GATE

DATA SCIENCE + CS & IT

Engineering Mathematics

SUPER 1500

Lec : 04 Calculus

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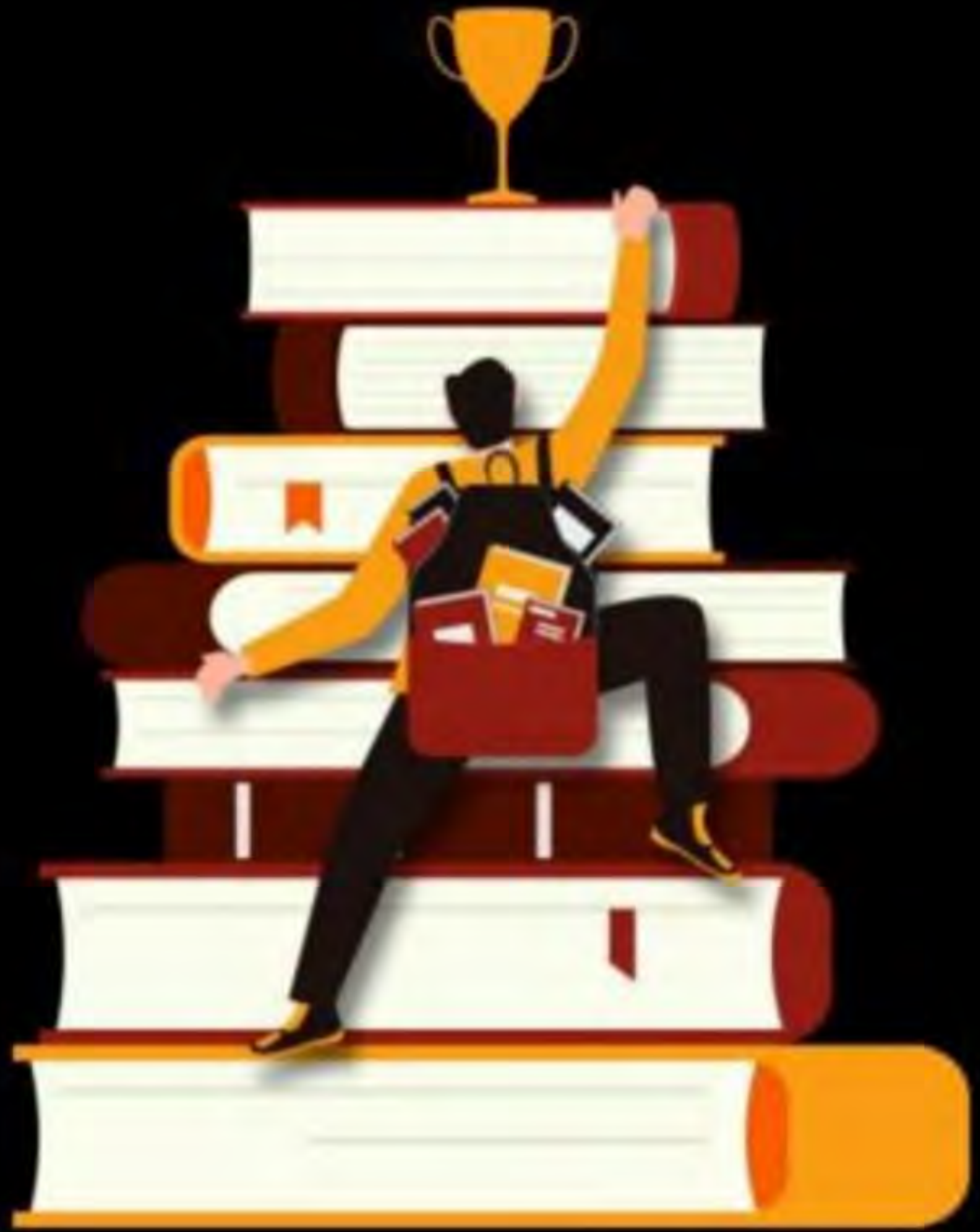
Topics *to be covered*

CALCULUS

1. DERIVATIVES
2. MAXIMA-MINIMA



DERIVATIVES

1. O.D $\longrightarrow y = f(x)$
2. P.D $\longrightarrow z = f(x, y)$
3. Chain Rule of P.D : $z \longrightarrow (x, y) \longrightarrow (r, s)$
4. Total D : $z \longrightarrow (x, y) \longrightarrow t$ (alone)
5. Jacobian : $(u, v) \longrightarrow (x, y)$

If $y = \sqrt{\tan x + \sqrt{\tan x + \sqrt{\tan x + \ldots\ldots}}}$ then $\frac{dy}{dx} =$

(a) $\frac{\tan x}{2y-1}$

(b) $\frac{\sqrt{\tan x}}{2y-1}$

(c) $\frac{\sec x}{2y-1}$

(d) $\frac{\sec^2 x}{2y-1}$

$y = \sqrt{\tan x + y}$

$y^2 = \tan x + y$

or $\boxed{y^2 - y = \tan x}$

$\frac{d}{dx}(y^2) - \frac{d}{dx}(y) = \frac{d}{dx}(\tan x)$

$2y\frac{dy}{dx} - \frac{dy}{dx} = \sec^2 x$

$\frac{dy}{dx} = \frac{\sec^2 x}{2y-1}$

If $u = x^3 + y^3$ where $x = a\cos t$, $y = b\sin t$, then $\frac{du}{dt} =$

(a) $-3a^3 \cos^2 t \sin t + 3b^3 \sin^2 t \cos t$

(b) $3a^3 \sin^2 t \cos t + 3b^3 \cos^2 t \sin t$

(c) $3b \sin^2 t \cos t + 3a^3 \sin^2 t \cos t$

(d) $-3a^3 \sin t + 3b^3 \cos^2 t \sin t$

M-I $u = a^3\cos^3 t + b^3 \sin^3 t$

$$\frac{du}{dt} = a^3\left(3\cos^2 t\,(-\sin t)\right) + b^3\left(3\sin^2 t \cdot \cos t\right)$$

$$= -3a^3\cos^2 t \sin t + 3b^3 \sin^2 t \cos t$$

M-II $\because u \to (x, y) \to t$ (alone)

So using T.D concept:

$$\boxed{du = \left(\frac{\partial u}{\partial x}\right)dx + \left(\frac{\partial u}{\partial y}\right)dy}$$

i.e. $\frac{du}{dt} = \left(\frac{\partial u}{\partial x}\right)\frac{dx}{dt} + \left(\frac{\partial u}{\partial y}\right)\frac{dy}{dt}$

$= (3x^2)(-a\sin t) + (3y^2)\,b\cos t$

$= 3(a^2\cos^2 t)(-a\sin t) + (3b^2\sin^2 t)\,b\cos t$

= i.e. (a)

2 Let $f(x) = e^{-|x|}$, where x is real. The value of $\frac{df}{dx} =$ at $x = -1$ is

(a) $-e$ (b) e

(c) $\frac{1}{e}$ (d) $-\frac{1}{e}$

$$f(x) = e^{-|x|} = \begin{cases} e^{x}, & x<0 \\ e^{-x}, & x>0 \end{cases}$$

$$f'(x) = \begin{cases} e^{x}, & x<0 \\ -e^{-x}, & x>0 \end{cases}$$

So $\left(\frac{df}{dx}\right)_{x=-1} = f'(-1) = e^{-1} = \frac{1}{e}$

Q $x^a \cdot y^b = (x+y)^{a+b}$ then $\frac{dy}{dx} = ?$

If $x^a y^b = (x+y)^{a+b}$ then $\frac{dy}{dx} =$

(a) $\frac{x}{y}$ (b) $\frac{1}{y}$

(c) $\frac{1}{x}$ (d) $\frac{y}{x}$ ✓

Sol: Taking log both sides

$\log x^a + \log y^b = \log (x+y)^{a+b}$

$a \log x + b \log y = (a+b) \log (x+y)$

$\frac{d}{dx}(a \log x) + \frac{d}{dx}(b \log y) = (a+b)\frac{d}{dx}\log(x+y)$

$a\left(\frac{1}{x}\right) + b\left(\frac{1}{y}\frac{dy}{dx}\right) = (a+b)\left(\frac{1}{x+y}\cdot\left(1+\frac{dy}{dx}\right)\right)$

$\left(\frac{b}{y} - \frac{a+b}{x+y}\right)\frac{dy}{dx} = \frac{a+b}{x+y} - \frac{a}{x}$

$\left(\frac{bx+by-ay-by}{y\cancel{(x+y)}}\right)y' = \frac{ax+bx-ax-ay}{x\cancel{(x+y)}}$

$\left(\frac{\cancel{bx-ay}}{y}\right)\frac{dy}{dx} = \left(\frac{\cancel{bx-ay}}{x}\right)$

$y' = \frac{y}{x}$

$x^{\sin y} = y^{\sin x}$, then $\frac{dy}{dx}$ is equal to

(a) $\frac{x^2 \cos x \log x - y \sin y}{x^2 \cos x \log x - x \sin x}$

(b) $\frac{y^2 \cos y \log y - x \sin x}{y^2 \cos y \log y - y \sin y}$

(c) $\frac{xy \cos x \cos y - y \sin y}{xy \cos x \cos y - x \sin x}$

(d) $\frac{xy \cos x \log y - y \sin y}{xy \log x \cos y - x \sin x}$

$x^{\sin y} = y^{\sin x}$

$\sin y . \log x = \sin x . \log y$

$\frac{d}{dx}(\sin y \log x) = \frac{d}{dx}(\sin x \log y)$

$\sin y\left(\frac{1}{x}\right) + \log x(\cos y\, y') = \sin x\left(\frac{1}{y} y'\right) + \log y(\cos x)$

$\left(\log x . \cos y - \frac{1}{y}\sin x\right) y' = \log y . \cos x - \frac{1}{x}\sin y$

$\left(\frac{y \log x \cos y - \sin x}{y}\right) y' = \frac{x \log y \cos x - \sin y}{x}$

$\frac{dy}{dx} = \frac{y}{x}\left[\frac{x \log y \cos x - \sin y}{y \log x \cos y - \sin x}\right]$ i.e (d)

Let $f(x, y) = \dfrac{ax^2 + by^2}{xy}$, where a and b are constants. If $\dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial y}$ at $x = 1$ and $y = 2$, then the relation between a and b is

(a) $a = \dfrac{b}{4}$ (b) $a = \dfrac{b}{2}$

(c) $a = 2b$ (d) $a = 4b$

$$\frac{\partial f}{\partial x} = \frac{xy(2ax) - (ax^2+by^2)(y)}{(xy)^2}$$

$$= \frac{ax^2y - by^3}{x^2y^2} \quad \text{---} \ 1$$

$$f_x(1,2) = \frac{2a - 8b}{4}$$

$$\frac{\partial f}{\partial y} = \frac{xy(2by) - (ax^2+by^2)(x)}{(xy)^2}$$

$$= \frac{bxy^2 - ax^3}{x^2y^2}$$

$$f_y(1,2) = \frac{4b-a}{4}$$

So $f_x = f_y$

$4b - a = 2a - 8b \Rightarrow a = 4b$

M-II $f = \dfrac{ax^2+by^2}{xy} = a\left(\dfrac{x}{y}\right)+b\left(\dfrac{y}{x}\right)$

$$f_x = a\left(\frac{1}{y}\right)+b\left(\frac{-y}{x^2}\right) \Rightarrow f_x(1,2) = \frac{a}{2}-2b$$

$$f_y = a\left(-\frac{x}{y^2}\right)+b\left(\frac{1}{x}\right) \Rightarrow f_y(1,2) = \frac{-a}{4}+b$$

ATQ, $\dfrac{a}{2}-2b = \dfrac{-a}{4}+b$

$$a = 4b$$

3 Consider the following expression:

$$\boxed{z = \sin(y + it) + \cos(y - it)}$$ i.e. $z = f(y, t)$

where z, y, and t are variables, and $i = \sqrt{-1}$ is a complex number. The partial differential equation derived from the above expression is

(a) $\frac{\partial^2 z}{\partial t^2} + \frac{\partial^2 z}{\partial y^2} = 0$ ✓

(b) $\frac{\partial^2 z}{\partial t^2} - \frac{\partial^2 z}{\partial y^2} = 0$

(c) $\frac{\partial z}{\partial t} - i\frac{\partial z}{\partial y} = 0$

(d) $\frac{\partial z}{\partial t} + i\frac{\partial z}{\partial y} = 0$

$$\frac{\partial z}{\partial y} = \cos(y+it) - \sin(y-it)$$

$$\frac{\partial^2 z}{\partial y^2} = -\sin(y+it) - \cos(y-it) = -z$$

$$\frac{\partial z}{\partial t} = \cos(y+it)(i) - \sin(y-it)(-i)$$

$$= i\{\cos(y+it) + \sin(y-it)\}$$

$$\frac{\partial^2 z}{\partial t^2} = i\{-\sin(y+it).i + \cos(y-it)(-i)\}$$

$$= -i^2\{\sin(y+it) + \cos(y-it)\} = z$$

So $$\frac{\partial^2 z}{\partial y^2} = -\frac{\partial^2 z}{\partial t^2}$$

Q. If $u = \sin^{-1}\left(\frac{x}{y}\right) + \cos^{-1}\left(\frac{y}{x}\right)$ then $\frac{u_x}{u_y} = ?$ (a) $\frac{x}{y}$ (b) $\frac{y}{x}$ (c) $-\frac{x}{y}$ (d) $-\frac{y}{x}$

Sol: $u = \sin^{-1}\left(\frac{x}{y}\right) + \frac{\pi}{2} - \sin^{-1}\left(\frac{y}{x}\right)$ —— (1)

$\because \boxed{\sin^{-1}\theta + \cos^{-1}\theta = \pi/2}$ & $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$

$$u_x = \frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-\frac{x^2}{y^2}}}\left(\frac{1}{y}\right) - \frac{1}{\sqrt{1-\frac{y^2}{x^2}}}\left(\frac{-y}{x^2}\right)$$

$$u_x = \frac{1}{\sqrt{y^2-x^2}} + \frac{y}{x}\frac{1}{\sqrt{x^2-y^2}}$$

$$u_x = \frac{x\sqrt{x^2-y^2} + y\sqrt{y^2-x^2}}{x\sqrt{x^2-y^2}\sqrt{y^2-x^2}}$$

Similarly

$$u_y = -\frac{x}{y}\frac{1}{\sqrt{y^2-x^2}} - \frac{1}{\sqrt{x^2-y^2}}$$

$$u_y = \frac{-x\sqrt{x^2-y^2} - y\sqrt{y^2-x^2}}{y\sqrt{y^2-x^2}\sqrt{x^2-y^2}}$$

So $$\frac{u_x}{u_y} = \frac{-1/x}{1/y} = -\frac{y}{x}$$

Let w = f(x, y), where x and y are functions of t.

Then according to the chain rule $\frac{dw}{dt}$ is equal to

(a) $\frac{dw}{dx}\frac{dx}{dt}+\frac{dw}{dy}\frac{dt}{dt}$ (b) $\frac{\partial w}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial t}$

(c) $\frac{\partial w}{\partial x}\frac{dx}{dt}+\frac{\partial w}{\partial y}\frac{dy}{dt}$ (d) $\frac{dw}{dx}\frac{\partial x}{\partial t}+\frac{dw}{dy}\frac{\partial y}{\partial t}$

$w \longrightarrow (x, y) \longrightarrow t$ (alone)

$$\boxed{dw = \left(\frac{\partial w}{\partial x}\right)dx + \left(\frac{\partial w}{\partial y}\right)dy}$$

$\Rightarrow \frac{dw}{dt} = ? = $ (c)

Let $z = x \sin y - y \sin x$. The total differential $dz =$ ________.

(a) $(\sin y + y \cos x)\, dx + (x \cos y + \sin x)\, dy$

(b) $(\sin y - y \cos x)\, dx + (x \cos y + \sin x)\, dy$

(c) $(\sin y + y \cos x)\, dx + (x \cos y - \sin x)\, dy$

(d) $(\sin y - y \cos x)\, dx + (x \cos y - \sin x)\, dy$

$$\because z \longrightarrow (x, y)$$

$$dz = \left(\frac{\partial z}{\partial x}\right) dx + \left(\frac{\partial z}{\partial y}\right) dy$$

$$= (\sin y - y \cos x)\, dx + (x \cos y - \sin x)\, dy$$

If $u = f(r, s)$ where $r = x + y$, $s = x - y$ then then

$u_x + u_y =$

(a) $2u_r$ (b) $2u_s$

(c) $-2u_r$ (d) $-2u_s$

$u \to (r, s) \to (x, y)$

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\left(\frac{\partial r}{\partial x}\right) + \frac{\partial u}{\partial s}\left(\frac{\partial s}{\partial x}\right) = u_r(1) + u_s(1)$$

$$u_y = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\left(\frac{\partial r}{\partial y}\right) + \frac{\partial u}{\partial s}\left(\frac{\partial s}{\partial y}\right) = u_r(1) + u_s(-1)$$

$$u_x + u_y = 2u_r$$

If $z = f(x, y)$ where $x = e^{u} + e^{-v}$, $y = e^{-u} - e^{v}$ then $z_u - z_v =$

(a) $xz_x - yz_y$ (b) $xz_x + yz_y$

(c) $xz_y + yz_x$ (d) $xz_y - yz_x$

HW

Q

$z = f(x,y)$, $x = e^{u} + e^{-v}$, $y = e^{-u} - e^{v}$ then $z_u - z_v = ?$

Sol: $z \longrightarrow (x,y) \longrightarrow (u,v)$ then

$$z_u = \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\left(\frac{\partial x}{\partial u}\right) + \frac{\partial z}{\partial y}\left(\frac{\partial y}{\partial u}\right) = z_x(e^{u}) + z_y(-e^{-u})$$

$$z_v = \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\left(\frac{\partial x}{\partial v}\right) + \frac{\partial z}{\partial y}\left(\frac{\partial y}{\partial v}\right) = z_x(-e^{-v}) + z_y(-e^{v})$$

$$z_u - z_v = z_x(e^{u} + e^{-v}) + z_y(-e^{-u} + e^{v})$$

$$= xz_x - yz_y$$

If $x = uv$, $y = \frac{u+v}{u-v}$, then $\frac{\partial(u,v)}{\partial(x,y)}$ is

(a) $\frac{(u-v)^2}{4uv}$ ✓ (b) $\frac{(u+v)^2}{4uv}$

(c) $\frac{(u-v)}{4uv}$ (d) $\cancel{\frac{(u+v)}{4uv}}$ $\frac{4uv}{(u-v)^2}$

$\because JJ' = 1 \Rightarrow J' = \frac{1}{J}$

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{\partial(x,y)/\partial(u,v)} = \frac{(u-v)^2}{4uv}$$

ATQ; $(x,y) \longrightarrow (u,v)$

so $J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

$$= \begin{vmatrix} v & u \\ \frac{-2v}{(u-v)^2} & \frac{2}{u-v} \end{vmatrix} = \frac{4uv}{(u-v)^2}$$

Q. If $u = x^2 - 2y$, $v = x + y + z$, $w = x - 2y + 3z$ then derivative of (u, v, w) with respect to (x, y, z) will be?

(a) $10x + 4$ (b) $2x + 2$ (c) $5x + 2$ (d) None

$\because (u, v, w) \longrightarrow (x, y, z)$ so

$C_2 \to C_2 - C_1$ & $C_3 \to C_3 - C_1$

$$J = \frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} = \begin{vmatrix} 2x & -2 & 0 \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = \begin{vmatrix} 2x & -2-2x & -2x \\ 1 & 0 & 0 \\ 1 & -3 & 2 \end{vmatrix}$$

$$= -\left[-4 - 4x - 6x\right] = -(-4 - 10x) = 10x + 4$$

E.Th: (1) $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu$ where u is Homog func^n of degree n

(2) $x^2u_{xx}+2xy\,u_{xy}+y^2u_{yy}=n(n-1)u$

(*) Consider $u=u(x,y)$ s.t $\boxed{u(\lambda x,\lambda y)=\lambda^n u(x,y)}$, $n\in R$

Imp

then u is called Homog func^n of degree n

(*) if $u=u(x,y,z)$ then $\boxed{x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=nu}$

where $u(\lambda x,\lambda y,\lambda z)=\lambda^n u(x,y,z)$

If $\sin u = \dfrac{x+2y+3z}{x^8+y^8+z^8}$ then $\Longrightarrow \because V(\lambda x, \lambda y, \lambda z) = \lambda^{-7} V(x,y,z)$

$x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} =$

(a) $\dfrac{1}{7}\tan u$

(b) $-7 \tan u$

(c) $\dfrac{1}{7}\sec u$

(d) $-\dfrac{1}{7}\tan u$

Let $\sin u = \boxed{v = \dfrac{x+2y+3z}{x^8+y^8+z^8}}$

So Applying E.Th for v

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} + z\frac{\partial v}{\partial z} = nv$$

$$x\frac{\partial}{\partial x}(\sin u) + y\frac{\partial}{\partial y}(\sin u) + z\frac{\partial}{\partial z}(\sin u) = -7v$$

$$x(\cos u)\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} + z\cos u\frac{\partial u}{\partial z} = -7\sin u$$

If $u = x^n f_1\left(\frac{x}{y}\right) + y^{-n} f_2\left(\frac{x}{y}\right)$ then $\Rightarrow u = \phi + \psi$

$\phi = x^n f_1\left(\frac{x}{y}\right)$, $\psi = y^{-n} f_2\left(\frac{x}{y}\right)$

$$\underbrace{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}} + \underbrace{x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}} =$$

(a) 0 (b) $n(n+1)u$

(c) n^2u (d) $n(n-1)u$

Theory

if $\boxed{u = \phi + \psi}$ where ϕ & ψ are Homo. func^n of degree n_1 & n_2 But u is not Homog then

① $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n_1\phi + n_2\psi$

② $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n_1(n_1-1)\phi + n_2(n_2-1)\psi$

Here ϕ & ψ are H. func^n of n & $-n$ Resp.

& u is Non Homog. $n_1 = n$ & $n_2 = -n$

So LHS $= (n_1\phi + n_2\psi) + n_1(n_1-1)\phi + n_2(n_2-1)\psi$

$= n\phi - n\psi + n(n-1)\phi + (-n)(-n-1)\psi$

$= n^2(\phi + \psi) = n^2u$

Proof of Part (1) :-

Note :- $u = \phi + \psi$

$\because \phi$ & ψ are Homo. fun^n so

$$x\phi_x + y\phi_y = n_1\phi$$

$$x\psi_x + y\psi_y = n_2\psi$$

$$\frac{\partial u}{\partial x} = \phi_x + \psi_x \quad \& \quad \frac{\partial u}{\partial y} = \phi_y + \psi_y$$

Now

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x(\phi_x + \psi_x) + y(\phi_y + \psi_y)$$

$$= (x\phi_x + y\phi_y) + (x\psi_x + y\psi_y)$$

$$= n_1\phi + n_2\psi$$

Hence proved

(P8) if $z = x^2y^4\sin^{-1}\left(\frac{x}{y}\right) + \log x - \log y$ then $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = ?$

(a) z

(b) $3z$

(c) $6z$

(d) None

$$= x^2y^4\sin^{-1}\left(\frac{x}{y}\right) + \log\left(\frac{x}{y}\right)$$

$$\boxed{z = \phi + \psi}$$

Here z is Non Homog But

ϕ is Homog of $n_1 = 6$

ψ is Homog of $n_2 = 0$

So $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = n_1\phi + n_2\psi = 6x^2y^4\sin^{-1}\left(\frac{x}{y}\right)$

$\neq 6z$

Doubt By Student :-

R.M.V Th : $f(x)$ is Cont & Diff s.t $f(a)=f(b) \neq 0$ then $\exists\, c \in (a,b)$; $f'(c)=0$

Here $x=a$ & $x=b$ are not the Roots of $f(x)$ But $x=c$ is Root of $f'(x)$

i.e Root of $g(x)=f'(x)$

Sp Case : if $f(x)$ is Cont and Diff s.t $\underbrace{f(a)=f(b)=0}$ so $\exists\, c \in (a,b)$ s.t $f'(c)=0$

Here $x=a$ & $x=b$ are the Roots of $f(x)$ & $x=c$ is Root of $f'(x)$

i.e Root of $g(x)=f'(x)$

i.e $\boxed{\text{Between any Roots of } f(x)\ \exists \text{ atleast one Root of } f'(x)}$

eg $f(x)=x^3-6x^2+11x-6$

$f(1)=0$
$f(2)=0$ so $\exists$ one Root of $f'(x)$ b/w 1 & 2
$f(3)=0$ & " " " $f'(x)$ b/w 2 & 3

Verification:

$f(x) = x^3 - 6x^2 + 11x - 6 \Rightarrow f'(x) = 3x^2 - 12x + 11$

$\Downarrow$

$x = 1, 2, 3$

$$x = \frac{12 \pm \sqrt{144 - 132}}{6}$$

$$x = 2 \pm \frac{\sqrt{12}}{6} = 2 \pm \sqrt{\frac{2}{3}}$$

is one Root is > 2 But < 3

& other Root is > 1 But < 2

Doubt By student $\longrightarrow$

$$\phi(s) = a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots a_n$$

$s=-\sigma$ is Triple Root of $\phi(s)$ then $(s+\sigma)^3$ is the Factor of $\phi(s)$

$$\phi(s) = (s+\sigma)^3 \text{ (Remaining Factor)}$$

$$\boxed{\phi(s) = (s+\sigma)^3 f(s)} \Rightarrow \phi(s) = 0 \text{ at } s=-\sigma$$

$$\phi'(s) = 0 \text{ at } s=-\sigma$$

$$\phi''(s) = 0 \text{ at } s=-\sigma$$

$$\phi'''(s) \neq 0 \text{ at } s=-\sigma$$

Thank

THANK

Keep Hustling!