t c_t = \frac{\hat{m}_{t-1} + (g_t - 1)}{g_t} = 1 \tag{36}$$

$$\text{7)} \quad \text{Production Function:} \quad y_t = \hat{z}_t k_t^{\theta} h_t^{1-\theta} \tag{37}$$

$$\text{8)} \quad \text{Labour Demand:} \quad w_t = (1-\theta) \hat{z}_t \left(\frac{k_t}{h_t} \right)^{\theta} \tag{38}$$

$$\text{9)} \quad \text{Capital Demand:} \quad r_t = \theta \hat{z}_t \left(\frac{k_t}{h_t} \right)^{\theta - 1} \tag{39}$$

$$\text{10)} \quad \text{Goods Market Clearing:} \quad y_t = c_t + x_t \tag{40}$$

2.3.2 Steady State

In order to input our model into DYNARE, we need to take all the equilibrium equations above and find their steady-states.

1) Technology: $\log(z) = 0 \implies z = 1$

2) Money Growth: $g = \bar{g}$

3) Euler: $\dfrac{1}{\beta} = 1 + r - \delta$

4) Labour Supply: $\dfrac{\beta}{gc} = \dfrac{B}{w}$

5) Budget Constraint: $\dfrac{1}{\hat{p}} = (r - \delta)k + wh$

6) CIA Constraint: $\hat{p}c = 1$

7) Production Function: $y = k^{\theta} h^{1-\theta}$

8) Labour Demand: $w = (1 - \theta) \left(\dfrac{k}{h}\right)^{\theta}$

9) Capital Demand: $r = \theta \left(\dfrac{k}{h}\right)^{\theta - 1}$

10) Goods Market Clearing: $y = c + x$

We must now simplify the system. As seen in the derivation below, we first start by isolating r, c, and $\hat{p}$. This allows us to replace these variables in the budget constraint giving us

$$c = (r - \delta)k + (1 - \theta)(r/\theta)^{\frac{\theta}{\theta - 1}} (r/\theta)^{\frac{1}{1-\theta}} k\,,$$

which we then use to isolate k. We can now use the firm FOCs to isolate h and w. Finally, the production function isolates y, and thus the market clearing equation isolates x.

$$r = \frac{1}{\beta} - (1-\delta)$$

$$c = \frac{\beta w}{gB}$$

$$\hat{p} = \frac{1}{c}$$

$$k = \frac{c}{r/\theta - \delta}$$

$$h = \left(\frac{r}{\theta}\right)^{\frac{1}{1-\theta}} k$$

$$w = (1-\theta)\left(\frac{r}{\theta}\right)^{\frac{\theta}{\theta-1}}$$

$$y = k^{\theta} h^{1-\theta}$$

$$x = y - c$$

2.3.3 Log-Linearisation

As Cooley & Hansen (1989) approximate their model up to the second order, we input the above system directly into Dynare for it to approximate to the second order as well. In order to demonstrate that we understand the process used, we log-linearise a few equations manually up to the first order, according to the general formula shown below. We chose to approximate the Euler equation, the labour supply condition, labour demand, money growth, and the CIA constraint.

Formula:

$$f(x_t, y_t) = 0$$

$$\frac{\partial f(x,y)}{\partial x} x\tilde{x}_t + \frac{\partial f(x,y)}{\partial y} y\tilde{y}_t = 0$$

Where (x_t, y_t) are the variables in levels, (x, y) are their steady-states, and $(\tilde{x}_t, \tilde{y}_t)$ are the percentage deviations from the steady-state. Formally, $\tilde{x}_t \equiv \frac{x_t - x}{x}$.

Euler:

$$\beta\frac{w_t}{w_{t+1}}([1+r_{t+1}]-\delta)-1=f(w_t,w_{t+1},[1+r_{t+1}])=0$$

$$\beta\frac{1}{w}(1+r-\delta)w\tilde{w}_t-\beta\frac{w}{w^2}(1+r-\delta)w\tilde{w}_{t+1}+\beta\frac{w}{w}(1+r)[1\widetilde{+r_{t+1}}]=0$$

$$\tilde{w}_t-\mathbb{E}_t(\tilde{w}_{t+1})+\beta r\mathbb{E}_t(\tilde{r}_{t+1})=0$$

Labour Supply:

$$\frac{\beta}{g_{t+1}c_{t+1}\hat{p}_{t+1}}-\frac{B}{w_t\hat{p}_t}=f(g_{t+1},c_{t+1},\hat{p}_{t+1},w_t,p_t)=0$$

$$-\frac{\beta}{g^2c\hat{p}}g\tilde{g}_{t+1}-\frac{\beta}{gc^2\hat{p}}c\tilde{c}_{t+1}-\frac{\beta}{gc\hat{p}^2}\hat{p}\tilde{\hat{p}}_{t+1}+\frac{B}{w^2\hat{p}}w\tilde{w}_t+\frac{B}{w\hat{p}^2}\hat{p}\tilde{\hat{p}}_t=0$$

$$\frac{\beta}{gc\hat{p}}\mathbb{E}_t(\tilde{g}_{t+1}+\tilde{c}_{t+1}+\tilde{\hat{p}}_{t+1})=\frac{\beta}{w\hat{p}}(\tilde{w}_t+\tilde{\hat{p}}_t)$$

$$\mathbb{E}_t(\tilde{g}_{t+1}+\tilde{c}_{t+1}+\tilde{\hat{p}}_{t+1})=\tilde{w}_t+\tilde{\hat{p}}_t$$

Labour Demand:

$$0=f(w_t,\hat{z}_t,k_t,h_t)=w_t-(1-\theta)\hat{z}_t\left(\frac{k_t}{h_t}\right)^\theta$$

$$\begin{aligned}0=w\tilde{w}_t-(1-\theta)\left(\frac{k}{h}\right)^\theta\hat{z}\tilde{\hat{z}}_t\\-\theta(1-\theta)\hat{z}\left(\frac{k}{h}\right)^{\theta-1}\frac{k}{h}\tilde{k}_t\\+\theta(1-\theta)\hat{z}\left(\frac{k}{h}\right)^{\theta-1}\frac{k}{h}\tilde{h}_t\end{aligned}$$

$$\tilde{w}_t-\tilde{\hat{z}}_t=\theta(\tilde{k}_t-\tilde{h}_t)$$

Money Growth: Since money growth is already linear, we just need to subtract the steady-state value from both sides of the equation in order to obtain the percentage deviations from the steady-state.

$$\log(g_{t+1})=\alpha\log(g_t)+(1-\alpha)\log(\bar{g})+\xi_{t+1}$$

$$\log(g_{t+1})-\log(\bar{g})=\alpha\log(g_t)+(1-\alpha)\log(\bar{g})-\log(\bar{g})+\xi_{t+1}$$

$$\tilde{g}_{t+1}=\alpha\tilde{g}_t+\xi_{t+1}$$

CIA Constraint

$$1 - \hat{p}_t c_t = f(\hat{p}_t, c_t) = 0$$
$$-c\hat{p}\tilde{\hat{p}}_t - \hat{p}c\tilde{c}_t = 0$$
$$\tilde{\hat{p}}_t + \tilde{c}_t = 0$$

2.4 Calibration

The last step before being able to simulate the modelled economy and analyse the results is to calibrate the parameters. We do so with the values described by the authors visible below. These are based on growth observations and the results of studies using microeconomic data, assuming periods are in quarters. Note that we will use various values for $\bar{g}$ to construct all the tables, though the “base” value we use is 1.024 (which is used to compute the IRFs).

$$\beta = 0.99$$
$$\theta = 0.36$$
$$\delta = 0.025$$
$$B = 2.86$$
$$\gamma = 0.95$$
$$\alpha = 0.48$$
$$\bar{g} = 1.024$$
$$\sigma_\epsilon = 0.00721$$
$$\sigma_\xi = 0.009$$

3 Results

3.1 Replication: Cyclical Properties

As mentioned in the introduction, COOLEY & HANSEN (1989) present two primary sets of results: the business cycle moments and the welfare costs of inflation. The first of these (Table 1) summarizes the cyclical properties of the model via standard deviations and correlations with output. This table is structured into four quadrants. The top-left quadrant reports historical U.S. statistical moments spanning 1955 Q3 to 1984 Q1.[1] Due to the proprietary nature of the original dataset, empirical price level data was omitted from this replication as we were not able to find a good source, though other aggregates were sourced from the FRED database[2]. The remaining quadrants display moments simulated from the model economy under three monetary regimes. The top-right quadrant assumes a constant money growth rate, whereas the bottom row utilizes the AR(1) process described in equation (6). We compare two different steady-state values of money growth, $\bar{g} = 1.015$ and $\bar{g} = 1.15$.

In the spirit of this exercise we want to stay as close to the authors' simulation-based methodology. Nonetheless, we do stray away from their original Value Function Iteration approach and instead utilize DYNARE's perturbation methods. Following the authors' protocol, we perform 50 simulations of 115 periods each. To ensure replicability across the different growth regimes, a fixed seed of 42 was applied. We extract the raw simulated series, apply the HP filter (with $\lambda = 1600$), and calculate the second-order statistical moments for each separate simulation. The values reported are the averages across these 50 realizations, with the standard deviation of the simulations reported in parentheses to capture sampling uncertainty.

Interestingly, the cyclical moments of the table's bottom two economies are nearly identical, despite our use of a second-order solution which should take into account smaller differences. This indicates that while the steady-state inflation rate $\bar{g}$ creates significant welfare distortions (shown in the next section), it does not materially alter the economy's second-order response to shocks. This is also due to the fact that the seed used in all three simulations is identical. The bottom two (autoregressive) economies therefore have the exact same shocks, while the top right economy has only a productivity shock (as money growth is constant). While COOLEY & HANSEN (1989) report slightly different values across regimes well within one standard deviation. These differences appear to be the result of sampling noise from using unique shock sequences. Thus, we cannot reject the idea that these two economies are almost identical in their shock responses, despite the different values of $\bar{g}$.[3]

Our findings confirm the authors' quite well as they state that "The cyclical properties of

[1] Note that productivity is simply defined as output divided by hours.

[2] Series used GNPC96 (Output), GPDIC1 (Investment), AWHMAN (Hours) and Consumption is the sum of PCEND (Consumption non durables) and PCES (Consumption durables).

[3] Using different seeds, we do get very small differences (well below one standard deviation). These are of the same magnitude as in COOLEY & HANSEN (1989) suggesting that they do not use the same "seed" as we do.

the economy are unaffected by the average growth rate ($\bar{g}$) of the money shock", which we found as well. All of this suggests that in the Cooley-Hansen framework, the magnitude of $\bar{g}$ primarily exerts a level effect on the steady-state rather than significantly altering the transmission mechanism of technology and monetary shocks.

3.1.1 Price Level Reconstruction

We note that the price level variable present in Table 1, is the nominal price level p and not the ratio $\hat{p}$. We however could not directly include the nominal price level nor the nominal money stock M_t in the model as they are non-stationary. This is due to the fact that they grow at an average rate of $\bar{g}$ each period, forever. This non-stationarity causes a problem in DYNARE as it solves models by perturbing around a stationary steady-state. Thus, these variables must be transformed as shown in (41) to be included in the model. To retrieve the nominal price level, we must then reconstruct it from the raw simulation data. Our first step is to take the log of money growth and create the money growth path by generating its cumulative sum, a vector of length 115 where each value is the sum of all past and the current $\log(g)$. This is then de-trended using the HP Filter to get the fluctuations of money growth[4]. Equation (42) shows that we can then simply add back $\log(M_t)$ to $\log(\hat{p}_t)$ to retrieve the log of the nominal price.

$$\hat{p}_t \equiv \frac{p_t}{M_t} \tag{41}$$

$$\log(\hat{p}_t) = \log(p_t) - \log(M_t) \tag{42}$$

3.2 Replication: Welfare Analysis

The authors proceed to do a welfare analysis by computing the increase in consumption needed to be as well off as under the Pareto optimal allocation. According to the authors this would be if there was no CIA constraint or alternatively if it is not binding, which occurs when $\bar{g} = \beta$ in the case of auto-regressive money growth. We ran the model five times with the given money growth rates 0.99, 1.00, 1.024, 1.19 and 1.41. As this table only uses steady-state values, we directly use DYNARE's analytical results for calculations.[5] The motivation behind this analysis are the results of Table 1, where the autoregressive economies reported identical values. This is because the money growth $\bar{g}$ primarily affects the steady-state values of the economy while not impacting the dynamics of shock responses.

We calculate the utility losses by comparing the steady-state utility to the Pareto optimal utility achievable when the CIA is not binding ($\bar{g} = \beta$)[6], while C^* and H^* are the steady-

[4] M_0 gets removed by the HP Filter as it is constant.

[5] We extract these steady-state values via a verbatim command into CSV files and load them into R to do the welfare analysis. As we only need steady-state values this is more than sufficient.

[6] This is essentially the Friedman Rule, as holding money has no opportunity cost. Importantly, the constraint still exists in the model but becomes non-binding.

Table 1: Standard Deviations and Correlations with Output

	Quarterly U.S. Time Series		Constant Growth Rate	
Series	Stan. Dev.	Corr w/ Output	Stan. Dev.	Corr w/ Output
Output	1.79	1.00	1.71 (0.18)	1.00 (0.00)
Consumption	0.85	0.11	0.49 (0.06)	0.88 (0.02)
Investment	7.43	0.92	5.44 (0.57)	0.99 (0.00)
Capital Stock			0.46 (0.09)	0.07 (0.05)
Hours	1.04	0.82	1.30 (0.14)	0.98 (0.00)
Productivity	1.11	0.84	0.49 (0.06)	0.88 (0.02)
Price Level			0.49 (0.06)	-0.88 (0.02)

	AR Growth (g=1.015)		AR Growth (g=1.15)	
Series	Stan. Dev.	Corr w/ Output	Stan. Dev.	Corr w/ Output
Output	1.79 (0.26)	1.00 (0.00)	1.79 (0.26)	1.00 (0.00)
Consumption	0.66 (0.08)	0.69 (0.05)	0.66 (0.08)	0.69 (0.05)
Investment	5.82 (0.85)	0.97 (0.01)	5.82 (0.85)	0.97 (0.01)
Capital Stock	0.50 (0.12)	0.06 (0.06)	0.50 (0.12)	0.06 (0.06)
Hours	1.36 (0.19)	0.98 (0.01)	1.36 (0.19)	0.98 (0.01)
Productivity	0.52 (0.09)	0.86 (0.02)	0.52 (0.09)	0.86 (0.02)
Price Level	1.96 (0.27)	-0.26 (0.16)	1.96 (0.27)	-0.26 (0.16)

Note: The figures in parentheses represent the sample standard errors of the statistical moments.

state consumption and labour hour values when the CIA constraint is binding (any $\bar{g} > \beta$). This gives us (43), where ΔC is the utility compensation needed to achieve Pareto Optimal Utility. We rearrange this equation and solve for ΔC, shown in equation (44). We then use this value to calculate the percentage increase needed to achieve $\bar{U}$, as shown in equation (45), which the authors use as a measure of welfare loss. The results for this are reported in Table 2.[7]

$$\bar{U} = \log\left(C^* + \Delta C\right) - BH^* \tag{43}$$

$$\Delta C = \exp\left(\bar{U} + BH^*\right) - C^* \tag{44}$$

[7]Note that the authors calculate values for both quarterly and monthly time intervals, as empirical evidence suggest that the CIA constraint holds for a month rather than a quarter. We only replicate the quarterly findings as we shifted our focus to more accurate replication.

$$\frac{\Delta C}{C} \times 100 \tag{45}$$

Table 2: Welfare Costs Associated with Different Annual Growth Rates of Money

	Annual Inflation Rate				
Variable	-4 Percent	0.0 Percent	10 Percent	100 Percent	400 Percent
$\bar{g} =$	β	1.000	1.024	1.190	1.410
Steady State					
Output	1.115	1.104	1.078	0.927	0.783
Consumption	0.829	0.821	0.801	0.690	0.582
Investment	0.286	0.283	0.276	0.238	0.201
Capital Stock	11.432	11.318	11.053	9.511	8.027
Hours	0.301	0.298	0.291	0.250	0.211
Welfare Costs					
$\Delta C/C \times 100$	0.000	0.144	0.520	4.014	10.215
$\Delta C/Y \times 100$	0.000	0.107	0.387	2.984	7.596

Notably, our results align perfectly with the original findings of Cooley & Hansen (1989). This high degree of consistency is expected, as Table 2 relies on steady-state calculations where numerical discrepancies between Value Function Iteration and Dynare's solution methods are negligible. Although subtle differences in computational precision surely exist, they do not manifest within the first three decimal places. Our findings thus provide a successful replication, reinforcing the robustness of the authors' original findings.

3.3 Impulse Response Functions

Additionally to the replication of the two tables present in the original paper, we include two Impulse Response Functions (IRFs) figures plotting the reaction of our main variables to a technology shock (increase in ϵ_t) and a money growth shock (increase in ξ_t).

The technology shock, shown in Figure 1, acts much in the same way as in a standard RBC model. We have that production, consumption, investment, labour hours, real wages, capital, and the real interest rate all increase sharply from the steady-state, before gradually returning to it. Output and investment increase sharply and early, as firms rush to take advantage of the lower relative cost of production which pushes up labour demand and thus real wages with it. Consumption, real wages, and capital exhibit a hump-shaped response likely explained by the path of the real interest rate. Since it starts off with a sharp increase, agents do not up their consumption fully as it becomes very beneficial to save. Later, once the interest rate goes down and even below the steady-state value (as the investment surge dies down), consumption increases much more reaching its peak before then gradually converging back to the steady-state. Capital reaches it peak a bit after consumption as it needs time to adjust (law of motion), while wages reach their peak at around the same moment as consumption. This increase in real wages can first be explained by higher labour demand due to higher marginal productivity of labour which pushes up real wages. Even though labour hours decrease gradually after the initial jump, wages continue rising as demand for goods (which translates into additional labour demand) increases with consumption. Hours increase by a lot as labour supply is infinitely elastic at the aggregate level (indivisible labour), but then decrease below steady-state as agents reduce their consumption and opt for more leisure as the technological shock disappears. The IRF for $\hat{p}_t$ exhibits the inverse behaviour of that of consumption, due to the inverse relationship showed in (36). We can rewrite this equation into $c_t = M_t/p_t$ which shows that the CIA constraint requires that the real stock of money is equal to the amount of consumption. Thus, the IRF for $\hat{p}_t$ can be interpreted as an increase in the real stock of money caused by the rise in consumption.

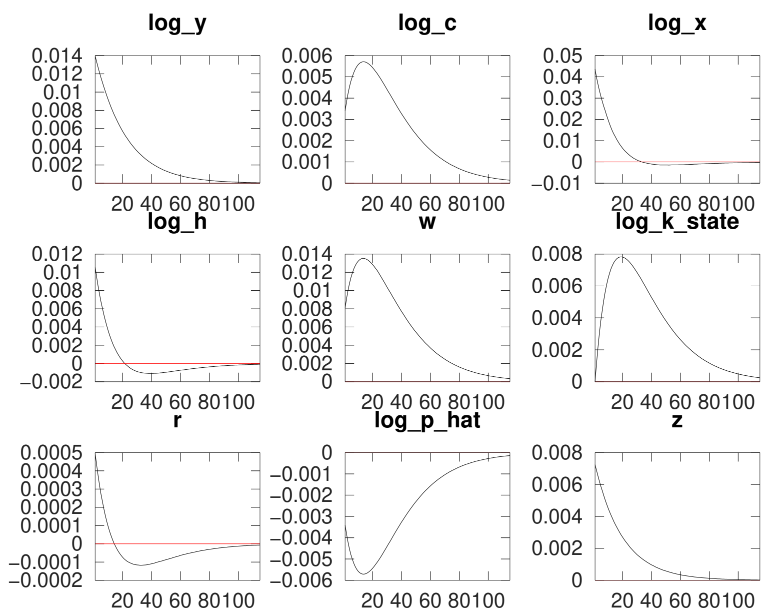

Figure 1: IRF from technology shock (ϵ_t)

The first thing we notice from the money growth shock, shown in Figure 2, is the very small impact relative to the previous technology shock. This is likely from the fact that the shock itself does not directly impact production or consumption like technology does, but instead only introduces a distortion. As seen from the replicated tables, introducing stochastic money growth does not change the statistical moments that much. The main mechanism of this shock is through the relative cost of consumption against all other assets. When money growth increases, there is inflation and a tax associated with it. Since the CIA constraint only applies to consumption of goods and not on leisure, capital, or investment, an increase in inflation makes goods relatively more expensive as agents must hold more deteriorating cash. This is why consumption and hours decrease in the IRFs, as agents substitute goods with leisure which reduces demand and explains why output declines and wages increase (less labour supply). In order to protect themselves against this inflation tax, agents also invest more into physical capital which is not subject to the cash constraint and thus unaffected by inflation. From the Fisher relationship, as inflation increases, we see the real interest rate decrease. Finally, $\hat{p}_t$ increases mechanically since its denominator, aggregate money supply, grows faster with a higher g_t.

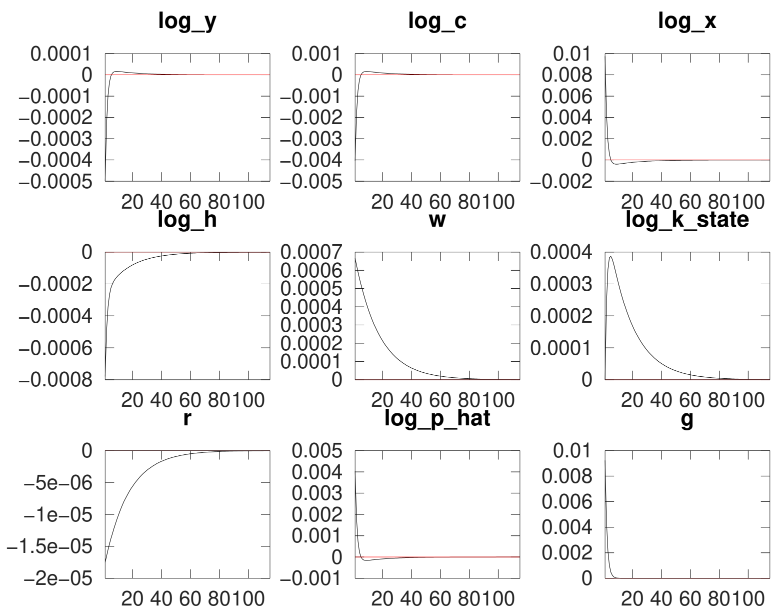

Figure 2: IRF from money growth shock (ξ_t)

The difference in the duration that the variables spend away from the steady-state between the technology shock and the money growth shock is likely due to the calibration. While the auto-regressive parameter for technology, γ, is equal to 0.95, the one for money growth, α, is set at 0.48 showing a much less persistent process. Changing the money growth parameter to 0.95 confirms this, as the shock then lasts about the same amount of time as for technology. However, very interestingly, the IRF of investment and capital suddenly invert in the case of high persistence, showing that investment actually lowers. A possible explanation for this is that if agents expect inflation to persist for much longer, they give up on consumption goods altogether, opting for a much sharper substitution to leisure. They would then also avoid protective investment, as they do not wish to consume goods later, expecting the tax to persist. Evidence for this is shown in Table 2, where a higher $\bar{g}$ leads to lower steady-state consumption, investment, and labour hours.

4 Self-Reflection

Our efforts in replicating Cooley & Hansen (1989) started with the analytical derivation of the model. The original authors went through the process of re-writing the entire problem in a single Bellman equation. We rather decided to opt for a more classic approach, where we derived each step individually, going from the households to the firms to market clearing and the equilibrium. As we ended up deriving the model quite early in the semester, we proceeded to put it into Dynare. It is here where we encountered quite a few problems, though they were mainly related to "rookie mistakes" in using Dynare. After fixing said issues, our first results were quite far from those of the original paper, especially for the standard deviations. After trying multiple solutions, what got our results to be quite close were the three following changes: First, we realised that all variables mentioned in Table 1 were in logs. Second, we realised that they were also de-trended using an HP filter. Third, we realised that the capital stock variable in the paper was actually capital stock at the beginning of the period. The problem being that Dynare automatically takes it at the end of the period. Including these three changes brought us very close to the original results, with the only variable which was far from the original paper being the price level. We realised that the reason for this was that we were taking $\hat{p}$, and not p. We could not, however, include p in the model as it is a non-stationary variable and would thus break the Dynare routines. Our solution was then to output the results and reconstruct p manually in R.

While we opted to using Dynare, this tool was not available at the time of the paper's publication. This made us have to determine whether the differences in our replication were due to different solution methods, as the author's use Value Function Iteration, or simply mistakes. Including the changes delineated above, we finally found that our steady-states were identical to the authors confirming that at least part of our solution method was correct. Then, using these values, we followed the utility loss calculation method and were ultimately able to exactly replicate Table 2.

Our inexperience also showed in the general handling of Dynare outputs. Cooley & Hansen (1989) is still a product of its time, reporting values for which it was not immediately obvious to us as to why they reported them, nor how to extract them. Table 1 particularly demanded quite a bit of trial and error. First, we replicated it using one 1000 period simulation which left us without the sample standard deviations (in parentheses in the table) and had us scrambling through .mat files on R to get the correct values. Our second approach was then to use 50 simulations, as the authors did.

Running these 50 simulations of 115 periods each, we observed large differences in the values of cyclical fluctuations. This made us realize that each individual simulated economy may depart substantially from the theoretical values. It took us some time to find a good way to extract the data from Dynare, but as we finished the pipeline, everything else was just routine work. By extracting the raw simulated paths, we were able to compute the mean moments of all 50 simulations and characterize the sampling uncertainty and therefore completely replicate Table 1. This process gave us a much clearer picture of why the authors reported their table in this format and with the chosen values.

The discovery that the cyclical moments for the two AR(1) economies were identical was initially puzzling. However, as we employed a second-order perturbation method and a fixed random seed, it became evident that the steady-state growth rate $\bar{g}$ does not materially alter the transmission of shocks in this framework. While Cooley & Hansen (1989) report slight variations across regimes, these are likely due to sampling noise from unique shock realizations. Because our results fall well within one standard deviation of the original estimates, we believe we have a successful replication. This confirmed to us what the authors briefly mentioned in their paper, which is that the inflation tax acts as a distortionary tax on the level of economic activity rather than a driver of cyclical volatility. Consequently, the welfare impacts are most visible in the steady-state differences in Table 2.

5 Conclusion

In short we have managed to recreate the full results from the Cooley & Hansen (1989) framework with some caveats. Our replication of Table 2 is completely identical while Table 1 diverges slightly, though lies within sampling uncertainty and therefore fits the original findings well. Even more precise replication would need the exact shocks they used to simulate their model.

We further extend their paper by producing impulse response functions and find that the ones for technology are similar to the ones of the broader RBC literature. The ones for money growth shocks are found to be essentially identical over different monetary regimes as the effect of the tax is on the level variables and therefore only really affect the steady-state while not really impacting volatility. While it is still interesting to study the IRF's, we can then understand why the original authors did not include any and focused on the steady-state effects.

Our exercise to replicate Cooley & Hansen's methodology with modern software showed us just how comparatively easy it is to do this type of analysis today compared to in 1989. Even though solution methods differ, the approximated solutions remain relatively similar, giving credence to the original paper's methodology.

6 References

Cooley, Thomas F & Hansen, Gary D (1989). "The inflation tax in a real business cycle model". In *The American Economic Review*, pp. 733–748.

Hansen, Gary D (1985). "Indivisible labor and the business cycle". In *Journal of monetary Economics* 16.3, pp. 309–327.