Exercise 15

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Question

We're asked to show the following two equations validity:

$$-\nabla\left(\frac{1}{r}\right) = \frac{\mathbf{x}}{r^3}, \tag{1}$$

and

$$r \neq 0 \implies \nabla^2\left(\frac{1}{r}\right) = 0. \tag{2}$$

In showing this, we will use Cartesian coordinates, as it's requested.

Answer

We begin by showing eq (1). We firstly note that $r = \sqrt{x^2+y^2+z^2}$, and using that we rewrite eq (1),

$$\begin{aligned}
-\nabla\left(\frac{1}{r}\right) &= -\nabla\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)\\
&= -\left(\hat{x}\partial_x + \hat{y}\partial_y + \hat{z}\partial_z\right)\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)\\
&= \hat{x}\frac{x}{\left(x^2+y^2+z^2\right)\big)^{\frac{3}{2}}} + \hat{y}\frac{y}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}} + \hat{z}\frac{z}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}\\
&= \frac{\left(\hat{x}x + \hat{y}y + \hat{z}z\right)}{r^3} = \frac{\mathbf{r}}{r^3}.
\end{aligned}$$

In the question, the vector $\mathbf{x}$ is the direction of the solution, but name suggest only a $\hat{x}$ direction, and thus I've renamed it to $\mathbf{r}$ to indicate a direction in space.

In order to show the validity of eq (2) we do the following. Supposing that $r \neq 0$:

$$\begin{aligned}
\nabla^2 \left(\frac{1}{r}\right) &= \left(\partial_x^2 + \partial_y^2 + \partial_z^2\right) \frac{1}{\sqrt{x^2 + y^2 + z^2}} \\
&= -\partial_x \left(x \cdot \left(x^2 + y^2 + z^2\right)^{\frac{-3}{2}} \right) \\
&- \partial_y \left(y \cdot \left(x^2 + y^2 + z^2\right)^{\frac{-3}{2}} \right) \\
&- \partial_z \left(z \cdot \left(x^2 + y^2 + z^2\right)^{\frac{-3}{2}} \right) \\
&= \frac{2x^2 - y^2 - z^2}{r^5} + \frac{2y^2 - x^2 - z^2}{r^5} + \frac{2z^2 - x^2 - y^2}{r^5} = 0.
\end{aligned}$$

And thus we've showed that validity of the two equations (1) and (2).