Module 9

Modern Navigation Systems

Satellite navigation using geostationary satellites

Module 9C

Navigation using multivariate Newton's method

Summary of Module 9

- Students will extend the algorithms used with sextants to the case of near-earth satellites, in particular geostationary satellites, thus introducing parallax (i.e, the ratio of the earth's radius *R* to the distance *r* from the center of the earth to a satellite or the moon). (9A)
- Students will compute their location on earth using plotting techniques along with an "assumed position" and values for elevation measurements to each of the satellites that correspond to the "actual position." Students will re-compute their location on earth using the sextant algorithm from Module 8 with the same information. (9B)
- **Students will compute their location on earth using multivariable Newton's method using angle measurements to the two satellites. Then, they will combine angle and azimuth "measurements" for only a single satellite, thus developing a "mixed-mode" algorithm that would correspond to, for example the use of a gravity based inclinometer and a magnetic compass. (9C)**

Multivariate Newton's Method

- This was covered in Module 4
- But now, the functions f(x,y) will be developed using the equations from Riccharia
- These are shown on the next slides (cf. Mod 9A)

Recap of Riccharia's equations

Finally, the following set of equations can be used to obtain satellite azimuth and elevation from a specified earth station:

$$\text{Right ascension, } \alpha = \arctan\left(y/x\right) \tag{B.43}$$

$$\text{Declination, } \delta = \arctan\left(\frac{z}{\sqrt{x^2 + y^2}}\right) \tag{B.44}$$

$$\text{Elevation, } \eta = \arctan\left(\frac{\sin\eta_s - \dfrac{R}{r}}{\cos\eta_s}\right) \tag{B.45}$$

where

$$\eta_s = \arcsin\left[\sin\delta\sin\theta_e + \cos\delta\cos\theta_e\cos\phi_{se}\right] \tag{B.46}$$

and R = Earth radius
r = satellite distance from Earth centre (use equation B.40c)
θ_e = earth station latitude
$\phi_{se} = \phi_s - \phi_e$
ϕ_s = satellite longitude
ϕ_e = earth station longitude.

Richharia's final equations

$$\text{Azimuth, } A = \arctan\left[\frac{\sin\phi_{se}}{\cos\theta_e\tan\delta - \sin\theta_e\cos\phi_{se}}\right] \qquad \text{(B.47)}$$

This is important

Use the convention given in chapter 2, section 2.6 to obtain the azimuth quadrant. *(see the next slide)*

Range

The distance ρ of a satellite from a given point on the Earth is given as

$$\rho = \sqrt{r^2 - R^2\cos^2\eta} - R\sin\eta \qquad \text{(B.48)}$$

The approach

- Define objective functions based on the azimuth and elevation angle equations, for example:

$$\tan(\mathrm{A}) = \frac{sin(\emptyset_{se})}{cos(\theta_e)tan(\delta) - sin(\theta_e)cos(\emptyset_{se})}$$

Then, write an objective function $f(\theta_e,\phi_{se})$ given by:

$f(\theta_e,\phi_{se}) = \tan(A) - \tan(A_{measured})$, where the measured term is a constant that disappears during differentiation.

The objective functions

- Objective functions can be written for elevation, azimuth, range, or other observable.
- They take the form of an expression for the theoretical value of the observable minus the measured value for that observable.
- Note that when the same observable is used for each of, for example, two satellites, the same objective function is used twice, but with different values of the measured parameter, one for each of the observed satellites.

Newton's method

- Newton's method is used iteratively to find the values of, in this case, θ_e and ϕ_{se} that make the functions f go to zero.
- The geostationary satellite case is important because $\phi_{se} = \phi_s - \phi_e$ is independent of time, a considerable simplification of the more general problem that will be addressed in module 10. Note that ϕ_s is a constant that disappears during differentiation.
- ϕ_e and θ_e, the longitude and latitude of the observer's position on the surface of the earth, are the unknown values being solved for.

The Jacobian matrix

- To implement Newton's method, it is necessary to compute the Jacobian matrix using partial differentiation.
- For the functions at hand, such as *tan(A)*, it is useful to use *implicit differentiation* followed by some algebra to yield the terms in the Jacobian matrix
    - o Implicit differentiation is covered in calculus books and on Wikipedia, and will seem familiar

Recapping Mod 4…

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

from Mod 4A becomes

$$x_{n+1} = x_n - \frac{F(x_n)}{J_F(x_n)}$$

Where the Jacobian matrix is

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial F_1}{\partial x_1} & \cdots & \dfrac{\partial F_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial F_m}{\partial x_1} & \cdots & \dfrac{\partial F_m}{\partial x_n} \end{bmatrix}$$

From Wikipedia

"In the formulation given above, one then has to left multiply with the inverse of the *k*-by-*k* Jacobian matrix $J_F(x_n)$ instead of dividing by $f\,'(x_n)$.
Rather than actually computing the inverse of this matrix, one can save time by solving the system of linear equations

$$J_F(x_n)(x_{n+1} - x_n) = -F(x_n)$$

for the unknown $x_{n+1} - x_n$."

http://en.wikipedia.org/wiki/Newton%27s_method#Nonlinear_systems_of_equations

Newton's method for multiple variables (cont'd)

For clarification,

$$J_F(x_n)(x_{n+1} - x_n) = -F(x_n)$$

means, for example, that in a two dimension problem F(x,y) is now represented as $F(x_1,x_2) = F(x_n)$, and the unknown quantity is now $(x_{n+1} - x_n)$.

Since the vector x_n is already known, the vector x_{n+1} is easily computed.

The importance of derivatives

- It is essential that students review their differential calculus, including
  - Differentiation of sine, cosine, tangent, etc.
  - Differentiation of *u*v*
  - Differentiation of *u/v*
- *And implicit differentiation*

The path forward

- Define the objective function *f* for elevation angles to a geostationary satellite
- Derive the Jacobian matrix
- Implement Newton's method using the same expression for *f* twice, but with the different elevation angles "measured" for each of the two satellites to be used in the two versions of *f*.

Once this works…

- Define the objective function *f* for the azimuth angle to a geostationary satellite
- Derive the Jacobian matrix for the *f* functions for elevation and azimuth, but in this case for a single satellite
- Implement Newton's method using the these objective functions, but with the az/el to a single satellite, as described in the assignment that follows.

This is a difficult and challenging pair of problems

- But, this method generalizes to “mixed-mode” navigation, where angles to a star and distances to a satellite can be combined to produce a position fix
- This leads to combinations such as using both GPS and inertial navigation together, where the different measurements “aid” each other to provide an extremely dynamic navigation capability for spaceflight and long distance aircraft flight.

Assignment 9.3

1. Use Newton's method for the geostationary satellite problem from 9A and 9B to compute a position fix iteratively using the elevation angle "measurements" to each satellite from the assumed position 40, -76.
2. Compare your results with the known location from which the "measured" angles were derived.
3. Use Newton's method again for the geostationary satellite problem from 9A and 9B to compute a position fix iteratively using the elevation and azimuth angle "measured" for the XM satellite at 115 West.
4. Compare your results with the known location from which the "measured" angle and azimuth to this satellite were derived.
5. Provide your thoughts on why Newton's method requires so many iterations as opposed to the Marc St. Hilaire algorithm and its graphical counterpart.

End of Mod 9C