Model Question Paper-1 with effect from 2019-20 (CBCS Scheme)

USN | | | | | | | | | | |

18MAT31

Third Semester B.E.Degree Examination

Transform Calculus, Fourier Series and Numerical Techniques

(Common to all Programmes)

Time: 3 Hrs   Max.Marks: 100

Note: Answer any FIVE full questions, choosing at least ONE question from each module.

Module-1

1. (a) Find the Laplace transform of (i) $\sqrt{e^{4(t+3)}} + e^{-2t}\sin 3t$ (ii) $te^{-3t}\sin 4t$ (iii) $(1-\cos t)/t$ **(10 Marks)**

(b) The square wave function $f(t)$ with period "a" is defined by $f(t) = \begin{cases} E, & 0 \le t < a/2 \\ -E, & a/2 \le t < a. \end{cases}$

Show that $L\{f(t)\} = (E/s)\tanh(as/4)$. **(05 Marks)**

(c) Employ Laplace transform to solve $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} - 4y = 2e^{-x}, y(0) = 1 = y'(0)$. **(05 Marks)**

OR

2. (a) Find (i) $L^{-1}\left\{\dfrac{3s+2}{s^2-s-2}\right\}$ (ii) $L^{-1}\{(s+5)/(s^2-6s+13)\}$ (iii) $L^{-1}[\cot^{-1}\{s/a\}]$ **(10 Marks)**

(b) Express $f(t) = \begin{cases} 1, & 0 \le t \le 1 \\ t, & t > 1 \end{cases}$ in terms Heaviside's unit step function and hence find its Laplace transform. **(05 Marks)**

(c) Find the inverse Laplace transform of $\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}$, using convolution theorem. **(05 Marks)**

Module-2

3. (a) Find the Fourier series expansion of $f(x) = \dfrac{\pi^2}{12} - \dfrac{x^2}{4}$ in $-\pi \le x \le \pi$. Hence deduce that

$\dfrac{1}{1^2} - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \dfrac{1}{4^2} + \ldots = \dfrac{\pi^2}{12}$. **(07 Marks)**

(b) Find the half-range cosine series of $f(x) = (x+1)^2$ the interval $0 \le x \le 1$. **(06 Marks)**

(c) Obtain the Fourier series of $f(x) = \begin{cases} l-x, & for\ 0 \le x \le l \\ 0, & for\ l \le x \le 2l \end{cases}$ Hence deduce that

$\dfrac{1}{1} - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \ldots = \dfrac{\pi}{4}$. **(07 Marks)**

OR

4. (a) The displacement y (in cms) of a machine part occurs due to the rotation of x radians is given below:

Rotation x (in radians)	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
Displacement y (in cms)	1.0	1.4	1.9	1.7	1.5	1.2	1.0

Expand y in terms of Fourier series up to second harmonics. **(07 Marks)**

(b) Find the half-range sine series of e^x the interval $0 \le x \le 1$. **(06 Marks)**

(c) Find the Fourier series expansion of $f(x)=|x|$ in $-\pi \le x \le \pi$. Hence deduce that

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ldots=\frac{\pi^2}{8}.$$ **(07 Marks)**

Module-3

5. (a) If $f(x)=\begin{cases}1-x^2, & for \quad |x| \le 1\\ 0 & for \ |x|>1\end{cases}$, find the infinite Fourier transform of $f(x)$ and hence evaluate

$$\int_0^\infty \frac{x\cos x-\sin x}{x^3}\cos\frac{x}{2}dx$$ **(07 Marks)**

(b) Find the Fourier cosine transform of $f(x)=e^{-2x}+4e^{-3x}$ **(06 Marks)**

(c) Solve: $u_{n+2}-3u_{n+1}+2u_n=2^n$, given $u_0=0, u_1=1$ by using z-transforms. **(07 Marks)**

OR

6. (a) Find the Fourier sine transform of $e^{-|x|}$. Hence show that $\int_0^\infty \frac{x\sin mx}{1+x^2}dx=\frac{\pi e^{-m}}{2}, m>0$. **(07 Marks)**

(b) Find the z-transform of $\cos[n\pi/2+\pi/4]$ **(06 Marks)**

(c) Find the inverse z-transform of $\dfrac{2z^2+3z}{(z+2)(z-4)}$ **(07 Marks)**

Module-4

7. (a) Solve $\dfrac{dy}{dx}=e^x-y, y(0)=1$ using Taylor's series method considering up to fourth degree terms and, find the value of $y(0.1)$. **(07 Marks)**

(b) Use Runge - Kutta method of fourth order to solve $(x+y)\dfrac{dy}{dx}=1, y(0.4)=1$, to find $y(0.5)$ (Take $h=0.1$). **(06 Marks)**

(c) Given that $\frac{dy}{dx}+\frac{y}{x}=\frac{1}{x^2}$ and $y(1)=1, y(1.1)=0.9960, y(1.2)=0.9860, \& \ y(1.3)=0.9720$ find $y(1.4)$, using Adam-Bashforth predictor-corrector method. **(07 Marks)**

OR

8. (a) Solve the differential equation $\frac{dy}{dx}=x\sqrt{y}$ under the initial condition $y(1)=1$, by using modified Euler's method at the point $x=1.4$. Perform three iterations at each step, taking $h=0.2$. **(07 Marks)**

(b) Use fourth order Runge - Kutta method, to find $y(0.1)$ with $h=0.1$, given

$$\frac{dy}{dx}+y+xy^2=0, y(0)=1,$$ **(06 Marks)**

(c) Apply Milne's predictor-corrector formulae to compute $y(0.3)$ given, $\frac{dy}{dx}=x+y^2$ with **(07 Marks)**

x	0.0	0.1	0.2	0.3
y	1.0000	1.1000	1.2310	1.4020

Module-5

9. (a) Solve $\frac{d^2y}{dx^2}-x^2\frac{dy}{dx}-2xy=1$, for $x=0.1$, correct to four decimal places, using initial conditions $y(0)=1, y'(0)=0$, using Runge - Kutta method, **(07 Marks)**

(b) Find the extremal of the functional $\int_0^1 (y'^2-y^2-y)e^{2x}dx$, that passes through the points $(0,0)$ and $(1,1/e)$. **(06 Marks)**

(c) A heavy cable hangs freely under gravity at two fixed points. Show that the shape of the cable is catenary. **(07 Marks)**

OR

10. (a) Apply Milne's predictor-corrector method to compute $y(0.4)$ given the differential equation $\frac{d^2y}{dx^2}=1+\frac{dy}{dx}$ and the following table of initial values: **(07 Marks)**

x	0	0.1	0.2	0.3
y	1	1.1103	1.2427	1.3990
y'	1	1.2103	1.4427	1.6990

(b) Derive Euler's equation in the standard form viz., $\frac{\partial f}{\partial y}-\frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right]=0$ **(06 Marks)**

(c) Find the extremal for the functional $\int_0^{\pi/2}(y^2-y'^2-2y\sin x)dx$; $y(0)=0, y(\pi/2)=1$. **(07 Marks)**

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Model Question Paper-2 with effect from 2019-20 (CBCS Scheme)

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Third Semester B.E.Degree Examination
Transform Calculus, Fourier Series and Numerical Techniques

(Common to all Programmes)

Time: 3 Hrs Max.Marks: 100

Note: Answer any FIVE full questions, choosing at least ONE question from each module.

Module-1

1. (a) Find the Laplace transform of : (i) $3^t + (4t+5)^3$ (ii) $te^{-4t}\sin 3t$ (ii) $(\cos at - \sin bt)/t$ **(10 Marks)**

(b) The triangular wave function $f(t)$ with period "$2a$" is defined by $f(t)=\begin{cases} t, & 0 \le t < a \\ 2a-t, & a \le t < 2a \end{cases}$

Show that $L\{f(t)\} = (1/s^2)\tanh(as/2)$. **(05 Marks)**

(c) Using Laplace transform method, solve $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 2y = 5\sin t$, $y(0) = 0 = y'(0)$. **(05 Marks)**

OR

2. (a) Find the inverse Laplace transform of (i) $\left\{\frac{1}{s(s+1)}\right\}$ (ii) $\{(s+1)/(s^2+6s+9)\}$

(iii) $[\log\{(s+a)/(s+b)\}]$ **(10 Marks)**

(b) Express $f(t)=\begin{cases} \sin t, & 0 < t \le \pi/2 \\ \cos t & t > \pi/2 \end{cases}$ in terms Heaviside's unit step function and hence find its Laplace transform. **(05 Marks)**

(c) Find the Laplace transform of $\frac{4}{(s^2+2s+5)^2}$, using convolution theorem. **(05 Marks)**

Module-2

3. (a) An alternating current $I(x)$ after passing through a rectifier has the form $I(x)=\begin{cases} I_0 \sin x, & for\ 0 \le x < \pi \\ 0, & for\ \pi < x \le 2\pi, \end{cases}$

where I_0 is the maximum current and the period is 2π. Express $I(x)$ as a Fourier series. **(07 Marks)**

(b) Find the half-range sine series of $f(x) = \frac{\sinh ax}{\sinh a\pi}$ the interval $(0, \pi)$. **(06 Marks)**

(c) Find the Fourier series expansion of $f(x) = x(1-x)(2-x)$ the interval $0 \le x \le 2$. Hence deduce the sum of the series that $\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \ldots$ **(07 Marks)**

OR

4. (a) In an electrical research laboratory, scientists have designed a generator which can generate the following currents at different time instant t, in the period T:

Time t (in sec)	0	$T/6$	$T/3$	$T/2$	$2T/3$	$5T/6$	T
$f(x)$	1.98	1.30	1.05	1.30	-0.88	-0.25	1.98

Determine the direct current part and amplitude of the first harmonic from the above data. **(07 Marks)**

(b) Find the half-range sine series of $f(x)=\begin{cases}\sin x & for\ \ 0\le x<\pi/4\\ \cos x, & for\ \ \pi/4<x\le\pi/2\end{cases}$ **(06 Marks)**

(c) Obtain the Fourier series of $f(x)=x(2\pi-x)$ valid in the interval $(0,2\pi)$. **(07 Marks)**

Module-3

5. (a) If $f(x)=\begin{cases}a^2-x^2, & for\ |x|\le a\\ 0 & for\ |x|>a\end{cases}$, find the infinite Fourier transform of $f(x)$ and hence evaluate $\int_0^\infty \frac{\sin x - x\cos x}{x^3}dx$ **(07 Marks)**

(b) Find the Fourier sine transform of $f(x)=\begin{cases}x, & if\ 0<x<1\\ 2-x, & if\ 1<x<2\\ 0, & if\ x>2\end{cases}$ **(06 Marks)**

(c) Solve $u_{n+2}+6u_{n+1}+9u_n=2^n, u_0=0=u_1$, by using z-transforms. **(07 Marks)**

OR

6. (a) If $f(x)=\begin{cases}1, & for\ |x|\le a\\ 0 & for\ |x|>a\end{cases}$, find the infinite Fourier transform of $f(x)$ and hence evaluate $\int_0^\infty \frac{\sin x}{x}dx$ **(07 Marks)**

(b) Find the z-transform of $2n+\sin(n\pi/4)+1$ **(06 Marks)**

(c) Find the inverse z-transform of $18z^2/[(2z-1)(4z+1)]$ **(07 Marks)**

Module-4

7. (a) Solve $\frac{dy}{dx}=x^3+y, y(1)=1$ using Taylor's series method considering up to fourth degree terms and, find the $y(1.1)$. **(07 Marks)**

(b) Use Runge - Kutta method of fourth order to solve $\frac{dy}{dx} = 3x + \frac{y}{2}$, $y(0) = 1$, to find $y(0.2)$. (Take $h = 0.2$). **(06 Marks)**

(c) Given $\frac{dy}{dx} = x^2(1+y)$ and $y(1) = 1, y(1.1) = 1.2330, y(1.2) = 1.5480, \& \ y(1.3) = 1.9790$ find $y(1.4)$, using Adam-Bashforth predictor-corrector method. **(07 Marks)**

OR

8. (a) Use modified Euler's method to compute $y(0.2)$, given $\frac{dy}{dx} - xy^2 = 0$ under the initial condition $y(0) = 2$. Perform three iterations at each step, taking $h = 0.1$. **(07 Marks)**

(b) Use fourth order Runge - Kutta method, to find $y(0.2)$ with $h = 0.2$, given $\frac{dy}{dx} = \sqrt{x+y}, y(0) = 1$, **(06Marks)**

(c) Apply Milne's predictor-corrector formulae to compute $y(2.0)$ given $\frac{dy}{dx} = \frac{1}{2}(x+y)$ with **(07 Marks)**

x	0.0	0.5	1.0	1.5
y	2.0000	2.6360	3.5950	4.9680

Module-5

9. (a) Using Runge - Kutta method, solve $\frac{d^2y}{dx^2} = x\left(\frac{dy}{dx}\right)^2 - y^2$, for $x = 0.2$, correct to four decimal places, using initial conditions $y(0) = 1, y'(0) = 0$. **(07 Marks)**

(b) Derive Euler's equation in the standard form viz., $\frac{\partial f}{\partial y} - \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right] = 0$ **(06 Marks)**

(c) Find the extremal of the functional $\int_0^{\pi}(y'^2 - y^2 + 4y\cos x)dx$; $y(0) = 0 = y(\pi)$ **(07 Marks)**

OR

10. (a) Given the differential equation $2\frac{d^2y}{dx^2} - \frac{dy}{dx} - 4x = 0$ and the following table of initial values:

x	1	1.1	1.2	1.3
y	2	2.2156	2.4649	2.7514
y'	2	2.3178	2.6725	2.0657

compute $y(1.4)$ by applying Milne's predictor-corrector method. **(07 Marks)**

(b) On what curves can the functional $\int_0^{\pi/2}(y'^2 - y^2 + 2xy)dx$, ; $y(0) = 0, y(\pi/2) = 0$ be extremized? **(06 Marks)**

(c) Prove that geodesics of a plane surface are straight lines. **(07 Marks)**

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UNIVERSITY QUESTION PAPER 1 & 2 SOLUTIONS

TCFS&NT (18MAT31)

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Module -1

1.

a) Find the Laplace transform of

i) $\sqrt{e^{4(t+3)}} + e^{-2t}\sin 3t$

let $f(t) = \sqrt{e^{4(t+3)}} + e^{-2t}\sin 3t$

$$= \left[e^{4(t+3)}\right]^{1/2} + e^{-2t}\sin 3t$$

$$= e^{2(t+3)} + e^{-2t}\sin 3t$$

$$= e^{2t+6} + e^{-2t}\sin 3t$$

$$f(t) = e^{2t}.e^{6} + e^{-2t}\sin 3t$$

$$\Rightarrow L[F(t)] = e^6 L[e^{2t}] + L[e^{-2t}\sin 3t]$$

$$= e^6 \frac{1}{s-2} + \left[\frac{3}{s^2+9}\right]_{s \to s+2}$$

$$= \frac{e^6}{s-2} + \frac{3}{(s+2)^2+9}$$

$$= \frac{e^6}{s-2} + \frac{3}{s^2+2s+4+9}$$

$$L[f(t)] = \frac{e^6}{s-2} + \frac{3}{s^2+2s+13}$$

ii) $te^{-3t}\sin 4t$

$$L[\sin 4t] = \frac{4}{s^2+16}$$

$$L[t\sin 4t] = (-1)^1 \frac{d}{ds}\left[\frac{4}{s^2+16}\right]$$

$$= -\left[\frac{(s^2+16)(0) - 4(2s)}{(s^2+16)^2}\right]$$

$$= -\left[\frac{-8s}{(s^2+16)^2}\right]$$

$$L[t\sin 4t] = \frac{8s}{(s^2+16)^2}$$

$$L[e^{-3t}t\sin 4t] = \left[\frac{8s}{(s^2+16)^2}\right]_{s \to s+3}$$

$$L[e^{-3t}t\sin 4t] = \frac{8(s+3)}{[(s+3)^2+16]^2} = \frac{8(s+3)}{(s^2+9+6s+16)^2}$$

$$L[e^{-3t} t \sin 4t] = \frac{8(s+3)}{[s^2+6s+25]^2}$$

iii) $(1-\cos t)/t$

$F(t) = 1-\cos t$

$$\Rightarrow L[F(t)] = L[1-\cos t]$$

$$= L[1] - L[\cos t]$$

$$f(s) = \frac{1}{s} - \frac{s}{s^2+1}$$

WKT

$$L\left[\frac{F(t)}{t}\right] = \int_s^\infty f(s)\, ds$$

$$= \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+1}\right) ds$$

$$= \int_s^\infty \frac{1}{s}\, ds - \int_s^\infty \frac{s}{s^2+1}\, ds$$

$$= [\log s]_s^\infty - \frac{1}{2}\int_s^\infty \frac{2s}{s^2+1}\, ds$$

$$= \left[[\log s] - \frac{1}{2}[\log(s^2+1)]\right]_s^\infty$$

$$= \left[\log s - \log(s^2+1)^{1/2}\right]_s^\infty$$

$$= \log\left[\frac{s}{(s^2+1)^{1/2}}\right]_s^\infty$$

$$= \log\left[\frac{s}{s\left(s+\frac{1}{s}\right)^{1/2}}\right]_s^\infty$$

$$= \log\left[\frac{1}{\left(s+\frac{1}{s}\right)^{1/2}}\right]_s^\infty$$

$$= \log 1 - \log\left[\frac{1}{\left(s+\frac{1}{s}\right)^{1/2}}\right].$$

$$= \left[0 - \log \frac{s^{1/2}}{(s^2+1)^{1/2}}\right] = -\log\left|\frac{s^{1/2}}{(s^2+1)^{1/2}}\right|$$

$$L\left[\frac{1-\cos t}{t}\right] = -\log\left|\frac{\sqrt{s}}{\sqrt{s^2+1}}\right|$$

1 b) The Square wave function $f(t)$ with period "a" is defined by $f(t) = \begin{cases} E, & 0 \le t < a/2 \\ -E, & a/2 \le t < a \end{cases}$ Show that

$$L\{f(t)\} = \frac{E}{s}\tanh\left(\frac{as}{4}\right)$$

Sol^n :- $f(t+T) = f(t+a)$

$$\Rightarrow \boxed{T = a}$$

WKT

$$L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-sa}}\int_0^a e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-as}}\left[\int_0^{a/2} e^{-st} f(t)\,dt + \int_{a/2}^{a} e^{-st} f(t)\,dt\right]$$

$$= \frac{1}{1-e^{-as}}\left[\int_0^{a/2} e^{-st} E\,dt + \int_{a/2}^{a} e^{-st}(-E)\,dt\right]$$

$$= \frac{E}{1-e^{-as}}\left[\int_0^{a/2} e^{-st}\,dt - \int_{a/2}^{a} e^{-st}\,dt\right]$$

$$= \frac{E}{1-e^{-as}}\left\{\left[\frac{e^{-st}}{-s}\right]_0^{a/2} - \left[\frac{e^{-st}}{-s}\right]_{a/2}^{a}\right\}$$

$$= \frac{E}{1-e^{-as}}\left\{-\frac{1}{s}\left[e^{-as/2} - e^{0}\right] + \frac{1}{s}\left[e^{-as} - e^{-as/2}\right]\right\}$$

$$= \frac{E}{s}\left(\frac{1}{1-e^{-as}}\right)\left\{-e^{-as/2} + 1 + e^{-as} - e^{-as/2}\right\}$$

$$= \frac{E}{s}\left(\frac{1}{1-e^{-as}}\right)\left\{1 - 2e^{-as/2} + e^{-as}\right\}$$

$$= \frac{E}{s}\left(\frac{1}{1-e^{-as}}\right)\left\{1^2 - 2(1)e^{-as/2} + (e^{-as/2})^2\right\}$$

$$= \frac{E}{s}\left(\frac{1}{1-e^{-as}}\right)\left(1 - e^{-as/2}\right)^2$$

$$= \frac{E}{s}\frac{(1-e^{-as/2})^2}{(1-e^{-as})}$$

$$= \frac{E}{s}\frac{(1-e^{-as/2})(1-e^{-as/2})}{[1^2 - (e^{-as/2})^2]}$$

$$= \frac{E}{s}\frac{(1-e^{-as/2})\cancel{(1-e^{-as/2})}}{\cancel{(1-e^{-as/2})}(1+e^{-as/2})}$$

$$= \frac{E}{s}\frac{(1-e^{-as/2})}{(1+e^{-as/2})} \quad x^{ly}\ e^{as/4} \text{ on } N^r \,\&\, D^r$$

$$= \frac{E}{s}\frac{(1-e^{-as/2})e^{as/4}}{(1+e^{-as/2})e^{as/4}}$$

$$= \frac{E}{s}\left[\frac{e^{as/4} - e^{-as/4}}{e^{as/4} + e^{-as/4}}\right] \qquad \tanh\theta = \frac{e^{\theta} - e^{-\theta}}{e^{\theta} + e^{-\theta}}$$

$$L[f(t)] = \frac{E}{s}\tanh\left(\frac{as}{4}\right)$$

1 c) Employ Laplace transform to solve

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 4y = 2e^{-x}, \quad y(0) = 1 = y'(0)$$

Soln: $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} - 4y = 2e^{-x}$

$$y''(x) - 3y'(x) - 4y(x) = 2e^{-x}$$

$$\Rightarrow L[y''(x)] - 3L[y'(x)] - 4L[y(x)] = L[e^{-x}]$$

$$\Rightarrow [s^2\bar{y}(s) - sy(0) - y'(0)] - 3[s\bar{y}(s) - y(0)] - 4\bar{y}(s) = 2\frac{1}{s+1}$$

$$\Rightarrow [s^2\bar{y}(s) - s - 1] - 3[s\bar{y}(s) - 1] - 4\bar{y}(s) = \frac{2}{s+1}$$

$$\Rightarrow s^2\bar{y}(s) - 3s\bar{y}(s) - 4\bar{y}(s) - s - 1 + 3 = \frac{2}{s+1}$$

$$\Rightarrow (s^2-3s-4)\bar{y}(s) - (s-2) = \frac{2}{s+1}$$

$$\Rightarrow (s^2-3s-4)\bar{y}(s) = \frac{2}{s+1} + (s-2)$$

$$\Rightarrow (s-4)(s+1)\bar{y}(s) = \frac{2+(s-2)(s+1)}{s+1}$$

$$\Rightarrow (s-4)(s+1)\bar{y}(s) = \frac{s^2-s}{s+1}$$

$$\Rightarrow \bar{y}(s) = \frac{s(s-1)}{(s+1)^2(s-4)} \rightarrow (1)$$

$$\frac{s(s-1)}{(s+1)^2(s-4)} = \frac{A}{(s+1)} + \frac{B}{(s+1)^2} + \frac{C}{s-4} \rightarrow (2)$$

$$s(s-1) = A(s+1)(s-4) + B(s-4) + C(s+1)^2$$

when $s=4$

$4(4-1) = C(4+1)^2$

$12 = C(25)$

$\boxed{C = \frac{12}{25}}$

when $s=-1$

$-1(-1-1) = B(-1-4)$

$-1(-2) = B(-5)$

$2 = B(-5)$

$\boxed{B = \frac{-2}{5}}$

Compare coefficients of s^2

$1 = A + C \Rightarrow A = 1 - C = 1 - \frac{12}{25} = \frac{25-12}{25}$ $\boxed{A = \frac{13}{25}}$

$$② \Rightarrow \frac{s(s-1)}{(s+1)^2(s-4)} = \frac{13/25}{s+1} - \frac{2/5}{(s+1)^2} + \frac{12/25}{s-4}$$

$$\bar{y}(s) = \frac{13}{25}\frac{1}{s+1} - \frac{2}{5}\frac{1}{(s+1)^2} + \frac{12}{25}\frac{1}{s-4}$$

$$L^{-1}[\bar{y}(s)] = \frac{13}{25} L^{-1}\left[\frac{1}{s+1}\right] - \frac{2}{5} L^{-1}\left[\frac{1}{(s+1)^2}\right] + \frac{12}{25} L^{-1}\left[\frac{1}{s-4}\right]$$

$$= \frac{13}{25} e^{-t} - \frac{2}{5} e^{-t} L^{-1}\left[\frac{1}{s^2}\right] + \frac{12}{25} e^{4t}$$

$$L^{-1}[\bar{y}(s)] = \frac{13}{25} e^{-t} - \frac{2}{5} t e^{-t} + \frac{12}{25} e^{4t}$$

$$y(t) = \frac{13}{25} e^{-t} - \frac{2}{5} t e^{-t} + \frac{12}{25} e^{4t}$$

2a) find

i) $L^{-1}\left\{\frac{3s+2}{s^2-s-2}\right\}$

Sol^n:- let $f(s) = \frac{3s+2}{s^2-s-2} = \frac{3s+2}{(s-2)(s+1)}$

$$\frac{3s+2}{(s-2)(s+1)} = \frac{A}{s-2} + \frac{B}{s+1} \longrightarrow ①$$

$$3s+2 = A(s+1) + B(s-2) \longrightarrow ②$$

when $s=2$

$② \Rightarrow 3(2)+2 = A(2+1)$

$8 = 3A \Rightarrow \boxed{A = \frac{8}{3}}$

when $s=-1$

$② \Rightarrow 3(-1)+2 = B(-1-2)$

$-1 = B(-3)$

$\boxed{B = 1/3}$

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$\therefore$ (1) $\Rightarrow$ $$\frac{3s+2}{(s-2)(s+1)} = \frac{8/3}{s-2} + \frac{1/3}{s+1}$$

$$L^{-1}\left[\frac{3s+2}{(s-2)(s+1)}\right] = \frac{8}{3} L^{-1}\left[\frac{1}{s-2}\right] + \frac{1}{3} L^{-1}\left[\frac{1}{s+1}\right]$$

$$L^{-1}[f(s)] = \frac{8}{3} e^{2t} + \frac{1}{3} e^{-t}$$

$$f(t) = \frac{1}{3}\left(8e^{2t} + e^{-t}\right)$$

ii) $L^{-1}\left\{(s+5) / (s^2-6s+13)\right\}$

$$f(s) = \frac{s+5}{s^2-6s+13}$$

$$= \frac{s+5}{\underbrace{s^2-2(s)(3)+3^2}-3^2+13}$$

$$= \frac{s+5}{(s-3)^2+4}$$

$$= \frac{(s-3)+3+5}{(s-3)^2+4}$$

$$f(s) = \frac{(s-3)+8}{(s-3)^2+4}$$

$$L^{-1}[f(s)] = L^{-1}\left[\frac{(s-3)+8}{(s-3)^2+4}\right]$$

$$= e^{3t} L^{-1}\left[\frac{s+8}{s^2+4}\right]$$

$$= e^{3t} L^{-1}\left[\frac{s}{s^2+4} + \frac{8}{s^2+4}\right]$$

$$L^{-1}[f(s)] = e^{3t}\left\{L^{-1}\left[\frac{s}{s^2+2^2}\right] + 8\,L^{-1}\left[\frac{1}{s^2+2^2}\right]\right\}$$

$$= e^{3t}\left(\cos 2t + 8\cdot\frac{1}{2}\sin 2t\right)$$

$$L^{-1}\left\{\frac{s+5}{s^2-6s+13}\right\} = e^{3t}\left(\cos 2t + 4\sin 2t\right)$$

iii) $L^{-1}\left[\cot^{-1}(s/a)\right]$

$$f(s) = \cot^{-1}(s/a)$$

diff w.r.t s

$$f'(s) = \frac{-1}{1+(s/a)^2}\cdot\frac{1}{a}$$

multiply with -1

$$-f'(s) = \frac{a}{a^2+s^2}$$

Taking inverse laplace on both sides

$$L^{-1}[-f'(s)] = L^{-1}\left[\frac{a}{s^2+a^2}\right]$$

$L[t f(t)] = -f'(s)$

$$t\,f(t) = \sin at$$

$L^{-1}[-f'(s)] = t\,f(t)$

$$\therefore \boxed{f(t) = \frac{\sin at}{t}}$$

2b) Express $f(t) = \begin{cases} 1, & 0 \le t \le 1 \\ t, & t > 1 \end{cases}$ in terms heavyside's unit step function and hence find its Laplace transform.

Sol^n :- $f(t) = 1 + (t-1)\,u(t-1)$

$$\Rightarrow L[f(t)] = L[1] + L[(t-1)\,u(t-1)] \longrightarrow (1)$$

let $g(t-1) = t-1$ $\qquad L[1] = \frac{1}{s}$

$g(t) = t$

$L[g(t)] = L[t]$

$\bar{g}(s) = \frac{1}{s^2}$

$\therefore L[g(t-1)\,u(t-1)] = e^{-s}\,\bar{g}(s)$

$L[(t-1)\,u(t-1)] = e^{-s}\frac{1}{s^2}$

$\therefore$ (1) $\Rightarrow$ $\bar{f}(s) = \frac{1}{s} + \frac{e^{-s}}{s^2}$

2c) Find the inverse laplace transform of $\frac{s^2}{(s^2+a^2)(s^2+b^2)}$ using convolution theorem.

Soln :- $\bar{f}(s)\cdot\bar{g}(s) = \frac{s^2}{(s^2+a^2)(s^2+b^2)} = \frac{s}{(s^2+a^2)}\cdot\frac{s}{(s^2+b^2)}$

$\bar{f}(s) = \frac{s}{s^2+a^2}$, $\bar{g}(s) = \frac{s}{s^2+b^2}$

$L^{-1}[\bar{f}(s)] = L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at = f(t)$

$L^{-1}[\bar{g}(s)] = L^{-1}\left[\frac{s}{s^2+b^2}\right] = \cos bt = g(t)$

$\therefore L^{-1}[\bar{f}(s)\cdot\bar{g}(s)] = \int_{u=0}^{t} f(u)\,g(t-u)\,du$

$= \int_{u=0}^{t} \cos au\cdot\cos(bt-bu)\,du$

$= \frac{1}{2}\int_{u=0}^{t} 2\cos au.\cos(bt-bu)\,du$

$= \frac{1}{2}\int_{u=0}^{t} \cos(au+bt-bu)+\cos(au-bt+bu)\,du$

$$= \frac{1}{2}\int_{u=0}^{t}\left\{\cos[(a-b)u+bt] + \cos[(a+b)u-bt]\right\} du$$

$$= \frac{1}{2}\left\{\frac{\sin[(a-b)u+bt]}{a-b} + \frac{\sin[(a+b)u-bt]}{a+b}\right\}_0^t$$

$$= \frac{1}{2}\left\{\left[\frac{\sin at}{a-b} + \frac{\sin at}{a+b}\right] - \left[\frac{\sin bt}{a-b} - \frac{\sin bt}{a+b}\right]\right\}$$

$$= \frac{1}{2}\left\{\left(\frac{1}{a-b} + \frac{1}{a+b}\right)\sin at - \left(\frac{1}{a-b} - \frac{1}{(a+b)}\right)\sin bt\right\}$$

$$= \frac{1}{2}\left[\frac{2a}{a^2-b^2}\sin at - \frac{2b}{a^2-b^2}\sin bt\right]$$

$$= \frac{a}{a^2-b^2}\sin at - \frac{b}{a^2-b^2}\sin bt$$

$$= \frac{1}{a^2-b^2}\left[a\sin at - b\sin bt\right], \quad a \neq b$$

Model question paper - 2

1 a) Find the laplace transform of

i) $3^t + (4t+5)^3$

$$f(t) = 3^t + (4t+5)^3$$

$$f(t) = e^{\log(3^t)} + (4t)^3 + 5^3 + 3(4t)^2 5 + 3(4t)(5)^2$$

$$f(t) = e^{(\log 3)t} + 64t^3 + 125 + 240t^2 + 300t$$

$$f(t) = e^{(\log 3)t} + 64t^3 + 240t^2 + 300t + 125$$

$$L[f(t)] = L\left[e^{(\log 3)t}\right] + 64\,L[t^3] + 240\,L[t^2] + 300\,L[t] + 125\,L[1]$$

$$L[f(t)] = \frac{1}{s-\log 3} + 64.\frac{6}{s^4} + 240\,\frac{2}{s^3} + 300.\frac{1}{s^2} + 125.\frac{1}{s}$$

$$f(s) = \frac{1}{s-\log 3} + \frac{384}{s^4} + \frac{480}{s^3} + \frac{300}{s^2} + \frac{125}{s}$$

ii) $t e^{-4t} \sin 3t$

$$L[\sin 3t] = \frac{3}{s^2+9}$$

$$L[t\sin 3t] = (-1)^1 \frac{d}{ds}\left[\frac{3}{s^2+9}\right]$$

$$= -\left[\frac{(s^2+9)(0) - 3(2s)}{(s^2+9)^2}\right]$$

$$= -\left[\frac{-6s}{(s^2+9)^2}\right]$$

$$L[t\sin 3t] = \frac{6s}{(s^2+9)^2}$$

$$L[e^{-4t} t\sin 3t] = \left[\frac{6s}{(s^2+9)^2}\right]_{s \to s+4}$$

$$= \frac{6(s+4)}{((s+4)^2+9)^2}$$

$$L[e^{-4t} t\sin 3t] = \frac{6(s+4)}{(s^2+8s+25)^2}$$

iii) $(\cos at - \cos bt)/t$

$$F(t) = \cos at - \cos bt$$

$$L[F(t)] = L[\cos at] - L[\cos bt]$$

$$f(s) = \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$$

WKT $\quad L\left[\frac{F(t)}{t}\right] = \int_s^\infty f(s)\, ds$

$$= \int_s^\infty \left[\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right] ds$$

$$= \frac{1}{2}\int_s^\infty \left[\frac{2s}{s^2+a^2} - \frac{2s}{s^2+b^2}\right] ds$$

$$= \frac{1}{2}\left[\log(s^2+a^2) - \log(s^2+b^2)\right]_s^\infty$$

$$= \frac{1}{2} \log\left[\frac{s^2+a^2}{s^2+b^2}\right]_s^\infty$$

$$= \frac{1}{2} \log\left[\frac{s^2\left(1+\frac{a^2}{s^2}\right)}{s^2\left(1+\frac{b^2}{s^2}\right)}\right]_s^\infty$$

$$= \frac{1}{2} \log\left[\frac{1+a^2/s^2}{1+b^2/s^2}\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\left(\frac{1+0}{1+0}\right) - \log\left(\frac{1+a^2/s^2}{1+b^2/s^2}\right)\right]$$

$$= -\frac{1}{2}\log\left[\frac{s^2+a^2}{s^2+b^2}\right]$$

$$= \log\left(\frac{s^2+a^2}{s^2+b^2}\right)^{-1/2}$$

$$L\left[\frac{\cos at - \cos bt}{t}\right] = \log\sqrt{\frac{s^2+b^2}{s^2+a^2}}$$

1b) The triangular wave function $f(t)$ with period "$2a$" is defined by $f(t) = \begin{cases} t & , 0 \le t < a \\ 2a-t & , a \le t < 2a \end{cases}$ Show that

$$L\{f(t)\} = \frac{1}{s^2}\tanh(as/2)$$

Soln :- $f(t+2a) = f(t) \Rightarrow T = 2a$

WKT
$$L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\, dt$$

$$L[f(t)] = \frac{1}{1-e^{-2as}}\int_0^{2a} e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-2as}}\left\{\int_0^a e^{-st} f(t)\, dt + \int_a^{2a} e^{-st} f(t)\, dt\right\}$$

$$= \frac{1}{1-e^{-2as}}\left\{\int_0^a t\,e^{-st}\,dt + \int_a^{2a}(2a-t)\,e^{-st}\,dt\right\}$$

$$\int_0^a te^{-st}\,dt = t\int_0^a e^{-st}\,dt - \int_0^a\left[1\int e^{-st}dt\right]dt$$

$$= -\frac{1}{s}\left[te^{-st}\right]_0^a + \frac{1}{s}\int_0^a e^{-st}\,dt$$

$$= -\frac{1}{s}\left[ae^{-as}-0\right] - \frac{1}{s^2}\left[e^{-as}-e^{0}\right]$$

$$= -\frac{1}{s}\left(ae^{-as}\right) - \frac{1}{s^2}\left(e^{-as}-1\right)$$

$$\int_0^a te^{-st}\,dt = \frac{1}{s^2} - \frac{1}{s^2}e^{-as} - \frac{a}{s}e^{-as}$$

$$\int_0^{2a}(2a-t)\,e^{-st}\,dt = (2a-t)\int_a^{2a}e^{-st}\,dt - \int_a^{2a}\left[(-1)\int e^{-st}dt\right]dt$$

$$= -\frac{1}{s}\left[(2a-t)\,e^{-st}\right]_a^{2a} - \frac{1}{s}\int_a^{2a}e^{-st}\,dt$$

$$= -\frac{1}{s}\left[0-ae^{-as}\right] + \frac{1}{s^2}\left[e^{-st}\right]_a^{2a}$$

$$= \frac{a}{s}e^{-as} + \frac{1}{s^2}\left[e^{-2as}-e^{-as}\right]$$

$$\int_0^{2a}(2a-t)e^{-st}\,dt = \frac{1}{s^2}e^{-2as} + \frac{a}{s}e^{-as} - \frac{1}{s^2}e^{-as}$$

$$L[F(t)] = \frac{1}{1-e^{-2as}}\left[\frac{1}{s^2} - \frac{1}{s^2}e^{-as} - \cancel{\frac{a}{s}e^{-as}} + \frac{1}{s^2}e^{-2as} + \cancel{\frac{a}{s}e^{-as}} - \frac{1}{s^2}e^{-as}\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\frac{1}{s^2} - \frac{2}{s^2}e^{-as} + \frac{1}{s^2}e^{-2as}\right]$$

$$= \frac{1}{1-e^{-2as}}\,\frac{1}{s^2}\left[1-2e^{-as}+e^{-2as}\right]$$

$$= \frac{1}{s^2}\,\frac{1}{1-e^{-2as}}\left[1^2 - 2(1)\,e^{-as} + (e^{-as})^2\right]$$

$$= \frac{1}{s^2} \frac{1}{1-e^{-2as}} \left[1-e^{-as}\right]^2$$

$$= \frac{1}{s^2} \frac{(1-e^{-as})^2}{1^2-(e^{-as})^2}$$

$$= \frac{1}{s^2} \frac{(1-e^{-as})^{\cancel{2}}}{\cancel{(1-e^{-as})}(1+e^{-as})}$$

$$= \frac{1}{s^2} \frac{(1-e^{-as})}{(1+e^{-as})} \quad x^{ly}\ e^{as/2} \text{ on } N^r \,\&\, D^r$$

$$= \frac{1}{s^2} \frac{(1-e^{-as})\,e^{as/2}}{(1+e^{-as})\,e^{as/2}}$$

$$= \frac{1}{s^2} \left[\frac{e^{as/2}-e^{-as/2}}{e^{as/2}+e^{-as/2}}\right]$$

$$L[f(t)] = \frac{1}{s^2} \tanh\left(\frac{as}{2}\right)$$

1c> Using laplace transform method, Solve

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 2y = 5\sin t, \quad y(0) = 0 = y'(0).$$

Sol^n

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 2y = 5\sin t$$

$$y''(t) + 2y'(t) + 2y(t) = 5\sin t$$

$$L[y''(t)] + 2L[y'(t)] + 2L[y(t)] = 5L[\sin t]$$

$$\left[s^2\bar{y}(s) - s y(0) - y'(0)\right] + 2\left[s\bar{y}(s) - y(0)\right] + 2\bar{y}(s) = 5\frac{1}{s^2+1}$$

$$s^2\bar{y}(s) + 2s\bar{y}(s) + 2\bar{y}(s) = \frac{5}{s^2+1}$$

$$(s^2+2s+2)\,\bar{y}(s) = \frac{5}{s^2+1}$$

$$\bar{y}(s) = \frac{5}{(s^2+1)(s^2+2s+2)}$$

let $$\frac{5}{(S^2+1)(S^2+2S+2)} = \frac{AS+B}{S^2+1} + \frac{CS+D}{S^2+2S+2}$$

$$5 = AS+B\,(S^2+2S+2) + (CS+D)\,(S^2+1)$$

$$5 = (A+C)S^3 + (2A+B+D)S^2 + (2A+2B+C)S + (2B+D)$$

comparing the coefficients on both sides, We get

$$A+C=0\ ,\ 2A+B+D=0\ ,\ 2A+2B+C=0\ ,\ 2B+D=5$$

$$A=-2\ ,\ B=1\ ,\ C=2\ ,\ D=3$$

$$\therefore \frac{5}{(S^2+1)(S^2+2S+2)} = \frac{-2S+1}{S^2+1} + \frac{2S+3}{S^2+2S+2}$$

$$y(t) = -2\,L^{-1}\left[\frac{S}{S^2+1}\right] + L^{-1}\left[\frac{1}{S^2+1}\right] + L^{-1}\left[\frac{2S+3}{S^2+2S+2}\right]$$

$$= -2\cos t + \sin t + L^{-1}\left\{\frac{2(S+1)+1}{(S+1)^2+1}\right\}$$

$$= -2\cos t + \sin t + e^{-t}\,L^{-1}\left\{\frac{2S+1}{S^2+1}\right\}$$

$$= -2\cos t + \sin t + e^{-t}\left[2L^{-1}\left(\frac{S}{S^2+1}\right) + L^{-1}\left(\frac{1}{S^2+1}\right)\right]$$

$$y(t) = -2\cos t + \sin t + e^{-t}\left[2\cos t + \sin t\right]$$

2 a) Find the inverse Laplace transform of

i) $\left\{\frac{1}{s(s+1)}\right\}$

Soln: let $f(s) = \frac{1}{s(s+1)}$

$$\frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1} \longrightarrow (1)$$

$$1 = A(s+1) + Bs \longrightarrow (2)$$

when $s=0$ when $s=-1$

$1 = A(0+1)$ $1 = B(-1)$

$\boxed{A=1}$ $\boxed{B=-1}$

$$\therefore (1) \Rightarrow \frac{1}{s(s+1)} = \frac{1}{s} + \frac{-1}{s+1}$$

$$L^{-1}\left[\frac{1}{s(s+1)}\right] = L^{-1}\left[\frac{1}{s}\right] - L^{-1}\left[\frac{1}{s+1}\right]$$

$$L^{-1}[f(s)] = 1 - e^{-t}$$

$$f(t) = 1 - e^{-t}$$

ii) $\left\{\frac{(s+1)}{(s^2+6s+9)}\right\}$

Soln: let $f(s) = \frac{s+1}{s^2+6s+9}$

$$= \frac{s+1}{(s+3)^2}$$

$$= \frac{s+3-3+1}{(s+3)^2}$$

$$f(s) = \frac{(s+3)-2}{(s+3)^2}$$

$$L^{-1}\left[\frac{s+1}{s^2+6s+9}\right] = L^{-1}\left[\frac{(s+3)-2}{(s+3)^2}\right]$$

$$L^{-1}[f(s)] = e^{-3t} L^{-1}\left[\frac{s-2}{s^2}\right]$$

$$= e^{-3t}\left\{L^{-1}\left[\frac{1}{s}\right] - 2L^{-1}\left[\frac{1}{s^2}\right]\right\}$$

$$f(t) = e^{-3t}\left[1-2t\right]$$

iii) $\log\left[\frac{(s+a)}{(s+b)}\right]$

Soln: let $f(s) = \log\left(\frac{s+a}{s+b}\right)$

$$f(s) = \log(s+a) - \log(s+b)$$

differentiate w.r.t s

$$f'(s) = \frac{1}{s+a} - \frac{1}{s+b}$$

x'ly −ve on both sides

$$-f'(s) = -\left[\frac{1}{s+a} - \frac{1}{s+b}\right]$$

$$-f'(s) = \frac{1}{s+b} - \frac{1}{s+a}$$

$$L^{-1}[-f'(s)] = L^{-1}\left[\frac{1}{s+b}\right] - L^{-1}\left[\frac{1}{s+a}\right]$$

$$t\,f(t) = e^{-bt} - e^{-at}$$

$$f(t) = \frac{e^{-bt} - e^{-at}}{t}$$

2b) Express $f(t) = \begin{cases} \sin t, & 0 < t \le \pi/2 \\ \cos t, & t > \pi/2 \end{cases}$ in terms of heaviside's unit step function and hence find its Laplace transform.

Sol^n:

$$f(t) = \sin t + [\cos t - \sin t]\, u(t-\pi/2)$$

$$L[f(t)] = L[\sin t] + L[\cos t - \sin t]\, u(t-\pi/2) \longrightarrow (1)$$

let $g(t-\pi/2) = \cos t - \sin t$

$$g(t) = \cos(t+\pi/2) - \sin(t+\pi/2)$$

$$= -\sin t - \cos t$$

$$L[g(t)] = -L[\sin t] - L[\cos t]$$

$$\bar{g}(s) = -\left[\frac{1}{s^2+1} + \frac{s}{s^2+1}\right]$$

$$\therefore L[g(t-\pi/2)\, u(t-\pi/2)] = e^{-\pi/2\, s} \cdot \bar{g}(s)$$

$$= e^{-\pi/2\, s} - \left[\frac{1}{s^2+1} + \frac{s}{s^2+1}\right]$$

$$\therefore (1) \Rightarrow \bar{f}(s) = \frac{1}{s^2+1} - e^{-\pi/2\, s}\left[\frac{1}{s^2+1} + \frac{s}{s^2+1}\right]$$

c) Find the Laplace transform of $\frac{4}{(s^2+2s+5)^2}$, using Convolution theorem.

Sol^n: given $\bar{f}(s) \cdot \bar{g}(s) = \frac{4}{(s^2+2s+5)^2}$

$$\bar{f}(s) \cdot \bar{g}(s) = \frac{2}{s^2+2s+5} \cdot \frac{2}{s^2+2s+5}$$

$$\bar{f}(s) = \frac{2}{(s+1)^2+4}, \quad \bar{g}(s) = \frac{2}{(s+1)^2+4}$$

$$\Rightarrow L^{-1}[\bar{f}(s)] = L^{-1}\left[\frac{2}{(s+1)^2+4}\right] = e^{-t} L^{-1}\left[\frac{2}{s^2+4}\right] = e^{-t}\sin 2t$$

$$= f(t)$$

$$L^{-1}[\bar{g}(s)] = L^{-1}\left[\frac{2}{(s+1)^2+4}\right] = e^{-t}L^{-1}\left[\frac{2}{s^2+4}\right] = e^{-t}\sin 2t = g(t)$$

$$\therefore f(t) = e^{-t}\sin 2t \qquad g(t) = e^{-t}\sin 2t$$

WKT $L^{-1}\left[\bar{f}(s)\cdot\bar{g}(s)\right] = f(t) * g(t)$

$$L^{-1}\left[\frac{4}{(s^2+2s+5)^2}\right] = \int_{u=0}^{t} f(u)\,g(t-u)\,du$$

$$= \int_{u=0}^{t} e^{-u}\sin 2u \; e^{-(t-u)}\sin(2t-2u)\,du$$

$$= \int_{u=0}^{t} \cancel{e^{-u}}\sin 2u \; e^{-t}\cdot\cancel{e^{u}}\sin(2t-2u)\,du$$

$$= e^{-t}\int_{u=0}^{t} \sin 2u\cdot\sin(2t-2u)\,du$$

$$= \frac{e^{-t}}{2}\int_{u=0}^{t} 2\sin 2u\sin(2t-2u)\,du$$

$$= \frac{e^{-t}}{2}\int_{u=0}^{t} \cos(2u-2t+2u) - \cos(2u+2t-2u)\,du$$

$$= \frac{e^{-t}}{2}\int_{u=0}^{t}\left[\cos(4u-2t) - \cos 2t\right]du$$

$$= \frac{e^{-t}}{2}\left\{\int_{u=0}^{t}\cos(4u-2t)\,du - \cos 2t\int_{0}^{t} 1\,du\right\}$$

$$= \frac{e^{-t}}{2}\left\{\left[\frac{\sin(4u-2t)}{4}\right]_{u=0}^{t} - t\cos 2t\right\}$$

$$= \frac{e^{-t}}{2}\left\{\frac{\sin 2t}{4} + \frac{\sin 2t}{4} - t\cos 2t\right\}$$

$$= \frac{e^{-t}}{2}\left[\frac{\sin 2t}{2} - t\cos 2t\right]$$

Module -3

5a) If $f(x) = \begin{cases} 1-x^2 & , \text{for } |x| \le 1 \\ 0 & \text{for } |x|>1 \end{cases}$, find the infinite Fourier transform of $f(x)$ and hence Evaluate $\int_0^\infty \frac{x\cos x - \sin x}{x^3} \cos\frac{x}{2}\, dx$

Soln: $f(x) = \begin{cases} 1-x^2 & , -1 \le x \le 1 \\ 0 & , x > 1 \end{cases}$

The fourier transform of $f(x)$ is $F[f(x)] = \int_{-\infty}^{\infty} e^{isx} F(x)\, dx$

$$= \int_{-\infty}^{-1} e^{-isx} F(x)\, dx + \int_{-1}^{1} e^{isx} F(x)\, dx + \int_{1}^{\infty} e^{isx} F(x)\, dx$$

$$= \int_{-1}^{1} e^{isx} F(x)\, dx$$

$$= \int_{-1}^{1} e^{isx} (1-x^2)\, dx$$

$$= \left\{ (1-x^2) \int_{-1}^{1} e^{isx} - \int_{-1}^{1} \left[(-2x) \int e^{isx}\, dx \right] dx \right\}$$

$$= \frac{1}{is} \left[(1-x^2) e^{isx} \right]_{-1}^{1} + \frac{2}{is} \int_{-1}^{1} x e^{isx}\, dx$$

$$= \frac{1}{is}(0-0) + \frac{2}{is} \left\{ x \int_{-1}^{1} e^{isx} - \int_{-1}^{1} \left[1 \int e^{isx}\, dx \right] dx \right\}$$

$$= 0 + \frac{2}{is} \left\{ \frac{1}{is} \left[x e^{isx} \right]_{-1}^{1} - \frac{1}{i^2 s^2} \left[e^{isx} \right]_{-1}^{1} \right\}$$

$$= \frac{2}{is} \left\{ \frac{1}{is} \left[e^{is} + e^{-is} \right] - \frac{1}{i^2 s^2} \left[e^{is} - e^{-is} \right] \right\}$$

$$= \frac{2}{is} \left\{ \frac{1}{is} \left[\cos s + i\cancel{\sin s} + \cos s - i\cancel{\sin s} \right] - \frac{1}{i^2 s^2} \left[\cancel{\cos s} + i\sin s - \cancel{\cos s} + i\sin s \right] \right\}$$

$$= \frac{2}{is}\left\{\frac{1}{is}(2\cos s) - \frac{1}{i^{\cancel{2}}s^2}(2\cancel{i}\sin s)\right\}$$

$$= \frac{4\cos s}{i^2 s^2} - \frac{4\sin s}{i^2 s^3}$$

$$= \frac{-4\cos s}{s^2} + \frac{4\sin s}{s^3}$$

$$= 4\left[\frac{\sin s}{s^3} - \frac{\cos s}{s^2}\right]$$

$$f(s) = 4\left[\frac{\sin s - s\cos s}{s^3}\right]$$

WKT The fourier inverse transform is

$$F^{-1}[f(s)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(s)\, e^{-isx}\, ds = F(x)$$

$$\int_{-\infty}^{\infty} f(s)\, e^{-isx}\, dx = 2\pi \begin{cases} 1-x^2 & -1 \le x \le 1 \\ 0 & , x > 1 \end{cases}$$

$$\int_{-\infty}^{\infty} 4\left[\frac{\sin s - s\cos s}{s^3}\right][\cos sx - i\sin sx] = 2\pi \begin{cases} 1-x^2, & -1 \le x \le 1 \\ 0, & x > 1 \end{cases}$$

$$\Rightarrow \int_{-\infty}^{\infty} 4\left[\frac{\sin s - s\cos s}{s^3}\right]\cos sx - i\int_{-\infty}^{\infty} 4\left[\frac{\sin s - s\cos s}{s^3}\right]\sin sx\, dx$$

$$= 2\pi \begin{cases} 1-x^2, & -1 \le x \le 1 \\ 0, & x > 1 \end{cases} + i(0)$$

$$\int_{-\infty}^{\infty} 4\left[\frac{\sin s - s\cos s}{s^3}\right]\cos sx\, dx = 2\pi \begin{cases} 1-x^2, & -1 \le x \le 1 \\ 0, & x > 1 \end{cases} \longrightarrow (1)$$

When $x = 1/2$

$$\therefore (1) \Rightarrow \int_{-\infty}^{\infty} 4\left(\frac{\sin s - s\cos s}{s^3}\right)\cos\frac{s}{2}\, ds = 2\pi\left(1-\frac{1}{4}\right) = \cancel{2}\pi\left(\frac{3}{\cancel{4}_{2}}\right) = 3\pi/2$$

$$\Rightarrow \int_{-\infty}^{\infty} 4\left(\frac{\sin s - s\cos s}{s^3}\right)\cos\frac{s}{2}\,ds = \frac{3\pi}{2}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin s - s\cos s}{s^3}\cos\frac{s}{2}\,ds = \frac{3\pi}{8}$$

$$\Rightarrow 2\int_{0}^{\infty} \frac{\sin s - s\cos s}{s^3}\cos\frac{s}{2}\,ds = \frac{3\pi}{8}$$

$$\Rightarrow \int_{0}^{\infty} \frac{\sin s - s\cos s}{s^3}\cos\frac{s}{2}\,ds = \frac{3\pi}{16}$$

$$\Rightarrow \int_{0}^{\infty} \frac{s\cos s - \sin s}{s^3}\cos\frac{s}{2}\,ds = \frac{-3\pi}{16}$$

$$\Rightarrow \int_{0}^{\infty} \frac{x\cos x - \sin x}{x^3}\cos\frac{x}{2}\,dx = -3\pi/16 \quad \text{when } s = x$$

5b) Find the fourier cosine transform of $f(x) = e^{-2x} + 4e^{-3x}$

Sol^n: WKT

$$F_c[F(x)] = \int_0^{\infty} F(x)\cos sx\,dx$$

$$= \int_0^{\infty}\left[e^{-2x} + 4e^{-3x}\right]\cos sx\,dx$$

$$= \int_0^{\infty} e^{-2x}\cos sx\,dx + 4\int_0^{\infty} e^{-3x}\cos sx\,dx$$

$$= \left\{\frac{e^{-2x}}{s^2+4}\left[-2\cos sx + s\sin(sx)\right]\right\}_0^{\infty} + 4\left\{\frac{e^{-3x}}{s^2+9}\left[-3\cos(sx) + s\sin(sx)\right]\right\}_0^{\infty}$$

$$= \left[0 - \frac{1}{s^2+4}(-2)\right] + 4\left[0 - \frac{1}{s^2+9}(-3)\right]$$

$$f_c(s) = \frac{2}{s^2+4} + \frac{12}{s^2+9}$$

5c) Solve : $u_{n+2} - 3u_{n+1} + 2u_n = 2^n$, given $u_0 = 0$, $u_1 = 1$ by using z-transforms.

Sol^n :- $\quad U_{n+2} - 3U_{n+1} + 2U_n = 2^n$

$$Z[U_{n+2}] - 3Z[U_{n+1}] + 2Z[U_n] = Z[2^n]$$

$$\Rightarrow z^2\left[\bar{u}(z) - u_0 - \frac{u_1}{z}\right] - 3z\left[\bar{u}(z) - u_0\right] + 2\bar{u}(z) = \frac{z}{z-2}$$

$$\Rightarrow z^2\left[\bar{u}(z) - 0 - \frac{1}{z}\right] - 3z\left[\bar{u}(z) - 0\right] + 2\bar{u}(z) = \frac{z}{z-2}$$

$$\Rightarrow z^2\bar{u}(z) - z^{\cancel{2}}\cdot\frac{1}{\cancel{z}} - 3z\bar{u}(z) + 2\bar{u}(z) = \frac{z}{z-2}$$

$$\Rightarrow z^2\bar{u}(z) - 3z\bar{u}(z) + 2\bar{u}(z) - z = \frac{z}{z-2}$$

$$\Rightarrow (z^2 - 3z + 2)\bar{u}(z) = \frac{z}{z-2} + z$$

$$\Rightarrow (z-1)(z-2)\bar{u}(z) = \frac{z^2 - z}{z-2}$$

$$\Rightarrow (z-\cancel{1})(z-2)\bar{u}(z) = \frac{z(z-\cancel{1})}{z-2}$$

$$\Rightarrow \bar{u}(z) = \frac{z}{(z-2)^2}$$

$$\bar{u}(z) = \frac{1}{2}\cdot\frac{2z}{(z-2)^2} \qquad z^{-1}[\bar{u}(z)] = \frac{1}{2}z^{-1}\left[\frac{2z}{(z-2)^2}\right]$$

$$U_n = \frac{1}{2}\, 2^n \cdot n$$

$$U_n = 2^{n-1}\cdot n$$

6a) Find the fourier sine transform of $e^{-|x|}$. Hence show that $\int_0^\infty \frac{x\sin mx}{1+x^2}dx = \frac{\pi}{2}e^{-m}$, $m>0$

Sol^n: Fourier sine transform is given by

$$F_S[F(x)] = \int_0^\infty F(x)\sin sx\, dx$$

$$= \int_0^\infty e^{-|x|}\sin sx\, dx \quad , \text{since } |x| = x, x>0$$

$$f_s(S) = \left[\frac{e^{-x}}{(-1)^2+s^2}[-1\sin sx - s\cos sx]\right]_0^\infty$$

$$= \left[0 - \frac{1}{1^2+s^2}[-1(0) - s(1)]\right]$$

$$= \frac{-1}{1^2+s^2}[-s]$$

$$f_s(S) = \frac{s}{1+s^2}$$

By inverse fourier sine transform we have,

$$\frac{2}{\pi}\int_0^\infty F_S(s)\sin sx\, ds = f(x)$$

$$\int_0^\infty \frac{s}{1+s^2}\sin sx\, ds = \frac{\pi}{2}f(x)$$

put $x = m$ where $m>0$ we have $f(x) = e^{-|m|} = e^{-m}$

$$\int_0^\infty \frac{s\sin ms}{1+s^2}ds = \frac{\pi}{2}e^{-m}$$

$$\boxed{\int_0^\infty \frac{x\sin mx}{1+x^2}dx = \frac{\pi}{2}e^{-m}}$$ when $s = x$

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6b) Find the z-transform of $\cos\left[\frac{n\pi}{2}+\frac{\pi}{4}\right]$

Soln: let $f(n) = \cos\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)$

$$f(n) = \cos\left(\frac{n\pi}{2}\right)\cos\frac{\pi}{4} - \sin\left(\frac{n\pi}{2}\right)\sin\frac{\pi}{4}$$

$$z[f(n)] = \frac{1}{\sqrt{2}}\, z\left[\cos\left(\frac{n\pi}{2}\right)\right] - \frac{1}{\sqrt{2}}\, z\left[\sin\left(\frac{n\pi}{2}\right)\right] \longrightarrow (1)$$

WKT $z[\cos n\theta] = \dfrac{z^2 - z\cos\theta}{z^2-2z\cos\theta+1}$

$$z[\cos n\pi/2] = \frac{z^2 - z\cos\pi/2}{z^2-2z\cos\pi/2+1}$$

$$= \frac{z^2-0}{z^2-0+1}$$

$$z[\cos n\pi/2] = \frac{z^2}{z^2+1}$$

$$z[\sin n\theta] = \frac{z\sin\theta}{z^2-2z\cos\theta+1}$$

$$z[\sin n\pi/2] = \frac{z\sin\pi/2}{z^2-2z\cos\pi/2+1}$$

$$z[\sin n\pi/2] = \frac{z}{z^2+1}$$

$\therefore (1) \Rightarrow$ $F(z) = \dfrac{1}{\sqrt{2}}\left(\dfrac{z^2}{z^2+1}\right) - \dfrac{1}{\sqrt{2}}\left(\dfrac{z}{z^2+1}\right)$

$$F(z) = \frac{z^2-z}{\sqrt{2}(z^2+1)}$$

6c) Find the inverse z-transform of $\dfrac{2z^2+3z}{(z+2)(z-4)}$

Soln: let $F(z) = \dfrac{2z^2+3z}{(z+2)(z-4)}$

$$\frac{F(z)}{z} = \frac{2z+3}{(z+2)(z-4)}$$

$$\frac{2z+3}{(z+2)(z-4)} = \frac{A}{z+2} + \frac{B}{z-4} \longrightarrow ①$$

$$2z+3 = A(z-4) + B(z+2) \longrightarrow ②$$

When $z = -2$

$$② \Rightarrow 2(-2)+3 = A(-2-4)$$

$$-1 = -6A$$

$$\boxed{A = 1/6}$$

When $z = 4$

$$② \Rightarrow 2(4)+3 = B(4+2)$$

$$11 = 6B$$

$$\boxed{B = 11/6}$$

$$\therefore ① \Rightarrow \frac{F(z)}{z} = \frac{1}{6} \cdot \frac{1}{z+2} + \frac{11}{6} \frac{1}{z-4}$$

$$F(z) = \frac{1}{6} \frac{z}{z+2} + \frac{11}{6} \frac{z}{z-4}$$

$$z^{-1}[F(z)] = \frac{1}{6} z^{-1}\left[\frac{z}{z+2}\right] + \frac{11}{6} z^{-1}\left[\frac{z}{z-4}\right]$$

$$z^{-1}[F(z)] = \frac{1}{6} \left[\frac{z}{z-(-2)}\right] + \frac{11}{6} z^{-1}\left[\frac{z}{z-4}\right]$$

$$f(n) = \frac{1}{6}(-2)^n + \frac{11}{6}(4)^n$$

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5a) If $f(x) = \begin{cases} a^2-x^2, & \text{for } |x|\le a \\ 0 & \text{for } |x|>a \end{cases}$, find the infinite fourier transform of $f(x)$ & hence evaluate $\int_0^\infty \frac{\sin x - x\cos x}{x^3}\,dx$

Soln:
$$f(x) = \begin{cases} a^2-x^2, & |x|\le a \\ 0, & |x|>a \end{cases}$$

$$f(x) = \begin{cases} a^2-x^2, & -a\le x\le a \\ 0, & x>a \end{cases}$$

WKT The fourier transform of $F(x)$ is
$$F[F(x)] = \int_{-\infty}^{\infty} e^{isx} F(x)\,dx$$

$$= \int_{-\infty}^{-a} e^{isx}F(x)\,dx + \int_{-a}^{a} e^{isx}F(x)\,dx + \int_{a}^{\infty} e^{isx}F(x)\,dx$$

$$= \int_{-a}^{a} e^{isx}(a^2-x^2)\,dx$$

$$= (a^2-x^2)\int_{-a}^{a} e^{isx}\,dx - \int_{-a}^{a}\left[(-2x)\int e^{isx}\,dx\right]dx$$

$$= \frac{1}{is}\left[(a^2-x^2)e^{isx}\right]_{-a}^{a} + \frac{2}{is}\int_{-a}^{a} x e^{isx}\,dx$$

$$= 0 + \frac{2}{is}\int_{-a}^{a} x e^{isx}\,dx$$

$$= \frac{2}{is}\left\{ x\int_{-a}^{a} e^{isx}\,dx - \int_{-a}^{a}\left[1\int e^{isx}\,dx\right]dx \right\}$$

$$= \frac{2}{is}\left\{ \frac{1}{is}\left[x e^{isx}\right]_{-a}^{a} - \frac{1}{i^2 s^2}\left[e^{isx}\right]_{-a}^{a} \right\}$$

$$= \frac{2}{is}\left\{ \frac{1}{is}\left[a e^{ias} - (-a)e^{-ias}\right] + \frac{1}{s^2}\left[e^{ias} - e^{-ias}\right] \right\}$$

$$= \frac{2}{is}\left\{ \frac{a}{is}\left[e^{ias}+e^{-ias}\right] + \frac{1}{s^2}\left[e^{ias}-e^{-ias}\right]\right\}$$

$$= \frac{2}{is}\left\{ \frac{a}{is}\left[\cos as + i\cancel{\sin as} + \cos as - i\cancel{\sin as}\right] + \frac{1}{s^2}\left[\cancel{\cos as} + i\sin as - \cancel{\cos as} + i\sin as\right]\right\}$$

$$= \frac{2}{is}\left\{ \frac{a}{is}\left[2\cos as\right] + \frac{1}{s^2}(2i\sin as)\right\}$$

$$= \frac{2}{is}\left\{ \frac{2a\cos(as)}{is} + \frac{2i\sin as}{s^2}\right\}$$

$$= \frac{4a\cos(as)}{i^2 s^2} + \frac{4\sin as}{s^3}$$

$$= \frac{4\sin as}{s^3} - \frac{4a\cos(as)}{s^2}$$

$$f(s) = 4\left[\frac{\sin as - as\cos(as)}{s^3}\right]$$

WKT Fourier inverse transform is given by

$$F^{-1}\left[f(s)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(s)\, e^{-isx}\, ds = F(x)$$

$$\Rightarrow \int_{-\infty}^{\infty} f(s)\, e^{-isx}\, dx = 2\pi \begin{cases} a^2-x^2, & -a \le x \le a \\ 0, & x > a \end{cases}$$

$$\Rightarrow \int_{-\infty}^{\infty} 4\left[\frac{\sin as - as\cos(as)}{s^3}\right]\left[\cos sx - i\sin sx\right] ds = 2\pi \begin{cases} a^2-x^2, & -a \le x \le a \\ 0, & x > a \end{cases}$$

$$\int_{-\infty}^{\infty} \frac{4\left[\sin as - as\cos(as)\right]}{s^3}\cos sx\, ds - i\int_{-\infty}^{\infty} \frac{4\left[\sin as - as\cos as\right]}{s^3}\sin(sx)\, ds = 2\pi \begin{cases} a^2-x^2, & -a \le x \le a \\ 0, & x > a \end{cases} + i(0)$$

$$\int_{-\infty}^{\infty} 4\,\frac{[\sin(as)-as\cos as]}{s^3}\cos(sx)\,dx = 2\pi\begin{cases} a^2-x^2, & -a\le x\le a \\ 0, & x>a\end{cases} \rightarrow (1)$$

When $x=0$

$$\therefore (1) \Rightarrow \int_{-\infty}^{\infty} \frac{4[\sin(as)-as\cos(as)]}{s^3}\,ds = 2\pi(a^2-0) = 2\pi a^2$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin(as)-as\cos(as)}{s^3}\,ds = \frac{2\pi a^2}{4}$$

$$\Rightarrow 2\int_{0}^{\infty} \frac{\sin(as)-as\cos(as)}{s^3}\,ds = \frac{\pi}{2}a^2$$

$$\Rightarrow \int_{0}^{\infty} \frac{\sin(as)-as\cos(as)}{s^3}\,ds = \frac{\pi}{4}a^2$$

$$\Rightarrow \int_{0}^{\infty} \frac{\sin s - s\cos s}{s^3}\,ds = \frac{\pi}{4} \quad \text{for } a=1$$

$$\Rightarrow \int_{0}^{\infty} \frac{\sin x - x\cos x}{x^3}\,dx = \frac{\pi}{4} \quad \text{for } s=x$$

5b) Find the fourier sine transform of $f(x)=\begin{cases} x, & \text{if } 0<x<1 \\ 2-x, & \text{if } 1<x<2 \\ 0, & \text{if } x>2\end{cases}$

Solⁿ:- WKT $F_s[f(x)] = f_s(s) = \int_0^{\infty} f(x)\sin(sx)\,dx$

$$\Rightarrow f_s(s) = \int_0^1 f(x)\sin sx\,dx + \int_1^2 f(x)\sin(sx)\,dx + \int_2^{\infty} f(x)\sin(sx)\,dx$$

$$f_s(s) = \int_0^1 x\sin(sx)\,dx + \int_1^2 (2-x)\sin(sx)\,dx \rightarrow (1)$$

$$\therefore \int_0^1 x\sin(sx)\,dx = \left[x\int_0^1 \sin sx\,dx - \int_0^1\left[1\int \sin sx\,dx\right]dx\right]$$

$$= -\frac{1}{s}[x\cos(sx)]_0^1 + \frac{1}{s^2}[\sin sx]_0^1$$

$$= -\frac{1}{s}[\cos s - 0] + \frac{1}{s^2}[\sin s - 0]$$

$$= -\frac{1}{s}\cos s + \frac{1}{s^2}\sin s$$

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$$\int_1^2 (2-x)\sin(sx)\,dx = (2-x)\int_1^2 \sin(sx)\,dx - \int_1^2 \left[(-1)\int \sin(sx)dx\right]dx$$

$$= -\frac{1}{s}\Big[(2-x)\cos sx\Big]_1^2 - \frac{1}{s^2}\Big[\sin sx\Big]_1^2$$

$$= -\frac{1}{s}[0-\cos s] - \frac{1}{s^2}[\sin 2s - \sin s]$$

$$= \frac{1}{s}\cos s - \frac{1}{s^2}\sin 2s + \frac{1}{s^2}\sin s$$

$\therefore$ ① $\Rightarrow$ $f_s(s) = -\cancel{\frac{1}{s}\cos s} + \frac{1}{s^2}\sin s + \cancel{\frac{1}{s}\cos s} - \frac{1}{s^2}\sin 2s + \frac{1}{s^2}\sin s$

$$f_s(s) = \frac{2}{s^2}\sin s - \frac{1}{s^2}\sin 2s$$

5c) Solve $u_{n+2} + 6u_{n+1} + 9u_n = 2^n$, $u_0 = 0 = u_1$, by using z-transforms.

Sol^n: given $u_{n+2} + 6u_{n+1} + 9u_n = 2^n$

$$Z[u_{n+2}] + 6Z[u_{n+1}] + 9Z[u_n] = Z[2^n]$$

$$z^2\left[\bar{u}(z) - u_0 - \frac{u_1}{z}\right] + 6z[\bar{u}(z) - u_0] + 9\bar{u}(z) = \frac{z}{z-2}$$

$$z^2\bar{u}(z) + 6z\bar{u}(z) + 9\bar{u}(z) = \frac{z}{z-2}$$

$$(z^2+6z+9)\,\bar{u}(z) = \frac{z}{z-2}$$

$$(z+3)^2\,\bar{u}(z) = \frac{z}{z-2}$$

$$\bar{u}(z) = \frac{z}{(z-2)(z+3)^2}$$

$$\frac{\bar{u}(z)}{z} = \frac{1}{(z-2)(z+3)^2} \longrightarrow ①$$

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$$\frac{1}{(z-2)(z+3)^2} = \frac{A}{z-2} + \frac{B}{z+3} + \frac{C}{(z+3)^2}$$

$$1 = A(z+3)^2 + B(z-2)(z+3) + C(z-2) \longrightarrow (2)$$

when $z=2$

$(2) \Rightarrow \quad 1 = 25A$

$$\boxed{A = \frac{1}{25}}$$

when $z=-3$

$(2) \Rightarrow \quad 1 = -5C$

$$\boxed{C = -\frac{1}{5}}$$

Compare coefficients of z^2

$A+B=0$

$B=-A$

$$\boxed{B = -\frac{1}{25}}$$

$$\therefore (1) \Rightarrow \frac{\bar{u}(z)}{z} = \frac{1}{25}\cdot\frac{1}{z-2} - \frac{1}{25}\,\frac{1}{z+3} - \frac{1}{5}\cdot\frac{1}{(z+3)^2}$$

$$\bar{u}(z) = \frac{1}{25}\,\frac{z}{z-2} - \frac{1}{25}\,\frac{z}{z+3} - \frac{1}{5}\,\frac{z}{(z+3)^2}$$

$$z^{-1}[\bar{u}(z)] = \frac{1}{25} z^{-1}\left[\frac{z}{z-2}\right] - \frac{1}{25} z^{-1}\left[\frac{z}{z-(-3)}\right] + \frac{1}{15} z^{-1}\left[\frac{-3z}{[z-(-3)]^2}\right]$$

$$u_n = \frac{1}{25}(2)^n - \frac{1}{25}(-3)^n + \frac{1}{15}(-3)^n\cdot n$$

6 a) If $f(x) = \begin{cases} 1, & \text{for } |x| \leq a \\ 0, & \text{for } |x| > a \end{cases}$, find the infinite fourier transform of $f(x)$ and hence Evaluate $\int_0^\infty \frac{\sin x}{x} dx$

Soln: given $f(x) = \begin{cases} 1, & |x| \leq a \\ 0, & |x| > a \end{cases}$

$$\Rightarrow f(x) = \begin{cases} 1, & -a < x \leq a \\ 0, & x > a \end{cases}$$

WKT

$$F[f(x)] = \int_{-\infty}^{\infty} e^{isx} f(x)\, dx$$

$$= \int_{-\infty}^{-a} e^{isx} f(x)\, dx + \int_{-a}^{a} e^{isx} f(x)\, dx + \int_{a}^{\infty} e^{isx} f(x)\, dx$$

$$= 0 + \int_{-a}^{a} e^{isx}\, 1\, dx + 0$$

$$F[f(x)] = \int_{-a}^{a} e^{isx}\, dx$$

$$= \left[\frac{e^{isx}}{is}\right]_{-a}^{a}$$

$$= \frac{1}{is}\left[e^{isx}\right]_{-a}^{a}$$

$$= \frac{1}{is}\left[e^{ais} - e^{-ais}\right]$$

$$= \frac{1}{is}\left[\cancel{\cos(as)} + i\sin(as) - \cancel{\cos(as)} + i\sin(as)\right]$$

$$= \frac{1}{\cancel{i}s}\left[2\cancel{i}\sin as\right]$$

$$f(s) = \frac{2}{s}\sin as$$

WKT $F^{-1}[f(s)] = f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx} f(s)\, ds$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[e^{isx}\ \frac{2}{s}\sin as\, ds\right] = \begin{cases}1, & -a<x\le a\\ 0, & x>a\end{cases}$$

$$\Rightarrow \int_{-\infty}^{\infty}\left\{\left[\frac{\cos(sx).\sin(as)}{s}\right] - i\left[\frac{\sin(as)-\sin(sx)}{s}\right]\right\}ds = \pi\begin{cases}1, & -a<x<a\\ 0, & x>a\end{cases}$$

$$\therefore \int_{-\infty}^{\infty}\frac{\cos(sx)\sin as}{s}\,ds = \pi\begin{cases}1, & -a<x<a\\ 0, & x>a\end{cases}$$

When $x=0$

$$① \Rightarrow \int_{-\infty}^{\infty}\frac{\cos 0 \sin as}{s}\,ds = \pi(1)$$

$$\int_{-\infty}^{\infty}\frac{\sin as}{s}\,ds = \pi$$

$$2\int_{0}^{\infty}\frac{\sin as}{s}\,ds = \pi$$

$$\int_{0}^{\infty}\frac{\sin as}{s}\,ds = \frac{\pi}{2}$$

$$\int_{0}^{\infty}\frac{\sin s}{s}\,ds = \frac{\pi}{2} \quad \text{for } a=1$$

$$\Rightarrow \int_{0}^{\infty}\frac{\sin x}{x}\,dx = \frac{\pi}{2} \quad \text{for } s=x$$

6b) Find the Z-transform of $2n + \sin\left(\frac{n\pi}{4}\right) + 1$

Sol^n: let $f(n) = 2n + \sin\left(\frac{n\pi}{4}\right) + 1$

$$Z[f(n)] = 2Z[n] + Z\left[\sin\left(\frac{n\pi}{4}\right)\right] + Z[1] \rightarrow ①$$

$$Z[n] = \frac{z}{(z-1)^2}$$

$$Z[\sin n\theta] = \frac{z\sin\theta}{z^2-2z\cos\theta+1}$$

$$Z\left[\sin\frac{n\pi}{4}\right] = \frac{z\sin\pi/4}{z^2-2z\cos\frac{\pi}{4}+1} = \frac{z/\sqrt{2}}{z^2-2z\left(1/\sqrt{2}\right)+1}$$

$$z\left[\sin\frac{n\pi}{4}\right] = \frac{z}{\sqrt{2}z^2 - 2z + \sqrt{2}}$$

$$z[1] = \frac{z}{z-1}$$

$$\therefore \text{①} \Rightarrow F(z) = \frac{2z}{(z-1)^2} + \frac{z}{\sqrt{2}z^2 - 2z + \sqrt{2}} + \frac{z}{z-1}$$

6c) Find the inverse z-transform of $18z^2/[(2z-1)(4z+1)]$

Sol^n: let $F(z) = \dfrac{18z^2}{(2z-1)(4z+1)}$

$$\frac{F(z)}{z} = \frac{18z}{(2z-1)(4z+1)} \longrightarrow \text{①}$$

$$\frac{18z}{(2z-1)(4z+1)} = \frac{A}{2z-1} + \frac{B}{4z+1}$$

$$\Rightarrow 18z = A(4z+1) + B(2z-1) \longrightarrow \text{②}$$

when $z = 1/2$

$$\text{②} \Rightarrow 9 = 3A$$

$$\boxed{A = 3}$$

when $z = -\frac{1}{4}$

$$\text{②} \Rightarrow 18(-1/4) = B[2(-1/4) - 1]$$

$$-9/2 = B[-1/2 - 1]$$

$$-9/2 = B(-3/2)$$

$$\boxed{B = 3}$$

$$\therefore \text{①} \Rightarrow \frac{F(z)}{z} = \frac{3}{2z-1} + \frac{3}{4z+1}$$

$$F(z) = 3 \cdot \frac{z}{2z-1} + 3 \cdot \frac{z}{4z+1}$$

$$F(z) = \frac{3}{2}\frac{z}{(z-1/2)} + \frac{3}{4}\frac{z}{(z+1/4)}$$

$$F(z) = \frac{3}{2}\frac{z}{(z-1/2)} + \frac{3}{4}\frac{z}{[z-(-1/4)]}$$

$$z^{-1}[F(z)] = \frac{3}{2} z^{-1}\left[\frac{z}{z-1/2}\right] + \frac{3}{4} z^{-1}\left[\frac{z}{z-(-1/4)}\right]$$

$$f(n) = \frac{3}{2}(1/2)^n + \frac{3}{4}(-1/4)^n$$

Module - 2

3 a) Find the fourier series expansion of $f(x) = \frac{\pi^2}{12} - \frac{x^2}{4}$ in $-\pi \le x \le \pi$. Hence deduce that $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$

Soln:
$$f(x) = \frac{\pi^2}{12} - \frac{x^2}{4}$$

$$f(-x) = \frac{\pi^2}{12} - \frac{(-x)^2}{4} = \frac{\pi^2}{12} - \frac{x^2}{4} = f(x)$$

$\therefore f(x)$ is an even function $\Rightarrow b_n = 0$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \longrightarrow (1)$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \frac{\pi^2}{12} - \frac{x^2}{4}\,dx$$

$$= \frac{2}{\pi}\left[\frac{\pi^2}{12}x - \frac{x^3}{12}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\pi^3}{12} - \frac{\pi^3}{12} - 0\right]$$

$$\boxed{a_0 = 0}$$

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^\pi \left(\frac{\pi^2}{12} - \frac{x^2}{4}\right)\cos nx\,dx$$

$$= \frac{2}{\pi}\left\{\left(\frac{\pi^2}{12} - \frac{x^2}{4}\right)\int_0^\pi \cos nx\,dx - \int_0^\pi\left[\left(-\frac{2x}{4}\right)\int \cos nx\,dx\right]dx\right\}$$

$$= \frac{2}{\pi}\left\{\frac{1}{n}\left[\left(\frac{\pi^2}{12} - \frac{x^2}{4}\right)\sin nx\right]_0^\pi + \frac{1}{2n}\int_0^\pi x\sin nx\,dx\right\}$$

$$= \frac{2}{\pi}\left\{\frac{1}{n}(0-0) + \frac{1}{2n}\left[x\int_0^\pi \sin nx\,dx - \int_0^\pi [1\cdot\int \sin nx]\,dx\right]\right\}$$

$$= \frac{2}{\pi}\left\{\frac{1}{2n}\left[\frac{-1}{n}[x\cos nx]_0^\pi + \frac{1}{n^2}[\sin nx]_0^\pi\right]\right\}$$

$$= \frac{2}{\pi}\left\{\frac{1}{2n}\left[-\frac{1}{n}[\pi\cos n\pi - 0] + \frac{1}{n^2}[\sin n\pi - \sin 0]\right]\right\}$$

$$= \frac{2}{\pi}\left\{\frac{-1}{2n^2}[\pi(-1)^n] + 0\right\}$$

$$= \frac{-(-1)^n}{n^2}$$

$$a_n = \frac{(-1)^{n+1}}{n^2}$$

(1) $\Rightarrow$ $$f(x) = 0 + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos nx$$

$$\frac{\pi^2}{12} - \frac{x^2}{4} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cdot\cos nx$$

let $x=0$

$$\frac{\pi^2}{12} - 0 = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cdot 1$$

$$\frac{\pi^2}{12} = \frac{(-1)^{1+1}}{1^2} + \frac{(-1)^{2+1}}{2^2} + \frac{(-1)^{3+1}}{3^2} + \frac{(-1)^{4+1}}{4^2} + \cdots$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

3b) Find the half - range cosine series of $f(x) = (x+1)^2$ in interval $0 \le x \le 1$.

Soln :- given $f(x) = (x+1)^2 \quad x \in [0,1]$

WKT the fourier half range cosine series of $f(x)$ in $[0,1]$

$\boxed{l = 1}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) \longrightarrow (1)$$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$= \frac{2}{1}\int_0^1 (x+1)^2\,dx$$

$$= 2\left[\frac{(x+1)^3}{3}\right]_0^1$$

$$= 2\left[\frac{8}{3} - \frac{1}{3}\right]$$

$$= 2\left[\frac{8-1}{3}\right]$$

$$= 2\left[\frac{7}{3}\right]$$

$$\boxed{a_0 = \frac{14}{3}}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{1}\int_0^1 (x+1)^2 \cos(n\pi x)\,dx$$

$$= 2\left\{ (x+1)^2 \int_0^1 \cos(n\pi x)\,dx - \int_0^1 \left[2(x+1)(1)\int \cos(n\pi x)\,dx\right]dx \right\}$$

$$= 2\left\{ \frac{1}{n\pi}\left[(x+1)^2 \sin(n\pi x)\right]_0^1 - \frac{2}{n\pi}\int_0^1 (x+1)\sin(n\pi x)\,dx \right\}$$

$$= 2\left\{\frac{1}{n\pi}[0-0] - \frac{2}{n\pi}\int_0^1 (x+1)\sin n\pi x\,dx\right\}$$

$$= \frac{-4}{n\pi}\int_0^1 (x+1)\sin(n\pi x)\,dx$$

$$= \frac{-4}{n\pi}\left\{(x+1)\int_0^1 \sin(n\pi x)\,dx - \int_0^1 [1.\int \sin n\pi x\,dx]\,dx\right\}$$

$$= \frac{-4}{n\pi}\left\{\frac{-1}{n\pi}\left[(x+1)\cos n\pi x\right]_0^1 + \frac{1}{n^2\pi^2}\left[\sin n\pi x\right]_0^1\right\}$$

$$= \frac{-4}{n\pi}\left\{-\frac{1}{n\pi}[2\cos n\pi - 1] + \frac{1}{n^2\pi^2}[\cancel{\sin n\pi}^{\,0} - 0]\right\}$$

$$= \frac{+4}{n^2\pi^2}\left\{[2(-1)^n - 1] + 0\right\}$$

$$a_n = \frac{4}{n^2\pi^2}[2(-1)^n - 1]$$

$$\therefore ① \Rightarrow (x+1)^2 = \frac{14/3}{2} + \sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}[2(-1)^n - 1]\cos n\pi x$$

$$(x+1)^2 = \frac{14}{6} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}[2(-1)^n - 1]\cos n\pi x$$

3 c) obtain the fourier series of $f(x) = \begin{cases} l-x, & \text{for } 0 \le x \le l \\ 0, & \text{for } l \le x \le 2l \end{cases}$

Soln:- given $f(x) = \begin{cases} l-x, & 0 \le x \le l \\ 0, & l \le x \le 2l \end{cases}$

The given function is neither even nor odd

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi x}{l}\right) \rightarrow ①$$

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx = \frac{1}{l}\int_0^{l} f(x)\,dx = \frac{1}{l}\int_0^{l}(l-x)\,dx$$

$$= \frac{1}{l}\left[lx - \frac{x^2}{2}\right]_0^l = \frac{1}{l}\left[l^2 - l^2/2\right] = \frac{1}{l}\left[l^2/2\right] = l/2$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\int_0^{l} (l-x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\left\{ \frac{l}{n\pi}\left[(l-x)\sin\frac{n\pi x}{l}\right]_0^l - \int_0^l \left[(-1)\frac{l}{n\pi}\sin\frac{n\pi x}{l}\,dx\right]dx \right\}$$

$$= \frac{1}{l}\left\{ (0-0) + \frac{l}{n\pi}\int_0^l \sin\frac{n\pi x}{l}\,dx \right\}$$

$$= \frac{1}{l}\left\{ \frac{-l^2}{n^2\pi^2}\left[\cos\frac{n\pi x}{l}\right]_0^l \right\}$$

$$= \frac{1}{l}\left\{ \frac{-l^2}{n^2\pi^2}\left[(-1)^n - 1\right] \right\}$$

$$a_n = \frac{l}{n^2\pi^2}\left[1-(-1)^n\right]$$

$$b_n = \frac{1}{l}\int_0^l (l-x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\left\{ \frac{-l}{n\pi}\left[(l-x)\cos\frac{n\pi x}{l}\right]_0^l - 0 \right\}$$

$$= \frac{1}{l}\left\{ \frac{-l}{n\pi}\left[0 - l(1)\right] \right\}$$

$$= \frac{1}{l}\left\{ \frac{l^2}{n\pi} \right\}$$

$$b_n = \frac{l}{n\pi}$$

$$\therefore (1) \Rightarrow f(x) = \frac{l}{4} + \sum_{n=1}^{\infty} \frac{l}{n^2\pi^2}\left[1-(-1)^n\right]\cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty}\frac{l}{n\pi}\sin\frac{n\pi x}{l}$$

4 a) The displacement y (in cms) of a machine part occurs due to the rotation of x radians is given below.

Rotation x (in radians)	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
Displacement y (in cms)	1.0	1.4	1.9	1.7	1.5	1.2	1.0

Expand y in terms of fourier Series upto Second harmonics.

x	y	cos x	cos 2x	sin x	sin 2x	y cos x	y cos 2x	y sin x	y sin 2x
0	1	1	1	0	0	1	1	0	0
60°	1.4	0.5	-0.5	0.866	0.866	0.7	-0.7	1.2124	1.2124
120	1.9	-0.5	-0.5	0.866	-0.866	-0.95	-0.95	1.6454	-1.6454
180	1.7	-1	1	0	0	-1.7	1.7	0	0
240	1.5	-0.5	-0.5	-0.866	0.866	-0.75	-0.75	-1.299	1.299
300	1.2	0.5	-0.5	-0.866	-0.866	0.6	-0.6	-1.0392	-1.0392
Total	8.7					-1.1	-0.3	0.5196	-0.1732

$a_1 = \frac{2}{N}\sum y\cos x = \frac{2}{6}(-1.1) = -0.367$ $\quad a_0 = \frac{2}{N}\sum y = \frac{2}{6}(8.7) = 2.9$

$a_2 = \frac{2}{N}\sum y\cos 2x = \frac{2}{6}(-0.3) = -0.1$

$b_1 = \frac{2}{N}\sum y\sin x = \frac{2}{6}(0.5196) = 0.1732$

$b_2 = \frac{2}{N}\sum y\sin 2x = \frac{2}{6}(-0.1732) = -0.0577$

Fourier Series upto second harmonics is given by

$$y = \frac{a_0}{2} + (a_1\cos x + b_1\sin x) + (a_2\cos 2x + b_2\sin 2x)$$

$$y = \frac{2.9}{2} + (-0.367\cos x + 0.1732\sin x) + (-0.1\cos 2x - 0.0577\sin 2x)$$

$$y = 1.45 + (-0.367\cos x + 0.1732\sin x) + (-0.1\cos 2x - 0.0577\sin 2x)$$

4c) Find the fourier Series Expansion of $f(x) = |x|$ in $-\pi \le x \le \pi$. Hence deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots = \frac{\pi^2}{8}$

Soln: $f(x) = |x| = \begin{cases} -x, & -\pi \le x \le 0 \\ x, & 0 \le x \le \pi \end{cases}$

$$\phi(x) = -x \quad , \quad \psi(x) = x$$

$$\phi(-x) = x = \psi(x)$$

$\therefore f(x)$ is an Even function $\Rightarrow b_n = 0$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \longrightarrow (1)$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x\,dx$$

$$= \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi}$$

$$\boxed{a_0 = \pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\left\{\int_0^{\pi} x\cos nx\,dx\right\}$$

$$= \frac{2}{\pi}\left\{x\int_0^{\pi}\cos nx\,dx - \int_0^{\pi}\left[1\int \cos nx\,dx\right]dx\right\}$$

$$= \frac{2}{\pi}\left\{\frac{1}{n}\left[x\sin nx\right]_0^{\pi} + \frac{1}{n^2}\left[\cos nx\right]_0^{\pi}\right\}$$

$$= \frac{2}{\pi}\left[\frac{1}{n^2}\left[\cos n\pi - 1\right]\right]$$

$$a_n = \frac{2}{n^2\pi}\left[(-1)^n - 1\right]$$

$$① \Rightarrow \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2}\left[(-1)^n - 1\right]\cos nx = \begin{cases} -x, & -\pi \le x \le 0 \\ x, & 0 \le x \le \pi \end{cases} \longrightarrow ②$$

when $x = 0$

$$② \Rightarrow \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} = 0$$

$$\frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} = -\pi/2$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} = \frac{-\pi^2}{4}$$

$$\Rightarrow \frac{-2}{1^2} + 0 - \frac{2}{3^2} + 0 - \frac{2}{5^2} + \cdots = \frac{-\pi^2}{4}$$

$$-2\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right] = -\frac{\pi^2}{4}$$

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}$$

4b) Find the half-range sine series of e^x in the interval $0 \le x \le 1$.

given $f(x) = e^x$

The half range sine series is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \longrightarrow ①$$

$$b_n = \frac{2}{l}\int_0^l f(x) \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{2}{1}\int_0^1 e^x \sin n\pi x\, dx$$

$$= 2\left[\frac{e^x}{1+n^2\pi^2}\left[\sin n\pi x - n\pi \cos n\pi x\right]\right]_0^1$$

$$= 2\left[\frac{e}{1+n^2\pi^2}\left[0 - n\pi(-1)^n\right] - \frac{1}{1+n^2\pi^2}\left[0 - n\pi\right]\right]$$

$$= \frac{2}{1+n^2\pi^2}\left[e\left[(-1)^{n+1} n\pi\right] + n\pi\right]$$

$$= \frac{2}{1+n^2\pi^2}\left[2.7183\, n\pi\,(-1)^{n+1} + n\pi\right]$$

$$= \frac{2n\pi}{1+n^2\pi^2}\left[2.7183 + 1\right]$$

$$= \frac{2n\pi}{1+n^2\pi^2}(3.7183)$$

$$b_n = \frac{7.436}{1+n^2\pi^2} n\pi$$

$$\therefore \text{①} \Rightarrow e^x = \sum_{n=1}^{\infty} \frac{7.436\, n\pi}{1+n^2\pi^2} \sin n\pi x$$

3 a) An alternating current $I(x)$ after passing through a rectifier has the form $I(x) = \begin{cases} I_0 \sin x, & \text{for } 0 \le x < \pi \\ 0, & \text{for } \pi < x \le 2\pi \end{cases}$, Where I_0 is the maximum current and the period is 2π. Express $I(x)$ as a fourier Series.

Sol^n: The fourier Series of period of 2π is given by

$$I = f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\theta + \sum_{n=1}^{\infty} b_n \sin n\theta \longrightarrow \text{①}$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\, d\theta$$

$$= \frac{I_0}{\pi}\int_0^{2\pi} \sin\theta\, d\theta = \frac{I_0}{\pi}\left\{\int_0^{\pi} \sin\theta\, d\theta + \int_{\pi}^{2\pi} \sin\theta\, d\theta\right\}$$

$$= \frac{I_0}{\pi}\int_0^{\pi} \sin\theta\, d\theta$$

$$a_0 = \frac{I_0}{\pi}\left[-\cos\theta\right]_0^{\pi}$$

$$a_0 = \frac{2I_0}{\pi}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta \, d\theta$$

$$= \frac{I_0}{\pi}\left\{\int_0^{\pi}\sin\theta\cos n\theta \, d\theta + \int_{\pi}^{2\pi} \overset{0.\cos n\theta}{\cancel{\sin\theta\cos n\theta}} \, d\theta\right\}$$

$$= \frac{I_0}{\pi}\int_0^{\pi}\sin\theta\cos n\theta \, d\theta$$

$$a_n = \frac{I_0}{\pi}\int_0^{\pi}\sin\theta\cos n\theta \, d\theta$$

Putting $n=1$ $\quad a_1 = \frac{I_0}{\pi}\int_0^{\pi}\sin\theta\cos\theta \, dx$

$$= \frac{I_0}{\pi}\int_0^{\pi}\frac{\sin 2x}{2}dx$$

$$a_1 = \frac{I_0}{2\pi}\left[-\frac{\cos 2x}{2}\right]_0^{\pi}$$

$$= \frac{-I_0}{4\pi}(\cos 2\pi - \cos 0)$$

$$= \frac{-I_0}{4\pi}(1-1)$$

$$a_1 = 0$$

and $\quad a_n = \frac{-I_0}{\pi(n^2-1)}\left\{1+(-1)^n\right\}$ for $n \neq 1$

$$b_n = \frac{I_0}{\pi}\int_0^{\pi}\sin\theta\sin n\theta \, d\theta$$

put $n=1$

$$b_1 = \frac{I_0}{\pi}\int_0^{\pi}\sin\theta\sin\theta \, d\theta$$

$$= \frac{I_0}{\pi}\int_0^{\pi}\sin^2\theta \, d\theta$$

$$b_1 = \frac{I_0}{\pi}\int_0^{\pi}\frac{1}{2}(1-\cos 2\theta)\,d\theta$$

$$= \frac{I_0}{2\pi}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi}$$

$$= \frac{I_0}{2\cancel{\pi}}(\cancel{\pi}-0)$$

$$b_1 = \frac{I_0}{2}$$

$$b_n = 0 \quad \text{for } n \neq 1$$

$$\therefore ① \Rightarrow I = f(\theta) = \frac{I_0}{\pi} + \sum_{n=2}^{\infty}\frac{-I_0}{\pi(n^2-1)}\{1+(-1)^n\}\cos n\theta + \frac{I_0}{2}\sin\theta$$

b) Find the half-range sine series of $f(x) = \frac{\sinh ax}{\sinh a\pi}$ in the interval $(0,\pi)$

Soln: given $f(x) = \frac{\sinh ax}{\sinh a\pi}$, $x \in (0,\pi)$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \longrightarrow ①$$

$$\therefore b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\frac{\sinh ax}{\sinh a\pi}\sin nx\,dx$$

$$= \frac{2}{\pi\sinh a\pi}\int_0^{\pi}\frac{e^{ax}-e^{-ax}}{2}\sin nx\,dx$$

$$= \frac{1}{\pi\sinh a\pi}\left\{\int_0^{\pi}e^{ax}\sin nx\,dx - \int_0^{\pi}e^{-ax}\sin nx\,dx\right\}$$

$$\therefore \int_0^{\pi}e^{ax}\sin nx\,dx = \left[\frac{e^{ax}}{a^2+n^2}\left[a\sin nx - n\cos nx\right]\right]_0^{\pi}$$

$$= \frac{e^{a\pi}}{a^2+n^2}\left[n(-1)^n\right] - \frac{e^0}{a^2+n^2}\left[0-n\right] = \frac{e^{a\pi}}{a^2+n^2} n(-1)^{n+1} + \frac{n}{a^2+n^2}$$

$$\int_0^\pi e^{-ax}\sin nx\,dx = \left[\frac{e^{-ax}}{(-a)^2+n^2}\left[-a\sin nx - n\cos nx\right]\right]_0^\pi$$

$$= \frac{e^{-a\pi}}{(-a)^2+n^2}\left[0-n(-1)^n\right] - \frac{e^0}{(-a)^2+n^2}\left[0-n\right]$$

$$= \frac{e^{-a\pi}}{(+a)^2+n^2}\left[n(-1)^{n+1}\right] + \frac{n}{a^2+n^2}$$

$$\therefore \quad b_n = \frac{1}{\pi \sin h a\pi}\left\{\frac{e^{a\pi}}{a^2+n^2} n(-1)^{n+1} + \frac{n}{a^2+n^2} - \frac{e^{-a\pi}}{a^2+n^2}\left[n(-1)^{n+1}\right] + \frac{n}{a^2+n^2}\right\}$$

$$\therefore \quad f(x) = \sum_{n=1}^{\infty} b_n \cdot \sin\frac{n\pi x}{l}$$

c> Find the fourier series expansion of $f(x) = x(1-x)(2-x)$ the interval $0 \le x \le 2$. Hence deduce the sum of the series that $\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \dots$

Solⁿ:- $f(x) = x(1-x)(2-x)$ $\qquad 2l = 2$

$f(x) = x^3 - 3x^2 + 2x$ $\qquad \boxed{l=1}$

$$f(2l-x) = f(2-x) = (2-x)^3 - 3(2-x)^3 + 2(2-x)$$

$$x(1-x)(2-x) = (2-x)(1-2+x)(\cancel{2}-\cancel{2}+x)$$

$$= x(2-x)(-1+x)$$

$$= -x(2-x)(1-x)$$

$$= -f(x)$$

4 a) In an electrical research laboratory, Scientists have designed a generator which can generate the following currents at different time instant t, in the period T:

time t (in sec)	0	T/6	T/3	T/2	2T/3	5T/6	T
f(x)	1.98	1.30	1.05	1.30	-0.88	-0.25	1.98

Determine the direct current part and amplitude of the first harmonic from the above data.

Sol^n :- Given the time t has been defined in the interval

$0 \le t \le T \Rightarrow 2l = T$

$l = T/2$ and removing the last term

We have

t	A	$\theta = \frac{2\pi t}{T}$	$\cos\theta$	$A\cos\theta$	$\sin\theta$	$A\sin\theta$
0	1.98	0	1	1.98	0	0
T/6	1.30	60	0.5	0.65	0.8660	1.1258
T/3	1.05	120	-0.5	-0.5250	0.86660	0.9093
T/2	1.30	180	-1	-1.30	0	0
2T/3	-0.88	240	-0.5	0.44	-0.86660	0.7621
5T/6	-0.25	300	0.5	-0.1250	-0.8660	0.2165
	Σ 4.53			1.12		3.0137

$$a_0 = \frac{2}{N}\Sigma A = \frac{2}{6}(4.53) = 1.5100$$

$$\frac{a_0}{2} = \frac{1.5100}{2} = \underline{0.75A}$$

$a_1 = \frac{2}{N}\Sigma A\cos\theta = \frac{2}{6}(1.12) = 0.3733$

$b_1 = \frac{2}{N}\Sigma A\sin\theta = \frac{2}{6}(3.0137) = 1.0046$

$\therefore$ Amplitude $= \sqrt{a_1^2+b_1^2} = \sqrt{(0.3733)^2+(1.0046)^2} = \underline{1.0717}$

4c) obtain the fourier series of $f(x) = x(2\pi - x)$ valid in the interval $(0, 2\pi)$

given $f(x) = x(2\pi - x)$, $x \in (0, 2\pi)$

$f(2\pi - x) = 2\pi - x(2\pi - (2\pi - x))$

$= 2\pi - x\,[\cancel{2\pi} - \cancel{2\pi} + x]$

$= x(2\pi - x)$

$= f(x)$

$\therefore$ $f(x)$ is an Even function $\Rightarrow b_n = 0$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \longrightarrow (1)$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x(2\pi - x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} 2\pi x - x^2\,dx$$

$$= \frac{2}{\pi}\left[\pi x^2 - \frac{x^3}{3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\pi^3 - \frac{\pi^3}{3}\right]$$

$$\boxed{a_0 = \frac{4}{3}\pi^2}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi}(2\pi x - x^2)\cos nx \, dx$$

$$= \frac{2}{\pi}\left[(2\pi x - x^2)\int_0^{\pi}\cos nx \, dx - \int_0^{\pi}(2\pi - 2x)\int \cos nx \, dx\right]$$

$$= \frac{2}{\pi}\left[\frac{1}{n}(2\pi x - x^2)\sin nx\right]_0^{\pi} + \frac{2}{n}\int_0^{\pi}(x-\pi)\sin nx \, dx$$

$$= \frac{4}{n\pi}\int_0^{\pi}(x-\pi)\sin nx \, dx$$

$$= \frac{4}{n\pi}\left\{(x-\pi)\int_0^{\pi}\sin nx \, dx - \int_0^{\pi}\left[1\int \sin nx \, dx\right]dx\right\}$$

$$= \frac{4}{n\pi}\left[-\frac{1}{n}(x-\pi)\cos nx\right]_0^{\pi} + \frac{1}{n^2}\left[\sin nx\right]_0^{\pi}$$

$$= \frac{4}{n\pi}\left[-\frac{1}{n}(0+\pi)(1) + \frac{1}{n^2}(0-0)\right]$$

$$= \frac{4}{n\pi}\left[-\pi/n\right]$$

$$\boxed{a_n = \frac{-4}{n^2}}$$

$$\therefore \text{①} \Rightarrow \quad x(2\pi - x) = \frac{4/3\,\pi^2}{2} + \sum_{n=1}^{\infty}\left(\frac{-4}{n^2}\right)\cos nx$$

$$2\pi x - x^2 = \frac{2}{3}\pi^2 - 4\sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx \longrightarrow \text{②}$$

when $x = 0$

$$\text{②} \Rightarrow \quad 0 = \frac{2\pi^2}{3} - 4\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$4\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{2}{3}\pi^2$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\cdots = \frac{\pi^2}{6}$$

when $x = \pi$

$$② \Rightarrow \quad 2\pi(\pi) - \pi^2 = \frac{2}{3}\pi^2 - 4\sum_{n=1}^{\infty} \frac{1}{n^2}\cos n\pi$$

$$\pi^2 = \frac{2}{3}\pi^2 - 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$-4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = \pi^2 - \frac{2}{3}\pi^2 = \frac{\pi^2}{3}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = \frac{-\pi^2}{12}$$

$$\Rightarrow \frac{-1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \cdots\cdots = -\frac{\pi^2}{12}$$

$$\Rightarrow \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\cdots = \frac{\pi^2}{12}$$

module - 4

7 a) Solve $\frac{dy}{dx} = e^x - y$, $y(0) = 1$ using Taylor's series method considering up to fourth degree terms and find the value of $y(0.1)$.

Soln:

$$\frac{dy}{dx} = y' = e^x - y \longrightarrow (1)$$

and $y(0) = 1$

$\Rightarrow x_0 = 0$, $y_0 = 1$

$y'(x_0) = e^{x_0} - y_0 = e^0 - 1 = 1 - 1 = 0$

$y''(x) = e^x - y' \Rightarrow y''(x_0) = e^{x_0} - y_0' = e^0 - 0 = 1 - 0 = 1$

$y'''(x) = e^x - y'' \Rightarrow y'''(x_0) = e^{x_0} - y_0'' = e^0 - (1) = 1 - 1 = 0$

$y^{IV}(x) = e^x - y''' \Rightarrow y^{IV}(x_0) = e^{x_0} - y_0''' = e^0 - 0 = 1 - 0 = 1$

WKT

$$y(x) = y(x_0) + \frac{(x-x_0)^1}{1!} y'(x_0) + \frac{(x-x_0)^2}{2!} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots$$

$$y(x) = 1 + \frac{(x-0)}{1!}(0) + \frac{(x-0)^2}{2!}(1) + \frac{(x-0)^3}{3!}(0) + \frac{(x-0)^4}{4!}(1) + \cdots$$

$$y(x) = 1 + \frac{x(0)}{1!} + \frac{x^2}{2!} + \frac{x}{3!}(0) + \frac{x^4}{4!} + \cdots$$

$$y(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \longrightarrow (2)$$

At $x = 0.1$

$$(2) \Rightarrow y(0.1) = 1 + \frac{(0.1)^2}{2} + \frac{(0.1)^4}{24}$$

$$y(0.1) = 1 + 0.0050 + 0$$

$$y(0.1) \cong 1.0050$$

b) Use Runge Kutta method of fourth order to solve $(x+y)\frac{dy}{dx} = 1$, $y(0.4) = 1$, to find $y(0.5)$ (Take $h = 0.1$).

Sol^n: $(x+y)\frac{dy}{dx} = 1$

$$\frac{dy}{dx} = \frac{1}{x+y} = f(x,y) \longrightarrow ①$$

and $y(0.4) = 1$

$x_0 = 0.4$, $y_0 = 1$, $h = 0.1$

$$K_1 = hf(x_0, y_0) = 0.1\, f(0.4, 1) = \frac{0.1}{0.4+1} = 0.0714$$

$$\boxed{K_1 = 0.0714}$$

$$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right) = 0.1\, f\left(0.4 + \frac{0.1}{2}, 1 + \frac{0.0714}{2}\right)$$

$$= 0.1\, f(0.45, 1.0357)$$

$$= 0.1\,(0.6731)$$

$$\boxed{K_2 = 0.0673}$$

$$K_3 = hf\left(x_0 + h/2, y_0 + \frac{K_2}{2}\right) = 0.1\, f\left(0.4 + \frac{0.1}{2}, 1 + \frac{0.0673}{2}\right)$$

$$= 0.1\, f(0.45, 1.0336)$$

$$= 0.1\,(0.6740)$$

$$\boxed{K_3 = 0.06740}$$

$$K_4 = hf(x_0 + h, y_0 + K_3) = 0.1\, f(0.5, 1.0674)$$

$$= 0.1\,(0.6379)$$

$$\boxed{K_4 = 0.06379}$$

WKT $y(x_0+h) = y(x_1) = y_0 + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$

$y(0.4+0.1) = y(0.5) = 1 + \frac{1}{6}[0.0714 + 2(0.0673) + 2(0.0674) + 0.06379]$

$$y(0.5) \cong \underline{1.0674}$$

c> Given that $\frac{dy}{dx}+\frac{y}{x}=\frac{1}{x^2}$ and $y(1)=1$, $y(1.1)=0.9960$, $y(1.2)=0.9860$ & $y(1.3)=0.9720$ find $y(1.4)$, using Adam - Bashforth predictor - corrector method.

Sol^n: Given $\frac{dy}{dx}+\frac{y}{x}=\frac{1}{x^2}$

$$\frac{dy}{dx}=\frac{1}{x^2}-\frac{y}{x}$$

$$\frac{dy}{dx}=\frac{1-xy}{x^2}=f(x,y)$$

$y(1)=1 \Rightarrow x_0=1$, $y_0=1$

$x_1=1.1$ $\quad y_1=0.9960$

$x_2=1.2$ $\quad y_2=0.9860$

$x_3=1.3$ $\quad y_3=0.9720$

$x_4=1.4$ $\quad y_4=?$ $\quad h=0.1$

$$f_0=f(x_0,y_0)=f(1,1)=\frac{1-x_0y_0}{x_0^2}=\frac{1-(1)(1)}{1^2}=0$$

$$f_1=\frac{1-x_1y_1}{x_1^2}=\frac{1-(1.1)(0.9960)}{(1.1)^2}=-0.0790$$

$$f_2=\frac{1-x_2y_2}{x_2^2}=\frac{1-(1.2)(0.9860)}{(1.2)^2}=-0.1272$$

$$f_3=\frac{1-x_3y_3}{x_3^2}=\frac{1-(1.3)(0.9720)}{(1.3)^2}=-0.1560$$

$$y_4^{(P)}=y_3+\frac{h}{24}\left[55f_3-59f_2+37f_1-9f_0\right]$$

$$=0.9720+\frac{0.1}{24}\left[-8.5800+7.5048-2.9230-0\right]$$

$$y_4^{(P)}=0.9553$$

$$\therefore f_4^{(P)} = \frac{1 - x_4 y_4^{(P)}}{x_4^2} = \frac{1-(1.4)(0.9553)}{(1.4)^2} = -0.1721$$

$$y_4^{(C)} = y_3 + \frac{h}{24}\left[9 f_4^{(P)} + 19 f_3 - 5 f_2 + f_1\right]$$

$$= 0.9720 + \frac{0.1}{24}\left[-1.5489 - 2.9640 + 0.6360 + 0.0790\right]$$

$$y_4^{(C)} = 0.9555$$

$$y(1.4) \cong \underline{\underline{0.9555}}$$

8 a) Solve the differential equation $\frac{dy}{dx} = x\sqrt{y}$ under the initial condition $y(1) = 1$, by using modified Euler's method at the point $x = 1.4$. Perform three iterations at each step, taking $h = 0.2$.

Sol^n : given $\frac{dy}{dx} = x\sqrt{y}$

$$y(1) = 1 \Rightarrow x_0 = 1 \quad y_0 = 1$$

$$h = 0.2 \qquad x_1 = x_0 + h = 1 + 0.2 = 1.2$$

To find $y(x_1) = y_1$

$$\therefore y(x_0 + h) = y_1^{(1)} = y(x_1) = y_0 + h f(x_0, y_0)$$

$$y_1^{(1)} = 1 + 0.2 f(1,1)$$

$$= 1 + 0.2(1)$$

$$y_1^{(1)} = 1.2000$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

$$= 1 + \frac{0.2}{2}\left[f(1,1) + f(1.2, 1.2)\right]$$

$$= 1 + 0.1\left[1 + 1.3145\right]$$

$$y_1^{(2)} = 1.23145$$

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1, y_1^{(2)})\right]$$

$$= 1 + \frac{0.2}{2}\left[f(1,1) + f(1.2, 1.2314)\right]$$

$$= 1 + 0.1\left[1 + 1.33162\right]$$

$$y_1^{(3)} = 1.23316$$

$$\therefore x_1 \simeq 1.2 \quad y_1 \approx 1.2331$$

To find $y(x_2) = y_2$

$$y(x_1+h) = y(1.2+0.2) = y(1.4)$$

$$y_2^{(1)} = y_1 + hf(x_1, y_1)$$

$$= 1.2331 + 0.2f(1.2, 1.2331)$$

$$= 1.2331 + 0.2\,(1.33254)$$

$$y_2^{(1)} = 1.4996$$

$$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1,y_1) + f(x_2, y_2^{(1)})\right]$$

$$= 1.2331 + \frac{0.2}{2}\left[f(1.2, 1.2331) + f(1.4, 1.4996)\right]$$

$$= 1.2331 + 0.1\left[1.3325 + 1.7144\right]$$

$$y_2^{(2)} = 1.53779$$

$$y_2^{(3)} = y_1 + \frac{h}{2}\left[f(x_1,y_1) + f(x_2, y_2^{(2)})\right]$$

$$= 1.2331 + \frac{0.2}{2}\left[f(1.2, 1.2331) + f(1.4, 1.5377)\right]$$

$$= 1.2331 + 0.1\left[1.3325 + 1.7360\right]$$

$$y_2^{(3)} = 1.5399$$

$$\therefore y(1.4) \simeq 1.5399$$

$$\boxed{x = 1.4,\ y = 1.5399}$$

b) Use fourth order Runge - kutta method, to find $y(0.1)$ with $h = 0.1$, given $\frac{dy}{dx} + y + xy^2 = 0$, $y(0) = 1$

given $\frac{dy}{dx} + y + xy^2 = 0$

$$\frac{dy}{dx} = -y - xy^2 = -(y + xy^2) = f(x, y)$$

$$y(0) = 1 \Rightarrow x_0 = 0, \; y_0 = 1 \quad h = 0.1$$

$$K_1 = hf(x_0, y_0) = 0.1 f(0, 1) = -0.1(1) = -0.1$$

$$K_2 = hf\left(x_0 + h/2, y_0 + \frac{K_1}{2}\right) = 0.1 f\left(\frac{0.1}{2}, 1 + \frac{(-0.1)}{2}\right)$$

$$= 0.1 f(0.0500, 0.9500)$$

$$= -0.1 \times 0.9951$$

$$K_2 = -0.0995$$

$$K_3 = hf\left(x_0 + h/2, y_0 + K_2/2\right) = 0.1 f\left[0.0500, 1 - \frac{0.0995}{2}\right]$$

$$= 0.1 f(0.0500, 0.9503)$$

$$= 0.1 \times -0.9955$$

$$K_3 = -0.0995$$

$$K_4 = hf(x_0 + h, y_0 + K_3) = 0.1 f(0.1, 0.9005) = -0.1 \times 0.9816 = -0.0982$$

WKT

$$y(x_0 + h) = y(x_1) = y_0 + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$$

$$y(x_1) = 1 + \frac{1}{6}[-0.1 + 2(-0.0995) + 2(-0.0995) + (-0.0982)]$$

$$= 1 + \frac{1}{6}[-0.5962]$$

$$= 1 - 0.0994$$

$$y(x_1) \simeq \underline{0.9006}$$

c) Apply milne's predictor - corrector formulae to compute y(0.3) given, $\frac{dy}{dx} = x + y^2$ with

x	0.0	0.1	0.2	0.3
y	1.0000	1.1000	1.2310	1.4020

Soln:

$$\frac{dy}{dx} = f(x,y) = x + y^2$$

$x_0 = 0.0 \quad y_0 = 1.0000$

$x_1 = 0.1 \quad y_1 = 1.1000$

$x_2 = 0.2 \quad y_2 = 1.2310$

$x_3 = 0.3 \quad y_3 = 1.4020$

$x_4 = 0.4 \quad y_4 = ? \quad h = 0.1$

$\therefore f_0 = x_0 + y_0^2 = 0 + 1^2 = 1$

$f_1 = x_1 + y_1^2 = 0.1 + 1.1^2 = 1.3100$

$f_2 = x_2 + y_2^2 = 0.2 + (1.2310)^2 = 1.7153$

$f_3 = x_3 + y_3^2 = 0.3 + (1.4020)^2 = 2.2656$

$$y_4^{(P)} = y(x_4) = y_0 + \frac{4h}{3}[2f_1 - f_2 + 2f_3]$$

$$= 1 + \frac{4(0.1)}{3}[2(1.31) - (1.7153) + 2(2.2656)]$$

$$y_4^{(P)} = 1.7247$$

$$f_4^{(P)} = x_4 + (y_4^P)^2 = 0.4 + (1.7247)^2 = 3.3745$$

$$y_4^{(C)} = y_2 + \frac{h}{3}[f_2 + 4f_3 + f_4^{(P)}]$$

$$= 1.231 + \frac{0.1}{3}[1.7153 + 4(2.2656) + 3.3745]$$

$$y_4^{(C)} = 1.7027$$

$$y(x_4) = y(0.4) \cong \underline{1.7027}$$

7a) Solve $\frac{dy}{dx} = x^3 + y$, $y(1) = 1$ using Taylor's series method considering up to fourth degree terms and find the $y(1.1)$.

Soln:- $\frac{dy}{dx} = x^3 + y$ $\quad y(1) = 1 \quad x_0 = 1 \quad y_0 = 1$

$$y' = x^3 + y \Rightarrow y'(x_0) = x_0^3 + y_0 = 1^3 + 1 = 2$$

$$y''(x) = 3x^2 + y' \Rightarrow y''(x_0) = 3x_0^2 + y_0' = 3(1)^2 + 2 = 5$$

$$y'''(x) = 6x + y'' \Rightarrow y'''(x_0) = 6x_0 + y_0'' = 6(1) + 5 = 11$$

$$y^{IV}(x) = 6 + y''' \Rightarrow y^{IV}(x_0) = 6 + 11 = 17$$

WKT

$$y(x) = y(x_0) + \frac{(x-x_0)}{1!}y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \frac{(x-x_0)^3}{3!}y'''(x_0) + \frac{(x-x_0)^4}{4!}y^{IV}(x_0) + \cdots$$

$$= 1 + \frac{(x-1)^2}{1} + \frac{(x-1)^2}{2}(5) + \frac{(x-1)^3}{6}(11) + \frac{(x-1)^4}{24}(17) + \cdots$$

$$y(1.1) = 1 + \frac{(0.1)^2}{1} + \frac{(0.1)^2}{2}(5) + \frac{11}{6}(0.1)^3 + (0.1)^4\frac{17}{24} + \cdots$$

$$y(1.1) = 1.22690$$

b) Use Runge-kutta method of fourth order to solve $\frac{dy}{dx} = 3x + \frac{y}{2}$, $y(0) = 1$, to find $y(0.2)$. (Take $h = 0.2$)

Soln: $\frac{dy}{dx} = 3x + \frac{y}{2} = f(x, y) \longrightarrow (1)$

$y(0) = 1$

$x_0 = 0 \quad y_0 = 1, \quad h = 0.2$

$K_1 = hf(x_0, y_0) = 0.2f(0,1) = 0.2(0.5) = 0.1$

$K_2 = hf\left(x_0+h/2,\ y_0+\frac{K_1}{2}\right) = 0.2f(0.1, 1.05) = 0.2(0.825)$

$K_2 = 0.165$

$K_3 = hf\left(x_0+h/2,\ y_0+\frac{K_2}{2}\right) = 0.2f(0.1, 1.0825)$

$= 0.2\times 0.84125$

$K_3 = 0.1682$

$K_4 = hf(x_0+h,\ y_0+K_3) = 0.2f(0.2, 1.1682)$

$= 0.2\times 1.1841$

$K_4 = 0.23682$

WKT

$y(x_1) = y_0 + \frac{1}{6}[K_1+2K_2+2K_3+K_4]$

$= 1+\frac{1}{6}[0.1+2\times 0.165+2\times 0.1682+0.23682]$

$= 1+0.1672$

$y(x_1) \cong \underline{1.1672}$

c) Given $\frac{dy}{dx} = x^2(1+y)$ and $y(1)=1$, $y(1.1)=1.2330$, $y(1.2) = 1.5480$ & $y(1.3)=1.9790$ find $y(1.4)$, using Adam - Bashforth predictor - corrector method.

$\underline{Sol^n}$: $\frac{dy}{dx} = x^2(1+y)$

$y(1)=1 \Rightarrow x_0=1, y_0=1$

$y(1.1) = 1.233 \Rightarrow x_1=1.1,\ y_1=1.233$

$y(1.2) = 1.548 \Rightarrow x_2=1.2,\ y_2=1.548$

$y(1.3) = 1.979 \Rightarrow x_3=1.3,\ y_3=1.979$

$y(1.4) = ? \quad h = 0.1$

$f_0 = f(x_0, y_0) = f(1,1) = 2$

$f_1 = f(x_1, y_1) = f(1.1, 1.233) = 2.7019$

$f_2 = f(x_2, y_2) = f(1.2, 1.548) = 3.6691$

$f_3 = f(x_3, y_3) = f(1.3, 1.979) = 5.0345$

WKT

$$y_4^{(P)} = y_3 + \frac{h}{24}\left[55f_3 - 59f_2 + 37f_1 - 9f_0\right]$$

$$= 1.979 + \frac{0.1}{24}\left[276.8975 - 216.4769 + 99.9703 - 18\right]$$

$$y_4^{(P)} = 2.5723$$

$$f_4^{(P)} = f(x_4, y_4^{(P)}) = f(1.4, 2.5723)$$

$$= 7.0017$$

$$y_4^{(c)} = y_3 + \frac{h}{24}\left[9f_4^{(P)} + 19f_3 - 5f_2 + f_1\right]$$

$$= 1.979 + \frac{0.1}{24}\left[9(2.5723) + 19(5.0345] - 5(3.6691) + 2.7019\right]$$

$$= 1.979 + 0.0042\left[103.1626\right]$$

$$y(x_4) = y(1.4) \cong \underline{\underline{2.5123}}$$

8a) Use modified Euler's method to compute $y(0.2)$, given $\frac{dy}{dx} - xy^2 = 0$ under the initial condition $y(0) = 2$. perform three iterations at each step, taking $h = 0.1$.

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Sol^n: given $\frac{dy}{dx} = xy^2 = f(x,y)$

$y(0) = 2 \Rightarrow x_0 = 0, y_0 = 2$

$h = 0.1$

To find $y(x_1) = y_1$

$y(x_0+h) = y(x_1) = y_1^{(1)} = y_0 + hf(x_0, y_0)$

$= 2 + 0.1\, f(0,2)$

$= 2 + 0.1(0)$

$y_1^{(1)} = 2$

$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0,y_0) + f(x_1, y_1^{(1)})]$

$= 2 + \frac{0.1}{2}[f(0.2) + f(0.1, 2)]$

$= 2 + 0.05[0 + 0.4]$

$y_1^{(2)} = 2.020$

$y_1^{(3)} = y_0 + \frac{h}{2}[f(x_0,y_0) + f(x_1, y_1^{(2)})]$

$= 2 + \frac{0.1}{2}[f(0,2) + f(0.1, 2.020)]$

$= 2 + 0.05[0 + 0.40804]$

$y_1^{(3)} = 2.0204$

$y(0.1) \cong 2.0204$

$\Rightarrow x_1 = 0.1 \quad y_1 = 2.0204$

To find $y(x_2) = y_2$

$\therefore y(x_1+h) = y(x_2) = y_2^{(1)} = y_1 + hf(x_1, y_1)$

$y_2^{(1)} = 2.0204 + \frac{0.1}{2}\, f(0.1, 2.0204)$

$y_2^{(1)} = 2.0612$

$$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(1)})\right]$$

$$= 2.0204 + \frac{0.1}{2}\left[f(0.1, 2.0204) + f(0.2, 2.0833)\right]$$

$$= 2.0204 + 0.05\left[0.4082 + 0.8497\right]$$

$$y_2^{(2)} = 2.0833$$

$$y_2^{(3)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(2)})\right]$$

$$= 2.0204 + \frac{0.1}{2}\left[f(0.1, 2.0204) + f(0.2, 2.0833)\right]$$

$$= 2.0204 + 0.05\left[0.4082 + 0.8680\right]$$

$$y_2^{(3)} = 2.0842$$

$$y(0.2) \cong \underline{\underline{2.0842}}$$

b) Use fourth Order Runge-kutta method, to find $y(0.2)$ with $h = 0.2$, given $\frac{dy}{dx} = \sqrt{x+y}$, $y(0) = 1$.

Sol^n: $\frac{dy}{dx} = \sqrt{x+y} = f(x,y)$

$y(0) = 1$

$\Rightarrow x_0 = 0, \; y_0 = 1$

$h = 0.2$

$$K_1 = hf(x_0, y_0) = 0.2f(0,1) = 0.2(1) = 0.2$$

$$K_2 = hf\left(x_0 + h/2, y_0 + \frac{K_1}{2}\right) = 0.2f\left(0.1, 1 + \frac{0.2}{2}\right)$$

$$= 0.2f(0.1, 1.1)$$

$$= 0.2 \times 1.0954$$

$$= 0.2191$$

$$k_3 = hf\left(x_0 + h/_2 \,,\, y_0 + k_2/_2\right) = 0.2\, f(0.1, 1.1096)$$
$$= 0.2 \times 1.0998$$
$$k_3 = 0.2200$$

$$k_4 = hf(x_0 + h,\, y_0 + k_3) = 0.2\, f(0.2, 1.2200)$$
$$= 0.2 \times 1.1916$$
$$k_4 = 0.2383$$

WKT

$$y(x_0+h) = y(x_1) = y_0 + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$$

$$y(0+0.2) = 1 + \frac{1}{6}\left[0.2 + 2(0.2191) + 2(0.2200) + 0.2383\right.$$

$$y(0.2) = 1 + \frac{1}{6}(1.3165)$$
$$= 1 + 0.2194$$
$$y(0.2) = y(x_1) \cong \underline{\underline{1.2194}}$$

c) Apply milne's predictor - corrector formulae to compute $y(2.0)$ given $\frac{dy}{dx} = \frac{1}{2}(x+y)$ with

x	0.0	0.5	1.0	1.5
y	2.0000	2.6360	3.5950	4.9680

Soln: given $\frac{dy}{dx} = \frac{1}{2}(x+y) = f(x,y)$

and $x_0 = 0 \quad y_0 = 2$

$x_1 = 0.5 \quad y_1 = 2.6360$

$x_2 = 1.0 \quad y_2 = 3.5950$

$x_3 = 1.5 \quad y_3 = 4.9680$

$x_4 = 2 \quad y_4 = ? \quad h = 0.5$

$$f_0 = \frac{x_0+y_0}{2} = \frac{0+2}{2} = 1$$

$$f_1 = \frac{x_1+y_1}{2} = \frac{0.5+2.6360}{2} = 1.5680$$

$$f_2 = \frac{x_2+y_2}{2} = \frac{1+3.5950}{2} = 2.2975$$

$$f_3 = \frac{x_3+y_3}{2} = \frac{1.5+4.9680}{2} = 3.2340$$

$$y_4^{(P)} = y_0 + \frac{4h}{3}\left[2f_1 - f_2 + 2f_3\right]$$

$$= 2 + \frac{4(0.5)}{3}\left[3.1360 - 2.2975 + 6.4680\right]$$

$$y_4^{(P)} = 6.8710$$

$$f_4^{(P)} = \frac{x_4 + y_4^{(P)}}{2} = \frac{2+6.8710}{2} = 4.4355$$

$$y_4^{(C)} = y_2 + \frac{h}{3}\left[f_2 + 4f_3 + f_4^{(P)}\right]$$

$$= 3.5950 + \frac{0.5}{3}\left[2.2975 + 12.9360 + 4.4355\right]$$

$$y_4^{(C)} = 6.8731$$

$$\Rightarrow y(2.0) \cong \underline{\underline{6.8731}}$$

Module-05

9 a) Solve $\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} - 2xy = 1$, for $x = 0.1$, correct to four decimal places, using initial conditions $y(0) = 1$, $y'(0) = 0$, using Runge-Kutta method.

Soln: Given $\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} - 2xy = 1 \longrightarrow (1)$

$$\frac{dy}{dx} = y' = z = f(x, y, z)$$

$$\Rightarrow \frac{dz}{dx} - x^2 z - 2xy = 1$$

$$\frac{dz}{dx} = 1 + 2xy + x^2 z = g(x, y, z)$$

and $y(0) = 1$, $y'(0) = 0$

$x_0 = 0$, $y_0 = 1$, $z_0 = 0 = y_0'$

but $h = 0.1$

$K_1 = hf(x_0, y_0, z_0) = 0.1 f(0, 1, 0) = 0.1(0) = 0$

$l_1 = hg(x_0, y_0, z_0) = 0.1 g(0, 1, 0) = 0.1(1) = 0.1$

$K_2 = hf(x_0 + h/2, y_0 + K_1/2, z_0 + l_1/2)$

$K_2 = 0.1 f(0.05, 1, 0.05) = (0.1)(0.05) = 0.005$

$l_2 = hg(x_0 + h/2, y_0 + K_1/2 + z_0 + l_1/2)$

$l_2 = 0.1 g(0.05, 1, 0.05) = (0.1)(1.1) = 0.11$

$K_3 = hf(x_0 + h/2, y_0 + K_2/2, z_0 + l_2/2)$

$K_3 = 0.1 f(0.05, 1.0025, 0.055)$

$K_3 = 0.1(0.055) = 0.0055$

$l_3 = 0.1 g(0.05, 1.0025, 0.055) = (0.1)(1.1)$

$l_3 = 0.11$

$K_4 = hf(x_0+h, y_0+K_3, z_0+l_3)$

$= 0.1f(0.1, 1.0055, 0.11)$

$= 0.1(0.11)$

$= 0.011$

WKT $y(x_1) = y_0 + \frac{1}{6}[K_1+2K_2+2K_3+K_4]$

$y(0.1) = 1+\frac{1}{6}[0+0.01+0.011+0.011]$

$y(0.1) \cong \underline{1.0053}$

9b) Find the extremal of the function $\int_0^1 (y'^2-y^2-ye^{2x})\,dx$ that passes through the points $(0,0)$ and $(1, 1/e)$

$\underline{Sol^n}$:- let $I = \int_{x_1}^{x_2} f(x,y,y')\,dx = \int_0^1 (y'^2-y^2-ye^{2x})\,dx$

$f(x,y,y') = y'^2-y^2-ye^{2x}$

$\frac{\partial f}{\partial y} = -2y-e^{2x}$

$\frac{\partial f}{\partial y'} = 2y'$

WKT $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$

$-2y-e^{2x}-\frac{d}{dx}(2y') = 0$

$\div$ by 2

$-y-\frac{1}{2}e^{2x} - y'' = 0$

$y''+y = -\frac{1}{2}e^{2x}$

$(D^2+1)y = -\frac{1}{2}e^{2x}$

The A.E is $m^2+1=0$

$m = 0 \pm 1i$

$$\boxed{y_c = c_1\cos x + c_2\sin x}$$

$$y_p = \frac{-\frac{1}{2}e^{2x}}{D^2+1}$$

$$= -\frac{1}{2}\frac{e^{2x}}{D^2+1}$$

$$= \frac{-1}{2}\frac{e^{2x}}{4+1}$$

$$\boxed{y_p = \frac{-e^{2x}}{10}}$$

$$\therefore\ y = y_c + y_p$$

$$y = c_1\cos x + c_2\sin x - \frac{e^{2x}}{10} \longrightarrow (1)$$

When $x=0 \Rightarrow y=0$

$$(1) \Rightarrow\ 0 = c_1(1) + 0 - \frac{1}{10}$$

$$c_1 = \frac{1}{10} = 0.1$$

when $x = 1 \Rightarrow y = 1/e$

$$1/e = c_1\cos 1 + c_2\sin 1 - \frac{e^2}{10}$$

$$0.3679 = (0.1)(0.5403) + c_2(0.8415) - 0.7389$$

$$c_2 = \frac{1.05277}{0.8415}$$

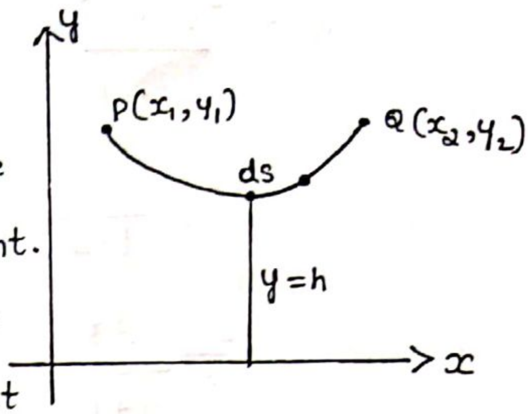
$$c_2 = 1.2570$$

$$\therefore (1) \Rightarrow\ y = 0.1\cos x + 1.2510\sin x - \frac{e^{2x}}{10}$$

c) A heavy cable hangs freely under gravity at two fixed points. Show that the shape of the cable is catenary.

Soln: let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be the two fixed points of the hanging cable.

let us consider an elementary arc length ds of the cable. let ρ be the density (mass/unit length) of the cable So that ρ ds is the mass of the element. If g is the acceleration due to gravity then the potential Energy of the element (m.g.h) is given by $(\rho\, ds)\cdot g \cdot y$ where x-axis is taken as the line of reference.



$\therefore$ Total potential Energy of the cable is given by

$$T = \int_P^Q (\rho\, ds)\cdot g\, y\, dx \qquad = \int_{x_1}^{x_2} \rho g y \frac{ds}{dx}\, dx$$

But , $\dfrac{ds}{dx} = \sqrt{1+y'^2}$

Here , $f(x,y,y') = (\rho g)\, y\sqrt{1+y'^2} = \text{const}\cdot y\sqrt{1+y'^2}$

$\therefore$ Euler's Equation $\quad f - y'\dfrac{\partial f}{\partial y'} = \text{constant}$

$$y\sqrt{1+y'^2} - y'\cdot\frac{y}{2\sqrt{1+y'^2}}\, 2y' = c$$

$$\frac{y(1+y'^2) - yy'^2}{\sqrt{1+y'^2}} = c \quad \underline{\text{or}} \quad c = \frac{y}{\sqrt{1+y'^2}}$$

$$y^2 = c^2(1+y'^2) \ \underline{\text{or}}$$

$$y'^2 = \frac{y^2-c^2}{c^2}$$

$$y' = \frac{dy}{dx} = \frac{\sqrt{y^2-c^2}}{c}$$

$$\underline{\text{or}} \quad \frac{dy}{\sqrt{y^2-c^2}} = \frac{1}{c}\, dx$$

$$\int \frac{dy}{\sqrt{y^2-c^2}} = \frac{1}{c}\int dx + K$$

$$\cosh^{-1}(y/c) = x/c + K$$

$$y/c = \cosh(x/c + K)$$

$$\therefore \boxed{y = c\cosh\left(\frac{x+a}{c}\right)}$$

where $a = Kc$. This is a catenary and it can be proved that this corresponds to the minimum value of T.

10 a) Apply milne's predictor - corrector method to compute $y(0.4)$ given the differential equation $\frac{d^2y}{dx^2} = 1 + \frac{dy}{dx}$ and the following table of initial values:

x	0	0.1	0.2	0.3
y	1	1.1103	1.2427	1.3990
y'	1	1.2103	1.4427	1.6990

Given $\frac{d^2y}{dx^2} = 1 + \frac{dy}{dx} \longrightarrow (1)$

let $\frac{dy}{dx} = y' = z = f(x,y,z)$

$(1) \Rightarrow \frac{dz}{dx} = 1+z = g(x,y,z)$

and given

$x_0 = 0 \qquad y_0 = 1 \qquad y_0' = z_0 = 1$

$x_1 = 0.1 \qquad y_1 = 1.1103 \qquad y_1' = z_1 = 1.2103$

$x_2 = 0.2 \qquad y_2 = 1.2427 \qquad y_2' = z_2 = 1.4427$

$x_3 = 0.3 \qquad y_3 = 1.3990 \qquad y_3' = z_3 = 1.6990$

$f_1 = f(x_1, y_1, z_1) = z_1 = 1.2103$

$f_2 = f(x_2, y_2, z_2) = z_2 = 1.4427$

$f_3 = f(x_3, y_3, z_3) = z_3 = 1.6990$

$$g_1 = g(x_1, y_1, z_1) = 1 + z_1 = 2.2103$$

$$g_2 = g(x_2, y_2, z_2) = 1 + z_2 = 2.4427$$

$$g_3 = g(x_3, y_3, z_3) = 1 + z_3 = 2.6990$$

$$y^P(x_4) = y_0 + \frac{4h}{3}\left[2f_1 - f_2 + 2f_3\right]$$

$$= 1 + \frac{4(0.1)}{3}\left[2.4206 - 1.4427 + 3.3980\right]$$

$$y_4^{(P)} = 1.5835$$

$$z_4^{(P)} = z_0 + \frac{4h}{3}\left[2g_1 - g_2 + 2g_3\right]$$

$$= 1 + \frac{4(0.1)}{3}\left[4.4206 - 2.4427 + 5.3980\right]$$

$$z_4^{(P)} = 1.9834$$

$$f_4^{(P)} = f(x_4, y_4^{(P)}, z_4^{(P)}) = z_4^{(P)} = 1.9834$$

$$y_4^{(C)} = y_2 + \frac{h}{3}\left[f_2 + 4f_3 + f_4^{(P)}\right]$$

$$y(0.4) = 1.2427 + \frac{0.1}{3}\left[1.4427 + 6.7960 + 1.9834\right]$$

$$y(0.4) \cong \underline{1.5834}$$

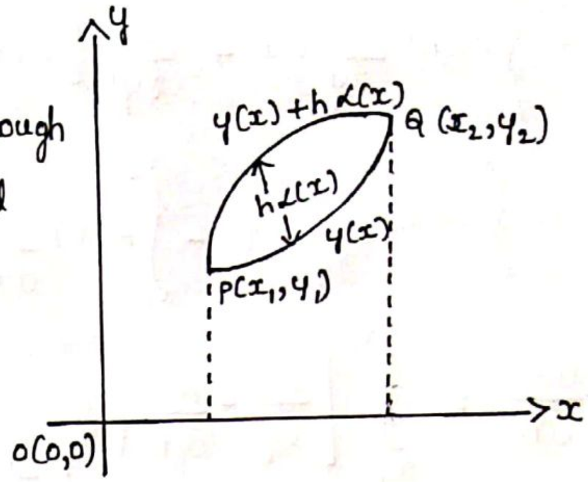
b) Derive Euler's Equation in the standard form

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right] = 0$$

Sol^n: Euler's Equation : A Necessary condition for the integral $I = \int_{x_1}^{x_2} f(x, y, y')\,dx$, where $y(x_1) = y_1$, $y(x_2) = y_2$ to be an Extremum is $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$

Proof :-

let the curve $y = y(x)$ passing through the points $p(x_1, y_1)$ and $Q(x_2, y_2)$ and make I an Extremum. Also let $y = y(x) + h\alpha(x)$ be the neighbouring curve passing through $p(x_1, y_1)$ & $Q(x_2, y_2)$ be an Extremum.



The curves both are coincide at P & Q that implies

$\alpha(x_1) = 0, \ \alpha(x_2) = 0$

Given $$I = \int_{x_1}^{x_2} f(x, y, y')\, dx \longrightarrow (1)$$

$$I = \int_{x_1}^{x_2} f\left(x, \frac{y}{y(x) + h\alpha(x)}, \frac{y'}{y'(x) + h\alpha'(x)}\right) dx$$

$$\frac{dI}{dh} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial x}\frac{\partial x}{\partial h} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial h} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial h}\right] dx$$

$$= \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial x}(0) + \frac{\partial f}{\partial y}\alpha(x) + \frac{\partial f}{\partial y'}\alpha'(x)\right] dx$$

$$= \int_{x_1}^{x_2} \frac{\partial f}{\partial y}\alpha(x) + \frac{\partial f}{\partial y'}\alpha'(x)\, dx$$

$$= \int_{x_1}^{x_2} \alpha(x) \frac{\partial f}{\partial y}\, dx + \int_{x_1}^{x_2} \frac{\partial f}{\partial y'}\alpha'(x)\, dx$$

$$= \int_{x_1}^{x_2} \alpha(x) \frac{\partial f}{\partial y}\, dx + \frac{\partial f}{\partial y'}\int_{x_1}^{x_2} \alpha'(x)\, dx - \int_{x_1}^{x_2} \left[\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\int \alpha'(x)\, dx\right] dx$$

$$= \int_{x_1}^{x_2} \alpha(x) \frac{\partial f}{\partial y}\, dx + \frac{\partial f}{\partial y}\left[\alpha(x)\right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right]\alpha(x)\, dx$$

$$= \int_{x_1}^{x_2} \alpha(x) \frac{\partial f}{\partial y} dx + \frac{\partial f}{\partial y'} \left[\alpha(x_2) - \alpha(x_1)\right] - \int_{x_1}^{x_2} \alpha(x) \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) dx$$

$$= \int_{x_1}^{x_2} \alpha(x) \frac{\partial f}{\partial y} dx - \int_{x_1}^{x_2} \alpha(x) \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) dx$$

$$\frac{dI}{dh} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right] \alpha(x)\, dx$$

For the Extremum of I, then $\frac{dI}{dh} = 0$

$$\Rightarrow \boxed{\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0}$$

c) Find the Extremal for the functional $\int_0^{\pi/2} (y^2 - y'^2 - 2y\sin x)\,dx$

$y(0) = 0, \quad y(\pi/2) = 1.$

Sol^n :-

$$I = \int_{x_1}^{x_2} f(x, y, y')\, dx = \int_0^{\pi/2} (y^2 - y'^2 - 2y\sin x)\,dx$$

$$f(x, y, y') = y^2 - y'^2 - 2y\sin x$$

WKT for the Extremum of I

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

$$(2y - 2\sin x) - \frac{d}{dx}(-2y') = 0$$

$$(y - \sin x) - \frac{d}{dx}\left(-\frac{dy}{dx}\right) = 0$$

$$y - \sin x + \frac{d^2y}{dx^2} = 0$$

$$\frac{d^2y}{dx^2} + y = \sin x$$

$(D^2+1)y = \sin x$ where $D = \frac{d}{dx}$

A.E is $m^2+1=0$

$m^2=-1$

$\boxed{m = 0 \pm i}$

$y_c = c_1\cos x + c_2 \sin x$

$y_p = \frac{\sin x}{D^2+1}$ $\qquad \frac{\sin ax}{D^2+a^2} = \frac{-x}{2a}\cos ax$

$= \frac{-x}{2(1)}\cos x$

$y_p = \frac{-x}{2}\cos x$

$y = y_c + y_p$

$y = c_1 \cos x + c_2 \sin x - x/_2 \cos x \longrightarrow (1)$

when $x=0 \Rightarrow y=0$

$(1) \Rightarrow 0 = c_1(1)$

$\boxed{c_1=0}$

When $x = \pi/_2 \Rightarrow y=1$

$(1) \Rightarrow 1 = c_1(0) + c_2(1)$

$\boxed{c_2=1}$

$(1) \Rightarrow y = \sin x - \frac{x}{2}\cos x$

9 a) Using Runge-Kutta method, solve $\frac{d^2y}{dx^2} = x\left(\frac{dy}{dx}\right)^2 - y^2$, for $x=0.2$, correct to four decimal places, using initial conditions $y(0)=1$, $y'(0)=0$

Sol^n: given $\frac{d^2y}{dx^2} = x\left(\frac{dy}{dx}\right)^2 - y^2$

$y'' = xy'^2 - y^2$

$$\frac{dy}{dx} = y' = z = f(x,y,z)$$

$$\frac{dz}{dx} = xz^2 - y^2 = g(x,y,z)$$

$x_0 = 0$, $y_0 = 1$, $y'(0) = z_0 = 0$ $h = 0.2$

$k_1 = hf(x_0, y_0, z_0) = 0.2f(0,1,0) = 0$

$l_1 = hg(x_0, y_0, z_0) = 0.2g(0,1,0) = -0.2$

$k_2 = hf(x_1, y_1, z_1) = 0.2f(0.1, 1, -0.1) = -0.02$

$l_2 = hg(x_1, y_1, z_1) = 0.2g(0.1, 1, -0.1) = -1.00$

$k_3 = hf(x_2, y_2, z_2) = 0.2f(0.1, 0.99, -0.5) = -0.1$

$l_3 = hg(x_2, y_2, z_2) = 0.2g(0.1, 0.99, -0.5) = -0.9551$

$k_4 = hf(x_3, y_3, z_3) = 0.2f(0.2, 0.95, -0.9551)$

$= 0.2(-0.9551)$

$k_4 = -0.1910$

$y(x_1) = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$y(0.2) = 1 + \frac{1}{6}[0 + 2(-0.02) + 2(-0.1) + (-0.1910)]$

$= 1 - 0.0718$

$y(0.2) = \underline{0.9282}$

b) Derive Euler's Equation in the standard form viz.,

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

c) Find the Extremal of the functional $\int_0^{\pi/2}(y'^2 - y^2 + 4y\cos x)\,dx$;
$y(0) = 0 = y(\pi/2)$

let $I = \int_{x_1}^{x_2} f(x,y,y')\,dx = \int_0^{\pi/2} (y'^2 - y^2 + 4y\cos x)\,dx$

$\therefore f(x,y,y') = y'^2 - y^2 + 4y\cos x$

$\frac{\partial f}{\partial y} = 0 - 2y + 4\cos x = -2y + 4\cos x$

$\frac{\partial f}{\partial y'} = 2y'$

WKT $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$

$-2y + 4\cos x - \frac{d}{dx}(2y') = 0$

$-y + 2\cos x - y'' = 0$

$\Rightarrow y'' + y = 2\cos x$

$(D^2+1)y = 2\cos x$

The A.E is $m^2+1=0$

$m^2 = -1$

$m = 0 \pm 1i$

$\therefore y_c = c_1\cos x + c_2\sin x$

$y_p = \frac{2\cos x}{D^2+1}$

$= \frac{2x}{2(1)}\sin x$

$y_p = x\sin x$

$\therefore y = y_c + y_p$

$y = c_1\cos x + c_2\sin x + x\sin x \longrightarrow (1)$

When $x=0 \Rightarrow y=0$

$(1) \Rightarrow 0 = c_1(1) + 0 + 0$

$\boxed{c_1 = 0}$

when $x = \pi/2 \Rightarrow y = 0$

$① \Rightarrow \quad 0 = c_1(0) + c_2(1) + \pi/2(1)$

$$\boxed{c_2 = -\pi/2}$$

$\therefore \; y = -\pi/2 \sin x + x \sin x$

10 a) Given the differential Equation $2\frac{d^2y}{dx^2} - \frac{dy}{dx} - 4x = 0$ and the following table of initial value:

x	1	1.1	1.2	1.3
y	2	2.2156	2.4649	2.7514
y'	2	2.3178	2.6725	2.0657

Compute $y(1.4)$ by applying milne's predictor- corrector method.

Sol^n: Given $2\frac{d^2y}{dx^2} - \frac{dy}{dx} - 4x = 0 \longrightarrow ①$

let $\frac{dy}{dx} = z = f(x,y,z)$

$① \Rightarrow \quad 2\frac{dz}{dx} - z - 4x = 0$

$$\frac{dz}{dx} = \frac{1}{2}(z+4x) = g(x,y,z)$$

$x_0 = 1 \qquad y_0 = 2 \qquad y_0' = z_0 = 2$

$x_1 = 1.1 \qquad y_1 = 2.2156 \qquad y_1' = z_1 = 2.3178$

$x_2 = 1.2 \qquad y_2 = 2.4649 \qquad y_2' = z_2 = 2.6725$

$x_3 = 1.3 \qquad y_3 = 2.7914 \qquad y_3' = z_3 = 2.0657$

$f_1 = f(x_1, y_1, z_1) = f(1.1, 2.2156, 2.3178) = 2.3178$

$f_2 = f(x_2, y_2, z_2) = f(1.2, 2.4649, 2.6725) = 2.6725$

$f_3 = f(x_3, y_3, z_3) = f(1.3, 2.7914, 2.0657) = 2.0657$

$g_1 = g(x_1, y_1, z_1) = g(1.1, 2.2156, 2.3178) = 3.3589$

$g_2 = g(x_2, y_2, z_2) = g(1.2, 2.4649, 2.6725) = 3.7363$

$g_3 = g(x_3, y_3, z_3) = g(1.3, 2.7914, 2.0657) = 3.6329$

$$y^P(x_4) = y_0 + \frac{4h}{3}[2f_1 - f_2 + 2f_3]$$

$$= 2 + \frac{4(0.1)}{3}[4.6356 - 2.6725 + 4.1314]$$

$$y_4^{(P)} = 2.8126$$

$$z_4^{(P)} = z_0 + \frac{4h}{3}[2g_1 - g_2 + 2g_3]$$

$$= 2 + \frac{4(0.1)}{3}[6.7178 - 3.7363 + 7.2658]$$

$$z_4^{(P)} = 3.3660$$

$$f_4^{(P)} = f(x_4, y_4^{(P)}, z_4^{(P)}) = z_4^{(P)} = 3.3660$$

$$y_4^{(C)} = y_2 + h/3\,[f_2 + 4f_3 + f_4^{(P)}]$$

$$= 2.4649 + \frac{0.1}{3}[2.6725 + 4(2.0657) + 3.3660]$$

$$= 2.4649 + 0.0333(14.3013)$$

$$\underline{y_4^{(C)} = 2.9416}$$

b) On what curves can the functional $\int_0^{\pi/2} (y'^2 - y^2 + 2xy)\,dx$; $y(0) = 0$, $y(\pi/2) = 0$ be Extremized ?

Sol^n: let $I = \int_0^{\pi/2} (y'^2 - y^2 + 2xy)\,dx = \int_{x_1}^{x_2} f(x, y, y')\,dx$

$$f(x, y, y') = y'^2 - y^2 + 2xy$$

$$\frac{\partial f}{\partial y} = 0 - 2y + 2x = 2x - 2y$$

$$\frac{\partial f}{\partial y'} = 2y'$$

WKT
$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

$$(2x-2y) - \frac{d}{dx}(2y') = 0$$

$$(x-y) - \frac{d}{dx}y' = 0$$

$$x-y-y'' = 0$$

$$y''+y = x$$

$$(D^2+1)y = x \quad , D = \frac{d}{dx}$$

The A.E is $m^2+1=0$

$$m^2 = -1$$

$$m = 0 \pm 1i$$

$$\therefore y_c = c_1\cos x + c_2 \sin x$$

$$y_p = \frac{x}{D^2+1}$$

$$y_p = (1+D^2)^{-1}x$$

$$y_p = (1-D^2+D^4-D^6+\dots\dots)x$$

$$y_p = x-0$$

$$y_p = x$$

$$\therefore y = y_c + y_p$$

$$y = c_1\cos x + c_2\sin x + x \longrightarrow (1)$$

when $x=0 \Rightarrow y=0$

$$(1) \Rightarrow 0 = c_1(1)+0+0$$

$$\boxed{c_1 = 0}$$

when $x = \pi/2 \Rightarrow y=0$

$$(1) \Rightarrow 0 = c_1(0) + c_2(1) + \pi/2$$

$$\boxed{c_2 = -\pi/2}$$

$$\therefore\ y = -\frac{\pi}{2}\sin x + x$$

c) prove that geodesics of a plane Surface are straight lenes.

Soln: let $S = \int_{x_1}^{x_2} \frac{\partial s}{\partial x}\, dx$

$$= \int_{x_1}^{x_2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_{x_1}^{x_2} \sqrt{1+y'^2}\, dx$$

$$f(x, y, y') = \sqrt{1+y'^2}$$

$$\therefore \frac{\partial f}{\partial y} = 0$$

$$\frac{\partial f}{\partial y} = \frac{1}{\cancel{2}\sqrt{1+y'^2}}\cancel{2}y'$$

$$\frac{\partial f}{\partial y} = \frac{y'}{\sqrt{1+y'^2}}$$

WKT $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$

$$0 - \frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right) = 0$$

$$\frac{d}{dx}\left[\frac{y'}{\sqrt{1+y'^2}}\right] = 0$$

$$\sqrt{1+y'^2}\, y'' - y'\frac{1}{\cancel{2}\sqrt{1+y'^2}}\cancel{2}y'y'' = 0$$

$$y''\sqrt{1+y'^2} - \frac{y'^2 y''}{\sqrt{1+y'^2}} = 0$$

$$y''(1+y'^2) - y'^2 y'' = 0$$

$$y'' + y'^2 y'' - y'^2 y'' = 0$$

$$\Rightarrow y'' = 0$$

$$D^2 y = 0 \quad , \quad D = \frac{d}{dx}$$

A.E is $m^2 = 0$

$$m = 0, 0$$

$$\therefore y = c_1 + c_2 x$$