Exercise 6

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Question

We wish to compute the flux of the vector field,

$$\mathbf{v}(\mathbf{x}) = 3xy\hat{x} + xz^2\hat{y} + y^3\hat{z},$$

through the unit sphere defined by $V \in \{r = 1, \theta \in [0, 2\pi), \phi \in [0, \pi)\}$. In order to compute this, we need to compute the right hand side of the the following equation:

$$\int_V \nabla \cdot \mathbf{B} dV = \oint_{\delta V} \mathbf{B} \cdot d\mathbf{A}. \tag{1}$$

Answer

We wish to calculate the above surface intergral, eq (1). The surface-element (vector) is given by $d\mathbf{A} = \hat{r}\ \sin(\theta) d\theta d\phi$, over the unit sphere. Using spherical coordinates we obtain the vector field $\mathbf{v}$ can be written as:

$$\begin{aligned}
v_x &= 3\sin^2(\theta)\sin(\phi)\cos(\phi),\\
v_y &= \sin(\theta)\cos^2(\theta)\cos(\phi),\\
v_z &= \sin^3(\theta)\sin^3(\phi).
\end{aligned}$$

Evaluating the integrand, $\mathbf{v} \cdot \hat{r}$ yields:

$$\begin{aligned}
\mathbf{v} \cdot \hat{r} = {} & 3\sin^3(\theta)\cos^2(\phi)\sin(\phi)\\
& + \sin^2(\theta)\cos^2(\theta)\cos(\phi)\sin(\phi)\\
& + \sin^3(\phi)\sin^3(\theta)\cos(\theta).
\end{aligned}$$

Evaluating the integral yields:

$$\begin{aligned}
\int_{\partial V} \mathbf{v} \cdot d\mathbf{A} = {} & 3\int_0^{\pi} d\theta \sin^4(\theta) \int_0^{2\pi} d\phi \left(\cos^2(\phi)\sin(\phi)\right)\\
& + \int_0^{\pi} d\theta \sin^3(\theta)\cos^2(\theta) \int_0^{2\pi} d\phi \left(\cos(\phi)\sin(\phi)\right)\\
& + \int_0^{\pi} d\theta \sin^4(\theta)\cos(\theta) \int_0^{2\pi} d\phi \left(\sin^3(\phi)\right)\\
& = 3\int_0^{\pi} d\theta \sin^4(\theta) \cdot 0\\
& + \int_0^{\pi} d\theta \sin^3(\theta)\cos^2(\theta) \cdot 0\\
& + \int_0^{\pi} d\theta \sin^4(\theta)\cos(\theta) \cdot 0.
\end{aligned}$$

Since all terms evalutes to zero, we can conlcude that the flux on the unit sphere given by $\mathbf{v}$ is zero, e.g.

$$\oint_{\partial V} \mathbf{v} \cdot d\mathbf{A} = 0.$$