GATE

DATA SCIENCE + CS & IT

Engineering Mathematics

SUPER 1500

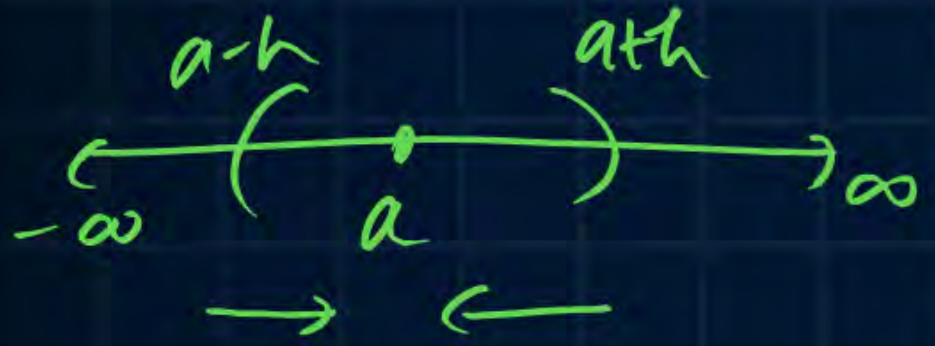
Lec : 02 Calculus

By - Dr. Puneet Sharma Sir

Topics *to be covered*

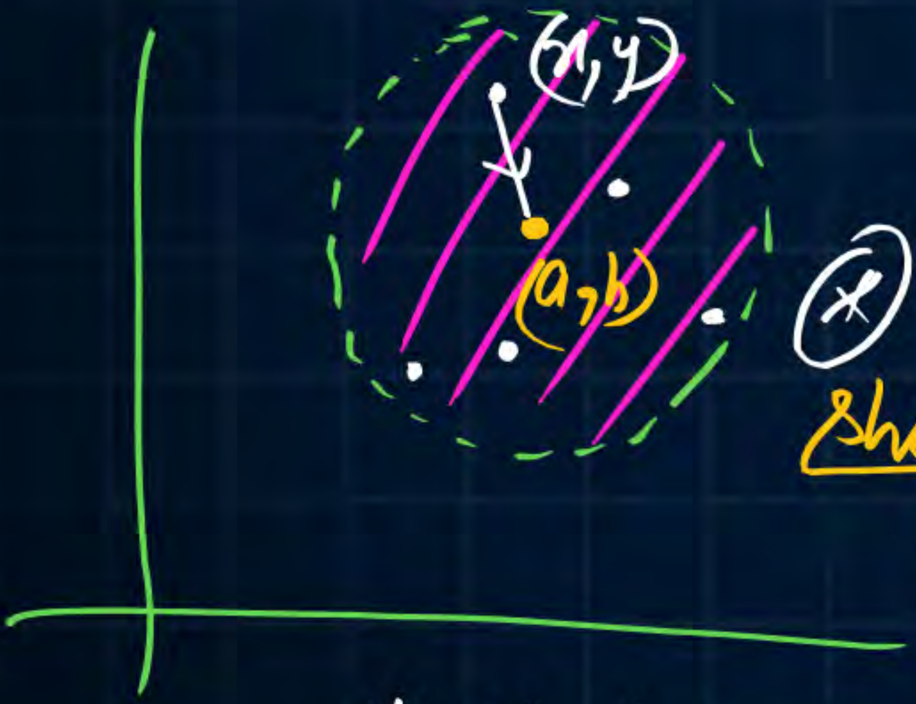
1. Double limit
2. Continuity
3. Differentiability
4. Taylor series & Maclaurin

Double limit Concept

① 

is Nbd of Real No of a means $(a-h, a+h)$

② Nbd of Point (a,b) means shaded Region in the Diag (i.e. Interior of circle)



if $(x,y) \to (a,b) \Rightarrow (x,y)$ lies in the Nbd of (a,b)

(*) for the existence of this limit, it must be unique all all paths

Shortcut :- Put $y = mx$ in the given Question.

(i) if limit depends upon m, we say that limit DNE

(ii) " " is free from m, " " " limit exist

this shortcut is useful when $(x,y) \to (0,0) \Rightarrow (x, mx) \to (0,0) \Rightarrow x \to 0$

Q. The value of $\lim\limits_{(x,y)\to(0,0)} \left(\dfrac{xy}{x^2+y^2}\right) = ?$ (a) 0 (b) $\frac{1}{2}$ (c) $\frac{1}{4}$ (d) DNE

Let $(x,y)\to(0,0)$ along $y=x$ then $(x,y)\to(0,0) \Rightarrow (x,x)\to(0,0)$ or $x\to 0$

$$\lim_{(x,y)\to(0,0)}\left(\frac{xy}{x^2+y^2}\right) = \lim_{x\to 0}\left(\frac{x^2}{x^2+x^2}\right) = \lim_{x\to 0}\left(\frac{1}{2}\right) = \frac{1}{2}$$

Now let us assume that $(x,y)\to(0,0)$ along $y=-2x$ then $(x,y)\to(0,0)$

$(x,-2x)\to(0,0) \Longrightarrow x\to 0$

$$\lim_{(x,y)\to(0,0)}\left(\frac{xy}{x^2+y^2}\right) = \lim_{x\to 0}\frac{x(-2x)}{x^2+(-2x)^2} = \lim_{x\to 0}\left(\frac{-2}{5}\right) = \frac{-2}{5}$$

i.e. limit is Not unique along two different paths Hence limit DNE.

M-II Put $y=mx$, $\lim\limits_{(x,y)\to(0,0)}\left(\dfrac{xy}{x^2+y^2}\right) = \lim\limits_{x\to 0}\left(\dfrac{x\cdot mx}{x^2+m^2x^2}\right) = \lim\limits_{x\to 0}\left(\dfrac{m}{1+m^2}\right) = \dfrac{m}{1+m^2}$ i.e. DNE

Q: $\lim\limits_{(x,y)\to(0,0)} \left(\frac{x^2y}{x^2+y^2}\right) = ? = \lim\limits_{x\to 0} \left(\frac{x^2 . mx}{x^2+m^2x^2}\right) = \lim\limits_{x\to 0} \left(\frac{mx}{1+m^2}\right) = 0$ (exist)

The value of $\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}$ is

(a) 0 (b) $\frac{1}{2}$

(c) 1 (d) ∞

$$\lim_{x\to 0} \frac{x^2 - x.mx}{\sqrt{x} - \sqrt{mx}} = \lim_{x\to 0} \frac{(1-m).x^{3/2}}{1-\sqrt{m}} = 0$$

Cont & Diff

Cont: $\lim_{x \to a} f(x) = f(a)$

or $\boxed{LHL = RHL = f(a)}$

Diff: if $LHD = RHD$

$LHD = LHL$ of $f'(x)$

$RHD = RHL$ of $f'(x)$

then $f(x)$ is called Diff

Q. The value of k for which

$$f(x)=\begin{cases}\dfrac{(4^x-1)^3}{\sin\left(\frac{x}{4}\right)\log\left(1+\frac{x^2}{3}\right)}, & x\neq 0\\ k, & x=0\end{cases}$$

is continuous at $x=0$

(a) $(\log 4)^3$ (b) $72\log 2$

(c) $12\log 4$ (d) $12(\log 4)^3$

w.k. that for cont at $x=0$

Exact Value = App. Value

$k=\lim f(x)$

$$k=\lim_{x\to 0} f(x)=\lim_{x\to 0}\frac{(4^x-1)^3}{\sin\left(\frac{x}{4}\right)\cdot\log\left(1+\frac{x^2}{3}\right)}=\frac{0}{0}\text{ form}$$

M-II

$$k=\lim_{x\to 0}\left[\frac{\left(\frac{4^x-1}{x}\right)^3\cdot(x^3)}{\frac{\sin\left(\frac{x}{4}\right)}{\left(\frac{x}{4}\right)}\cdot\left[\frac{\log\left(1+\frac{x^2}{3}\right)}{\frac{x^2}{3}}\right]\left(\frac{x}{4}\right)\cdot\left(\frac{x^2}{3}\right)}\right]$$

$$=\frac{(\log 4)^3}{1\times 1}\times 12=12(\log 4)^3$$

Q. If $f(x)=\begin{cases} \dfrac{ae^{x}-b\cos x+ce^{-x}}{x\sin x}, & x\neq 0 \\ 2, & x=0 \end{cases}$

is continuous everywhere then correct option is?

(a) $a=2,\ b=1,\ c=1$ ✗

(b) $a=1,\ b=-2,\ c=1$

(c) $a=1,\ b=1,\ c=2$ ✗

(d) $a=1,\ b=2,\ c=1$ ✓

for cont at $x=0$, $f(0)=\lim_{x\to 0} f(x)$

$$\lim_{x\to 0}\left[\frac{ae^{x}-b\cos x+ce^{-x}}{x\sin x}\right]=2$$

$$\frac{a-b+c}{0}\Rightarrow a-b+c=0 \quad \text{--- (1)}$$

$$\lim_{x\to 0}\left[\frac{ae^{x}+b\sin x-ce^{-x}}{x\cos x+\sin x}\right]=2$$

$$\frac{a-c}{0}=2\Rightarrow a-c=0 \quad \text{--- (2)}$$

$$\lim_{x\to 0}\left[\frac{ae^{x}+b\cos x+ce^{-x}}{x(-\sin x)+\cos x+\cos x}\right]=2$$

$$\frac{a+b+c}{2}=2\Rightarrow a+b+c=4 \quad \text{--- (3)}$$

$$\Longrightarrow a=1,\ b=2,\ c=1$$

The function $f(x) = |x| + |x-1|$ for real x,

(a) is both continuous and differentiable at $x = 0$ and $x = 1$.

(b) is not continuous but is differentiable at $x = 0$ and $x = 1$.

(c) is continuous but not differentiable at $x = 0$ and $x = 1$.

(d) is neither continuous not is differentiable at $x = 0$ and $x = 1$.

$$f(x) = |x| + |x-1| = \begin{cases} -x-(x-1), & x<0 \\ x-(x-1), & 0<x<1 \\ x+(x-1), & x>1 \end{cases}$$

$$\because |x| = \begin{cases} -x, & x<0 \\ +x, & x>0 \end{cases}$$

$$\text{i.e. } f(x) = \begin{cases} 1-2x, & x\le 0 \\ 1, & 0<x<1 \\ 2x-1, & x\ge 1 \end{cases}$$

At $x=0$: LHL $=1$, RHL $=1$, $f(0)=1$ i.e. Cont

At $x=1$: LHL $=1$, RHL $=1$, $f(1)=1$ i.e. Cont

$$f'(x) = \begin{cases} -2, & x<0 \\ 0, & 0<x<1 \\ 2, & x>1 \end{cases}$$

At $x=0$: LHD $=-2$, RHD $=0$ i.e. Not Diff at $x=0$

At $x=1$: LHD $=0$, RHD $=2$ i.e. " " at $x=1$

Which of the following function is differentiable at $x = 0$?

(a) $f(x) = |x|$  (b) $f(x) = |x| + |x-1|$ $\Rightarrow$ Previously Solved

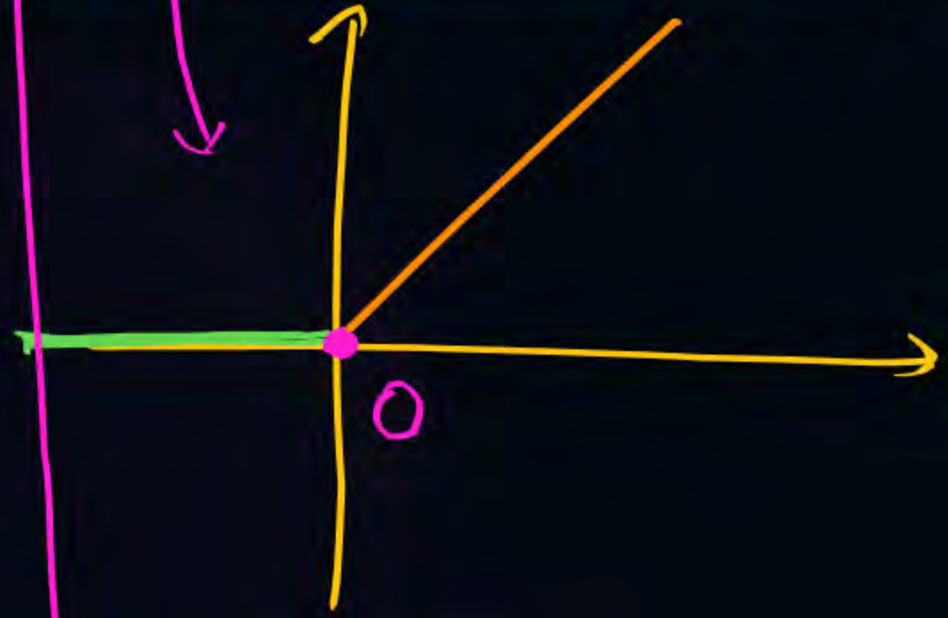
(c) $f(x) = x|x|$ (d) $f(x) = \begin{cases} 0 \text{ if, } & x \le 0 \\ x \text{ if, } & x > 0 \end{cases}$ $\Rightarrow f'(x) = \begin{cases} 0, & x<0 \\ 1, & x>0 \end{cases}$ LHD $= 0$, RHD $= 1$

$f(x) = x|x|$

$f(x) = \begin{cases} -x^2, & x<0 \\ x^2, & x>0 \end{cases}$

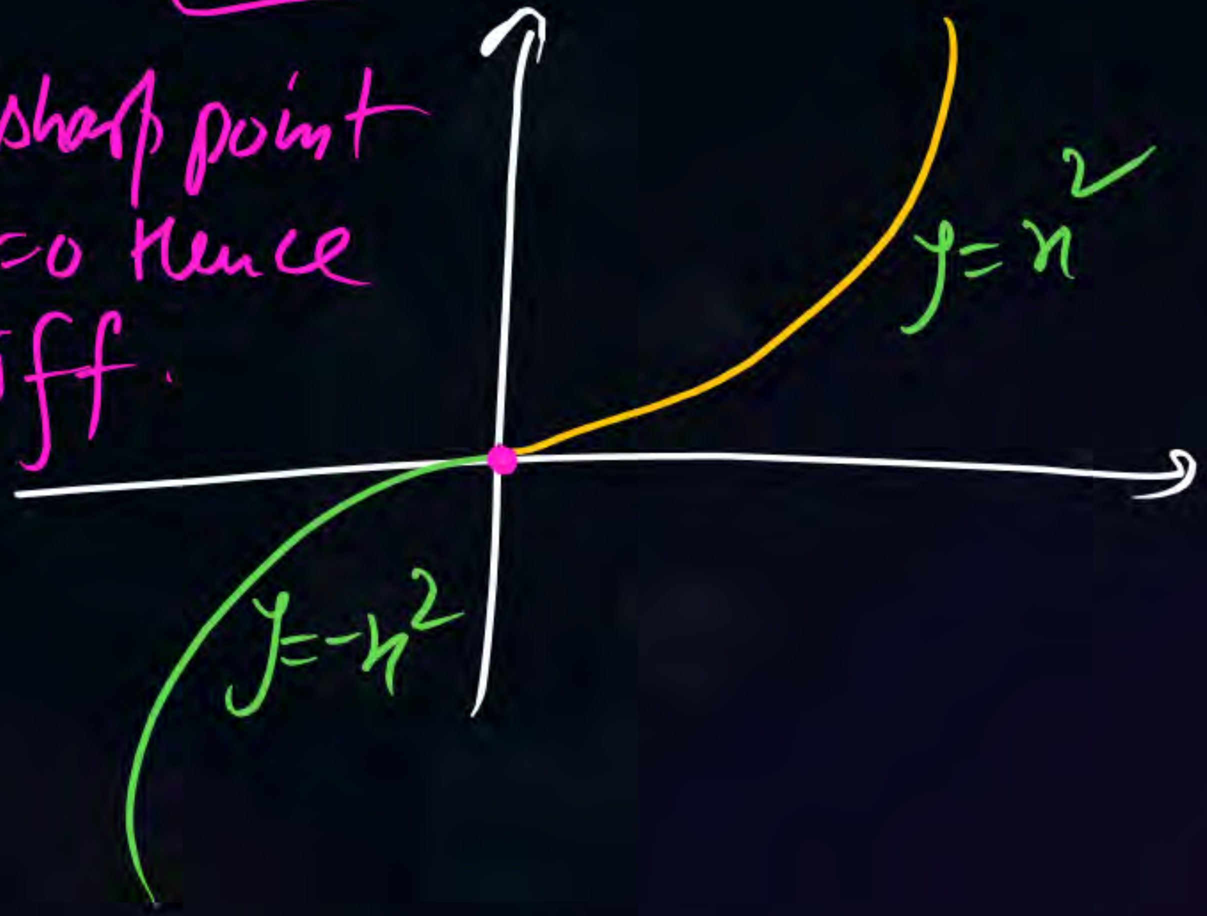
$f'(x) = \begin{cases} -2x, & x<0 \\ 2x, & x>0 \end{cases}$ LHD $= 0$, RHD $= 0$

So Diff at $x=0$



Graph of $\boxed{f(x) = x|x|}$

No sharp point at $x=0$ hence Diff.



Which one of the following function is continuous at $x = 3$?

(a) $f(x)=\begin{cases} 2, & \text{if } x=3 \\ x-1, & \text{if } x>3 \\ \dfrac{x+3}{3}, & \text{if } x<3 \end{cases}$ $f(3)=2$, RHL $=2$, LHL $=2$ ✓

(b) $f(x)=\begin{cases} 4, & \text{if } =3 \\ 8-x, & \text{if } x\neq 3 \end{cases}$ $\because$ App V $\neq$ E. Value $(=5)$ $(=4)$ ✗

(c) $f(x)=\begin{cases} x+3, & \text{if } x\le 3 \\ x-4, & \text{if } x>3 \end{cases}$ LHL $=6$, RHL $=-1$ ✗

(d) $f(x)=\dfrac{1}{x^3-27}, \text{ if } x\neq 3$ exact Value DNE ✗

The values of a and b for which the function

$$f(x) = \begin{cases} 2x+1, \text{ if } x \le 1 \\ ax^2+b \text{ if } 1 < x < 3 \\ 5x+2a \text{ if } x \ge 3 \end{cases}$$ is continuous every where

(a) $a = 2, b = 1$ (b) $a = 1, b = 2$

(c) $a = 3, b = 2$ (d) $a = 2, b = 3$

At $x=1$

$LHL = 3$

$RHL = a+b$

$f(1) = 3$

ie $a+b=3$

At $x=3$

$LHL = 9a+b$

$RHL = 15+2a$

$f(3) = 15+2a$

ie $15+2a = 9a+b$

$\Rightarrow 7a+b=15$

$f'(x) = \begin{cases} 2\alpha x+\beta \\ 3\alpha x^2+2\beta x+5\cos x \end{cases}$

A real function

$$f(x) = \begin{cases} \alpha x^2 + \beta x, & for \ x < 0 \\ \alpha x^3 + \beta x^2 + 5\sin x, & x \geq 0 \end{cases}.$$

If $f(x)$ is twice differentiable then

(a) $\alpha = 1, \beta = 0$ ✗ (b) $\alpha = 1, \beta = 5$

(c) $\alpha = 5, \beta = -10$ ✗ (d) $\alpha = 5, \beta = 5$ ✓

$f'(x) = \begin{cases} 2\alpha x+\beta & , x<0 \\ 3\alpha x^2+2\beta x+5\cos x & , x>0 \end{cases}$

LHD $= \beta$, RHD $= 5$ $\Rightarrow \beta = 5$

$f''(x) = \begin{cases} 2\alpha & , x<0 \\ 6\alpha x+2\beta-5\sin x & , x>0 \end{cases}$

LHD $= 2\alpha$, RHD $= 2\beta$ $\Rightarrow \alpha = \beta = 5$

TAYLOR & MACLAURIN SERIES

T. Series: $f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \ldots$

this series gives approx value of $f(x)$ in the Nbd of $x = a$

Maclaurin Series: $f(x) = f(0) + x f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots$

eg $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$

eg $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$

Q. In the Neighbourhood of $x=1$, $f(x)$ has the Power Series Expansion of the type $f(x) = 1 + (1-x) + (1-x)^2 + (1-x)^3 + \cdots\cdots \infty$ then $f(x)$ will be?

(a) $\frac{1}{x-1}$

(b) $\frac{1}{1-x}$

(c) $\frac{1}{x}$ ✓

(d) $\frac{x-1}{x+1}$

W.K.That Sum of Infinite GP is

$$a + ar + ar^2 + ar^3 + \cdots\cdots = \frac{a}{1-r}$$

Put $a=1$, then

$$1 + r + r^2 + r^3 + \cdots\cdots = \frac{1}{1-r} = (1-r)^{-1}$$

Put $r = (1-x)$

$$1 + (1-x) + (1-x)^2 + (1-x)^3 + \cdots\cdots = \frac{1}{1-(1-x)} = \frac{1}{x} = f(x)$$

Q. In the power series Expansion of $f(x) = \frac{x-1}{x+1}$ about $x=1$, 3^{rd} term will be?

(a) $(x-1)^2/2$

(b) $(x-1)^2/4$

(c) $(x-1)^3/4$

(d) $(x-1)^3/8$ ✓

Put $(x-1) = t \Rightarrow x = t+1$

So $f(x) = \frac{x-1}{x+1} = \frac{t}{t+2} = \frac{(t+2)-2}{t+2} = 1 - \frac{2}{t+2}$

$= 1 - \frac{1}{\left(1+\frac{t}{2}\right)} = 1 - \left(1+\frac{t}{2}\right)^{-1}$

$= 1 - \left\{1 + \left(-\frac{t}{2}\right) + \left(-\frac{t}{2}\right)^2 + \left(-\frac{t}{2}\right)^3 + \cdots\cdots\right\}$

$= \frac{t}{2} - \frac{t^2}{4} + \frac{t^3}{8} - \cdots\cdots$ So $T_3 = \frac{t^3}{8} = \frac{(x-1)^3}{8}$

The Taylor series expansion of $\dfrac{\sin x}{x-\pi}$ at $x = \pi$ is given by

(a) $1+\dfrac{(x-\pi)^2}{3!}+\ldots$ (b) $-1-\dfrac{(x-\pi)^2}{3!}+\ldots$

(c) $1-\dfrac{(x-\pi)^2}{3!}+\ldots$ (d) $-1+\dfrac{(x-\pi)^2}{3!}+\ldots$

M-I Using fundamental concept of Taylor series → Irritating

M-II Let $f(x)=\dfrac{g(x)}{x-\pi}=\dfrac{\sin x}{x-\pi}$

Now Try to find T.S.Exp of $g(x)$ in the Nbd of π & then divide it by $(x-\pi)$ you will get ans → little bit lengthy

M-III $f(x)=\dfrac{\sin x}{x-\pi}=\dfrac{\sin(\pi+(x-\pi))}{x-\pi}=\dfrac{-\sin(x-\pi)}{x-\pi}=\dfrac{-1}{x-\pi}\left[\sin(x-\pi)\right]$

x lies in the Nbd of π

i.e. $x\to\pi$

or $(x-\pi)\to 0$

$$=\frac{-1}{x-\pi}\left[(x-\pi)-\frac{(x-\pi)^3}{3!}+\frac{(x-\pi)^5}{5!}-\cdots\right]$$

$$=-1+\frac{(x-\pi)^2}{3!}-\frac{(x-\pi)^4}{5!}+\cdots$$

Let $f(x) = e^{x+x^2}$ for real x. From among the following, choose the Taylor series approximation of $f(x)$ around $x = 0$, which includes all powers of x less than or equal to 3.

(a) $1 + x + x^2 + x^3$

(b) $1 + x + \frac{3}{2}x^2 + x^3$

(c) $1 + x + \frac{3}{2}x^2 + \frac{7}{6}x^3$

(d) $1 + x + 3x^2 + 7x^3$

M-I use Standard Result of Maclaurin series and then proceed → time taking.

M-II $f(x) = e^{x+x^2}$; $x \to 0$

$f'(x) = e^{x+x^2}(1+2x)$

$f''(x) = e^{x+x^2}(1+2x)^2 + e^{x+x^2}(2)$

$f'''(x) = e^{x+x^2}(1+2x)^3 + e^{x+x^2}(2(1+2x))2 + 2e^{x+x^2}(1+2x)$

$f'''(0) = 1 + 4 + 2 = 7$

Coeff of $x^3 = \frac{f'''(0)}{3!} = \frac{7}{6}$

M-III $f(x) = e^{x+x^2}$

$= e^x \cdot e^{x^2}$

$= (\ldots\ldots)(\ldots\ldots)$

Now compare coeff of $x^3 = ?$

The integer n for which

$$\lim_{x\to 0}\frac{(\cos x-1)(\cos x-e^x)}{x^n}$$ is a finite non zero number is

(a) 1 (b) 2 (c) 3 (d) 4

$$\lim_{x\to 0}\frac{(\cos x-1)(\cos x-e^x)}{x^n} = \text{finite Number}$$

$$=\lim_{x\to 0}\frac{\left[\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}\cdots\right)-1\right]\left[\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}\cdots\right)-\left(1+x+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots\right)\right]}{x^n}=\text{finite}$$

$$\lim_{x\to 0}\left[\frac{\left(-\frac{x^2}{2}+\frac{x^4}{24}\cdots\right)\left(-x-x^2+\cdots\right)}{x^n}\right]$$

$$\lim_{x\to 0}\left[\frac{\frac{x^3}{2}+(\ )x^4+(\ )x^5+\cdots}{x^n}\right]=\text{finite} \Rightarrow n=3$$

$$\lim_{x\to 0}\left[\frac{(\cos x-1)(\cos x-e^{x})}{x^{n}}\right]=\text{finite}$$

let $n=3$

$$\lim_{x\to 0} -\frac{(1-\cos x)}{x^{2}}\times\left(\frac{\cos x-e^{x}}{x}\right)$$

$$=\lim_{x\to 0} -\left(\frac{1-\cos x}{x^{2}}\right)\left(\frac{-\sin x-e^{x}}{1}\right)$$

$$=\quad -\frac{1}{2}\times(-1)=\frac{1}{2}=\text{finite}$$

i.e. our assumption is correct, we can take $n=3$

Convergence & DIVERGENCE of Infinite series

- If sum of an Infinite is finite then series is called Convergent
- " " " " is $+\infty$ or $-\infty$ " " " Divergent.
- " " " " is Neither finite, Nor Infinite

ie values lies in some Range then series is called oscillatory

eg the series $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ is ?

(a) conv
(b) Div
(c) oscillatory
(d) None

$\simeq \sin x$ = An oscillatory funcn

$\because -1 \le \sin x \le 1$

The series $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges to

(a) 2 ln 2 (b) $\sqrt{2}$

(c) 2 (d) e

$$\sum_{n=0}^{\infty} \frac{1}{n!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots\cdots = ?$$

W.K. that $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\cdots = e^x$

Put $x = 1$ $1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots\cdots = e^1$

Q The series $1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots\cdots = ? = \sin(1) \equiv \sin\left(\frac{\pi}{3}\right)$

(a) Conv. (b) Div (c) oscillatory, (d) None

= finite Hence Conv.

Q. The series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\dots\dots\infty$ Converges at ______?

(a) 1.5 (b) 1 (c) Any Number b/n 1 & 2 (d) Not Converge

w.k. that $-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\dots\dots = \log(1-x)$

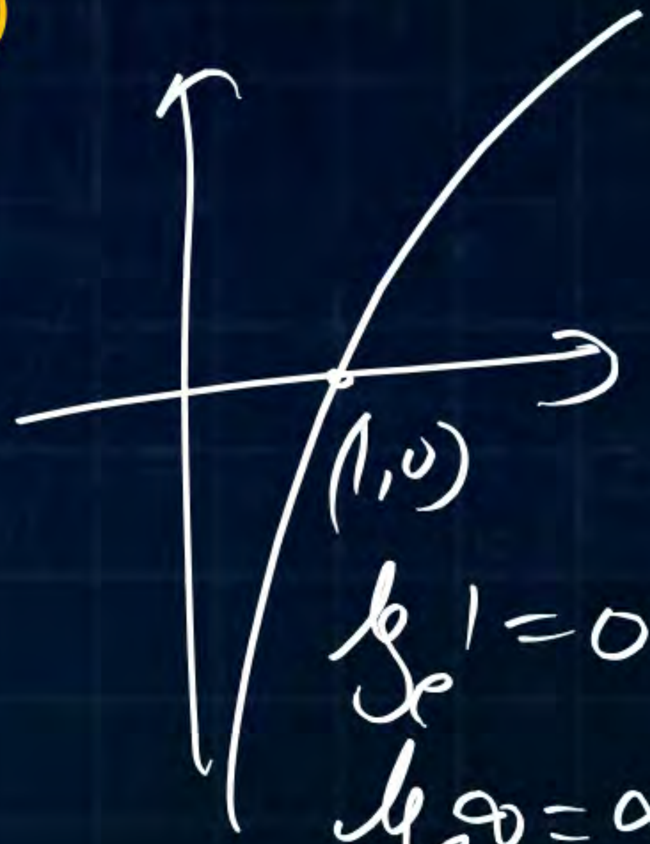
Put $x=1$

$$-1-\frac{1}{2}-\frac{1}{3}-\frac{1}{4}-\dots\dots = \log(1-1)$$

$$-\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots\right) = -\infty$$

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots = \infty$$

= Divergent



$\log_e 1 = 0$

$\log_e \infty = \infty$

$\log_e 0 = -\infty$

Q. if α is the repeated root of $ax^2+bx+c=0$ then

$$\lim_{x\to\alpha}\left[\frac{\tan(ax^2+bx+c)}{(x-\alpha)^2}\right] = ? = \lim_{x\to\alpha}\left[\frac{\tan\left(a(x-\alpha)^2\right)}{a(x-\alpha)^2}\right]\cdot a = 1\times a = a$$

(a) a

(b) b

(c) c

(d) a+b+c

$$\boxed{f(x) = ax^2+bx+c}$$

$$f(x) = a(x-\alpha)^2 \Rightarrow f(\alpha) = 0$$

$$f'(\alpha) = 2a(x-\alpha) \Rightarrow f'(\alpha) = 0$$

logic behind this concept comes from Rolle's Th.

Thank
THANK

Keep Hustling!