Handin 3

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Problem 1

Consider the one-dimensional Ising-model.

a)

In the thermodynamics limit and $H = 0$, write an expression of the free energy $\mathcal{F}$. Determine the low and high temperature limits of $\mathcal{F}$ and explain what they imply.

Answer: The partition function for the Ising model is given by:

$$Z = \sum_{\{S\}} \exp\left[h\sum_i S_i + K\sum_{\langle ij\rangle} S_iS_j\right],$$

and since $H = 0$, the first term is zero.

$$Z = \sum_{\{S\}} \exp\left[K\sum_{\langle ij\rangle} S_iS_j\right].$$

One rewrite the second sum in the following manner,

$$\begin{aligned}
Z &= \sum_{\{S\}} \exp\left[K\sum_{i=1}^{N-1} S_iS_{i+1}\right] \\
&= 2\left(2\cosh(K)\right)^{N-1}.
\end{aligned}$$

The free energy is then given by:

$$\begin{aligned}
\mathcal{F} &= -k_bT\ln(Z) \\
&= -k_bT\ln\left(2\left(2\cosh(K)\right)^{N-1}\right) \\
&= -k_bT\left[\ln(2) + (N-1)\ln\left(2\cosh(K)\right)\right] \\
&= -k_bT\left[\ln(2) + (N-1)\ln\left(e^K + e^{-K}\right)\right] \\
&= -k_bT\left[\ln(2) + (N-1)\ln\left[e^K\left(1 + e^{-2K}\right)\right]\right] \\
&= -k_bT\left[\ln(2) + (N-1)\left(K + \ln\left(1 + e^{-2K}\right)\right)\right].
\end{aligned}$$

The low and high temperature limits are given by:

$$\begin{aligned}
&\lim_{T\to 0}\lim_{N\to\infty}\left[-k_bT\left[\ln(2)+(N-1)\left(K+\ln\left(1+e^{-2K}\right)\right)\right]\right]\\
&=\lim_{T\to 0}\lim_{N\to\infty}\left[-k_bT\left[N\left(\frac{J}{k_bT}+\ln\left(1+e^{-\frac{2J}{k_bT}}\right)\right)\right]\right]\\
&=\lim_{T\to 0}\lim_{N\to\infty}\left[-k_bT\left[N\left(\frac{J}{k_bT}+\ln\left(1+e^{-\frac{2J}{k_bT}}\right)\right)\right]\right]\\
&=\lim_{N\to\infty}\left[-JN\right]=-\infty,\\
&\lim_{T\to \infty}\lim_{N\to\infty}\left(-k_bT\left[\ln(2)+(N-1)\left(K+\ln\left(1+e^{-2K}\right)\right)\right]\right)=-\infty,
\end{aligned}$$

In the thermodynamic limit, the free energy diverges.

b)

Obtain expressions for the average energy $\langle E\rangle$ and the heat-capacity C using the expression of $\mathcal{F}$.

Answer: The average energy is defined by:

$$\begin{aligned}
\langle E\rangle &= -\frac{1}{Z}\frac{\partial Z}{\partial\beta}\\
&= -\frac{1}{\left(2\cosh(J\beta)\right)^{N-1}}\frac{\partial}{\partial\beta}\left[\left(2\cosh(J\beta)\right)^{N-1}\right]\\
&= -\frac{2}{\left(2\cosh(J\beta)\right)^{N-1}}\left(N-1\right)J\sinh(J\beta)\left(2\cosh(J\beta)\right)^{N-2}\\
&= -J\left(N-1\right)\tanh(J\beta).
\end{aligned}$$

The heat capacity is defined by:

$$\begin{aligned}
C &= \frac{\partial\langle E\rangle}{\partial T}\\
&= \frac{\partial}{\partial\beta}\left(-J\left(N-1\right)\tanh\left(\frac{J}{k_bT}\right)\right)\\
&= \frac{J^2}{k_bT^2}\frac{(N-1)}{\cosh^2\left(\frac{J}{k_bT}\right)}.
\end{aligned}$$

c)

Consider now the Ising model in a field and write an expression for $\mathcal{F}$ (again in the thermodynamic limit).

Answer: Now H is non-zero, and thus the partition function is given by:

$$Z=\sum_{\{S\}}\exp\left[h\sum_i S_i+K\sum_{\langle ij\rangle}S_iS_j\right].$$

Furthermore, with introducing the transfer matrix, one obtains that the free energy can be written as:

$$
\begin{aligned}
\mathcal{F} &= -k_b T N \ln\left[e^K\left(\cosh(h) + \sqrt{\sinh^2(h) + e^{-4K}}\right)\right] \\
&= -JN - k_b T N \ln\left[\cosh(h) + \sqrt{\sinh^2(h) + e^{-4K}}\right].
\end{aligned}
$$

d)

Compute the magnetization per spin M. What is M in the limit $T \to 0$?

Answer: One computes the following expression for the magnetization per spin:

$$
\begin{aligned}
M &= \frac{1}{N}\frac{\partial \mathcal{F}}{\partial H} \\
&= \frac{1}{k_b T}\frac{\partial}{\partial h}\left(-J - k_b T \ln\left[\cosh(h) + \sqrt{\sinh^2(h) + e^{-4K}}\right]\right) \\
&= \frac{\sinh^2(h)}{\cosh^2(h) + e^{-4K}}.
\end{aligned}
$$

Problem 2

Consider the $d = 1$ Ising-model with periodic boundary conditions.

a)

Construct the matrix $\mathbf{S}$ which diagonalises the transfer matrix $\mathbf{T}$. You will find it useful to write the matrix elements in terms of the variable θ given by:

$$
\coth(2\theta) = e^{2K}\sinh(h).
$$

Answer: Firstly, we define the transfer matrix $\mathbf{T}$ as:

$$
\mathbf{T} = \begin{pmatrix} e^{K+h} & e^{-K} \\ e^{-K} & e^{-h+K} \end{pmatrix}.
$$

A matrix $\mathbf{S}$ that diagonalises $\mathbf{T}$ is given by:

$$
\mathbf{T}' = \mathbf{S}^{-1}\mathbf{T}\mathbf{S} \tag{1}
$$

$$
\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \mathbf{S}^{-1}\begin{pmatrix} e^{K+h} & e^{-K} \\ e^{-K} & e^{-h+K} \end{pmatrix}\mathbf{S}. \tag{2}
$$

One needs to find the eigenvalues of $\mathbf{T}$, which is achieved by solving the characteristic equation:

$$
\begin{aligned}
\det(T - \lambda I) &= \begin{vmatrix} e^{K+h} - \lambda & e^{-K} \\ e^{-K} & e^{-h+K} - \lambda \end{vmatrix} \\
&= \left(e^{K+h} - \lambda\right)\cdot\left(e^{K-h} - \lambda\right) - e^{-2K} \\
&= \lambda^2 - \left(e^{K+h} + e^{K-h}\right)\lambda + e^{2K} - e^{-2K}
\end{aligned}
$$

The roots of this equation are given by:

$$
\begin{aligned}
\lambda_{1,2} &= \frac{e^{K+h}+e^{K-h}}{2} \pm \sqrt{\left(\frac{e^{K+h}+e^{K-h}}{2}\right)^2 - e^{2K} + e^{-2K}} \\
&= e^K \left[\cosh(h) \pm \sqrt{\sinh^2(h) + e^{-4K}}\right].
\end{aligned}
$$

Thus, λ_1 and λ_2 are the eigenvalues of $\mathbf{T}$, i.e. $\lambda_{1,2} = e^K \left[\cosh(h) \pm \sqrt{\sinh^2(h) + e^{-4K}}\right]$. Furthermore, we seek the eigenvectors corresponding to the eigenvalues of $\mathbf{T}$. The eigenvectors are given by:

$$
\begin{aligned}
(\lambda_1 \mathbf{I} - \mathbf{T})\mathbf{v}_1 = \mathbf{0}, \\
(\lambda_2 \mathbf{I} - \mathbf{T})\mathbf{v}_2 = \mathbf{0}.
\end{aligned}
$$

Looking at the first equation, one obtains:

$$
\begin{gathered}
\begin{pmatrix} \lambda_1 - e^{K+h} & -e^{-K} \\ -e^{-K} & \lambda_1 - e^{-h+K} \end{pmatrix} \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
v_{11}\left(\lambda_1 - e^{K+h}\right) - v_{12}e^{-K} = 0 \\
v_{12}\left(\lambda_1 - e^{-h+K}\right) - v_{11}e^{-K} = 0
\end{gathered}
$$

This implies that one can write the eigenvector of λ_1 as:

$$
\mathbf{v}_1 = \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix} = \begin{pmatrix} e^{-K} \\ -e^K \sinh(h) + \sqrt{e^{2K}\sinh^2(h) + e^{-2k}} \end{pmatrix},
$$

and similarly, the eigenvector of λ_2 is given by:

$$
\mathbf{v}_2 = \begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix} = \begin{pmatrix} -e^K \sinh(h) + \sqrt{e^{2K}\sinh^2(h) + e^{-2k}} \\ -e^{-K} \end{pmatrix}.
$$

Using $\cot(2\theta) = e^{2K}\sinh(h)$, on the term $-e^k\sinh(h) + \sqrt{e^{2K}\sinh^2(h) + e^{-2K}}$, one gets:

$$
\begin{aligned}
-e^k\sinh(h) + \sqrt{e^{2K}\sinh^2(h) + e^{-2K}} &= e^{-K}\left(-e^{2K}\sinh(h) + \sqrt{\cot^2(2\theta)+1}\right) \\
&= e^{-K}\tan(\theta).
\end{aligned}
$$

Thus, the matrix $\mathbf{S}$ can be written as, after normalizing the eigenvectors, i.e $\mathbf{v_i}/\|\mathbf{v_i}\|$:

$$
\mathbf{S} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.
$$

b)

Derive the relation:

$$
\langle S_i \rangle = \frac{\operatorname{Tr}\left(\mathbf{S}^{-1}\sigma_z \mathbf{S}\mathbf{T}'^N\right)}{Z_N},
$$

and use your answer from part a) to show that $\langle S_i \rangle = \cos(2\theta)$ as $N \to \infty$. Similarly compute $\langle S_i S_j \rangle$ and hence show that in the thermodynamics limit:

$$\begin{aligned} G(i, i+j) &= \langle S_i S_j \rangle - \langle S_i \rangle \langle S_j \rangle \\ &= \sin^2(2\theta) \left(\frac{\lambda_2}{\lambda_1} \right)^j . \end{aligned}$$

Answer: We begin by stating that the average spin of a particle i is given by:

$$\begin{aligned} \langle S_i \rangle &= \frac{1}{Z_n} \sum_{S_1} ... \sum_{S_N} T_{S_1,S_2} ... T_{S_{i-1},S_i} S_i ... T_{S_{N-1},S_N} \\ &= \frac{1}{Z_n} \sum_{S_1} ... \sum_{S_N} ... \mathbf{T} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{=\sigma_z} \mathbf{T} ... \mathbf{T} \sigma_z \mathbf{T}, \end{aligned}$$

This then yields that the average spin of particle i can be written as:

$$\langle S_i \rangle = \frac{\mathrm{Tr}\left(\sigma_z \mathbf{T}^N\right)}{Z_n}.$$

From this, one then inverts eq (2), such that $\mathbf{T} = \mathbf{S}\mathbf{T}'\mathbf{S}^{-1}$, and thus:

$$\langle S_i \rangle = \frac{\mathrm{Tr}\left(\sigma_z \left[\mathbf{S}\mathbf{T}'\mathbf{S}^{-1}\right]^N\right)}{Z_n}.$$

From this, one expands the $(\mathbf{S}T'\mathbf{S}^{-1})^N$ term and utilizes that $\mathbf{S}^{-1} \cdot \mathbf{S} = \mathbf{I}$, such that one obtain:

$$\begin{aligned} \langle S_i \rangle &= \frac{\mathrm{Tr}\left(\sigma_z \mathbf{S}\mathbf{T}'\mathbf{S}^{-1}\mathbf{S}T' ... \mathbf{S}T'\mathbf{S}^{-1}\right)}{Z_N} \\ \langle S_i \rangle &= \frac{\mathrm{Tr}\left(\mathbf{S}^{-1}\sigma_z \mathbf{S}\mathbf{T}'^N\right)}{Z_N}. \end{aligned}$$

In order to evaluate $\langle S_i \rangle$ in the thermodynamic limit, one has to evaluate the expression $\mathbf{S}^{-1}\sigma_z\mathbf{S}$

$$\begin{aligned} \mathbf{S}^{-1}\sigma_z\mathbf{S} &= \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \\ &= \begin{pmatrix} \cos(2\theta) & -\sin(2\theta) \\ -\sin(2\theta) & -\cos(2\theta) \end{pmatrix}. \end{aligned}$$

Thus, $\langle S_i \rangle$ can be expressed as:

$$\begin{aligned} \langle S_i \rangle &= \frac{\mathrm{Tr}\left(\mathbf{S}^{-1}\sigma_z \mathbf{S}\mathbf{T}'^N\right)}{Z_N} \\ &= \frac{\mathrm{Tr}\left(\underbrace{\begin{pmatrix} \cos(2\theta) & -\sin(2\theta) \\ -\sin(2\theta) & -\cos(2\theta) \end{pmatrix}}_{=\delta} \cdot \begin{pmatrix} \lambda_1^N & 0 \\ 0 & \lambda_2^N \end{pmatrix} \right)}{\mathrm{Tr}\left(T^N\right)} \end{aligned}$$

In the thermodynamic limit, this then simplifies to:

$$\langle S_i \rangle = \cos(2\theta).$$

The correlation term $\langle S_i S_j \rangle$ is then given by:

$$\langle S_i S_j \rangle = \frac{1}{\mathrm{Tr}\left(T^N\right)} \left(\mathbf{T}'^i \cdot \delta \mathbf{T}'^j \delta \mathbf{T}'^{N-i-j}\right).$$

This, in the thermodynamic limit, can be written as:

$$\langle S_i S_j \rangle = \cos^2(2\theta) + \sin^2(2\theta) \left(\frac{\lambda_2}{\lambda_1}\right)^j.$$

Finally, the correlation function $G(i, i+j)$ is given by:

$$\begin{aligned} G(i, i+j) &= \langle S_i S_j \rangle - \langle S_i \rangle \langle S_j \rangle \\ &= \cos^2(2\theta) + \sin^2(2\theta) \left(\frac{\lambda_2}{\lambda_1}\right)^j - \cos^2(2\theta) \\ &= \sin^2(2\theta) \left(\frac{\lambda_2}{\lambda_1}\right)^j. \end{aligned}$$

c)

Calculate the isothermal suspectability χ_T from the formula given for $M(H)$ in the text. Verify explicitly that $\chi_T = \beta \sum_j G(i, i+j)$ (in the thermodynamic limit the sum runs from $-\infty \to \infty$).

Answer: The isothermal susceptibility is defined by:

$$\chi_T = \frac{\partial M}{\partial H}.$$

The magnetization per spin is given by:

$$M = \sum_{i=1}^{N} \langle S_i \rangle = N \cos(2\theta).$$

Thus, the isothermal susceptibility is given by:

$$\begin{aligned} \chi_T = \frac{\partial M}{\partial H} &= N \frac{\partial \cos(2\theta)}{\partial H} \\ &= N\beta \frac{\partial \cos(2\theta)}{\partial h} \\ &= -N\beta \sin(2\theta) \frac{\partial 2\theta}{\partial h}. \end{aligned}$$

We now need to find an explicit expression for θ which one does from $\cot(2\theta) = e^{2K} \sinh(h)$, such that:

$$\theta = \frac{1}{2} \cot^{-1}\left(e^{2K} \sinh(h)\right).$$

This then yields:

$$\begin{aligned} \chi_T &= -N\beta \sin(2\theta) \frac{\partial \theta}{\partial h} \\ &= -2N\beta \sin(2\theta) \frac{\partial}{\partial h} \left(\frac{1}{2} \cot^{-1}\left(e^{2K} \sinh(h)\right)\right) \\ &= N\beta \sin^3(2\theta)\, e^{2K} \cosh(h). \end{aligned}$$

To prove that the magnetic susceptibility is given by $\chi_T = \sum_j G(i, i+j)$, one needs to evaluate the sum:

$$\begin{aligned}
\sum_j G(i,i+j) &= \sum_j \sin^2(2\theta) \left(\frac{\lambda_2}{\lambda_1}\right)^j \\
&= N \sin^2(2\theta) \left(\frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2}\right) \\
&= N \sin^2(2\theta) \left(\frac{e^{-K}\sqrt{\cot^2(2\theta)+1}}{e^K \cosh(h)}\right)^{-1} \\
&= N \sin^2(2\theta) \left(\sin(2\theta) e^{2K} \cosh(h)\right) \\
&= N \sin^3(2\theta)\, e^{2K} \cosh(h).
\end{aligned}$$

Since the $\beta G(i,i+j) = \chi_T$ one has proved that the identity holds.

d)

We will now consider what happens at the boundaries. Consider the partition function:

$$Z(h,K) = \sum_{S_1} \dots \sum_{S_N} \exp\left[h\left(S_1 + \dots + S_N\right) + K\left(S_1S_2 + \dots + S_{N-1}S_N\right)\right].$$

In this case, the partition function is not simply $Z = \mathrm{Tr}\left(\mathbf{T}'\right)^N$. Work out what the correct expression is, you will have to introduce a new matrix in addition to $\mathbf{T}$, and show that the free energy is given by:

$$\mathcal{F} = N f_b(h,K) + f_s(h,K) + F_{f_s}(N,h,K),$$

where f_b is the bulk free energy, f_s is the surface free energy and F_{f_s} is an intrinsic finite size contribution that depends on the system size as $e^{-C(h,K)N}$, where C is function.

Answer: We begin by rewriting the partition function:

$$\begin{aligned}
Z(h,K) &= \sum_{S_1} \dots \sum_{S_N} \exp\left[h\left(S_1 + \dots + S_N\right) + K\left(S_1S_2 + \dots + S_{N-1}S_N\right)\right] \\
&= \sum_{S_1} \dots \sum_{S_N} e^{\frac{h}{2}(S_1+S_2)+KS_1S_2} \dots e^{\frac{h}{2}(S_{N-1}S_N)+KS_{N-1}S_N} e^{\frac{h}{2}(S_NS_1)} \\
&= \mathrm{Tr}\left(\underbrace{\begin{pmatrix} e^{\frac{h}{2}+K} & e^{-\frac{h}{2}} \\ e^{-\frac{h}{2}} & e^{\frac{h}{2}+K} \end{pmatrix}^{N-1}}_{\mathbf{T}^{N-1}} \cdot \underbrace{\begin{pmatrix} e^h & 1 \\ 1 & e^{-h} \end{pmatrix}}_{\mathbf{T_b}}\right).
\end{aligned}$$

From here, one diagonalises the matrix $\mathbf{T}$ and $\mathbf{T_b}$, such that:

$$\begin{aligned}
\mathbf{T} &= \mathbf{S}^{-1}\mathbf{T}'\mathbf{S}, \\
\mathbf{T_b} &= \mathbf{S_b}^{-1}\mathbf{T_b}'\mathbf{S_b} \\
\implies Z(h,K) &= \mathrm{Tr}\left[\left(\mathbf{S}^{-1}\mathbf{T}'\mathbf{S}\right)^{N-1} \mathbf{S}^{-1}\mathbf{T_b}\mathbf{S}\right] \\
&= \mathrm{Tr}\left[\begin{pmatrix} \lambda_1^{N-1} & 0 \\ 0 & \lambda_2^{N-1} \end{pmatrix} \cdot \mathbf{T_b}'\right]
\end{aligned}$$

One computes $\mathbf{T_b}'$ such that:

$$\begin{aligned}
\mathbf{T_b}' &= \mathbf{S_b}^{-1}\mathbf{T_b}\mathbf{S_b} \\
&= \mathbf{S_b}^{-1}\begin{pmatrix} e^h & 1 \\ 1 & e^{-h} \end{pmatrix}\mathbf{S_b} \\
&= \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \cdot \begin{pmatrix} e^h & 1 \\ 1 & e^{-h} \end{pmatrix} \cdot \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix},
\end{aligned}$$

writing out the matrix multiplication, one obtains, where c is $\cos(\theta)$, and $s = \sin(\theta)$:

$$\mathbf{T_b}' = \begin{pmatrix} c^2 e^{K+h} + 2cse^{-K} + s^2 e^{K-h} & (c^2 - s^2)e^{-K} - sc\left(e^{K+h} - e^{K-h}\right) \\ (c^2 - s^2)e^{-K} - sc\left(e^{K+h} - e^{K-h}\right) & c^2 e^{K-h} - 2cse^{-K} + s^2 e^{K+h} \end{pmatrix}$$

Thus, one has derived the correction term to the partition function. In order to compute the free energy, one has to simplify the partition function further:

$$\begin{aligned}
Z(h,K) = \mathrm{Tr}\left(\mathbf{T}'^{N-1}\mathbf{T_b}'\right) &= \lambda_1^{N-1}\left[\cosh(h) + \cos(2\theta)\sinh(h) + \sin(2\theta)\right] \\
&+ \lambda_2^{N-1}\left[\cos(2\theta)\cosh(h) - \sinh(h) - \sin(2\theta)\right].
\end{aligned}$$

From this, one computes the free energy as:

$$\begin{aligned}
\mathcal{F} &= -k_b T\ln(Z) = -k_b T\ln\Big[\lambda_1^{N-1}\left[\cosh(h) + \cos(2\theta)\sinh(h) + \sin(2\theta)\right] \\
&+\lambda_2^{N-1}\left[\cos(2\theta)\cosh(h) - \sinh(h) - \sin(2\theta)\right]\Big] \\
&= -k_b T\,[N-1]\ln(\lambda_1) - k_b T\ln\left[\cosh(h) + \cos(2\theta)\sinh(h) + \sin(2\theta)\right] \\
&- k_b T\ln\left[1 + \left(\frac{\lambda_2}{\lambda_1}\right)^{N-1}\frac{\cosh(h) + \cos(2\theta)\sinh(h) + \sin(2\theta)}{\cos(2\theta)\cosh(h) - \sinh(h) - \sin(2\theta)}\right].
\end{aligned}$$

In the thermodynamic limit $N \to \infty$ one can write the free energy as:

$$\begin{aligned}
\mathcal{F} &= N f_b(h,K) + f_s(h,K) + F_{f_s}(N,h,K), \\
f_b(h,K) &= -k_b T\ln(\lambda_1), \\
f_s(h,K) &= -k_b T\ln\left[\cosh(h) + \cos(2\theta)\sinh(h) + \sin(2\theta)\right], \\
F_{f_s}(N,h,K) &= -k_b T\ln\left[1 + \frac{\lambda_2}{\lambda_1}^{N-1}\frac{\cosh(h) + \cos(2\theta)\sinh(h) + \sin(2\theta)}{\cos(2\theta)\cosh(h) - \sinh(h) - \sin(2\theta)}\right].
\end{aligned}$$

In order to prove that F_{f_s} is an intrinsic finite size contribution, one has to show that it depends on the system size as $e^{-C(h,K)N}$. One can write F_{f_s} as:

$$\begin{aligned}
F_{f_s}(N,h,K) &= k_b T\ln\left[1 + \left(\frac{\lambda_2}{\lambda_1}\right)^{N-1}\frac{\cosh(h) + \cos(2\theta)\sinh(h) + \sin(2\theta)}{\cos(2\theta)\cosh(h) - \sinh(h) - \sin(2\theta)}\right] \\
&= \exp\left[\ln\left[k_b T\ln\left[1 + \left(\frac{\lambda_2}{\lambda_1}\right)^{N-1}\frac{\cosh(h) + \cos(2\theta)\sinh(h) + \sin(2\theta)}{\cos(2\theta)\cosh(h) - \sinh(h) - \sin(2\theta)}\right]\right]\right] \\
&\sim \exp\left[N\ln\left[\frac{\lambda_2}{\lambda_1}\right]\right].
\end{aligned}$$

Thus, one has showed that, in the thermodynamic limit, F_{f_s} depends on the system size as $e^{-C(h,K)N}$.

e)

Check that in the case $h = 0$ and $N \to \infty$, your result for the surface free energy agrees with that obtained from $\lim_{N\to\infty} \mathcal{F}_N^{free} - \mathcal{F}_N^{periodic}$.

Answer: We rewrite the expression for the surface free energy f_s previously obtained, with hyperbolic functions:

$$\begin{aligned} f_s(h,K) &= -k_bT \ln\left[\cosh(h) + \cos(2\theta)\sinh(h) + \sin(2\theta)\right] \\ &= -k_bT \ln\left[\frac{\cosh(h) + \frac{\sinh^2(h)}{\sqrt{\sinh^2(h)+e^{-4K}}} + \frac{e^{-2K}}{\sqrt{\sinh^2(h)+e^{-4K}}}}{e^K\left(\cosh(h) + \sqrt{\sinh^2(h)+e^{-4k}}\right)}\right]. \end{aligned}$$

In the case $h = 0$, this reduces to:

$$\begin{aligned} f_s(0,K) &= -k_bT \ln\left[\frac{1 + \frac{0}{\sqrt{0+e^{-4K}}} + \frac{e^{-2K}}{\sqrt{0+e^{-4K}}}}{e^K\left(1+\sqrt{0+e^{-4k}}\right)}\right] \\ &= -k_bT \ln\left[\frac{2}{e^K + e^{-K}}\right] = -k_bT \ln\left[\frac{1}{\cosh(h)}\right]. \end{aligned}$$

Thus, we now evaluate the limit:

$$\begin{aligned} f_s &= \lim_{N\to\infty}\left(\mathcal{F}_N^{free} - \mathcal{F}_N^{periodic}\right) \\ &= \lim_{N\to\infty}\left(-k_bT(N-1)\ln(2\cosh(K)) - k_bT\ln(2) - k_bT\ln\left[(2\cosh(K))^N + (2\sinh(K))^N\right]\right) \\ &= -k_bT \lim_{N\to\infty}\left[(N-1)\ln(2\cosh(K)) + \ln(2) - \left(\ln(2\cosh(K))^N + \ln\left(1+\tanh(K)^N\right)\right)\right] \\ &= -k_bT\left(\ln(2\cosh(K)) + \ln(2)\right) = -k_bT\frac{1}{\cosh(K)}. \end{aligned}$$

Thus, we have showed that the surface energy that we obtained previously holds.