# Module-2

## Properties of the Integers: Mathematical Induction

### Special sets of numbers

$Z$ - the set of integers: $\{0, 1, -1, 2, -2, \cdots\}$

$N$ - the set of nonnegative integers or natural nos.: $\{0, 1, 2, 3, \ldots\}$

$Z^+$ - the set of positive integers: $\{1, 2, 3, \ldots\} = \{x \in Z \mid x > 0\}$

$Q$ - the set of rational numbers: $\{a/b \mid a, b \in Z, b \neq 0\}$

$Q^+$ - the set of positive rational numbers

$Q^*$ - the set of nonzero rational numbers

$R$ - the set of real numbers

$R^+$ - the set of positive real numbers

$R^*$ - the set of nonzero real numbers.

$C$ - the set of complex numbers: $\{x + iy \mid x, y \in R, i^2 = -1\}$

$C^*$ - the set of nonzero complex numbers

$Z_n$ - $\{0, 1, 2, \ldots n-1\}$, for $n \in Z^+$

- In this chapter, we examine a special property exhibited by the subset of positive integers ($Z^+$). This property will enable us to establish certain mathematical formulas and theorems by using a technique called mathematical induction.

<u>The Well-ordering principle</u>: Every nonempty subset of $Z^+$ contains a smallest element. In other words, $Z^+$ is well ordered.



<u>Theorem 1</u>:

<u>Finite Induction Principle or Principle of Mathematical Induction</u>:

Let $S(n)$ denotes an open mathematical statement that involves one or more occurrences of the variable $n$, which represents a positive integer.

a) If $S(1)$ is true;

b) If $S(k)$ is true, then $S(k+1)$ is true. where, $k$ is arbitrarily chosen, $k \in Z^+$.

<u>Proof</u>: Let $S(n)$ be an open statement satisfying conditions (a) and (b). ——— (1)

Let, $F = \{t \in Z^+ \mid S(t) \text{ is false}\}$ and we have to prove that $F = \phi$ ($\phi$ is null set)

(Meaning, we have to prove that there is no such element $t$ such that $S(t)$ is false.)

Let us assume the contradiction $F \neq \phi$ is true

$F \neq \phi$ ——— (2)

Then by the well-ordering principle, F has least element '$s$'.

Since $S(1)$ is true, (according to (1)), it follows that $s \neq 1$. $\because s \in F$ & $F \neq \phi$

$\therefore s > 1$ and $(s-1) \notin F$ as '$s$' is the smallest element of F.

But, $s-1 \in Z^+$ and $S(s-1)$ is true. as $(s-1) \notin F$

By condition (b), it follows that $S((s-1)+1) = S(s)$ is true.

but $s \in F$ and we proved that $S(s)$ is true.

This contradicts our assumption that $F \neq \phi$.

$F = \phi$ because there exists at least one element $s$ such that $S(s)$ is true.

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In the Theorem (1) statement, the condition in part (a) is referred to as the <u>basis step</u>, and in part (b) is called the <u>inductive step</u>.

- Choice of 1 in the first condition is not mandatory.
- we need open statement $S(n)$ to be true for some first element $n_0 \in Z$ so that the induction process has a starting place.
- Truth of $S(n_0)$ for basis step is necessary.
- The set $Z^+$ is union with $\{0\}$ or any <u>finite sets</u> of negative integers is well-ordered.
- Under these circumstances, we express the Finite Induction Principle, using quantifiers, as

$$\boxed{[S(n_0) \wedge [\forall k \geq n_0 [S(k) \Rightarrow S(k+1)]]] \Rightarrow \forall n \geq n_0 \, S(n)}$$

Prove the following statements by mathematical Induction

1) $\forall\, n \in Z^+, \sum_{i=1}^{n} i = 1+2+\cdots+n = \frac{(n)(n+1)}{2} = S(n)$

i) Basis step : Let $n=1$

$$\therefore S(1) = \sum_{i=1}^{1} i = \frac{1(2)}{2} = 1$$

$\therefore S(1)$ is true



ii) Inductive step

We assume the truth of $S(k)$ for some $k \in Z^+$

$$S(k) = \sum_{i=1}^{k} i = \frac{k(k+1)}{2}$$

Now, we want to deduce the truth of

$$S(k+1) = \sum_{i=1}^{k+1} i = 1+2+\cdots+k+(k+1)$$

$$= \sum_{i=1}^{k} i + (k+1)$$

$$= \frac{k(k+1)}{2} + (k+1)$$

$$= \frac{k(k+1)+2(k+1)}{2}$$

$$S(k+1) = \frac{(k+1)(k+2)}{2}$$

By the Principle of Mathematical Induction, $S(n)$ is true for all $n \in Z^+$.

2) $\forall\, n \in Z^+, \sum_{i=1}^{n} i^2 = 1^2+2^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$

Let $S(n) = \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

i) Basis step Let $n=1$

$$S(1) = \sum_{i=1}^{1} i^2 = \frac{1(1+1)(2+1)}{6} = 1$$

$\therefore S(1)$ is true

Inductive step: Assume that for some $k$, $k \in Z^+$, $S(k)$ is true.

$$S(k) = \sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$$

we want to deduce the truth of $S(k+1) = \frac{(k+1)(k+2)(2k+3)}{6}$

$$S(k+1) = \sum_{i=1}^{k+1} i^2 = 1^2+2^2+\cdots+k^2+(k+1)^2$$

$$= \sum_{i=1}^{k} i^2 + (k+1)^2$$

$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

$$= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6}$$

$$= \frac{(k+1)(2k^2+7k+6)}{6}$$

$$= \frac{(k+1)(2k^2+4k+3k+6)}{6}$$

$$= \frac{(k+1)\,2k(k+2)+3(k+2)}{6}$$

$$S(k+1) = \frac{(k+1)(k+2)(2k+3)}{6}$$

$\therefore S(n)\ \forall n \in Z^+$ is true.

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3) $\forall n\ n \in Z^+$, $\sum_{i=1}^{n}(2i-1)^2 = 1^2+3^2+5^2+\cdots+(2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$

Let $S(n) = \sum_{i=1}^{n}(2i-1)^2 = \frac{n(2n-1)(2n+1)}{3}$

Basis step: Let $n=1$

$$S(1) = \frac{1(1)(3)}{3} = 1$$

$\therefore S(1)$ is true

<u>Inductive step</u>: we assume that $S(k)$ is true for some $k \in Z^+$.

$$S(k) = \sum_{i=1}^{k} (2k-1)^2 = \frac{k(2k-1)(2k+1)}{3}$$

we have to deduce the truth of

$$S(k+1) = \frac{(k+1)(2(k+1)-1)(2(k+1)+1)}{3}$$

$$= \frac{(k+1)(2k+1)(2k+3)}{3}$$



$$S(k+1) = \sum_{i=1}^{k+1} (2(k+1)-1)^2$$

$$= \cancel{\sum_{i=1}^{k+1}} \; 1^2 + 3^2 + \cdots + (2k-1)^2 + (2k+1)^2$$

$$= \frac{k(2k+1)(2k-1)}{3} + (2k+1)^2$$

$$= \frac{k(2k+1)(2k-1) + 3(2k+1)^2}{3}$$

$$= \frac{(2k+1)(k(2k-1) + 3(2k+1))}{3}$$

$$= \frac{(2k+1)(2k^2 - k + 6k + 3)}{3} = \frac{(2k+1)(2k^2 + 5k + 3)}{3}$$

$$= \frac{(2k+1)(2k^2 + 2k + 3k + 3)}{3} = \frac{(2k+1)\; 2k(k+1) + 3(k+1)}{3}$$

$$S(k+1) = \frac{(2k+1)(2k+3)(k+1)}{3}$$

$\therefore$ $S(n)$ is true.

1) $\forall n\ n \in Z^+$, $\sum_{i=1}^{n} n(n+1) = 1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \cdots + n(n+1)$

$$= \frac{n(n+1)(n+2)}{3}$$

Let $S(n) = \sum_{i=1}^{n} n(n+1) = \frac{n(n+1)(n+2)}{3}$

Basis step: Let $n=1$. $\therefore S(1) = \frac{1(2)(3)}{3} = 2$

$\therefore S(1)$ is true

Inductive step: Assume that $S(n)$ is true for some $k \in Z^+$

$$\therefore S(k) = \sum_{i=1}^{k} k(k+1) = \frac{k(k+1)(k+2)}{3}$$

we have to deduce the truth of

$$S(k+1) = \sum_{i=1}^{k+1} (k+1)(k+2) = \frac{(k+1)(k+2)(k+3)}{3}$$

$$S(k+1) = 1\cdot 2 + 2\cdot 3 + \cdots + k(k+1) + (k+1)(k+2)$$

$$= \sum_{i=1}^{k} k(k+1) + (k+1)(k+2)$$

$$= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2)$$

$$= \frac{k(k+1)(k+2) + 3(k+1)(k+2)}{3}$$

$$S(k+1) = \frac{(k+1)(k+2)(k+3)}{3}$$

$\therefore S(n)$ is true for $n \in Z^+$ by Principle of Mathematical Induction

5) By mathematical Induction, prove that $n! \geq 2^{n-1}$ $\forall n \in Z^+$

$S(n): n! \geq 2^{n-1} \ \forall n \in Z^+$

<u>Basis step:</u> for $n=1$, $S(1) = 1! = 1$

$1 \geq 2^0$ which is true. $\therefore S(1)$ is true

<u>Inductive step:</u> Assume that $S(n)$ is true for some $k \in Z^+$

$\therefore k! \geq 2^{k-1}$

or

$2^{k-1} \leq k!$ —— (1)

for $S(k+1)$, we have to p.t, $2^k \leq (k+1)!$

$2^k = 2 \cdot 2^{k-1} \leq 2 \cdot k!$ — according to (1)

$\leq (k+1) k!$ $\quad 2 \leq k+1$ for $k \geq 1$

$\leq (k+1)!$

$\therefore 2^k \leq (k+1)!$ or

$(k+1)! \geq 2^k$

$\therefore S(n)$ is true



6) P.T. $4n < (n^2-7)$ for all positive integers $n \geq 6$

$S(n): 4n < (n^2-7)$ $\forall n \geq n_0$ where $n_0 = 6$

<u>Basis step:</u> Let $n = n_0 = 6$

$S(n_0) = S(6) : 4 \cdot 6 < 36 - 7$

$: 24 < 29$

$\therefore S(n_0)$ is true

<u>Inductive step:</u> $S(n)$ is true for some $k \in Z^+$ & $k \geq 6$.

$\therefore$ we get $4k < (k^2 - 7)$ —— (1)

we have to deduce the truth of $S(k+1)$

i.e., $4(k+1) < ((k+1)^2 - 7)$

$4(k+1) = 4k+4$

$< (k^2-7) + (2k+1)$

[because, when $k \geq 6$, we have $2k+1$
$= 13 > 4$]

$(k+1)^2 - 7$

$k^2 + (2k+1) - 7$

$\cancel{k^2+2k+6}$

$\therefore 4(k+1) < (k+1)^2 - 7$

$\therefore S(n)$ is true.

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7) P.T. $2^n > n^2 \; \forall n \; n \in Z^+$ and $n > 4$ & $n \geq 5$

$S(n): 2^n > n^2 \; \forall \; n \in Z^+$ & $n \geq 5$

Basis step: Let $n = n_0 = 5$

$\therefore S(n_0) = S(5): \; 2^5 > 5^2$

i.e., $32 > 25$ which is true

$\therefore S(n_0)$ is true

Inductive step: $S(n)$ is true for some $k \in Z^+$ & $k \geq 5$

$S(k): \; 2^k > k^2$ ——(1)

we have to deduce the truth of

$2^{k+1} > (k+1)^2$

$2^{k+1} = 2^k \cdot 2 > k^2 \cdot 2$ (according to (1))

$\cancel{2k^2 = k^2 + k^2}$

$\therefore 2^{k+1} > 2k^2$

$> k^2 + k^2 > k^2 + 2k+1$

$2^{k+1} > (k+1)^2$

$(k+1)^2 = k^2 + 2k + 1$

$k \geq 5$

$5^2 > 2(5) + 1$

$25 > 11$

$\therefore k^2 > (2k+1)$

8) P.T. for $n \in Z^+$, 5 divides $n^5 - n$

Let $S(n)$: 5 divides $n^5 - n$

Basis step: Let $n=1$ $S(1)$:

5 divides $1^5 - 1$; 5 divides 0.

$\therefore S(1)$ is true

Inductive step: Assume that $S(n)$ is true for some $k \in Z^+$.

$\therefore$ 5 divides $k^5 - k$

ie., $k^5 - k$ is a multiple of 5



In other words, $k^5 - k = 5m$ for $m \in Z^+$ —— (1)

We have to deduce the truth of $S(k+1)$

$$(k+1)^5 - (k+1) = (k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1) - (k+1)$$
$$= (k^5 - k) + 5(k^4 + 10k^3 + 10k^2 + 5k)$$
$$= 5m + 5(k^4 + 10k^3 + 10k^2 + 5k) \text{ —— By (1)}$$
$$= 5(m + k^4 + 10k^3 + 10k^2 + 5k)$$
$$= 5p \quad \text{where, } p = m + k^4 + 10k^3 + 10k^2 + 5k$$

This shows that $(k+1)^2 - (k+1)$ is a multiple of 5.

$\therefore S(n)$ is true

) P.T. every positive integer $n \geq 24$ can be written as sum of 5's and/or 7's.

$S(n)$: $n$ can be written as sum of 5's &/or 7's $\forall n \geq 24$

Basis step: $S(24)$

$24 = (7+7) + (5+5)$

$\therefore S(24)$ is true

<u>Inductive step:</u> Assume that $S(n)$ is true for some $k$
$k \in Z^+$ & $k \geq 24$.

$$k = (7+7+\cdots) + (5+5+\cdots).$$

Suppose this representation of $k$ has '$r$' number of 7's and '$s$' number of 5's. & $r \geq 2$ & $s \geq 2$.
Using this representation of $k$, we find that

$$k+1 = \underbrace{(7+7+\cdots)}_{r} + \underbrace{(5+5+\cdots)}_{s} + 1$$

$$= \underbrace{(7+7+\cdots)}_{(r-2)} + (7+7) + \underbrace{(5+5+\cdots)}_{s} + 1$$

$$= \underbrace{(7+7+\cdots)}_{(r-2)} + \underbrace{(5+5+\cdots)}_{(s+3)}$$

$\left[\begin{array}{l} 7+7+1=15 \\ =5+5+5 \end{array}\right]$

This shows that $S(k+1)$ is a sum of 7's & 5's.
$\therefore$ $S(n)$ is true [111y do it for $n \geq 14$ in terms of using only 3's and/or 8's as summands]

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Q) Let $a_0 = 1$, $a_1 = 2$, $a_2 = 3$ & $a_n = a_{n-1} + a_{n-2} + a_{n-3}$ for $n \geq 3$. P.T. $a_n \leq 3^n \ \forall\ n \in Z^+$

$S(n) : a_n \leq 3^n$

<u>Basis step:</u>

$a_0 = 1 \leq 3^0$, $a_1 = 2 \leq 3$, $a_2 = 3 \leq 9$

$\therefore$ $S(n)$ is true for $n = 0, 1, 2$.

<u>Induction step:</u> Assume that $S(n)$ is true for $k \in Z^+$ & $k \geq 3$ $\quad a_k \leq 3^k$

$$a_{k+1} = \cancel{3^{k+1}}\ a_k + a_{k-1} + a_{k-2}$$
$$\leq 3^k + 3^{k-1} + 3^{k-2}$$
$$\leq 3^k + 3^k + 3^k \qquad (\because 3^{k-1} \leq 3^k \ \&\ 3^{k-2} \leq 3^k)$$
$$\leq 3 \cdot 3^k$$
$$a_{k+1} \leq 3^{k+1}$$

$\therefore$ $S(n)$ is true.

Thus, $S(n)$ is true for $n_0, n_0+1, n_0+2, \ldots, n_1$

<u>Inductive step:</u> Assume that $S(n)$ is true for

$n_0, n_0+1, n_0+2, \ldots \underbrace{n_0+4}_{n_1}, \ldots, k.$

where $k \geq n_1 = 28.$

Then, $S(k-4)$ i.e., $S(28-4)$ is a sum of 7's and/or 5's

$\therefore k+1 = (k-4)+5$ is also a sum of 7's and/or 5's.

$\therefore S(k+1)$ is true.

Hence, by the alternative form of the principle of Induction, it follows that $S(n)$ is true for all integers $k \geq 24.$

11) Let $H_1 = 1$, $H_2 = 1 + \frac{1}{2}$, $H_3 = 1 + \frac{1}{2} + \frac{1}{3}$, ...

$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. $\forall\ n \in Z^+$

P.T. $\sum_{i=1}^{n} H_i = (n+1) H_n - n$

**Sol^n:** $S(n)$: $\sum_{i=1}^{n} H_i = (n+1) H_n - n$

**Basis step:** Let $n = 1$

$S(1)$: $H_1 = 2H_1 - 1$
$= 2 \times 1 - 1 = 1$ ($H_1 = 1$ is given)

$\therefore S(1)$ is true

**Inductive step:** Assume that $S(k)$ is true for $k \in Z^+$

$$\sum_{i=1}^{k} H_i = (k+1) H_k - k$$

$$\boxed{\begin{aligned} H_{k+1} &= 1 + \tfrac{1}{2} + \cdots + \tfrac{1}{k} + \tfrac{1}{k+1} \\ H_{k+1} &= H_k + \tfrac{1}{k+1} \\ \therefore H_k &= H_{k+1} - \tfrac{1}{k+1} \end{aligned}}$$

$$S(k+1) = \sum_{i=1}^{k+1} H_i = \sum_{i=1}^{k} H_i + H_{k+1}$$

$$= (k+1) H_k - k + H_{k+1}$$

$$= (k+1)\left[H_{k+1} - \frac{1}{(k+1)}\right] - k + H_{k+1}$$

$$= (k+1) H_{k+1} - \frac{k+1}{k+1} - k + H_{k+1}$$

$S(k+1)$ $= (k+2) H_{k+1} - (k+1)$.

$\therefore S(n)$ is true.

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Alternative solution

**Basis Step:** $n_0 = 24 = 7+7+5+5$

$n_0 + 1 = 25 = 5+5+5+5+5$

$n_0 + 2 = 26 = 7+7+7+5$

$n_0 + 3 = 27 = 7+5+5+5+5$

$n_0 + 4 = 28 = 7+7+7+7$

Let $n_1 =$

# RECURSIVE DEFINITIONS

**Sequence:** A sequence is an arrangement of elements of a set in order as the first element, the second element, and so on.

Eg: $a_1, a_2, \ldots, a_n, \ldots$ is a sequence in which $a_1$ is the first element, $a_2$ is the second element, …, $a_n$ is the $n^{th}$ element, and so on.

- The element $a_n$ is called the general term of the sequence, and the sequence is denoted by $\{$
- In a sequence, if the first term is $a_0$, then the second term is $a_1$, and $a_{n-1}$ will be the $n^{th}$ term
- The sequences of this type is denoted by $\{a_{n-}$

For describing a sequence, two methods are commonly u

1) The explicit method (The explicit formula)

- The general term of the sequence is calculated by an explicit formula.

Eg: Consider the integer sequence $b_0, b_1, \ldots, b_n$ where $b_n = 2n$ for all $n \in N$.

Here, we find that

$b_0 = 2 \cdot 0 = 0 \quad b_1 = 2 \cdot 1 = 2 \quad b_2 = 2 \cdot 2 = 4 \quad b_3 = 2 \cdot 3 = 6.$

If we need to determine $b_6$, simply calculate $b_6 = 2 \cdot 6 = 12$ without the need to calculate the value of $b_n$ for any other $n \in N$. (The explicit formula is $b_n = 2n$)



2) Recursive Method

- First few terms of the sequence are explicitly mentioned and the general term is specified through the formula to obtain new terms of the sequence from the terms already known.

Eg:- Consider the integer sequence $b_1, b_2, \ldots, b_n$ where $b_n = 2n \ \forall\ n \in N$.

Another way of specifying this sequence is
$$b_n = b_{n-1} + 2 \text{ for } n \geq 2 \text{ and } b_1 = 2$$

- This sequence is described by a recursive method.
- Recursive definition of a sequence should consist of two parts.
  - → First part - first few terms of the sequence must be indicated explicitly (base for the second part)
  - → Second part - obtain new terms of the sequence from the terms already known by a rule called recursive rule.

- If the recursive rule requires the use of 'k' number preceding terms for obtaining a new term, then the first 'k' terms of the sequence must be indicated explicitly in the first part of the defn.

## Examples

1) obtain a recursive definition for the sequence $\{a_n\}$ in each of the following cases.

i) $a_n = 5n$ ii) $a_n = 6^n$ iii) $a_n = 3n + 7$

iv) $a_n = n(n+2)$ v) $a_n = n^2$ vi) $a_n = 2 - (-1)^n$

i) $a_1 = 5,\ a_2 = 10,\ a_3 = 15, \ldots$

$\boxed{a_n = a_{n-1} + 5}$, for $n \geq 2$. [Here, $a_n$ is obtained fr previous term $a_{n-1}$

ii) $a_n = 6^n$

$\therefore a_1 = 6,\ a_2 = 6^2,\ a_3 = 6^3, \ldots$

$\boxed{a_n = a_{n-1} \times 6}$ for $n \geq 2$

iii) $a_n = 3n+7$

$a_1 = 10,\ a_2 = 13,\ a_3 = 16,\ a_4 = 19, \ldots$

$\therefore \boxed{a_n = a_{n-1} + 3}$, for $n \geq 2$

iv) $a_n = n(n+2)$

$a_1 = 3,\ a_2 = 8,\ a_3 = 15,\ a_4 = 24, \ldots$

$a_2 - a_1 = 5 = 2\times1+3,\quad a_3 - a_2 = 7 = 2\times2+3,$

$a_4 - a_3 = 9 = 2\times3+3,\quad a_5 - a_4 = 11 = 2\times4+3$

$\therefore \boxed{a_{n+1} - a_n = 2n+3}$, for $n \geq 1$

<OR>

$a_2 - a_1 = 5 = 2\times2+1,\quad a_3 - a_2 = 7 = 2\times3+1,\quad a_4 - a_3 = 9 = 2\times4$

$\therefore \boxed{a_n - a_{n-1} = 2n+1}$ for $n \geq 2$

v) $a_n = n^2$

$a_1 = 1,\quad a_2 = 2^2,\quad a_3 = 9,\quad a_4 = 16, \ldots$

$a_2 - a_1 = 3 = 2\times 1 + 1,\quad a_3 - a_2 = 5 = 2\times 2 + 1,$

$a_4 - a_3 = 7 = 2\times 3 + 1$

$\therefore \boxed{a_{n+1} - a_n = 2n+1}$ for $n \geq 1$

vi) $a_n = 2 - (-1)^n$

$a_1 = 3,\quad a_2 = 1,\quad a_3 = 3,\quad a_4 = 1,\quad a_5 = 3,\quad a_6 = 1, \ldots$

~~$a_2 - a_1 = -2,\quad a_4 - a_3 = -2,\quad a_5 - a_4 =$~~

$a_2 - a_1 = -2,\quad a_3 - a_2 = 2,\quad a_4 - a_3 = -2$

$\therefore \boxed{a_{n+1} - a_n = 2(-1)^n}$

Here, we have alternate +ve and −ve 2's.
When the second term (ie, $a_n$) is odd term (such as $a_1, a_3, a_5, \ldots$), it takes negative value.
When the second term (ie, $a_n$) is even term, (such as $a_2, a_4, \ldots$), it takes positive value.

Also, ~~$(-1)^{k+1}$ positive~~

$(-1)^{\text{even number}} = \text{positive number}$

$(-1)^{\text{odd number}} = \text{negative number}$

$\therefore$ in general it is $(-1)^n$ to get alternate −ve & +ve numbers.

2) A Sequence $\{a_n\}$ is defined recursively by

$$a_1 = 4,\ a_n = a_{n-1} + n \quad \text{for } n \geq 2$$

Sol$^n$

$$a_n = a_{n-1} + n$$

$$= [[a_{n-2}] + n-1] + n$$

$$= a_{n-3} + (n-2) + (n-1) + n$$

$$= a_{n-4} + (n-3) + (n-2) + (n-1) + n$$

$$\vdots$$

$$= a_1 + 2 + 3 + 4 + \cdots + n$$

$a_3 = a_2 + 3$
$= (a_1 + 2) + 3$

$a_4 = a_3 + 4$
$= (a_1 + 2) + 3$

$a_5 = a_4 + 5$
$= a_1 + 2 + 3 +$
$\cdots$

$a_n = a_1 + 2 + 3 +$

We have $a_1 = 4$

$$\therefore a_n = 4 + 2 + 3 + 4 + \cdots + n$$

$$= 3 + [1 + 2 + 3 + \cdots + n] = 3 + \frac{n(n+1)}{2}$$

$\therefore$ explicit formula for $a_n$ is

$$\boxed{a_n = 3 + \frac{n(n+1)}{2}}$$



3) Find an explicit def$^n$ of the sequence defined recursively by

$$a_1 = 7,\ a_n = 2a_{n-1} + 1 \quad \text{for } n \geq 2$$

Sol

$$a_n = 2a_{n-1} + 1$$

$$a_{n-1} = 2a_{n-2} + 1$$

$$a_n = 2\left[2a_{n-2} + 1\right] + 1$$

$$= 2\left[2\{2a_{n-3} + 1\} + 1\right] + 1$$

$$= 2^3 a_{n-3} + 2^2 + 2 + 1$$

$$\vdots$$

$$= 2^{n-1} a_{n-(n-1)} + 2^{n-2} + 2^{n-3} + \cdots + 2^2 + 2 + 1$$

$$= 2^{n-1} a_1 + 2^{n-2} + 2^{n-3} + \cdots + 2^2 + 2 + 1$$

we have $a_1 = 7$

$$\therefore a_n = 2^{n-1} \cdot 7 + 2^{n-2} + 2^{n-3} + \cdots + 2^2 + 2 + 1$$

we have standard result

$$\left[1 + a + a^2 + \cdots + a^{n-1} = \frac{a^n - 1}{a - 1} \text{ for } a > 1\right]$$

$$\therefore a_n = 7 \times 2^{n-1} + 2^{(n-1)} - 1 = 2^{n-1}\left[7 + 1\right] - 1$$

$$\boxed{a_n = (8 \times 2^{n-1}) - 1}$$

4) If $F_0, F_1, F_2, \ldots$ are Fibonacci numbers, prove that $\sum_{i=0}^{n} F_i^2 = F_n \times F_{n+1}$ for all positive integers $n$.

Sol$^n$ [Note: Fibonacci no's are recursively defined by, $F_0 = 0$, $F_1 = 1$ & $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$
$F_2 = 1 + 0 = 1$, $F_3 = F_2 + F_1 = 2$, $F_4 = F_3 + F_2 = 3$ so on]

Let us consider,

$$\sum_{i=0}^{1} F_i^2 = F_0^2 + F_1^2 = 0^2 + 1^2 = 1 = 1 \times 1 = F_1 \times F_2$$

This ~~requires~~ the verifies the required result for $n=1$

Inductive step: Assume the result is true for $n = k$, $k \geq$

$$\sum_{i=0}^{k} F_i^2 = F_k \times F_{k+1}$$

Consequently,

$$\sum_{i=0}^{k+1} F_i^2 = \sum_{i=0}^{k} F_i^2 + F_{k+1}^2$$

$$= F_k \times F_{k+1} + F_{k+1}^2$$

$$= F_{k+1} \times (F_k + F_{k+1})$$

$$\sum_{i=0}^{k+1} F_i^2 = F_{k+1} \times F_{k+2}$$

[$\because$ according to the defn of Fibonacci series, $F_{k+2} = F_{k+1} + F_k$]

Thus the result is true for $n = k+1$.

Hence, by mathematical induction, ~~the~~ we have proved that

$$\sum_{i=0}^{n} F_i^2 = F_n \times F_{n+1} \quad \forall \text{ +ve integers } n.$$

5) Lucas numbers are defined recursively by,

$L_0 = 2,\ L_1 = 1,\ L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$

Evaluate $L_2$ to $L_{10}$

$L_2 = L_1 + L_0 = 1 + 2 = 3$

$L_3 = L_2 + L_1 = 3 + 1 = 4$

$L_4 = L_3 + L_2 = 4 + 3 = 7$

$L_5 = L_4 + L_3 = 7 + 4 = 11$

$L_6 = L_5 + L_4 = 11 + 7 = 18$

$L_7 = L_6 + L_5 = 18 + 11 = 29$

$L_8 = L_7 + L_6 = 29 + 18 = 47$

$L_9 = L_8 + L_7 = 47 + 29 = 76$

$L_{10} = L_9 + L_8 = 76 + 47 = 123$

---

6) If $F_i$'s are Fibonacci numbers and $L_i$'s are Lucas
(*) numbers, prove that

$$L_n = F_{n-1} + F_{n+1} \text{ for all positive integers } n.$$

Soln W.k.t,

$F_0 = 0,\ F_1 = 1,\ F_2 = 1,\ F_3 = 2, \ldots\ F_n = F_{n-1} + F_{n-2}$

$L_0 = 0,\ L_1\ 3,\ L_3 = 4,\ L_4 = 7, \ldots\ L_n = L_{n-1} + L_{n-2}$

$L_1 = 1 = 0 + 1 = F_0 + F_2$

$L_2 = 3 = 1 + 2 = F_1 + F_3$

$L_3 = 4 = 1 + 3 = F_2 + F_4$

Hence the result is true for $n = 1, 2, 3$.

Assume that the result is true for $n = 1, 2, \ldots, k$ for $k \geq 2$.

$\therefore \boxed{L_k = F_{k-1} + F_{k+1}}$

$\cancel{L_{k+1} 7 F_k}$

we have,

$$L_{k+1} = \underbrace{L_k}_{(F_{k-1} + F_{k+1})} + \underbrace{L_{k-1}}_{(F_{k-2} + F_k)} \quad (\text{by defn of } L_n)$$

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$$L_{k+1} = (F_{k-1} + F_{k+1}) + (F_{k-2} + F_k)$$ [by our assumption]

$$= (F_{k-1} + F_{k-2}) + (F_k + F_{k+1})$$

$$\boxed{L_{k+1} = F_k + F_{k+2}}$$ (by the definition of $F_n$)

This shows that the result is true for $n = k+1$.

$\therefore$ by mathematical induction method, we proved that

$$L_n = F_{n-1} + F_{n+1}$$

(*) 7) For the Fibonacci sequence, $F_0, F_1, F_2, \ldots$, prove that

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

Soln For $n=0$,

$$F_0 = \frac{1}{\sqrt{5}}\left[1 - 1\right] = 0$$

For $n=1$,

$$F_1 = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)\right] = \frac{1}{\sqrt{5}}\left[\frac{2\sqrt{5}}{2}\right] = 1$$

$\therefore$ It is true for $n = 0, 1$

Assume that the result is true for $n = 0, 1, 2, \ldots, k$ where $k \geq 1$.

According to the definition of Fibonacci series

$$F_{k+1} = F_k + F_{k-1}$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right] + \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k-1} - \left(\frac{1-\sqrt{5}}{2}\right.\right.$$

(using the assumption made)

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k} + \left(\frac{1+\sqrt{5}}{2}\right)^{k-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k} - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\right]$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k-1}\left\{\frac{1+\sqrt{5}}{2}+1\right\} - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\left\{\frac{1-\sqrt{5}}{2}+1\right\}\right]$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k-1}\left\{\frac{3+\sqrt{5}}{2}\right\}\times\frac{2}{2} - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\left\{\frac{3-\sqrt{5}}{2}\right\}\times\frac{2}{2}\right]$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k-1}\left\{\frac{6+2\sqrt{5}}{4}\right\} - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\left\{\frac{6-2\sqrt{5}}{4}\right\}\right]$$

$$\left[\left(\frac{1+\sqrt{5}}{2}\right)^{2} = \frac{1+5+2\sqrt{5}}{4} = \frac{6+2\sqrt{5}}{4}\right]$$

$$\therefore F_{k+1} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k-1}\left(\frac{1+\sqrt{5}}{2}\right)^{2} - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\left(\frac{1-\sqrt{5}}{2}\right)^{2}\right]$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\right]$$

This shows that the required result is true for $n=k+1$. Hence, by mathematical induction (alternative form), the result is true for all non negative integers $n$.

8) (*) If $F_0, F_1, \ldots, F_n$ are Fibonacci numbers, prove that

$$\sum_{i=1}^{n} \frac{F_{i-1}}{2^i} = 1 - \frac{F_{n+2}}{2^n}$$ for all positive integers $n$.

Soln For $n=1$, the required result is

$$\frac{F_0}{2^1} = 1 - \frac{F_3}{2^1} = 1 - 1 = 0 \qquad (F_0 = 0,\ F_3 = 2)$$

$\therefore$ It is true for $n=1$.

Assume that the result is true for $n = k \geq 1$.

$$\sum_{i=1}^{k} \frac{F_{i-1}}{2^i} = 1 - \frac{F_{k+2}}{2^k} \quad \text{——— (1)}$$

We have to prove that

$$\sum_{i=1}^{k+1} \frac{F_{i-1}}{2^i} = 1 - \frac{F_{k+3}}{2^{k+1}}$$

w.k.t,

$$\sum_{i=1}^{k+1} \frac{F_{i-1}}{2^i} = \sum_{i=1}^{k} \frac{F_{i-1}}{2^i} + \frac{F_k}{2^{k+1}}$$

$$= \left(1 - \frac{F_{k+2}}{2^k}\right) + \frac{F_k}{2^{k+1}} \quad \text{(by the assumption made (1))}$$

$$= 1 - \frac{F_{k+2}}{2^k} \cdot \frac{2}{2} + \frac{F_k}{2^{k+1}}$$

$$= 1 - \frac{1}{2^{k+1}}\left[2F_{k+2} - F_k\right]$$

$$= 1 - \frac{1}{2^{k+1}}\left[(F_{k+2} - F_k) + F_{k+2}\right]$$

$$= 1 - \frac{1}{2^{k+1}}\left[F_{k+1} + F_{k+2}\right]$$

$F_{k+2} = F_k + F_{k+1}$

$F_{k+1} = F_{k+2} - F_k$

We know that,

$$F_{k+3} = F_{k+2} + F_{k+1}$$

By substituting this, we get

$$\sum_{i=1}^{k+1} \frac{F_{i-1}}{2^i} = 1 - \frac{1}{2^{k+1}}\left(F_{k+3}\right)$$

The result is true for $n = k+1$.

Hence, by mathematical induction, we have proved that

$$\sum_{i=1}^{n} \frac{F_{i-1}}{2^i} = 1 - \frac{1}{2^n} F_{n+2}$$

# UNIT 2: FUNDAMENTAL PRINCIPLES OF COUNTING



## The rule of sum:

If a first task can be performed in 'm' ways, while a second task can be performed in 'n' ways, and the two tasks cannot be performed simultaneously, then performing either task can be accomplished in any one of $(m+n)$ ways.

**Example:** A computer science instructor who has, five introductory books each on C++, FORTRAN, Java, and Pascal can recommend any of these 20 books to a student who is interested in learning a first programming language.

## The rule of Product:

If a procedure can be broken down into first and second stages, and if there are 'm' possible outcomes for the first stage and if, for each of these outcomes, there are 'n' possible outcomes for the second stage, then the total procedure can be carried out, in the designated order, in $(mn)$ ways.

- This rule is also referred to as the principle of choice.



## Examples:

### The Rule of Sum

1) There are 16 boys and 18 girls in a class. In how many ways you can select one of these students as the class representative?

Soln: No. of ways of selecting a boy is 16

No. of ways of selecting a girl is 18

∴ total no. of ways of selecting a student (boy or girl) is $16 + 18 = \underline{34}$

2) Suppose $T_1$ is the task of selecting a prime number less than 10 and $T_2$ is the task of selecting an even less than 10, in how many ways either $T_1$ or $T_2$ can be done?

Soln: $T_1$ can be performed in 4 ways (selecting 2 or 3 or 5 or 7)
$T_2$ can be performed in 4 ways (selecting 2 or 4 or 6 or 8)
Since 2 is both a prime and an even number less than 10, the task $T_1$ or $T_2$ can be performed in
$4+4-1 = 7$ ways.

---

Examples on The Rule of Product

1) If a person has 3 shirts and 5 ties, in how many ways he can choose a shirt and a tie?

Soln: Person has 3 shirts namely p, q, and r
Person has 5 ties namely a, b, c, d, and e
Possibilities of 1 shirt and 1 tie are
(p, a), (p, b), (p, c), (p, d), (p, e)
(q, a), (q, b), (q, c), (q, d), (q, e)
(r, a), (r, b), (r, c), (r, d), (r, e)
∴ he has $3 \times 5 = 15$ ways of choosing a shirt and a tie.

2) In how many ways it is possible to construct sequences of four symbols in which first 2 are English letters and the next 2 are single digit numbers, and no letter and digit can be repeated?

Soln: No. of English letters is 26
No. of single digits is 10 (0 to 9)

Constructing 4 symbols ~~3~~ - 4 tasks namely $T_1, T_2, T_3,$

$T_1 \rightarrow$ selecting ~~26~~ a letter among 26 letters is done in 26 ways

$T_2 \rightarrow$ If $T_1$ is already selected, then it cannot be used again. $\therefore$ second letter can be selected in 25 ways.

$T_3 \rightarrow$ First digit can be selected in 10 ways.

$T_4 \rightarrow$ ~~Seto~~ second digit can be selected in 9 ways.

$\therefore$ we can construct the sequence in

$26 \times 25 \times 10 \times 9 = 58500$ ways.



3) A restaurant sells 6 South Indian dishes, 4 North Indian dishes, 3 hot beverages, and 2 cold beverages. In how many ways, a visitor can select one South Indian dish and one hot beverage or 1 North Indian dish and a cold beverage?

Soln: First choice in $6 \times 3 = 18$ ways (Product Rule)

Second choice in $4 \times 2 = 8$ ways (Product Rule)

$\therefore$ the total number of ways the visitor can select his/her meal is $18 + 8 = 26$ (Sum Rule)

4) There are four bus routes between places A and B, and three bus routes between places B and C. Find the number of ways a person can make round trip from A to A via B if he does not use a route more than once.

Soln: A to B in 4 ways

B to C in 3 ways

C to B in $(3-1) = 2$ ways (does not take the same route)

B to A in $(4-1) = 3$ ways.

∴ the number of ways he can make the round trip is $4 \times 3 \times 2 \times 3 = 72$.

5) Let A be a set with $n$ elements. In how many different sequences, each of length $r$, can be formed using the elements from A if the elements in the sequence may be repeated?

Sol^n: The sequence has length '$r$'. Each place in the sequence can be filled in '$n$' different ways.

i.e, 1st place  2nd place  3rd place ... $r^{th}$ place.

n ways  n ways  n ways ... n ways.

$n \times n \times n \ldots \times n$ [$r$ times]

∴ There are $n^r$ ways of filling the $r$ places in the sequence. [or there are $n^r$ possible sequences]

6) Find the number of binary sequences of length $n$.

Sol^n:- Binary sequence - either 0 or 1.

Sequence of length $n$ contains $n$ position

∴ No. of ways of filling '$n$' positions is $2^n$.

7) A bit is 0 or 1. A byte is a sequence of 8 bits.
Find i) the number of bytes

ii) No. of bytes that begin with 11 and end with 11

iii) No. of bytes that begin with 11 and do not end with 11

iv) No. of bytes that begin with 11 or end with 11.

Solⁿ i) Each byte contains 8 bits. ∴ length is 8.
Each bit is 0 or 1 (2 possible elements)
∴ number of bytes is $2^8 = \underline{\underline{256}}$

ii) Byte beginning & ending with 11

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 1 | | | | | 1 | 1 |

There are 4 open positions. These can be filled in
$2^4 = 16$ ways.
∴ There are 16 bytes that begin ~~with~~ and end
with 11

iii) Bytes begin with 11 but do not end with 1

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 1 | | | | | | |

6 open positions

These can be filled in $2^6 = 64$ ways.
∴ there are 64 bytes that begin with 11.
But, there are 16 bytes that begin and end with
11 (ii)
∴ Bytes ~~that~~ begin with 11 and do not begin with
11 is $64 - 16 = \underline{48}$

iv) The no of bytes begin with 11 is 64 — (1)
The no of bytes end with 11 is 64 — (2)
The no of bytes begin and end with 11 is 16 — (3)
∴ The number of bytes <u>begin or end</u> with 11 is
(1) + (2) − (but not both) (3)
i.e, $64 + 64 - 16 = 112$.

$|A \cup B| = |A| + |B| - |A \cap$

## PERMUTATIONS

Suppose that we are given 'n' distinct objects and wish to arrange 'r' of these objects in a line. There are n ways of choosing the first object, (n-1) ways of choosing second element, ..., finally $(n-r+1)$ ways of choosing $r^{th}$ object.

Then, it follows by the product rule of counting that the number of different arrangements or permutations is $n(n-1)(n-2)\ldots(n-r+1)$.

We denote this by $P(n,r)$ or $^{n}P_{r}$ and is referred to as the number of permutations of size 'r' of 'n' objects.

$$P(n,r) = n(n-1)(n-2)\ldots(n-r+1)$$

$$P(n,r) = \frac{n(n-1)(n-2)\ldots(n-r+1)(n-r)(n-r-1)\ldots 2\cdot 1}{(n-r)(n-r-1)(n-r-2)\ldots 2\cdot 1}$$

$$\boxed{P(n,r) = \frac{n!}{(n-r)!}}$$

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Examples:

1) How many different words of length 4 can be formed using the letters of word FLOWER?

Soln: There are 6 letters and all are distinct.

∴ required no. of words is

$$6P_4 = \frac{6!}{(6-4)!} = \frac{6\times5\times4\times3\times2\times1}{2\times1} = 360$$

2) Find the number of permutations of the letters of the word SUCCESS.

Soln: There are 7 letters. There are 3 S's, 2 C's, 1 U & 1 E.

∴ Required no of permutation is

$$\frac{7!}{3!\cdot2!\cdot1!\cdot1!} \Rightarrow \frac{\text{arrangement of all 7 letters which such repetitive letters i.e., } 7P_7}{\text{repetitive S} \times \text{repetitive C}}$$

$$= \frac{7\times6\times5\times\cancel{4}^{2}\times\cancel{3\times2\times1}}{\cancel{3\times2}\times\cancel{2}\times1} = 420$$

3) Find the number of permutations of the letters of the word MASSASAUGA. In how many of these, all four A's are together? How many of them begin with S?

Soln: The given word has 10 letters with 4 A's, 3 S's, and M, U, & G are appeared one

∴ the required no of permutations are

$$\frac{10!}{4!\,3!\,1!\,1!\,1!} = 25,200$$

If, in a permutation, all A's are to be together, we treat all of A's as one single letter.

∴ letters to be permuted are (AAAA), S, S, S, M, U, G.

∴ No of permutation is $\dfrac{7!}{1!\,3!\,1!\,1!\,1!} = 840$

For permutations to begin with S,
once the first letter is decided, then there are 9 positions to fill, where two are S, four are A.

∴ No of such permutations is

$$\frac{9!}{2!\,4!\,1!\,1!\,1!} = 7560$$

8) How many positive integers n can we form using the ~~letters~~ digits 3, 4, 4, 5, 5, 6, 7 if we want to exceed 5,000,000 ?

Soln: Here n must be of the form

$n = x_1 x_2 x_3 x_4 x_5 x_6 x_7$ [Since 5,000,000 has 7 digits]

$x_1$ can take either 5, or 6 or 7 to exceed 5,000,000.

~~There are~~ ~~the~~ $x_i$'s.

There are two 4's and two 5's.

If $x_1 = 5$, Then $x_2 x_3 x_4 x_5 x_6 x_7$ arrangement has two 4's and only one 5.

∴ No of arrangement is

$$\frac{6!}{2!\,1!\,1!\,1!\,1!} = 360$$

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If $x_1 = 6$, then the arrangement $x_2 x_3 x_4 x_5 x_6 x_7$ has two 4's and two 5's.

$\therefore$ No of such arrangements is

$$\frac{6!}{2!\,2!} = 180$$

If $x_1 = 7$, the number of arrangement is

$$\frac{6!}{2!\,2!} = 180$$

By the sum rule, the number of n's of the desired type is $360 + 180 + 180 = 720$.

6) Find the value of $n$ so that $2P(n,2) + 50 = P(2n$

Soln $2P(n,2) + 50 = P(2n, 2)$

$$2 \times \frac{n!}{(n-2)!} + 50 = \frac{(2n)!}{(2n-2)!}$$

$$2 \times \frac{n \times (n-1) \times \cancel{(n-2) \times \cdots \times 2 \times 1}}{\cancel{(n-2)(n-3) \cdots (2)(1)}} + 50 = \frac{2n \times (2n-1) \cancel{(2n-2) \cdots}}{\cancel{(2n-2)(2n-3) \cdots 2 \times}}$$

$$2n(n-1) + 50 = 2n(2n-1)$$

$$\cancel{2n(n-1) + 2n(2n-1) =}$$

$$2n(2n-1) - 2n(n-1) = 50$$

$$2n[2n - \cancel{1} - n + \cancel{1}] = 50$$

$$2n^2 = 50 \quad \text{or} \quad n^2 = 25$$

$\therefore$ $n = 5$ or $-5$. Since $n$ cannot be negative,

$$\boxed{n = 5}$$

7) It is required to seat 5 men and 4 women in a row so that the women occupy the even places. How many such arrangements are possible?

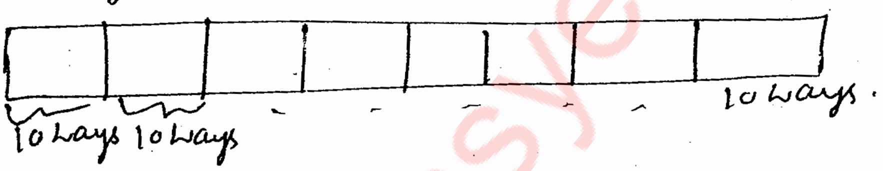
Solⁿ: 5 men may be seated in 5! ways.
4 women may be seated in 4! ways.

Corresponding to each arrangement of the men there is an arrangement of the women, of the desired type.
∴ total no. of arrangements of the is

$$5! \times 4! = 120 \times 24 = \underline{2880}$$

8) How many 8-digit telephone numbers have one or more repeated digits?

Solⁿ:- 8 digit number with repetition of integers




∴ $10^8$ ways.

There are $10^8$ ~~numbers~~ 8-digit numbers with repetition of integers. Of these, ${}^{10}P_8$ numbers do not contain repetitions.

∴ Numbers with one or more repeated digits are

$$\boxed{10^8 - {}^{10}P_8}$$

## COMBINATIONS

Selecting a set of $r$ objects from a set of $n \geq r$ objects without regard to order is called a combination of $r$ objects. It is denoted as

$$C(n,r) \equiv {}^nC_r \equiv \binom{n}{r} \quad \text{and}$$

$$C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{(n-r)!\,r!} \quad \text{for } 0 \leq r \leq n.$$

Examples:

1) How many committees of five with a given chairperson can be selected from 12 persons?

Sol$^n$:- The chairperson can be chosen in 12 ways, and following this, the other four on the committee can be selected in ${}^{11}C_4$ ways.

∴ possible numbers of such committees is

$$12 \times {}^{11}C_4 = \underline{3960}$$

2) Find the number of committees of 5 that can be selected from 7 men and 5 ~~men~~ women if the committee is to consist of at least 1 man and at least 1 woman

Sol$^n$:- Committees of 5 among 12 persons can be done ${}^{12}C_5$ ways.

Committee consisting of only men (5 among 7 men) is

$${}^7C_5 = 21$$

Committee consisting of only women (5 among 5 ~~men~~ women) is

$${}^5C_5 = 1$$

∴ Committee of at least 1 man & 1 woman is

$${}^{12}C_5 - {}^7C_5 - {}^5C_1 = \underline{\underline{770}}$$

3) The housing office has decided to appoint, for each floor, one male and one female residential advisor. How many different pairs of advisors can be selected for a seven floor building from 12 male and 15 female candidates?

Sol^n There are 7 floors.

~~12~~ 7 male candidates can be appointed among 12 in $^{12}C_7$ ways

7 female candidates can be appointed among 15 in $^{15}C_7$ ways.

∴ by product rule, the total number of possible pairs of advisors of the required type is,

$$^{12}C_7 \times {}^{15}C_7 = \frac{12!}{5!\,7!} \times \frac{15!}{8!\,7!} = 5,096,520$$

4) A woman has 11 relatives and she wishes to invite 5 of them to dinner. In how many ways can she invite them in the following situations?

i) There is no restriction on the choice

ii) Two particular persons will not attend separately.

iii) Two particular persons will not attend together.

Sol^n i) 5 out of 11 relatives can be invited in

$^{11}C_5 = 462$ ways

ii) Two persons will not attend separately.

Case 1: They both are invited. Then 3 have to be selected from remaining 9.

$$^9C_3 = \frac{9!}{6!\,3!} = 84 \text{ ways}$$

Case 2: Both of them are not invited. Then 5 have to be selected from remaining 9

$$^9C_5 = \frac{9!}{4!\,5!} = 126 \text{ ways}$$

$\therefore$ total no of ways in which invitees can be selected is
$84 + 126 = 210$

iii) Two persons say A & B will not attend together.

Case 1 : A is invited and B is not.

A is already selected and B is also selected (not to invite)

$\therefore$ 2 persons out of 11 are selected.

$\therefore$ From the remaining 9, 4 have to be selected in

$$9_{C_4} = \frac{9!}{5!4!} = 126 \text{ ways}$$



Case 2: B is invited and A is not.

$9_{C_4} = 126$ ways $\parallel^{al}$ to case 1

Case 3: A & B both are not invited

Need to select 5 out of remaining 9 in
$9_{C_5}$ ways i.e., 126 ways.

$\therefore$ total no of ways in which the invitees can be selected is $126 + 126 + 126 = 378$

5) Find the number of 5-digit positive integers such that in each of them every digit is greater than the digit to the right.

Soln: 5 distinct digits can be selected among 10 digits in $\binom{10}{5}$ ways.

once ~~these~~ digits are selected, there is only one way of arranging them in a decreasing order.

$\therefore$ no. of 5-digit +ve integers of the required type is

$$1 \times \frac{10!}{5!5!} = \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5!}{5! \times 5 \times 4 \times 3 \times 2} = 252 \text{ ways}$$

6) From seven consonants and five vowels, how many sets consisting of four different consonents and three different vowels can be formed?

Solⁿ:- 4 different consonents among 7 can be selected in

$\binom{7}{4} = \frac{7!}{4!\,3!}$ ways.

3 different vowels among 5 can be selected in

$\binom{5}{3} = \frac{5!}{2!\,3!}$ ways.

- Now, there are 7 different letters. Number of arrangement of these selected 7 digits is 7! (By the Rule of Product)

∴ number of possible sets is

$$\frac{7!}{4!\,3!} \times \frac{5!}{2!\,3!} \times 7! = 1,764,000$$

7) A party is attented by n persons. If each person in the party shakes hands with all the others in the party, find the number of hand shakes.

Solⁿ: Total number of handshakes is equal to the number of combinations of two persons that can be selected from 'n' persons. This is equal to

$$^nC_2 \text{ or } \binom{n}{2} = \frac{n!}{(n-2)!\,2!} = \frac{n \cdot n-1 \cdot \cancel{(n-2)}!}{\cancel{(n-2)}!\,2!}$$

$$= \frac{1}{2}n(n-1)$$

8) a) How many diagonals are there in a regular polygon with n sides?

b) Which regular polygon has the same number of diagonals as sides?

a) A regular polygon has n sides and n vertices.

- Any two vertices determine either a side or a diagonal

∴ we need to find all combinations of two ~~sides~~ vertices among 'n' vertices. And is equal to $^nC_2$. But $^nC_2$ has both sides and diagonals. Since regular polygon has 'n' sides, it has $(^nC_2 - n)$ diagonals.

$$^nC_2 - n = \frac{n!}{(n-2)!\,2!} - n = \frac{n(n-1)}{2} - n = \frac{1}{2}n(n-3).$$

b) Number of sides (n) is equal to the number of diagonals $\frac{1}{2}n(n-3)$

$$n = \frac{1}{2}n(n-3)$$

$$n^2 - 5n = 0 \quad \text{or} \quad n(n-5) = 0.$$

∴ n = 5. It must be a pentagon.

## Binomial Theorem

One of the basic properties of $\binom{n}{r}$ is that, it is the coefficient of $x^r y^{n-r}$ in the expansion of the expression $(x+y)^n$, where $x$ and $y$ are any real numbers.

ie, $$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}$$

This result is the <u>Binomial Theorem</u> for a positive integral index.

The numbers $\binom{n}{r}$ for $r=0,1,2,\ldots,n$ in the above results are known as the <u>binomial coefficients</u>.

<u>Examples:</u>

1) Prove the following identities for a positive integer $n$

i) $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$

<u>Soln</u>: The Binomial theorem for a positive integral index

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r} \quad \text{—(1)}$$

When $x = y = 1$,

(1) becomes $2^n = \sum_{r=0}^{n} \binom{n}{r} 1^r \cdot 1^{n-r} = \sum_{r=0}^{n} \binom{n}{r} = \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$

ii) $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} + \cdots + (-1)^n \binom{n}{n} = 0$

<u>Soln</u>: W.k.t

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}$$

When $x=-1$ & $y=1$, we get

$$0 = \sum_{r=0}^{n} \binom{n}{r} (-1)^r = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} + \cdots (-1)^r \binom{n}{n}$$

2) Find the coefficient of $x^9y^3$ in the expansion of $(2x-3y)^{12}$

Sol$^n$: By the binomial theorem,

$$(2x-3y)^{12} = \sum_{r=0}^{12} \binom{12}{r} \cdot (2x)^r (-3y)^{12-r}$$

In this expansion, the coefficient of $x^9y^3$ corresponds to $r=9$

$$\binom{12}{9} \cdot (2x)^9 (-3y)^{12-9} = \underline{1946}$$

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## Combinations with repetitions

Suppose we wish to select, with repetition, a combination of 'r' objects from a set of 'n' distinct objects. The number of such selections is given by,

$$C(n+r-1, r) = \binom{n+r-1}{r} = \frac{(n+r-1)!}{r!\,(n-1)!}$$

$$= \binom{r+n-1}{n-1} = C(r+n-1, n-1)$$

&

$C(n+r-1, r) = C(r+n-1, n-1)$ represents the number of combinations of 'n' distinct objects, taken 'r' at a time, with repetitions allowed.

- $C(n+r-1, r) = C(r+n-1, n-1)$ represents number of ways in which r identical objects can be distributed among 'n' distinct containers.
- It also represents the number of nonnegative integer solutions of the equation

$$x_1 + x_2 + \dots + x_n = r.$$

1) A bag contains coins of seven different denominations, at least 12 coins in each denomination. In how many ways can we select a dozen coins from the bag?

Sol^n: $r=12$ i.e., choosing 12 coins from the bag

$n=7$ i.e., 7 distinct denominations

$$C(n+r-1,r) = C(7+12-1,12) = C(18,12)$$

$$= \frac{18\times17\times16\times15\times14\times13\times\cancel{12!}}{\cancel{12!}\ 6\times5\times4\times3\times2} = 18,564$$

2) In how many ways can we distribute 10 identical marbles among 6 distinct containers?

Sol^n: $r=10$, $n=6$

$$C(6+10-1,10) = C(15,10) = \frac{15!}{10!\,5!}$$

$$= 3003$$



3) Find the number of non-negative integer solutions for the equation

$$x_1+x_2+x_3+x_4+x_5=8.$$

Sol^n: We have,

$$x_1+x_2+\cdots+x_n=r$$ which is represented by

$$C(n+r-1,r)$$

Here, $n=5$ & $r=8$

$$C(5+8-1,8) = C(12,8) = 495$$

# Fundamental Principles of counting

## Solutions to VTU Question Papers

1) Maruti Cars come in four models, twelve colours, three engine types, and two transmission types. How many distinct Maruti cars can be manufactured? Of these how many have the same colour?

Sol^n: We can solve this using the Rule of Product

No. of cars with distinct features } = (4 models) × (12 colours) × (3 engine types) × (2 trans^n types) = 4 × 12 × 3 × 2

= 288 different cars



No of cars with same colour } = (4 models) × (1 colour) × (3 engine types) × (2 trans^n types) = 4 × 1 × 3 × 2

= 24 cars have same colour.

2) i) How many arrangements are there for all the letters in SOCIOLOGICAL.
ii) In how many of the arrangements in part (i) are A and G adjacent?
iii) In how many of the arrangements in part (i) are all the vowels adjacent?

1) Sol^n: i) SOCIOLOGICAL. No. of letters is 12

| Letter | Frequency |
|---|---|
| O | 3 |
| C | 2 |
| I | 2 |
| L | 2 |

∴ No. of ways to arrange such sequence is

$$\frac{12!}{3!\,2!\,2!\,2!} = 9979200$$

ii) If A & G are adjacent, AG or GA can be treated as single letter

Case 1: When A is before G (AG)

$$\frac{11!}{2!\,3!\,2!\,2!} = 831600 \text{ ways}$$

Case 2: When G is before A. (GA)

$$\frac{11!}{3!\,2!\,2!\,2!} = 831600 \text{ ways}$$

∴ Total number of arrangements with A & G adjacent is

~~813~~ 831600 + 831600 = 1663200 ways

iii) There are 6 vowels in the given sequence.

O O O I I A

We have letters,

SCLGCL $\boxed{\text{OOOIIA}}$ → This can be treated as single letter

No. of ways to arrange of this type is

(no. of ways to arrange letters in sequence SCLGCL☐) X (no. of ways of arranging only vowels in sequence OOOIIA)

$$= \frac{7!}{2!\,2!} \times \frac{6!}{3!\,2!}$$

= 75,600 ways

3) How many positive integers n can be formed using the digits 3, 4, 4, 5, 5, 6, 7 if we want 'n' to exceed 5,000,000?

[This problem is already solved in notes. Please refer the same]

4) In a certain implementation of the programming language Pascal, an identifier consists of single letter or a letter followed by upto seven symbols, which may be letters or digits. There are 36 reserved words. How many distinct identifiers possible in this version of Pascal?

Soln: Identifier contains 8 positions.

First position can be filled in 26 ways (letter)

Remaining seven positions can be filled in 36 ways (26 alphabets and 10 digits)



∴ Total possible identifiers

$$= \underbrace{26}_{\text{single letter}} + \underbrace{26 \times 36 \times 36 \times 36 \times 36 \times 36 \times 36 \times 36}_{8 \text{ symbols}} - \underbrace{36}_{\text{reserved words}}$$

$$= 26 + 26 \times 36^7 - 36$$

$$= 26 \times 36^7 - 10$$

5) i) How many distinct four digit integers can be formed from the digits 1, 3, 3, 7, 7, 8?

ii) Find the no of arrangements of the letters in TALLAHASSEE which have no adjacent A's.

Solns i) No. of four digit integers (distinct) $= \dfrac{{}^6P_4}{2!\,2!} = \dfrac{6 \times 5 \times 4 \times 3}{2 \times 2}$

$= 90$ such integers

ii) TALLAHASSEE

No. of arrangements without A's is (TLLHSSEE)

$$\frac{8!}{2!\,2!\,2!} = 5040 \text{ ways}$$

There are 9 locations in above sequence for 3-A's to be placed without adjacent A's.

$$\uparrow T \uparrow L \uparrow L \uparrow H \uparrow S \uparrow S \uparrow E \uparrow E \uparrow$$

∴ number of ways to place 3 A's in 9 locations is

$$^9C_3 \text{ or } \binom{9}{3} = \frac{9!}{(9-3)!\,3!} = \frac{\cancel{9}^3 \times \cancel{8}^4 \times 7 \times \cancel{6 \times 5 \times 4 \times 3 \times 2 \times 1}}{\cancel{6!}\ \cancel{3} \times \cancel{2} \times 1}$$

$$= 84 \text{ ways}$$

For one arrangement of sequence T L L H S S E E without A's, there are 84 ways to insert 3 A's.

∴ for 5040 possible sequences, there are

$$5040 \times 84 = 423360 \text{ ways.}$$



6) How many 9-letter words can be formed using the letters of the word DIFFICULT.
[Please refer notes for the solution]

7) Determine the number of 6-digit integers (No leading 0's) in which
i) No digit is repeated
ii) No digit is repeated and it is even
iii) No digit is repeated and is divisible by 5.

Sol<sup></sup>

Soln i) No digit is repeated - Condn is that the number should not start with 0.

$$\underbrace{9}_{\substack{\text{no zero} \\ (1,2,\dots,9)}} \times \underbrace{9}_{\substack{(0,1,2,\dots,9) \\ \text{but except} \\ 1^{st}\text{ digit}}} \times 8 \times 7 \times 6 \times 5 = 136080 \text{ ways}$$

ii) Condn are no digit is repeated, no. should be even, & number is not started with 0

Numbers ending with 0:

$9 \times 8 \times 7 \times 6 \times 5 = 15120$ ways.



Numbers ending with 2:

$$\underbrace{8}_{\substack{\text{all nos except} \\ 0 \text{ and } 2}} \times \underbrace{8}_{\substack{\text{all nos except } 2 \\ \text{and } 1^{st} \text{ digit}}} \times \underbrace{7}_{\substack{\text{all nos except } 2 \\ \text{and } 1^{st},\ 2^{nd} \text{ digit}}} \times 6 \times 5 = 13440 \text{ ways}$$

& so o

IIIly for numbers ending with 4 = $8 \times 8 \times 7 \times 6 \times 5 = 13440$ wa

—————— " —————— 6 = $8 \times 8 \times 7 \times 6 \times 5 = 13440$ wa

—————— " —————— 8 = $8 \times 8 \times 7 \times 6 \times 5 = 13440$ wa

$\therefore$ total no of even numbers with no repetition of digits is $15120 + 4 \times 13440 = 68880$

iii) Numbers ending with 0 and ending with 5 are divisible by 5.

Calculate these using the logic of (ii)

Numbers ending with 0 : $9 \times 8 \times 7 \times 6 \times 5 = 15120$ ways

—————— " —————— 5 : $8 \times 8 \times 7 \times 6 \times 5 = 13440$ ways.

$\therefore$ total numbers divisible by 5 $= 15120 + 13440 = 28560$

8) How many arrangements of the letters in MISSISSIPPI have no consecutive S's?

Soln: Total letters present are 11

No. of arrangements without S's are (eg MIIIPPI)

$$\frac{7!}{4!\,2!} = \underline{105 \text{ ways}}$$

There are 4 S's which can be placed between the letters $\uparrow M \uparrow I \uparrow I \uparrow I \uparrow P \uparrow P \uparrow I \uparrow$ and there are 8 locations to place 4 S's

This ~~can~~ be done in

$$^8C_4 \text{ or } \binom{8}{4} = \frac{8!}{(8-4)!\,4!} = \frac{\cancel{8}^{2}\times 7\times \cancel{6}^{2}\times 5\times \cancel{4!}}{\cancel{4!}\times \cancel{4}_1\times \cancel{3}_1\times \cancel{2}} = \underline{70 \text{ ways}}$$

$\therefore$ Total number of such arrangements are:

$$105 \times 70 = \underline{7350 \text{ ways}}$$

**20.** Prove the following identities:

(i) $C(2n,2) = 2C(n,2) + n^2$

(ii) $C(3n,3) = 3C(n,3) + 6nC(n,2) + n^3$

(iii) $C(n+r+1,r) = C(n+1,1) + C(n+2,2) + \cdots + C(n+r,2)$

(iv) $C(n+1,r+1) = C(r,r) + C(r+1,2) + \cdots C(n,r)$

(v) $n\,C(m+n,m) = (m+1)\,C\,(m+n,m+1)$

(vi) $C(m+n,n) = \sum_{r=0}^{n} C(m,r)C(n,r)$

(vii) $C(n+r+1,r) = \sum_{k=0}^{r} C(n+k,k)$

**Answers**

**1.** $C(9,5)\times C(15,4)$ **2.** $C(20,3)\times C(30,4)$ **3.** $C(5,4)+C(6,4)$ **4.** $C(10,6)$

**5.** (a) $C(5,3)\times C(21,5)\times 8!$ (b) $C(4,2)\times C(20,4)\times 6!$ (c) $C(4,2)\times C(19,3)\times 8!$

**6.** $C(21,3)\times C(5,2)\times 5!$ **7.** 450 **8.** $\sum_{r=1}^{5}\binom{9}{r}\binom{6}{r+1}$

**9.** $C(15,3)\times 3^{12}$ **10.** 39600 **11.** 980

**12.** 7350 **13.** (i) 120 (ii) 50 (iii) 110

**14.** (i) 28 (ii) 70 (iii) 28 (iv) 37 **15.** (a) 350 (b) 150 (c) 105

**16.** (a) 350 (b) 21 (c) 980 (d) 1176 **17.** 32, (a) 5 (b) 10 **18.** 55 **19.** 16

### 5.3.1 Binomial and Multinomial Theorems

One of the basic properties of $C(n,r) \equiv \binom{n}{r}$ is that it is the coefficient of $x^r y^{n-r}$ in the expansion of the expression $(x+y)^n$, where $x$ and $y$ are any real numbers. In other words,

$$(x+y)^n = \sum_{r=0}^{n}\binom{n}{r}x^r y^{n-r}$$

This result is known as the *Binomial Theorem for a positive integral index.*

The numbers $\binom{n}{r}$ for $r = 0, 1, 2, \ldots n$ in the above result are known as the *binomial coefficients.*

The student is already familiar with the proof by mathematical induction of the above-mentioned binomial theorem.

The following is a generalization of the binomial theorem, known as the *Multinomial Theorem**.

**Theorem** : *For positive integers n and t, the coefficient of* $x_1^{n_1}x_2^{n_2}x_3^{n_3}\dots x_t^{n_t}$ *in the expansion of* $(x_1 + x_2 + \cdots + x_t)^n$ *is*

$$\frac{n!}{n_1!\; n_2!\; n_3! \dots n_t!}$$

*where each* $n_i$ *is a nonnegative integer* $\le n$, *and* $n_1 + n_2 + n_3 + \cdots + n_t = n$.

**Proof:** We note that in the expansion of $(x_1 + x_2 + \cdots + x_t)^n$ the coefficient of $x_1^{n_1}x_2^{n_2}\dots x_t^{n_t}$ is the number of ways we can select $x_1$ from $n_1$ of the $n$ factors, $x_2$ from $n_2$ of the $n - n_1$ remaining factors, $x_3$ from $n_3$ of the $n - n_1 - n_2$ remaining factors, and so on. Therefore, this coefficient is, by the product rule,

$$\begin{aligned}
&C(n, n_1)\cdot C(n-n_1, n_2)\cdot C(n-n_1-n_2, n_3)\cdot\cdots\cdot C(n-n_1-n_2-\cdots-n_{t-1}, n_t)\\
&= \frac{n!}{n_1!\,(n-n_1)!}\cdot\frac{(n-n_1)!}{n_2!\,(n-n_1-n_2)!}\cdot\frac{(n-n_1-n_2)!}{n_3!(n-n_1-n_2-n_3)!}\\
&\qquad\cdots\frac{(n-n_1-n_2-\cdots-n_{t-1})!}{n_t!\,(n-n_1-n_2-\cdots-n_{t-1}-n_t)!}\\
&= \frac{n!}{n_1!\; n_2!\; n_3!\;\cdots n_t!}
\end{aligned}$$

This proves the required result. •

**Note:** Another way of stating the Multinomial theorem is: The general term in the expansion of $(x_1 + x_2 + x_3 + \cdots x_t)^n$ is $\frac{n!}{n_1!\,n_2!\cdots n_t!}x_1^{n_1}x_2^{n_2}\cdots x_t^{n_t}$, where $n_1, n_2, \dots n_t$ are nonnegative integers not exceeding $n$ and $n_1 + n_2 + n_3 + \cdots + n_t = n$.

The expression $\frac{n!}{n_1!\,n_2!\cdots n_t!}$ is also written as

$$\binom{n}{n_1, n_2, n_3, \dots n_t}$$

and is called a *multinomial coefficient*.

*The case $t = 2$ of the multinomial theorem corresponds to the Binomial theorem.

**Example 1.** *Prove the following identities for a positive integer n:*

(i) $\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}=2^n$

(ii) $\binom{n}{0}-\binom{n}{1}+\binom{n}{2}+\cdots+(-1)^n\binom{n}{n}=0$

► The Binomial theorem for a positive integral index $n$ reads

$$(x+y)^n=\sum_{r=0}^{n}\binom{n}{r}x^r y^{n-r}.$$

When $x = y = 1$, this becomes

$$2^n=\sum_{r=0}^{n}\binom{n}{r}1^r\cdot 1^{n-r}=\sum_{r=0}^{n}\binom{n}{r}=\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n},$$

and when $x = -1$ and $y = 1$, we get

$$0=\sum_{r=0}^{n}\binom{n}{r}(-1)^r=\binom{n}{0}-\binom{n}{1}+\binom{n}{2}+\cdots+(-1)^n\binom{n}{n}.$$ ■

**Example 2.** *Find the coefficient of* $x^9y^3$ *in the expansion of* $(2x-3y)^{12}$.

► We have, by the binomial theorem,

$$(2x-3y)^{12}=\sum_{r=0}^{12}\binom{12}{r}\cdot(2x)^r(-3y)^{12-r}$$

$$=\sum_{r=0}^{12}\binom{12}{r}2^r(-3)^{12-r}x^r y^{12-r}$$

In this expansion, the coefficient of $x^9y^3$ (which corresponds to $r = 9$) is

$$\binom{12}{9}2^9(-3)^3=-2^9\times 3^3\times\frac{12!}{9!\,3!}=-2^9\times 3^3\times\frac{12\times 11\times 10}{6}$$

$$=-2^{10}\times 3^3\times 11\times 10=1946.$$ ■

**Example 3.** *Evaluate :* $\binom{12}{5,3,2,2}$.

► We have

$$\binom{12}{5,3,2,2} = \frac{12!}{5!\ 3!\ 2!\ 2!} = 166320. \quad ■$$

**Example 4.** *Find the term which contains $x^{11}$ and $y^4$ in the expansion of $(2x^3 - 3xy^2 + z^2)^6$.*

► By the multinomial theorem, the general term in the given expansion is

$$\binom{6}{n_1,\ n_2,\ n_3}(2x^3)^{n_1}(-3xy^2)^{n_2}(z^2)^{n_3} = \binom{6}{n_1,\ n_2,\ n_3}2^{n_1}(-3)^{n_2}x^{3n_1+n_2}y^{2n_2}z^{2n_3}$$

Thus, for the term containing $x^{11}$ and $y^4$ we should have $3n_1 + n_2 = 11$ and $2n_2 = 4$, so that $n_1 = 3$ and $n_2 = 2$. Since $n_1 + n_2 + n_3 = 6$, we should then have $n_3 = 1$. Accordingly, the term containing $x^{11}$ and $y^4$ is

$$\binom{6}{3,\ 2,\ 1}2^3(-3)^2x^{11}y^4z^2 = \left\{\frac{6!}{3!\ 2!\ 1!} \times 8 \times 9\right\}x^{11}y^4z^2 = 4320x^{11}y^4z^2. \quad ■$$

**Example 5.** *Determine the coefficient of*

(i) *$xyz^2$ in the expansion of $(2x - y - z)^4$, and*

(ii) *$a^2b^3c^2d^5$ in the expansion of $(a + 2b - 3c + 2d + 5)^{16}$.*

► (i) By the multinomial theorem, we note that the general term in the expansion of $(2x - y - z)^4$ is

$$\binom{4}{n_1,\ n_2,\ n_3}(2x)^{n_1}(-y)^{n_2}(-z)^{n_3}.$$

For $n_1 = 1, n_2 = 1$ and $n_3 = 2$, this reads

$$\begin{aligned}\binom{4}{1,\ 1,\ 2}(2x)(-y)(-z)^2 &= \binom{4}{1,\ 1,\ 2} \times 2 \times (-1) \times (-1)^2 xyz^2 \\ &= -2 \times \binom{4}{1,\ 1,\ 2}xyz^2 \\ &= -2 \times \frac{4!}{1!\ 1!\ 2!}xyz^2 = -24xyz^2.\end{aligned}$$

Thus, the required coefficient is $-24$.

(ii) By the multinomial theorem, we note that the general term in the expansion of $(a + 2b - 3c + 2d + 5)^{16}$ is

$$\binom{16}{n_1,\ n_2,\ n_3,\ n_4,\ n_5}(a)^{n_1}(2b)^{n_2}(-3c)^{n_3}(2d)^{n_4}(5)^{n_5}.$$

For $n_1 = 2, n_2 = 3, n_3 = 2, n_4 = 5$ and $n_5 = 16 - (2 + 3 + 2 + 5) = 4$, this becomes

$$\binom{16}{2,3,2,5,4} a^2(2b)^3(-3c)^2(2d)^5 5^4 = \binom{16}{2,3,2,5,4} \times 2^3 \times (-3)^2 \times 2^5 \times 5^4 \times a^2b^3c^2d^5$$
$$= 2^8 \times 3^2 \times 5^4 \times \frac{16!}{2!\ 3!\ 2!\ 5!\ 4!} a^2b^3c^2d^5$$
$$= 3 \times 2^5 \times 5^3 \times \frac{16!}{(4!)^2} a^2b^3c^2d^5$$

Thus, the required coefficient is

$$\frac{16! \times 2^5 \times 5^3 \times 3}{(4!)^2}.$$

## Exercises

**1.** Find the coefficient of

(i) $x^9y^3$ in the expansion of $(x + 2y)^{12}$ (ii) $x^5y^2$ in the expansion of $(2x - 3y)^7$.

**2.** Show that $\binom{n}{n_1, n_2} = \binom{n}{n_1} = \binom{n}{n_2}$.

**3.** Compute the following:

(i) $\binom{7}{2,3,2}$ (ii) $\binom{8}{4,2,2,0}$ (iii) $\binom{10}{5,3,2,2}$.

**4.** Find the coefficient of:

(i) $xyz^5$ in the expansion of $(x + y + z)^7$.

(ii) $xyz^{-2}$ in the expansion of $(x - 2y + 3z^{-1})^4$.

(iii) $x^3z^4$ in the expansion of $(x + y + z)^7$.

(iv) $x^3y^3z^2$ in the expansion of $(2x - 3y + 5z)^8$.

(v) $w^3x^2yz^2$ in the expansion of $(2w - x + 3y - 2z)^8$.

(vi) $x_1^2x_3x_4^3x_5^4$ in the expansion of $(x_1 + x_2 + x_3 + x_4 + x_5)^{10}$

**5.** Prove that, if $n$ is a nonnegative integer,

$$\frac{1}{2}\{(1+x)^n + (1-x)^n\} = \binom{n}{0} + \binom{n}{2}x^2 + \cdots + \binom{n}{k}x^k,$$

where $k = \begin{cases} n \text{ if } n \text{ is even} \\ n-1 \text{ if } n \text{ is odd.} \end{cases}$

Deduce that (with $k$ defined as above)

$$\binom{n}{0} + \binom{n}{2} + \cdots + \binom{n}{k} = \begin{cases} 2^{n-1} \text{ for } n > 0 \\ 1 \text{ if } n = 0. \end{cases}$$

**Answers**

**1.** (i) 1760 (ii) 6048

**3.** (i) 210 (ii) 420 (iii) meaningless (why ?)

**4.** (i) 42 (ii) −216 (iii) 35 (iv) −3024000 (v) 161280 (vi) 12600

## 5.3.2 Combinations with Repetitions

Suppose we wish to select, with *repetition*, a combination of $r$ objects from a set of $n$ distinct objects. The number of such selections is given by*

$$\begin{aligned} C(n+r-1, r) &\equiv \binom{n+r-1}{r} = \frac{(n+r-1)!}{r!\,(n-1)!} \\ &= \binom{r+n-1}{n-1} \equiv C(r+n-1, n-1) \end{aligned}$$

In other words, $C(n+r-1, r) = C(r+n-1, n-1)$ represents the number of combinations of $n$ distinct objects, taken $r$ at a time, with repetitions allowed.

The following are other interpretations of this number:

(i) $C(n+r-1, r) = C(r+n-1, n-1)$ represents the number of ways in which $r$ identical objects can be distributed among $n$ distinct containers.

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*For proofs, see Example 11, Section 7.1.2 and Example 8, Section 8.4.2.

(ii) $C(n+r-1, r) = C(r+n-1, n-1)$ represents the number of nonnegative integer solutions[†] of the equation

$$x_1 + x_2 + \cdots + x_n = r.$$

**Example 1.** *A bag contains coins of seven different denominations, with at least one dozen coins in each denomination. In how many ways can we select a dozen coins from the bag?*

▶ The selection consists in choosing *with repetitions*, $r = 12$ coins of $n = 7$ distinct denominations. The number of ways of making this selection is

$$C(7+12-1, 12) = C(18, 12) = \frac{18!}{12!\,6!} = 18,564. \qquad \blacksquare$$

**Example 2.** *In how many ways can we distribute 10 identical marbles among 6 distinct containers?*

▶ The required number is

$$C(6+10-1, 10) = C(15, 10) = \frac{15!}{10!\,5!} = 3003. \qquad \blacksquare$$

**Example 3.** *Find the number of nonnegative integer solutions of the equation*

$$x_1 + x_2 + x_3 + x_4 + x_5 = 8.$$

▶ The required number is

$$C(5+8-1, 8) = C(12, 8) = 495. \qquad \blacksquare$$

**Example 4.** *Find the number of distinct terms in the expansion of*

$$(x_1 + x_2 + x_3 + x_4 + x_5)^{16}.$$

▶ Every term in the expansion is of the form (by multinomial theorem)

$$\binom{16}{n_1,\ n_2,\ n_3,\ n_4,\ n_5} x_1^{n_1} x_2^{n_2} x_3^{n_3} x_4^{n_4} x_5^{n_5}$$

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[†]A nonnegative integer solution of the equation $x_1 + x_2 + \cdots + x_n = r$ is an $n$-tuple $(x_1, x_2, x_3, \ldots x_n)$, where $x_1, x_2, \ldots x_n$ are nonnegative integers whose sum is $r$.

where each $n_i$ is a nonnegative integer, and these $n_i$s sum to 16. Therefore, the number of distinct terms in the expansion is precisely equal to the number of nonnegative integer solutions of the equation

$$n_1 + n_2 + n_3 + n_4 + n_5 = 16.$$

This number is

$$C(5 + 16 - 1, 16) = C(20, 16) = 4845. \qquad \blacksquare$$

**Example 5.** *Find the number of nonnegative integer solutions of the inequality*

$$x_1 + x_2 + x_3 + \cdots + x_6 < 10.$$

► We have to find the number of nonnegative integer solutions of the equation

$$x_1 + x_2 + x_3 + \cdots + x_6 = 9 - x_7$$

where $9 - x_7 \leq 9$ so that $x_7$ is a nonnegative integer. Thus, the required number is the number of nonnegative solutions of the equation

$$x_1 + x_2 + x_3 + \cdots + x_7 = 9.$$

This number is

$$C(7 + 9 - 1, 9) = C(15, 9) = \frac{15!}{9!\ 6!} = 5005. \qquad \blacksquare$$

**Example 6.** *Find the number of integer solutions of*

$$x_1 + x_2 + x_3 + x_4 + x_5 = 30$$

*where* $x_1 \geq 2,\ x_2 \geq 3,\ x_3 \geq 4,\ x_4 \geq 2,\ x_5 \geq 0.$

► Let us set $y_1 = x_1 - 2,\ y_2 = x_2 - 3,\ y_3 = x_3 - 4,\ y_4 = x_4 - 2,\ y_5 = x_5$. Then $y_1, y_2, \ldots y_5$ are all nonnegative integers. When written in terms of $y's$, the given equation reads

$$(y_1 + 2) + (y_2 + 3) + (y_3 + 4) + (y_4 + 2) + y_5 = 30, \quad \text{or} \quad y_1 + y_2 + y_3 + y_4 + y_5 = 19.$$

The number of nonnegative integer solutions of this equation is the required number, and the number is

$$C(5 + 19 - 1, 19) = C(23, 19) = \frac{23!}{19!\ 4!} = 8855. \qquad \blacksquare$$

**Example 7.** *In how many ways can we distribute* 12 *identical pencils to* 5 *children so that every child gets at least* 1 *pencil?*

▶ First, we distribute one pencil to each child. Then, there remain 7 pencils to be distributed. The number of ways of distributing these 7 pencils to 5 children is the required number. This number is

$$C(5+7-1,7) = C(11,7) = \frac{11!}{7!\,4!} = 330.$$ ■

**Example 8.** *A total amount of Rs.* 1500 *is to be distributed to* 3 *poor students A, B, C of a class. In how many ways the distribution can be made in multiples of Rs.* 100 (i) *if everyone of these must get at least Rs.* 300*?* (ii) *if A must get at least Rs.* 500*, and B and C must get at least Rs.* 400 *each?*

▶ Taking Rs. 100 as a unit, there are 15 units for distribution.

In case (i), each of the three students must get at least 3 units. Let us first distribute 3 units to each of the 3 students. Then there remain 6 units for distribution. The number of ways of distributing these 6 units to $A$, $B$, $C$ is the required number (in this case). This number is $C(3+6-1,6) = C(8,6) = 28$.

In case (ii), $A$ must get at least 5 units, $B$ and $C$ must get at least 4 units each. Let us distribute 5 units to $A$ and 4 units to each of $B$ and $C$. Then there remain 2 units for distribution. Accordingly, the number of ways of making the distribution in this case is $C(3+2-1,2) = C(4,2) = 6$. ■

**Example 9.** *In how many ways can we distribute 7 apples and 6 oranges among 4 children so that each child gets at least 1 apple?*

▶ Suppose we first give 1 apple to each child. This exhausts 4 apples. The remaining 3 apples can be distributed among the 4 children in $C(4+3-1,3) = C(6,3)$ ways. Also, 6 oranges can be distributed among the 4 children in $C(4+6-1,6) = C(9,6)$ ways. Therefore, by the product rule, the number of ways of distributing the given fruits under the given condition is

$$C(6,3) \times C(9,6) = \frac{6!}{3!\,3!} \times \frac{9!}{6!\,3!} = 20 \times 84 = 1680.$$ ■

**Example 10.** *Find the number of ways of giving 10 identical gift boxes to 6 persons A, B, C, D, E, F in such a way that the total number of boxes given to A and B together does not exceed 4.*

▶ Of the 10 boxes, suppose $r$ boxes are given to $A$ and $B$ together. Then $0 \le r \le 4$. The number of ways of giving $r$ boxes to $A$ and $B$ is

$$C(2 + r - 1, r) = C(r + 1, r) = r + 1. \qquad \text{(i)}$$

The number of ways in which the remaining $(10 - r)$ boxes can be given to $C$, $D$, $E$, $F$ is

$$C(4 + (10 - r) + 1, (10 - r)) = C(13 - r, 10 - r) = C(13 - r, 3). \qquad \text{(ii)}$$

Consequently, the number of ways in which $r$ boxes may be given to $A$ and $B$ and $10 - r$ boxes to $C$, $D$, $E$, $F$ is, by the product rule,

$$(r + 1) \times C(13 - r, 3). \qquad \text{(iii)}$$

Since $0 \le r \le 4$, the total number of ways in which the boxes may be given is, by the sum rule,

$$\sum_{r=0}^{4} (r + 1) \times C(13 - r, 3).$$ ■

**Example 11.** *A message is made up of* 12 *different symbols and is to be transmitted through a communication channel. In addition to the* 12 *symbols, the transmitter will also send a total of* 45 *blank spaces between the symbols, with at least three spaces between each pair of consecutive symbols. In how many ways can the transmitter send such a message?*

▶ The 12 symbols can be arranged in 12! ways. For each of these arrangements, there are 11 positions between the 12 symbols. Since there must be at least three spaces between successive symbols, 33 of the 45 spaces will be used up. The remaining 12 spaces are to be accommodated in 11 positions. This can be done in $C(11 + 12 - 1, 12) = C(22, 12)$ ways. Consequently, by the product rule, the required number is

$$12! \times C(22, 12) = \frac{22!}{10!} = 3.097445 \times 10^{14}.$$ ■

**Example 12.** *Show that $C(n-1+r, r)$ represents the number of binary numbers that contains $(n-1)$ 1's and r 0's.*

► A binary number that contains $(n-1)$ 1's and $r$ 0's, has $n-1+r$ positions and is determined by $r$ positions of 0's. The number of such binary numbers is therefore $C(n-1+r, r)$. ■

**Example 13.** *Given positive integers $m, n$ with $m \geq n$, show that the number of ways to distribute m identical objects into n distinct containers such that each container gets at least r objects, where $r \leq (m/n)$, is*

$$C(m-1+(1-r)n, n-1).$$

► Suppose we place $r$ of the $m$ identical objects into each of the $n$ distinct containers. Then, there remain $(m-nr)$ identical objects to be distributed into $n$ distinct containers. The number of ways of doing this is the required number. This number is

$$\begin{aligned} C(n+(m-nr)-1, m-nr) &= C(n+(m-nr)-1, n-1)^{*} \\ &= C(m-1+(1-r)n, n-1) \end{aligned}$$ ■

## Exercises

1. In how many ways can 20 similar books be placed on 5 different shelves?
2. Find the number of ways of placing 8 identical balls in 5 numbered boxes.
3. Determine the number of nonnegative integer solutions of the equation $x_1 + x_2 + x_3 + x_4 = 7$.
4. Find the number of distinct terms in the expansion of $(w+x+y+z)^{10}$.
5. How many integer solutions are there to $x_1 + x_2 + x_3 + x_4 + x_5 = 20$ where each $x_i \geq 2$?
6. How many integer solutions are there to $x_1 + x_2 + x_3 + x_4 + x_5 = 20$, where $x_1 \geq 3$, $x_2 \geq 2$, $x_3 \geq 4$, $x_4 \geq 6$, $x_5 \geq 0$?

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*Recall that $C(n, r) = C(n, n-r)$.