Handin 7

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Warmup

a)

$$\beta(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

Compute $\beta(4,3)$

$$\begin{aligned}\beta(4,3) &= \frac{\Gamma(4)\Gamma(3)}{\Gamma(7)} \\ &= \frac{3!\cdot 2!}{6!} = \frac{2}{4\cdot 5\cdot 6} = \frac{1}{60}\end{aligned}$$

b)

Calculate the general solution of $\ln(1+i)$. One begin by stating $1+i=z$, and thus z can be expressed in euler form.

$$\begin{aligned}\ln(1+i) &= \ln(z) \\ &= \ln\left(\sqrt{2}\exp\left[i\arctan(1)\right]\right) \\ &= \ln\left(\sqrt{2}\exp\left[i\frac{\pi}{4}\right]\right) \\ &= \sqrt{2}\left(i\frac{\pi}{4} + n2\pi i\right); \quad n\in\mathbb{Z} \qquad (1)\end{aligned}$$

Complex integral with branch-cut

a)

The function $\ln(z)$ is not contionous in the complex plane, due to eq (1), since n is an integer; $\ln(z)$ does a phase-shift of 2π for every integer n. Now suppose a function $f(z) = z^\alpha \ \forall\alpha\in[0,\infty)$. One wish to find whether such a function is contionous:

$$f(z) = z^\alpha = \left(r\exp[i\theta]\right)^\alpha = r^\alpha\exp[i\theta\alpha]$$

The function $f(z) = z^\alpha$ is contionous for all $\alpha\in\mathbf{Z}^+$ but for every other value of α is is not contionous, and we get a branch-point.

b)

$$I = \int_0^\infty \frac{dx}{x^\alpha(x+p)} \qquad (2)$$

I

To check whether the expression (2) converges one firstly check the behaviour of the integrand as $x \to 0$ and $x \to \infty$.

$$I = \int_0^\infty \frac{dx}{x^{\alpha+1} + x^\alpha p}.$$

As the limit approches infinity, the integral converges for $\alpha > 1$ but when the limit approaches zero, it diverges for all $\alpha \notin [0, 1)$. Thus the integral converges for $\alpha \in [0, 1)$ and diverges for $\alpha \notin [0, 1)$.

II

Considered the contour from figure (11.26) in Arfken, one computes the following integral:

$$\oint_C \frac{dz}{z^\alpha(z+p)} = 0.$$

One computes the find the singularities of the integrand:

$$\begin{aligned} &z^\alpha(z+p) = 0 \\ &\implies z = 0, \quad z = -p \end{aligned}$$

Computing the residues at the singularties, at $z = -p$ which is a single pole, one finds the following:

$$\text{Res}(f, -p) \lim_{z \to -p} (z+p) \frac{1}{z^\alpha(z+p)} = \frac{1}{(-p)^\alpha} = \frac{1}{(-1)^\alpha \cdot (p)^\alpha}$$

Given the countour, we avoid the point $z = 0$, and thus the residue at $z = 0$ is zero. Using this one computes the following:

$$\begin{aligned} \oint_C \frac{dz}{z^\alpha(z+p)} &= \int_0^\infty \frac{dr}{r^\alpha(r+p)} + \int_\infty^0 \frac{e^{-2\pi i\alpha} dr}{r^\alpha(r+p)} \\ &= \int_0^\infty \frac{dr}{r^\alpha(r+p)} - e^{-2\pi i\alpha} \int_0^\infty \frac{dr}{r^\alpha(r+p)} \\ &= (1 - e^{-2\pi i\alpha}) I = 2\pi i \sum_i \text{Res}(f, z_i) \\ \implies I &= \frac{2\pi i \sum_i \text{Res}(f, z_i)}{1 - e^{-2\pi i\alpha}} \\ &= 2\pi \frac{1}{p^\alpha} \left(\frac{1}{\frac{\sin(\pi \cdot \alpha)}{2\pi}} \right) = \frac{\csc(\pi\alpha)}{p^\alpha} \\ \text{Reflection formula:} \quad &= \frac{\Gamma(\alpha)\Gamma(1-\alpha)}{p^\alpha}. \end{aligned}$$

This is the identity one wished to show.

c)

$$J = \int_0^\infty \frac{\ln(x) dx}{a^2 + x^2}$$

I

The complex logarithm for $\ln(x)$ for $x \in (-\infty, 0)$ is defined as:

$$\ln(x) = \ln(|x|) + i\pi.$$

In other words, there exists a phase-shift, which results in result to lie in the complex plane.

II

Firstly one defines the following:

$$\oint \frac{\ln(z)}{z^2 + a^2} dz. \tag{3}$$

Secondly, one computes the singularities of the integrand:

$$\begin{aligned} z_0 &= 0 \\ z_{1,2} &= \pm ai \end{aligned}$$

The objective is not to define a contour which encloses one of the singularities but not both, and avoids the branch-cut. One defines the following contour:

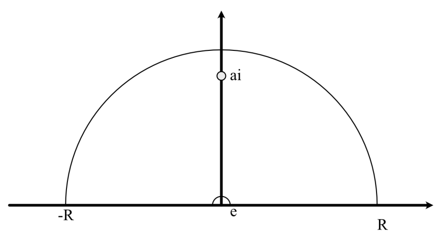

Figure 1: The contour C, but the right side of the branch-cut is not included.

Thus the closed contour integral becomes:

$$\begin{aligned} \oint_C \frac{\ln(z)}{z^2 + a^2} dz &= \lim_{R\to\infty, \epsilon\to 0} \int_\epsilon^R \frac{\ln(x)}{x^2 + a^2} dx + \int_{arc} f(z) dz + \int_{-R}^{-\epsilon} \frac{\ln(z)}{z^2 + a^2} dz \\ &= \lim_{R\to\infty, \epsilon\to 0} \int_\epsilon^R \frac{\ln(x)}{x^2 + a^2} dx - \int_{-\epsilon}^{-R} \frac{\ln(re^{i\theta})}{r^2 + a^2} dr \\ &= \lim_{R\to\infty, \epsilon\to 0} \int_\epsilon^R \frac{\ln(x)}{x^2 + a^2} dx + \int_\epsilon^R \frac{\ln(re^{-i\theta})}{r^2 + a^2} dr \\ &= \lim_{R\to\infty, \epsilon\to 0} \int_\epsilon^R \frac{\ln(x)}{x^2 + a^2} dx + \int_\epsilon^R \frac{\ln(r) - i\theta}{z^2 + a^2} dr \\ &= \lim_{R\to\infty, \epsilon\to 0} \int_\epsilon^R \frac{\ln(x)}{x^2 + a^2} dx + \int_\epsilon^R \frac{\ln(r)}{r^2 + a^2} dr - \int_\epsilon^R \frac{i\theta}{z^2 + a^2} \\ &= \lim_{\epsilon\to 0} 2 \cdot I - \int_\epsilon^R \frac{i\theta}{z^2 + a^2} dz \end{aligned}$$

In this instance, θ is a fixed value close to π

$$\lim_{\epsilon\to 0} \frac{i}{2} \oint \frac{\pi - \epsilon}{z^2 + a^2} dz = \lim_{\epsilon\to 0} \frac{i(\pi - \epsilon)}{2} \left(2\pi i \frac{1}{2ai}\right) = \frac{i\pi^2}{2a}$$

Thus one has the following:

$$\begin{aligned}\oint_c f(z)dz &= 2I - \frac{i\pi^2}{2a} = \pi\frac{\ln(ai)}{a}\\ \implies I &= \frac{1}{2}\left(\pi\frac{\ln(ai)}{a} + \frac{i\pi^2}{2a}\right)\\ &= \frac{1}{2}\left(\pi\frac{\ln(a)}{a} - \frac{i\pi^2}{2a} + \frac{i\pi^2}{2a}\right) = \frac{\ln(a)\pi}{2a}\end{aligned}$$

Integration of Bessel-functions

$$I_n = \int_0^\infty J_n(x)dx$$

a)

The Bessel-function of the first kind can be defined as:

$$J_n(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(s+n)!} \cdot \left(\frac{x}{2}\right)^{n+2s}.$$

To find the behavior as $x \to 0$ and $x \to \infty$ one defines the following limits, (where n is a fixed integer):

$$\begin{aligned}\lim_{x\to x_i} J_n(x) &= \lim_{x\to x_i}\left[\sum_{s=0}^{\infty} \frac{(-1)^s}{s!(s+n)!} \cdot \left(\frac{x}{2}\right)^{n+2s}\right]\\ &= \lim_{x\to x_i}\left[\sum_{s=0}^{\infty} \frac{x^n}{(s+n)!} \cdot \underbrace{\frac{(-1)^2}{s!}\left(\frac{x}{2}\right)^{2s}}_{p_s(x)}\right]\\ &\implies \lim_{x\to 0} J_n(x) = 0\\ &\implies \lim_{x\to 0} J_0(x) = 1\\ &\implies \lim_{x\to \infty} J_n(x) = 0\end{aligned}$$

b)

Show $I_1 = -[J_0(x)]_0^\infty = 1$. The special case for $n = 0$ is given by:

$$\begin{aligned}J_1(x) &= -J_0'(x)\\ \implies \int_0^\infty J_1(x) &= -\int_0^\infty J_0'(x)dx\\ I_1 &= -\left[J_0(x)\right]_{x=0}^{\infty}\\ &= -(0-1) = 1\end{aligned}$$

c)

One wishes to show$I_{n-1} = I_{n+1}$; in order to accomplish this, one defines the following integral: $Q = I_{n+1} - I_{n-1}$. Under the assumption that Q is non-zero, we do the following computation:

$$\begin{aligned}
Q = I_{n+1} - I_{n-1} &= \int_0^\infty J_{n+1}(x) - J_{n-1}(x) dx \\
&= \int_0^\infty dx \left[J_{n+1}(x) - J_{n-1}(x)\right] \\
&= \int_0^\infty dx \left(2 \cdot J_n'(x)\right) \\
&= 2\Big[J_n(x)\Big]_{x=0}^\infty = 0.
\end{aligned}$$

We've disproven the previous assumption, and thus $I_{n-1} = I_{n+1}$.

d)

Compute I_0;

$$\begin{aligned}
I_0 &= \int_0^\infty J_0(x) dx \\
&= \mathbf{R}\frac{1}{2\pi}\int_0^\infty dx \left(\int_0^{2\pi} d\theta \left(\exp\left[ix\cos(\theta)\right]\right)\right) \\
\text{Fubinis theorem} : &= \mathbf{R}\frac{1}{2\pi}\int_0^{2\pi} d\theta \left(\int_0^\infty dx \left(\exp\left[ix\cos(\theta)\right]\right)\right) \\
&= \mathbf{R}\frac{1}{2\pi}\int_0^{2\pi} d\theta \left[\left(\frac{\exp\left[ix\cos(\theta)\right]}{i\cos\theta}\right)\right]_{x=0}^\infty \\
&= \frac{1}{2\pi}\Big[\theta\Big]_{\theta=0}^{2\pi} = 1
\end{aligned}$$