Module 3

Modern Navigation Systems

Spherical trigonometry and perturbation theory for solving great and small circle problems on or near the surface of the earth.

Module 3C

The Mason-Dixon Computation

Summary of Module 3

- Students will learn to apply perturbation theory and spherical trigonometry to the problems of surveying and of comparing great circle and small circle routes for navigation on or near the surface of the earth.
- Students will repeat the spherical trigonometry computations performed by Charles Mason when surveying the Mason-Dixon line, and will compute the maximum difference in separation between great circle and small circle arcs that intersect
- Students will apply perturbation theory to the simplification of the equations of small and great circles. Students will apply perturbation theory to the simplification of the equations of small and great circles.
- Students will learn the nuances of Mercator and Lambert Conformal Conic map projections, with particular emphasis on the critical differences between aviation and nautical charts.

The Mason-Dixon survey

- By two different royal charters from England, the southern border of Pennsylvania and the northern border of Maryland were not the same line
- In particular, William Penn moved his line south, exceeding the royal charter, in order to have a deep water port on the Delaware River
  - This is now the city of Philadelphia, and home of the deep-water Philadelphia Naval Shipyard
  - Which is across I-95 from the stadium where the Philadelphia Eagles play
- Mason and Dixon were hired by the British Royal Society to survey a new line agreed to by a Commission of persons from each state
- This line survives to this day, with 20th century modifications to cede a “no-man’s land” called the “wedge” to Delaware

Mason and Dixon were required to survey:

- A line of constant latitude from Fenwick Island to the Chesapeake Bay and to determine its “midpoint”.
- A circle of radius 12 miles centered at a church tower in the New Castle.
- A line from the “midpoint” that forms a perfect tangent to the circle.
- A line pointing due north from the circle, starting a few miles clockwise along the circle from the tangent point, with this line intersecting:
- The line of constant latitude we call the Mason-Dixon Line, which serves as the border between Maryland and Pennsylvania.
- Note that Delaware did not exist at the time.

The Wedge

- When Delaware was created, a wedge-shaped section of land was inadvertently left over that did not belong to Pennsylvania, Maryland, or Delaware
- This “wedge” became part of Delaware in 1921.

A map…

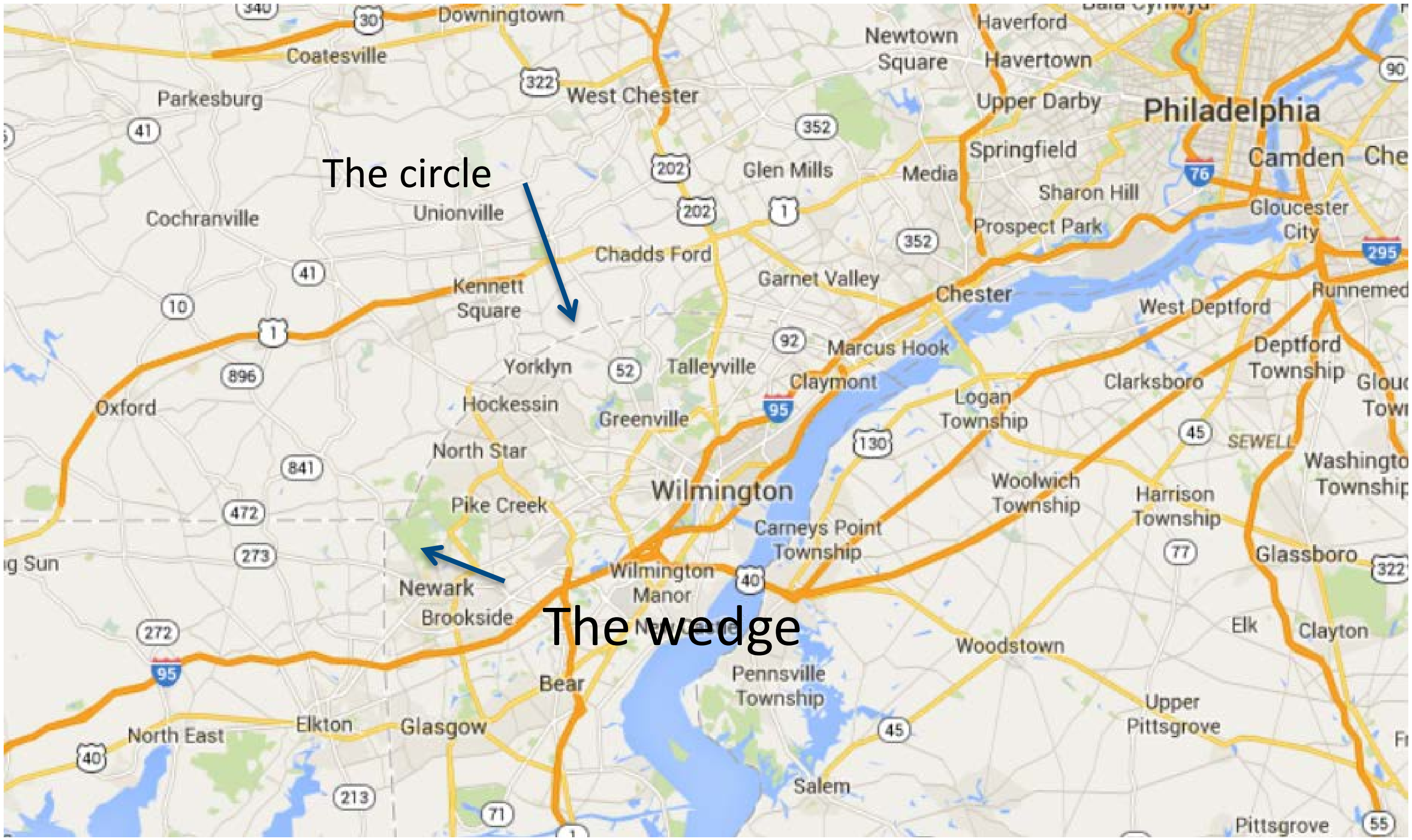
The modern-day border, from Google maps.

Note the circle that still forms part of the border, but is now a border with Delaware.

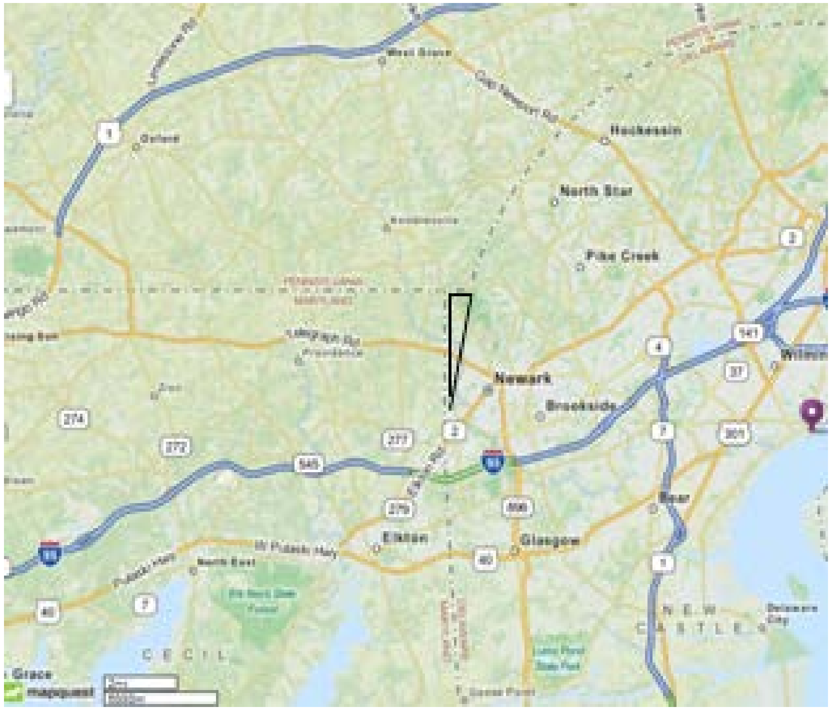
The bottom of the tangent line

These markers are located at the intersection of the tangent line and the Fenwick Island line, forming the border at the southwest corner of Delaware.

Origin of the wedge



Close-up of the Wedge



Original field notes showing the survey of the tangent line

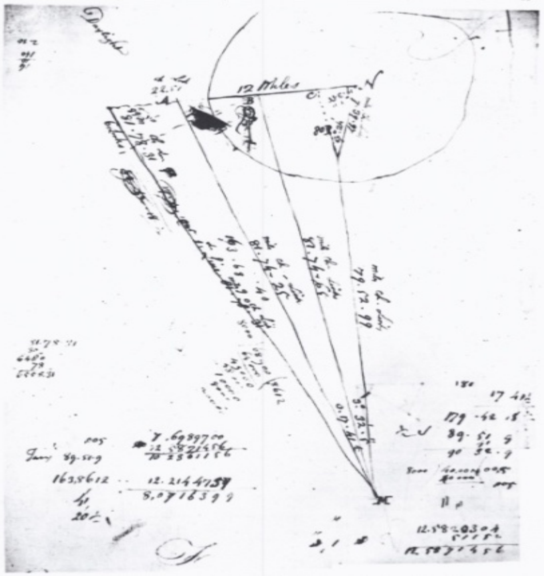


FIG. 5. Surveyors' diagram illustrating location of the Tangent Line (Journal, page 24).

Mason & Dixon’s actual journal, located in Archives II in College Park, MD after having been found in a basement in Nova Scotia in 1860 and purchased from the Canadians in 1877 for $500 in gold.

A modern depiction of the wedge

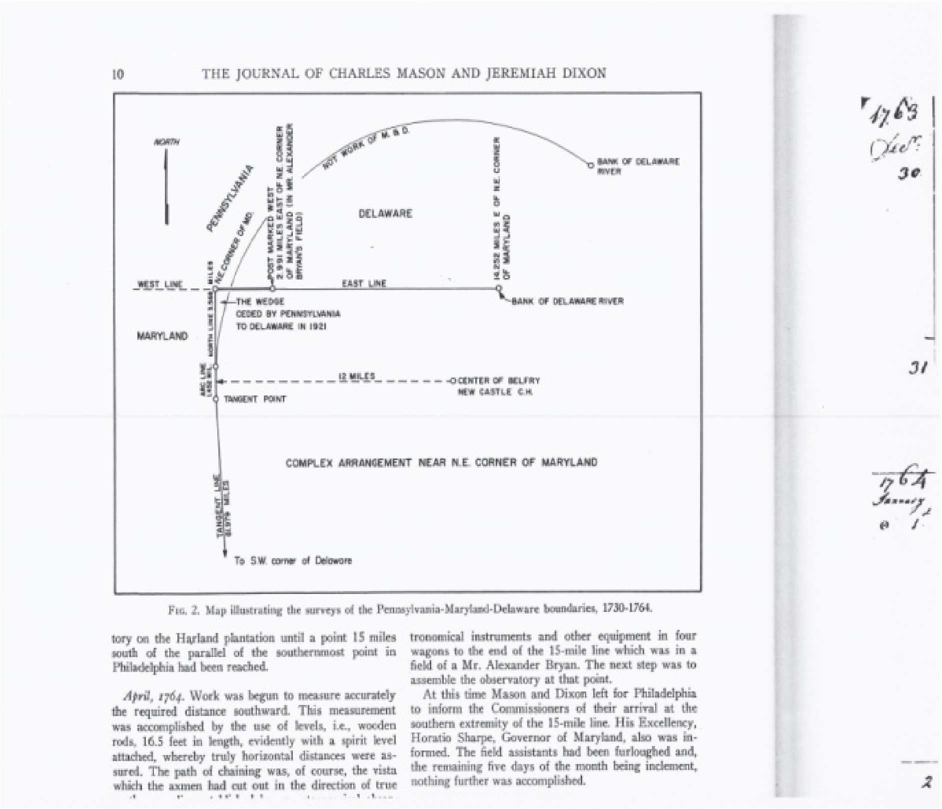

10 THE JOURNAL OF CHARLES MASON AND JEREMIAH DIXON

FIG. 2. Map illustrating the surveys of the Pennsylvania-Maryland-Delaware boundaries, 1730-1764.

tory on the Harland plantation until a point 15 miles south of the parallel of the southernmost point in Philadelphia had been reached.

April, 1764. Work was begun to measure accurately the required distance southward. This measurement was accomplished by the use of levels, i.e., wooden rods, 16.5 feet in length, evidently with a spirit level attached, whereby truly horizontal distances were assured. The path of chaining was, of course, the vista which the axmen had cut out in the direction of true

tronomical instruments and other equipment in four wagons to the end of the 15-mile line which was in a field of a Mr. Alexander Bryan. The next step was to assemble the observatory at that point.

At this time Mason and Dixon left for Philadelphia to inform the Commissioners of their arrival at the southern extremity of the 15-mile line. His Excellency, Horatio Sharpe, Governor of Maryland, also was informed. The field assistants had been furloughed and, the remaining five days of the month being inclement, nothing further was accomplished.

From C. Mason and J. Dixon,
The Journal of Charles Mason
and Jeremiah Dixon (Memoirs of
the American Philosophical Society,
v. 76), American
Philosophical Society, 1969.

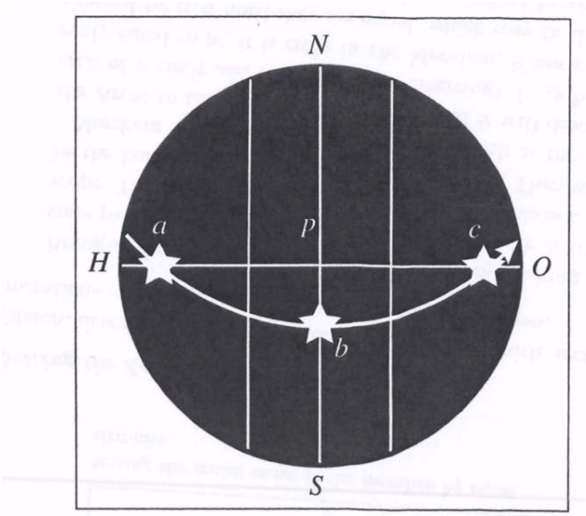
What does the Mason-Dixon line have to do with modern navigation systems?

- Mason and Dixon used the stars to determine true north
  - They did not use the north star, which isn't precise enough for determination of true north
  - Instead, they used a chronometer and equal-time observations of the transit of a star observed using a telescope
  - The transit-mounted telescope points due north when the time it takes for a star to rise above a reference "horizon" to its maximum height in the field of view of the telescope is equal to the time required to descend back to the reference horizon

The method of equal altitudes

The concept is illustrated in a drawing excerpted from the book by Danson [1], which In turn is based on an illustration provided by Mason and Dixon in their journal of the survey.

When the time required for a star (any star in the northern sky) to move from point *a* to point *b* was equal to the time required for the star to move from point *b* to point *c*, the transit was known to be pointing due north.



Setting the zenith sector in the meridian by equal altitudes.

[1]See the supplemental text by Danson.

- The method of equal altitudes, from Danson [1].

- The image is inverted because of the telescope lens used in the “zenith sector”, or transit

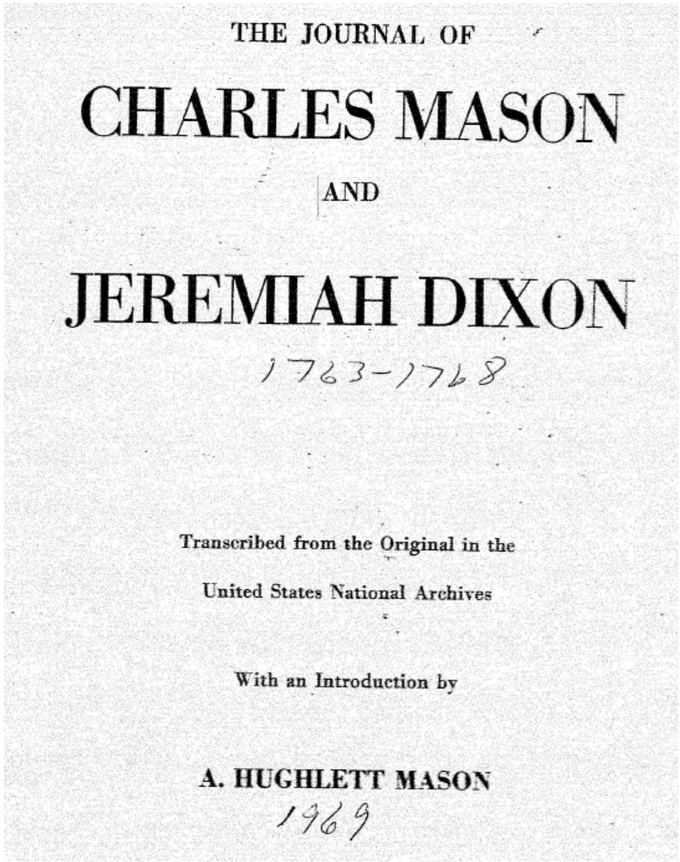
The importance of knowing “north”

- When a tripod-mounted telescope is properly leveled and pointed to true north, the rotation of the telescope to a line of any desired heading can be computed using the equations of spherical trigonometry
- However, since lines of light form great circles, a further adjustment to an optically surveyed line must be made
- Mason and Dixon spent four years doing this in order to change great circle lines into lines of constant heading and/or latitude.
- Andrew Ellicott and Benjamin Banneker used similar methods to survey the city of Washington

How to survey a line of constant latitude

- Choose an arc length of, for instance, 10 minutes of arc.
- Locate the surveying instrument at a point on the desired line of constant latitude.
- Identify true north using celestial observations.
- Compute the bearing from true north that produces a great circle that re-intersects the line of constant latitude at the distance of 10 arc-minutes.
- Rotate the instrument to this bearing, and then use candles at night, held by distant observers, to survey this great circle.
- Compute the distance between the great circle and line of constant latitude at convenient increments of distance.
- Using “chains” (the measuring tapes of the time), survey the line of constant latitude with reference to the surveyed great circle.
- The details of this are shown on the next slide.

Standing in the footsteps of Mason & Dixon



THE JOURNAL OF

CHARLES MASON

AND

JEREMIAH DIXON

1763–1768

Transcribed from the Original in the

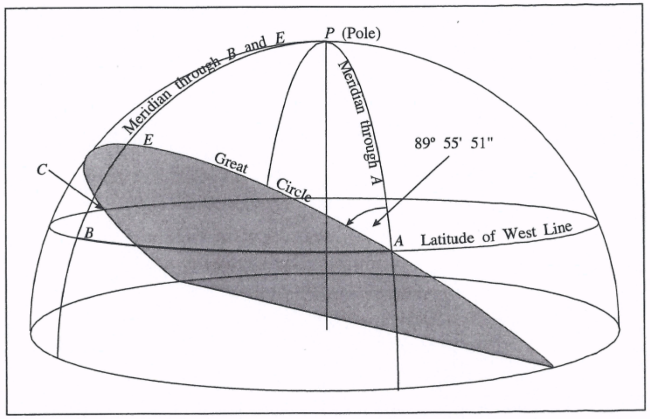
United States National Archives

With an Introduction by

A. HUGHLETT MASON

1969

Spherical Right Triangles



A 10-minute arc of the great circle. The great circle is the arc passing through *A*, *E*, and *C*. The distance *B* to *E* equals 1,714 feet.

From Danson, the triangle AEP is a spherical right triangle and is formed by great circle segments. Since E is the midpoint of the arc AC, the included angle AEP is a right angle. The figure Illustrates the geometry used to survey the Mason-Dixon line.

Note that AEB is a segment of a small circle.

Note also that the sum of the internal angles of a right spherical triangle is greater than 180 degrees.

March 1, 1765

1765
March
1 Began to prepare for running the Western Line: the method of proceeding as follows.
Let P be the Pole, ABCD the Parallel of Latitude to be drawn.
AC the arch of a great circle. At pleasure suppose = 10 minutes
(which we shall set out with on the first station, and in order to
find the direction AC, there is given in the Right Angled Spherical Triangle EPA
AP = Complement of Latitude = $50^{\circ}\ 16'\ 42''.6$ } Hence Angle PAE = $89^{\circ}\ 55'\ 51''$ =
AE = One-half AC = 5'
the angle from the North Westward : and to lay off this angle with the
Transit Instrument by the Stars; Let P be the Pole
Z the Zenith and S the place of the star. Then in the oblique angled
Spherical Triangle SPZ, there is given
SP = the star's distance from the Pole
ZP = the Complement of the Latitude
Angle SZP = $89^{\circ}\ 55'\ 51''$ = the star's azimuth from the North when it will
be on the direction AEC above. To find the angle SPZ or angle at the
Pole when the star is on the said azimuth.

Assignment 3.1-1 Use Napier's rules to compute 89 55 51

March 1, 1765, cont'd

1765
March
1 The angle SPZ being added to the star's Right Ascension: if to the Westward of the meridian or subtracted if the Star is to the Eastward; gives the Right Ascension of the Mid-Heaven, when the star is upon the azimuth Required. In this manner the Right Ascension of the Mid-Heaven for different stars is as follows. -----Next to find by the clock when the star will be on the said azimuth, two equal altitudes of the same star before the time are observed, whence the time is gained. At this instant of time the Middle Wire is brought to bisect the star, and in that position
(The axis of the Telescope, etc., being Horizontal) the vertical axis is made fast: Then the Telescope is brought parallel to the Horizon, and a Mark set by the help of a candle (at the distance of 1/2 or 3/4 of a mile) so that the middle wire bisects it.
In this manner we proceed with 3 or 4 different stars and find that at the distance of 1/2 a mile the extremes of the distances of the marks made by the different stars will not in general exceed 5 or 6 inches.

The line AC being run with the Transit Instrument, at C we set up the Sector to prove or correct the work, by observing the Zenith Distances of the same stars that were observed at the point A.

At C, we find a new direction as before, etc., etc.

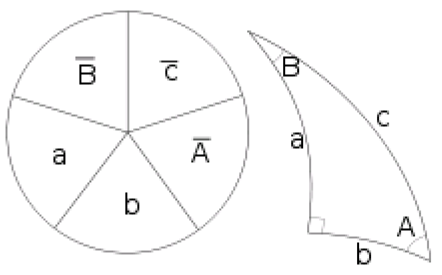
The greatest distance EB to be laid off from the right Line AEC when AE = 5' is 17.14 feet.
2 Computing the star Azimuths, etc., for the direction Westward
3 (Sunday)
4
19 Cloudy, Heavy rains, etc.

Assignment 3.1-2 Use Napier's rules to compute 17.14 feet

(Napier's rules are re-stated in the slides that follow.)

Napier’s Pentagon

Napier's Pentagon



Napier's Circle shows the relations of parts of a right spherical triangle

Napier’s Pentagon, showing a mnemonic for the angular trigonometry of right spherical triangles (from Wikipedia).

A and B are angles, where C is a right angle.

a, b, and c are arc lengths, specified in radians for a sphere of radius 1.

Napier’s rules define how a, b, and c are explained on the next slide.

N.B., the complement bars mean that, for example, complement(B) = 90 degrees - B

Napier’s rules

First write in a circle the six parts of the triangle (three vertex angles, three arc angles for the sides): for the triangle shown above left this gives *aCbAcB*. Next replace the parts which are not adjacent to C (that is *A, c, B*) by their complements and then delete the angle C from the list. The remaining parts are as shown in [previously]. For any choice of three contiguous parts, one (the *middle* part) will be adjacent to two parts and opposite the other two parts.

The ten Napier's Rules are given by

- sine of the middle part = the product of the tangents of the adjacent parts
- sine of the middle part = the product of the cosines of the opposite parts

From http://en.wikipedia.org/wiki/Spherical_trigonometry

Also from Wikipedia (ibid)

- The full set of rules for the right spherical triangle is (Todhunter,[1] Art.62)

$$
\begin{array}{llll}
\text{(R1)} & \cos c = \cos a\, \cos b, & \text{(R6)} & \tan b = \cos A\, \tan c, \\
\text{(R2)} & \sin a = \sin A\, \sin c, & \text{(R7)} & \tan a = \cos B\, \tan c, \\
\text{(R3)} & \sin b = \sin B\, \sin c, & \text{(R8)} & \cos A = \sin B\, \cos a, \\
\text{(R4)} & \tan a = \tan A\, \sin b, & \text{(R9)} & \cos B = \sin A\, \cos b, \\
\text{(R5)} & \tan b = \tan B\, \sin a, & \text{(R10)} & \cos c = \cot A\, \cot B.
\end{array}
$$

[1]Todhunter, I. (1886). *Spherical Trigonometry* (5th ed.). MacMillan.

Assignment 3.2

- 3.2.1 Use Napier's rules to compute 89 55 51
- 3.2.2 Derive 17.14 feet for AE = 5' of arc

End of Mod 3C