Exercise 12

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Date: October 28, 2024

Question

We wish to derive:

$$\nabla^2\mathbf{E} - \frac{1}{c^2}\partial_t^2\mathbf{E} = 0,$$

using Maxwells equations:

$$\begin{aligned}
&\nabla \cdot \mathbf{B} = 0,\\
&\nabla \times \mathbf{E} + \partial_t\mathbf{B} = 0,\\
&\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\varrho,\\
&\nabla \times \mathbf{B} - \frac{1}{c^2}\partial_t\mathbf{E} = \mu_0\mathbf{J},
\end{aligned}$$

Answer

In this setting, $\varrho = 0$ and $\mathbf{J} = 0$. Using this, we compute the following:

$$\begin{aligned}
\nabla \times \mathbf{E} + \partial_t\mathbf{B} &= 0\\
\nabla \times \mathbf{E} &= -\partial_t\mathbf{B}\\
\implies \nabla \times \nabla \times \mathbf{E} &= -\nabla \times \partial_t\mathbf{B}\\
-\nabla^2\mathbf{E} + \nabla(\nabla \cdot \mathbf{E}) &= -\partial_t\nabla \times \mathbf{B}\\
&= \lceil \nabla \cdot \mathbf{E} = 0 \rfloor\\
\nabla^2\mathbf{E} &= \partial_t\nabla \times \mathbf{B}
\end{aligned}$$

Since $\mathbf{J} = 0$ in this setting, one can rewrite Maxwells-Ampéres law in the following way,

$$\nabla \times \mathbf{B} = \frac{1}{c^2}\partial_t\mathbf{E},$$

subsituting this in the above expression, one obtains:

$$\begin{aligned}
\nabla^2\mathbf{E} &= \partial_t\nabla \times \mathbf{B}\\
&= \frac{1}{c^2}\partial_t^2\mathbf{E}\\
\implies \nabla^2\mathbf{E} &- \frac{1}{c^2}\partial_t^2\mathbf{E} = 0.
\end{aligned}$$

Thus, we've showed the desired result.