GATE

CRASH COURSE

ALL BRANCHES

Engineering Mathematics

Calculus (Part 02)
(Lec 05)

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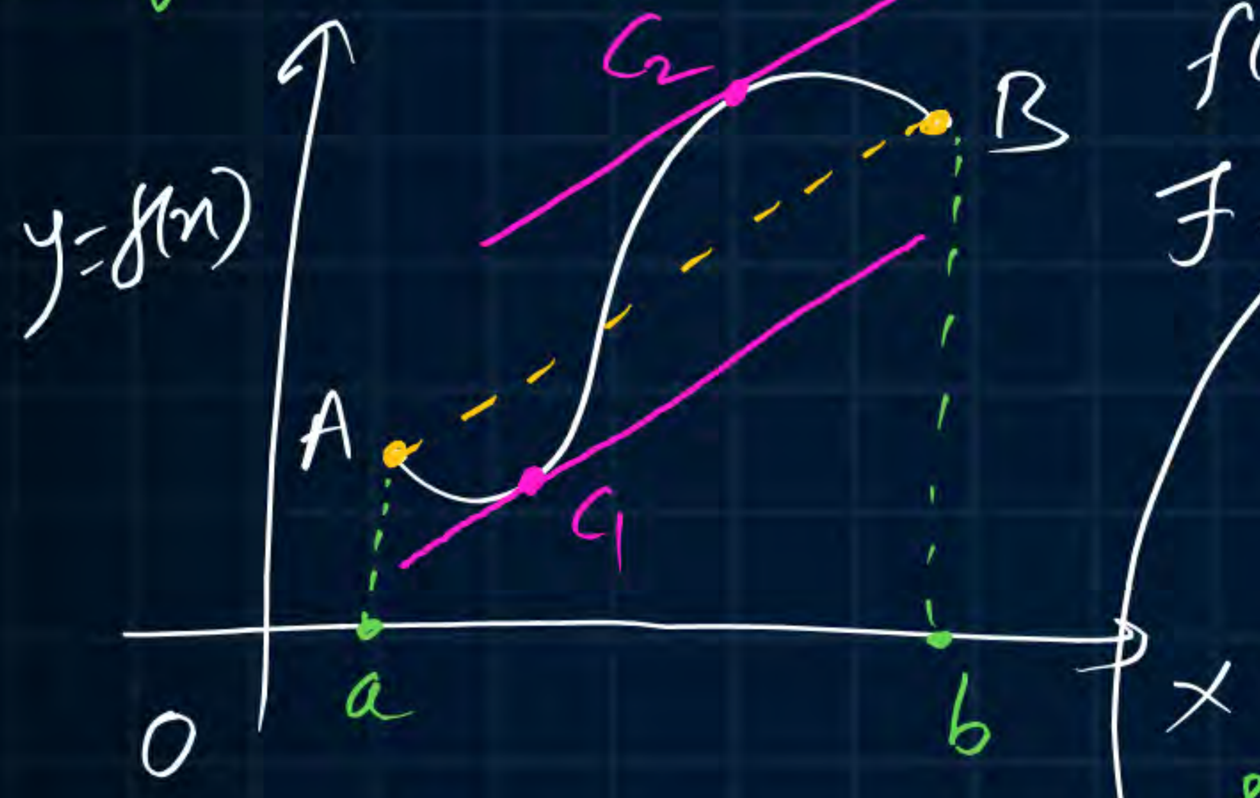
Topics *to be covered*

100%

1. Mean Value Theorems
2. Increasing Decreasing funcn
3. Maxima - Minima (Single Variable)
4. " " (two variables)

Lagrange's M.V. Th :-



$A(a, f(a))$

$B(b, f(b))$

Consider $f(x)$ is defined in $[a, b]$ s.t

$f(x)$ is continuous as well as Diff then

∃ atleast one point c in b/w a & b for which

Slope of chord $AB = \dfrac{f(b)-f(a)}{b-a}$

Slope of tangent at 'c' $= f'(c)$

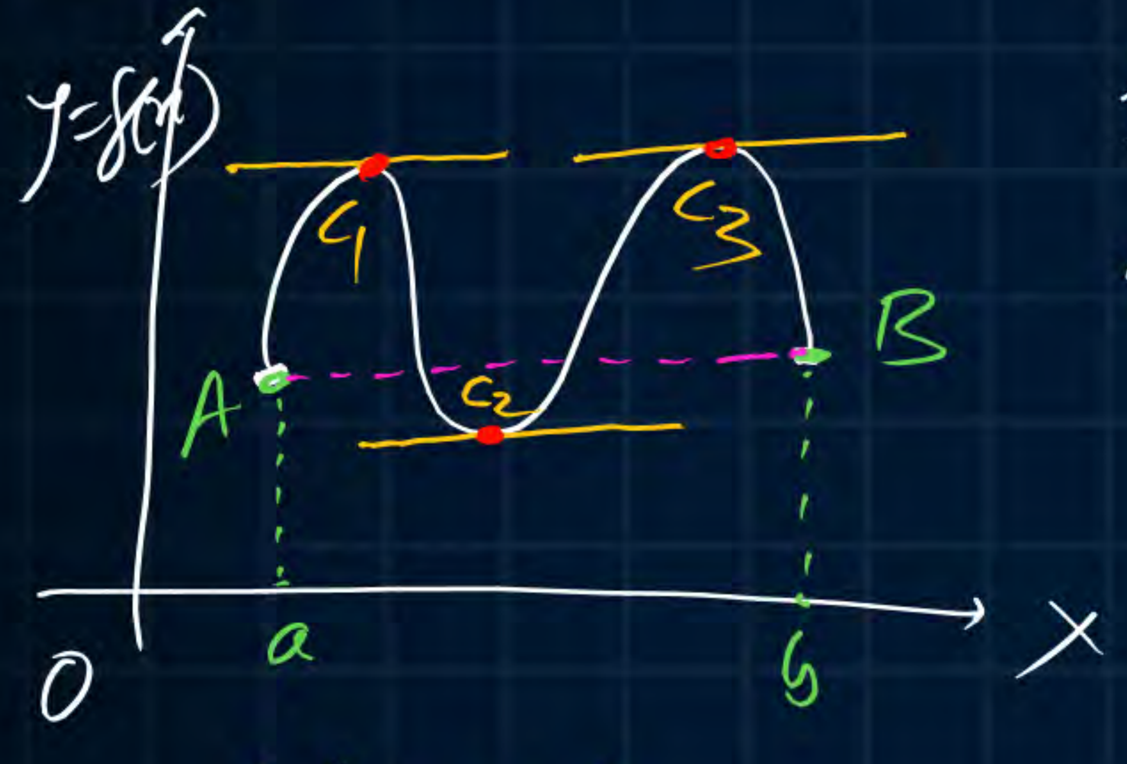
So A.T. Diagram,

$$\boxed{f'(c) = \frac{f(b)-f(a)}{b-a}}$$

i.e Slope of tangent at c = Slope of chord AB.

i.e tangent at c is $\|^{r}$ to the chord AB

Rolle's M.V.Th ⁿ



$A(a, f(a))$

$B(b, f(b))$

Consider $f(x)$ is defined $[a, b]$ s.t

$f(x)$ is Cont & diff & also we have $f(a)=f(b)$

Then $\exists$ at least one c b/n a & b where

tangent is $\|^r$ to x axis

or Horizontal tangent exist

or $\boxed{f'(c)=0}$

Cauchy M.V.Th^{n}: Consider $f(x)$ & $g(x)$ are two func^{n} defined in $[a,b]$

s.t ① Both $f(x)$ & $g(x)$ are cont. $[a,b]$

② " " " are diff (a,b)

③ $g'(x) \neq 0 \ \forall x \in (a,b)$

then $\exists$ exist atleast one c in b/w a & b for which

$$\boxed{\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}}$$

Q. $f(x) = \dfrac{\sin x}{e^x}$; $[0, \pi]$ then using R.Th the value of $c = ?$

(a) $\pi/2$

(b) $\pi/4$

(c) $\pi/6$

(d) 0

$f(x)$ is cont as well as diff (obvious)

& $f(0) = 0 = f(\pi)$ i.e all 3 R. Cond^n are satisfied

So By R.Th $f'(c) = 0$

$$\frac{\cos c - \sin c}{e^c} = 0$$

$\sin c = \cos c$

$\tan c = 1 \Rightarrow c = \frac{\pi}{4}$

$f(x) = \dfrac{\sin x}{e^x}$

$$f'(x) = \frac{e^x(\cos x) - \sin x(e^x)}{(e^x)^2}$$

$$= \frac{\cos x - \sin x}{e^x}$$

Q: The Point of the Curve $\boxed{f(x) = \cos x - 1}$, $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ where tangent is $\parallel^r$ to x axis?

(a) $\left(\frac{\pi}{2}, 1\right)$ ✗

(b) $(\pi, 2)$

(c) $(\pi, -2)$ ✓

(d) $\left(\frac{\pi}{4}, \pi\right)$ ✗

it is cont as well Diff (obvious) & $f\left(\frac{\pi}{2}\right) = -1 = f\left(\frac{3\pi}{2}\right)$

So By R.Th, $f'(c) = 0$

$$[-\sin c] = 0$$

$$c = n\pi, n \in I$$

$$c = \ldots -2\pi, -\pi, 0, \pi, 2\pi, 3\pi \ldots$$

only $c = \pi$ lies in the given Domain

$f(c) = \cos c - 1$

$= \cos \pi - 1 = -1 - 1 = -2$

i.e. $(c, f(c)) = (\pi, -2)$

Q. For $y=\sqrt{x-2}$, $[2,3]$ find the eqⁿ of tangent which is $\parallel^r$ to chord joining end points?

Sol: $\boxed{f(x)=\sqrt{x-2}}$ — ①

$$f'(x)=\frac{1}{2\sqrt{x-2}}$$

$a=2$, $f(2)=0$

$b=3$, $f(3)=1$

using LMVTh

$$f'(c)=\frac{f(3)-f(2)}{3-2}$$

$m=1$

$$\frac{1}{2\sqrt{c-2}}=\frac{1-0}{1}$$

$$\sqrt{c-2}=\frac{1}{2}$$

$$c-2=\frac{1}{4}$$

$$c=\frac{9}{4}$$

$$f(c)=\sqrt{c-2}=\sqrt{\frac{9}{4}-2}=\frac{1}{2}$$

So Point where tangent is $\parallel^r$ to chord

$$=(c,f(c))=\left(\frac{9}{4},\frac{1}{2}\right)$$

R. equⁿ of tangent is

$$y-\frac{1}{2}=m\left(x-\frac{9}{4}\right)$$

$$\frac{2y-1}{2}=\frac{4x-9}{4}$$

$$4y-4=8x-18$$

$$8x-4y-14=0 \Rightarrow \boxed{4x-2y=7}$$

Q. Consider $f(x) = \log x$ & $g(x) = \log\left(\frac{1}{x}\right)$ defined in $[1, 2]$ then using C.M.V.Th, $c = ?$

(a) 1

(b) 1.5

(c) 1.25

(d) Any value b/w 1 & 2

(e) 2

$f(x) = \log x$, $g(x) = -\log x$

$f'(x) = \frac{1}{x}$ $g'(x) = -\frac{1}{x} \Rightarrow g'(x) \neq 0 \; \forall x \in (1, 2)$

By C.M.V.Th, $\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$

$\frac{f(2) - f(1)}{g(2) - g(1)} = \left(\frac{1/c}{-1/c}\right) \Rightarrow \frac{\log 2 - 0}{-\log 2 - 0} = -1 \Rightarrow -1 = -1$

Identity

By applying Lagranges mean value for the function $f(x)=(1+x)\log(1+x)$ on [0, 1] the value of $c \in (0, 1)$ is

(a) $\frac{4}{e}$ (b) $\frac{1}{e}$

(c) $\frac{4-e}{e}$ (d) $\frac{1-e}{e}$

$f(x) = (1+x)\log(1+x)$; $[0,1]$

$a=0,\ f(a)=f(0)=0$

$b=1,\ f(b)=2\log 2$

ie $(a, f(a)) = (0,0)$

$(b, f(b)) = (1, 2\log 2)$

slope of chord AB is

$m = \frac{2\log 2 - 0}{1-0}$

$f'(c) = 2\log 2$

$f(x) = (1+x)\log(1+x)$

$f'(x) = (1+x)\left[\frac{1}{1+x}\right] + \log(1+x)[1]$

$= 1 + \log(1+x)$

$f'(c) = 2\log 2$

$1 + \log(1+c) = 2\log 2$

$\log(1+c) = \log(2)^2 - 1$

$\log(1+c) = \log_e 4 - \log_e e$

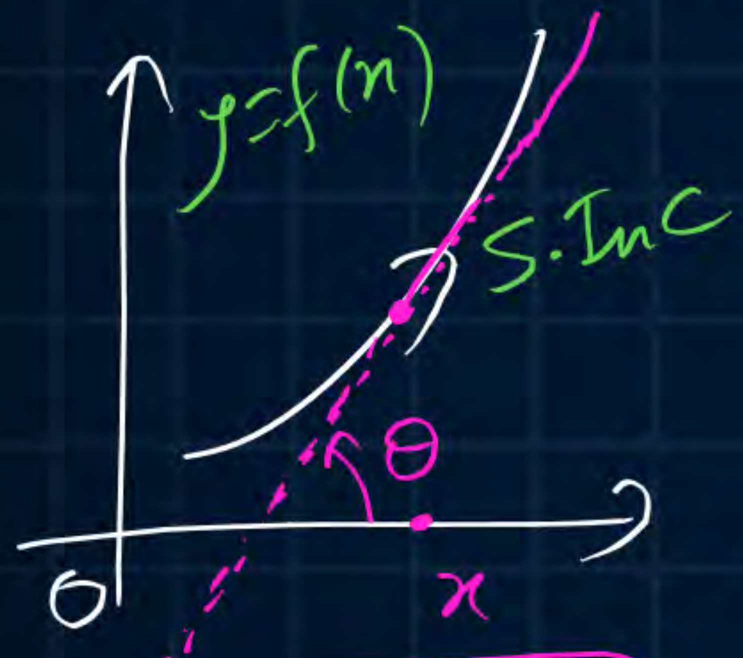
$\log(1+c) = \log\left(\frac{4}{e}\right)$

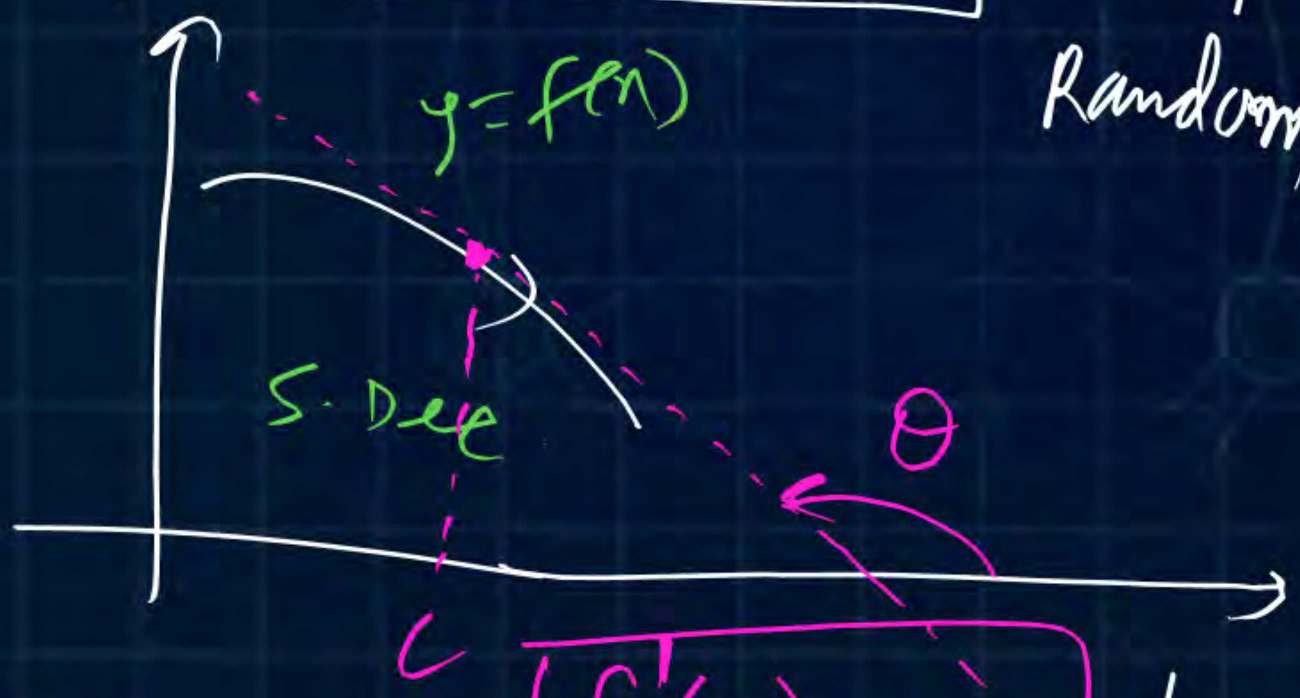
$1 + c = \frac{4}{e}$

$c = \frac{4}{e} - 1 = \frac{4-e}{e}$

MAXIMA - MINIMA (of single variable)

(*) W.K. that for $y = f(x)$, $\frac{dy}{dx} = \boxed{f'(x) = \tan\theta}$ = slope of tangent at any Random point x



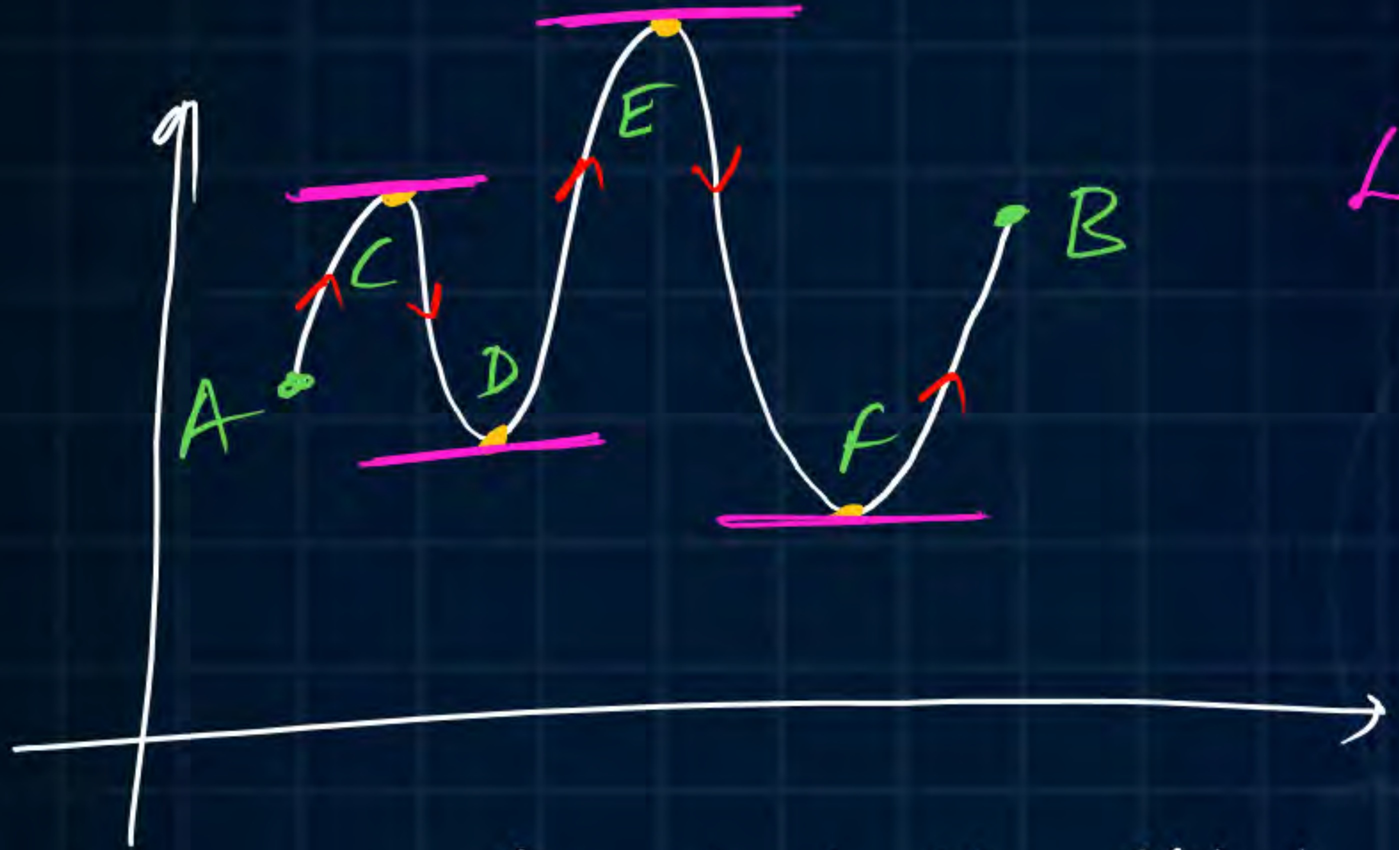


$\boxed{f'(x) > 0}$ always $\qquad \boxed{f'(x) < 0}$

Hence for Inc. func^n $\Longleftrightarrow$ $f'(x) > 0$ always

& for Dec func^n $\Longleftrightarrow$ $f'(x) < 0$ "

& for Horizontal tangent $\boxed{f'(x) = 0}$



Local Max Points = C, E, B

" " Values = $f(C), f(E), f(B)$

Max. Value = $f(E)$

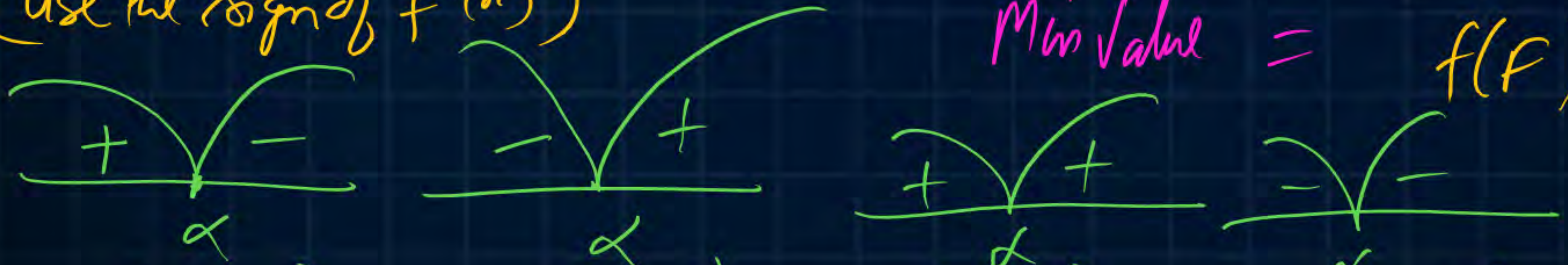
Local min Points = A, D, F

" " Values = $f(A), f(D), f(F)$

Min Value = $f(F)$

Short Cut Method to check Max-Min. :-

(use the sign of $f'(x)$)



(Maxima) (minima) (NMNM) (NMNM)

Turning Points / Critical Point :-

Put $\boxed{f'(x)=0}$ & try to find x

say it is α then

$x=\alpha$ is called T. Point.

i.e N. Cond^n for Maxima-Minima is $\boxed{f'(x)=0}$

(*) Corner Points are not T. Points (True)

(*) Extreme Points :- Points where Maxima or minima occurs called E. Points.

(*) Corner Points are Extreme Points (T)

(*) Shortcut Method of finding Maxima or Minima is applicable only for T. Points.

(*) Max-Min occurs alternately (T)

(*) Long Method :-

Let $x=\alpha$ is the T. Point then

$f'(\alpha)=0$ &
- $f''(\alpha)<0 \Rightarrow$ Maxima
- $f''(\alpha)>0 \Rightarrow$ Minima

The maximum value of

$f(x) = x^3 - 9x^2 + 24x + 5$ in the interval $[1, 6]$ is

(a) 21 (b) 25

(c) 41 (d) 46

$f'(x) = 3x^2 - 18x + 24$

$= 3(x^2 - 6x + 8)$

$\boxed{f'(x) = 3(x-4)(x-2)}$

T. Points are $f'(x) = 0$

$x = 2 \;\&\; 4$

No. of Extremas = Four

+	−	+

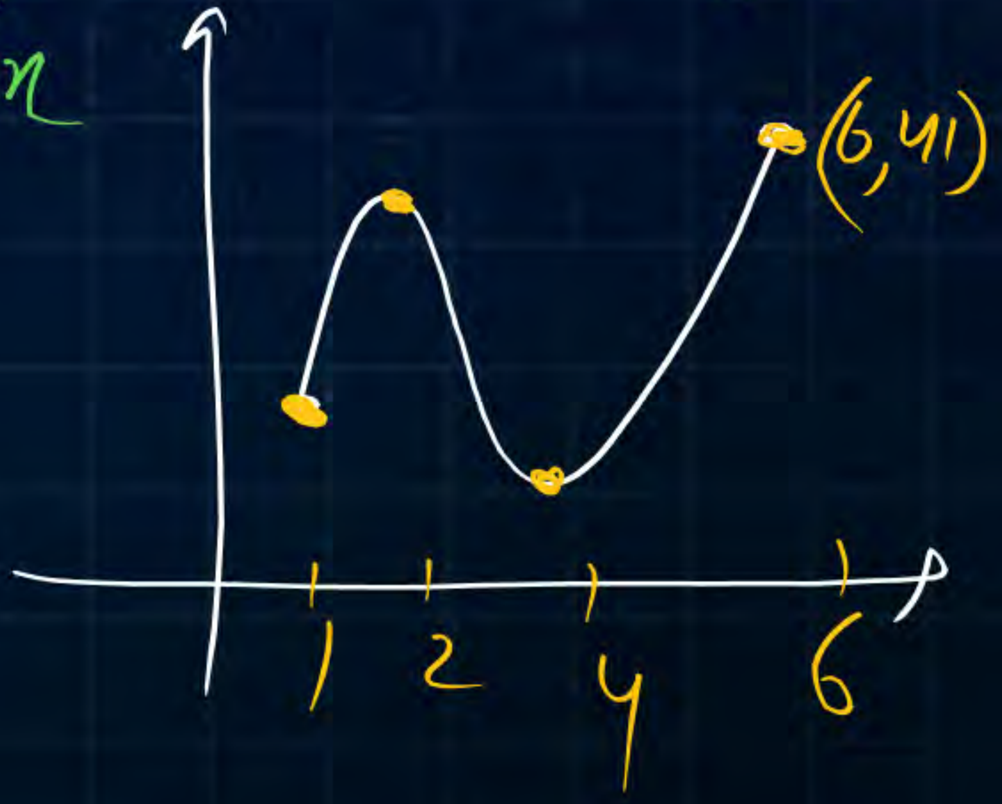
1 2 4 6

min max min max

$f(2) = 25$

$f(6) = 41$

So Points of Max: $x = 2$, $x = 6$



The maximum value of $(-\infty, \infty)$

$f(x) = x^3 - 9x^2 + 24x + 5$ is ____.

(a) 21

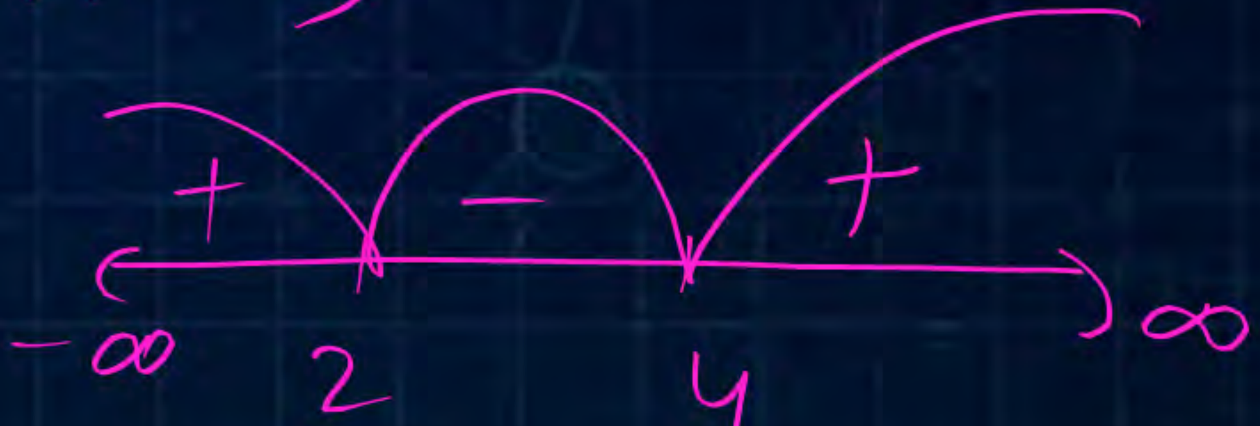
(b) 25

(c) 41

(d) DNE ✓

$f'(x) = 3(x-4)(x-2)$

$x = 2, 4$



Max

minima

No. of Extremas = two

The maximum value of $f(x) = 2x^3 - 9x^2 + 12x - 3$ in the interval $0 \le x \le 3$ is $\underline{\ 6\ }$.

$f'(x) = 6x^2 - 18x + 12$

$= 6(x^2 - 3x + 2)$

$\boxed{f'(x) = 6(x-1)(x-2)}$

for T-Points, $f'(x) = 0$

$x = 1 \,\&\, 2$

So Points of Maxima are $x = 1 \,\&\, 3$

& Local Max values $\to f(1) = 2$, $f(3) = 6$

So Absolute Max Value $= \underline{6}$ & it occurs at $x = 3$

No. of Extremas $= 4$

(min & max both)

A point on the curve is said to be an extremum if it is a local minimum (or) a local maximum. The number of distinct extrema for the curve $3x^4 - 16x^3 + 24x^2 + 37$ is ______.

(a) 0 (b) 1

(c) 2 (d) 3

if only $x=0$ is Extreme point.

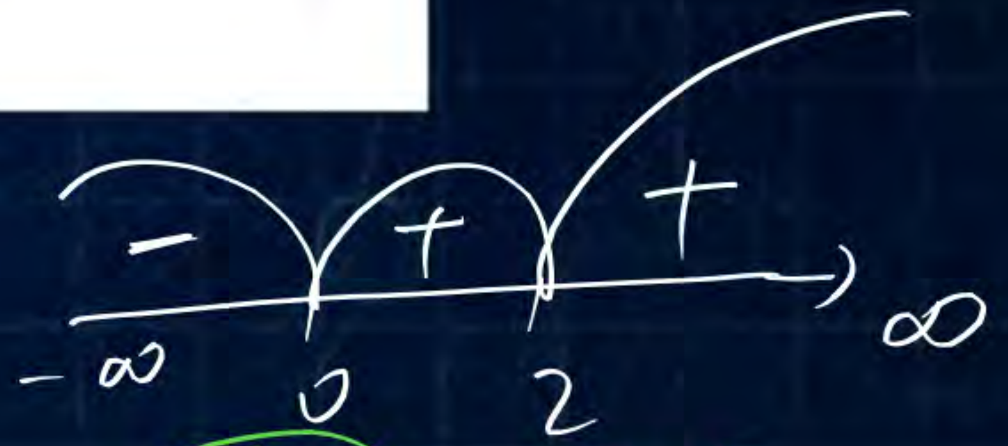
$$f(x) = 3x^4 - 16x^3 + 24x^2 + 37 \; ; \; (-\infty, \infty)$$

$$f'(x) = 12x^3 - 48x^2 + 48x$$

$$= 12x(x^2 - 4x + 4)$$

$$\boxed{f'(x) = 12x(x-2)^2}$$

T. Points; $x = 0, 2$



min NMNM

$f'(-1) = 12(-1)(-1-2)^2 = -ve$

$f'(1) = 12(1)(1-2)^2 = +ve$

$f'(3) = 12(3)(3-2)^2 = +ve$

∴ we have only one Extrema which occurs at $x = 0$

The minimum value of the function $f(x) = \frac{1}{3}x\left(x^2-3\right)$ in the interval $-100 \le x \le 100$ occurs at x = ________.

(a) 1

(b) -1

(c) -100

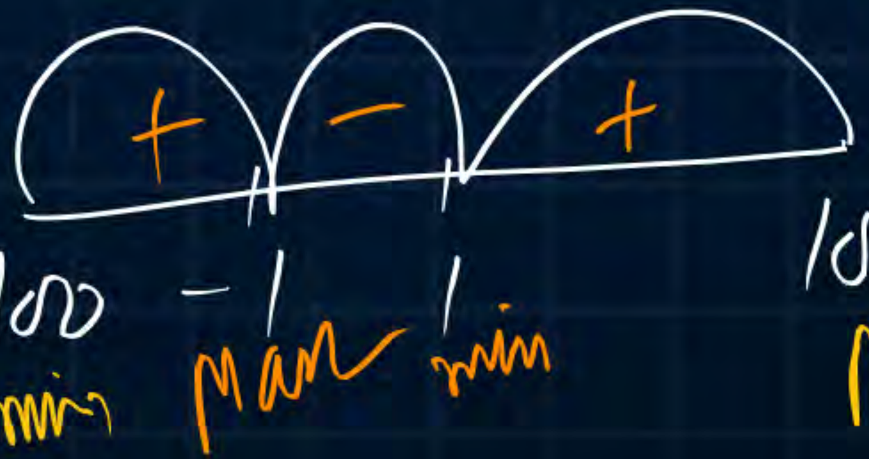
(d) None

$$\boxed{f(x) = \frac{1}{3}x(x^2-3)} = \frac{1}{3}(x^3 - 3x)$$

$$f'(x) = \frac{1}{3}\left[3x^2 - 3\right] = \boxed{x^2 - 1}$$

T. Points, Put $f'(x) = 0$

$x = \pm 1$



$$\boxed{f'(x) = (x+1)(x-1)}$$

Minima occur at $x = 1$, $x = -100$

Min Values: $f(1) = -2/3$, $f(-100) = <<0$

Maxima - Minima of func^n of two Variables

$$\left[z = f(x,y) \right] \rightarrow \begin{cases} \text{Point} = (x,y) \\ \text{Value} = z \end{cases}$$

N. Cond^n for Maxima - Minima

$$\boxed{\frac{\partial z}{\partial x} = 0 \quad (1) \quad , \quad \frac{\partial z}{\partial y} = 0 \quad (2)}$$

Critical Points → Points obtained by solving (1) & (2) are called Critical Points. let after solving we get $x = a$ & $y = b$ then $P(a,b)$ is C.P.

S. Cond^n for Max-Minima → func^n must satisfy Lagrange's Cond^n.

Lagrange's Cond^n :-

Let $\left(\frac{\partial^2 z}{\partial x^2}\right)_P = r$, $\left(\frac{\partial^2 z}{\partial x \partial y}\right)_P = s$, $\left(\frac{\partial^2 z}{\partial y^2}\right)_P = t$

(1) if $rt - s^2 > 0$ & $r > 0$ then $P(a,b)$ is Point of Minima

(2) if $rt - s^2 > 0$ & $\boxed{r < 0}$ " " " " Maxima

(3) if $rt - s^2 < 0$ then " " " Inflexion ($\simeq$ Saddle Point)

(4) if $rt - s^2 = 0$ then Not possible to check Nature of Point in terms of Maxima or Minima

(*) Saddle point $\Longrightarrow$ we have Neither Maxima Nor minima

$\not\Leftarrow$

Given a function $f(x, y) = 4x^2 + 6y^2 - 8x - 4y + 8$. The optimum value of $f(x, y)$ is

(a) a minimum equal to 10/3

(b) a maximum equal to 10/3

(c) a minimum equal to 8/3

(d) a maximum equal to 8/3

Optimum Value
→ Extreme Value
{Min Value & Max Value both}

$z = f(x, y) = 4x^2 + 6y^2 - 8x - 4y + 8$

$\frac{\partial z}{\partial x} = 8x - 8$, $\frac{\partial^2 z}{\partial x^2} = 8$

$\frac{\partial z}{\partial y} = 12y - 4$, $\frac{\partial^2 z}{\partial y^2} = 12$

$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}(12y - 4) = 0$

Critical Points;

$\frac{\partial z}{\partial x} = 0$ & $\frac{\partial z}{\partial y} = 0$

$x = 1$ & $y = \frac{1}{3}$

i.e. $P(1, \frac{1}{3})$ is Critical Point.

$r = (z_{xx})_P = 8$

$s = (z_{xy})_P = 0$

$t = (z_{yy})_P = 12$

$\because r = 8 > 0$

& $rt - s^2 = (8)(12) - (0)^2$
$= 96 > 0$

so $P(1, \frac{1}{3})$ is Point of Minima

Min Value $(z) = (f(x, y))_P = \left[4x^2 + 6y^2 - 8x - 4y + 8\right]_P = \cdots = \frac{10}{3}$

$$\boxed{z = 2x + \frac{12000}{xy} + y + 5}$$

$$z_x = 2 - \frac{12000}{x^2 y}$$

$$z_y = \frac{-12000}{xy^2} + 1$$

For C.Points ; $\frac{\partial z}{\partial x} = 0$ & $\frac{\partial z}{\partial y} = 0$

$$2 - \frac{12000}{x^2 y} = 0 \quad \& \quad \frac{-12000}{xy^2} + 1 = 0$$

$$\boxed{x^2 y = 6000} \; (1) \quad \& \quad \boxed{xy^2 = 12000} \; (2)$$

$$\frac{x^2 y}{xy^2} = \frac{6000}{12000} \Rightarrow \frac{x}{y} = \frac{1}{2}$$

The total cost (C_T) of an equipment in terms of the operation variables x and y is

$$C_T = 2x + \frac{12000}{xy} + y + 5$$

The optimal value of C_T, rounded to 1 decimal place, is ______.

i.e. $x : y = 1 : 2$; $x = k$ (let), $y = 2k$

So By (1), $(k)^2 (2k) = 6000 \Rightarrow k^3 = 3000$

$k = 14.4$; $x = k = 14.4$, $y = 2k = 28.8$

$$z = 2(14.4) + \frac{12000}{(14.4)(28.8)} + 28.8 + 5$$

$$= 91.5$$

Thank
THANK

Keep Hustling!