Exercise 61

Author : Andreas Evensen

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Question

Given the vector potential,

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{x}}{|\mathbf{x}|^3},$$

we want to write the magnetic field **B** as

$$\mathbf{B} = \frac{\mu_0}{4\pi}\frac{\mathbf{3xm}\cdot\mathbf{x} - r^2\mathbf{m}}{r^5}$$

Answer

We seek to do this with Einstien notation, and thus we first write write **A** in this notation,

$$A_i(x) = \frac{\mu_0}{4\pi|x|^3}\epsilon_{ijk}m_jx_k.$$

To find **B** we simply compute the the crossproduct, e.g. $\epsilon_{ijk}\partial_jA_k$,

$$\begin{aligned}
B_i = \epsilon_{ijk}\partial_jA_k &= \frac{\mu_0}{4\pi}\epsilon_{ijk}\partial_j\left(\epsilon_{kmn}\frac{m_mx_n}{|x|^3}\right)\\
&= \frac{\mu_0}{4\pi}\epsilon_{ijk}\epsilon_{kmn}\partial_j\left(\frac{m_mx_n}{|x^3|}\right)\\
&= \frac{\mu_0}{4\pi}\left(\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}\right)\partial_j\left(\frac{m_mx_n}{|x^3|}\right)\\
&= \frac{\mu_0}{4\pi}\left(\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}\right)\left(\delta_{jn}\frac{m_m}{|x|^3} - 3\frac{m_mx_nx_j}{|x|^5}\right)\\
&= \frac{\mu_0}{4\pi}\left(2\frac{m_i}{|x|^3} - 3\frac{m_ix_jx_j}{|x|^5} + 3\frac{m_jx_ix_j}{|x|^5}\right)\\
&= \frac{\mu_0}{4\pi}\left(\frac{3m_jx_jx_i - m_ix_jx_j}{|x|^5}\right)
\end{aligned}$$

And thus we've used index notation in order to obtain the magnetic field **B** from the vector potential **A**.