GATE

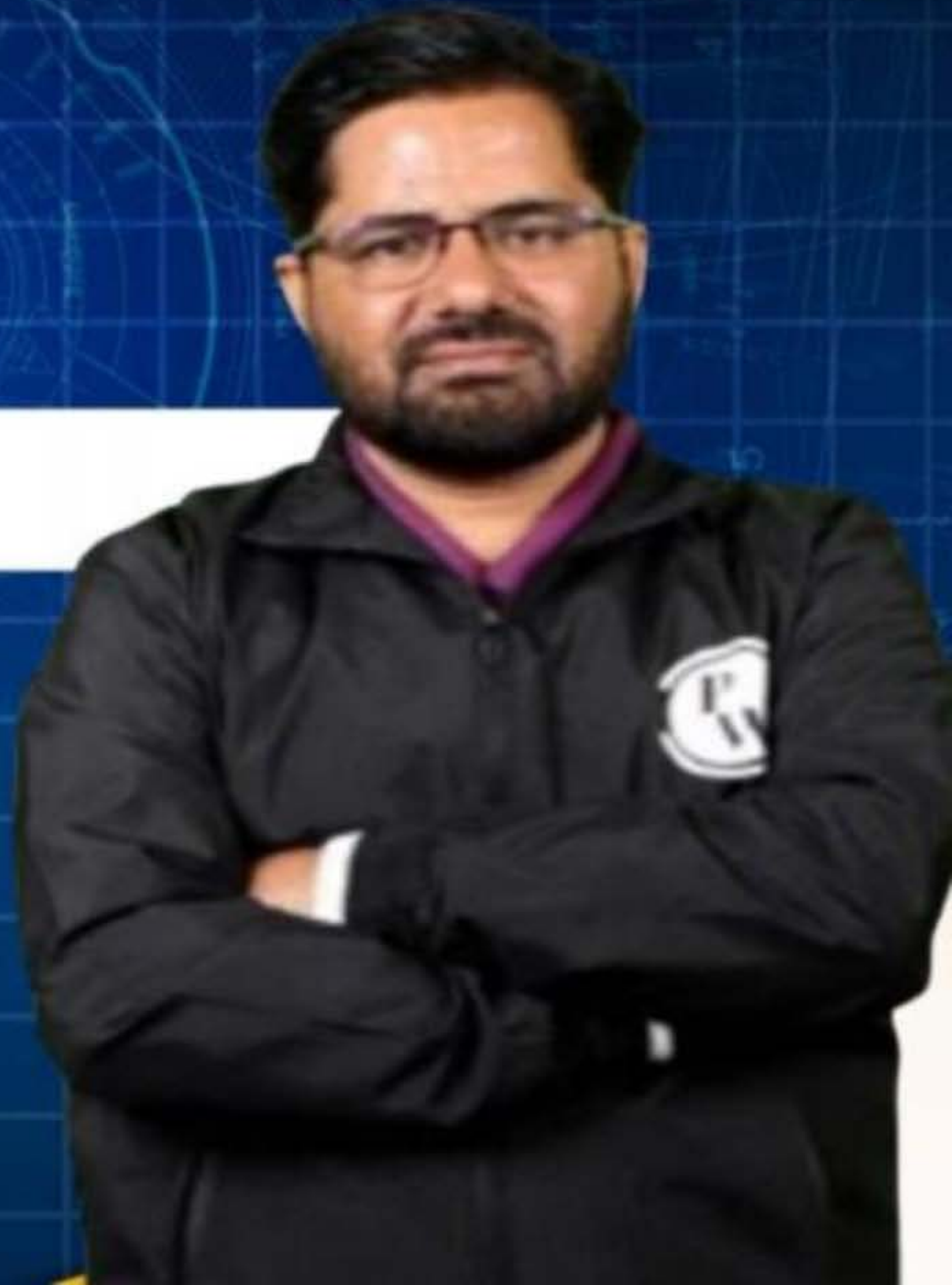
DATA SCIENCE + CS & IT

Engineering Mathematics

SUPER 1500

Lec : 01 Calculus

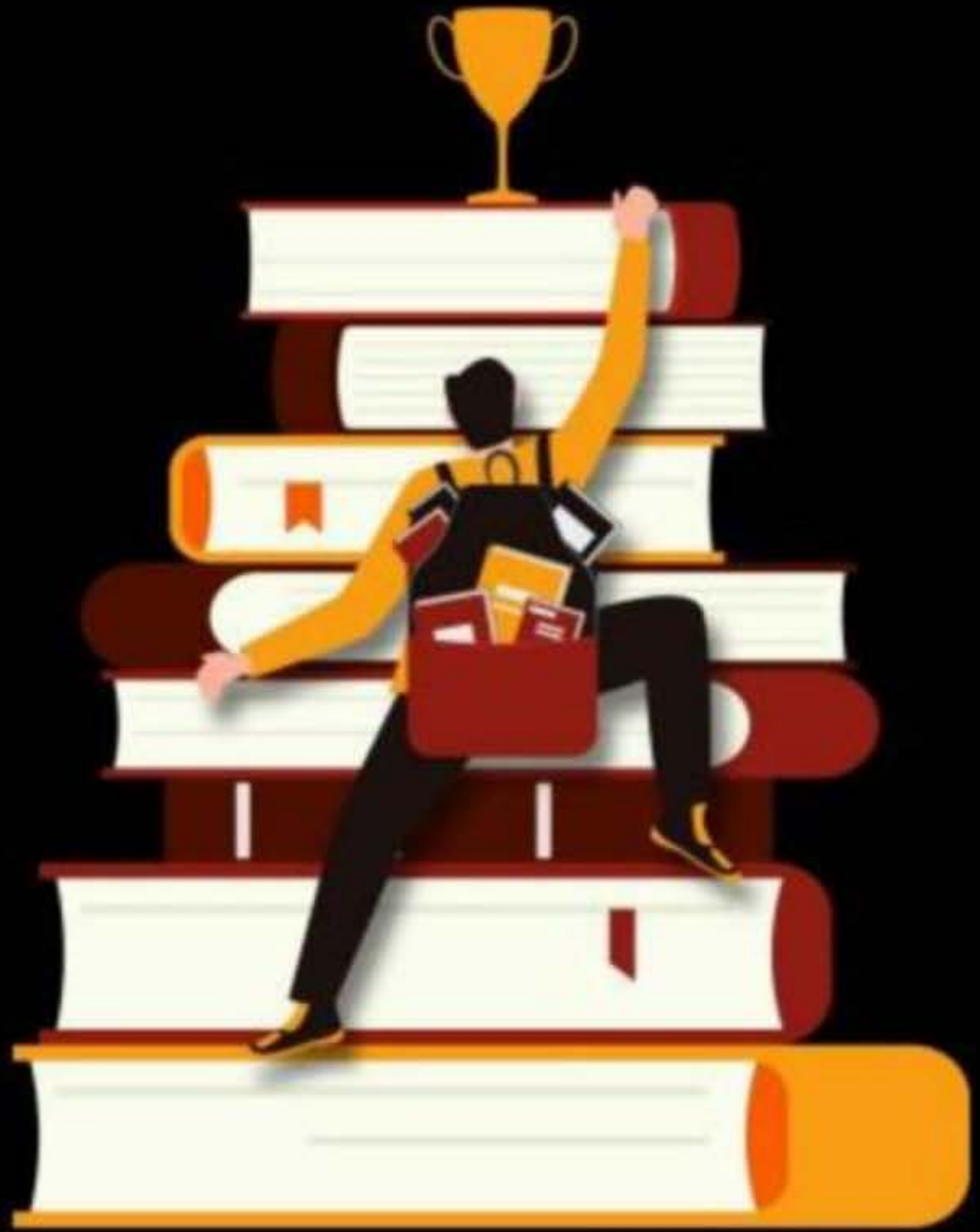
By - Dr. Puneet Sharma Sir



Topics to be covered

CALCULUS

Limits ($\approx$ 100% Chance)



PLANNER

Calculus → 4 days

Linear-1 → 4 days

Linear-2 → 2 days

Prob & Stats-1 → 6 days

" " -2 → 2 days

Except SUNDAY

Total = 18 days

(1) Direct Substitution

(2) Factorisation

(3) Rationalisation

(4) INDETERMINATE FORMS

$\left(\frac{0}{0}, \frac{\infty}{\infty}, 0\times\infty, \infty-\infty, 0^0, \infty^0, 1^\infty\right)$

(5) Standard Results

(6) Common sense

Q: $\lim_{n\to\infty}\left\{\dfrac{(1-2)+(3-4)+(5-6)+(7-8)+\dots\dots+((2n-1)-2n)}{\sqrt{n^2+1}+\sqrt{n^2-1}}\right\}=?$ (a) 0.5 (b) -1 (c) 0 $\cancel{(d)}$ -0.5

$$\lim_{n\to\infty}\frac{(-1)+(-1)+(-1)+\dots\dots+(-1)\ \{n \text{ times}\}}{\sqrt{n^2+1}+\sqrt{n^2-1}}$$

$$\lim_{n\to\infty}\frac{(-n)}{n\left[\sqrt{1+\frac{1}{n^2}}+\sqrt{1-\frac{1}{n^2}}\right]}$$

$$=\frac{-1}{\sqrt{1+0}+\sqrt{1-0}}=\frac{-1}{2}$$

Q. $\lim_{n\to\infty}\left\{\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\dots\dots+\frac{1}{(2n+1)(2n+3)}\right\}=?$ (a) 0 (b) -0.5 (c) 0.5 (d) DNE

$$=\lim_{n\to\infty}\left[\frac{1}{2}\left\{\frac{1}{1}-\frac{1}{3}\right\}+\frac{1}{2}\left\{\frac{1}{3}-\frac{1}{5}\right\}+\frac{1}{2}\left\{\frac{1}{5}-\frac{1}{7}\right\}+\dots\dots+\frac{1}{2}\left\{\frac{1}{2n+1}-\frac{1}{2n+3}\right\}\right]$$

$$=\frac{1}{2}\lim_{n\to\infty}\left[\left(1-\cancel{\frac{1}{3}}\right)+\left(\cancel{\frac{1}{3}}-\cancel{\frac{1}{5}}\right)+\left(\cancel{\frac{1}{5}}-\cancel{\frac{1}{7}}\right)+\left(\cancel{\frac{1}{7}}-\right)\dots\dots+\left(\cancel{\frac{1}{2n+1}}-\frac{1}{2n+3}\right)\right]$$

$$=\frac{1}{2}\lim_{n\to\infty}\left[1-\frac{1}{2n+3}\right]=\frac{1}{2}\left\{1-\frac{1}{\infty}\right]=\frac{1}{2}(1-0)=0.5$$

Q. $\lim_{n\to\infty}\left\{\dfrac{(n+2)!+(n+1)!}{(n+2)!-(n-1)!}\right\}=?$ (a) $1/2$ (b) n ~~(c)~~ 1 (d) ∞

W.K.T. $\boxed{n! = n(n-1)!} = n(n-1)(n-2)! = \cdots\cdots$

$\Rightarrow (n+2)! = (n+2)(n+1)\,n\,(n-1)!$

$$\lim_{n\to\infty}\left[\frac{(n+2)(n+1)!+(n+1)!}{(n+2)(n+1)!-(n-1)!}\right]$$

$$\lim_{n\to\infty}\frac{(n+1)!}{(n+1)!}\left[\frac{(n+2)+1}{(n+2)-\frac{(n-1)!}{(n+1)!}}\right]$$

$$\lim_{n\to\infty}\left[\frac{n+2+1}{(n+2)-\frac{1}{(n+1)n}}\right]$$

$$\lim_{n\to\infty}\left[\frac{1+\frac{1}{n+2}}{1-\frac{1}{(n+2)(n+1)n}}\right]=\frac{1+0}{1-0}$$

$$=1$$

Q. $\lim_{x\to\infty} \sqrt{x}\left\{\sqrt{x+2}-\sqrt{x}\right\} = ?$ (a) 1, (b) ∞, (c) 0 (d) $\frac{1}{2}$

$$= \lim_{x\to\infty} \sqrt{x}\left[\left(\sqrt{x+2}-\sqrt{x}\right)\times\left\{\frac{\sqrt{x+2}+\sqrt{x}}{\sqrt{x+2}+\sqrt{x}}\right\}\right]$$

$$= \lim_{x\to\infty} \sqrt{x}\left[\frac{(x+2)-(x)}{\sqrt{x+2}+\sqrt{x}}\right] = \lim_{x\to\infty} \frac{\cancel{\sqrt{x}}}{\cancel{\sqrt{x}}}\left[\frac{2}{\sqrt{1+\frac{2}{x}}+1}\right]$$

$$= \frac{2}{\sqrt{1+0}+1} = \frac{2}{2} = 1$$

Q. $\lim_{x\to 1}\left(\frac{1}{x^2+x-2}-\frac{x}{x^3-1}\right)=?$ (a) $\frac{1}{3}$ (b) -9 (c) $-\frac{1}{3}$ (d) $-\frac{1}{9}$

Sol: $\lim_{x\to 1}\left\{\frac{(x^3-1)-x(x^2+x-2)}{(x^2+x-2)(x^3-1)}\right\} = \lim_{x\to 1}\left\{\frac{x^3-1-x^3-x^2+2x}{\underbrace{(x+2)(x-1)}\,\underbrace{(x-1)(x^2+1+x)}}\right\}$

$= \lim_{x\to 1}\left[\frac{-1-x^2+2x}{(x+2)(x-1)^2(x^2+1+x)}\right] = \lim_{x\to 1}\left[\frac{-(x-1)^2}{(x+2)(x-1)^2(x^2+1+x)}\right]$

$= \frac{-1}{(3)(3)} = -\frac{1}{9}$

Q: $\lim_{x\to 0}\left\{\frac{-\sin x}{2\sin x + x\cos x}\right\} = \underline{\quad ? \quad}$ $= \frac{0}{0}$ form.

M-I

$$= \lim_{x\to 0}\left[\frac{-\cos x}{2\cos x + \{x(-\sin x) + (1)\cos x\}}\right]$$

$$= \lim_{x\to 0}\left(\frac{-\cos x}{3\cos x - x\sin x}\right) = \frac{-1}{3-0} = -\frac{1}{3}$$

M-II

$$\lim_{x\to 0}\left\{\frac{-\sin x}{x\left\{2\frac{\sin x}{x} + \cos x\right\}}\right\} = \lim_{x\to 0}\left[\frac{-\left(\frac{\sin x}{x}\right)}{2\left(\frac{\sin x}{x}\right) + \cos x}\right] = \frac{-(1)}{2(1)+1}$$

$$= -1/3$$

Q. $\lim_{x\to 0}\left\{\dfrac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}\right\} = ?$ (a) 0 (b) 1 (c) -1 (d) DNE

(M-I) it is in $\frac{\infty}{\infty}$ from so use L-Hospital's Rule.

~~it is in $\frac{\infty}{\infty}$ from so use L-Hospital's Rule~~

we can't use L-Hosp. Rule $\because f'(x)$ is Not Defined at $x=0$

$\because$ LHL $\neq$ RHL

so limit DNE

(M-II)

$-\infty \longleftarrow (\; 0 \;) \longrightarrow \infty$

$$\text{RHL} = \lim_{x\to 0^+} f(x) = \lim_{x\to(0+h)}\left(\frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}\right) = \lim_{h\to 0}\left[\frac{e^{\frac{1}{h}}-1}{e^{\frac{1}{h}}+1}\right] = \lim_{h\to 0}\left[\frac{1-e^{-\frac{1}{h}}}{1+e^{-\frac{1}{h}}}\right] = \frac{1-0}{1+0} = 1$$

$$\text{LHL} = \lim_{x\to 0^-} f(x) = \lim_{x\to(0-h)}\left(\frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}\right) = \lim_{h\to 0}\left(\frac{e^{-\frac{1}{h}}-1}{e^{-\frac{1}{h}}+1}\right) = \frac{e^{-\infty}-1}{e^{-\infty}+1} = \frac{0-1}{0+1} = -1$$

$(h>0)$

(M-II) $$\lim_{x\to 0} \frac{\left(\frac{e^{\frac{1}{x}}-1}{1/x}\right)}{\left(\frac{e^{\frac{1}{x}}+1}{1/x}\right)} = ???$$

??

$$\because \lim_{x\to 0}\left(\frac{e^{x}-1}{x}\right)=1$$

Q→ $\lim_{x\to 0^+}\{1+\tan^2 x\}^{\frac{1}{2x}} = ?$ (a) $\sqrt{e}$ (b) e (c) e^2 (d) e^{-2} (e) 1

it is in 1^∞ form so taking log;

Let $K = \lim_{x\to 0^+}\left(1+\tan^2 x\right)^{\frac{1}{2x}}$

$$\log K = \lim_{x\to 0^+}\left[\log\left(1+\tan^2 x\right)^{\frac{1}{2x}}\right]$$

$$= \frac{1}{2}\left[\lim_{x\to 0^+} \cdot \frac{\log(1+\tan^2 x)}{x}\right] = \frac{0}{0}$$

$$= \frac{1}{2}\lim_{x\to 0^+}\left(\frac{\left(\frac{1}{1+\tan^2 x}\right)\left(2\tan x \sec^2 x\right)}{1}\right)$$

$$= \frac{1}{2}\lim_{x\to 0}\left(2\tan x\right) = 0$$

$$\log K = 0 \Rightarrow K = e^0 = 1$$

Q. $K = \lim_{x \to 0} \left\{ \frac{1^x + 2^x + 4^x}{3} \right\}^{\frac{1}{x}} = ? = 1^\infty$ form

Ans (2)

$$\log K = \lim_{x \to 0} \log \left(\frac{1^x + 2^x + 4^x}{3} \right)^{\frac{1}{x}}$$

$$= \lim_{x \to 0} \left[\frac{\log \left(\frac{1^x + 2^x + 4^x}{3} \right)}{x} \right] = \frac{0}{0}$$

$$= \lim_{x \to 0} \left[\frac{\frac{1}{\left(\frac{1^x + 2^x + 4^x}{3} \right)} \cdot \frac{\left(1^x \log_e 1 + 2^x \log_e 2 + 4^x \log_e 4 \right)}{3}}{1} \right]$$

$$= \lim_{x \to 0} \left(\frac{2^x \log 2 + 4^x \log 4}{1^x + 2^x + 4^x} \right)$$

$$= \frac{\log 2 + \log 4}{3} = \frac{1}{3} \log 8$$

$$\log K = \log (8)^{1/3} = \log 2$$

$$\boxed{K = 2}$$

Q. $\lim_{x\to\infty}\left(\frac{x+3}{x+4}\right)^{x+5} = ?$ (a) e (b) -1 (c) $\cancel{e^{-1}}$ (d) 1

let $K = \lim_{x\to\infty}\left(\frac{x+3}{x+4}\right)^{x+5}$

it is in 1^∞ form $\because \lim_{x\to\infty}\left(\frac{1+\frac{3}{x}}{1+\frac{4}{x}}\right)^{x+5} = \left(\frac{1+0}{1+0}\right)^\infty = 1^\infty$ form

M-I $\log_e K = \lim_{x\to\infty} \log\left\{\frac{x+3}{x+4}\right\}^{x+5}$

$= \lim_{x\to\infty}(x+5)\cdot\log\left\{\frac{x+3}{x+4}\right\} = (0\times\infty)$

$= \lim_{x\to\infty}\left[\frac{\log\left(\frac{x+3}{x+4}\right)}{\left(1/x+5\right)}\right] = \frac{0}{0}$ form

= use L-Hosp Rule .... $= -1 \Rightarrow K = e^{-1}$

M-II $\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x = e^a$

$\lim_{x\to\infty}\left(\frac{x+3}{x+4}\right)^{x+5} = \lim_{x\to\infty}\left(\frac{x+3}{x+4}\right)^x \cdot \lim_{x\to\infty}\left(\frac{x+3}{x+4}\right)^5$

$= \lim_{x\to\infty}\frac{\left(1+\frac{3}{x}\right)^x}{\left(1+\frac{4}{x}\right)^x} \times \lim_{x\to\infty}\frac{\left(1+\frac{3}{x}\right)^5}{\left(1+\frac{4}{x}\right)^5}$

$= \frac{e^3}{e^4} \times \frac{(1+0)^5}{(1+0)^5} = \frac{1}{e} = e^{-1}$

Q. $\lim_{x \to \frac{\pi}{2}} \left\{ \frac{1+\cos 2x}{(\pi-2x)^2} \right\} = ?$ (a) 0.5 (b) $\frac{\pi}{2}$ (c) 0 (d) DNE

Put $\boxed{(\pi-2x) = t}$ & when $x \to \frac{\pi}{2}$

$t \to 0$

$$\lim_{t \to 0} \left[\frac{1+\cos(\pi-t)}{t^2} \right]$$

$$= \lim_{t \to 0} \left(\frac{1-\cos t}{t^2} \right) = \frac{1}{2}$$

w.k. that

$$\cancel{\lim_{x \to 0} \left(\frac{1-\cos x}{x^2} \right) = \frac{1}{2}}$$

$$\Rightarrow \cancel{\lim_{x \to \frac{\pi}{2}} \left[\frac{1-\cos 2x}{(\pi-2x)^2} \right] = \frac{\pi}{2}}$$

Q. If α is the repeated Root of $ax^2+bx+c=0$ then $\lim_{x\to\alpha}\left[\frac{\tan(ax^2+bx+c)}{(x-\alpha)^2}\right]=?$

(a): a ✓

(b): b

(c): c

(d): a+b+c

w.k.that $\boxed{ax^2+bx+c=0 \Rightarrow a(x-\alpha)^2=0}$

eg $4x^2-24x+36=0$

Here $x=3$ is the Repeated Root.

$4(x^2-6x+9)=0$

$4(x-3)^2=0$

& w.k.that $\lim_{x\to 0}\left(\frac{\tan x}{x}\right)=1$

$$=\lim_{x\to\alpha}\left[\frac{\tan a(x-\alpha)^2}{a.(x-\alpha)^2}\right].a$$

$$=\lim_{a(x-\alpha)^2\to 0}\left[\frac{\tan a(x-\alpha)^2}{a(x-\alpha)^2}\right].a$$

$$=1\times a=a$$

If $f(a) = a^2$, $\varphi(a) = b^2$; $f'(a) = \varphi'(a)$, then

$\lim\limits_{x \to a} \dfrac{\sqrt{f(x)} - a}{\sqrt{\varphi(x)} - b}$ is equal to $= \frac{0}{0}$ form

(a) $\frac{a}{b}$

(b) $\frac{b}{a}$

(c) 0

(d) ∞

$$= \lim_{x \to a} \left(\frac{\frac{1}{2\sqrt{f(x)}} f'(x)}{\frac{1}{2\sqrt{\varphi(x)}} \varphi'(x)} \right) = \lim_{x \to a} \left(\frac{f'(x) \cdot \sqrt{\varphi(x)}}{\varphi'(x) \sqrt{f(x)}} \right) = \frac{\cancel{f'(a)} \cdot \sqrt{\varphi(a)}}{\cancel{\varphi'(a)} \sqrt{f(a)}}$$

$$= \frac{\sqrt{b^2}}{\sqrt{a^2}} = b/a$$

$$\lim_{x\to\infty}\left(e^{\frac{1}{5x}}-1\right)\left(5x+\frac{x}{5}\sin\frac{1}{x}\right) = \_\_\_\_$$

Ans = 1

Put $\frac{1}{5x} = y$

when $x \to \infty$, $y \to 0$

$x = \frac{1}{5y}$ & $\frac{1}{x} = 5y$

$$\lim_{y\to 0}\left(e^{y}-1\right)\left[\frac{1}{y}-\frac{1}{25y}\sin(5y)\right]$$

$$= \lim_{y\to 0}\left(\frac{e^{y}-1}{y}\right)\cdot\left[1-\frac{1}{25}\sin(5y)\right]$$

$$= 1(1-0) = 1$$

$$\lim_{x\to 0} e^x (\cos x)^{\frac{1}{\sin^2 x}} =$$

(a) 1

(b) $e^{-1/2}$

(c) $e^{1/2}$

(d) e

let $K_1 = (\cos x)^{\frac{1}{\sin^2 x}}$ (1^∞)

$$\log K_1 = \lim_{x\to 0}\left(\frac{\log \cos x}{\sin^2 x}\right)$$

$$= \lim_{x\to 0} \frac{\frac{1}{\cos x}(-\sin x)}{2\sin x \cos x}$$

$$= \lim_{x\to 0}\left(\frac{-1}{2\cos^2 x}\right) = -\frac{1}{2}$$

$$K_1 = e^{-1/2} = \frac{1}{\sqrt{e}}$$

Hence $K = \lim_{x\to 0}(e^x) \cdot \lim_{x\to 0}(\cos x)^{\frac{1}{\sin^2 x}}$

$$= 1 \times \frac{1}{\sqrt{e}}$$

$$\lim_{x\to 0}\frac{x(e^x-1)+2(\cos x-1)}{(1-\cos x)} = \underline{\quad 0 \quad}$$

(MI)

$$= \lim_{x\to 0}\frac{x(e^x-1)-2(1-\cos x)}{(1-\cos x)}$$

$$= \lim_{x\to 0}\left[\frac{\left\{\left(\frac{e^x-1}{x}\right)-2\left(\frac{1-\cos x}{x^2}\right)\right\}}{\left(\frac{1-\cos x}{x^2}\right)}\right]$$

$$= \frac{1-2\left(\frac{1}{2}\right)}{1/2} = \frac{0}{1/2} = 0$$

(M-II) use L-Hosp Rule

------ Ans $= 0$

If $\lim_{x\to 0}\frac{x(1+a\cos x)-b\sin x}{x^3}=1$

Question is having typing Error.

(a) $a=-\frac{5}{2}, b=-\frac{1}{2}$ (b) $a=-\frac{3}{2}, b=-\frac{1}{2}$ (c) $a=-\frac{3}{2}, b=-\frac{5}{2}$ (d) $a=-\frac{5}{2}, b=-\frac{3}{2}$

$$\lim_{x\to 0}\left[\frac{x(1+a\cos x)-b\sin x}{x^3}\right]=1$$

it is in $\frac{0}{0}$ form so,

$$\lim_{x\to 0}\left[\frac{[x(-a\sin x)+(1+a\cos x)]-b\cos x}{3x^2}\right]=1$$

$$\Rightarrow \frac{0+(1+a)-b}{0}=1 \Rightarrow a-b=-1$$

Now again it is in $\frac{0}{0}$ form so

$$\lim_{x\to 0}\frac{-ax\sin x+1+a\cos x-b\cos x}{3x^2}=1$$

$$\lim_{x\to 0}\frac{-a(x\cos x+\sin x)-a\sin x+b\sin x}{6x}=1$$

$$\lim_{x\to 0}\frac{-a(-x\sin x+\cos x+\cos x)-a\cos x+b\cos x}{6}=1$$

$$\frac{-2a-a+b}{6}=1 \Rightarrow \boxed{-3a+b=6}$$

So Ans = (d)

Thank

THANK

Keep Hustling!