Module 2C

The vector cross product, angular momentum, and Kepler's laws

In this module

- We
  - Define the vector cross product
  - Use the cross product to define angular momentum
  - Note that conservation of angular momentum yields Kepler's Laws, which are then summarized

Momentum

- From Newtonian mechanics, students should remember that kinetic energy is $E = \frac{1}{2}mv^2$ and that momentum is $p = mv$.
- Energy is always conserved, although *work* may be converted to *heat* and vice versa.
- In an elastic collision (meaning the interaction between two bodies or entities, such as the moon and the earth), momentum is also conserved.
- Of interest here, however, is angular momentum.

Angular momentum

- Angular momentum L is defined as

$$\mathbf{L} = \mathbf{r} \times m\mathbf{v}$$

where the use of bold letters denotes vectors!

r represents the instantaneous radius-of-rotation vector of a body of scalar mass *m* moving with a vector velocity ***v*** (where the vector ***v*** defines magnitude <u>and</u> direction)

The multiplication sign "x" represents the vector cross product of the vectors ***r*** and ***v*** to form the angular momentum vector ***L***.

The vector cross product

If we define a Cartesian coordinate system where the unit vectors **x**, **y**, and **z** are now represented by the letters **i**, **j**, and **k**, the cross product of two vectors **u** and **v** is given by:

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Where u_1, v_1, etc. represent the scalar magnitudes of the **i**, **j**, and **k** components of the vectors ***u*** and ***v***, and the determinant shown above is evaluated in the usual manner.

The cross product, continued

- Note that the cross product yields a vector, not a scalar (whereas the dot product yields a scalar).
- Two vectors define a plane, and the cross product of the same two vectors yields a vector that is at right angles, or normal, to this plane.
- The dot product is sometimes called the “inner” product, whereas the cross product is sometimes called the “outer” product.

Circular orbits

- For an object orbiting another object in a circular trajectory, such as a GPS satellite in a circular orbit around the earth, the velocity vector can be represented as $\boldsymbol{\omega} r$, where $\boldsymbol{\omega}$ is the orbital velocity vector, in units of radians per second, and r is now the scalar distance from the center of the orbit to the satellite.
- The magnitude of the angular momentum becomes $m\omega r^2$
- Angular momentum is conserved!

Elliptical orbits and Kepler’s laws

- The definition of angular momentum generalizes to elliptical orbits in which ***r*** and ω now vary with time according to Kepler’s Laws:
  - Periodic solutions to the equation for conservation of angular momentum are ellipses
  - The orbital trajectory carve out equal areas in equal times
  - The orbital period squared is proportional to the radius of the orbit cubed (or the length of the semi-major axis, for an elliptical orbit).
- We will encounter these features in much greater detail in future modules!

Assignment 2.3

2.3.1 Compute the cross product of the vectors 2***i*** + 3***j*** and 5***i*** – 7***j***. Show that the cross product points in the ***k*** direction. **i**, ***j***, and ***k*** are unit vectors in the x, y, and z directions.

2.3.2 Explain, mathematically, the change in rotation rate of an ice skater as he/she brings his/her arms close to their body during a spin.

2.3.3 Explain how tightrope walkers reverse direction on a tightrope while holding a balancing pole. Why do such poles bend?

End of Mod 2C