Exercise 46

Author : Andreas Evensen

Date: September 27, 2024

Question

We're asked to show that the following is true:

$$\int_{-1}^{1} dx \left[P_{l'}(x)P_l(x)\right] = \frac{2}{2l+1}\delta_{l',l}, \tag{1}$$

which implies that the Legendre polynomials are orthagonal.

Answer

We're given a set of equations to our disposal, that will aid us in the search.

$$Y_{l,0} = \sqrt{\frac{2l+1}{4\pi}}P_l(\cos(\theta)), \tag{2}$$

$$\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \left[Y^*_{l',m'}(\theta,\phi)Y_{l,m}(\theta,\phi)\sin(\theta)\right] = \delta_{l',l}\delta_{m',m}. \tag{3}$$

If we use substitution of variables on eq (1) with $x = \cos(\theta)$, we obtain the following:

$$\int_{-1}^{1} dx \left[P_{l'}(x)P_l(x)\right] = -\int_{-\pi}^{0} d\theta \sin(\theta)\left[P_{l'}(\cos(\theta))P_l(\cos(\theta))\right]$$

Since $Y_{l,0}$ is real $Y_{l,0} = Y^*_{l,0}$. Furthermore, since $Y_{l,0}$ is axially symmetric, if we integrate with respect to ϕ we obtain a factor 2π. Using these facts we can write the following

$$-\int_{-\pi}^{0} d\theta \sin(\theta)\left[P_{l'}(\cos(\theta))P_l(\cos(\theta))\right] = \frac{4\pi}{2l+1}\frac{1}{2\pi}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin(\theta)\left[Y^*_{l',0}(\theta,\phi)Y_{l,0}(\theta,\phi)\right].$$

We end up with the following:

$$\begin{aligned}\int_{-1}^{1} dx \left[P_{l'}(x)P_l(x)\right] &= \frac{2}{2l+1}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin(\theta)\left[Y^*_{l',0}(\theta,\phi)Y_{l,0}(\theta,\phi)\right] \\ &= \frac{2}{2l+1}\delta_{l',l}\delta_{0,0} = \frac{2}{2l+1}\delta_{l',l}.\end{aligned}$$

And thus we've proved the orthogonality of the Legendre polynomails.