Species population

Solving the Lotka-Volterra equations

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May 28, 2024

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- Can one use this information to model the population of any two or more species?

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3. The rate of which the population for the two species increases/decreases is proportional to their current population.
4. The ecological properties remains unchanged.
5. Species Y always 'hunt' species X.
6. Each of the two species population are described by only time, i.e. the entire population of Y partake in 'hunting' species X.

Theory

These assumptions gives rise to the following equations:

$$
\begin{aligned}
\frac{dx}{dt} &= \alpha x - \beta xy \\
\frac{dy}{dt} &= \delta xy - \gamma y.
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- Can be modifed to the Arditi–Ginzburg equations.

Method

Rewriting our system we obtain:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \underbrace{\begin{pmatrix} \alpha - \beta^* y & -\beta^* x \\ \delta^* y & \delta^* x - \gamma \end{pmatrix}}_{\mathbf{J}^*(x,y)} \cdot \begin{pmatrix} x \\ y \end{pmatrix}, \tag{2}$$

where $\beta^* = 0.5\beta$, $\delta^* = 0.5\delta$.

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1. Fixed point at $(0, 0)$ (Saddle point).
2. Fixed point at $\left(\frac{\gamma}{\delta^*}, \frac{\alpha}{\beta^*}\right)$ (Oscillatory behavior).

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Solutions to eq 1, can be obtained via Explicit Euler, or any order Runge-Kutta method.

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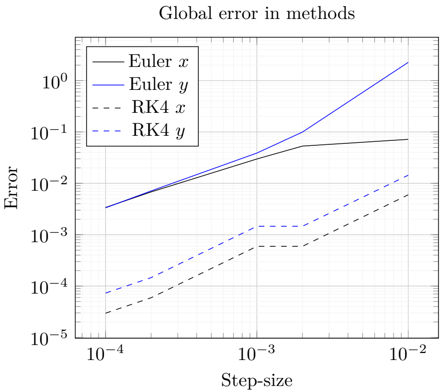
- $$\begin{aligned} k_1 &= f(t_n, x_n) \\ k_2 &= f\left(t_n + \frac{dt}{2}, x_n + dt \cdot \frac{k_1}{2}\right) \\ k_3 &= f\left(t_n + \frac{dt}{2}, x_n + dt \cdot \frac{k_2}{2}\right) \\ k_4 &= f\left(t_n + dt, x_n + dt \cdot k_3\right) \\ x_{n+1} &= x_n + \frac{dt}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right). \end{aligned} \tag{3}$$

Accuracy

Comparison of the two method



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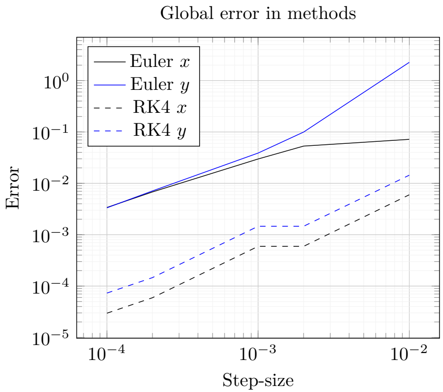
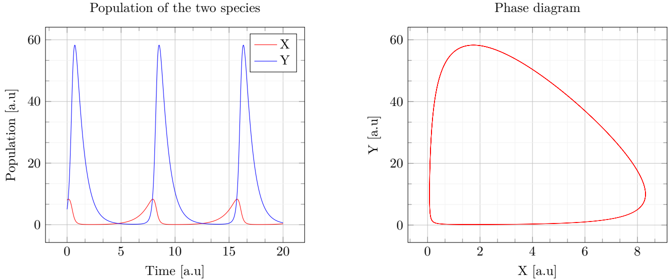


- Choose Runge-Kutta method with $dt = 0.001$.

Solutions

Variables: $\alpha = 1.0$, $\beta = 0.1$, $\delta = 0.8$, and $\gamma = 1.4$.
Initial condition: $x(0) = 8$, $y(0) = 10$.

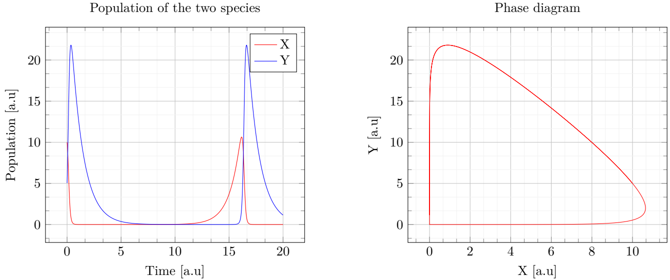


(a) Population in time

(b) Phase diagram

Figure: Population and phase diagram

Variables: $\alpha = 1.0$, $\beta = 0.5$, $\delta = 1.0$, and $\gamma = 0.9$.
Initial condition: $x(0) = 10$, $y(0) = 5$.



(a) Population in time (b) Phase diagram

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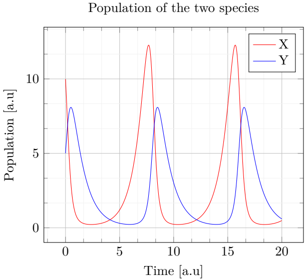
Solutions

Variables: $\alpha = 1.1$, $\beta = 0.5$, $\delta = 0.3$, and $\gamma = 0.9$.
Initial condition: $x(0) = 10$, $y(0) = 5$.



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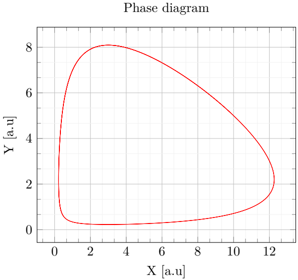
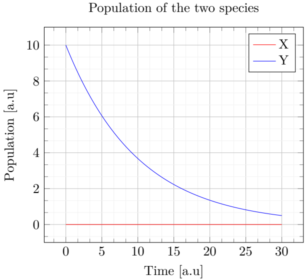


(b) Phase diagram

Figure: Population and phase diagram

Extinction

Variables: $\alpha = 1.5$, $\beta = 0.1$, $\delta = 0.1$, and $\gamma = 0.1$.
Initial condition: $x(0) = 0$, $y(0) = 10$.



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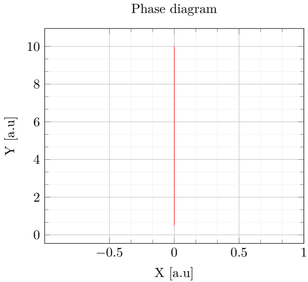

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New set of equations

Suppose we instead want three species, X, Y, and Z.

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- $$\begin{aligned} \frac{dx}{dt} &= \alpha x - \beta xz \\ \frac{dy}{dt} &= \eta xy - \chi yz \\ \frac{dz}{dt} &= \delta xz - \gamma z. \end{aligned} \tag{4}$$

Three species

This gives rise to a problem.

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- Problem becomes more stiff.

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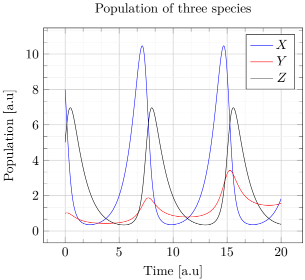
-

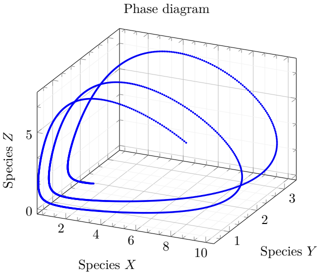
α	β	δ	γ	η	χ
1.1	0.5	0.3	0.9	0.1	0.1

Three species solution

Initial condition: $x(0) = 10$, $y(0) = 1$, and $z(0) = 5$.



(a) Population in time



(b) Phase diagram

Figure: Population and phase diagram

Generalized Lotka-Volterra equations

Suppose now instead of protection, we say that one herbivore and two carnivores.

$$\begin{aligned}\frac{dx}{dt} &= \alpha x - \beta xy - \zeta xy \\ \frac{dy}{dt} &= \eta xy - \chi yz \\ \frac{dz}{dt} &= \delta xz + \iota yz - \gamma z.\end{aligned} \tag{5}$$

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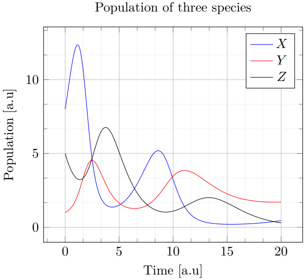

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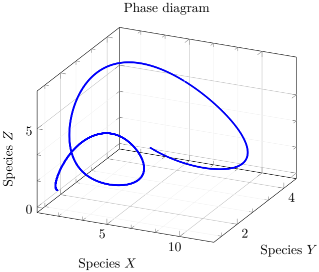

(b) Phase diagram

Figure: Population and phase diagram for generalized Lotka-Volterra solution

Time dependent variables

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- $$\frac{dx}{dt} = \alpha(t)x - \beta(t)xy$$
  $$\frac{dy}{dt} = \delta(t)xy - \gamma(t)y. \tag{6}$$

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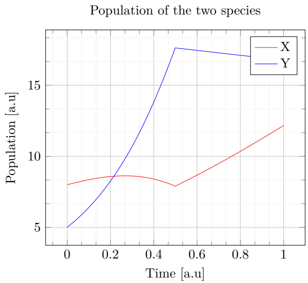
- $$\begin{aligned}\alpha(t) &= \alpha_1 \cos(t) + \alpha_1 \\ \beta(t) &= \mathrm{H}^{-1}(t) \cdot \beta_1 \\ \delta(t) &= \mathrm{H}^{-1}(t) \cdot \delta_1 \\ \gamma(t) &= \mathrm{H}(t) \cdot \gamma_1\end{aligned}$$

Time dependent variables



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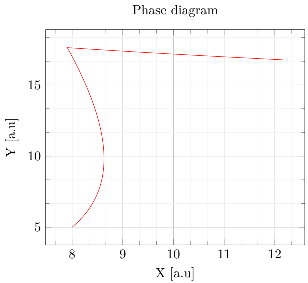

(b) Phase diagram

Figure: Population and phase diagram with time dependent variables

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- Questions?

Thanks

Thanks for listening