GATE

CRASH COURSE

ALL BRANCHES

Engineering Mathematics

Numerical Techniques
(Lec 12)

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Topics *to be covered*

Numerical Techniques (1 or 2M)

1. N-R method
2. T-Rule
3. Simpson's $\frac{1}{3}$rd Rule
4. Euler Method

$\left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\}$ 99.9%

N. Tech

1. Soln of linear Equn's
   1. Gauss-Jacobi Method
   2. Gauss-Seidal Method
2. Soln of Non linear Equn's & (Transcedental Equn)
   1. Bisection M
   2. Regula-Falsi M
   3. Newton Raphson M
3. N. Diff
   - F. Diff
   - B "
   - I "
   - N.D. Diff f.
4. N. Int
   1. T-Rule
   2. Simpson's $\frac{1}{3}^{rd}$ Rule
   3. " $\frac{3}{8}^{th}$ Rule
5. Soln of D Equn
   1. Forward Euler M
   2. Back " "
   3. M " "
   4. R-K M of 4th order

N-R Method $\rightarrow$ Consider $\boxed{f(x)=0}$ is the given Equ^n & we want to find $\alpha = ?$ for which $f(\alpha)=0$ & α lies near x_0.

then we will 1st find approximated values of α as follows,

Iteration 1: $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

Iteration 2: $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

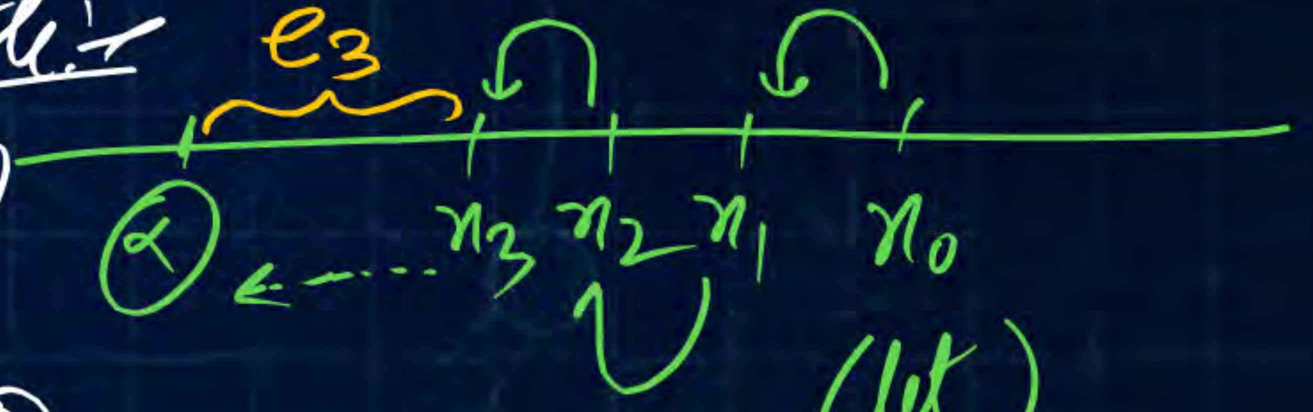
Iteration 3: $x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)}$

$\vdots$

Iterative Equ^n: $\boxed{x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}}$

Note:-

(1)



(2) Here $x_1, x_2, x_3 \ldots\ldots$ (let) α

i.e. x_1, x_2, x_3 are approximated values of α, which are improving resp.

(3) x_0 = Initial Assumption of α

α = Exact soln of your equ^{n}s.

(*) Error = |App Value - Exact Value|

$$\boxed{e_n = |x_n - \alpha|}$$

Starting from $x_0 = 1$, one step of Newton-Raphson method in solving the equation $x^3 + 3x - 7 = 0$ gives the next value (x_1) as

(a) $x_1 = 0.5$ (b) $x_1 = 1.406$

(c) $x_1 = 1.5$ (d) $x_1 = 2$

$f(x) = x^3 + 3x - 7, \quad x_0 = 1$

$f'(x) = 3x^2 + 3$

IT(1): $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{f(1)}{f'(1)}$

$x_1 = 1 - \left(\frac{1^3 + 3 - 7}{3(1)^2 + 3}\right)$

$= 1 - \frac{(-3)}{6}$

$= 1 + \frac{1}{2} = 1.5$

Let $x^2 - 117 = 0$. The iterative steps for the solution using Newton-Raphson's method is given by

(M-II)

(a) $x_{k+1} = \frac{1}{2}\left(x_k + \frac{117}{x_k}\right)$ $\Rightarrow \alpha = \frac{1}{2}\left(\alpha + \frac{117}{\alpha}\right)$

$2\alpha = \alpha + \frac{117}{\alpha}$

(b) $x_{k+1} = x_k - \frac{117}{x_k}$ $\qquad \alpha = \frac{117}{\alpha}$

(c) $x_{k+1} = x_k - \frac{x_k}{117}$ $\qquad \alpha^2 - 117 = 0$

(d) $x_{k+1} = x_k - \frac{1}{2}\left(x_k + \frac{117}{x_k}\right)$ $\qquad$ or $\boxed{x^2 - 117 = 0}$

$x^2 - 117 = 0 \Rightarrow x = \sqrt{117} = $ sq. of 117

let $\boxed{f(x) = x^2 - 117}$, $\boxed{f'(x) = 2x}$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \left[\frac{(x_n^2 - 117)}{2x_n}\right]$$

$$x_{n+1} = \frac{1}{2}\left[x_n + \frac{117}{x_n}\right]$$

Iteration

The recursion relation to solve $x = e^{-x}$ using Newton-Raphson method is

(a) $x_{n+1} = e^{-x_n}$

(b) $x_{n+1} = x_n - e^{-x_n}$

(c) $x_{n+1} = (1+x_n)\dfrac{e^{-x_n}}{1+e^{-x_n}}$

(d) $x_{n+1} = \dfrac{x_n^2 - e^{-x_n}(1+x_n) - 1}{x_n - e^{-x_n}}$

$\boxed{f(x) = x - e^{-x}}$ $\quad f'(x) = 1 + e^{-x}$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \left(\frac{x_n - e^{-x_n}}{1 + e^{-x_n}}\right)$$

$= \text{C}$

Solve the equation $x = 10\cos(x)$ using the Newton-Raphson method. The initial guess is $x = \pi/4$. The value of the predicted root after the first iteration, up to second decimal, is ________.

$f(x) = x - 10\cos x \;,\; x_0 = \frac{\pi}{4}$

$f'(x) = 1 + 10\sin x$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = \frac{\pi}{4} - \frac{f(\pi/4)}{f'(\pi/4)} = \frac{\pi}{4} - \frac{\left(\frac{\pi}{4} - \frac{10}{\sqrt{2}}\right)}{\left(1 + \frac{10}{\sqrt{2}}\right)} = 1.564$$

ESE

What is the cube root of 1468 to 3 decimal places?

(a) 11.340
(b) 11.353
(c) 11.365 ✓
(d) 11.382

Let $(1468)^{\frac{1}{3}} = \alpha$

i.e. $\alpha = (1468)^{\frac{1}{3}}$

$\alpha^3 = 1468$

$\alpha^3 - 1468 = 0$

So we can take, $\boxed{f(x) = x^3 - 1468 = 0}$, $f'(x) = 3x^2$

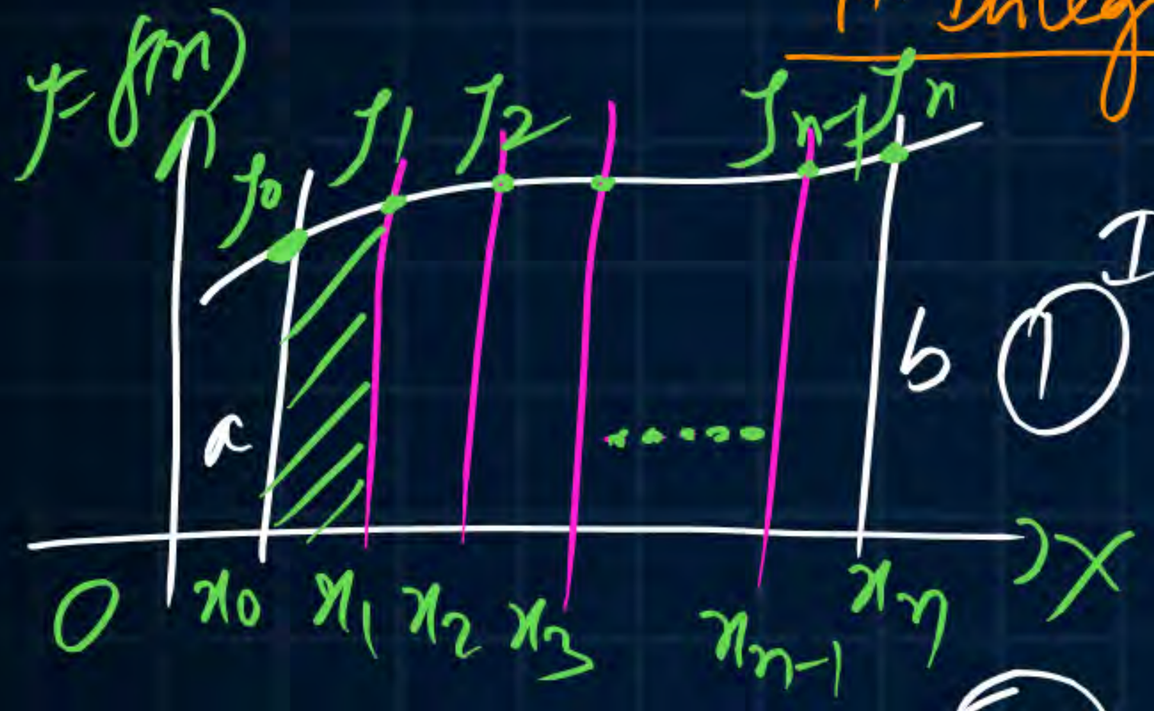
Let $x_0 = 11.3$

So $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

$= 11.3 - \dfrac{f(11.3)}{f'(11.3)}$

$= 11.3 - \left[\dfrac{(11.3)^3 - 1468}{3(11.3)^2}\right] = 11.365$

N. Integration



(1) $I=\int_a^b f(x)\,dx =$

→ M-I → use pure maths concepts

→ M-II → use N. Tech → T-Rule; $\frac{1}{3}^{rd}$ Rule

(2)

	$=a$	$=a+h$	$=a+2h$	$a+3h$			$=b$
$x=$	x_0	x_1	x_2	x_3	-----	x_{n-1}	x_n
$y=$	y_0	y_1	y_2	y_3	-----	y_{n-1}	y_n

(3) $I=\int_a^b f(x)\,dx=\int_{x_0}^{x_n} f(x)\,dx$

→ $I=\frac{h}{2}\left[y_0+y_n+2(y_1+y_2+y_3+\dots+y_{n-1})\right]$ T-Rule

→ $I=\frac{h}{3}\left[y_0+y_n+4(y_1+y_3+y_5+\dots)+2(y_2+y_4+y_6+\dots)\right]$ = Simpson's $\frac{1}{3}^{rd}$ Rule

(4) h → step size / size of the interval.

$I=\frac{3h}{8}\left[y_0+y_n+3(y_1+y_2+y_4+y_5+\dots)+2(y_3+y_6+\dots)\right]$

(*) $\frac{3h}{8}$:- $I = \frac{3h}{8}\left[y_0 + y_n + 3(y_1 + y_2 + y_4 + y_5 + y_7 + \cdots) + 2(y_3 + y_6 + \cdots)\right]$

(*) $n \rightarrow$ No. of subintervals & $(n+1) \rightarrow$ No. of points in Table

(*) $h \rightarrow$ step size / size of the interval.

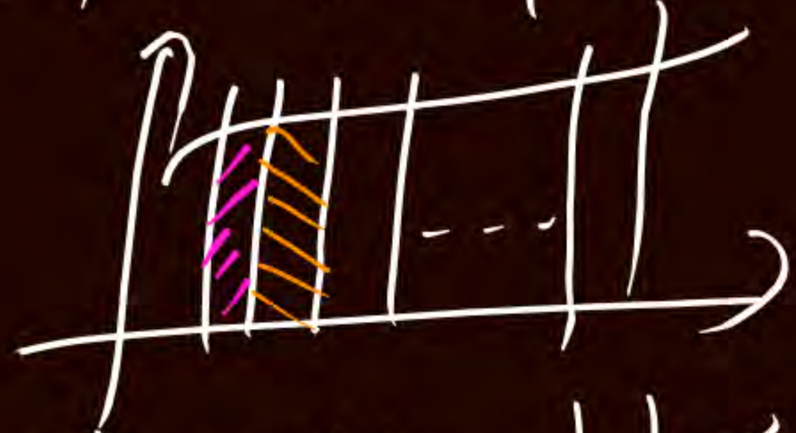
(*) $\boxed{b = a + nh} \Rightarrow h = \frac{b-a}{n}$ is $h \propto \frac{1}{n}$

n = No. of steps / No of iterations

(*) In this chapter possible values of n are as follows;

for T-Rule $\longrightarrow$ $n = 1, 2, 3, 4, 5, 6 \ldots\ldots$

for $\frac{1}{3}$rd Rule $\longrightarrow$ $n = 2, 4, 6, 8, 10 \ldots\ldots$



for $\frac{3}{8}$th Rule $\longrightarrow$ $n = 3, 6, 9, 12 \ldots\ldots$

The table below gives values of a function F(x) obtained for values of x at intervals of 0.25.

$h = 0.25,$

x	0	0.25	0.5	0.75	1.0
F(x)	1	0.9412	0.8	0.64	0.50
	y_0	y_1	y_2	y_3	y_4

The value of the integral of the function between the limits 0 to 1 using Simpson's rule is → $\frac{1}{3}$rd Rule

(a) 0.7854 (b) 2.3562

(c) 3.1416 (d) 7.5000

$$I = \frac{h}{3}\left[y_0 + y_4 + 4(y_1 + y_3) + 2(y_2)\right]$$

$$= \frac{0.25}{3}\left[1 + 0.5 + 4(0.9412 + 0.64) + 2(0.8)\right] = 0.7854 \text{ is } (a)$$

Match the CORRECT pairs

Numerical Integration Scheme	Order of Fitting Polynomial
P. Simpson's 3/8 Rule	1. First
Q. Trapezoidal Rule	2. Second
R. Simpson's 1/3 Rule	3. Third

(a) P – 2, Q – 1, R – 3

(b) P – 3, Q – 2, R – 1

(c) P – 1, Q – 2, R – 3

(d) P – 3, Q – 1, R – 2

Simpson's $\frac{1}{3}$ rule is used to integrate the function $f(x) = \frac{3}{5}x^2 + \frac{9}{5}$ between x = 0 and x = 1 using the least number of equal sub intervals. The value of integral is ______ Ans = 2

$f(x) = \frac{3}{5}x^2 + \frac{9}{5}$

In $\frac{1}{3}$rd Rule, $n_{min} = 2$

$h = \frac{b-a}{n} = \frac{1-0}{2} = 0.5$

	x_0	x_1	x_2
$x =$	0	0.5	1
$f =$	9/5	39/20	12/5
	y_0	y_1	y_2

By $\frac{1}{3}$rd Rule: $I_s = \frac{h}{3}\left[y_0 + y_2 + 4(y_1) + 2(0)\right]$

$= \frac{0.5}{3}\left[\frac{9}{5} + \frac{12}{5} + 4\left(\frac{39}{20}\right)\right]$

$= \frac{1}{6}(12) = 2$

Using a unit step size, the value of integral

$h=1$

$\int_1^2 x \ln x \, dx$ by trapezoidal rule is 0.693

$a=1,\ b=2,\ \boxed{f(x)=x\ln x}$

& $h=1$ (given)

	x_0	x_1
$x=$	1	2
$y=$	0	$2\ln 2$
	y_0	y_1

By T-Rule;

$$I=\frac{h}{2}\left[y_0+y_1+2(0)\right]$$

$$=\frac{1}{2}(0+2\ln 2)=\ln 2=0.693$$

The error in numerically computing the integral $\int_0^{\pi} (\sin x + \cos x)\,dx$ using the trapezoidal rule with three intervals of equal length between 0 and π is ______.

$n = 3$

$$I_E = \int_0^{\pi} (\sin x + \cos x)\,dx$$

$$= (-\cos x + \sin x)_0^{\pi}$$

$$= (-(-1)+0) - (-1+0) = 2$$

So Error $= |\text{App} - \text{Exact}|$

$= 0.185$

$\boxed{f(x) = \sin x + \cos x}$, $a = 0, b = \pi, n = 3$

$$h = \frac{b-a}{n} = \frac{\pi - 0}{3} = \pi/3$$

	x_0	x_1	x_2	x_3
$x =$	0	$\pi/3$	$2\pi/3$	π
$y =$	1	$\frac{\sqrt{3}+1}{2}$	$\frac{\sqrt{3}-1}{2}$	-1
	y_0	y_1	y_2	y_3

$$I = \frac{h}{3}\left[y_0 + y_3 + 2(y_1 + y_2)\right] = \,? \quad (\text{App Value})$$

Match the items in columns I and II.

Column I	Column II
P. Gauss-Seidel method	1. Interpolation
Q. Forward Newton Gauss method	2. Non-linear differential equations
R. Runge-Kutta method	3. Numerical integration
S. Trapezoidal Rule	4. Linear algebraic equations

(a) P – 1, Q – 4, R – 3, S – 2

(b) P – 1, Q – 4, R – 2, S – 3

(c) P – 1, Q – 3, R – 2, S – 4

(d) P – 4, Q – 1, R – 2, S – 3

Matching exercise choose the correct one out of the alternatives A, B, C, D

Group-I

P. 2^{nd} order differential equations — 1

Q. Non-linear algebraic equations — 2

R. Linear algebraic equations — 3

S. Numerical integration → 4

Group-II

(1) Runge-Kutta method

(2) Newton-Raphson method

(3) Gauss Elimination

(4) Simpson's Rule

(a) P-3, Q-2, R-4, S-1 (b) P-2, Q-4, R-3, S-1

(c) P-1, Q-2, R-3, S-4 (d) P-1, Q-3, R-2, S-4

N. Sol of O.D.Eq $= y$

Consider $\boxed{\frac{dy}{dx} = f(x,y)}$; $y(x_0) = y_0$, step size $= h$

① Forward Euler Method

(Explicit E.M)

It ① $y_1 = y_0 + h f(x_0, y_0)$

It ② $y_2 = y_1 + h f(x_1, y_1)$

It ③ $y_3 = y_2 + h f(x_2, y_2)$

⋮

$$\boxed{y_{n+1} = y_n + h f(x_n, y_n)}$$

② Backward Euler Method

(Implicit E.M)

It ① $y_1 = y_0 + h f(x_1, y_1)$

" ② $y_2 = y_0 + h f(x_2, y_2)$

" ③ $y_3 = y_2 + h f(x_3, y_3)$

⋮

$$\boxed{y_{n+1} = y_n + h f(x_{n+1}, y_{n+1})}$$

Consider the equation $\frac{du}{dt} = 3t^2 + 1$ with $u = 0$ at $t = 0$. This is numerically solved by using the forward Euler method with a step size. $\Delta t = 2$. The absolute error in the solution in the end of the first time step is ________.

$$\frac{du}{dt} = \boxed{f(t, u) = 3t^2 + 1}$$

$$u = u(t) \begin{cases} t_0 = 0 \\ u_0 = 0 \end{cases}$$

$$h = \Delta t = 2$$

$$\frac{du}{dt} = (3t^2 + 1)$$

$$\int du = \int (3t^2 + 1)\, dt + C$$

$$u = t^3 + t + C$$

$$u_E = u(2) = 10 = \text{Exact.}$$

$$u_1 = u_0 + h\, f(t_0, u_0)$$

$$= 0 + 2 f(0, 0)$$

$$= 2(0 + 1) = 2 \; (\text{App})$$

	t_0	t_1
$t =$	0	2
$u =$	0	?
	u_0	u_1

$$\text{Error} = |u_A - u_E| = ? = 8$$

Consider the differential equation

$$\frac{dy}{dx} = \boxed{4(x+2) - y} = f(x,y)$$

For the initial condition $y = 3$ at $x = 1$, the value of y at $x = 1.4$ obtained using Euler's method with a step-size of 0.2 is ________. (round off to one decimal place)

Here $\frac{dy}{dx} = f(x,y) = \boxed{4x - y + 8}$

$y(1) = 3$ → $x_0 = 1$, $y_0 = 3$

$y(1.4) = ? = y_2 = ?$, $h = 0.2$

	x_0	x_1	x_2
$x=$	1	1.2	1.4
$y=$	3	–	?
	y_0	y_1	y_2

$y_1 = y_0 + h f(x_0, y_0)$

$= 3 + 0.2 f(1, 3)$

$= 3 + 0.2(4 - 3 + 8)$

$= \frac{24}{5} = 4.8$

$y_2 = y_1 + h f(x_1, y_1)$

$= 4.8 + 0.2 f(1.2, 4.8)$

$= 4.8 + \frac{1}{5}\left[4 \times 1.2 - 4.8 + 8\right]$

$= 6.4$

Q. Also evaluate $y(1.4) = ?$ using Backward E. Method ?

	x_0	x_1	x_2
$x =$	1	1.2	1.4
$y =$	3	—	?
	y_0	y_1	y_2

, $h = 0.2$ & $\frac{dy}{dx} = f(x,y) = \boxed{4x - y + 8}$

It (1):

$$y_1 = y_0 + h f(x_1, y_1)$$

$$y_1 = y_0 + h\left[4x_1 - y_1 + 8\right]$$

$$(1+h)y_1 = y_0 + 4hx_1 + 8h$$

$$y_1 = \frac{y_0 + 4hx_1 + 8h}{1+h} = 4.63$$

It (2):

$$y_2 = y_1 + h f(x_2, y_2)$$

$$y_2 = y_1 + h(4x_2 - y_2 + 8)$$

$$(1+h)y_2 = y_1 + 4hx_2 + 8h$$

$$y_2 = \left(\frac{y_1 + 4hx_2 + 8h}{1+h}\right) = \ldots\ldots = 6.12 \text{ Ans}$$

Beta-Gamma

① $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = B(m,n) = \dfrac{\Gamma m\,\Gamma n}{\Gamma m+n}$

$\boxed{\Gamma\tfrac{1}{2} = \sqrt{\pi}}$

② $\int_0^\infty e^{-x} x^{n-1}\,dx = \Gamma n$

③ $\int_0^{\pi/2} \sin^m\theta \cos^n\theta \, d\theta = \dfrac{\Gamma\frac{m+1}{2}\,\Gamma\frac{n+1}{2}}{2\,\Gamma\frac{m+n+2}{2}}$

④ $\Gamma n+1 = \begin{cases} n!, & n\in N \\ n\Gamma n, & n\in Q \end{cases}$

$\Gamma 5 = ? = 4! = 24$

$\Gamma 8 = ? = 7! = ?$

$\Gamma\frac{3}{2} = ? = \Gamma\frac{1}{2}+1 = \frac{1}{2}\Gamma\frac{1}{2}$

Q $\Gamma\frac{7}{2} = \Gamma\frac{5}{2}+1 = \frac{5}{2}\Gamma\frac{5}{2} = \frac{5}{2}\Gamma\frac{3}{2}+1 = \frac{5}{2}\cdot\frac{3}{2}\Gamma\frac{3}{2} = \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}$ //

Q. $I = \int_0^1 x^5 (1-x^3)^{10} \, dx = ?$

Put $x^3 = y \Rightarrow x = y^{\frac{1}{3}}$

$dx = \frac{1}{3} \cdot y^{-\frac{2}{3}} \cdot dy$

$$\boxed{\int_0^1 x^{m-1}(1-x)^{n-1} \, dx = B(m,n)}$$

& $B(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}$

$= \int_0^1 (y^{\frac{1}{3}})^5 (1-y)^{10} \cdot \frac{1}{3} y^{-\frac{2}{3}} \cdot dy$

$= \frac{1}{3} \int_0^1 y^{\left(\frac{5}{3}-\frac{2}{3}\right)} (1-y)^{10} \, dy$

$= \frac{1}{3} \int_0^1 y^1 \cdot (1-y)^{10} \, dy = \frac{1}{3} \int_0^1 y^{2-1} \cdot (1-y)^{11-1} \, dy$

$= \frac{1}{3} B(2,11) = \frac{1}{3} \left(\frac{\Gamma 2 \, \Gamma 11}{\Gamma 2+11} \right) = \frac{1! \, (10!)}{3 \, (12!)}$

$= \frac{1}{396}$

$\int_0^1 x^6\sqrt{1-x^2}\,dx =$

(a) $\frac{5\pi}{256}$ (b) $\frac{5\pi}{128}$

(c) $\frac{5\pi}{512}$ (d) $\frac{3\pi}{512}$

Put $x = \sin\theta$

$dx = \cos\theta\, d\theta$

At $x=0,\ \theta = 0$

At $x=1,\ \theta = \pi/2$

$$I = \int_0^{\pi/2} \sin^6\theta \sqrt{1-\sin^2\theta}\cdot\cos\theta\, d\theta$$

$$= \int_0^{\pi/2} \sin^6\theta\cdot\cos^2\theta\, d\theta = ?$$

$m=6,\ n=2$

$$I = \frac{\Gamma\frac{6+1}{2}\,\Gamma\frac{2+1}{2}}{2\,\Gamma\frac{6+2+2}{2}} = \frac{\Gamma\frac{7}{2}\,\Gamma\frac{3}{2}}{2\Gamma 5}$$

$$\frac{\left(\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}\right)\cdot\left(\frac{1}{2}\Gamma\frac{1}{2}\right)}{2(4!)}$$

$$= \frac{15\pi}{32\times 24} = \frac{5\pi}{32\times 8}$$

$\int_0^\infty e^{-y^3} \cdot y^{1/2} dy =$

(a) $\sqrt{\pi}$ (b) $\frac{\sqrt{\pi}}{3}$ ✓

(c) $\frac{\sqrt{\pi}}{2}$ (d) 0

Put $y^3 = t \Rightarrow y = t^{\frac{1}{3}}$

$dy = \frac{1}{3} t^{-\frac{2}{3}} dt$

$$\boxed{\int_0^\infty e^{-x} \cdot x^{n-1} dx = \overline{|n}}$$

$$I = \int_0^\infty e^{-y^3} \cdot y^{\frac{1}{2}} dy = ?$$

$$= \int_0^\infty e^{-t} (t^{1/3})^{\frac{1}{2}} \cdot \frac{1}{3} t^{-2/3} \cdot dt$$

$$= \frac{1}{3} \int_0^\infty e^{-t} \cdot t^{-\frac{1}{2}} dt = \frac{1}{3} \int_0^\infty e^{-t} \cdot t^{\frac{1}{2}-1} dt = \frac{1}{3} \overline{|\tfrac{1}{2}} = \frac{\sqrt{\pi}}{3}$$

Thank

THANK

Keep Hustling!