Handin 8

Author : Andreas Evensen

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Note:

In this exercise sheet the Fourier transform, and it's inverse transform is defined below:

$$\text{Transform: } \tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$$

$$\text{Inverse transform: } f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\omega)e^{i\omega t}d\omega$$

Warmup

a)

Using Rodrigues formula

$$L_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n,$$

and recursion formula:

$$(2l+1)(1-x^2)^{\frac{1}{2}}P_l^m(x) = P_{l-1}^{m+1}(x) - P_{l+1}^{m+1}(x) \tag{1}$$

one wish to compute $P_2^1(x)$. Rearranging eq (1) and setting $l = 1$ and $m = 0$ gives:

$$\begin{aligned} P_{l+1}^{m+1}(x) &= -(2l+1)(1-x^2)^{\frac{1}{2}}P_l^m(x) + P_{l-1}^{m+1}(x) \\ P_2^1(x) &= -(2+1)(1-x^2)^{\frac{1}{2}}P_1^0(x) + P_0^1(x) \\ &= -3\sqrt{1-x^2}P_1(x) + (-1)^1\sqrt{1-x^2}\frac{d}{dx}P_0(x) \\ &= -3\sqrt{1-x^2}P_1(x) = -3x\sqrt{1-x^2} \end{aligned}$$

b)

The Heaviside function defined by: $H(x) = \frac{1}{2} + \frac{1}{2}sign(x)$. Computing the Fourier transform is done by the following:

$$\begin{aligned} \tilde{H}(\omega) &= \int_{-\infty}^{\infty} dt\left(H(s)e^{-i\omega t}\right) \\ &= \lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon}\frac{1}{2}e^{-i\omega t} + \int_{\epsilon}^{\infty} dt\left(e^{-i\omega t}\right) \\ &= \lim_{\epsilon\to 0}\left[\frac{1}{-2i\omega}e^{-i\omega t}\right]_{t=-\epsilon}^{\epsilon} + \left[\frac{e^{-i\omega t}}{-i\omega}\right]_{t=\epsilon}^{\infty} \\ \tilde{H}(\omega) &= p.v\left(\frac{i}{\omega}\right) + \pi\delta(\omega), \end{aligned}$$

where $p.v(x)$ denotes Cauchy's principal value of x. The Laplace transform is thus defined by:

$$
\begin{aligned}
\hat{H}(s) &= \int_0^\infty e^{-sx} H(x) dx \\
&= \lim_{\epsilon \to 0} \left(\underbrace{\frac{1}{2} \int_0^\epsilon e^{-sx} dx}_{=0} + \int_\epsilon^\infty e^{-sx} dx \right) \\
\hat{H}(s) &= \lim_{\epsilon \to 0} \left[\frac{e^{-sx}}{-s} \right]_{x=\epsilon}^\infty = \frac{1}{s}.
\end{aligned}
$$

c)

$$
f(t) = \begin{cases} 1 - t; t \in [0, 1] \\ t + 1; t \in [-1, 0) \\ 0;\ \text{otherwise} \end{cases}
$$

$$
\begin{aligned}
\tilde{f}(\omega) &= \int_{-\infty}^\infty dt \left(f(t) e^{-i\omega t} \right) \\
&= \int_{-1}^0 (t+1) e^{-i\omega t} dt + \int_0^1 (1-t) e^{-i\omega t} dt \\
&= \int_{-1}^0 e^{-i\omega t} dt + \int_{-1}^0 t e^{-i\omega t} dt + \int_0^1 e^{-i\omega t} dt - \int_0^1 t e^{-i\omega t} dt \\
&= \int_{-1}^1 e^{-i\omega t} dt + \int_{-1}^0 t e^{-i\omega t} dt - \int_0^1 t e^{-i\omega t} dt \\
&= \left[\frac{e^{-i\omega t}}{-i\omega} \right]_{-1}^1 + \left[\frac{t e^{i\omega t}}{-i\omega} \right]_{-1}^1 - \int_{-1}^0 \frac{e^{-i\omega t}}{-i\omega} dt - \left[\frac{t e^{-i\omega t}}{-i\omega} \right]_0^1 + \int_0^1 \frac{e^{-i\omega t}}{-i\omega} dt \\
&= \left[\frac{e^{-i\omega t}}{-i\omega} \right]_{-1}^1 + \left[\frac{t e^{i\omega t}}{-i\omega} \right]_{-1}^1 - \left[\frac{e^{-i\omega t}}{-\omega^2} \right]_{-1}^0 - \left[\frac{t e^{-i\omega t}}{-i\omega} \right]_0^1 + \left[\frac{e^{-i\omega t}}{-\omega^2} \right]_0^1 \\
&= \frac{4}{\omega} \sin(\omega) + \frac{2}{\omega^2} \left(\cos(\omega) - 1 \right)
\end{aligned}
$$

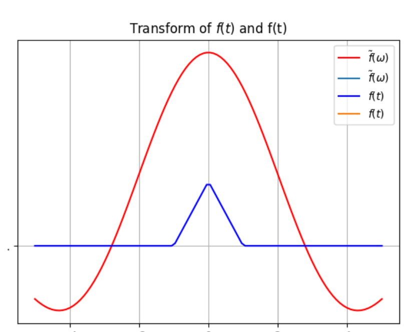

Figure 1: Visualization of $f(t)$ and $\tilde{f}(\omega)$

Evaluation of infinite series

a)

$$S = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Constructing a function $f(z) = \pi \cot(\pi z) \cdot z^{-2}$, one can use the residue theorem to find the sum of the series, and the following contour

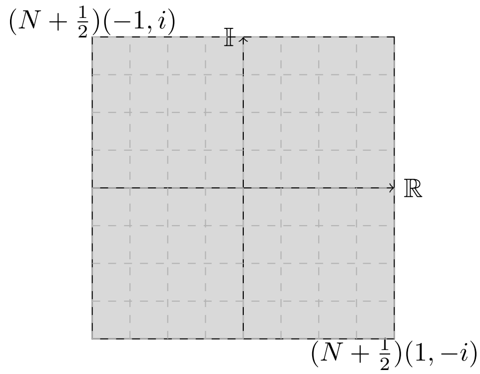

Figure 2: Contour used to find the sum of the series.

At $z = 0$ the function, $f(z)$ has a simple pole and one does the Taylor expansion of $cot(z)$ around $z = 0$:

$$f(z) = \frac{\pi}{z^2}\left(1 - \frac{(\pi z)^2}{2!} + \frac{(\pi z)^4}{4!} - ...\right) = \left(\pi z - \frac{(\pi z)^3}{3!} + ...\right) \cdot \left(... + \frac{a_{-1}}{\pi z} + a_1(\pi z) + ...\right),$$

then $a_{-1} = \frac{-\pi^2}{3}$. Using this, one has the following situation:

$$\begin{aligned}
\frac{\pi^2}{3} &= \sum_{n=-\infty}^{\infty} \frac{1}{n^2} \\
&= \sum_{n=-\infty}^{1} \frac{1}{n^2} + \sum_{n=1}^{\infty} \frac{1}{n^2} \\
&= 2\sum_{n=1}^{\infty} \frac{1}{n^2} \\
\implies \sum_{n=1}^{\infty} \frac{1}{n^2} &= \frac{\pi^2}{6}
\end{aligned}$$

b)

One wish to prove the following identity:

$$\sum_{n=0}^{\infty} \frac{1}{n^2 + a^2} = \frac{\pi}{2a}\coth(\pi a) + \frac{1}{2a^2} \quad ; a > 0.$$

In order to accomplish this, construct the following function: $f(z) = (z^2 + a^2)^{-1}$ which has simple poles at $z = \pm ai$, and uses the contour defined in 2:

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} f(n) &= -\sum_{z_i} \mathrm{Res}(\tilde{f}, z_i) \\
&= -\left(\lim_{z \to ai} \frac{(z-ai)\pi\cot(\pi a)}{(z-ai)\cdot(z+ai)} + \lim_{z \to -ai} \frac{(z+ai)\pi\cot(\pi a)}{(z+ai)\cdot(z-ai)} \right) \\
&= -\left(\frac{\pi\cot(\pi ai)}{2ai} + \frac{\pi\cot(-\pi ai)}{-2ai} \right) \\
&= \frac{\pi\coth(\pi a)}{a}
\end{aligned}$$

Then one does the following identification:

$$\begin{aligned}
\frac{\pi\coth(\pi a)}{a} &= \sum_{n=-\infty}^{\infty} \frac{1}{n^2+a^2} \\
&= \sum_{n=-\infty}^{-1} \frac{1}{n^2+a^2} + \sum_{n=1}^{\infty} \frac{1}{n^2+a^2} + f(0) \\
&= 2\sum_{n=1}^{\infty} \frac{1}{n^2+a^2} + f(0) \\
\implies \sum_{n=1}^{\infty} \frac{1}{n^2+a^2} &= \frac{1}{2}\left(\frac{\pi\coth(\pi a)}{a} - f(0) \right) \\
&= \frac{\pi\coth(\pi a)}{2a} - \frac{1}{2a^2}
\end{aligned}$$

Laplace transform of the sine cardinal

a)

One wish to show the following identity, where $\hat{f}(x)$ is the Laplace transform of $f(t)$:

$$\int_s^{\infty} dx \left(\hat{f}(x) \right) = \mathcal{L}\left[\frac{f(t)}{t} \right](s)$$

In order to accomplish this, one suppose the following:

$$\begin{aligned}
\int_s^{\infty} dx \left(\hat{f}(x) \right) &= \int_s^{\infty} dx \int_0^{\infty} dt \left(e^{-xt} f(t) \right) \\
\text{Fubinis theorem:} \quad &= \int_0^{\infty} dx \left(f(x) \right) \int_s^{\infty} dt \left(e^{-xt} \right) \\
&= \int_0^{\infty} dx \left(\frac{f(x)}{x} \right) e^{-sx} = \mathcal{L}\left[\frac{f(t)}{t} \right](s)
\end{aligned}$$

b)

One wish to prove the following transformation:

$$\mathcal{L}\left[\frac{\sinh(at)}{at} \right](s) = \frac{1}{a}\coth^{-1}\left(\frac{s}{a} \right)$$

In order to achieve this, one does the following, using the result from a):

$$
\begin{aligned}
g(t) &= \frac{\sinh(at)}{a} \implies \hat{g}(s) = \frac{1}{s^2 - a^2} \\
\mathcal{L}\left(\frac{\sinh(at)}{at}\right)(s) &= \cdot\mathcal{L}\left(\frac{g(x)}{x}\right)(s) \\
&= \int_s^\infty \hat{g}(x)dx = \int_s^\infty \frac{1}{x^2 - a^2}dx \\
&= \int_{\frac{s}{a}}^\infty \frac{a}{a^2u^2 - a^2}du = -\frac{1}{a}\underbrace{\int_{\frac{s}{a}}^\infty \frac{1}{-u^2+1}du}_{\coth^{-1}(\frac{s}{a});\left|\frac{s}{a}\right|>1} \\
&= -\frac{1}{a}\coth^{-1}\left(\frac{s}{a}\right)
\end{aligned}
$$

c)

Noting that $g(t) = \frac{\sin(at)}{a}$, and $\hat{g}(s) = \frac{1}{x^2+a^2}$, one uses the same trick as before to prove the following identity:

$$
\mathcal{L}\left[\frac{\sin(at)}{at}\right](x) = \frac{1}{a}\cot^{-1}\left(\frac{s}{a}\right).
$$

One does the following set of computations to find the result:

$$
\begin{aligned}
\mathcal{L}\left(\frac{\sin(at)}{at}\right) &= \int_s^\infty \frac{1}{x^2 + a^2}dx \\
&= \frac{1}{a}\int_{\frac{s}{a}}^\infty \frac{1}{u^2+1}du \\
&= \frac{1}{a}\cot^{-1}\left(\frac{s}{a}\right)
\end{aligned}
$$