

FK7058: Handin2

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Consider Ising model, with the following Hamiltonian:

$$-\mathcal{H} = H(S_1 + S_2) + JS_1S_2,$$

where $J > 0$ and $S_i = \pm 1 \forall i$.

1)

Enumerate all the microstates of the system as well as the probabilities of these microstates.

Answer: All the possible states are the combination of the two spins S_1 and S_2 .

1. $S_1 = 1$ and $S_2 = 1$.
2. $S_1 = 1$ and $S_2 = -1$.
3. $S_1 = -1$ and $S_2 = 1$.
4. $S_1 = -1$ and $S_2 = -1$.

There is an equal probability for each state and since there are four microstates, the probability of each state is $1/4$.

2)

Calculate the partition function.

Answer: The partition function is given as:

$$\begin{aligned} \sum_i e^{-\beta E_i} &= e^{\beta(2H+J)} + 2e^{\beta J} + e^{\beta(-2H+J)} \\ &= e^{\beta J} \left[2 + e^{2\beta H} + e^{-2\beta H} \right] \\ &= e^{\beta J} \left[2 + 2 \cosh(2\beta H) \right] \end{aligned}$$

3)

How does the partition function (and hence the free energy) change if we replace H by $-H$?

Answer: If instead H is replaced by $-H$ the partition function becomes:

$$\begin{aligned} Z &= \sum_i e^{-\beta E_i} \\ &= e^{-\beta E_1} + e^{-\beta E_2} + e^{-\beta E_3} + e^{-\beta E_4} \\ &= e^{-\beta(-2H+J)} + 2e^{\beta J} + e^{-\beta(2H+J)} \\ &= e^{\beta J} \left[2 + 2 \cosh(2\beta H) \right]. \end{aligned}$$

The partition function does not change, and thus, neither does the free energy.

4)

Calculate the average energy of this system.

Answer: The average energy of the system is given by:

$$\begin{aligned}
 \langle E \rangle &= \frac{1}{Z} \frac{\partial Z}{\partial \beta} \\
 &= -\frac{1}{Z} \frac{\partial}{\partial \beta} \left[e^{\beta J} [2 + 2 \cosh(2\beta H)] \right] \\
 &= -\frac{1}{Z} \left[e^{\beta J} J [2 + 2 \cosh(2\beta H)] + e^{\beta J} [4H \sinh(2\beta H)] \right] \\
 &= -\frac{1}{Z} e^{\beta J} \left[J [2 + 2 \cosh(2\beta H)] + [4H \sinh(2\beta H)] \right] \\
 &= -\frac{J (2 + 2 \cosh(2\beta H)) + 4H \sinh(2\beta H)}{2 + 2 \cosh(2\beta H)} \\
 &= -[J + 2H \tanh(\beta H)]
 \end{aligned}$$

5)

Calculate the magnetization per unit spin $M = \langle \frac{M_1 + M_2}{2} \rangle$. How is M related to the free energy of the system? What is M at $H = 0$?

Answer:

6)

What is M in the limit $T \rightarrow 0$?

Answer:

7)

Calculate $\langle S_1 \cdot S_2 \rangle - \langle S_1 \rangle \langle S_2 \rangle$ the connected correlation function.

Answer: We begin by computing the average of S_1 and S_2 :

$$\begin{aligned}
 \langle S_1 \rangle &= \sum_i S_1^{(i)} \cdot p_i = \frac{1}{4} - \frac{1}{4} = 0, \\
 \langle S_2 \rangle &= \sum_i S_2^{(i)} \cdot p_i = \frac{1}{4} - \frac{1}{4} = 0.
 \end{aligned}$$

8)

Show that, if we change the Hamiltonian of the system so that:

$$-\mathcal{H} = H_1 S_1 + H_2 S_2 + J S_1 S_2,$$

the connected correlation function can be written as the second derivative of the free energy.

Answer: