Exercise 72

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Question

We wish to show that $V_\beta T^{\beta\alpha}$ is a contravarient vector. And what is different when having $\partial_\beta T^{\beta\alpha}$?

Answer

Firstly, we postulate that the transform of $V_\beta(x) = V'_\beta(x')$, and the same goes for $T^{\beta\alpha}(x) = T'^{\beta\alpha}(x')$. Therefore, we apply the transforms, and analyze the result.

$$\begin{aligned}
V'_\beta(x') &= (\Lambda^{-1})^\gamma_\beta V_\gamma(x) \\
T'^{\beta\alpha}(x') &= \Lambda^\beta_\mu \Lambda^\alpha_\nu T^{\mu\nu}(x) \\
V'_\beta T'^{\beta\alpha} &= (\Lambda^{-1})^\gamma_\beta V_\gamma(x) \Lambda^\beta_\mu \Lambda^\alpha_\nu T^{\mu\nu}(x) \\
&= \left(\Lambda^{-1}\Lambda\right)^\gamma_\mu V_\gamma(x) \Lambda^\alpha_\nu T^{\mu\nu}(x) \\
&= \Lambda^\alpha_\nu V_\beta(x) T^{\beta\nu}(x)
\end{aligned}$$

At this point, we see that $\Lambda^\alpha_\nu V_\beta(x) T^{\beta\nu}(x) = \left(V_\beta(x) T^{\beta\mu}(x)\right)'$, and thus it behaves like a contravarient vector, implying that it is a contravarient vector. If we do the same for $\partial_\beta T^{\beta\alpha}$ we instead obtain,

$$\begin{aligned}
\partial'_\beta &= (\Lambda^{-1})^\gamma_\beta \partial_\gamma, \quad \partial_\gamma \equiv \frac{\partial}{\partial x^\gamma}, \\
T'^{\beta\alpha}(x') &= \Lambda^\beta_\mu \Lambda^\alpha_\nu T^{\mu\nu}(x), \\
\partial'_\beta T'^{\beta\alpha} &= \left(\Lambda^{-1}\right)^\gamma_\beta \partial_\gamma \left(\Lambda^\beta_\mu \Lambda^\alpha_\nu T^{\mu\nu}\right) \\
&= \left(\Lambda^{-1}\right)^\gamma_\beta \left[\partial_\gamma \left(\Lambda^\beta_\mu \Lambda^\alpha_\nu\right) T^{\mu\nu} + \Lambda^\beta_\mu \Lambda^\alpha_\nu \partial_\gamma \left(T^{\mu\nu}\right)\right] \\
&= \left(\Lambda^{-1}\right)^\gamma_\beta \partial_\gamma \left(\Lambda^\beta_\mu \Lambda^\alpha_\nu\right) T^{\mu\nu} + \Lambda^\alpha_\nu \partial_\gamma \left(T^{\gamma\nu}\right).
\end{aligned}$$

The second term behaves like a contravarient transform, however the first term is not as simple, and thus it does not behave like the ordinary tensor transform.