GATE

CRASH COURSE

ALL BRANCHES

Engineering Mathematics

Complex Analysis (Part 02)
(Lec 09)

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Topics *to be covered*

Complex Analysis (Part 2)

1. Cauchy Integral Th
2. " Residue Th (99%)
3. Singularities (Pole)

C-R equns: for $w = f(z) = u + iv$

$$\Rightarrow \boxed{u_x = v_y \ \& \ u_y = -v_x}$$

Singularities of complex func^n $w = f(z)$ $\rightarrow$ z_0 is called singular point of $f(z)$ if it is not analytic at z_0 but it should analytic at every point that lies in the Nbd of z_0.

Note: By putting $D^r = 0$ we can calculate singularities (Poles)

eg $\boxed{f(z) = \dfrac{(z-1)(z+2)}{(z-2)(z-3)^2(z-i)^3}}$

Singularities are Put $D^r = 0$

$\Rightarrow$ $z = 2$ (Simple Pole)

$z = 3$ (Pole of order 2)

$z = i$ (,, ,, 3)

eg $f(z) = \dfrac{\sin z}{z}$ then $z = 0$ is pole of order ?

Sol: $f(z) = \dfrac{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots}{z}$

$= 1 - \dfrac{z^2}{3!} + \dfrac{z^4}{5!} - \dfrac{z^6}{7!} + \cdots$

$z = 0$ is singular point but Not Pole it is called Removable sing.

Methods of Calculating Residues -

Case I: if z_0 is the simple pole then

$$\text{Res of } f(z) \text{ (at } z=z_0) = \boxed{\lim_{z\to z_0}(z-z_0)f(z)}$$

Case II: if z_0 is the pole of order m then

$$\text{Res of } f(z) \; (z=z_0) = \boxed{\frac{1}{(m-1)!}\left[\frac{d^{m-1}}{dz^{m-1}}(z-z_0)^m f(z)\right]_{z=z_0}}$$

Complex Integration

Cauchy Integral Th-r

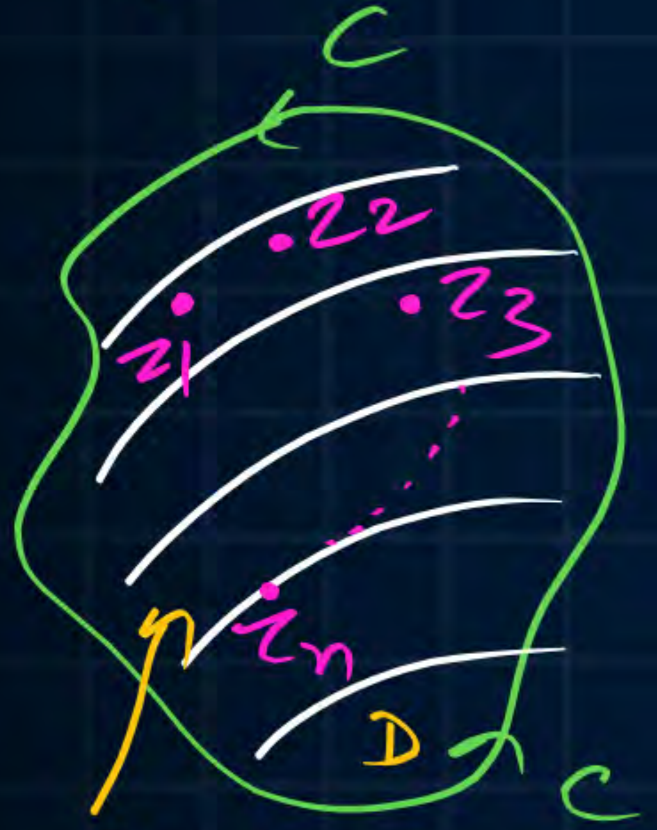
f(z) = Analytic

Let f(z) is analytic with in the Region D bounded by closed contour c then

$$\boxed{\oint_c f(z)\,dz = 0}$$

Cauchy-Residue Th^m

Let $f(z)$ is Non Analytic at finite Number of poles with in the Region D bounded by closed Contour C then



$f(z)$ = Non Analytic

$$\boxed{\oint_C f(z)\,dz = 2\pi i\left[\text{Sum of Residues at Poles that lies inside } C\right]}$$

$$= 2\pi i\left[R_1 + R_2 + R_3 + \dots + R_n\right]$$

Given $X(z) = \dfrac{z}{(z-a)^2}$ with $|z| > a$, the residue of $X(z).z^{n-1}$ at $z = a$ for $n \geq 0$ will be

(a) a^{n-1}

(b) a^n

(c) $n\ a^n$

(d) $n\ a^{n-1}$

Let $f(z) = X(z).z^{n-1} = \dfrac{z}{(z-a)^2}.(z^{n-1}) = \dfrac{z^n}{(z-a)^2}$ Here $z=a$ is pole of order 2

So $\underset{(z=a)}{\text{Res}}\, f(z) = \dfrac{1}{(2-1)!}\left[\dfrac{d^{2-1}}{dz^{2-1}}(z-a)^2 f(z)\right]_{z=a} = \left[\dfrac{d}{dz}(z-a)^2\dfrac{z^n}{(z-a)^2}\right]_{z=a} = \left(nz^{n-1}\right)_{z=a} = na^{n-1}$

$m=2$

(d)

Poles are

$z = 2$ (Simple)

$z = 1$ (Double)

The sum of residues of $f(z) = \dfrac{2z}{(z-1)^2(z-2)}$ at its singular point is

(a) -8 (b) -4

(c) 0 (d) 4

$$R_1 = \underset{(z=2)}{\operatorname{Res}} f(z) = \lim_{z \to 2} (z-2) f(z) = \lim_{z \to 2} \frac{2z}{(z-1)^2} = \frac{4}{(1)^2} = 4$$

$$R_2 = \underset{\substack{(z=1)\\ m=2}}{\operatorname{Res}} f(z) = \frac{1}{(2-1)!}\left[\frac{d^{2-1}}{dz^{2-1}}(z-1)^2 f(z)\right]_{z=1} = \left[\frac{d}{dz}\left(\frac{2z}{z-2}\right)\right]_{z=1} = \left[\frac{(z-2)(2) - 2z(1)}{(z-2)^2}\right]_{z=1} = -4$$

Hence Req Sum $= R_1 + R_2 = 4 - 4 = 0$

The residue of $f(z) = \dfrac{z^3}{(z-1)^4(z-2)(z-3)}$ at $z = 3$ is

(a) -8

(b) $\dfrac{101}{16}$

(c) 0

(d) $\dfrac{27}{16}$

$$\underset{(z=3)}{\operatorname{Res}} f(z) = \lim_{z\to 3} (z-3) f(z) = \lim_{z\to 3} \left[\frac{z^3}{(z-1)^4(z-3)}\right] = \cdots = \frac{27}{16}$$

For the function $\frac{\sin z}{z^3}$ of a complex variable z, the point $z = 0$ is

(a) a pole of order 3 (b) a pole of order 2

(c) a pole of order 1 (d) not a singularity

$$f(z) = \frac{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots}{z^3}$$

$$= \frac{1}{z^2}\left(1 - \frac{1}{3!} + \frac{z^2}{5!} - \frac{z^4}{7!} + \cdots\right)$$

i.e. $z=0$ is pole of order 2

$C : |z| = 1$

$f(z) = \sec z = \dfrac{1}{\cos z}$

for Poles; put

$\cos z = 0$

$z = \ldots\ldots -\dfrac{3\pi}{2}, -\dfrac{\pi}{2}, \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{5\pi}{2} \ldots\ldots$

$z = (2n+1)\dfrac{\pi}{2}, \; n \in I$

i.e these are the sing of $f(z)$

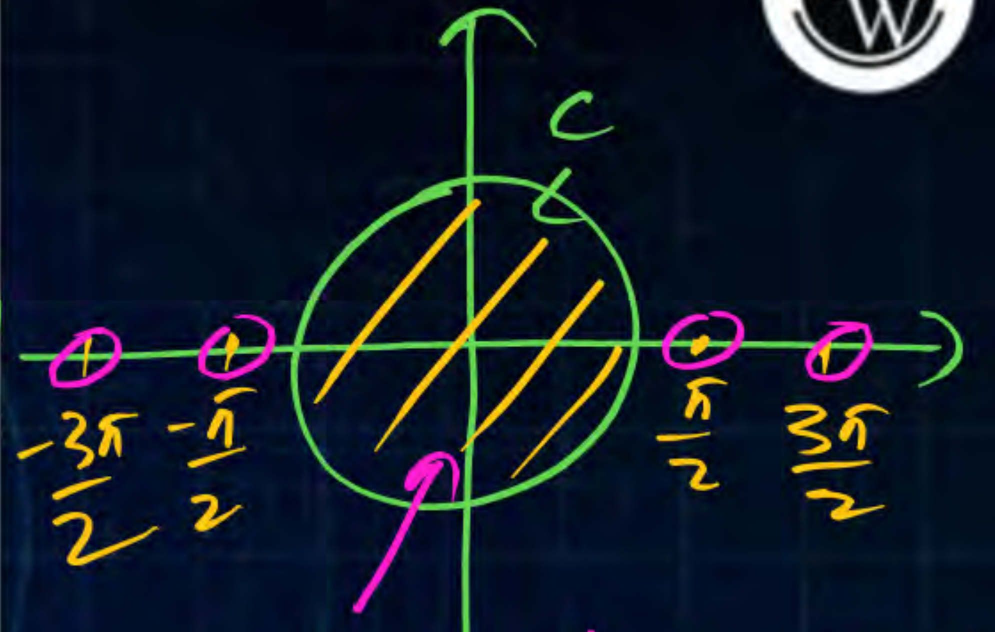
Consider likely applicability of Cauchy's integral theorem to evaluate the following integral counter clockwise around the unit circle C, $I = \oint_C \sec z \, dz$, z being a complex variable. The value of I will be

(a) $I = 0$, singularities set $= \phi$

(b) $I = 0$, singularities set $= \left\{\pm \dfrac{2n+1}{2}\pi; n = 0, 1, 2 \ldots\right\}$

(c) $I = \dfrac{\pi}{2}$, singularities set $= \{\pm n\pi; n = 0, 1, 2 \ldots\}$

(d) None of the above



$f(z) =$ Analytic inside C

So By C.I.Th,

$I = \oint_C f(z)\, dz = 0$

Let $f(s) = f(z) = \frac{1}{z^2 - 1}$

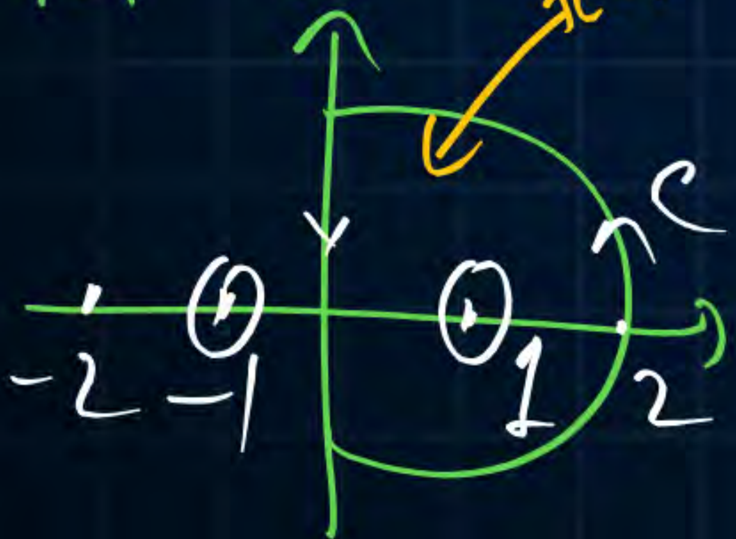
$$f(z) = \frac{1}{(z-1)(z+1)}$$

its poles are $z = -1$

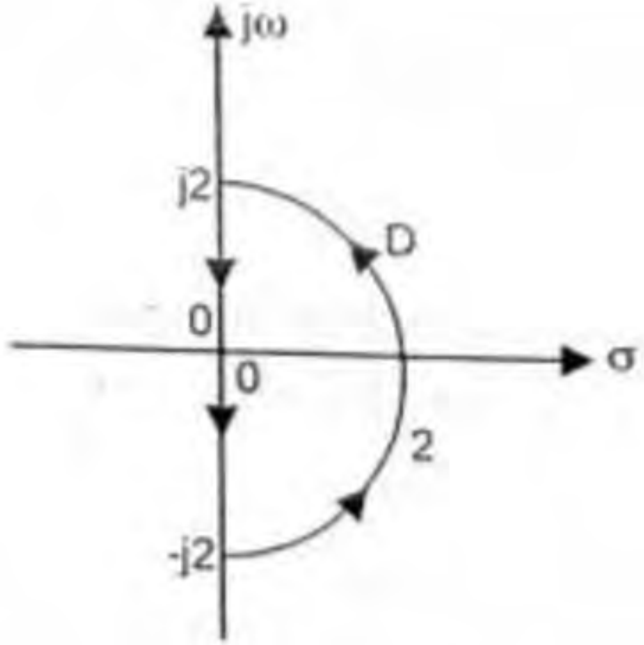
$z = 1$

both are simple poles

$|z| = 2$



If the semicircular contour D of radius 2 is as shown in the figure, then the value of the integral $\oint_D \frac{1}{(s^2-1)} ds$ is where $s = \sigma + j\omega$



(a) $j\pi$ ✓

(b) $-j\pi$

(c) $-\pi$

(d) π

∵ only $z = 1$ lies inside C

so $\operatorname{Res}_{(z=1)} f(z) = \lim_{z \to 1} (z-1) f(z)$

$= \lim_{z \to 1} \left(\frac{1}{z+1}\right) = \frac{1}{2}$

By C.R.Th,

$\oint_C f(z)\, dz = 2\pi i \left(\frac{1}{2}\right)$

$= \pi i$

$f(z) = \frac{(1-z)}{(z+1)(z+3)}$

So Poles are $z = -1, -3$

$C: |z+1| = 1$ — rad $= 1$; Centre $= -1 = (-1, 0)$

Given $f(z) = \frac{1}{z+1} - \frac{2}{z+3}$. If C is a counterclockwise path in the z-plane such that $|z + 1| = 1$, the value of $\frac{1}{2\pi j}\oint_C f(z)dz$ is

(a) −2 (b) −1

(c) 1 (d) 2

$\underset{(z=-1)}{\text{Res}\, f(z)} = \lim_{z\to -1} (z+1) f(z)$

$= \lim_{z\to -1} \left(\frac{1-2}{z+3}\right) = \frac{+2}{2} = +1$

So By C.R. Th.

$I = \frac{1}{2\pi i}\oint_C f(z)dz$

$= \frac{1}{2\pi i}\left[2\pi i\,(+1)\right]$

$= +1$

Let $z = x + iy$ be a complex variable. Consider the contour integration is performed along the unit circle in anticlockwise direction. Which one of the following statements is NOT TRUE?

(a) The residue of $\frac{z}{z^2-1}$ at $z = 1$ is 1/2 (T)

(b) $\oint_C z^2 dz = 0$ (T) By C.I.Th

(c) $\frac{1}{2\pi i}\oint_C \frac{1}{z} dz = 1$ (T) By C.R.Th

(d) $\bar{z}$ (complex conjugate of z) is an analytic function. ✓

$C : |z| = 1$

(a) →

$\text{Res } f(z)$ at $(z = 1)$

$= \lim_{z \to 1} \left(\frac{z}{z+1}\right) = \frac{1}{2}$

(b) $w = z^2$

$f(z) = (x^2 - y^2) + i(2xy)$

$u = x^2 - y^2,\ v = 2xy$

$\because \boxed{u_x = v_y \ \& \ u_y = -v_x}$

$\Rightarrow 2x = 2x \ \& \ 2y = -(-2y)$

(c) $f(z) = \frac{1}{z}$

so Pole is $z = 0$

$f(z) = N.A$

$\text{Res } f(z)_{(z=0)} = 1$

so By C.R.Th, $I = 1$

(d):

$f(z) = \bar{z}$

$u + iv = x - iy$ $\to$ $u = x$, $v = -y$

$\boxed{u_x = v_y \ \& \ u_y = -v_x}$

$1 = -1$ & $0 = 0$

i.e. $f(z) = \bar{z}$ is No where Analytic

Note:

∃ Infinite No of Singular points for $f(z) = \bar{z}$ F

If C is a circle of radius r with centre z_0, in the complex z-plane and if n is a non-zero integer then $\oint_C \frac{dz}{(z-z_0)^{n+1}}$ equals

(a) $2\pi nj$

(b) 0

(c) $\frac{nj}{2\pi}$

(d) $2\pi n$

By - C. R. Th

$C: |z-z_0| = r$

$f(z) = \frac{1}{(z-z_0)^{n+1}}$

Pole is $z = z_0$

it is a pole of order $(n+1)$

i.e. $m = n+1$

$$\operatorname{Res}_{(z=z_0)} f(z) = \frac{1}{((n+1)-1)!}\left[\frac{d^n}{dz^n}(z-z_0)^{n+1}\cdot f(z)\right]_{z=z_0} = \frac{1}{n!}\left[\frac{d^n}{dz^n}(1)\right]_{z=z_0} = 0$$

M-II $\quad I = \oint_C \frac{1}{(z-z_0)^{n+1}}\,dz = ? = \oint_C \frac{1}{(re^{i\theta})^{n+1}}(re^{i\theta}\cdot i\,d\theta)$

$C: |z-z_0| = r \Longrightarrow (z-z_0) = r\cdot e^{i\theta}$

i.e $z = z_0 + re^{i\theta}$

$dz = 0 + r\,e^{i\theta}\cdot i\,d\theta$

$0 \le \theta \le 2\pi$

w.k.that $\left[z = re^{i\theta}\right] \Longleftrightarrow \left[|z| = r\right]$

$i\int e^{-ni\theta - i\theta + i\theta}\,d\theta$

$i\int_0^{2\pi} e^{-ni\theta}\,d\theta = i\left[\frac{e^{-ni\theta}}{-ni}\right]_0^{2\pi}$

$= -\frac{1}{n}\left[e^{-2\pi i} - 1\right]$

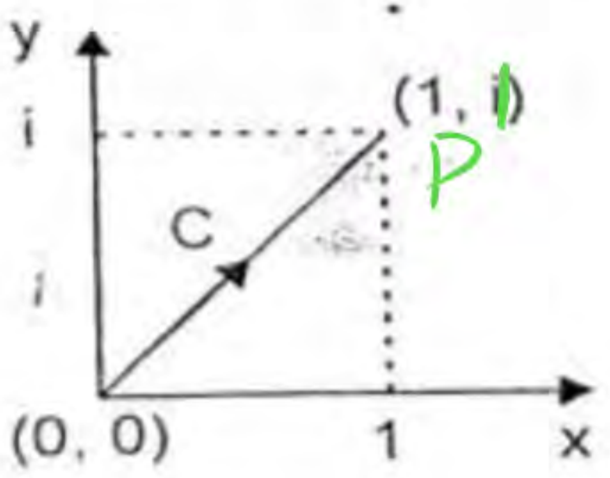
$= -\frac{1}{n}\left(\frac{1}{e^{2\pi i}} - 1\right) = \frac{1}{n}(1-1)$

$= 0$

The integral $\oint f(z)\,dz$ evaluated around the unit circle on the complex plane for $f(z)=\dfrac{\cos z}{z}$ is

(a) $2\pi i$ (b) $4\pi i$ (c) $-2\pi i$ (d) 0

Consider the line integral $I = \int_c (x^2 + iy^2)\,dz$, where $z = x + iy$. The line c is shown in the figure below



The value of I is

(a) $\frac{1}{2}i$

(b) $\frac{2}{3}i$ ✓

(c) $\frac{3}{4}i$

(d) $\frac{4}{5}i$

C : OP (Open Contour)

eqn of OP is $y = x$

$z = x + iy$

$dz = dx + i\,dy$

$dz = dx + i(dx)$

$dz = (1+i)\,dx$

$0 \le x \le 1$

$\therefore I = \int_c (x^2 + iy^2)\,dz = \int (x^2 + ix^2)(1+i)\,dx = (1+i)^2 \int_{x=0}^{1} x^2\,dx = 2i\left(\frac{x^3}{3}\right)_0^1 = \frac{2i}{3}$

PYQ

Given $i = \sqrt{-1}$, what will be the evaluation of the definite integral $\int_0^{\pi/2} \frac{\cos x + i\sin x}{\cos x - i\sin x} dx$?

(a) 0

(b) 2

(c) $-i$

(d) i

$$I = \int_0^{\pi/2} \frac{e^{ix}}{e^{-ix}} dx = \int_0^{\pi/2} e^{2ix} dx = \left(\frac{e^{2ix}}{2i}\right)_0^{\pi/2} = \frac{1}{2i}\left(e^{\pi i} - 1\right)$$

$$= \frac{1}{2i}(-1-1) = \frac{-1}{i} = \frac{i^2}{i} = i$$

Change of order of Integration →

If V-Strip is given, then for changing the order of integration, Draw H-Strip and put the limits accordingly & vice-versa

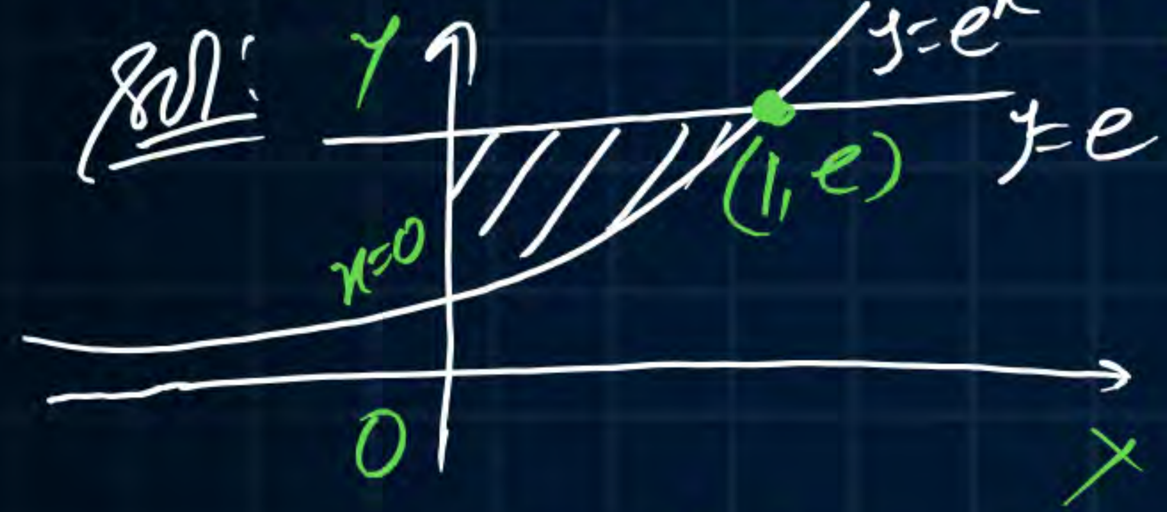
* V-Strip ← y has Variable limits & x has constant limits
  H-Strip ← x " " " , y " " "

$$I = \int_{x=a}^{b}\int_{y=f(x)}^{g(x)} f(x,y)\,dy\,dx = \int_{y=c}^{d}\int_{x=\phi(y)}^{\psi(y)} f(x,y)\,dx\,dy$$

Q: Find the Area bounded by $y=e^x$, $y=e$ with y axis?

(a) e (b) 1 (c) $e-1$ (d) $2e$

Sol:

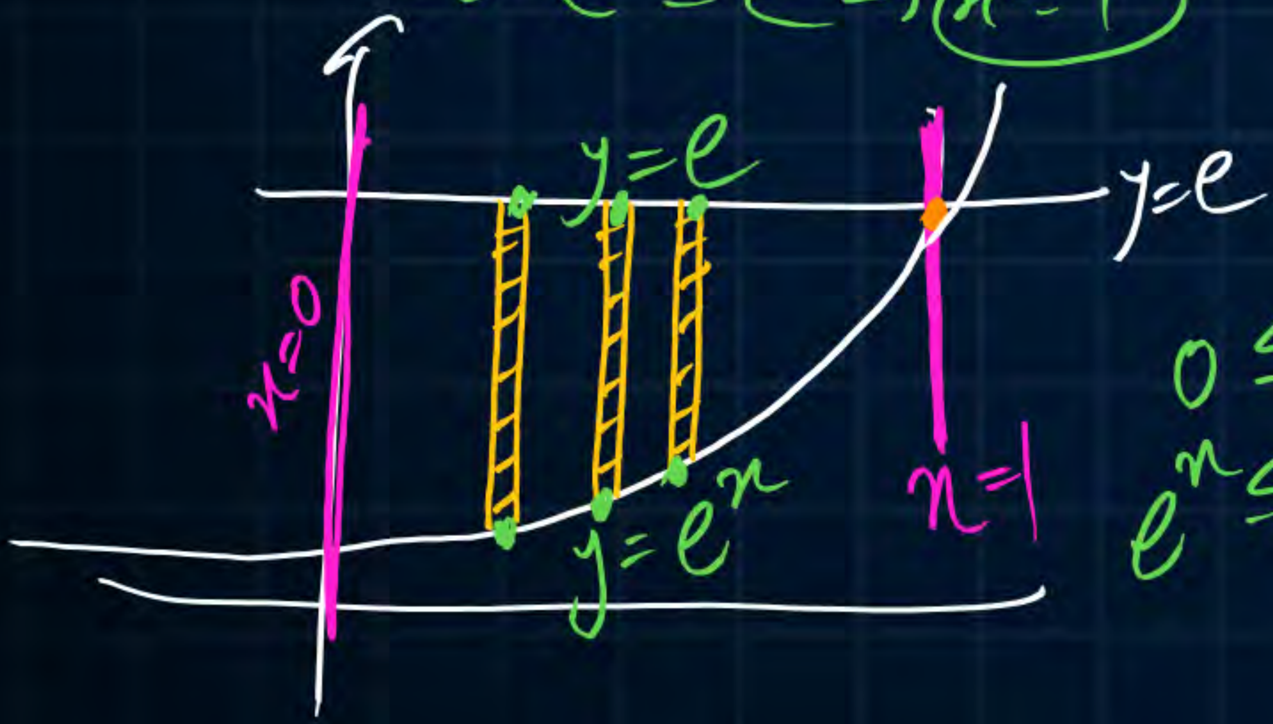


$$\boxed{\text{Area} = \iint (1)\,dy\,dx}$$

$$= \int_{x=0}^{1} \int_{y=e^x}^{e} (1)\,dy\,dx$$

$y=e^x$ & $y=e$

$\Rightarrow e^x = e \Rightarrow x=1$



$0 \le x \le 1$

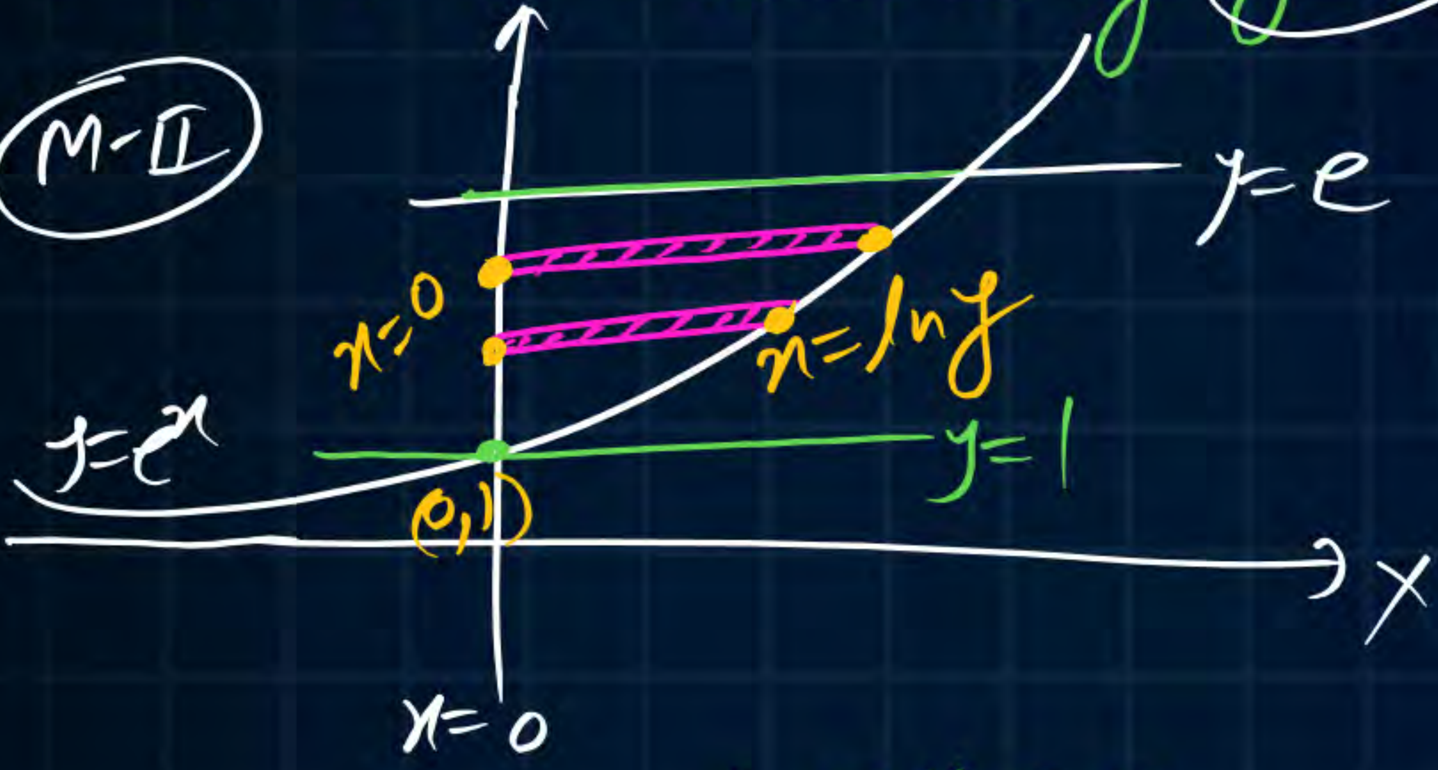
$e^x \le y \le e$

$$= \int_{x=0}^{1} (e-e^x)\,dx = \left(ex - e^x\right)\Big|_{x=0}^{1}$$

$$= (e\times 1 - e^1) - (e\times 0 - e^0) = 1$$

Q: Find the Area bounded by $y=e^x$, $y=e$ with Y axis? (using H-Strip)

(M-II)



$$1 \le y \le e$$
$$0 \le x \le \ln y$$

$$\text{Req Area} = \iint (1)\,dy\,dx \quad (\text{using V-Strip})$$

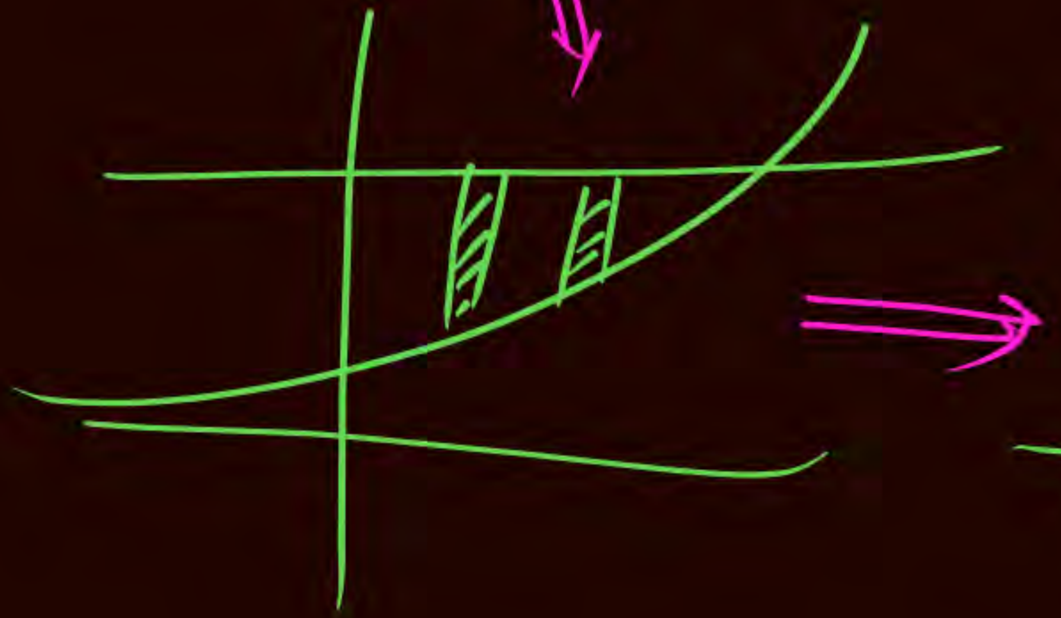
$$= \iint (1)\,dx\,dy \quad (\text{using H-Strip})$$

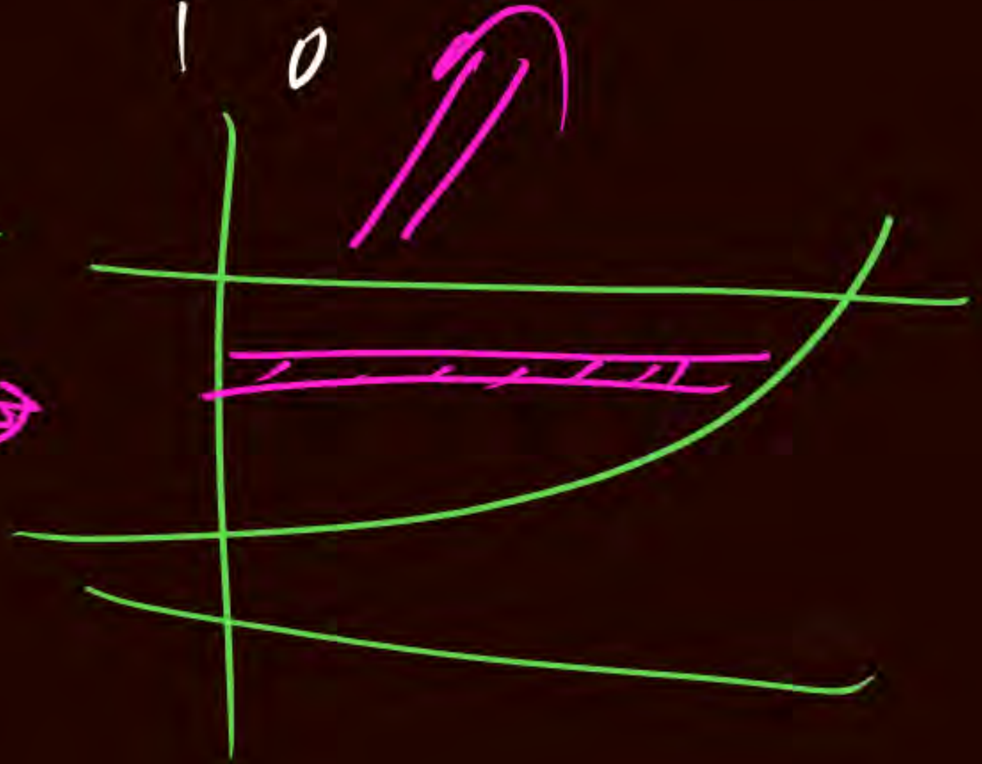
$$= \int_{y=1}^{e}\int_{x=0}^{\ln y} (1)\,dx\,dy = \int_{y=1}^{e} (\ln y)\,dy$$

$$= \Big(y\ln y - y\Big)_{y=1}^{e} = (e-e)-(0-1) = 1$$

Q: By changing the order of integration $\int_0^1 \int_{e^x}^{e} f(x,y)\,dy\,dx$ converts into $\int_p^q \int_r^s f(x,y)\,dx\,dy$ then $s = ?$

Sol: $I = \int_0^1 \int_{e^x}^{e} f(x,y)\,dy\,dx = \int_1^e \int_0^{\ln y} f(x,y)\,dx\,dy \Rightarrow s = \ln y$





Q: By changing the order of integration evaluate $\int_0^\infty \int_0^\infty e^{-xy} \sin px \, dy \, dx = ?$

(M-I) w/o changing order of integration let us try to calculate

Given strip is vertical so $I = \int_{x=0}^{\infty} \left(\int_{y=0}^{\infty} e^{-xy} \, dy \right) \sin px \, dx = \int_{x=0}^{\infty} \left\{ \frac{e^{-xy}}{-x} \right\}_0^\infty \sin px \, dx$

$= \int_0^\infty \left\{ -\frac{1}{x} (0-1) \sin px \, dx \right. = \int_0^\infty \left(\frac{\sin px}{x} \right) dx = ???$ —— (1)

(M-II) (By changing the order of integration) →

$I = \int_0^\infty \int_0^\infty e^{-xy} \sin px . \, dx \, dy = \int_0^\infty \left\{ \frac{e^{-xy}}{y^2+p^2} (-y \sin px - p \cos px) \right\}_{x=0}^{\infty} (dy) = \int_0^\infty \frac{p}{y^2+p^2} dy$

$$I = p\left\{\frac{1}{p}\tan^{-1}\left(\frac{y}{p}\right)\right\}_0^\infty = \tan^{-1}(\infty) - \tan^{-1}(0) = \frac{\pi}{2} \longrightarrow (2)$$

Note: By (1) & (2), $\boxed{\int_0^\infty \left(\frac{\sin px}{x}\right) dx = \frac{\pi}{2}}$

$$\int_0^\infty \left(\frac{\sin x}{x}\right) dx = ? = \frac{\pi}{2}$$

Note:

w.k. that $L\left\{\frac{\sin px}{x}\right\} = \tan^{-1}\left(\frac{p}{s}\right)$

$$\int_0^\infty e^{-sx}\left(\frac{\sin px}{x}\right) dx = \tan^{-1}\left(\frac{p}{s}\right)$$

$s \to 0$

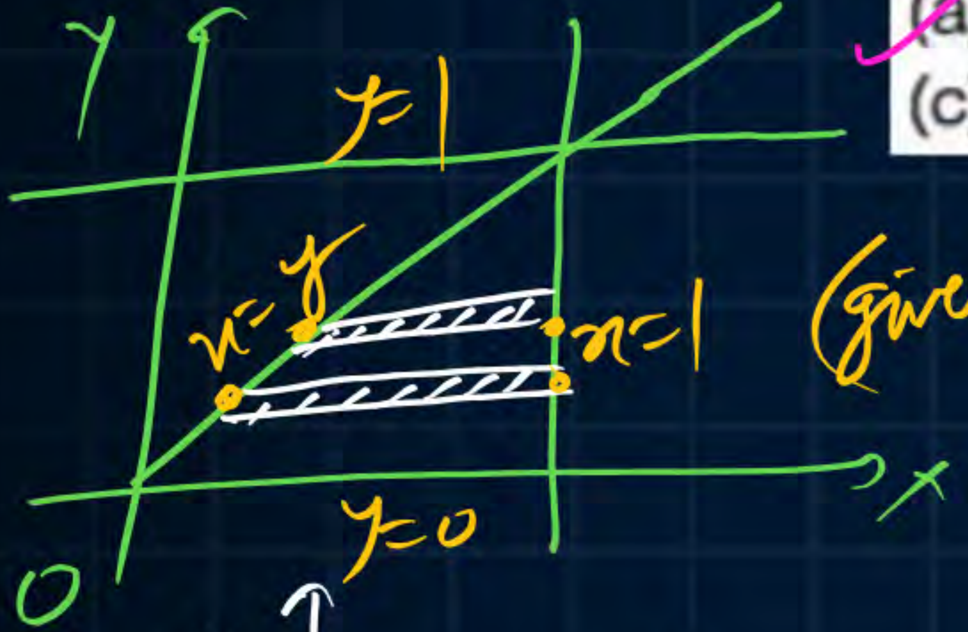
$$\int_0^\infty \left(\frac{\sin px}{x}\right) dx = \pi/2$$

Let $\int_0^1\int_y^1 xy\sin(xy)\,dx\,dy = \int_0^1\int_a^b xy\sin xy\,dy\,dx$

(a) $a=0, b=x$ (b) $a=1, b=x$

(c) $a=0, b=1$ (d) $a=-1, b=x$

$y=0,\ y=1,\ x=y,\ x=1$



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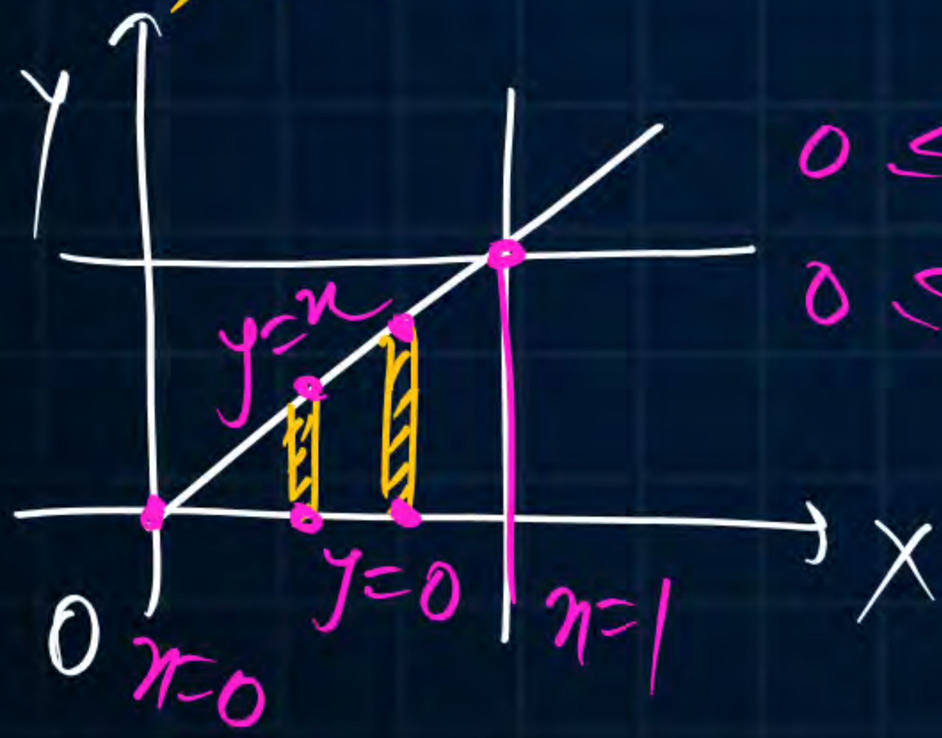

$0 \le x \le 1$

$0 \le y \le x$

$I = \int_0^1 \int_y^1 xy \sin(xy)\,dx\,dy$

$= \int_0^1 \int_{0}^{x} (xy)\sin(xy)\,dy\,dx$

$x = b$

$0 = a$

Thank

THANK

Keep Hustling!