GATE

DATA SCIENCE

Engineering Mathematics

SUPER 1500

Lec : 02 Linear - 2

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Topics *to be covered*

LINEAR ALGEBRA – II

1. Partition Matrix

2. Vector space & subspace

PROJECTION MATRIX (along a subspace spanned by set of vectors) →

Let W is the subspace spanned by set of vectors in A

& we want to to find projection of B onto the subspace spanned by A

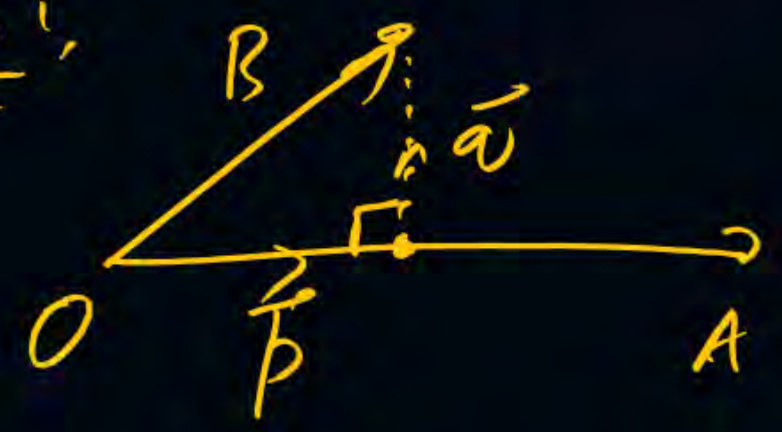
then Projection Mat is given as.

$$\boxed{P = A(A^TA)^{-1}A^T}$$ & Projection vector is $$\boxed{\bar{p} = PB}$$

PROJECTION MATRIX (along a vector $A_{n\times 1}$ or ONTO the line A) →

$$P = \left(\frac{AA^T}{A^TA}\right)_{n\times n}$$ and Projection vector of B onto A is $$\boxed{\bar{p} = PB}$$

Note:



$\vec{p}$ = projection vector of B onto A

& it is given as $\boxed{\vec{p} = PB}$

Similarly $\vec{q}$ = orthogonal complement of B onto A

& it is given as $\boxed{\vec{q} = B - \vec{p}}$ Learn.

$\because$ $\boxed{\vec{p} + \vec{q} = \vec{B}}$

or $P\vec{B} + Q\vec{B} = \vec{B}$

i.e. $\boxed{P + Q = I}$

i.e. orthogonal projection

Mat of B onto A is given as

$\boxed{Q = I - P}$ Learn.

Properties of Projection Matrix → $\boxed{P = A(A^TA)^{-1}A^T}$ & $\boxed{\tilde{p} = PB}$

① $\boxed{\text{Trace of Proj. Mat (onto the line)} = 1}$

& Trace of Proj Mat (onto the subspace) is Not Necessarily one

② Projection Mat is always Symmetric as well as Idempotent

(whether onto the line or onto the subspace)

③ Det of Projection Mat (except identity Matrix) is always = 0

(Doesn't matter, whether onto the line or onto the subspace)

eg $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ → $I^2 = I$ idempotent
→ $I^T = I$ symm

So I mat is also Projection Mat

(4) if A is Invertible i.e $|A| \neq 0$ & A^{-1} exist then $\boxed{P = I}$, $\boxed{\vec{p} = B}$ itself

(5) if B is orthogonal to the Columns of A then $P =$ No idea But $\boxed{\vec{p} = \vec{0}}$

(6) If B is LD on Columns of A then $P =$ No idea But $\boxed{\vec{p} = B}$ itself

Given a vector $V = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and a subspace W spanned by $W = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, what is the projection matrix P that projects any vector onto W, and what is the projection of V onto W?

(a) $P = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}$, Projection of $V = \begin{pmatrix} 2.5 \\ 2.5 \end{pmatrix}$

(b) $P = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}$, Projection of $V = \begin{pmatrix} 1.5 \\ 1.5 \end{pmatrix}$

(c) $P = \begin{pmatrix} 0.5 & 2.5 \\ 1.5 & 1.5 \end{pmatrix}$, Projection of $V = \begin{pmatrix} 1.5 \\ 1.5 \end{pmatrix}$

(d) $P = \begin{pmatrix} 0.5 & 0.25 \\ 1.5 & 1.25 \end{pmatrix}$, Projection of $V = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$

P $\quad \vec{p}$

$$P = \frac{WW^T}{W^TW} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}_{2\times 2}$$

$Tr = 1$

$|P| = 0$

$\lambda = 0, 1$

$\rho(P) = 1$

$$\vec{p} = P.V = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5/2 \\ 5/2 \end{bmatrix}$$

P is Diagonalizable $\because X_1$ & X_2 are LI

$$W^TW = \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 2$$

$$WW^T = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

11 In a 2-dimensional space R^2, consider the subspace W spanned by the vector $V_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Let P be the projection matrix onto W. Whcih of the following vectors is the image of $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ under projection P?

P

$\vec{b}$

(a) $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ (b) $\begin{bmatrix} 3 \\ 3 \end{bmatrix}$ (c) $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ (d) $\begin{bmatrix} 7/2 \\ 7/2 \end{bmatrix}$

$$w = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow P = \frac{ww^T}{w^Tw} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$ (already Calculated in Previous Quest)

$$\vec{p} = PB = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 7/2 \\ 7/2 \end{bmatrix}$$

For a 2 × 2 projection matrix P that projects onto the line $y=x$, what is the matrix P?

(a) $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

(b) $\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$

Along line $y=x$, Random vector $A=\begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow P=\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

0 Which of the following matrices represents a projection onto the line L with direction vector $d = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$?

(a) $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

(b) $\begin{bmatrix} \frac{1}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{4}{5} \end{bmatrix}$

(c) $\begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{bmatrix}$

(d) $\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$

$d^T d = [1\ 2]\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 5$

$d d^T = \begin{bmatrix} 1 \\ 2 \end{bmatrix}[1\ 2] = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

So $P = \frac{d d^T}{d^T d} = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

$Tr(P) = 1$

$|P| = 0$

$\lambda = 0, 1$ $\quad X_1 \,\&\, X_2$ are LI

$\rho(P) = 1$

Consider the vectors space R^3 and the subspace W spanned by vectors $W_1 = \begin{pmatrix}1\\1\\0\end{pmatrix}$ and $W_2 = \begin{pmatrix}0\\1\\1\end{pmatrix}$. What is the projection matrix P that projects any vector onto W and what is the projection of $V = \begin{pmatrix}3\\4\\5\end{pmatrix}$ onto W?

P $\vec{p}$

(a) $P = \begin{bmatrix}2 & 1 & 1\\1 & 2 & 1\\-1 & 1 & 1\end{bmatrix}$, Projection of $V = \begin{bmatrix}5\\4\\3\end{bmatrix}$ ✗

(b) $P = \frac{1}{3}\begin{bmatrix}2 & 1 & -1\\1 & 2 & 1\\-1 & 1 & 2\end{bmatrix}$, Projection of $V = \frac{1}{3}\begin{bmatrix}5\\16\\11\end{bmatrix}$ ✓

(c) $P = \begin{bmatrix}1 & 1 & 0\\0 & 1 & 0\\1 & 1 & 1\end{bmatrix}$, Projection of $V = \begin{bmatrix}2\\3\\4\end{bmatrix}$ ✗

(d) None of the above

$W = [W_1 W_2] = \begin{bmatrix}1 & 0\\1 & 1\\0 & 1\end{bmatrix} \Rightarrow W^TW = \begin{bmatrix}1 & 1 & 0\\0 & 1 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\1 & 1\\0 & 1\end{bmatrix} = \begin{bmatrix}2 & 1\\1 & 2\end{bmatrix}_{2\times 2}$

$(W^TW)^{-1} = \frac{1}{3}\begin{bmatrix}2 & -1\\-1 & 2\end{bmatrix}$

$\boxed{P = W(W^TW)^{-1}W^T}$

$= \underbrace{\begin{bmatrix}1 & 0\\1 & 1\\0 & 1\end{bmatrix}}_{3\times 2} \underbrace{\frac{1}{3}\begin{bmatrix}2 & -1\\-1 & 2\end{bmatrix}}_{2\times 2} \underbrace{\begin{bmatrix}1 & 1 & 0\\0 & 1 & 1\end{bmatrix}}_{2\times 3} = \frac{1}{3}\begin{bmatrix}2 & 1 & -1\\1 & 2 & 1\\-1 & 1 & 2\end{bmatrix}$

& $\vec{p} = PV = \frac{1}{3}\begin{bmatrix}5\\16\\11\end{bmatrix}$

GM of 1 = No. of L.I. of $(A-\lambda I)$
= two

for $\lambda = 1$, x_2 & x_3 exist

$Tr(P) = 2$, $|P| = 0$

$\lambda = 0, 1, 1$, $\rho(P) = 2$

Given a 3 × 3 projection matrix P that projects onto the plane spanned by the vectors $[1, 0, 1]^T$ and $[0, 1, 1]^T$. What will be the norm of $P\begin{bmatrix}1\\1\\1\end{bmatrix}$?

$A = \begin{bmatrix}1 & 0\\0 & 1\\1 & 1\end{bmatrix} \Rightarrow A^TA = \begin{bmatrix}1 & 0 & 1\\0 & 1 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 1\\1 & 1\end{bmatrix} = \begin{bmatrix}2 & 1\\1 & 2\end{bmatrix}$

$P = A(A^TA)^{-1}A^T = \underbrace{\begin{bmatrix}1 & 0\\0 & 1\\1 & 1\end{bmatrix}}_{3\times 2}\frac{1}{3}\underbrace{\begin{bmatrix}2 & -1\\-1 & 2\end{bmatrix}}_{2\times 2}\underbrace{\begin{bmatrix}1 & 0 & 1\\0 & 1 & 1\end{bmatrix}}_{2\times 3}$

(a) 1

(b) $\sqrt{2}$

(c) $\sqrt{3}$

(d) $2\sqrt{\frac{2}{3}}$

HW

$= \text{-------} = \frac{1}{3}\begin{bmatrix}2 & -1 & 1\\-1 & 2 & 1\\1 & 1 & 2\end{bmatrix}$

$\vec{p} = PB = \frac{1}{3}\begin{bmatrix}2 & -1 & 1\\-1 & 2 & 1\\1 & 1 & 2\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} = \frac{1}{3}\begin{bmatrix}2\\2\\4\end{bmatrix}$

$\|\vec{p}\| = \sqrt{p^Tp} = \sqrt{\frac{4}{9}+\frac{4}{9}+\frac{16}{9}} = \sqrt{\frac{24}{9}} = \sqrt{\frac{8}{3}} = \frac{2\sqrt{2}}{\sqrt{3}}$

Note: $Tr = 2$, $|P| = 0$, $\lambda = 0, 1, 1$, $\rho(P) = 2$ → GM of 1 = two.

In a data analysis scenario you have a data set represented by a 4×4 matrix X, where each row is a data point and each column is a feature. You want to project the data onto a 2-D subspace to visualise it. If the projection matrix P is a 4×4 matrix given by : $P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

Which of the following statements about the projected data is correct?

(a) The projected data will lie in the space spanned by vectors $[1\ 1\ 0\ 0]^T$ and $[0\ 0\ 1\ 1]^T$.

(b) The projection matrix P projects the data onto the x-axis and y-axis in the original feature space.

(c) The projected data will be identical to the original data matrix X.

(d) The projection matrix P effectively reduces the dimensionability of the data to 2 dimensions, and the basis vectors are aligned with the original features.

$A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \Rightarrow A^T A = ?$ then

$P = A(A^T A)^{-1} A^T = \cdots = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$ i.e. (a) is correct.

Note: $Tr(P) = 2$, $|P| = 0$, $\rho(P) = 2$, $\lambda = 0, 0, 1, 1$

GM of 0 = two & GM of 1 = two.

i.e. P is diagonalizable

$P = M^{-1} D M$ where M = Modal Mat of P

$= M^{-1} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} M$

$= \begin{bmatrix} X_1 & X_2 & X_3 & X_4 \end{bmatrix}$

$= LI$.

For a projection matrix P that projects onto a subspace spanned by $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ in R^2, what will be det (P)?

(a) 0 (b) 0.5

(c) 1 (d) 2

M-I $A = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow A^T A = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 5$

$P = \dfrac{AA^T}{A^T A} = \dfrac{1}{5} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

$|P| = 0$

& $Tr(P) = \frac{1}{5} + \frac{4}{5} = 1$

M-II using direct property that $|P| = 0$

If P is a 2×2 projection matrix such that $P\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}2\\4\end{bmatrix}$ and $P\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}1\\0.5\end{bmatrix}$, what is det (P)?

(a) 1.5

(b) 0.25

(c) 0.5

(d) No Valid Projection Matrix P

M-I Let $P=\begin{bmatrix}a & b\\c & d\end{bmatrix}$

$P\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}2\\4\end{bmatrix}$

$\Rightarrow$ get the eqn in terms of a, b, c, d

& $P\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}1\\1/2\end{bmatrix}$

$\begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}1\\1/2\end{bmatrix}$

get the eqn in terms of a, b, c, d

Now solve these eqn's & find $P=\begin{bmatrix}- & -\\- & -\end{bmatrix}=??$

M-II using Direct Prop that $|P|=0$

so only choice is (d)

If P is a 3×3 projection matrix that projects onto a 2D subspace, what is the characteristic polynomial P?

(a) $\lambda^3 - 2\lambda^2 + \lambda$ (b) $\lambda^3 - 2\lambda^2 - \lambda$

(c) $\lambda^3 - 3\lambda^2 + 2\lambda$ (d) $\lambda^3 - \lambda^2$

$P = \begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix}_{3\times 3} \Rightarrow \because P$ is idempotent so $\lambda = 0$ or 1

& $|P| = 0 \Rightarrow \lambda = 0$

$\therefore$ E. Values of 3×3 Projection Mat are $\lambda = 0, 1, 1$

C. Eqn is $(\lambda - 0)(\lambda - 1)(\lambda - 1) = 0$

$\lambda(\lambda^2 - 2\lambda + 1) = 0$

$\lambda^3 - 2\lambda^2 + \lambda = 0$ ie Char Poly is (a)

Let P be a projection matrix. Which of the following properties does not necessarily hold?

(a) P is idempotent. (T)

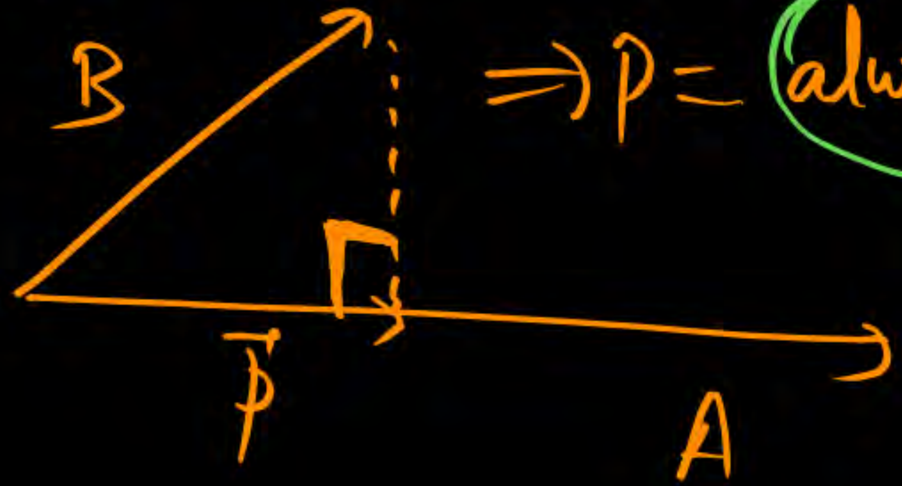
(b) The eigen values of P are either 0 or 1. (T)

(c) P is always diagonalizable. (T)

(d) P is always symmetric. ✓

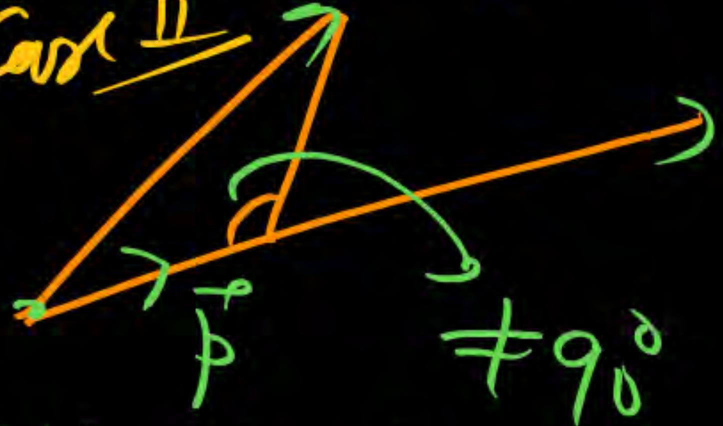
Projection Mat will be symm only for orthogonal Projections.

Case I



$\Rightarrow P =$ always symm

Case II



it is not orthogonal Projection

$P =$ Not Necessarily symm.

MSQ

For a matrix A with orthonormal columns, which of the following represents the orthogonal projection matrix onto the column space of A?

$n \times m$

90° projection

(a) A^TA

(b) AA^T

(c) $A(A^TA)^{-1}A^T$

(d) $AA^T(AA^T)^{-1}$

(e) I

A = orthonormal columns i.e. A is an Orthogonal Mat i.e. $\boxed{AA^T = A^TA = I}$

Sq. Mat.

$$\boxed{P = A(A^TA)^{-1}A^T} \text{ i.e. (c)}$$
$$= A(I)^{-1}A^T$$
$$= AA^T \text{ ie (b)}$$
$$= A^TA \text{ ie (a)}$$
$$= I \text{ ie (e)}$$

(d) $P = A(A^TA)^{-1}A^T \neq AA^T(AA^T)^{-1}$

so (d) is wrong

VECTOR SPACE

Def^n:→ Any set of vectors satisfying following properties is called V. space V

Imp

① if X & $Y \in V$ then $X+Y \in V$ (Closure property of V. addition)

② if X & $Y \in V$ then $X+Y=Y+X$ (Commutative prop of Addition)

③ if $X, Y, Z \in V$ then $X+(Y+Z)=(X+Y)+Z$ (Associative prop of Addition)

Imp

④ if $X \in V$ then $\exists\ 0 \in V$ st $X+0=X$ (Additive Identity)

⑤ if $X \in V$ then $\exists\ -X \in V$ st $X+(-X)=0$ (Additive Inverse)

Imp

(6) if $X \in V$ & c is any scalar then $c.X \in V$ (ie closure prop for scalar Multi)

(7) if $X \in V$ & λ, μ are scalars then $\lambda(\mu X) = (\lambda\mu)X$ (ie Associative prop of S-Multi)

(8) if $X \in V$ then $1.X \in V$ (multiplicative identity of S-Multi)

(9) if X & $Y \in V$ & λ is any scalar then $\lambda(X+Y) = \lambda X + \lambda Y$ (Distributive property)

(10) if $X \in V$ & λ, μ are scalars then $(\lambda+\mu)X = \lambda X + \mu X$ (Distributive prop)

Any set of Vectors satisfying above 10 properties called Vector space.

Subspace → Let V is a vector space & let $W \subset V$ i.e W is any subset of V s.t W is itself a V.space then W is called subspace of V

i.e W must satisfy all the 10 properties of V-space.

Concept of SPANNING in V-space → Let W is a vector space spanned by S then $\boxed{W = L\{S\}}$ $\Rightarrow$ Any Member of W can be expressed as a Linear Combination of vectors in S

Short cut to check subspace → "if X_1 & $X_2 \in W$ s.t $\boxed{CX_1 + X_2 \in}$ W then W is subspace"

OR

W is a subspace of any vector space if (i) it is closed under vector addition
(ii) it is closed under Scalar Multiplication

BASIS of Vector space → Let V is any Vector space & let S is any set of vectors then S is Called Basis of Vector space V if

① S spans V or $\boxed{L\{S\} = V}$

② S must Contains LI vectors

for eg $S = \left\{ \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{=X_1}, \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{=X_2} \right\}$ is Basis for R^2 ∵ $L\{S\} = R^2$; X_1 & X_2 are LI vectors

& $S = \left\{ \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{X_1 =}, \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{X_2 =}, \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{X_3 =} \right\}$ is Basis for R^3 ∵ $L\{S\} = R^3$; X_1, X_2, X_3 are LI vectors

∵ $S = [X_1 X_2 X_3]$ & $|S| \neq 0$

DIMENSION of Vector space → $\boxed{\dim(V) = \text{Number of elements in Basis of } V}$

for eg $\dim(R^2) = 2$, & $\dim(R^3) = 3$

PROPERTIES of BASIS & DIMENSION →

① Basis of any vector space is not unique (justified in Next example)

eg for R^3, standard Basis is $\underset{S_1}{\{\epsilon_1, \epsilon_2, \epsilon_3\}} = \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$

② if dimension of any vector space is n then any set of n LI vectors forms Basis for V

or if $\boxed{\dim(V) = n}$ then $\boxed{\text{Any set of } n \text{ LI vectors forms Basis of } V}$

eg $\because \dim(R^3) = 3 \implies$ Any set of 3 LI vectors forms Basis for R^3

Ex: for $V = R^3$, S_1 is Basis, $S_2 = \{(1\,2\,1)', (2\,1\,{-4})', (3\,{-2}\,1)'\}$ is Basis $\because |S_2| \neq 0$

$S_3 = \{(1\,2\,3)', (4\,5\,6)', (3\,4\,2)'\}$ is also Basis $\because |S_3| \neq 0$ But $S_4 = \{(1\,2\,3)', (4\,5\,6)', (3\,4\,5)'\}$ is not Basis $\because |S_4| = 0$

Q) Let M_n denote the vector space of all $n\times n$ real matrices. Consider the following subsets of M_n

MSQ

1. $W_1 = \{A \in M_n \mid A^2 = I\}$, where I is the identity matrix.
2. $W_2 = \{A \in M_n \mid \text{rank}(A) = 1\}$
3. $W_3 = \{A \in M_n \mid \text{trace}(A) = 0\}$
4. $W_4 = \{A+B \mid A \in M_n, B \text{ is a fixed matrix in } M_n\}$

Which of the following statements is correct?

(a) W_1 is a linear subspace of M_n (F)

(b) W_2 is a linear subspace of M_n.

c) W_3 is a linear subspace of M_n.

d) W_4 is a linear subspace of M_n only if B is a zero matrix.

1. $W_1 = \{A \in M_n : A^2 = I\}$, let $A, B \in M_n$ i.e. $A^2 = I$ & $B^2 = I$

Now $(A+B)^2 = A^2 + B^2 + AB + BA$

$= I + I + AB + BA \neq I$ i.e. $A+B \notin W_1$ i.e. Not closed under vector addition

(b) $W_2 = \{A \in M_n : \rho(A) = 1\}$,

let $A, B \in W_2$ i.e. $\rho(A) = 1$ & $\rho(B) = 1$

Now $\rho(A+B) \leq \rho(A) + \rho(B)$

$\rho(A+B) \leq 2$

i.e. there is a chance that $\rho(A+B) \neq 1$

so $A+B \notin W_2$ so Not closed under vector addition

$W_3 = \{A \in M_n : Tr(A) = 0\}$

let $A, B \in W_3$ i.e. $Tr(A) = 0$ & $Tr(B) = 0$

Now $Tr(A+B) = Tr(A) + Tr(B) = 0$

i.e. $A+B \in W_3$ i.e. closed under vector addition.

Now, $Tr(cA) = c\,Tr(A) = c \times 0 = 0$

i.e. $cA \in W_3$ so closed under scalar Multiplication

Hence W_3 is a subspace.

(d) $W_4 = \{(A+B) \in M_n$ where B is a fixed Matrix$\}$

for W_4 to be a subspace, W_4 must contains Additive identity

i.e Null Matrix

if $B \neq 0$ Then W_4 does not have Additive identity & Hence

W_4 to be a Subspace B must be Null Mat i.e $B = 0$

or B = fixed Matrix.

Q) Let S_1, S_2, S_3 be sets of real-valued functions defined as:

MSQ

$S_1 = \{ f \mid f(3) = 0 \}$

$S_2 = \{ g \mid g(x) = x+1 \text{ for all } x \in R \}$

$S_3 = \{ h \mid h(x) = c \text{ for some instant } c \in K \}$

Which of the following statements is correct?

a) S_1 is a vector space & S_2 is not a vector space. (T)

b) S_2 is a vector space & S_3 is not a vector space.

c) S_1 & S_3 are vector spaces, but S_2 is not a vector space (T)

d) S_2 & S_3 are vector spaces but S_1 is not a vector space.

$S_2 = \{ g : g(x) = x+1 \}$

let $g, h \in S_2$ i.e. $g(x) = x+1$

$h(x) = x+1$

Now $(g+h)(x) = g(x) + h(x)$

$= (x+1)+(x+1) \neq x+1$

So $g+h \notin S_2$ So S_2 is not a V.Space

$S = \{ f : f(3) = 0 \}$, let $f, g \in S_1$ i.e. $f(3) = g(3) = 0$

Now $(f+g)(3) = f(3) + g(3) = 0+0 = 0 \Rightarrow f+g \in S_1$

Now $(cf)(3) = c f(3) = c(0) = 0 \Rightarrow cf \in S_1$ So S_1 is V.Space.

$S_3 = \{ h(x) ;\ h(x) = c \}$, let h & $g \in S_3$ ie $h(x) = c$ & $g(x) = c$

Now $(h+g)(x) = h(x) + g(x) = c + c = 2c = c_1$ i.e const

$\Rightarrow (h+g) \in S_3$

Now $(\alpha h)(x) = \alpha\, h(x) = \alpha \cdot c = c_2$ i.e const.

$\Rightarrow \alpha h \in S_3$ hence S_3 is V. space

Q) What is the dimension of the vector space of all $n \times n$ real symmetric matrices where the sum of all off-diagonal elements is zero (for $n \geq 2$)

a) $\frac{n^2+n}{2} - 1$ (b) n^2+n+2

c) $\frac{n^2+n}{2} - 2$ (d) $\frac{n^2-n+2}{2}$

Total element in $A_{n\times n} = n^2$

No. of Diag. elements $= n$

No of off diag elements $= (n^2-n)$

No of off diag elements that lies above Main diag $= \left(\frac{n^2-n}{2}\right)$

Total Ind. Entries $= n + \left(\frac{n^2-n}{2}\right)$

$= \frac{n(n+1)}{2}$

Dimension of V.Sp formed by above def^n

= No. of Ind entries in above Mat.

But we have one more Restriction,

Sum of all off Diag elements $= 0$

So Req No. of Ind entries $= \frac{n(n+1)}{2} - 1$

Hence Dim (V.Sp) $= \frac{n(n+1)}{2} - 1$

$$A_{3\times 3} = \begin{bmatrix} -1 & 2 & 4 \\ 2 & 2 & -6 \\ 4 & -6 & 3 \end{bmatrix} = \text{Symm. Mat}$$

Total Ind entries $= 5$

Put $n = 3$ in $\frac{n(n+1)}{2} - 1 = \frac{3\times 4}{2} - 1$

$= 6 - 1$

$= 5$

Q) Let K_n be the space of all $n \times n$ matrices $A = a_{ij}$ with entries in $\mathbb{R}$ satisfying the following conditions.

1. $a_{ij} = a_{rj}$, $i+j = r+j$
2. $a_{ij} = a_{is}$, $i+j = i+s$

What is the dimension of K_n as a vector space over $\mathbb{R}$?

a) n^2 (b) $n+1$ (c) $2n-1$ (d) $2n+1$

$$A = [a_{ij}]_{n\times n} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & & a_{2n} \\ \vdots & \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nn} \end{bmatrix}$$

$i+j$ varies from 2 to $2n$ (i.e. No of entries $= 2n-1$)

i.e. No of Ind Entries $= (2n-1)$

$P = \begin{bmatrix} -10 \\ -1 \\ 3 \end{bmatrix}^T$, $Q = \begin{bmatrix} -2 \\ -5 \\ 9 \end{bmatrix}^T$ and $R = \begin{bmatrix} 2 \\ -7 \\ 12 \end{bmatrix}^T$ are three vectors. An orthogonal set of vectors having a span that contains P, Q, R is

(a) $\begin{bmatrix} -6 \\ -3 \\ 6 \end{bmatrix}$, $\begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix}$ it is the only O-set of vectors

(b) $\begin{bmatrix} -4 \\ 2 \\ 4 \end{bmatrix}$ $\begin{bmatrix} 5 \\ 7 \\ -11 \end{bmatrix}$ $\begin{bmatrix} 8 \\ 2 \\ -3 \end{bmatrix}$ × Not orthogonal

(c) $\begin{bmatrix} 6 \\ 7 \\ -1 \end{bmatrix}$ $\begin{bmatrix} -3 \\ 2 \\ -2 \end{bmatrix}$ $\begin{bmatrix} 3 \\ 9 \\ -4 \end{bmatrix}$ × " "

(d) $\begin{bmatrix} 4 \\ 3 \\ 11 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 31 \\ 3 \end{bmatrix}$ $\begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix}$ × " "

Span:

$P = X_1 - X_2$

$Q = X_1 + X_2$

$R = X_1 + 2X_2$

i.e. P, Q, R are in the Linear Span of X_1 & X_2

The following vector is linearly dependent upon the solution to the previous problem

(a) $\begin{bmatrix} 8 \\ 9 \\ 3 \end{bmatrix}$

(b) $\begin{bmatrix} -2 \\ -17 \\ 30 \end{bmatrix}$

(c) $\begin{bmatrix} 4 \\ 4 \\ 5 \end{bmatrix}$

(d) $\begin{bmatrix} 13 \\ 2 \\ -3 \end{bmatrix}$

$x_1 = \begin{bmatrix} -6 \\ -3 \\ 6 \end{bmatrix}$, $x_2 = \begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix}$

let us take (b):

$$A = \begin{array}{c} \begin{bmatrix} -6 & 4 & -2 \\ -3 & -2 & -17 \\ 6 & 3 & 30 \end{bmatrix} \\ \quad x_1 \quad\; x_2 \quad\; x_3 \end{array} \Rightarrow |A| = 0$$

So x_1, x_2 & x_3 are LD

i.e. x_3 is LD on x_1 & x_2


Thank
You
Keep Hustling!