Exercise 40

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Date: September 26, 2024

Question

We with to apply the laplace operator,

$$\frac{1}{\sqrt{g}}\partial_i\left(\sqrt{g}g^{ij}\partial_j\right),$$

to the two functions $f(r) = r^n$, and $g(r) = r^{-n-1}$, and show that they behave in the same way.

Answer

Firstly, we compute as the determinant of $g_{ij} = r^4\sin^2(\theta)$. Secondly, since g_{ij} is a real diagonal matrix, we obtain the elements:

$$g^{rr} = \frac{1}{g_{rr}}, \quad g^{\theta\theta} = \frac{1}{g_{\theta\theta}}, \quad g^{\phi\phi} = \frac{1}{g_{\phi\phi}}.$$

The off-diagonal elements are given as zero. Since both functions f and g and only dependent on r, we obtain:

$$\begin{aligned}
\nabla^2 f(r) &= \frac{1}{r^2\sin(\theta)}\partial_r(r^2\sin(\theta)\partial_r f) = \frac{1}{r^2}\partial_r\left(r^2 n r^{n-1}\right) \\
&= \frac{n}{r^2}\left(2r^n + r^2(n-1)r^{n-2}\right) = n\left(2r^{n-2} + (n-1)r^{n-2}\right) \\
&= \left(n^2+n\right)r^{n-2} = \left(n^2+n\right)\frac{f(r)}{r^2} \\
\nabla^2 g(r) &= \frac{1}{r^2\sin(\theta)}\partial_r(r^2\sin(\theta)\partial_r g) = \frac{1}{r^2}\partial_r\left(r^2(-n-1)r^{-n-2}\right) \\
&= \frac{(-n-1)}{r^2}\left(2r^{-n-1} + (-n-2)r^{n-1}\right) \\
&= \left(n^2+n\right)r^{-n-3} = \left(n^2+n\right)\frac{g(r)}{r^2}
\end{aligned}$$

The expressions behave in the same mannor. The same prefactor is located in the two expressions, and one obtains the original factor divided by the same factor.