MODULE - 1

FUNDAMENTALS OF LOGIC

PROPOSITION: It is a statement or declaration which, in a given content, can be either true or false, but not both.

Eg: i) France borders Belgium (with value true)
ii) 8 is an even number. (with value true)
iii) $4+3=7$ (value is true)
iv) 3 is an even number (with value false)
v) Mumbai is in Karnataka (———, ———)

- Note that all sentences are not propositions
  Eg: i) Consider a triangle ABC
  ii) What an amazing day!

- The truth or falsity of a proposition is called its truth value.

- If proposition p is true, its truth value is 1, if p is false, its truth value is 0.

LOGICAL CONNECTIVES

- With the given propositions, the new propositions are obtained by using phrases like, not, or, and, if then, if and only if etc. Such words or phrases are called logical connectives.

- The new propositions obtained by the use of logical connectives are called compound propositions.

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- The original propositions from which a compound proposition is obtained are called the components or the primitives of the compound propositions.
- Propositions that do not contain any logical connective are called simple propositions.

Negation: A proposition obtained by inserting the word 'not' at an appropriate place in a given proposition is called the negation of the given proposition. It is denoted as $\neg p$ for any proposition p.

Eg:- p: 3 is an odd number

$\neg p$: 3 is not an odd number.

- Truth table for negation

p	$\neg p$
0	1
1	0

Conjunction: A compound proposition obtained by combining 2 given propositions with 'and' in between is called the conjunction of the given propositions. It is denoted as $p \wedge q$.

Eg: p: $\sqrt{2}$ is an irrational number

q: $2+5=7$

$p \wedge q$: $\sqrt{2}$ is an irrational number and $2+5=7$.

- Truth table for conjunction $p \wedge q$

p	q	$p \wedge q$
0	0	0
0	1	0
1	0	0
1	1	1

Disjunction: A compound proposition obtained by combining 2 given propositions by inserting 'or' in between them is called the disjunction of the given propositions. It is denoted as $p \vee q$

Eg: p: Triangles have 3 sides

q: Bangalore is in Karnataka

$p \vee q$: Triangles have 3 sides or Bangalore is in Karnataka.

Truth table for disjunction

p	q	$p \vee q$
0	0	0
0	1	1
1	0	1
1	1	1

Exclusive Disjunction: -

If a compound proposition p or q is true ~~but not both~~ when either p or q is true but not both, then it is said to be exclusive disjunction, denoted by $p \underline{\vee} q$.

Eg:- p: $\sqrt{2}$ is an irrational number

q: $2+3=5$

$p \underline{\vee} q$: Either $\sqrt{2}$ is an irrational number or $2+3=5$, but not both

Truth table for exclusive disjunction.

p	q	$p \underline{\vee} q$
0	0	0
0	1	1
1	0	1
1	1	0

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Conditional (Implication)

A compound proposition obtained by combining 2 propositions by the words 'if' and 'then' at appropriate places is called a conditional or an implication.

- It is denoted as $p \to q$ (if p, then q)

Eg: p: I weigh more than 120 pounds

q: I shall enroll in an exercise class.

$p \to q$: If I weigh more than 120 pounds, then I shall enroll in an exercise class.

- Truth table for conditional:

p	q	$p \to q$
0	0	1
0	1	1
1	0	0
1	1	1

Bidirectional (Double implication)

Let p & q are the two propositions, then the conjunction (and) of the conditionals $p \to q$ and $q \to p$ is called the biconditional of p and q.

It is denoted as $p \leftrightarrow q$

$\therefore p \leftrightarrow q \Leftrightarrow (p \to q) \wedge (q \to p)$

$\to$ $p \leftrightarrow q$ is read as "If p, then q and if q, then p" (or) "p if and only if q".

Eg:- p: I shall enroll in an exercise class

q: I weigh more than 120 pounds.

$p \leftrightarrow q$: I shall enroll in an exercise class if and only if I weigh more than 120 pounds.

Truth table for bidirectional

p	q	$p \rightarrow q$	$q \rightarrow p$	$p \leftrightarrow q$
0	0	1	1	1
0	1	1	0	0
1	0	0	1	0
1	1	1	1	1

Problems

1) Let s, t, & u denote the following primitive statements.
   s: Phyllis goes out for a walk
   t: The moon is out
   u: It is snowing.
   Express the following compound propositions in words

i) $(t \wedge \neg u) \rightarrow s$: If the moon is out and it is not snowing, then Phyllis goes out for a walk.

ii) $t \rightarrow (\neg u \rightarrow s)$: If the moon is out, then it is not raining, Phyllis goes ~~out~~ for a walk.

iii) $\neg (s \leftrightarrow (u \vee t))$: It is not the case that Phyllis goes out for a walk if and only if it is snowing or the moon is out.

iv) $s \leftrightarrow t$: Phyllis goes out for a walk if and only if the moon is out.

v) $(u \wedge \neg t) \rightarrow \neg s$: If it is snowing and the moon is not out, then Phyllis does not go out for a walk.

vi) $u \wedge s$: It is snowing and phyllis goes out for a walk.

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2) Construct the truth tables for the following compound propositions: i) $p \wedge \neg q$ ii) $p \to \neg q$

Sol^n: i)

p	q	$\neg q$	$p \wedge \neg q$
0	0	1	0
0	1	0	0
1	0	1	1
1	1	0	0

ii)

p	q	$\neg q$	$p \to \neg q$
0	0	1	1
0	1	0	1
1	0	1	1
1	1	0	0

3) Let p and q be primitive statements for which the implication $p \to q$ is false. Determine the truth values of the following i) $p \wedge q$ ii) $\neg p \vee q$ iii) $q \to p$ iv) $\neg q \to \neg p$

Sol^n: Given that $p \to q$ is false. Recall the truth table of implication. $p \to q$ is false only when $p = 1$ and $q = 0$. So for $p \to q = 0$, $p = 1$ & $q = 0$

i) $p \wedge q \Rightarrow 1 \wedge 0 = 0$ $\therefore$ $p \wedge q$ is false.

ii) $\neg p \vee q \Rightarrow \neg(1) \vee 0 \Rightarrow 0 \vee 0 = 0$. $\therefore$ $\neg p \vee q$ is false.

iii) $q \to p \Rightarrow 0 \to 1 = 1$ $\therefore$ $q \to p$ is true

iv) $\neg q \to \neg p \Rightarrow \neg(0) \to \neg(1) \Rightarrow 1 \to 0 = 0$

$\therefore$ $\neg q \to \neg p$ is false.

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4) Let p, q, and r be propositions having truth values 0, 0, and 1 respectively. Find the truth values of the following:-

i) $(p \vee q) \vee r$ ii) $(p \wedge q) \wedge r$ iii) $(p \wedge q) \to r$

iv) $p \to (q \wedge r)$ v) $p \wedge (r \to q)$, vi) $p \to (q \to \neg r)$

Sol^n: Given $p=0$, $q=0$, $r=1$

i) $(p \vee q) \vee r \Rightarrow (0 \vee 0) \vee 1 \Rightarrow 0 \vee 1 = 1$ - true

ii) $(p \wedge q) \wedge r \Rightarrow 0 \wedge 1 = 0$ - false

iii) $(p \wedge q) \to r \Rightarrow 0 \to 1 = 1$ - true

iv) $p \to (q \wedge r) \Rightarrow 0 \to (0 \wedge 1) \Rightarrow 0 \to 0 = 1$ - true

v) $p \wedge (r \to q) \Rightarrow 0 \wedge (1 \to 0) \Rightarrow 0 \wedge 0 = 0$, - false.

vi) $p \to (q \to \neg r) \Rightarrow 0 \to (0 \to 0) \Rightarrow 0 \to 0 = 1$ - true

5) Find the possible truth values of p, q & r in the following cases

i) $p \to (q \vee r)$ is false.

i.e., $p \to q \vee r = 0$. This is possible when $p=1$ & $q \vee r = 0$. $q \vee r$ is false when $q=0$ & $r=0$

$\therefore$ the truth values are, $p=1$, $q=0$, & $r=0$.

ii) $p \wedge (q \to r)$ is true

i.e., $p \wedge (q \to r) = 1$. This is possible when $p=1$ and $q \to r = 1$. In 3 cases, $q \to r = 1$.

q	r	$q \to r$
0	0	1
0	1	1
1	1	1

$\therefore$ possible truth values are,

p	q	r
1	0	0
1	0	1
1	1	1

In all these cases, $p \wedge (q \to r)$ is true.

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6) Construct the truth tables for the following:
i) $(p \vee q) \wedge r$ ii) $p \vee (q \vee r)$ iii) $(p \wedge q) \rightarrow \neg r$
iv) $q \wedge (\neg r \rightarrow p)$ v) $p \wedge (\neg p \vee q)$

Soln:

i)

p	q	r	$p \vee q$	$(p \vee q) \wedge r$
0	0	0	0	0
0	0	1	0	0
0	1	0	1	0
0	1	1	1	1
1	0	0	1	0
1	0	1	1	1
1	1	0	1	0
1	1	1	1	1

Construct truth tables for all other compound propositions in the same way.

7) If a proposition q has truth value 1, determine all truth value assignments for the primitive propositions p, r & s for which the truth value of the following proposition is 1.

$$[q \rightarrow \{(\neg p \vee r) \wedge \neg s\}] \wedge \{\neg s \rightarrow (\neg r \wedge q)\}$$

Soln:

7th Soln: Given $q = 1$ and

$$[q \to \{(\neg p \vee r) \wedge (\neg s)\}] \wedge [\neg s \to (\neg r \wedge q)] = 1$$

$\therefore$ $q \to \{(\neg p \vee r) \wedge (\neg s)\} = 1$ —(1) and

$\neg s \to (\neg r \wedge q) = 1$ —(2)

Substituting $q = 1$,

$1 \to \{(\neg p \vee r) \wedge \neg s\} = 1$. This is true only when $\{(\neg p \vee r) \wedge \neg s\} = 1$

$\therefore$ $\neg p \vee r = 1$ and $\neg s = 1$

If $\neg s = 1$, $\boxed{s = 0}$

Substituting $\neg s = 1$ in (2),

$1 \to (\neg r \wedge q) = 1$. This is true only when

$\neg r \wedge q = 1$.

$\therefore$ $\neg r = 1$ & $q = 1$

$\Rightarrow \boxed{r = 0}$

We have $\neg p \vee r = 1$

$r = 0$ $\therefore$ $\neg p = 1$ $\Rightarrow \boxed{p = 0}$

$\therefore$ the truth values of the primitive propositions are $p = 0, q = 1, r = 0, s = 0$

8) Indicate how many rows are needed for the truth table of the compound proposition $(p \vee \neg q) \leftrightarrow \{(\neg r \wedge s) \to t\}$. Find the truth value of this proposition if p and q, r are true and s, t are false.

Soln: The given compound proposition has 5 primitives.

$\therefore$ the number of rows needed for the truth table is $\boxed{2^5 = 32}$.

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Given truth values are, $p=1, q=0, r=1, s=0, t=0$.
Substituting these in compound statements,

$$(p \vee \neg q) \leftrightarrow \{(\neg r \wedge s) \rightarrow t\}$$

$$\Rightarrow (1 \vee \neg 0) \leftrightarrow \{(\neg 1 \wedge 0) \rightarrow 0\}$$

$$\Rightarrow 1 \leftrightarrow (0 \rightarrow 0) \Rightarrow 1 \leftrightarrow 1$$

$$\Leftrightarrow (1 \rightarrow 1) \wedge (1 \rightarrow 1) = 1$$

$\therefore$ truth value of $(p \vee \neg q) \leftrightarrow \{(\neg r \wedge s) \rightarrow t\}$ is true.

9) Give the conjunction and disjunction of p & q and indicate the truth value.

i) p: 4 is a perfect square.
q: 27 is a prime number

ii) p: 5 is divisible by 2.
q: 7 is a multiple of 5

Soln: i) Conjunction $p \wedge q \Rightarrow$ 4 is a perfect square and 27 is a prime number.

$p=1$ and $q=0$ $\therefore$ $p \wedge q$ is false.

Disjunction $p \vee q \Rightarrow$ 4 is a perfect square or 27 is a prime number.

$p=1$ & $q=0$ $\therefore$ $p \vee q$ is true.

ii) Conjunction $p \wedge q \Rightarrow$ 5 is divisible by 2 and 7 is a multiple of 5.

$p=0$ & $q=0$ $\therefore$ $p \wedge q = 0$ or false.

Disjunction $p \vee q \Rightarrow$ 5 is divisible by 2 or 7 is a multiple of 5

$p=0$ & $q=0$ $\therefore$ $p \vee q = 0$ or false.

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10) From the information given in each of the following determine the truth value required.

i) $p \wedge q$ is false and p is true. Find truth value of q.

ii) $p \rightarrow q$ is true, q is false, find the truth value of p

iii) $p \leftrightarrow q$ is true, p is false, find truth value of q.

Soln: i) $p \wedge q = 0$ & $p = 1$

$\therefore 1 \wedge q = 0 \quad \therefore \boxed{q = 0}$

ii) $p \rightarrow q = 1$, $q = 0$. This is possible only when $\boxed{p = 0}$

iii) $p \leftrightarrow q = 1$ & $p = 0$. Recall the truth table of biconditional. $p \leftrightarrow q$ is true only when both p & q are true or both p & q are false. Given p is false $\therefore \boxed{q = 0}$

TAUTOLOGIES & CONTRADICTIONS

TAUTOLOGY: A compound proposition which is always true regardless of the truth values of its components is called a tautology.

CONTRADICTION: A compound proposition which is always false regardless of the truth values of its components is called a contradiction or absurdity.

CONTINGENCY: A compound proposition that can be true or false is called a contingency. Contingency is a compound proposition which is neither tautology nor a contradiction.

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1) Prove that the compound proposition $p \vee \neg p$ is a tautology and $p \wedge \neg p$ is a contradiction.

Soln: Construct the truth table as follows:-

p	$\neg p$	$\neg p \vee p$	$p \wedge \neg p$
0	1	1	0
1	0	1	0

Truth value of $\neg p \vee p$ is always true. $\therefore$ $\neg p \vee p$ is a tautology.

Truth value of $p \wedge \neg p$ is always false. $\therefore$ $\neg p \wedge p$ is a contradiction.

2) Show that, for any 2 propositions p and q

i) $(p \veebar q) \vee (p \leftrightarrow q)$ is a tautology

ii) $(p \veebar q) \wedge (p \leftrightarrow q)$ is a contradiction

iii) $(p \veebar q) \wedge (p \rightarrow q)$ is a contingency

iv) $p \rightarrow (p \vee q)$ is a tautology

v) $p \wedge (\neg p \wedge q)$ is a contradiction.

Soln: i)

p	q	$p \veebar q$	$p \leftrightarrow q$	$(p \veebar q) \vee (p \leftrightarrow q)$
0	0	0	1	1
0	1	1	0	1
1	0	1	0	1
1	1	0	1	1

All the truth values are true. $\therefore$ the given compound proposition is a tautology.

ii)

p	q	$p \underline{\vee} q$	$p \leftrightarrow q$	$(p \underline{\vee} q) \wedge (p \leftrightarrow q)$
0	0	0	1	0
0	1	1	0	0
1	0	1	0	0
1	1	0	1	0

All the truth values are zero. $\therefore$ it is contradiction.

iii)

p	q	$p \underline{\vee} q$	$p \to q$	$(p \underline{\vee} q) \wedge (p \to q)$
0	0	0	1	0
0	1	1	1	1
1	0	1	0	0
1	1	0	1	0

Truth values are 0's & 1's. $\therefore$ it is neither tautology nor contradiction. $\therefore$ it is contingency.

iv)

p	q	$p \vee q$	$p \to (p \vee q)$
0	0	0	1
0	1	1	1
1	0	1	1
1	1	1	1

All the truth values are true. $\therefore$ it is a tautology.

v)

p	q	$\neg p$	$\neg p \wedge q$	$p \wedge (\neg p \wedge q)$
0	0	1	0	0
0	1	1	1	0
1	0	0	0	0
1	1	0	0	0

All the truth values are false. $\therefore$ the given compound proposition is a contradiction.

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3) Show that the truth values of the following compound propositions are independent of the truth values of their components.

i) $\{p \wedge (p \rightarrow q)\} \rightarrow q$ ii) $(p \rightarrow q) \leftrightarrow (\neg p \vee q)$

Soln: Here, we need to prove that truth values of the compound propositions are always 1 or always 0 regardless of the truth values of p and q. Hence, we need to prove that i) & ii) are tautology or contradiction.

i)

p	q	$p \rightarrow q$	$p \wedge (p \rightarrow q)$	$p \wedge (p \rightarrow q) \rightarrow q$
0	0	1	0	1
0	1	1	0	1
1	0	0	0	1
1	1	1	1	1

All the truth values are 1 regardless of the truth values of p & q. Hence, proved.

ii)

p	q	$p \rightarrow q$	$\neg p$	$\neg p \vee q$	$(p \rightarrow q) \leftrightarrow (\neg p \vee q)$
0	0	1	1	1	1
0	1	1	1	1	1
1	0	0	0	0	1
1	1	1	0	1	1

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4) Find the possible truth values of p, q, r, s, t for which the following are contradictions.

So i) $[(p \wedge q) \wedge r] \rightarrow (s \vee t)$ ii) $[p \wedge (q \wedge r)] \rightarrow (s \veebar t)$

Soln: i) Given $[(p \wedge q) \wedge r] \rightarrow (s \vee t) = 0$

$[p \rightarrow q = 0$ only when $p = 1$ & $q = 0]$

$\therefore (p \wedge q) \wedge r = 1$ and $s \vee t = 0$

$(p \wedge q) \wedge r = 1$ only when $p \wedge q = 1$ and $\boxed{r = 1}$

$p \wedge q = 1$ only when $\boxed{p = q = 1}$

$s \vee t = 0$ only when $\boxed{s = t = 0}$

$\therefore$ possible truth values are $p = 1, q = 1, r = 1, s = 0, t = 0$

ii) Given $[(p \wedge (q \wedge r)] \rightarrow (s \veebar t) = 0$

$\therefore [p \wedge (q \wedge r)] = 1$ and $s \veebar t = 0$

$[p \wedge (q \wedge r)] = 1$ only when $p = q = r = 1$

$s \veebar t = 0$ only when $s = t = 0$ or $s = t = 1$.

$\therefore$ possible truth values are,

p	q	r	s	t
1	1	1	1	1
1	1	1	0	0

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LOGICAL EQUIVALENCE

Two propositions u and v are said to be logically equivalent whenever u & v have the same truth value or the biconditional $u \Leftrightarrow v$ is always a tautology.

- The logical equivalence is denoted by $\Leftrightarrow$

Problems

1) For any two propositions p & q, prove that $p \rightarrow q$ is logically equivalent to $\neg p \vee q$.

Soln: We need to prove $(p \rightarrow q) \Leftrightarrow (\neg p \vee q)$

p	q	$p \rightarrow q$	$\neg p$	$\neg p \vee q$
0	0	1	1	1
0	1	1	1	1
1	0	0	0	0
1	1	1	0	1

Column 3 & 5 have same truth values.

$\therefore (p \rightarrow q) \Leftrightarrow (\neg p \vee q)$

2) Prove that $p \veebar q$ and $(p \vee q) \wedge (\neg p \vee \neg q)$ are logically equivalent.

p	q	$p \veebar q$	$p \vee q$	$\neg p$	$\neg q$	$(p \vee q) \wedge (\neg p \vee \neg q)$
0	0	0	0	1	1	0
0	1	1	1	1	0	1
1	0	1	1	0	1	1
1	1	0	1	0	0	0

Column 3 and 7 have same truth values.

Hence they are logically equivalent.

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3) Prove that for any propositions p, q, r,

$$[p \to (q \wedge r)] \Leftrightarrow [(p \to q) \wedge (p \to r)]$$

Soln:

p	q	r	$q \wedge r$	$p \to (q \wedge r)$	$p \to q$	$p \to r$	$(p \to q) \wedge (p \to r)$
0	0	0	0	1	1	1	1
0	0	1	0	1	1	1	1
0	1	0	0	1	1	1	1
0	1	1	1	1	1	1	1
1	0	0	0	0	0	0	0
1	0	1	0	0	0	1	0
1	1	0	0	0	1	0	0
1	1	1	1	1	1	1	1
				*			*

Column 5 and 8 have same truth values. ∴ they are logically equivalent.

4) Prove that, for any 3 propositions p, q, r,

$$[(p \vee q) \to r] \Leftrightarrow \{(p \to r) \wedge (q \to r)\}$$

p	q	r	$p \vee q$	$(p \vee q) \to r$	$p \to r$	$q \to r$	$(p \to r) \wedge (q \to r)$
0	0	0	0	1	1	1	1
0	0	1	0	1	1	1	1
0	1	0	1	0	1	0	0
0	1	1	1	1	1	1	1
1	0	0	1	0	0	1	0
1	0	1	1	1	1	1	1
1	1	0	1	0	0	0	0
1	1	1	1	1	1	1	1
				*			*

Column 5 and 8 have same truth values. ∴ they are logically equivalent.

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5) Show that $p \wedge (\neg q \vee r)$ and $p \vee (q \wedge \neg r)$ are not logically equivalent.

p	q	r	$\neg q$	$\neg q \vee r$	$p \wedge (\neg q \vee r)$	$\neg r$	$q \wedge \neg r$	$p \vee (q \wedge \neg r)$
0	0	0	1	1	0	1	0	0
0	0	1	1	1	0	0	0	0
0	1	0	0	0	0	1	1	1
0	1	1	0	1	0	0	0	0
1	0	0	1	1	1	1	0	1
1	0	1	1	1	1	0	0	1
1	1	0	0	0	0	1	1	1
1	1	1	0	1	1	0	0	1
					*			*

Column 6 and 9 have different truth values.

$$\therefore \; p \wedge (\neg q \vee r) \not\Leftrightarrow p \vee (q \wedge \neg r)$$

THE LAWS OF LOGIC

For any primitive statements p, q, r, any tautology T_0, and any contradiction F_0, the following laws hold good.

1) Law of double negation $\quad \neg\neg p \Leftrightarrow p$

2) DeMorgan's Laws
   i) $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$
   ii) $\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$

3) Commutative Laws
   i) $(p \vee q) \Leftrightarrow (q \vee p)$
   ii) $(q \quad p) \Leftrightarrow (q \wedge p)$

4) Associative Laws
   i) $p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$
   ii) $p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$

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5) Distributive Laws

i) $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$

ii) $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

6) Idempotent Laws

i) $(p \vee p) \Leftrightarrow p$

ii) $(p \wedge p) \Leftrightarrow p$

7) Identity Laws

i) $(p \vee F_0) \Leftrightarrow p$

ii) $(p \wedge T_0) \Leftrightarrow p$

8) Inverse Laws

i) $(p \vee \neg p) \Leftrightarrow T_0$

ii) $(p \wedge \neg p) \Leftrightarrow F_0$

9) Domination Laws

i) $p \vee T_0 \Leftrightarrow T_0$

ii) $p \wedge F_0 \Leftrightarrow F_0$

10) Absorption Laws

i) $p \vee (p \wedge q) \Leftrightarrow p$

ii) $p \wedge (p \vee q) \Leftrightarrow p$

[Prove all the laws using truth table]

<u>Some important relations</u>

i) $p \to q \Leftrightarrow \neg p \vee q$

ii) $\neg(p \to q) \Leftrightarrow p \wedge \neg q$

[prove this using truth table]

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TRANSITIVE AND SUBSTITUTION RULES

1) If u, v, w are propositions such that $u \Leftrightarrow v$ and $v \Leftrightarrow w$, then $u \Leftrightarrow w$. This is known as "Transitive rule"

2) Suppose that a compound proposition 'u' is a tautology and p is a component of u. If we replace each occurrence of p in u by a proposition q, then the resulting compound proposition v is also a tautology. This is called a "Substitution Rule".

3) Suppose that u is a compound proposition which contains a proposition p. Let q be a proposition such that $q \Leftrightarrow p$. Suppose we replace one or more occurrences of p by q and obtain a compound proposition, v, then $v \Leftrightarrow u$. This is also called a "Substitution Rule"

Law of negation for a conditional

1) Let x be a specific number. Write the negation of the following conditional.

"If x is an integer, then x is a rational number".

Solⁿ: Let p: x is an integer
q: x is a rational number.

negation of $p \to q$ $\quad \neg(p \to q) \Leftrightarrow \neg(\neg p \vee q)$

$\Leftrightarrow p \wedge \neg q$

(using DeMorgan's Law)

$\therefore \neg(p \to q)$ is "x is an integer and x is not a rational number"

2) Let x be a specified number. Write the negation of the following proposition.
"If x is not a real number, then it is not a rational number and not an irrational number"

Sol^n: Let p: x is a real number
q: x is a rational number
r: x is an irrational number

Given that $\neg p \rightarrow (\neg q \wedge \neg r)$. Its negation is

$$\neg(\neg p \rightarrow (\neg q \wedge \neg r))$$
$$\Leftrightarrow \neg(\neg\neg p \vee (\neg q \wedge \neg r))$$
$$\Leftrightarrow \neg(p \vee (\neg q \wedge \neg r)) \text{ — law of double negation}$$
$$\Leftrightarrow \neg p \wedge \neg(\neg q \wedge \neg r) \text{ — DeMorgan's law}$$
$$\Leftrightarrow \neg p \wedge (q \vee r) \text{ — DeMorgan's law \& law of double negation.}$$

$\therefore$ the negation of the given proposition is
"x is not a real number and it is a rational number or an irrational number"

Problems on Logical Equivalence

1) Prove the following logical equivalences without using truth tables.

i) $p \vee [p \wedge (p \vee q)] \Leftrightarrow p$

ii) $[p \vee q \vee (\neg p \wedge \neg q \wedge r)] \Leftrightarrow (p \vee q \vee r)$

iii) $[(\neg p \vee \neg q) \rightarrow (p \wedge q \wedge r)] \Leftrightarrow p \wedge q$.

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Soln: i) $p \vee [p \wedge (p \vee q)]$

$\Leftrightarrow p \vee p$ — Absorption Law

$\Leftrightarrow p$ — Idempotent Law.

ii) $[p \vee q \vee (\neg p \wedge \neg q \wedge r)]$

$\Leftrightarrow (p \vee q) \vee (\neg(p \vee q) \wedge r)$ — DeMorgan's law

$\Leftrightarrow \underbrace{[(p \vee q) \vee \neg(p \vee q)]}$ $\wedge$ $(p \vee q) \vee r$ — Distributive law (expansion)

$\Leftrightarrow T_0 \wedge (p \vee q \vee r)$ — Inverse Law

$\Leftrightarrow (p \vee q \vee r)$ — Identity law

$\therefore [p \vee q \vee (\neg p \wedge \neg q \wedge r)] \Leftrightarrow (p \vee q \vee r)$

iii) $[(\neg p \vee \neg q) \rightarrow (p \wedge q \wedge r)]$

$\Leftrightarrow \neg(p \wedge q) \rightarrow (p \wedge q \wedge r)$ — DeMorgan's law

$\Leftrightarrow \neg\neg(p \wedge q) \vee (p \wedge q \wedge r)$ — The fact that $p \rightarrow q \Leftrightarrow \neg p \vee r$

$\Leftrightarrow (p \wedge q) \vee (p \wedge q \wedge r)$ — Law of double negation

$\Leftrightarrow x \vee (x \wedge r)$ where $x = p \wedge q$

$\Leftrightarrow x$ — Absorption law

$= (p \wedge q)$

$\therefore [(\neg p \vee \neg q) \rightarrow (p \wedge q \wedge r)] \Leftrightarrow (p \wedge q)$

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2) Prove the following logical equivalences

i) $[(p\vee q)\wedge(p\vee\neg q)]\vee q \Leftrightarrow p\vee q$

$[(p\vee q)\wedge(p\vee\neg q)]\vee q$

$\Leftrightarrow [p\vee(q\wedge\neg q)]\vee q$ — Distributive law

$\Leftrightarrow (p\vee F_0)\vee q$ — Inverse law

$\Leftrightarrow \underline{p\vee q}$ — Identity law

ii) $(p\to q)\wedge[\neg q\wedge(r\vee\neg q)] \Leftrightarrow \neg(q\vee p)$

$(p\to q)\wedge[(\neg q\wedge(r\vee\neg q))]$

$\Leftrightarrow (p\to q)\wedge[\neg q\wedge(\neg q\vee r)]$ — Commutative law

$\Leftrightarrow (p\to q)\wedge[\neg q]$ — Absorption law

$\Leftrightarrow (\neg p\vee q)\wedge(\neg q)$ — The fact that $p\to q\Leftrightarrow\neg p\vee q$

$\Leftrightarrow (\neg p\wedge\neg q)\vee(q\wedge\neg q)$ — Distributive law

$\Leftrightarrow (\neg p\wedge\neg q)\vee F_0$ — Inverse law

$\Leftrightarrow \neg(p\vee q)\vee F_0$ — Demorgan's law

$\Leftrightarrow \neg(p\vee q)$ — Identity law.

iii) $p\to(q\to r) \Leftrightarrow (p\wedge q)\to r$

$p\to(q\to r)$

$\Leftrightarrow \neg p\vee(q\to r)$ — $\because$ $p\to q\Leftrightarrow\neg p\vee q$

$\Leftrightarrow \neg p\vee(\neg q\vee r)$ — ——— " ———

$\Leftrightarrow (\neg p\vee\neg q)\vee r$ — Associative law

$\Leftrightarrow \neg(p\wedge q)\vee r$ — Demorgan's law

$\Leftrightarrow (p\wedge q)\to r$ — $\because$ $\neg p\vee q\Leftrightarrow p\to q$

iv) $[\neg p\wedge(\neg q\wedge r)]\vee(q\wedge r)\vee(p\wedge r)\Leftrightarrow r$

$[\neg p\wedge(\neg q\wedge r)]\vee(q\wedge r)\vee(p\wedge r)$

$\Leftrightarrow [(\neg p\wedge\neg q)\wedge r]\vee(r\wedge q)\vee(r\wedge p)$ — Commutative & associative laws.

$\Leftrightarrow (\neg(p\vee q)\wedge r)\vee[r\wedge(q\vee p)]$ — Demorgan's law & Distributive law.

$\Leftrightarrow [r\wedge(\neg(p\vee q))]\vee[r\wedge(q\vee p)]$ — Commutative law

$\Leftrightarrow r \wedge [\neg(p \vee q) \vee (p \vee q)]$ - Distributive law

$\Leftrightarrow r \wedge T_0$ - Inverse law

$\Leftrightarrow r$ - Identity law

v) $\neg[\{(p \vee q) \wedge r\} \rightarrow \neg q] \Leftrightarrow \neg(\neg\{(p \vee q) \wedge r\} \vee \neg q)$
$\Leftrightarrow q \wedge r$

LHS,

$\neg[\{(p \vee q) \wedge r\} \rightarrow \neg q]$

$\Leftrightarrow (p \vee q) \wedge r \wedge q$ — $\because \neg(p \rightarrow q) \Leftrightarrow p \wedge \neg q$ & law of double negation

$\Leftrightarrow (p \vee q) \wedge (r \wedge q)$

$\Leftrightarrow [(p \vee q) \wedge q] \wedge r$ - Commutative and associative

$\Leftrightarrow q \wedge r$ — (1) - Absorptive law

Consider LHS,

$\neg\{(p \vee q) \wedge r\} \rightarrow \neg q$

$\Leftrightarrow \neg[\neg\{(p \vee q) \wedge r\} \vee \neg q]$ - $\because p \rightarrow q \Leftrightarrow \neg p \vee q$

$\Leftrightarrow q \wedge r$ by (1) Hence proved.

3) Prove that $[(p \vee q) \wedge \neg\{\neg p \wedge (\neg q \vee \neg r)\}] \vee (\neg p \wedge \neg q) \vee (\neg p \wedge \neg r)$ is a tautology without using truth table

Soln:

$[(p \vee q) \wedge \neg\{\neg p \wedge (\underbrace{\neg q \vee \neg r}))\}] \vee (\underbrace{\neg p \wedge \neg q}) \vee (\underbrace{\neg p \wedge \neg r})$

$\Leftrightarrow [(p \vee q) \wedge \underbrace{\neg\{\neg p \wedge \neg(q \wedge r)\}}] \vee \neg(p \vee q) \vee \neg(p \vee r)$ - DeMorgan's law

$\Leftrightarrow \underbrace{[(p \vee q) \wedge \{p \vee (q \wedge r)\}]} \vee \neg(p \vee q) \vee \neg(p \vee r)$ - DeMorgan's law

$\Leftrightarrow \{p \vee [q \wedge (q \wedge r)]\} \vee \neg(p \vee q) \vee \neg(p \vee r)$ — Distributive

$\Leftrightarrow \{p \vee (q \wedge r)\} \vee \neg\{(p \vee q) \wedge (p \vee r)\}$ — Idempotent and Demorgans law.

$\Leftrightarrow \{p \vee (q \wedge r)\} \vee \neg\{p \vee (q \wedge r)\}$ — Distributive law

$= u \vee \neg u$ where $u = p \vee (q \wedge r)$

$\Leftrightarrow T_0$ — By Inverse law.

$\therefore$ the given compound proposition is a tautology.

DUALITY

[Refer question bank solution for definition and problems] (At the end of this notes)

CONVERSE, INVERSE AND CONTRAPOSITIVE

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Consider a conditional $p \to q$, then,

1) $q \to p$ is called the converse of $p \to q$
2) $\neg p \to \neg q$ is called the inverse of $p \to q$
3) $\neg q \to \neg p$ is called the contrapositive of $p \to q$.

Truth Table:

p	q	$p \to q$	$q \to p$	$\neg p$	$\neg q$	$\neg p \to \neg q$	$\neg q \to \neg p$
0	0	1	1	1	1	1	1
0	1	1	0	1	0	0	1
1	0	0	1	0	1	1	0
1	1	1	1	0	0	1	1

$\therefore$ $p \to q \Leftrightarrow \neg q \to \neg p$ i.e., Conditional $\Leftrightarrow$ Contrapositive

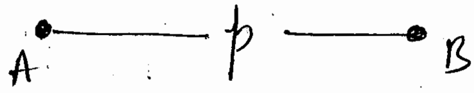
and $q \to p \Leftrightarrow \neg p \to \neg q$ i.e., Converse $\Leftrightarrow$ inverse.

APPLICATION TO SWITCHING NETWORKS

- Switching network is made up of wires and switches connecting two terminals, say A & B.
- Each switch is open (so that no current flows through it) or closed (so that the current flows through it)
- Value 0 – when open
  Value 1 – when closed
- We can relate switches and their states (open or closed) with the propositions and their truth values.

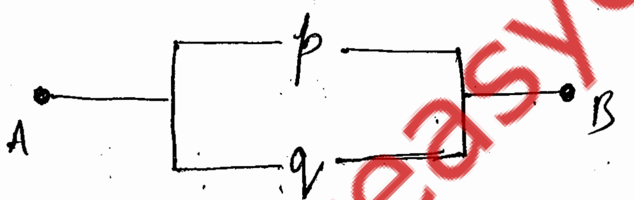
Examples

1.



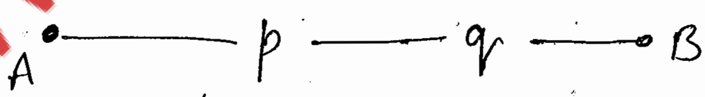
This has only one switch p. If it is closed, value is 1, otherwise 0.

2.



This switching n/w is a <u>parallel network</u> consisting of 2 switches p & q. Current flows from terminal A to B if <u>p or q or both</u> are closed. [current flows even if one switch is open]. This n/w is represented as <u>$p \vee q$</u>.

3.



This switching n/w is a <u>series network</u>. This is consisting of 2 switches p & q in which the current flows from terminal A to B only when both p & q are closed. This n/w is represented as <u>$p \wedge q$</u>.

Problems - [Refer question bank and solution at the end of this notes]

THE CONNECTIVES NAND, NOR.

- The compound proposition $\neg(p \wedge q)$ is read as "Not p and q" and is denoted by $(p \uparrow q)$. The symbol '$\uparrow$' is called the NAND connective.
- The compound proposition $\neg(p \vee q)$ is read as "Not p or q" and is denoted by $(p \downarrow q)$. The symbol '$\downarrow$' is called the NOR connective.

$$\therefore \boxed{\begin{array}{l} \text{NAND} - (p \uparrow q) = \neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q \\ \text{NOR} - (p \downarrow q) = \neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q \end{array}}$$

$\therefore$ $p \uparrow q$ & $p \downarrow q$ are duals of each other.

Truth table:

p	q	$p \uparrow q$	$p \downarrow q$
0	0	1	1
0	1	1	0
1	0	1	0
1	1	0	0

Problems:-

1) For any propositions p, q, prove the following.
   i) $\neg(p \downarrow q) \Leftrightarrow (\neg p \uparrow \neg q)$
   ii) $\neg(p \uparrow q) \Leftrightarrow (\neg p \downarrow \neg q)$

i) $\neg(p \downarrow q) \Leftrightarrow \neg\{\neg(p \vee q)\}$

$\Leftrightarrow \neg(\neg p \wedge \neg q)$ — DeMorgan's law

$\Leftrightarrow \neg p \uparrow \neg q$ Hence proved

ii) $\neg(p \uparrow q) \Leftrightarrow \neg(\neg(p \wedge q)) \Leftrightarrow \neg(\neg p \vee \neg q)$

$\Leftrightarrow \neg p \downarrow \neg q$. Hence proved.

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2) Express the following propositions in terms of only NAND and NOR connectives.

i) $\neg p$ ii) $p \wedge q$ iii) $p \vee q$ iv) $p \rightarrow q$ v) $p \leftrightarrow q$

Soln: By Idempotent law, $p \wedge q \Leftrightarrow p$ & $p \vee p \Leftrightarrow p$.

i) $\neg p \Leftrightarrow \neg(p \wedge q) \Leftrightarrow p \uparrow q$ or

$\neg p \Leftrightarrow \neg(p \vee q) \Leftrightarrow p \downarrow q$ —— (1)

ii) $p \wedge q \Leftrightarrow \neg\neg(p \wedge q)$ - Law of double negation

$\Leftrightarrow \neg(\neg p \vee \neg q)$ - Demorgan's law

$\Leftrightarrow (\neg p \vee \neg q) \uparrow (\neg p \vee \neg q)$ - by (1)

$\Leftrightarrow \neg(p \wedge q) \uparrow \neg(p \wedge q)$ - Demorgan's

$\Leftrightarrow (p \uparrow q) \uparrow (p \uparrow q)$ —— (2)

iii) $p \vee q \Leftrightarrow \neg\neg(p \vee q)$

$\Leftrightarrow \neg(\neg p \wedge \neg q)$

$\Leftrightarrow (\neg p \wedge \neg q) \downarrow (\neg p \wedge \neg q)$

$\Leftrightarrow (p \downarrow q) \downarrow (p \downarrow q)$ —— (3)

iv) $p \rightarrow q \Leftrightarrow \neg p \vee q \Leftrightarrow \neg\neg(\neg p \vee q)$

$\Leftrightarrow \neg(p \wedge \neg q)$

$\Leftrightarrow p \uparrow \neg q$

$\Leftrightarrow p \uparrow (q \uparrow q)$ - by (1) —— (4)

v) $p \leftrightarrow q \Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p)$

$\Leftrightarrow [(p \rightarrow q) \uparrow (q \rightarrow p)] \uparrow [(p \rightarrow q) \uparrow (q \rightarrow p)]$ - by (2)

$\Leftrightarrow [\{p \uparrow (q \uparrow q)\} \uparrow \{q \uparrow (p \uparrow p)\}] \uparrow$

$[\{p \uparrow (q \uparrow q)\} \uparrow \{q \uparrow (p \uparrow p)\}]$ - by (4)

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3) For any propositions $p, q, \& r$, prove the following.

i) $p \uparrow (q \uparrow r) \Leftrightarrow \neg p \vee (q \wedge r)$ ii) $(p \uparrow q) \uparrow r \Leftrightarrow (p \wedge q) \vee \neg r$

iii) $p \downarrow (q \downarrow r) \Leftrightarrow \neg p \wedge (q \vee r)$ iv) $(p \downarrow q) \downarrow r \Leftrightarrow (p \vee q) \wedge \neg r$

Soln: i) $p \uparrow (q \uparrow r) \Leftrightarrow \neg [p \wedge \{\neg (q \wedge r)\}]$

$\Leftrightarrow \neg p \vee (q \wedge r)$ - Demorgan's & double negation

ii) $(p \uparrow q) \uparrow r \Leftrightarrow \neg [\{\neg (p \wedge q)\} \wedge r]$

$\Leftrightarrow (p \wedge q) \vee \neg r$ - Demorgan's & double negation.

iii) $p \downarrow (q \downarrow r) \Leftrightarrow \neg [p \vee \{\neg (q \vee r)\}]$

$\Leftrightarrow \neg p \wedge (q \vee r)$ —— " ——

iv) $(p \downarrow q) \downarrow r \Leftrightarrow \neg [\{\neg (p \vee q)\} \vee r]$

$\Leftrightarrow (p \vee q) \wedge \neg r$ —— " ——

Note: $p \uparrow (q \uparrow r) \not\Leftrightarrow (p \uparrow q) \uparrow r$ & $p \downarrow (q \downarrow r) \not\Leftrightarrow (p \downarrow q) \downarrow r$.

$\therefore$ $\uparrow$ and $\downarrow$ are not associative.

LOGICAL IMPLICATION:-

Consider a conditional $p \to q$ where p & q are related in a way that the truth value of q depends on the truth value of p and vice versa. Such conditionals are called as hypothetical or implicative statements.

When a hypothetical statement $p \to q$ is such that q is true whenever p is true, then we say that p logically implies q. This is symbolically represented as $p \Rightarrow q$. where '$\Rightarrow$' denotes implication

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when $p \Rightarrow q$ is true always, then $p \to q$ is a tautology. So, in this case we say that the conditional $p \to q$ is a logical implication.

If $p \to q$ is not a tautology, then $p \to q$ is not a logical implication. So, we write $p \not\Rightarrow q$ [p does not imply q], which means q need not be true when p is true.

NECESSARY AND SUFFICIENT CONDITIONS.

Consider two propositions p & q whose truth values are interrelated. Then, for $p \to q$ to be a logical implication, the following statements hold good.

i) $p \Rightarrow q$
ii) p is sufficient for q
iii) q is necessary for p.

1) Give the necessary and sufficient condition for the following conditionals.

a) If a quadrilateral is a parallelogram, then its diagonals bisect each other.

b) If a real number x^2 is greater than zero, then x is not equal to zero

c) If a triangle is not isosceles then it is not equilateral.

Soln:

a) p: quadrilateral is a parallelogram
q: Quadrilaterals diagonals bisect each other.
∴ the given statement can be symbolically represented as $p \to q$.

w.k.t, p is sufficient for q and
q is necessary for p.

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∴ A <u>necessary condition</u> for a quadrilateral to be a parallelogram is that its diagonals bisect each other.

<u>A sufficient condition</u> for the diagonals of a quadrilateral to bisect each other is that the quadrilateral is a parallelogram.

[$|||^{ly}$ do it for (b) & (c)]

2) Give the contrapositive of $[p \to (q \to r)]$ with
a) only one occurrence of $\to$
b) no occurrence of $\to$

<u>Soln</u>: a) Contrapositive of $p \to (q \to r)$ is

$$\neg (q \to r) \to \neg p \Leftrightarrow \neg\neg(q \to r) \vee \neg p$$

$$\Leftrightarrow (q \to r) \vee \neg p.$$

b) Considering the result of (a),

$$(q \to r) \vee \neg p \Leftrightarrow (\neg q \vee r) \vee \neg p$$

$$\Leftrightarrow \neg p \vee \neg q \vee r.$$

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RULES OF INFERENCE

Rule of Inference	Logical Implication	Name of the rule
1) p q $\therefore p \wedge q$	—	Rule of conjunction
2) $p \wedge q$ $\therefore p$	$(p \wedge q) \to p$	Rule of conjunctive simplification
3) p $\therefore p \vee q$	$p \to (p \vee q)$	Rule of Disjunctive amplification
4) p $p \to q$ $\therefore q$	$[p \wedge (p \to q)] \to q$	Rule of Detatchment or Modus Ponens.
5) $p \to q$ $\neg q$ $\therefore \neg p$	$[(p \to q) \wedge \neg q] \to \neg p$	Modus Tollens.
6) $p \to q$ $q \to r$ $\therefore p \to r$	$[(p \to q) \wedge (q \to r)] \to (p \to r)$	Law of Syllogism.
7) $p \vee q$ $\neg p$ $\therefore q$	$[(p \vee q) \wedge \neg p] \to q$	Rule of disjunctive syllogism.
8) $\neg p \to F_0$ $\therefore p$	$(\neg p \to F_0) \to p$	Rule of contradiction.

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ARGUMENT: PREMISES | HYPOTHESIS, CONCLUSION

Consider a set of propositions $p_1, p_2, \ldots, p_n$ and q. Then a compound proposition of the form $[p_1 \wedge p_2 \wedge p_3 \wedge \ldots \wedge p_n] \rightarrow q$ is called an argument.
Here $p_1, p_2, \ldots p_n$ are called the premises | hypothesis of the argument and q is called the conclusion of the argument.

- This argument is represented in a tabular form.

$$\begin{array}{l} p_1 \\ p_2 \\ p_3 \\ \vdots \\ p_n \\ \hline \therefore q \end{array}$$

- The argument is said to be valid if whenever each of the premises $p_1, p_2, \ldots p_n$ is true then the conclusion q is true.
or in other words, if $(p_1 \wedge p_2 \ldots \wedge p_n) \rightarrow q$ is valid when $(p_1 \wedge p_2 \wedge \ldots \wedge p_n) \Rightarrow q$.

- In an argument, the premises are always considered to be true (and hence the name hypothesis), whereas the conclusion may be true or false
- The conclusion is true only in the case of a valid argument.
- We use the rules of inference to establish the validity of the arguments.

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1) Test whether the following argument is valid.

If Sachin hits a century, then he gets a free car

Sachin hits a century

$\therefore$ Sachin gets a free car.

Soln: Let p: Sachin hits a century

q: Sachin gets a free car

$\therefore$ the given argument is symbolically represented as

$$\begin{array}{l} p \to q \\ p \\ \hline \therefore q \end{array}$$

In the view of Modus Ponens rules, this is a valid argument.

2) Test whether the following argument is valid.

If Sachin hits a century, then he gets a free car

Sachin gets a free car

$\therefore$ Sachin has hit a century.

Let p: Sachin hits a century

q: Sachin gets a free car.

$\therefore$ Symbolically,

$$\begin{array}{l} p \to q \\ q \\ \hline \therefore p \end{array}$$

Recall the truth table of conditional.

If $p \to q$ is true, & q is true, p can be true or false. i.e., $0 \to 1 = 1$ & $1 \to 1 = 1$

$\therefore$ we cannot say p is true. Also there is no rule of inference that asserts p is true.

$\therefore$ it is not a valid argument

[Sachin might have got a free car as his birthday gift]

3) Test the validity of the following argument.

I will become famous or I will not become a musician
I will become a musician

$\therefore$ I will become famous.

Soln: Let p: I will become famous
q: I will become a musician

Symbolically, $\begin{array}{l} p \vee \neg q \\ q \\ \hline \therefore p \end{array} \iff \begin{array}{l} \neg q \vee p \\ q \\ \hline \therefore p \end{array}$

This is equivalent to $\begin{array}{l} q \to p \\ q \\ \hline \therefore p \end{array}$ $[\because p \to q \iff \neg p \vee q]$

$\therefore$ In the view of Modus Ponens, the argument is valid.

4) Test the validity of the following argument

Rita is baking cake

If Rita is baking cake, then she is not practicing her flute.

If Rita is not practicing her flute, then her father will not buy her a car.

Therefore, Rita's father will not buy her a car.

Soln: Let p: Rita is baking cake
q: Rita is practicing her flute
r: Rita's father will buy her a car.

Symbolically, $\begin{array}{l} p \to \neg q \\ q \to \neg r \\ \hline \therefore \neg r \end{array}$

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Steps	Reasons
1) $p \to \neg q$	Premise
2) $\neg q \to \neg r$	Premise
3) $p \to \neg r$	Step (1),(2) and law of syllogism
4) p	Premise
5) $\therefore \neg r$	Steps (3) & (4), & Modus Ponens.

$\therefore$ the argument is valid.

5) Test whether the arguments are valid.

i)
$$\begin{array}{l} p \to r \\ r \to s \\ t \vee \neg s \\ \neg t \vee u \\ \neg u \\ \hline \therefore \neg p \end{array}$$

ii)
$$\begin{array}{l} \neg p \leftrightarrow q \\ q \to r \\ \neg r \\ \hline \therefore p \end{array}$$

iii)
$$\begin{array}{l} (\neg p \vee \neg q) \to (r \wedge s) \\ r \to t \\ \neg t \\ \hline \therefore p \end{array}$$

[For solution, refer Question Bank and solution at the end of this notes.]

iv)
$$\begin{array}{l} p \to q \\ q \to (r \wedge s) \\ \neg r \to (\neg t \vee u) \\ p \wedge t \\ \hline \therefore u \end{array}$$

v)
$$\begin{array}{l} a \vee (b \to a) \\ \neg a \wedge c \\ \hline \therefore \neg b \end{array}$$

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6) Demonstrate the validity of the following argument

$$\begin{array}{l} p \to r \\ \neg p \to q \\ q \to s \\ \hline \therefore \neg r \to s \end{array}$$

Sol^n:

Steps	Reason
1) $p \to r$	Premise
2) $\neg r \to \neg p$	Contrapositive of (1)
3) $\neg p \to q$	Premise
4) $\neg r \to q$	Steps (2) & (3) and law of syllogism.
5) $q \to s$	Premise.
6) $\therefore \neg r \to s$	Steps (4) & (5) and law of syllogism.

$\therefore$ the given argument is valid.

7) $m \vee n$
$\neg m \wedge F_0$
$\therefore n$

Sol^n:

Steps	Reason
1) $m \vee n$	Premise
2) $\neg m \wedge F_0$	Premise
3) $\neg m$	Step (2) & conjunctive simplification
4) $\therefore n$	Steps (1) & (3) & disjunctive syllogism.

$\therefore$ the argument is valid.

8) $a \to b$
$a \vee (c \wedge d)$
$\neg b \wedge \neg e$
$\therefore c$

Sol^n:

Steps	Reason
1) $a \to b$	Premise
2) $a \vee (c \wedge d)$	Premise
3) $\neg b \wedge \neg e$	Premise
4) $\neg b$	Step (3) & conjunctive simplification
5) $\neg a$	Steps (1) & (4) & Modus Tollens
6) $c \wedge d$	Steps (2) & (5) & disjunctive syllogism
7) $\therefore c$	Step (6) & conjunctive simplification.

9) $(k \vee l) \to (m \vee n)$
$(m \vee n) \to f \wedge p$
k

$\therefore f$

Soln:

Steps	Reasons
1) $(k \vee l) \to (m \vee n)$	Premise
2) $(m \vee n) \to (f \wedge p)$	Premise
3) k	Premise
*4) $k \vee l$	Step(3) & disjunctive amplification
5) $(m \vee n)$	Steps(1) & (4) & Modus Ponens
6) $(f \wedge p)$	Steps(2) & (5) & Modus Ponens
7) $\therefore f$	Step (6) & Conjunctive Simplification

(* If k is true $k \vee$ anything is also true)
$\therefore$ the given argument is valid.

10) $(q \vee \neg r) \vee s$
$\neg q \vee (r \wedge \neg q)$

$\therefore r \to s$

Soln:

Steps	Reasons
1) $(q \vee \neg r) \vee s$	Premise
2) $\neg q \vee (r \wedge \neg q)$	Premise
3) $(\neg q \vee r) \wedge (\neg q \vee \neg q)$	Step(2), and distributive law
4) $(\neg q \vee r) \wedge \neg q$	Step(3) & Idempotent law
5) $\neg q$	Step(4) and Absorption law
6) $q \vee (\neg r \vee s)$	Step(1) and Associative law.
7) $\neg r \vee s$	Step(5) & (6) and Disjunctive Syllogism.
8) $\therefore r \to s$	Step(7) & the fact that $r \to s \Leftrightarrow \neg r \vee s$.

$\therefore$ the given argument is valid.

11) $(t \rightarrow e) \wedge (a \rightarrow l)$

$\therefore (t \wedge a) \rightarrow (e \wedge l)$

Soln:

Steps	Reasons
1) $(t \rightarrow e) \wedge (a \rightarrow l)$	Primise
2) $t \rightarrow e$	Step (1) and conjunctive simplification
3) $\neg t \vee e$	Step(2) and the fact that $p \rightarrow q \Leftrightarrow \neg p \vee q$.
4) $(\neg t \vee e) \vee \neg a$	Step (3) and disjunctive amplification
5) $(\neg a \vee \neg t) \vee e$	Step (4), commutative & associative law.
6) $\neg(t \wedge a) \vee e$	Step (5), Demorgan's & commutative law.
7) $a \rightarrow l$	Step (1), rule of conjunctive simplification
8) $\neg a \vee l$	Step(7), & the fact that $p \rightarrow q \Leftrightarrow \neg p \vee q$.
9) $(\neg a \vee l) \vee \neg t$	Step (8), & rule of disjunctive amplification.
10) $(\neg a \vee \neg t) \vee l$	Step (9) & associative law.
11) $\neg(t \wedge a) \vee l$	Step (10) & Demorgan's and commutative law.
12) $[\neg(t \wedge a) \vee e] \wedge [\neg(t \wedge a) \vee l]$	Steps (6) & (11) and conjunction rule.
13) $\neg(t \wedge a) \vee (e \wedge l)$	Step (12) & distributive law
14) $\therefore (t \wedge a) \rightarrow (e \wedge l)$	Step (13) & the fact that $p \rightarrow q \Leftrightarrow \neg p \vee q$.

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12) $(\neg p \vee q) \rightarrow r$

$r \rightarrow (s \vee t)$

$\neg s \wedge \neg u$

$\neg u \rightarrow \neg t$

$\therefore p$

Soln:

Steps	Reasons
1) $\neg s \wedge \neg u$	Premise
2) $\neg u$	Step (1) & rule of conjunctive simplification
3) $\neg u \rightarrow \neg t$	Premise
4) $\neg t$	Steps (2) & (3), rule of Modus Ponens
5) $\neg u \wedge \neg s$	Step (1), commutative law.
6) $\neg s$	Rule (5), conjunctive simplification
7) $\neg s \wedge \neg t$	Step (4), (6), & rule of conjunction
8) $r \rightarrow (s \vee t)$	Premise
9) $\neg (s \vee t) \rightarrow \neg r$	Step (8), and contrapositive.
10) $(\neg s \wedge \neg t) \rightarrow \neg r$	Step (9), DeMorgan's law.
11) $\neg r$	Step (7) & (10), Modus Ponens
12) $(\neg p \vee q) \rightarrow r$	Premise
13) $\neg r \rightarrow \neg(\neg p \vee q)$	Step (12) & contrapositive
14) $\neg r \rightarrow (p \wedge \neg q)$	Step (13) & DeMorgan's law
15) $p \wedge \neg q$	Steps (14), (11), Modus Ponens
16) $\therefore p$	Step (15) & rule of conjunctive simplification.

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USE OF QUATIFIERS

Open statement: It is a declarative statement which contains one or more variables.

- It is not a statement, but when the variables in it are replaced by certain allowable choices, it can be called as a statement.

Eg:- i) $x+5=10$ ii) $x^3 < 100$

- open statements containing a variable are denoted by $p(x)$, $q(x)$, etc. Hence 'x' is called a free variable.

Eg: $p(x): x+5=10$. If $x=5$, $p(5)=10$

$\therefore$ the truth value of $p(5)$ is true.

Negation $\neg p(x)$

Conjunction $p(x) \wedge q(x)$

Disjunction $p(x) \vee q(x)$

Conditional $p(x) \rightarrow q(x)$

Biconditional $p(x) \leftrightarrow q(x)$

1) Suppose the universe consists of all integers, consider the following open statements:

$p(x): x \leq 3$, $q(x): x+1$ is odd $r(x): x > 0$

Give the truth values of the following.

i) $p(2)$ ii) $\neg q(4)$ iii) $p(-1) \wedge q(1)$ iv) $\neg p(3) \vee r(0)$

v) $p(0) \rightarrow q(0)$ vi) $p(1) \leftrightarrow \neg q(2)$ vii) $p(4) \vee [q(1) \wedge r(2)]$

viii) $p(2) \wedge [q(0) \vee \neg r(2)]$

Soln: i) $p(2): 2 \leq 3$ is true

ii) $\neg q(4): 4+1=5$ is not odd - false.

iii) $p(-1) \wedge q(1)$.

$p(-1): -1 \leq 3$ - true

$q(1): 1+1=2$ is odd - false.

$\therefore p(-1) \wedge q(1)$ is false.

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iv) $\neg p(3) \vee r(0)$

$\neg p(3)$: $3 \nleq 3$ is false

$r(0)$: $0 > 0$ is false

$\therefore$ $\neg p(3) \vee r(0)$ is false.

v) $p(0) \rightarrow q(0)$

$p(0)$: $0 \leq 3$ – true

$q(0) = 1$ is odd – true

$\therefore$ $p(0) \rightarrow q(0)$ is true

vi) $p(1) \leftrightarrow \neg q(2)$

$p(1)$: $1 \leq 3$ – true

$\neg q(2)$: $2+1=3$ is not odd – false.

$\therefore$ $p(1) \leftrightarrow \neg q(2)$ is false.

vii) $p(4) \vee [q(1) \wedge r(2)]$

$p(4)$: $4 \leq 3$ is false

$q(1)$: 2 is odd is false

$r(2)$: $2 > 0$ is true

$\therefore$ $p(4) \vee (q(1) \wedge r(2))$ = false $\vee$ (false $\wedge$ true) is false.

viii) $p(2) \wedge [q(0) \vee \neg r(2)]$

$p(2)$: $2 \leq 3$ is true

$q(0)$: 1 is odd is true.

$\neg r(2)$: $2 \ngtr 0$ is false.

$\therefore$ $p(2) \wedge [q(0) \vee \neg r(2)]$ is true

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QUANTIFIERS

Statements which contain phrases that are associated with the idea of quantity, are called as quantifiers.

- Types of quantifiers

i) Universal Quantifier - Denoted by $\forall x$ and is read as "for all x", "for any x", "for each x", "for every x".

$\forall x, y$ - "for all x, y"

Eg:- $\forall x \in S, p(x)$ - It is read as "for all x belong to S, where $p(x)$ is the open statement. The variable x is called as a bound variable.

ii) Existential Quantifiers - Denoted by $\exists x$ which is read as "for some x", "for at least one x", "there exists an x".

Eg:- $\exists x \in S, p(x)$ - "there exists x which belongs to S and $p(x)$ is an open statement.

Problems:

1) For the universe of all integers, let

$p(x): x > 0$

$q(x): x$ is even

$r(x): x$ is a perfect square

$s(x): x$ is divisible by 3

$t(x): x$ is divisible by 7

Write the following quantified statements in the symbolic form.

i) At least one integer is even

Soln: $\exists x, q(x)$

2) There exists a positive integer that is even.

$\exists x, [p(x) \wedge q(x)]$

3) Some even integers are divisible by 3

$\exists x\, [q(x) \wedge s(x)]$

4) Every integer is either even or odd

$\forall x\, [q(x) \underline{\vee} \neg q(x)]$

5) If x is even and a perfect square, then x is not divisible by 3.

$\forall x\, [\{q(x) \wedge r(x)\} \rightarrow \neg s(x)]$

6) If x is odd or is not divisible by 7, then x is divisible by 3.

$\forall x\, [\{\neg q(x) \vee \neg t(x)\} \rightarrow s(x)]$

2) Consider the open statements given below.

$p(x)$: $x > 0$, $q(x) = x$ is even, $r(x)$: x is a perfect square

$s(x)$: x is divisible by 3. $t(x)$: x is divisible by 7.

Express each of the following symbolic statements in words and indicate its truth value.

i) $\forall x\, [r(x) \rightarrow p(x)]$

Soln: For any integer x, if x is a perfect square, then $x > 0$

Let $x = 0$, $r(0)$: 0 is a perfect square

$\therefore r(0) =$ true

$p(0)$: $0 > 0$ is false.

Hence, $\forall x\, [r(x) \rightarrow p(x)]$ is false.

ii) $\exists x\, [s(x) \wedge \neg q(x)]$

Soln: There exists an integer x such that x is divisible by 3 and x is not even.

Let $x = 9$

$s(9)$: 9 is divisible by 3 – it is true

$\neg q(9)$: 9 is not even – it is true

$\therefore \exists x\,[s(x) \wedge \neg q(x)]$ is true

iii) $\forall x\,[\neg r(x)]$

Sol^n: For any integer x, x is not a perfect square.

Let $x = 4$, $r(4)$: 4 is a perfect square, it is true

if $r(4)$ is true, then $\neg r(4)$ is false.

$\therefore \forall x\,[\neg r(x)]$ is false.

iv) $\forall x\,[r(x) \vee t(x)]$

Sol^n: For any integer x, x is a perfect square or x is divisible by 7.

Let $x = 2$. $r(2)$: 2 is a perfect square – it is false.

$t(2)$: 2 is divisible by 7 – it is false.

$\therefore \forall x\,[r(x) \vee t(x)]$ is false.

3) Consider the following open statements with R as the universe.

$p(x): |x| > 3$, $q(x): x > 3$. Find the truth value of the statement $\forall x\,[p(x) \rightarrow q(x)]$. Also write the converse, inverse and contrapositive of this statement and find their truth values.

Sol^n: $p(x): |x| > 3$, $q(x): x > 3$

Let $x = -4$.

$p(-4): |-4| = 4 > 3$ is true

$q(-4): -4 > 3$ is false

true $\rightarrow$ false = false. $\therefore \forall x\,[p(x) \rightarrow q(x)]$ is false.

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The converse of $\forall x [p(x) \rightarrow q(x)]$ is $\forall x [q(x) \rightarrow p(x)]$

For every real number greater than 3 has its absolute value greater than 3.

Let $x = -4$, $q(-4) : -4 > 3$ is false.

$p(-4) : |-4| > 3$ is true

false $\rightarrow$ true = true.

$\therefore \forall x [q(x) \rightarrow p(x)]$ is true

The inverse of the given statement is $\forall x [\neg p(x) \rightarrow \neg q(x)]$

For every real number x, if $|x| \leq 3$, then $x \leq 3$.

Let $x = 2$,

$\neg p(x) : |2| \not> 3$ is true

$\neg q(x) : 2 \not> 3$ is true

$\therefore \forall x [\neg p(x) \rightarrow \neg q(x)]$ is true.

The contrapositive of the given statement is

$\forall x [\neg q(x) \rightarrow \neg p(x)]$

For every real number x, if x is not greater than or equal to 3, then it has its magnitude not greater than or equal to 3.

Let $x = -4$

$q(-4) : -4 > 3$ is false $\therefore \neg q(-4)$ is true

$p(-4) : |-4| > 3$ is true $\therefore \neg p(-4)$ is false

true $\rightarrow$ false = false

$\therefore \forall x [\neg q(x) \rightarrow \neg p(x)]$ is false.

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4) Consider the following statements with the set of all real nos as the universe.

$p(x): x \geq 0$, $q(x): x^2 \geq 0$, $r(x): x^2-3x-4=0$

$s(x): x^2-3>0$. Determine the truthness or falsity of the following statements.

i) $\exists x[p(x) \wedge q(x)]$

Soln: Let $x=2$

$p(2): 2 \geq 0$, $q(2): 4 \geq 0$ both are true

$\therefore \exists x[p(x) \wedge q(x)]$ is true.

ii) $\forall x[p(x) \rightarrow q(x)]$

Let $x=-1$. $p(x): -1 \geq 0$ is false.

$q(x): 1 \geq 0$ is true

For any real number, $q(x)$ is true.

true $\rightarrow$ true $=$ true, false $\rightarrow$ true $=$ true.

$\therefore \forall x[p(x) \rightarrow q(x)]$ is true.

iii) $\forall x[q(x) \rightarrow s(x)]$

Soln: Let $x=1$.

$q(1): 1 \geq 0$ is true

$s(1): -2>0$ is false.

$\therefore \forall x[q(x) \rightarrow s(x)]$ is false.

iv) $\forall x[r(x) \vee s(x)]$

Soln: Let $x=1$

$r(1): -7=0$ is false

$s(1): -2>0$ is false

$\therefore \forall x[r(x) \vee s(x)]$ is false.

v) $\exists x[p(x) \wedge r(x)]$

Soln Let $x=4$. $p(4): 4 \geq 0$ is true

$r(4): 4^2-(3)(4)-4=0$ is true

$\therefore \exists x[p(x) \wedge r(x)]$ is true

vi) $\forall x[r(x) \rightarrow p(x)]$

Soln: let $x=-1$ $r(-1): 0=0$ is true, $p(-1): -1 \geq 0$ is false.

$\therefore \forall x[r(x) \rightarrow p(x)]$ is false.

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Rule of Negation for quantified statement

$$\boxed{\begin{array}{l}\neg[\forall x\, p(x)] \Leftrightarrow \exists x[\neg p(x)] \\ \neg[\exists x\, p(x)] \Leftrightarrow \forall x[\neg p(x)]\end{array}}$$

1) Find the negation of the following statements when Z is the universe.

$p(x)$: x is odd $q(x)$: x^2-1 is even.

$\forall x\,[p(x) \to q(x)]$

Negation of $\forall x\,[p(x)\to q(x)]$ is

$\neg\,[\forall x\{p(x) \to q(x)\}]$

$\Leftrightarrow \exists x\,[\neg\{p(x) \to q(x)\}]$

$\Leftrightarrow \exists x\,[\neg\{\neg p(x) \vee q(x)\}]$

$$\boxed{\neg[\forall x\{p(x)\to q(x)\}] \Leftrightarrow \exists x\,[\,p(x) \wedge \neg q(x)]}$$

Negation in words "there exists an x such that x is odd and x^2-1 is not even"

$\forall x\,[p(x) \to q(x)]$ is true"

$\exists x\,[\,p(x) \wedge \neg q(x)]$ is false.

2) Negate & simplify each of the following

i) $\exists x\,[p(x) \vee q(x)]$ ii) $\forall x\,[p(x) \wedge \neg q(x)]$

iii) $\forall x\,[p(x) \to q(x)]$ iv) $\exists x\,[\{p(x) \vee q(x)\} \to r(x)]$

Soln: [Refer Question Bank and solution at the end of this notes]

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3) Write the following proposition in symbolic form and find its negation.

"If all triangles are right-angled, then no triangle is equiangular".

Solⁿ: Let $p(x)$: x is right-angled triangle

$q(x)$: x is equiangular

The given proposition is symbolically represented as

$$\forall x\, p(x) \rightarrow \forall x(\neg q(x))$$

Negation is $\neg[\forall x\, p(x) \rightarrow \forall x(\neg q(x))]$

$$\Leftrightarrow \neg[\neg \forall x\, p(x) \vee \forall x(\neg q(x))]$$

$$\Leftrightarrow \forall x\, p(x) \wedge \neg\{\forall x[\neg q(x)]\} \text{ - De'morgans law}$$

$$\Leftrightarrow \forall x\, p(x) \wedge \exists x\{\neg(\neg q(x))\} \text{ - Negation law}$$

$$\Leftrightarrow \forall x\, p(x) \wedge \exists x\, q(x)$$

In words, "All triangles are right-angled and some triangles are equiangular".

4) Write the negation of each of the following.

1) For all integers n, if n is not divisible by 2, then n is odd.

Solⁿ: Let $p(x)$: n is divisible by 2

$q(x)$: n is odd.

Given statement is $\forall x[\neg p(x) \rightarrow q(x)]$

negation is $\neg[\forall x\{\neg p(x) \rightarrow q(x)\}]$

$$\Leftrightarrow \exists x[\neg\{\neg p(x) \rightarrow q(x)\}]$$

$$\Leftrightarrow \exists x[\neg\{p(x) \vee q(x)\}]$$

$$\Leftrightarrow \exists x[\neg p(x) \wedge \neg q(x)]$$

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"For some integer n, n is not divisible by 2 and n is not odd".

ii) If k, m, n are any integers where $(k-m)$ and $(m-n)$ are odd then $(k-n)$ is even

Soln: $p(x): (k-m)$ is odd
$q(x): (m-n)$ is odd
$r(x): (k-n)$ is even

$$\forall k, m, n\, [p(x) \land q(x)] \rightarrow r(x)$$

Its negation, $\neg [\forall k, m, n \{p(x) \land q(x)\} \rightarrow r(x)]$

$$\Leftrightarrow \exists k, m, n\, [\neg \{p(x) \land q(x) \rightarrow r(x)\}]$$

$$\Leftrightarrow \exists k, m, n\, [\neg \{\neg (p(x) \land q(x)) \lor r(x)\}]$$

$$\Leftrightarrow \exists k, m, n\, [\{p(x) \land q(x)\} \land \neg r(x)]$$

In words, "There exist integers k, m, n such that $(k-m)$, and $(m-n)$ are odd and $(k-n)$ is not even".

iii) For all real numbers x, if $|x-3| > 7$ then $-4 < x < 10$

iv) If x is a real number where $x^2 > 16$, then $x < -4$ or $x > 4$.

Solve them similarly.

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Quantified Statements with more than one variable

$\forall x \forall y \ p(x,y)$

$\exists x \exists y \ p(x,y)$

$\forall x \exists y \ p(x,y)$

$\exists x \forall y \ p(x,y)$

Note:

$$\boxed{\begin{array}{l} \forall x \forall y \ p(x,y) \Leftrightarrow \forall y \forall x \ p(x,y) \\ \exists x \exists y \ p(x,y) \Leftrightarrow \exists y \exists x \ p(x,y) \end{array}}$$

Example: If $p(x,y) : x + y = 1$

→ $\forall x \exists y \ p(x,y)$ is read as
"for all integers x, there exists an integer y such that $x+y=1$"

This is a true statement. For any x, there exists $y = 1-x$ such that $x+y=1$

→ $\exists y \forall x \ p(x,y)$ is read as
"There exists some integer y such that for all integers x, $x+y=1$".

This is a false statement. If we consider any y, then x is restricted to be, $x = 1-y$. ∴ the phrase "for all" doesn't hold for x.

∴ we note that $\forall x \exists y \ p(x,y) \not\Leftrightarrow \exists y \forall x \ p(x,y)$

1) Find the negation of $\forall x \exists y \left[\{p(x,y) \wedge q(x,y)\} \rightarrow r(x,y)\right]$

Sol: $\neg\left[\forall x \exists y \{[p(x,y) \wedge q(x,y)] \rightarrow r(x,y)\}\right]$

$\Leftrightarrow \exists x \left[\neg\left[\exists y \{[p(x,y) \wedge q(x,y)] \rightarrow r(x,y)\}\right]\right]$

$\Leftrightarrow \exists x \forall y \left[\neg \{(p(x,y) \wedge q(x,y)) \rightarrow r(x,y)\}\right]$

$\Leftrightarrow \exists x \forall y \left[\neg \{\neg(p(x,y) \wedge q(x,y)) \vee r(x,y)\}\right]$

$\Leftrightarrow \exists x \forall y \left[p(x,y) \wedge q(x,y) \wedge \neg r(x,y)\right]$

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Some important relations.

$$\exists x\,[p(x) \wedge q(x)] \Rightarrow \exists x\, p(x) \wedge \exists x\, q(x)$$
$$\exists x\,[p(x) \vee q(x)] \Leftrightarrow \exists x\, p(x) \vee \exists x\, q(x)$$
$$\forall x\,[p(x) \wedge q(x)] \Leftrightarrow \forall x\, p(x) \wedge \forall x\, q(x)$$
$$\forall x\, p(x) \vee \forall x\, q(x) \Rightarrow \forall x\,[p(x) \vee q(x)]$$

THE RULE OF UNIVERSAL SPECIFICATION

- If an open statement becomes true for all replacement by the numbers in a given universe, then that open statement is true for any specific individual number in that universe.
- If $p(x)$ is an open statement for a given universe and if $\forall x\, p(x)$ is true, then $p(a)$ is true for each a in the universe.

THE RULE OF UNIVERSAL GENERALIZATION

- If an open statement $p(x)$ is proved to be true when x is replaced by an arbitrarily chosen element 'a' from universe, then the universally quantified statement $\forall x\, p(x)$ is true.

Problems:

[Important problems are solved in question bank solution.]

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Extra Problems : verify whether the arguments are valid.

1) All mathematics professors have studied Calculus
Leona ~~has~~ is a mathematics professor.
Therefore, Leona has studied Calculus.

Sol^n: $m(x)$: x is a mathematics professor
$c(x)$: x has studied Calculus.
Symbolic form of the argument is

$$\begin{array}{l} \forall x\,[m(x) \to c(x)] \\ m(l) \\ \hline \therefore c(l) \end{array}$$

Steps	Reasons
1) $\forall x[m(x) \to c(x)]$	Premise
2) $m(l) \to c(l)$	step(1), Rule of universal specification
3) $m(l)$	Premise
4) $\therefore c(l)$	steps (2) & (3) & Modus Ponens

$\therefore$ the given argument is valid.

2) In Δ^{le} xyz there is no pair of angles of equal measure
If a Δ^{le} has two sides of equal length, then it is isosceles.
If a Δ^{le} is isosceles, then it has two angles of equal measure.
Therefore, Δ^{le} xyz has no two sides of equal length.

Sol^n: Let $p(t)$: Δ^{le} t has two sides of equal length.
$q(t)$: t is an isosceles triangle.
$r(t)$: t has two sides of equal measure

Let c denote Δle xyz. Symbolically,

$$\neg r(c).$$
$$\forall t\,[p(t) \rightarrow q(t)]$$
$$\forall t\,[q(t) \rightarrow r(t)]$$
$$\overline{\qquad\qquad\qquad}$$
$$\therefore \neg p(c)$$

Steps	Reason
1) $\forall t\,[p(t) \rightarrow q(t)]$	Premise
2) $p(c) \rightarrow q(c)$	step (1) & Rule of universal specification
3) $\forall t\,[q(t) \rightarrow r(t)]$	Premise
4) $q(c) \rightarrow r(c)$	step (2) & Rule of universal specification
5) $\neg r(c)$	Premise
6) $p(c) \rightarrow r(c)$	Steps (2), (4), Law of Syllogism.
7) $\therefore \neg p(c)$	Steps (6), (5), Modus Tollens.

$\therefore$ the given argument is valid.

---

3) All squares have four sides
Quadrilateral EFGH is not a square
Therefore quadrilateral EFGH is does not have four sides.

Soln: $p(x)$: x is a square
$q(x)$: x has four sides.

Let c denote the quadrilateral EFGH

$$\forall x\,[p(x) \rightarrow q(x)]$$
$$\neg p(c)$$
$$\overline{\qquad\qquad\qquad}$$
$$\therefore \neg q(c)$$

Steps	Reason
1) $\forall x\,[p(x) \rightarrow q(x)]$	Premise
2) $p(c) \rightarrow q(c)$	step (1) & Rule of universal specification.
3) $\neg p(c)$	Premise
4) $\therefore \neg q(c)$	Step (3) & Modus Tollens.

$\therefore$ it is valid

4) $\forall x\,[p(x) \to q(x)]$
$\forall x\,[q(x) \to r(x)]$
$\therefore \forall x\,[p(x) \to r(x)]$

Steps	Reasons
1) $\forall x\,[p(x) \to q(x)]$	Premise
2) $p(a) \to q(a)$	step (1) & Rule of universal specification
3) $\forall x\,[q(x) \to r(x)]$	Premise
4) $q(a) \to r(a)$	step (3) & Rule of universal specification
5) $p(a) \to r(a)$	steps (2),(4) and Rule of Syllogism.
6) $\forall x\,[p(x) \to r(x)]$	step 5 & Rule of universal generalization.

$\therefore$ the given argument is valid.

5) $\exists x\,[p(x) \land \neg q(x)]$
$\forall x\,[p(x) \to r(x)]$
$\therefore \exists x\,[r(x) \land \neg q(x)]$

Soln:

Steps	Reasons
1) $\exists x\,[p(x) \land \neg q(x)]$	Premise
2) $\forall x\,[p(x) \to r(x)]$	Premise
3) $p(a) \land \neg q(a)$	Step (1) & Rule of universal specn.
4) $p(a) \to r(a)$	step (2) & ——— " ———
5) $p(a)$	step (3) & rule of conjunctive simplification
6) $r(a)$	steps (4),(5) & Modus Ponens.
7) $\neg q(a)$	step (3) & rule of conjunctive simplification
8) $r(a) \land \neg q(a)$	steps (5) & (7) & conjunction rule.
9) $\therefore \exists x\,[r(x) \land \neg q(x)]$	step 8 and rule of universal generalization.

$\therefore$ the given argument is valid.

6) $\forall x\ [v(x) \rightarrow w(x)]$

$\forall x\ [w(x) \rightarrow \neg y(x)]$

$\therefore \forall x\ [y(x) \rightarrow \neg v(x)]$

Sol^n: Steps — Reason

1) $\forall x\ [v(x) \rightarrow w(x)]$ — Premise
2) $v(a) \rightarrow w(a)$ — step (1) & rule of universal specification
3) $\forall x\ [w(x) \rightarrow \neg y(x)]$ — Premise
4) $w(a) \rightarrow \neg y(a)$ — step (2) & rule of universal specification
5) $v(a) \rightarrow \neg y(a)$ — Steps (2) & (5), rule of syllogism.
6) $y(a) \rightarrow \neg v(a)$ — step (5) & its contrapositive.
7) $\therefore \forall x\ [y(x) \rightarrow \neg v(x)]$ — step (6) and rule of universal generalization.

---

7) $\forall x \forall y\ [p(x,y) \rightarrow q(x,y)]$

$\neg q(a,b)$

$\therefore \exists x, \exists y\ [\neg p(x,y)]$

Sol^n: Steps — Reason

1) $\forall x \forall y\ [p(x,y) \rightarrow q(x,y)]$ — Premise
2) $p(a,b) \rightarrow q(a,b)$ — step (1) & rule of US.
3) $\neg q(a,b)$ — Premise.
4) $\neg p(a,b)$ — Steps (2) & (3), Modus Tollens
5) $\therefore \exists x \exists y\ [\neg p(a,b)]$ — step (4) & rule of UG.

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8) $\forall x\,[b(x) \rightarrow \neg c(x)]$
$\exists x\,[c(x) \wedge d(x)]$

$\therefore \exists x\,[d(x) \wedge \neg b(x)]$

Soln:

Steps	Reason
1) $\forall x\,[b(x) \rightarrow \neg c(x)]$	Premise
2) $b(a) \rightarrow \neg c(a)$	Step (1) and rule of US
3) $\exists x\,[c(x) \wedge d(x)]$	~~Step~~ Premise
4) $c(a) \wedge d(a)$	Step (3) & rule of US
5) $c(a)$	Step (4) & conjunctive simplification
6) $\neg b(a)$	Steps (2),(5) & Modus Tollens
7) $d(a)$	Step (4) & conjunctive simplification
8) $d(a) \wedge \neg b(a)$	Steps (7),(6) & Rule of conjunction
9) $\therefore \exists x\,[d(x) \wedge \neg b(x)]$	Step (8) & rule of UG

METHODS OF PROOFS AND DISPROOF

- Given a conditional $p \to q$, the process of establishing that the conditional is true by using the rules/laws of logic and other known facts constitutes a proof ~~as~~ of the conditional.
- The process of establishing that a conditional is false is called as disproof.

Types of Proofs:-

1) Direct Proof: The direct proof of proving the conditional $p \to q$ is true is:
   i) Hypothesis : First assume that p is true.
   ii) Analysis: Starting with hypothesis, employing the rules/laws of logic and other known facts, infer that q is true.
   iii) Conclusion: $p \to q$ is true.

2) Indirect Proof: steps are
   1. $p \to q \Leftrightarrow \neg q \to \neg p$ - known fact.
   2. Assume $\neg q$ is true
   3. With the help of rules/laws of logic & other known facts, infer p is false. $\therefore \neg p$ is true.
   4. If $\neg q$ & $\neg p$ is true, $\neg q \to \neg p$ is true
      $\therefore p \to q$ is also true.

3) Proof by contradiction: steps are:
   1. Hypothesis: Assume that $p \to q$ is false.
      $p \to q$ is false only when p is true & q is false.
   2. Analysis: Starting with the hypothesis that q is false, employing rules/laws of logic, and

other known facts, infer that p is false. This contradicts the assumption that "p is true"

3) Conclusion: we infer that $p \to q$ is true because of the contradiction arrived in the analysis step.

Types of Disproofs - 1) Disproof by Contradiction

- We prove that the conditional $p \to q$ is false. Steps:-

i) Hypothesis: Assume that p is true & q is true and hence $p \to q$ is true.

ii) Analysis: using laws of logic / other known facts show that our assumption (hypothesis step) is wrong and hence $p \to q$ is false. This disproves the given statement.

2) Disproof by counterexample

- We know that the quantified statement $\forall x\ p(x)$ is false if for any one element a $p(a)$ is false. Hence, take one case such that $p(x)$ is false and hence the given proposition is false.

PROBLEMS -

Problems are solved in question bank solution. Along with that, solve other problems also.

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Discrete Mathematical Structures - Module-1 ②

1) Define a proposition, tautology and contradiction. P.T. for any propositions p, q, r, the compound proposition $[(p \to q) \wedge (q \to r)] \to (p \to r)$ is a tautology.

2) Define converse, inverse and contrapositive of a conditional with truth table.

3) State the converse, inverse, and contrapositive of the following statement

"If a triangle is not isosceles, then it is not equilateral".

4) Prove the logical equivalence by using the laws of logic

i) $(p \to q) \wedge (\neg q \wedge [r \vee \neg q]) \Leftrightarrow \neg(q \vee p)$

ii) $(\neg p \vee q) \wedge (p \wedge (p \wedge q)) \Leftrightarrow p \wedge q$

iii) $[(\sim p \vee \sim q) \to (p \wedge q \wedge r)] \Leftrightarrow p \wedge q$

iv) $[p \to (q \to r)] \Leftrightarrow (p \wedge q) \to r$

5) Define the dual of logic statements. Write duals of the following propositions:

i) $(p \to q) \to r$ ii) $p \to (q \to r)$

6) Verify the principle of duality for the following equivalence.

$$[(\neg(p \wedge q) \to \neg p) \vee (\neg p \vee q)] \Leftrightarrow (\neg p \vee q)$$

7) Establish the validity of the argument

a)

$$\begin{array}{l} \neg p \leftrightarrow q \\ q \to r \\ \neg r \\ \hline \therefore p \end{array}$$

b)

$$\begin{array}{l} (\neg p \vee \neg q) \to (r \wedge s) \\ r \to t \\ \neg t \\ \hline \therefore p \end{array}$$

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(2)

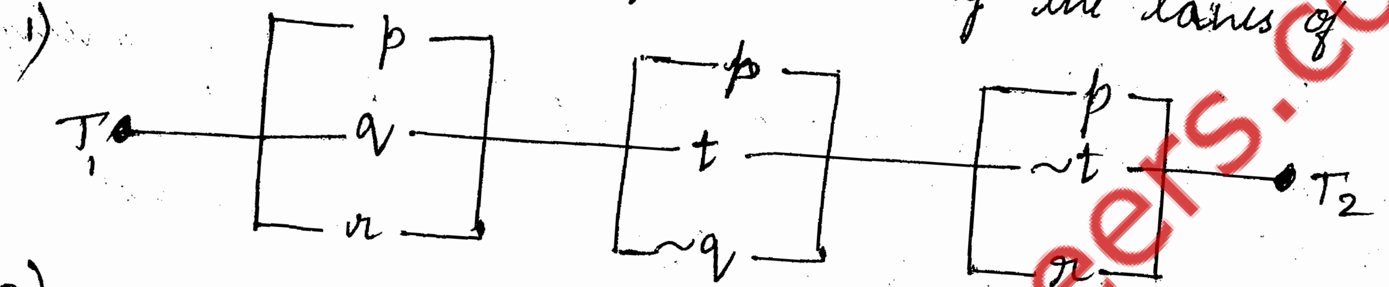
c) $p \to r$
$r \to s$
$t \vee \neg s$
$\neg t \vee u$
$\neg u$
$\therefore \neg p$

d) $p \to q$
$q \to (r \wedge s)$
$\neg r \to (\neg t \vee u)$
$p \wedge t$
$\therefore u$

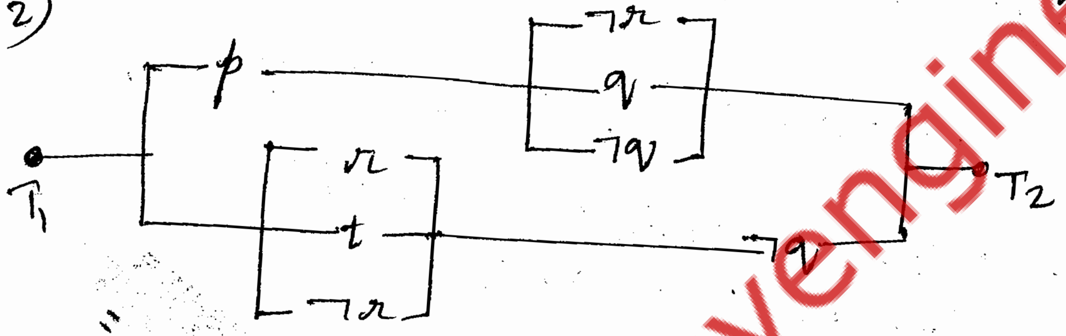
e) $a \vee (b \to a)$
$\neg a \wedge c$
$\therefore \neg b$

8) Simplify the switching n/w using the laws of logic

1)

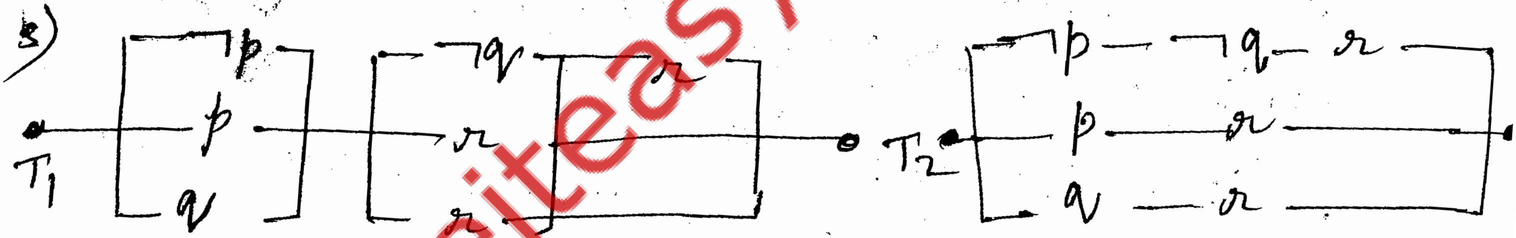


2)



Parallel ckts – OR
Series – " – AND

3)



) Verify whether the argument is valid.

i) NO engineering students of 1st ~~and~~ or 2nd semester studies logic
Anil is an engineering student who studies logic

$\therefore$ Anil is not in second semester

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ii) No junior or senior is enrolled in a physical education class.

Sam is enrolled in a physical education class.

Therefore, Sam is not a senior.

iii) No engineering student is bad in studies

Jeff is not bad in studies

Therefore, Jeff is an engineering student

iv) $\forall x\,[p(x) \vee q(x)]$

$\forall x\,[\{\neg p(x) \wedge q(x)\} \rightarrow r(x)]$

$\therefore\ \forall x\,[\neg r(x) \rightarrow p(x)]$

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) Give direct proof of the statement "The square of an odd integer is an odd integer"

) Prove that for all integers 'k' and 'l' if k and l are both odd then $(k+l)$ is even.

Give i) a direct proof ii) an indirect proof iii) proof by contradiction for the following statement "If m is an even integer, then $(m+7)$ is odd

) Negate and simplify the following:

i) $\exists x\,[p(x) \vee q(x)]$ ii) $\forall x\,[p(x) \rightarrow q(x)]$

iii) $\forall x\,[p(x) \wedge \neg q(x)]$ iv) $\exists x\,[(p(x) \vee q(x)) \rightarrow r(x)]$.

FUNDAMENTALS OF LOGIC
QUESTION BANK SOLUTION

1) **Proposition**: It is a statement or declaration which, in a given content, can be said to be either true or false, but not both.

Eg: 1) Spain borders Belgium
2) Birmingham is in UK
3) 2 is a prime number
4) $2+3=5$

Tautology: A compound statement/proposition which is always true regardless of the truth values of its components is called a tautology.

Contradiction: A compound proposition which is always false regardless of the truth values of its components is called a contradiction.

2) Let $[(p \to q) \wedge (q \to r)] \to (p \to r) = A$

Truth Table

p	q	r	$p \to q$	$q \to r$	$(p\to q)\wedge(q \to r)$	$p \to r$	A
0	0	0	1	1	1	1	1
0	0	1	1	1	1	1	1
0	1	0	1	0	0	1	1
0	1	1	1	0	0	1	1
1	0	0	0	1	0	0	1
1	0	1	0	1	0	1	1
1	1	0	1	0	0	0	1
1	1	1	1	1	1	1	1

The values of $[(p \to q) \wedge (q \to r)] \to (p \to r)$ is always 1.
Hence, it is tautology

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> Consider a conditional $p \to q$, then:
i) $q \to p$ is called the <u>converse</u> of $p \to q$
ii) $\neg p \to \neg q$ is called the <u>inverse</u> of $p \to q$
iii) $\neg q \to \neg p$ is called the <u>contrapositive</u> of $p \to q$

<u>Truth Tables</u>

p	q	$p \to q$	$q \to p$	$\neg p$	$\neg q$	$\neg p \to \neg q$	$\neg q \to \neg p$
0	0	1	1	1	1	1	1
0	1	1	0	1	0	0	1
1	0	0	1	0	1	1	0
1	1	1	1	0	0	1	1

Let p: Triangle is not isosceles
q: Triangle is not equilatera

<u>$p \to q$</u> : If a triangle is not isosceles, then it is not equilateral.

<u>$q \to p$</u> : If a triangle is not equilateral, then it is not isosceles.

<u>$\neg p \to \neg q$</u>: If a triangle is isosceles, then it is equilateral.

<u>$\neg q \to \neg p$</u>: If triangle is equilateral, then it is isosceles.

i) $(p \to q) \wedge [\neg q \wedge (r \vee \neg q)] \Leftrightarrow \neg(q \vee p)$

$(p \to q) \wedge [\neg q \wedge (r \vee \neg q)] \Leftrightarrow (p \to q) \wedge [\neg q \wedge (\neg q \vee r)]$
— by commutative law.

$\Leftrightarrow (p \to q) \wedge \neg q$, by Absorption law

$\Leftrightarrow (\neg p \vee q) \wedge \neg q$, $\because p \to q \Leftrightarrow \neg p \vee q$

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$\Leftrightarrow (\neg p \wedge \neg q) \vee (q \wedge \neg q)$, Expansion by Distributive Law

$\Leftrightarrow (\neg p \wedge \neg q) \vee F_0$, Inverse law

$\Leftrightarrow \neg(p \vee q) \vee F_0$, De'Morgan's law

$\Leftrightarrow \neg(p \vee q)$, Identity law.

ii) $(\neg p \vee q) \wedge (p \wedge (p \wedge q)) \Leftrightarrow p \wedge q$

$(\neg p \vee q) \wedge (p \wedge (p \wedge q)) \Leftrightarrow (\neg p \vee q) \wedge ((p \wedge p) \wedge q)$

– ~~Commul~~ Associative law.

$\Leftrightarrow (\neg p \vee q) \wedge (p \wedge q)$, Idempotent law.

$\Leftrightarrow ((\neg p \vee q) \wedge q) \wedge p$, ~~Absorption~~ Associative law.

$\Leftrightarrow (\underbrace{q \wedge (q \vee \neg p)}) \wedge p$, Associative law.

$\Leftrightarrow q \wedge p$, Absorption law.

$\Leftrightarrow p \wedge q$, Commutative law.

iii) $[(\sim p \vee \sim q) \rightarrow (p \wedge q \wedge r)] \Leftrightarrow p \wedge q$

$[(\neg p \vee \neg q) \rightarrow (p \wedge q \wedge r)] \Leftrightarrow \neg(\neg p \vee \neg q) \vee (p \wedge q \wedge r)$

$\because p \rightarrow q \Leftrightarrow \neg p \vee q$

$\Leftrightarrow (p \wedge q) \vee [(p \wedge q) \wedge r]$, De'morgan's law

$\Leftrightarrow (p \wedge q)$, By Absorption law.

iv) $[p \rightarrow (q \rightarrow r)] \Leftrightarrow (p \wedge q) \rightarrow r$

$[p \rightarrow (q \rightarrow r)] \Leftrightarrow \neg p \vee (\neg q \vee r)$ $\because p \rightarrow q \Leftrightarrow \neg p \vee q$

$\Leftrightarrow (\neg p \vee \neg q) \vee r$, Associative law.

$\Leftrightarrow \neg(p \wedge q) \vee r$, Demorgan's law.

$\Leftrightarrow (p \wedge q) \rightarrow r$ $\because p \rightarrow q \Leftrightarrow \neg p \vee q$

5) <u>Duality</u>: Suppose we have a compound proposition 'u' which contains the connectives $\wedge$ and $\vee$, and also contains T_0 & F_0. then the new compound proposition 'v' is obtained by replacing each occurrence of $\wedge$ and $\vee$ in 'u' by $\vee$ and $\wedge$ in 'v', and also replace T_0 and F_0 in 'u' by F_0 and T_0 respectively. the resultant compound proposition 'v' is the dual of u and is denoted by u^d.

i) $(p \to q) \to r$

$$((p\to q)\to r)^d \Leftrightarrow (\neg(\neg p \vee q) \vee r)^d$$

$$\Leftrightarrow ((p \wedge \neg q) \vee r)^d \Leftrightarrow (p \vee \neg q) \wedge r$$

ii) $p \to (q \to r)$

$$(p \to (q\to r))^d \Leftrightarrow (\neg p \vee (\neg q \vee r))^d$$

$$\Leftrightarrow (\neg p \vee \neg q \vee r)^d$$

$$\Leftrightarrow \neg p \wedge \neg q \wedge r$$

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) For any two propositions u and v. if $u \Leftrightarrow v$ then, $u^d \Leftrightarrow v^d$. This is known as "The Principle of Duality"

we have,

$\neg(p\wedge q) \to (\neg p) \vee (\neg p \vee q)) \Leftrightarrow (\neg p \vee q)$

Let $u = \neg(p \wedge q) \to$

$((p\wedge q) \vee \neg p) \vee (\neg p \vee q)$

$(p \wedge q) \vee \neg p \vee \neg p \vee q$

$(p\wedge q) \vee q \vee \neg p$

$q \vee \neg p$

We have,

$(\neg(p\wedge q) \to \neg p) \vee (\neg p \vee q) \Leftrightarrow (\neg p \vee q)$

Let $u = (\neg(p\wedge q) \to \neg p) \vee (\neg p \vee q)$ and

$v = \neg p \vee q$

$(v)^d = \neg p \wedge q$ ———— (1)

We have to prove that $u^d \Leftrightarrow v^d$.

$u = (\neg(p \wedge q) \to \neg p) \vee (\neg p \vee q)$

$\Leftrightarrow [[\neg\neg(p\wedge q)] \vee \neg p] \vee (\neg p \vee q)$ $\because p \to q \Leftrightarrow \neg p \vee q$

$\Leftrightarrow (p \wedge q) \vee \neg p \vee \neg p \vee q$

$\therefore u^d \Leftrightarrow (p \vee q) \wedge \neg p \wedge \neg p \wedge q$

$\Leftrightarrow (p \vee q) \wedge \neg p \wedge q$ Idempotent law

$\Leftrightarrow (p \vee q) \wedge q \wedge \neg p$ Associative law.

$\Leftrightarrow q \wedge \neg p$ Absorption law.

$\Leftrightarrow \neg p \wedge q$ commutative law.

$\Leftrightarrow v^d$ By (1)

$\therefore u^d \Leftrightarrow v^d$ if $u \Leftrightarrow v$.

7) a) $\neg p \leftrightarrow q$

$q \to r$

$\neg r$

$\therefore p$

Steps	Reason
(1) $\neg p \leftrightarrow q$	Premise
(2) $(\neg p \to q) \wedge (q \to \neg p)$	Step 1 & $p \leftrightarrow q \Leftrightarrow (p \to q) \wedge (q \to p)$
(3) $(\neg p \to q)$	Step 2 & Rule of conjunctive simplification
(4) $q \to r$	Premise
(5) $\neg p \to r$	Steps (3) (4), & law of syllogism.
(6) $\neg r$	Premise.
(7) $\neg(\neg p)$	Steps (5), (6), & Modus Tollens
(8) $\therefore p$	Step 7 and law of double negation

$$\begin{array}{l} (\neg p \vee \neg q) \to (r \wedge s) \\ r \to t \\ \neg t \\ \hline \therefore p \end{array}$$

Steps	Reason
1) $r \to t$	Premise
2) $\neg t$	Premise
3) $\neg r$	Steps (1) & (2), Modus Tollens
4) $\neg r \vee \neg s$	Step (3), Rule of disjunctive amplification
5) $\neg(r \wedge s)$	Step (4), DeMorgan's law.
6) $(\neg p \vee \neg q) \to (r \wedge s)$	Premise
7) $(\neg p \vee \neg q)$	Step (6), (5), Modus Pones
8) $(p \wedge q)$	Step (7), DeMorgan's law
9) $\therefore p$	Step (8), Rule of conjunctive simplification.

c) $p \rightarrow r$
$r \rightarrow s$
$t \vee \neg s$
$\neg t \vee u$
$\neg u$

$\therefore \neg p$

Steps	Reason
1) $p \rightarrow r$	Premise
2) $r \rightarrow s$	Premise
3) $p \rightarrow s$	Steps (1), (2), & law of syllogism
4) $t \vee \neg s$	Premise
5) $\neg s \vee t$	Step (4) Commutative law
6) $s \rightarrow t$	Step (5) & the fact $p \rightarrow q \Leftrightarrow \neg p \vee q$
7) $p \rightarrow t$	Steps (3) & (6), law of syllogism
8) $\neg t \vee u$	Premise
9) $t \rightarrow u$	Step (8) & the fact $p \rightarrow q \Leftrightarrow \neg p \vee q$
10) $p \rightarrow u$	Steps (7), (9), law of syllogism
11) $\neg u$	Premise
12) $\therefore \neg p$	Steps (10), 11, and Modus Tollens

d) $p \rightarrow q$
$q \rightarrow (r \wedge s)$
$\neg r \rightarrow (\neg t \vee u)$
$p \wedge t$

$\therefore u$

Steps	Reason
1) $p \to q$	Premise
2) $q \to (r \wedge s)$	Premise
3) $p \to (r \wedge s)$	steps (1), (2), & law of syllogism.
4) $p \wedge t$	Premise
5) $t \wedge p$	Step (4), commutative law
6) p	Step (5), conjunctive simplification
7) $r \wedge s$	Steps (3), (6), Modus Pones
8) r	Step (7), rule of conjunctive simplification
9) $\neg r \vee (\neg t \vee u)$	Premise
10) $(\neg r \vee \neg t) \vee u$	Step (9), Associative law
11) $\neg (r \wedge t) \vee u$	Step (10), De Morgan's law
12) t	Step (5) rule of conjunctive simplification
13) $r \wedge t$	Steps (8), (12), & rule of conjunction
14) $\therefore u$	Steps (11), (13), & rule of disjunctive syllogism.

$a \vee (b \to a)$

$\neg a \wedge c$

$\therefore \neg b$

Steps	Reason
1) $a \vee (b \to a)$	Premise
2) $\neg a \wedge c$	Premise
3) $\neg a$	Step (2), conjunctive Simplification
4) $b \to a$	Step (1) & (3), Disjunctive Syllogism.
5) $\therefore \neg b$	Steps (3), (4) & Modus Tollens.

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8) Switching Circuits

1) We write the given network as:

$$(p \vee q \vee r) \wedge (p \vee t \vee \neg q) \wedge (p \vee \neg t \vee r)$$

$\Leftrightarrow p \vee [(q \vee r) \wedge (t \vee \neg q) \wedge (\neg t \vee r)]$ - distributive law

$\Leftrightarrow p \vee [(t \vee \neg q) \wedge (q \vee r) \wedge (\neg t \vee r)]$ commutative law

$\Leftrightarrow p \vee [(t \vee \neg q) \wedge \{r \vee (q \wedge \neg t)\}]$ - Distributive law

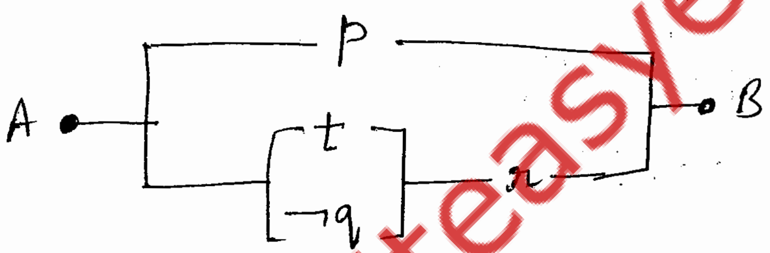
$\Leftrightarrow p \vee [\{(t \vee \neg q) \wedge r\} \vee \{(t \vee \neg q) \wedge (\neg t \wedge q)\}]$. Distributive law

$\Leftrightarrow p \vee [\{(t \vee \neg q) \wedge r\} \vee \{(t \vee \neg q) \wedge \neg (t \vee \neg q)\}]$ Demorgan's law

$\Leftrightarrow p \vee [\{(t \vee \neg q) \wedge r\} \vee F_0]$ Inverse law

$\Leftrightarrow p \vee \{(t \vee \neg q) \wedge r\}$ Identity law

$\therefore$ the simplified n/w is



2) The given network is written as

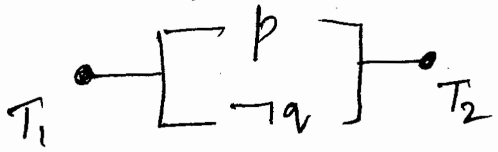
$$[p \wedge (\neg r \vee q \vee \neg q)] \vee [(r \vee t \vee \neg r) \wedge \neg q]$$

$\Leftrightarrow [p \wedge (\neg r \vee T_0)] \vee [(t \vee T_0) \wedge \neg q]$ Inverse law

$\Leftrightarrow [p \wedge T_0] \vee [\neg q \wedge T_0]$ Domination law

$\Leftrightarrow p \vee \neg q$ Identity laws

the simplified n/w is



) The given n/w is represented as

$$(\neg p \wedge \neg q \wedge r) \vee (p \wedge r) \vee (q \wedge r)$$

$\Leftrightarrow r \wedge [(\neg p \wedge \neg q) \vee p \vee q]$ - distributive law

$\Leftrightarrow r \wedge [\neg(p \vee q) \vee (p \vee q)]$ - DeMorgan's law

$\Leftrightarrow r \wedge [T_0]$ Inverse law

$\Leftrightarrow r$ Identity law.

$\therefore$ The simplified n/w is

T_1 o—— r ——o T_2

) i) Let $p(x)$: x is in 1st semester

$q(x)$: x is in 2nd semester

$r(x)$: x studies logic

$\therefore$ symbolic form of the given argument is

$$\forall x [\{p(x) \wedge q(x)\} \rightarrow \neg r(x)]$$

$$r(a)$$

$$\therefore \neg q(a)$$

Steps	Reason
$\forall x [\{p(x) \wedge q(x)\} \rightarrow \neg r(x)]$	Premise
$\{p(a) \wedge q(a)\} \rightarrow \neg r(a)$	step (1) and rule of universal specification
$r(a) \rightarrow \neg(p(a) \wedge q(a))$	step (2) and contrapositive
$r(a) \rightarrow (\neg p(a) \vee \neg q(a))$	step (3) & DeMorgan's law
$r(a)$	Premise
$\neg p(a) \vee \neg q(a)$	

) $\therefore$ we cannot deduce $\neg p(a)$. Invalid argument.

ii) Refer class notes

iii) Let $p(x)$: x is engineering student
$q(x)$: x is bad in studies
~~$r(x)$~~

∴ The symbolic form of the given argument

$$\forall x\,[p(x) \to \neg q(x)]$$
$$\neg q(j)$$
$$\therefore p(j)$$

Steps	Reason
1) $\forall x\,[p(x) \to \neg q(x)]$	Premise
2) $p(j) \to \neg q(j)$	Step (1) & rule of universal specification
3) $\neg q(j)$	Premise

4) Here we cannot deduce $p(j)$. Hence, the given argument is not valid

iv) Steps	Premise
1) $\forall x\,[p(x) \vee q(x)]$	Premise
2) $p(a) \vee q(a)$	Step (1) & rule of universal specification
3) $\forall x\,[\{\neg p(x) \wedge q(x)\} \to r(x)]$	Premise
4) $\{\neg p(a) \wedge q(a)\} \to r(a)$	Step (2) & rule of universal specification
5) $\neg r(a) \to \neg(\neg p(a) \wedge q(a))$	step (4) & contrapositive.
6) $\neg r(a) \to (p(a) \wedge \neg q(a))$	Demorgan's law on step (5)

(Assume that $\neg r(a)$ is true)

7) $\neg r(a)$	Premise assumed	
8) $p(a) \vee \neg q(a)$	Steps (6) & (7), modus pones	
9) $[p(a) \vee q(a)] \wedge [p(a) \vee \neg q(a)]$	Steps (2), (8) & rule of conjunction	
10) $p(a) \vee [q(a) \wedge \neg q(a)]$	Step (9) and distributive law	
11) $p(a) \vee F_0$	step (10), inverse law	
12) $p(a)$	Step (11) & identity laws	
13) $\neg r(a) \rightarrow p(a)$ *	Steps (7) & (12) & conditional	* when $\neg r(a)$ is true, given premises imply
14) $\forall x [\neg r(x) \rightarrow p(x)]$	Step (13) & rule of universal generalization.	

) Direct proof

"The square of an odd integer, is an odd integer"

- Assume that 'n' is an odd integer. Then,

$n = 2k+1$ for some integer k.

$\therefore \quad n^2 = (2k+1)^2 = 4k^2 + 4k + 1$

$= 2(2k^2 + 2k) + 1$

Let $p = 2k^2 + 2k$ where p is some integer

$\therefore \quad n^2 = 2p + 1$

This proves that n^2 is an odd integer.

) Take any two integers k and l, & assume that they are odd.

$k = 2m+1$ & $l = 2n+1$ for some integers m, & n.

$\therefore \quad k + l = (2m+1) + (2n+1) = 2(m+n+1)$

$\therefore \quad k+l$ is even number

11) i) Direct Proof:

Let p: m is even q: m+7 is odd.

Assume q is true. i.e, m+7 is odd.

$\therefore$ $m+7 = 2k+1 \Rightarrow m = 2k-6 = 2(k-3)$

which is divisible by 2.

Hence, m is even

$\therefore$ $p \to q$ is true.

ii) Indirect Proof:

We need to prove that $\neg q \to \neg p$ is true

assume $\neg q$ is true $\Rightarrow$ q is false $\Rightarrow$ m+7 is even.

$\therefore$ $m+7 = 2k \Rightarrow m = 2k-7$

$\Rightarrow$ m is not divisible by 2. or m is odd.

$\therefore$ p is false $\Rightarrow \neg p$ is true.

$\therefore$ $\neg q \to \neg p$ is true.

Thus, the given statement is proved using indirect proof.

iii) Proof by contradiction

Assume $p \to q$ is false.

This implies p is true and q is false.

Now, if q is false, m+7 is not odd.

$\Rightarrow$ m+7 is even

i.e, $m+7 = 2k$

$m = 2k-7 = (2k-8)+1$

$\boxed{m = 2(k-4)+1}$ $\Rightarrow$ m is not divisible by 2 or m is odd.

$\Rightarrow$ p is false. This contradicts the assumption that p is true

$\therefore$ $p \to q$ cannot be false.
Using proof by contradiction, we proved that $p \to q$ is true

$\Rightarrow$ i) $\exists x [p(x) \vee q(x)]$

$\neg \exists x [p(x) \vee q(x)]$

$\Leftrightarrow \forall x [\neg (p(x) \vee q(x))]$

$\Leftrightarrow \forall x [\neg p(x) \wedge \neg q(x)]$

ii) $\forall x [p(x) \to q(x)]$

$\neg \forall x [p(x) \to q(x)]$

$\Leftrightarrow \exists x [\neg (p(x) \to q(x))]$

$\Leftrightarrow \exists x [\neg (\neg p(x) \vee q(x))]$

$\Leftrightarrow \exists x [p(x) \wedge \neg q(x)]$

iii) $\forall x [p(x) \wedge \neg q(x)]$

$\neg \forall x [p(x) \wedge \neg q(x)]$

$\Leftrightarrow \exists x [\neg (p(x) \wedge \neg q(x))]$

$\Leftrightarrow \exists x [\neg p(x) \vee q(x)]$

v) $\exists x [(p(x) \vee q(x)) \to r(x)]$

$\neg \exists x [(p(x) \vee q(x)) \to r(x)]$

$\Leftrightarrow \forall x [\neg (\{p(x) \vee q(x)\} \to r(x))]$

$\Leftrightarrow \forall x [\neg (\neg (p(x) \vee q(x)) \vee r(x))]$

$\Leftrightarrow \forall x [\neg (\neg p(x) \wedge \neg q(x)) \vee r(x)] \Leftrightarrow \forall x [\neg (\neg p(x) \vee \neg q(x)) \vee r(x)]$

$\forall x [p(x)]$

$\Leftrightarrow \forall x [\neg (\neg p(x) \vee \neg q(x)) \wedge \neg r(x)]$

$\Leftrightarrow \forall x [p(x) \wedge q(x) \wedge \neg r(x)]$