Exercise 7

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Question

One wish to prove the following identity,

$$\int_V (\nabla T) dV = \oint_{\delta V} T\hat{n} dA,$$

for any arbitrary function T that is suitable.

Answer

Let $\mathbf{v} = \mathbf{c}T$, where $\mathbf{c}$ is an arbitrary constant vector-field. Using product rules, we obtain that the divergence of $\mathbf{v}$ can be expressed as follows:

$$\nabla \cdot \mathbf{v} = T(\nabla \cdot \mathbf{c}) + \mathbf{c} \cdot \nabla T = \mathbf{c} \cdot \nabla T$$

Applying Gauss theorem to our constructed vector field $\mathbf{v}$ yields:

$$\begin{aligned}
&\int_V (\nabla \cdot \mathbf{v}) dV = \int_V \Big[\mathbf{c} \cdot \nabla T\Big] dV = \mathbf{c} \cdot \int_V \nabla T dV = \oint_{\delta V} \mathbf{v} \cdot \hat{n} dA \\
&\mathbf{c} \cdot \int_V \nabla T dV = \mathbf{c} \cdot \oint_{\delta V} T\hat{n} dA
\end{aligned}$$

Since $\mathbf{c}$ was a constructed constant vector, we can find it's norm, e.g. $||\mathbf{c}||$, and divide by it

$$\begin{aligned}
\frac{\mathbf{c}}{||\mathbf{c}||} \cdot \int_V \nabla T dV &= \frac{\mathbf{c}}{||\mathbf{c}||} \cdot \oint_{\delta V} T\hat{n} dA. \\
\hat{c} \cdot \int_V \nabla T dV &= \hat{c} \cdot \oint_{\delta V} T\hat{n} dA.
\end{aligned}$$

Since $\hat{c}$ has arbitrary direction, the above most hold for each component of $\hat{c}$. And thus we obtain:

$$\int_V (\nabla T) dV = \oint_{\delta V} T\hat{n} dA.$$

This concludes the proof.