Exercise 38

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Question

We wish to prove that the Laplace equation, in cylindrical coordinates, can be expressed as:

$$\nabla^2\Phi = \left(\partial_x^2 + \partial_y^2 + \partial_z^2\right)\Phi = \frac{1}{\rho}\partial_\rho\left(\rho\partial_\rho\Phi\right) + \frac{1}{\rho^2}\partial_\theta^2\Phi + \partial_z^2\Phi = 0. \tag{1}$$

Answer

We firstly begin by defining ρ and θ as:

$$\rho = \sqrt{x^2+y^2}, \quad \tan(\theta) = \frac{y}{x}.$$

Then $x = \rho\cos(\theta)$, and $y = \rho\sin(\theta)$. We're asked to use the chain rule in order to confirm eq (1), and thus we begin by computing the first derivatives.

$$\begin{aligned}
\partial_x &= \left(\partial_x(\rho)\partial_\rho + \partial_x(\theta)\partial_\theta\right)\\
&= \left(\cos(\theta)\partial_\rho - \frac{\sin(\theta)}{\rho}\partial_\theta\right)\\
\partial_y &= \left(\partial_y(\rho)\partial_\rho + \partial_y(\theta)\partial_\theta\right)\\
&= \left(\sin(\theta)\partial_\rho + \frac{\cos(\theta)}{\rho}\partial_\theta\right)\\
\partial_z &= \partial_z.
\end{aligned}$$

Applying the derivative once again yeilds:

$$\begin{aligned}
\partial_x^2 &= \left(\cos(\theta)\partial_\rho - \frac{\sin(\theta)}{\rho}\partial_\theta\right)\left(\cos(\theta)\partial_\rho - \frac{\sin(\theta)}{\rho}\partial_\theta\right)\\
&= \cos^2(\theta)\partial_\rho^2\\
\partial_y^2 &= \left(\sin(\theta)\partial_\rho + \frac{\cos(\theta)}{\rho}\partial_\theta\right)\left(\sin(\theta)\partial_\rho + \frac{\cos(\theta)}{\rho}\partial_\theta\right)\\
\partial_z^2 &= \partial_z^2.
\end{aligned}$$

From the above, it's visible that the outer cross-term vanishes in the sum of $\partial_x^2 + \partial_y^2$, and thus we're left with:

$$\begin{aligned}
\partial_x^2 + \partial_y^2 + \partial_z^2 = \cos^2(\theta)\partial_\rho^2 &- \frac{\sin(\theta)}{\rho}\partial_\theta\left(\cos(\theta)\partial_\rho\right) + \frac{\sin(\theta)}{\rho^2}\partial_\theta\left(\sin(\theta)\partial_\theta\right)\\
&+ \sin^2(\theta)\partial_\rho^2 + \frac{\cos(\theta)}{\rho}\partial_\theta\left(\sin(\theta)\partial_\rho\right) + \frac{\cos(\theta)}{\rho^2}\partial_\theta\left(\cos(\theta)\partial_\theta\right) + \partial_z^2\\
&= \partial_\rho^2 + \frac{1}{\rho}\partial_\rho + \frac{1}{\rho^2}\partial_\theta^2 + \partial_z^2.
\end{aligned}$$

We now recognize that the expression $\partial_\rho^2 + \rho^{-1}\partial_\rho$ is the definition of the derivative $\partial_\rho(\rho\partial_\rho)$, and thus we can write the above as:

$$\nabla^2 = \partial_x^2 + \partial_y^2 + \partial_z^2 = \frac{1}{\rho}\partial_\rho\left(\rho\partial_\rho\right) + \frac{1}{\rho^2}\partial_\theta^2 + \partial_z^2.$$

This concludes the proof of Laplace equation in cylindrical coordinates using the chain rule.