GATE

CRASH COURSE

ALL BRANCHES

Engineering Mathematics

Calculus (Part 01)
(Lec 04)

By - Dr. Puneet Sharma Sir

Topics *to be covered*

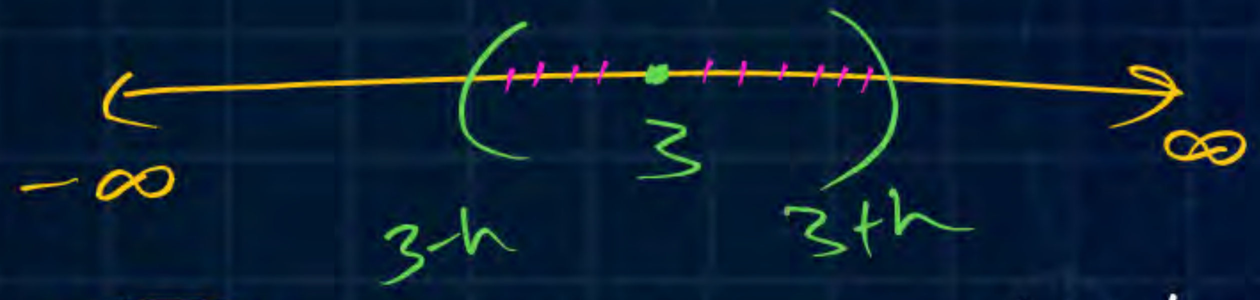
CALCULUS

1. Limits
2. Continuity
3. Differentiability
4. Taylor & Maclaurin series

LIMITS

(*) $x \to 3 \Rightarrow$ x lies in the Neighbourhood of 3 i.e $x \in (3-h, 3+h)$

or $3-h < x < 3+h$



where h = very small +ve No.

(*) At $x=3$, $f(3)$ = functional Value / Exact Value

when $x \to 3$, $f(3)$ = limiting value / Approx Value

eg $f(x) = x^2 - 5x + 1 \Rightarrow f(3) = 3^2 - 5(3) + 1 = -5$ (exact Value)

eg $f(x) = \dfrac{x^2-9}{x-3} \Rightarrow f(3) = \dfrac{9-9}{3-3} = \dfrac{0}{0} = ? =$ DNE / so $\lim\limits_{x\to 3} f(x) = \lim\limits_{x\to 3}\left(\dfrac{x^2-9}{x-3}\right) = 6$

$$f(x)=\frac{x^2-9}{x-3}=\boxed{\frac{(x-3)(x+3)}{(x-3)}}$$

$$\text{At } x=3,\ f(3)=\frac{(3-3)(3+3)}{(3-3)}=\frac{0\times 6}{0}=\frac{0}{0}=\ ?=\text{DNE}$$

while in the Nbd of 3 i.e

when $x \to 3$

let $x=2.999 \Rightarrow LHL=f(2.999)=\dfrac{(\cancel{-0.001})(5.999)}{(\cancel{-0.001})}\simeq 6$

let $x=3.001 \Rightarrow RHL=f(3.001)=\dfrac{(\cancel{0.001})(6.001)}{(\cancel{0.001})}\simeq 6$

$\because$ LHL = RHL, so limit exist & it's value $=6$

M-II

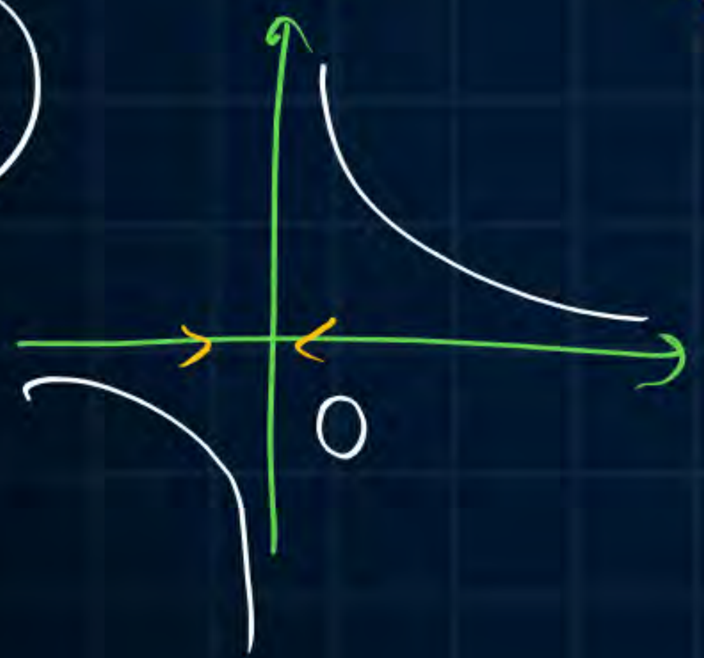
$$\lim_{x\to 3}\left(\frac{x^2-9}{x-3}\right)=\frac{0}{0}=\lim_{x\to 3}\left(\frac{2x-0}{1-0}\right)=\lim_{x\to 3}(2x)=2\times 3=6$$

(*) limit exist $\Longrightarrow$ It is possible to find unique, Constant Value

or if $\boxed{LHL = RHL}$ then we say that limit exist.

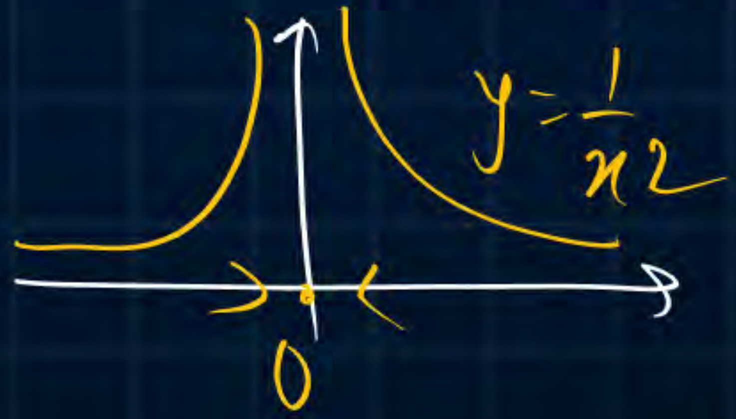
(*) $\dfrac{\text{something}}{0} = \begin{cases} +\infty \\ -\infty \end{cases}$ i.e. $\boxed{\dfrac{\text{something}}{0} = N.D}$ & $\boxed{\dfrac{\text{something}}{\infty} \cong 0}$ Assumption

$y = \dfrac{1}{x}$



(*) $\boxed{\infty + \infty = \infty}$, $\boxed{\infty \times \infty = \infty}$, $\boxed{\infty^{\infty} = \infty}$

Infinity is Not unique



for eg $\lim_{x \to 0} \left(\dfrac{1}{x}\right) = ? = \left.\begin{matrix} +\infty \\ \text{or} \\ -\infty \end{matrix}\right\}$ Not unique $= DNE$

for eg $\lim_{x \to 0} \left(\dfrac{1}{x^2}\right) = ? = \infty \rightarrow$ Not Unique $= DNE$

(*) $\boxed{\infty + \infty = \infty}$, $\boxed{\infty^{\infty} = \infty}$, $\boxed{1^{\text{something}} = 1}$, $\boxed{a^0 = 1}$

But $\infty - \infty = \text{IND}$, But $\infty^0 = \text{IND}$, $1^{\infty} = \text{IND}$, But $0^0 = \text{IND}$

$\boxed{0 \times a = 0}$, $\boxed{\dfrac{\text{something}}{0} = \text{ND}}$, $\boxed{\dfrac{\text{something}}{\infty} = 0}$

But $0 \times \infty = \text{IND}$, But $\dfrac{0}{0} = \text{IND}$, But $\dfrac{\infty}{\infty} = \text{IND}$

INDETERMINATE form → this form has Multiple answers.

for eg $\dfrac{0}{0} = 1, 2, 3, -4, \dfrac{5}{2}, \sqrt{7} \ldots\ldots$ i.e Multiple ans exist.

eg $\dfrac{\infty}{\infty} = ? = \dfrac{1/0}{1/0} = \dfrac{0}{0}$ = already IND form.

L-Hospital's Rule → applicable only for $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form.

Methods of evaluating limits →

1. By Direct Substitution (Best Method)
2. By Factorisation
3. By Rationalisation (useful in $\infty - \infty$ form)
4. By IND Form Concept ($\frac{0}{0}, \frac{\infty}{\infty}, 0 \times \infty, \infty - \infty, 0^0, \infty^0, 1^\infty$)
5. By Standard Results

$\frac{0}{0} = 1, 2, -3, \frac{1}{4}, 5, -7 \ldots\ldots$

$0^0 =$ IND form $\Rightarrow e^1, e^{-2}, e^3, e^{\frac{1}{4}}, e^{57}, e^0, e^5 \ldots\ldots$

$= 1$

Q: $\lim_{x\to 2}\left(\frac{x^2-4}{x+3}\right) = ?$

$= \frac{2^2-4}{2+3} = \frac{0}{5} = 0$ Ans

Q: $\lim_{x\to 0}\left(\frac{1}{x}\right) = ? = +\infty, -\infty$

i.e. DNE

Q: $\lim_{x\to 0}\left(\frac{\sqrt{1+x}+\sqrt{1-x}}{1+x}\right) = ?$

$= \frac{\sqrt{1+0}+\sqrt{1-0}}{1+0} = \frac{1+1}{1} = \frac{2}{1} = 2$

Q: $\lim_{x\to 0}\left(\frac{1}{x^2}\right) = ? = \infty$ i.e. not unique

$=$ DNE

$$\lim_{n\to\infty}\left(\frac{1}{n^3+1}+\frac{4}{n^3+1}+\frac{9}{n^3+1}+\ldots\ldots+\frac{n^2}{(n^3+1)}\right)$$ is equal to

(a) 1 (b) $\frac{2}{3}$ (c) $\frac{1}{3}$ ✓ (d) 0

$$\lim_{n\to\infty}\left[\frac{1^2+2^2+3^2+\ldots\ldots+n^2}{n^3+1}\right]=\lim_{n\to\infty}\left[\frac{\frac{n(n+1)(2n+1)}{6}}{n^3+1}\right]$$

$$=\lim_{n\to\infty}\frac{1}{6}\left[\frac{n^3\left\{\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\right\}}{n^3\left(1+\frac{1}{n^3}\right)}\right]=\frac{1}{6}\frac{\left(1+\frac{1}{\infty}\right)\left(2+\frac{1}{\infty}\right)}{\left(1+\frac{1}{\infty}\right)}=\frac{(1+0)(2+0)}{6(1+0)}$$

$$=\frac{2}{6}=\frac{1}{3}$$

Factorisation →

Q. $\lim_{x\to 2}\left\{\dfrac{x^3-6x^2+11x-6}{x^2-6x+8}\right\} = ? = \dfrac{0}{0}$ form

M-I

$$= \lim_{x\to 2}\cdot\left[\frac{(x-1)\cancel{(x-2)}(x-3)}{\cancel{(x-2)}(x-4)}\right] = \lim_{x\to 2}\frac{(x-1)(x-3)}{(x-4)} = \frac{1\times(-1)}{(-2)} = \frac{1}{2}$$

M-II

Using L-Hosp Rule, $\lim_{x\to 2}\left(\dfrac{3x^2-12x+11}{2x-6}\right) = \dfrac{12-24+11}{4-6} = \dfrac{1}{2}$

Q. $\lim_{x\to 1}\left(\dfrac{x^{\frac{1}{4}}-1}{x^{\frac{1}{3}}-1}\right)=\,?$ $=\dfrac{1^{\frac{1}{4}}-1}{1^{\frac{1}{3}}-1}=\dfrac{1-1}{1-1}=\dfrac{0}{0}$

Put $x=y^{12}$

when $x\to 1$

$y\to 1$

So $\lim_{x\to 1}\left(\dfrac{x^{\frac{1}{4}}-1}{x^{\frac{1}{3}}-1}\right)=\lim_{y\to 1}\left(\dfrac{y^{3}-1}{y^{4}-1}\right)=\dfrac{0}{0}$

$$=\lim_{y\to 1}\frac{(y-1)(y^2+1+y)}{(y^2-1)(y^2+1)}$$

$$=\lim_{y\to 1}\left(\frac{y^2+1+y}{(y+1)(y^2+1)}\right)=\frac{3}{2\times 2}=\frac{3}{4}$$

$$\lim_{n\to\infty}\left(\sqrt{n^2+n}-\sqrt{n^2+1}\right) \text{ is } \underline{(\infty-\infty) \text{ form.}}$$

$$\lim_{n\to\infty}\left(\underbrace{\sqrt{n^2+n}}_{=a}-\underbrace{\sqrt{n^2+1}}_{=b}\right)\left[\frac{\underbrace{\sqrt{n^2+n}}_{=a}+\underbrace{\sqrt{n^2+1}}_{=b}}{\sqrt{n^2+n}+\sqrt{n^2+1}}\right]$$

$$=\lim_{n\to\infty}\left(\frac{(n^2+n)-(n^2+1)}{\sqrt{n^2+n}+\sqrt{n^2+1}}\right)=\lim_{n\to\infty}\left[\frac{n-1}{n\sqrt{1+\frac{n}{n^2}}+n\sqrt{1+\frac{1}{n^2}}}\right]$$

$$=\lim_{n\to\infty}\frac{1-\frac{1}{n}}{\sqrt{1+\frac{1}{n}}+\sqrt{1+\frac{1}{n^2}}}=\frac{1-0}{\sqrt{1+0}+\sqrt{1+0}}=\frac{1}{1+1}=\frac{1}{2}$$

L-Hospital's Rule $\rightarrow$

Q. $\lim_{x\to 0}\left[\frac{(1-x)^n-1}{x}\right] = ? = \frac{0}{0}$ form

(a) n (b) 0

(c) ∞ ~~(d) None~~

$$= \lim_{x\to 0}\left[\frac{n(1-x)^{n-1}(0-1)-0}{1}\right] = \lim_{x\to 0}\left[-n(1-x)^{n-1}\right] = -n$$

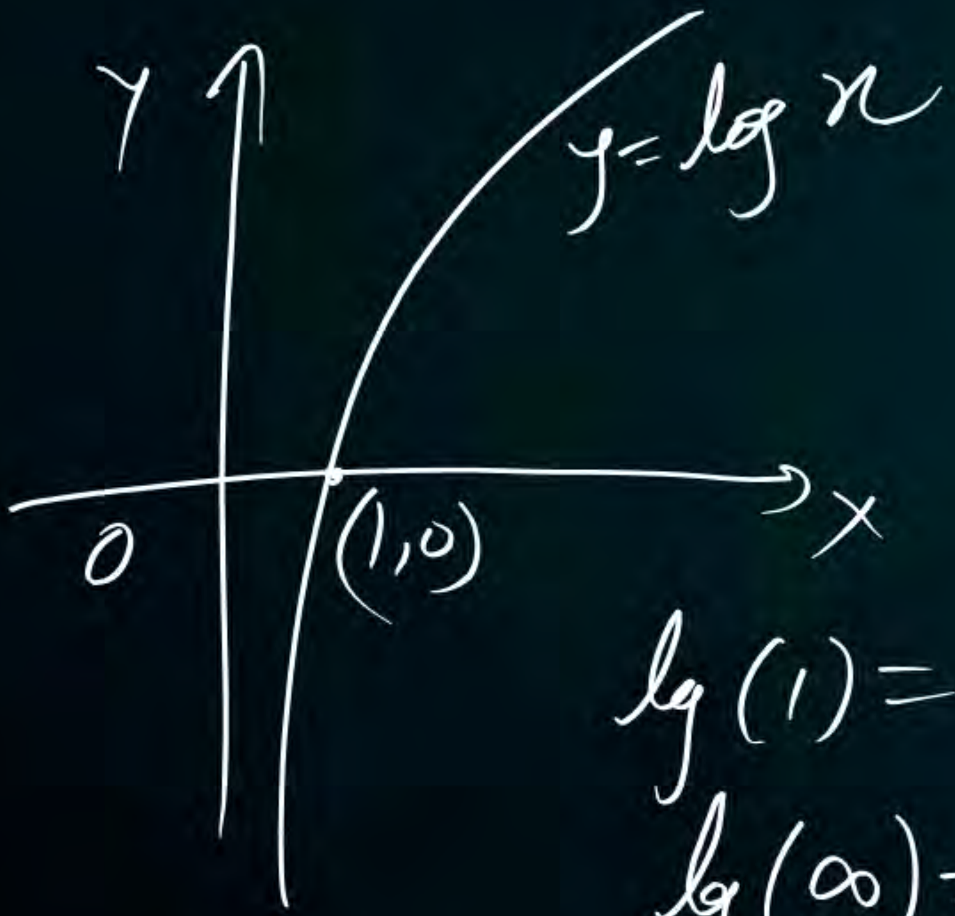
Q. $\lim_{x\to 0}\left\{\frac{e^{\sin x}-1}{x}\right\} = ?$ $= \frac{0}{0}$ form

$$= \lim_{x\to 0}\left[\frac{e^{\sin x}(\cos x) - 0}{1}\right] = e^{0}(\cos 0) = 1\times 1 = 1$$

Q. $\lim_{x\to 0} (\tan x \cdot \log x) = ?$

$0 \times (-\infty)$

$$= \lim_{x\to 0}\left[\frac{\log x}{\cot x}\right] = \left(\frac{\infty}{\infty}\right) = \lim_{x\to 0}\left[\frac{\left(\frac{1}{x}\right)}{-\operatorname{cosec}^2 x}\right]$$



$\log(1) = 0$

$\log(\infty) = +\infty$

$\log(0) = -\infty$

$$= \lim_{x\to 0}\left(\frac{-\sin^2 x}{x}\right) = \frac{0}{0}$$

$$= -\lim_{x\to 0}\left(\frac{\sin x}{x}\right) \times \lim_{x\to 0}(\sin x)$$

$$= -1 \times 0 = 0$$

M-II $\lim_{x\to 0}\left(\frac{-2\sin x\cos x}{1}\right) = -2 \times 0 \times 1 = 0$

Q. $\lim_{x\to 0} (\sin x)^{\tan x} = ? = 0^0$ form. $(\because 5^0 = 1, 7^0 = 1, 8^0 = 1$ But $0^0 \neq 1)$

Let $K = \lim_{x\to 0} (\sin x)^{\tan x}$

$$\log K = \lim_{x\to 0} \log(\sin x)^{\tan x}$$

$$= \lim_{x\to 0} \tan x . (\log \sin x)$$

$$(0 \times \infty)$$

$$= \lim_{x\to 0} \left[\frac{\log \sin x}{\cot x}\right] = \frac{\infty}{\infty} \text{ form}$$

$$= \lim_{x\to 0} \left[\frac{\frac{1}{\sin x}(\cos x)}{-\operatorname{cosec}^2 x}\right]$$

$$= \lim_{x\to 0} -\left(\frac{\cos x . \sin^2 x}{\sin x}\right)$$

$$= \lim_{x\to 0} -(\cos x \times \sin x) = -1 \times 0 = 0$$

$$\log_e K = 0 \Rightarrow \boxed{K = e^0 = 1}$$

Q. $\lim_{x \to 0} (\cos 2x)^{\frac{1}{x^2}} = ? \simeq 1^{\infty}$ form $\{ 1^{20} = 1, 1^{30} = 1, 1^{22} = 1$, But $1^{\infty} \neq 1$?

Let $k = \lim_{x \to 0} (\cos 2x)^{\frac{1}{x^2}}$

$$\log k = \lim_{x \to 0} \log (\cos 2x)^{\frac{1}{x^2}}$$

$$= \lim_{x \to 0} \frac{1}{x^2} \log (\cos 2x)$$

$$= \lim_{x \to 0} \left[\frac{\log (\cos 2x)}{x^2} \right] \simeq \frac{0}{0} \text{ form}$$

$$\log k = \lim_{x \to 0} \left[\frac{\frac{1}{\cos 2x}(-\sin 2x)(2)}{2x} \right]$$

$$= -2 \cdot \lim_{x \to 0} \left(\frac{\sin 2x}{2x} \right) \times \lim_{x \to 0} \left(\frac{1}{\cos 2x} \right)$$

$$\log_e k = -2(1)\left(\frac{1}{1}\right) = -2$$

$$k = e^{-2} = \frac{1}{e^2}$$

Standard Results $\rightarrow$ Q. $\lim\limits_{\alpha \to 0}\left(\frac{x^{\alpha}-1}{\alpha}\right) = ? = \log_e x$ PYQ

① $\lim\limits_{x \to 0}\left(\frac{\sin x}{x}\right) = 1$, ③ $\lim\limits_{x \to \infty}\left(\frac{\sin x}{x}\right) = \frac{\sin \infty}{\infty} = \frac{\text{No b/w } -1 \;\&\; 1}{\infty} = 0$

② $\lim\limits_{x \to 0}\left(\frac{\cos x}{x}\right) = \frac{\cos 0}{0} = \frac{1}{0} = \text{DNE}$ ④ $\lim\limits_{x \to \infty}\left(\frac{\cos x}{x}\right) = \frac{\cos \infty}{\infty} = \frac{\text{No b/w } -1 \;\&\; 1}{\infty} = 0$

⑤ $\lim\limits_{x \to 0}\left(\frac{a^{x}-1}{x}\right) = \log_e a$ eg $\lim\limits_{x \to 0}\left(\frac{e^{x}-1}{x}\right) = 1$

⑥ $\lim\limits_{x \to 0}\left(\frac{1-\cos x}{x^2}\right) = \frac{1}{2}$, ⑦ $\lim\limits_{x \to 0}\left[\frac{\log(1+x)}{x}\right] = 1$

⑧ $\lim\limits_{x \to 0}(1+ax)^{\frac{1}{x}} = e^{a}$

eg $\lim\limits_{x \to 0}(1+x)^{\frac{1}{x}} = e$

⑨ $\lim\limits_{x \to \infty}\left(1+\frac{a}{x}\right)^{x} = e^{a}$

eg $\lim\limits_{x \to \infty}\left(1+\frac{1}{x}\right)^{x} = e$

All the above Results are based on $\frac{0}{0}$ form except 3rd, 4th, 8th & 9th.

Q→ $\lim_{x\to 0}\left(\frac{\log(1+x^3)}{\sin^3 x}\right) = ?$: $\simeq \frac{0}{0}$ form, use L-Hospital's Rule → lengthy

M-II $\lim_{x\to 0}\left(\frac{\log(1+x^3)}{x^3}\right) \times \lim_{x\to 0}\left(\frac{x^3}{\sin^3 x}\right)$

$= \lim_{x^3\to 0}\left(\frac{\log(1+x^3)}{x^3}\right) \times \lim_{x\to 0}\left(\frac{x}{\sin x}\right)^3$

$= 1 \times 1^3 = 1$

Let $x = 0.1$ i.e. $x \to 0$

$\Rightarrow x^3 = 0.001$ i.e. $x^3 \to 0$

Q: $\lim_{x \to 0} \left[\frac{\sin(\pi \cos^2 x)}{x^2}\right] = ?$ $= \pi$

HWQ

M-I → using L-Hosp Rule → HW

M-II → using Standard Results → HW

Continuity → $f(x)$ is said to be cont if $\boxed{\lim_{x \to a} f(x) = f(a)}$

limiting value = functional value

(exist in the Nbd of a) (Exist at a)

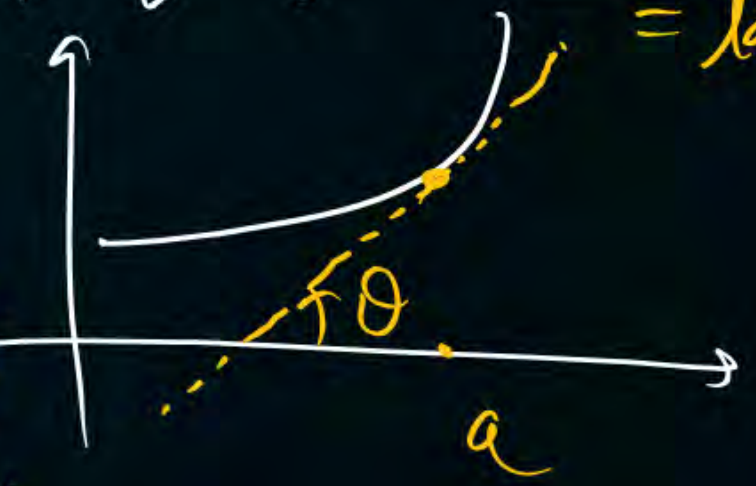
(for cont. func^n $f(x)$ graph will not break)

OR $\boxed{(LHL = RHL) = f(a)}$

DIFFERENTIABILITY → $f(x)$ is said to be diff at $x=a$ if $\lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}\right)$ exist.

& value of this limit is called Derivative of $f(x)$ at $x=a$

i.e $\left(\frac{dy}{dx}\right)_{x=a} = \boxed{f'(a) = \lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}\right)}$ = slope of tangent at $x=a$ $= \tan\theta$



⊛ LHD = LHL of $f'(a)$ & when $\boxed{LHD = RHD}$

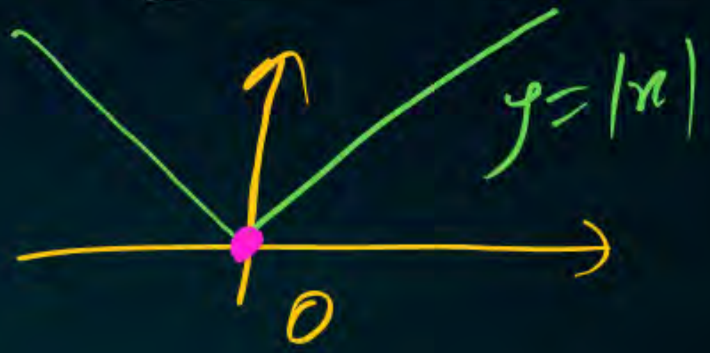
RHD = RHL of $f'(a)$ then $f(x)$ is called Diff at $x=a$

⊛ N. Cond^n for differentiability : $f(x)$ must be continuous

Sufficient Cond^n " " " : $\lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}\right)$ exist.

(*) At Sharp point, $f(x)$ is said to be Non Diff

eg $f(x) = |x| = \begin{cases} -x, & x < 0 \\ +x, & x \geq 0 \end{cases}$ =



i.e At $x=0$, $f(x) = |x|$ is Continuous but not Differentiable

(M-II) $f(x) = \begin{cases} -x, & x<0 \\ x, & x \geq 0 \end{cases}$

LHL = 0
RHL = 0
f(0) = 0

i.e Cont at $x=0$

$f'(x) = \begin{cases} -1, & x<0 \\ 1, & x>0 \end{cases}$

LHD = −1
RHD = 1

$\because$ LHD ≠ RHD
So Not Diff.

The values of a and b for which the function

$$f(x) = \begin{cases} 2x+1, \text{ if } x \le 1 \\ ax^2+b \text{ if } 1 < x < 3 \text{ is continuous every} \\ 5x+2a \text{ if } x \ge 3 \end{cases}$$

where

(a) a = 2, b = 1 (b) a = 1, b = 2

(c) a = 3, b = 2 (d) a = 2, b = 3

$$f(x) = \begin{cases} 2x+1, & x \le 1 \\ ax^2+b, & 1<x<3 \\ 5x+2a, & x \ge 3 \end{cases}$$

To check Cont at x=1

$LHL = RHL = f(1)$

$2(1)+1 = a(1)^2+b = 3$

i.e. $\boxed{a+b=3}$

Again Cont at x=3;

$LHL = RHL = f(3)$

$a(3)^2+b = 5(3)+2(a) = 15+2a$

$\boxed{7a+b=15}$

HW8

Which one of the following function is continuous at x = 3?

(a) $f(x)=\begin{cases}2, \text{ if } x=3\\ x-1, \text{ if } x>3\\ \frac{x+3}{3}, \text{ if } x<3\end{cases}$

(b) $f(x)=\begin{cases}4, \text{ if } =3\\ 8-x, \text{ if } x\neq 3\end{cases}$

(c) $f(x)=\begin{cases}x+3, \text{ if } x\leq 3\\ x-4, \text{ if } x>3\end{cases}$

(d) $f(x)=\frac{1}{x^3-27}, \text{ if } x\neq 3$

PYQ

The function $y = |2 - 3x|$

(a) is continuous $\forall x \in R$ and differentiable $\forall x \in R$

(b) is continuous $\forall x \in R$ and differentiable $\forall x \in R$ except at $x = 3/2$

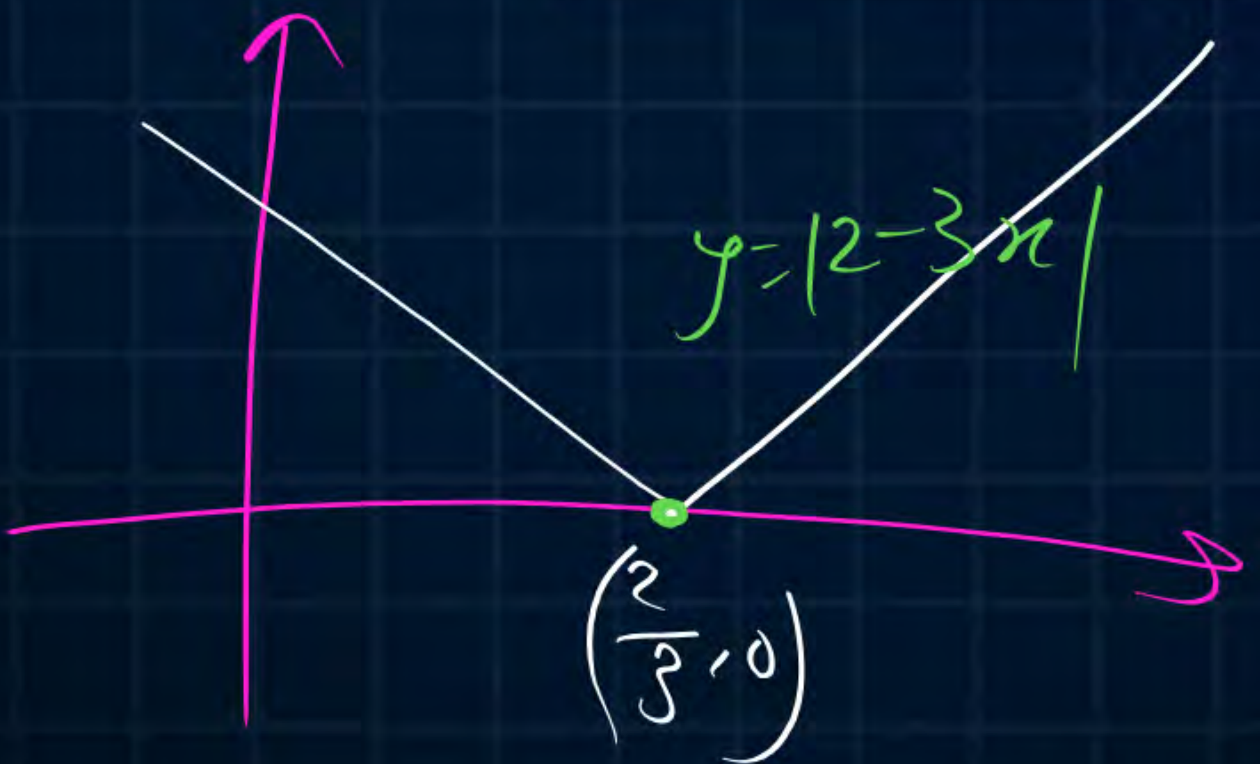
(c) is continuous $\forall x \in R$ and differentiable $\forall x \in R$ except at $x = 2/3$

(d) is continuous $\forall x \in R$ except $x = 3$ and differentiable $\forall x \in R$

$|2-3x| = 0$

$2-3x = 0$

$x = \frac{2}{3}$



A real function

$$f(x)=\begin{cases}\alpha x^2+\beta x, & \text{for } x<0\\ \alpha x^3+\beta x^2+5\sin x, & x\geq 0\end{cases}.$$

If $f(x)$ is twice differentiable then

(a) $\alpha = 1, \beta = 0$ (b) $\alpha = 1, \beta = 5$

(c) $\alpha = 5, \beta = -10$ (d) $\alpha = 5, \beta = 5$ ✓

$$f'(x)=\begin{cases}2\alpha x+\beta, & x<0\\ 3\alpha x^2+2\beta x+5\cos x, & x>0\end{cases}$$

LHD $= \beta$

RHD $= 5$

$\Rightarrow \beta = 5$

$$f''(x)=\begin{cases}2\alpha, & x<0\\ 6\alpha x+2\beta-5\sin x, & x>0\end{cases}$$

LHD $= 2\alpha$

RHD $= 2\beta$

$\Rightarrow \alpha = \beta = 5$

Neighbourhood of Real Number $a \rightarrow = (a-h, a+h)$

(*) x lies in the Nbd of 'a' $\Rightarrow x \in (a-h, a+h)$

or $a-h < x < a+h$

or $x \rightarrow a$

TAYLOR SERIES: $f(x)$ can be expanded (in the Nbd of a) as follows, $\Rightarrow x \to a$

$$f(x) = \underbrace{\underbrace{f(a) + (x-a)f'(a)}_{\text{Linear form}} + \frac{(x-a)^2}{2!}f''(a)}_{\text{Q. Form}} + \frac{(x-a)^3}{3!}f'''(a) + \cdots$$

MACLAURIN SERIES: it is T.S. Exp of $f(x)$ in the Nbd of 0 $\Rightarrow x \to 0$

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots$$

Some Important Maclaurin Series →

① $a^x = 1 + x(\log_e a) + \frac{x^2}{2!}(\log_e a)^2 + \frac{x^3}{3!}(\log_e a)^3 + \cdots\cdots$

✓ $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\cdots$ & $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} \cdots\cdots$

② $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\cdots$

$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots\cdots$

③ $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\cdots$

$\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots\cdots = \frac{e^x - e^{-x}}{2}$

④ $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\cdots$

$\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots\cdots = \frac{e^x + e^{-x}}{2}$

for $x \to 0$

$\sin x \simeq x$ $\quad$ $\cos x \simeq 1$

Q. In the power series Expansion of $f(x) = \frac{x-1}{x+1}$, about $x=1$, 3^{rd} term will be?

(a) $(x-1)^2/2$

(b) $(x-1)^2/4$

(c) $(x-1)^3/8$

(d) $(x-1)^3/4$

In the Nbd of $x=1$, T.S. Exp is

$$f(x) = f(1) + (x-1) f'(1) + \frac{(x-1)^2}{2!} f''(1) + \frac{(x-1)^3}{3!} f'''(1) + \cdots$$

3^{rd} term.

HW.

So $T_3 = \cdots\cdots = $ Ans is (c)

M-II $f(x) = \frac{x-1}{x+1} = \frac{t}{t+2} = \frac{t+2-2}{(t+2)} = 1 - \frac{2}{t+2}$

Put $x-1 = t$

$x = t+1$

$$= 1 - \frac{1}{\left(\frac{t}{2}+1\right)} = 1 - \left(1+\frac{t}{2}\right)^{-1} = 1 - \left[1 - \frac{t}{2} + \frac{t^2}{4} - \frac{t^3}{8} + \cdots\right]$$

$$= \frac{t}{2} - \frac{t^2}{4} + \frac{t^3}{8} - \cdots$$ i.e. $T_3 = \frac{t^3}{8} = \frac{(x-1)^3}{8}$

The quadratic approximation of $f(x) = x^3 - 3x^2 - 5$ at the point $x = 0$ is

(a) $3x^2 - 6x - 5$ (b) $-3x^2 - 5$ (e) $3x^2 - 6x$

(c) $-3x^2 + 6x - 5$ (d) $3x^2 - 5$ (f) $6x^2 - 6$

$f(x) = x^3 - 3x^2 - 5 \Rightarrow f(0) = -5$

$f'(x) = 3x^2 - 6x \Rightarrow f'(0) = 0$

$f''(x) = 6x - 6 \Rightarrow f''(0) = -6$

$\left.\begin{array}{l} f'''(x) = 6 \\ f^{IV}(x) = 0 \end{array}\right\}$ No Need to Calculate

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So T.S. Exp of $f(x)$ in the Nbd of $x=0$ is given as

$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) +$ Neglect.

$= -5 + x(0) + \frac{x^2}{2!}(-6)$

$= -3x^2 - 5$

if $x <<< 1$ then $\coth(x)$ can be approximated as ?

(a) x

(b) $1/x$

(c) x^2

(d) $1/x^2$

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{\frac{e^x + e^{-x}}{2}}{\frac{e^x - e^{-x}}{2}} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

$$= \frac{1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots}{x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots} \simeq \frac{1 + \text{neglect}}{x + \text{neglect}}$$

$$= \frac{1}{x}$$

Thank

THANK

Keep Hustling!