CS 577 - Greedy

Marc Renault

Department of Computer Sciences
University of Wisconsin – Madison

Spring 2022
TopHat Join Code: 997116

WISCONSIN
UNIVERSITY OF WISCONSIN–MADISON

Greedy

Greedy Algorithms

What is a Greedy Algorithm (greedy)?

- Typically, thought of as a *heuristic* that is locally optimal.

Greedy Algorithms

What is a Greedy Algorithm (greedy)?

- Typically, thought of as a *heuristic* that is locally optimal.
- Is greedy always the best?

GREEDY ALGORITHMS

What is a Greedy Algorithm (GREEDY)?

- Typically, thought of as a *heuristic* that is locally optimal.
- Is GREEDY always the best? No, but a good place to start.

GREEDY ALGORITHMS

What is a Greedy Algorithm (GREEDY)?

- Typically, thought of as a *heuristic* that is locally optimal.
- Is GREEDY always the best? No, but a good place to start.
- This notion has yet to be fully formalized, and it often problem specific.

Greedy Algorithms

What is a Greedy Algorithm (greedy)?

- Typically, thought of as a *heuristic* that is locally optimal.
- Is greedy always the best? No, but a good place to start.
- This notion has yet to be fully formalized, and it often problem specific.

Definition from Priority Algorithms

A greedy algorithm is an algorithm that processes the input in a specified order. For each request in the input, the greedy algorithm processes it so as to minimize (resp. maximize) the objective, assuming that the request is the last request.

Greedy Algorithms

What is a Greedy Algorithm (greedy)?

- Typically, thought of as a *heuristic* that is locally optimal.
- Is greedy always the best? No, but a good place to start.
- This notion has yet to be fully formalized, and it often problem specific.

Definition from Priority Algorithms

A greedy algorithm is an algorithm that processes the input in a specified order. For each request in the input, the greedy algorithm processes it so as to minimize (resp. maximize) the objective, assuming that the request is the last request.

For a given problem, there may be many greedy algorithms.

Is greedy Optimal?

Not always: Bin Packing Problem

- Bins of size 1, and requests of size $(0, 1]$.

Is greedy Optimal?

Not always: Bin Packing Problem

- Bins of size 1, and requests of size $(0, 1]$.
- Objective: Pack the items in the minimum number of bins.

IS GREEDY OPTIMAL?

Not always: Bin Packing Problem

- Bins of size 1, and requests of size $(0, 1]$.
- Objective: Pack the items in the minimum number of bins.
- Greedy heuristic: FIRST FIT INCREASING (FFI)

Is greedy Optimal?

Not always: Bin Packing Problem

- Bins of size 1, and requests of size $(0, 1]$.
- Objective: Pack the items in the minimum number of bins.
- Greedy heuristic: First Fit Increasing (ffi)

Non-optimal example:

IS GREEDY OPTIMAL?

Not always: Bin Packing Problem

- Bins of size 1, and requests of size $(0, 1]$.
- Objective: Pack the items in the minimum number of bins.
- Greedy heuristic: FIRST FIT INCREASING (FFI)

Non-optimal example:

- $\sigma = \langle 1/2 - \varepsilon, 1/2 - \varepsilon, 1/2 + \varepsilon, 1/2 + \varepsilon \rangle$

Is greedy Optimal?

Not always: Bin Packing Problem

- Bins of size 1, and requests of size $(0, 1]$.
- Objective: Pack the items in the minimum number of bins.
- Greedy heuristic: First Fit Increasing (ffi)

Non-optimal example:

- $\sigma = \langle 1/2 - \varepsilon, 1/2 - \varepsilon, 1/2 + \varepsilon, 1/2 + \varepsilon \rangle$
- ffi:

Is greedy Optimal?

Not always: Bin Packing Problem

- Bins of size 1, and requests of size $(0, 1]$.
- Objective: Pack the items in the minimum number of bins.
- Greedy heuristic: First Fit Increasing (ffi)

Non-optimal example:

- $\sigma = \langle 1/2 - \varepsilon, 1/2 - \varepsilon, 1/2 + \varepsilon, 1/2 + \varepsilon \rangle$
- ffi: 3 bins

IS GREEDY OPTIMAL?

Not always: Bin Packing Problem

- Bins of size 1, and requests of size $(0, 1]$.
- Objective: Pack the items in the minimum number of bins.
- Greedy heuristic: FIRST FIT INCREASING (FFI)

Non-optimal example:

- $\sigma = \langle 1/2 - \varepsilon, 1/2 - \varepsilon, 1/2 + \varepsilon, 1/2 + \varepsilon \rangle$
- FFI: 3 bins
- OPT:

Is greedy Optimal?

Not always: Bin Packing Problem

- Bins of size 1, and requests of size $(0, 1]$.
- Objective: Pack the items in the minimum number of bins.
- Greedy heuristic: First Fit Increasing (ffi)

Non-optimal example:

- $\sigma = \langle 1/2 - \varepsilon, 1/2 - \varepsilon, 1/2 + \varepsilon, 1/2 + \varepsilon \rangle$
- ffi: 3 bins
- opt: 2 bins

Is greedy Optimal?

Not always: Bin Packing Problem

- Bins of size 1, and requests of size $(0, 1]$.
- Objective: Pack the items in the minimum number of bins.
- Greedy heuristic: First Fit Increasing (ffi)

Non-optimal example:

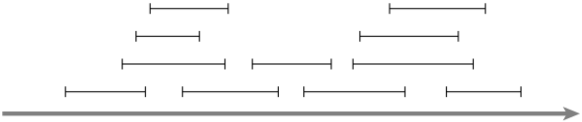
- $\sigma = \langle 1/2 - \varepsilon, 1/2 - \varepsilon, 1/2 + \varepsilon, 1/2 + \varepsilon \rangle$
- ffi: 3 bins
- opt: 2 bins

Techniques for showing that greedy is optimal:

- Always stays ahead
- Exchange argument

Stays Ahead: Interval Scheduling

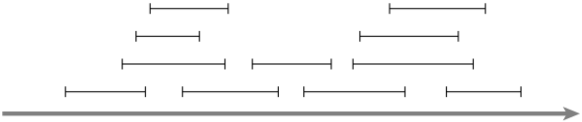
Interval Scheduling



Problem Definition

- Requests: $\sigma = \{r_1, \cdots, r_n\}$

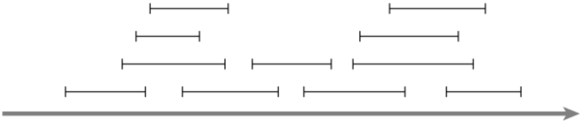
Interval Scheduling



Problem Definition

- Requests: $\sigma = \{r_1, \cdots, r_n\}$
- A request $r_i = (s_i, f_i)$, where s_i is the start time and f_i is the finish time.

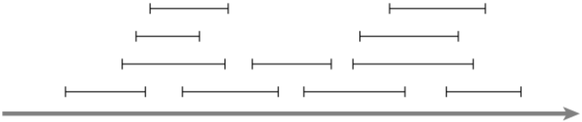
Interval Scheduling



Problem Definition

- Requests: $\sigma = \{r_1, \cdots, r_n\}$
- A request $r_i = (s_i, f_i)$, where s_i is the start time and f_i is the finish time.
- Objective: Produce a *compatible* schedule S that has maximum cardinality.

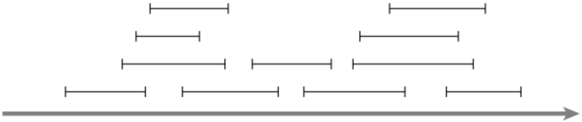
Interval Scheduling



Problem Definition

- Requests: $\sigma = \{r_1, \cdots, r_n\}$
- A request $r_i = (s_i, f_i)$, where s_i is the start time and f_i is the finish time.
- Objective: Produce a *compatible* schedule S that has maximum cardinality.
- Compatible schedule S: $\forall r_i, r_j \in S, f_i \leq s_j \vee f_j \leq s_i$.

Interval Scheduling



Problem Definition

- Requests: $\sigma = \{r_1, \cdots, r_n\}$
- A request $r_i = (s_i, f_i)$, where s_i is the start time and f_i is the finish time.
- Objective: Produce a *compatible* schedule S that has maximum cardinality.
- Compatible schedule S: $\forall r_i, r_j \in S, f_i \leq s_j \vee f_j \leq s_i$.

TopHat Discussion 1: What greedy heuristic might work?

Greedy Algorithms for Interval Scheduling

Heuristic 1: Earliest First

Schedule a compatible request with the earliest start time.

Greedy Algorithms for Interval Scheduling

Heuristic 1: Earliest First

Schedule a compatible request with the earliest start time.

Optimal?

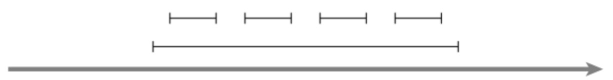
Greedy Algorithms for Interval Scheduling

Heuristic 1: Earliest First

Schedule a compatible request with the earliest start time.

Optimal?

Counter-example:



Greedy Algorithms for Interval Scheduling

Heuristic 2: Smallest Interval

Schedule a compatible request r_i with the smallest interval $(f_i - s_i)$.

Greedy Algorithms for Interval Scheduling

Heuristic 2: Smallest Interval

Schedule a compatible request r_i with the smallest interval $(f_i - s_i)$.

Optimal?

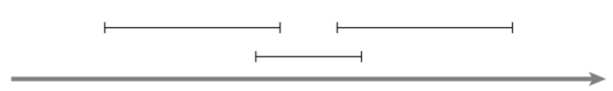
Greedy Algorithms for Interval Scheduling

Heuristic 2: Smallest Interval

Schedule a compatible request r_i with the smallest interval $(f_i - s_i)$.

Optimal?

Counter-example:



Greedy Algorithms for Interval Scheduling

Heuristic 3: Fewest Conflicts

Schedule a compatible request with the fewest remaining conflicts.

Greedy Algorithms for Interval Scheduling

Heuristic 3: Fewest Conflicts

Schedule a compatible request with the fewest remaining conflicts.

Optimal?

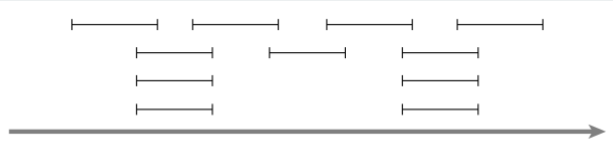
Greedy Algorithms for Interval Scheduling

Heuristic 3: Fewest Conflicts

Schedule a compatible request with the fewest remaining conflicts.

Optimal?

Counter-example:



Greedy Algorithms for Interval Scheduling

Heuristic 4: Finish First

Schedule a compatible request with the smallest finish time.

Greedy Algorithms for Interval Scheduling

Heuristic 4: Finish First

Schedule a compatible request with the smallest finish time.

Optimal?

Greedy Algorithms for Interval Scheduling

Heuristic 4: Finish First

Schedule a compatible request with the smallest finish time.

Optimal?

Counter-example? Let's try and prove it.

Exercise: Formalize the algorithm (pseudocode)

Heuristic 4: Finish First

Exercise: Formalize the algorithm (pseudocode)

Heuristic 4: Finish First

Algorithm: FinishFirst

Let S be an initially empty set.
while *σ is not empty* **do**
  Choose $r_i \in \sigma$ with the smallest finish time (break ties arbitrarily).
  Add r_i to S.
  Remove all incompatible request in σ.
end
return S

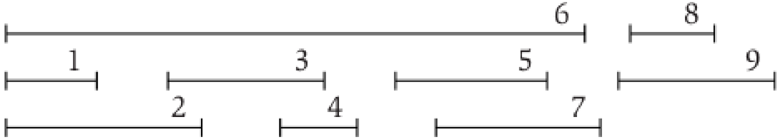
Exercise: Formalize the algorithm (pseudocode)

Heuristic 4: Finish First

Algorithm: FinishFirst

Let S be an initially empty set.
while *σ is not empty* **do**
  Choose $r_i \in \sigma$ with the smallest finish time (break ties arbitrarily).
  Add r_i to S.
  Remove all incompatible request in σ.
end
return S

Sample Run (TopHat Q1: What is $|S|$?)



Analysis of FinishFirst

Observation 1

Immediate from the definition of FinishFirst, S is compatible.

Analysis of FinishFirst

Observation 1

Immediate from the definition of FinishFirst, S is compatible.

Showing Optimality

Let S^* be an optimal solution.

- We can show the strong claim that $S = S^*$.

Analysis of FinishFirst

Observation 1

Immediate from the definition of FinishFirst, S is compatible.

Showing Optimality

Let S^* be an optimal solution.

- We can show the strong claim that $S = S^*$.
- Can there be multiple S^*?

Analysis of FinishFirst

Observation 1

Immediate from the definition of FinishFirst, S is compatible.

Showing Optimality

Let S^* be an optimal solution.

- We can show the strong claim that $S = S^*$.
- Can there be multiple S^*? Yes.

Analysis of FinishFirst

Observation 1

Immediate from the definition of FinishFirst, S is compatible.

Showing Optimality

Let S^* be an optimal solution.

- We can show the strong claim that $S = S^*$.
- Can there be multiple S^*? Yes.
- Hence, we can show the weaker claim of $|S| = |S^*|$ for this problem.

Analysis of FinishFirst

Observation 1

Immediate from the definition of FinishFirst, S is compatible.

Showing Optimality

Let S^* be an optimal solution.

- We can show the strong claim that $S = S^*$.
- Can there be multiple S^*? Yes.
- Hence, we can show the weaker claim of $|S| = |S^*|$ for this problem.
- Technique: "Always stays ahead"

Analysis of FinishFirst

Observation 1

Immediate from the definition of FinishFirst, S is compatible.

Showing Optimality

Let S^* be an optimal solution.

- We can show the strong claim that $S = S^*$.
- Can there be multiple S^*? Yes.
- Hence, we can show the weaker claim of $|S| = |S^*|$ for this problem.
- Technique: "Always stays ahead"
  - At every time step i, $|S_i| \geq |S_i^*|$.

Stay Ahead Analysis

- Label $S = \langle i_1, \ldots, i_k \rangle$ such that $f_{i_u} < f_{i_v}$ for $u < v$.
- Label $S^* = \langle j_1, \ldots, j_m \rangle$ such that $f_{j_u} < f_{j_v}$ for $u < v$.

Stay Ahead Analysis

- Label $S = \langle i_1, \ldots, i_k \rangle$ such that $f_{i_u} < f_{i_v}$ for $u < v$.
- Label $S^* = \langle j_1, \ldots, j_m \rangle$ such that $f_{j_u} < f_{j_v}$ for $u < v$.

Lemma 1

For all i_r, j_r with $r \leq k$, we have $f_{i_r} \leq f_{j_r}$

Proof.

Stay Ahead Analysis

- Label $S = \langle i_1, \ldots, i_k \rangle$ such that $f_{i_u} < f_{i_v}$ for $u < v$.
- Label $S^* = \langle j_1, \ldots, j_m \rangle$ such that $f_{j_u} < f_{j_v}$ for $u < v$.

Lemma 1

For all i_r, j_r with $r \leq k$, we have $f_{i_r} \leq f_{j_r}$

Proof.

The proof is by induction.

- For $r = 1$, the claim is true as FinishFirst first selects the request with the earliest finish time.

Stay Ahead Analysis

- Label $S = \langle i_1, \ldots, i_k \rangle$ such that $f_{i_u} < f_{i_v}$ for $u < v$.
- Label $S^* = \langle j_1, \ldots, j_m \rangle$ such that $f_{j_u} < f_{j_v}$ for $u < v$.

Lemma 1

For all i_r, j_r with $r \leq k$, we have $f_{i_r} \leq f_{j_r}$

Proof.

The proof is by induction.

- For $r = 1$, the claim is true as FinishFirst first selects the request with the earliest finish time.
- Assume true for $r - 1$.
  - By the induction hypothesis, we have that $f_{i_{r-1}} \leq f_{j_{r-1}}$.
  - The only way for S to fall behind S^* would be for FinishFirst to choose a request q with $f_q \geq f_{j_r}$, but this is a contradiction.

□

Stay Ahead Analysis

- Label $S = \langle i_1, \ldots, i_k \rangle$ such that $f_{i_u} < f_{i_v}$ for $u < v$.
- Label $S^* = \langle j_1, \ldots, j_m \rangle$ such that $f_{j_u} < f_{j_v}$ for $u < v$.

Lemma 1

For all i_r, j_r with $r \leq k$, we have $f_{i_r} \leq f_{j_r}$

The optimality of FinishFirst, essentially, follows immediately from Lemma 1.

FinishFirst is Optimal

- Label $S = \langle i_1, \ldots, i_k \rangle$ such that $f_{i_u} < f_{i_v}$ for $u < v$.
- Label $S^* = \langle j_1, \ldots, j_m \rangle$ such that $f_{j_u} < f_{j_v}$ for $u < v$.

Theorem 2

FinishFirst produces an optimal schedule.

Proof.

FinishFirst is Optimal

- Label $S = \langle i_1, \ldots, i_k \rangle$ such that $f_{i_u} < f_{i_v}$ for $u < v$.
- Label $S^* = \langle j_1, \ldots, j_m \rangle$ such that $f_{j_u} < f_{j_v}$ for $u < v$.

Theorem 2

FinishFirst produces an optimal schedule.

Proof.

By way of contradiction, assume that $|S^*| > |S|$. This implies that $m > k$. Lemma 1 shows that FinishFirst is ahead for all the k requests. That means it would be able to add the $(k+1)$-st item of S^*. As it did not, this contradicts the definition of FinishFirst. $\square$

Implementation and Running Time

Algorithm: FinishFirst

Let S be an initially empty set.
while *σ is not empty* **do**
  Choose $r_i \in \sigma$ with the smallest finish time (break ties arbitrarily).
  Add r_i to S.
  Remove all incompatible request in σ.
end
return S

Implementation Details

Implementation and Running Time

Algorithm: FinishFirst

Let S be an initially empty set.
while *σ is not empty* **do**
    Choose $r_i \in \sigma$ with the smallest finish time (break ties arbitrarily).
    Add r_i to S.
    Remove all incompatible request in σ.
end
return S

Implementation Details

- Choose request with smallest finish time:
- Remove incompatible requests:

Implementation and Running Time

Algorithm: FinishFirst

Let S be an initially empty set.
while *σ is not empty* **do**
  Choose $r_i \in \sigma$ with the smallest finish time (break ties arbitrarily).
  Add r_i to S.
  Remove all incompatible request in σ.
end
return S

Implementation Details

- Choose request with smallest finish time: Before processing, sort requests: $O(n \log n)$.
- Remove incompatible requests:

Implementation and Running Time

Algorithm: FinishFirst

Let S be an initially empty set.
while *σ is not empty* **do**
    Choose $r_i \in \sigma$ with the smallest finish time (break ties arbitrarily).
    Add r_i to S.
    Remove all incompatible request in σ.
end
return S

Implementation Details

- Choose request with smallest finish time: Before processing, sort requests: $O(n \log n)$.
- Remove incompatible requests: Advance in sorted order until a request with a compatible start time.

Implementation and Running Time

Algorithm: FinishFirst

Let S be an initially empty set.
while *σ is not empty* **do**
  Choose $r_i \in \sigma$ with the smallest finish time (break ties arbitrarily).
  Add r_i to S.
  Remove all incompatible request in σ.
end
return S

Implementation Details

- Choose request with smallest finish time: Before processing, sort requests: $O(n \log n)$.
- Remove incompatible requests: Advance in sorted order until a request with a compatible start time.

Overall:

$O(n \log n) + O(n) = O(n \log n)$

Interval Extensions

- Online variant: Requests are presented in a specific order to the algorithm. At request i, the algorithm does not know n nor $r_{i+1}, \ldots, r_n$.

Interval Extensions

- Online variant: Requests are presented in a specific order to the algorithm. At request i, the algorithm does not know n nor $r_{i+1}, \ldots, r_n$.
- Add a value to the intervals (online/offline). Now objective is to maximize the total value of scheduled intervals.

Interval Extensions

- Online variant: Requests are presented in a specific order to the algorithm. At request i, the algorithm does not know n nor $r_{i+1}, \ldots, r_n$.
- Add a value to the intervals (online/offline). Now objective is to maximize the total value of scheduled intervals.
- Scheduling all intervals: Interval Colouring Problem.

Interval Extensions

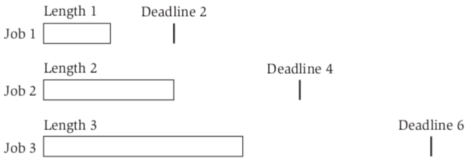
- Online variant: Requests are presented in a specific order to the algorithm. At request i, the algorithm does not know n nor $r_{i+1}, \ldots, r_n$.
- Add a value to the intervals (online/offline). Now objective is to maximize the total value of scheduled intervals.
- Scheduling all intervals: Interval Colouring Problem.
  - Unlimited resources and the algorithm must produce multiple compatible schedules that cover all the requests (without duplicates between the schedules).

Interval Extensions

- Online variant: Requests are presented in a specific order to the algorithm. At request i, the algorithm does not know n nor $r_{i+1}, \dots, r_n$.
- Add a value to the intervals (online/offline). Now objective is to maximize the total value of scheduled intervals.
- Scheduling all intervals: Interval Colouring Problem.
  - Unlimited resources and the algorithm must produce multiple compatible schedules that cover all the requests (without duplicates between the schedules).
  - Objective: Minimize the number of schedules.

Exchange Argument: Minimize Max Lateness

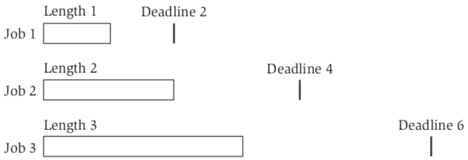
Scheduling Problem: Minimize Lateness



Problem Definition

- n jobs and a single machine that can process one job at a time

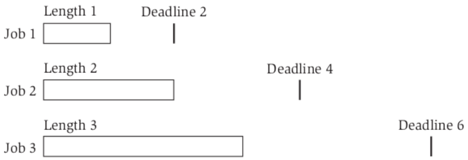
Scheduling Problem: Minimize Lateness



Problem Definition

- n jobs and a single machine that can process one job at a time
- For job i:

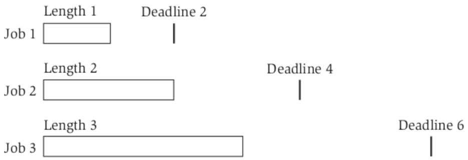
Scheduling Problem: Minimize Lateness



Problem Definition

- n jobs and a single machine that can process one job at a time
- For job i:
  - t_i is the processing time, d_i is the deadline.

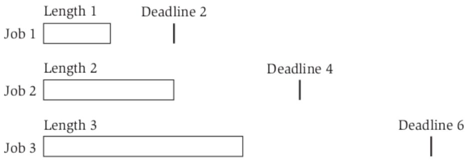
Scheduling Problem: Minimize Lateness



Problem Definition

- n jobs and a single machine that can process one job at a time
- For job i:
  - t_i is the processing time, d_i is the deadline.
  - Lateness $l_i = f_i - d_i$ if finish time $f_i > d_i$; 0 otherwise.

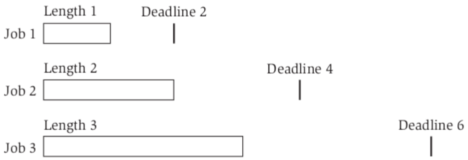
Scheduling Problem: Minimize Lateness



Problem Definition

- n jobs and a single machine that can process one job at a time
- For job i:
  - t_i is the processing time, d_i is the deadline.
  - Lateness $l_i = f_i - d_i$ if finish time $f_i > d_i$; 0 otherwise.
- Objective: Build a schedule for all the jobs that minimizes the max lateness.

Scheduling Problem: Minimize Lateness



Problem Definition

- n jobs and a single machine that can process one job at a time
- For job i:
  - t_i is the processing time, d_i is the deadline.
  - Lateness $l_i = f_i - d_i$ if finish time $f_i > d_i$; 0 otherwise.
- Objective: Build a schedule for all the jobs that minimizes the max lateness.

TopHat Discussion 2: What greedy heuristic might work?

Greedy Algorithms for Interval Scheduling

Heuristic 1: Increasing processing time.

Schedule jobs by increasing t_i.

Greedy Algorithms for Interval Scheduling

Heuristic 1: Increasing processing time.

Schedule jobs by increasing t_i.

Optimal?

Greedy Algorithms for Interval Scheduling

Heuristic 1: Increasing processing time.

Schedule jobs by increasing t_i.

Optimal?

Counter-example: Jobs (t_i, d_i): $\{(1, 100), (10, 10)\}$

Greedy Algorithms for Interval Scheduling

Heuristic 2: Increasing slack.

Schedule by increasing $d_i - t_i$.

Greedy Algorithms for Interval Scheduling

Heuristic 2: Increasing slack.

Schedule by increasing $d_i - t_i$.

Optimal?

Greedy Algorithms for Interval Scheduling

Heuristic 2: Increasing slack.

Schedule by increasing $d_i - t_i$.

Optimal?

Counter-example:

$$\text{Jobs } (t_i, d_i)\text{: } \{(1,2), (10,10)\}$$

Greedy Algorithms for Interval Scheduling

Heuristic 3: Earliest deadline first.

Schedule by increasing d_i.

Greedy Algorithms for Interval Scheduling

Heuristic 3: Earliest deadline first.

Schedule by increasing d_i.

Optimal?

Greedy Algorithms for Interval Scheduling

Heuristic 3: Earliest deadline first.

Schedule by increasing d_i.

Optimal?

Counter-example? Let's try and prove it.

Exercise: Formalize the algorithm (pseudocode)

Heuristic 3: Earliest deadline first.

Exercise: Formalize the algorithm (pseudocode)

Heuristic 3: Earliest deadline first.

Algorithm: EDF

Let J be the set of jobs.
Let S be an initially empty list.
while *J is not empty* **do**
  Choose $j \in J$ with the smallest d_i (break ties arbitrarily).
  Append j to S.
end
return S

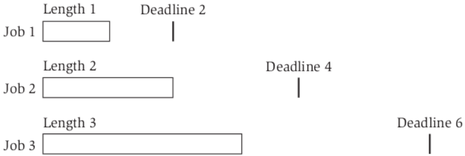
Exercise: Formalize the algorithm (pseudocode)

Heuristic 3: Earliest deadline first.

Algorithm: EDF

Let J be the set of jobs.
Let S be an initially empty list.
while *J is not empty* **do**
  Choose $j \in J$ with the smallest d_i (break ties arbitrarily).
  Append j to S.
end
return S

Sample Run (TopHat Q1: What is max lateness?)



Analysis of edf

Observation 2

There is an optimal schedule with no idle time.

Analysis of EDF

Observation 2

There is an optimal schedule with no idle time.

Showing Optimality

Let S^* be an optimal solution.

- Is it sufficient to show that $|S| = |S^*|$?

Analysis of EDF

Observation 2

There is an optimal schedule with no idle time.

Showing Optimality

Let S^* be an optimal solution.

- Is it sufficient to show that $|S| = |S^*|$? No.

Analysis of EDF

Observation 2

There is an optimal schedule with no idle time.

Showing Optimality

Let S^* be an optimal solution.

- Is it sufficient to show that $|S| = |S^*|$? No.
- Can there be multiple S^*?

Analysis of EDF

Observation 2

There is an optimal schedule with no idle time.

Showing Optimality

Let S^* be an optimal solution.

- Is it sufficient to show that $|S| = |S^*|$? No.
- Can there be multiple S^*? Yes.

Analysis of EDF

Observation 2

There is an optimal schedule with no idle time.

Showing Optimality

Let S^* be an optimal solution.

- Is it sufficient to show that $|S| = |S^*|$? No.
- Can there be multiple S^*? Yes.
- We need to show either $S = S^*$, or $S \equiv S^*$ for max lateness.

Analysis of EDF

Observation 2

There is an optimal schedule with no idle time.

Showing Optimality

Let S^* be an optimal solution.

- Is it sufficient to show that $|S| = |S^*|$? No.
- Can there be multiple S^*? Yes.
- We need to show either $S = S^*$, or $S \equiv S^*$ for max lateness.
- Technique: "Exchange Argument"

Analysis of EDF

Observation 2

There is an optimal schedule with no idle time.

Showing Optimality

Let S^* be an optimal solution.

- Is it sufficient to show that $|S| = |S^*|$? No.
- Can there be multiple S^*? Yes.
- We need to show either $S = S^*$, or $S \equiv S^*$ for max lateness.
- Technique: "Exchange Argument"
  - Start with an optimal solution S^* and transform it over a series of steps to something equivalent to S while maintaining optimality.

Analysis of EDF

Observation 2

There is an optimal schedule with no idle time.

Showing Optimality

Let S^* be an optimal solution.

- Is it sufficient to show that $|S| = |S^*|$? No.
- Can there be multiple S^*? Yes.
- We need to show either $S = S^*$, or $S \equiv S^*$ for max lateness.
- Technique: "Exchange Argument"
  - Start with an optimal solution S^* and transform it over a series of steps to something equivalent to S while maintaining optimality.
  - $S^* \equiv S_1 \equiv S_2 \equiv \cdots \equiv S$ for max lateness.

Exchange Argument Analysis

Definition 3

A schedule A has an *inversion* if the are jobs i and j with i scheduled before j and $d_j < d_i$.

Exchange Argument Analysis

Definition 3

A schedule A has an *inversion* if the are jobs i and j with i scheduled before j and $d_j < d_i$.

Lemma 4

All schedules with no inversions and no idle time have the same lateness.

Exchange Argument Analysis

Definition 3

A schedule A has an *inversion* if the are jobs i and j with i scheduled before j and $d_j < d_i$.

Lemma 4

All schedules with no inversions and no idle time have the same lateness.

Proof.

Exchange Argument Analysis

Definition 3

A schedule A has an *inversion* if the are jobs i and j with i scheduled before j and $d_j < d_i$.

Lemma 4

All schedules with no inversions and no idle time have the same lateness.

Proof.

- Only vary in jobs with the same deadline.
- Jobs with same deadline must sequential.
- Ordering of jobs with same deadline won't change lateness.

□

Analysis of EDF

Theorem 5

There is an optimal schedule that has no inversions and no idle time.

Analysis of edf

Theorem 5

There is an optimal schedule that has no inversions and no idle time.

Proof.

Analysis of EDF

Theorem 5

There is an optimal schedule that has no inversions and no idle time.

Proof.

- If S^* has an inversion, then there is a pair of jobs i and j such that j is scheduled immediately after i and has $d_j < d_i$.

Analysis of EDF

Theorem 5

There is an optimal schedule that has no inversions and no idle time.

Proof.

- If S^* has an inversion, then there is a pair of jobs i and j such that j is scheduled immediately after i and has $d_j < d_i$.
- We will swap i and j to create a new schedule S'. Note that S' has one less inversion than S^*.

Analysis of EDF

Theorem 5

There is an optimal schedule that has no inversions and no idle time.

Proof.

- If S^* has an inversion, then there is a pair of jobs i and j such that j is scheduled immediately after i and has $d_j < d_i$.
- We will swap i and j to create a new schedule S'. Note that S' has one less inversion than S^*.
- We need to show that S' has the same max lateness as S^*:

Analysis of EDF

Theorem 5

There is an optimal schedule that has no inversions and no idle time.

Proof.

- If S^* has an inversion, then there is a pair of jobs i and j such that j is scheduled immediately after i and has $d_j < d_i$.
- We will swap i and j to create a new schedule S'. Note that S' has one less inversion than S^*.
- We need to show that S' has the same max lateness as S^*:
  - Swapping i and j means that l'_j (lateness in S') is less than that in S^*.

Analysis of EDF

Theorem 5

There is an optimal schedule that has no inversions and no idle time.

Proof.

- If S^* has an inversion, then there is a pair of jobs i and j such that j is scheduled immediately after i and has $d_j < d_i$.
- We will swap i and j to create a new schedule S'. Note that S' has one less inversion than S^*.
- We need to show that S' has the same max lateness as S^*:
  - Swapping i and j means that l'_j (lateness in S') is less than that in S^*.
  - Lateness of i may increase, but: $l'_i = f'_i - d_i = f^*_j - d_i \leq f^*_j - d_j = l^*_j$.

Analysis of EDF

Theorem 5

There is an optimal schedule that has no inversions and no idle time.

Proof.

- If S^* has an inversion, then there is a pair of jobs i and j such that j is scheduled immediately after i and has $d_j < d_i$.
- We will swap i and j to create a new schedule S'. Note that S' has one less inversion than S^*.
- We need to show that S' has the same max lateness as S^*:
  - Swapping i and j means that l'_j (lateness in S') is less than that in S^*.
  - Lateness of i may increase, but: $l'_i = f'_i - d_i = f^*_j - d_i \leq f^*_j - d_j = l^*_j$.
- Let $S^* := S'$ and repeat until no more inversions.

$\square$

EDF is Optimal

Corollary 6

EDF produces an optimal schedule.

Proof.

EDF is Optimal

Corollary 6

EDF produces an optimal schedule.

Proof.

- EDF produces a schedule with no inversions and no idle time.
- From Theorem 5, there is an optimal schedule with no inversions and no idle time.
- Lemma 4 shows that these two schedules have the same max lateness.

□

EDF is Optimal

Corollary 6

EDF produces an optimal schedule.

Proof.

- EDF produces a schedule with no inversions and no idle time.
- From Theorem 5, there is an optimal schedule with no inversions and no idle time.
- Lemma 4 shows that these two schedules have the same max lateness.

□

Run time: .

EDF is Optimal

Corollary 6

EDF produces an optimal schedule.

Proof.

- EDF produces a schedule with no inversions and no idle time.
- From Theorem 5, there is an optimal schedule with no inversions and no idle time.
- Lemma 4 shows that these two schedules have the same max lateness.

□

Run time: Sort the jobs by deadline: $O(n \log n)$.

Shortest Path

Finding the Shortest Path

Problem Definition

We have a directed graph $G = (V, E)$, where $|V| = n$ and $|E| = m$ and a node s that has a path to every other node in V. For each edge e, $\ell_e \geq 0$ is the length of the edge.

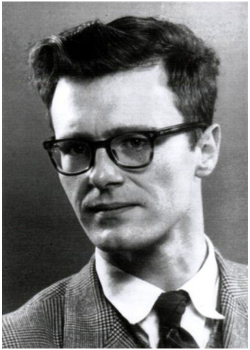
- What is the shortest path from s to each other node?

Finding the Shortest Path

Problem Definition

We have a directed graph $G = (V, E)$, where $|V| = n$ and $|E| = m$ and a node s that has a path to every other node in V. For each edge e, $\ell_e \geq 0$ is the length of the edge.

- What is the shortest path from s to each other node?



Edsger Dijkstra, 1956
Dijkstra's shortest path fame

Dijkstra's

Algorithm: *Dijkstra's*

Let S be the set of explored nodes.
For each $u \in S$, we store a distance value $d(u)$.
Initialize $S = \{s\}$ and $d(s) = 0$
while $S \neq V$ **do**

  Choose $v \notin S$ with at least one incoming edge originating from a node in S with the smallest

$$d'(v) = \min_{e=(u,v):u\in S} \{d(u) + \ell_e\}$$

  Append v to S and define $d(v) = d'(v)$.

end

Dijkstra's

Algorithm: *Dijkstra's*

Let S be the set of explored nodes.
For each $u \in S$, we store a distance value $d(u)$.
Initialize $S = \{s\}$ and $d(s) = 0$
while $S \neq V$ **do**
    Choose $v \notin S$ with at least one incoming edge originating from a node in S with the smallest

$$d'(v) = \min_{e=(u,v):u \in S} \{d(u) + \ell_e\}$$

    Append v to S and define $d(v) = d'(v)$.
end

How is it greedy?

Dijkstra's

Algorithm: *Dijkstra's*

Let S be the set of explored nodes.
For each $u \in S$, we store a distance value $d(u)$.
Initialize $S = \{s\}$ and $d(s) = 0$
while $S \neq V$ **do**
    Choose $v \notin S$ with at least one incoming edge originating from a node in S with the smallest

$$d'(v) = \min_{e=(u,v):u \in S} \{d(u) + \ell_e\}$$

    Append v to S and define $d(v) = d'(v)$.
end

How is it greedy?
TopHat 3: Which technique to prove optimality?

Correctness of Dijkstra's

Theorem 7

Consider the S at any point in the execution of Dijkstra's. For each $u \in S$, the path P_u is a shortest $s - u$ path.

Correctness of Dijkstra's

Theorem 7

Consider the S at any point in the execution of Dijkstra's. For each $u \in S$, the path P_u is a shortest $s - u$ path.

Proof.

Correctness of Dijkstra's

Theorem 7

Consider the S at any point in the execution of Dijkstra's. For each $u \in S$, the path P_u is a shortest $s - u$ path.

Proof.

By induction on the size of S.

Correctness of Dijkstra's

Theorem 7

Consider the S at any point in the execution of Dijkstra's. For each $u \in S$, the path P_u is a shortest $s - u$ path.

Proof.

By induction on the size of S.

- For $|S| = 1$, the claim follows trivially as $S = \{s\}$.

Correctness of Dijkstra's

Theorem 7

Consider the S at any point in the execution of Dijkstra's. For each $u \in S$, the path P_u is a shortest $s - u$ path.

Proof.

By induction on the size of S.

- For $|S| = 1$, the claim follows trivially as $S = \{s\}$.
- By the induction hypothesis, for $|S| = k$, P_u is the shortest $s - u$ path for all $u \in S$.

Correctness of Dijkstra's

Theorem 7

Consider the S at any point in the execution of Dijkstra's. For each $u \in S$, the path P_u is a shortest $s - u$ path.

Proof.

By induction on the size of S.

- In step $k + 1$, we add v.
  - By definition, P_v is shortest path connected to S by one edge.

Correctness of Dijkstra's

Theorem 7

Consider the S at any point in the execution of Dijkstra's. For each $u \in S$, the path P_u is a shortest $s - u$ path.

Proof.

By induction on the size of S.

- In step $k + 1$, we add v.
  - By definition, P_v is shortest path connected to S by one edge.
  - Since P_u is a shortest path to u, P_v is the shortest path to v when considering only the nodes of S.

Correctness of Dijkstra's

Theorem 7

Consider the S at any point in the execution of Dijkstra's. For each $u \in S$, the path P_u is a shortest $s - u$ path.

Proof.

By induction on the size of S.

- In step $k + 1$, we add v.
  - By definition, P_v is shortest path connected to S by one edge.
  - Since P_u is a shortest path to u, P_v is the shortest path to v when considering only the nodes of S.
  - Moreover, there cannot be a shorter path to v passing through another node $y \notin S$ else y that would be added at $k + 1$.

□

Dijkstra's Observations

Algorithm: *Dijkstra's*

Let S be the set of explored nodes.
For each $u \in S$, we store a distance value $d(u)$.
Initialize $S = \{s\}$ and $d(s) = 0$
while $S \neq V$ **do**
  Choose $v \notin S$ with at least one incoming edge originating from a node in S with the smallest $d'(v) = \min_{e=(u,v):u \in S}\{d(u) + \ell_e\}$
  Append v to S and define $d(v) = d'(v)$.
end

- Negative edge weights, where does it fail?

Dijkstra's Observations

Algorithm: *Dijkstra's*

Let S be the set of explored nodes.
For each $u \in S$, we store a distance value $d(u)$.
Initialize $S = \{s\}$ and $d(s) = 0$
while $S \neq V$ **do**
  Choose $v \notin S$ with at least one incoming edge originating from a node in S with the smallest $d'(v) = \min_{e=(u,v):u \in S}\{d(u) + \ell_e\}$
  Append v to S and define $d(v) = d'(v)$.
end

- Negative edge weights, where does it fail?
- TopHat 4: It is graph exploration, what kind of exploration?

Dijkstra's Observations

Algorithm: *Dijkstra's*

Let S be the set of explored nodes.
For each $u \in S$, we store a distance value $d(u)$.
Initialize $S = \{s\}$ and $d(s) = 0$
while $S \neq V$ **do**
  Choose $v \notin S$ with at least one incoming edge originating from a node in S with the smallest $d'(v) = \min_{e=(u,v):u \in S}\{d(u) + \ell_e\}$
  Append v to S and define $d(v) = d'(v)$.
end

- Negative edge weights, where does it fail?
- TopHat 4: It is graph exploration, what kind of exploration?
  - Weighted (continuous) BFS

Implementation and Run Time of Dijkstra's

Algorithm: *Dijkstra's*

Let S be the set of explored nodes.
For each $u \in S$, we store a distance value $d(u)$.
Initialize $S = \{s\}$ and $d(s) = 0$
while $S \neq V$ **do**
  Choose $v \notin S$ with at least one incoming edge originating from a node in S with the smallest $d'(v) = \min_{e=(u,v):u \in S}\{d(u) + \ell_e\}$
  Append v to S and define $d(v) = d'(v)$.
end

- TopHat 5: Number of iterations of the loop?

Implementation and Run Time of Dijkstra's

Algorithm: *Dijkstra's*

Let S be the set of explored nodes.
For each $u \in S$, we store a distance value $d(u)$.
Initialize $S = \{s\}$ and $d(s) = 0$
while $S \neq V$ **do**
  Choose $v \notin S$ with at least one incoming edge originating from a node in S with the smallest $d'(v) = \min_{e=(u,v):u \in S}\{d(u) + \ell_e\}$
  Append v to S and define $d(v) = d'(v)$.
end

- TopHat 5: Number of iterations of the loop? $n - 1$

Implementation and Run Time of Dijkstra's

Algorithm: *Dijkstra's*

Let S be the set of explored nodes.
For each $u \in S$, we store a distance value $d(u)$.
Initialize $S = \{s\}$ and $d(s) = 0$
while $S \neq V$ **do**
  Choose $v \notin S$ with at least one incoming edge originating from a node in S with the smallest $d'(v) = \min_{e=(u,v):u \in S}\{d(u) + \ell_e\}$
  Append v to S and define $d(v) = d'(v)$.
end

- TopHat 5: Number of iterations of the loop? $n - 1$
- Key Operations:

Implementation and Run Time of Dijkstra's

Algorithm: *Dijkstra's*

Let S be the set of explored nodes.
For each $u \in S$, we store a distance value $d(u)$.
Initialize $S = \{s\}$ and $d(s) = 0$
while $S \neq V$ **do**
    Choose $v \notin S$ with at least one incoming edge originating from a node in S with the smallest $d'(v) = \min_{e=(u,v):u \in S}\{d(u) + \ell_e\}$
    Append v to S and define $d(v) = d'(v)$.
end

- TopHat 5: Number of iterations of the loop?
  $n - 1$
- Key Operations:
  - Finding the min: Easy in $O(m)$

Implementation and Run Time of Dijkstra's

Algorithm: *Dijkstra's*

Let S be the set of explored nodes.
For each $u \in S$, we store a distance value $d(u)$.
Initialize $S = \{s\}$ and $d(s) = 0$
while $S \neq V$ **do**
  Choose $v \notin S$ with at least one incoming edge originating from a node in S with the smallest $d'(v) = \min_{e=(u,v):u \in S}\{d(u) + \ell_e\}$
  Append v to S and define $d(v) = d'(v)$.
end

- TopHat 5: Number of iterations of the loop?
  $n - 1$
- Key Operations:
  - Finding the min: Easy in $O(m)$
- Overall: $O(mn)$

Implementation and Run Time of Dijkstra's

Algorithm: *Dijkstra's*

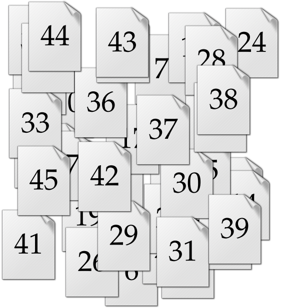
Let S be the set of explored nodes.
For each $u \in S$, we store a distance value $d(u)$.
Initialize $S = \{s\}$ and $d(s) = 0$
while $S \neq V$ **do**
  Choose $v \notin S$ with at least one incoming edge originating from a node in S with the smallest $d'(v) = \min_{e=(u,v):u \in S}\{d(u) + \ell_e\}$
  Append v to S and define $d(v) = d'(v)$.
end

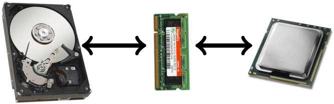
- TopHat 5: Number of iterations of the loop? $n - 1$
- Key Operations:
  - Finding the min: Easy in $O(m)$
- Overall: $O(mn)$
- How can we get $O(m \log n)$?

Paging

Paging Problem

$\mathcal{U}$:





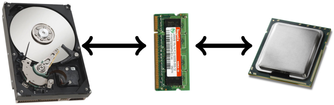
Cache:

Requests:

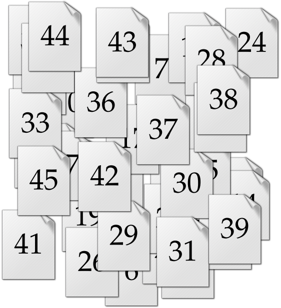
Definition

- $\mathcal{U}$: universe of pages ($|\mathcal{U}| > k$).
- Cache of size k.
- Requests are to the pages of $\mathcal{U}$.
- Goal: Minimize the number of page faults (requests to pages not in the cache).

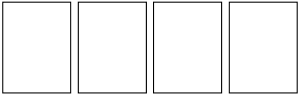
Paging Problem



$\mathcal{U}$:



Cache:



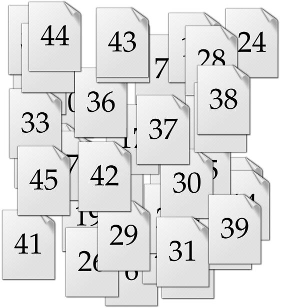
Requests:

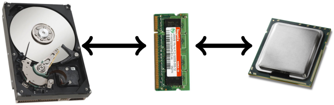
Definition

- $\mathcal{U}$: universe of pages ($|\mathcal{U}| > k$).
- Cache of size k.
- Requests are to the pages of $\mathcal{U}$.
- Goal: Minimize the number of page faults (requests to pages not in the cache).

Paging Problem

$\mathcal{U}$:





Cache: 12

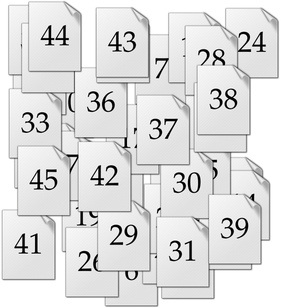
Requests: 12

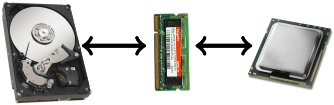
Definition

- $\mathcal{U}$: universe of pages ($|\mathcal{U}| > k$).
- Cache of size k.
- Requests are to the pages of $\mathcal{U}$.
- Goal: Minimize the number of page faults (requests to pages not in the cache).

Paging Problem

$\mathcal{U}$:





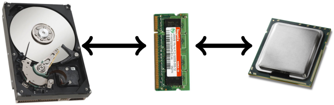
Cache: 12

Requests: 12 3

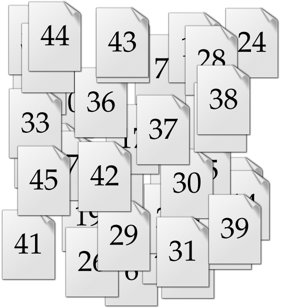
Definition

- $\mathcal{U}$: universe of pages ($|\mathcal{U}| > k$).
- Cache of size k.
- Requests are to the pages of $\mathcal{U}$.
- Goal: Minimize the number of page faults (requests to pages not in the cache).

Paging Problem



$\mathcal{U}$:



Cache: 12 3

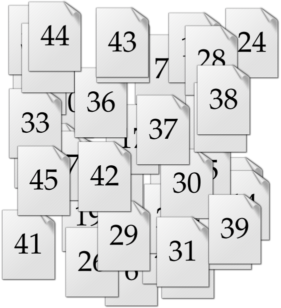
Requests: 12 3

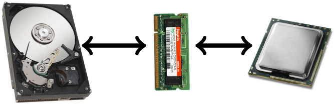
Definition

- $\mathcal{U}$: universe of pages ($|\mathcal{U}| > k$).
- Cache of size k.
- Requests are to the pages of $\mathcal{U}$.
- Goal: Minimize the number of page faults (requests to pages not in the cache).

Paging Problem

$\mathcal{U}$:





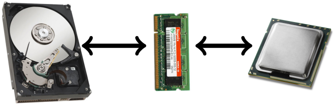
Cache: 12 3

Requests: 12 3 33

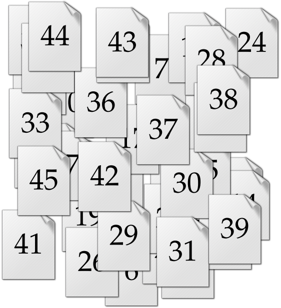
Definition

- $\mathcal{U}$: universe of pages ($|\mathcal{U}| > k$).
- Cache of size k.
- Requests are to the pages of $\mathcal{U}$.
- Goal: Minimize the number of page faults (requests to pages not in the cache).

Paging Problem



$\mathcal{U}$:



Cache: 12 3 33

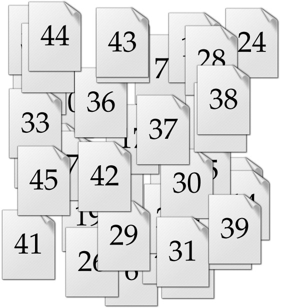
Requests: 12 3 33

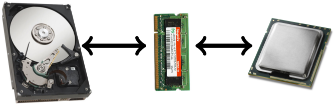
Definition

- $\mathcal{U}$: universe of pages ($|\mathcal{U}| > k$).
- Cache of size k.
- Requests are to the pages of $\mathcal{U}$.
- Goal: Minimize the number of page faults (requests to pages not in the cache).

Paging Problem

$\mathcal{U}$:





Cache: 12 3 33

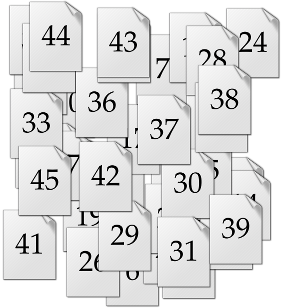
Requests: 12 3 33 14

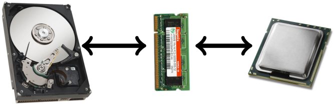
Definition

- $\mathcal{U}$: universe of pages ($|\mathcal{U}| > k$).
- Cache of size k.
- Requests are to the pages of $\mathcal{U}$.
- Goal: Minimize the number of page faults (requests to pages not in the cache).

Paging Problem

$\mathcal{U}$:





Cache: 12 3 33 14

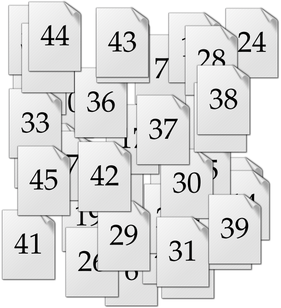
Requests: 12 3 33 14

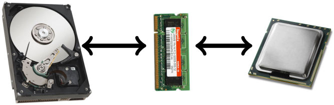
Definition

- $\mathcal{U}$: universe of pages ($|\mathcal{U}| > k$).
- Cache of size k.
- Requests are to the pages of $\mathcal{U}$.
- Goal: Minimize the number of page faults (requests to pages not in the cache).

Paging Problem

$\mathcal{U}$:





Cache: 12 3 33 14

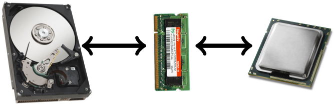
Requests:

12 3 33 14 12

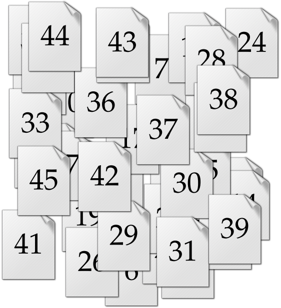
Definition

- $\mathcal{U}$: universe of pages ($|\mathcal{U}| > k$).
- Cache of size k.
- Requests are to the pages of $\mathcal{U}$.
- Goal: Minimize the number of page faults (requests to pages not in the cache).

Paging Problem



$\mathcal{U}$:



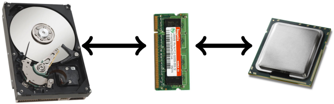
Cache: 12 3 33 14

Requests: 12 3 33 14 12 20

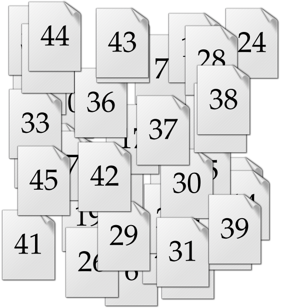
Definition

- $\mathcal{U}$: universe of pages ($|\mathcal{U}| > k$).
- Cache of size k.
- Requests are to the pages of $\mathcal{U}$.
- Goal: Minimize the number of page faults (requests to pages not in the cache).

Paging Problem



$\mathcal{U}$:



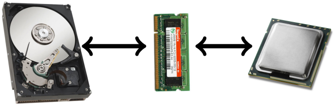
Cache: 12 | 3 | 33 | 14

Requests: 12 | 3 | 33 | 14 | 12 | 20

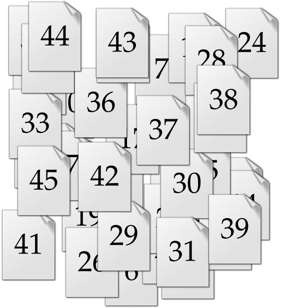
Eviction Strategies

- When designing an algorithm, we are picking an eviction strategy.

Paging Problem



$\mathcal{U}$:



Cache: 12 3 33 14

Requests: 12 3 33 14 12 20

Eviction Strategies

- When designing an algorithm, we are picking an eviction strategy.
- In the offline version, the algorithm knows the request sequence. What might be a good eviction strategy?

Offline Greedy Algorithm

Farthest-in-Future (FF)

Evict the page whose next request is the furthest into the future.

Offline Greedy Algorithm

Farthest-in-Future (FF)

Evict the page whose next request is the furthest into the future.

Small Run:

- $\mathcal{U} = \{a, b, c\}$
- $k = 2$
- $\sigma = \langle a, b, c, b, c, a, b \rangle$

Offline Greedy Algorithm

Farthest-in-Future (FF)

Evict the page whose next request is the furthest into the future.

Small Run:

- $\mathcal{U} = \{a, b, c\}$
- $k = 2$
- $\sigma = \langle a, b, c, b, c, a, b \rangle$
- TopHat 6: How many faults in small run?

OFFLINE GREEDY ALGORITHM

Farthest-in-Future (FF)

Evict the page whose next request is the furthest into the future.

Small Run:

- $\mathcal{U} = \{a, b, c\}$
- $k = 2$
- $\sigma = \langle a, b, c, b, c, a, b \rangle$
- TopHat 6: How many faults in small run?

TopHat 7: Which strategy to prove optimality?

Proving FF Optimality

Exchange Argument

Theorem 8

Let S be a schedule for the n request that make the same eviction decisions as S_{FF} for the first j items. Then, there is a schedule S' that makes the same eviction requests as S_{FF} for the first $j+1$ items with no more faults than S.

PROVING FF OPTIMALITY

EXCHANGE ARGUMENT

Theorem 8

Let S be a schedule for the n request that make the same eviction decisions as S_{FF} for the first j items. Then, there is a schedule S' that makes the same eviction requests as S_{FF} for the first $j+1$ items with no more faults than S.

Proof.

Proving FF Optimality

Exchange Argument

Theorem 8

Let S be a schedule for the n request that make the same eviction decisions as S_{FF} for the first j items. Then, there is a schedule S' that makes the same eviction requests as S_{FF} for the first $j+1$ items with no more faults than S.

Proof.

- If on request $j+1$, S behaves as S_{FF}. Then define S' as S and the claim follows.

Proving FF Optimality

Exchange Argument

Theorem 8

Let S be a schedule for the n request that make the same eviction decisions as S_{FF} for the first j items. Then, there is a schedule S' that makes the same eviction requests as S_{FF} for the first $j+1$ items with no more faults than S.

Proof.

- If on request $j+1$, S behaves as S_{FF}. Then define S' as S and the claim follows.
- Otherwise, say S evicts u and S_{FF} evicts v. We will build S' by following S_{FF} for the first $j+1$ requests. Note that the number of faults are the same for S and S' up to $j+1$, and the caches match except for u and v.

Proving FF Optimality

Exchange Argument

Theorem 8

Let S be a schedule for the n request that make the same eviction decisions as S_{FF} for the first j items. Then, there is a schedule S' that makes the same eviction requests as S_{FF} for the first $j+1$ items with no more faults than S.

Proof.

- From $j+2$ onward, S' follows S until either:

Proving FF Optimality

Exchange Argument

Theorem 8

Let S be a schedule for the n request that make the same eviction decisions as S_{FF} for the first j items. Then, there is a schedule S' that makes the same eviction requests as S_{FF} for the first $j+1$ items with no more faults than S.

Proof.

- From $j+2$ onward, S' follows S until either:
  1. S evicts v. In this case, S' evicts u.

Proving FF Optimality

Exchange Argument

Theorem 8

Let S be a schedule for the n request that make the same eviction decisions as S_{FF} for the first j items. Then, there is a schedule S' that makes the same eviction requests as S_{FF} for the first $j+1$ items with no more faults than S.

Proof.

- From $j+2$ onward, S' follows S until either:
  1. S evicts v. In this case, S' evicts u.
  2. S evicts $g \neq v$ to bring u into the cache. In this case, S' evicts g and brings in v.

Proving FF Optimality

Exchange Argument

Theorem 8

Let S be a schedule for the n request that make the same eviction decisions as S_{FF} for the first j items. Then, there is a schedule S' that makes the same eviction requests as S_{FF} for the first $j+1$ items with no more faults than S.

Proof.

- From $j+2$ onward, S' follows S until either:
  1. S evicts v. In this case, S' evicts u.
  2. S evicts $g \neq v$ to bring u into the cache. In this case, S' evicts g and brings in v.
- In either case, both S and S' have a page fault, and afterwards their cache match.

□

Proving FF Optimality

Exchange Argument

Theorem 8

Let S be a schedule for the n request that make the same eviction decisions as S_{FF} for the first j items. Then, there is a schedule S' that makes the same eviction requests as S_{FF} for the first $j+1$ items with no more faults than S.

How do we get optimality of S_{FF} from Theorem 8?

Proving FF Optimality

Exchange Argument

Theorem 8

Let S be a schedule for the n request that make the same eviction decisions as S_{FF} for the first j items. Then, there is a schedule S' that makes the same eviction requests as S_{FF} for the first $j+1$ items with no more faults than S.

How do we get optimality of S_{FF} from Theorem 8?

By induction: We begin with the optimal schedule S^* and inductively apply Theorem 8 for $j = 1, 2, 3, \ldots, n$, which after the n iterations, produces S_{FF}.

MST

Minimum Spanning Tree Problem

MST Problem

Let $G = (V, E)$ be a connected graph, where $|V| = n$ and $|E| = m$. For each edge e, $c_e > 0$ is the cost of the edge.

- Find an edge set $F \subseteq E$ with minimum cost that keeps the graph connected. That is, F should minimize $\sum_{e \in F} c_e$.

Minimum Spanning Tree Problem

MST Problem

Let $G = (V, E)$ be a connected graph, where $|V| = n$ and $|E| = m$. For each edge e, $c_e > 0$ is the cost of the edge.

- Find an edge set $F \subseteq E$ with minimum cost that keeps the graph connected. That is, F should minimize $\sum_{e \in F} c_e$.

Observation 3

Let $T = (V, F)$ be a minimum-cost solution to the problem described above. Then, T is a tree.

Minimum Spanning Tree Problem

MST Problem

Let $G = (V, E)$ be a connected graph, where $|V| = n$ and $|E| = m$. For each edge e, $c_e > 0$ is the cost of the edge.

- Find an edge set $F \subseteq E$ with minimum cost that keeps the graph connected. That is, F should minimize $\sum_{e \in F} c_e$.

Observation 3

Let $T = (V, F)$ be a minimum-cost solution to the problem described above. Then, T is a tree.

Proof.

Minimum Spanning Tree Problem

MST Problem

Let $G = (V, E)$ be a connected graph, where $|V| = n$ and $|E| = m$. For each edge e, $c_e > 0$ is the cost of the edge.

- Find an edge set $F \subseteq E$ with minimum cost that keeps the graph connected. That is, F should minimize $\sum_{e \in F} c_e$.

Observation 3

Let $T = (V, F)$ be a minimum-cost solution to the problem described above. Then, T is a tree.

Proof.

- By the definition of the problem, T must be connected.
- By way of contradiction, assume that T has a cycle C. Remove any edge from C resulting in a graph T'. T' is still connect and has a cost less than T.

$\square$

Algorithm Design

TopHat Discussion 3: What greedy heuristic might work?

Algorithm Design

TopHat Discussion 3: What greedy heuristic might work?

Kruskal's (1956) Algorithm

- Sort edges by cost from lowest to highest.
- Insert edges unless insertion would create a cycle.

Algorithm Design

TopHat Discussion 3: What greedy heuristic might work?

Kruskal's (1956) Algorithm

- Sort edges by cost from lowest to highest.
- Insert edges unless insertion would create a cycle.

Jarník's (1929), Kruskal's (1956), Prim's (1957), Loberman and Weinberger (1957), Dijkstra's (1958) Algorithm

- Initialize a node set S with an arbitrary node s.
- Keep the least expensive edge as long as it does not create a cycle.

Algorithm Design

TopHat Discussion 3: What greedy heuristic might work?

Kruskal's (1956) Algorithm

- Sort edges by cost from lowest to highest.
- Insert edges unless insertion would create a cycle.

Prim's (1957) Algorithm

- Initialize a node set S with an arbitrary node s.
- Keep the least expensive edge as long as it does not create a cycle.

Algorithm Design

TopHat Discussion 3: What greedy heuristic might work?

Kruskal's (1956) Algorithm

- Sort edges by cost from lowest to highest.
- Insert edges unless insertion would create a cycle.

Prim's (1957) Algorithm

- Initialize a node set S with an arbitrary node s.
- Keep the least expensive edge as long as it does not create a cycle.

Kruskal's (1956) Algorithm

- Sort edges by cost from highest to lowest.
- Remove edges unless graph would become disconnected.

Algorithm Design

TopHat Discussion 3: What greedy heuristic might work?

Kruskal's (1956) Algorithm

- Sort edges by cost from lowest to highest.
- Insert edges unless insertion would create a cycle.

Prim's (1957) Algorithm

- Initialize a node set S with an arbitrary node s.
- Keep the least expensive edge as long as it does not create a cycle.

Reverse-Delete (Kruskal's 1956) Algorithm

- Sort edges by cost from highest to lowest.
- Remove edges unless graph would become disconnected.

Assume Distinct Weights

WLOG (without loss of generality)

Theorem 9

(HW Q2) If all edge weights in a connected graph are distinct, then G has a unique MST.

Assume Distinct Weights

WLOG (without loss of generality)

Theorem 9

(HW Q2) If all edge weights in a connected graph are distinct, then G has a unique MST.

Observation 4

All we need is a consistent tie-breaker when $c_{e_1} = c_{e_2}$ for some pair of edges. I.e. based on the labels of the vertices of $e_1 \cup e_2$.

Assume Distinct Weights

WLOG (without loss of generality)

Theorem 9

(HW Q2) If all edge weights in a connected graph are distinct, then G has a unique MST.

Observation 4

All we need is a consistent tie-breaker when $c_{e_1} = c_{e_2}$ for some pair of edges. I.e. based on the labels of the vertices of $e_1 \cup e_2$.

Assumption: all edge weights are distinct.

Analyzing MST Heuristics

Lemma 10

Let $S \subset V$ be an non-empty proper subset of the nodes, and let $e = (v, w)$ be the minimum cost edge connecting S and $V \setminus S$. Then, every MST contains e.

Analyzing MST Heuristics

Lemma 10

Let $S \subset V$ be an non-empty proper subset of the nodes, and let $e = (v, w)$ be the minimum cost edge connecting S and $V \setminus S$. Then, every MST contains e.

Proof.

Analyzing MST Heuristics

Lemma 10

Let $S \subset V$ be an non-empty proper subset of the nodes, and let $e = (v, w)$ be the minimum cost edge connecting S and $V \setminus S$. Then, every MST contains e.

Proof.

By exchange argument:

Analyzing MST Heuristics

Lemma 10

Let $S \subset V$ be an non-empty proper subset of the nodes, and let $e = (v, w)$ be the minimum cost edge connecting S and $V \setminus S$. Then, every MST contains e.

Proof.

By exchange argument:

- Let T be a spanning tree that does not contain e.

Analyzing MST Heuristics

Lemma 10

Let $S \subset V$ be an non-empty proper subset of the nodes, and let $e = (v, w)$ be the minimum cost edge connecting S and $V \setminus S$. Then, every MST contains e.

Proof.

By exchange argument:

- Let T be a spanning tree that does not contain e.
- Let $e' = (v', w')$, where e' **is in** $P_{v,w} \in T$**,** $v' \in S$**, and** $w' \in V \setminus S$.

Analyzing MST Heuristics

Lemma 10

Let $S \subset V$ be an non-empty proper subset of the nodes, and let $e = (v, w)$ be the minimum cost edge connecting S and $V \setminus S$. Then, every MST contains e.

Proof.

By exchange argument:

- Let T be a spanning tree that does not contain e.
- Let $e' = (v', w')$, where e' **is in** $P_{v,w} \in T$**,** $v' \in S$**, and** $w' \in V \setminus S$.
- Let $T' = T \setminus e' \cup e$.

Analyzing MST Heuristics

Lemma 10

Let $S \subset V$ be an non-empty proper subset of the nodes, and let $e = (v, w)$ be the minimum cost edge connecting S and $V \setminus S$. Then, every MST contains e.

Proof.

By exchange argument:

- Let T be a spanning tree that does not contain e.
- Let $e' = (v', w')$, where e' **is in** $P_{v,w} \in T$**,** $v' \in S$**, and** $w' \in V \setminus S$.
- Let $T' = T \setminus e' \cup e$.
- T' is connected as e is a $P_{v,w} \in T'$.

Analyzing MST Heuristics

Lemma 10

Let $S \subset V$ be an non-empty proper subset of the nodes, and let $e = (v, w)$ be the minimum cost edge connecting S and $V \setminus S$. Then, every MST contains e.

Proof.

By exchange argument:

- Let T be a spanning tree that does not contain e.
- Let $e' = (v', w')$, where e' **is in** $P_{v,w} \in T$**,** $v' \in S$**, and** $w' \in V \setminus S$.
- Let $T' = T \setminus e' \cup e$.
- T' is connected as e is a $P_{v,w} \in T'$.
- Since $c_e < c_{e'}$, cost of T' is less than T.

$\square$

Kruskal's Algorithm is Optimal

Kruskal's (1956) Algorithm

- Sort edges by cost from lowest to highest.
- Insert edges unless insertion would create a cycle.

Theorem 11

Kruskal's Algorithm produces an MST.

Kruskal's Algorithm is Optimal

Kruskal's (1956) Algorithm

- Sort edges by cost from lowest to highest.
- Insert edges unless insertion would create a cycle.

Theorem 11

Kruskal's Algorithm produces an MST.

Proof.

Kruskal's Algorithm is Optimal

Kruskal's (1956) Algorithm

- Sort edges by cost from lowest to highest.
- Insert edges unless insertion would create a cycle.

Theorem 11

Kruskal's Algorithm produces an MST.

Proof.

- Let $e = (v, w)$ be the edge added at any step i.

Kruskal's Algorithm is Optimal

Kruskal's (1956) Algorithm

- Sort edges by cost from lowest to highest.
- Insert edges unless insertion would create a cycle.

Theorem 11

Kruskal's Algorithm produces an MST.

Proof.

- Let $e = (v, w)$ be the edge added at any step i.
- Since e does not create a cycle, $v \in S$ and $w \notin S$ (WLOG).

Kruskal's Algorithm is Optimal

Kruskal's (1956) Algorithm

- Sort edges by cost from lowest to highest.
- Insert edges unless insertion would create a cycle.

Theorem 11

Kruskal's Algorithm produces an MST.

Proof.

- Let $e = (v, w)$ be the edge added at any step i.
- Since e does not create a cycle, $v \in S$ and $w \notin S$ (WLOG).
- As c_e is the minimum cost edge, the claim follows from Lemma 10.

□

Prim's Algorithm is Optimal

Prim's (1957) Algorithm

- Initialize a node set S with an arbitrary node s.
- Keep the least expensive edge as long as it does not create a cycle.

Theorem 12

Prim's Algorithm produces an MST.

Prim's Algorithm is Optimal

Prim's (1957) Algorithm

- Initialize a node set S with an arbitrary node s.
- Keep the least expensive edge as long as it does not create a cycle.

Theorem 12

Prim's Algorithm produces an MST.

Proof.

Prim's Algorithm is Optimal

Prim's (1957) Algorithm

- Initialize a node set S with an arbitrary node s.
- Keep the least expensive edge as long as it does not create a cycle.

Theorem 12

Prim's Algorithm produces an MST.

Proof.

- Immediate from Lemma 10.

Prim's Algorithm is Optimal

Prim's (1957) Algorithm

- Initialize a node set S with an arbitrary node s.
- Keep the least expensive edge as long as it does not create a cycle.

Theorem 12

Prim's Algorithm produces an MST.

Proof.

- Immediate from Lemma 10.
- That is, Prim's algorithm does exactly what Lemma 10 describes.

□

Reverse-Delete is Optimal

Reverse-Delete (Kruskal's 1956) Algorithm

- Sort edges by cost from highest to lowest.
- Remove edges unless graph would become disconnected.

How should we prove that it produces an MST?

Reverse-Delete is Optimal

Lemma 13

Let C be any cycle in G, and let e be the most expensive edge of C. Then, e is not in any MST of G.

Reverse-Delete is Optimal

Lemma 13

Let C be any cycle in G, and let e be the most expensive edge of C. Then, e is not in any MST of G.

Proof.

Reverse-Delete is Optimal

Lemma 13

Let C be any cycle in G, and let e be the most expensive edge of C. Then, e is not in any MST of G.

Proof.

- Let T be a spanning tree that does contain e.

Reverse-Delete is Optimal

Lemma 13

Let C be any cycle in G, and let e be the most expensive edge of C. Then, e is not in any MST of G.

Proof.

- Let T be a spanning tree that does contain e.
- Let S and $V \setminus S$ be the nodes of the connected components after removing e from T.

Reverse-Delete is Optimal

Lemma 13

Let C be any cycle in G, and let e be the most expensive edge of C. Then, e is not in any MST of G.

Proof.

- Let T be a spanning tree that does contain e.
- Let S and $V \setminus S$ be the nodes of the connected components after removing e from T.
- Let e' be an edge in C that connects S and $V \setminus S$.

Reverse-Delete is Optimal

Lemma 13

Let C be any cycle in G, and let e be the most expensive edge of C. Then, e is not in any MST of G.

Proof.

- Let T be a spanning tree that does contain e.
- Let S and $V \setminus S$ be the nodes of the connected components after removing e from T.
- Let e' be an edge in C that connects S and $V \setminus S$.
- Let $T' = T \setminus e \cup e'$.

Reverse-Delete is Optimal

Lemma 13

Let C be any cycle in G, and let e be the most expensive edge of C. Then, e is not in any MST of G.

Proof.

- Let T be a spanning tree that does contain e.
- Let S and $V \setminus S$ be the nodes of the connected components after removing e from T.
- Let e' be an edge in C that connects S and $V \setminus S$.
- Let $T' = T \setminus e \cup e'$.
- T' is connected as e' reconnects S and $V \setminus S$.

Reverse-Delete is Optimal

Lemma 13

Let C be any cycle in G, and let e be the most expensive edge of C. Then, e is not in any MST of G.

Proof.

- Let T be a spanning tree that does contain e.
- Let S and $V \setminus S$ be the nodes of the connected components after removing e from T.
- Let e' be an edge in C that connects S and $V \setminus S$.
- Let $T' = T \setminus e \cup e'$.
- T' is connected as e' reconnects S and $V \setminus S$.
- Since $c_e > c_{e'}$, cost of T' is less than T.

$\square$

Reverse-Delete is Optimal

Lemma 13

Let C be any cycle in G, and let e be the most expensive edge of C. Then, e is not in any MST of G.

Theorem 14

Reverse-Delete Algorithm produces an MST.

Reverse-Delete is Optimal

Lemma 13

Let C be any cycle in G, and let e be the most expensive edge of C. Then, e is not in any MST of G.

Theorem 14

Reverse-Delete Algorithm produces an MST.

Proof.

Reverse-Delete is Optimal

Lemma 13

Let C be any cycle in G, and let e be the most expensive edge of C. Then, e is not in any MST of G.

Theorem 14

Reverse-Delete Algorithm produces an MST.

Proof.

- Let $e = (v, w)$ be an edge removed at any step i.

Reverse-Delete is Optimal

Lemma 13

Let C be any cycle in G, and let e be the most expensive edge of C. Then, e is not in any MST of G.

Theorem 14

Reverse-Delete Algorithm produces an MST.

Proof.

- Let $e = (v, w)$ be an edge removed at any step i.
- By definition e, belongs to a cycle C.

Reverse-Delete is Optimal

Lemma 13

Let C be any cycle in G, and let e be the most expensive edge of C. Then, e is not in any MST of G.

Theorem 14

Reverse-Delete Algorithm produces an MST.

Proof.

- Let $e = (v, w)$ be an edge removed at any step i.
- By definition e, belongs to a cycle C.
- As c_e is the maximum cost edge of C, the claim follows from Lemma 13.

$\square$

Implementing Prim's Algorithm

Prim's (1957) Algorithm

- Initialize a node set S with an arbitrary node s.
- Keep the least expensive edge as long as it does not create a cycle.

Key Operations

Implementing Prim's Algorithm

Prim's (1957) Algorithm

- Initialize a node set S with an arbitrary node s.
- Keep the least expensive edge as long as it does not create a cycle.

Key Operations

- Retrieve the minimum valued edge between S and $V \setminus S$.

Implementing Prim's Algorithm

Prim's (1957) Algorithm

- Initialize a node set S with an arbitrary node s.
- Keep the least expensive edge as long as it does not create a cycle.

Key Operations

- Retrieve the minimum valued edge between S and $V \setminus S$.
- Prim's and Dijkstra's have nearly identical implementations (but different minimizers)!

Implementing Prim's Algorithm

Prim's (1957) Algorithm

- Initialize a node set S with an arbitrary node s.
- Keep the least expensive edge as long as it does not create a cycle.

Key Operations

- Retrieve the minimum valued edge between S and $V \setminus S$.
- Prim's and Dijkstra's have nearly identical implementations (but different minimizers)!

Priority Queue (min-heap)

- `ExtractMin` ($O(1)$): $n - 1$ times.
- `ChangeKey` ($O(\log(n))$): m times.

Overall: $O(m \log(n))$

Implementing Kruskal's Algorithm

Kruskal's (1956) Algorithm

- Sort edges by cost from lowest to highest.
- Insert edges unless insertion would create a cycle.

Key Operations

Implementing Kruskal's Algorithm

Kruskal's (1956) Algorithm

- Sort edges by cost from lowest to highest.
- Insert edges unless insertion would create a cycle.

Key Operations

- Sorting the edges: $(O(m \log m)$ and, since $m \leq n^2$, $O(m \log n))$.

Implementing Kruskal's Algorithm

Kruskal's (1956) Algorithm

- Sort edges by cost from lowest to highest.
- Insert edges unless insertion would create a cycle.

Key Operations

- Sorting the edges: $(O(m \log m)$ and, since $m \leq n^2$, $O(m \log n))$.
- Maintain sets of connected components that we merge.

Implementing Kruskal's Algorithm

Kruskal's (1956) Algorithm

- Sort edges by cost from lowest to highest.
- Insert edges unless insertion would create a cycle.

Key Operations

- Sorting the edges: ($O(m \log m)$ and, since $m \leq n^2$, $O(m \log n)$).
- Maintain sets of connected components that we merge.
- Initialize one set per node: $O(n)$.

Implementing Kruskal's Algorithm

Kruskal's (1956) Algorithm

- Sort edges by cost from lowest to highest.
- Insert edges unless insertion would create a cycle.

Key Operations

- Sorting the edges: ($O(m \log m)$ and, since $m \leq n^2$, $O(m \log n)$).
- Maintain sets of connected components that we merge.
- Initialize one set per node: $O(n)$.

Union-Find Data Structure

- `Find(x)`: Finds the set containing x. ($O(\log n)$ can be $O(\alpha(n))$)
- `Union(x,y)`: Joins two sets x and y. ($O(1)$)

Union-Find / Disjoint-Set

Key Operations

- `Find(x)`: Finds the set containing x. ($O(\log n)$ can be $O(\alpha(n))$)
- `Union(x,y)`: Joins two sets x and y. ($O(1)$)

Union-Find / Disjoint-Set

Key Operations

- `Find(x)`: Finds the set containing x. ($O(\log n)$ can be $O(\alpha(n))$)
- `Union(x,y)`: Joins two sets x and y. ($O(1)$)

Basic Container

`node`	`rank`	`parent`

Union-Find / Disjoint-Set

Key Operations

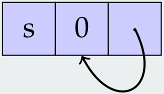
- `Find(x)`: Finds the set containing x. ($O(\log n)$ can be $O(\alpha(n))$)
- `Union(x,y)`: Joins two sets x and y. ($O(1)$)

Basic Container

`node`	`rank`	`parent`

Initializing Data Structure for Kruskal's

For each node s, create a singleton set. That is each container has rank 0 and points to itself.



Union-Find Operations

`Find`(x): $O(\log n)$

Union-Find Operations

`Find`(x): $O(\log n)$

- If x`.parent` points to x, return x.
- Else `Find`(x`.parent`)

Union-Find Operations

`Find(`x`)`: $O(\log n)$

- If x`.parent` points to x, return x.
- Else `Find(`x`.parent)`
- $O(\log n)$ requires balanced trees.

Union-Find Operations

`Find`(x): $O(\log n)$

- If x`.parent` points to x, return x.
- Else `Find`(x`.parent`)
- $O(\log n)$ requires balanced trees.
- $O(\alpha(n))$ with *path compression*.

Union-Find Operations

`Find(`x`)`: $O(\log n)$

- If x`.parent` points to x, return x.
- Else `Find(`x`.parent)`
- $O(\log n)$ requires balanced trees.
- $O(\alpha(n))$ with *path compression*.

`Union(`x`,`y`)`: $O(1)$

Union-Find Operations

Find(x): $O(\log n)$

- If x.parent points to x, return x.
- Else Find(x.parent)
- $O(\log n)$ requires balanced trees.
- $O(\alpha(n))$ with *path compression*.

Union(x,y): $O(1)$

- (WLOG) x.rank $\geq y$.rank:
  y.parent $= x$

Union-Find Operations

`Find(`x`)`: $O(\log n)$

- If x`.parent` points to x, return x.
- Else `Find(`x`.parent)`
- $O(\log n)$ requires balanced trees.
- $O(\alpha(n))$ with *path compression*.

`Union(`x`,`y`)`: $O(1)$

- (WLOG) x`.rank` $\geq y$`.rank`:
  y`.parent` $= x$
- If x`.rank` $= y$`.rank`:
  x`.rank` $:= x$`.rank + 1`

Union-Find Operations

`Find(`x`)`: $O(\log n)$

- If x`.parent` points to x, return x.
- Else `Find(`x`.parent)`
- $O(\log n)$ requires balanced trees.
- $O(\alpha(n))$ with *path compression*.

`Union(`x`,`y`)`: $O(1)$

- (WLOG) x`.rank` $\geq y$`.rank`:
  y`.parent` $= x$
- If x`.rank` $= y$`.rank`:
  x`.rank` $:= x$`.rank + 1`
- By using rank, we maintain balanced sets if we start with balanced sets.

Implementing Kruskal's Algorithm

Kruskal's (1956) Algorithm

- Sort edges by cost from lowest to highest.
- Insert edges unless insertion would create a cycle.

Key Operations

- Sorting the edges: ($O(m \log m)$ and, since $m \leq n^2$, $O(m \log n)$).
- Maintain sets of connected components that we merge.
- Initialize one set per node: $O(n)$.

Union-Find Data Structure
TH: How many Find and Unions?

- `Find(x)`: Finds the set containing x.
- `Union(x,y)`: Joins two sets x and y.

Implementing Kruskal's Algorithm

Kruskal's (1956) Algorithm

- Sort edges by cost from lowest to highest.
- Insert edges unless insertion would create a cycle.

Key Operations

- Sorting the edges: $(O(m \log m)$ and, since $m \leq n^2$, $O(m \log n))$.
- Maintain sets of connected components that we merge.
- Initialize one set per node: $O(n)$.

Union-Find Data Structure

- `Find(x)`: $2m$ times $O(\log n)$ (can be $O(\alpha(n))$).
- `Union(x,y)`: $n-1$ times $O(1)$.

Graph Exploration Overview

BFS and DFS

- Traverses a graph G starting from some node s.
- Builds a tree T.
- No guarantee on any distance measure.

Graph Exploration Overview

BFS and DFS

- Traverses a graph G starting from some node s.
- Builds a tree T.
- No guarantee on any distance measure.

Dijktra's

- Traverses a graph starting from some node s.
- Builds a tree T.
- All s to u paths in T are the shortest such path in G.

Graph Exploration Overview

BFS and DFS

- Traverses a graph G starting from some node s.
- Builds a tree T.
- No guarantee on any distance measure.

Dijktra's

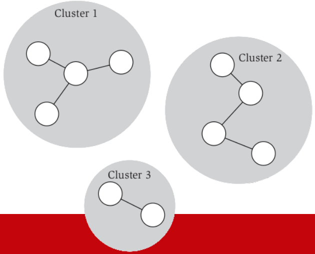
- Traverses a graph starting from some node s.
- Builds a tree T.
- All s to u paths in T are the shortest such path in G.

MST Algorithms

- Explores a graph G edges.
- Builds a tree T.
- T is minimum cost to connect all nodes in G.

Clustering

k-Clustering



Maximizing Spacing Problem

- A universe $\mathcal{U} := \{p_1, \ldots, p_n\}$ of n objects.
- Distance function $d : \mathcal{U} \times \mathcal{U} \to \mathbb{R}$ such that, for all $p_i, p_j \in \mathcal{U}$:
  - $d(p_i, p_i) = 0$
  - $d(p_i, p_j) > 0$
  - $d(p_i, p_j) = d(p_j, p_i)$
- Objective: Partition $\mathcal{U}$ into k non-empty groups $\mathcal{C} := C_1, \ldots, C_k$ with maximum spacing:

$$\text{maximize} \min_{C_i, C_j \in \mathcal{C}} \min_{u \in C_i, v \in C_j} d(u, v)$$

Algorithm Design

TopHat Discussion 4: What greedy approach might work?

Algorithm Design

Algorithm

- Build an MST.
- Remove $k - 1$ largest edges.

Algorithm Design

Algorithm

- Build an MST.
- Remove $k - 1$ largest edges.

k-Clusters at max spacing?

- Start with a tree, remove $k - 1$ edges: We get a forest of k trees.
- By definition largest edges are removed so max spacing.

Algorithm Design

Algorithm

- Build an MST.
- Remove $k-1$ largest edges.

k-Clusters at max spacing?

- Start with a tree, remove $k-1$ edges: We get a forest of k trees.
- By definition largest edges are removed so max spacing.

TopHat Q10: Which MST algorithm?

Algorithm Design

Algorithm

- Build an MST.
- Remove $k-1$ largest edges.

k-Clusters at max spacing?

- Start with a tree, remove $k-1$ edges: We get a forest of k trees.
- By definition largest edges are removed so max spacing.

TopHat Q10: Which MST algorithm?

Kruskal's ($O(m \log n)$ which is $O(n^2 \log n)$ for clustering):

- Merge sets from lowest to most expensive edges.
- Stop when we have k sets.

Prefix Codes

Binary Encoding

Fixed-Width Encoding

- Set of symbols $S := \{a, b, c, d, e\}$.
- Encoding function $\gamma : S \to \{0, 1\}^k$.
  $\gamma(S) := \{000, 001, 010, 011, 100\}$.
- Ex. ASCII

Binary Encoding

Fixed-Width Encoding

- Set of symbols $S := \{a, b, c, d, e\}$.
- Encoding function $\gamma : S \to \{0, 1\}^k$.
  $\gamma(S) := \{000, 001, 010, 011, 100\}$.
- Ex. ASCII
- TopHat Q11: Decode 000010.

Binary Encoding

Fixed-Width Encoding

- Set of symbols $S := \{a, b, c, d, e\}$.
- Encoding function $\gamma : S \to \{0, 1\}^k$.
  $\gamma(S) := \{000, 001, 010, 011, 100\}$.
- Ex. ASCII
- TopHat Q11: Decode 000010.

Variable-Width Encoding

- Set of symbols $S := \{a, b, c, d, e\}$.
- Encoding function $\gamma : S \to \{0, 1\}^*$.
  $\gamma(S) := \{0, 1, 10, 01, 11\}$.

Binary Encoding

Fixed-Width Encoding

- Set of symbols $S := \{a, b, c, d, e\}$.
- Encoding function $\gamma : S \to \{0, 1\}^k$.
  $\gamma(S) := \{000, 001, 010, 011, 100\}$.
- Ex. ASCII
- TopHat Q11: Decode 000010.

Variable-Width Encoding

- Set of symbols $S := \{a, b, c, d, e\}$.
- Encoding function $\gamma : S \to \{0, 1\}^*$.
  $\gamma(S) := \{0, 1, 10, 01, 11\}$.
- TopHat Q12: How many ways to decode 0010?

Unique Variable-Width Encodings

Prefix Codes

Encoding of S such that no encoding of a symbol in S is a prefix of another.

- Set of symbols $S := \{a, b, c, d, e\}$.
- Encoding function $\gamma : S \to \{0, 1\}^*$.
  $\gamma(S) := \{11, 01, 001, 000, 100\}$.

Unique Variable-Width Encodings

Prefix Codes

Encoding of S such that no encoding of a symbol in S is a prefix of another.

- Set of symbols $S := \{a, b, c, d, e\}$.
- Encoding function $\gamma : S \to \{0, 1\}^*$.
  $\gamma(S) := \{11, 01, 001, 000, 100\}$.
- 0010 invalid sequence

Unique Variable-Width Encodings

Prefix Codes

Encoding of S such that no encoding of a symbol in S is a prefix of another.

- Set of symbols $S := \{a, b, c, d, e\}$.
- Encoding function $\gamma : S \to \{0, 1\}^*$.
  $\gamma(S) := \{11, 01, 001, 000, 100\}$.
- 0010 invalid sequence
- TopHat 13: Decode 1101.

Unique Variable-Width Encodings

Prefix Codes

Encoding of S such that no encoding of a symbol in S is a prefix of another.

- Set of symbols $S := \{a, b, c, d, e\}$.
- Encoding function $\gamma : S \to \{0, 1\}^*$.
  $\gamma(S) := \{11, 01, 001, 000, 100\}$.

Easy Decoding

Scan left to right, once an encoding is matched, output symbol.

Unique Variable-Width Encodings

Prefix Codes

Encoding of S such that no encoding of a symbol in S is a prefix of another.

- Set of symbols $S := \{a, b, c, d, e\}$.
- Encoding function $\gamma : S \to \{0, 1\}^*$.
  $\gamma(S) := \{11, 01, 001, 000, 100\}$.

Easy Decoding

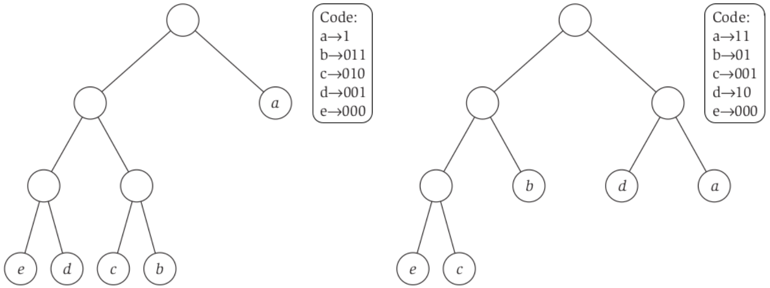
Scan left to right, once an encoding is matched, output symbol.

Optimal Prefix Codes

- For a set of symbols S, let f_x denote the frequency of x in the text to be encoded.
- Average bits $\text{ABL}(\gamma) := \sum_{x \in S} f_x \cdot |\gamma(x)|$.
- Goal: Find γ that minimizes ABL.

Algorithm Design

Prefix Binary Trees



Optimal Prefix Tree is Full

Theorem 15

The binary tree corresponding to the optimal prefix code is full.

Optimal Prefix Tree is Full

Theorem 15

The binary tree corresponding to the optimal prefix code is full.

Proof.

Optimal Prefix Tree is Full

Theorem 15

The binary tree corresponding to the optimal prefix code is full.

Proof.

By exchange argument:

- Let T be an optimal prefix tree with a node u with one child v.

Optimal Prefix Tree is Full

Theorem 15

The binary tree corresponding to the optimal prefix code is full.

Proof.

By exchange argument:

- Let T be an optimal prefix tree with a node u with one child v.
- Let T' be T with u replaced with v.

Optimal Prefix Tree is Full

Theorem 15

The binary tree corresponding to the optimal prefix code is full.

Proof.

By exchange argument:

- Let T be an optimal prefix tree with a node u with one child v.
- Let T' be T with u replaced with v.
- Distance to v decreases by 1 in T', a contradiction.

□

Top-Down Approach

Algorithm

- Split S into two sets such that the sets frequency are $1/2$ the total frequency.
- Recurse on new sets until singletons.

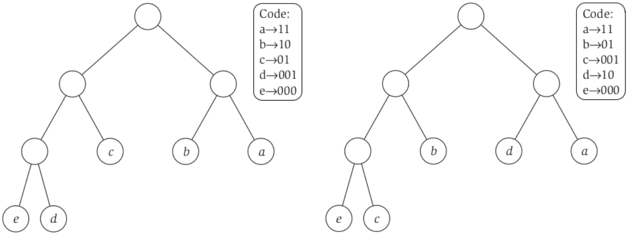
Top-Down Approach

Algorithm

- Split S into two sets such that the sets frequency are 1/2 the total frequency.
- Recurse on new sets until singletons.

$$f_a = .32, f_b = .25, f_c = .2, f_d = .18, f_e = .05$$

$\textsc{abl}(\textsc{opt}) = 2.23$ $\textsc{abl}(\text{TopDown}) = 2.25$



What if we knew the optimal tree?

Let T^* be the optimal (unlabelled) prefix tree.

What if we knew the optimal tree?

Let T^* be the optimal (unlabelled) prefix tree.

Lemma 16

Let u, v be leaves of $T*$ such that $depth(u) < depth(v)$, where u is labelled with y and v is labelled with z. Then, $f_y \geq f_z$.

What if we knew the optimal tree?

Let T^* be the optimal (unlabelled) prefix tree.

Lemma 16

Let u, v be leaves of $T*$ such that $depth(u) < depth(v)$, where u is labelled with y and v is labelled with z. Then, $f_y \geq f_z$.

Proof.

What if we knew the optimal tree?

Let T^* be the optimal (unlabelled) prefix tree.

Lemma 16

Let u, v be leaves of $T*$ such that $depth(u) < depth(v)$, where u is labelled with y and v is labelled with z. Then, $f_y \geq f_z$.

Proof.

If $f_y < f_z$, exchange the labelling of y and z. Since depth$(u) <$ depth(v), ABL(T^*) must decrease with the new labelling. □

What if we knew the optimal tree?

Let T^* be the optimal (unlabelled) prefix tree.

Lemma 16

Let u, v be leaves of $T*$ such that $depth(u) < depth(v)$, where u is labelled with y and v is labelled with z. Then, $f_y \geq f_z$.

Labelling T*

What if we knew the optimal tree?

Let T^* be the optimal (unlabelled) prefix tree.

Lemma 16

Let u, v be leaves of $T*$ such that $depth(u) < depth(v)$, where u is labelled with y and v is labelled with z. Then, $f_y \geq f_z$.

Labelling T*

- Order symbols by increasing frequency.
- Assign them to leaves of T^* by decreasing depth.

What if we knew the optimal tree?

Let T^* be the optimal (unlabelled) prefix tree.

Lemma 16

Let u, v be leaves of $T*$ such that $depth(u) < depth(v)$, where u is labelled with y and v is labelled with z. Then, $f_y \geq f_z$.

Labelling T*

- Order symbols by increasing frequency.
- Assign them to leaves of T^* by decreasing depth.

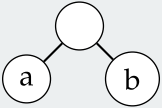
Observation 5

In T^*, the lowest frequency letters are siblings.

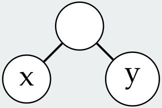
Bottom-Up Approach

Huffman Code

Huffman's Algorithm

- If $|S| = 2$, return 

- Let x and y be the lowest frequency symbols.
- Set $S := S \setminus \{x, y\} \cup \{xy\}$ and $f_{xy} = f_x + f_y$.
- $T :=$ recurse on S.
- Replace 

with 

- return T

Huffman Codes are Optimal

Lemma 17

Let T' be the tree at the $(k-1)$-st step, and let T be the tree at the k-th step. $\textsc{abl}(T') = \textsc{abl}(T) - f_w$, where w is the symbol replaced in the k-th step by y and z.

Huffman Codes are Optimal

Lemma 17

Let T' be the tree at the $(k-1)$-st step, and let T be the tree at the k-th step. $\textsc{abl}(T') = \textsc{abl}(T) - f_w$, where w is the symbol replaced in the k-th step by y and z.

Proof.

$$\begin{aligned}
\textsc{abl}(T) &= \sum_{x \in S} f_x \cdot \text{depth}(x) \\
&= f_y \cdot \text{depth}(y) + f_z \cdot \text{depth}(z) + \sum_{x \in S; x \notin \{y,z\}} f_x \cdot \text{depth}(x) \\
&= f_w + f_w \cdot \text{depth}(w) + \sum_{x \in S \setminus \{y,z\}} f_x \cdot \text{depth}(x) \\
&= f_w + \textsc{abl}(T')
\end{aligned}$$

□

Huffman Codes are Optimal

Lemma 17

Let T' be the tree at the $(k-1)$-st step, and let T be the tree at the k-th step. $\textsc{abl}(T') = \textsc{abl}(T) - f_w$, where w is the symbol replaced in the k-th step by y and z.

Theorem 18

Huffman Algorithm is optimal.

Proof.

Huffman Codes are Optimal

Lemma 17

Let T' be the tree at the $(k-1)$-st step, and let T be the tree at the k-th step. $\textsc{abl}(T') = \textsc{abl}(T) - f_w$, where w is the symbol replaced in the k-th step by y and z.

Theorem 18

Huffman Algorithm is optimal.

Proof.

By induction:

HUFFMAN CODES ARE OPTIMAL

Lemma 17

Let T' be the tree at the $(k-1)$-st step, and let T be the tree at the k-th step. $\textsc{abl}(T') = \textsc{abl}(T) - f_w$, where w is the symbol replaced in the k-th step by y and z.

Theorem 18

Huffman Algorithm is optimal.

Proof.

By induction:

- Base case $|S| = 2$

Huffman Codes are Optimal

Lemma 17

Let T' be the tree at the $(k-1)$-st step, and let T be the tree at the k-th step. $\textsc{abl}(T') = \textsc{abl}(T) - f_w$, where w is the symbol replaced in the k-th step by y and z.

Theorem 18

Huffman Algorithm is optimal.

Proof.

By induction:

- Base case $|S| = 2$
- Inductive step: We have T. By way of contradiction, assume $\textsc{abl}(Z) \leq \textsc{abl}(T)$.

Huffman Codes are Optimal

Lemma 17

Let T' be the tree at the $(k-1)$-st step, and let T be the tree at the k-th step. $\textsc{abl}(T') = \textsc{abl}(T) - f_w$, where w is the symbol replaced in the k-th step by y and z.

Theorem 18

Huffman Algorithm is optimal.

Proof.

By induction:

- We observed that y and z are siblings. Hence:

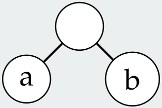
$$\begin{aligned} & \textsc{abl}(Z) < \textsc{abl}(T) \\ \iff & \textsc{abl}(Z') + f_w < \textsc{abl}(T') + f_w, \text{ by Lemma 17} \\ \iff & \textsc{abl}(Z') < \textsc{abl}(T'), \text{ a contradiction.} \end{aligned}$$

$\square$

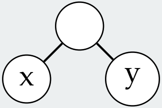
Bottom-Up Approach

Huffman Code

Huffman's Algorithm

- If $|S| = 2$, return 

- Let x and y be the lowest frequency symbols.
- Set $S := S \setminus \{x, y\} \cup \{xy\}$ and $f_{xy} = f_x + f_y$.
- $T :=$ recurse on S.
- Replace 

with 

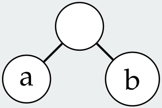
- return T

Runtime:

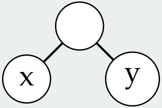
Bottom-Up Approach

Huffman Code

Huffman's Algorithm

- If $|S| = 2$, return 

- Let x and y be the lowest frequency symbols.
- Set $S := S \setminus \{x, y\} \cup \{xy\}$ and $f_{xy} = f_x + f_y$.
- $T :=$ recurse on S.
- Replace 

with 

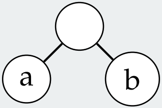
- return T

Runtime: $|S| - 1$ recursions with find min over $|S_i|$ elements

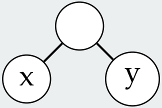
Bottom-Up Approach

Huffman Code

Huffman's Algorithm

- If $|S| = 2$, return 

- Let x and y be the lowest frequency symbols.
- Set $S := S \setminus \{x, y\} \cup \{xy\}$ and $f_{xy} = f_x + f_y$.
- $T :=$ recurse on S.
- Replace 

with 

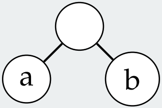
- return T

Runtime: $O(|S|^2)$

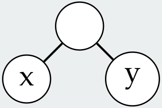
Bottom-Up Approach

Huffman Code

Huffman's Algorithm

- If $|S| = 2$, return 

- Let x and y be the lowest frequency symbols.
- Set $S := S \setminus \{x, y\} \cup \{xy\}$ and $f_{xy} = f_x + f_y$.
- $T :=$ recurse on S.
- Replace 

with 

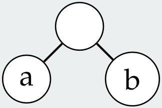
- return T

Runtime: $O(|S|^2)$
what about $O(|S| \log |S|)$?

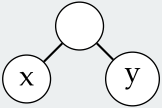
Bottom-Up Approach

Huffman Code

Huffman's Algorithm

- If $|S| = 2$, return 

- Let x and y be the lowest frequency symbols.
- Set $S := S \setminus \{x, y\} \cup \{xy\}$ and $f_{xy} = f_x + f_y$.
- $T :=$ recurse on S.
- Replace 

with 

- return T

Runtime: $O(|S|^2)$
what about $O(|S| \log |S|)$? Priority Queue (min-heap)

Appendix

References

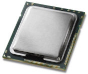
Image Sources I

https://www.cse.unsw.edu.au/~cs1521/17s2/lecs/notices/slide068.html

http://mediablogrueil.blogspot.fr/2012/11/one-page-design-effet-de-mode-ou-reel.html

http://www.culturizame.es/articulo/nuestro-pequeno-diccionario-de-tecnologia



http://computer-help-tips.blogspot.fr/2011/04/different-types-of-computer-processors.html

Image Sources II

https://brand.wisc.edu/web/logos/