Exercise 18

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Question

We wish to prove that every analytic function is a solution *Laplace equation*,

$$\nabla^2 f = 0.$$

Answer

We have that $\Theta(x, y) = \Theta(x + iy) = \Theta(z)$, by the Cauchy-Riemann equations, we can write the first order partial derivatives as the following, which they must follow in order to be analytic:

$$\begin{aligned} \partial_x u &= \partial_y v \\ \partial_y u &= -\partial_x v, \end{aligned}$$

where $u = \mathcal{R}(\Theta)$ and $v = \mathcal{I}(\Theta)$. Calculating the second order derivatives yields:

$$\begin{aligned} \frac{1}{\partial x}\left(\frac{\partial u}{\partial x}\right) &= \frac{\partial^2 v}{\partial xy}, \\ \frac{1}{\partial y}\left(\frac{\partial u}{\partial y}\right) &= -\frac{\partial^2 v}{\partial xy}, \\ \frac{1}{\partial x}\left(\frac{\partial v}{\partial x}\right) &= -\frac{\partial^2 u}{\partial xy}, \\ \frac{1}{\partial y}\left(\frac{\partial v}{\partial y}\right) &= \frac{\partial^2 u}{\partial xy}. \end{aligned}$$

Thus, one has for any arbitrary analytic function f:

$$\nabla^2 f = \frac{\partial^2 v}{\partial xy} - \frac{\partial^2 v}{\partial xy} - \frac{\partial^2 u}{\partial xy} + \frac{\partial^2 u}{\partial xy} = 0.$$