Handin 5

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Question 1

Say that we have a uniform probability distribution for values of $x \in [0, 1]$. We want to transform such a distribution to a distribution for $\pi \in [0, \pi]$, such that the probability distribution is given by $p_1(\phi) = \frac{1}{2}\sin(\phi)$. We can do this by using the method of importance sampling. One have that the probability distribution for x is given by $p_0(x) = 1$. Then we can find the probability distribution for ϕ by using the following relation:

$$\begin{aligned} p_1(\phi) &= p_0(x)\left|\frac{dx}{d\phi}\right| \\ &= \left|\frac{dx}{d\phi}\right|. \end{aligned}$$

We can find the relation between x and ϕ by integrating the probability distribution $p_1(\phi)$:

$$\begin{aligned} u(\phi) &= \int_{\phi_0}^{\phi} p_1(\phi')d\phi' \\ &= \int_{\phi_0}^{\phi} \frac{1}{2}\sin(\phi')d\phi' \\ &= -\frac{1}{2}\cos(\phi')\Big|_{\phi_0}^{\phi} \\ &= \frac{1}{2}\big(\cos(\phi_0) - \cos(\phi)\big). \end{aligned}$$

We now want to find the inverse of $u(\phi)$, which is given by:

$$\begin{aligned} 2u &= \cos(\phi_0) - \cos(\phi) \\ \cos(\phi) &= \cos(\phi_0) - 2u \\ \phi &= \arccos(\cos(\phi_0) - 2u). \end{aligned}$$

Thus, any random number u describes a random value ϕ.

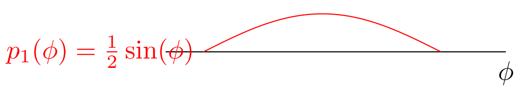

Figure 1: Probability distribution 1

Question 2

The method of importance sampling is a method to sample from a probability distribution $p(x)$, by sampling from a different distribution $q(x)$, which is easier to sample from. The idea is to sample from $q(x)$, and then use the samples to estimate the expectation value of some function $f(x)$ with respect to $p(x)$. We do this by looking at the integral of the approximated probability distribution, as the slope of the integrated function is proportional to the probability distribution.

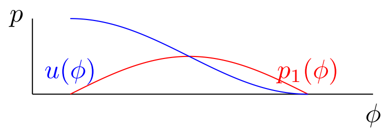

Figure 2: $p_1(\phi)$ and $u(\phi)$

We can in the above figure see that the probability slope of the integrated function $u(\phi)$ is proportional to the probability distribution $p_1(\phi)$.

Question 3

Now suppose $p_2(\phi) = \frac{1}{2}\sin^2(\phi)$. We wish to sample the $p_2(\phi)$ distribution based on random number obtained by $p_1(\phi)$. We can do this by using the method of importance sampling. One has that the probability distribution for $p_2(\phi)$ is given by:

$$\begin{aligned} p_2(\phi) &= p_1(\phi)\left|\frac{du}{d\phi}\right| \\ &= \frac{1}{2}\sin(\phi)\left|\frac{du}{d\phi}\right|. \end{aligned}$$

We can find the relation between u and ϕ by integrating the probability distribution $p_2(\phi)$:

$$\begin{aligned} v(\phi) &= \int_{\phi_0}^{\phi} p_2(\phi')d\phi' \\ &= \int_{\phi_0}^{\phi} \frac{1}{2}\sin^2(\phi')d\phi' \\ &= \frac{1}{4}\left(\phi - \sin(\phi)\cos(\phi)\right). \end{aligned}$$

We now want to find the inverse of $v(\phi)$, which is given by:

$$\begin{aligned} 4v &= \phi - \sin(\phi)\cos(\phi) \\ \phi &= \arcsin\left(\frac{4v}{\sqrt{16v^2+1}}\right). \end{aligned}$$

Thus, any random number v describes a random value ϕ.

Question 4

Suppose now, that instead we have a random uniform distribution $p_3(\phi) = \frac{1}{2}$. How does one then obtain a random number ϕ_{rand}? We, have that the probability distribution for $p_3(\phi)$ is given by:

$$\begin{aligned} p_3(\phi) &= p_1(\phi)\left|\frac{du}{d\phi}\right| \\ &= \frac{1}{2}\sin(\phi)\left|\frac{du}{d\phi}\right|. \end{aligned}$$

We can find the relation between u and ϕ by integrating the probability distribution $p_3(\phi)$:

$$
\begin{aligned}
w(\phi) &= \int_{\phi_0}^{\phi} p_3(\phi')d\phi' \\
&= \int_{\phi_0}^{\phi} \frac{1}{2}d\phi' \\
&= \frac{1}{2}\phi.
\end{aligned}
$$

One now want to find the inverse of $w(\phi)$, which is given by:

$$
\begin{aligned}
2u &= \phi \\
\phi &= 2u.
\end{aligned}
$$

Thus, any random number u describes a random value ϕ. The efficiency of the last method is not as good as the previous method, one has to complete more samples to obtain a good estimate of the probability distribution; whereas the previous method requires fewer samples to obtain a good estimate of the probability distribution.