Andreas Evensen

Author : Andreas Evensen

Date: September 16, 2024

Question

We're asked compute the following integral:

$$\int \frac{\bar{\rho} \cdot (\bar{\rho} + \bar{n})}{\rho^3 \left(|\bar{\rho} + \bar{n}|^3 \right)} d^3\rho, \tag{1}$$

and verify that the value is 4π.

Answer

We begin by rewriting our expression

$$\begin{aligned}
\int \frac{\bar{\rho} \cdot (\bar{\rho} + \bar{n})}{\rho^3 \left(|\bar{\rho} + \bar{n}|^3 \right)} d^3\rho &= \int \frac{\bar{\rho}}{\rho^3} \left(-\nabla_\rho \left(\frac{1}{\rho + n} \right) \right) d^3\rho \\
&= \int \left(\nabla_\rho \left(\frac{1}{\rho} \right) \right) \cdot \left(\nabla_\rho \left(\frac{1}{\rho + n} \right) \right) d^3\rho,
\end{aligned}$$

performing partial integration[1], we obtain:

$$\begin{aligned}
\int \left(\nabla_\rho \left(\frac{1}{\rho} \right) \right) \cdot \left(\nabla_\rho \left(\frac{1}{\rho + n} \right) \right) d^3\rho = \int d\rho^3 &\left[\nabla_\rho \cdot \left(\frac{1}{|\bar{\rho} + \bar{n}|} \nabla_\rho \left(\frac{1}{\rho} \right) \right) \right] \\
- \int d\rho^3 &\left[\left(\frac{1}{|\bar{\rho} + \bar{n}|} \right) \nabla_\rho^2 \left(\frac{1}{\rho} \right) \right].
\end{aligned}$$

We now evaluate the two sperate integrals. Using the divergence theorem we obtain the following for the first term:

$$\int d\rho^3 \left[\nabla_\rho \cdot \left(\frac{1}{|\bar{\rho} + \bar{n}|} \nabla_\rho \left(\frac{1}{\rho} \right) \right) \right] = \oint d\bar{a} \left[\frac{1}{|\bar{\rho} + \bar{n}|} \nabla_\rho \left(\frac{1}{\rho} \right) \right] = 0.$$

We now evaluate the second integral. We note that $\bar{n}$ is a unit-vector, and we use this in the last equality.

$$-\int d\rho^3 \left[\left(\frac{1}{|\bar{\rho} + \bar{n}|} \right) \nabla_\rho^2 \left(\frac{1}{\rho} \right) \right] = \lceil \nabla^2 (x^{-1}) = -4\pi\delta(x) \rfloor = \int d\rho^3 \left[\left(\frac{1}{|\bar{\rho} + \bar{n}|} \right) 4\pi\delta(\rho) \right] = 4\pi.$$

This concludes the exercise. We've verified the value of the integral (1) to be 4π.

[1] also known as integration by parts