Module 4

Modern Navigation Systems

Newton’s Method for Solving Systems of Nonlinear Equations

Module 4A

Overview, Newton’s Method in 1 Dimension

Summary of Module 4

- Students will learn to Newton's method in one dimension to solve the nonlinear problem $M = E - e \sin E$, an important problem with respect to the orbits of satellites and planets
- Students will extend Newton's method to the solution of a system of nonlinear equations
- The use of Newton's method in future modules will be summarized briefly

The fundamental theorem of Algebra

- The **fundamental theorem of algebra** states that every non-constant single-variable polynomial with complex coefficients has at least one complex root. This includes polynomials with real coefficients, since every real number is a complex number with zero imaginary part.
- Equivalently (by definition), the theorem states that the field of complex numbers is algebraically closed.
- The theorem is also stated as follows: every non-zero, single-variable, degree *n* polynomial with complex coefficients has, counted with multiplicity, exactly *n* roots. The equivalence of the two statements can be proven through the use of successive polynomial division.
- In spite of its name, there is no purely algebraic proof of the theorem, since any proof must use the completeness of the reals (or some other equivalent formulation of completeness), which is not an algebraic concept. Additionally, it is not fundamental for modern algebra; its name was given at a time when the study of algebra was mainly concerned with the solutions of polynomial equations with real or complex coefficients.
- **From *http://en.wikipedia.org/wiki/Fundamental_theorem_of_algebra***

Why do we care?

- Because of the spherical geometry, many navigation problems become the problem of solving a system of nonlinear equations
  - For which perturbation theory needs to be supplanted by iterative perturbation methods
  - This becomes the problem of finding the roots of polynomials using iterative techniques

For example…

- Consider the orbital equation *M = E – e sinE,* where e is a constant that is known
  - *This equation is nonlinear*
  - *If E is known, solving for M is trivial*
  - *If M is known, solving for E is nontrivial*
- *To find E, solve for the roots of the equation f(E) = E - esinE -M*
  - *Since sin(E) can be expressed as a polynomial by using a Taylor series, we know that f(E) has a solution*
  - *More importantly, this solution can be found using iterative techniques*
    - *Of which Newton's method is the classic (although not necessarily the most efficient) approach*

What is $M = E - e \sin E$

- As you will see in a future module, M is the mean anomaly of a two dimensional elliptical Keplerian orbit, such as:
  - The moon around the earth
  - A satellite around the earth
  - The earth around the sun
  - The moons around Jupiter
  - Etc.

What is M?

- The mean anomaly is defined as an imaginary point, given as an angle, on a <u>circle</u>, and is a simple function of time
- E maps the position on the circle to a corresponding position on an <u>ellipse</u>
- This mapping is nonlinear, except in the case where $e = 0$, and the ellipse becomes the circle that is defined by M

Figure 5.5 from the text

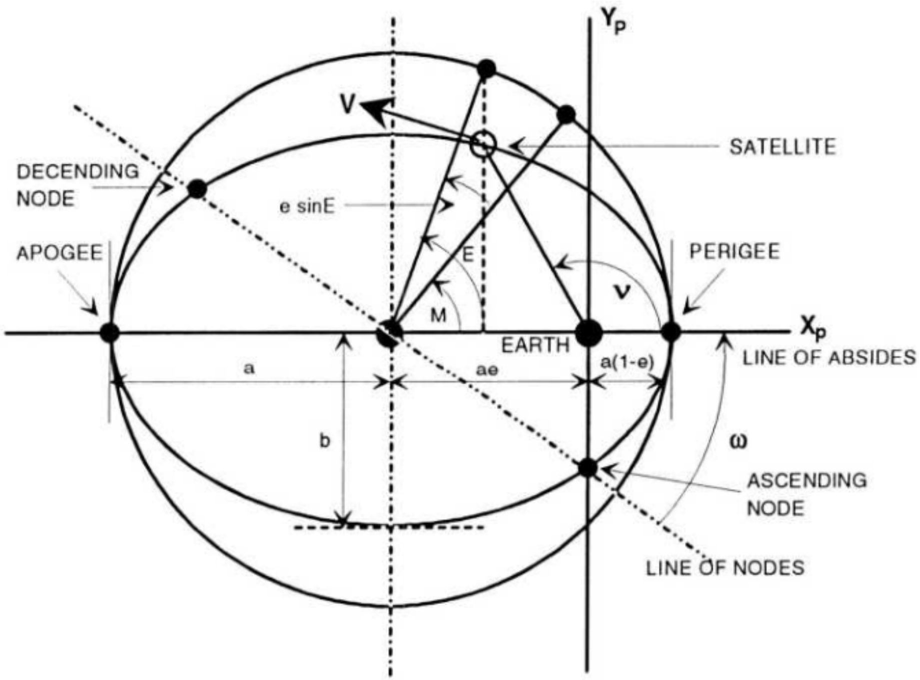


Figure 5.5 The elliptical orbit.

This rather cluttered diagram from the text depicts an elliptical orbit inscribed within a circle. The open circle shows the instantaneous position of a satellite within the elliptical orbit. Note that its position can be specified by the angle E, the *elliptic anomaly*.

The angle M is the *mean*, or average *anomaly*. It is easily computed. Then, E can be computed using the techniques developed in this module.

Greenwich Mean Time

- The earth's orbit around the sun is elliptical
- Because of Kepler's second law, the earth moves slower near apogee than it does near perigee.
- For reasons of convenience, clock time is linked to M, not E
    - Hence the name "Greenwich Mean Time"
- This will be described in detail in Module 5

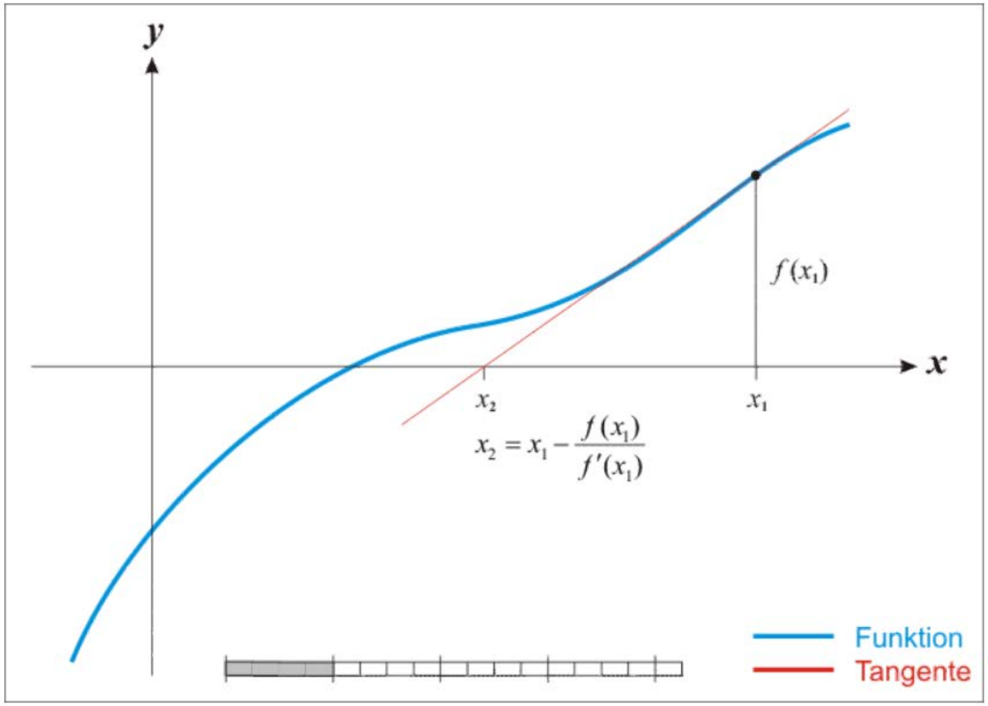
Summary of Newton's Method

- A technique for solving nonlinear equations using an iterative technique
- It depends on the behavior of the derivatives of the equation being solved
- Also depends on the starting point of the computation
- The number of iterations required for convergence can be significant (e.g., in the hundreds)

The fundamental approach

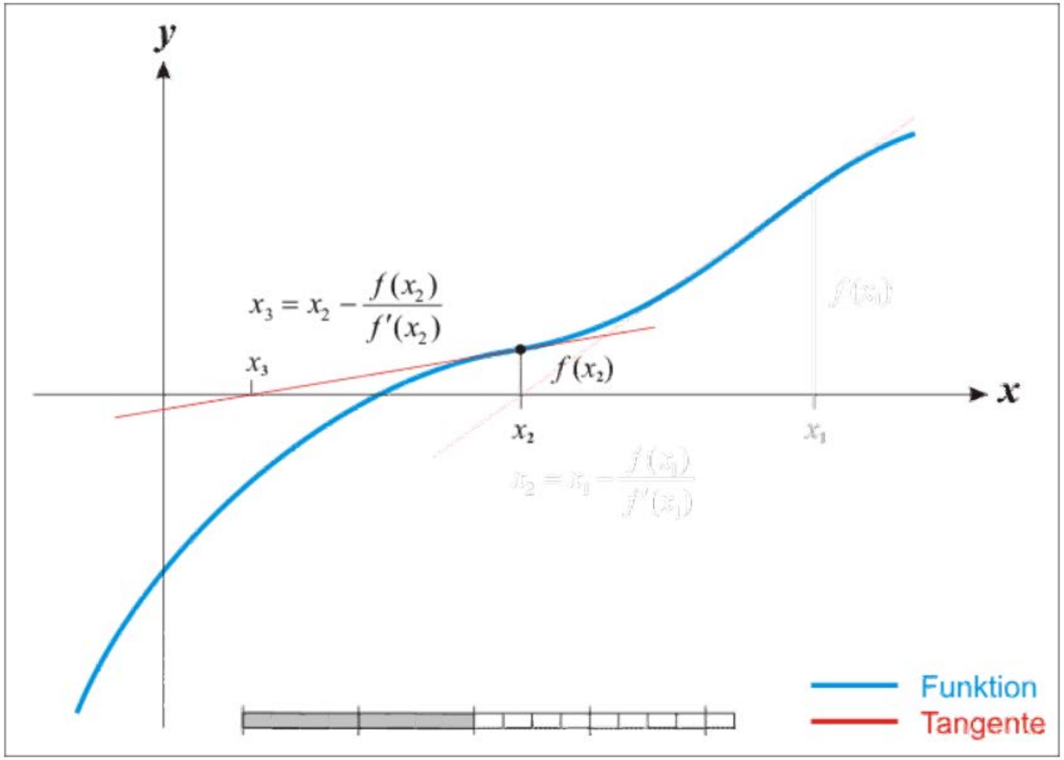
- Example: find the roots of a polynomial f(x)
- Write as f(x) = 0
- Guess an initial value $x = x_0$
- Compute the derivative $f'(x_0)$
- Solve iteratively for the value of x_n that makes $f(x_n) = 0$.
- This value is the desired solution to the problem

First iteration of Newton's method in one variable

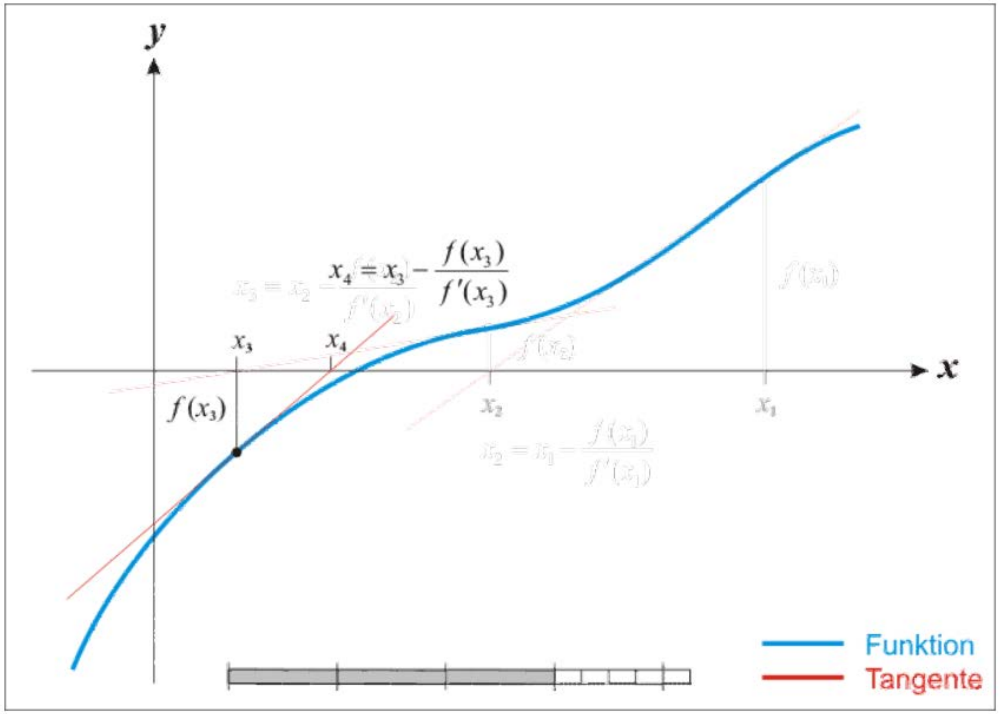


From Wikipedia:http://upload.wikimedia.org/wikipedia/commons/e/e0/NewtonIteration_Ani.gif

Second iteration of Newton's method in one variable



Third iteration of Newton's method in one variable



Convergence of Newton's Method

- Fundamental theorem of algebra
  - n^{th} order polynomial has n roots (which may include complex numbered roots)
- Depends on the first and subsequent (n-1) guesses
- Depends on the slope at $f(x_n)$ not being equal to 0
- Polynomial root finders address these problems to
  - Find all of the roots
  - Avoid getting stuck in an endless-loop, underflow, or overflow situation

Rate of convergence

- The method is slow for navigation algorithms (requires many iterations) because small values of the derivative J’(x) make a large numerical difference in the putative solution to the problem from one iteration to the next
- Whereas, the desired result is for the each iteration to bring one significantly closer to the actual solution to the problem

Other issues

- The ability of Newton's method to find all of the roots of a polynomial depends on the starting guesses for the parameter x at the beginning of the search for each root of a polynomial
- Fortunately, for navigation problems, the issue is solving a system of low order polynomials, rather than finding all of the roots of a single, high order polynomial

Application to navigation problems

- Start with y = f(x), where y is typically a known value (e.g., a measurement) and x is the unknown parameter that one wishes to know
  - M = E – e sin E, where M and e are known, but E is not
- Rewrite as f(x) – y = 0
  - f(E) = E – e sin E – M
- Guess an initial value E_0
- Compute the derivative $f'(E_0)$
- Solve iteratively for the value of E_n that makes $f(E_n) = 0$.
- This value is the desired solution to the problem

Assignment 4.1

4.1.1 Use Newton's method to find the value of x for which the curve

$y = 2x + 3x^2$ is equal to 0

4.1.2 For M = E - e sin E, solve for E when M = π/2 and e = .3

End of Mod 4A