Module 7

Modern Navigation Systems

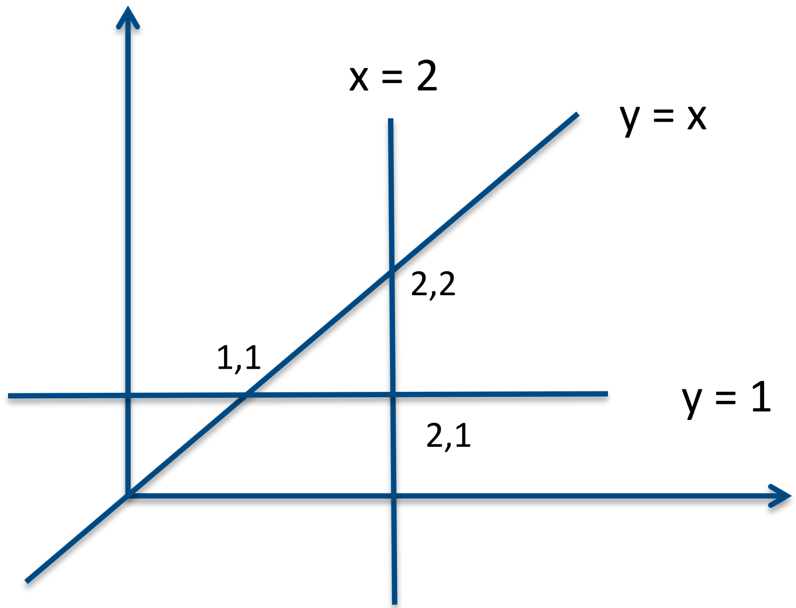
Lines of position and weighted least squares

Module 7B
Least Squares

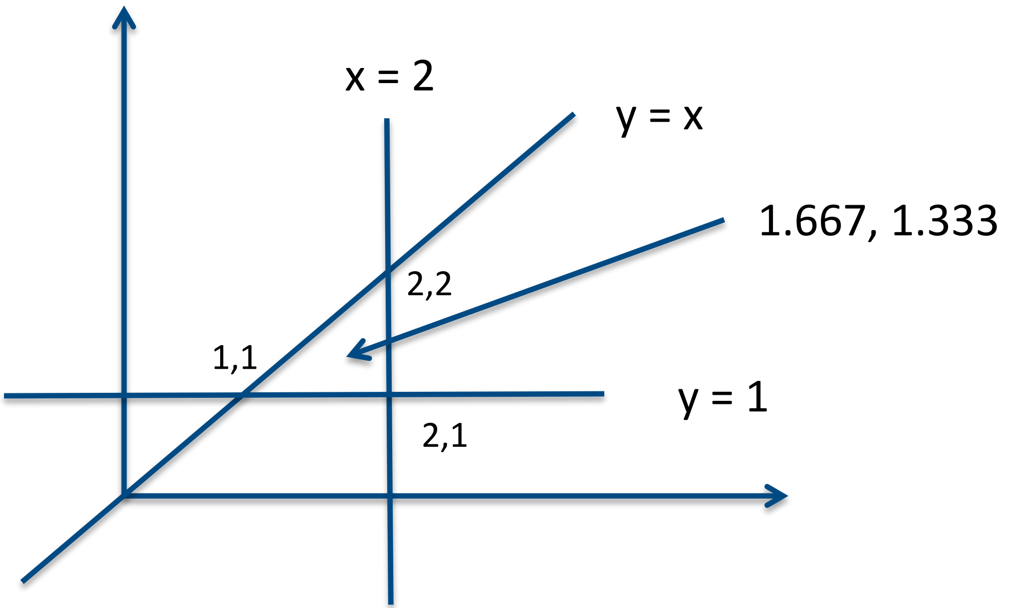
Summary of Module 7

- Students will combine the perturbation analysis and position determination skills from previous modules to define lines of position. The goal is to use the techniques of linear algebra to compute the best estimate of the point in three dimensional space at which the lines intersect. This will yield a "position fix". (7A)
- **The techniques are then extended to the case where the problem is over-specified. That is, *m* measurements are made with respect to the solution of an *n*-dimensional set of equations, where $m > n$. This leads to the least-squares solution of the problem. (7B)**
- The concept of measurement uncertainty is introduced, leading to the technique of weighted least squares. This leads to the concept of the Kalman filter, which is introduced briefly. (7C)

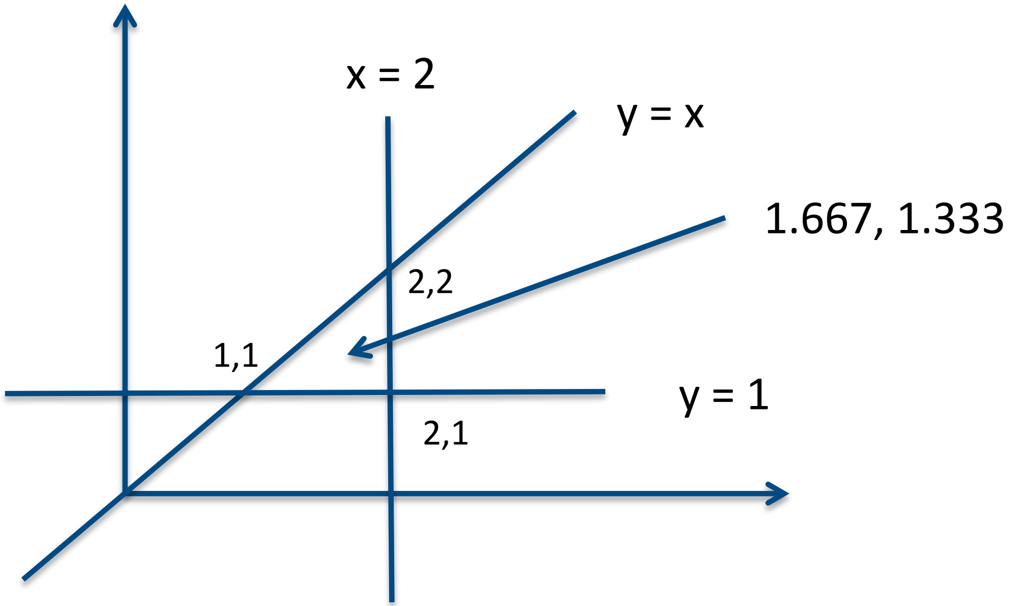
Solve 3 equations in 2 unknowns that don't intersect at the same point:



By inspection, the best overall fit is somewhere in the center of the triangle, perhaps at (1.667, 1.333)…



Solve this using matrices

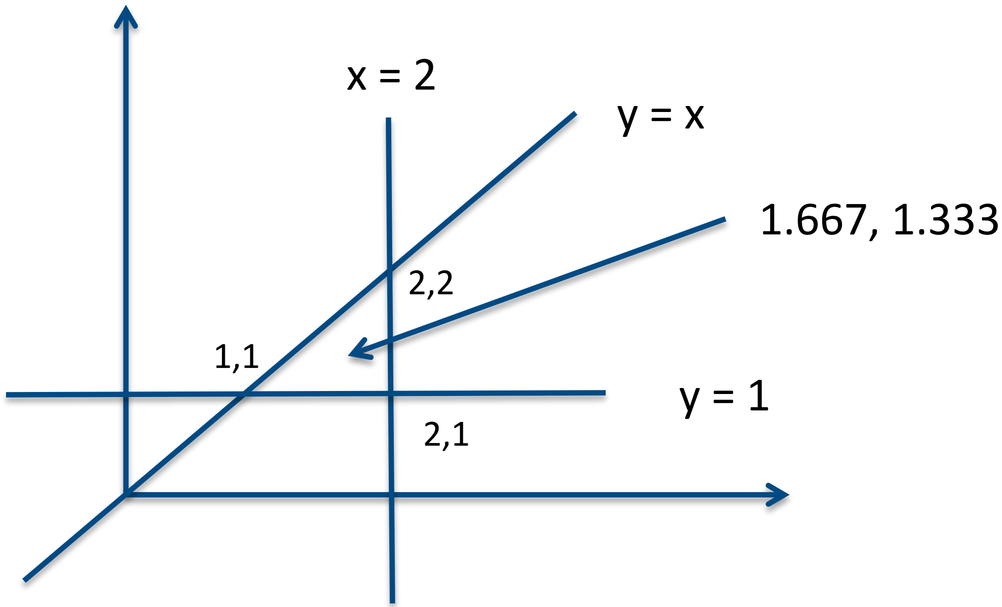


Define the lines using $Ax = B$, where A is a matrix and x and B are vectors.

A will have 3 rows and two columns, making the system of equations unsolvable.

Solve the system $A^TAx = A^TB$ instead. This will yield the least squares solution, which minimizes the sum of the squares of the minimum distances from the solution point to each of the original lines.

Solve this using matrices, cont'd



Three equations in 2 unknowns:

$0x + y = 1$

$x + 0y = 2$

$x - y = 0$

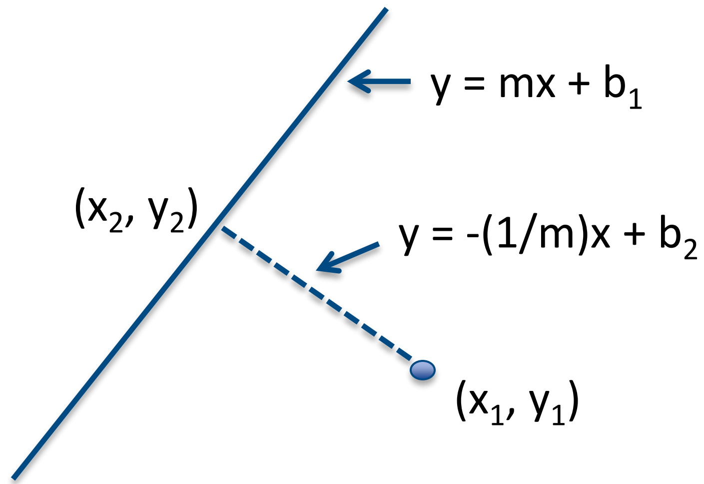
$$A := \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & -1 \end{pmatrix} \qquad B := \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

$$A^T = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$$

$$x := \left(A^T \cdot A\right)^{-1} \cdot A^T \cdot B$$

$$x = \begin{pmatrix} 1.667 \\ 1.333 \end{pmatrix}$$

Compute the distance from a point to a line



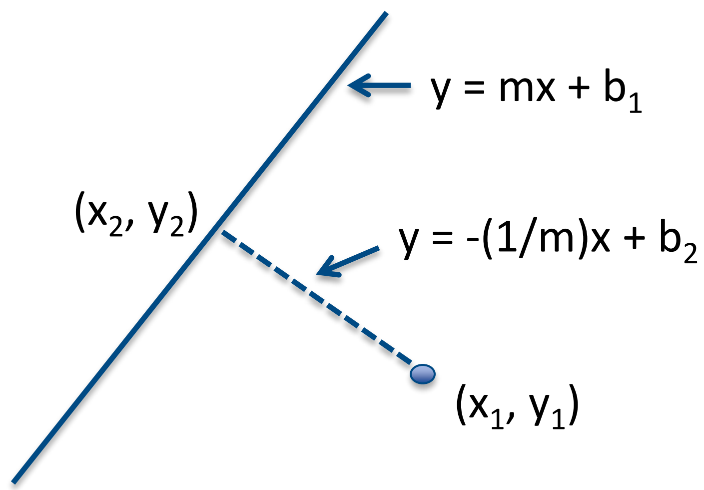
m and b_1 are known, as are x_1 and y_1.

Slope of dashed line is equal to –(1/m), because the shortest path to the solid line must hit the solid line at a right angle, which yields $m_1m_2 = -1$.

Substitute to solve for b_2.

Solve the two equations in two unknowns to find x_2, y_2.

For example:



m and b_1 are known, as are x_1 and y_1.

Slope of dashed line is equal to –(1/m), because the shortest path to the solid line must hit the solid line at a right angle, which yields $m_1m_2 = -1$.

Substitute to solve for b_2.

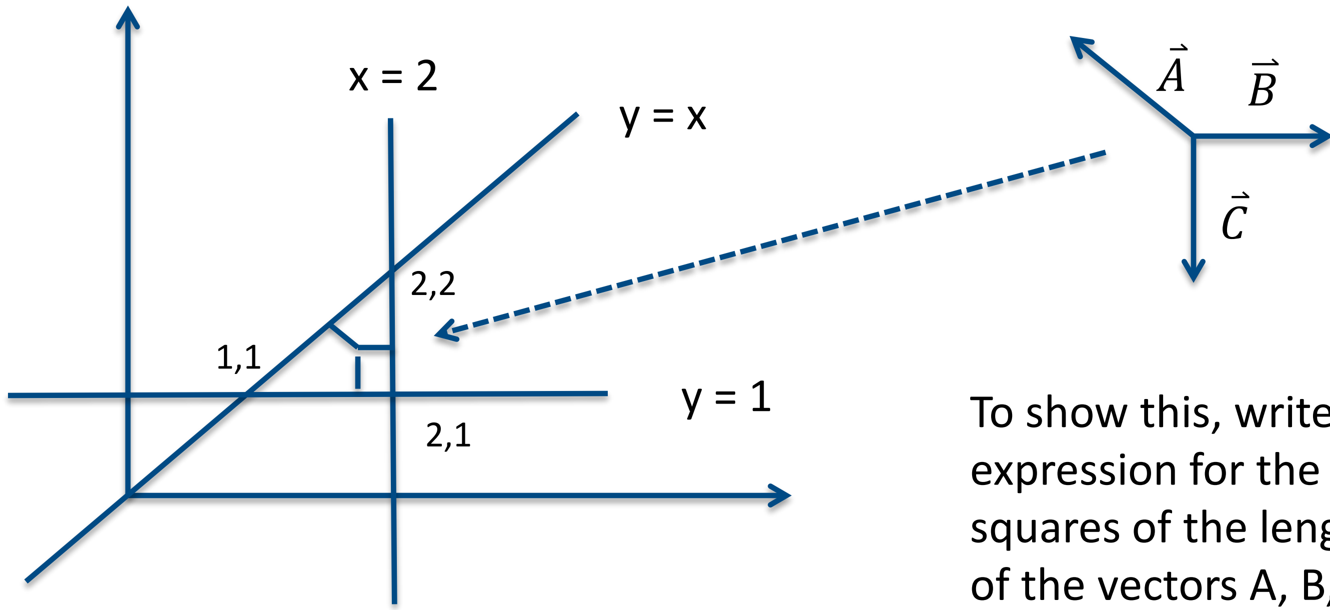
Solve the two equations in two unknowns to find x_2, y_2.

Let m = 1, thus -1/m = -1.

Let x_1, y_1 = (1.667, 1.333).

Then x_2, y_2 = (1.5,1.5)

The point x_1, y_1 minimizes the sum of the squares of the distances from this point to each of the three lines



To show this, write an expression for the sum of the squares of the lengths of the vectors A, B, and C.

Set the derivative of this expression equal to 0, and solve for the point x_1, y_1.

Least squares

- Notice that minimization of the sum of squares requires taking partial derivatives of a quadratic function f(x,y,z,…).
- When differentiating a quadratic function, one obtains a set of linear equations that can be solved using matrix techniques.
- Note the connection to minimization of energy functions in physics.

Assignment 7.2

7.2.1 Invent a system of three equations in two unknowns. Graph the equations and find the least squares solution. Plot the solution on your graph.

7.2.2 Using the example in the presentation, compute the coordinates x_1, y_1 by differentiating the expression for the sum of the squares of the distances to each of the three lines.

End of Mod 7B