Handin4 FK7048

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Warm up problems

a)

Suppose the following PDE:

$$y'' = x^2 + 4.$$

We wish to solve this with Greens functions with the boundary-conditions $y(0) = y'(L) = 0$; thus we can rewrite the above equation to the following:

$$y'' = f(x); \quad f(x) = x^2 + 4.$$

We thus wish to find the Greens function $G(x, s)$ such that:

$$\begin{aligned}
y(x) &= \int_0^x G_1(x,s)f(s)ds + \int_x^L G_2(x,s)f(s)ds \\
&= \int_0^x (-s)f(s)ds + \int_x^L (-x)f(s)ds \\
&= \left[-\frac{s^4}{4} - 2s^2\right]_0^x - x\left[\frac{s^3}{3} + 4s\right]_x^L \\
&= -\Big[\frac{x^4}{4} + 2x^2\Big] - x\Big[\frac{L^3}{3} + 4L - \frac{x^3}{3} - 4x\Big] \\
&= 2x^2 + \frac{x^4}{12} - \Big[\frac{xL^3 + 12xL}{3}\Big]
\end{aligned}$$

b)

We wish to find a general seperable solution to the following PDE:

$$\frac{\partial u}{\partial t} - \lambda\frac{\partial^2 u}{\partial x^2} = 0.$$

We thus assume that u can be decomposed in the following manner:

$$\begin{aligned}
u(x,t) &= X(x)T(t) \\
\implies \frac{T'}{T} &= \lambda\frac{X''}{X} = c = \mu^2.
\end{aligned}$$

With this we can assume the following, since $\lambda > 0$:

$$\begin{aligned}
T(t) &= c_1 \exp\big[-\frac{\lambda}{\mu^2}t\big] + c_2 \\
X(x) &= A\cos(\frac{x}{\mu}) + B\sin\big(\frac{x}{\mu}\big) \\
\implies u(x,t) &= e^{-\frac{\lambda}{\mu^2}t}\Big(\tilde{A}\cos(\frac{x}{\mu}) + \tilde{B}\sin(\frac{x}{\mu})\Big) \qquad (1)
\end{aligned}$$

We test this solution by inserting it into the PDE:

$$\frac{\partial}{\partial t}\left[e^{-\frac{\lambda}{\mu^2}t}\Big(\tilde{A}\cos(\frac{x}{\mu})+\tilde{B}\sin(\frac{x}{\mu})\Big)\right]-\lambda\frac{\partial^2}{\partial x^2}\left[e^{-\frac{\lambda}{\mu^2}t}\Big(\tilde{A}\cos(\frac{x}{\mu})+\tilde{B}\sin(\frac{x}{\mu})\Big)\right]$$
$$=-\frac{\lambda}{\mu^2}e^{-\frac{\lambda}{\mu^2}t}\Big(\tilde{A}\cos(\frac{x}{\mu})+\tilde{B}\sin(\frac{x}{\mu})\Big)+\frac{\lambda}{\mu^2}e^{-\frac{\lambda}{\mu^2}t}\Big(\tilde{A}\cos(\frac{x}{\mu})+\tilde{B}\sin(\frac{x}{\mu})\Big)=0.$$

Thus eq (1) is a solution to the PDE, we now find the constants by the initial-condtions:

$$u(0,t)=e^{-\frac{\lambda}{\mu^2}t}\Big(\tilde{A}\Big)=0\implies\tilde{A}=0.$$
$$u(L,t)=e^{-\frac{\lambda}{\mu^2}t}\Big(\tilde{B}\sin\big(\frac{L}{\mu}\big)\Big)=0\implies\mu=\frac{n\pi}{L},\quad n=0,1,2,3,\ldots$$

Thus the final solution for the PDE is the following:

$$u_n(x,t)=\tilde{B}\exp\left[-\frac{\lambda\cdot L^2}{\big(n\pi\big)^2}\right]\sin\left(\frac{x\cdot L}{n\pi}\right)$$

Laplace equation

Suppose the following PDE:

$$\frac{1}{r^2}\frac{\partial^2}{\partial r^2}\Big(r^2\frac{\partial\phi}{\partial r}\Big)+\frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial\theta}\Big(\sin(\theta)\frac{\partial\phi}{\partial\theta}\Big)+\frac{1}{r^2\sin^2(\theta)}\frac{\partial^2\phi}{\partial\varphi^2}=0, \tag{2}$$
$$u_r(R,\theta)=0;\quad\forall\theta\in[0,\pi],$$

where $\phi=\phi(\rho,z)$. Given a slip condition we have the following boundary-conditions:

$$u_r(R,\theta)=0;\quad\forall\theta\in[0,\pi].$$

Moreover, we recall Laplace's equation:

$$0=\left[\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}\right]u\quad\text{In cartesian coordinates}$$

a)

We wish to find boundary condtion for $r\to\infty$ such that $\mathbf{u}=U\hat{z}$ in polar coordinates, i.e we wish to find the following functions, $\alpha(\theta)$ and $\beta(\theta)$ such that the following holds:

$$\lim_{r\to\infty}\mathbf{u}(r,\theta)=U\left[\alpha(\theta)\hat{r}+\beta(\theta)\hat{\theta}\right]=U\hat{z}$$

We recall that the unit-vectors can be written as, when treating the y component as zero:

$$\hat{r}=\cos(\theta)\hat{z}+\sin(\theta)\hat{x},$$
$$\hat{\theta}=\cos(\theta)\hat{x}-\sin(\theta)\hat{z}.$$

Hence, we have two equations, one for the $\hat{x}$ direction and one for $\hat{z}$ direction. This thus implies $\alpha(\theta)=\cos(\theta)$ and $\beta(\theta)=-\sin(\theta)$.

b)

Suppose the following function to be a solution to the PDE (2):

$$\phi(r,\theta) = A \cdot r\cos(\theta) + \frac{B}{r} + \frac{C\cos(\theta)}{r^2}.$$

We wish to find the constants A, B and C by the boundary-conditions implied by eq (2), and the limit condition defined above. In order to do so, we first compute the gradient of the function $\phi(r,\theta)$, where the φ component is zero:

$$\begin{aligned}\vec{\nabla}\phi(r,\theta) &= \left[\frac{\partial}{\partial r}\Big(Ar\cos(\theta) + \frac{B}{r} + \frac{C\cos(\theta)}{r^2}\Big)\hat{r} + \frac{1}{r}\frac{\partial}{\partial\theta}\Big(Ar\cos(\theta) + \frac{B}{r} + \frac{C\cos(\theta)}{r^2}\Big)\hat{\theta}\right] \\ &= \left[\Big(A\cos(\theta) - \frac{B}{r^2} - \frac{2C\cos(\theta)}{r^3}\Big)\hat{r} - \Big(A\sin(\theta) + \frac{C\sin(\theta)}{r^3}\Big)\hat{\theta}\right]\end{aligned}$$

If we look at the $\hat{r}$ component we have by the boundary-conditions that:

$$\begin{aligned}\hat{r}\cdot\vec{\nabla}\phi(R,\theta) &= A\cos(\theta) - \frac{B}{R^2} - \frac{2C\cos(\theta)}{R^3} = 0 \\ \implies \cos(\theta)&\underbrace{\Big(A - \frac{2C}{R^3}\Big)}_{=0} = 0 \quad \forall\theta\in[0,\pi]\end{aligned}$$

Thus we have that $B = 0$ and $A = \frac{2C}{R^3}$, we now use the second boundary-condition:

$$\begin{aligned}&\lim_{r\to\infty}\vec{\nabla}\phi = U\hat{z} \\ \implies &\lim_{r\to\infty}\cdot\vec{\nabla}\phi = \lim_{r\to\infty}\left[\frac{\hat{z}}{\cos(\theta)}\Big(A\cos(\theta) - \frac{C\cos(\theta)}{2r^3}\Big)\right] \\ &\lim_{r\to\infty}\left[A - \frac{B}{2r^3}\right]\hat{z} = U\hat{z}.\end{aligned}$$

Thus we have that $A = U$, and $C = \frac{1}{2}UR^3$. Moreover, this is a valid solution to the PDE posed in eq (2), which has a physical interpretation of a static field with both a monopole and a dipole contribution; however the monopole contribution is zero.

c)

The velocity field is given by $\mathbf{u}(r,\theta)$, which thus is given by the gradient of $\phi(r,\theta)$:

$$\mathbf{u}(r,\theta) = \vec{\nabla}\phi(r,\theta) = \left[\Big(U\cos(\theta) - \frac{UR^3\cos(\theta)}{r^3}\Big)\hat{r} - \Big(U\sin(\theta) + \frac{UR^3\sin(\theta)}{2r^3}\Big)\hat{\theta}\right]$$

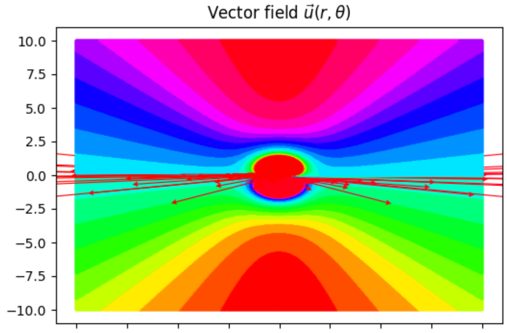


Figure 1: The velocity field $\mathbf{u}$

The velocity field $\mathbf{u}$ shows that the velocity becomes constant as r grows, which was in accordane to our boundary-conditions. Moreover, it's expanding around a singularity point which also is to be expected. It symmetric around $\hat{\theta}$ which also is to be expected from a physical perspective.

Wave equation

Suppose the following PDE;

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2}$$

$$\text{Initial-conditions: } \begin{cases} u(x,0) = q(x) \\ \frac{\partial u}{\partial t}(x,0) = p(x) \end{cases} \quad ; \quad \forall x \in \mathbb{R}.$$

It has a solution to the initial value problem, given by:

$$u(x,t) = \frac{1}{2}\Big(q(x+ct) + q(x-ct)\Big) + \frac{1}{2c}\int_{x-ct}^{x+ct} p(s)ds \tag{3}$$

a)

Given the following definitions of $p(x)$ and $q(x)$:

$$\begin{cases} q(x) & = \left(1 - \frac{|x|}{L}\right) H\left(1 - \frac{|x|}{L}\right) \\ p(x) & = 0 \end{cases},$$

where $H(x)$ is the Heaviside operator. In the function (3), we analyse the integral given the functions $q(x)$, and $p(x)$ defined above:

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left[\left(1 - \frac{|x+ct|}{L}\right) H\left(1 - \frac{|x+ct|}{L}\right) + \left(1 - \frac{|x-ct|}{L}\right) H\left(1 - \frac{|x-ct|}{L}\right)\right] \\ &+ \frac{1}{2}\int_{x-ct}^{x+ct} 0 ds \\ u(x,t) &= \frac{1}{2}\left[\left(1 - \frac{|x+ct|}{L}\right) H\left(1 - \frac{|x+ct|}{L}\right) + \left(1 - \frac{|x-ct|}{L}\right) H\left(1 - \frac{|x-ct|}{L}\right)\right] \end{aligned}$$

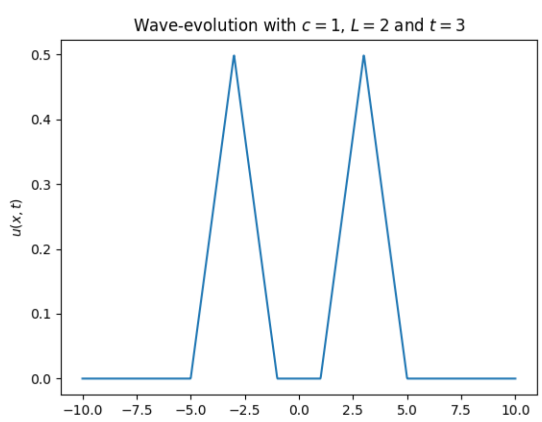


Figure 2: Time-evolution of the propagating wave

As seen in the figure above, the wave propagates with a velocity c in both directions of the origin, $x_0 = 0$. I wont lose amplitude as it propagates in time, but rather the amplitude is constant. The physical meaning of this is a wave propagating through a medium without friction, e.g vaccum.

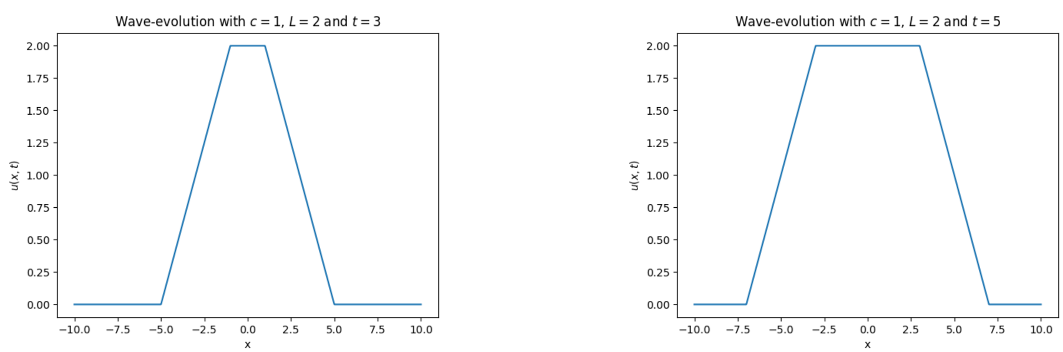
b)

Given the following definitions of $p(x)$ and $q(x)$:

$$\begin{cases} q(x) & = 0 \\ p(x) & = H(x+L)H(L-x) \end{cases}.$$

Using these definitions in the function (3), yields the following:

$$\begin{aligned} u(x,t) &= \frac{1}{2}\Big(0\Big) + \frac{1}{2c}\int_{x-ct}^{x+ct} d\tilde{x}\Big[H(\tilde{x}+L)H(L-\tilde{x})\Big] \\ &= \frac{1}{2c}\Bigg[(\tilde{x}+L)H(\tilde{x}+L)H(L-\tilde{x}) + 2LH(\tilde{x}-L)\Bigg]_{\tilde{x}=x-ct}^{\tilde{x}=x+ct} \end{aligned}$$



(a) Wave propagating at $t = 3$.

(b) Wave propagating at $t = 5$.

Figure 3: Time-evolution of the wave

Again the amplitude does not change when time increases but rather the extent of the wave increases. In time it would reach a steady state of which has a constant ampltiude extending, for $t \to \infty$ for all $x \in \mathbb{R}$.

Proving the primitive function

In order to prove the primitive function of the Heaviside operator, we recall the following identity:

$$\delta(x) = \frac{d}{dx}\Big(H(x)\Big).$$

Taking the derivative of the primitiv-function yields:

$$
\begin{aligned}
\frac{d}{dx}\big(g(x)\big) &= \frac{d}{dx}\Big[(x+L)H(x+L)H(L-x)\Big] + \frac{d}{dx}\Big[2LH(x-L)\Big] \\
&= H(x+L)H(L-x) + (x+L)H'(x+L)H(L-x) - H(x+L)H'(L-x) \\
&+ 2LH'(x-L) \\
&= H(x+L)\Big(H(L-x) - H'(L-x)\Big) + (x+L)H'(x+L)H(L-x) + 2LH'(x-L) \\
&= H(x+L)\Big(H(L-x) - \underbrace{\delta(L-x)}_{=0}\Big) + \underbrace{(x+L)\delta(x+L)H(L-x)}_{=0} + \underbrace{2L\delta(x-L)}_{=0} \\
&= H(x+L)H(L-x).
\end{aligned}
$$

c)

As seen by the figures above: fig 2 and 3a, there is a difference between the two waves. The first waves propagates through space as a transversing wave, while the second wave propagates through space and accumulates the amplitude. One can be seen as plucking a string, the first wave, whilst the second wave can be viewed as a wave that leaves a transvering trace through space.

This can be viewed by eq (3), since the second wave is just the ingral of the expression for $p(x)$. Thus the second wave can be viewed as a super-position of many small waves, which in turn have a constant amplitude.