Handin 3

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The system

Suppose two systems:

$$x_A(t+1) = \frac{1}{2}x_A(t) - 3,$$

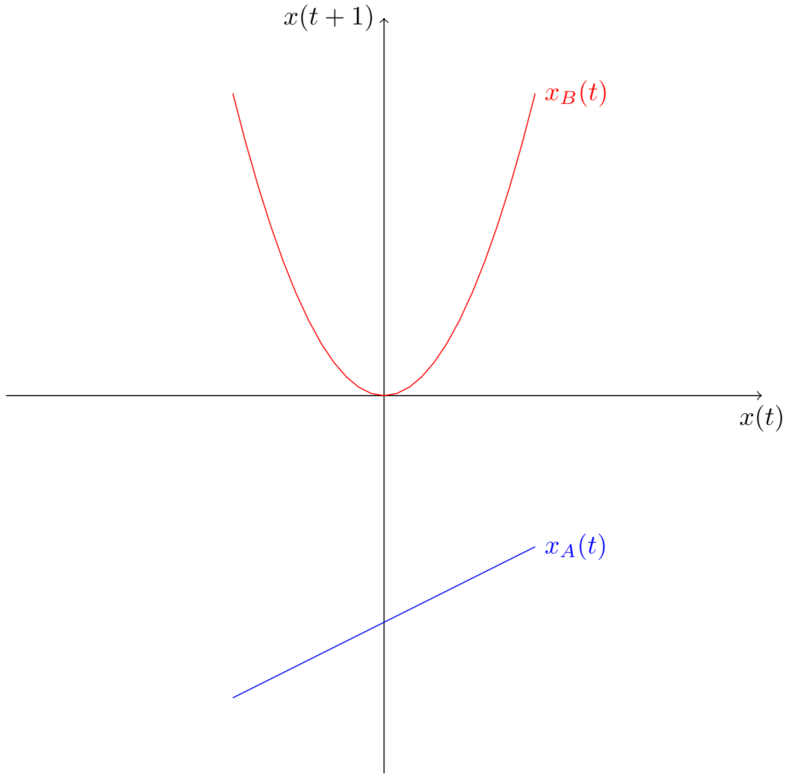
and

$$x_B(t+1) = x_B(t)x_B(t) = x_B(t)^2.$$

Question 1

Sketch $x_i(t+1)$ for the two systems in terms of x_i.

Answer: We begin by looking at the expression themselves, and from that we obtain the following



Question 2

How does the value of $\lim_{t\to\infty} x_i$ depend on the initial condition for $x(0)$ for the two systems.

Answer: The system can only converge in the limit of $t \to \infty$ at points where $x(t+1) = x(t)$, and thus for system B, we see that the only way B converges is when $x_B(0) = 0$; all other initial conditions diverges system B. For system A, we need to find the expression in terms of $x_A(0)$.

$$
\begin{aligned}
x_A(t+3) &= \frac{1}{2}x_A(t+2) - 3 \\
&= \frac{1}{2}\left(\frac{1}{2}x_A(t+1) - 3\right) - 3 \\
&= \frac{1}{2}\left(\frac{1}{2}\left(\frac{1}{2}x_A(t) - 3\right) - 3\right) - 3 \\
&= \left(\frac{1}{2}\right)^3 x_A(t) - \left(\frac{1}{2}\right)^2 3 - \left(\frac{1}{2}\right) 3 - 3 \\
&= \left(\frac{1}{2}\right)^3 x_A(t) - 3\left(1 + \frac{1}{2} + \frac{1}{4}\right).
\end{aligned}
$$

Thus, if we express t to be 0 and our added value tend towards infinity we have the following scenario:

$$
\begin{aligned}
&\lim_{t\to\infty} \left(\frac{1}{2}\right)^t x_A(0) - 3\left(1 + \frac{1}{2} + \frac{1}{4} + ...\right) \\
&\lim_{t\to\infty} \left(\frac{1}{2}\right)^t x_A(0) - 3 = -3.
\end{aligned}
$$

Thus, system A converges independent on the initial condition.

Question 3

For the first system, consider some initial condition $x_A(0)$ and a nearby condition, such that $\tilde{x}_A(0) = x_A(0) + \delta x_A(0)$. Calculate the difference in the iterations.

Answer: We begin by computing the first iteration, i.e. $x_A(0)$ and $\tilde{x}_A(0)$

$$
\begin{aligned}
x_A(1) &= \frac{1}{2}x_A(0) - 3, \\
\tilde{x}_A(1) &= \frac{1}{2}\left(x_A(0) + \delta x_A(0)\right) - 3.
\end{aligned}
$$

The difference in this iteration is $x_A(1) - \tilde{x}_A(1) = -\frac{\delta x_A(0)}{2}$. The next iteration becomes.

$$
\begin{aligned}
x_A(2) &= \frac{1}{2}x_A(1) - 3 \\
&= \frac{1}{4}x_A(0) - \frac{3}{2} - 3, \\
\tilde{x}_A(2) &= \frac{1}{2}\tilde{x}_A(1) - 3 \\
&= \frac{1}{4}\tilde{x}_A(0) + \frac{\tilde{x}_A(0)}{4} - \frac{3}{2} - 3.
\end{aligned}
$$

The difference is then $-\frac{\delta x_A(0)}{4}$; computing the third iteration is then:

$$\begin{aligned}
x_A(3) &= \frac{1}{2}x_A(2) - 3 \\
&= \frac{1}{8}x_A(0) - \frac{3}{4} - \frac{3}{2} - 3, \\
\tilde{x}_A(3) &= \frac{1}{2}\tilde{x}_A(2) - 3 \\
&= \frac{1}{8}x_A(0) + \frac{\delta x_A(0)}{8} - \frac{3}{4} - \frac{3}{2} - 3.
\end{aligned}$$

The difference in this iteration is then $-\frac{\delta x_A(0)}{8}$; thus as the number of iterations goes towards infinity, the difference tends towards zero as:

$$\lim_{t\to\infty} x_A(t) - \tilde{x}_A(t) = \lim_{t\to\infty} -\frac{\tilde{x}_A(0)}{t} = 0.$$

Question 4

The largest Lyapunov exponent is the exponential rate at which infinitesimally close initial conditions separate, i.e.

$$\lambda = \lim_{t\to\infty} \frac{1}{t} \ln\left|\frac{\delta x(t)}{\delta x(0)}\right|.$$

What is the Lyapunov exponent of system A? Note that $\delta x(0)$ is truly infinitesimal.

Answer: We use the result that we found previously:

$$\begin{aligned}
\lambda &= \lim_{t\to\infty} \frac{1}{t} \ln\left|\frac{\delta x_A(t)}{\delta x_A(0)}\right| \\
&= \lim_{t\to\infty} \frac{1}{t} \ln\left|\frac{-\delta x_A(0)}{t}\frac{1}{\delta x_A(0)}\right| \\
&= \lim_{t\to\infty} \frac{1}{t} \ln\left|\frac{1}{t}\right| \\
&= 0.
\end{aligned}$$

This result shows that the method, for system A, is the volume of the system is conserved.