Exercise 26

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Question

We're asked to show the following statement:

$$\partial_t \omega = \frac{1}{\mu_0} \nabla \cdot (\mathbf{B} \times \mathbf{E}) - \mathbf{E} \cdot \mathbf{J}. \tag{1}$$

Answer

Firstly, we'll use eq (104) in the compendium, stateing the following:

$$\omega(\mathbf{x}, t) = \frac{1}{2} \left(\epsilon_0 \mathbf{E} \cdot \mathbf{E} + \frac{1}{\mu_0} \mathbf{B} \cdot \mathbf{B} \right).$$

Performing the derivative yields the following:

$$\begin{aligned}
\partial_t \omega(\mathbf{x}, t) &= \partial_t \left[\frac{1}{2} \left(\epsilon_0 \mathbf{E} \cdot \mathbf{E} + \frac{1}{\mu_0} \mathbf{B} \cdot \mathbf{B} \right) \right] \\
&= \frac{1}{2} \left[\epsilon_0 \partial_t \left(\mathbf{E} \cdot \mathbf{E} \right) + \frac{1}{\mu_0} \partial_t \left(\mathbf{B} \cdot \mathbf{B} \right) \right] \\
&= \frac{1}{2} \left[\epsilon_0 \left(2\mathbf{E} \cdot \partial_t \mathbf{E} \right) + \frac{1}{\mu_0} \left(2\mathbf{B} \cdot \partial_t \mathbf{B} \right) \right]
\end{aligned}$$

Using *Faraday's law* and *Ampère-Maxwell's law* we obtain:

$$\begin{aligned}
\partial_t \omega(\mathbf{x}, t) &= \epsilon_0 c^2 \big(\mathbf{E} \cdot (\nabla \times \mathbf{B} - \mu_0 \mathbf{J}) \big) - \frac{1}{\mu_0} \big(\mathbf{B} \cdot (\nabla \times \mathbf{E}) \big) \\
&= \frac{1}{\mu_0} \big(\mathbf{E} \cdot (\nabla \times \mathbf{B}) - \mathbf{B} \cdot (\nabla \times \mathbf{E}) \big) - \mathbf{E} \cdot \mathbf{J} \\
&= \frac{1}{\mu_0} \big((\nabla \times \mathbf{B}) \cdot \mathbf{E} - \mathbf{B} \cdot (\nabla \times \mathbf{E}) \big) - \mathbf{E} \cdot \mathbf{J}
\end{aligned}$$

Using the identity $\nabla \cdot \mathbf{A} \times \mathbf{E} = (\nabla \times \mathbf{A}) \cdot \mathbf{B} - \mathbf{A} \cdot (\nabla \times \mathbf{B})$, we obtain:

$$\partial_t \omega(\mathbf{x}, t) = \frac{1}{\mu_0} \nabla \cdot (\mathbf{B} \times \mathbf{E}) - \mathbf{E} \cdot \mathbf{J}.$$

This concludes the equality.