Handin3

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1 Bi-harmonic equation

Consider the Navier-Cauchy equation in the absence of anybody forces - the only forces are at boundaries

$$(1-2\sigma)\nabla^2\mathbf{u} + \nabla\nabla\cdot u = 0$$

where $\mathbf{u}$ is the deformation field. Show that $\mathbf{u}$ satisfies the bi-harmonic equation

$$\nabla^4\mathbf{u} = 0,$$

even if there is a constant gravitational force acting on the body. It is useful to remember that although $\mathbf{u}$ may satisfy a bi-harmonic equation any solution of the bi-harmonic equation is not a solution of the Navier–Cauchy equation. Now show that if $\mathbf{v}$ is a solution of the b-harmonic equation then a general solution for the Navier–Cauchy equation (without volume force) can be written as

$$\mathbf{u} = \nabla^2\mathbf{v} - \frac{1}{2(1-\sigma)}\nabla\nabla\cdot\mathbf{u}.$$

Hint

Assume $\mathbf{u}$ is on the form

$$\mathbf{u} = \nabla^2\mathbf{v} + A\nabla\nabla\cdot\mathbf{v},$$

where $\mathbf{v}$ solves the bi-harmonic equation. Then substitute this $\mathbf{u}$ into the Navier–Cauchy equation (without volume force) to find A.

Answer

Question 2

What is the equation of deformation of a small volume in a homogeneous and isotropic material in three dimensions?

Answer

Question 3

Now consider the kind of deformation where $u_z = 0$ everywhere in a three-dimensional body (that is homogeneous and isotropic with elastic coefficients Y and σ) and u_x and u_y are functions of x and y alone (not functions of z). Note that the dimension on the body along the z direction remains constant. Hence, although $u_z = 0$ everywhere, $\sigma_{z,z}$ is not zero.

a)

What is the equations of equilibrium involving the stresses $\sigma_{i,j}$ where i, j are x and y.

Answer

b)

Show that for a scalar function $\psi(x, y)$ the equations of equilibrium are satisfied by the following choice

$$\sigma_{x,x} = \frac{\partial^2 \psi(x,y)}{\partial y^2} \tag{1}$$

$$\sigma_{x,y} = -\frac{\partial^2 \psi(x,y)}{\partial y \partial x} \tag{2}$$

$$\sigma_{y,y} = \frac{\partial^2 \psi(x,y)}{\partial x^2} \tag{3}$$

$$\tag{4}$$

Answer

c)

Find out what kind of partial differential equation ψ obeys. *Hint:* Write the stress in terms of strains

$$\sigma_{\alpha,\beta} = \frac{Y}{1+\sigma}\left(s_{\alpha,\beta} + \frac{\sigma}{1-2\sigma} s_{\kappa,\kappa}\delta_{\alpha,\beta}\right).$$

Take trace on both sides. The function ψ is known as the Airy stress function.

Answer

Contact mechanics

A sphere made of isotropic material with Young's modulus Y and Poisson's ratio σ and mass M with uniform density falls from a height h on a flat surface. The gravitational acceleration g is constant. The surface is very hard compared to the sphere such that you can ignore any deformation of the surface. Estimate the amount of time the sphere stays in contact with the surface.

Answer

Problem for extra credit

A cylinder made of isotropic material with Young's modulus Y and Poisson's ratio σ is resting on a flat surface. Ignore the deformations of the surface. If the cylinder has mass per unit length m and uniform gravitational acceleration is g calculate the area of contact. Note that we did contact of spheres in class. You need to generalize the ideas to contact of cylinder with a flat surface.

Answer