GATE

CRASH COURSE

ALL BRANCHES

Engineering Mathematics

Complex Analysis (Part 01)
(Lec 08)

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Topics *to be covered*

1. General Properties
2. Complex function
3. Analytic func^n (C-R equ^{n}s)
4. Construction of an Analytic func^n

The line integral of function $F = yzi$, in the counter $\Rightarrow$ ACW

clockwise direction, along the circle $x^2 + y^2 = 1$ at

$z = 1$ is

on XY plane $\Rightarrow$ $\boxed{\hat{n} = \hat{k}}$

(a) -2π

(b) $-\pi$

(c) π

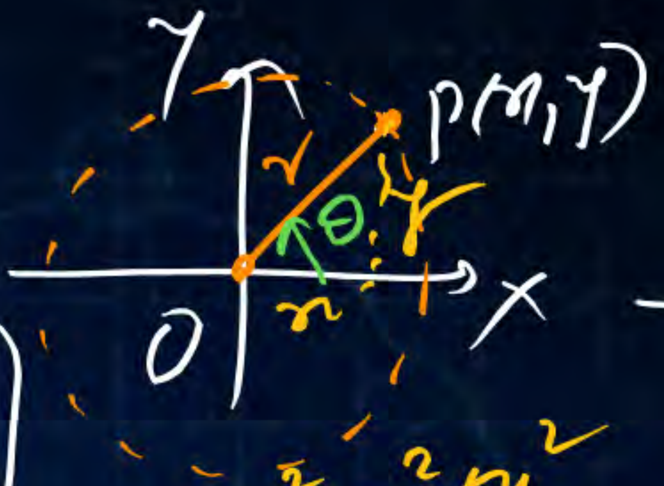
(d) 2π

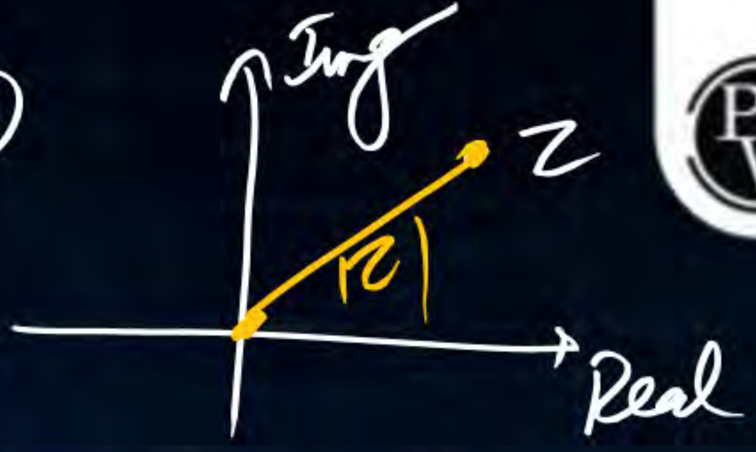
$$\vec{f} = yz\hat{i} \Rightarrow \text{curl}\,\vec{f} = \vec{\nabla} \times \vec{f} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & 0 & 0 \end{vmatrix} = 0\hat{i} - \hat{j}(-y) + \hat{k}(-z)$$

$$\text{curl}\,\vec{f} \cdot \hat{n} = (0\hat{i} - y\hat{j} - z\hat{k}) \cdot \hat{k} = -z$$

$$LI = \oint_C \vec{f} \cdot d\vec{r} = \iint_S (-z)\, dS = -1 \times \text{Area of } S = -\pi(1)^2 = -\pi$$

① Complex Number → $z = x + iy = (x, y)$





$$\boxed{z = x + iy = (x, y) = x\hat{i} + y\hat{j} = \begin{bmatrix} x \\ y \end{bmatrix}}$$

$r^2 = x^2 + y^2$ $\qquad r = |z|$,

$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \text{Arg}(z)$

eg $2 + 3i = (2, 3)$

$-5i = 0 + (-5)i = (0, -5)$

$2 = 2 + (0)i = (2, 0)$

$3 - 4i = (3, -4)$

② if $z = x + iy \Rightarrow r = |z| = \sqrt{x^2 + y^2}$

& $\theta = \text{Arg}(z) = \tan^{-1}\left(\frac{y}{x}\right)$

③ $z = x + iy$

- x = Real part of $z = \dfrac{z + \bar{z}}{2}$
- y = Img part of $z = \dfrac{z - \bar{z}}{2i}$

& $\boxed{\bar{z} = x - iy}$

④ $\boxed{i = \sqrt{-1},\ i^2 = -1,\ i^3 = -i,\ i^4 = 1}$

To Find Principle Values of θ → { these lies in $(-\pi, \pi]$ } i.e $-\pi < \theta \le \pi$

Let us take $\alpha = \tan^{-1}\left|\frac{y}{x}\right|$ & then take θ as follows;

- II: $\theta = (\pi - \alpha)$
- I: $\theta = \alpha$
- III: $\theta = -(\pi - \alpha)$
- IV: $\theta = -\alpha$

Q. Find the Arg. of $z_1 = 1+i$ (I) & $z_2 = -1-i$ ⇒ III Quad.

Sol. $\alpha = \tan^{-1}\left|\frac{y}{x}\right| = \tan^{-1}(1) = \frac{\pi}{4}$

So $\theta_1 = \alpha = \pi/4$ & $\theta_2 = -(\pi - \alpha) = -\frac{3\pi}{4}$

Q. If $z = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ (II) then Find $|z|$ & $\theta = ?$

Sol. $|z| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$

$\alpha = \tan^{-1}\left|\frac{\sqrt{3}/2}{-1/2}\right| = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3} \Rightarrow \theta = \pi - \alpha = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

Euler Notation: it is given as;

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta} \quad \text{Learn.} \quad (1)$$

$$\& \; e^{-i\theta} = \cos\theta - i\sin\theta \quad (2)$$

i.e. $\boxed{1 = e^{2\pi i}, \; -1 = e^{\pi i}, \; i = e^{\frac{\pi}{2}i}, \; -i = e^{-\frac{\pi}{2}i}}$

De Moivre's Th:-

$$(\cos\theta \pm i\sin\theta)^n = \cos(n\theta) \pm i\sin(n\theta)$$

Proof: $LHS = (e^{i\theta})^n = e^{i(n\theta)}$

$$= \cos(n\theta) + i\sin(n\theta)$$

$$= RHS$$

Evaluate the values of ?

$$e^{2\pi i} = ? = \cos(2\pi) + i\sin(2\pi) = 1$$

$$e^{\pi i} = ? = \cos(\pi) + i\sin(\pi) = -1$$

$$e^{\frac{\pi}{2}i} = ? = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) = i$$

$$e^{-\frac{\pi}{2}i} = ? = \cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right) = -i$$

By (1) & (2) $\boxed{\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}}$ & $\boxed{\sin\theta = \frac{e^{+i\theta} - e^{-i\theta}}{2i}}$

eg $e^{\frac{3\pi}{2}i} - e^{-\frac{3\pi}{2}i} = ? = 2i\sin\frac{3\pi}{2} = -2i$

$$Z = x+iy = r(\cos\theta + i\sin\theta) = \boxed{r e^{i\theta}}$$

Best presentation

where $r = |z|$, $\theta = \mathrm{Arg}(z)$, $\boxed{|e^{i\theta}| = 1}$ $(\because \cos^2\theta + \sin^2\theta = 1)$

eg $Z_1 = 1+i$, $r_1 = \sqrt{2}$, $\theta_1 = \frac{\pi}{4} \Rightarrow Z_1 = r_1 e^{i\theta_1} = \sqrt{2}\, e^{\frac{\pi}{4} i}$

$Z_2 = -1-i$, $r_2 = \sqrt{2}$, $\theta_2 = \frac{-3\pi}{4} \Rightarrow Z_2 = r_2 e^{i\theta_2} = \sqrt{2}\, e^{\left(-\frac{3\pi}{4}\right) i}$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\cosh\theta = \frac{e^{\theta} + e^{-\theta}}{2}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2}$$

$$\sinh\theta = \frac{e^{\theta} - e^{-\theta}}{2}$$

(*) Equⁿ of circle in Complex form → Let C is the circle having centre at (x_0, y_0) & rad $= r$

then $(x-x_0)^2+(y-y_0)^2 = r^2$ (Cartesian form)

Let $z_0 = x_0+iy_0 \simeq (x_0,y_0)$ & $z = x+iy \simeq (x,y)$

then $z-z_0 = (x-x_0)+i(y-y_0)$

& $|z-z_0| = \sqrt{(x-x_0)^2+(y-y_0)^2}$

$|z-z_0|^2 = (x-x_0)^2+(y-y_0)^2$

$|z-z_0|^2 = r^2$

$\Rightarrow \boxed{|z-z_0| = r}$ — Centre $= z_0 \simeq (x_0,y_0)$, rad $= r$

eg $|z| = 3$ circle — centre $= 0 \simeq (0,0)$, rad $= 3$

eg $|z-2+3i| = 2$ — Centre $= 2-3i \simeq (2,-3)$, rad $= 2$

The least positive integer n such that $\left(\frac{2i}{1+i}\right)^n$ is a positive integer is

(a) 2 (b) 4 (c) 8 (d) 16

$$z=\left(\frac{2i}{1+i}\right)^n=\left(\frac{2i}{1+i}\times\frac{1-i}{1-i}\right)^n=\left[\frac{2i(1-i)}{1-i^2}\right]^n=\left[i(1-i)\right]^n=(i-i^2)^n=(1+i)^n$$

for $n=2$ $z=(1+i)^2=1+i^2+2i=2i\notin Z^+$

$n=4$, $z=(1+i)^4=(1+i)^2(1+i)^2$

$=(2i)(2i)$

$=4i^2=-4\notin Z^+$

Similarly check for $n=8$ we will get ans.

M-II $z=\left(\frac{2i}{1+i}\right)^n=(1+i)^n$

$$z=\left(\sqrt{2}e^{i\frac{\pi}{4}}\right)^n=2^{\frac{n}{2}}e^{\frac{n\pi}{4}i}$$

for $n=8$, $z=2^{\frac{8}{2}}e^{\frac{8\pi}{4}i}=2^4e^{2\pi i}=16$

i.e. $z\in I^+$

$z = \frac{\sqrt{3}}{2} + \frac{i}{2} = I$

$r = |z| = \sqrt{\frac{3}{4} + \frac{1}{4}} = 1$

$\theta = \alpha = \tan^{-1}\left(\frac{1/2}{\sqrt{3}/2}\right) = \frac{\pi}{6}$

If a complex number $z = \frac{\sqrt{3}}{2} + i\frac{1}{2}$ then z^4 is

(a) $2\sqrt{2} + 2i$

(b) $-\frac{1}{2} + i\frac{\sqrt{3}}{2}$

(c) $\frac{\sqrt{3}}{2} - i\frac{1}{2}$

(d) $\frac{\sqrt{3}}{8} - i\frac{1}{8}$

So $z = \frac{\sqrt{3}}{2} + \frac{i}{2} = re^{i\theta} = 1 \cdot e^{i\left(\frac{\pi}{6}\right)}$

So $z^4 = \left(e^{i\frac{\pi}{6}}\right)^4 = e^{i\left(\frac{2\pi}{3}\right)} = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)$

$= -\frac{1}{2} + i\left(\frac{\sqrt{3}}{2}\right)$

Consider the circle $|z-5-5i|=2$ in the complex number plane (x, y) with $z = x + iy$. The minimum distance from the origin to the circle is

(a) $5\sqrt{2}-2$ (b) $\sqrt{54}$

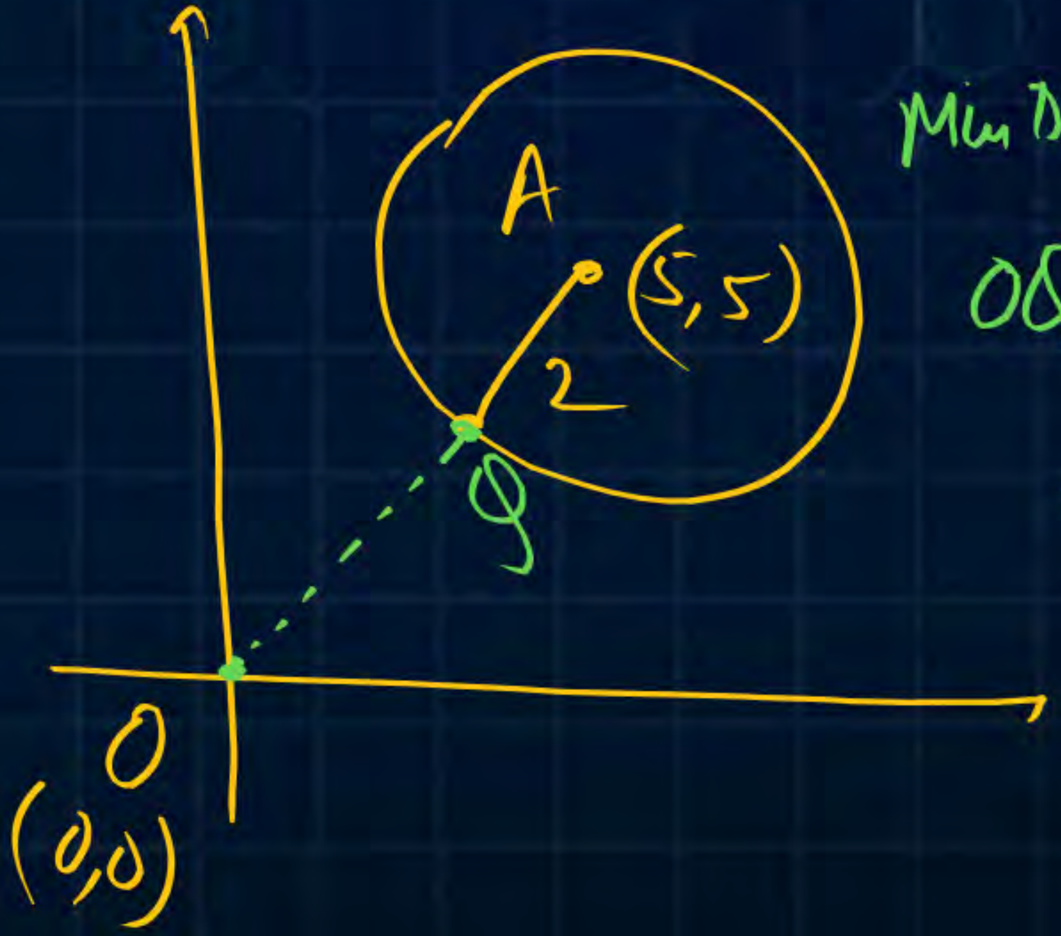
(c) $\sqrt{34}$ (d) $5\sqrt{2}$

$|z-5-5i|=2$

$|z-(5+5i)|=2$

Centre $=(5+5i)=(5,5)$

rad $=2$



Min Dist $= OQ$

$OQ = OA - AQ$

$= \sqrt{(5-0)^2+(5-0)^2} - 2$

$= 5\sqrt{2} - 2$

(*) Cube Roots of unity:- Consider $\boxed{z^3-1=0} \Rightarrow z=(1)^{1/3}$ → 1, ω, ω^2

where $\omega = -\frac{1}{2}+i\frac{\sqrt{3}}{2}$ & $\omega^2 = -\frac{1}{2}-i\frac{\sqrt{3}}{2}$ & $|\omega| = |\omega^2| = 1$

$= e^{i\left(\frac{2\pi}{3}\right)}$ $\qquad = e^{i\left(\frac{4\pi}{3}\right)}$ $\qquad \because \omega^2 = \omega\omega = e^{i\left(\frac{2\pi}{3}\right)} \cdot e^{i\left(\frac{2\pi}{3}\right)} = e^{i\left(\frac{4\pi}{3}\right)}$

(*) $\overline{\omega} = \omega^2$, $\overline{\omega^2} = \omega$, $\boxed{\omega^3 = 1}$, $\boxed{1+\omega+\omega^2 = 0}$

(*) Cube Roots of (-1):- Consider $\boxed{z^3+1=0} \Rightarrow z=(-1)^{1/3}$ → -1, $-\omega$, $-\omega^2$

where ω is defined as above.

(*) $z = x+iy$ then $|z|^2 = x^2+y^2$, & $\boxed{z^2 = (x^2-y^2)+i(2xy)}$ learn.

If a complex number ω satisfies the equation $\omega^3 = 1$ then value of $1 + \omega + \frac{1}{\omega}$ is

(a) 0 (b) 1

(c) 2 (d) 4

$$1 + w + \frac{1}{w} = 1 + w + \frac{w^2}{w^3}$$

$$= 1 + w + \frac{w^2}{1}$$

$$= 0$$

Complex func^n

$z = 2+3i$ (Comp Number)

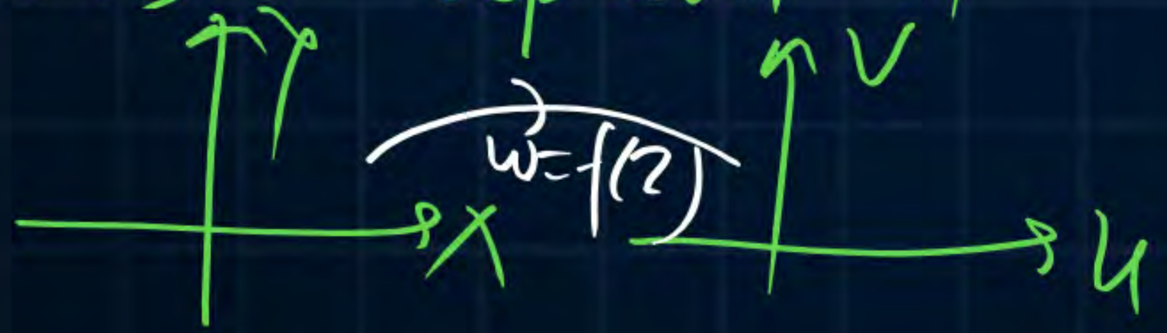
$z = x+iy$ (Comp Variable)

$w = f(z)$ (Comp func^n)

$$\boxed{u+iv = f(x+iy)} \quad \begin{cases} u = u(x,y) \\ v = v(x,y) \end{cases}$$

$(x,y) \to$ Independent Variables

$(u,v) \to$ Dependent "



The complex number $z = x + iy$ which satisfies the equation $\left|\frac{z-5i}{z+5i}\right| = 1$ lies on

(a) The x-axis

(b) The line $y = 5$

(c) A circle through the origin

(d) None of these

ATQ, $\left|\frac{z-5i}{z+5i}\right| = 1$, where $z = x+iy$

$|z-5i| = |z+5i|$

$|(x+iy)-5i|^2 = |(x+iy)+5i|^2$

$|x+i(y-5)|^2 = |x+i(y+5)|^2$

$x^2+(y-5)^2 = x^2+(y+5)^2$

$y^2+25-10y = y^2+25+10y$

$20y = 0 \Rightarrow y=0$ i.e. X axis

The locus represented by $|z-1|=|z+i|$ is

(a) The circle of radius 1

(b) An ellipse with foci at 1 and $-i$

(c) A circle through the origin

(d) Line passing through origin

$|z-1|=|z+i|$ where $z=x+iy$

$|(x-1)+iy|^2=|x+i(y+1)|^2$

$(x-1)^2+y^2=x^2+(y+1)^2$

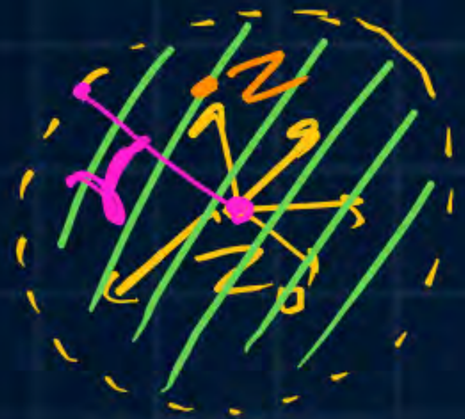
$x^2+1-2x+y^2=x^2+y^2+1+2y$

$x+y=0 \Rightarrow y=-x$

ANALYTIC funcⁿ

Neighbourhood of comp No (z_0) → open disc i.e

Nbd of z_0 is $\boxed{|z-z_0| < r}$



Derivative of Complex funcⁿ $w = f(z)$ (at z_0) → $f(z)$ is said to diff at z_0

if $\lim\limits_{z \to z_0} \left(\dfrac{f(z) - f(z_0)}{z - z_0} \right)$ exist & value of this limit is called $f'(z_0)$

≃ Not an Easy task ∵ $\lim\limits_{(x,y) \to (x_0, y_0)} \Big(\quad \Big]$ ≃ Double limit Concept.

ANALYTIC (at z_0) → $w=f(z)$ is called Analytic at z_0 if

"it is not only differentiable at z_0 but it should also diff at every point that lies in the Nbd of z_0"

[It is almost impossible to check Analyticity using the defn of differentiability.] ← V.V. tough

Hence we would take the help of C-R equns to check Analyticity.

C-R equn Let $f(z)=u+iv$ then $\dfrac{\partial u}{\partial x}=\dfrac{\partial v}{\partial y}$ & $\dfrac{\partial u}{\partial y}=-\dfrac{\partial v}{\partial x}$

or $\boxed{u_x=v_y \text{ \& } u_y=-v_x}$

$u = x^2 + ay^2$

$v = bxy$

If $f(z) = \underbrace{(x^2 + ay^2)}_{=u} + i\underbrace{bxy}_{v}$ is a complex analytic function of $z = x + iy$, where $i = \sqrt{-1}$, then

(a) $a = -1, b = -1$

(b) $a = -1, b = 2$

(c) $a = 1, b = 2$

(d) $a = 2, b = 2$

By C-R equⁿ; $u_x = v_y$ & $u_y = -v_x$

$2x = bx$ $\quad$ $2ay = -by$

$b = 2$ $\quad$ $2a = -2$

$a = -1$

F(Z) is a function of the complex variable z = x + iy given by

$$F(Z) = iz + k\,\mathrm{Re}(Z) + i\,\mathrm{Im}(Z)$$

For what value of k will F(Z) satisfy the Cauchy-Riemann equations?

(a) 0 (b) 1

(c) −1 (d) y

$$F(z) = i(x+iy) + k(x) + i(y)$$
$$= (kx - y) + i(x+y)$$

i.e. $u = kx - y$ & $v = x + y$

By CR eq^n; $u_x = v_y$ & $u_y = -v_x$

$k = 1$ & $-1 = -1$

$|z| \le 1$

interior & boundary of unit circle

$|z| = 1$

Which one of the following functions are analytic in the region $|z| \le 1$?

(a) $\frac{z^2-1}{z+2}$ $z = -2$

(b) $\frac{z^2-1}{z+i(0.5)}$ $z = \frac{-i}{2}$

(c) $\frac{z^2-1}{z}$

$z = 0$

(d) $\frac{z^2-1}{z-0.5}$

$z = \frac{1}{2}$

Construction of an Analytic func^n (Short Cut method)

(MILNE THOMSON method) $f(z) = u + iv$

Case I: (if Real part u is given)

step ① Find $u_x = ? = \phi_1(x, y)$ (let)

" ② find $\phi_1(z, 0) = ?$

" ③ find $u_y = ? = \phi_2(x, y)$ (let)

" ④ Find $\phi_2(z, 0) = ?$

" ⑤ $f(z) = \int (\phi_1 - i\phi_2)\, dz + c$

Case II: (if Img part v is given) →

(i) find $v_x = ? = \phi_2(x, y)$ (let)

(ii) find $\phi_2(z, 0) = ?$

(iii) find $v_y = ? = \phi_1(x, y)$ (let)

(iv) find $\phi_1(z, 0) = ?$

(v) $f(z) = \int (\phi_1 + i\phi_2)\, dz + C$

Total Derivative :-

$z \to (x, y)$

$$dz = \left(\frac{\partial z}{\partial x}\right)dx + \left(\frac{\partial z}{\partial y}\right)dy$$

Potential function ϕ is given as $\phi = x^2 - y^2$. What will be the stream function (ψ) with the condition $\psi = 0$ at $x = y = 0$?

(a) $2xy$
(b) $x^2 + y^2$
(c) $x^2 - y^2$
(d) $2x^2y^2$

Or

if $u = x^2 - y^2$

then $v = ?$, $v(0,0) = 0$

if $f(z) = u + iv$ is an Analytic func^n

(M-I) Here $v \to (x, y)$

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy \quad \left\{\begin{matrix}\text{By C-R equ}^n \\ v_x = v_y \ \& \ u_y = -v_x\end{matrix}\right\}$$

$$= \left(-\frac{\partial u}{\partial y}\right)dx + \left(\frac{\partial u}{\partial x}\right)dy$$

$$= -(-2y)dx + (2x)dy$$

i.e. $dv = 2(y\,dx + x\,dy)$

$$\int dv = 2\int d(xy) + c$$

$$v = 2(xy) + c$$

At $(0,0)$, $v = 2xy$

Q: $\boxed{w = \phi + i\psi}$ where $\phi = x^2 - y^2$ then $\psi = ?$ if $\psi(0,0) = 0$

ϕ → vel. pot $\qquad$ ψ → stream funcⁿ

$f(z) = u + iv$ where $u = x^2 - y^2$ then $v = ?$ if $v(0,0) = 0$

Sol: (i) $u_x = 2x = \phi_1(x,y)$

(ii) $\phi_1(z,0) = 2z$

(iii) $u_y = -2y = \phi_2(x,y)$

(iv) $\phi_2(z,0) = 0$

(v) $f(z) = \int (\phi_1 - i\phi_2)\,dz + C$

$f(z) = \int [2z - i(0)]\,dz + C = 2\int z\,dz + C$

$f(z) = 2\left(\frac{z^2}{2}\right) + C = z^2 + C$

$u + iv = (x+iy)^2 + C$

$u + iv = (x^2 - y^2) + i(2xy) + C$

$u = x^2 - y^2$

$v = 2xy + C$

$u = x^2 - y^2$

$v = ?$

$$U = (x^2 - y^2) + i(2xy) + c_1$$

$$= (x^2 - y^2) + i(2xy) + iC$$

$$= (x^2 - y^2) + i(2xy + C)$$

Let $c_1 = iC$

$\because c_1$ and C are Arbitrary const

If $f(x + iy) = \underbrace{x^3 - 3xy^2}_{u} + i\underbrace{\phi(x, y)}_{v}$ where $i = \sqrt{-1}$

and $f(x + iy)$ is an analytic function then $\phi(x, y)$ is

(a) $y^3 - 3x^2y$

(b) $3x^2y - y^3$ ✓

(c) $x^4 - 4x^2y$

(d) $xy - y^2$

use case I of M-Th method → HW → (b)

Tel: $\boxed{\text{dr puneet sir pw}}$

Thank

THANK

Keep Hustling!