Module 7

Modern Navigation Systems

Lines of position and weighted least squares

Module 7A
Lines of Position

Summary of Module 7

- **Students will combine the perturbation analysis and position determination skills from previous modules to define lines of position. The goal is to use the techniques of linear algebra to compute the best estimate of the point in three dimensional space at which the lines intersect. This will yield a "position fix". (7A)**
- The techniques are then extended to the case where the problem is over-specified. That is, m measurements are made with respect to the solution of an n-dimensional set of equations, where $m > n$. This leads to the least-squares solution of the problem. (7B)
- The concept of measurement uncertainty is introduced, leading to the technique of weighted least squares. This leads to the concept of the Kalman filter, which is introduced briefly. (7C)

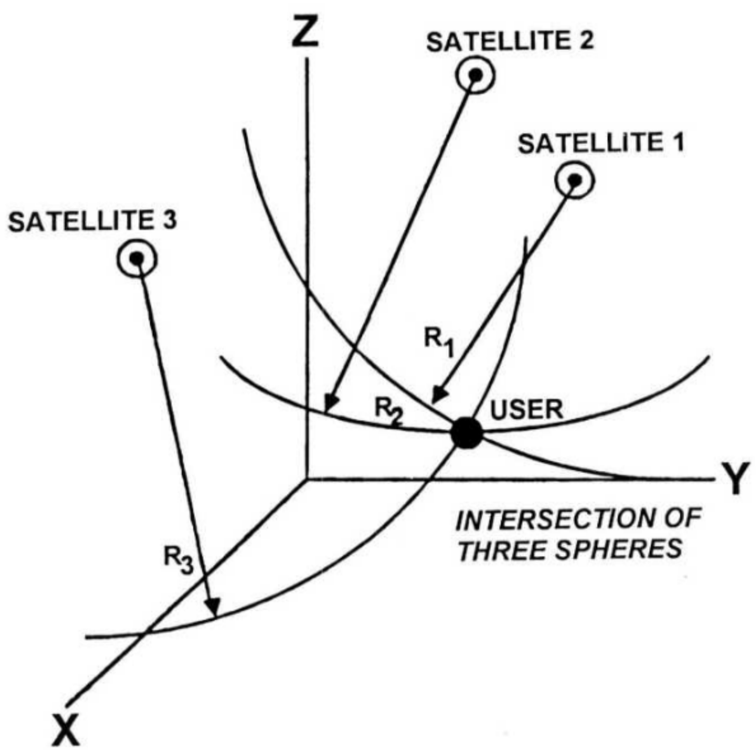

Figure 5.2 Ranging satellite radio-navigation solution.

Lines of position

GPS is used for navigation not only on the surface of the earth, but above the ground and in space. Hence, the circles of position, as represented in chapter 5 of the primary text by Kayton & Fried and reproduced here, become *spheres* of position. The point where the spheres intersect is the location of the “user,” as shown in the diagram to the left. Spheres from 3 satellites are shown, as the navigation solution is in three dimensions, providing values for latitude, longitude, and altitude.

In general, time offset due to errors in the GPS receiver’s internal clock must also be determined. This is why GPS fixes typically require signals from 4 satellites, and require the solution of 4 x 4 matrices.

Using perturbation theory, the lines of intersection derived from the tangents to the spheres are functions of x, y, and z. However, the techniques shown in the following slides work perfectly well for three dimensional and, with qualification, four and higher dimensional systems of linear equations.

Simultaneous Linear Equations

- N equations in N unknowns
- Use Cramer's rule, matrix inversion, or other techniques to solve the equations
  - e.g., *LU* decomposition with back substitution as explained in "Numerical Recipes in C," or in "Intro to Applied Mathematics" by Gilbert Strang

Vector notation

- Rewrite $a_1x + b_1y = c_1$ and $a_2x + b_2y = c_2$ as
  - $A\vec{x} = \vec{b}$ where x and y are captured in the vector $\vec{x}$ and A is the coefficient matrix
- Use MatLab, MathCad, or other application to compute solutions to the equations
- Or, solve them “by hand” using Cramer’s rule

Mathcad using matrix inversion

Solve:

2x + 3y = 5

7x - 5y = 13

$$A := \begin{pmatrix} 2 & 3 \\ 7 & -5 \end{pmatrix}$$

$$b := \begin{pmatrix} 5 \\ 13 \end{pmatrix}$$

$$x := A^{-1} \cdot b$$

$$x = \begin{pmatrix} 2.065 \\ 0.29 \end{pmatrix}$$

Check the solution:

$$b := A \cdot x$$

$$b = \begin{pmatrix} 5 \\ 13 \end{pmatrix}$$

Note the navigation problems are typically 4 x 4 matrices or lower, and if a computer is available, computational efficiency is often unimportant. Hence, matrix inversion is fine.

However, for an embedded application, such as GPS in a cell phone with multiple updates per second, processor resources and battery life are critical drivers for efficiency, and better algorithms are called for.

Cramer's rule using Mathcad

Solve:

2x + 3y = 5

7x - 5y = 13

$$A := \begin{pmatrix} 2 & 3 \\ 7 & -5 \end{pmatrix} \qquad X := \begin{pmatrix} 5 & 3 \\ 13 & -5 \end{pmatrix} \qquad Y := \begin{pmatrix} 2 & 5 \\ 7 & 13 \end{pmatrix}$$

$$|A| = -31 \qquad |X| = -64 \qquad |Y| = -9$$

$$x := \frac{|X|}{|A|} \qquad y := \frac{|Y|}{|A|}$$

$$x = 2.065 \qquad y = 0.29$$

Expansion by minors

- For square matrices up to 3 x 3, “zig-zag” multiplication and summing of forward and reverse diagonals works for evaluating determinants
  - This is typically the case for solving Maxwell’s equations, where curl of a vector requires evaluation of a 3 x 3 determinant
- But, for 4 x 4 and higher, expansion by minors is needed
  - This is laborious if done by hand
- 4 x 4 matrices are needed if one is solving for x, y, z, and time, time offset (GPS), or sextant index error (celestial)
- Higher order matrices are used
  - in hybrid systems which combine, for example, inertial and radio measurements
  - with Kalman filter techniques

Assignment 7.1

7.1.1 Solve a 3 x 3 system of simultaneous linear equations that you invent, using Cramer's rule *by hand*. If you know how to invert a matrix *by hand*, or if you have the software that is capable of matrix inversion, use that technique as well. Then, check your results, *by hand*. **It is important to break your dependency on digital computers!**

End of Mod 7A