GATE

DATA SCIENCE + CS & IT

Engineering Mathematics

SUPER 1500

Lec : 05 Calculus

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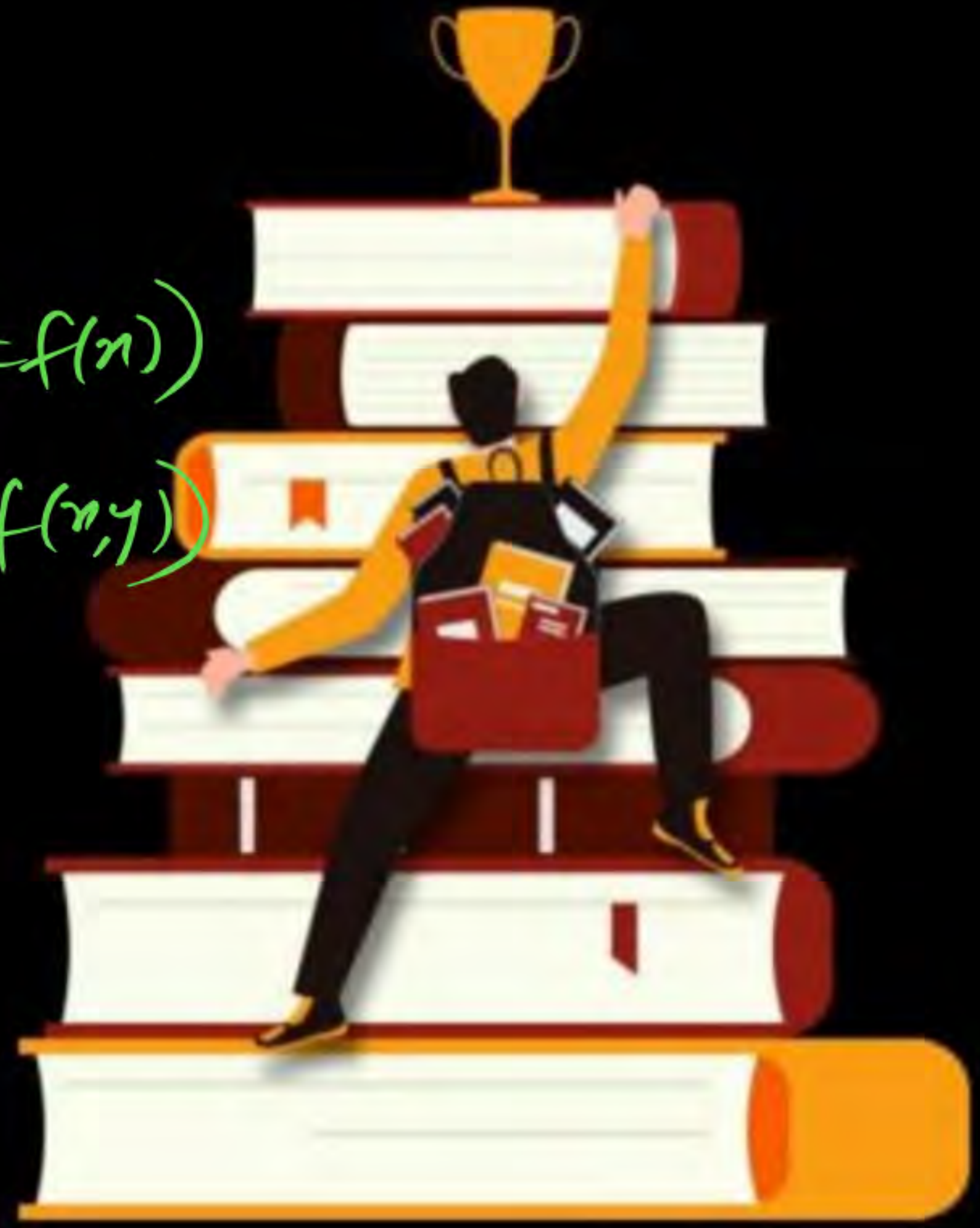
Topics *to be covered*

CALCULUS

(1) Maxima-Minima of Curve $(y=f(x))$

(2) " " of surface $(z=f(x,y))$



MAXIMA-MINIMA of $y=f(x)$

(1) Point $= x$ & Value $= y$, (2) N. Cond^n: $\boxed{f'(x)=0 \text{ or undefined}}$

(3) S. Cond^n: (M-I) 1^{st} Derivative test (M-II) 2^{nd} Derivative test.

(4) we will also check the Maxima-Minima at Corner Points.

	+	−	+	+	−
a	c_1	c_2	c_3	c_4	b
m	M	m	NM	M	m

(5) Concave up: $f''(x)>0$ & Concave Down $f''(x)<0$

(6) S. Inc: $f'(x)>0$ & S. Dec $f'(x)<0$

(7) Saddle Point $\Rightarrow$ $\exists$ N.M. N.M

$\not\Leftarrow$

The maximum value of

$y = f(x) = x^3 - 9x^2 + 24x + 5$ in the interval $[1, 6]$ is

(a) 21 (b) 25

(c) 41 (d) 46

$f'(x) = 3x^2 - 18x + 24$

$= 3(x^2 - 6x + 8)$

$$\boxed{f'(x) = 3(x-4)(x-2)}$$

T. Points are $x = 2, 4$

$+ \quad - \quad +$

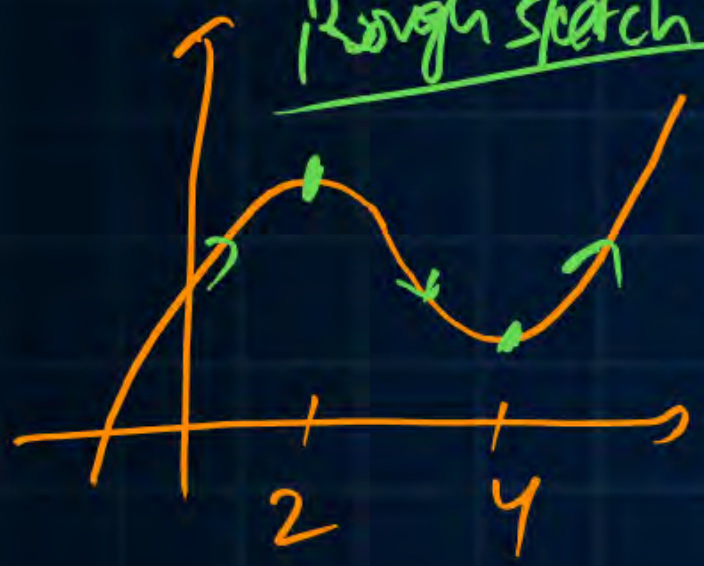
1 min 2 Max 4 (min) 6 (Max)

Max Value will occur

at $x = 2$, $f(2) = 25$

$x = 6$, $f(6) = 41$

Rough sketch



The maximum value of $(-\infty, \infty)$

$f(x) = x^3 - 9x^2 + 24x + 5$ is ____.

(a) 21 (b) 25

(c) 41 (d) DNE

T. Points are $x = 2$ & 4

$\boxed{f'(x) = 3(x-2)(x-4)}$

So Local Max Value

$= f(2) = 25$

$f'(1) = +ve$

$f'(3) = -ve$

$f'(5) = +ve$

Q. Max Value of $f(x) = -x^2 + 4x - 4$ will be?

(a) ∞ (b) 0

(c) 25 (d) 2

Q: Max Value of $f(x) = -x^2+4x-4$ will be (a) ∞ (b) 0 (c) 25 (d) 2

Sol: $f(x) = -(x^2-4x+4)$; $(-\infty,\infty)$

$y = f(x) = -(x-2)^2$

Max Value occurs at $x=2$

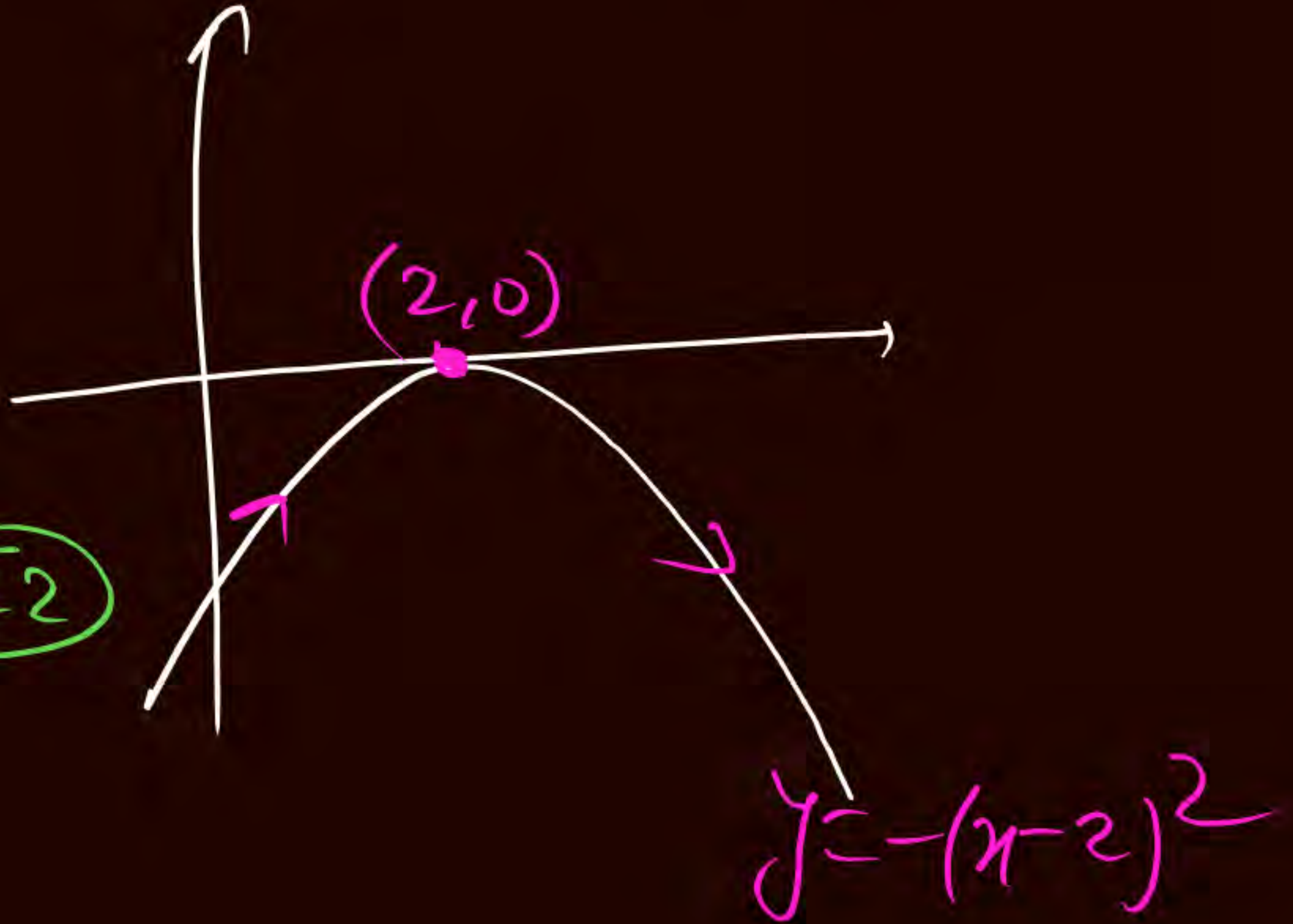
& Value $= f(2) = 0$

M-II

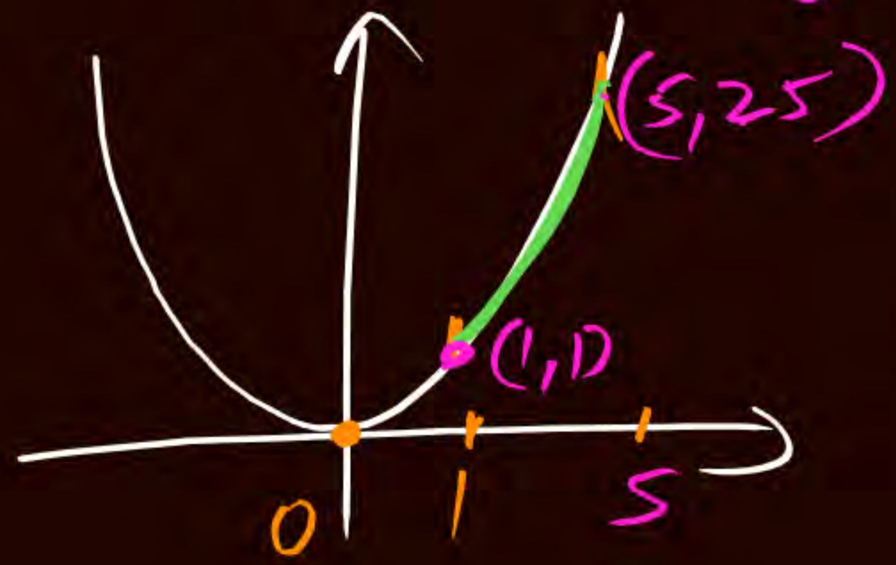
$\boxed{f'(x) = -2(x-2)}$ So T.P is $x=2$

$-\infty \quad + \quad 2 \quad - \quad \infty$

(Max)



Q: the Max value of $y = x^2$ in $[1, 5]$ will be ? ________



Min Value = 1

Max " = 25

The maximum value of $f(x) = 2x^3 - 9x^2 + 12x - 3$ in the interval $0 \le x \le 3$ is ______.

$f(x) = 2x^3 - 9x^2 + 12x - 3$

$f'(x) = 6x^2 - 18x + 12$

$= 6(x^2 - 3x + 2)$

$$\boxed{f'(x) = 6(x-1)(x-2)}$$

T.P are $x = 1$ & 2

	$+$		$-$		$+$	
0		1		2		3
min		Max		min		Max

$f(1) = 2 - 9 + 12 - 3 = 2$

$f(3) = 2(27) - 9(9) + 12(3) - 3 = 6$

A point on the curve is said to be an extremum if it is a local minimum (or) a local maximum. The number of distinct extrema for the curve $3x^4 - 16x^3 + 24x^2 + 37$ is ______.

(a) 0 (b) 1

(c) 2 (d) 3

$$f(x) = 3x^4 - 16x^3 + 24x^2 + 37$$

$$f'(x) = 12x^3 - 48x^2 + 48x$$

$$= 12x(x^2 - 4x + 4)$$

$$\boxed{f'(x) = 12x(x-2)^2}$$

T. Points are $x = 0, 2, 2$

The range of values of k for which the function $f(x) = (k^2 - 4)x^2 + 6x^3 + 8x^4$ has a local maxima at point $x = 0$ is

(a) $k < -2$ or $k > 2$ (b) $k \le -2$ or $k \ge 2$

(c) $-2 < k < 2$ (d) $-2 \le k \le 2$

$f(x) = (k^2-4)x^2 + 6x^3 + 8x^4$

$\boxed{f'(x) = (k^2-4)(2x) + 18x^2 + 32x^3}$

$f''(x) = (k^2-4)(2) + 36x + 96x^2$

At $x=0$, using N. Cond^n $f'(0) = 0$

$0 = 0$

S. Cond^n at $x=0$ is $f''(0) < 0$

$2(k^2-4) + 0 + 0 < 0$

$(k-2)(k+2) < 0$

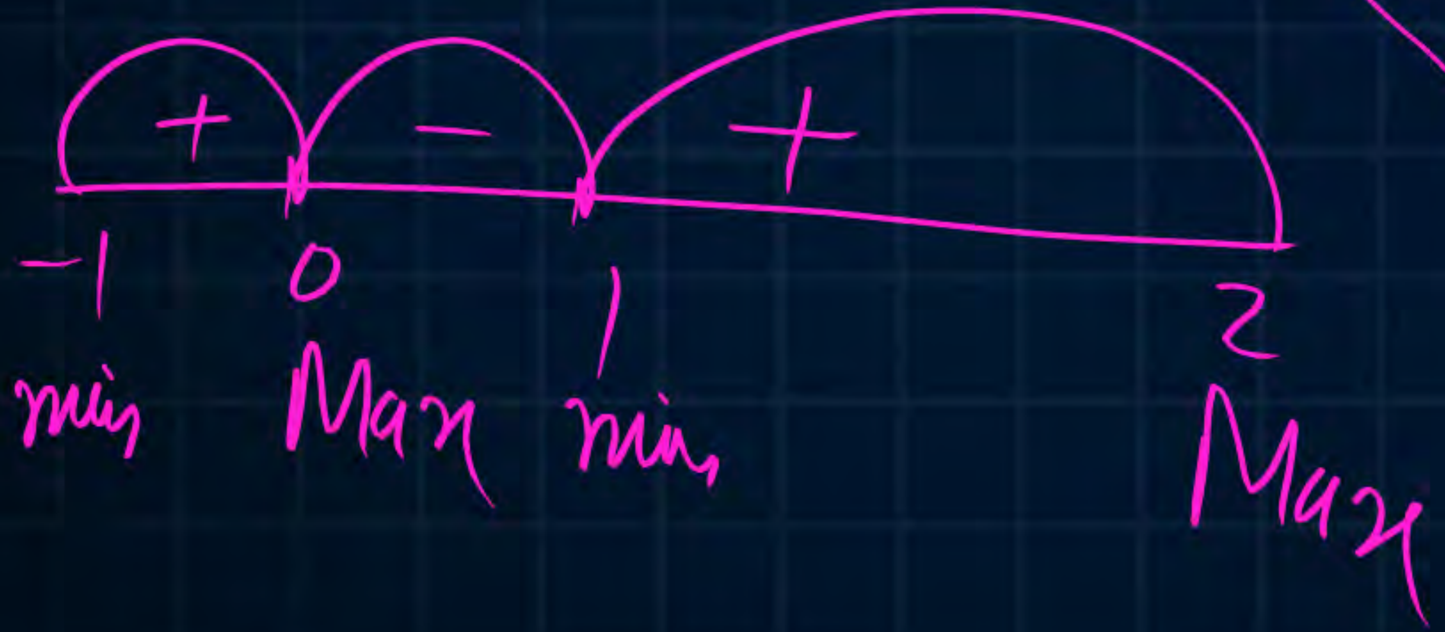
$-2 < k < 2$

Consider the function $f(x) = 2x^3 - 3x^2$ in the domain $[-1, 2]$. The global minimum of $f(x)$ is ____

$f(x) = x^2(2x-3)$

$f'(x) = 6x^2 - 6x = \boxed{6x(x-1)}$

T.P $\quad x = 0, 1$



L-min $= f(-1) = -5$ = global min

L.min $= f(1) = -1$ = local min

As x varies from − 1 to + 3, which one of the following describes the behavior of the function $f(x) = x^3 - 3x^2 + 1$?

(a) f(x) increases monotonically.

(b) f(x) increases, then decreases and increases again.

(c) f(x) decreases, then increases and decreases again.

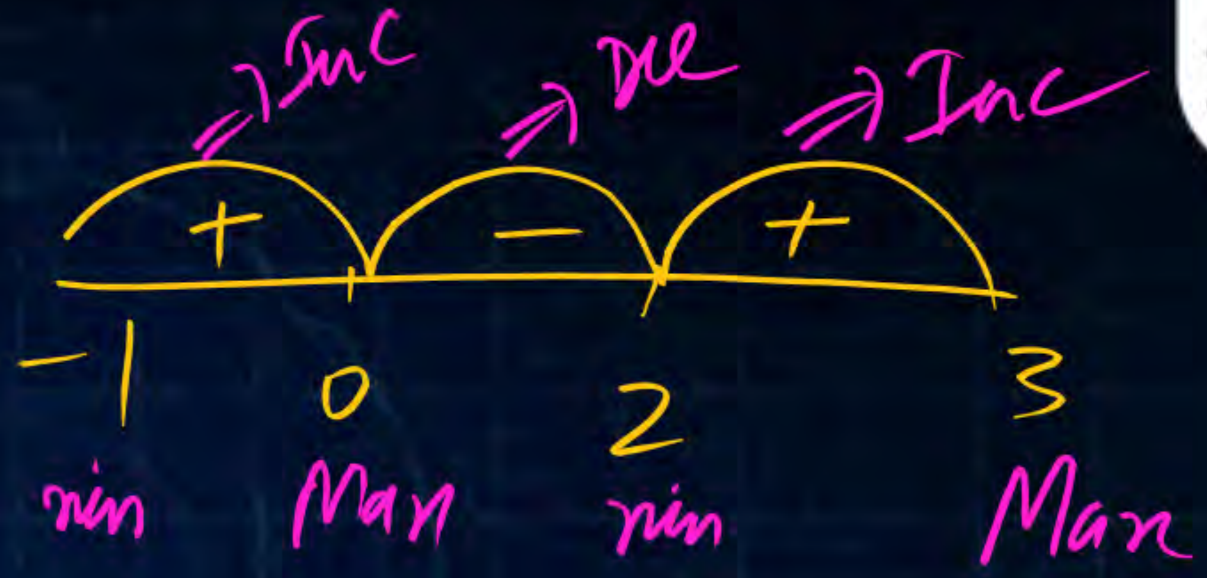
(d) f(x) increases and then decreases.

$$f(x) = x^3 - 3x^2 + 1$$

$$f'(x) = 3x^2 - 6x$$

$$\boxed{f'(x) = 3x(x-2)}$$

T P are $x = 0, 2$



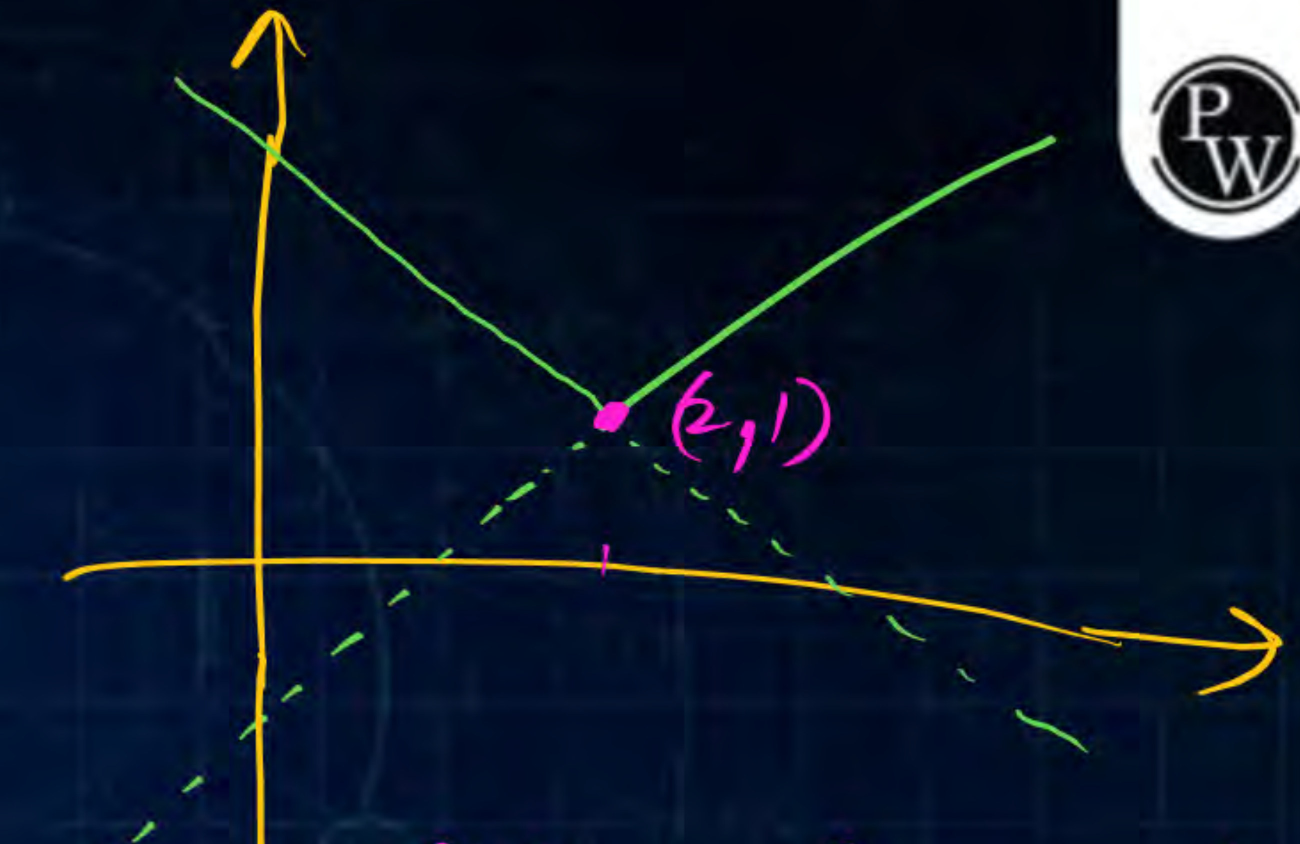
Let max $\{a, b\}$ denote the maximum of two real numbers a and b. Which of the following statement(s) is/are TRUE about the function $f(x) = \max\{3 - x, x - 1\}$?

(a) It is continuous on its domain.

(b) It has a local minimum at $x = 2$.

(c) It has a local maximum at $x = 2$.

(d) It is differentiable on its domain.



$f(x) = \max\{3-x, x-1\}$

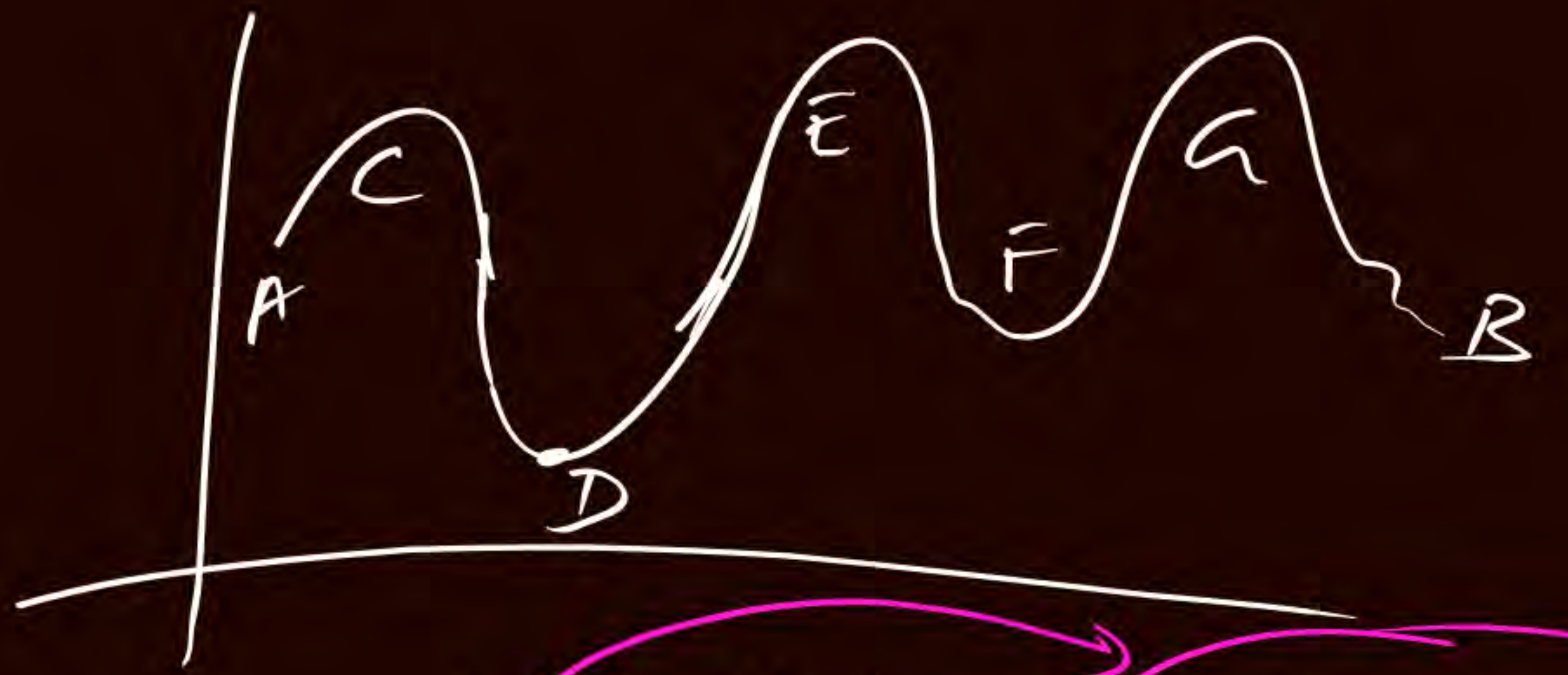
$$f(x) = \begin{cases} 3-x, & x<2 \\ x-1, & x>2 \end{cases}$$

$y = 3-x$ & $y = x-1$

$x+y=3$ $\quad$ $x-y=1$

$\frac{x}{3}+\frac{y}{3}=1$ $\quad$ $\frac{x}{1}+\frac{y}{-1}=1$

$x-1 = 3-x \Rightarrow x=2$ & $y=1$



Local Minima : A, D, F, B

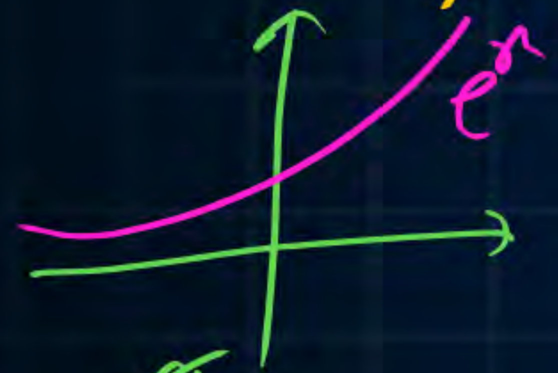
Global " : D

For real values of x, the minimum value of th[e] function f(x) = exp(x) + exp(− x) is

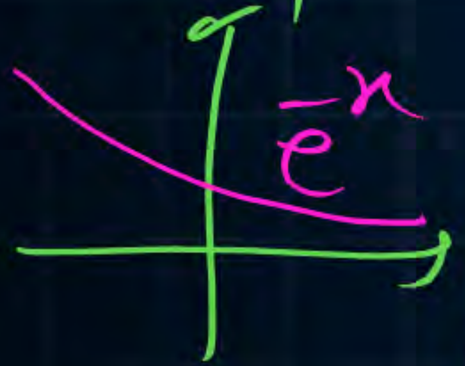
(a) 2 ✓ (b) 1

(c) 0.5 (d) 0

$$f(x) = e^x + e^{-x}$$



$$= e^x + \frac{1}{e^x}$$



$$= R^+ + \frac{1}{R^+}$$

$$\boxed{f(x) \geq 2} \Rightarrow f_{min} = 2$$

Th: if $x \in R^+$ then $\boxed{x + \frac{1}{x} \geq 2}$

if $x = 1$ then $1 + \frac{1}{1} = 2$

if $x = 3$ then $3 + \frac{1}{3} > 2$

if $x = 2$ then $2 + \frac{1}{2} > 2$

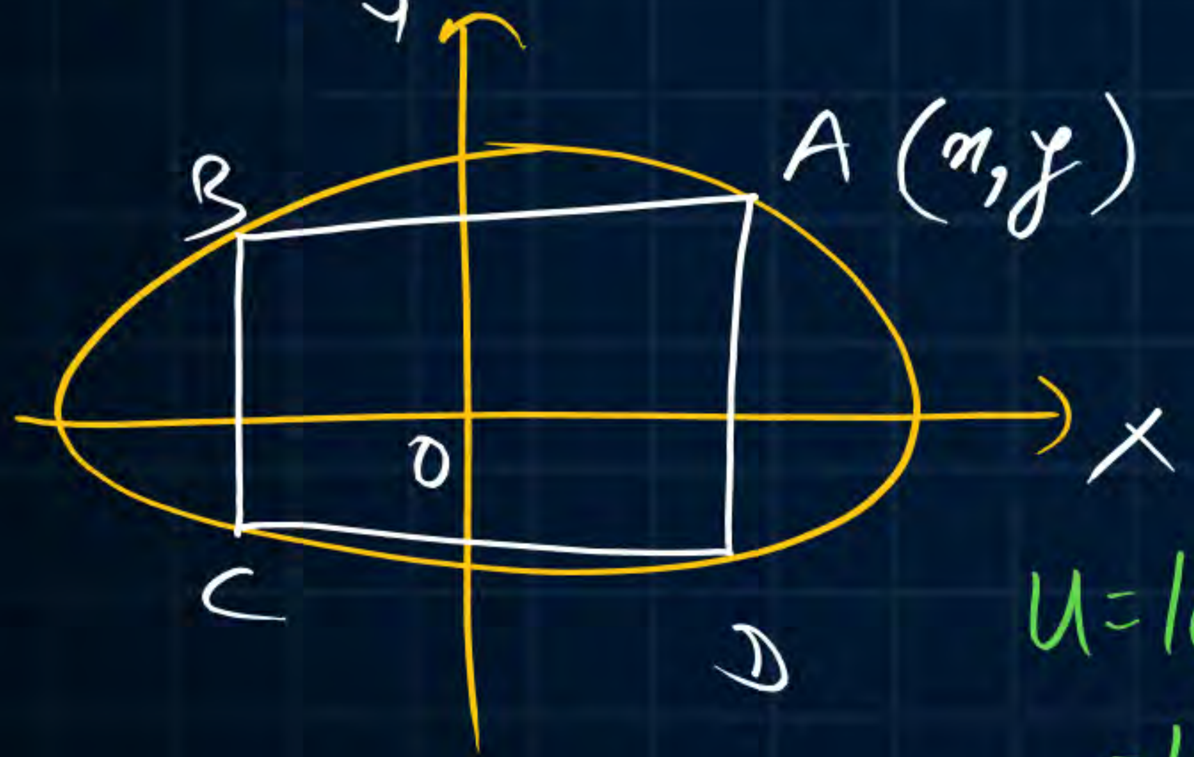
if $x = 0.2$, then $0.2 + \frac{1}{0.2} = 0.2 + 5 > 2$

. . . . . .

The maximum area (in square units) of a rectangle whose vertices lie on the ellipse $x^2+4y^2=1$ is______.

(M-I) $x^2+4y^2=1$

$\frac{x^2}{1^2}+\frac{y^2}{(1/2)^2}=1$



$A = (AB)\times(AD) = (2x)(2y) = 4xy$ — (1)

Let $U = A^2 = 16x^2y^2$ where $A=\sqrt{U}$

or $A_{max} = \sqrt{U_{max}} = ?$

Now Try to Calculate Max Value of U

$U = 16x^2y^2$

$= 16\left(x^2\left(\frac{1-x^2}{4}\right)\right)$

$\boxed{U = 4(x^2-x^4)}$ — (1)

$\frac{du}{dx}=0$

$4(2x-4x^3)=0$

$8x(1-2x^2)=0$

$x=0, \pm\frac{1}{\sqrt{2}}$

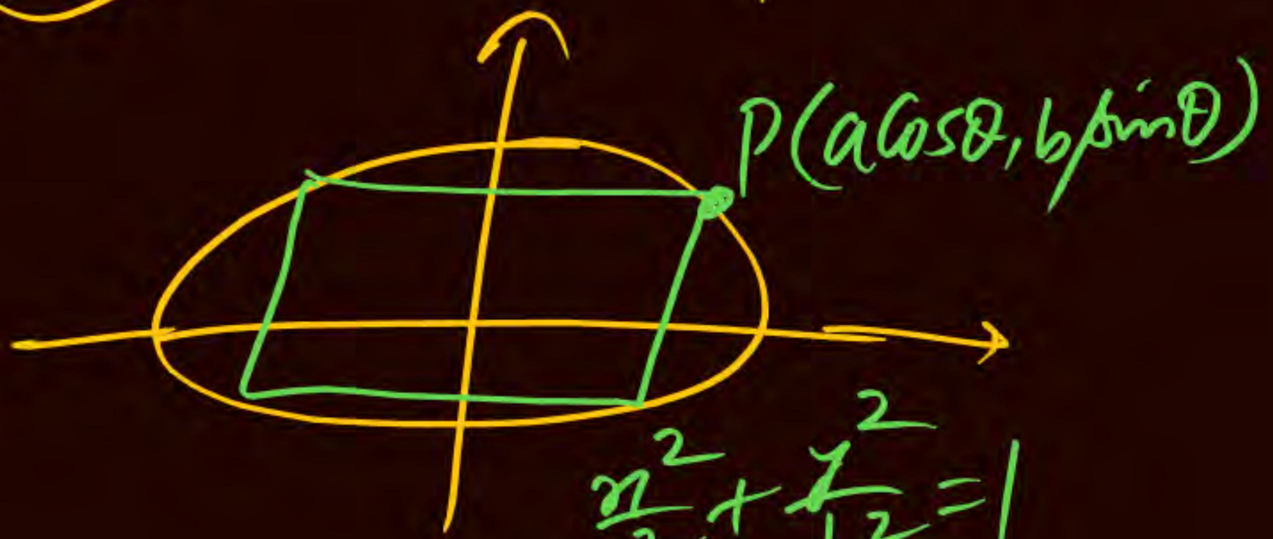
$\frac{d^2u}{dx^2} = (8-48x^2)$

$\because \left(\frac{d^2u}{dx^2}\right)_{x=\frac{1}{\sqrt{2}}} = -ve \Rightarrow u$ will be max at $x = \frac{1}{\sqrt{2}}$

& $U_{max} = \left(4(x^2 - x^4)\right)_{x=\frac{1}{\sqrt{2}}} = 4\left(\frac{1}{2} - \frac{1}{4}\right) = 4\left(\frac{2-1}{4}\right) = 1$

So $A_{max} = \sqrt{U_{max}} = \sqrt{1} = 1$

M-II



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Sol: $x^2+4y^2=1 \Rightarrow \frac{x^2}{1^2}+\frac{y^2}{(1/2)^2}=1$

So Random Point of Ellipse is $P\left(1\cos\theta, \frac{1}{2}\sin\theta\right)$

Area $= L \times B = (2\cos\theta)(\sin\theta) = \sin 2\theta$

$\boxed{A=\sin 2\theta} \Rightarrow \frac{dA}{d\theta}=2\cos 2\theta$, $\frac{d^2A}{d\theta^2}=-4\sin 2\theta$

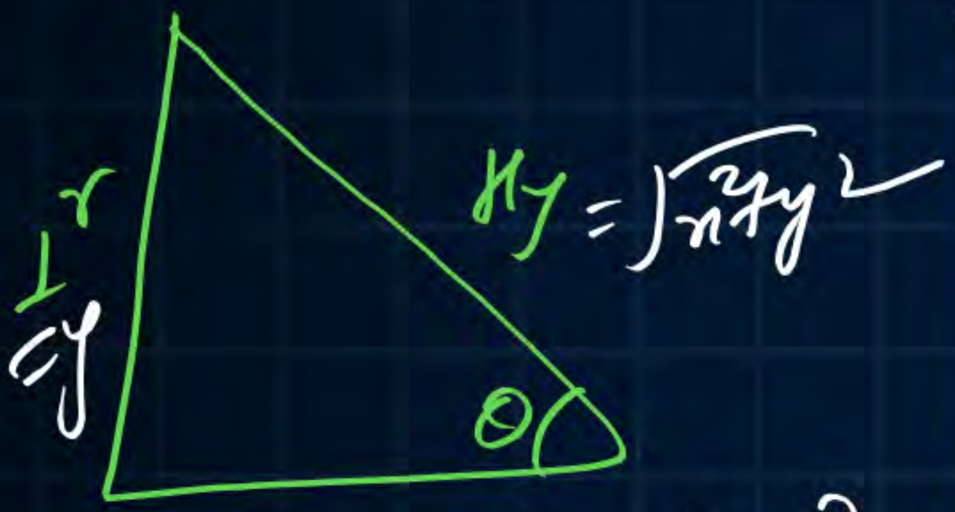
T-Points $\frac{dA}{d\theta}=0 \Rightarrow 2\theta=\frac{\pi}{2} \Rightarrow \theta=\frac{\pi}{4}$

$\left(\frac{d^2A}{d\theta^2}\right)_{\theta=\frac{\pi}{4}}=-4<0$ so $\theta=\frac{\pi}{4}$ is Point of Maxima

& $\left(\text{Max Value of } A\right)_{\theta=\pi/4}=\sin 2\left(\frac{\pi}{4}\right)=1$

For a right angled triangle, if the sum of the length of the hypotenuse and a side is kept constant, in order to have maximum area of the triangle, the angle between the hypotensuse and the side is

(a) 120° (b) 60°
(c) 30° (d) 45°



$B = x$, $\theta = ?$

AT Q $\boxed{\sqrt{x^2+y^2} + x = K}$ (Const)

$x^2+y^2 = (K-x)^2$

$x^2+y^2 = K^2+x^2-2Kx$

$y^2 = K^2-2Kx$ — (1)

$A = \frac{1}{2}xy \Rightarrow u = A^2 = \frac{1}{4}(x^2y^2) = \frac{1}{4}x^2(K^2-2Kx)$

For Max Area, u must be Max $\because A = \sqrt{u}$

$\Rightarrow \frac{du}{dx} = 0$

$\frac{1}{4}\left[2K^2x - 2K(3x^2)\right] = 0$

$Kx - 3x^2 = 0$

$x = K/3 \Rightarrow y = \frac{K}{\sqrt{3}}$

$\tan\theta = \frac{y}{x} = \frac{K/\sqrt{3}}{K/3}$

$\tan\theta = \sqrt{3} \Rightarrow \theta = 60°$

$f(x) = e^{-x} \sin(100x)$

The best approximation of the minimum value attained by $e^{-x}\sin(100x)$ for $x > 0$ is ______

w.k.that e^{-x} is always +ve i.e $e^{-x} \geq 0$ i.e Min Value of $e^{-x} > 0$

So Min Value of $f(x)$ depends on $(\sin 100x)$

& w.k.that, Min of $\sin(100x) = -1$ $\quad (\because -1 \leq \sin x \leq 1)$

$\sin 100x = \sin\left(\frac{3\pi}{2}\right)$

$100x = \frac{3\pi}{2}$

i.e Point of Minima is $x = \frac{3\pi}{200}$

& Min Value $= f\left(\frac{3\pi}{200}\right) = e^{-\frac{3\pi}{200}}(-1)$

$= -e^{-\frac{3\pi}{200}} = -0.954$

Maxima - Minima of func^n of two Variables

$$\left[z = f(x, y) \right]$$

Point $= (x, y)$

Value $= z$

S. Cond^n :-

1. if $rt - s^2 > 0$ & $r > 0$ then $P(a, b)$ is Point of minima
2. if $rt - s^2 > 0$ & $r < 0$ " " " of Maxima
3. if $rt - s^2 < 0$ then " " " of Inflexion
4. if $rt - s^2 = 0$ then Case fails

N. Cond^n :- $\dfrac{\partial z}{\partial x} = 0$ & $\dfrac{\partial z}{\partial y} = 0$

The total cost (C_T) of an equipment in terms of the operation variables x and y is

$$z = C_T = 2x + \frac{12000}{xy} + y + 5$$

The optimal value of C_T, rounded to 1 decimal place, is _____.

$z_x = 2 - \frac{12000}{x^2 y}$

$z_{xx} = \frac{24000}{x^3 y}$

$z_y = -\frac{12000}{xy^2} + 1$

$z_{yy} = \frac{24000}{xy^3}$

$z_{xy} = \frac{12000}{x^2 y^2}$

$z_x = 0$ & $z_y = 0$

$2 - \frac{12000}{x^2 y} = 0$ —(1) & $\frac{-12000}{xy^2} + 1 = 0$ —(2)

$x^2 y = 6000$ & $xy^2 = 12000$

$\boxed{\frac{x}{y} = \frac{1}{2}}$ $x = \alpha$ (let), $y = 2\alpha$ (let)

By (1); $2 = \frac{12000}{\alpha^3}$

$\alpha^3 = 6000 \Rightarrow \alpha = 14.2$

Hence $x = \alpha = 14.3$

$y = 2k = 28.6$

Hence $C_T = \left(2x + \frac{12000}{xy} + y + 5\right)_{(14.2,\, 28.4)}$

$= 91.5$

The function $f(x, y) = 2x^4 + y^2 - x^2 - 2y$ has a relative_____.

(a) maxima at $\left(\frac{1}{2}, 1\right)$ (b) minima at $\left(\frac{1}{2}, 1\right)$

(c) maxima at (0, 1) (d) minima at (0, 1)

$Z = 2x^4 + y^2 - x^2 - 2y$

$\frac{\partial z}{\partial x} = 8x^3 - 2x$, $\frac{\partial^2 z}{\partial x^2} = 24x - 2$

$\frac{\partial z}{\partial y} = 2y - 2$, $\frac{\partial^2 z}{\partial y^2} = 2$

$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}(2y-2) = 0$

C. Points are: $\frac{\partial z}{\partial x} = 0$ & $\frac{\partial z}{\partial y} = 0$

$2x(4x^2-1) = 0$ & $2(y-1) = 0$

$x = 0, \pm\frac{1}{2}$, $y = 1$

So C. Points are $(0,1), \left(\frac{1}{2}, 1\right), \left(-\frac{1}{2}, 1\right)$

At $P(0,1) \rightarrow r = (24x-2)_P = -2$, $t = 2$, $s = 0$

$\because rt - s^2 = (-2)(2) - (0)^2 < 0 \Rightarrow$ saddle point.

At $Q\left(\frac{1}{2}, 1\right) \rightarrow r = (24x-2)_Q = 10$, $t = 2$, $s = 0$

$rt - s^2 = (10)(2) - (0)^2 > 0$ & $r > 0$

Find the absolute maxima and minima of the value function

$z = f(x, y) = x^2 - xy - y^2 - 6x + 2$ on the rectangular plate $0 \le x \le 5$, $-3 \le y \le 0$

$$\frac{\partial z}{\partial x} = 2x - y - 6, \quad \frac{\partial^2 z}{\partial x^2} = 2$$

$$\frac{\partial z}{\partial y} = -x - 2y, \quad \frac{\partial^2 z}{\partial y^2} = -2$$

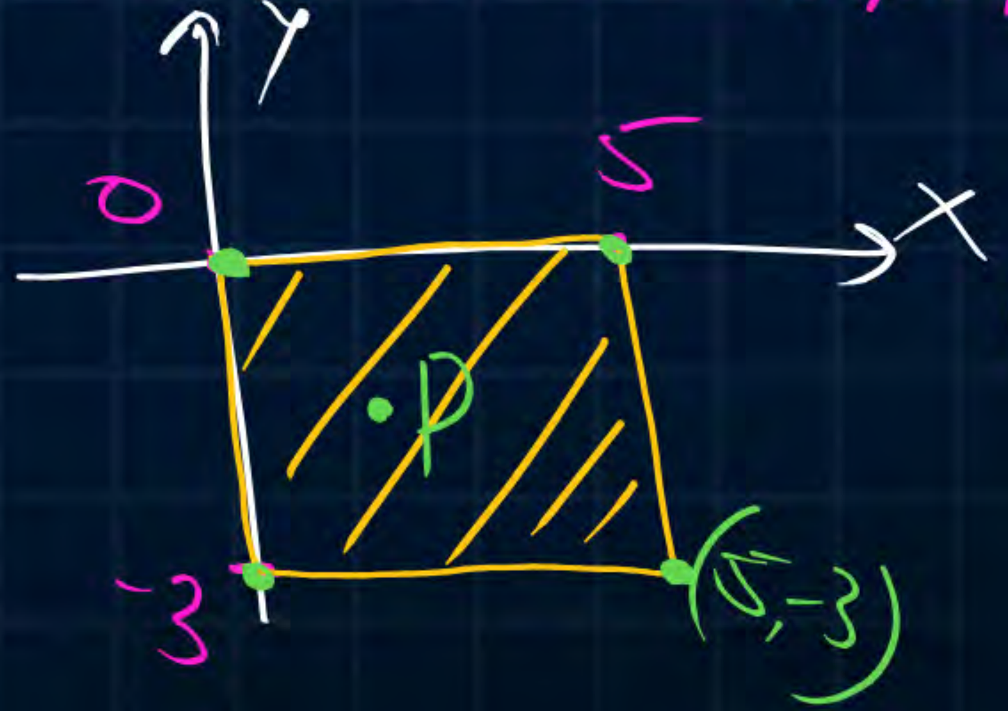
$$\frac{\partial^2 z}{\partial x \partial y} = -1$$

C. Points are $\frac{\partial z}{\partial x} = 0$ & $\frac{\partial z}{\partial y} = 0$

$2x - y - 6 = 0$ & $-x - 2y = 0$

$$\left.\begin{aligned} 2x - y &= 6 \\ x + 2y &= 0 \end{aligned}\right\} \Rightarrow x = \frac{12}{5}, \; y = \frac{-6}{5} \quad \therefore P\left(\frac{12}{5}, \frac{-6}{5}\right) \text{ C. Point}$$

Now At P; $rt - s^2 < 0$ so $P\left(\frac{12}{5}, \frac{-6}{5}\right)$ is Saddle point



Corner Points are

$(0,0)$ $(5,0)$, $(0,-3)$

$(5,-3)$

$$f(x,y) = x^2 - xy - y^2 - 6x + 2$$

$$f(0,0) = 0-0-0-0+2 = 2$$

$$f(5,0) = 25-0-0-30+2 = -3$$

$$f(0,-3) = 0-0-9-0+2 = -7$$ (Global Min value) & it occurs at $(0,-3)$

$$f(5,-3) = 25+15-9-30+2 = 3$$ (" Max value) & " " at $(5,-3)$

Thank

THANK

Keep Hustling!