Exercise 41

Author : Andreas Evensen

Date: September 25, 2024

Question

We wish to show that the following expression is true:

$$\nabla^2 f = \frac{1}{\sqrt{g}}\partial_i\left(\sqrt{g}g^{ij}\partial_j f\right), \tag{1}$$

where $g = \det(g_{ij})$ and the tensor g_{ij} is defined by, in spherical coordinates $(x^1, x^2, x^3) = (r, \theta, \phi)$:

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2(\theta) \end{pmatrix}.$$

Answer

Firstly, we compute as the determinant of $g_{ij} = r^4\sin^2(\theta)$. Secondly, since g_{ij} is a real diagonal matrix, we obtain the elements:

$$g^{rr} = \frac{1}{g_{rr}}, \quad g^{\theta\theta} = \frac{1}{g_{\theta\theta}}, \quad g^{\phi\phi} = \frac{1}{g_{\phi\phi}}.$$

The off-diagonal elements are given as zero. Expanding the einstein notation from eq (1), we obtain the following,

$$\begin{aligned}
r &\implies \frac{1}{r^2\sin(\theta)}\left(\partial_r\left(r^2\sin(\theta)\left(\partial_r\right)\right)\right) \\
&= \frac{1}{r^2\sin(\theta)}\left(2r\sin(\theta)\partial_r + r^2\sin(\theta)\partial_r^2\right) = \frac{2}{r}\partial_r + \partial_r^2, \\
\theta &\implies \frac{1}{r^2\sin(\theta)}\left(\partial_\theta\left(r^2\sin(\theta)\left(\frac{1}{r^2}\partial_\theta\right)\right)\right) \\
&= \frac{1}{r^2\sin(\theta)}\left(\cos(\theta)\partial_\theta + \sin(\theta)\partial_\theta^2\right), \\
\phi &\implies \frac{1}{r^2\sin(\theta)}\left(\partial_\phi\left(r^2\sin(\theta)\left(\frac{1}{r^2\sin^2(\theta)}\partial_\phi\right)\right)\right) = \frac{1}{r^2\sin^2(\theta)}\partial_\phi^2.
\end{aligned}$$

Summing over the terms calculated yields:

$$\begin{aligned}
\frac{1}{\sqrt{g}}\partial_i\left(\sqrt{g}g^{ij}\partial_j\right) &= \frac{2}{r}\partial_r + \partial_r^2 + \frac{1}{r^2\sin(\theta)}\left(\cos(\theta)\partial_\theta + \sin(\theta)\partial_\theta^2\right) + \frac{1}{r^2\sin^2(\theta)}\partial_\phi^2 \\
&= \frac{1}{r^2}\partial_r\left(r^2\partial_r\right) + \frac{1}{r^2\sin(\theta)}\partial_\theta\left(\sin(\theta)\partial_\theta\right) + \frac{1}{r^2\sin^2(\theta)}\partial_\phi^2 = \nabla^2.
\end{aligned}$$

And thus we've verified the claim that eq (1) indeed holds for spherical coordinates.