GATE

DATA SCIENCE + CS & IT

Engineering Mathematics

SUPER 1500

Lec : 03 Linear - 1

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Topics to be covered

LINEAR ALGEBRA

1. Non Homogeneous System
2. Homogeneous System
3. Eigen Values

Non-Homogeneous System

$$\boxed{A_{m\times n} X_{n\times 1} = B_{m\times 1}}$$

m = No of eqn^s

n = No of variables

Rank method

$(m>n,\ m=n,\ m<n)$

1. if $\rho(A)=\rho(A:B)=n \Rightarrow$ unique sol
2. if $\rho(A)=\rho(A:B)<n \Rightarrow \infty$ sol
3. if $\rho(A)\neq\rho(A:B) \Rightarrow$ No sol

Matrix Method (only for $m=n$)

1. if $|A|\neq 0 \Rightarrow$ unique sol.
2. if $|A|=0$, $(\text{adj}A)B=0 \Rightarrow \infty$ sol.
3. if $|A|=0$, $(\text{adj}A)B\neq 0 \Rightarrow$ No sol.

(*) N Cond^n for Consistency is $\boxed{\rho(A)=\rho(A:B)}$

(*) underdetermined system can not have unique solution

If the system of n linear equations in n unknowns has more thanone solution, then its associated matrix ______.

(a) has rank < n (b) has rank = n

(c) has rank > n (d) has rank one

$A_{n\times n} X = B_{n\times 1}$

$\Rightarrow \infty$ sol^n $\Rightarrow \boxed{\rho(A) = \rho(A:B) < n}$

Let M be an $m \times n (m < n)$ matrix with rank m. Then

(a) for every b in R^m, $Mx = b$ has unique solutions.

(b) for every b in R^m, $Mx = b$ has a solution but it is not unique.

(c) there exists be $b \in R^m$ for which $Mx = b$ has no solution.

(d) None of the above

$\boxed{A_{m\times n} X_{n\times 1} = B_{m\times 1}}$, $m < n$

underdetermined system

$\rho(A) \le m$

i.e. $\rho(A) < n$ (By common sense)

i.e. $\rho(A) <$ No of variables $\Rightarrow \infty$ sol^n

$\rho(A) \ne \rho(A:B)$

eg $M_{2\times 3} X_{3\times 1} = B_{2\times 1}$

$[M:B]_{2\times 4} \Rightarrow \rho(M) = 2$ given

$\rho(M:B) \le 2 \Rightarrow \rho(M:B) = 2$

Given a system of equations :

$x + 2y + 2z = b_1$

$5x + y + 3z = b_2$

Which of the following is true regarding its solution ?

(a) The system has a unique solution for any given b_1 and b_2

(b) The system will have infinitely many solutions for any given b_1 and b_2

(c) Whether or not a solution exists depends on the given b_1 and b_2

(d) The system would have no solution for any values of b_1 and b_2

$$[A:B] = \left[\begin{array}{ccc|c} 1 & 2 & 2 & b_1 \\ 5 & 1 & 3 & b_2 \end{array}\right]_{2\times 4}$$

$$= \left[\begin{array}{ccc|c} 1 & 2 & 2 & b_1 \\ 0 & -9 & -7 & (b_2 - 5b_1) \end{array}\right]_{2\times 4}$$

$\rho(A) = 2$ & $\rho(A:B) = 2$

for ∞ sol^n $\because$ $\rho(A) = \rho(A:B) <$ No of Variables

The system of eqations

$$x + y + z = 6$$
$$\left.\begin{aligned} x + 4y + 6z &= 20 \\ x + 4y + \lambda z &= \mu \end{aligned}\right\}$$

has no solution for values of λ and μ given by

(a) $\lambda = 6, \mu = 20$ (b) $\lambda = 6, \mu \neq 20$

(c) $\mu \neq 6, \mu = 20$ (d) $\mu \neq 20, \lambda \neq 6$

$$[A:B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 4 & 6 & 20 \\ 1 & 4 & \lambda & \mu \end{array}\right] \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 3 & 5 & 14 \\ 0 & 3 & (\lambda-1) & (\mu-6) \end{array}\right]$$

PIVOTS

$$\xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 3 & 5 & 14 \\ 0 & 0 & (\lambda-6) & (\mu-20) \end{array}\right]_{3\times 4}$$

3×3

for No/Sol $\begin{cases} \rho(A) = 2 \Rightarrow \lambda = 6 \\ \rho(A:B) = 3 \Rightarrow \mu \neq 20 \end{cases}$

M-II $[A:B] \xrightarrow{R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 3 & 5 & 14 \\ 1 & 4 & \lambda & \mu \end{array}\right]$

$\xrightarrow{R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 3 & 5 & 14 \\ 1 & 1 & (\lambda-5) & (\mu-14) \end{array}\right]$

E-operations $R_2 \to R_2 - R_1$

$R_3 \to R_3 - R_2$

??

The system of equations :

$$2x + y = 5$$

$$x - 3y = -1$$

$$3x + 4y = k$$

is consistent when k is ______

(a) 1 (b) 2

(c) 5 (d) 10

$$[A:B] = \left[\begin{array}{cc|c} 2 & 1 & 5 \\ 1 & -3 & -1 \\ 3 & 4 & k \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_1} \left[\begin{array}{cc|c} 1 & -3 & -1 \\ 2 & 1 & 5 \\ 3 & 4 & k \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & -3 & -1 \\ 0 & 7 & 7 \\ 0 & 13 & k+3 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & -3 & -1 \\ 0 & 1 & 1 \\ 0 & 13 & k+3 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - 13R_2} \left[\begin{array}{cc|c} 1 & -3 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & k-10 \end{array}\right]$$

For consistency: $\rho(A) = \rho(A:B)$

$\because \rho(A) = 2$ so $\rho(A:B) = 2 \Rightarrow k = 10$

For what values of α and β, the following simultaneous equations have an infinite number of solutions?

$$x + y + z = 5$$

$$x + 3y + 3z = 9$$

$$x + 2y + \alpha z = \beta$$

(a) 2, 7
(b) 3, 8
(c) 7, 2
(d) 8, 3

$$[A:B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 1 & 3 & 3 & 9 \\ 1 & 2 & \alpha & \beta \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 2 & 2 & 4 \\ 0 & 1 & \alpha-1 & \beta-5 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & \alpha-1 & \beta-5 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & (\alpha-2) & (\beta-7) \end{array}\right]$$

for ∞ sol^n: $\rho(A) = \rho(A:B) < 3$

$\Rightarrow \alpha = 2, \beta = 7$

For what value of a, If any, will the following system of equations in x, y and z have a solution = unique sol.

$$2x + 3y = 4$$
$$x + y + z = 4$$
$$x + 2y - z = a$$

(a) Any real number
(b) 0
(c) 1
(d) There is no such value

$(adj A) B = ? \neq 0$ No sol.

$(adj A) B = 0 \Rightarrow \infty$ sol.

$$(A : B) = \left[\begin{array}{ccc|c} 2 & 3 & 0 & 4 \\ 1 & 1 & 1 & 4 \\ 1 & 2 & -1 & a \end{array}\right]$$

$$|A| = \begin{vmatrix} 2 & 3 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & -1 \end{vmatrix} = \text{------} = 0 \Rightarrow \text{unique sol DNE}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & - & - & - \\ 0 & 0 & 0 & ? \end{array}\right]$$

ie this system either have ∞ sol or No sol

$\Rightarrow \rho(A) = 2$

M-II

$$[A:B] = \left[\begin{array}{ccc|c} 2 & 3 & 0 & 4 \\ 1 & 1 & 1 & 4 \\ 1 & 2 & -1 & a \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 2 & 3 & 0 & 4 \\ 1 & 2 & -1 & a \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & -2 & -4 \\ 0 & 1 & -2 & a-4 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & -2 & -4 \\ 0 & 0 & 0 & a \end{array}\right]$$

for $a=0$; $\rho(A) = \rho(A:B) = 2 \Rightarrow \infty$ sol.

for $a \neq 0$, $\rho(A) \neq \rho(A:B) \Rightarrow$ No sol.

Homogeneous System

① unique Sol^n ≡ Zero Sol^n ≡ Trivial Sol^n always exist → is $X = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

② ∞ Sol^n ≡ Non Zero Sol^n ≡ Non Trivial Sol^n also exist, $X = \begin{bmatrix} \text{Non Zero} \\ \text{elements} \end{bmatrix}$

(*) $\rho(A)$ = No. of Variables ⟹ unique Sol^n. ⟸ if $|A| \neq 0$

$\rho(A)$ < " " ⟹ ∞ Sol^n ⟸ if $|A| = 0$

(*) underdetermined Homog System always exist ∞ Sol^n.

(*) $\boxed{\text{Nullity} = \text{No. of Columns} - \rho(A)}$

No. of LI Sol^n of Homog System $(AX = 0)$ = No. of Columns $- \rho(A)$

Let A be 3 × 3 matrix with rank 2. Then AX = 0 has

(a) only Trivial Solution (b) one Ind Solution

(c) two Ind Sol (d) three Ind Sol.

$$A_{3\times 3} X_{3\times 1} = O_{3\times 1} \longrightarrow (1)$$

$$\underbrace{\text{Number of LI Sol}^n}_{N(A)} \text{ of } (1) = \text{No. of } C - \rho(A)$$

$$= 3 - 2 = 1$$

The nullity of system of quations:

$$x_1 + x_2 - x_3 + x_4 = 0$$

$$2x_1 + 3x_2 + x_3 + 4x_4 = 0$$

$$3x_1 + 2x_2 - 6x_3 + x_4 = 0$$

(a) 1 (b) 2

(c) 3 (d) 4

$$A = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 2 & 3 & 1 & 4 \\ 3 & 2 & -6 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & -1 & -3 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 2$

so $N(A) = \text{No of } C - \rho(A)$

$= 4 - 2 = 2$

The value of α for which the system of equation

$$x + y + z = 0$$

$$y + 2z = 0$$

$$\alpha x + z = 0$$

has more than one solution is $\rightarrow$ ∞ soln

(a) -1 (b) 0

(c) $\frac{1}{2}$ (d) 1

for ∞ soln $|A| = 0$

$$\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ \alpha & 0 & 1 \end{vmatrix} = 0$$

$$1[1-0] - 0[?] + \alpha[2-1] = 0$$

$$1 + \alpha = 0 \Rightarrow \alpha = -1$$

The set of equations :

$$\lambda x - y + (\cos\theta)z = 0$$
$$3x + y + 2z = 0$$
$$(\cos\theta)x + y + 2z = 0$$

$0 \le \theta \le 2\pi$ was non-trivial solution → ∞ soln!

(a) For no value of λ and θ

(b) For all values of λ and θ

(c) For no values of λ and only two values of θ

(d) None of these

$$|A| = \begin{vmatrix} \lambda & -1 & \cos\theta \\ 3 & 1 & 2 \\ \cos\theta & 1 & 2 \end{vmatrix}$$

$$= \lambda[0] - 3[-2 - \cos\theta] + \cos\theta[-2 - \cos\theta]$$

$$= 6 + 3\cos\theta - 2\cos\theta - \cos^2\theta$$

$$= 6 + \cos\theta - \cos^2\theta$$

i.e. $\boxed{|A| = 6 + \cos\theta - \cos^2\theta}$

$= 6 + (\text{Value b/w } -1 \,\&\, 1) - (\text{Value b/w } 0 \,\&\, 1)$

i.e. $|A| \neq 0 \Rightarrow$ unique soln exist

So in this system we will never get ∞ soln.

The number of linearly independent solutions of the system of equations

$$\begin{bmatrix} 1 & 0 & 2 \\ 1 & -1 & 0 \\ 2 & -2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = O \text{ is equal to}$$

(a) 1 (b) 2

(c) 3 (d) 0

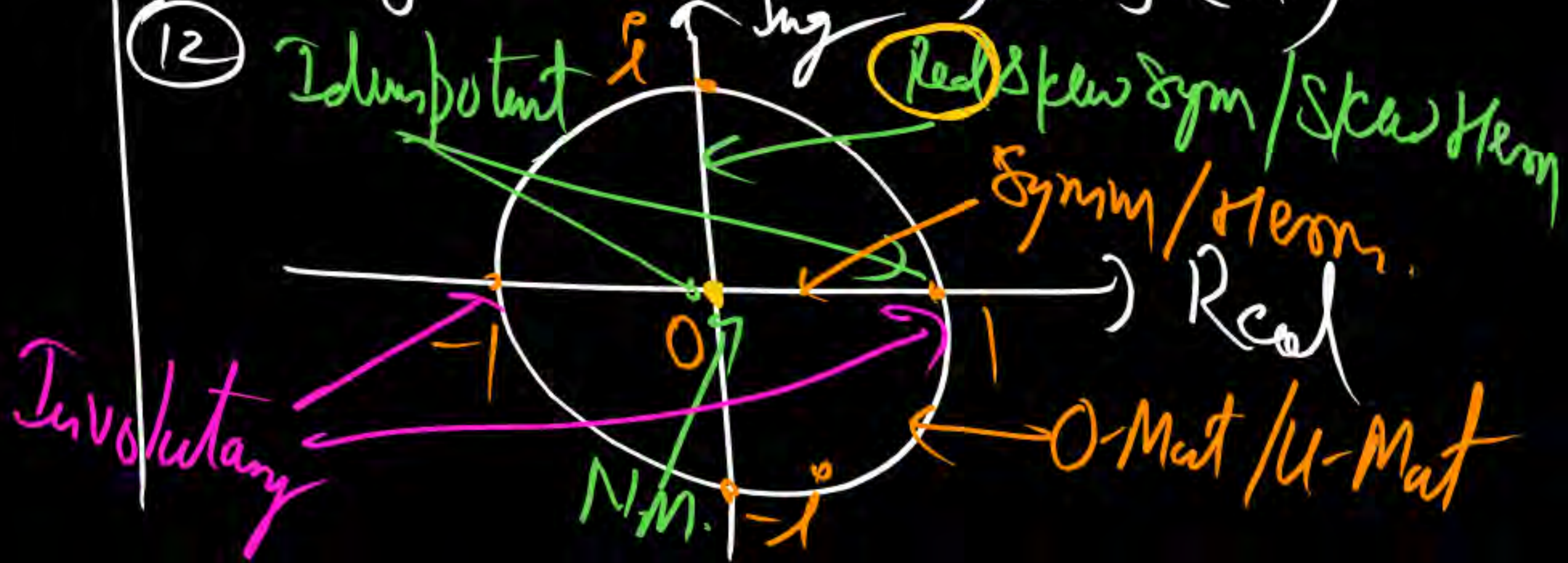
$N(A) = ? = \text{No. of } C - \rho(A)$

$= 3 - 2 = 1$

EIGEN VALUES

$\boxed{AX=\lambda X} \Rightarrow (A-\lambda I)X=0$

(*) C-Eqn is $|A-\lambda I|=0$, (*) $X \to$ Non Zero E Vector, (*) $A \to$ Sq. Mat

1. Sum of E Values $= Tr(A)$
2. Product " " $= |A|$
3. $(\lambda=0) \Longleftrightarrow (|A|=0)$
4. U.T.M/LTM/D.M $\Rightarrow \lambda =$ Diag elements
5. $A^m \to \lambda^m$
6. $A^{-1} \Rightarrow \frac{1}{\lambda}$
7. adj $A \to \frac{|A|}{\lambda}$
8. $A^T \to$ Same as A
9. Don't apply E operations in given Mat while calculating E Values
10. $\lambda \to A \Rightarrow (A^3-5A^2+7A+2I) = (A \to \lambda)$
11. No of Non Zero E Values $\leq \rho(A)$
12. 

① U-Mat of Real Nos is $\underline{\text{orthogonal}}$ also But converse is not necessarily True

② H-Mat of " " is $\underline{\text{Symm.}}$ also " "

③ Skew H-Mat of " " is $\underline{\text{skew symm.}}$ also " "

$$A = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}, \quad A = \begin{bmatrix} 2 & 4 & 5 \\ 4 & -3 & 6 \\ 5 & 6 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 4 & 5 \\ -4 & 0 & 6 \\ -5 & -6 & 0 \end{bmatrix}$$

$\because AA^\theta = I \Rightarrow$ U-Mat

$\because AA^T = I \Rightarrow$ O-Mat

Herm as well as Symm also

Skew Herm as well as Skew Symm.

Eg $A = \left[\right.$

$$A = \begin{bmatrix} 2 & 5 & 2+i \\ 5 & -3 & 7 \\ 2+i & 7 & 1 \end{bmatrix}$$ Symm but Not Herm Mat

The eigen values of A = $\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$ are ______.

(a) Purely imaginary (b) Zero

(c) Real (d) None of the above

C.Equ^n is $|A - \lambda I| = 0$

$$\lambda^2 - (Tr(A))\lambda + (|A|) = 0$$

$$\lambda^2 - (0)\lambda + (-1) = 0$$

$$\lambda^2 - 1 = 0$$

$$\lambda = \pm 1 \text{ (Real)}$$

$\because$ A is Herm. Mat $\Rightarrow \lambda = \pm 1$

Here A is Skew Symm But not in Real form that's why we are not getting Purely img E Values.

If a 3 × 3 real skew-symmetric matrix has an eigen value 2i, then one of the remaining eigen value is _____.

(a) $\frac{1}{2i}$ (b) $-\frac{1}{2i}$

(c) 0 (d) 1

Note: $A_{n\times n}$ skew symm then

(1) $|A| = \begin{cases} 0, & n = \text{odd} \Rightarrow \lambda = 0 \\ \text{perfect sq}, & n = \text{even} \end{cases}$

(2) H. Mat of Real Nos is symm. also.
But converse is not necessary true

$\lambda_1 = +2i$

$\Rightarrow \lambda_2 = -2i = \frac{2i}{-1} = \frac{2i}{i^2} = \frac{2}{i}$

$\begin{cases} \lambda_1 + \lambda_2 + \lambda_3 = 0 \\ 2i + (-2i) + 0 = 0 \end{cases}$

$0 \to 0$ (Purely Real)

$0 \to 0 + 0i = 0i$ (Purely Img)

(*) Comp Roots occur in pair if Coeff are Real

$\lambda_1 = 2i$, $\lambda_2 = -2i$, $\Longrightarrow$ for Real Skew Symm of 3×3 another E Value must be $\lambda = 0$

(*) 0 is considered as both Purely Real as well as Purely Img.

If scalar λ is a characteristic root of the matrix A, then $(A - \lambda I)$ is ______.

(a) Singular matrix (b) Non-singualr matrix

(c) Diagonal matrix (d) None of the above

$A - \lambda I$ = Char Matrix

$|A - \lambda I|$ = Char Polynomial

$|A - \lambda I| = 0$ Char Equⁿs

$AX = \lambda X$

$(A - \lambda I)X = 0$

for the Existence of Non Zero E. Vector

$\boxed{|A - \lambda I| = 0} \Rightarrow (A - \lambda I)$ is singular

Let M be a skew-symmetric, orthogonal real matrix. Then only possible eigen values of M are ______.

(a) $-1, 1$ ✗

(b) $-i, i$ ✓ $\Rightarrow |i| = 1$ & $|-i| = 1$

(c) 0

(d) $1, i$ ✗

Real Skew Symm $\Rightarrow$ Purely Imag

O-Mat $\Rightarrow$ Unit Modulus

The eigen values of $\begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & -1 \\ 3 & 1 & 0 \end{bmatrix}$ are ______.

= Skew symm formed by Real Nos

$\Rightarrow$ Purely Img E Values

$\Rightarrow \boxed{\lambda = 0, \pm \alpha i}$

& $Tr(A) = 0 + (\alpha i) + (-\alpha i)$

$= 0$

(a) –1, –2 2
(b) –2, –1, 3
(c) 2, 2, 3
(d) None of the above

Thank

THANK

Keep Hustling!