Exercise 57

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Question

We wish to show the following:

$$\int \mathbf{n}\cos(\gamma)d\Omega = \frac{4\pi}{3}\frac{\mathbf{x}'}{r'}. \tag{1}$$

Answer

We remember the following:

$$\begin{aligned}
\mathbf{n} &= \sin(\theta)\cos(\phi)\hat{x} + \sin(\theta)\sin(\phi)\hat{y} + \cos(\theta)\hat{z},\\
\cos(\gamma) &= \mathbf{n}\cdot\mathbf{n}' = \cos(\theta)\cos(\theta') + \sin(\theta)\sin(\theta')\cos(\phi-\phi'),\\
d\Omega &= d\theta d\phi\sin(\theta).
\end{aligned}$$

$$\begin{aligned}
\int \mathbf{n}\cos(\gamma)d\omega &= \underbrace{\hat{x}\int d\Omega\left(\sin(\theta)\cos(\phi)\cos(\gamma)\right)}_{=I_x}\\
&+ \underbrace{\hat{y}\int d\Omega\left(\sin(\theta)\sin(\phi)\cos(\gamma)\right)}_{=I_y}\\
&+ \underbrace{\hat{z}\int d\Omega\left(\cos(\theta)\cos(\gamma)\right)}_{=I_z}
\end{aligned}$$

Evaluating the three integrals, I_x, I_y and I_z is done via trigonometry

$$\begin{aligned}
I_x &= \cos(\theta')\underbrace{\int_0^{2\pi} d\phi(\cos(\phi))}_{=0}\int_0^{\pi} d\theta\left(\sin^2(\theta)\cos(\theta)\right)\\
&+ \sin(\theta')\int_0^{2\pi} d\phi(\cos(\phi-\phi')\cos(\phi))\int_0^{\pi} d\theta\left(\sin^3(\theta)\right)\\
&= \frac{4}{3}\sin(\theta')\int_0^{2\pi} d\phi\left(\cos(\phi)\left[\cos(\phi)\cos(\phi') + \sin(\phi)\sin(\phi')\right]\right)\\
&= \frac{4}{3}\sin(\theta')\left[\cos(\phi')\int_0^{2\pi}\cos^2(\phi) + \sin(\phi')\int_0^{2\pi}\cos(\phi)\sin(\phi)\right]\\
&= \frac{4}{3}\sin(\theta')\left(\cos(\phi')\pi + 0\right)
\end{aligned}$$

$$
\begin{aligned}
I_y &= \cos(\theta') \underbrace{\int_0^{2\pi} d\phi (\sin(\phi))}_{=0} \int_0^{\pi} d\theta \left(\sin^2(\theta)\cos(\theta)\right) \\
&+ \sin(\theta') \int_0^{2\pi} d\phi \cos(\phi - \phi') \sin(\phi) \int_0^{\pi} d\theta \sin^3(\theta) \\
&= \sin(\theta') \int_0^{2\pi} d\phi \left(\sin(\phi)\cos(\phi)\cos(\phi') + \sin(\phi)\sin(\phi')\right) \int_0^{\pi} d\theta \sin^3(\theta) \\
&= \frac{4}{3} \sin(\theta') \left[\cos(\phi') \int_0^{2\pi} d\phi \left(\sin(\phi)\cos(\phi)\right) + \sin(\phi') \int_0^{2\pi} d\phi \left(\sin^2(\phi)\right)\right] \\
&= \frac{4}{3} \sin(\theta') \left[0 + \pi \sin(\phi')\right] \\
I_z &= \cos(\theta') \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \left(\sin(\theta)\cos^2(\theta)\right) \\
&+ \sin(\theta') \int_0^{2\pi} d\phi \left(\cos(\phi - \phi')\right) \int_0^{\pi} d\theta \left(\sin^2(\theta)\cos(\theta)\right) \\
&= \cos(\theta') \left[2\pi \frac{2}{3} + 0\right]
\end{aligned}
$$

Adding the three expressions, and taking into account the direction, we're left with

$$
\begin{aligned}
I_x\hat{x} + I_y\hat{y} + I_z\hat{z} &= \frac{4\pi}{3} \left[\sin(\theta')\cos(\phi')\hat{x} + \sin(\theta')\sin(\phi')\hat{y} + \cos(\theta')\hat{z}\right] \\
&= \frac{4\pi}{3}\mathbf{n}' = \frac{4\pi}{3} \frac{\mathbf{x}'}{|\mathbf{x}'|}.
\end{aligned}
$$

And thus, we've confirmed the expression.