GATE

DATA SCIENCE + CS & IT

Engineering Mathematics

SUPER 1500

Lec : 02 Probability

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Topics *to be covered*

PROBABILITY

R.Exp – Not sure about the outcome

Dipuneet sir PW

S.Space – Total possible outcomes of R.Exp in set form

Event i.e – Any subset of S.Space is called Event.

eg $S_{Die} = \{1, 2, 3, 4, 5, 6\}$, Pair of Dice $S = \begin{Bmatrix} (11) & (12) & (13) & \cdots & (16) \\ (21) & (22) & & & (26) \\ \vdots & & & & \\ (61) & (62) & \cdots & & (66) \end{Bmatrix}$

$n(S) = 6$

$n(S) = \underset{D_1}{6} \times \underset{D_2}{6} = 36$ pair

eg $S_{coin} = \{H, T\} \Rightarrow n(S) = 2$,

eg Three Coins are tossed simultaneously then $\Rightarrow n(S) = \underset{C_1}{2} \times \underset{C_2}{2} \times \underset{C_3}{2} = 8$ Triplets

$S = \{(HHH), (HHT), (HTH), (HTT), (THH), (THT), (TTH), (TTT)\}$

$S = \{1, 2, 3, 4, 5, 6\}$

Event: eg $E_1 = \{\text{odd No.}\} = \{1, 3, 5\} \Rightarrow n(E) = 3$

$E_2 = \{\text{No} \geq 3 \text{ occurs}\} = \{3, 4, 5, 6\} \Rightarrow n(E) = 4$ & so on ...

eg $S = \begin{Bmatrix} (11)\ (12)\ (13) \ldots (16) \\ (21)\ (22) \ldots (66) \end{Bmatrix} \Rightarrow E_1 = \{\text{odd no. of } 1^{st} \text{ Die}\}$

$n(S) = 36$

$= \begin{Bmatrix} (12)(13) \ldots (16) \\ (31)(32) \ldots (36) \\ (51)(52) \ldots (56) \end{Bmatrix} \Rightarrow n(E_1) = 18$

Impossible Event: $\because \phi \subset S \Rightarrow \phi$ is also an Event & $P(\phi) = 0$

Sure Event: $\because S \subseteq S \Rightarrow S$,, ,, ,, Event & $P(S) = 1$

$$0 \leq P(E) \leq 1$$

(M-I) $P(E) = \frac{n(E)}{n(S)} = \frac{\text{No. of elements in Fav Event}}{\text{No. of elements in S. Space}}$ (By Counting the elements in E & S)

(M-II) $\text{Req Prob} = \frac{\text{Fav No of Cases}}{\text{Total No. of Cases}}$ (Using concept of P & C)

(M-III) Using some standard Results & Standard Def^n.

Note - A coin is tossed 4 times

4 coins are tossed simultaneously

$\Rightarrow$ Both have same sample space

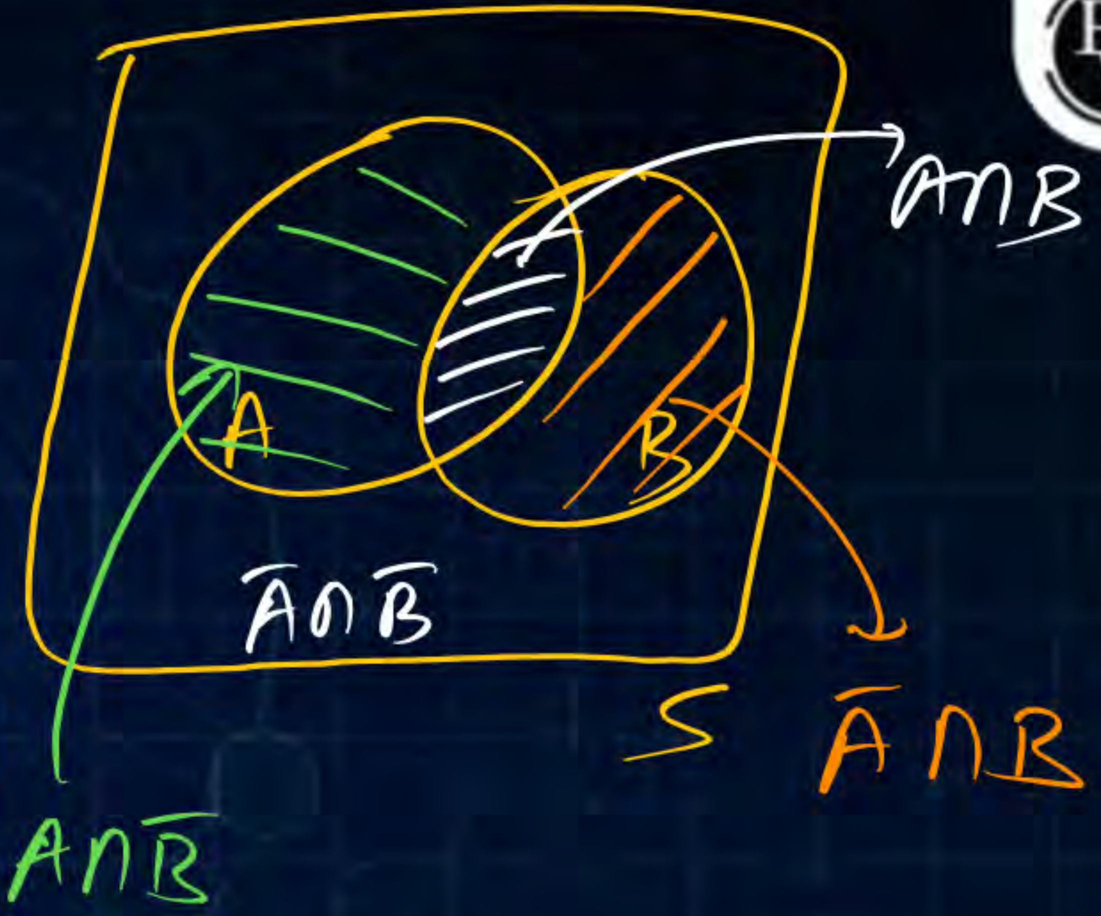
$$S = \{\underset{=1}{(HHHH)}\ \underset{=4}{(HHHT)}\ \underset{=6}{(HHTT)}\ \underset{=4}{(HTTT)}\ \underset{=1}{(TTTT)}\}$$

$$n(S) = \underline{2} \times \underline{2} \times \underline{2} \times \underline{2} = 16 \text{ Quadruplets}$$

Note:

1. Either A or B or Both $= ? = A \cup B$
2. Both A & B $= ? = A \cap B$
3. Neither A Nor B $= ? = \bar{A} \cap \bar{B}$

$$\boxed{(A \cap \bar{B}) \cup (\bar{A} \cap B) \cup (A \cap B) = A \cup B}$$



(*) Addition theorem of Prob. $\rightarrow$ $\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$

(*) Multi theorem of Prob $\rightarrow$ $P(A \cap B) = P(A/B) \cdot P(B)$

(*) $\boxed{P(\bar{A} \cap \bar{B}) = 1 - P(A \cup B)}$

Three fair cubical dice are thrown simultaneously. The probability that all three dice have the same number on the faces showing up is (up to third decimal place) ______________

$S = \{(111), (112) \ldots (116), (211) \ldots (666)\} \Rightarrow n(S) = 6^3 = 216$

$fav = \{(111)(222) \ldots (666)\} \Rightarrow n(A) = 6$

$\text{Req Prob} = \frac{6}{216} = \frac{1}{36} = 0.027$

A bag contains 4 black balls and 3 white balls. Balls are picked up one by one at random and placed in row. The chance that the balls are alternatively of different colours is ____.

[Ans: [illegible]]

M-I Total arrangements $= \dfrac{7!}{4!\,3!} = 35$

Fav Arrangement $= 1$

(B W B W B W B) ✓

OR

(B W B W B W W) ✗

So Req Prob $= \dfrac{f}{T} = \dfrac{1}{35}$

M-II Req Prob = ?

= B W B W B W B

$= \left(\dfrac{\cancel{4}}{7} \times \dfrac{\cancel{3}}{\cancel{6}} \times \dfrac{\cancel{3}}{5} \times \dfrac{\cancel{2}}{\cancel{4}} \times \dfrac{\cancel{2}}{\cancel{3}} \times \dfrac{1}{\cancel{2}} \times \dfrac{1}{1}\right)$

$= \dfrac{1}{35}$ ✓

Suppose that each of three men at a party throws his hat into the centre of the room. The hats are first mixed up and then each man randomly selects a hat. What is the probability that none of the three men selects his own hat?

(a) $\frac{1}{6}$

(b) $\frac{1}{3}$

(c) $\frac{2}{3}$

(d) 0

Dearrangements (for $n=3$) $= 3!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}\right) = 2$

Total arrangement $= 3! = 6$
(RNA)

Req Prob $= \frac{2}{6} = \frac{1}{3}$

Three different prizes have to be distributed among 4 different students. Each student could get 0 to 3 prizes. If all the prizes were distributed. Find, the probability that exactly 2 students did not receive a prize.

(a) $\frac{3}{16}$ (b) $\frac{5}{21}$ (c) $\frac{9}{16}$ (d) $\frac{9}{32}$

Total Distributions $= \underset{P_1}{4} \times \underset{P_2}{4} \times \underset{P_3}{4} = 64$ ways.

No. of ways of selecting those students who are getting Prizes $= {}^4C_2$

No of ways of Distributing 3 prizes to exactly 2 students

$= {}^4C_2 \times \left(\underset{P_1}{2} \times \underset{P_2}{2} \times \underset{P_3}{2} - 2\right) = {}^4C_2(2^3 - 2)$

Prob $= \frac{f}{T} = \frac{6 \times 6}{64} = \frac{9}{16}$

Candidates were asked to come to an interview with 3 pens each. Black, Blue, green and red were the permitted pen colours that the candidate could bring. The probability that a candidate comes with all 3 pens having the same colour is ______.

Total ways of Carrying Pen = (3 Pens of Diff Colours) or (All 3 pens of same Colour) or (2 Pens of Same Colour & 1 Pen is of another Colour)

$$= {}^4C_3 + {}^4C_1 + {}^4C_1 \times {}^3C_1 = 4+4+12 = 20 \text{ ways.}$$

fav ways $= {}^4C_1 = 4$ ways So Prob $= \frac{f}{T} = \frac{4}{20} = \frac{1}{5} = 0.2$

In a population of N families, 50% of the families have three children, 30% of the families have two children and the remaining families have one child. What is the probability that a randomly picked child belongs to a family with two children?

(a) 3/23
(b) 6/23
(c) 3/10
(d) 3/5

Sol

Total families = N

Total children in N families = ?

$= \left(\frac{50N}{100}\right)\times 3 + \left(\frac{30N}{100}\right)\times 2 + \left(\frac{20N}{100}\right)\times 1$

& fav. Cases $= \left(\frac{30N}{100}\right)\times 2$

So Req Prob $= \frac{f}{T}$

$= \frac{\left(\frac{30N}{100}\right)\times 2}{\text{Total}} = \frac{6}{23}$

M-II Let N = 100 families.

Total Children

$= 50\times 3 + 30\times 2 + 20\times 1$

$= 150 + 60 + 20 = 230$

Prob $= \frac{f}{T} = \frac{{}^{60}C_1}{{}^{230}C_1} = \frac{60}{230} = \frac{6}{23}$

Child

family with 3 Child

Family with 2 Child

familly with one child

Total Case = ?

A party of n persons takes their seats at random at a round table, then the probability that two specified person do not sit together is

(a) $\dfrac{2}{n-1}$ (b) $\dfrac{n-3}{n-1}$

(c) $\dfrac{n-2}{n-1}$ (d) $\dfrac{1}{n-1}$

Total Circular Arrangements $= (n-1)!$

Sol:

Two Particular Persons are Seated together $= (n-2)! \times 2!$

Req Prob $= 1 - P(\text{always together})$

$$= 1 - \frac{(n-2)! \times 2!}{(n-1)!}$$

$$= 1 - \frac{2}{n-1} = \frac{n-3}{n-1}$$

Mutually Exclusive Events → if $\boxed{A \cap B = \phi}$ then $\Longleftrightarrow$ (A & B are called M.E)

i.e. for ME Events A & B we have $P(A \cap B) = 0$

& $P(A \cup B) = P(A) + P(B) - 0$

Independent Events → if $\boxed{P(A \cap B) = P(A) \cdot P(B)}$ $\Longleftrightarrow$ then (A & B are Independent)

& $P(A \cup B) = P(A) + P(B) - P(A) \cdot P(B)$

ME

eg $S = \{1, 2, 3, 4, 5, 6\}$

$E_1 = \{1, 3, 5\}$, $E_2 = \{2, 4, 6\}$

$\because E_1 \cap E_2 = \phi \Rightarrow P(E_1 \cap E_2) = 0$

Ind → $S = \{H, T\}$, $D = \{1, 2, 3, 4, 5, 6\}$

$E_1 = \{H\}$, $E_2 = \{2, 4, 6\}$

$P(E_1 \cap E_2) = \frac{1}{2} \times \frac{3}{6} = \frac{1}{4}$

~~Two~~ dice are tossed. One dice is regular $=$ fair and the other is biased $=$ unfair with probabilities $P(1) = P(6) = 1/6$, $P(2) = P(4) = 0$ and $P(3) = P(5) = 1/3$. The probability of obtaining a sum of 4 is

(a) 1/9 (b) 1/12 (c) 1/18 (d) 1/24

$S = \{2, 3, 4, 5, 6, \ldots 11, 12\}$

Req Prob $= P[\text{Sum} = 4]$

$= P\{\underset{F\ Un}{(2\,2)} \text{ or } \underset{F\ Un}{(1\,3)} \text{ or } \underset{F\ Un}{(3\,1)}\}$

$= \left(\frac{1}{6} \times 0\right) + \left(\frac{1}{6} \times \frac{1}{3}\right) + \left(\frac{1}{6} \times \frac{1}{6}\right)$

$= 0 + \frac{1}{18} + \frac{1}{36} = \frac{3}{36} = \frac{1}{12}$

A loaded dice has following probability distribution of occurrences

unfair Die

Dice Value	1	2	3	4	5	6
Probability	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$

If three identical dice as the above are thrown, the probability of occurrence of values 1, 5 and 6 on the three dice is

(a) Same as that of occurrence of 3, 4, 5

(b) Same as that of occurrence of 1, 2, 5

(c) 1/128

(d) 5/8 [redacted]

All 3 Dice are Ind

$P(1 \& 5 \& 6) = P(A \cap B \cap C) = \frac{1}{4} \times \frac{1}{8} \times \frac{1}{4} = \frac{1}{128}$

(a) $P(3 \cap 4 \cap 5) = \frac{1}{8} \times \frac{1}{8} \times \frac{1}{8} \neq \frac{1}{128}$

(b) $P(1 \cap 2 \cap 5) = \frac{1}{4} \times \frac{1}{8} \times \frac{1}{8} \neq \frac{1}{128}$

A fair dice is rolled twice. The probability that an odd number will follow an even number is

(a) $\frac{1}{2}$

(b) $\frac{1}{6}$

(c) $\frac{1}{3}$

(d) $\frac{1}{4}$ ✓

$P(E \cap O)$

$\frac{3}{6} \times \frac{3}{6} = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

If $P(A) = 0.8$, $P(B) = 0.9$, $P(AB) = p$, then which one of the following is correct?

(a) $0.72 \le p \le 0.8$

(b) $0.7 \le p \le 0.8$

(c) $0.72 < p < 0.8$

(d) $0.7 < p < 0.8$

w.k. that $P(A \cup B) \le 1$

$P(A) + P(B) - P(A \cap B) \le 1$

$0.8 + 0.9 - p \le 1$

$0.7 \le p$

$\because P(A) < P(B)$ (given)

$\Rightarrow P(A \cap B) \le \min\{P(A), P(B)\}$

or $p \le P(A)$

ie $p \le 0.8$

Hence $0.7 \le p \le 0.8$

A man can kill a bird once in a three shots. On this assumption he fires three shots. What is the chance that a bird is killed?

(M-I) $P(K) = \frac{1}{3}$, $P(\bar{K}) = \frac{2}{3}$, Each Shot is Ind.

Req Prob = (Bird is killed in 1^{st} shot) $= \frac{1}{3}$

or

(Not killed in 1^{st} shot But killed in 2^{nd} shot) $= \frac{2}{3} \times \frac{1}{3}$

or

(Not killed in 1^{st} two shots & killed in 3^{rd} shot) $= \frac{2}{3} \times \frac{2}{3} \times \frac{1}{3}$

So Req Ans $= \frac{1}{3} + \frac{2}{9} + \frac{4}{27}$

$= \frac{9}{27} + \frac{6}{27} + \frac{4}{27}$

$= \frac{19}{27}$

(M-II) P(Bird is killed) = ? = P(B is killed in at least one shot killed)

$= 1 - P(\text{No shot Hit the Bird}) = 1 - \left(\frac{2}{3} \times \frac{2}{3} \times \frac{2}{3}\right) = \frac{19}{27}$

M-III Req Prob $= {}^3C_1\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)^2 + {}^3C_2\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^1 + {}^3C_3\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^0$

$= ?$

Let $X \in (0, 1)$ and $Y \in (0, 1)$ be two independent binary random variables. If $P(X = 0) = p$ and $P(Y = 0) = q$, then $P(X + Y \geq 1)$ is equal to

(a) $pq + (1-p)(1-q)$

(b) pq

(c) $p(1-q)$

(d) $1-pq$

$$P(0,0) + \underbrace{P(0,1) + P(1,0) + P(1,1)}_{(X+Y \geq 1)} = 1$$

$X = \{0, 1\}, \; Y = \{0, 1\}$

$S = \{(X, Y)\} = \{(0\,0), (0\,1), (1\,0), (1\,1)\}$

$P(X+Y \geq 1) = P[(X,Y) = (0\,1), (1\,0), (1\,1)]$

$= 1 - P[(X,Y) = (0,0)]$

$= 1 - P(X=0) \cdot P(Y=0)$

$= 1 - pq$

Conditional Prob

$P(A/B) = \frac{P(A\cap B)}{P(B)}$ = it is the prob of A when B has already occurred

i.e. " " " A while B is the condⁿ.

or $\boxed{P(A\cap B) = P(A/B)\cdot P(B)}$

Note: if A & B are Ind. Events then condⁿ has No meaning

$\because P(A/B) = ? = \frac{P(A\cap B)}{P(B)} = \frac{P(A)\cdot P(B)}{P(B)} = P(A)$

A box contains 4 while balls and 3 red balls. In succession, two balls are randomly selected and removed from the box. Given that the first removed ball is white, the probability that the second removed ball is red is

(a) 1/3 (b) 3/7

(c) 1/2 (d) 4/7

M-I Req Prob $= P(W \& R)$ given

$= 1 \times \frac{3}{6} = \frac{1}{2}$

M-II original S.Sp $= \{(WW), (WR), (RW), (RR)\}$

R.S.Sp $= \{(WW), (WR)\} = 2$

fav $= \{(WR)\} = 1$

Cond^n Prob $= \frac{fav}{R.SSp} = \frac{1}{2}$

Consider two independent random variables X and Y with identical distributions. The variables X and Y take values 0, 1 and 2 with probability 1/2, 1/4 and 1/4 respectively. What is the conditional probability $P(X+Y=2 / X-y=0)$?

(a) 0 (b) 1/16

(c) 1/6 (d) 1

$P(X=0)=P(Y=0)=\frac{1}{2}$

$P(X=1)=P(Y=1)=\frac{1}{4}$

$P(X=2)=P(Y=2)=\frac{1}{4}$

$X=\{0,1,2\}=Y$

$P(X+Y=2 / X-Y=0)=P\left(\frac{X+Y=2}{X=Y}\right)$

$=\frac{P(11)}{P(00)+P(11)+P(22)}$

$=\frac{\frac{1}{4}\times\frac{1}{4}}{\frac{1}{2}\times\frac{1}{2}+\frac{1}{4}\times\frac{1}{4}+\frac{1}{4}\times\frac{1}{4}}=\frac{1}{6}$

BAYE'S Theorem (Inverse Prob Th)

- This Th is applicable to Solve Tough Questions of Conditional Prob.
  ??
- Here we will try to cross check something according to Condⁿ.
- ME & Exhaustive Events: $[P(E_1)+P(E_2)+P(E_3)=1]$

  $\Leftrightarrow$ [E_1, E_2, E_3 are ME & Exhaustive.]

  $E_i \cap E_j = \phi$ $E_1 \cup E_2 \cup E_3 = S$

There are two identical locks with two identical keys and the key are among six ~~different~~ ones which a person carries in his pocket. In hurry he drops one key somewhere. Then the probability that the locks can still opened by drawing one key at random is equal to

(a) $\frac{1}{3}$ (b) $\frac{5}{6}$

(c) $\frac{1}{12}$ (d) $\frac{1}{30}$

[redacted]

Law of Total Prob

key (2 Id & 4 Diff)

$\frac{2}{6}$ — Identical Lost (E_1): (1 Id & 4 D)

$\frac{4}{6}$ — Id. Not Lost (E_2): (2 I & 3 D)

Lock Open A: $\frac{1}{5}$ $\frac{2}{5}$

$$P(A) = \frac{2}{6} \times \frac{1}{5} + \frac{4}{6} \times \frac{2}{5}$$

$$= \frac{1}{3}$$

A letter is known to have come either from TATANAGAR or CALCUTTA. On the envelope just two consecutive letters TA are visible. What is the probability that the letter has come from Calcutta ____

A = { Two vissible letters are TA }

TATANAGAR

i.e 9 letters will form 8 pair

$P(A/E_1) = \frac{2}{8}$

Letter

$\frac{1}{2}$ Coming from TATA (E_1)

$\frac{1}{2}$ Coming from CALCUTTA (E_2)

A: $\frac{2}{8}$ $\qquad$ $\frac{1}{7}$

$P(A) = \frac{1}{2} \times \frac{2}{8} + \frac{1}{2} \times \frac{1}{7} = ?$

$P(E_2/A) = \frac{\frac{1}{2} \times \frac{1}{7}}{P(A)} =$

CALCUTTA

i.e 8 letters will form 7 pair

$P(A/E_2) = \frac{1}{7}$

Thank

you

Keep Hustling!