Exercise 36

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Question

We wish to prove that any arbitrary function f can be expressed as:

$$f(x) = \sum_n c_n u_n(x), \tag{1}$$

where where u_n forms an orthonormal basis. The coefficients c_n are then defined by

$$c_n = \int_a^b u_n^*(x) f(x) dx. \tag{2}$$

Answer

Suppose we have an infinite set of vectors, that form an orhornomal basis, $\{u_n(x)\}_n$ defined on $I \in [a, b]$, then it fullfills the following:

$$\int_a^b u_i^*(x) u_j(x) dx = \delta_{i,j},$$

and

$$\sum_{i=1}^{\infty} u_i(x) u_i^*(y) = \delta(x, y). \tag{3}$$

Using the definition of the coefficient c_n, and the second property we expand eq (1):

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} c_n u_n(x) \\ &= \sum_{n=1}^{\infty} \left(\int_a^b u_n^*(x') f(x') dx' \right) u_n(x) \end{aligned}$$

Rearraging the term, and exploiting commutativity we obtain the following:

$$\begin{aligned} f(x) &= \int_a^b dx' \left(f(x') \sum_{n=1}^{\infty} u_n(x) u_n^*(x') \right) = \left\lceil \sum_{i=1}^{\infty} u_i(x) u_i^*(y) = \delta(x, y) \right\rfloor \\ &= \int_a^b dx' \left[f(x') \delta(x, x') \right] = f(x = x') \end{aligned}$$

And thus we've proved (1)