Handin 1, FK7048

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Task 1

a)

Determine the type of the following ODE and solve it:

$$(2x - 3y)dx + (2y - 3x)dy = 0.$$

Firstly, the type of the ODE is that of homogenous equation; we solve by doing the following substitution: $y = vx$ which leads to the following expression:

$$\begin{aligned}
(2xv - 3x)(xdv + dxv) + (2x - 3xv)dx &= 0\\
\frac{(2x^2v - 3x^2)}{x}dv + \frac{(2xv^2 - 3xv - 2x - 3xv)}{x}dx &= 0\\
-\Big(\frac{2v-3}{2v^2 - 6v + 2}\Big)dv &= \Big(\frac{1}{x}\Big)dx\\
\implies -\int dv\Big(\frac{2v-3}{2v^2-6v+2}\Big) &= \int dx\Big(\frac{1}{x}\Big)\\
-\underbrace{\int dv\Big(\frac{2v-3}{2v^2-6v+2}\Big)}_{u=2v^2-6v+2;\quad du=(4v-6)dv} &= -\frac{\ln\big(\big|2v^2 - 3v + 1\big|\big)}{2} + c_2 = \ln(x) + c_1
\end{aligned}$$

$$-\frac{\ln\left(\left|2\left(\frac{y}{x}\right)^2 - 3\frac{y}{x} + 1\right|\right)}{2} + c_2 = \ln(x) + c_1$$

This is an implicit solution. *Note: This is also an exact ODE.*

b)

Determine the type of the following ODE and solve it:

$$\frac{dy}{dx} = \cos^2(x)\sin(x)\sec(y).$$

We begin by rewriting the equation:

$$\frac{dy}{dx} = \frac{\cos^2(x)\sin(x)}{\cos(y)},$$

Now it's clear that the equation is of type 'seperable' equation.

$$\cos(y)dy = \cos^2(x)\sin(x)dx$$
$$\int_0^y \cos(y')dy' = \int_0^x \cos^2(x')\sin(x')dx'$$
$$\sin(y) + c_1 = \lfloor u = \cos(x); du = -\sin(x)dx \rceil = -\int u^2 du = -\frac{u^3}{3} + c_2$$

The solution becomes the trivial $y = \arcsin\left(c - \frac{\cos^3(x)}{3}\right)$

Task 2

a)

The ODE that we seek to solve is the following:

$$-\vec{\nabla}P + \rho(P)\vec{g} = \mathbf{0}.$$

Given that the system under investigation is that of the Earth's atmosphere we can make the assumption that the pressure P is only a function of height and thus treating this with only one parameter, namely z; the ODE then becomes:

$$\begin{aligned} -\frac{d}{dz}(P_z) + g \cdot \rho(P_z) &= 0, \\ \rho(P_z) &= \frac{1}{g}\frac{dP_z}{dz}. \qquad (1) \\ g\int_0^z \rho(P)dz' &= \int_{P_0}^P dP'. \end{aligned}$$

b)

The ideal gas law states as follows:

$$\begin{aligned} PV &= Nk_bT, \\ \implies P\frac{m}{\rho} &= Nk_bT, \\ \implies \rho(P) &= \frac{Pm}{Nk_bT} = P \cdot \alpha \qquad (2) \end{aligned}$$

where $\alpha = \frac{m}{Nk_bT}$.

c)

Using the result from (1) into (2) and rearranging yields the following:

$$\begin{aligned} \int_{P_0}^P dP'\Big(\frac{1}{P'}\Big) &= \int_0^z dz'\Big(g \cdot \alpha\Big), \\ \ln\left(\frac{P}{P_0}\right) &= g\alpha \cdot z, \\ P(z) &= P_0 \exp\Big[g\alpha \cdot z\Big] \end{aligned}$$

The expression $g \cdot \alpha$ simply then becomes $-\frac{1}{h}$ which gives the final expression:

$$P(z) = P_0 \cdot e^{\frac{-z}{h}}.$$

To show this, we look at the following,

$$
\begin{aligned}
g \cdot \alpha &= \frac{gm}{Nk_bT} \\
&= \Big[\frac{\frac{\mathrm{m}}{\mathrm{s}^2} \cdot \mathrm{kg}}{\mathrm{J} \cdot \mathrm{K}^{-1} \cdot \mathrm{K}}\Big] \\
&= \Big[\frac{\mathrm{m}}{\mathrm{s}^2} \cdot \frac{\mathrm{kg}}{\mathrm{J}}\Big] \\
&= \Big[\frac{\mathrm{m}}{\mathrm{s}^2} \cdot \frac{\mathrm{kg}}{\mathrm{kg} \cdot \mathrm{m}^2\mathrm{s}^{-2}}\Big] \\
&= \Big[\mathrm{m}^{-1}\Big]
\end{aligned}
$$

Hence, the unit-test checks out.

d)

Rewriting (1) yields the following expression for $-h^{-1}$:

$$
-h^{-1} = g \cdot \frac{M}{RT}
$$

such that R is the ideal gas constant and T is the base temperature. Doing this computation gives the value $h^{-1} = 0.00012524$, which is fairly plausable. This means that as one approaches the limit $\lim_{z\to\infty} \exp[-\frac{z}{h}] = 0$.

e)

If the temperature is difference is non-negliable, we get the following ODE:

$$
\begin{aligned}
\int_{P_0}^{P} dP' \Big(\frac{1}{P'}\Big) &= \int_0^z dz' \Big(g \cdot \frac{m}{Nk_bT(z)}\Big) \\
&= \frac{g}{Nk_b} \int_0^z dz' \Big(\frac{1}{T_0 - \beta z}\Big) \\
\ln\Big(\frac{P}{P_0}\Big) &= \frac{g}{Nk_b(-\beta)} \ln\Big(\frac{T_0 - \beta z}{T_0}\Big) \\
P(z) &= P_0 \exp\Big[\frac{g}{Nk_b(-\beta)}\Big] \cdot \Big(\frac{T_0 - \beta z}{T_0}\Big)
\end{aligned}
$$

Task 3: Parallel RLC Circuit

a)

We can show this by Kirchhoff's laws. The totalt voltage V is given by the following:

$$
E\cos(\omega t) = R \cdot I + U
$$

where R is the resistance and I is the total current over the circuit. I can be decomposed over as $I = I_1 + I_2$ where I_1 is the current over the conductor and I_2 is the current over the inductor.

$$
\begin{aligned}
E\cos(\omega t) &= R \cdot (I_1 + I_2) + U \\
&= R \cdot I_1 + R \cdot I_2 + U \\
&= R \cdot I_1 + \Big(-L\frac{dI}{dt}\Big)
\end{aligned}
$$

b)

$$\ddot{u}(t) + \frac{1}{RC}\dot{u}(t) + \frac{1}{LC}u(t) = \frac{1}{RC}\dot{e}(t)$$

In order to first start to solve this second order non-homogenous differential-equation we'll solve the characteristic polynomial, providing the ansats $u(t) = \exp(rt)$:

$$\exp(rt)\Big[r^2 + \alpha r + \beta\Big] = 0$$

such that $\alpha = \frac{1}{RC}$ and $\beta = \frac{1}{LC}$. Solving for r_i gives the following:

$$\begin{aligned} r_1 &= -\frac{\alpha}{2} + \sqrt{\frac{\alpha^2}{2} - \beta}, \\ r_2 &= -\frac{\alpha}{2} - \sqrt{\frac{\alpha^2}{2} - \beta}. \end{aligned}$$

The homogenous, the first part of the solution then becomes

$$\begin{aligned} u_h(t) &= c_1 \exp\left[(-\frac{\alpha}{2} + \sqrt{\frac{\alpha^2}{2} - \beta})t\right] + c_2 \exp\left[\Big(-\frac{\alpha}{2} - \sqrt{\frac{\alpha^2}{2} - \beta}\Big)t\right], \\ u_h(t) &= c_1 \exp\left[\Big(-\sigma\omega_0 + \sqrt{\omega_0^2(\sigma^2 - 1)}\Big)t\right] + c_2 \exp\left[\Big(-\sigma\omega_0 - \sqrt{\omega_0^2(\sigma^2 - 1)}\Big)t\right]; \quad \sigma \in [0, 1] \end{aligned}$$

c)

In order to find the particular solution we make the ansats that the particular solution $u_p(t)$ is on the following form: $u_p(t) = A\exp[i\omega t]$; differating twice yields;

$$\begin{aligned} u_p(t) &= A\exp[i\omega t], \\ \dot{u}_p(t) &= i\omega A\exp[i\omega t] \\ \ddot{u}_p(t) &= -\omega^2 A\exp[i\omega t]. \end{aligned}$$

Plugging this into the ODE, and instead finding $\tilde{u}_p(t)$ yields;

$$\frac{d^2}{dt^2}\big(\tilde{u}(t)\big) + \frac{\omega_0}{Q}\frac{d}{dt}\big(\tilde{u}(t)\big) + \omega_0^2\tilde{u}(t) = \frac{\omega_0 E\omega}{Q}\exp[i\omega t]$$

We make the ansats $\tilde{u}(t) = A(\omega)\exp[i\omega t]$,

$$\begin{aligned} -\omega^2 A(\omega)\exp[i\omega t] + i\omega\frac{\omega_0}{Q}A(\omega)\exp[i\omega t] + \omega_0^2 A(\omega)\exp[i\omega t] &= \frac{\omega_0 E\omega}{Q}\exp[i\omega t] \\ A(\omega)\exp[i\omega t]\left[-\omega^2 + i\omega\frac{\omega_0}{Q} + \omega_0^2\right] &= \frac{\omega_0 E\omega}{Q}\exp[i\omega t] \end{aligned}$$

$$\implies A(\omega) = \frac{\omega_0 E\omega}{Q\Big[(\omega_0^2 - \omega^2) + i\omega\frac{\omega_0}{Q}\Big]}$$

$$A(\omega) = \frac{\omega_0 E\omega}{Q}\left[\frac{\Big((\omega_0^2 - \omega^2) - \frac{i\omega\omega_0}{Q}\Big)}{(\omega_0^2 - \omega^2)^2 - \omega^2\frac{\omega_0^2}{Q^2}}\right]$$

$$A(\omega) = \frac{\omega_0 E}{Q}\left[\frac{\Big((\omega_0^2 - \omega^2) - \frac{i\omega\omega_0}{Q}\Big)}{(\omega_0^2 - \omega^2)^2 - \omega^2\frac{\omega_0^2}{Q^2}}\right]$$

$$= \frac{\omega_0 E\omega}{Q}\left[\frac{\exp[-i\theta]}{\Big(\omega_0^2 - \omega^2)^2 - \omega^2\frac{\omega_0^2}{Q^2}\Big)^{1/2}}\right]$$

The solution thus becomes:

$$u_p(t) = \mathrm{Re}\left(\frac{\omega_0 E\omega}{Q\Big(\omega_0^2 - \omega^2)^2 - \omega^2\frac{\omega_0^2}{Q^2}\Big)^{1/2}}\exp\Big[i(\omega t - \theta)\Big]\right)$$

$$= \frac{\omega_0 E\omega}{Q\Big(\omega_0^2 - \omega^2)^2 - \omega^2\frac{\omega_0^2}{Q^2}\Big)^{1/2}} \cdot \cos(\omega t - \theta)$$

d)

$$\lim_{\omega \to \omega_0 \omega} \frac{\omega_0 E}{Q\Big(\omega_0^2 - \omega^2)^2 - \omega^2\frac{\omega_0^2}{Q^2}\Big)^{1/2}} \cdot \cos(\omega t - \theta) = \text{undefined}$$

This becomes undefined since there is a discharge in the system.

Task 4: Thickness distribution of ICE

$$\int_{h_1}^{h_2} dh\Big(g(h,t)\Big) = \frac{A(h_1, h_2)}{\mathcal{A}} \tag{3}$$

$$\frac{d^2}{dh^2}g(h) + \Big(\frac{1}{H} - \frac{q}{h}\Big)\frac{d}{dh}g(h) + \frac{q}{h^2}g(h) = 0 \tag{4}$$

a)

We ought to justify that the method of Frobenius is a valid method to solving this ODE.

b)

We seek to find the indicial equation and the recurrence relation for all of the coefficients in the series solution. In order to do this, we firstly states the g can we written in the following manner:

$$g(h) = \sum_{n=0}^{\infty} a_n(h - h_0)^{(n+r)}$$

$$g'(h) = \sum_{n=0}^{\infty} \cdot(n + r) \cdot a_n(h - h_0)^{(n+r-1)}$$

$$g''(h) = \sum_{n=0}^{\infty}(n + r - 1) \cdot (n + r) \cdot a_n(h - h_0)^{(n+r-2)}$$

Plugging this into (4) we get the following, when $h_0 = 0$:

$$
\begin{aligned}
&\left[\sum_{n=0}^{\infty}(n+r-1)\cdot(n+r)\cdot a_n(h)^{(n+r-2)}\right] \\
&+\left(\frac{1}{H}-\frac{q}{h}\right)\left[\sum_{n=0}^{\infty}(n+r)\cdot a_n(h)^{(n+r-1)}\right] \\
&+\frac{q}{h^2}\left[\sum_{n=0}^{\infty}a_n(h)^{(n+r)}\right]=0, \\
\Longrightarrow &\sum_{n=0}^{\infty}\left[\Big((n+r)(n+r-1)-(n+r)q+q\Big)\right]a_n(h)^{n+r-2}+\sum_{n=0}^{\infty}a_n(h)^{n+r-1}\frac{1}{H}(n+r)=0 \\
\Longrightarrow &\sum_{m=-1}^{\infty}\left[\Big((m+r+1)(m+r)-(m+r+1)q+q\Big)\right]a_{m+1}(h)^{m+r-1}+\sum_{n=0}^{\infty}a_n(h)^{n+r-1}\frac{1}{H}(n+r)=0 \\
&=\sum_{m=0}^{\infty}\left[\Big((m+r+1)(m+r)-(m+r+1)q+q\Big)a_{m+1}+\frac{m+r}{H}a_m\right]h^{m+r-1} \\
&+\underbrace{a_0\Big((r)(r-1)-(r)q+q\Big)h^{r-2}}_{\text{Indicial equation}}=0 \qquad (5)
\end{aligned}
$$

Solving for r, via quadratic formula yields:

$$
\begin{aligned}
r &= +\frac{(1+q)}{2}\pm\frac{1}{2}\sqrt{\Big(1+q\Big)^2-4q} \\
&= \frac{(1+q)}{2}\pm\frac{1}{2}\sqrt{(1-q)^2} \\
&= \frac{(1+q)}{2}\pm\frac{1-q}{2} \\
r &= \{1,q\}
\end{aligned}
$$

The recursive relation becomes, from (5):

$$
a_{m+1}=-a_m\cdot\frac{m+r}{H\cdot\Big((m+r+1)(m+r)-(m+r+1)q+q\Big)}
$$

c)

Choosing $r = q$ makes the recursive relation become:

$$\begin{aligned}
\frac{a_{m+1}}{a_m} &= \frac{1}{H} \cdot \left[\frac{m+1+q}{(m+1+1)(m+q) - (m+1+q)q + q} \right] \\
&= -\frac{1}{H} \frac{1}{m+1}, \\
a_1 &= -\frac{1}{H} \frac{1}{1} a_0, \\
a_2 &= -\frac{1}{H} \frac{1}{2} a_1 = \frac{1}{H^2} \frac{1}{2 \cdot 1} a_0, \\
a_3 &= -\frac{1}{H} \frac{1}{3} a_2 = -\frac{1}{H^3} \frac{1}{3 \cdot 2 \cdot 1} a_0 \\
\implies a_n &= \frac{a_0}{H^n} \cdot \frac{(-1)^n}{(n)!} \\
\implies g_{p,1}(h) &= a_0 \cdot h^q \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{h}{H} \right)^n \\
&= a_0 \cdot h^q \exp \left[-\frac{h}{H} \right]
\end{aligned}$$

Using the normalization condition $\int dh(g(h)) = 1$ we get:

$$\begin{aligned}
1 &= a_0 \int_0^{\infty} dh \Bigg(\underbrace{h^q \exp \left[-\frac{h}{H} \right]}_{\int dx [h^q \cdot e^{-h/H}] = -\gamma(q+1) H^{q+1}} \Bigg) \\
\implies a_0 &= -\frac{1}{\gamma(q+1) H^{q+1}}
\end{aligned}$$