Mathematical Methods: FK7048

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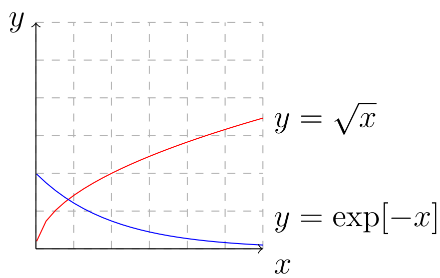
Date: October 27, 2023



Contents

This document is a summary of the course FK7048 given at autumn 2023. It includes examples taught by the professor and personal notes. No references are included; this is not an "official" document.

1 Ordinary Differential Equations

In this course, one covers four different types of Ordinary Differential equations: Seperable first order differential equations, Exact differentail equations, Homogenous differential equations, and linear differential equations; all of them are first order differential equations. The general form of any order differential equation is:

$$\sum_{i=1}^{n} a_{n-i} \frac{d^{n-i}}{x^{n-i}} y(x) = 0.$$

The following sections cover the different types of differential equations, and how to solve them. This will be for the general first order ode, written on the form:

$$a_1(x) \cdot y'(x) = f(x, y) \tag{1}$$

1.1 Seperable ODE's

Seperable ODEs are ODE's that can be seperated into two sets of functions:

$$y'(x) = f(x, y) = \frac{P(x, y)}{Q(x, y)}$$

In the case of P and Q being seperable, one means that both can be written explicitly as a function of x or y, i.e. $P(x)$ and $Q(y)$ or vice versa. In that case, one can write the following:

$$Q(y)dy = P(x)dx, \text{ or } \frac{dy}{P(y)} = \frac{dx}{Q(x)}.$$

One can then integrate both sides and solve for y as a function of x; *Note:* when integrating, one has to use reference integration, i.e. form y_0 to y and x_0 to x.

1.2 Exact ODE's

Suppose that $f(x, y)$ can't be written as two seperate functions only depending on a single variable, but rather as a function of both x and y:

$$\frac{dy}{dx} = f(x, y) = \frac{P(x.y)}{Q(x, y)},$$

then one can write it on the following form:

$$P(x, y)dx + Q(x, y)dy = 0.$$

Using the fact that mixed derivatives are equal one can solve this problem as a three-step method:

$$\frac{\partial \psi}{\partial y} = Q(x, y) \quad \& \quad \frac{\partial \psi}{\partial x} = P(x, y),$$

for an arbitrary function $\psi(x, y)$.

- Evaluate, if true then we proceed, and it might be possible to solve via this method:

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} = \frac{\partial^2 \psi}{\partial x \partial y}$$

- Evaluate:

$$\frac{\partial \psi}{\partial x} = P(x, y).$$

  Don't miss any integration constant or functions of y.

- Evaluate:

$$\begin{aligned} & Q(x,y)dy + P(x,y)dx = 0 \\ \implies & \int \frac{\partial \psi}{\partial y} dy + \int \frac{\partial \psi}{\partial x} dx = c \\ \implies & \psi(x,y) = \frac{c}{2} \end{aligned}$$

It might not be possible to write it on an explicit form, but rather an implicit form, i.e. $F(x, y) = c$.

1.3 Homogenous ODE's

In this case, one does not mean that the $f(x, y)$ term in the general case (1) is 0 but rather the following:

$$f(kx, ky) = k^n \cdot f(x, y).$$

In order tos solve this type of ODE, one use the following:

$$\begin{aligned} & Q(x,y)dy + P(x,y)dx = 0 \\ \implies & xQ^*(x,v)dv + [P^*(x,v) + v]dx = 0 \\ \implies & xP^*(v)dv + Q^*(v)dx = 0 \\ \implies & \frac{P^*(v)}{Q^*(v)} dv = \frac{1}{x} dx \end{aligned}$$

Note: the variable change. An example of this is the following:

$$\begin{aligned} & \frac{dy}{dx} = -\frac{x}{x+y} \\ \implies & (x+y)dy + xdx = 0 \quad [y = xv] \\ \implies & (x+xv)(xdv + vdx) + xdx = 0 \\ \implies & x(x+v)dv = -(v + v^2 + 1)dx. \end{aligned}$$

And form this point it's trivial to solve. The entire point of this method is to make the ODE seperable, but in order to do so, a clever subsitution is required.

1.4 Linear ODE's

Linear ODE's have the following form:

$$\boxed{\frac{dy}{dx} + p(x)y = q(x).}$$

These are solved via the so called integrating factor:

$$\alpha(x) = \exp\left[\int p(x)dx\right]$$
$$\implies y(x) = \int \frac{\alpha(x)q(x) - c}{\alpha(x)}dx$$

An example of this would be the ODE $y'(x) + 2y = 2e^{2x}$:

$$\begin{aligned}
a(x) &= \int p(x)dx = exp\left[\int 2dx\right] = e^{2x} \\
y(x) &= \int \frac{\alpha(x)q(x) - c}{\alpha(x)}dx \\
&= \int \frac{e^{2x}2e^{2x} - c}{e^{2x}}dx \\
&= \frac{1}{2}e^{2x} + c_1e^{-2x} + c_2.
\end{aligned}$$

Using boundary conditions one then finds the two integration constants.

1.5 Solving higer order ODE's

Higher ODE's can be harder to solve, but a good way to start is by making the ansats $y(x) = e^{rx}$ and find the roots of the characteristic equation; this is only possible when the ODE has constant coefficients, i.e. that the constants infront of the terms is indenpendent of x. If the root are real and distinct, then the solution is given by, for a second order problem: In this instance it doesn't matter if the roots are real, plugging them back into the problem will yield a cosinus and sinus expansion.

$$y(x) = Ae^{r_1x} + Be^{r_2x}.$$

However, if the roots are the same, we instead get:

$$y(x) = Ae^{r_1x} + xBe^{r_1x}.$$

If the problem is inhomgenous, i.e. that it's source-function $f(x) \neq 0$, then the total solution is given by particular solution added to the homogenous solution, i.e. $y(x) = y_p(x) + y_h(x)$. From here it's important to note that there exists something called singularity points; where the functions $p(x)$ and/or $q(x)$ diverges.

- **Regular singular points**: where $p(x)$ and $q(x)$ approaches ∞, but where $(x-x_0)p(x)$ and $(x-x_0)^2q(x)$ don't.
- **Irregular singular points**: where $p(x)$ and $q(x)$ approaches ∞, but where $(x-x_0)p(x)$ and $(x-x_0)^2q(x)$ also does.

1.6 Power series solutions

There exists two methods of solving via a power-series ansats, either the power-series ansats or the Frobenius anstats.

$$\text{Frobenius:}\ \boxed{y(x)=\sum_{n=0}^{\infty}a_nx^{n+r}}, \tag{2}$$

$$\text{Power series:}\ \boxed{y(x)=\sum_{n=0}^{\infty}a_nx^{n}}. \tag{3}$$

The method of this is to differentiate the expressions twice, reindex to find write everything on same exponent, and then find the recursion relation. In doing so, one can compare them to Taylor-expansions and find the corresponding functions. In the case of the Frobenius ansats, one has to find the indicial equation, is the $n=0$ term of the expansion: the roots of the indicial equation, which are used to substitute the unknown r in the series-expansion. Frobenius method can only be applied to regular singular points whilst the power series ansats can be applied to both regular and irregular singular points.

1.7 Finding additional solutions

Suppose that one finds one of the solutions to a problem, such that $y(x)=y_1(x)+y_2(x)$, then one can use Abel's theorem to find the second solution:

$$y_2(x)=ky_1(x)\int\left(\frac{\exp\left[-\int_{x_0}^{x}p(\omega)d\omega\right]}{\big(y_1(x)\big)^2}dx\right),$$

where k is an arbitrary constant. This is only valid for linear ODE's. But also, the Wronskian of two solutions is given by:

$$\begin{aligned}W(y_1,y_2)&=k\exp\left[\int_{x_0}^{x}p(\omega)d\omega\right]\\&=\begin{vmatrix}y_1&y_2\\y_1'&y_2'\end{vmatrix}=y_1y_2'-y_2y_1'.\end{aligned}$$

If the Wronskian is zero, then the two solutions are linearly dependent (this is actually not always the case but for the scope of this course this assumption can be made), and one can't use Abel's theorem to find the second solution. However, if the wronskien is nonzero, one can find an additional solution via Abels' theorem and variation of parameters, sec 1.7.1.

1.7.1 Variation of parameters

Suppose we have the following:

$$y''(x) + p(x)y'(x) + q(x)y(x) = F(x).$$

Finding the particular solution can be done via the methods above, however the particular solution can also be stated to have the following form:

$$y_p(x) = A_1(x)y_1(x) + A_2(x)y_2(x),$$

where $y_i(x)$ are the solutions to the homogeneous problem, and $A_i(x)$ are unknown functions. These functions can be found via the following:

$$\Longrightarrow \boxed{A_1(x) = \int -\frac{1}{W(y_1, y_2)} y_2(x)F(x)dx},$$

$$\Longrightarrow \boxed{A_2(x) = \int \frac{1}{W(y_1, y_2)} y_1(x)F(x)dx},$$

where W is the Wronskian. This is often used to generate the particular solution for more problematic source-functions; the general solution is outlined in the example below:

$$\begin{aligned}
y'' - y &= \frac{x}{e^x} \\
y_h(x) &= c_1 e^x + c_2 e^{-x} \\
W(e^x, e^{-x}) &= e^x \cdot -e^{-x} - e^{-x} \cdot e^x = -2 \neq 0 \\
y_p(x) &= A_1(x)e^x + A_2(x)e^{-x} \\
A_1(x) &= \int \frac{e^{-x}}{2} \cdot \frac{x}{e^x} dx = \frac{1}{2} \int e^{-2x} x dx \\
&= \frac{1}{2}\left(-\frac{1}{4}e^{-2x}(2x+1)\right) \\
A_2(x) &= -\int \frac{e^x}{2} \cdot \frac{x}{e^x} dx = -\frac{1}{2} \int x dx \\
&= -\frac{1}{2}\frac{x^2}{2} \\
y(x) = y_h(x) + y_p(x) &= c_1 y_1(x) + c_2 y_2(x) + A_1(x)y_1(x) + A_2(x)y_2(x)
\end{aligned}$$

2 Storm-Louiville equation

Suppose an ODE on the following form:

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + [q(x) + \lambda r(x)]y = 0, \tag{4}$$

the function $r(x)$ is called the weighting function and is defined as:

$$\int_a^b r(x)y_n(x)y_m(x)dx = \delta_{nm},$$

where $y_i(x)$ are the eigenfunctions of the problem. When doing SL problems, one finds that the solutions are eigenfunctions. Moreover, if λ in eq (4) is equal to zero, then the problem becomes self-adjoint. That is the following:

$$\boxed{\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y = 0,} \tag{5}$$

$$\boxed{p(x)y'' + p'(x)y'(x) + q(x)y = 0,}$$

2.1 Bondary conditions

There exists a few classifications on boundary conditions:

- **Dirichlet**: $y(a) = y(b) = 0$.
- **Neumann**: $y'(a) = y'(b) = 0$.
- **Robin**: $y'(a) = 0$ and $y(b) = 0$.

In this list above, all the boundary conditions are homogenous, however that does not have to be the case; they can also be non-homogenous, i.e. that they are non-zero: a function or a constant.

2.2 Converting an ODE into SL form

Many ODEs' can be converted into SL form using the following method:

$$\begin{aligned}
p_0(x)\psi'' + p_1(x)\psi' + p_2(x)\psi &= \lambda\psi \\
w(x) &= \frac{1}{p_0(x)}\exp\left[\int\frac{p_1(x)}{p_0(x)}dx\right] \\
\implies \lambda\psi w(x) &= w(x)\left(p_0(x)\psi'' + p_1(x)\psi' + p_2(x)\psi\right) \\
\implies \bar{p}_0' &= \frac{d}{dx}\exp\left[\frac{p_0}{p_0}\int\frac{p_1}{p_0}dx\right] = \bar{p}_1
\end{aligned}$$

$\bar{p}_0$ is now the first term of the expansion, which is the weighing-function times the original-expression, and the second term is computed in accordance to the above expression. An example of this would be the following:

$$xy'' + (1-x)y' + 1 = 0$$
$$w(x) = \frac{1}{x}\exp[\int \frac{1-x}{x}dx] = e^{-x}$$

3 Partial Differential Equations

There exists mulitple different types of ordinary differential equations, however in this course only linear PDEs' were covered and those being that of: Hyperbolic-, Parabolic-, and Elliptic- PDEs'. A second order PDE is defined as the following:

$$A(x,y)f_{xx} + 2B(x,y)f_{xy} + C(x,y)f_{yy} + D(x,y)f_x + E(x,y)f_y + F(x,y)f = G(x,y). \quad (6)$$

We compute the different types of PDE's in accordance to the list below:

- **Hyperbolic**: $B^2 - AC > 0$.
- **Elliplitc**: $B^2 - AC < 0$
- **Parabolic**: $B^2 - AC = 0$.

The different types of PDEs' have different behaviors, and different methods of solving them. The following sections will cover the different types of PDEs' and how to solve them.

3.1 Different common PDE's

The Laplace-equation, eq (7) is defined below and is an example of an elliptic PDE:

- **Laplace-equation**:

  $$\Delta f(x,y,z) = \nabla^2 f(x,y,z) = 0. \quad (7)$$

  This is obviously a second order PDE, but since $B = 0$ given the general case eq (6) it's an elliptic equaption.

- **Poission-equation**:

  $$f(x,y,z) = \nabla^2 u(x,y,z) \quad (8)$$

  This is also an elliptic PDE.

- **Heat-equation**:

$$\nabla^2 f = \frac{1}{a}\frac{\partial f}{\partial t}$$

This is also an elliptic PDE.

- **Wave-equation**:

$$\nabla^2 f = \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}$$

This is also an elliptic PDE.

- **Schrödingers-equation**:

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = i\hbar\frac{\partial\psi}{\partial t}$$
$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi$$

- **Klein Gordons-equation**:

$$\Box\psi = -\mu^2\psi$$
$$\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$$

3.2 Solving PDE's

There exists numerous ways of solving PDE's, however most PDE's are solved via the aid of computers via methods such as: method of lines. However, there exists a few methods for solving PDE's by hand: Seperation of variables, Fourier transforms 7.1, method of characteristic and so on. This section is divided such that they cover each of the different methods covered in this course.

3.2.1 Method of characteristic

Suppose a first order PDE on the following form:

$$A(x,y)u_x + B(x,y)u_y + C(x,y)u = D(x,y).$$

Suppose the case when A and B are constants and the others are zero:

$$au_x + bu_y = 0.$$
$$\implies [a,b]\cdot\begin{pmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\end{pmatrix}u$$

Using some clever change of variable: $s = ax + by$ and $t = bx - ay$ one can solve this problem. The method of characteristic is a method of solving first order PDE's, and is done by the following:

$$\frac{\partial u}{\partial s} = 0$$
$$\frac{\partial u}{\partial t} = u(t)$$

An example of this would be:

$$3u_x + 4u_y = 0 \quad \& \quad u(0, y) = \sin(y).$$

This is then written on the following form:

$$(3, 4) \cdot \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} u$$

Using the variable change: $s = 3x + 4y$ and $t = 4x - 3y$ one has that:

$$\begin{aligned} &u(t) = u(4x - 3y) \\ &u(0, y) = \sin(y) \implies u(x, y) = \sin(4y - 3x) \end{aligned}$$

In the cases where A and B are not constants but rather functions one instead has:

$$A(x, y)u_x + B(x, y)u_y = 0.$$
$$\implies [A(x, y), B(x, y)] \cdot \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} u$$

This instead yields characteristic curves, and the solution is given by:

$$\frac{dy}{dx} = \frac{B(x, y)}{A(x, y)}.$$

There also exists a plug and play variant of this solution method: Suppose that one has the following PDE:

$$A(x, y)u_x + B(x, y)u_y = C(x, y).$$

Then the solution can be found via the following, which is called the **Lagrange-Charpit Equation**:

$$\boxed{\frac{dx}{A(x, y, z)} = \frac{dy}{B(x, y, z)} = \frac{dz}{C(x, y, z)}} \tag{9}$$

Note this method only works for first order PDE's, and if the variables are non-coupled. If they are coupled, one has to de-couple them first by taking the derivatives multiple times to get rid of the coupling.

3.2.2 Seperation of variables

Some PDE's can be solved using the method of seperation of variables; which effectivly takes and PDE to and ODE; this is a very powerful technique which takes the form:

$$u(x, y) = X(x)Y(y).$$

This is then inserted into the PDE and using an example one would get the following, the Laplace-equation (7):

$$\begin{aligned} &u_{xx} + u_{yy} = 0 \\ \implies &\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = 0 \\ \implies &\frac{X''}{X} = -\frac{Y''}{Y} = \lambda^2 \end{aligned}$$

This means that the rate of change is the same, which is just some constant, thus one can write the following:

$$u(x, y) = X(x)Y(y) = \Big(c_1 e^{-\lambda x} + c_2 e^{\lambda x}\Big) \cdot \big(c_3 \sin(\lambda y) + c_4 \cos(\lambda y)\big)$$

Note: many of the PDE's in time are evaluated in the so-called *steady state*, i.e. that the time-derivative is zero, e.g. that the system has reached an equalibrium. Also, not every system is seperable; it puts constraints on the system.

3.3 Notes

The Hessian of a system:

$$H = \begin{vmatrix} \frac{\partial^2}{\partial x^2} & \frac{\partial^2}{\partial x \partial y} & \frac{\partial^2}{\partial x \partial z} \\ \frac{\partial^2}{\partial x \partial y} & \frac{\partial^2}{\partial y^2} & \frac{\partial^2}{\partial y \partial z} \\ \frac{\partial^2}{\partial x \partial z} & \frac{\partial^2}{\partial y \partial z} & \frac{\partial^2}{\partial z^2} \end{vmatrix}$$

If the eigenvalues of the Hessian H are positive and real, then there exists a local minimum, if it's negative and real, then there exists a local maximum and if it's zero then there exists a saddle-point.

4 Green's Functions

Green's function are a way to solve ordinary differnetial equations, and can be used to solve partial differnetial equations however that is not covered in this course. These functions are defined as the solution to the following equation:

$$\begin{aligned} &\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y(x) = f(x), \qquad (10) \\ &\mathcal{L}G(x, s) = \delta(x - s), \end{aligned}$$

where $\mathcal{L}$ is a linear operator defined by the problem at hand. One solves the two set of equations:

$$\text{Continuity condition: } \mathcal{G}_1(s,s) = \mathcal{G}_2(s,s)$$
$$\text{Discontinuity condition: } \left.\frac{d\mathcal{G}_2}{dx}(x,s)\right|_{x=s} - \left.\frac{d\mathcal{G}}{dx}(x,s)\right|_{x=s} = \frac{1}{p(x)}.$$

Green's functions are used to solve inhomogenous differential equations on the form (10). And the solution is given in totalt to be:

$$\boxed{y(x) = \int_0^x \mathcal{G}_1^*(x,s)f(s)ds + \int_x^L \mathcal{G}_2^*(x,s)f(s)ds},$$
$$\boxed{\mathcal{G}(x,s) = \begin{cases} \mathcal{G}_1(x,s), & 0 \in [0,s) \\ \mathcal{G}_2(x,s), & x \geq s. \end{cases}}$$

Note: in this case the complex conjugation yields: $\mathcal{G}^*(x,s) = \mathcal{G}(s,x)$, i.e. that one switched the integration boundary or the terms in the expression.

Example

Suppose the following problem:

$$-y''(x) = \sin(\pi x); \quad y(0) = y(1) = 0.$$

The homogenous solution is given by $y = ax + b$, so that is the Green's function. Dividing the Green's function into two regions, one finds two coefficients using the boundary conditions:

$$\mathcal{G}_1(0,s) = b_1 = 0 \implies \mathcal{G}_1(x,s) = a_1 x$$
$$\mathcal{G}(1,s) = a_2 + b_2 = 0 \implies \mathcal{G}_2(x,s) = a_2(1-x)$$

Using the continuity and discontinuity conditions:

$$\begin{aligned} \text{Continuity: } & a_1 s = a_2(1-s) \\ \text{Discontinuity: } & -a_2 - a_1 = -1 \\ \implies & a_1 = 1 - a_2 \quad \& \quad a_1 s = (1-a_1)(1-s) \\ & a_1 = 1 - s \quad \& \quad a_2 = s \\ \implies & \mathcal{G}(x,s) = \begin{cases} (1-s)x; & x \in [0,s) \\ s(1-x); & x \in [s,1] \end{cases} \end{aligned}$$

Putting this into the integral from above, one has the following:

$$\begin{aligned} y(x) &= \int_0^x \mathcal{G}_1^*(x,s)f(s)ds + \int_x^L \mathcal{G}_2^*(x,s)f(s)ds \\ &= \int_0^x (1-x)s\sin(\pi s)ds + \int_x^1 x(1-s)\sin(\pi s)ds \\ &= (1-x)\int_0^x s\sin(\pi s)ds + x\int_x^1 (1-s)\sin(\pi s)ds \\ &= \frac{\sin(\pi x)}{\pi^2} \end{aligned}$$

Note: in this example, since the problem is in SL form, the symmetri behind $\mathcal{G}_1$ and $\mathcal{G}_2$ yields $\mathcal{G}_1^*(x,s) = \mathcal{G}_2(x,s)$.

Example

Suppos the following problem:

$$y'' + y = f(t); \quad y(0) = y'(0) = 0.$$

Again, as prior, find the general formula for the homogenous solution:

$$y'' + y = 0 \implies y = a\cos(x) + b\sin(x).$$

Applying the boundary conditions: which only affect the first region of the Green's function:

$$\begin{aligned} \mathcal{G}_1(0,s) &= a_1\cos(x) + b_1\sin(x) = 0 \implies a_1 = 0 \\ \mathcal{G}_1'(0,s) &= b_1\cos(0) = 0 \implies b_1 = 0 \\ \implies \mathcal{G}(x,s) &= \begin{cases} 0; & x \in [0,s) \\ a_2\cos(x) + b_2\sin(x); & x \in [s,L] \end{cases} \end{aligned}$$

Appying the two conditions one has:

$$\begin{aligned} 0 &= a_2\cos(s) + b_2\sin(s) \\ &- a_2\sin(s) + b_2\cos(s) = 1 \\ \implies \mathcal{G}_2(x,s) &= -\sin(s)\cos(x) + \cos(s)\sin(x) = \sin(x-s) \end{aligned}$$

The final step is to solve the integral:

$$y(x) = \int_0^\infty \mathcal{G}^*(x,s)f(s)ds = \int_x^\infty \mathcal{G}_2^*(x,s)f(s)ds = \int_0^x \sin(x-s)f(s)ds$$

Note: the boundary change as discussed above with the change of integration.

5 Complex Analysis

A complex number z can be defined as $z = a + bi$ or $z(x, y) = u(x, y) + i(v, y)$. In doing so, one finds that the differentiability of such a function z would be the following:

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},} \tag{11}$$

which is also known as the Cauchy-Riemann equations. Furthermore, if a function $f(z)$ can be written on the following form, the derivatives hold:

$$\begin{aligned} f(z) &= u(x, y) + iv(x, y), \\ \frac{df}{dz} &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \\ &= \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}. \end{aligned}$$

Note: it might be more usefull to write the function $f(z)$ on euler-from, where $r = \sqrt{u(x, y)^2 + v(x, y)^2}$ and $\theta = \arctan\left(\frac{v(x,y)}{u(x,y)}\right)$; such that $f(z) = re^{i\theta}$. The laplacian ∇^2 in the complex space is a bit more tricky and is defined by:

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial v^2} = \frac{\partial^2 v}{\partial y \partial x} - \frac{\partial^2 v}{\partial x \partial y} = 0.}$$

If the function z is differentiable, then it is also analytic. A function $f(z)$ can be analytic around points: singularities. There exists two kinds of singularities: essential-singularities and poles. Essential singularities are when the function $f(z)$ diverges fast enough to be expanded into a Laurent series.

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n. \tag{12}$$

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(\tilde{z})}{(\tilde{z} - z_0)^{n+1}} d\tilde{z}$$

The a_{-1} coefficient is known as the residue of the function $f(z)$. *Note:* a good trick so solving problems in complex is via a change of coordinates, from cartesian to polar on euler form, i.e. $z = x + iy = re^{i\theta}$.

5.1 Residue

The residue is important when computing complex contour integrals:

$$\boxed{\oint_\Gamma f(z)dz = 2\pi i \sum_i \mathrm{Res}(f, z_i),}$$

where Γ is the closed contour and z_i are the singularities of the function $f(z)$; note not essential-singularities, but rather poles of different order. There exists three ways to compute the residue of a function $f(z)$: Finding the Laurent series, or the following two methods:

$$\boxed{\text{Res}(f, z_i) = \lim_{z \to z_i} (z - z_i) f(z),} \tag{13}$$

$$\boxed{\text{Res}(f, z_i) = \lim_{z \to z_i} \frac{1}{(n-1)!} \frac{d^{(n-1)}}{dz^{(n-1)}} (z - z_i)^n f(z).} \tag{14}$$

In eq (13) one can compute the residue of a simple pole, and in eq (14) one can compute the residue of a pole of order n. An important trick is to deconstruct the function f such that one can find the Taylor/Mclaren-series of each component, and then find the residue of the $(z - z_0)^{-1}$ term. Also, if one tries to use eq (13) and one gets ∞ the reason it's because it's a higher order pole and one should use eq (14) instead.

5.2 Contour Integral tricks

Suppose one has a funciton f which is analytic expect at a point z_0, and one wants to compute the ingral, one can decompose the closed integral to the elements along a certain path Γ and the elements around the point z_0:

$$\begin{aligned} \oint_{\partial\gamma} f(z)dz &= 2\pi i \text{Res}(f, z_0) \\ &= \sum_i \int_{\partial\Gamma_i} f(z)dz. \end{aligned}$$

Often, the decomposed integrals might be easier to compute than the original integral, or maybe one of the integrals is the one that is wanted. Moreover, Cauchy's integral formula states the following:

$$\text{Cauchy's integral formula: } \boxed{f(z_0) = \frac{1}{2\pi i} \oint_\Gamma \frac{f(z)}{z - z_0} dz},$$
$$\boxed{f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_\Gamma \frac{f(z)}{(z - z_0)^{n+1}} dz}$$

where z_0 is a singularity point. An example of the usefullness of this could be the following:

$$\begin{aligned} &\oint_\Gamma \frac{\sin^2(z)}{(z-a)^4} dz \\ \implies &f(z) = \sin^2(z) \\ \implies &\oint_\Gamma \frac{\sin^2(z)}{(z-a)^4} dz = \frac{2\pi i}{3!} f^{(3)}(z = a) \end{aligned}$$

Example

Suppose the following integral:

$$\int_0^\infty \frac{dx}{x^3 x+1} dx.$$

Suppose one solves this by construcing a path-integral from $(0,0)\times(R,0)\times(re^{2i\theta/3})$ which is closed (Γ), then $z^3 = r^3 e^{i\theta 3}$. Using this one has the following:

$$\begin{aligned}
\oint_\Gamma \frac{1}{z^3+1} dz &= \lim_{R\to\infty} \int_0^R \frac{1}{x^3+1} dx + \int_{arc} + \int_R^0 \frac{e^{2i\pi/3}}{z^3 e^{2i\pi}} \\
&= (1-e^{i2\pi/3}) \int_0^\infty \frac{dx}{x^3+1} = 2\pi i \sum_i \text{Res}(f, z_i) \\
&= \frac{2\pi}{3\sqrt{3}}
\end{aligned}$$

5.3 Moreras Theorem

Moreras theorem is equvilent to the fundamental theorem of calculus, but in the complex plane. It states the following: If there exists a contionous function $f(z)$ on a simply connected in some region Γ, then:

$$\boxed{\oint_\Gamma f(z) dz = 0,}$$

then $f(z)$ is analytic everywhere in Γ. The implication of this is that $F'(z) = \frac{F(z_1)-F(z_0)}{z_1-z_0}$ which is equvilent to the fundamental theorem of calculus.

5.4 Branch cut

When doing complex integration one can stumble upon a distinct problem, that a function is not properly defined for a certain value; or that it's not unqiuely defined. A common example is the function $f(z) = \ln(z)\cdot g(z)$, where $\ln(z)$ is, in a complex sense, not symemtric. The function $f(z)$ can only be solved with a branch-cut, i.e. that the contour is defined in a specific manner. Suppose the following integral, where $p \in (0,1)$:

$$I = \int_0^\infty \frac{x^p}{x^2+1} dx.$$

Turning this integral to the complex plane one would have the following:

$$\oint_\Gamma \frac{z^p}{z^2+1} dz.$$

However, there exists a problem; we have a branch-cut at $z = 0$ since $z^p = r^p e^{2i\pi p}$. One thus have to do the following analysis:

$$\oint_{\Gamma} \frac{z^p}{z^2+1} dz = \int_{\partial\Gamma_1} \frac{r^p}{r^2+1} dz + \int_{\partial\Gamma_2} \frac{z^p}{z^2+1} dz + \int_{\partial\Gamma_3} \frac{(re^{i2\pi})}{z^2+1} dz + \int_{\delta r} \frac{z^p}{z^2+1} dz$$

One sees that the integral over $\partial\Gamma_1$ is the integral one wants, but also $\partial\Gamma_3$ is the same with a phaseshift. Thus, one can compute the integral as the following:

$$\begin{aligned}(1-e^{2\pi ip})I = \oint_{\Gamma} \frac{z^p}{z^2+1} &= 2\pi i \sum_i \mathrm{Res}(f, z_i),\\ \implies I = \frac{1}{1-e^{2\pi ip}} &\sum_i \mathrm{Res}(f, z_i).\end{aligned}$$

From this point, the only thing left is to compute the residues of the function $f(z)$, and since both are poles of the first kind one can use eq (13) to compute the residues. *Note:* when computing around a branch-cut, we avoid the essential-singuarlity by constructing a countour that avoids the point in itself.

Suppose that one has two functions f and g which are defined to be analytic on two domains U_1 and U_2 respectivly. If at the intersection there exists atleast one **accumulation point**, $U = U_1 \cap U_2$, one has that $f(z) = g(z)$, then one can define the following function:

$$h(z) = \begin{cases} f(z); \forall z \in U_1 \\ g(z); \forall z \in U_2 \cap U_1^C \end{cases}$$

where U_1^C is the compliment domain. The Laruent expansion of the two functions are then defined as:

$$\begin{aligned} f(z) &= \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n \\ g(z) &= \sum_{n=-\infty}^{\infty} b_n (z-z_1)^n \end{aligned}$$

Then at the point accumulation point the following holds for all n:

$$\begin{aligned} a_n(z-z_0)^n &= b_n(z-z_1)^n \\ \implies \psi(z) = f(z) - g(z) &= \sum_{n=-\infty}^{\infty} (a_n - b_n)(z-z_0)^n = 0. \end{aligned}$$

5.5 Cauchys principal value

This is a method of given values to undetermined integrals which does not necessarily converge. Suppose that one has the following integral:

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)dx$$
$$= \lim_{R\to\infty} F(x)\Big|_{x=-R}^{R}$$

An example of this could be the following:

$$I = \int_0^\infty \frac{\sin(x)}{x}dx = \int_0^\infty \frac{e^{ix}-e^{-ix}}{2ix}dx$$
$$= \int_0^\infty \frac{e^{ix}}{2ix}dx - \int_0^\infty \frac{e^{-ix}}{2ix}dx$$
$$= \int_0^\infty \frac{e^{ix}}{2ix}dx + \int_{-\infty}^0 \frac{e^{ix}}{2ix}dx$$
$$= \int_{-\infty}^\infty \frac{e^{ix}}{2ix}dx$$

Using the complex integral of the same expression one can compute the value of the original expression:

$$\lim_{R\to\infty} \oint_{\Gamma(R)} \frac{1}{2i}\frac{e^{iz}}{z}dz = \int_0^\pi \frac{1}{2i}\frac{re^{i\theta}}{re^{i\theta}}d\theta + \int_\pi^{2\pi} \frac{1}{2i}\frac{re^{i\theta}}{re^{i\theta}}d\theta = \pi.$$

Thus, the integral one first wanted is equal to half of that i.e. $I = \frac{\pi}{2}$.

5.6 Meromorphic functions & Winding numbers

A function $f(z)$ is said to meromorphic at any point z_0 forall z in the close neighbourhood of z_0 if the function can be decomposed as the following:

$$f(z) = \frac{g(z)}{h(z)}.$$

If the two functions g and h are analytic at the point z_0 then the function f is said to meromorphic around z_0. Another way to define it is to say that h can only be 0 at countable points. Meromorphic functions can be written as a 'non-infinite' Laruent series:

$$f(z) = \sum_{n=-N}^{\infty} a_n(z-z_0)^n.$$

This result is used in the **Mittag-Leffler theorem**. The theorem states: Assuming a function f which is analaytic around the origin and only has countable singularities, N in total, which are simple poles at z_i with residues a_i, then the function f can be written as the following:

$$\boxed{f(z) = f(0) + \sum_{i=1}^{N} a_i \left(\frac{z_i}{z - z_i} + \frac{1}{z_i} \right)}.$$

For this to hold the function f has to be bounded, that is $f(z) \leq M$ for a constant M in the contour that encompases all singularities.

Winding numbers are defined in laymans terms of how many times the function curls around itself. Assuming a function $f(z)$ which is analytic in some region Ω and a closed contour Γ in Ω which does not pass through any singularities of $f(z)$, then the winding number is defined as:

$$\boxed{\oint_\Gamma \frac{f'(z)}{f(z)} dz = 2\pi i w_n,}$$

where w_n is the winding number. Now it should be identified that the winding-number w_n is the sum of the residues, i.e. $w_n = \sum_i \mathrm{Res}(f, z_i)$ of the function in the integrand of the contour integral. This connects to **Rouche's theorem** which states: If Γ is a closed contour and both the functions $f(z)$ and $g(z)$ are analytic inside and on Γ, and $|f(z)| > |g(z)|$ on Γ, then $f(z)$ and $f(z) + g(z)$ have the same number of zeros inside Γ.

6 Bessel functions

There exists multiple different kinds of Bessel-functions. Bessel-functions are solutions to the Bessel differential equation and often occur in problems with cylindrical symmetri:

$$\boxed{x^2 y'' + x y' + (x^2 - n^2) y = 0.} \tag{15}$$

The term n in the above equation is the so-called order of the equation, not to be confused with the order of the ODE. Bessel functions of the first kind are defined as:

$$\boxed{J_n(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(s+n)!} \left(\frac{x}{2} \right)^{2s+n}.}$$

They are found by solving the Bessel-equation, eq (15) with either a power-series solution or a Frobenius solution. One important property of Bessel functions is that they are orthogonal:

$$\int_0^1 x J_n(\lambda_{n,m} x) J_n(\lambda_{n,l} x) dx = \frac{1}{2} \delta_{m,l},$$

moreover, they have the following property, excluding the $n = 0$ case:

$$J_{n-1}(x) - J_{n+1}(x) = 2J'_n(x),$$
$$J_{n+1}(x) + J_{n-1}(x) = \frac{2n}{x} J_n(x).$$

The moddified Bessel functions are the Bessel-function with imaginary argument, i.e. the moddified Bessel-function of the first kind is defined as the following:

$$\mathcal{J}_n(ix) = \sum_{k=1}^{\infty} \frac{(-1)^k}{k!\Gamma(k+n+1)} \cdot \left(\frac{ix}{2}\right)^{2k-n}$$

The integral representation of the Bessel-function of the first kind allows for easier computation of the expression of $J_n(x)$:

$$J_n(x) = \frac{1}{\pi} \int_0^{\pi} \cos\big(n\tau - x\sin(\tau)\big)\, d\tau = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp\big[i(n\tau - x\sin(\tau))\big]\, d\tau,$$

and for the zeroth order Bessel-function one has the following:

$$J_0(x) = \frac{1}{\pi} \int_0^{\pi} \cos\big(x\sin(\tau)\big)\, d\tau.$$

6.1 Legendre's polynomials

The Legendre poylynomials, and the associate Legendre polynomials are solutions to the Legendre differential equation:

$$(1-x^2)\frac{d^2}{dx^2}P_l^m(x) - 2x\frac{d}{dx}P_l^m(x) + \left[l(l+1) - \frac{m^2}{1-x^2}\right] P_l^m(x) = 0. \tag{16}$$

Note: in this course, m has been set to zero when solving the above equation, and thus is written on the following form:

$$(1-x^2)p''(x) - 2xy'(x) + l(l+1)p(x) = 0.$$

This equation is solved by the Legendre polynomials which can be derived from Rodrigues' formula:

$$L_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l,$$

which then gives the following solution to the associated polynomial:

$$\boxed{P_l^m(x) = (-1)^m(1-x^2)^{m/2}\frac{d^m}{dx^m}P_l(x)}$$

6.2 Generating functions

To solve certain problems one uses the concept of generating functions to solve the problem at hand; there exist multiple different generating functions however, the following is a known one:

$$\boxed{g(x,t) = \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n.}$$

6.3 Other special functions

There exists other special functions, such as the Hermite polynomials, the Laguerre polynomials, and the Chebyshev polynomials. These will not be covered in this recap sheet. However, special functions such as the Gamma function, the Beta function and the Di-Gamma and Poly-Gamma functions can be defined in the following manner:

The Gamma-function Γ.

$$\begin{aligned}
\Gamma(z) &= \int_0^{\infty} t^{z-1}e^{-t}dt, \\
&= \lim_{n\to\infty} \frac{n!n^z}{z(z+1)(z+2)\cdots(z+n)} \\
\Gamma(z) &= (z-1)! \text{ if } z \in \mathbb{N} \\
\text{Identity: } \Gamma(z+1) &= z\Gamma(z), \\
\text{Reflection: } \Gamma(z+1)\Gamma(1-z) &= \frac{\pi}{\sin(\pi z)},
\end{aligned}$$

The beta-function β.

$$\begin{aligned}
\beta(z,\omega) &= \int_0^1 t^{z-1}(1-t)^{\omega-1}dt, \\
&= \frac{\Gamma(z)\Gamma(\omega)}{\Gamma(z+\omega)} \text{ if z \& } \omega \text{ are integers}
\end{aligned}$$

The Di-Gamma function ψ.

$$\boxed{\psi(z) = \frac{d}{dz}\big(\ln(\Gamma(z))\big) = \frac{\Gamma'(z)}{\Gamma(z)}}$$

7 Transforms

There exists numerous transforms, however in the course only linear transforms were considered; the transforms of Laplace and Fourier were taught. Both being linear operators. There exists other linear transforms, such as the Mellin transform, the Hankel transform, the Hilbert transform, the Z-transform, and the Radon transform. However, these will not be covered in this recap sheet.

7.1 Fourier Transform

The Fourier transform is defined as the operator $\mathcal{L}$ operating on a function $f(t)$ to be the $\tilde{f}(\omega) = \mathcal{L}\big(f(t)\big)(\omega)$. This is then defined as:

$$\text{Transform: } \boxed{\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt,}$$
$$\text{Inverse: } \boxed{f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(\omega)e^{i\omega t}d\omega.}$$

In order for the Fourier transform to exist, the function $f(t)$ must be absolutely integrable, i.e. $\int_{-\infty}^{\infty}|f(t)|dt < \infty$; moreover, the function f has to be in $\mathbb{L}^1$. The transform has the following properties:

$$\begin{aligned}
&1\text{: } \frac{df}{dx} \implies iw\tilde{f}(\omega), \\
&2\text{: } \frac{d^n f}{dx^n} \implies (iw)^n\tilde{f}(\omega), \\
&3\text{: } (f \otimes g)(t) \implies \tilde{f}\cdot\tilde{g}, \\
&4\text{: } f(at) \implies \frac{1}{|a|}\tilde{f}\left(\frac{\omega}{a}\right),
\end{aligned} \tag{17}$$

An example of the Fourier transform is the χ function:

$$\chi(-1,1) = \begin{cases} 1, & |x| < 1, \\ 0, & \text{elsewhere} \end{cases}$$
$$\tilde{\chi}(\omega) = \int_{-1}^{1} e^{-i\omega t}dt = \frac{2\sin(\omega)}{\omega}.$$

7.2 Laplace Transform

Another transform is the Laplace transform, $\hat{f}(s) = \mathcal{L}(f(t))(s)$, which is defined as:

$$\hat{f}(s) = \int_0^\infty f(t)e^{-st}dt$$
$$f(t) = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{\gamma-iT}^{\gamma+iT} f(s)e^{st}ds,$$

where γ is the real part of s. The Laplace transform has the following properties:

$$1:\ f'(t) \implies s\hat{f}(s) - \hat{f}(0^+),$$
$$2:\ f(t)\cdot t^{-1} \implies \int_s^\infty \hat{f}(x)dx,$$
$$3:\ f(t)e^{at} \implies \hat{f}(s-a),$$
$$4:\ f(at) \implies \frac{1}{|a|}\hat{f}\left(\frac{s}{a}\right),$$
$$5:\ tf(t) \implies -\frac{d}{ds}\hat{f}(s).$$

The Laplace transform is a linear operator, and is defined for $t \geq 0$; since it's a linear operator, $\mathcal{L}(f+g)(s) = \hat{f} + \hat{g}$, and multiplication by a constant can be taken outside the operator.

8 Usefull identifies

$$\boxed{\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}}$$

$$\boxed{n! = \frac{(2n)!!}{2^n}}$$

Euler-Mascheroni constant: $\gamma = 0.5772...$

The fourier transform of a gaussian is another gaussian, proved via completion of squares;

$$\boxed{f(t) = e^{at^2}}$$

$$\boxed{\tilde{f}(\omega) = \sqrt{\frac{2}{a}}\exp\left[-\frac{\omega^2}{4a}\right]}$$

Common expansions, centered around zero:

$$\boxed{\text{Taylor expansion} f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n}$$

$$\boxed{\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ... = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}}$$

$$\boxed{\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ... = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}}$$

$$\boxed{e^x = 1 + x + \frac{x^2}{2!} + ... = \sum_{n=0}^{\infty} \frac{x^n}{n!}}$$

$$\boxed{\tan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} + ... = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)}}$$

$$\boxed{\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + ... = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}}$$

$$\boxed{\tanh^{-1}(x) = x + \frac{x^3}{3} + \frac{x^5}{5} + ... = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}}$$

$$\boxed{\int g'(x) \cdot h(x) dx = \left[g(x) \cdot h(x) \right] - \int g(x) \cdot h'(x) dx}$$

$$\boxed{\int_0^{\infty} t^n e^{-st} dt = \frac{\Gamma(n+1)}{s^{n+1}}}$$

9 Practice Exam

Problem 1)

Solve the ODE:

$$y'' + 4y = e^x; \quad y(0) = y'(0) = \frac{2}{5}.$$

Solution

Making the ansats $y = e^{rx}$ gives the characteristic equation:

$$e^{rx}(r^2 + 4) = 0 \implies r = \pm 2i.$$
$$\implies y_h(x) = a\cos(2x) + b\sin(2x).$$

The particular solution can be found by the method of undetermined coefficients:

$$y_p(x) = Ae^x \implies y_p'(x) = Ae^x \implies y_p''(x) = Ae^x.$$

Substituting into the ODE gives:

$$Ae^x + 4Ae^x = e^x \implies A = \frac{1}{5}.$$

Thus, the general solution is:

$$y(x) = a\cos(2x) + b\sin(2x) + \frac{1}{5}e^x.$$

Applying the initial conditions gives:

$$y(0) = a + \frac{1}{5} = \frac{2}{5} \implies a = \frac{1}{5}$$
$$y'(0) = -2b + \frac{1}{5} = \frac{2}{5} \implies b = -\frac{1}{10}.$$

Thus, the solution is:

$$y(x) = \frac{1}{5}\cos(2x) - \frac{1}{10}\sin(2x) + \frac{1}{5}e^x.$$

Problem 2)

Explain/Sketch-out (in a few equations and sentences) the following methods for solving 1st Ordinary ODEs:

1. Seperation of variables
2. Exact differential equations
3. Homogeneous ODEs
4. Linear ODE's

Solution

1. **Seperation of variables**: Assuming a form $\frac{dy}{dx} = \frac{p(x,y)}{q(x,y)}$ where one only depends on one variable, then moving to either side and integrate.

2. **Exact differential equations**: Is a first order ODE where $\frac{dy}{dx} = f(x, y) = \frac{g(x,y)}{h(x,y)}$. In exact differential equations, neither g nor h can be written as a function of only one variable and thus one has to construct the following $\psi_y = g$ and $\psi_x = h$ and then solve the ODE $\frac{d\psi}{dx} = h$.

3. **Homogeneous ODEs**: This type of ODE does not refer to the source-function being 0 but rather the type of ODE where the function can be written on the followig form $f(kx, ky) = k^n f(x, y)$. Solving such a system one makes a variable substitution and solves the system with the transformed variables.

4. **Linear ODEs**: This type of ODE has the following form: $y' + p(x)y = q(x)$, where $p(x)$ and $q(x)$ are functions of x. This type of ODE can be solved by the method of integrating factors, where one multiplies the ODE by the integrating factor $\alpha(x) = e^{\int p(x)dx}$.

Problem 3)

Show that:

$$(1 - x^2)y'' - xy' + n^2 y = 0,$$

can be but put on SL-from with the following substitution: $\sqrt{(1 - x^2)}^{-1}$. Also, explain the orthangonaility condition of the SL problem under the transformation.

Solution

There exists more than one way of doing this, one would be to compute the weighting-function; if the weighting is the same as the suggested transformation then the transformation is valid; another way would be to plug it into the equation by multiplying both sides by the transform: However, in this instance the weighting-function option is opted:

$$\begin{aligned} w(x) &= \frac{1}{1 - x^2} \exp\left[\int \frac{-x}{1 - x^2} dx\right] \\ &= \frac{1}{1 - x^2} \exp\left[\frac{\ln(x^2 - 1)}{2}\right] = \frac{\sqrt{x^2 - 1}}{1 - x^2} \end{aligned}$$

Problem 4)

Solve via the method of Frobenius:

$$x^2y'' - 2xy' + y = 0.$$

Solution

Ansats, centered around $x = 0$:

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} a_n x^{n+r}, \\ y'(x) &= \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}, \\ y''(x) &= \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}. \end{aligned}$$

Substiting this into the ode yields:

$$\begin{aligned} & x^2 \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2} - 2x \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0 \\ & \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r} - 2a_n (n+r) x^{n+r} + a_n x^{n+r} = 0 \\ & \sum_{n=0}^{\infty} \big[(n+r)(n+r-1) - 2(n+r) + 1\big] a_n x^{n+r} = 0 \\ & \underbrace{\big[r(r-1) - 2r + 1\big] a_0 x^r}_{\text{Indicial eq.}} + \sum_{n=1}^{\infty} \big[(n+r)(n+r-1) - 2(n+r) + 1\big] a_n x^{n+r} = 0 \\ & r^2 - 3r + 1 = 0 \implies r = \frac{1}{2}\{3 + \sqrt{5}, 3 - \sqrt{5}\} \end{aligned}$$

Problem 5)

Use Green's functions to solve the following ODE:

$$y'' = x^2 + 4; \quad y(0) = y'(L) = 0.$$

Solution

Being by stating that we have the following form:

$$\begin{aligned}
&p(x)y'' + q(x)y' + r(x)y = 0. \\
&\implies \mathcal{G}_2'(s,s) - \mathcal{G}_1'(s,s) = 1 \quad \& \quad \mathcal{G}_2(s,s) = \mathcal{G}_1(s,s) \\
&\mathcal{L}(\mathcal{G}) = G''(x) = \delta(x-s) \\
&\mathcal{G}_1(x,s) = A_1x + B_1 \quad \& \quad \mathcal{G}_2(x,s) = A_2x + B_2 \\
&\mathcal{G}_1(0,s) = 0 \implies B_1 = 0 \quad \& \quad \mathcal{G}_2'(L,s) = 0 \implies A_2 = 0 \\
&\mathcal{G}_1(s,s) = \mathcal{G}_2(s,s) \implies A_1 s = B_2 \\
&G_2'(s,s) - G_1'(s,s) = 0 - A_1 = 1 \implies A_1 = -1 \\
&\implies \mathcal{G}(x,s) = \begin{cases} -x; & x \in [0,s) \\ -s; & x \in [s,L] \end{cases}
\end{aligned}$$

Thus, the solution is given by:

$$\begin{aligned}
y(x) &= \int_0^x -x \cdot (s^2+4)ds + \int_x^L -s \cdot (s^2+4)ds \\
&= -x\left[\frac{s^3}{3} + 4s\right]_{s=0}^{x} - \left[\frac{s^4}{4} + 2s^2\right]_{s=x}^{L} \\
&= -x\left[\frac{x^3}{3} + 4x\right] - \left[\frac{L^4}{4} + 2L^2\right] + \left[\frac{x^4}{4} + 2x^2\right]
\end{aligned}$$

The integration boundaries should be switched in the two integrals above.

Problem 6)

Calculate the contour integral:

$$\oint_\Gamma \frac{4z}{z^2(z+1)^2} dz$$

where Γ is a circle of radius 3.

Solution

Rewriting the integral:

$$\oint_\Gamma \frac{4z}{z^2(z+1)^2} dz = \oint_\Gamma \frac{4}{z(z+1)^2} dz.$$

Finding the roots of the denominator:

$$z(z+1)^2 = 0 \implies z = 0, -1.$$

where both roots are contained within the contour and 0 is a simple pole and -1 is a double pole. The residues can be computed in accordance to eq (13) and (14) and yields:

$$\text{Res}(f, 0) = \lim_{z \to 0} \frac{z4}{z(z-1)^2} = \frac{4}{1} = 4$$
$$\text{Res}(f, -1) = \lim_{z \to -1} \frac{d}{dz}(z+1)^2 \frac{4}{z(z+1)^2} = \lim_{z \to -1} -\frac{4}{z^2} = -4$$

Using the theroem of residues gives:

$$\oint_\Gamma \frac{4z}{z^2(z+1)} dz = 2\pi i \left(\text{Res}(f, 0) + \text{Res}(f, -1)\right) = 2\pi i(4-4) = 0$$

Problem 7)

Calculate the general solution to $\ln(1+i)$ and describe a method for obtaining a simple solution.

Solution

We begin by saying that $z = 1 + 1i$ and compute the euler form:

$$z = r^{i\theta}; \quad r = \sqrt{2} \quad \& \quad \theta = \arctan(1) = \frac{\pi}{4}.$$

Thus, the general solution is given by:

$$\ln(1+i) = \ln(\sqrt{2}) + i\left(\frac{\pi}{4} + 2\pi n\right).$$

To obtain a simple solution one can use the following identity:

Problem 8)

Using Rodrigues formula

$$L_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n,$$

and recursion formula:

$$(2l+1)(1-x^2)^{\frac{1}{2}} P_l^m(x) = P_{l-1}^{m+1}(x) - P_{l+1}^{m+1}(x) \tag{18}$$

one wish to compute $P_2^1(x)$.

Solution

Rearranging eq (18) and setting $l = 1$ and $m = 0$ gives:

$$\begin{aligned}
P_{l+1}^{m+1}(x) &= -(2l+1)(1-x^2)^{\frac{1}{2}} P_l^m(x) + P_{l-1}^{m+1}(x) \\
P_2^1(x) &= -(2+1)(1-x^2)^{\frac{1}{2}} P_1^0(x) + P_0^1(x) \\
&= -3\sqrt{1-x^2} P_1(x) + (-1)^1 \sqrt{1-x^2} \frac{d}{dx} P_0(x) \\
&= -3\sqrt{1-x^2} P_1(x) = -3x\sqrt{1-x^2}
\end{aligned}$$

Problem 9)

Provide a general solution to the following PDE:

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} + \frac{\partial \psi}{\partial z} = x + y$$

Solution

One recoginize the following:

$$[1, 1, 1] \cdot \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \psi = x + y$$

one could also say $\nabla \cdot \psi = x + y$. Next step?

Problem 10)

Calculate the Fourier and Laplace Transforms (if they are defined) of the function

$$f(x) = 1 - 2x; \quad x \in \left[0, \frac{1}{2}\right].$$

Solution

$$
\begin{aligned}
\hat{f}(s) &= \int_0^\infty f(t)e^{-st}dt = \int_0^{\frac{1}{2}} (1-2t)e^{-st}dt \\
&= \int_0^{\frac{1}{2}} e^{-st}dt - 2\int_0^{\frac{1}{2}} te^{-st}dt \\
&= \left[\frac{e^{-st}}{-s}\right]_{t=0}^{\frac{1}{2}} - 2\left[\frac{te^{-st}}{-s}\right]_{t=0}^{\frac{1}{2}} + 2\int_0^{\frac{1}{2}} \frac{e^{-st}}{s}dt \\
&= \frac{e^{-0.5s}-1}{-s} - \frac{e^{-0.5s}}{-s} + 2\left[\frac{e^{-st}}{s^2}\right]_{t=0}^{\frac{1}{2}} \\
&= \frac{1}{s} + 2\left[\frac{e^{-0.5s}-1}{s^2}\right]
\end{aligned}
$$

$$
\begin{aligned}
\tilde{f}(\omega) &= \int_{-\infty}^\infty f(t)e^{-i\omega t}dt = \int_0^{\frac{1}{2}} (1-2t)e^{-i\omega t}dt \\
&= \int_0^{\frac{1}{2}} e^{-i\omega t}dt - 2\int_0^{\frac{1}{2}} te^{-i\omega t}dt \\
&= \left[\frac{e^{-i\omega t}}{-i\omega}\right]_{t=0}^{\frac{1}{2}} - 2\left[\frac{te^{-i\omega t}}{-i\omega}\right]_0^{\frac{1}{2}} + 2\int_0^{\frac{1}{2}} \frac{e^{-i\omega t}}{-i\omega}dt \\
&= \frac{i}{\omega}\left(e^{i0.5\omega}-1\right) - \frac{ie^{i0.5\omega}}{\omega} + 2\left[\frac{e^{i\omega t}}{-\omega^2}\right]_0^{\frac{1}{2}} \\
&= \frac{i}{\omega} - \frac{2}{\omega}\left(e^{-i0.5\omega}-1\right)
\end{aligned}
$$

Problem 11)

Solve the PDE:

$$
a\frac{\partial^2\psi}{\partial x^2} = \frac{\partial\psi}{\partial t}.
$$

For an infinitely long rod in one dimension, $x \in (-\infty, \infty)$, for a solution with a heat pulse: $\psi_0(x) = A\delta(x)$.

Solution

Rewrite the function initial-condition: $\psi(x,0) = A\delta(x)$. Assume that ψ can be written as a fourier transform where the dimensional component is transformed.

$$\tilde{\psi}(k,t) = \int_{-\infty}^{\infty} \psi(x,t)e^{-ikx}dx$$

Using eq (17) one has:

$$a\left(-k^2 \cdot \tilde{\psi}(k,t)\right) = \frac{\partial}{\partial t}\tilde{\psi}(k,t).$$

This is now an ODE of the first kind which can be solved as follows:

$$\tilde{\psi}(k,t) = \tilde{\psi}(k,0)e^{-ak^2t}.$$

Either transform back, or transform the initial-condition.

$$\tilde{\psi}(k,0) = \int_{-\infty}^{\infty} A\delta(x)e^{-ikx}dx = A$$

Thus the solution, in k-space is given by: $\tilde{\psi}(k,t) = Ae^{-ak^2t}$. Transforming back will yield an indefinite interal known as the error-function.

10 Previous exam 2021

Problem 1

Provide the formula for 4 of the 5 following equations and give an example of a physical scenario in which they may be employed:

1. **Poisson's equation**:
2. **Heat equation**:
3. **Schrödinger's equation**:
4. **Klein Gordon equation**:
5. **Wave-equation**:

Solution

1. **Poisson's equation**: $\nabla^2 u = f$, when dealing with electrostatics.
2. **Heat equation**: $\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$, when dealing with heat diffusion and heat transfer.
3. **Schrödinger's equation**: $-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi$ when dealing with quantum mechanics; e.g. the hydrogen atom.
4. **Klein Gordon equation**: $(\partial_\mu \partial^\mu + m^2)\phi = 0$, when dealing with relativistic quantum mechanics; e.g. the Klein Gordon field.
5. **Wave-equation**: $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ when dealing with waves; e.g. sound waves.

Problem 2

Solve the differential equation:

$$2xydx + (x^2 + 3y^2)dy = 0.$$

Note: You will only be able to find an implicit solution for x and y (but your solution should not include their derivatives).

Solution

This is a homogeneous ODE, so we can use the substitution $y = vx$ to get:

$$\begin{aligned}
&dy = vdx + xdv \\
\implies &v(dx) + (x^2 + 3v^2x^2)(vdx + xdv) = 0 \\
\implies &vdx + x^2(1 + 3v^2)(vdx + xdv) = 0
\end{aligned}$$

Problem 3

Use contour integration to evaluate the integral:

$$\int_0^\infty \frac{1}{x^3+1}dx.$$

Solution

Firstly one needs to find the residues of the integrand of:

$$\oint_\Gamma \frac{1}{z^3+1}dz,$$

which is done by the following:

$$z^3 - 1 = (z+1)(z^2+z+1) = 0 \implies z = -1, \frac{1}{2}\{1+i\sqrt{3}, 1-i\sqrt{3}\}$$

which is a simple pole. The residue is then given by eq (13):

$$\begin{aligned}
\text{Res}(f,-1) &= \lim_{z\to -1} \frac{(z+1)}{(z+1)(z^2+z+1)} = 1. \\
\text{Res}(f, \frac{1}{2} + i\frac{\sqrt{3}}{2}) &= \lim_{z\to z_0} \frac{(z-z_0)}{(z+1)(z-z_0)(z-z_1)} = \frac{1}{z_0+1}\cdot\frac{1}{(z_0-z_1)}
\end{aligned}$$

One then defines a contour that goes from 0 to R on the real axis, and then a quarter-circle of radius R in the upper half plane, where R goes to infinity. Then:

$$\oint_\Gamma \frac{1}{z^3+1}dz = \int_0^R \frac{1}{x^3+1}dx + \underbrace{\int_{\Gamma_R} \frac{1}{z^3+1}dz}_{=0}$$

$$\begin{aligned}
\lim_{R\to\infty}\int_0^R \frac{1}{x^3+1}dx &= 2\pi i \frac{1}{\frac{1}{2}+i\frac{\sqrt{3}}{2}+1}\cdot\frac{1}{(\frac{1}{2}+i\frac{\sqrt{3}}{2}-\frac{1}{2}-i\frac{\sqrt{3}}{2})} \\
&= 2\pi i \frac{1}{\frac{1}{2}+\frac{i\sqrt{3}}{2}}\cdot\frac{1}{i\sqrt{3}} \\
&= 2\pi i \frac{1}{(\frac{i\sqrt{3}}{2}-\frac{3}{2})}
\end{aligned}$$

Problem 4

Write down one example that fulfills the definition for each of the following ODEs, you do not need to solve them:

1. An ODE that is not linear, but is homogeneous:

2. Has an order of 4:
3. Is of degree 2:
4. Is linear, but not homogeneous:

Solution

Remember that homogenuity is given by: $f(kx) = k^n f(x)$.

1. **An ODE that is not linear, but is homogeneous:** $y' = -\frac{x}{y+1}$
2. **Has an order of 4:** $y^{(4)} + 1 = 0$.
3. **Is of degree 2:** $y' = \sqrt{1+x}$
4. **Is linear, but not homogeneous:** $y' + 2y = 2e^{2x}$.

Problem 5

Find a general solution for this partial differential equation

$$\frac{\partial u}{\partial t} - \lambda \frac{\partial^2 u}{\partial x^2} = 0,$$

where λ is taken to be a positive number and the solution fulfills boundary conditions $u(0,t) = u(L,t) = 0$.

Solution

Using separation of variables, and moving one term to the other side:

$$\begin{aligned} T'(t)X(x) &= \lambda T(t)X''(x) \\ \frac{T'(t)}{T(t)} &= \lambda \frac{X''(x)}{X(x)} = \mu^2 \\ \implies T(t) &= Ae^{\mu^2 t} \\ \implies X(x) &= a_1 \sin\left(\frac{\mu}{\sqrt{\lambda}}x\right) + a_2 \cos\left(\frac{\mu}{\sqrt{\lambda}}x\right) \end{aligned}$$

The coefficient a_2 is zero from the $u(0,t) = 0$ boundary condition. The other boundary condition gives:

$$X(L) = a_1 \sin\left(\frac{\mu}{\sqrt{\lambda}}L\right) = 0 \implies \frac{\mu}{\sqrt{\lambda}}L = n\pi \implies \mu_n = \frac{n\pi}{L}\sqrt{\lambda}$$

Thus the final solution is given by:

$$u_n(x,t) = Ae^{\mu^2 t}\sin(\mu_n x)$$

Problem 6

Take a break! You are more than halfway done. Don't put anything here (or potentially make a doodle in the space provided below) and you receive 5 points.

Solution

Problem 7

Provide all poles and calculate all residues for:

$$f(z) = \frac{e^{iz}}{(z^2+1)(z+2)^2}.$$

It may be useful to remember:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint \frac{f(z)}{(z-z_0)^{n+1}} dz.$$

Solution

Rewriting the function as:

$$f(z) = \frac{e^{iz}}{(z+i)(z-i)(z+2)^2}.$$

From there one sees that there exists three poles, $z = i, -i, -2$, the first of the two poles are simple poles and last is a second order pole.

$$\begin{aligned}
\mathrm{Res}(f, i) &= \lim_{z\to i} \frac{(z-i)e^{iz}}{(z-i)(z+i)(z+2)^2} = \frac{e^{-1}}{2i(i+2)^2} \\
&= \frac{e^{-1}}{2i(3+4i)} = \frac{e^{-1}}{6i-4} \\
\mathrm{Res}(f, -i) &= \lim_{z\to -i} \frac{(z+i)e^{iz}}{(z-i)(z+i)(z+2)^2} = \frac{e}{-2i(-i+2)^2} \\
&= \frac{ie}{6-8i} \\
\mathrm{Res}(f, -2) &= \lim_{z\to -2} \frac{d}{dz}\left(\frac{(z-2)^2 e^{iz}}{(z-i)(z+i)(z-2)^2}\right) \\
&= \lim_{z\to -2} \frac{d}{dz}\left(\frac{e^{iz}}{(z+i)(z-i)}\right) \\
&= \lim_{z\to -2} \frac{ie^{iz}(z^2+2iz+1)}{(z^2+1)^2} \\
&= \frac{ie^{-2i}(4-4i+1)}{(4+1)^2} = \frac{ie^{-2i}(3-4i)}{25}
\end{aligned}$$

Problem 8

Calculate the Fourier transform of the triangle pulse:

$$f(t) = \begin{cases} 1 - t; & t \in [0, 1] \\ 1 + t; & t \in [-1, 0) \end{cases}$$

Solution

Using the following definitions:

$$\begin{aligned} \tilde{f}(\omega) &= \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \\ f(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(\omega) e^{i\omega t} d\omega \end{aligned}$$

The transform is computed as:

$$\begin{aligned} \tilde{f}(\omega) &= \int_{-\infty}^{\infty} dt \left(f(t) e^{-i\omega t} \right) \\ &= \int_{-1}^{0} (t+1) e^{-i\omega t} dt + \int_{0}^{1} (1-t) e^{-i\omega t} dt \\ &= \int_{-1}^{0} e^{-i\omega t} dt + \int_{-1}^{0} t e^{-i\omega t} dt + \int_{0}^{1} e^{-i\omega t} dt - \int_{0}^{1} t e^{-i\omega t} dt \\ &= \int_{-1}^{1} e^{-i\omega t} dt + \int_{-1}^{0} t e^{-i\omega t} dt - \int_{0}^{1} t e^{-i\omega t} dt \\ &= \left[\frac{e^{-i\omega t}}{-i\omega} \right]_{-1}^{1} + \left[\frac{t e^{i\omega t}}{-i\omega} \right]_{-1}^{1} - \int_{-1}^{0} \frac{e^{-i\omega t}}{-i\omega} dt - \left[\frac{t e^{-i\omega t}}{-i\omega} \right]_{0}^{1} + \int_{0}^{1} \frac{e^{-i\omega t}}{-i\omega} dt \\ &= \left[\frac{e^{-i\omega t}}{-i\omega} \right]_{-1}^{1} + \left[\frac{t e^{i\omega t}}{-i\omega} \right]_{-1}^{1} - \left[\frac{e^{-i\omega t}}{-\omega^2} \right]_{-1}^{0} - \left[\frac{t e^{-i\omega t}}{-i\omega} \right]_{0}^{1} + \left[\frac{e^{-i\omega t}}{-\omega^2} \right]_{0}^{1} \\ &= \frac{4}{\omega} \sin(\omega) + \frac{2}{\omega^2} \left(\cos(\omega) - 1 \right) \end{aligned}$$

Problem 9

The beta function is defined as:

$$\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Compute $\beta(3, 4)$.

Solution

Since both are integers, one can use the factorial expression:

$$\beta(3,4) = \frac{2!3!}{6!} = \frac{2}{6 \cdot 5 \cdot 4} = \frac{1}{6 \cdot 5 \cdot 2} = \frac{1}{60}$$

Problem 10

Use the Method of Characteristics to solve:

$$xu_x + yu_y = 2x$$

with the boundary conditions: $u(x, x^2) = x$.

Solution

Using the Lagrange-Charpit equations, eq (9):

$$\boxed{\frac{dx}{x} = \frac{dy}{y}} = \frac{du}{2x}.$$

Using the framed equations one gets:

$$\begin{aligned} \frac{dy}{y} &= \frac{dx}{x} \\ \implies \ln(y) &= \ln(x) + \tilde{c}_1 \\ \implies y &= c_1 x \quad c_1 = \frac{y}{x} \end{aligned}$$

Taking the any other combination of the equations, since the boundary condition contains x one uses the x derivative:

$$\frac{dx}{x} = \frac{du}{2x}$$
$$2x + c_2 = u \implies c_2 = u - 2x$$

Combining this gives:

$$\begin{aligned} G(c_1) &= c_2 \\ G\left(\frac{y}{x}\right) &= u - 2x \end{aligned}$$

Using the condition posed by $u(x, x^2) = x$ gives that the y-component can be written as x^2 and thus one has the following:

$$G\left(\frac{y}{x}\right) = G\left(\frac{x^2}{x}\right) = G(x) = u(x, x^2) - 2x = (x) - 2x = -x$$

So the function $G(x)$ only returns the negative of the argument. Thus the solution is given by:

$$u(x, y) = -\left(\frac{y}{x}\right) + 2x$$

$\boxed{\text{Previous exam 2021 is concluded here.}}$

11 Previous exam 2022

Problem 1

Solve the differential equation:

$$x\frac{dy}{dx} + 2y = 10x^2; \quad y(1) = 3.$$

Solution

Solving the homogenous equation:

$$\begin{aligned}
xy' + 2y &= 0 \\
xy' &= -2y \\
\frac{dy}{y} &= -\frac{2}{x}dx \\
\ln(y) &= -2\ln(x) + c_1 \\
y &= e^{-2\ln(x)+c1} \\
&= \frac{c}{x^2}
\end{aligned}$$

The particular solution is expected to be on the form: $y_p(x) = ax^2 + bx + c$, so we can insert this into the equation:

$$\begin{aligned}
x \cdot (2ax) + 2 \cdot (ax^2 + bx + c) &= 10x^2 \\
2ax^2 + 2bx + 2c &= 10x^2 \implies b = c = 0 \quad \& \quad a = 5
\end{aligned}$$

Thus the total solution is given by:

$$\begin{aligned}
y(x) &= \frac{c}{x^2} + 5x^2 \\
y(1) &= c + 5 = 3 \implies c = -2 \\
y(x) &= \frac{-2}{x^2} + 5x^2
\end{aligned}$$

Problem 2

Use a Green's function approach to solve the differential equation:

$$y'' = 2x + 1; \quad y(0) = y'(L) = 0.$$

Solution

$$\begin{aligned}
\mathcal{L} &= \frac{d^2}{dx^2} \\
\mathcal{L}(y) &= 2x+1 \\
\mathcal{L}\Big(\mathcal{G}(x,s)\Big) &= \delta(x-s) \\
\mathcal{G}(x,s) &= ax+b \\
\mathcal{G}_1(0,s) &= a_1\cdot 0 + b_1 = 0 \implies b_1 = 0 \\
\mathcal{G}_2'(L,s) &= a_2 = 0 \implies a_2 = 0 \\
\mathcal{G}_1(s,s) &= \mathcal{G}_2(s,s) \implies a_1 s = b_2 \\
\mathcal{G}_2'(s,s) - \mathcal{G}_1'(s,s) &= 1 \implies 0 - a_1 = 1 \implies a_1 = -1 \\
\mathcal{G}(x,s) &= \begin{cases} -x; x\in[0,s] \\ -s; x\in(s,L] \end{cases}
\end{aligned}$$

The solution is then given by:

$$y(x) = \int_0^x (2s+1)(-x)ds + \int_x^L (2s+1)(-s)ds$$

Problem 3

Write down one equation that fulfills the following definitions. You do not need to solve equations that you write down:

1. An ODE that is in Sturm-Liouville form. Explain why it is in Sturm Liouville form:
2. A 2nd order PDE that could be solved using the Method of Characteristics:
3. An ODE that is inhomogeneous:
4. A contour integral that has a pole, but would evaluate to 0. Specify both the integral and the contour that you chose.
5. A complex function that is continuous but nowhere differentiable.

Solution

1. An ODE that is in Sturm-Liouville form. Explain why it is in Sturm Liouville form: The equation:

   $$y'' + y = 0$$

   is in Sturm-Liouville form, since it can be written as:

   $$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y = 0,$$

where both $p(x) = q(x) = 1$.

2. A 2nd order PDE that could be solved using the Method of Characteristics:

$$xu_{xx} + yu_{yy} = 1.$$

3. An ODE that is inhomogeneous: $y'' + y = x$ is inhomogeneous due to the x term.

4. A contour integral that has a pole, but would evaluate to 0. Specify both the integral and the contour that you chose: Suppose the function $f(z) = \sin(z) \cdot z^{-1}$, which has a single pole at $z = 0$ but the closed contour integarl, along a unit circle, of the function is 0.

5. A complex function that is continuous but nowhere differentiable: the function $f(z) = \bar{z}$ is continous everywhere, but nowhere differentiable due to the restriction given by the Cauchy-Riemann equations, eq (11).

Problem 4

Solve the differential equation using the Method of Frobenius:

$$4xy'' + 2y' + y = 0.$$

Remember to write out full solutions for y1 and y2. These solutions may still be infinite sums.

Solution

$$y'' + \frac{1}{2x}y' + \frac{1}{4x}y = 0$$
$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$$
$$y'(x) = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}$$
$$y''(x) = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$$
$$\implies \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2} + \frac{1}{2}\sum_{n=0}^{\infty} a_n (n+r) x^{n+r-2} + \frac{1}{4}\sum_{n=0}^{\infty} a_n x^{n+r-1} = 0$$

Problem 5

Solve:

$$\int_{1}^{\infty} \frac{\sin(x)}{x^2+3}\delta(x-4)dx.$$

Solution

$$\int_{-\infty}^{\infty} f(x)\delta(x-x_0)dx = f(x_0)$$

$$\implies \int_{1}^{\infty} \frac{\sin(x)}{x^2+3}\delta(x-4)dx = \frac{\sin(4)}{19}$$

Problem 6

Solve the integral:

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1}dx$$

Solution

Taking the contour from 0 to R with a semicircle to $-R$ and then back to 0:

$$\oint_{\Gamma} \frac{1}{z^2+1}dz = \oint_{\Gamma} \frac{1}{(z+i)(z-i)}dz = 2\pi i \cdot \frac{1}{2i} = \pi$$

Problem 7

Calculate the Fourier transform of the following function:

$$f(t) = \begin{cases} 1; x \in [-T, 0] \\ -1; x \in (0, T] \\ 0; \text{else} \end{cases}$$

Solution

Using the definition comprised in the recap notes one has the following:

$$\begin{aligned} \tilde{f}(\omega) &= \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt = \int_{-T}^{0} e^{-i\omega t}dt - \int_{0}^{T} e^{-i\omega t}dt \\ &= \int_{-T}^{0} e^{i\omega t} + e^{-i\omega t}dt = 2\int_{-T}^{0} \cos(\omega t)dt = 2\left[\frac{\sin(\omega t)}{\omega}\right]_{-T}^{0} = \frac{2}{\omega}\big(1-\sin(\omega T)\big) \end{aligned}$$

Problem 8

Two important relations for the Bessel function are:

$$
\begin{aligned}
J_{n-1}(x) + J_{n+1}(x) &= \frac{2n}{x} J_n(x), \\
J_{n-1}(x) - J_{n+1}(x) &= 2J_n'(x).
\end{aligned}
$$

Use these to prove:

$$
\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)
$$

Solution

In order to show the above relationship one does the following:

$$
\begin{aligned}
\frac{2n}{x} J_n(x) - 2J_n'(x) &= J_{n-1}(x) + J_{n+1}(x) - \big(J_{n-1}(x) - J_{n+1}(x)\big) = 2J_{n+1}(x) \\
\implies J_n'(x) - \frac{n}{x} J_n(x) &= -J_{n+1}(x) \\
x^{-n}\left(J_n'(x) - nx^{-1} J_n(x)\right) &= -x^{-n} J_{n+1}(x) \\
x^{-n} J_n'(x) - nx^{-n-1} J_n(x) &= -x^{-n} J_{n+1}(x) \\
\frac{d}{dx}\left[x^{-n} J_n(x)\right] &= -x^{-n} J_{n+1}(x) \quad \blacksquare
\end{aligned}
$$

Problem 9

A particle is defined in a 3D rectangular box, of lengths $a \times b \times c$. The potential of the particle is 0 inside the box, but is infinite outside the box. Using Schrödingers's equation in steady state:

$$
-\frac{\hbar^2}{2m} \nabla^2 \psi + V\psi = E\psi,
$$

what is the lowest energy state that this particle can have?

Solution

Suppos the following:

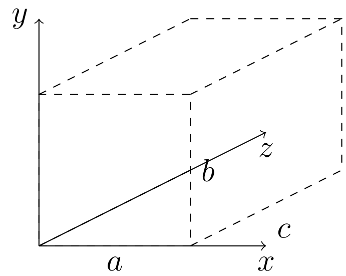


Figure 1: Overview

The wavefunction is seperable by nature, so one can write the following, with the natural boundary conditions:

$$\psi_{n,m,l}(x, y, z) = c \sin\left(\frac{n\pi}{a}x\right) \sin\left(\frac{m\pi}{b}y\right) \sin\left(\frac{l\pi}{c}z\right)$$

$\boxed{\text{This summerizes both the previous exam and the compendium}}$