GATE

DATA SCIENCE + CS & IT

Engineering Mathematics

SUPER 1500

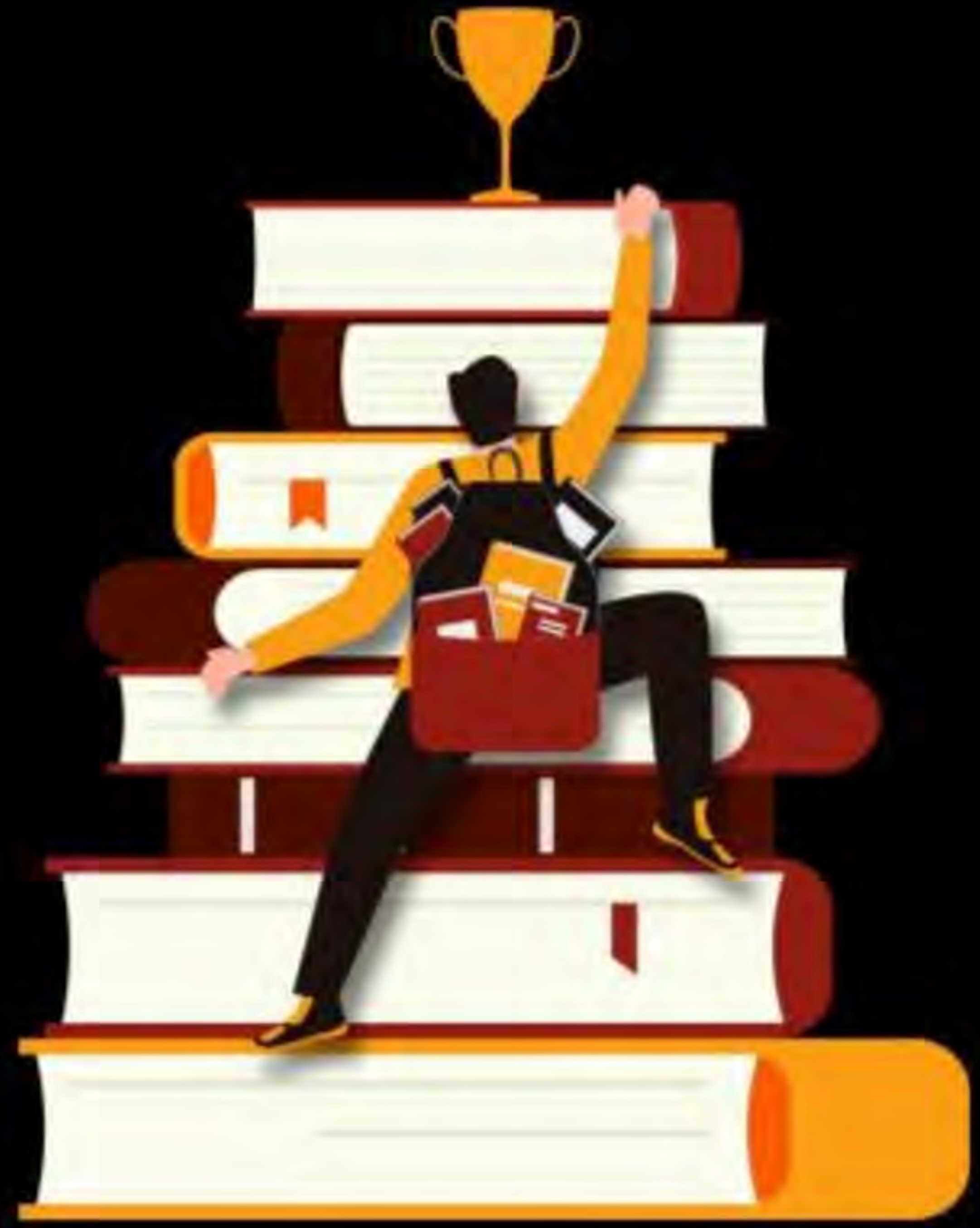
Lec : 02 Linear - 1

By - Dr. Puneet Sharma Sir

Topics *to be covered*

LINEAR ALGEBRA

1. Types of MATRICES
2. RANK
3. VECTORS



Types of MATRICES

(1) Singular Mat $\rightarrow$ if $|A| = 0$

(2) Non Sing Mat $\rightarrow$ if $|A| \neq 0$

(3) Invertible Mat $\rightarrow$ if A^{-1} exist.

& N. Condⁿ is $|A| \neq 0$ & Defⁿ is

$$A^{-1} = \frac{\operatorname{adj} A}{|A|} = \frac{(\operatorname{cof} A)^T}{|A|}$$

(4) Symm. Mat: if $A^T = A$

(5) Skew Symm: if $A^T = -A$

(6) Hermitian: if $A^\theta = A$

(7) Skew Herm: if $A^\theta = -A$

(8) Tranjugate Mat: $A^\theta = (\overline{A^T})$

(9) Orthogonal Mat: if $AA^T = I$ or $(A^{-1} = A^T)$

(10) Unitary Mat: if $AA^\theta = I$ or $(A^{-1} = A^\theta)$

(11) Real Mat: if $\bar{A} = A$ or $(A^\theta = A^T)$

(12) Idempotent: if $A^2 = A$

(13) Involutary: if $A^2 = I$

(14) Nilpotent: if $A^k = 0$

(15) U.T.M: if $A = [a_{ij}]$ st $a_{ij} = 0 \ \forall\ i > j$

(16) L.T.M: if $A = [a_{ij}]$ st $a_{ij} = 0 \ \forall\ i < j$

(17) Diag Mat:

$$a_{ij} = \begin{cases} 0 & , \forall\ i \neq j \\ \text{at least one element is Non Zero} & , \forall\ i = j \end{cases}$$

(18) Scalar: $a_{ij} = \begin{cases} 0, & i \neq j \\ k, & i = j \end{cases}$

(19) $\operatorname{Trace}(A) = \sum a_{ii}$

An n × n array V is defined as follows v[i, j] = i − j for all i, j, 1 ≤ i, j ≤ n then the sum of the elements of the array V is

(a) 0 (b) n − 1

(c) $n^2 - 3n + 2$ (d) n (n + 1)

$V_{ij} = i - j \ \forall\ i \neq j$, let $n = 3$

$$V = [V_{ij}]_{3\times3} = \begin{bmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{bmatrix}_{3\times3}$$

$$= \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix} \Rightarrow \sum_{i=1}^{3} V_{ij} = 0$$

Note: Sum of all the elements in skew symm Mat = 0

The number of different n × n symmetric matrices with each elements being either 0 or 1 is

(a) $2^n \Rightarrow 2^2 = 4$ ✗

(b) $2^{n^2} = 2^{2^2} = 2^4 = 16$ ✗

(c) $2^{\frac{n^2+n}{2}} = 2^{\frac{4+2}{2}} = 2^3 = 8$ ✓

(d) $2^{\frac{n^2-n}{2}}$ $\quad 2^{\frac{4-2}{2}} = 2^1$ ✗

M-II

Consider $A_{2\times2} = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$

No. of places to fill $= \begin{bmatrix} \text{two ways} & \text{two ways} \\ ? & \text{two ways} \end{bmatrix}$

$= 2\times2\times2 = 8$ ways.

M-I W.K. that, Max No. of different elements required to construct Symm Mat of $n\times n = \frac{n(n+1)}{2}$

ie No. of places in Mat that have to be filled by 0 & 1 $= \frac{n(n+1)}{2}$

Req Ans $= \underset{1^{st}}{2} \times \underset{2^{nd}}{2} \times \underset{3^{rd}}{2} \times \underset{4^{th}}{2} \cdots\cdots \underset{}{2} \times \underset{\left(\frac{n(n+1)}{2}\right)^{th}}{2} = 2^{\frac{n(n+1)}{2}}$

The number of linearly independent entries in a skew symmetric matrix of order n equals.

different

(a) $n = 3$?

(b) $\frac{n(n+1)}{2} = \frac{3\times 4}{2} = 6$

(c) $\frac{n(n-1)}{2} = 3$

(d) $n^2 - 1 = 9 - 1 = 8$

M-I No. of elements in any sq. Mat $A_{n\times n} = n^2$

But for skew symm $A = \begin{bmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ -\beta & -\gamma & 0 \end{bmatrix}_{3\times 3}$

No. of Ind entries in skew symm Mat of $n\times n = \frac{\text{Total Entries} - \text{Diag Entries}}{2} = \frac{n^2 - n}{2}$

M-II $A_{3\times 3} = \begin{bmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ -\beta & -\gamma & 0 \end{bmatrix}$

i.e. 3 Ind entries are required for 3×3 skew symm Mat

let $A = [\ \ ]_{4\times 4} \Rightarrow$ No. of diff entries $= 6$

If A and B are two symmetric matrices. Then consider the following staements.

(i) A + B is symmetric (T)

(ii) AB is symmetric (F)

(iii) AB + BA is symmetric (T)

(a) Only (i) is true

(b) (i) and (ii) are true

(c) (i) and (iii) are true

(d) (i), (ii) and (iii) are true.

??

A & B, $A^T = A$ & $B^T = B$ (given)

Now,

(i) $(A+B)^T =$

(ii) $(AB)^T = B^T A^T = BA \neq AB$ (in general)

??

$(AB+BA)^T = B^T A^T + A^T B^T$

$= BA + AB$

$= AB + BA$

A, B be any $n \times n$ matrices. Then choose incorrect statement.

(a) AA^T is symmetric (T)

(b) $A + A^T$ is symmetric (T)

(c) A^TA is symmetric (T)

(d) $A - A^T$ symmetric (F)

W.K. that If A is any Mat → $(AA^T)_{n\times n}$ is symm, $(A^TA)_{n\times n}$ is symm.

& $(A-A^T)^T = A^T - (A^T)^T = A^T - A$

$= -(A-A^T)$

Skew symm.

Q: if $A_{m\times n}$ then → AA^T is symm, A^TA is symm

(a) Both T (b) Both False

(c) 1^{st} T, 2^{n} F (d) 1^{st} F & 2^{nd} T

$A_{m\times n}$ then $A^T_{n\times m}$ → $(AA^T)_{m\times m}$, $(A^TA)_{n\times n}$

Although order are different But both are symm.

If T is Idempdotent then $T^k = T$ for $\Rightarrow T^2 = T$

(a) $k = 2$

(b) All integer k

(c) All positive integer $k \geq 2$ ✓

(d) All of the above

$\because T^2 = T$

$\Rightarrow T^3 = T^2 \cdot T = T \cdot T = T^2 = T$

& $T^4 = T^3 \cdot T = T \cdot T = T^2 = T$

For the given orthogonal matrix Q.

$$Q = \begin{bmatrix} 3/7 & 2/7 & 6/7 \\ -6/7 & 3/7 & 2/7 \\ 2/7 & 6/7 & -3/7 \end{bmatrix}$$

The inverse is

(a) $\begin{bmatrix} 3/7 & 2/7 & 6/7 \\ -6/7 & 3/7 & 2/7 \\ 2/7 & 6/7 & -3/7 \end{bmatrix}$

(b) $\begin{bmatrix} -3/7 & -2/7 & -6/7 \\ 6/7 & -3/7 & -2/7 \\ -2/7 & -6/7 & 3/7 \end{bmatrix}$

(c) $\begin{bmatrix} 3/7 & -6/7 & 2/7 \\ 2/7 & 3/7 & 6/7 \\ 6/7 & 2/7 & -3/7 \end{bmatrix}$

(d) $\begin{bmatrix} -3/7 & -6/7 & -2/7 \\ -2/7 & -3/7 & -6/7 \\ -6/7 & -2/7 & 3/7 \end{bmatrix}$

For O-Mat: $QQ^T = I \Rightarrow Q^{-1} = Q^T$

A matrix $A = [a_{ij}]_{n \times n}$ is said to be lower triangular if

(a) $a_{ij} = 0$ for $i > j$
(b) $a_{ij} = 0$ for $i < j$
(c) $a_{ij} = 0$ for $i \geq j$
(d) $a_{ij} = 0$ for $i \leq j$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \text{L.T.M}$$

Note:- for upper Diag elements $i < j$ & for LTM $a_{ij} = 0 \ \forall \ i < j$
for lower " " $i > j$ & for U.T.M $a_{ij} = 0 \ \forall \ i > j$
for Diag elements $i = j$
for off Diag elements $i \neq j$

Find all the matrices that commute with the matrix

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = A$

(a) $B = \frac{1}{2}\begin{bmatrix} 2a & +2b \\ 3b & 2a+3b \end{bmatrix}$

(b) $B = -\frac{1}{2}\begin{bmatrix} -2a & a \\ -3b & a+b \end{bmatrix}$

(c) $B = \frac{1}{2}\begin{bmatrix} a & a+b \\ b & b \end{bmatrix}$

(d) $B = \frac{1}{2}\begin{bmatrix} b & a+c \\ a & a+b \end{bmatrix}$

$AB = BA$

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$\begin{bmatrix} (a+2c) & (b+2d) \\ (3a+4c) & (3b+4d) \end{bmatrix} = \begin{bmatrix} (a+3b) & (2a+4b) \\ (c+3d) & (2c+4d) \end{bmatrix}$

$\Rightarrow a+2c = a+3b$ & $b+2d = 2a+4b$

$2c = 3b$ & $2d = 2a+3b$

$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2a & 2b \\ 2c & 2d \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2a & 2b \\ 3b & (2a+3b) \end{bmatrix}$

i.e (a) is correct

RANK OF MATRICES

Def^n (1) $\rho(A)_{6\times 7} = 3$

- $\exists$ at least one Non Sing submatrix of 3×3
- Every Sq. submatrix of 4×4, 5×5, 6×6 are singular

Def^n (2) $\rho(A_{6\times 7}) = 3$

- $\exists$ at Most 3 LI Rows
- $\exists$ at most 3 LI Columns

(*) ρ(Echelon form) = No. of Non Zero Rows.

If for a matrix, rank equals both the number of rows and number of columns, then the matrix is called

(a) non-singular (b) singular

(c) transpose (d) minor

ATQ, $\boxed{\rho(A) = \text{No. of Rows} = \text{No of Columns}}$ learn

Let $A_{3\times3}$ $\left\{\begin{array}{l}\text{No. of Rows} = 3\\ \text{No of Column} = 3\end{array}\right\} \Rightarrow \rho(A) = 3 \Rightarrow |A| \neq 0$

i.e. A must be Non sing.

The rank of the matrix $\begin{bmatrix} 1 & 4 & 8 & 7 \\ 0 & 0 & 3 & 0 \\ 4 & 2 & 3 & 1 \\ 3 & 12 & 24 & 2 \end{bmatrix}$

(a) 3 (b) 1

(c) 2 (d) 4

$$\xrightarrow[R_4 \to R_4 - 3R_1]{R_3 \to R_3 - 4R_1} \begin{pmatrix} 1 & 4 & 8 & 7 \\ 0 & 0 & 3 & 0 \\ 0 & -14 & -29 & -27 \\ 0 & 0 & 0 & -19 \end{pmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & 4 & 8 & 7 \\ 0 & -14 & -29 & -27 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -19 \end{pmatrix} = \text{Echelon form}$$

$\rho(A) = 4$

M-II Expanding along R_2

$$|A| = -3 \begin{vmatrix} 1 & 4 & 7 \\ 4 & 2 & 1 \\ 3 & 12 & 2 \end{vmatrix} = -3\{1(4-12) - 4(8-3) + 7(48-6)\}$$

$$= -3\{-8 - 20 + 42 \times 7\}$$

$\neq 0$ Non-sing $\Rightarrow \rho(A) = 4$

The rank of 4 × 4 skew-symmetric matrix

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & -1 & 0 \end{bmatrix}_{4\times 4}$$ is ______.

$=0 \quad =0 \quad =0 \quad \neq 0$

$\rho(A) = 4 \quad \because |A| = \text{perfect sq.} \neq 0$ i.e. non-singular

for skew symm Mat $A_{n\times n}$

$$|A| = \begin{cases} 0 & , n = \text{odd} \\ \text{perfect sq.} & , n = \text{even} \end{cases}$$

HW: Try to find it's $|A| = ?$

$\boxed{|A| = 4}$

The rank of a 3×3 matrix C (= AB), found by multiplying a non-zero column matrix A of size 3×1 and a non-zero row matrix B of size 1×3, is

(a) 0
(b) 1
(c) 2
(d) 3

$A = \begin{bmatrix} - \\ - \\ - \end{bmatrix}_{3\times 1}$, $B = \begin{bmatrix} - & - & - \end{bmatrix}_{1\times 3}$

$\rho(A) = 1 \qquad \rho(B) = 1$

$AB = \begin{bmatrix} \\ \\ \end{bmatrix}\begin{bmatrix} & & \end{bmatrix} = \begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix}_{3\times 3}$

$\boxed{\rho(AB) = 1}$

$\because \rho(AB) \le \min\{\rho(A), \rho(B)\}$
$\le \min\{1, 1\}$
$= 1$

Let $P = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$ and $Q = \begin{bmatrix} -1 & -2 & -1 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{bmatrix}$ be two matrices. Then the rank of P + Q is______.

WRONG APP

$\cancel{\rho(P+Q) = \rho(P) + \rho(Q)}$

$= ??$

But $\rho(A+B) \le \rho(A) + \rho(B)$

$P+Q = \begin{bmatrix} 0 & -1 & -2 \\ 8 & 9 & 10 \\ 8 & 8 & 8 \end{bmatrix}_{3\times 3}$ → Non sing of 2×2

$C_2 \to C_2 - C_1$
$C_3 \to C_3 - C_1$

then $|P+Q| = \begin{vmatrix} 0 & -1 & -2 \\ 8 & 1 & 2 \\ 8 & 0 & 0 \end{vmatrix}$

$= 8\left[-2+2\right] = 0$

$\because |P+Q| = 0 \Rightarrow \rho(P+Q) \neq 3$

So $\rho(P+Q) = 2$

The rank of the matrix $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$ is ____.

(a) 1
(b) 2
(c) 3
(d) 4

$$\xrightarrow[\substack{R_3 - R_2 \\ R_2 - R_1}]{R_4 - R_3} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \end{bmatrix} \xrightarrow[R_4 - R_2]{R_3 - R_2} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 4 & 4 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{4\times 4}$$

2×2 ⟹ Non sing Mat exist

$\rho(A) = 2$

If A and B are matrices of same order then

(a) $\rho(A + B) \leq p(A) + \rho(B)$ (Direct property)

(b) $\rho(A + B) \geq p(A) + \rho(B)$

(c) $\rho(A + B) = p(A) + \rho(B)$

(d) None of the above

The rank of a matrix A is defined as ______.

(a) The number of zero rows in A.

(b) The number of linearly dependent rows (or columns) in A.

(c) The number of linearly indepndent rows (columns) in A.

(d) The number of such columns which has been obtained by linear combination of some other columns in A............

The rank of the matrix $\begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 3 & 6 & 9 & 12 \\ 0 & 5 & 10 & 15 & 20 \end{bmatrix}$ is

______.

(a) 0 (b) 1

(c) 2 (d) 3

$$\xrightarrow[R_3 - 5R_1]{R_2 - 3R_1} \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\rho(A) =$ one

If $A = \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix}$ then which of the following is false.

(a) $\rho(A) = 1$; if $a = b = c$

(b) $\rho(A) = 2$; if $a = b \neq c$

(c) $\rho(A) = 3$; if $a \neq b \neq c$

(d) $\rho(A)$ is always 3

w.k.that

$$\boxed{|A| = (a-b)(b-c)(c-a)}$$

if $a = b = c \Rightarrow A = \begin{bmatrix} 1 & 1 & 1 \\ a & a & a \\ a & a & a \end{bmatrix} \Rightarrow \rho(A) = 1$

if $a = b \neq c \Rightarrow A = \begin{bmatrix} 1 & 1 & 1 \\ a & a & c \\ a^2 & a^2 & c^2 \end{bmatrix}$ → Non sing of 2×2 exist

$\because C_1 = C_2 \Rightarrow |A| = 0$, $\rho(A) \neq 3$

$\because \exists$ Non sing of $2\times2 \Rightarrow \rho(A) = 2$

(iii) if $a \neq b \neq c \Rightarrow |A| \neq 0$

i.e A is Non sing $\Rightarrow \rho(A) = 3$

Rank of a skew symmetric matrix cannot be

(a) 1 (b) 2

(c) 4 (d) 0

eg $A = \begin{bmatrix} 0 & 5 \\ -5 & 0 \end{bmatrix}_{2\times 2}$

= Non Sing.

$\rho(A) = 2$

eg $A = \begin{bmatrix} 0 & 5 \\ 0 & 0 \end{bmatrix}$

$|A| = 0$

$\rho(A) = 1$

$\because$ A is Not skew symm.

eg $A = \begin{bmatrix} 0 \end{bmatrix}_{1\times 1}$

$\rho(A) = 0$

LEARN By ❤

Nature of VECTORS

$x_1, x_2, x_3 \dots x_r$ are the given vectors then

Consider $A = [x_1 \; x_2 \; x_3 \dots x_r]$

General Method

(i) if $\rho(A) =$ No. of vectors $\Rightarrow$ vectors are LI

(ii) if $\rho(A) <$ " " $\Rightarrow$ vectors are LD

Tricky Method (applicable when A is sq. Mat)

(i) if $|A| \neq 0 \Rightarrow$ vectors are LI

(ii) if $|A| = 0 \Rightarrow$ " " LD

Consider the following two statements:

I. The maximum number of linearly independent column vectors of a matrix A is called the rank of A. (T)

II. If A is an $n \times n$ square matrix, it will be nonsingular if rank $A = n$. (T)

With reference to the above statements, which of the following applies?

(a) Both the statements are false

(b) Both the statements are true

(c) I is true but II is false

(d) I is false but II is true

$q_1, q_2, q_3 \ldots q_m$ are n-dimensional vectors with $m < n$. This set of vectors is linearly dependent. Q is the matrix with $q_1, q_2, q_3 \ldots q_m$ as the columns. the rank of Q is

(a) less than n (b) m (F)

(c) between m and n (d) n (e) None

$$q_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}, \quad q_2 = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix}_{n\times 1}, \quad q_3 = ? \ldots$$

$$A = Q = [q_1 \, q_2 \, q_3 \ldots q_m]_{n\times m}$$, $m < n$ & vectors are LD

$\Rightarrow$ By Prop of Rank, $\rho(Q) \leq \min\{n, m\}$

$\Rightarrow \rho(Q) \leq m$ But Vectors are LD so $\rho(Q) <$ No of vectors

ie $\rho(Q) < m$

ie $\rho(Q) < n$ (By Common Sense)

let $m = 3$ $\Rightarrow$ No. of vectors, $n = 4$, $A = \begin{bmatrix} & \end{bmatrix}_{4\times 3}$

$$\rho(A) \le 3 \quad \text{(By prop of Rank)}$$

But vectors are LD so $\rho(A) <$ No of vectors

$$\Rightarrow \boxed{\rho(A) < 3} \quad \text{Ans.}$$

So By Common Sense, $\rho(A) < 4 \; (= n)$

$S = \{x_1 \; x_2 \; x_3\}$

$= \begin{bmatrix} & & \end{bmatrix}_{4\times 3}$

$\rho(A) \le 3$

But for LD vectors

$\boxed{\rho(A) < 3} = m$

ie $\rho(A) < m$

or $\rho(A) < n$ also

Thank
THANK

Keep Hustling!