Module 6

Modern Navigation Systems

Orbit Propagators

Module 6A
Building an orbit propagator

Summary of Module 6

- Students will combine the perturbation analysis and position determination skills from previous modules into a numerical orbit propagator that can be used to predict the effects of thrust and delta-velocity (delta-V) on the position and velocity of an object orbiting the earth or other body, including the effects of inhomogeneous gravitational fields, drag, and the gravitational forces of multiple moons and planets.
- The study of the ocean tides will be used as an example of a three-body problem that requires a numerical orbit propagator. The world's largest mechanical tide computer, located 12 miles from JHU/APL at the NOAA building in Silver Spring, MD will be used as an example of a mechanical computer for performing orbital computations.

The need for orbit propagators

- If asked to plot an orbit, students typically plot an ellipse based on values of *M, e, sqrt(a)*, and occasionally t_p (time of perigee passage).
- However, Keplerian orbits can only be derived for an object orbiting in a homogenous gravitational field that can be approximated as a point source with a single center of mass.
- This excludes an analytic solution for three-body problems (e.g., the earth, a satellite, and the moon), or orbits around the earth that consider the non-symmetric mass distribution around the earth's center of mass.
- To generate accurate orbital predictions, it is necessary to start with an accurate gravity model and solve the differential equation $F = ma = -GmM/r^{2,}$ taking into account the mass distribution in physical space of M(x,y,z) and the corresponding adjustments to the distance r.
- For purposes here, we will derive an orbit propagator based on a uniform mass M, and show that it yields the Keplerian results.

As perspective…

- Satellite ToolKit, STK, has a student edition that is available for free download. This version does not include detailed gravity models or the full graphics capabilities.
- The “full-up” commercial version, with all of the available options, costs more than $75,000.

Also note:

- The orbit propagator technique uses the same algorithms that an inertial navigation system does
  - The algorithms shown in the following slides use single and double integration of acceleration to predict velocity and position
  - Inertial sensors and systems will be described in a future module.
- Read chapter 7 of the text to see how this is done.

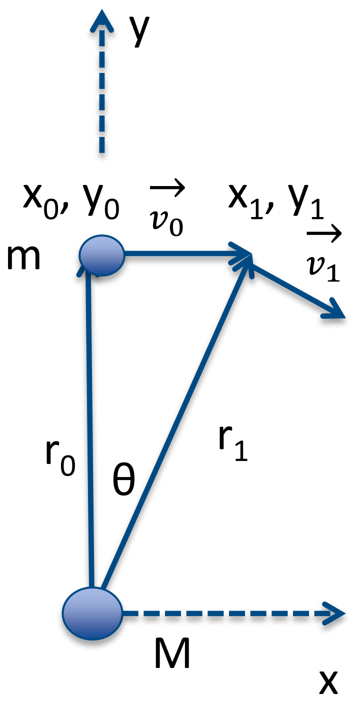
Consider this universal road sign

This ubiquitous road sign captures the essence of an orbit Propagator.

The truck tips over because of the torque generated as the tires follow the road but the trailer continues in its pre-turn direction.

The equations that describe the centripetal forces involved are identical to those we will use for the orbit propagator.

The incremental equations of motion



A satellite of mass m moving in the horizontal direction at velocity v_0 experiences a force in the –y direction given by $-GmM/r^2$

The acceleration is computed by dividing this force by the mass m of the satellite

This can be resolved into x and y components ax and ay and integrated once to compute v_{1x} and v_{1y} and twice to compute x_1, y_1, and r_1.

Note that this is a two dimensional problem.

To create a propagator

- Solve the incremental problem, for vanishingly small increments of θ to compute new values of *x, y, r,* and velocity *v*.
- Apply a rotation of axes so that the new values of x and y are oriented into x and y directions so that the results of the first iteration can be repeated.
- Iterate indefinitely.

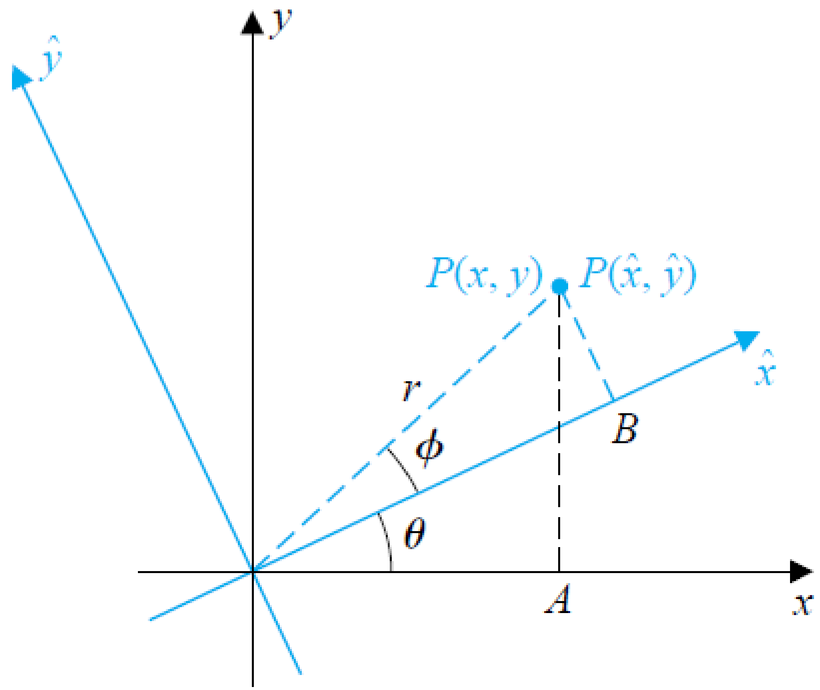
Advantages of using a propagator

- In addition to including multiple gravitational sources (which we won't do here) and other effects (such as atmospheric drag)
  - It is possible to add velocity changes due to engine or thruster burns, thus using the propagator to predict changes to the orbit
- This will be demonstrated using an Excel spreadsheet that will be made available to students as an executable and easily modified file for experimentation

The approach to be used here

- Developing an orbital propagator is time consuming and tedious, but exploring its properties is not.
- The instructor’s demo will be used to demonstrate the effects of various Δv (delta-velocity) changes on orbits
- Of particular interest will be
  - Demonstrating the requirements for a circular orbit
  - Demonstration of Kepler’s laws
  - Numerical computation of the Shuler period
  - Understanding the universal truck tip-over sign
  - Illustration of a geostationary transfer orbit
  - Illustration of the Apollo 13 free-return orbit
  - Illustration of the concept of escape velocity

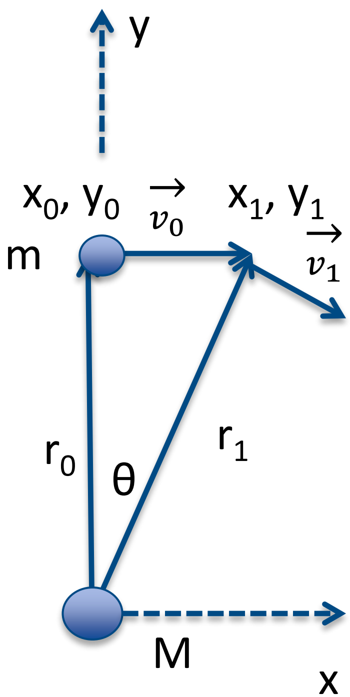
Rotation of axes



The appropriate equations are easily derived by using polar coordinates to write
P(x,y) as r cosθ, r sinθ, and
P(x',y') as r cos(θ + ϕ) and r sin(θ + ϕ)

From: http://math.sci.ccny.cuny.edu/document/show/2685

The fundamental computation



The simulation starts with the assumption that the velocity $v_y = 0$ and that v_x is at a known initial value v_0. However, there are acceleration components in both the x and y directions. These components combine to form the net acceleration *a*, which is in the $-r_1$ direction at the increment of time *dt* and increment of angle θ.

In the limit of small θ, θ is approximated by $\tan \theta = v_0 t / r_0$, where $r_0 = y_0$ = the instantaneous radius of the orbit at the beginning of the simulation

Since θ is small, $\tan \theta \sim \theta$

This yields the accelerations $a_x = -g \sin \theta$ and $a_y = -g \cos \theta$ (Note: g assumes that the orbit is at or near the surface of the earth; in the actual propagator, g is replaced by GM/r^2, where r is the instantaneous distance from the satellite to the center of mass of the earth, or of the mass distribution that the satellite is orbiting around)

Note again that this is a two dimensional problem.

The next step is to compute v_{1x}, v_{1y}, x_1, and y_1 by single and double integration, taking into account suitable constants of integration.

Integrating the acceleration

- This is the first step to understanding inertial navigation, which is the topic of a future module
- Assume $\theta = {}^{v_0 t}/_{r}$ and $a_x = -{}^{g v_0 t}/_{r}$ and $a_y = -g$ because $\sin\theta \approx \theta$ and $\cos\theta \approx 1$ in the small angle limit
- This yields $v_x = v_0 - {}^{g v_0 t^2}/_{2r}$ and $v_y = -gt$

Continuing…

- Solving for ${v_x}^2 + {v_y}^2$ yields, to second order, a constraint that yields the solution that $v_{new} = v_0$
- This constraint is that $mg = {m{v_0}^2}/_{r}$, which is the centripetal force on an object accelerating at $g = 9.8{m}/_{sec^2}$ in a circular trajectory

Re-consider this universal road sign

If a curved road with a radius r is banked at an angle θ, the equivalent gravitational force is mg $\sin\theta = mv_0^2/r$.

The velocity v_0 is the velocity at which a vehicle can stay on the banked curve, even if it is layered with a sheet of ice and there is no friction.

With friction, v_0 is the velocity at which there will be no torque on the vehicle, and a top heavy truck, for example, will not tip over while rounding the curve.

Assignment 6.1

6.1.1 Solve for ${v_x}^2 + {v_y}^2$ to second order, to show that $v_{new} = v_0$

6.1.2 Integrate v_x and v_y to show that $x^2 + y^2 = {r_0}^2$

6.1.3 For the constraint that $mg = {}^{m{v_0}^2}\!/_{r}$, compute v_0 if the acceleration due to gravity is $g = 9.8 m/_{sec^2}$ and *r* is the radius of the earth

6.1.4 Show that ${}^{2\pi r_{earth}}\!/_{v_0} = T_{orbit} = 84.4\, minutes$, the period of a Schuler pendulum.

End of Mod 6A