Exercise 5

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Question

We're asked to compute the following:

$$\int_C \nabla f \cdot d\mathbf{l} = f(\mathbf{x}_2) - f(\mathbf{x_1})$$

where f and C are defined as follows:

$$\begin{aligned} f(x, y, z) &= x^2 y e^z, \\ C \in \sigma(t) &= (\sin(t), \cos(t), t);\ t \in [0, 2\pi] \end{aligned}$$

Answer

Firstly, we need to compute the integral, and use the result to ensure that the identity is correct. In order to do that, we compute the gradient of f,

$$\nabla f = (2xye^z)\,\hat{x} + \left(x^2 e^z\right)\hat{y} + \left(x^2 y e^z\right)\hat{z}.$$

Using the parametrization we obtain the following integral:

$$\begin{aligned} \int_C \nabla f(\mathbf{x}) d\mathbf{l} &= \int_0^{2\pi} \nabla f(\sigma(t)) \cdot \dot{\sigma}(t) dt \\ &= \int_0^{2\pi} dt \left(\left(2\sin(t)\cos(t)e^t)\hat{x} + \left(\sin^2(t)e^t\right)\hat{y} + \left(\sin^2(t)\cos(t)e^t\right)\hat{z} \right) \cdot \left(\cos(t)\hat{x} - \sin(t)\hat{y} + \hat{z}\right)\right) \\ &= \int_0^{2\pi} dt \left(2\cos^2(t)\sin(t)e^t\cos(t) - \sin^3(t)e^t + \sin^2(t)\cos(t)e^t\right) \\ &= \int_0^{2\pi} dt \left(\frac{d}{dt}\left[\cos(t)\sin^2(t)e^t\right]\right) \end{aligned}$$

Since the integrand evaluate at the points 0 and 2π are zero, so does the entire integral. We now evaluate

$$\begin{aligned} f(\mathbf{x}_2) - f(\mathbf{x}_1) &= f(\sigma(2\pi)) - f(\sigma(0)) \\ &= \cos(2\pi)\sin^2(2\pi)e^{2\pi} - \cos(0)\sin^2(0)e^0 = 0. \end{aligned}$$

The right hand side of the integral evalutes to the integral and thus we have proved the identity by case.