# Handin 3, Fk 7048

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Date: September 19, 2023

## Warmup

### a)

Suppose we have the following PDE:

$$x \cdot u_x + y \cdot u_y = 2x,$$

then we can write down the following:

$$\frac{dx}{x} = \frac{dy}{y} = \frac{du}{2x}.$$

Computing the following yields:

$$\begin{aligned} \frac{dx}{x} &= \frac{dy}{y} \\ \implies \ln(x) + \tilde{c}_1 &= \ln(y) \\ x \cdot c_1 &= y \\ \implies c_1 &= \frac{y}{x}. \end{aligned}$$

Similarly we have:

$$\begin{aligned} \frac{dx}{x} &= \frac{du}{2x} \\ \implies 2x + c_2 &= u \\ \implies c_2 &= u - 2x. \end{aligned}$$

We can now write down the following:

$$\begin{aligned} G(c_1) &= c_2 \\ \implies G\left(\frac{y}{x}\right) &= u - 2x. \end{aligned}$$

The boundary condition is given by:

$$u(x, x^2) = x,$$

which implies the following:

$$\begin{aligned} G\left(\frac{y}{x}\right) = G(x) &= u - 2x \\ \implies u(x,y) &= -\left(\frac{y}{x}\right) + 2x. \end{aligned}$$

# Problem 1

## 1)

Suppose we have the following PDE:

$$\frac{\partial u}{\partial t} = \sigma \frac{\partial^2 u}{\partial x^2}$$

### a)

We wish to determine the dimensions of $\sigma$ which we do by the following:

$$\begin{aligned}
\frac{1}{T} &= \sigma \frac{1}{L^2}, \\
[z] &= 1 \triangleq x t^{\alpha} \sigma^{\beta} \\
1 &= L T^{\alpha} \left(\frac{L^2}{T}\right)^{\beta}, \\
T^{\beta-\alpha} &= L^{2\beta+\alpha} \\
\implies \alpha &= \beta = -\frac{1}{2}.
\end{aligned}$$

### b)

We suppose we can decompose $u(x,t)$ as the following:

$$u(x,t) = t^{\delta} \cdot f(z); \quad z \triangleq \frac{x}{\sqrt{t\sigma}}.$$

We thus compute the derivatives as follows:

$$\begin{aligned}
\frac{\partial z}{\partial x} &= \frac{1}{\sqrt{t\sigma}} \\
\frac{\partial z}{\partial t} &= \frac{-1}{2} \cdot \frac{x}{t^{3/2}\sqrt{\sigma}} \\
\frac{\partial}{\partial t} &= \frac{\partial}{\partial z} \cdot \frac{\partial z}{\partial t} \\
&= \frac{\partial}{\partial z} \cdot \left(\frac{-1}{2} \cdot \frac{x}{t^{3/2}\sqrt{\sigma}}\right) \\
&= \frac{\partial}{\partial z} \cdot \left(\frac{-1}{2} \frac{z}{t}\right) = \frac{-z}{2t} \frac{\partial}{\partial z}, \\
\frac{\partial^2}{\partial x^2} &= \underbrace{\frac{\partial}{\partial z} \cdot \frac{\partial z}{\partial x}}_{\to u_x} \cdot \left(\underbrace{\frac{\partial}{\partial z} \cdot \frac{\partial z}{\partial x}}_{\to u_x}\right) \\
&= \frac{\partial}{\partial z} \cdot \frac{1}{\sqrt{t\sigma}} \cdot \left(\frac{\partial}{\partial z} \cdot \frac{1}{\sqrt{t\sigma}}\right) \\
&= \frac{1}{t\sigma} \frac{\partial^2}{\partial z^2}.
\end{aligned}$$

Applying these transformations yields the following:

$$\begin{aligned}
\frac{\partial}{\partial t}\left(t^{\delta} \cdot f(z)\right) &= \frac{1}{t} \frac{\partial^2}{\partial z^2}\left(t^{\delta} f(z)\right) \\
\delta t^{\delta-1} f(z) - \frac{z t^{\delta-1}}{2} \frac{df}{dz} &= t^{\delta} \frac{d^2 f}{dz^2} \\
\implies \frac{d^2 f}{dz^2} + \frac{z}{2} \frac{df}{dz} - \delta f &= 0.
\end{aligned}$$

**c)**

Setting $\delta = -\frac{1}{2}$ and solving with method of Frobenius yields the following:

$$\begin{aligned}
&\frac{d^2 f}{dz^2} + \frac{z}{2}\frac{df}{dz} + \frac{1}{2}f = 0. \\
&f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^{n+r} \\
&f'(z) = \sum_{n=0}^{\infty} a_n (n+r)(z - z_0)^{n+r-1} \\
&f''(z) = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1)(z - z_0)^{n+r-2} \\
\implies &\sum_{n=0}^{\infty} a_n (n+r)(n+r-1)\cdot(z)^{n+r-2} + \frac{1}{2}\sum_{n=0}^{\infty} a_n \cdot (n+r+1)\cdot(z)^{n+r} = 0. \\
m = n-2 \implies &\sum_{m=-2}^{\infty} a_{m+2}(m+r+2)(m+r+1)\cdot(z)^{m+r} + \frac{1}{2}\sum_{n=0}^{\infty} a_n \cdot (n+r+1)\cdot(z)^{n+r} = 0,
\end{aligned}$$

$$\underbrace{a_{-2}(r)(r-1)z^{r-2}}_{=0} + \underbrace{a_{-1}(r)(r+1)z^{r-1}}_{=0} + \sum_{n=0}^{\infty} z^{n+r}\underbrace{\left[a_{n+2}(n+r+1)(n+r+2) + \frac{a_n}{2}(n+r+1)\right]}_{=0} = 0.$$

There exists three cases for $r$: $r_1 = 0$, $r_2 = -1$ and $r_3 = 1$. We will discard the solution for $r_2 = -1$. The recursive relation from above is given by:

$$\begin{aligned}
\frac{a_{n+2}}{a_n} &= -\frac{n+r+1}{2(n+r+1)(n+r+2)} = -\frac{1}{2(n+r+2)} \\
r = 0 \text{ gives:} \quad a_{n+2} &= -\frac{a_n}{2(n+2)} \\
a_2 &= -\frac{a_0}{2(2)} = -\frac{a_0}{4} \\
a_4 &= -\frac{a_2}{2(2+2)} = \frac{a_0}{4\cdot 8} = -\frac{a_0}{32} \\
a_6 &= -\frac{a_4}{12} = \frac{a_0}{12\cdot 32} = \frac{a_0}{384} \\
a_{n+2} &= -\frac{a_n}{2(2+n)} = \frac{a_{n-2}}{2^2(n+2)(n)} = -\frac{a_{n-4}}{2^3(n+2)(n)(n-2)} \\
&= \Big(-1\Big)^{n+1}\frac{a_0}{2^n(n+2)!!} \\
a_{2n} &= (-1)^n \cdot \frac{\left(\frac{1}{4}\right)^n}{n!} a_0 \\
\implies f(z) &= a_0 \exp\left[-\frac{z^2}{4}\right]
\end{aligned}$$

## 2)

Suppose the following equation:

$$
\begin{aligned}
u(x,y)\frac{\partial u}{\partial x} + v(x,y)\frac{\partial u}{\partial y} &= \lambda\frac{\partial^2 u}{\partial y^2} \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0.
\end{aligned} \tag{1}
$$

## a)

Given the following transformations:

$$
\begin{aligned}
u &= \frac{\partial w}{\partial y}, \\
v &= -\frac{\partial w}{\partial x},
\end{aligned}
$$

we wish to transform eq (1) into a PDE for $w(x,y)$. We compute the following:

$$
\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial y}\right) = \frac{\partial^2 w}{\partial x \partial y}, \\
\frac{\partial u}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y}\right) = \frac{\partial^2 w}{\partial y^2}.
\end{aligned}
$$

Plugging this into the original ODE, eq (1), yields the following:

$$
\frac{\partial w}{\partial y}\cdot\frac{\partial^2 w}{\partial x \partial y} - \frac{\partial w}{\partial x}\cdot\frac{\partial^2 w}{\partial y^2} = \lambda\frac{\partial^3 w}{\partial y^3}.
$$

The above equation is now a third order PDE of $w(x,y)$. Moreover, we have by the boundary condition:

$$
\lim_{y\to\infty}\frac{\partial w}{\partial y} = U,
$$

where $U$ is constant; thus $[w] = [U][y]$; since $U$ is a constant flow, or flux, we can impose that the dimensions of $w$ is that of flux per unit length.

## b)

We wish to find: $[x], [y], [\lambda]$ and $[w]$ and show the following:

$$
[y] = \left[\sqrt{\frac{\lambda x}{U}}\right], \quad [w] = \left[\sqrt{U\lambda x}\right].
$$

Using what we found in the previous exercise we have the following:

$$
\begin{aligned}
&[w] = [U][y] \\
\implies &\left[\frac{w}{y}\cdot\frac{w}{xy}\right] = \left[\frac{w}{x}\cdot\frac{w}{y^2}\right] = \left[\lambda\cdot\frac{w}{y^3}\right] \\
\implies &\left[\frac{w^2}{xy^2}\right] = \left[\lambda\frac{w}{y^3}\right] \\
\implies &[y] = \left[\sqrt{\frac{\lambda y x}{w}}\right] = \left[\sqrt{\frac{\lambda x}{U}}\right] \\
\implies &[w]^2 = \left[\lambda\frac{wx}{y}\right] \implies [w] = \left[\sqrt{Ux\lambda}\right]
\end{aligned}
$$

## c)

We look for solutions on the form:

$$w(x,y) = \sqrt{U\lambda x} f(\eta); \quad \eta = y\sqrt{\frac{U}{\lambda x}}.$$

We want to find a third order differential equation in terms of $f(\eta)$, and we obtain this by the following computations:

$$\begin{aligned}
&\frac{\partial w}{\partial y} \cdot \frac{\partial^2 w}{\partial x \partial y} - \frac{\partial w}{\partial x} \cdot \frac{\partial^2 w}{\partial y^2} = \lambda \frac{\partial^3 w}{\partial y^3} \\
&\frac{\partial w}{\partial y} = \frac{\partial}{\partial y}\Big(\sqrt{U\lambda x} f(\eta)\Big) = \sqrt{U\lambda x} f'(\eta) \cdot \sqrt{\frac{U}{\lambda x}} \\
&= U f'(\eta) \\
&\frac{\partial^2 w}{\partial y^2} = \frac{\partial}{\partial y}\Big(\frac{\partial w}{\partial y}\Big) = \frac{\partial}{\partial y}\Big(U f'(\eta)\Big) \\
&= \sqrt{\frac{U^3}{\lambda x}} f''(\eta) \\
&\frac{\partial^3 w}{\partial y^3} = \frac{\partial}{\partial y}\Big(\sqrt{\frac{U^3}{\lambda x}} f''(\eta)\Big) \\
&= \frac{U^2}{\lambda x} f'''(\eta) \\
&\frac{\partial w}{\partial x} = \frac{\partial}{\partial x}\Big(\sqrt{U\lambda x} f(\eta)\Big) = f(\eta)\frac{\partial}{\partial x}\Big(\sqrt{U\lambda x}\Big) + \sqrt{U\lambda x} f'(\eta)\frac{\partial}{\partial x}\Big(y\sqrt{\frac{U}{\lambda x}}\Big) \\
&= f(\eta)\sqrt{\lambda U}\frac{1}{2 \cdot x^{\frac{1}{2}}} + \sqrt{U\lambda x} f'(\eta)\sqrt{\frac{U}{\lambda x}}\frac{-y}{2 \cdot x^{\frac{3}{2}}} \\
&= \frac{\sqrt{U\lambda}}{2 \cdot x^{\frac{1}{2}}} f(\eta) - \frac{yU}{2 \cdot x} f''(\eta) \\
&\frac{\partial^2 w}{\partial x \partial y} = \frac{\partial}{\partial x}\Big(U f'(\eta)\Big) = -U f''(\eta)\sqrt{\frac{U}{\lambda}}\frac{1}{2 \cdot x^{\frac{3}{2}}}
\end{aligned}$$

Using the properties of the partial derivatives we have the following PDE in terms of $w(x,y)$:

$$\begin{aligned}
&\frac{\partial w}{\partial y} \cdot \frac{\partial^2 w}{\partial x \partial y} - \frac{\partial w}{\partial x} \cdot \frac{\partial^2 w}{\partial y^2} = \lambda \frac{\partial^3 w}{\partial y^3} \\
&- \Big(U f'(\eta) \cdot U f''(\eta)\sqrt{\frac{U}{\lambda}}\frac{1}{2 \cdot x^{\frac{3}{2}}}\Big) \\
&- \Big(f(\eta)\sqrt{\lambda U}\frac{1}{2 \cdot x^{\frac{1}{2}}} + \sqrt{U\lambda x} f'(\eta)\sqrt{\frac{U}{\lambda x}}\frac{-y}{2 \cdot x^{\frac{3}{2}}}\Big) \cdot \left(\sqrt{\frac{U^3}{\lambda x}} f''(\eta)\right) = \frac{U^2}{x} f''' \\
&\implies 2f'''(\eta) = -f'(\eta) \cdot f(\eta),
\end{aligned}$$

Which now is the so called Blasius equation.

# Problem 2

Suppose the following PDE:

$$\begin{aligned}
&y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial z} = u(x,y) - 1; \\
&u(x, x = 2x) = x^2 + y + 1 \quad \forall (x,y) \in \mathbb{R}^2
\end{aligned}$$

Using $x = t$, $y = 2t$ and $u = (t+1)^2$ we get the following:

$$\begin{aligned} x + y = t + 2t = 3t \implies t = \frac{1}{3}(x+y) \\ \implies u = (t+1)^2 \\ u(x,y) = \Big(\frac{1}{3}(x+y)+1\Big)^2 \end{aligned}$$

This solution solves the PDE with the boundary conditions and due to uniqueness of solutions, this is the only solution to the PDE.

# Problem 3

## a)

We wish to find the general solution to the differential equation:

$$y''(x) + 2y'(x) + y(x) = 0.$$

We look for solutions on the form; $y(x) = e^{rx}$, which yields the following:

$$\begin{aligned} e^{rx}\Big(r^2 + 2r + 1\Big) = 0 \\ \implies r = -1. \end{aligned}$$

Since we have a double root (multiplicity) we have the following solution for the homogenous solution:

$$y_h(x) = c_1 e^{-x} + c_2 x e^{-x} + c_3.$$

## b)

$$\begin{aligned} y''(x) + 2y'(x) + y(x) = f(x); \\ f(x) = \begin{cases} \sin(x); x \in [0, 2\pi] \\ 0; x \notin [0, 2\pi] \end{cases} \end{aligned}$$

One notice that $f(0) = f(2\pi) = 0$ and thus we can divide the solution into two regions. Firstly, we will look for a particular solution in the region $x \in [0, 2\pi]$.

$$\begin{aligned} y_p(x) &= A\cos(x) + B\sin(x), \\ y_p'(x) &= -A\sin(x) + B\cos(x) \\ y_p''(x) &= -A\cos(x) - B\sin(x) \\ \implies -A\cos(x) - B\sin(x) &+ 2\Big(-A\sin(x) + B\cos(x)\Big) + A\cos(x) + B\sin(x) = \sin(x) \\ \implies -2A\sin(x) &= \sin(x) \quad A = -\frac{1}{2}. \\ y_p(x) &= -\frac{1}{2}\cos(x) \end{aligned}$$

Thus the particular solution in the region $x \in [0, 2\pi]$ is given by:

$$y = y_h + y_p = c_1 e^{-x} + c_2 x e^{-x} + c_3 - \frac{1}{2}\cos(x).$$

The boundary conditions yields that $c_3 = \frac{1}{2}$ which gives the final solution:

$$y(x) = c_1 e^{-x} + c_2 x e^{-x} + \frac{1}{2} - \frac{1}{2}\cos(x).$$

The function $y(x)$ is $C^{(\infty)}$-smooth and thus is differentiable in every point on $x \in \mathbb{R}$.