Modern Navigation Systems

Module 4B

Newton's Method in Multiple Dimensions

Newton's Method for multiple equations

- x is now a vector x_1 to x_n
- The derivative f'(x) is now replaced by a matrix of derivatives called the Jacobian matrix

Specifically,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

from Mod 4A becomes

$$x_{n+1} = x_n - \frac{F(x_n)}{J_F(x_n)}$$

Where the Jacobian matrix is

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial F_1}{\partial x_1} & \cdots & \dfrac{\partial F_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial F_m}{\partial x_1} & \cdots & \dfrac{\partial F_m}{\partial x_n} \end{bmatrix}$$

From Wikipedia

"In the formulation given above, one then has to left multiply with the inverse of the *k*-by-*k* Jacobian matrix $J_F(x_n)$ instead of dividing by $f\,'(x_n)$. Rather than actually computing the inverse of this matrix, one can save time by solving the system of linear equations

$$J_F(x_n)(x_{n+1} - x_n) = -F(x_n)$$

for the unknown $x_{n+1} - x_n$."

http://en.wikipedia.org/wiki/Newton%27s_method#Nonlinear_systems_of_equations

Newton's method for multiple variables (cont'd)

For clarification,

$$J_F(x_n)(x_{n+1} - x_n) = -F(x_n)$$

means, for example, that in a two dimension problem F(x,y) is now represented as $F(x_1,x_2) = F(x_n)$, and the unknown quantity is now $(x_{n+1} - x_n)$.

Since the vector x_n is already known, the vector x_{n+1} is easily computed.

The importance of derivatives

- It is essential that students review there differential calculus, including
  - Differentiation of sine, cosine, tangent, etc.
  - Differentiation of *u*v*
  - Differentiation of *u/v*

When computing spread sheets…

- In order to avoid divide-by-zero problems, compute derivatives at, for example, $x = .00001$ instead of $x = 0$.

A two dimensional example

- A circle is given by the equation
  - $(x-x_0)^2 + (y - y_0)^2 = r^2$
- Rewrite this as
  - $F(x_0,y_0) = (x - x_0)^2 + (y - y_0)^2 - r^2 = 0$
- Pick two points on the circle
- Define the corresponding equations in terms of $F_1(x_0,y_0)$ and $F_2(x_0,y_0)$
- Use Newton's method to solve for the center of the circle x_0,y_0.
- Note that x and y are known values. In navigation problems, these will be measurements of, for example, the angles of two or more stars above the horizon or the ranges to two or more GPS satellites.

Specifics of the computation

- In the MathCad computations that follow, take note of how the system of simultaneous linear equations is developed, what the unknowns are, and how the iterations are performed.
- Note also that the starting guess for the process determines which of the multiple solutions, if a problem has multiple solutions, Newton's method converges to.

$x0 := 10$

$y0 := 20$

$X := (x0 \;\; y0)$

$X = (10 \;\; 20)$

$A11 := -2 \cdot (1 - x0)$

$A12 := -2 \cdot (4 - y0)$

$A21 := -2 \cdot (3 - x0)$

$A22 := -2(2 - y0)$

$A := \begin{pmatrix} A11 & A12 \\ A21 & A22 \end{pmatrix}$

$A = \begin{pmatrix} 18 & 32 \\ 14 & 36 \end{pmatrix}$

$B1 := 4 - (1 - x0)^2 - (4 - y0)^2$

$B2 := 4 - (3 - x0)^2 - (2 - y0)^2$

$B := (B1 \;\; B2)$

$A^{-1} = \begin{pmatrix} 0.18 & -0.16 \\ -0.07 & 0.09 \end{pmatrix}$ $\quad A \cdot A^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad B^T = \begin{pmatrix} -333 \\ -369 \end{pmatrix}$ $\quad DeltaX := A^{-1} \cdot B^T$

$DeltaX = \begin{pmatrix} -0.9 \\ -9.9 \end{pmatrix}$ $\quad Xnew := X + DeltaX^T$

$Xnew = (9.1 \;\; 10.1)$

Take Xnew to the top of the spread sheet as the new values for x0 and y0.
Repeat the algorithm until the values in Xnew do not change significantly from one iteration to the next.

Mathcad iteration starting at the point 10,20.

The guesses for x_0 and y_0 of 10 and 20 are arbitrary.
The functions F are given by the vector B.
The Jacobian matrix is given by the matrix A.
The system of simultaneous equations is solved using Matrix inversion for convenience; this method is not computationally efficient for large problems.

Iteration #2

$x0 := 9.1$

$y0 := 10.1$

$X := (x0 \;\; y0)$

$X = (9.1 \;\; 10.1)$

$A11 := -2\cdot(1 - x0)$

$A12 := -2\cdot(4 - y0)$

$A21 := -2\cdot(3 - x0)$

$A22 := -2(2 - y0)$

$$A := \begin{pmatrix} A11 & A12 \\ A21 & A22 \end{pmatrix}$$

$$A = \begin{pmatrix} 16.2 & 12.2 \\ 12.2 & 16.2 \end{pmatrix}$$

$B1 := 4 - (1 - x0)^2 - (4 - y0)^2$

$B2 := 4 - (3 - x0)^2 - (2 - y0)^2$

$B := (B1 \;\; B2)$

$$A^{-1} = \begin{pmatrix} 0.143 & -0.107 \\ -0.107 & 0.143 \end{pmatrix} \quad A\cdot A^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad B^T = \begin{pmatrix} -98.82 \\ -98.82 \end{pmatrix} \quad \text{DeltaX} := A^{-1}\cdot B^T$$

$$\text{DeltaX} = \begin{pmatrix} -3.48 \\ -3.48 \end{pmatrix} \quad \text{Xnew} := X + \text{DeltaX}^T$$

$\text{Xnew} = (5.62 \;\; 6.62)$

Take Xnew to the top of the spread sheet as the new values for x0 and y0. Repeat the algorithm until the values in Xnew do not change significantly from one iteration to the next.

Iteration #3

$x0 := 5.62$

$y0 := 6.62$

$X := (x0 \;\; y0)$

$X = (5.62 \;\; 6.62)$

$A11 := -2\cdot(1 - x0)$

$A12 := -2\cdot(4 - y0)$

$A21 := -2\cdot(3 - x0)$

$A22 := -2(2 - y0)$

$$A := \begin{pmatrix} A11 & A12 \\ A21 & A22 \end{pmatrix}$$

$$A = \begin{pmatrix} 9.24 & 5.24 \\ 5.24 & 9.24 \end{pmatrix}$$

$B1 := 4 - (1 - x0)^2 - (4 - y0)^2$

$B2 := 4 - (3 - x0)^2 - (2 - y0)^2$

$B := (B1 \;\; B2)$

$$A^{-1} = \begin{pmatrix} 0.16 & -0.09 \\ -0.09 & 0.16 \end{pmatrix} \quad A\cdot A^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad B^T = \begin{pmatrix} -24.209 \\ -24.209 \end{pmatrix} \quad eltaX := A^{-1}\cdot B^T$$

$$DeltaX = \begin{pmatrix} -1.672 \\ -1.672 \end{pmatrix} \quad Xnew := X + DeltaX^T$$

$Xnew = (3.948 \;\; 4.948)$

Take Xnew to the top of the spread sheet as the new values for x0 and y0. Repeat the algorithm until the values in Xnew do not change significantly from one iteration to the next.

Iteration #4

$x0 := 3.948$

$y0 := 4.948$

$X := (x0 \;\; y0)$

$X = (3.948 \;\; 4.948)$

$A11 := -2 \cdot (1 - x0)$

$A12 := -2 \cdot (4 - y0)$

$A21 := -2 \cdot (3 - x0)$

$A22 := -2(2 - y0)$

$$A := \begin{pmatrix} A11 & A12 \\ A21 & A22 \end{pmatrix}$$

$$A = \begin{pmatrix} 5.896 & 1.896 \\ 1.896 & 5.896 \end{pmatrix}$$

$B1 := 4 - (1 - x0)^2 - (4 - y0)^2$

$B2 := 4 - (3 - x0)^2 - (2 - y0)^2$

$B := (B1 \;\; B2)$

$$A^{-1} = \begin{pmatrix} 0.189 & -0.061 \\ -0.061 & 0.189 \end{pmatrix} \quad A \cdot A^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad B^T = \begin{pmatrix} -5.589 \\ -5.589 \end{pmatrix} \quad DeltaX := A^{-1} \cdot B^T$$

$$DeltaX = \begin{pmatrix} -0.717 \\ -0.717 \end{pmatrix} \quad Xnew := X + DeltaX^T$$

$Xnew = (3.231 \;\; 4.231)$

Take Xnew to the top of the spread sheet as the new values for x0 and y0. Repeat the algorithm until the values in Xnew do not change significantly from one iteration to the next.

Iteration #5

$x0 := 3.231$

$y0 := 4.231$

$X := (x0 \ \ y0)$

$X = (3.231 \ \ 4.231)$

$A11 := -2\cdot(1 - x0)$

$A12 := -2\cdot(4 - y0)$

$A21 := -2\cdot(3 - x0)$

$A22 := -2(2 - y0)$

$$A := \begin{pmatrix} A11 & A12 \\ A21 & A22 \end{pmatrix}$$

$$A = \begin{pmatrix} 4.462 & 0.462 \\ 0.462 & 4.462 \end{pmatrix}$$

$B1 := 4 - (1 - x0)^2 - (4 - y0)^2$

$B2 := 4 - (3 - x0)^2 - (2 - y0)^2$

$B := (B1 \ \ B2)$

$$A^{-1} = \begin{pmatrix} 0.227 & -0.023 \\ -0.023 & 0.227 \end{pmatrix} \quad A\cdot A^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad B^T = \begin{pmatrix} -1.031 \\ -1.031 \end{pmatrix} \quad \text{DeltaX} := A^{-1}\cdot B^T$$

$$\text{DeltaX} = \begin{pmatrix} -0.209 \\ -0.209 \end{pmatrix} \quad \text{Xnew} := X + \text{DeltaX}^T$$

$\text{Xnew} = (3.022 \ \ 4.022)$

Take Xnew to the top of the spread sheet as the new values for x0 and y0.
Repeat the algorithm until the values in Xnew do not change significantly from one iteration to the next.

$x0 := 3.022$

$y0 := 4.022$

$X := (x0 \;\; y0)$

$X = (3.022 \;\; 4.022)$

$A11 := -2\cdot(1 - x0)$

$A12 := -2\cdot(4 - y0)$

$A21 := -2\cdot(3 - x0)$

$A22 := -2(2 - y0)$

$$A := \begin{pmatrix} A11 & A12 \\ A21 & A22 \end{pmatrix}$$

$$A = \begin{pmatrix} 4.044 & 0.044 \\ 0.044 & 4.044 \end{pmatrix}$$

$B1 := 4 - (1 - x0)^2 - (4 - y0)^2$

$B2 := 4 - (3 - x0)^2 - (2 - y0)^2$

$B := (B1 \;\; B2)$

$$A^{-1} = \begin{pmatrix} 0.247 & -2.691 \times 10^{-3} \\ -2.691 \times 10^{-3} & 0.247 \end{pmatrix} \qquad B^T = \begin{pmatrix} -0.089 \\ -0.089 \end{pmatrix} \qquad DeltaX := A^{-1}\cdot B^T$$

$$DeltaX = \begin{pmatrix} -0.022 \\ -0.022 \end{pmatrix} \qquad Xnew := X + DeltaX^T$$

$Xnew = (3 \;\; 4)$

Take Xnew to the top of the spread sheet as the new values for x0 and y0. Repeat the algorithm until the values in Xnew do not change significantly from one iteration to the next.

Iteration #6

Note that the algorithm has Converged to the point (3,4).

Mathcad iteration starting at the point -5,-7.

$x0 := -5$

$y0 := -7$

$X := \begin{pmatrix} x0 & y0 \end{pmatrix}$

$X = \begin{pmatrix} -5 & -7 \end{pmatrix}$

$A11 := -2\cdot(1 - x0)$

$A12 := -2\cdot(4 - y0)$

$A21 := -2\cdot(3 - x0)$

$A22 := -2(2 - y0)$

$A := \begin{pmatrix} A11 & A12 \\ A21 & A22 \end{pmatrix}$

$A = \begin{pmatrix} -12 & -22 \\ -16 & -18 \end{pmatrix}$

$B1 := 4 - (1 - x0)^2 - (4 - y0)^2$

$B2 := 4 - (3 - x0)^2 - (2 - y0)^2$

$B := \begin{pmatrix} B1 & B2 \end{pmatrix}$

$A^{-1} = \begin{pmatrix} 0.132 & -0.162 \\ -0.118 & 0.088 \end{pmatrix}$ $\quad A\cdot A^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad B^T = \begin{pmatrix} -153 \\ -141 \end{pmatrix}$ $\quad DeltaX := A^{-1}\cdot B^T$

$DeltaX = \begin{pmatrix} 2.559 \\ 5.559 \end{pmatrix}$ $\quad Xnew := X + DeltaX^T$

$Xnew = \begin{pmatrix} -2.441 & -1.441 \end{pmatrix}$

Take Xnew to the top of the spread sheet as the new values for x0 and y0.
Repeat the algorithm until the values in Xnew do not change significantly from one iteration to the next.

Iteration #2

$x0 := -2.441$

$y0 := -1.441$

$X := (x0 \;\; y0)$

$X = (-2.441 \;\; -1.441)$

$A11 := -2 \cdot (1 - x0)$

$A12 := -2 \cdot (4 - y0)$

$A21 := -2 \cdot (3 - x0)$

$A22 := -2(2 - y0)$

$$A := \begin{pmatrix} A11 & A12 \\ A21 & A22 \end{pmatrix}$$

$$A = \begin{pmatrix} -6.882 & -10.882 \\ -10.882 & -6.882 \end{pmatrix}$$

$B1 := 4 - (1 - x0)^2 - (4 - y0)^2$

$B2 := 4 - (3 - x0)^2 - (2 - y0)^2$

$B := (B1 \;\; B2)$

$$A^{-1} = \begin{pmatrix} 0.097 & -0.153 \\ -0.153 & 0.097 \end{pmatrix} \quad A \cdot A^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad B^T = \begin{pmatrix} -37.445 \\ -37.445 \end{pmatrix} \quad eltaX := A^{-1} \cdot B^T$$

$$DeltaX = \begin{pmatrix} 2.108 \\ 2.108 \end{pmatrix} \quad Xnew := X + DeltaX^T$$

$Xnew = (-0.333 \;\; 0.667)$

Take Xnew to the top of the spread sheet as the new values for x0 and y0.
Repeat the algorithm until the values in Xnew do not change significantly from one iteration to the next.

Iteration #3

$x0 := -.333$

$y0 := .667$

$X := (x0\ \ y0)$

$X = (-0.333\ \ 0.667)$

$A11 := -2\cdot(1 - x0)$

$A12 := -2\cdot(4 - y0)$

$A21 := -2\cdot(3 - x0)$

$A22 := -2(2 - y0)$

$$A := \begin{pmatrix} A11 & A12 \\ A21 & A22 \end{pmatrix}$$

$$A = \begin{pmatrix} -2.666 & -6.666 \\ -6.666 & -2.666 \end{pmatrix}$$

$B1 := 4 - (1 - x0)^2 - (4 - y0)^2$

$B2 := 4 - (3 - x0)^2 - (2 - y0)^2$

$B := (B1\ \ B2)$

$$A^{-1} = \begin{pmatrix} 0.071 & -0.179 \\ -0.179 & 0.071 \end{pmatrix} \quad A\cdot A^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad B^T = \begin{pmatrix} -8.886 \\ -8.886 \end{pmatrix} \quad DeltaX := A^{-1}\cdot B^T$$

$$DeltaX = \begin{pmatrix} 0.952 \\ 0.952 \end{pmatrix} \quad Xnew := X + DeltaX^T$$

$Xnew = (0.619\ \ 1.619)$

Take Xnew to the top of the spread sheet as the new values for x0 and y0. Repeat the algorithm until the values in Xnew do not change significantly from one iteration to the next.

Iteration #4

$x0 := .619$

$y0 := 1.619$

$X := (x0 \;\; y0)$

$X = (0.619 \;\; 1.619)$

$A11 := -2\cdot(1 - x0)$

$A12 := -2\cdot(4 - y0)$

$A21 := -2\cdot(3 - x0)$

$A22 := -2(2 - y0)$

$$A := \begin{pmatrix} A11 & A12 \\ A21 & A22 \end{pmatrix}$$

$$A = \begin{pmatrix} -0.762 & -4.762 \\ -4.762 & -0.762 \end{pmatrix}$$

$B1 := 4 - (1 - x0)^2 - (4 - y0)^2$

$B2 := 4 - (3 - x0)^2 - (2 - y0)^2$

$B := (B1 \;\; B2)$

$$A^{-1} = \begin{pmatrix} 0.034 & -0.216 \\ -0.216 & 0.034 \end{pmatrix} \quad A\cdot A^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad B^T = \begin{pmatrix} -1.814 \\ -1.814 \end{pmatrix} \quad DeltaX := A^{-1}\cdot B^T$$

$$DeltaX = \begin{pmatrix} 0.328 \\ 0.328 \end{pmatrix} \quad Xnew := X + DeltaX^T$$

$Xnew = (0.947 \;\; 1.947)$

Take Xnew to the top of the spread sheet as the new values for x0 and y0.
Repeat the algorithm until the values in Xnew do not change significantly from one iteration to the next.

Iteration #5

$x0 := .947$

$y0 := 1.947$

$X := (x0 \;\; y0)$

$X = (0.947 \;\; 1.947)$

$A11 := -2\cdot(1 - x0)$

$A12 := -2\cdot(4 - y0)$

$A21 := -2\cdot(3 - x0)$

$A22 := -2(2 - y0)$

$$A := \begin{pmatrix} A11 & A12 \\ A21 & A22 \end{pmatrix}$$

$$A = \begin{pmatrix} -0.106 & -4.106 \\ -4.106 & -0.106 \end{pmatrix}$$

$B1 := 4 - (1 - x0)^2 - (4 - y0)^2$

$B2 := 4 - (3 - x0)^2 - (2 - y0)^2$

$B := (B1 \;\; B2)$

$$A^{-1} = \begin{pmatrix} 6.292 \times 10^{-3} & -0.244 \\ -0.244 & 6.292 \times 10^{-3} \end{pmatrix} \begin{matrix} 1 & 0 \\ & 1 \end{matrix}\Bigg) \qquad B^T = \begin{pmatrix} -0.218 \\ -0.218 \end{pmatrix} \qquad DeltaX := A^{-1}\cdot B^T$$

$$DeltaX = \begin{pmatrix} 0.052 \\ 0.052 \end{pmatrix} \qquad Xnew := X + DeltaX^T$$

$Xnew = (0.999 \;\; 1.999)$

Take Xnew to the top of the spread sheet as the new values for x0 and y0.
Repeat the algorithm until the values in Xnew do not change significantly from one iteration to the next.

$x0 := .999$

$y0 := 1.999$

$X := (x0 \ \ y0)$

$X = (0.999 \ \ 1.999)$

$A11 := -2\cdot(1 - x0)$

$A12 := -2\cdot(4 - y0)$

$A21 := -2\cdot(3 - x0)$

$A22 := -2(2 - y0)$

$$A := \begin{pmatrix} A11 & A12 \\ A21 & A22 \end{pmatrix}$$

$$A = \begin{pmatrix} -2\times10^{-3} & -4.002 \\ -4.002 & -2\times10^{-3} \end{pmatrix}$$

$B1 := 4 - (1 - x0)^2 - (4 - y0)^2$

$B2 := 4 - (3 - x0)^2 - (2 - y0)^2$

$B := (B1 \ \ B2)$

$$A^{-1} = \begin{pmatrix} 1.249\times10^{-4} & -0.25 \\ -0.25 & 1.249\times10^{-4} \end{pmatrix} \quad \begin{matrix} 1 & 0 \\ ) & 1 \end{matrix}\Bigg) \qquad B^T = \begin{pmatrix} -4.002\times10^{-3} \\ -4.002\times10^{-3} \end{pmatrix} \quad := A^{-1}\cdot B^T$$

$$DeltaX = \begin{pmatrix} 9.995\times10^{-4} \\ 9.995\times10^{-4} \end{pmatrix} \quad v := X + DeltaX^T$$

$Xnew = (1 \ \ 2)$

Take Xnew to the top of the spread sheet as the new values for x0 and y0.
Repeat the algorithm until the values in Xnew do not change significantly from one iteration to the next.

Iteration #6

Note the convergence to
the center of the second circle
that passes through the two original
data points.

In two or three dimensions:

- The matrix multiplications are easily done by hand or programmed into a spread sheet.
- But in 4 or more dimensions, the matrices must be expanded by "minors". It is in this situation that tools like MatLab, which is short for "Matrix Laboratory", become invaluable.
- MathCad is especially nice, because of the manner in which matrices can be displayed

Assignment 4.2

4.2.1 Rewrite the Newton's method equations for the 2 dimensional case for yourself explicitly, without using matrix notation. Doing this yourself instead of simply being a spectator to the instructor's example problems will be of significant help when completing assignments from future modules.

4.2.2 Define two circles in a two dimensional plane that overlap. Specify the centers of the two circles as points x_1, y_1 and x_2, y_2. Use Newton's method to find the two points x_{01}, y_{01} and x_{02}, y_{02} at which the circles intersect. This is how GPS works. Specifically, the locations of the GPS satellites are known. A GPS receiver measures the distance to each satellite, thus defining a circle of position on the surface of the earth that corresponds to each satellite. The points at which the circles from multiple satellites intersect are computed, thus defining one's location on the surface of the earth.

End of Module 4B