GATE

DATA SCIENCE + CS & IT

Engineering Mathematics

SUPER 1500

Lec : 03 Calculus

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Topics *to be covered*

CALCULUS (part 3)

① Nature of Roots

② M.V.Th

(i) R.Th

(ii) L.M.V.Th

(iii) C.M.V.Th.

(iv) LMVTh for Integrals

NATURE of ROOTS

⊛ n^{th} degree poly has exactly n Roots whether Real or Complex.

⊛ An odd degree poly with Real coeff has at least one Real Root (∵ Complex Roots occurs in pair)

⊛ Complex Roots occurs in pair only if coeff are Real

eg
$$x^2 - (i+1)x + i = 0$$
$$x^2 - ix - x + i = 0$$
$$x(x-i) - 1(x-i) = 0$$
$$(x-i)(x-1) = 0$$
$$x = 1 \ \& \ i$$

⊛ n^{th} degree poly bends at most $(n-1)$ times

that's why it has at most $(n-1)$ extremas (Max or Min, both)

eg $f(x) = (x^2-4)^2$

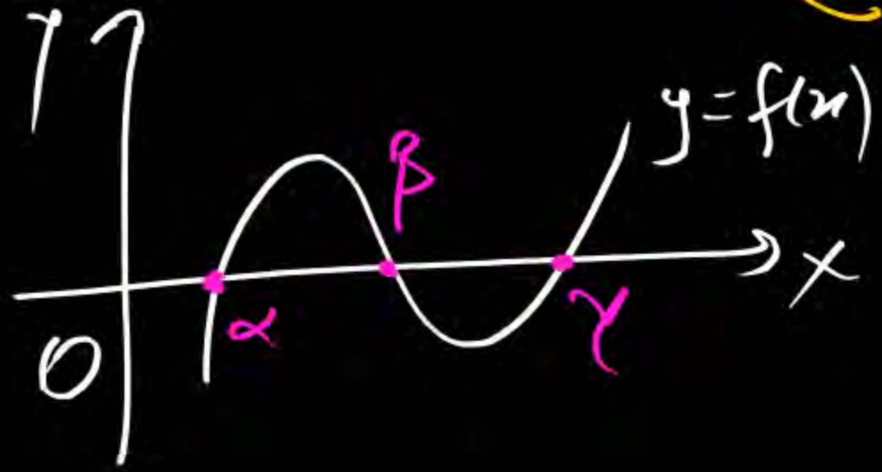
Qe A Cubic Poly with Real Coeff has 3 zero crossings & 2 extremas

(*) Descartes Rule of Sign :→ Consider $f(x)$ is a given polynomial

(i) Number of +ve Real Roots of $f(x)$ $\leq$ No. of times Sign changes in $f(x)$

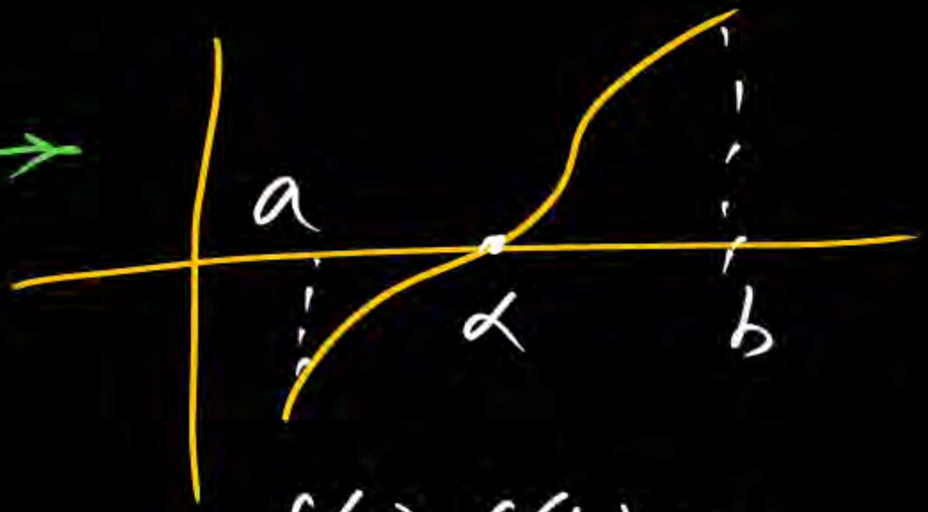
(ii) " " -ve " " " " $\leq$ " " " " in $f(-x)$

(*) Roots / Solutions → Real Roots = Zero Crossing ; Comp Roots = Zeros

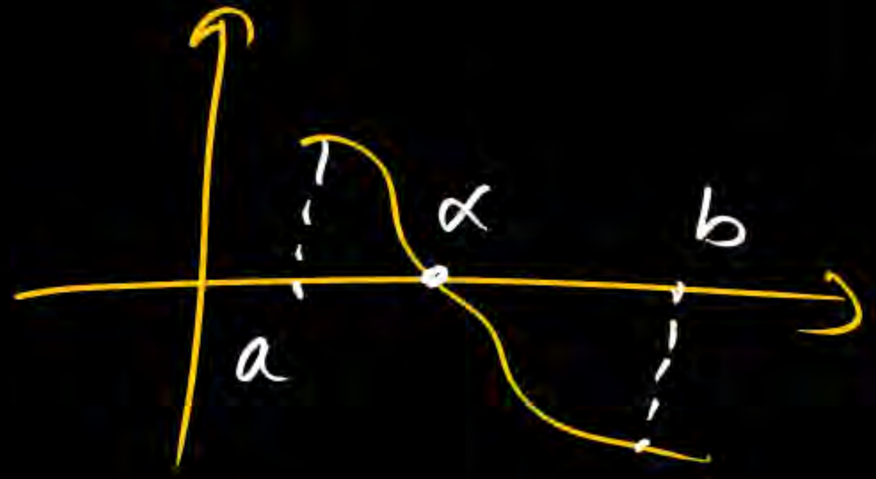


Real Roots → those points where graph of $f(x)$ cuts x axis are called Real Roots i.e $\boxed{f(\alpha) = f(\beta) = f(\gamma) = 0}$

BOLZANO THEOREM →



$f(a) \cdot f(b) < 0$



$f(a) \cdot f(b) < 0$

whenever $f(a)$ & $f(b)$ are of opposite sign then $\exists$ at least one Root α of $f(x)$ b/w a & b

OR

$$\boxed{\text{if } f(a) \cdot f(b) < 0 \text{ then } \exists\, \alpha \in (0,1) \text{ for which } f(\alpha) = 0}$$

Q- Choose the possible correct options for $f(x) = x^9 + 5x^3 - x^2 + 7x + 2$

MSQ

(a) $f(x)$ has at most 2 +ve roots

(b) $f(x)$ has at most 3 -ve roots

(c) $f(x)$ has at least 4 complex roots

(d) $f(x)$ has at least one real root

No. of +ve real roots ≤ 2 (No. of times sign changes in $f(x)$)

Now $f(-x) = (-x)^9 + 5(-x)^3 - (-x)^2 + 7(-x) + 2$

$= -x^9 - 5x^3 - x^2 - 7x + 2$

No. of -ve real roots ≤ 1 (No. of times sign changes in $f(-x)$)

No. of complex roots ≥ 6

The polynomial $p(x) = x^5 + x + 2$ has

(a) all real roots

(b) 3 real and 2 complex roots

(c) 1 real and 4 complex roots

(d) all complex roots

$\because$ p(x) is a ploynomial odd degree

A polynomial $f(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x - a_0$ with all coefficients positive has

(a) no real roots

(b) no negative real root

(c) odd number of real roots

(d) at least one positive and one negative real root

No. of +ve Real Roots ≤ 1

Now $f(-x) = a_4x^4 - a_3x^3 + a_2x^2 - a_1x - a_0$

No of -ve Real Roots ≤ 3

3 A polynomial $\phi(s) = a_n s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_o$ of degree $n > 3$ with constant real coefficients a_n, $a_{n-1}, \dots a_o$ has triple roots at $s = -\sigma$. Which one of the following conditions must be satisfied?

(a) $\phi(s) = 0$ at all the three values of s satisfying $s^3 + \sigma^3 = 0$

(b) $\phi(s) = 0$,

$$\frac{d\phi(s)}{ds} = 0, \text{ and } \frac{d^2\phi(s)}{ds^2} = 0 \text{ at } s = -\sigma$$

(c) $\phi(s) = 0$,

$$\frac{d^2\phi(s)}{ds^2} = 0, \text{ and } \frac{d^4\phi(s)}{ds^4} = 0 \text{ at } s = -\sigma$$

(d) $\phi(s) = 0$, and $\dfrac{d^3\phi(s)}{ds^3} = 0$ at $s = -\sigma$

$\Rightarrow \boxed{\phi(s) = (s+\sigma)^3 f(s)} \Rightarrow \phi(-\sigma) = 0$

$\phi'(s) = 3(s+\sigma)^2 f(s) + (s+\sigma)^3 f'(s)$

$\Rightarrow \phi'(-\sigma) = 0$

$\phi''(s) = 6(s+\sigma)f(s) + 3(s+\sigma)^2 f'(s)$
$+ 3(s+\sigma)^2 f'(s) + (s+\sigma)^3 f''(s)$

$\phi''(-\sigma) = 0$

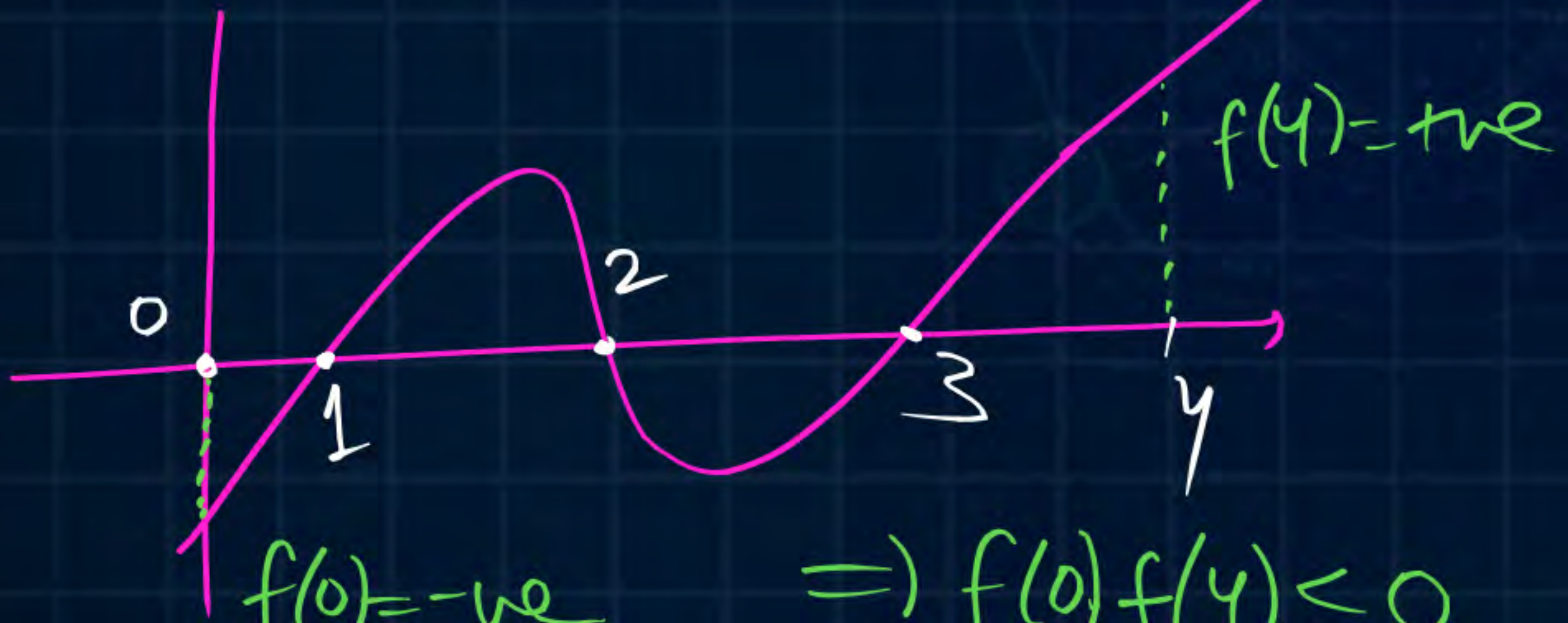
$\phi'''(s) = 6f(s) + (s+\sigma)[\text{-----}]$

$\phi'''(-\sigma) = 6f(-\sigma) + 0$
$\neq 0$

A non-zero polynomial $f(x)$ of degree 3 has roots at $x = 1$, $x = 2$ and $x = 3$. Which one of the following must be TRUE?

(a) $f(0)\,f(4) < 0$ (b) $f(0)\,f(4) > 0$

(c) $f(0) + f(4) > 0$ (d) $f(0) + f(4) < 0$



$\Rightarrow f(0)\,f(4) < 0$

MSQ

A function f(x) is continuous in interval [0, 2]. It is known that f(0) = f(2) = –1 and f(1) = 1. Which one of the following statements must be true ?

(a) There exists a 'y' in the interval (0, 1) such that $\boxed{f(y) = f(y + 1)}$

(b) For every 'y' in the interval (0, 1), f(y) = f(2–y)

(c) the maximum value of the function in the interval (0, 2) is 1

(d) There exists a 'y' in the interval (0, 1) such that f(y) = –f(2 – y).

f(x) is defined in [0, 2]

s.t. $f(0)=f(2)=-1$ & $f(1)=1$

Let us cross check (a)

Assume $\boxed{\phi(y)=f(y)-f(y+1)}$

Now $\phi(0)=f(0)-f(1)=-1-(1)=-ve$

$\phi(1)=f(1)-f(2)=1-(-1)=+ve$

$\because \phi(0).\phi(1)<0$ so By B.Th $\exists\, \alpha\in(0,1)$ s.t. $\phi(\alpha)=0$

i.e. $f(\alpha)-f(\alpha+1)=0$

i.e. $\boxed{f(\alpha)=f(\alpha+1) \text{ where } \alpha\in(0,1)}$

$\boxed{\phi(y)=f(y)+f(2-y)}$

$\phi(0)=f(0)+f(2)=(-1)+(-1)$
$=-ve$

$\phi(1)=f(1)+f(1)=1+1=+ve$

$\because \phi(0)\phi(1)<0$

So By B.Th $\exists\, \alpha\in(0,1)$ s.t.

$\phi(\alpha)=0 \Rightarrow \boxed{f(y)=-f(2-y)}$

MEAN VALUE THEOREMS

R.M.V.Th: if $f(x)$ is Cont, Diff and $f(a)=f(b)$ then $\exists\ c \in (a,b)$ s.t $\boxed{f'(c)=0}$

ie "$\exists$ one Root c of $f'(x)$"

sp case: if $\boxed{f(a)=f(b)=0}$ then $\exists\ c \in (a,b)$ s.t $\boxed{f'(c)=0}$

{$x=a$ & $x=b$ are Roots of $f(x)$} {$x=c$ is the Root of $f'(x)$}

$\boxed{\text{" Between any two Roots of } f(x), \exists \text{ at least one Root of } f'(x) \text{ "}}$

LMVTh: if $f(x)$ is Cont as well as Diff then $\exists\ c \in (a,b)$ s.t $\boxed{\dfrac{f(b)-f(a)}{b-a}=f'(c)}$

C.M.V.Th: if $f(x)$ & $g(x)$ both are Cont as well as Diff s.t $g'(x) \neq 0$

then $\exists\, c \in (a,b)$ where $\boxed{\dfrac{f(b)-f(a)}{g(b)-g(a)} = \dfrac{f'(c)}{g'(c)}}$

L.M.V.Th for Integrals: if $f(x)$ is Cont and Diff then $\boxed{f(c) = \dfrac{1}{b-a}\int_a^b f(x)\,dx}$

$\forall\, c \in (a,b)$

Here Mean Value $\equiv$ Average Value of $f(x)$ b/w a & b

$\equiv$ " Height " " " "

Q. If $f(x) = e^x(\sin x - \cos x)$; $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$ then one of root of $f'(x)$ will be ?

(a) $\pi/2$

(b) $3\pi/4$

(c) π ✓

(d) $\pi/4$

$\because \boxed{f\left(\frac{\pi}{4}\right) = 0 = f\left(\frac{5\pi}{4}\right)}$ so By R.Th $f'(c) = 0$.

$f'(x) = e^x(\sin x - \cos x) + e^x(\cos x + \sin x)$

By R.Th. $f'(c) = 0$

$e^c(\sin c - \cos c) + e^c(\cos c + \sin c) = 0$

$e^c(2\sin c) = 0$

$\sin c = 0 \Rightarrow c = n\pi = \ldots -2\pi, -\pi, 0, \pi, 2\pi \ldots$

ie $x = \pi$ is the Root of $f'(x)$ while $\frac{\pi}{4}$ & $\frac{5\pi}{4}$ are the Roots of $f(x)$

Q-e / MSQ: if $f(x)=x^3-6x^2+11x-6$ defined in $[1,3]$ then Roots of $f'(x)$ lies in

(a) $(1,2)$ ✓

(b) $(-\infty,1)\cup(3,\infty)$

(c) $(2,3)$ ✓

(d) $(-1,1)$

$f(1)=1-6+11-6=0$

$f(3)=3^3-6(3)^2+11(3)-6=27-54+33-6=0$

i.e. $x=1$ & $x=3$ are Roots of $f(x)$

So one Root of $f'(x)$ lies in between 1 & 3

Again At $x=2$, $f(2)=2^3-6(2)^2+11(2)-6=0$

i.e. $x=2$ is also a Root of $f(x)$

So one Root of $f'(x)$ lies in b/w 1 & 2

& another " " " " 2 & 3

(By Sp. Conclusions of R.Th)

Consider $p(s) = s^3 + a_2 s^2 + a_1 s + a_0$ with all real coefficients. It is known that its derivative $p'(s)$ has no real roots. The number of real roots of $p(s)$ is

(a) 0

(b) 1

(c) 2

(d) 3

ATQ, $P'(s)$ has No Real Root

$\Rightarrow$ $P(s)$ Can not have Two Roots

Explanation: $\because$ $P(s)$ is odd degree poly (with Real Coeff) so $P(s)$ must have one Real Root

Now let us assume that $P(s)$ has two Real Roots

So By Sp Case of R-Th, $P'(s)$ also has one Root

But it is Contradiction according to the information given in Q.

Which of the following function satisfies all the conditions of Rolle's Theorem in the interval [0, 1]

(a) $f(x) = \tan \pi x$ ✗

(b) $f(x) = \begin{cases} x, & 0 \le x < \frac{1}{2} \\ 1-x, & \frac{1}{2} \le x \le 1 \end{cases}$ ✗

~~(c)~~ $f(x) = x^2$ ✗

(d) $f(x) = \sqrt{x(1-x)}$ ✓

(c): $f(x) = x^2$; $[0, 1]$

$f(0) = 0$, $f(1) = 1$

$\because f(0) \neq f(1)$ so (c) ✗

Let us take (b)

$f(0) = 0$, $f(1) = 0$

i.e. $f(0) = f(1)$

But $f'(x)$ is Not defined at $x = \frac{1}{2}$

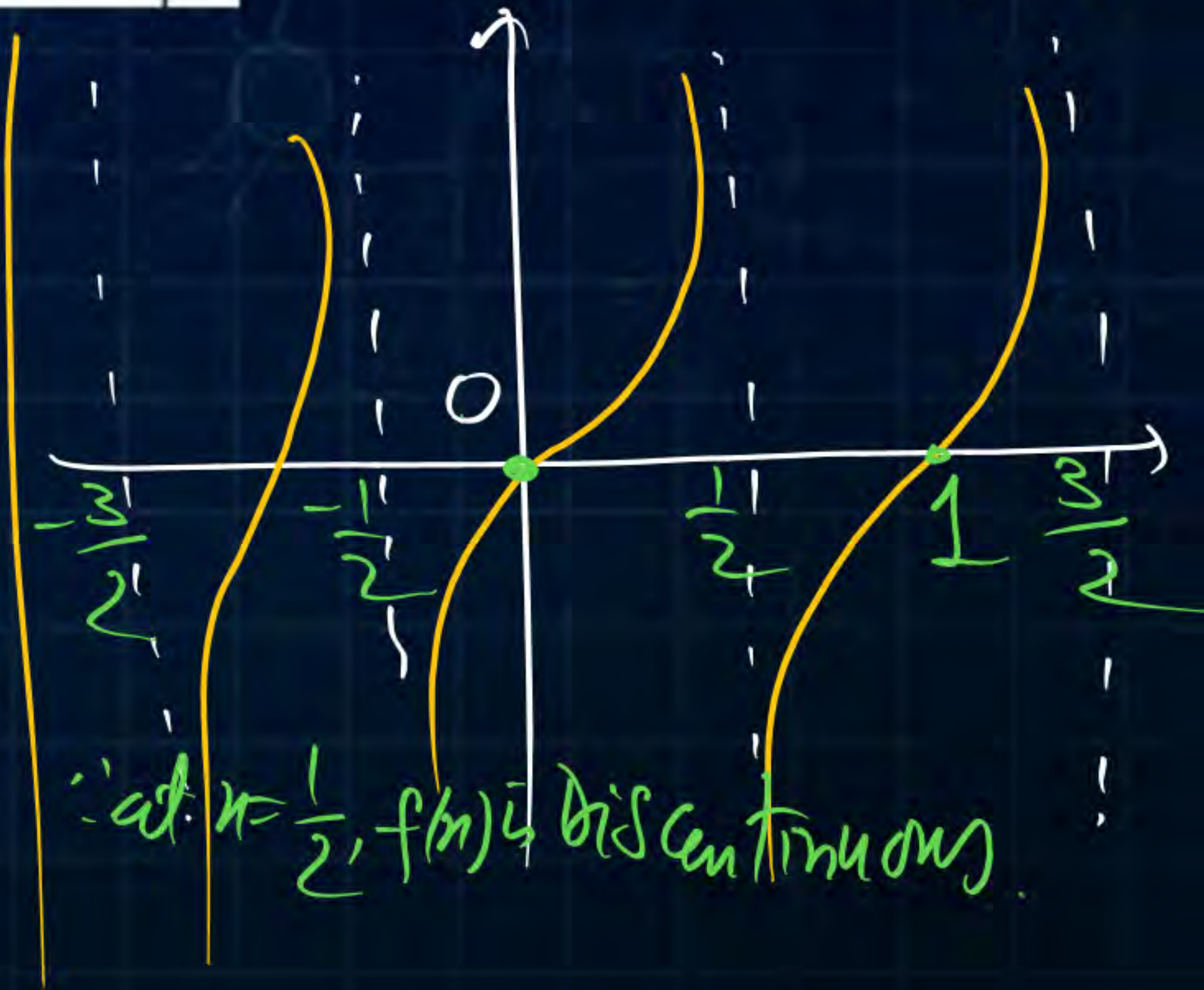
$\because f'(x) = \begin{cases} 1, & 0 < x < \frac{1}{2} \\ -1, & \frac{1}{2} < x < 1 \end{cases}$ LHD $= -1$, RHD $= 1$

i.e. $f(x)$ is Not Diff at $x = \frac{1}{2}$

Let us take (a):

Period of $\tan x = \pi$

So period of $\tan(\pi x) = \frac{\pi}{\pi} = 1$



$\because$ at $x = \frac{1}{2}$, $f(x)$ is Discontinuous.

(d):

$f(x) = \sqrt{x(1-x)}$ $[0,1]$

$f'(x)$ = Defined for every value in $(0,1)$

$\because$ Diff is defined in open interval so $x=0$ & $x=1$ will not disturb us in R-Th

$$f'(x) = \frac{1}{2\sqrt{x(1-x)}} \frac{d}{dx}\left(x(1-x)\right) = \frac{1-2x}{2\sqrt{x(1-x)}}$$

$f(0) = 0$, $f(1) = 0$ i.e. $f(0) = f(1)$

Q. for the function $f(x)=|x|$, Lagrange's Mean Value is not applicable in ?

(a) $1 \le x \le 3$

(b) $x < -1$ or $x > 1$

(c) $0 < x < 1$

(d) $-2 < x < 2$

$f(x) = |x|$

At $x=0$, $f'(x) = \text{DNE}$

Q. For the function $f(x) = \sin\left(\frac{1}{x}\right)$ Lagrange's Mean Value is applicable in ?

(a) $[-3, 3]$

(b) $[-2, 5]$

(c) $[2, 3]$

(d) $[-1, 4]$

At $x = 0$, $f(x)$ is Not Differentiable

A function $y = 5x^2 + 10x$ is defined over an open interval $x = (1, 2)$. At least at one point in this interval, $\frac{dy}{dx}$ is exactly 25

$y = f(x) = 5x^2 + 10x$ = Poly of 2 degree < Cont, Diff.

By L.M.V.Th,
$$f'(c) = \frac{f(2) - f(1)}{2-1} = \frac{(5(2)^2 + 10(2)) - (5(1)^2 + 10(1))}{2-1}$$

$$\left(\frac{dy}{dx}\right)_{x=c} = 25$$ where c lies in b/w 1 & 2

By applying Lagranges mean value for the function $f(x) = (1+x)\log(1+x)$ on [0, 1] the value of $c \in (0, 1)$ is

(a) $\frac{4}{e}$ (b) $\frac{1}{e}$

(c) $\frac{4-e}{e}$ (d) $\frac{1-e}{e}$

$$\boxed{f(x) = (1+x)\log(1+x)}$$

$$f(0) = (1+0)(\log 1+0) = 0$$

$$f(1) = (1+1)\log(1+1) = 2\log 2$$

$$f'(x) = \frac{1+x}{1+x} + \log(1+x)$$

$$\frac{dy}{dx} = 1 + \log(1+x)$$

By L.M.V.Th, $\frac{f(1)-f(0)}{1-0} = f'(c)$

$$\frac{2\log 2 - 0}{1-0} = f'(c)$$

i.e. $\left(\frac{dy}{dx}\right)_{x=c} = 2\log 2$

So $1 + \log(1+c) = 2\log 2$

$$\log(1+c) = \log(2)^2 - \log_e e = \log\left(\frac{4}{e}\right)$$

$$1 + c = \frac{4}{e} \Rightarrow c = \frac{4}{e} - 1 = \frac{4-e}{e}$$

$f(0) = 2$, $f'(x) = \frac{1}{5-x^2}$

& $f(x)$ is defined in $[0, 1]$

If f(0) = 2.0 and $f'(x) = \frac{1}{5-x^2}$, then the lower and upper bounds of f(1) estimated by the mean value theorem are

(a) 1.9, 2.2
(b) 2.2, 2.25
(c) 2.25, 2.5
(d) none of the above

i.e. Min and Max value of $\phi(x)$ occurs at 1 & 2 respectively

$$f''(x) = \frac{-1}{(5-x^2)^2}(-2x) = \frac{2x}{(5-x^2)^2}$$

Let $\boxed{\phi(x) = f'(x)}$ then $\phi'(x) = f''(x)$

i.e. $\phi(x) = \frac{1}{5-x^2}$ & $\phi'(x) = \frac{2x}{(5-x^2)^2}$

By observation, $\phi'(x) > 0$ always in $(0, 1)$

$\Rightarrow$ $\phi(x)$ is S. Inc in $(0, 1)$

By L.Th $f'(c) = \frac{f(1) - f(0)}{1-0} = \frac{f(1) - 2}{1-0}$

i.e. $\boxed{f'(c) = f(1) - 2} = \phi(c)$

By Common Sense;

$\text{Min}\,\phi(x) < \phi(x) < \text{Max}\,\phi(x)$

or $\text{Min}\,\phi(c) < \phi(c) < \text{Max}\,\phi(c)$

$\frac{1}{5} < f(1) - 2 < \frac{1}{4}$

$2.20 < f(1) < 2.25$

$$\boxed{\phi(x) = f'(x)} = \frac{1}{5-x^2} \quad \begin{cases} f'(0) = \frac{1}{5} = 0.20 \\ f'(1) = \frac{1}{5-1} = 0.25 \end{cases}$$

$$\phi'(x) = f''(x) = \frac{2x}{(5-x^2)^2}, \quad f(0) = 2$$

$\because \phi'(x) > 0$ always in $(0,1)$

$\Rightarrow \phi(x)$ will be S. Inc in $(0,1)$

So $\text{Min } \phi(x) = \phi(0) = f'(0) = 0.20$

$\text{Max } \phi(x) = \phi(1) = f'(1) = 0.25$

& By L.M.V.Th, $f'(c) = \dfrac{f(1) - f(0)}{1-0} = f(1) - 2$

$$\boxed{\phi(c) = f(1) - 2}$$

By Common Sense ;

$$\text{Min } \phi(x) < \phi(c) < \text{Max } \phi(x)$$

$$0.20 < f(1) - 2 < 0.25$$

$$\boxed{2.20 < f(1) < 2.25}$$

ie lower Bound of $f(1) = 2.20$

upper " " $f(1) = 2.25$

Q. Two functions $f(x) = |x-1|$, $g(x) = \ln x$ are given then using Cauchy's MVTh evaluate a number $\xi \in [2,5]$ correct upto two decimal places 3.27

Let $\xi = c$ ☺ so by C.M.V.Th.

$$\frac{f(5)-f(2)}{g(5)-g(2)} = \frac{f'(c)}{g'(c)}$$

$$\frac{4-1}{\ln 5 - \ln 2} = \frac{1}{1/c}$$

$$c = 3.27$$

$$f(x) = |x-1| = \begin{cases} -(x-1), & x < 1 \\ +(x-1) & x > 1 \end{cases}$$

$$f'(x) = \begin{cases} -1, & x < 1 \\ 1, & x > 1 \end{cases} \Rightarrow f'(c) = 1$$

$\because c \in (2,5)$ i.e. $c > 1$

so $f'(c) = 1$

Q: if $f(x) = \log x$, $g(x) = \log\left(\frac{1}{x}\right)$ then find c using Cauchy's M.V.Th in $[1, 2]$?

(a) 1.33

(b) 1.25

(c) 1.50

(d) ✓ Any Value b/n 1 & 2

$f(x) = \log x$, $g(x) = -\log x$

$f'(c) = \frac{1}{c}$, $g'(c) = -\frac{1}{c}$

By C.M.V.Th,
$$\frac{f'(c)}{g'(c)} = \frac{f(2) - f(1)}{g(2) - g(1)}$$

$$\frac{1/c}{-1/c} = \frac{\log 2 - \log 1}{-\log 2 - (-\log 1)}$$

$-1 = -1$ i.e identity for c is any Value b/n 1 & 2

Q. The point on the curve $y = \frac{4}{x^2}$ b/n $\underline{-4}$ & $\underline{-2}$ where we can find mean value of the function?

a   b

ⓐ $(-2\sqrt{2}, -0.5)$

ⓑ $(-2\sqrt{2}, 0.5)$ ✓

ⓒ $(2\sqrt{2}, -0.5)$

ⓓ $(2\sqrt{2}, 0.5)$

Let the required point in $(c, f(c))$

By M.V.Th of Integrals,

$$f(c) = \frac{1}{(-2)-(-4)} \int_{-4}^{-2} f(x)\,dx$$

$$= \frac{1}{2} \int_{-4}^{2} \left(\frac{4}{x^2}\right) dx = \frac{1}{2}$$

$f(c) = \frac{1}{2}$

$\frac{4}{c^2} = \frac{1}{2}$

$c^2 = 8 \Rightarrow c = \pm 2\sqrt{2}$

$c = -2\sqrt{2}$ ✓ & $c = 2\sqrt{2}$ ✗

∵ c must lies in a & b

i.e c lies in b/n -4 & -2

i.e $c = -ve$

Q. The Abscissa of the point, where we can find Mean Value of the function $f(x) = 5x^4+2$ when $-1 \le x \le 2$, is ?

(a) 1.50

(b) 13

(c) -1.21

(d) 1.21

By M.V.Th of integrals,

$$f(c) = \frac{1}{2-(-1)} \int_{-1}^{2} f(x)\,dx$$

$$= \frac{1}{3} \int_{-1}^{2} (5x^4+2)\,dx$$

$$= \frac{1}{3} \left(x^5 + 2x\right)_{-1}^{2}$$

$$f(c) = \frac{1}{3}\left[(32+4) - (-1-2)\right] = \frac{39}{3}$$

$$f(c) = 13$$

$$5c^4 + 2 = 13$$

$$c^4 = \frac{11}{5} \Rightarrow c = \left(\frac{11}{5}\right)^{\frac{1}{4}} = 1.21$$

Thank

THANK

Keep Hustling!