GATE

CRASH COURSE

ALL BRANCHES

Engineering Mathematics

Differential Equation (Part 01)
(Lec 10)

By - Dr. Puneet Sharma Sir

Topics *to be covered*

DIFFERENTIAL EQUATIONS (Part 1)

1. order / Degree
2. Solving Methods of 1^{st} order D Eq
   - → (i) Variable separable
   - → (ii) L.D Eq of 1^{st} order (IF)
   - → (iii) Homog D Eq
   - → (iv) Exact D Eq
5. → (V) observation

D.Eqn → O.D.Eq → Consist only O-Derivatives

P.D.Eq → " " Partial "

order of D Eqn → The highest order Derivative occurring in a DEq is called it's order

Degree of D Eqn → Degree is the exponent of highest order Derivative when D.Eq is represented in the polynomial form in terms of Derivatives

eg
$$\frac{d^2y}{dx^2} = \sqrt{\left(\frac{dy}{dx}\right)^3 + y} \Rightarrow \boxed{\left(\frac{d^2y}{dx^2}\right)^{2} = \left(\frac{dy}{dx}\right)^{3} + y}$$

I   III

order = 2, Degree = 2, NL

Q: $y = px + \sqrt{ap^2+b^2}$ where $p = \frac{dy}{dx}$, a & b are const. ?

$$(y - px) = \sqrt{a^2p^2+b^2}$$

$$(y-px)^2 = a^2p^2+b^2$$

$$y^2 + p^2x^2 - 2xyp = a^2p^2+b^2$$

$$(x^2-a^2)p^2 - 2xyp + y^2 - b^2 = 0$$

$$(x^2-a^2)\left(\frac{dy}{dx}\right)^2 - 2xy\frac{dy}{dx} + y^2 - b^2 = 0$$

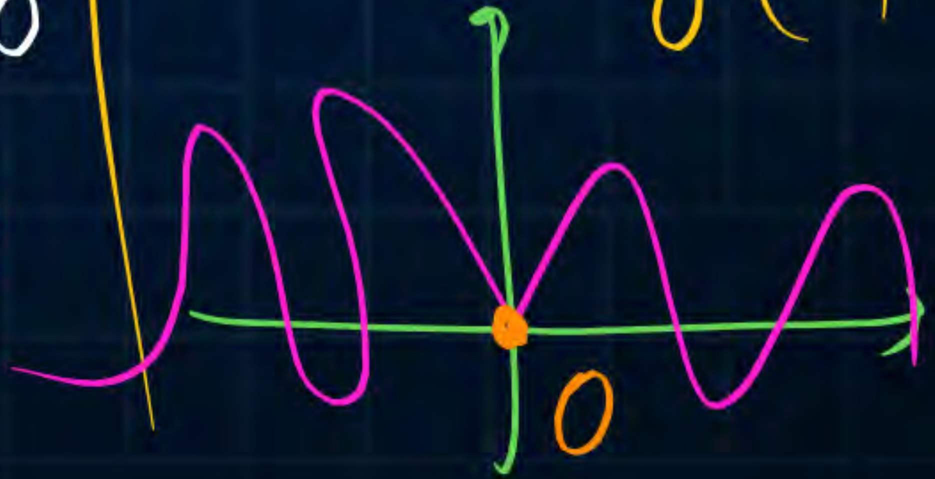
Order = 1, Degree = 2 NL

$fog(x) = f(g(x)) = ?$

$f(x) = |x|$, $\boxed{g(x) = \sin x}$

$gof(x) = g(f(x))$

$= g(|x|) = \sin|x|$



Formation of D Eqn's

$\boxed{y = c_1 x + c_2 x^2}$; c_1, c_2 A. Const. — (1)

$y' = c_1 + 2c_2 x \Rightarrow y' = c_1 + xy''$

$y'' = 2c_2$

$c_2 = \frac{1}{2} y''$ $\qquad$ $c_1 = y' - xy''$

$y = (y' - xy'')x + \frac{1}{2} y''(x^2)$

$y = xy' - x^2 y'' + \frac{x^2}{2} y''$

$\frac{x^2}{2} y'' - xy' + y = 0 \Rightarrow \boxed{x^2 y'' - 2xy' + 2y = 0}$ — (2)

Linear

O = 2

Deg = 1

eg check which of the following is not a sol of $\boxed{x^2 y'' - 2xy' + 2y = 0}$

(a) $y = Ax + Bx^2$

(b) $y = 3x^2$

(c) $y = 2x - 5x^2$

(d) $y = 2x + x^3$ ✓

(b) $y' = 6x, y'' = 6$

$x^2(6) - 2x(6x) + 2(3x^2) = 0$

$6x^2 - 12x^2 + 6x^2 = 0$

$12x^2 - 12x^2 = 0$

$0 = 0$

☺

Conclusions: ① No of A Const in G. Sol = order of D Eq.

② To form a D Eq we can diff it → Easy task

③ there should not be any A Const in D. Eq. (T)

④ there should not be any Derivative in the Sol of D Eq. (T)

⑤ Solution of D Eqⁿ: The Relationship b/w y & x satisfying the given D Eq is called Sol.

Types:
- G. Sol. → Consist A. Const
- P. Sol. → it has No A. Const.

⑥ To find the Sol of D Eq we should integrate it → Tough task

The order and degree of the differential equation

$$\frac{d^3y}{dx^3} + 4\sqrt{\left(\frac{dy}{dx}\right)^3 + y^2} = 0$$ are respectively

(a) 3 and 2 (b) 2 and 3

(c) 3 and 3 (d) 3 and 1

order = 3, Degree = 2

eg $\frac{dy}{dx} + 4y^2 = \sin\left(\frac{dy}{dx}\right)$?? $\Rightarrow$ O = 1, Degree = N.D

$\Rightarrow \frac{dy}{dx} + 4y^2 = \frac{dy}{dx} - \frac{(dy/dx)^3}{3!} + \frac{(dy/dx)^5}{5!} + \cdots$

Which one of the following differential equations has a solution given by the function

$y = 5\sin\left(3x + \frac{\pi}{3}\right)$

(a) $\frac{dy}{dx} - \frac{5}{3}\cos(3x) = 0$

(b) $\frac{dy}{dx} + \frac{5}{3}\cos(3x) = 0$

(c) $\frac{d^2y}{d^2x} + 9y = 0$ ✓

(d) $\frac{d^2y}{d^2x} - 9y = 0$

$$y = 5\sin\left(3x + \frac{\pi}{3}\right) \longleftarrow (1)$$

$$\frac{dy}{dx} = 5\cos\left(3x + \frac{\pi}{3}\right)(3)$$

$$\frac{d^2y}{dx^2} = 15\left[-\sin\left(3x + \frac{\pi}{3}\right)\cdot(3)\right]$$

$$= -45\sin\left(3x + \frac{\pi}{3}\right)$$

$$= -9\left[5\sin\left(3x + \frac{\pi}{3}\right)\right]$$

$$y'' = -9y \Rightarrow y'' + 9y = 0$$

Which of the following is a solution to the differe[ntial]

equation $\frac{dx(t)}{dt} + 3x(t) = 0$?

(a) $x(t) = 3e^{-t}$ ✗

(b) $x(t) = 2e^{-3t}$ ✓

(c) $x(t) = \frac{-3}{2}t^2$ ✗

(d) $x(t) = 3t^2$ ✗

Let us take

$x = 2e^{-3t}$

$\frac{dx}{dt} = 2(-3e^{-3t}) = -3(2e^{-3t}) = -3x$

$\frac{dx}{dt} + 3x = 0$ ie (b) ✓

M-II

$\frac{dx}{dt} = -3x$

$\int \frac{dx}{x} = -3\int dt + C_1$

$\log x = -3t + \log C$

$\log_e\left(\frac{x}{C}\right) = -3t$

$\frac{x}{C} = e^{-3t}$

$\boxed{x = Ce^{-3t}}$

The solution for the following differential equation with boundary conditions $y(0) = 2$ and $y^1(1) = -3$ is? Where $\frac{d^2y}{dx^2} = 3x - 2$

(a) $y = \frac{x^3}{3} - \frac{x^2}{2} = 3x - 2$

(b) $y = 3x^3 - \frac{x^2}{2} - 5x + 2$

(c) $y = \frac{x^3}{2} - x^2 - \frac{5x}{2} + 2$

(d) $y = x^3 - \frac{x^2}{2} + 5x + \frac{3}{2}$

$$\frac{d^2y}{dx^2} = 3x - 2$$

$$\int\left(\frac{d^2y}{dx^2}\right)dx = \int(3x-2)\,dx + C_1$$

$$\frac{dy}{dx} = \frac{3x^2}{2} - 2x + C_1$$

$$\int\left(\frac{dy}{dx}\right)dx = \int\left(\frac{3}{2}x^2 - 2x + C_1\right)dx + C_2$$

$$y = \frac{x^3}{2} - x^2 + C_1x + C_2$$

Gate

Consider the following second order linear differential equation $\int \frac{d^2y}{dx^2} = \int -12x^2 + 24x - 20$

The boundary condition are at $x = 0$, $y = 5$ and at $x = 2$, $y = 21$

The value of y at $x = 1$ is ______ 18

Solve: $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$

$$\frac{dy}{dx} + y + \int y\,dx = C$$

$$\frac{dy}{dx} + y + (\,??\,) = C$$

V.V.V. Imp Point

$$y' = -4x^3 + 12x^2 - 20x + c_1$$

$$\boxed{y = -x^4 + 4x^3 - 10x^2 + c_1 x + c_2} \text{ — (1)}$$

$y(0) = 5 \Rightarrow$

$y(2) = 21 \Rightarrow 21 = -16 + 32 - 40 + 2c_1 + c_2$

$21 = -14 + c_2 \Rightarrow$

$y(1) = ? = 18$

VARIABLE-SEPARABLE Method

The general solution of the differential equa

$\frac{dy}{dx}=\frac{1+\cos 2y}{1-\cos 2x}$ is

(a) $\tan y-\cot x=c$ (c is a constant)

(b) $\tan x-\cot y=c$ (c is a constant)

(c) $\tan y+\cot x=c$ (c is a constant)

(d) $\tan x+\cot y=c$ (c is a constant)

$$\frac{dy}{dx}=\frac{2\cos^2 y}{2\sin^2 x}$$

$$\int\frac{1}{\cos^2 y}dy=\int\frac{1}{\sin^2 x}dx+c$$

$$\tan y=-\cot y+C$$

$$\cos 2A=\begin{cases}2\cos^2 A-1\\ 1-2\sin^2 A\\ \cos^2 A-\sin^2 A\end{cases}$$

Match each differential equation in Group I to its family of solution curves from Group II

Group I	Group II
A. $\frac{dy}{dx} = \frac{y}{x}$	1. Circles
B. $\frac{dy}{dx} = \frac{-y}{x}$	2. Straight lines
C. $\frac{dy}{dx} = \frac{x}{y}$	3. Hyperbolas
D. $\frac{dy}{dx} = \frac{-x}{y}$	

(a) A-2, B-3, C-3, D-1

(b) A-1, B-3, C-2, D-1

(c) A-2, B-1, C-3, D-3

(d) A-3, B-2, C-1, D-2

$\frac{dy}{dx} = \frac{y}{x}$

$\frac{1}{y} dy = \frac{dx}{x}$

$\log y = \log x + \log C$

$\log\left(\frac{y}{x}\right) = \log C$

$y = Cx$

straight lines

$\frac{dy}{dx} = \frac{-y}{x}$

$+\int \frac{1}{y} dy = -\int \frac{1}{x} dx + \log C$

$+\log y = -\log x + \log C$

$xy = C$ Rectangular Hyperbola

(C) $\frac{dy}{dx} = \frac{x}{y} \Rightarrow \int y\, dy = \int x\, dx + C$

$\frac{y^2}{2} = \frac{x^2}{2} + C$

$-\frac{x^2}{2} + \frac{y^2}{2} = C$

The general solution of the differential equation $\frac{dy}{dx} = \cos(x+y)$, with c as a constant, is

(a) $y + \sin(x+y) = x + c$

(b) $\tan\left(\frac{x+y}{2}\right) = y + c$

(c) $\cos\left(\frac{x+y}{2}\right) = x + c$

(d) $\tan\left(\frac{x+y}{2}\right) = x + c$

Put $x+y=t \Rightarrow 1+\frac{dy}{dx}=\frac{dt}{dx}$

$\because \frac{d}{dx}(x+y)=\frac{d}{dx}(t)$

$1+\frac{dy}{dx}=\frac{dt}{dx}$

$\frac{dy}{dx}=\cos(x+y)$

$\left(\frac{dt}{dx}-1\right)=\cos(t)$

$\frac{dt}{dx}=1+\cos t$

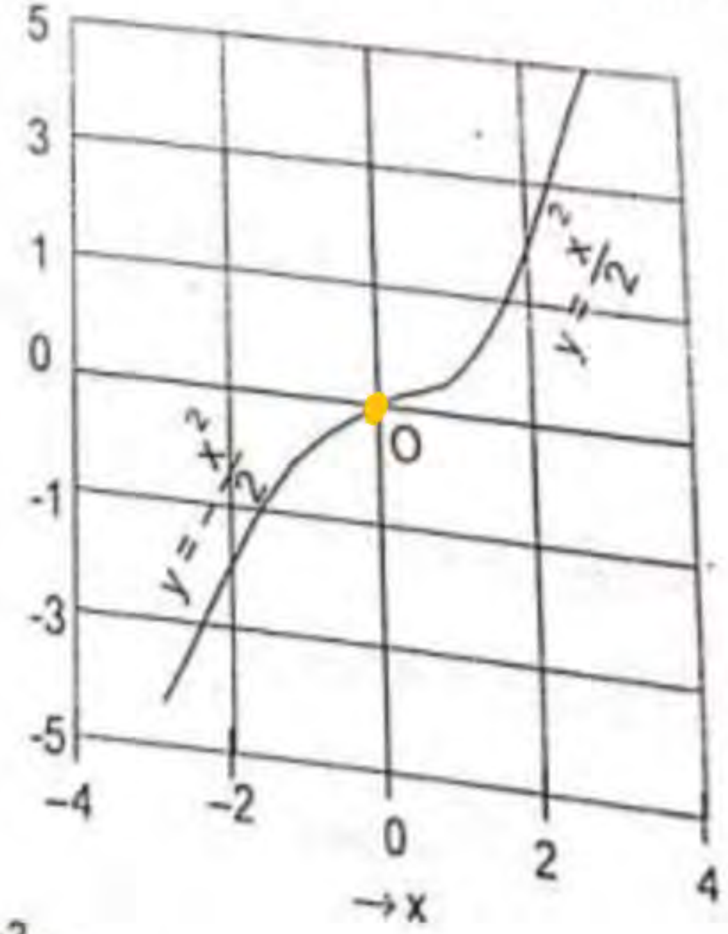
$\frac{dt}{dx}=2\cos^2\frac{t}{2}$

$\int \sec^2\frac{t}{2}\,dt = 2\int dx + c_1$

$\frac{\tan\frac{t}{2}}{1/2}=2x+c_1$

$\tan\left(\frac{x+y}{2}\right)=x+c$

The figure shows the plot of y as a function of x. The function shown is the solution of the differential equation (assuming all initial conditions to be zero) is



(a) $\frac{d^2y}{dx^2} = 1$

(b) $\frac{dy}{dx} = +x$

(c) $\frac{dy}{dx} = -x$

(d) $\frac{dy}{dx} = |x|$

$$y = \begin{cases} -\frac{x^2}{2}, & x < 0 \\ \frac{x^2}{2}, & x > 0 \end{cases}$$

$$\frac{dy}{dx} = \begin{cases} -x, & x < 0 \\ x, & x > 0 \end{cases} \Rightarrow \boxed{\frac{dy}{dx} = |x|}$$

LINEAR D.Eq of 1^{st} order

Type I: it's G. Form is $\boxed{\dfrac{dy}{dx} + Py = Q}$ — (1) where P & Q are func^n of $\underline{x}$ alone

it is LDE in $\underline{y}$ & $\underline{x}$

& it's I.F $= e^{\int P\,dx}$ & it's solution is

$$\boxed{y(\text{IF}) = \int Q(\text{IF})\,dx + C}$$

Type II: it is given as $\dfrac{dx}{dy} + Px = Q$ where P & Q are func^n of $\underline{y}$ alone

it is LDE in $\underline{x}$ & $\underline{y}$

& $\text{IF} = e^{\int P\,dy}$ & Sol is $\boxed{x(\text{IF}) = \int Q(\text{IF})\,dx + C}$

Bernoullie D Equⁿ – it can be converted into linear form. [let the soln is $y = f(x)$]

Non linear D Eq
① If degree is more than 1
② If exponent of y is other than one
③ If exponent of any derivative is other than one
④ If product of y & it's derivative exist.

Linear D Eq – when ant D Eq is free from all the above 4 properties then it is called L D Eq.

$$\left(\frac{d^3x}{dt^3}\right) + 2x^5 + x\frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^2 = 4t^3$$

$x = f(t)$

O = 3, Dg = 1, NL

The general solution of the differential equation

$$\frac{dy}{dx} + x\sin 2y = x^3\cos^2 y \text{ is}$$

(a) $\tan y = \frac{1}{2}(x^2 - 1) + c\cdot e^{-x^2}$

(b) $\tan y = (x^2 - 2) + c\cdot e^{-x^2}$

(c) $\tan y = (x^2 - 1) + c\cdot e^{-x^2}$

(d) $\cot y = \frac{1}{2}(x^2 - 1) + c\cdot e^{x^2}$

$\frac{dy}{dx} + x\sin 2y = x^3\cos^2 y$ — (1) N.L

$\frac{1}{\cos^2 y}\frac{dy}{dx} + \frac{x\cdot 2\sin y\cos y}{\cos^2 y} = x^3$

$\sec^2 y\frac{dy}{dx} + 2x(\tan y) = x^3$

Put $\tan y = u \Rightarrow \sec^2 y\frac{dy}{dx} = \frac{du}{dx}$

$\frac{du}{dx} + 2x(u) = x^3$

i.e. $\boxed{\frac{du}{dx} + (2x).u = (x^3)}$ — (2)

it is LDE in $\underline{u}$ & $\underline{x}$

$IF = e^{\int P dx} = e^{x^2}$

Sol is $u(e^{x^2}) = \int (x^3)(e^{x^2})dx + C$

$$u(e^{x^2}) = \int x^3 \cdot e^{x^2} dx + c$$

$$x^2 = t \Rightarrow x\,dx = \frac{dt}{2}$$

$$u(e^{x^2}) = \int x^2 \cdot e^{x^2} \cdot (x\,dx) + c$$

$$= \int t \cdot e^t \cdot \frac{dt}{2} + c$$

$$u(e^t) = \frac{1}{2}(te^t - e^t) + c$$

$$u = \frac{t}{2} - \frac{1}{2} + ce^{-t} = \frac{x^2}{2} - \frac{1}{2} + ce^{-x^2}$$

The general solution of $(x^3y^2 + xy)\frac{dx}{dy} = 1$ is

(a) $\frac{-1}{y} = x^2 - 2 + c \cdot e^{-x^2/2}$

(b) $\frac{1}{y} = x^2 + 2 + c \cdot e^{-x^2/2}$

(c) $\frac{1}{y} = x^2 + 2 + c \cdot e^{x^2/2}$

(d) $\frac{1}{y} = x^2 + 1 + c \cdot e^{-x^2/2}$

$\frac{dy}{dx} = x^3y^2 + xy$ — (1)

$\boxed{\frac{dy}{dx} - xy = x^3y^2}$ N.L

$\frac{1}{y^2}\frac{dy}{dx} - x\left(\frac{1}{y}\right) = x^3$ — (2)

Put $\frac{1}{y} = u \Rightarrow -\frac{1}{y^2}\frac{dy}{dx} = \frac{du}{dx}$

So By (2) $\left(-\frac{du}{dx}\right) - x(u) = x^3$

$\boxed{\frac{du}{dx} + (x)u = (-x^3)}$ — (3) linear

$IF = e^{\int x\,dx} = e^{\frac{x^2}{2}}$

Sol is $u\left(e^{\frac{x^2}{2}}\right) = \int (-x^3)\left(e^{\frac{x^2}{2}}\right)dx + c$

The solution of the differential equation $x^2\frac{dy}{dx}+2xy-x+1=0$ given that at $x=1$, $y=0$ is — (1)

(a) $\frac{1}{2}-\frac{1}{x}+\frac{1}{2x^2}$ (b) $\frac{1}{2}-\frac{1}{x}-\frac{1}{2x^2}$

(c) $\frac{1}{2}+\frac{1}{x}+\frac{1}{2x^2}$ (d) $-\frac{1}{2}+\frac{1}{x}+\frac{1}{2x^2}$

$$\frac{dy}{dx}+\left(\frac{2}{x}\right)y=\frac{x-1}{x^2} \quad \text{— (1)}$$

$$P=\frac{2}{x},\ Q=\frac{x-1}{x^2}$$

$$IF=e^{\int P\,dx}=e^{\int\frac{2}{x}dx}=e^{2\ln x}=e^{\ln(x^2)}=x^2$$

Hence G.Sol is, $y(IF)=\int Q(IF)\,dx+C$

$$y(x^2)=\int\left(\frac{x-1}{x^2}\right)(x^2)\,dx+C$$

$$y(x^2)=\frac{x^2}{2}-x+C$$

$$\boxed{y=\frac{1}{2}-\frac{1}{x}+\frac{C}{x^2}} \quad \text{— (2)}$$

$$y(1)=0 \Rightarrow 0=\frac{1}{2}-1+C \Rightarrow C=\frac{1}{2}$$

$$y=\frac{1}{2}-\frac{1}{x}+\frac{1}{2x^2}$$

The solution of the differential equation

$$(1+y^2)dx = (\tan^{-1}y - x)dy \quad (1)$$

is

(a) $x = \tan^{-1}y + 1 + ce^{-\tan^{-1}y}$

(b) $x = \tan^{-1}y - 1 + ce^{-\tan^{-1}y}$

(c) $x = \frac{1}{2}\tan^{-1}y - 1 + ce^{-\tan^{-1}y}$

(d) $x = \frac{1}{2}\tan^{-1}y + 1 + ce^{-\tan^{-1}y}$

$$\frac{dx}{dy} = \frac{\tan^{-1}y - x}{1+y^2}$$

$$\frac{dx}{dy} = \frac{\tan^{-1}y}{1+y^2} - \frac{x}{1+y^2}$$

$$\frac{dx}{dy} + \underbrace{\left(\frac{1}{1+y^2}\right)}_{=P}x = \underbrace{\left(\frac{\tan^{-1}y}{1+y^2}\right)}_{=Q}$$

$$IF = e^{\int P\,dy} = e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1}y}$$

Sol of (1) is $x(IF) = \int Q(IF)\,dy + c$

$$x\left(e^{\tan^{-1}y}\right) = \int \left(\frac{\tan^{-1}y}{1+y^2}\right) \cdot e^{\tan^{-1}y}\,dy$$

$$x(e^t) = \int t \cdot e^t\,dt + c$$

$$xe^t = te^t - e^t + c \Rightarrow x = t - 1 + ce^{-t}$$

Consider the differential equation $(t^2 - 81)\frac{dy}{dt} + 5t\,y = \sin(t)$ with $y(1) = 2\pi$. There exists a unique solution for this differential equation when t belongs to the interval

(a) $(-2, 2)$ (b) $(-10, 10)$

(c) $(-10, 2)$ (d) $(0, 10)$

$$t^2 - 81 = 0 \Rightarrow t = \pm 9$$

$$\frac{dy}{dt} + \underbrace{\left(\frac{5t}{t^2-81}\right)}_{=P} y = \underbrace{\frac{\sin t}{t^2-81}}_{=Q} \text{ --- } (1)$$

As of Now, it is LDE in y & t

IF = ?

G. Sol = ?

B. Cond^n = ?

then we will get ans.

Time taking

check.

Thank

THANK

Keep Hustling!