Exercise 1

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1 Question

We wish to prove the equation of *charge conservation*:

$$\partial_t \rho + \nabla \cdot \mathbf{J} = 0.$$

2 Solution

We can use *Ampère-Maxwells law* to solve this very trivially. The law is being formulated in the following equation:

$$\nabla \times \mathbf{B} - \frac{1}{c^2}\partial_t \mathbf{E} = \mu_0 \mathbf{J}.$$

Isolating the curl of the magnetic field we obtain:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{1}{c^2}\partial_t \mathbf{E}.$$

Taking the divergence of the above expression yields the following:

$$\begin{aligned}\nabla \cdot (\nabla \times \mathbf{B}) &= \nabla \cdot \left(\boldsymbol{\mu_0} \mathbf{J} + \frac{\mathbf{1}}{\mathbf{c^2}}\boldsymbol{\partial_t} \mathbf{E}\right) \\ &= \lceil \frac{1}{c^2} = \mu_0 \epsilon_0 \rfloor = \mu_0 \nabla \cdot (\mathbf{J} + \boldsymbol{\epsilon_0 \partial_t} \mathbf{E}).\end{aligned}$$

We now note that the divergence of a curl is always zero and we divide by μ_0, which gives us:

$$0 = \nabla \cdot \mathbf{J} + \nabla \cdot (\boldsymbol{\epsilon_0 \partial_t} \mathbf{E}).$$

Using Gauss law, we can replace the divergence of the electric field with the charge density and cancel out the ϵ_0 terms, and thus we obtain:

$$\begin{aligned}0 &= \nabla \cdot \mathbf{J} + \partial_t \left(\nabla \cdot \boldsymbol{\epsilon_0} \mathbf{E}\right) \\ 0 &= \nabla \cdot \mathbf{J} + \partial_t \left(\epsilon_0 \frac{1}{\epsilon_0}\rho\right) \\ &= \nabla \cdot \mathbf{J} + \partial_t \rho.\end{aligned}$$

This concludes the proof of *charge conservation.*