Exercise 16

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Question

We're asked to prove the following entry equation, fifth entry about delta functions in Jackson,

$$\delta(f(x)) = \sum_i \frac{1}{\left|\frac{df}{dx}(x_i)\right|}\delta(x - x_i).$$

Answer

To prove this, one wants to provide a way of re-writing $\delta(f(x))$. To obtain this one needs to ensure that $f(x)$ then becomes a bijection, and to accomplish this one creates intervals that makes up the entire real line $\mathbb{R}$. Hence $\cup_{i=1}^{\infty}\Big(I_i\Big) = \mathbb{R}$ where $I_i = (k_i, j_i)$. From this point, one can now restrict $k_i < j_i$ and that at least some of the intervals I_i contains zero. Thus the following is true,

$$\int_{-\infty}^{\infty}\phi(x)\delta(f(x))dx = \sum_{i=1}^{\infty}\int_{I_i}\phi(x)\delta(f(x))dx; x \in I_i$$

Since one now has ensured bijection on the strips I_i, the inverse function $f^{-1}(t) = x$ exists and hence the following can be applied,

$$f(x) = t, \quad f^{-1}(t) = x \implies dx = \frac{dt}{\left|f'(x)\right|}$$

$$\text{Hence} \quad \int_{-\infty}^{\infty}\phi(x)\delta(f(x))dx = \sum_{i=1}^{\infty}\int_{I_i}\frac{\phi(f^{-1}(t))}{\left|f'(f^{-1}(t))\right|}\delta(t)dt = \sum_{\{x_i\}}\frac{\phi(x_i)}{\left|f'(x_i)\right|}.$$

Lastly, we then realize that $\phi(x_i)$ can be expressed as a dirac delta, e.g. we obtain the requsted form:

$$\delta(f(x)) = \sum_i \frac{1}{\left|f'(x_i)\right|}\delta(x - x_i).$$