Module 3 : Relations and Functions

→ Cartesian Products and Relations

→ zero - one Matrices and Digraphs.

→ operations on Relations

- composition of Relations

→ Properties of Relations

→ Equivalence Relations and Partitions.

→ Partial orders – Hasse Diagrams.

→ 1) functions

2) • Plain, one-to-one, onto functions.

→ 5) The Pigeon-hole Principle

→ 3) function composition

→ 4) Inverse functions.

Module - 3 Relations and Functions

Cartesian Product of sets :

Let A and B be 2 sets. Then the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$, is called Cartesian Product (or) cross Product (or) Product set of A and B and is denoted by $A \times B$.

Thus $A \times B = \{(a,b) \mid a \in A \text{ and } b \in B\}$.

Note: 1) $A \times B$ is not same as $B \times A$, because

$B \times A = \{(b,a) \mid b \in B \text{ and } a \in A\}$.

$\therefore (a,b) \neq (b,a)$ in general.

2) If (a, b) and (c, d) are ordered pairs, then $(a,b) = (c,d)$ if and only if $a = c$, $b = d$.

3) If A and B are finite sets with $|A| = m$, $|B| = n$, then

$|A \times B| = mn = |A||B|$.

4) $|P(A \times B)| = 2^{mn}$ i.e $A \times B$ has 2^{mn} subsets where $|A| = m$, $|B| = n$. for ex: if A has 5 elements, B has 6 elements then $A \times B$ has $2^{5 \times 6} = 2^{30}$ subsets.

Problems :

1) Let $A = \{1,2,3,4\}$ $B = \{3,4,5,6\}$ $C = \{2,4,6\}$. find $A \times B$, $B \times A$, $A \times (B \cup C)$, $(A \cap B) \times C$, $(A \times B) \cap (B \times C)$, $(A \times B) - (B \times C)$.

Soln :- $A \times B = \{(1,3)(1,4)(1,5)(1,6)\ (2,3)(2,4)(2,5)(2,6)\ (3,3)(3,4)$
$(3,5)(3,6)\ (4,3)(4,4)(4,5)(4,6)\}$

$B \times A = \{(3,1)(3,2)(3,3)(3,4)(4,1)(4,2)(4,3)(4,4)(5,1)(5,2)(5,3)$
$(5,4)(6,1)(6,2)(6,3)(6,4)\}$

$B \cup C = \{2,3,4,5,6\}$.

$A \times (B \cup C) = \{(1,2)(1,3)(1,4)(1,5)(1,5)\ (2,2)(2,3)(2,4)(2,5)(2,6)(3,2)$
$(3,3)(3,4)(3,5)(3,6)\ (4,2)(4,3)(4,4)(4,5)(4,6)\}$

$(A\cap B) = \{3,4\}$, $C = \{2,4,6\}$.

$(A\cap B)\times C = \{(3,2)(3,4)(3,6)(4,2)(4,4)(4,6)\}$.

$(B\times C) = \{(3,2)(3,4)(3,6)(4,2)(4,4)(4,6)(5,2)(5,4)(5,6)(6,2)$
$(6,4)(6,6)\}$.

$\therefore (A\times B)\cap(B\times C) = \{(3,4)(3,6)(4,4)(4,6)\}$.

$(A\times B)-(B\times C) = \{(1,3)(1,4)(1,5)(1,6)(2,3)(2,4)(2,5)(2,6)(3,3)$
$(3,5)(4,3)(4,5)\}$.

elements in $(A\times B)$
but not in $(B\times C)$.

2) For any non-empty sets A, B, C, prove the following results:

i) $A\times(B\cup C) = (A\times B)\cup(A\times C)$

ii) $(A\cup B)\times C = (A\times C)\cup(B\times C)$

iii) $A\times(B\cap C) = (A\times B)\cap(A\times C)$.

iv) $(A\cap B)\times C = (A\times C)\cap(B\times C)$

v) $A\times(B-C) = (A\times B)-(A\times C)$.

Soln:- For any ordered pair (x,y), we have

i) $(x,y)\in\{A\times(B\cup C)\} \Leftrightarrow x\in A$ and $y\in B\cup C$

$\Leftrightarrow x\in A$ and, $y\in B$ or $y\in C$.

$\Leftrightarrow x\in A$ and $y\in B$, or $x\in A$ and $y\in C$.

$\Leftrightarrow (x,y)\in A\times B$ or $(x,y)\in A\times C$

$\Leftrightarrow (x,y)\in\{(A\times B)\cup(A\times C)\}$.

Thus $A\times(B\cup C) = (A\times B)\cup(A\times C)$.

ii) $(x,y)\in\{(A\cup B)\times C\} \Leftrightarrow x\in A\cup B$, and $y\in C$.

$\Leftrightarrow x\in A$ or $x\in B$ and $y\in C$

$\Leftrightarrow x\in A$ and $y\in C$ or $x\in B$ and $y\in C$.

$\Leftrightarrow (x,y)\in A\times C$ or $(x,y)\in B\times C$.

$\Leftrightarrow (x,y)\in\{(A\times C)\cup(B\times C)\}$.

Thus $(A\cup B)\times C = (A\times C)\cup(B\times C)$.

iii) $(x,y) \in \{A \times (B \cap C)\} \Leftrightarrow x \in A$ and $y \in B \cap C$.

$\Leftrightarrow x \in A$ and $y \in B$ and $y \in C$.

$\Leftrightarrow x \in A$ and $y \in B$, and $x \in A$ and $y \in C$.

$\Leftrightarrow (x,y) \in A \times B$ and $(x,y) \in A \times C$.

$\Leftrightarrow (x,y) \in \{(A \times B) \cap (A \times C)\}$

Thus $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

iv) $(x,y) \in (A \cap B) \times C \Leftrightarrow x \in A \cap B$ and, $y \in C$.

$\Leftrightarrow x \in A$ and $x \in B$, and $y \in C$.

$\Leftrightarrow x \in A$ and $y \in C$ and $x \in B$ and $y \in C$

$\Leftrightarrow (x,y) \in (A \times C)$ and $(x,y) \in (B \times C)$

$\Leftrightarrow (x,y) \in \{(A \times C) \cap (B \times C)\}$.

Thus $(A \cap B) \times C = (A \times C) \cap (B \times C)$.

v) $A \times (B - C) = (A \times B) - (A \times C)$.

$(x,y) \in \{A \times (B - C)\} \Leftrightarrow x \in A$ and $y \in B - C$.

$\Leftrightarrow x \in A$ and $y \in B$, and $y \notin C$.

$\Leftrightarrow x \in A$ and $y \in B$, and $x \in A$ and $y \notin C$

$\Leftrightarrow (x,y) \in (A \times B)$ and $(x,y) \notin (A \times C)$

$\Leftrightarrow (x,y) \in \{(A \times B) - (A \times C)\}$.

Thus $A \times (B - C) = (A \times B) - (A \times C)$.

3) suppose $A, B, C \subseteq Z \times Z$ with $A = \{(x,y) \mid y = 5x - 1\}$, $B = \{(x,y) \mid y = 6x\}$, $C = \{(x,y) \mid 3x - y = -7\}$.

Find i) $A \cap B$ ii) $B \cap C$ iii) $\overline{A} \cup \overline{C}$ iv) $\overline{B} \cup \overline{C}$.

Soln:- i) $(x,y) \in A \cap B \Leftrightarrow (x,y) \in A$ and $(x,y) \in B$.

$\Leftrightarrow y = 5x - 1$ and $y = 6x$.

$\Leftrightarrow 5x - 1 = y = 6x$.

$\Leftrightarrow x = -1, y = -6$.

$\therefore A \cap B = \{(-1, -6)\}$.

ii) $(x,y) \in B \cap C \iff (x,y) \in B$ and $(x,y) \in C$.

$\iff y = 6x$ and $3x - y = -7$.

$\iff y = 6x$ and $y = 3x + 7$

$\iff 6x = y = 3x + 7$

$\iff 3x = 7$ i.e. $x = 7/3$

which is not possible, because $x \in \mathbb{Z}$.

Thus $(B \cap C) = \phi$.

iii) we have $\bar{A} \cup \bar{C} = \overline{A \cap C}$

$\Rightarrow \overline{\bar{A} \cup \bar{C}} = \overline{\overline{A \cap C}} = A \cap C$.

$(x,y) \in A \cap C \iff (x,y) \in A$ and $(x,y) \in C$.

$\iff y = 5x - 1$ and $3x - y = -7$.

$\iff y = 5x - 1$ and $y = 3x + 7$

$\iff 5x - 1 = y = 3x + 7$

$\iff x = 4, \ y = 19$.

Thus $\overline{\bar{A} \cup \bar{C}} = A \cap C = \{(4,19)\}$.

iv) we have $\bar{B} \cup \bar{C} = \overline{B \cap C}$.

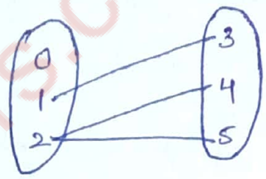
from (ii), $B \cap C = \phi$.

$\therefore \overline{B \cap C} = \bar{B} \cup \bar{C} = \mathbb{Z} \times \mathbb{Z}$ (universal set).

Relations :- Let A and B be 2 sets. Then a subset of $A \times B$ is called a binary relation or just a relation from A to B. Thus (if R is a relation from A to B, then R is a set of ordered pairs (a,b) where $a \in A$ and $b \in B$), and
→ need not be all

conversely if R is a set of ordered pairs (a,b) where $a \in A$ and $b \in B$, then R is a relation from A to B.

If $(a,b) \in R$, we say that "a is related to b by R". this is denoted by aRb. If R is a relation from A to A ie R is a subset of $A \times A$, we say that R is a binary relation on A.



(a)

Ex :- $A = \{0,1,2\}$ and $B = \{3,4,5\}$

Let $R = \{(1,3)\,(2,4)\,(2,5)\}$

R is a subset of $A \times B$. This is represented by the diagram (a) and this diagram is called the arrow diagram.

Note :- If A has 'm' elements and B has 'n' elements, then the no. of relations from A to B is 2^{mn}.

Problems :-

1) Let $A = \{1,2,3,4,6\}$ and R be the relation on A defined by $(a,b) \in R$ if and only if a is a multiple of b. write down R as a set of ordered pairs.

Repeated later

Soln :- $R = \{(a,b) \mid a,b \in A$ and a is a multiple of b$\}$

$R = \{(1,1)\,(2,1)\,(2,2)\,(3,1)\,(3,3)\,(4,1)\,(4,2)\,(4,4)\,(6,1)\,(6,2)\,(6,3)\,(6,6)\}$

2) Let A and B be finite sets with $|B| = 3$. If there are 4096 relations from A to B, what is $|A| = ?$

Soln :- If $|A| = m$, $|B| = n$, then there are 2^{mn} relations from A to B.

Given $|B| = n = 3$. and $2^{mn} = 4096$.

$\therefore \; 2^{3m} = 4096$

$\Rightarrow \quad 3m \log_e 2 = \log_e 4096$

$\Rightarrow \quad m = \dfrac{\log_e 4096}{3 \times \log_e 2} = 4$

Thus $|A| = 4$.

3) Let $A = \{1, 2, 3\}$ and $B = \{2, 4, 5\}$. Determine the following:

i) $|A \times B|$.

ii) No. of relations from A to B.

iii) No. of binary relations on A.

iv) No. of relations from A to B that contain $(1,2)$ & $(1,5)$.

v) No. of relations from A, B that contain exactly 5 ordered pairs

vi) No. of binary relations on A that contain atleast 7 ordered pairs.

<u>soln</u>:- Given $|A| = m = 3$, $|B| = n = 3$.

i) $|A \times B| = mn = 9$.

ii) No. of relations from A to B is $2^{mn} = 2^9 = 512$.

iii) No. of binary relations on A is $2^{mm} = 2^{m^2} = 2^9 = 512$.

iv) Let $R_1 = \{(1,2)(1,5)\}$. Every relation from A to B that contains the elements $(1,2)$ and $(1,5)$ is of the form $R_1 \cup R_2$, where R_2 is a subset of $\overline{R_1}$ in $A \times B$.

$\therefore$ No. of such relations $=$ no. of subsets of $\overline{R_1}$.

$= 2^7 \qquad (\because |\overline{R_1}| = |A \times B| - |R_1| = 9 - 2 = 7)$

$= 128$

Thus there are 128 no. of relations from A to B that contain the elements $(1,2)$ and $(1,5)$.

v) Since $A \times B$ contains 9 ordered pairs, the no. of relations from A to B that contain exactly 5 ordered pairs = no. of ways of choosing 5 ordered pairs from 9 ordered pairs.

This no. is ${}^9C_5 = 126$.

vi) III^{ly}, the no. of binary relations on A that contain atleast 7 elements (ordered pairs) is ${}^9C_7 + {}^9C_8 + {}^9C_9 = 46$.

Matrix of a relation :- Consider the sets $A = \{a_1, a_2 \dots a_m\}$, $B = \{b_1, b_2 \dots b_n\}$ of orders m and n resp. Then $A \times B$ contains all ordered pairs of the form (a_i, b_j) $1 \le i \le m$, $1 \le j \le n$ which are mn in number.

Let R be a relation from A to B so that R is a subset of $A \times B$. Let $m_{ij} = (a_i, b_j)$ and

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R. \end{cases} \rightarrow \text{zero-one matrix.}$$

The $m \times n$ matrix formed by m_{ij} is called the relation matrix for R or the adjacency matrix or zero-one matrix for R and is denoted by M_R or $M(R)$.

Rows of M_R correspond to elements of A and columns correspond to elements of B.

when $B = A$, then M_R is an $n \times n$ matrix whose elements are

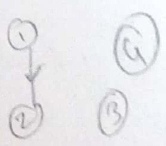
$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, a_j) \in R \\ 0 & \text{if } (a_i, a_j) \notin R. \end{cases}$$

Ex :- 1) $A = \{1, 2, 3, 4\}$ $B = \{4, 5\}$ and $R = \{(1,4)\ (2,5)\}$

$$\therefore M_R = \begin{array}{c c} & \begin{array}{cc} 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \end{array}$$

2) $A = \{p, q, r\}$ $R = \{(p,p)\ (p,q)\ (r,r)\ (r,q)\}$ then

$$M_R = \begin{array}{c c} & \begin{array}{ccc} p & q & r \end{array} \\ \begin{array}{c} p \\ q \\ r \end{array} & \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$



→ is defined only on $R: A \to A$, never on $R: A \to B$

Digraph of a relation :- Let R be a relation on a finite set A.

To get the digraph of R, we follow the following procedure :-

1. Draw a small circle or bullet for each of the elements of A and label it with the corresponding element of A. These circles are called vertices or nodes.

2. Draw an arrow, called an edge from a vertex x to a vertex y if and only if $(x,y) \in R$.

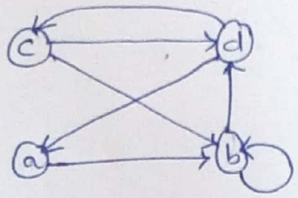
<u>Note</u> :- 1) In a digraph, a vertex from which an edge leaves is called the origin or the <u>source</u> for that edge and a vertex where an edge ends is called the <u>terminus</u> for that edge.

2) A vertex which is neither a source nor a terminus of an edge is known as as <u>isolated vertex</u>.

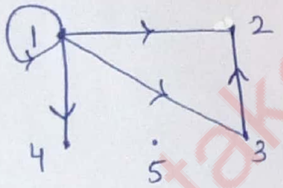
3) An edge for which the source and the terminus are one and the same vertex is called a <u>loop</u>.

4) The no. of edges (arrows) terminating at a vertex is called the <u>in-degree</u> of that vertex and the no. of edges (arrows) leaving a vertex is called the <u>out-degree</u> of that vertex.

<u>Ex</u> :- 1) $A = \{a, b, c, d\}$ $R = \{(a,b)\ (b,b)\ (b,d)\ (c,b)\ (c,d)\ (d,a)\ (d,c)\}$



2) $A = \{1, 2, 3, 4, 5\}$ $R = \{(1,1)\ (1,2)\ (1,3)\ (1,4)\ (3,2)\}$



<u>Operations on Relations</u> :

1) <u>Union and Intersection of Relations</u> :

Given the relations R_1 and R_2 from a set A to a set B, the union of R_1 and R_2, denoted by $R_1 \cup R_2$, is defined as a relation from A to B with the property that $(a,b) \in R_1 \cup R_2$ iff $(a,b) \in R_1$ or $(a,b) \in R_2$.

The intersection of R_1 and R_2, denoted by $R_1 \cap R_2$, is defined as a relation from A to B with the property that $(a,b) \in R_1 \cap R_2$ iff $(a,b) \in R_1$ and $(a,b) \in R_2$.

Complement of a Relation :

Given a relation R from a set A to a set B, the complement of R, denoted by $\overline{R}$, is defined as a relation from A to B with the property that $(a,b) \in \overline{R}$ iff $(a,b) \notin R$.

Converse of a Relation :-

Given a relation R from a set A to a set B, the converse of R denoted by R^c, is defined as a relation from B to A with the property that $(a,b) \in R^c$ iff $(b,a) \in R$.

NOTE : 1) If M_R is the matrix of R, then $(M_R)^T$, the transpose of M_R, is the matrix of R^c.

{(a,b) (c,d)} R^c ((b,a)) (d,c)}

2) $(R^c)^c = R$.

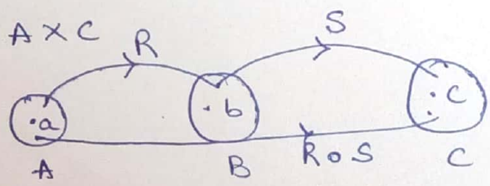
Composition of Relations :-

consider a relation R from a set A to a set B and a relation S from the set B to the set C. Then the product or the composition of R and S is a relation from the set A to the set C, denoted by $R \circ S$ and is defined as if $a \in A$ and $c \in C$ then $(a,c) \in R \circ S$ iff there is some b in B such that $(a,b) \in R$ and $(b,c) \in S$

i.e. $R \circ S = \{(a,c) \mid a \in A,\ c \in C$ and $\exists\, b \in B$ with $(a,b) \in R$ and $(b,c) \in S\}$.

NOTE :- 1) $R \subseteq A \times B$, $S \subseteq B \times C$ & $R \circ S \subseteq A \times C$

(a,b) (b,c)



2) $R \circ S \neq S \circ R$.

3) If R is a relation on A, then $R \circ R$ is a relation on A, denoted by R^2 and $(R \circ R) \circ R$ is also a relation on A, denoted by R^3.

4) Let R be a relation from A to B and S be a relation from B to C, then the matrices of R, S and $R \circ S$ satisfy $M(R) \cdot M(S) = M(R \circ S)$

5) $M(R^2) = [M(R)]^2$ and $M[R^n] = [M(R)]^n$, $n \in \mathbb{Z}^+$.

6) Let A, B, C, D be the sets and R, S, T be the relations from A to B, B to C and C to D resp., then $R \circ (S \circ T) = (R \circ S) \circ T$.

Problems :-

1) Let $A = \{1, 2, 3, 4\}$ and let R be the relation on A defined by $x R y$ iff $y = 2x$.

i) write down R as a set of ordered pairs

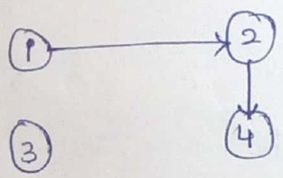
ii) Draw the digraph of R.

iii) Determine the in-degrees and out-degrees of the vertices in the digraph. iv) write the matrix of R.

Soln :- i) For $x, y \in A$, $(x, y) \in R$ iff $y = 2x$.

$\therefore \quad R = \{(1, 2) \ (2, 4)\}$.

ii) Digraph of R is as shown below:



iii)

vertices	Indegree	out degree
1	0	1
2	1	1
3	0	0
4	1	0

iv) $$M(R) = M_R = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

2) Let $A = \{1, 2, 3, 4, 6\}$ and R be the relation on A defined by $a R b$ iff a is a multiple of b.

i) Represent the relation R as a set of ordered pairs.

ii) Draw its digraph.

iii) Determine the indegrees and outdegrees of the vertices.

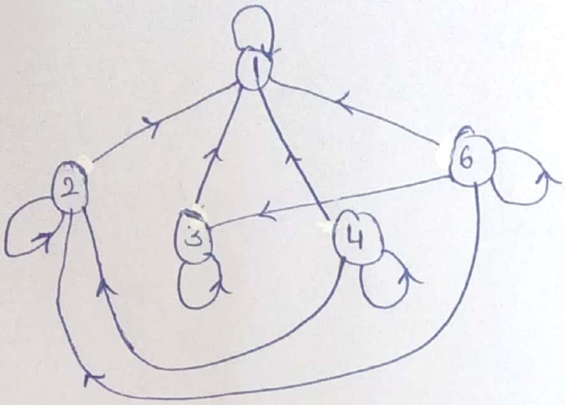
iv) write the matrix of R.

Soln :- Given $(a, b) \in R$ iff a is a multiple of b.

i) $R = \{(1,1) \ (2,1) \ (2,2) \ (3,1) \ (3,3) \ (4,1) \ (4,2) \ (4,4) \ (6,1) \ (6,2)$
$(6,3) \ (6,6)\}$.

iv) $M(R) = \begin{array}{c|c} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 6 \end{array} \\ \hline \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 6 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix} \end{array}$

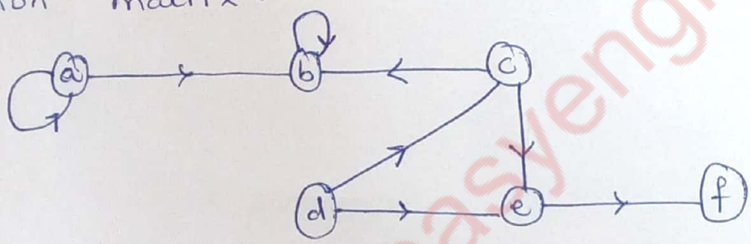
ii)



iii)

vertices	Indegree	out degree
1	5	1
2	3	2
3	2	2
4	1	3
6	1	4

3) For $A = \{a, b, c, d, e, f\}$ the digraph in the fig below rep-resents a relation on A. Determine R and the associated relation matrix.



<u>soln</u> :- $R = \{(a,a)\ (a,b)\ (b,b)\ (c,b)\ (c,e)\ (d,c)\ (d,e)\ (e,f)\}$.

$M(R) = \begin{array}{c|c} & \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \hline \begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \end{array} & \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$

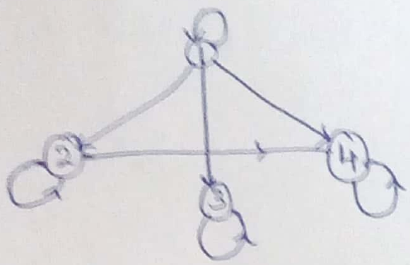
4) Let $A = \{1, 2, 3, 4\}$ and R be a relation on A defined by $(a,b) \in$ R iff $a \mid b$ (a divides b). write down R as a set of ordered pairs. write the matrix of this relation and draw the digraph of R. Also find indegree and outdegree of all the vertices. Also find R^2, R^3 and matrices of R^2 and R^3 and digraphs.

P.T.O.

Soln:- Given $A = \{1,2,3,4\}$ and R is the relation on A defined by $(a,b) \in R$ iff $a|b$.

$\therefore R = \{(1,1)\ (1,2)\ (1,3)\ (1,4)\ (2,2)\ (2,4)\ (3,3)\ (4,4)\}$

$$M(R) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$



Vertices	Indegree	out degree
1	1	4
2	2	2
3	2	1
4	3	1

$R^2 = R \circ R = \{(1,1)\ (1,2)\ (1,3)\ (1,4)\ (2,2)\ (2,4)\ (3,3)\ (4,4)\}$

$\circ$

$\{(1,1)\ (1,2)\ (1,3)\ (1,4)\ (2,2)\ (2,4)\ (3,3)\ (4,4)\}$

$R^2 = \{(1,1)\ (1,2)\ (1,3)\ (1,4)\ (2,2)\ (2,4)\ (3,3)\ (4,4)\}$

$\therefore \boxed{R^2 = R}$

$R^3 = R^2 \circ R = R \circ R = R^2 = R.$

$\therefore \boxed{R^3 = R}$

Since R^2 and R^3 are same as R, matrices of R^2 and R^3 are same as matrix of R and digraphs of R^2 and R^3 are same of digraphs of R.

Do 6^{th}, then 5^{th}.

5) Let $A = \{1,2,3,4\}$, $B = \{w,x,y,z\}$ and $C = \{5,6,7\}$. Also let R_1 be a relation from A to B; R_2 and R_3 be relations from B to C defined by

$R_1 = \{(1,x)\ (2,x)\ (3,y)\ (3,z)\}$

$R_2 = \{(w,5)\ (x,6)\}$ $\quad R_3 = \{(w,5)\ (w,6)\}$.

i) Find $R_1 \circ R_2$ and $R_1 \circ R_3$.

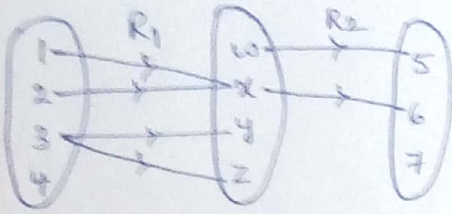
ii) Find $M(R_1)$, $M(R_2)$ and $M(R_1 \circ R_2)$

iii) Verify that $M(R_1 \circ R_2) = M(R_1) \cdot M(R_2)$

Soln :- i) $(1,x) \in R_1$ and $(x,6) \in R_2 \Rightarrow (1,6) \in R_1 \circ R_2$

$(2,x) \in R_1$ and $(x,6) \in R_2 \Rightarrow (2,6) \in R_1 \circ R_2$.

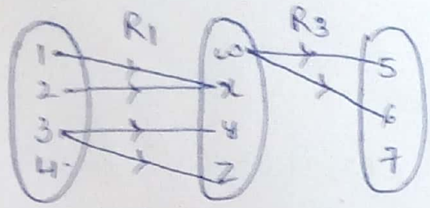
$\therefore R_1 \circ R_2 = \{(1,6)\ (2,6)\}$. (see fig (i))



$R_1 \circ R_2$

There is no element $(a,b) \in R_1$ such that $(b,c) \in R_3$

$\therefore R_1 \circ R_3 = \{\ \} = \phi$ (see fig (ii))



$R_1 \circ R_3$

ii) $M(R_1) = \begin{array}{c|cccc} & w & x & y & z \\ \hline 1 & 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 1 & 1 \\ 4 & 0 & 0 & 0 & 0 \end{array}$

$M(R_2) = \begin{array}{c|ccc} & 5 & 6 & 7 \\ \hline w & 1 & 0 & 0 \\ x & 0 & 1 & 0 \\ y & 0 & 0 & 0 \\ z & 0 & 0 & 0 \end{array}$

$M(R_1 \circ R_2) = \begin{array}{c|ccc} & 5 & 6 & 7 \\ \hline 1 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{array}$

iii) consider $M(R_1) \cdot M(R_2) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

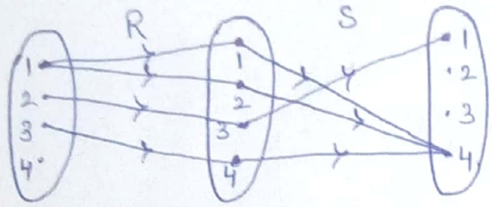
$= M(R_1 \circ R_2)$.

6) Let $A = \{1,2,3,4\}$ and $R = \{(1,1)\ (1,2)\ (2,3)\ (3,4)\}$, $S = \{(3,1)(4,4)(2,4)(1,4)\}$ be relations on A. Determine the relations $R \circ S$, $S \circ R$, $R \circ (R \circ S)$, $R \circ (S \circ R)$, $S \circ (R \circ S)$, $S \circ (S \circ R)$, R^2 and S^2.

P.T.O.

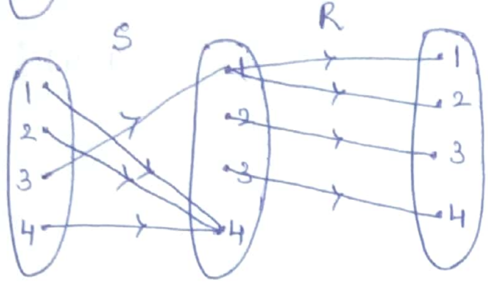
Sol^n :- $R \circ S = \{(1,4)(2,1)(3,4)\}$

$R \circ S$:



$S \circ R = \{(3,1)\ (3,2)\}$

$S \circ R$:



$$R \circ (R \circ S) = \{(1,1)\ (1,2)\ (2,3)\ (3,4)\} \circ \{(1,4)\ (2,1)\ (3,4)\}$$
$$= \{(1,4)\ (1,1)\ (2,4)\}$$

$$R \circ (S \circ R) = \{(1,1)\ (1,2)\ (2,3)\ (3,4)\} \circ \{(3,1)\ (3,2)\}$$
$$= \{(2,1)\ (2,2)\}$$

$$S \circ (R \circ S) = \{(3,1)\ (4,4)\ (2,4)\ (1,4)\} \circ \{(1,4)\ (2,1)\ (3,4)\}$$
$$= \{(3,4)\}$$

$$S \circ (S \circ R) = \{(3,1)\ (4,4)\ (2,4)\ (1,4)\} \circ \{(3,1)\ (3,2)\}$$
$$= \phi = \{\ \}$$

$$R^2 = R \circ R = \{(1,1)\ (1,2)\ (2,3)\ (3,4)\} \circ \{(1,1)\ (1,2)\ (2,3)\ (3,4)\}$$
$$= \{(1,1)\ (1,2)\ (1,3)\ (2,4)\}$$

$$S^2 = S \circ S = \{(3,1)\ (4,4)\ (2,4)\ (1,4)\} \circ \{(3,1)\ (4,4)\ (2,4)\ (1,4)\}$$
$$= \{(3,4)\ (4,4)\ (2,4)\ (1,4)\}.$$

7) If $A = \{1,2,3,4\}$ and R is a relation on A defined by $R = \{(1,2)\ (1,3)\ (2,4)\ (3,2)\ (3,3)\ (3,4)\}$. find R^2 and R^3. write down their digraphs. Find $M(R)$, $M(R^2)$, $M(R^3)$. verify that $M(R^2) = [M(R)]^2$ and $M(R^3) = [M(R)]^3$.

Sol^n :- $R^2 = R \circ R = \{(1,2)\ (1,3)\ (2,4)\ (3,2)\ (3,3)\ (3,4)\}$

$$\circ$$

$$\{(1,2)\ (1,3)\ (2,4)\ (3,2)\ (3,3)\ (3,4)\}$$

$\therefore\ R^2 = \{(1,4)\ (1,2)\ (1,3)\ (3,4)\ (3,2)\ (3,3)\}$.

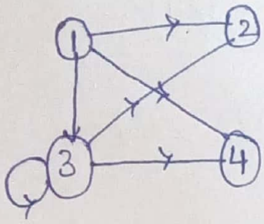
$R^3 = R^2 \circ R = \{(1,4)(1,2)\ (1,3)\ (3,4)\ (3,2)\ (3,3)\}$

$\circ$

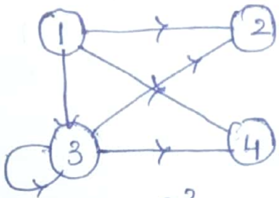
$\{(1,2)\ (1,3)\ (2,4)(3,2)\ (3,3)\ (3,4)\}$

$R^3 = \{(1,4)\ (1,2)\ (1,3)\ ,\ (3,4)\ (3,2)\ (3,3)\}$.

The digraphs of R^2 and R^3 are as shown below:



R^2



R^3

$$M(R) = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad M(R^2) = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad M(R^3) = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

by matrix multiplication with the stipulation that $1+1=1$, we find

$$[M(R)]^2 = M(R)\cdot M(R) = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

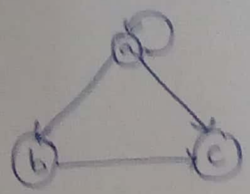
$$= \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = M(R^2).$$

and $[M(R)]^3 = M(R)\cdot M(R^2) = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

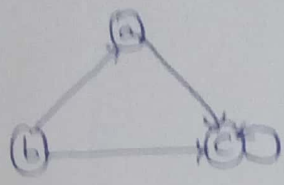
$$= \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = M(R^3).$$

8) The digraphs of two relations R and S on the set $A = \{a, b, c\}$ are given below. Draw the digraphs of $\overline{R}$, $R \cup S$, $R \cap S$ and R^c.

R:



S:



Soln :- From the digraphs,

$R = \{(a,a)\ (a,b)\ (a,c)\ (b,c)\}$ & $S = \{(a,c)\ (b,a)\ (b,c)\ (c,c)\}$.

$\therefore\ \overline{R} = \{(b,a)\ (b,b)\ (c,a)\ (c,b)\ (c,c)\}$

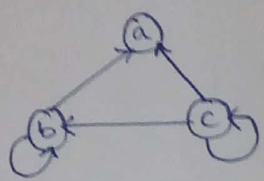
$R \cup S = \{(a,a)\ (a,b)\ (a,c)\ (b,a)\ (b,c)\ (c,c)\}$.

$R \cap S = \{(a,c)\ (b,c)\}$

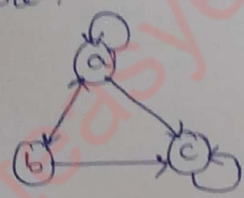
$R^c = \{(a,a)\ (b,a)\ (c,a)\ (c,b)\}$.

The digraphs of these are:

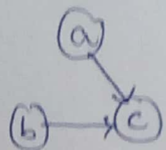
$\overline{R}$:



$R \cup S$:



$R \cap S$:



R^c:

Properties of Relations :

1) Reflexive relation : A relation R on a set A is said to be reflexive if $(a,a) \in R \quad \forall \overrightarrow{a}^{\text{all } a} \in A$, i.e. $aRa, \forall a \in A$.

R is not reflexive (i.e. non-reflexive) if $\exists$ some $a \in A$ such that $(a,a) \notin R$.

Ex :- 1) $\leq, =, \geq$ are reflexive relations on the set of all real no's, $\because a = a$ for $\forall a \in R$ where as $<, >$ are not reflexive relations on the set of all real no's.

2) If $A = \{1,2,3,4\}$, then $R = \{(1,1)\ (2,2)\ (3,3)\}$ is not non-reflexive b'coz $4 \in A$ but $(4,4) \notin R$.

2) Irreflexive relation :- A relation R on a set A is said to be irreflexive if $(a,a) \notin R, \quad \forall \overrightarrow{a}^{\text{all } a} \in A$. i.e. there is no element of A related to itself

Ex :- '$<$', '$>$' are irreflexive on the set of all real no's.

Note :- 1) A non reflexive relation need not be irreflexive.

2) A relation can be neither reflexive nor irreflexive.

3) Symmetric Relation :- A relation R on a set A is said to be symmetric whenever $(a,b) \in R$, $(b,a) \in R \quad \forall a, b \in A$.

4) Asymmetric relation :- A relation R on a set A is said to be asymmetric whenever $(a,b) \in R$, $(b,a) \notin R \quad \forall a, b \in A$.

5) Antisymmetric relation :- A relation R on a set A is said to be anti-symmetric if whenever $(a,b) \in R$ and $(b,a) \in R$ then $a = b$. i.e. $(a,b) \in R$, and $a \neq b$ then $(b,a) \notin R$

Ex :- '$\leq$' is antisymmetric on the set of all real numbers, b'coz if $a \leq b$ and $b \leq a$, then $a = b$.

Note : 1) Asymmetric and antisymmetric relations are not same.

2) A rel^n can be both symmetric and antisymmetric. It can be neither symmetric nor antisymmetric.

6) Transitive relation :- A relation R on a set A is said to be transitive if whenever $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R \ \forall \ a,b,c \in A$.

Ex :- '$\leq$', '$\geq$' are transitive on the set of all real no's, b'coz if $a \leq b$ and $b \leq c$ then $a \leq c$ $\forall \ a,b,c \in \mathbb{R}$ (real no's) and if $a \geq b$ and $b \geq c$ then $a \geq c$.

Note :- R is not transitive if there exist $a,b,c \in A$ such that $(a,b) \in R$ and $(b,c) \in R$ but $(a,c) \notin R$.

— x —

Equivalence relation :- A relation R on a set A is said to be an equivalence relation on A if R is reflexive, symmetric and transitive on A.

Ex :- ' = ' is an equivalence relation on the set of all real no's.

Equivalence classes :- Let R be an equivalence relation on a set A and $a \in A$. Then the set of all those elements of A which are related to a by R is called the equivalence class of a with respect to R. This equivalence class is denoted by R(a) or [a] or $\bar{a}$. Thus $\bar{a} = [a] = R(a) = \{x \in A \mid (x,a) \in R\}$.

$\Downarrow$ xRa

Ex :- $R = \{(1,1)(1,2)(2,1)(2,2)(3,3)\}$ $[1] = \{1,2\}$. $[3] = \{3\}$ $[2] = \{1,2\}$

Partition of a set : Let A be a non empty set. suppose there exist non empty subsets $A_1, A_2, A_3 \ldots A_K$ of A such that the following two conditions hold :

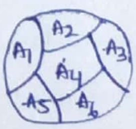
i) A is union of $A_1, A_2, \ldots A_K$ i.e. $A = A_1 \cup A_2 \cup \ldots \cup A_K$.

ii) Any 2 of the subsets $A_1, A_2 \ldots A_K$ are disjoint i.e. $A_i \cap A_j = \phi$ for $i \neq j$.

Then $P = \{A_1, A_2, \ldots A_K\}$ is called a Partition of A.

Also $A_1, A_2 \ldots A_K$ are called the blocks or cells of the Partition.

A Partition of a set with 6 blocks (cells) is as shown below:



Problems:

1) Let $A = \{1, 2, 3\}$. Determine the nature of the following relations on A.

i) $R_1 = \{(1,2)\ (2,1)\ (1,3)\ (3,1)\}$.

- Irreflexive
- symmetric
- Non-transitive.

ii) $R_2 = \{(1,1)\ (2,2)\ (3,3)\ (2,3)\}$.

- Reflexive
- Transitive
- Asymmetric (not symmetric)

iii) $R_3 = \{(1,1)\ (2,2)\ (3,3)\}$

- Reflexive
- symmetric.
- transitive

} Equivalence relation

iv) $R_4 = \{(1,1)\ (2,2)\ (3,3)\ (2,3)\ (3,2)\}$

- Reflexive
- symmetric
- Transitive

} Equivalence Relation

v) $R_5 = \{(1,1)\ (2,3)\ (3,3)\}$

- Non-reflexive or Irreflexive
- not symmetric ie Asymmetric.
- Transitive

vi) $R_6 = \{(2,3)\ (3,4)\ (2,4)\}$.

- Transitive
- Irreflexive.
- Not symmetric

vii) $R_7 = \{(1,3)\ (3,2)\}$

- Irreflexive
- non-transitive
- not symmetric

P.T.O.

2) If $A = \{1,2,3,4\}$ and R be a relation on A. Give an example of a relation for each of the following:

i) reflexive and symmetric, but not transitive.

ii) reflexive and transitive, but not symmetric

iii) symmetric and transitive, but not reflexive.

Soln:- i) $\{(1,1)\ (2,2)\ (3,3)\ (4,4)\ (1,2)\ (2,1)\ (2,3)\ (3,2)\}$.

ii) $\{(1,1)\ (2,2)\ (3,3)\ (4,4)\ (1,2)\}$

iii) $\{(1,1)\ (2,2)\ (1,2)\ (2,1)\}$.

3) Let $A = \{1,2,3,4\}$ and $R = \{(1,1)(1,2)(2,1)(2,2)(3,4)(4,3)(3,3)(4,4)\}$ be a relation on A. verify that R is an equivalence relation.

Soln:- To verify R is an equivalence relation, we have to show that R is reflexive, symmetric and transitive.

i) $(1,1)\ (2,2)\ (3,3)\ (4,4)$ belong to R. ie $(a,a) \in R\ \forall\ a \in A$.

$\therefore$ R is reflexive.

ii) $(1,2), (2,1) \in R$ and $(3,4)\ (4,3) \in R$

ie if $(a,b) \in R$, then $(b,a) \in R$ for $a,b \in A$.

$\therefore$ R is symmetric.

iii) $(1,2)\ (2,1)\ (1,1) \in R$, $(2,1)\ (1,2)\ (2,2) \in R$,

$(4,3)\ (3,4)\ (4,4) \in R$.

ie if $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R\ \forall\ a,b,c \in A$

$\therefore$ R is transitive.

Thus R is an equivalence relation.

Do this problem after 7^{th}, combining with 8^{th}.

4) Let $A = \{1,2,3,4,5,6,7,8,9,10,11,12\}$. On this set define the relation R by $(x,y) \in R$ iff $x-y$ is a multiple of 5.
QP
verify that R is an equivalence relation.

Soln:- i) for any $x \in A$, we have $x - x = 0$ is a multiple of 5

$(\because 0 = 5 \times 0)$ ie $(x,x) \in R$.

$\therefore$ R is reflexive.

ii) for any $x, y \in A$,

if $(x,y) \in R$ then $x-y = 5k$ for some integer k.

$\Rightarrow$ $y-x = 5(-k)$. so that $(y,x) \in R$.

$\therefore$ R is symmetric.

iii) for any $x,y,z \in A$, if $(x,y) \in R$ and $(y,z) \in R$, then

$x-y = 5k_1$ and $y-z = 5k_2$ for some integers k_1 and k_2.

$$\therefore \quad x-z = (x-y)-(z-y)$$
$$= (x-y)+(y-z)$$
$$= 5k_1 + 5k_2 = 5(k_1+k_2)$$

$\therefore$ $(x,z) \in R$. $\Rightarrow$ R is Transitive.

Thus R is an equivalence relation.

5) If $A = A_1 \cup A_2 \cup A_3$, where $A_1 = \{1,2\}$, $A_2 = \{2,3,4\}$ and $A_3 = \{5\}$

Define the relation R on A by xRy iff x and y are in the same set A_i, $i = 1,2,3$. Is R an equivalence relation?

Sol:- $A = A_1 \cup A_2 \cup A_3 = \{1,2,3,4,5\}$

$R = \{(1,1)\ (1,2)\ (2,1)\ (2,2)\ (2,3)\ (2,4)\ (3,2)\ (3,3)\ (3,4)\ (4,2)\ (4,3)$
$(4,4)\ (5,5)\}$

i) R is reflexive $\because$ $(a,a) \in R$ $\forall a \in A$.

ii) R is symmetric $\because$ whenever $(a,b) \in R$, $(b,a) \in R$ $\forall a,b \in A$.

iii) R is not transitive $\because$ $(1,2) \in R$, $(2,3) \in R$ but $(1,3) \notin R$.

$\therefore$ R is not an equivalence relation.

6) for a fixed integer $n > 1$, Prove that the relation "Congruent modulo n" is an equivalence relation on the set of all integers $\mathbb{Z}$.

Sol:- For any $a, b \in \mathbb{Z}$, "a is congruent to b modulo n" $[a \equiv b \pmod n]$

if $a-b$ is a multiple of n. ie $a-b = kn$ for some $k \in \mathbb{Z}$.

Let R be a relation on $\mathbb{Z}$ defined by aRb iff $a \equiv b \pmod n$

i) we have $a-a = 0 = 0 \times n$. $\Rightarrow$ $a \equiv a \pmod n$.

$\therefore$ aRa.

$\Rightarrow$ R is reflexive.

ii) Since aRb, $a \equiv b \pmod n$

$\therefore a - b = kn$

$\Rightarrow b - a = (-k)n$

$\Rightarrow b \equiv a \pmod n$ $(\because k \in Z, -k \in Z)$

$\therefore bRa$

Thus whenever aRb, bRa $\forall a, b \in Z$.

$\therefore R$ is symmetric.

iii) Let aRb and bRc.

ie $a \equiv b \pmod n$ and $b \equiv c \pmod n$.

ie $a - b = k_1 n \rightarrow ①$ and $b - c = k_2 n \rightarrow ②$, $k_1, k_2 \in Z$.

① + ② $\Rightarrow$ $a - c = k_1 n + k_2 n$

$a - c = (k_1 + k_2) n$.

$\Rightarrow a \equiv c \pmod n$ $(\because k_1 + k_2 \in Z)$

$\therefore aRc$.

Thus aRb and bRc $\Rightarrow aRc$, $\forall a, b, c \in Z$.

$\therefore R$ is transitive.

Thus R is an equivalence relation.

7) For the equivalence relation, $R = \{(1,1)(1,2)(2,1)(2,2)(3,4)(4,3)(3,3)(4,4)\}$ defined on the set $A = \{1,2,3,4\}$. Determine the Partition induced.

Soln :- The equivalence classes of the elements of A w.r.t R are

$[1] = \{1, 2\}$ $[2] = \{1, 2\}$ $[3] = \{3, 4\}$ $[4] = \{3, 4\}$.

Of these, only $[1]$ and $[3]$ are distinct.

$\therefore$ Partition $P = \{[1], [3]\} = \{\{1,2\}\{3,4\}\}$ is the Partition of the given A induced by R.

[Observe that $A = [1] \cup [3] = \{1,2\} \cup \{3,4\} = \{1,2,3,4\}$]

P.T.O.

8) Find the Partition of A induced by R, given
$A = \{1,2,3,4,5,6,7,8,9,10,11,12\}$.

Soln:- $R = \{(x,y) \in R$ iff $x-y$ is a multiple of $5\}$.

i.e. $R = \{(1,1)(2,2)\ldots(12,12)\ (6,1)(7,2)(8,3)(9,4)(10,5)(11,6)(12,7)(11,1)(12,2)(1,6)$
$(2,7)(3,8)(4,9)(5,10)(6,11)(7,12)(1,11)(2,12)\}$. (or) [see (*)]

∴ Equivalence classes are

$[1] = \{1,6,11\}$ $[2] = \{2,7,12\}$ $[3] = \{3,8\}$

$[4] = \{4,9\}$ $[5] = \{5,10\}$.

All these classes are distinct.

∴ $P = \{[1],[2],[3],[4],[5]\}$ is the Partition of A induced by R, and $A = \{1,6,11\} \cup \{2,7,12\} \cup \{3,8\} \cup \{4,9\} \cup \{5,10\}$.

9) Let $A = \{1,2,3,4,5,6,7\}$ and R be the equivalence relation on A that induces the Partition $A = \{1,2\} \cup \{3\} \cup \{4,5,7\} \cup \{6\}$. Find R.

Soln:- Given Partition has 4 blocks : $\{1,2\}$, $\{3\}$, $\{4,5,7\}$, $\{6\}$.

Let R be the equivalence relation ~~including~~ inducing this Partition.

Since 1, 2 are in the same block, we have 1R1, 1R2, 2R1, 2R2.

Since 3 belongs to block [3], 3R3.

Since 4, 5, 7 belongs to same block,

4R4, 4R5, 4R7, 5R4, 5R5, 5R7, 7R4, 7R5, 7R7.

Since 6 belongs to [6], 6R6.

∴ $R = \{(1,1)(1,2)(2,1)(2,2)(3,3)(4,4)(4,5)(4,7)(5,4)(5,5)(5,7)(7,4)$
$(7,5)(7,7)(6,6)\}$.

(*) Prob 8) by defn $[a] = \{x \in A \mid xRa\}$.

∴ $[1] = \{x \in A \mid xR1\} = \{x \in A \mid x-1$ is a multiple of $5\}$

$= \{1,6,11\}$.

10) Let $A = \{1,2,3,4,5\}$. Define a relation R on $A \times A$ by $(x_1, y_1) R (x_2, y_2)$ if and only if $x_1 + y_1 = x_2 + y_2$.

i) verify that R is an equivalence relation on $A \times A$.

ii) Determine the equivalence classes $[(1,3)]$, $[(2,4)]$ and $[(1,1)]$.

iii) Determine the partition of $A \times A$ induced by R.

Soln:-

i) a) for any $(x,y) \in A \times A$, we have

$$x+y = x+y.$$

$\Rightarrow (x,y) R (x,y)$. $\Rightarrow$ R is reflexive.

b) for any $(x_1,y_1), (x_2,y_2) \in A \times A$,

Suppose $(x_1,y_1) R (x_2,y_2)$ then $x_1+y_1 = x_2+y_2$.

$\Rightarrow x_2+y_2 = x_1+y_1$

$\Rightarrow (x_2,y_2) R (x_1,y_1)$

$\therefore$ R is symmetric.

c) for any $(x_1,y_1), (x_2,y_2), (x_3,y_3) \in A \times A$,

Suppose $(x_1,y_1) R (x_2,y_2)$ and $(x_2,y_2) R (x_3,y_3)$

then $x_1+y_1 = x_2+y_2$ and $x_2+y_2 = x_3+y_3$

$\Rightarrow x_1+y_1 = x_3+y_3$.

$\Rightarrow (x_1,y_1) R (x_3,y_3)$.

$\therefore$ R is transitive.

Thus R is an equivalence relation.

ii) we have $[(1,3)] = \{(x,y) \in A \times A \mid (x,y) R (1,3)\}$

$= \{(x,y) \in A \times A \mid x+y = 1+3 = 4\}$

$= \{(1,3)\ (2,2)\ (3,1)\}$ $[\because A = \{1,2,3,4,5\}]$

$|||^{ly}$ $[(2,4)] = \{(1,5)\ (2,4)\ (3,3)\ (4,2)\ (5,1)\}$

$[(1,1)] = \{(1,1)\}$

iii) To determine the partition induced by R, we have to first find the equivalence classes of all elements (x,y) of $A \times A$ w.r.t R.

P.T.O.

we have $[(1,1)] = \{(1,1)\}$.

$[(1,2)] = \{(1,2)\ (2,1)\} = [(2,1)]$

$[(1,3)] = \{(1,3)\ (3,1), (2,2)\} = [(3,1)] = [(2,2)]$

$[(1,4)] = \{(1,4)\ (4,1)\ (2,3)\ (3,2)\} = [(4,1)] = [(2,3)] = [(3,2)]$

$\cancel{[(2,4)] = \{(3,4)\ (4,2)\ (3,3)}$

$[(1,5)] = \{(1,5)\ (5,1)\ (3,3)\ (2,4)\ (4,2)\} = [(5,1)] = [(3,3)] = [(2,4)] = [(4,2)]$

$[(2,5)] = \{(2,5)\ (5,2)\ (3,4)\ (4,3)\} = [(5,2)] = [(3,4)] = [(4,3)]$

$[(3,5)] = \{(3,5)\ (5,3)\ (4,4)\} = [(4,4)] = [(5,3)]$.

$[(4,5)] = \{(4,5)\ (5,4)\} = [(5,4)]$

$[(5,5)] = \{(5,5)\}$.

Thus $[(1,1)]$, $[(1,2)]$, $[(1,3)]$, $[(1,4)]$, $[(1,5)]$, $[(2,5)]$, $[(3,5)]$, $[(4,5)]$, $[(5,5)]$ are the only distinct equivalence classes of A X A on R.

$\therefore$ Partition of A X A induced by R is represented by
$P = \{[(1,1)], [(1,2)], [(1,3)], [(1,4)], [(1,5)], [(2,5)], [(3,5)], [(4,5)], [(5,5)]\}$.

$\left(A \times A = [(1,1)] \cup [(1,2)] \cup [(1,3)] \cup [(1,4)] \cup [(1,5)] \cup [(2,5)] \cup [(3,5)] \cup [(4,5)] \cup [(5,5)]\right)$.

11) Find the number of equivalence relations that can be defined on a finite set A with $|A| = 6$. $\longrightarrow$ DO in final later

soln:-
Note:- The no. of possible ways to assign 'm' distinct objects into 'n' identical places with empty places allowed is given by the formula $p(m) = \sum_{i=1}^{n} S(m, i)$ for $m \geq n$, where $S(m,n)$ is the Stirling number of 2nd kind given by

$$S(m,n) = \frac{1}{n!} \sum_{k=0}^{n} (-1)^k \left({}^{n}C_{n-k}\right) (n-k)^m \quad \text{for } m \geq n.$$

This no. represents the no. of ways of arranging 'm' objects into 'n' distinct containers with no container left empty. (Place)

Soln :- Since $|A| = 6$, the partition of A can have atmost 6 cells. Treating the elements of A as objects (ie $m = 6$) and cells as containers (ie $n = 6$), the no. of partitions having k cells is $S(6,k)$. Since k varies from 1 to 6, the total no. of different partitions of A is

$$p(6) = \sum_{i=1}^{6} S(6,i) = S(6,1) + S(6,2) + S(6,3) + S(6,4) + S(6,5) + S(6,6)$$

$$\therefore \quad S(m,n) = \frac{1}{n!} \sum_{k=0}^{n} (-1)^k \left({}^nC_{n-k}\right)(n-k)^m .$$

$$S(6,1) = \frac{1}{1!} \sum_{k=0}^{1} (-1)^k \left({}^1C_{1-k}\right)(1-k)^6$$

$$= 1 \cdot {}^1C_1 \cdot 1^6 + (-1)\,{}^1C_0\,(0)^6$$

$$= 1 + 0 = 1 .$$

$$S(6,2) = \frac{1}{2!} \sum_{k=0}^{2} (-1)^k \left({}^2C_{2-k}\right)(2-k)^6$$

$$= \frac{1}{2}\left[1 \cdot 2^6 - {}^2C_1 \cdot 1^6 + {}^2C_0\,(0)^6\right].$$

$$= \frac{1}{2}\left[2^6 - 2\right] = 31 .$$

$$S(6,3) = \frac{1}{3!} \sum_{k=0}^{3} (-1)^k \left({}^3C_{3-k}\right)(3-k)^6$$

$$= \frac{1}{6}\left[3^6 - 3 \times 2^6 + 3 \times 1^6\right] = 90 .$$

$$S(6,4) = \frac{1}{4!} \sum_{k=0}^{4} (-1)^k \left({}^4C_{4-k}\right)(4-k)^6$$

$$= \frac{1}{24}\left[4^6 - 4 \times 3^6 + 6 \times 2^6 - 4 \times 1^6\right] = 65$$

$$S(6,5) = \frac{1}{5!} \sum_{k=0}^{5} (-1)^k \left({}^5C_{5-k}\right)(5-k)^6$$

$$= \frac{1}{120}\left[5^6 - 5 \times 4^6 + 10 \times 3^6 - 10 \times 2^6 + 5 \times 1^6\right] = 15$$

$$S(6,6) = \frac{1}{6!} \sum_{k=0}^{6} (-1)^k \left({}^6C_{6-k}\right)(6-k)^6$$

$$= \frac{1}{720}\left[6^6 - 6 \times 5^6 + 15 \times 4^6 - 20 \times 3^6 + 15 \times 2^6 - 6 \times 1^6 + 0\right] = 1$$

$\therefore$ No. of Partitions of A is

$P(6) = 1 + 31 + 90 + 65 + 15 + 1 = 203$ ie 203 equivalence relations can be defined on A.

Partial Orders :- A relation R on a set A is said to be a partial order on A, if R is reflexive, antisymmetric and transitive on A. A set A with a partial order defined on it is called a Partially order set (or) a **Poset** and is denoted by the pair (A, R).

Ex :- The relation '$\leq$' on the set of all integers is a Partial order. $\therefore (Z, \leq)$ is a Poset. The relation '$\geq$' on the set of all integers is a partial order. $\therefore (Z, \geq)$ is a Poset.

Total order : Let R be a partial order on a set A. Then R is called a total order on A, if for all $x, y \in A$, either xRy or yRx. In this case, the Poset (A, R) is called a totally ordered set.

Ex :- The Partial order relation '$\leq$' is a total order on the set of all real numbers $\mathbb{R}$ because for any $x, y \in \mathbb{R}$, we have $x \leq y$ or $y \leq x$. $\therefore (\mathbb{R}, \leq)$ is a totally ordered set.

Note :- Every total order is a partial order but every partial order need not be a total order.

Hasse Diagrams :-

For a Partial order relation on a finite set, we can draw digraph of a Partial order.

1. Since a Partial order is reflexive, at every vertex of digraph of Partial order, there will be a loop. While drawing the digraph of partial order, we do not show the loops explicitly. They will be automatically understood by convention.
2. In the digraph of Partial order, if there is an edge from vertex a to b and an edge from vertex b to c, then there will be an edge from a to c (b'coz of transitivity). But we do not exhibit an edge from a to c explicitly. It will be automatically understood by convention.

3. To simplify the format of the digraph of a partial order, we represent the vertices by dots and draw the digraph in such a way that all edges point upward. with this convention we need not put arrows in the edges.

The digraph of a partial order drawn by adopting the conventions indicated above is called a Poset diagram or the Hasse digram for the Partial order.

Problems :-

1) Let $A = \{1,2,3,4\}$ and $R = \{(1,1)(1,2)(2,2)(2,4)(1,3)(3,3)(3,4),(1,4)(4,4)\}$. Verify that R is a Partial order on A. Also write down the Hasse diagram for R.

Soln:- i) we see that $(a,a) \in R \ \forall \ a \in A$. Hence R is reflexive on A.

ii) if $(a,b) \in R$ and $a \neq b$, then we see that $(b,a) \notin R$, $\forall \ a,b \in A$.

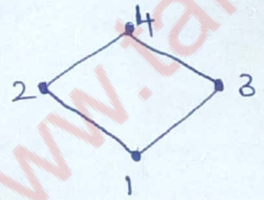
$\therefore$ R is antisymmetric.

iii) if $(a,b) \in R$ and $(b,c) \in R$ then we see that $(a,c) \in R$.

$\therefore$ R is transitive.

Thus R is a partial order on A. i.e. (A,R) is a Poset.

The Hasse Diagram for R is as shown below:



→ No Δ^{le} shud come in Hasse diagram.

→ 2 no's which are related to eachother shud not be in same line

2) Let R be a relation on the set $A = \{1,2,3,4\}$ defined by xRy iff x divides y. Prove that (A,R) is a Poset. Draw its Hasse diagram.

HW

Soln:. $R = \{(x,y) \mid x,y \in A \text{ and } x \text{ divides } y\}$

i.e. $R = \{(1,1)(1,2)(1,3)(1,4)(2,2)(2,4)(3,3)(4,4)\}$.

i) we see that $(a,a) \in R \ \forall \ a \in A$. ⇒ R is reflexive on A.

ii) If $(a,b) \in R$ and $a \neq b$, then we see that $(b,a) \notin R$.

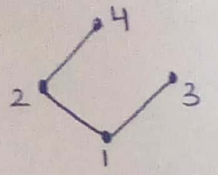
$\therefore$ R is antisymmetric on A.

iii) if $(a,b) \in R$ and $(b,c) \in R$, then we see that $(a,c) \in R$.

$\therefore$ R is transitive.

Thus R is a partial order on A. i.e. (A, R) is a Poset.

The Hasse diagram for R is as shown below:



3) Let $A = \{1, 2, 3, 4, 6, 8, 12\}$. On A, define the partial ordering relation R by xRy iff $x \mid y$. Prove that R is a partial order on A. Draw the Hasse diagram for R.

Soln:. $R = \{(1,1)(1,2)(1,3)(1,4)(1,6)(1,8)(1,12)\ (2,2)(2,4)(2,6)(2,8)(2,12),$
$(3,3)(3,6)(3,12)(4,4)(4,8)(4,12)(6,6)(6,12)\ (8,8)\ (12,12)\}$.

i) we see that $(a,a) \in R\ \forall\ a \in A$. $\Rightarrow$ R is reflexive.

ii) if $(a,b) \in R$ and $a \neq b$, then we see that $(b,a) \notin R$.

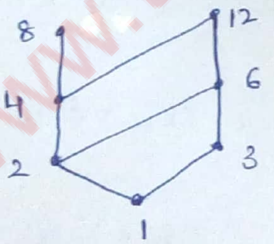
$\therefore$ R is antisymmetric.

iii) if $(a,b) \in R$ and $(b,c) \in R$, then we see that $(a,c) \in R$.

$\therefore$ R is transitive.

Thus R is a Partial order on A i.e. (A, R) is a Poset.

The Hasse diagram is as below:



4) Draw the Hasse diagram representing the positive divisors of 36.

Soln:- The set of all positive divisors of 36 are:

$D_{36} = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$.

The relation R of divisibility i.e. aRb iff a divides b, is a Partial order on this set.

we note that, under R,

1 is related to all elements of D_{36}.

2 is '' —— 2, 4, 6, 12, 18, 36

3 '' —— 3, 6, 9, 12, 18, 36.

4 '' —— 4, 12, 36.

6 '' —— 6, 12, 18, 36.

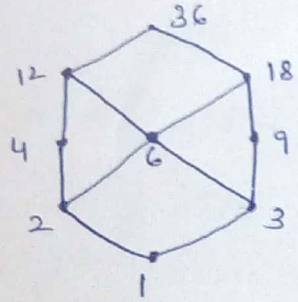
9 '' —— 9, 18, 36

12 '' —— '' 12, 36

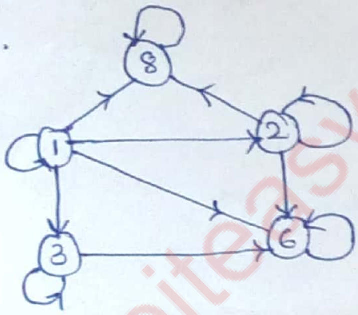
18 '' —— '' 18, 36

36 '' —— 36.

Hasse diagram :-



5) The digraph for a relation on the set $A = \{1, 2, 3, 6, 8\}$ is as shown below: Verify that (A, R) is a Poset and write down its Hasse diagram.



Soln:- $R = \{(1,1)(1,2)(1,3)(1,6)(1,8)\ (2,2)(2,6)(2,8)(3,3)(3,6)(6,6)(8,8)\}$.

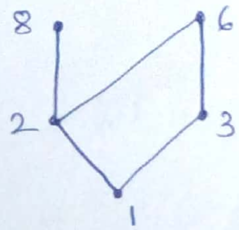
i) R is reflexive b'coz $(a,a) \in R\ \forall\ a \in A$.

ii) R is antisymmetric b'coz if $(a,b) \in R$ and $a \neq b$, then we see that $(b,a) \notin R$.

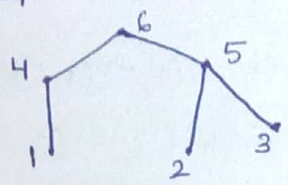
iii) R is transitive b'coz if $(a,b) \in R$ and $(b,c) \in R$, we see that $(a,c) \in R$.

Thus R is a partial order on A. ie (A,R) is a Poset.

Hasse diagram :



6) The Hasse diagram of a partial order R on the set A = {1, 2, 3, 4, 5, 6} is as given below. Write down R as a subset of A×A. construct its digraph. Also determine the matrix of the Partial order.

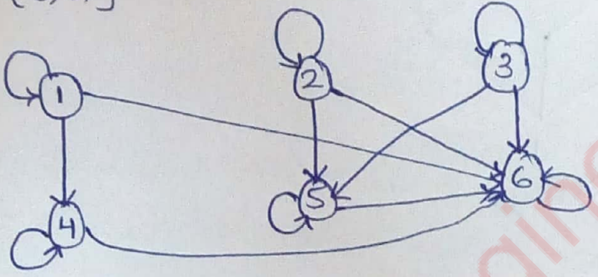


Soln :- from the diagram,

1R1, 1R4, 1R6, 2R2, 2R5, 2R6,
3R3, 3R5, 3R6, 4R4, 4R6, 5R5, 5R6, 6R6.

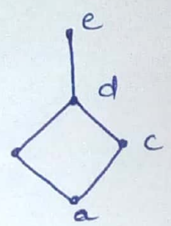
∴ R = {(1,1) (1,4) (1,6) (2,2) (2,5) (2,6) (3,3) (3,5) (3,6) (4,4) (4,6) (5,5) (5,6) (6,6)}

Digraph :-



$$M(R) = M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

7) For A = {a, b, c, d, e}, the Hasse diagram for the Poset (A, R) is as shown below:



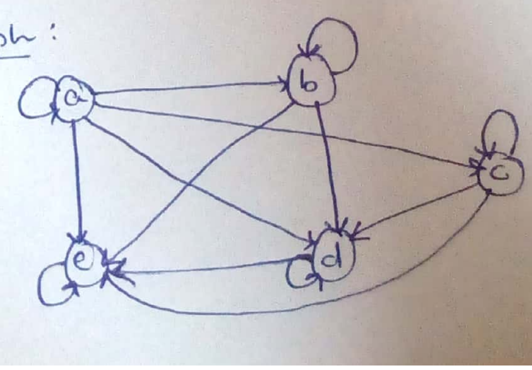
i) Determine the relation matrix for R.

ii) Construct the digraph for R.

Soln :- R = {(a,a) (a,b) (a,c) (a,d) (a,e) (b,b) (b,d) (b,e) (c,c) (c,d) (c,e) (d,d) (d,e) (e,e)}.

$$\text{i) } M(R) = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

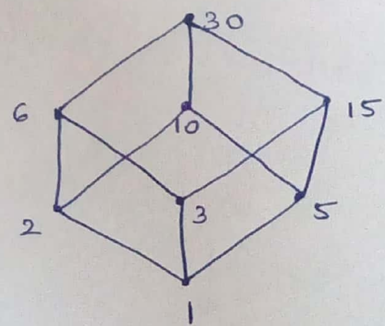
ii) Digraph :



8) In the following cases, consider the partial order of divisibility on the set A. Draw the Hasse diagram for the poset and determine whether the poset is totally ordered or not.

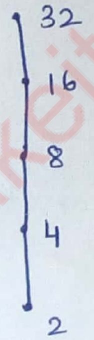
(i) $A = \{1, 2, 3, 5, 6, 10, 15, 30\}$ ii) $A = \{2, 4, 8, 16, 32\}$.

Soln:- i) $R = \{(1,1)(1,2)(1,3)(1,5)(1,6)(1,10)(1,15)(1,30)(2,2)(2,6)(2,10)(2,30)$
$(3,3)(3,6)(3,15)(3,30)(5,5)(5,10)(5,15)(5,30)(6,6)(6,30)$
$(10,10)(10,30)(15,15)(15,30)(30,30)\}$



(A, R) is not ~~totaally~~ ordered set because if we consider 2, 3 in A neither 2 divides 3 nor 3 divides 2.

ii) $R = \{(2,2)(2,4)(2,8)(2,16)(2,32)(4,4)(4,8)(4,16)(4,32)(8,8)$
$(8,16)(8,32)(16,16)(16,32)(32,32)\}$.



(A, R) is totally ordered set.

Extremal elements in Poset : Consider a Poset (A, R), we define some special elements called extremal elements that may exist in A.

1) An element $a \in A$ is called a maximal element if there exists no x in A other than 'a' such that aRx (bottom → top) i.e 'a' is maximal element if and only if in Hasse diagram of R, no edge starts at 'a'.

2) An element $a \in A$ is called a minimal element if there exists no x in A other than 'a' such that xRa i.e 'a' is minimal element if and only if in Hasse diagram of R, no edge terminates at 'a'.

3) An element $a \in A$ is called a greatest element of A if xRa $\forall x \in A$. i.e all elements of A shud be related to 'a'.

4) An element $a \in A$ is called a least element of A if aRx $\forall x \in A$ i.e 'a' shud be related to all elements of A.

5) Let $B \subseteq A$. Then an element $a \in A$ is called an upper bound of B if xRa, $\forall x \in B$. i.e all elements of subset B shud be related to 'a'.

6) Let $B \subseteq A$. Then an element $a \in A$ is called a lower bound of B if aRx, $\forall x \in B$. i.e a shud be related to all elements of subset B.

7) Let $B \subseteq A$. Then an element $a \in A$ is called the Least Upper bound (LUB) of B if
   i) a is an upper bound of B and
   ii) If a' is also an upper bound of B, then aRa'

8) Let $B \subseteq A$. Then an element $a \in A$ is called the greatest lower bound (GLB) of B if
   i) a is a lower bound of B and
   ii) If a' is also a lower bound of B, then $a'Ra$.

Note:- LUB is also called Supremum and
GLB is " ——— Infimum.

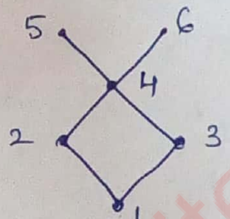
Lattices :- Let (A, R) be a Poset. This is called a Lattice if $\forall\ x, y \in A$, the elements $LUB\{x, y\}$ and $GLB\{x, y\}$ exist in A.

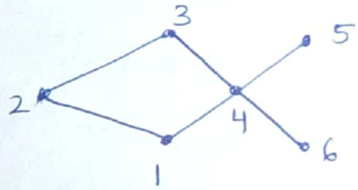
Ex :- 1) $(N, \leq)$ is a lattice. for all x, y in N, $LUB\{x, y\} = \max\{x, y\}$.
$GLB\{x, y\} = \min\{x, y\}$ and
Both of these belong to N.

2) $(Z^+, |)$ is a lattice where $|$ is the divisibility relation.
For all x, y in Z^+, $GLB\{x, y\} = \gcd(x, y)$ and
$LUB\{x, y\} = \text{lcm}(x, y)$. Both of these belong to Z^+.

Problems :-

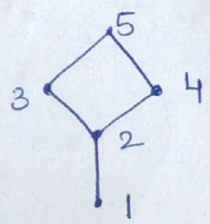
1) For the Poset (A, R) represented by the Hasse diagram, find
i) maximal ii) minimal iii) greatest and iv) least element(s)

a) 

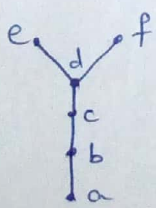
b) 

Soln :- a) 5 and 6 are maximal elements
1 is a minimal element.
1 is the least element ($\because$ 1 is related to all elements of A)
No greatest element ($\because$ not 5 $\because$ 6 is not related to 5
not 6 $\because$ 5 " 6)

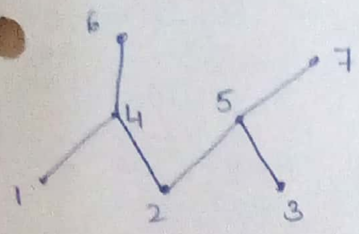
b) Maximal : 3, 5 Greatest : No greatest ($\because$ $5 \not R 3$ and $3 \not R 5$)
Minimal : 1, 6 Least : No least ($\because$ $1 \not R 6$ & $6 \not R 1$)

c) 

Maximal : 5
Minimal : 1
Greatest : 5 ($\because$ 1R5, 2R5, 3R5, 4R5, 5R5)
Least : 1 ($\because$ 1R1, 1R2, 1R3, 1R4, 1R5)

d) 

Maximal : e, f
Minimal : a
greatest : No
least : a

e) 

Maximal : 6, 7   greatest : No

Minimal : 1, 2, 3   least : No.

2) Consider the Hasse diagram of a Poset (A, R) given below:

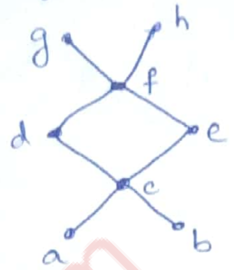
If B = {c, d, e}, find (if they exist)

i) all upper bounds of B.

ii) all lower bounds of B

iii) the least upper bound of B

iv) the greatest lower bound of B.



Soln :- i) All of c, d, e which are in B are related to f, g, h.

∴ f, g, h are upper bounds of B.

ii) The elements a, b, c are related to all of c, d, e which are in B. ∴ a, b, c are lower bounds of B.

iii) The upper bound f of B is related to the other upper bounds g and h of B. ∴ f is the LUB of B. i.e LUB(B) = f

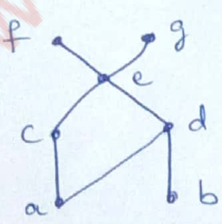
iv) The lower bounds a and b of B are related to the lower bound c of B. ∴ c is the GLB of B. i.e GLB(B) = c

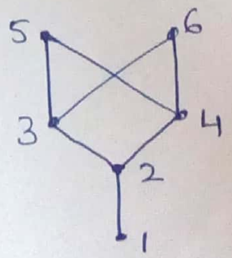
3) For the Posets shown in the following Hasse diagrams, find

i) all upper bounds   ii) all lower bounds

iii) LUB   and   iv) GLB of the set B, where

B = {c, d, e} in case (a) and B = {3, 4, 5} in case (b).

(a) 

(b) 

Soln :- (a) i) All of c, d, e which are in B are related to e, f, g.

∴ e, f, g are upper bounds of B.

ii) The element a is related to all of c, d, e which are in B. ∴ a is a lower bound of B.

iii) The upper bound e of B is related to the other upper bounds f and g of B.

$\therefore$ e is the LUB of B. i.e LUB(B) = e.

iv) GLB(B) = a.

b) i) 5 is the upper bound of B. (not 6 $\because$ 5 $\not R$ 6)

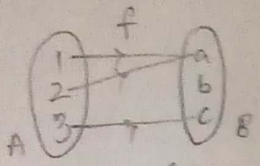
ii) 1, 2 are the lower bounds of B.

iii) There is no LUB of B.

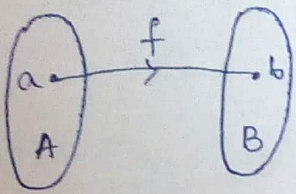
iv) GLB(B) = 2.

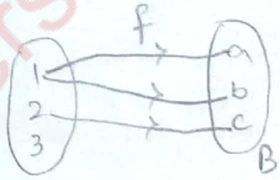
Functions

Defn:- Let A and B be non empty sets. Then a function (or a mapping) from A to B is a relation from A to B such that for each a in A, there exists a unique b in B such that $(a,b) \in f$. Then we write $b = f(a)$. Here b is called the image of a and a is called the preimage of b, under f. The element a is also called an argument of the function f and $b = f(a)$ is called the value of the function f for the argument a. A function f from A to B is denoted by $f : A \to B$. The pictorial representation of f is as below:



a f^n.





A rel^n not a f^n.

"Every function is a relation but every relation is not a f^n".

B'coz, if R is a relation from A to B then an element of A can be related to 2 elements of B under R. But under a function, an element of A can be related to only one element of B.

For the function $f : A \to B$, A is called the domain of f and B is called the co-domain of f. The subset of B consisting of the images of all elements of A under f is called the range of f, denoted by $f(A)$.

Note:-

1) Every a in A belong to some pair $(a,b) \in f$ and if $(a,b_1) \in f$ and $(a,b_2) \in f$ then $b_1 = b_2$.
2) An element $b \in B$ need not have a preimage in A, under f.
3) Two different elements of A can have the same image in B, under f.

4) The statements $(a,b) \in f$, afb and $b = f(a)$ are equivalent.

5) If g is a function from A to B, then $f = g$ iff $f(a) = g(a)$, $\forall\ a \in A$.

6) The range of $f : A \to B$ is given by $f(A) = \{f(x) / x \in A\}$ and $f(A)$ is a subset of B.

7) For $f : A \to B$, if $A_1 \subseteq A$ and $f(A_1)$ is defined by $f(A_1) = \{f(x) / x \in A_1\}$, then $f(A_1) \subseteq f(A)$. Here $f(A_1)$ is called the image of A_1, under f.

8) For $f : A \to B$, if $b \in B$ and $f^{-1}(b)$ is defined by $f^{-1}(b) = \{x \in A \mid f(x) = b\}$, then $f^{-1}(b) \subseteq A$. Here $f^{-1}(b)$ is called the preimage of b, under f.

9) For $f : A \to B$, if $B_1 \subseteq B$ and $f^{-1}(B_1)$ is defined by $f^{-1}(B_1) = \{x \in A \mid f(x) \in B_1\}$, then $f^{-1}(B_1) \subseteq A$. Here $f^{-1}(B_1)$ is called the Preimage of B_1, under f.

Types of functions :-

1) Identity function :- A f^n f on a set A is an identity function if the image of every element of A (under f) is itself and is denoted by I_A.

ie $f : A \to A$ (only, not $A \to B$) is such that $f(a) = a$ $\forall\ a \in A$.

In case of identity function, $f(A) = A$.

2) Constant function :- A f^n $f : A \to B$ is called a constant f^n if $f(a) = c$ $\forall\ a \in A$. ie f is a constant f^n if image of every element of A is same in B, and in this case $f(A) = \{c\}$.

P.T.O

3) **onto function (surjective f^n) :-** A f^n $f : A \to B$ is said to be an onto f^n from A to B if for every element b of B, there exists an element a of A such that $f(a) = b$.

i.e., f is an onto f^n from A to B if every element b of B has a preimage in A. i.e. range of $f = B$.

no element in B shud be free.

4) **One-to-one function :- (Injective f^n) :-** A f^n $f : A \to B$ is said to be an one-to-one f^n from A to B if different elements of A have different images in B under f.

i.e. if $f(a_1) = f(a_2)$ then $a_1 = a_2$ where $a_1, a_2 \in A$.

(or) if $a_1 \neq a_2$ then $f(a_1) \neq f(a_2)$ (by contrapositive).

5) **One-to-one correspondence (Bijective f^n or Bijection) :-** A f^n $f : A \to B$ is said to be an one-to-one correspondence if f is both one-to-one and onto. If $f : A \to B$ is a bijective f^n, then every element of A has a unique image in B and every element of B has a unique preimage in A.

Properties of functions :-

1) Let $f : X \to Y$ be a f^n and A and B be arbitrary non-empty sub sets of X, then

i) If $A \subseteq B$, then $f(A) \subseteq f(B)$

ii) $f(A \cup B) = f(A) \cup f(B)$.

iii) $f(A \cap B) \subseteq f(A) \cap f(B)$ and the equality holds if f is 1-1.

2) Let A and B be finite sets and f be a f^n from A to B, then the following are true :

i) If f is one-to-one then $|A| \leq |B|$.

ii) If f is onto then $|B| \leq |A|$.

iii) If f is bijective then $|A| = |B|$.

3) Suppose $|A| = |B|$ and $f : A \to B$, then f is one-to-one iff f is onto.

4) If $f: A \to B$ and $|A| = |B|$, then f is bijective iff f is one-to-one or onto.

5) If $f: A \to B$, $|A| = m$ and $|B| = n$ then there are n^m functions from A to B and if $m \le n$ then there are $\frac{n!}{(n-m)!}$ one-to-one functions from A to B.

Stirling number of second kind :-

Let A and B be finite sets with $|A| = m$ and $|B| = n$, where $m \geq n$. Then the number of onto functions from A to B is given by the formula :

$$p(m,n) = \sum_{k=0}^{n} (-1)^k \left({}^nC_{n-k}\right)(n-k)^m$$

(If $m < n$, then there are no onto fn from A to B)

with $p(m,n)$ given by the above formula, the no. $\left[\frac{p(m,n)}{n!}\right]$ is called the Stirling number of the second kind and is denoted by $S(m,n)$.

i) The no. of possible ways to assign 'm' distinct objects to 'n' identical places (containers) with no place (container) left empt is given by

$$S(m,n) = \frac{p(m,n)}{n!} = \frac{1}{n!} \sum_{k=0}^{n} (-1)^k \left({}^nC_{n-k}\right)(n-k)^m, \text{ for } m \geq n$$

ii) The no. of possible ways to assign 'm' distinct objects to 'n' identical places with empty places allowed is given by

$$p(m) = \sum_{i=1}^{n} S(m,i), \quad \text{for } m \geq n.$$

Problems :

1) Let $A = \{1,2,3,4\}$. Determine whether or not the following relations on A are fns :

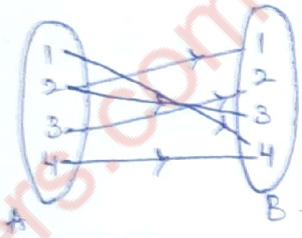
i) $f = \{(2,3)\ (1,4)\ (2,1)\ (3,2)\ (4,4)\}$

ii) $g = \{(2,1)\ (3,4)\ (1,4)\ (2,1)\ (4,4)\}$

Soln :-

i) we see that $(2,3) \in f$ and $(2,1) \in f$ ie the element 2 is related to two different elements 3 and 1, under f.

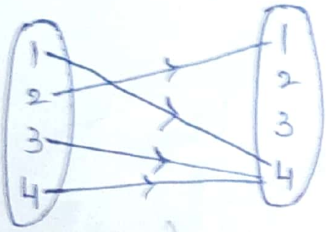
$\therefore$ f is not a fn.



ii) we see that, under g, every element of A is related to a unique element of A.

$\therefore$ g is a fn from A to A.

Range of $g = g(A) = \{1,4\}$



((2,1) appears twice. This has no special significance)

2) Let $A = \{0, \pm 1, \pm 2, \pm 3\}$. consider $f : A \to \mathbb{R}$ (where $\mathbb{R}$ is the set of all real numbers) defined by $f(x) = x^3 - 2x^2 + 3x + 1$ for $x \in A$. Find the range of f.

Soln :- $f(0) = 1$.

$f(1) = 1^3 - 2(1)^2 + 3(1) + 1 = 3$

|||^{ly} $f(2) = 7$, $f(3) = 19$, $f(-1) = -5$, $f(-2) = -21$, $f(-3) = -53$.

$\therefore$ Range of $f = \{-53, -21, -5, 1, 3, 7, 19\}$.

3) Let $A = \{1,2,3,4,5,6\}$ and $B = \{6,7,8,9,10\}$. If a fn $f : A \to B$ is defined by $f = \{(1,7)\ (2,7)\ (3,8)\ (4,6)\ (5,9)\ (6,9)\}$. Determine $f^{-1}(6)$ and $f^{-1}(9)$. If $B_1 = \{7,8\}$ and $B_2 = \{8,9,10\}$ find $f^{-1}(B_1)$ and $f^{-1}(B_2)$.

Soln :- By defn,

$$f^{-1}(b) = \{x \in A \mid f(x) = b\}$$

$\therefore$ $f^{-1}(6) = \{x \in A \mid f(x) = 6\} = \{4\}$.

$f^{-1}(9) = \{x \in A \mid f(x) = 9\} = \{5, 6\}$.

For any $B_1 \subseteq B$, by defn

$$f^{-1}(B_1) = \{x \in A \mid f(x) \in B_1\}.$$

for $B_1 = \{7, 8\}$, $f(x) \in B_1$ when $f(x) = 7$ and $f(x) = 8$.

From the defn of f, we see that $f(x) = 7$ when $x = 1$ & $x = 2$, and $f(x) = 8$ when $x = 3$.

$\therefore$ $f^{-1}(B_1) = \{1, 2, 3\}$.

IIIly $f^{-1}(B_2) = \{x \in A \mid f(x) \in B_2\} = \{3, 5, 6\}$.

4) Let $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \begin{cases} 3x - 5 & \text{for } x > 0 \\ -3x + 1 & \text{for } x \leq 0. \end{cases}$

i) Determine $f(0)$, $f(-1)$, $f\left(\frac{5}{3}\right)$, $f(-5/3)$

ii) Find $f^{-1}(0)$, $f^{-1}(1)$, $f^{-1}(-1)$, $f^{-1}(3)$, $f^{-1}(-3)$, $f^{-1}(-6)$.

iii) what are $f^{-1}([-5, 5])$ and $f^{-1}([-6, 5])$?

Soln :- i) $f(0) = -3(0) + 1 = 1$.

$f(-1) = -3(-1) + 1 = 4$.

$f(5/3) = 3\left(\frac{5}{3}\right) - 5 = 0$.

$f(-5/3) = -3\left(\frac{-5}{3}\right) + 1 = 6$.

ii) By defn $f^{-1}(b) = \{x \in \mathbb{R} \mid f(x) = b\}$.

a) $f^{-1}(0) = \{x \in \mathbb{R} \mid f(x) = 0\}$.

Consider $f(x) = 0$.

$3x - 5 = 0 \quad (x > 0)$ and $-3x + 1 = 0 \quad (x \leq 0)$

$3x = 5$ $\qquad$ $3x = 1$

$x = 5/3 > 0$ possible. $\qquad$ $x = 1/3 > 0$ # not possible.

Thus $f^{-1}(0) = \{5/3\}$.

b) $f^{-1}(1) = \{x \in \mathbb{R} \mid f(x) = 1\}$

$f(x) = 1 \Rightarrow 3x - 5 = 1 \; (x > 0)$ and $-3x + 1 = 1 \; (x \leq 0)$

$\Rightarrow 3x = 6$ $\qquad$ $3x = 0$

$x = 2 > 0$ ✓ $\qquad$ $x = 0$ ✓

$\therefore f^{-1}(1) = \{0, 2\}$.

c) $f^{-1}(-1) = \{x \in \mathbb{R} \mid f(x) = -1\}$

$f(x) = -1 \Rightarrow 3x - 5 = -1 \; (x > 0)$ and $-3x + 1 = -1 \; (x \leq 0)$

$-3x = -2$

$x = 2/3$ ✗.

$\Rightarrow 3x = 4$

$x = 4/3$ ✓

$\therefore f^{-1}(-1) = \{4/3\}$.

d) $f^{-1}(3) = \{x \in \mathbb{R} \mid f(x) = 3\}$

$f(x) = 3 \Rightarrow 3x - 5 = 3 \; (x > 0)$ and $-3x + 1 = 3 \; (x \leq 0)$

$-3x = 2$

$3x = 8$ $\qquad$ $x = -2/3$ ✓

$x = 8/3$ ✓

$\therefore f^{-1}(3) = \{-\frac{2}{3}, \frac{8}{3}\}$.

e) $f^{-1}(-3) = \{x \in \mathbb{R} \mid f(x) = -3\}$

$f(x) = -3 \Rightarrow 3x - 5 = -3 \; (x > 0)$ and $-3x + 1 = -3 \; (x \leq 0)$

$-3x = -4$

$\Rightarrow 3x = 2$ $\qquad$ $x = 4/3$ ✗

$x = 2/3$ ✓

$\therefore f^{-1}(-3) = \{2/3\}$

f) $f^{-1}(-6) = \{x \in \mathbb{R} \mid f(x) = -6\}$

$f(x) = -6 \Rightarrow 3x - 5 = -6 \; (x > 0)$ and $-3x + 1 = -6 \; (x \leq 0)$

$-3x = -7$

$\Rightarrow 3x = -1$ $\qquad$ $x = 7/3$ ✗.

$x = -1/3$ ✗

$\therefore f^{-1}(-6) = \{\} = \phi$.

iii) a) $f^{-1}([-5,5]) = \{x \in \mathbb{R} \mid f(x) \in [-5,5]\}$.
$= \{x \in \mathbb{R} \mid -5 \le f(x) \le 5\}$.

for $f(x) = 3x - 5 \quad (x > 0)$

$-5 \le 3x - 5 \le 5$.

add 5 throughout

$0 \le 3x \le 10$.

$\div$ by 3.

$0 \le x \le \frac{10}{3}$

for $f(x) = -3x + 1 \quad (x \le 0)$

$-5 \le -3x + 1 \le 5$.

add -1 throughout

$-6 \le -3x \le 4$.

$\div$ by 3.

$-2 \le -x \le \frac{4}{3}$.

$\times^{\text{ply}}$ by -1

$2 \geq x \geq -4/3$.

ie $-\frac{4}{3} \le x \le 2$.

Combining, $f^{-1}([-5,5]) = \{x \in \mathbb{R} \mid -\frac{4}{3} \le x \le 2 \text{ or } 0 < x \le 10/3\}$

$= \{x \in \mathbb{R} \mid -\frac{4}{3} \le x \le 10/3\}$ ie $x \in \left[\frac{-4}{3}, \frac{10}{3}\right]$.

$= \left[\frac{-4}{3}, \frac{10}{3}\right]$.

b) $f^{-1}([-6,5]) = \{x \in \mathbb{R} \mid f(x) \in [-6,5]\}$

$= \{x \in \mathbb{R} \mid -6 \le f(x) \le 5\}$.

for $f(x) = 3x - 5 \quad (x > 0)$

$-6 \le 3x - 5 \le 5$

add 5

$-1 \le 3x \le 10$.

$\div 3$

$-\frac{1}{3} \le x \le \frac{10}{3}$

for $f(x) = -3x + 1 \quad (x \le 0)$

$-6 \le -3x + 1 \le 5$

add -1

$-7 \le -3x \le 4$

$\div 3$

$-7/3 \le -x \le 4/3$

$\times^{\text{ply}}$ by -1.

$7/3 \geq x \geq -4/3$

ie $-\frac{4}{3} \le x \le 7/3$.

combining, $f^{-1}([-6,5]) = \{x \in \mathbb{R} \mid -\frac{4}{3} \le x \le 7/3 \text{ or } -\frac{1}{3} \le x \le 10/3\}$

$= \{x \in \mathbb{R} \mid -\frac{4}{3} \le x \le \frac{10}{3}\}$

$= \left[\frac{-4}{3}, \frac{10}{3}\right]$.

DO 6^{th}, then 5^{th}.

5) a) Let A and B be finite sets with $|A| = m$ and $|B| = n$. Find how many functions are possible from A to B?

b) If there are 2187 functions from A to B and $|B| = 3$, what is $|A|$?

Soln:- a) Let $A = \{a_1, a_2 \dots a_m\}$ and $B = \{b_1, b_2 \dots b_n\}$. then a fn $f : A \to B$ is of the form

$f = \{(a_1, x), (a_2, x), \dots (a_m, x)\}$, where x stands for b_j for some j.

Since there are 'n' no. of b_j's, there are 'n' choices for x in each of the 'm' ordered pairs belonging to f.

$\therefore$ Total no. of choices for x is

$n \times n \times \dots \times n$ (m factors) $= n^m$.

Thus there are $n^m = |B|^{|A|}$ possible fns from A to B.

b) Given $|B| = n = 3$, and $n^m = 2187$.

$\therefore \quad 3^m = 2187$

$\Rightarrow \quad m \log_e 3 = \log_e 2187$

$\Rightarrow \quad m = |A| = 7$

6) Let $f : A \to B$ be a function. Let C and D be arbitrary nonempty subsets of B. Prove the following:

i) $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$ ii) $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$.

iii) $f^{-1}(\overline{C}) = \overline{f^{-1}(C)}$.

Soln:- i) for any $x \in A$,

$x \in f^{-1}(C \cup D) \Leftrightarrow f(x) \in \{C \cup D\}$

$\Leftrightarrow f(x) \in C$ or $f(x) \in D$.

$\Leftrightarrow x \in f^{-1}(C)$ or $x \in f^{-1}(D)$

$\Leftrightarrow x \in \{f^{-1}(C) \cup f^{-1}(D)\}$

$\therefore \quad f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$.

ii) for any $x \in A$,

$x \in f^{-1}(C \cap D) \Leftrightarrow f(x) \in \{C \cap D\}$

$\Leftrightarrow f(x) \in C$ and $f(x) \in D$.

$\Leftrightarrow x \in f^{-1}(C)$ and $x \in f^{-1}(D)$

$\Leftrightarrow x \in \{f^{-1}(C) \cap f^{-1}(D)\}$

$\therefore \quad f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$.

iii) for any $x \in A$,

$x \in f^{-1}(\overline{C}) \Leftrightarrow f(x) \in \overline{C}$

$\Leftrightarrow f(x) \notin C$

$\Leftrightarrow x \notin f^{-1}(C)$.

$\Leftrightarrow x \in \overline{f^{-1}(C)}$.

$\therefore \quad f^{-1}(\overline{C}) = \overline{f^{-1}(C)}$.

7) How many 1-1 functions are possible from $A \rightarrow B$ where $|A| = m$ & $|B| = n$, if there are 60 1-1 functions from $A \rightarrow B$ and $|A| = 3$, $|B| = ?$.

Soln :- No. of 1-1 functions from A to B is $\frac{n!}{(n-m)!}$

Given $\frac{n!}{(n-m)!} = 60$, $|A| = m = 3$, $|B| = n = ?$.

$\frac{n!}{(n-3)!} = 60$.

$\Rightarrow \frac{n \times (n-1) \times (n-3)!}{(n-3)!} = 60$

$\Rightarrow n(n-1)(n-2) = 60$.

$\Rightarrow 5 \times 4 \times 3 = 60$

$\Rightarrow \boxed{n = 5}$

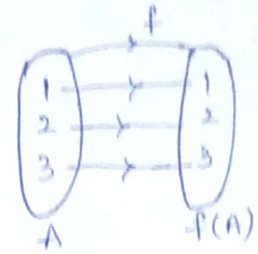
ie $|B| = n = 5$.

P.T.O

8) Find the nature of the following fns defined on $A = \{1, 2, 3\}$.

i) $f = \{(1,1)\,(2,2)\,(3,3)\}$ ii) $g = \{(1,2)\,(2,2)\,(3,2)\}$.

iii) $h = \{(1,2)(2,3)(3,1)\}$.

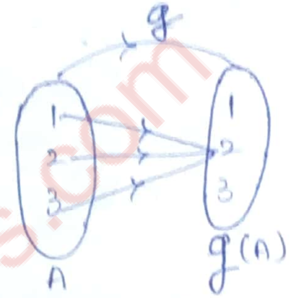


Soln :- i) for every $a \in A$, $(a, a) \in f$ ie a fa.

ie $A = f(A)$

ie image of every element in A is itself.

$\therefore$ f is the identity fn on A.

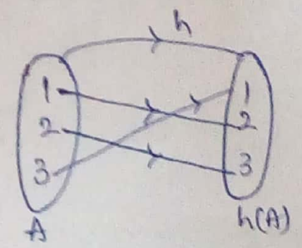


ii) we see that every $a \in A$ has 2 as its image.

ie $g(1) = 2$, $g(2) = 2$, $g(3) = 2$

$\therefore$ g is a constant fn on A.

iii)



we see that, every element of A has a unique image and every element of A has a unique preimage, under h.

$\therefore$ h is both one-to-one and onto.

$\therefore$ h is 1-1 correspondence.

9) Find whether the following fns from A to B are 1-1, onto:

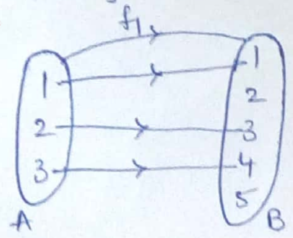
i) $f_1 = \{(1,1)\,(2,3)\,(3,4)\}$ for $A = \{1,2,3\}$, $B = \{1,2,3,4,5\}$.

ii) $f_2 = \{(1,1)\,(2,3)\,(3,4)\,(4,2)\}$ for $A = B = \{1,2,3,4\}$

iii) $A = \{a, b, c\}$ $B = \{1, 2, 3, 4\}$, $f_3 = \{(a,1)\,(b,1)\,(c,4)\}$.

iv) $A = \{1, 2, 3, 4\}$ $B = \{a, b, c, d\}$, $f_4 = \{(1,a)\,(2,a)\,(3,d)\,(4,c)\}$.

Soln:- i)

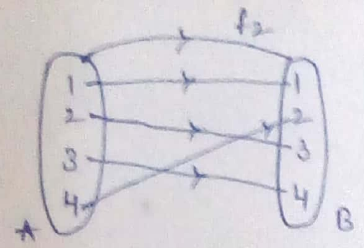


under f_1, every element of A has a unique image in B and no two elements of A have the same image in B.

$\therefore$ f is one-to-one.

under f_1, the elements 2 & 5 of B has no pre-image in A. $\therefore$ f is not onto.

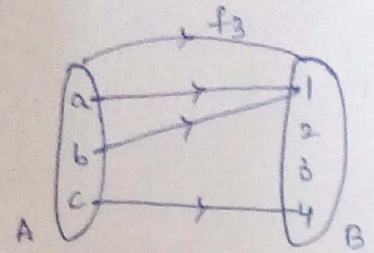
ii)



f_2 is both 1–1 & onto

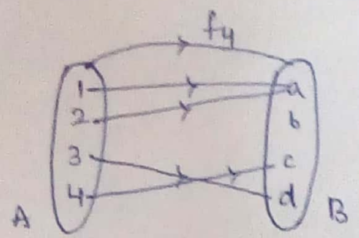
$\therefore$ f_2 is bijective.

iii)



f_3 is neither 1–1 nor onto.

iv)



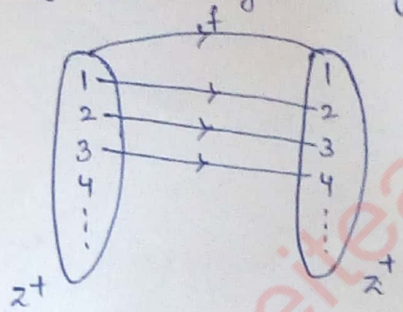
f_4 is neither 1–1 nor onto.

10) Let $f, g : z^+ \to z^+$ where $\forall x \in z^+$, $f(x) = x+1$, $g(x) = \max(1, x-1)$

i) what is the range of f? ii) Is f 1–1? iii) Is f onto?

iv) what is the range of g? v) Is g 1–1? vi) Is g onto?

Soln :-

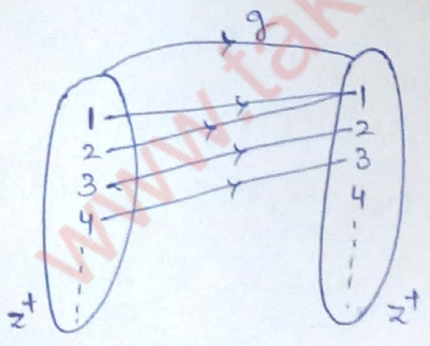


i) Range of $f = \{2, 3, 4 \ldots\}$

$= z^+ - \{1\}$.

ii) f is 1–1

iii) f is not onto $\because$ $f^{-1}(1)$ does not exist.



$g(x) = \max(1, x-1)$

$g(1) = \max(1, 0) = 1$.

$g(2) = \max(1, 1) = 1$

$g(3) = \max(1, 2) = 2$

$g(4) = \max(1, 3) = 3$ etc

iv) Range of $g = \{1, 2, 3 \ldots\} = z^+$.

v) g is not 1–1

vi) g is onto.

Do 15^{th} after this

P.T.O.

11) Let $A = \{1, 2, 3, 4\}$ and $B = \{1, 2, 3, 4, 5, 6\}$.

i) Find how many fns are there from A to B. How many of these are 1-1? How many are onto?

ii) Find how many fns are there from B to A. How many of these are 1-1? How many are onto?

Soln :- Here $|A| = m = 4$ and $|B| = n = 6$.

i) The no. of fns possible from A to B is $|B|^{|A|} = n^m = 6^4 = 1296$.

The no. of 1-1 fns possible from A to B is

$$\frac{n!}{(n-m)!} = \frac{6!}{2!} = 360.$$

There is no onto fn from A to B. ($\because m < n$)

ii) The no. of fns possible from B to A is $|A|^{|B|} = m^n = 4^6 = 4096$.

There is no 1-1 fn from B to A ($\because m \not> n$).

The no. of onto fns from B to A is

$$p(n,m) = \sum_{k=0}^{m} (-1)^k \left({}^mC_{m-k}\right)(m-k)^n$$

$$p(6,4) = \sum_{k=0}^{4} (-1)^k \left({}^4C_{4-k}\right)(4-k)^6$$

$$= {}^4C_4 \times 4^6 - {}^4C_3 \times 3^6 + {}^4C_2 \times 2^6 - {}^4C_1 \times 1^6$$

$$= 1560.$$

Note:- 1) 1-1 → $\frac{n!}{(n-m)!}$ for $m \le n$

2) onto → $p(m,n)$ for $m \ge n$.

12) There are 6 programmers who can assist eight executives. In how many ways can the executives be assisted so that each programmer assists atleast one executive?

use f^n def^n concept
no 'a' in A shud be free

soln :- all programmer shud have an executive

Let A denote the set of executives and B denote the set of programmers. $\therefore$ $|A| = m = 8$, $|B| = n = 6$.

P.T.O.

Thus Required no. = no. of onto fns from A to B.

$\Rightarrow p(m,n) = p(8,6) = (6!) \times S(8,6)$.

no need do like last problem

$\therefore S(8,6) = \frac{1}{6!} \sum_{k=0}^{6} (-1)^k \left({}^6C_{(6-k)}\right)(6-k)^8$

$= \frac{1}{6!}\left\{ {}^6C_6 \times 6^8 - {}^6C_5 \times 5^8 + {}^6C_4 \times 4^8 - {}^6C_3 \times 3^8 + {}^6C_2 \times 2^8 - {}^6C_1 \times 1^8 \right\}$

$= 266$.

$\therefore p(8,6) = 6! \times 266 = 191520$.

13) Find the no. of ways of distributing 4 distinct objects among 3 identical containers, with some container(s) possible empty.

Sol^n:- Here, no. of objects is $m=4$ and no. of containers is $n=3$.

$\therefore$ Req. no. $= p(m) = \sum_{i=1}^{n} S(m,i)$

$p(4) = \sum_{i=1}^{3} S(4,i) = S(4,1) + S(4,2) + S(4,3) \to (1)$

$S(4,1) = \frac{1}{1!} \sum_{k=0}^{1} (-1)^k \left({}^1C_{1-k}\right)(1-k)^4 = 1$.

$S(4,2) = \frac{1}{2!} \sum_{k=0}^{2} (-1)^k \left({}^2C_{2-k}\right)(2-k)^4 = \frac{1}{2}\left[2^4 - {}^2C_1 \times 1^4\right] = 7$.

$|||^{ly}$ $S(4,3) = 6$.

$\therefore (1) \Rightarrow$ Req. no. $= 1+7+6 = 14$.

14) Find the no. of equivalence relations that can be defined on a finite set A with $|A| = 6$.

Don't do

Sol^n:- Since $|A| = m = 6$, a partition of A can have atmost 6 cells. Treating the elements of A as objects & cells as containers, the no. of partitions having k cells is $S(6,k)$. Since k varies from 1 to 6, the total no. of different partitions of A is

$$p(m) = \sum_{i=1}^{n} S(m,i)$$

$\therefore p(6) = \sum_{i=1}^{6} S(6,i)$

$p(6) = S(6,1) + S(6,2) + S(6,3) + S(6,4) + S(6,5) + S(6,6)$ $\rightarrow$ (1)

we know that $S(m,n) = \frac{1}{n!} \sum_{k=0}^{n} (-1)^k ({}^nC_{n-k}) (n-k)^m$

$S(6,1) = \frac{1}{1!} \sum_{k=0}^{1} (-1)^k \, {}^1C_{1-k} \, (1-k)^6 = 1$.

$S(6,2) = \frac{1}{2!} \sum_{k=0}^{2} (-1)^k \, {}^2C_{2-k} \, (2-k)^6 = \frac{1}{2}\left[{}^2C_2 \times 2^6 - {}^2C_1 \times 1^6\right] = 31$

$S(6,3) = \frac{1}{3!} \sum_{k=0}^{3} (-1)^k \, {}^3C_{3-k} \, (3-k)^6 = \frac{1}{6}\left[{}^3C_3 \times 3^6 - {}^3C_2 \times 2^6 + {}^3C_1 \times 1^6\right] = 90$

$S(6,4) = 65$, $\quad S(6,5) = 15$, $\quad S(6,6) = 1$.

$\therefore$ No. of partitions of A is

(1) $\Rightarrow$ $p(6) = 203$.

Since each partition of A corresponds to an equivalence relⁿ on A, it follows that if $|A| = 6$, then 203 equivalence relations can be defined on A.

15) Let $A = \mathbb{R}$, $B = \{x \mid x$ is real and $x \geq 0\}$. Is the fⁿ $f: A \rightarrow B$ defined by $f(a) = a^2$ an onto fⁿ? Is it a 1-1 fⁿ?.

Solⁿ:- i) Take any $b \in B$, then b is non-negative real no.

Since $f(a) = a^2 \Rightarrow b = a^2 \Rightarrow a = \pm\sqrt{b} \in A$ $(\because A = \mathbb{R})$.

we see that, since $f(a) = a^2$,

$f(\sqrt{b}) = (\sqrt{b})^2 = b$.

$f(-\sqrt{b}) = (-\sqrt{b})^2 = b$.

Thus $\pm\sqrt{b}$ are the preimages of b under f.

Since 'b' is arbitrary element of B, every elemence in B has a preimage in A.

$\therefore$ f is onto fⁿ.

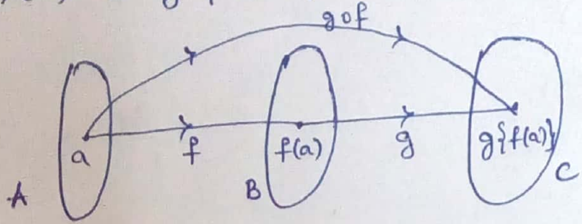
ii) Since $b \in B$ has 2 preimages $\pm\sqrt{b} \in A$ under f,

f is not one-to-one.

Composition of functions :-

Consider 3 non-empty sets A, B, C and the fns $f : A \to B$ and $g : B \to C$. The composition (or product) of these 2 functions is defined as $g \circ f : A \to C$ with

$(g \circ f)(a) = g\{f(a)\} \quad \forall \ a \in A$.



Note :- 1) For $f : A \to A$, $f^1 = f$, $f^2 = f \circ f$, $f^3 = f \circ f^2$ & so on.

Thus $f^1 = f$, $f^n = f \circ f^{n-1}$.

→ done in Problem(2)

2) Let $f : A \to B$ and $g : B \to C$, then

i) If f and g are one-to-one, then $g \circ f$ is also one-to-one

ii) If $g \circ f$ is one-to-one, then f is one-to-one.

iii) If f and g are onto, then $g \circ f$ is also onto.

iv) If $g \circ f$ is onto, then g is onto.

3) The composition of 2 fns is not commutative ie $g \circ f \neq f \circ g$.

Problems :-

1) Let $f : A \to B$, $g : B \to C$ and $h : C \to D$ be 3 functions.

Then $(h \circ g) \circ f = h \circ (g \circ f)$.

Soln :- Since $f : A \to B$, $g : B \to C$ and $h : C \to D$, we have both $(h \circ g) \circ f$ and $h \circ (g \circ f)$ are fns from A to D.

For any $x \in A$,

$[(h \circ g) \circ f](x) = (h \circ g)[f(x)] = (h \circ g)(y)$ where $y = f(x)$

$= h[g(y)] = h(z)$ where $z = g(y)$.

$\hookrightarrow$ ①

$[h \circ (g \circ f)](x) = h[(g \circ f)(x)]$

$= h[g\{f(x)\}]$

$= h[g(y)] = h(z) \rightarrow$ ②

Thus $[(h \circ g) \circ f](x) = [h \circ (g \circ f)](x)$ for every $x \in A$.

$\therefore (h \circ g) \circ f = h \circ (g \circ f) \Rightarrow$ Compositions of 3 fns is associative.

2) Let $f : A \to B$ and $g : B \to C$, Prove that

i) If f and g are 1-1, then $g \circ f$ is 1-1.

ii) If $g \circ f$ is 1-1, then f is 1-1.

iii) If f and g are onto, then $g \circ f$ is onto.

iv) If $g \circ f$ is onto, then g is onto.

Soln:- Let $a_1, a_2 \in A$.

i) Suppose $(g \circ f)(a_1) = (g \circ f)(a_2)$.

$\Rightarrow g(f(a_1)) = g(f(a_2))$

$\Rightarrow g(b_1) = g(b_2)$ $[\because f : A \to B \Rightarrow b_1 = f(a_1)$ & $b_2 = f(a_2)]$

$\Rightarrow b_1 = b_2$ $(\because g$ is 1-1$)$

$\Rightarrow f(a_1) = f(a_2)$

$\Rightarrow a_1 = a_2$ $(\because f$ is 1-1$)$

Thus if $(g \circ f)(a_1) = (g \circ f)(a_2)$, then $a_1 = a_2$.

$\Rightarrow g \circ f$ is 1-1.

ii) Let $f(a_1) = f(a_2)$.

$\Rightarrow g(f(a_1)) = g(f(a_2))$

$\Rightarrow (g \circ f)(a_1) = (g \circ f)(a_2)$

$\Rightarrow a_1 = a_2$ $(\because g \circ f$ is 1-1$)$

Thus if $f(a_1) = f(a_2)$, then $a_1 = a_2$

$\Rightarrow f$ is 1-1.

iii) f is onto $\Rightarrow \forall\, b \in B, \exists\, a \in A$ such that $b = f(a)$ $\to$ (1)

g is onto $\Rightarrow \forall\, c \in C, \exists\, b \in B$ such that $c = g(b)$ $\to$ (2)

from (2), $c = g(b) = g(f(a))$ (using (1))

$c = (g \circ f)(a)$

$\therefore \forall\, c \in C, \exists\, a \in A$ such that $c = (g \circ f)(a) \Rightarrow g \circ f$ is onto.

iv) $g \circ f$ is onto $\Rightarrow \forall\, c \in C, \exists\, a \in A$ such that $c = (g \circ f)(a)$.

$\Rightarrow c = g(f(a))$

$= g(b)$, $b = f(a) \in B$.

$\therefore \forall\, c \in C, \exists\, b = f(a) \in B$ such that $c = g(b)$

$\therefore g$ is onto.

3) Let $A = \{1,2,3,4\}$ and $f : A \to A$ is a fn defined by
$f = \{(1,2)(2,2)(3,1)(4,3)\}$. Find f^2.

Soln :- Given : $f(1) = 2$, $f(2) = 2$, $f(3) = 1$, $f(4) = 3$.

$\therefore$ $f^2(1) = f \circ f(1) = f(f(1)) = f(2) = 2$.

$f^2(2) = f \circ f(2) = f(f(2)) = f(2) = 2$.

$f^2(3) = f \circ f(3) = f(f(3)) = f(1) = 2$

$f^2(4) = f \circ f(4) = f(f(4)) = f(3) = 1$.

i.e. $f^2 = \{(1,2)(2,2)(3,2)(4,1)\}$.

4) Consider the fns f and g defined by $f(x) = x^3$ and $g(x) = x^2+1$, $\forall x \in \mathbb{R}$, find $g \circ f$, $f \circ g$, f^2 and g^2.

Soln :- $(g \circ f)(x) = g\{f(x)\} = g(x^3) = (x^3)^2 + 1 = x^6 + 1$.

$(f \circ g)(x) = f\{g(x)\} = f(x^2+1) = (x^2+1)^3 = (x^2)^3 + 1^3 + 3 \cdot x^2 \cdot 1 (x^2+1)$
$= x^6 + 1 + 3x^4 + 3x^2$

$f^2(x) = (f \circ f)(x) = f\{f(x)\} = f(x^3) = (x^3)^3 = x^9$

$g^2(x) = (g \circ g)(x) = g\{g(x)\} = g(x^2+1) = (x^2+1)^2 + 1 = x^4 + 2 + 2x^2$.

5) Let $A = \{1,2,3,4\}$ and $B = \{a,b,c\}$ and $C = \{w,x,y,z\}$ with
$f : A \to B$ and $g : B \to C$ given by
$f = \{(1,a)(2,a)(3,b)(4,c)\}$ and $g = \{(a,x)(b,y)(c,z)\}$. Find $g \circ f$.

Soln :- By data, $f(1) = a$ and $g(a) = x$
$f(2) = a$ $\qquad g(b) = y$
$f(3) = b$ $\qquad g(c) = z$.
$f(4) = c$

Since $f : A \to B$ and $g : B \to C$, $g \circ f : A \to C$.

$\therefore$ $g \circ f(1) = g(f(1)) = g(a) = x$.

$g \circ f(2) = g(f(2)) = g(a) = x$.

$g \circ f(3) = g(f(3)) = g(b) = y$.

$g \circ f(4) = g(f(4)) = g(c) = z$.

Thus $g \circ f = \{(1,x)(2,x)(3,y)(4,z)\}$.

P.T.O.

6) Let f, g, h be fns from $\mathbb{Z}$ to $\mathbb{Z}$ defined by

$$f(x) = x-1, \quad g(x) = 3x, \quad h(x) = \begin{cases} 0, & \text{if } x \text{ is even} \\ 1, & \text{if } x \text{ is odd.} \end{cases}$$

Determine $(f\circ(g\circ h))(x)$ and $((f\circ g)\circ h)(x)$ and verify that $f\circ(g\circ h) = (f\circ g)\circ h$.

Soln: we have $(g\circ h)(x) = g\{h(x)\} = 3h(x)$.

$$\begin{aligned} \therefore (f\circ(g\circ h))(x) &= f\{(g\circ h)(x)\} \\ &= f\{g\{h(x)\}\}. \\ &= f\{3h(x)\} = 3h(x) - 1. \\ &= \begin{cases} -1 & \text{if } x \text{ is even} \\ 2 & \text{if } x \text{ is odd.} \end{cases} \longrightarrow (1) \end{aligned}$$

$$\boxed{\cancel{((f\circ g)\circ h)(x) = (f\circ g)\{h(x)\} = f\{g\{h(x)}}}$$

we have $(f\circ g)(x) = f\{g(x)\} = g(x) - 1 = 3x - 1$.

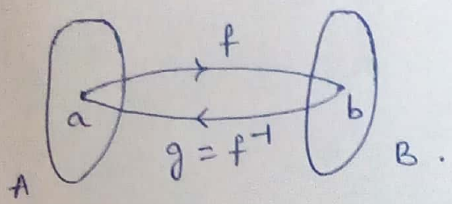
$$\begin{aligned} \therefore ((f\circ g)\circ h)(x) &= (f\circ g)\{h(x)\} \\ &= 3h(x) - 1 = \begin{cases} -1 & \text{if } x \text{ is even} \\ 2 & \text{if } x \text{ is odd.} \end{cases} \longrightarrow (2) \end{aligned}$$

from (1) & (2),

$$f\circ(g\circ h) = (f\circ g)\circ h.$$

Invertible functions :- A f^n $f : A \to B$ is said to be invertible if there exists a f^n $g : B \to A$ such that $g \circ f = I_A$ and $f \circ g = I_B$, where I_A is the Identity f^n on A and I_B is the Identity f^n on B. Then g is called an inverse of f and we write $g = f^{-1}$.

[since $f : A \to B$, $g : B \to A$
$g \circ f : A \to A$
$f \circ g : B \to B$



Note :-

1) If a f^n $f : A \to B$ is invertible then it has a unique inverse. further, if $f(a) = b$ then $f^{-1}(b) = a$.
2) If f is invertible, then $f(a) = b$ and $a = f^{-1}(b)$ are equivalent.
3) If $f = \{ (a, b) \mid a \in A, b \in B \}$ is invertible, then $f^{-1} = \{ (b, a) \mid b \in B, a \in A \}$ and conversely.
4) If f is invertible then f^{-1} is invertible and $(f^{-1})^{-1} = f$.
5) A f^n $f : A \to B$ is invertible iff it is one-to-one and onto.
6) If $f : A \to B$ and $g : B \to C$ are invertible f^ns then $g \circ f : A \to C$ is also invertible and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.
7) Let A and B be finite sets with $|A| = |B|$ and f be a f^n from A to B. Then the following statements are equivalent
   i) f is one-to-one.
   ii) f is onto.
   iii) f is invertible.

P.T.O

Problems :-

1) Suppose $f : A \to B$ is investible, then

Prove that f has unique inverse. further, p.T if $f(a) = b$ then

$a = f^{-1}(b)$.

Soln :- Given f is invertible.

Let g be inverse of f. ie $g = f^{-1}$.

$\therefore g \circ f = I_A$ and $f \circ g = I_B$.

Let us assume there is one more inverse for f, say h.

$\therefore h \circ f = I_A$ and $f \circ h = I_B$.

consider $h \circ (f \circ g) = (h \circ f) \circ g$.

$h \circ I_B = I_A \circ g$.

$\therefore h = g$ #

$\therefore f$ has unique inverse.

Further, $f(a) = b$.

$\Rightarrow g(f(a)) = g(b)$

$\Rightarrow g \circ f (a) = g(b)$

$\Rightarrow I_A (a) = g(b)$

$\Rightarrow a = g(b)$

$\Rightarrow a = f^{-1}(b)$ $(\because g = f^{-1})$

2) A f^n $f : A \to B$ is investible iff it is one-to-one and onto.

QP

Soln :- Let $f : A \to B$ be a investible f^n, then there exists a unique f^n $g : B \to A$ such that $g \circ f = I_A$ and $f \circ g = I_B$.

TPT f is one-to-one.

Let $f(a_1) = f(a_2)$

$\Rightarrow g(f(a_1)) = g(f(a_2))$

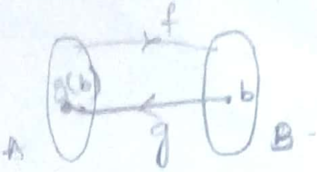
$\Rightarrow g \circ f (a_1) = g \circ f (a_2)$

$\Rightarrow I_A (a_1) = I_A (a_2)$

$\Rightarrow a_1 = a_2$

$\therefore f$ is 1-1.

T f is onto.



Take any $b \in B$, then $g(b) \in A$.

and $b = I_B(b) = (f \circ g)(b) = f\{g(b)\}$

Thus b is the image of an element $g(b) \in A$ under f.

$\therefore$ f is onto.

Conversely, Suppose f is 1-1 and onto. $\therefore$ If is bijective.

$\therefore$ for every $b \in B$ there exists a unique $a \in A$ such that $f(a) = b$.

Consider a f^n $g: B \to A$ defined by $g(b) = a$.

then $(f \circ g)(b) = f\{g(b)\} = f(a) = b = I_B(b)$

$(g \circ f)(a) = g\{f(a)\} = g(b) = a = I_A(a)$.

$\therefore$ f is Invertible with g as the Inverse.

3) If $f: A \to B$ and $g: B \to C$ are invertible f^{ns}, then Prove that $g \circ f : A \to C$ is also invertible and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

QP

Soln:- Suppose f and g are invertible.

$\therefore$ f and g are 1-1 and onto.

$\therefore$ $g \circ f$ is also 1-1 and onto.

Hence $g \circ f$ is Invertible.

Next, we have $f: A \to B$, $g: B \to C$, $g \circ f : A \to C$.

$\therefore$ $f^{-1}: B \to A$, $g^{-1}: C \to B$, and let $h = f^{-1} \circ g^{-1} : C \to A$.

consider $h \circ (g \circ f) = (f^{-1} \circ g^{-1}) \circ (g \circ f)$

$= f^{-1} \circ (g^{-1} \circ g) \circ f = f^{-1} \circ I_B \circ f = f^{-1} \circ f = I_A \to$ (1)

consider $(g \circ f) \circ h = (g \circ f) \circ (f^{-1} \circ g^{-1})$

$= g \circ (f \circ f^{-1}) \circ g^{-1} = g \circ I_A \circ g^{-1} = g \circ g^{-1} = I_B \to$ (2)

from (1) & (2), h is the inverse of $g \circ f$.

ie $h = (g \circ f)^{-1}$.

Thus $(g \circ f)^{-1} = h = f^{-1} \circ g^{-1}$.

4) Let $A = \{1,2,3,4\}$ and f and g be fns from A to A given by $f = \{(1,4)(2,1)(3,2)(4,3)\}$ and $g = \{(1,2)(2,3)(3,4)(4,1)\}$. Prove that f and g are inverses of each other.

Soln :- By data, $f(1) = 4$, $f(2) = 1$, $f(3) = 2$, $f(4) = 3$

$g(1) = 2$, $g(2) = 3$, $g(3) = 4$, $g(4) = 1$.

we have
$$(g \circ f)(1) = g\{f(1)\} = g(4) = 1 = I_A(1).$$
$$(g \circ f)(2) = g\{f(2)\} = g(1) = 2 = I_A(2)$$
$$(g \circ f)(3) = g\{f(3)\} = g(2) = 3 = I_A(3).$$
$$(g \circ f)(4) = g\{f(4)\} = g(3) = 4 = I_A(4).$$

and
$$(f \circ g)(1) = f\{g(1)\} = f(2) = 1 = I_A(1).$$
$$(f \circ g)(2) = f\{g(2)\} = f(3) = 2 = I_A(2)$$
$$(f \circ g)(3) = f\{g(3)\} = f(4) = 3 = I_A(3).$$
$$(f \circ g)(4) = f\{g(4)\} = f(1) = 4 = I_A(4).$$

Thus $\forall\, x \in A$, $(g \circ f)(x) = I_A(x)$ and $(f \circ g)(x) = I_A(x)$.

$\Rightarrow$ g is an inverse of f and f is an Inverse of g.

ie f and g are inverses of eachother.

5) Let $A = B = C = \mathbb{R}$ and $f : A \to B$ and $g : B \to C$ be defined by

QP $f(a) = 2a+1$ and $g(b) = \frac{1}{3}b$, $\forall\, a \in A$, $\forall\, b \in B$.

compute $g \circ f$ and show that $g \circ f$ is investible. find $(g \circ f)^{-1}$.

Soln :- we have $(g \circ f)(a) = g\{f(a)\} = g(2a+1) = \frac{1}{3}(2a+1)$.

Thus $g \circ f : A \to C$ is defined by $(g \circ f)(a) = \frac{1}{3}(2a+1)$

we have $b = f(a) = 2a+1 \Rightarrow a = \frac{b-1}{2}$.

$f(a_1) = f(a_2)$
$\Rightarrow 2a_1+1 = 2a_2+1$
$\Rightarrow 2a_1 = 2a_2 \Rightarrow a_1 = a_2$
$\Rightarrow f$ is 1-1

$\therefore$ $a = f^{-1}(b) = \frac{b-1}{2}$ $\Rightarrow f$ is onto $\Rightarrow$ f is investible.

$g(b_1) = g(b_2)$
$\Rightarrow \frac{b_1}{3} = \frac{b_2}{3}$
$\Rightarrow b_1 = b_2$
$\Rightarrow g$ is 1-1

Also $c = g(b) = \frac{1}{3}b$. $\Rightarrow b = 3c$.

$\therefore$ $b = g^{-1}(c) = 3c$ $\Rightarrow g$ is onto $\Rightarrow$ g is investible.

Thus $g \circ f$ is investible and its inverse is given by
$$(g \circ f)^{-1}(c) = (f^{-1} \circ g^{-1})(c) = f^{-1}\{g^{-1}(c)\} = f^{-1}(3c) = \frac{3c-1}{2}$$

$$\therefore \cancel{(g\circ f)^{-1}(c) = (f^{-1}\circ g^{-1})(c) = f^{-1}\{g^{-1}(c)\} = f^{-1}\{3c\} = \frac{1}{2}(3c-1)}$$

6) Let $A = B = \mathbb{R}$, the set of all real no.'s, and the fns

QP $f : A \to B$ and $g : B \to A$ be defined by

$f(x) = 2x^3 - 1,\ \forall\, x \in A$; $g(y) = \left\{\frac{1}{2}(y+1)\right\}^{1/3}$, $\forall\, y \in B$.

Show that each of f and g is the inverse of the other.

Soln :- for any $x \in A$,

$$(g\circ f)(x) = g\{f(x)\} = g(y) \quad \text{where } y = f(x)$$
$$= \left\{\frac{1}{2}(y+1)\right\}^{1/3}$$
$$= \left\{\frac{1}{2}(2x^3-1+1)\right\}^{1/3} \quad (\because\ y = f(x) = 2x^3-1)$$
$$= x.$$

$\therefore\ g\circ f = I_A$.

for any $y \in B$,

$$(f\circ g)(y) = f\{g(y)\}$$
$$= f(z) \quad \text{where } z = g(y)$$
$$= 2z^3 - 1$$
$$= 2(g(y))^3 - 1 = 2\left(\left\{\frac{1}{2}(y+1)\right\}^{1/3}\right)^3 - 1$$
$$= 2\left\{\frac{1}{2}(y+1)\right\} - 1 = y+1-1 = y.$$

$\therefore\ f\circ g = I_B$.

$\therefore$ each of f and g is an invertible fn.

$\Rightarrow$ f and g are inverses of each other.

THE PIGEONHOLE PRINCIPLE

Statement :- If 'm' pigeons occupy 'n' pigeonholes and if $m > n$, then two or more pigeons occupy the same pigeonhole.

(OR)

If 'm' pigeons occupy 'n' pigeonholes where $m > n$, then atleast one pigeonhole must contain two or more pigeons in it.

Ex :- If 8 children are born in the same week, then two or more children are born on the same day of the week.

Generalised pigeon hole Principle :-

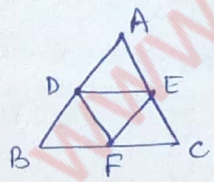
If 'm' pigeons occupy 'n' pigeonholes and $m > n$, then atleast one pigeonhole must contain $p+1$ or more pigeons in it,

where $p = \left\lfloor \frac{(m-1)}{n} \right\rfloor$

Problems :-

1) ABC is an equilateral Δ^{le} whose sides are of length 1 cm each. If we select 5 points inside the Δ^{le}, prove that atleast 2 of these points are such that the distance b/w them is less that $1/2$ cm.

Soln :-



Consider the equilateral Δ^{le} ABC whose sides are of length 1 cm each. Consider the Δ^{le} DEF formed by the midpoints of the sides AB, AC and BC resp (see fig). This divides Δ^{le} ABC into 4 small equilateral Δ^{les}, whose length of each side is $1/2$ cm.

Let us treat 5 points as pigeons and 4 small equilateral Δ^{les} as pigeon holes, then by pigeon hole principle, atleast one small Δ^{le} contains 2 or more points and the distance b/w such points is less than $1/2$ cm.

2) A bag contains 12 pair of socks (each pair in different color).
If a person draws the socks one by one at random,
determine at most how many draws are required to get
atleast one pair of matched socks.

Soln :- Let n be the no. of draws.
For $n \le 12$, it is possible that the socks drawn are of different
colors ∵ there are 12 colors.
For $n = 13$, all socks cannot have different colors – atleast two
must have same color.
Let us treat 13 as the no. of pigeons and 12 colors as 12 pigeonholes.
∴ atmost 13 draws are required to have atleast 1 pair of
socks of the same color.

3) If 5 colors are used to paint 26 doors, prove that atleast
6 doors will have same color.

Soln:- Consider 26 doors as pigeons and 5 colors as pigeon
holes. By generalised pigeon hole principle, atleast 1 color
must be assigned to $p+1$ or more doors.

ie $p+1 = \left\lfloor \frac{m-1}{n} \right\rfloor + 1 = \left\lfloor \frac{26-1}{5} \right\rfloor + 1 = 5+1 = 6.$

Do 5th one here.

4) How many persons must be chosen in order that atleast five
of them will have birthdays in the same calender month ?

Soln:- Let 'm' be the no. of persons. Number of months
over which the birthdays are distributed is $n = 12$.
The least no. of persons having birthday in the same month
is $5 = p+1$.

ie $p+1 = 5 \Rightarrow \left\lfloor \frac{m-1}{n} \right\rfloor + 1 = 5 \Rightarrow \left\lfloor \frac{m-1}{12} \right\rfloor = 4$

$\Rightarrow m-1 = 48 \Rightarrow \boxed{m = 49}$

$\therefore$ No. of persons is 49 (at the least).

5) Prove that if 30 dictionaries in a library contain a total of 61,327 pages, then atleast one of the dictionaries must have atleast 2045 pages.

Soln: Consider 61,327 pages as pigeons ie $m = 61,327$

and 30 dictionaries as pigeon holes ie $n = 30$.

By using the generalised pigeonhole principle, atleast 1 dictionary must contain $p+1$ or more pages.

ie $p+1 = \left\lfloor \frac{m-1}{n} \right\rfloor + 1 = \left\lfloor \frac{61,327-1}{30} \right\rfloor + 1 = \lfloor 2044.2 \rfloor + 1$

$= 2044 + 1$

$= 2045$.

This proves the required result.

6) If any $n+1$ numbers are chosen from 1 to $2n$, then Show that atleast one pair add to $2n+1$.

Soln:- Let us consider the following sets:

$A_1 = \{1, 2n\}$, $A_2 = \{2, 2n-1\}$, $A_3 = \{3, 2n-2\}$ .....

$A_{n-1} = \{n-1, n+2\}$, $A_n = \{n, n+1\}$.

These are the only sets containing 2 no's from 1 to $2n$ whose sum is $2n+1$.

Since there are only n sets, two of the $n+1$ chosen numbers belong to same set (by pigeon hole principle) and sum of these 2 no's $= 2n+1$.

P.T.O.

7) Prove that if 101 integers are selected from the set $S = \{1, 2, 3, \dots 200\}$, then atleast 2 of these are such that one divides the other.

Soln:- we have $S = \{1, 2, 3, \dots 200\}$.

Let $X = \{1, 3, 5 \dots 199\}$. $\Rightarrow |X| = 100$.

Any element n in the set S can be written as

$n = 2^k \times x$ where k is an integer ≥ 0 and $x \in X$.

Ex:- $1 = 2^0 \times 1$.

$2 = 2^1 \times 1$

$3 = 2^0 \times 3$

$4 = 2^2 \times 1$ etc

Considering 101 elements as pigeons and elements in X as pigeonhole, by pigeonhole principle, atleast 2 elements out of 101 must be selected to same $x \in X$.

Let a and b be such elements, then $a = 2^{k_1} x$, $b = 2^{k_2} x$, where k_1 and k_2 are integers ≥ 0.

Thus if $k_1 \leq k_2$ then a divides b.

if $k_1 > k_2$ then b divides a.
(ie $k_2 < k_1$)

8) Shirts numbered consecutively from 1 to 20 are worn by 20 students of a class. when any 3 of these students are chosen to be a debating team from the class, the sum of their shirt numbers is used as the code no. of the team. Show that if any 8 of the 20 are selected, then from these 8 we may form atleast 2 different teams having the same code no.

P.T.O.

~~Soln:- No. of ways of choosing 3 students out of 8 is $^8C_3 = 56$. This is the no of teams possible. Smallest code no. is $1+2+3 = 6$.~~

Soln:- From the 8 of the 20 students selected, the no. of teams of 3 students that can be formed is $^8C_3 = 56$.

Smallest possible code no. is $1+2+3=6$.

Largest " ———— " $18+19+20 = 57$.

$\therefore$ code no.'s vary from 6 to 57 (inclusive) and there are 52 in no.

Let us consider no. of teams as pigeons i.e. $m = 56$.

and the no. of codes as pigeon holes i.e. $n = 52$.

$\therefore$ By pigeon hole principle, atleast 2 different teams will have the same code no, i.e. atleast 1 code must be assigned to 2 or more teams.