CS 577 - Graphs

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WISCONSIN
UNIVERSITY OF WISCONSIN–MADISON

Graphs

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- Directed Acyclic Graph (DAG)
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- Forests

Trees

Definition

- A connected graph without cycles.
- A single node may be designated as the *root* of the tree.
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Properties of a tree T

1. If $|V| \geq 2$, (unrooted) T has at least 2 leaves.
2. For all nodes u and v, there exists one path between them in T.
3. $|V| = |E| + 1$ for $|V| \geq 1$.

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TopHat 1

Is P_{10} a tree?

What can be represented by graphs?

- Transportation networks
- Communication networks
- Information networks
- Social networks
- Dependency networks

Connectivity

Graph Connectivity

Problem: s-t connectivity

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Connected Components

Let $H \subset G$ be a subgraphg of G. If H is connected and there are no edges between H and $G \setminus H$. Then, H is a connected component of G.

Graph Exploration/Traversal

Determining *s*-*t* Connectivity

Requires an algorithm that explores or traverses the graph by considering the edges of the graph to find all nodes connected to s.

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Graph Encodings and Implementation

Representations

- **Adjacency matrix**: $|V|$ by $|V|$ matrix with a 1 if nodes are adjacent.
- **Adjacency list**: For each node, list adjacent nodes.
- **Edge list**: List of all node pairs representing the edges (plus list of nodes).
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TopHat 2

Which graph representation would be best suited?

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- At step i: $O(|E_i| \cdot (\log |R_i| + \log |R_i|) + \log |R_i|)$, assuming R is a self-balancing BST.

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What is this algorithm lacking?

Breadth-First Search (BFS)

Process

- Also referred to as graph flooding.
- Let L_i be all the neighbours at a distance i from s.
- Starting from $i = 0$, visit all the nodes (not previously visited) in L_i. Increment i and repeat.

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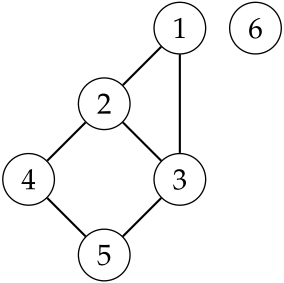
TopHat 3: This process engenders a BFS tree. Start at 1 and draw such a tree for the following.



Depth-First Search (DFS)

Recursive Process starting at s

- Mark s as visited.
- For each $(s, u) \in E$ where u has not been visited, do DFS(u).

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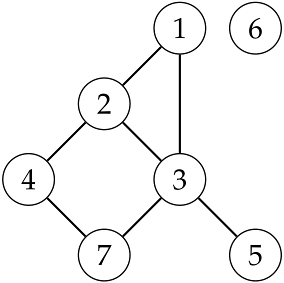

Implementing BFS and DFS

TopHat 5

Which graph representation would be best for BFS and DFS?

IMPLEMENTING BFS AND DFS

TopHat 5

Which graph representation would be best for BFS and DFS? Why?

Implementing BFS and DFS

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Implementing BFS and DFS

Algorithm: BFS(S)

```
Initialize v[u] = false for all nodes
Set v[s] = true
Add s to tree T
Add s to queue Q
while Q is not empty do
    u = dequeue(Q)
    foreach neighbour r of u do
        if !v[r] then
            Add (u, r) to T
            Set v[r] = true
            Enqueue v
        end
    end
end
return T
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Algorithm: DFS(S)

Initialize $v[u]$ = false and $p[u]$ = null for all nodes
Push s to stack S
while *S is not empty* **do**
    $u = \text{pop}(S)$
    if $!v[u]$ **then**
        Add $(p[u], u)$ to T
        Set $v[u]$ = true
        foreach *neighbour r of u* **do**
            Push r to stack S
            Set $p[r] = u$
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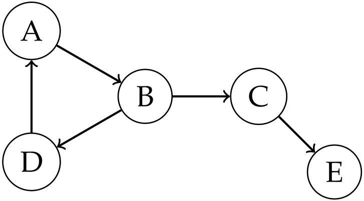
Runtime: $O(|E| + |V|)$

Strongly Connected Components

Directed Graphs

Directed Graph

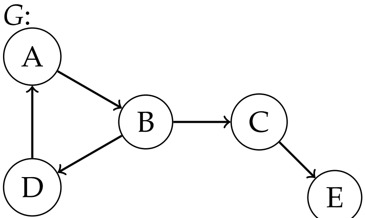
- In a directed graph, the edges have a direction and are often called *arcs*.
- I.e. (u, v) is different than (v, u).



Strong Connectivity

Mutually Reachable

- A pair of nodes (u, v) in a directed graph are *mutually reachable* if there is a path from u to v, and from v to u.
- Note: This property is transitive: if (u, v) and (v, w) are both mutually reachable, then u, w is mutually reachable.



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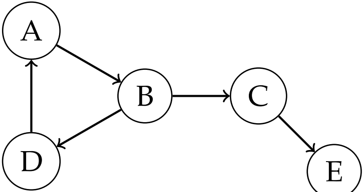
Strongly Connected

A directed graph is *strongly connected* if, for every pair of nodes (u, v), u and v are mutually reachable.

G:



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Testing for Mutually Reachable

How might we check if (u, v) is mutually reachable?

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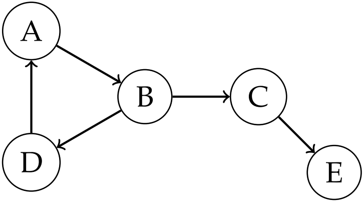

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Check if DFS/BFS from u reach v, and DFS/BFS from v reaches u.

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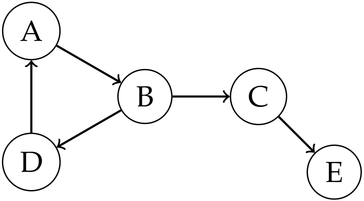

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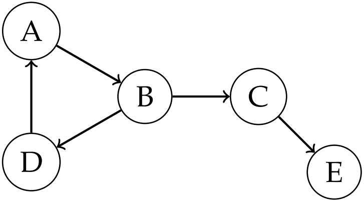

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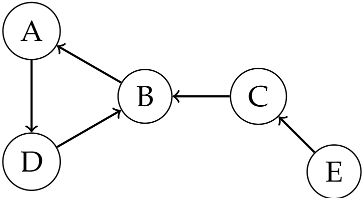

Strongly Connected Components

Strongly Connected Component (SCC)

A maximal strongly connected subgraph.

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TopHat 6: How many SCC in G?

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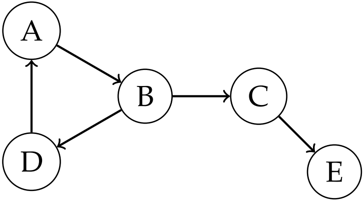

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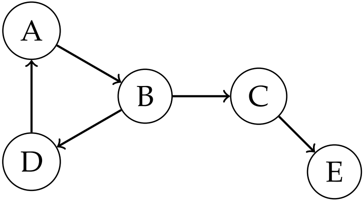

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Problem

Find the SCCs in a digraph G.

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Kosaraju's Algorithm

1. Populate a stack S with a DFS on G.
2. Build G^{REV} for G, and set all nodes to unvisited.
3. While S is not empty:
    1. Pop node v from S.
    2. If v is unvisited, run DFS on G^{REV} from v to extract an SCC.

Strongly Connected Components

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TopHat 7: What is the time complexity of Kosaraju's Algorithm?

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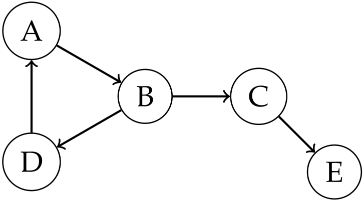
Topological Ordering

Directed Graphs

Directed Graph

- In a directed graph, the edges have a direction and are often called *arcs*.
- I.e. (u, v) is different than (v, u).



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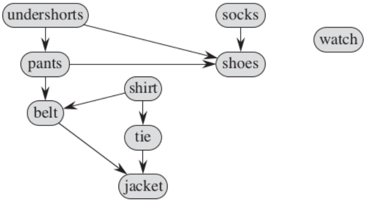
Directed Acyclic Graph (DAG)

- A directed graph with no directed cycles.
- Precedence relationships.

Getting dressed:



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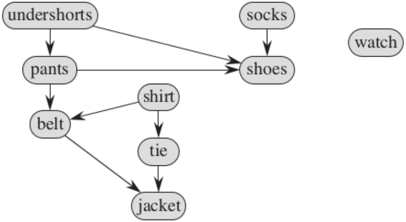

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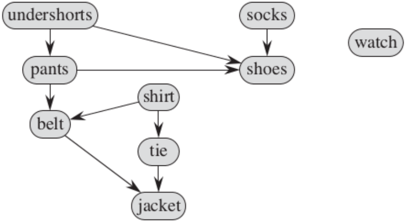

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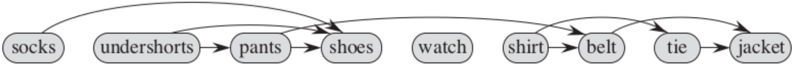

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- After visiting $|V| + 1$ nodes, by the Pigeon Hole principle, we have visited some node w twice $\implies$ G contains a cycle.

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- The Key Property allows us to show that all DAGs have a topological ordering.
- Prove it by induction.
- Does the inductive proof imply an algorithm to build a topological ordering from a DAG? If so, what is it?

Appendix

References

Image Sources I

WISCONSIN UNIVERSITY OF WISCONSIN–MADISON https://brand.wisc.edu/web/logos/