向量空间

Vector Spaces

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Spaces and Subspaces

- Many mathematical entities that were considered to be quite different from matrices were in fact quite similar.
- For example, objects such as points in the plane $\Re^2$ and $\Re^3$, polynomials, continuous functions, and differentiable functions satisfy the same additive properties and scalar multiplication properties given for matrices.
- Rather than studying each topic separately, it is more efficient and productive to study many topics at one time by studying the common properties that they satisfy.
- This eventually led to the axiomatic definition of a vector space.
- A vector space involves four thingstwo sets $\mathcal{V}$ and $\mathcal{F}$, and two algebraic operations called vector addition and scalar multiplication.

Vector Space Definition

The set $\mathcal{V}$ is called a ***vector space over*** $\mathcal{F}$ when the vector addition and scalar multiplication operations satisfy the following properties.

(A1) $\mathbf{x}+\mathbf{y} \in \mathcal{V}$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{V}$. This is called the ***closure property for vector addition.***

(A2) $(\mathbf{x}+\mathbf{y})+\mathbf{z} = \mathbf{x}+(\mathbf{y}+\mathbf{z})$ for every $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{V}$.

(A3) $\mathbf{x}+\mathbf{y} = \mathbf{y}+\mathbf{x}$ for every $\mathbf{x}, \mathbf{y} \in \mathcal{V}$.

(A4) There is an element $\mathbf{0} \in \mathcal{V}$ such that $\mathbf{x}+\mathbf{0} = \mathbf{x}$ for every $\mathbf{x} \in \mathcal{V}$.

(A5) For each $\mathbf{x} \in \mathcal{V}$, there is an element $(-\mathbf{x}) \in \mathcal{V}$ such that $\mathbf{x}+(-\mathbf{x}) = \mathbf{0}$.

(M1) $\alpha\mathbf{x} \in \mathcal{V}$ for all $\alpha \in \mathcal{F}$ and $\mathbf{x} \in \mathcal{V}$. This is the ***closure property for scalar multiplication.***

(M2) $(\alpha\beta)\mathbf{x} = \alpha(\beta\mathbf{x})$ for all $\alpha, \beta \in \mathcal{F}$ and every $\mathbf{x} \in \mathcal{V}$.

(M3) $\alpha(\mathbf{x}+\mathbf{y}) = \alpha\mathbf{x}+\alpha\mathbf{y}$ for every $\alpha \in \mathcal{F}$ and all $\mathbf{x}, \mathbf{y} \in \mathcal{V}$.

(M4) $(\alpha+\beta)\mathbf{x} = \alpha\mathbf{x}+\beta\mathbf{x}$ for all $\alpha, \beta \in \mathcal{F}$ and every $\mathbf{x} \in \mathcal{V}$.

(M5) $1\mathbf{x} = \mathbf{x}$ for every $\mathbf{x} \in \mathcal{V}$.

- The formal definition of a vector space stipulates how these four things relate to each other.
- $\mathcal{V}$ is a nonempty set of objects called ***vectors***. Although $\mathcal{V}$ can be quite general, we will usually consider $\mathcal{V}$ to be a set of n-tuples or a set of matrices.
- $\mathcal{F}$ is a scalar fieldfor us $\mathcal{F}$ is either the field $\Re$ of real numbers or the field $\mathcal{C}$ of complex numbers.
- Vector addition (denoted by $x + y$) is an operation between elements of $\mathcal{V}$.
- Scalar multiplication (denoted by αx) is an operation between elements of $\mathcal{F}$ and $\mathcal{V}$.

Example 1

The set $\Re^{m\times n}$ of $m \times n$ real matrices is a vector space over $\Re$.
The set $\mathcal{C}^{m\times n}$ of $m \times n$ real matrices is a vector space over $\mathcal{C}$.

Example 2

The ***real coordinate spaces***

$$\Re^{1\times n} = \{(x_1, x_2, \cdots, x_n), x_i \in \Re\}$$

$$\Re^{n\times 1} = \{(x_1, x_2, \cdots, x_n)^T, x_i \in \Re\}$$

Example 3

With function addition and scalar multiplication defined by

$$(f+g)(x) = f(x) + g(x) \text{ and } (\alpha f)(x) = \alpha f(x),$$

the following sets are vector spaces over $\Re$:

1. *The set of functions mapping the interval [0, 1] into $\Re$.*
2. *The set of all real-valued continuous functions defined on [0, 1].*
3. *The set of real-valued functions that are differentiable on [0, 1].*
4. *The set of all polynomials with real coefficients.*

Subspaces

Let $\mathcal{S}$ be a nonempty subset of a vector space $\mathcal{V}$ over $\mathcal{F}$ (symbolically, $\mathcal{S} \subseteq \mathcal{V}$). If $\mathcal{S}$ is also a vector space over $\mathcal{F}$ using the same addition and scalar multiplication operations, then $\mathcal{S}$ is said to be a ***subspace*** of $\mathcal{V}$. It's not necessary to check all 10 of the defining conditions in order to determine if a subset is also a subspace—only the closure conditions **(A1)** and **(M1)** need to be considered. That is, a nonempty subset $\mathcal{S}$ of a vector space $\mathcal{V}$ is a subspace of $\mathcal{V}$ if and only if

$$\textbf{(A1)} \qquad \mathbf{x}, \mathbf{y} \in \mathcal{S} \implies \mathbf{x} + \mathbf{y} \in \mathcal{S}$$

and

$$\textbf{(M1)} \qquad \mathbf{x} \in \mathcal{S} \implies \alpha\mathbf{x} \in \mathcal{S} \text{ for all } \alpha \in \mathcal{F}.$$

Example 4

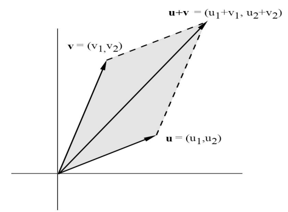
Given a vector space $\mathcal{V}$, the set $\mathcal{Z} = 0$ containing only the zero vector is a subspace of $\mathcal{V}$. This subspace is called the ***trivial subspace***.

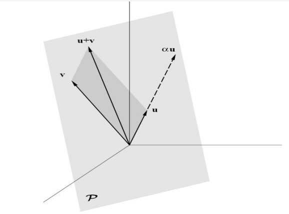
Example 5

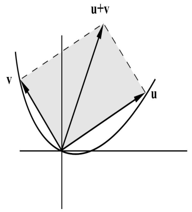
Straight lines through the origin in $\Re^2$ and $\Re^3$ are subspaces.

Example 6

In $\Re^3$, Planes through the origin are also subspaces.







- Questions:
  - What about straight lines not through the origin?
  - What about curved lines through the origin?

- Visual interpretation: Subspaces are the flat surfaces passing through the origin.
- For a set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_r\}$ from a vector space $\mathcal{V}$, the set of all possible linear combinations of the $\mathbf{v}_i$'s is denoted by

$$span(S) = \{\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_r\mathbf{v}_r | \alpha_i \in \mathcal{F}\}.$$

- Notice that $span(S)$ is a subspace of $\mathcal{V}$.
- In fact, all subspaces of $\Re^n$ are of the type $span(S)$.
- If $\mathbf{u} \neq 0$ is a vector in $\Re^3$, then $span\{\mathbf{u}\}$ is the straight line passing through the origin and $\mathbf{u}$.
- If $S = \{\mathbf{u}, \mathbf{v}\}$, where $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors in $\Re^3$, $span(S)$ is the plane passing through the origin and the points $\mathbf{u}$ and $\mathbf{v}$.

Spanning Sets

- For a set of vectors $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$, the subspace

$$span(\mathcal{S}) = \{\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_r\mathbf{v}_r\}$$

  generated by forming all linear combinations of vectors from $\mathcal{S}$ is called the ***space spanned by*** $\mathcal{S}$.
- If $\mathcal{V}$ is a vector space such that $\mathcal{V} = span(\mathcal{S})$, we say $\mathcal{S}$ is a ***spanning set*** for $\mathcal{V}$. In other words, $\mathcal{S}$ ***spans*** $\mathcal{V}$ whenever each vector in $\mathcal{V}$ is a linear combination of vectors from $\mathcal{S}$.

Example 7

1. The unit vectors $\{e_1, e_2, \cdots, e_n\}$ form a spanning set for $\mathfrak{R}^n$.
2. The finite set $\{1, x, x^2, \cdots, x^n\}$ spans the space of all polynomials such that deg $p(x) \leq n$, and the infinite set $\{1, x, x^2, \cdots\}$ spans the space of all polynomials.

Example 8

For a set of vectors $S = \{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n\}$ *from a subspace* $\mathcal{V} \subseteq \mathfrak{R}^{m\times 1}$*, let* $\mathbf{A}$ *be the matrix containing the* $\mathbf{a}_i$*'s as its columns.* S *spans* $\mathcal{V}$ *if and only if for each* $\mathbf{b} \in \mathcal{V}$*, there corresponds a column* x *such that* $\mathbf{A}x = \mathbf{b}$*.*

- This simple observation often is quite helpful. For example, to test whether or not $S = \{(1,1,1), (1,-1,-1), (3,1,1)\}$ spans $\mathfrak{R}^3$.
  - Just place these row as columns in a matrix $\mathbf{A}$.
  - Check "Is the system $\mathbf{Ax} = \mathbf{b}$ consistent for every $\mathbf{b} \in \mathfrak{R}^3$?

$$\begin{pmatrix} 1 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

  - As we know, $\mathbf{Ax} = \mathbf{b}$ is consistent if and only if $rank[\mathbf{A}|\mathbf{b}] = rank(\mathbf{A})$.
  - In this case, $rank(\mathbf{A}) = 2$, but $rank[\mathbf{A}|\mathbf{b}] = 3$ for some $\mathbf{b}$ (e.g., $b_1 = 0$, $b_2 = 1$, $b_3 = 0$), so S doesn't span $\mathfrak{R}^3$.

Sum of Subspaces

If $\mathcal{X}$ and $\mathcal{Y}$ are subspaces of a vector space $\mathcal{V}$, then the ***sum*** of $\mathcal{X}$ and $\mathcal{Y}$ is defined to be the set of all possible sums of vectors from $\mathcal{X}$ with vectors from $\mathcal{Y}$. That is,

$$\mathcal{X} + \mathcal{Y} = \{\mathbf{x} + \mathbf{y} \mid \mathbf{x} \in \mathcal{X} \text{ and } \mathbf{y} \in \mathcal{Y}\}.$$

- The sum $\mathcal{X} + \mathcal{Y}$ is also a subspace of $\mathcal{V}$.
- If S_X, S_Y span $\mathcal{X}, \mathcal{Y}$, then $S_X \bigcup S_Y$ spans $\mathcal{X} + \mathcal{Y}$.

Example 9

If $\mathcal{X} \subseteq \Re^2$ and $\mathcal{Y} \subseteq \Re^2$ are subspaces defined by two different lines through the origin, then $\mathcal{X} + \mathcal{Y} = \Re^2$. This follows from the parallelogram law.

Four Fundamental Subspaces

- Subspace are intimately related to linear functions as explain below.

Subspaces and Linear Functions

For a linear function f mapping $\Re^n$ into $\Re^m$, let $\mathcal{R}(f)$ denote the ***range*** of f. That is, $\mathcal{R}(f) = \{f(\mathbf{x}) \,|\, \mathbf{x} \in \Re^n\} \subseteq \Re^m$ is the set of all "images" as $\mathbf{x}$ varies freely over $\Re^n$.

- The range of every linear function $f : \Re^n \to \Re^m$ is a subspace of $\Re^m$, and every subspace of $\Re^m$ is the range of some linear function.

For this reason, subspaces of $\Re^m$ are sometimes called ***linear spaces***.

- This result means that every matrix $\mathbf{A} \in \Re^{m\times n}$ generates a subspace of $\Re^m$ by means of the range of the linear function $f(x) = \mathbf{A}x$.

Proof. If $f : \Re^n \to \Re^m$ is a linear function, then the range of f is a subspace of $\Re^m$ because the closure properties **(A1)** and **(M1)** are satisfied. Establish **(A1)** by showing that $\mathbf{y}_1, \mathbf{y}_2 \in \mathcal{R}(f) \Rightarrow \mathbf{y}_1 + \mathbf{y}_2 \in \mathcal{R}(f)$. If $\mathbf{y}_1, \mathbf{y}_2 \in \mathcal{R}(f)$, then there must be vectors $\mathbf{x}_1, \mathbf{x}_2 \in \Re^n$ such that $\mathbf{y}_1 = f(\mathbf{x}_1)$ and $\mathbf{y}_2 = f(\mathbf{x}_2)$, so it follows from the linearity of f that

$$\mathbf{y}_1 + \mathbf{y}_2 = f(\mathbf{x}_1) + f(\mathbf{x}_2) = f(\mathbf{x}_1 + \mathbf{x}_2) \in \mathcal{R}(f).$$

Similarly, establish **(M1)** by showing that if $\mathbf{y} \in \mathcal{R}(f)$, then $\alpha\mathbf{y} \in \mathcal{R}(f)$ for all scalars α by using the definition of range along with the linearity of f to write

$$\mathbf{y} \in \mathcal{R}(f) \Longrightarrow \mathbf{y} = f(\mathbf{x}) \text{ for some } \mathbf{x} \in \Re^n \Longrightarrow \alpha\mathbf{y} = \alpha f(\mathbf{x}) = f(\alpha\mathbf{x}) \in \mathcal{R}(f).$$

Now prove that every subspace $\mathcal{V}$ of $\Re^m$ is the range of some linear function $f : \Re^n \to \Re^m$. Suppose that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a spanning set for $\mathcal{V}$ so that

$$\mathcal{V} = \{\alpha_1\mathbf{v}_1 + \cdots + \alpha_n\mathbf{v}_n \mid \alpha_i \in \mathcal{R}\}.$$

Stack the $\mathbf{v}_i$'s as columns in a matrix $\mathbf{A}_{m\times n} = \big(\mathbf{v}_1 \mid \mathbf{v}_2 \mid \cdots \mid \mathbf{v}_n\big)$, and put the α_i's in an $n \times 1$ column $\mathbf{x} = (\alpha_1, \alpha_2, \ldots, \alpha_n)^T$ to write

$$\alpha_1\mathbf{v}_1 + \cdots + \alpha_n\mathbf{v}_n = (\mathbf{v}_1 \mid \mathbf{v}_2 \mid \cdots \mid \mathbf{v}_n) \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \mathbf{A}\mathbf{x}.$$

The function $f(\mathbf{x}) = \mathbf{A}\mathbf{x}$ is linear and we have that
$\mathcal{R}(f) = \{\mathbf{A}\mathbf{x} \mid \mathbf{x} \in \Re^{n\times 1}\} = \{\alpha_1\mathbf{v}_1 + \cdots + \alpha_n\mathbf{v}_n \mid \alpha_i \in \mathcal{R}\} = \mathcal{V}$. ∎

Range Spaces

- Likewise, the transpose of $\mathbf{A}$ defines a subspace of $\mathfrak{R}^n$ by means of the range of the linear function $f(x) = \mathbf{A}^T x$.
- These two "range spaces" are two of the four fundamental subspace defined by a matrix.

Range Spaces

The ***range of a matrix*** $\mathbf{A} \in \Re^{m \times n}$ is defined to be the subspace $R(\mathbf{A})$ of $\Re^m$ that is generated by the range of $f(\mathbf{x}) = \mathbf{A}\mathbf{x}$. That is,

$$R(\mathbf{A}) = \{\mathbf{A}\mathbf{x} \,|\, \mathbf{x} \in \Re^n\} \subseteq \Re^m.$$

Similarly, the range of $\mathbf{A}^T$ is the subspace of $\Re^n$ defined by

$$R\left(\mathbf{A}^T\right) = \{\mathbf{A}^T\mathbf{y} \,|\, \mathbf{y} \in \Re^m\} \subseteq \Re^n.$$

Because $R(\mathbf{A})$ is the set of all "images" of vectors $\mathbf{x} \in \Re^m$ under transformation by $\mathbf{A}$, some people call $R(\mathbf{A})$ the ***image space*** of $\mathbf{A}$.

- As we know, that every matrixvector product $\mathbf{A}x$ (i.e., every image) is a linear combination of the columns of $\mathbf{A}$ provides a useful characterization of the range spaces.
- Therefore, $R(\mathbf{A})$ is nothing more than the space spanned by the columns of $\mathbf{A}$. $R(\mathbf{A})$ is often called the column space of $\mathbf{A}$.

Column and Row Spaces

For $\mathbf{A} \in \Re^{m\times n}$, the following statements are true.

- $R(\mathbf{A}) =$ the space spanned by the columns of $\mathbf{A}$ (column space).
- $R(\mathbf{A}^T) =$ the space spanned by the rows of $\mathbf{A}$ (row space).
- $\mathbf{b} \in R(\mathbf{A}) \Longleftrightarrow \mathbf{b} = \mathbf{A}\mathbf{x}$ for some $\mathbf{x}$.
- $\mathbf{a} \in R(\mathbf{A}^T) \Longleftrightarrow \mathbf{a}^T = \mathbf{y}^T\mathbf{A}$ for some $\mathbf{y}^T$.

Nullspace

By considering the linear functions $f(\mathbf{x}) = \mathbf{Ax}$ and $g(\mathbf{y}) = \mathbf{A}^T\mathbf{y}$, the other two fundamental subspaces defined by $\mathbf{A} \in \Re^{m\times n}$ are obtained. They are $\mathcal{N}(f) = \{\mathbf{x}_{n\times 1} \mid \mathbf{Ax} = \mathbf{0}\} \subseteq \Re^n$ and $\mathcal{N}(g) = \{\mathbf{y}_{m\times 1} \mid \mathbf{A}^\mathbf{T}\mathbf{y} = \mathbf{0}\} \subseteq \Re^m$.

Nullspace

- For an $m \times n$ matrix $\mathbf{A}$, the set $N(\mathbf{A}) = \{\mathbf{x}_{n\times 1} \mid \mathbf{Ax} = \mathbf{0}\} \subseteq \Re^n$ is called the ***nullspace*** of $\mathbf{A}$. In other words, $N(\mathbf{A})$ is simply the set of all solutions to the homogeneous system $\mathbf{Ax} = \mathbf{0}$.
- The set $N\left(\mathbf{A}^T\right) = \left\{\mathbf{y}_{m\times 1} \mid \mathbf{A}^T\mathbf{y} = \mathbf{0}\right\} \subseteq \Re^m$ is called the ***left-hand nullspace*** of $\mathbf{A}$ because $N\left(\mathbf{A}^T\right)$ is the set of all solutions to the left-hand homogeneous system $\mathbf{y}^T\mathbf{A} = \mathbf{0}^T$.

Summary

The four fundamental subspaces associated with $\mathbf{A}_{m\times n}$ are as follows.

- The range or column space: $R(\mathbf{A}) = \{\mathbf{Ax}\} \subseteq \Re^m$.
- The row space or left-hand range: $R(\mathbf{A}^T) = \{\mathbf{A}^T\mathbf{y}\} \subseteq \Re^n$.
- The nullspace: $N(\mathbf{A}) = \{\mathbf{x} \mid \mathbf{Ax} = \mathbf{0}\} \subseteq \Re^n$.
- The left-hand nullspace: $N(\mathbf{A}^T) = \{\mathbf{y} \mid \mathbf{A}^T\mathbf{y} = \mathbf{0}\} \subseteq \Re^m$

Let $\mathbf{P}$ be a nonsingular matrix such that $\mathbf{PA} = \mathbf{U}$, where $\mathbf{U}$ is in row echelon form, and suppose $rank(\mathbf{A}) = r$.

- Spanning set for $R(\mathbf{A})$ = the basic columns in $\mathbf{A}$.
- Spanning set for $R(\mathbf{A}^T)$ = the nonzero rows in $\mathbf{U}$.
- Spanning set for $N(\mathbf{A})$ =the $\mathbf{h}_i$'s in the general solution of $\mathbf{Ax} = \mathbf{0}$.
- Spanning set for $N(\mathbf{A}^T)$ = the last $m - r$ rows of $\mathbf{P}$.

If $\mathbf{A}$ and $\mathbf{B}$ have the same shape, then

- $\mathbf{A} \overset{\text{row}}{\sim} \mathbf{B} \Longleftrightarrow N(\mathbf{A}) = N(\mathbf{B}) \Longleftrightarrow R(\mathbf{A}^T) = R(\mathbf{B}^T)$.
- $\mathbf{A} \overset{\text{col}}{\sim} \mathbf{B} \Longleftrightarrow R(\mathbf{A}) = R(\mathbf{B}) \Longleftrightarrow N(\mathbf{A}^T) = N(\mathbf{B}^T)$.

Linear Independence

- For a given set of vectors $\{\mathbf{v}_1,\ \mathbf{v}_2, \cdots \mathbf{v}_n\}$, there may or may not exist dependency relationships in the sense that it may not be possible to express one vector as a linear combination of others.
- Consider two sets of vectors

$$\mathcal{A} = \left\{ \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 9 \\ -3 \\ 4 \end{pmatrix} \right\},$$

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

- For the first set, $\mathbf{v_3} = 3\mathbf{v_1} + 2\mathbf{v_2}$, i.e., $3\mathbf{v_1} + 2\mathbf{v_2} - \mathbf{v_3} = 0$.
- For the second set, there are no solutions for α_1, α_2 and α_3 in the homogeneous equation $\alpha_1\mathbf{v_1} + \alpha_2\mathbf{v_2} + \alpha_3\mathbf{v_3} = 0$, other than the trivial solution $\alpha_1 = \alpha_2 = \alpha_3 = 0$.

Linear Independence

A set of vectors $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is said to be a ***linearly independent set*** whenever the only solution for the scalars α_i in the homogeneous equation

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n = \mathbf{0}$$

is the trivial solution $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$.

- Whenever there is a nontrivial solution for α's (i.e. at least one $\alpha_i \neq 0$), the set $\mathcal{S}$ is said to be a **Linearly dependent set**.
- Linearly independent sets are those that contain no dependency relations,
- Linearly dependent sets are those in which at least one vector is a combination of the others.
- The empty set is always linearly independent.

- How to determine whether or not a set of vectors is linearly independent?
- Example

$$\mathcal{S} = \left\{ \mathbf{v_1} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \ \mathbf{v_2} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \ \mathbf{v_3} = \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix} \right\}.$$

- Solution: determine whether or not there exists a nontrivial solution for the following homogeneous equation:

$$\alpha_1 \mathbf{v_1} + \alpha_2 \mathbf{v_2} + \alpha_3 \mathbf{v_3} = (0, 0, 0)^T,$$

- Equivalently, $\mathbf{A}\boldsymbol{\alpha} = \mathbf{0}$, where

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 5 \\ 2 & 0 & 6 \\ 1 & 2 & 7 \end{pmatrix}, \ \boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)^T, \ \mathbf{0} = (0, 0, 0)^T$$

Linear Independence and Matrices

Let $\mathbf{A}$ be an $m \times n$ matrix.

- Each of the following statements is equivalent to saying that the columns of $\mathbf{A}$ form a linearly independent set.
  - $N(\mathbf{A}) = \{\mathbf{0}\}$.
  - $rank(\mathbf{A}) = n$.
- Each of the following statements is equivalent to saying that the rows of $\mathbf{A}$ form a linearly independent set.
  - $N(\mathbf{A}^T) = \{\mathbf{0}\}$.
  - $rank(\mathbf{A}) = m$.
- When $\mathbf{A}$ is a square matrix, each of the following statements is equivalent to saying that $\mathbf{A}$ is nonsingular.
  - The columns of $\mathbf{A}$ form a linearly independent set.
  - The rows of $\mathbf{A}$ form a linearly independent set.

Special Types of Matrices

Diagonally dominant matrices

- A matrix $\mathcal{A}_{n\times n}$ is said to be ***diagonally dominant*** whenever

$$|a_{ii}| > \sum_{j=1, j\neq i}^{n} |a_{ij}| \ for\ each\ i = 1, 2, \cdots, n.$$

- The magnitude of each diagonal entry exceeds the sum of the magnitudes of the off-diagonal entries in the corresponding row.
- Diagonally dominant matrices occur naturally in a wide variety of practical applications.
- In 1900, Minkowski discovered that **all diagonally dominant matrices are nonsingular**.
- The strategy is to prove that if $\mathbf{A}$ is diagonally dominant, then $N(\mathbf{A}) = \{0\}$.

Vandermonde Matrices

- Matrices of the form

$$\mathbf{V}_{m\times n} = \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & x_m & x_m^2 & \cdots & x_m^{n-1} \end{pmatrix}$$

- This is named in honor of the French mathematician: **Alexandre Theophile Vandermonde (1735-1796)**.
- Columns constitute a linearly independent set whenever $n \leq m$.
- Problem: Given a set of m points $\mathcal{S} = \{(x_1, y_1), (x_2, y_2), \cdots, (x_m, y_m)\}$ in which the x_i's are distinct, there is a unique polynomial

$$l(t) = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \cdots + \alpha_{m-1} t^{m-1}$$

of degree $m-1$ that passes through each point in $\mathcal{S}$.

- $l(t)$ must be given by

$$l(t) = \sum_{i=1}^{m} \left(y_i \frac{\prod_{j \neq i}^{m} (t - x_j)}{\prod_{j \neq i}^{m} (x_i - x_j)} \right).$$

- The polynomial $l(t)$ is known as the ***Lagrange interpolation polynomial*** of degree $m - 1$.
- If $rank(\mathbf{A}_{m \times n}) < n$, the columns of $\mathbf{A}$ must be a dependent set.
- For such matrices, we often wish to extract a ***maximal linearly independent subset*** of columns.
- i.e., a linearly independent set containing as many columns from $\mathbf{A}$ as possible.
- Although there can be several ways to make such a selection, the basic columns in $\mathbf{A}$ always constitute one solution.

Maximal Independent Subsets

If $rank(\mathbf{A}_{m\times n}) = r$, then the following statements hold.

- Any maximal independent subset of columns from $\mathbf{A}$ contains exactly r columns.
- Any maximal independent subset of rows from $\mathbf{A}$ contains exactly r rows.
- In particular, the r basic columns in $\mathbf{A}$ constitute one maximal independent subset of columns from $\mathbf{A}$.

Basic Facts of Independence

For a nonempty set of vectors $\mathcal{S} = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ in a space $\mathcal{V}$, the following statements are true.

- If $\mathcal{S}$ contains a linearly dependent subset, then $\mathcal{S}$ itself must be linearly dependent.
- If $\mathcal{S}$ is linearly independent, then every subset of $\mathcal{S}$ is also linearly independent.
- If $\mathcal{S}$ is linearly independent and if $\mathbf{v} \in \mathcal{V}$, then the ***extension set*** $\mathcal{S}_{ext} = \mathcal{S} \cup \{\mathbf{v}\}$ is linearly independent if and only if $\mathbf{v} \notin span(\mathcal{S})$.
- If $\mathcal{S} \subseteq \Re^m$ and if $n > m$, then $\mathcal{S}$ must be linearly dependent.

Wornski matrix

- Let $\mathcal{V}$ be the vector space of real-valued functions of a real variable, and let $\mathcal{S} = \{f_1(x),\ f_2(x), \cdots, f_n(x), \}$ be a set of functions that are $n-1$ times differentiable.
- The Wornski matrix is defined to be

$$\mathbf{W} = \begin{pmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{pmatrix}$$

- If there is at least one point $x = x_0$ such that $\mathbf{w}(x_0)$ is nonsingular, $\mathcal{S}$ must be a linearly independent set.
- For example, to verify that the set of polynomials $\mathcal{P} = 1, x, x^2, \cdots, x^n$ is linearly independent, just observe that the associated Wronski matrix.

Basis and Dimension

- A linearly independent spanning set for a vector space $\mathcal{V}$ is called a ***basis*** of $\mathcal{V}$.
- It can be proven that every vector space $\mathcal{V}$ possesses a basis.
- Just as in the case of spanning sets, a space can possess many different bases.
  1. The unit vectors $\mathcal{S} = \{\mathbf{e_1}, \mathbf{e_2}, \cdots, \mathbf{e_n}\}$ in $\mathfrak{R}^n$ are a basis for $\mathfrak{R}^n$. This is called the standard basis for $\mathfrak{R}^n$.
  2. The set $\{1, x, x^2, \cdots, x^n\}$ is a basis for the vector space of polynomials having degree n or less.
  3. The infinite set $\{1, x, x^2, \cdots\}$ is a basis for the vector space of all polynomials.
  4. If $\mathbf{A}$ is an $n \times n$ nonsingular matrix, then the set of rows in $\mathbf{A}$ as well as the set of columns from $\mathbf{A}$ constitute a basis for $\mathfrak{R}^n$.
  5. For the trivial vector space $\mathcal{Z} = \{0\}$, there is no nonempty linearly independent spanning set. Consequently, the empty set is considered to be a basis for $\mathcal{Z}$.

- Spaces that possess a basis containing an infinite number of vectors are referred to as **infinite-dimensional spaces**.
- Those that have a finite basis are called **finite-dimensional spaces**.

Characterizations of a Basis

Let $\mathcal{V}$ be a subspace of $\Re^m$, and let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\} \subseteq \mathcal{V}$. The following statements are equivalent.

- $\mathcal{B}$ is a basis for $\mathcal{V}$.
- $\mathcal{B}$ is a minimal spanning set for $\mathcal{V}$.
- $\mathcal{B}$ is a maximal linearly independent subset of $\mathcal{V}$.

- Although a space $\mathcal{V}$ can have many different bases, all bases for $\mathcal{V}$ contain the same number of vectors.
- This number is quite important, which is called the dimension of $\mathcal{V}$.

Dimension

The ***dimension*** of a vector space $\mathcal{V}$ is defined to be

$$\begin{aligned} \dim \mathcal{V} &= \text{number of vectors in any basis for } \mathcal{V} \\ &= \text{number of vectors in any minimal spanning set for } \mathcal{V} \\ &= \text{number of vectors in any maximal independent subset of } \mathcal{V}. \end{aligned}$$

- Some examples:
  1. $\dim\mathfrak{R}^3 = 3$ because the three unit vectors $\{e_1, e_2, e_3\}$ constitute a basis for $\mathfrak{R}^3$.
  2. If $\mathcal{L}$ is a line through the origin in $\mathfrak{R}^3$, then $\dim\mathcal{L} = 1$ because a basis for $\mathcal{L}$ consists of any nonzero vector lying along $\mathcal{L}$.
  3. In $\mathcal{P}$ is a plane through the origin in $\mathfrak{R}^3$, then $\dim\mathcal{P} = 2$ because a minimal spanning set for $\mathcal{P}$ must contain two vector from $\mathcal{P}$.
  4. If $\mathcal{Z} = \{0\}$ is the trivial subspace, then $\dim\mathcal{Z} = 0$ because the basis for this space is the empty set.
- In a loose sense the dimension of a space is a measure of the amount of "stuff" in the space
- Dimension is in terms of **degrees of freedom**
  1. In the trivial space $\mathcal{Z}$, there are no degrees of freedom: you can move nowhere.
  2. For a line, there is one degree of freedom:length.
  3. In a plane there are two degree of freedom: length and width.
  4. In $\mathfrak{R}^3$ there are three degrees of freedom: length, width and height.

- It is important not to confuse the dimension of a vector space $\mathcal{V}$ with the number of components contained in the individual vector.
- For example, $\dim \mathcal{P} = 2$, but the individual vector in $\mathcal{P}$ each have three components.
- There is the relationship between the dimension and the number of components contained in the individual vectors.

Subspace Dimension

For vector spaces $\mathcal{M}$ and $\mathcal{N}$ such that $\mathcal{M} \subseteq \mathcal{N}$, the following statements are true.

- $\dim \mathcal{M} \leq \dim \mathcal{N}$.
- If $\dim \mathcal{M} = \dim \mathcal{N}$, then $\mathcal{M} = \mathcal{N}$.

Fundamental Subspaces—Dimension and Bases

For an $m \times n$ matrix of real numbers such that $rank(\mathbf{A}) = r$,

- $\dim R(\mathbf{A}) = r$,
- $\dim N(\mathbf{A}) = n - r$,
- $\dim R(\mathbf{A}^T) = r$,
- $\dim N(\mathbf{A}^T) = m - r$.

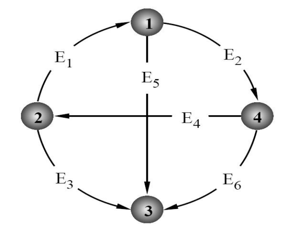
Let $\mathbf{P}$ be a nonsingular matrix such that $\mathbf{PA} = \mathbf{U}$ is in row echelon form, and let $\mathcal{H}$ be the set of $\mathbf{h}_i$'s appearing in the general solution of $\mathbf{Ax} = \mathbf{0}$.

- The basic columns of $\mathbf{A}$ form a basis for $R(\mathbf{A})$.
- The nonzero rows of $\mathbf{U}$ form a basis for $R(\mathbf{A}^T)$.
- The set $\mathcal{H}$ is a basis for $N(\mathbf{A})$.
- The last $m - r$ rows of $\mathbf{P}$ form a basis for $N(\mathbf{A}^T)$.

For matrices with complex entries, the above statements remain valid provided that $\mathbf{A}^T$ is replaced with $\mathbf{A}^*$.

Rank and Connectivity

- A set of points (or nodes), $\{N_1, N_2, \cdots, N_m\}$, together with a set of paths (or edges), $\{E_1, E_2, \cdots, E_m\}$, between the nodes is called a **graph**.
- A **connected graph** is one in which there is a sequence of edges linking any pair of nodes.
- A directed graph is one in which each edge has been assigned a direction.
- There is a close relationship between the graph connectivity and matrix rank.
- The incidence matrix associated with a directed graph containing m nodes and n edges is defined to be the $m \times n$ matrix $\mathbf{E}$ whose $e_{k,j}$ is
    - 1 if edge E_j is directed toward node N_k;
    - -1 if edge E_j is directed away from node N_k;
    - 0 if edge E_j neither begins nor ends at node N_k.



$$\mathbf{E} = \begin{array}{c} \\ N_1 \\ N_2 \\ N_3 \\ N_4 \end{array} \begin{array}{c} \begin{array}{cccccc} E_1 & E_2 & E_3 & E_4 & E_5 & E_6 \end{array} \\ \begin{pmatrix} 1 & -1 & 0 & 0 & -1 & 0 \\ -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 & 0 & -1 \end{pmatrix} \end{array}.$$

- Each column in $\mathbf{E}$ must contain exactly tow nonzero entries: $+1$ and -1. Consequently, all column sums are zero.
- $rank(\mathbf{E}) = rank(\mathbf{E^T}) = m - dimN(\mathbf{E^T}) \leq m - 1.$

Rank and Connectivity

Let $\mathcal{G}$ be a graph containing m nodes. If $\mathcal{G}$ is undirected, arbitrarily assign directions to the edges to make $\mathcal{G}$ directed, and let $\mathbf{E}$ be the corresponding incidence matrix.

- $\mathcal{G}$ is connected if and only if $rank(\mathbf{E}) = m - 1$.

- If $\mathcal{X}$ and $\mathcal{Y}$ are subspaces of a vector space $\mathcal{V}$, then

$$dim(\mathcal{X}+\mathcal{Y}) = dim\mathcal{X} + dim\mathcal{Y} - dim(\mathcal{X}\bigcap\mathcal{Y}).$$

- $rank(\mathbf{A}+\mathbf{B}) \leq rank(\mathbf{A}) + rank(\mathbf{B})$

Proof.

Observe that

$$R(\mathbf{A}+\mathbf{B}) \subseteq R(\mathbf{A}) + R(\mathbf{B}).$$

Recall that if $\mathcal{M} \subseteq \mathcal{N}$, then $dim\mathcal{M} \leq dim\mathcal{N}$, we have

$$\begin{aligned} rank(\mathbf{A}+\mathbf{B}) &= dimR(\mathbf{A}+\mathbf{B}) \leq dim(R(\mathbf{A}) + R(\mathbf{B})) \\ &= dimR(\mathbf{A}) + dimR(\mathbf{B}) - dim(R(\mathbf{A})\bigcap R(\mathbf{B})) \\ &\leq dimR(\mathbf{A}) + dimR(\mathbf{B}) = rank(\mathbf{A}) + rank(\mathbf{B}) \end{aligned}$$

$\square$

More About Rank

- Rank is invariant under multiplication by a nonsingular matrix.
- If $\mathbf{P}$ and $\mathbf{Q}$ are nonsingular matrices such that the product $\mathbf{PAQ}$ is defined, then

$$rank(\mathbf{A}) = rank(\mathbf{PAQ}) = rank(\mathbf{PA}) = rank(\mathbf{AQ}).$$

- However, multiplication by rectangular or singular matrices can alter the rank.

Rank of a Product

If $\mathbf{A}$ is $m \times n$ and $\mathbf{B}$ is $n \times p$, then

$$rank(\mathbf{AB}) = rank(\mathbf{B}) - \dim N(\mathbf{A}) \cap R(\mathbf{B}).$$

- Although $rank(\mathbf{A})$ and $rank(\mathbf{B})$ are frequently or can be estimated, the term $dimN(\mathbf{A})\bigcap R(\mathbf{B})$ can be costly to obtain.
- Upper and lower bounds for $rank(\mathbf{AB})$ depends only on $rank(\mathbf{A})$ and $rank(\mathbf{B})$.

Bounds on the Rank of a Product

If $\mathbf{A}$ is $m \times n$ and $\mathbf{B}$ is $n \times p$, then

- $rank(\mathbf{AB}) \le \min\{rank(\mathbf{A}),\ rank(\mathbf{B})\}$,
- $rank(\mathbf{A}) + rank(\mathbf{B}) - n \le rank(\mathbf{AB})$.

- The products A^TA and AA^T and their complex counterparts A^*A and AA^* deserve special attention because they naturally appear in a wide variety of applications.

Products $\mathbf{A}^T\mathbf{A}$ and $\mathbf{A}\mathbf{A}^T$

For $\mathbf{A} \in \Re^{m\times n}$, the following statements are true.

- $rank\left(\mathbf{A}^T\mathbf{A}\right) = rank\left(\mathbf{A}\right) = rank\left(\mathbf{A}\mathbf{A}^T\right)$.
- $R\left(\mathbf{A}^T\mathbf{A}\right) = R\left(\mathbf{A}^T\right)$ and $R\left(\mathbf{A}\mathbf{A}^T\right) = R\left(\mathbf{A}\right)$.
- $N\left(\mathbf{A}^T\mathbf{A}\right) = N\left(\mathbf{A}\right)$ and $N\left(\mathbf{A}\mathbf{A}^T\right) = N\left(\mathbf{A}^T\right)$.

For $\mathbf{A} \in \mathcal{C}^{m\times n}$, the transpose operation $(\star)^T$ must be replaced by the conjugate transpose operation $(\star)^*$.

- Consider an $m \times n$ system of equations $\mathbf{A}x = b$ that may or may not be consistent.
- Multiplying on the left-hand side by $\mathbf{A}^T$ produces the $n \times n$ system: $\mathbf{A}^T\mathbf{A}x = \mathbf{A}^Tb$ called **the associated system of normal equations**.
- First, the normal equations are always consistent, regardless of whether or not the original system is consistent because $\mathbf{A}^Tb \in R(\mathbf{A}^T) = R(\mathbf{A}^TA)$.

- If $\mathbf{A}x = b$ happens to be consistent, then $\mathbf{A}x = b$ and $\mathbf{A}^T\mathbf{A}x = \mathbf{A}^Tb$ have the same solution set.
- If $\mathbf{A}x = b$ is consistent and has a unique solution, the same is true for $\mathbf{A}^T\mathbf{A}x = \mathbf{A}^Tb$, and the unique solution common to both system is

$$x = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^Tb.$$

- There is one outstanding question: what do the solutions of the normal equations $\mathbf{A}^T\mathbf{A}x = \mathbf{A}^Tb$ represent when the original system $\mathbf{A}x = b$ is not consistent?
- Use of the product $\mathbf{A}^T\mathbf{A}$ or the normal equations is not recommended for numerical computation.
- Any sensitivity to small perturbations that is present in the underlying matrix $\mathbf{A}$ is magnified by forming the product $\mathbf{A}^T\mathbf{A}$.
- Nevertheless, the normal equations are an important theoretical idea that leads to practical tools of fundamental importance such as the method least squares.

Normal Equations

- For an $m \times n$ system $\mathbf{Ax} = \mathbf{b}$, the associated system of ***normal equations*** is defined to be the $n \times n$ system $\mathbf{A}^T\mathbf{Ax} = \mathbf{A}^T\mathbf{b}$.
- $\mathbf{A}^T\mathbf{Ax} = \mathbf{A}^T\mathbf{b}$ is always consistent, even when $\mathbf{Ax} = \mathbf{b}$ is not consistent.
- When $\mathbf{Ax} = \mathbf{b}$ is consistent, its solution set agrees with that of $\mathbf{A}^T\mathbf{Ax} = \mathbf{A}^T\mathbf{b}$. the normal equations provide least squares solutions to $\mathbf{Ax} = \mathbf{b}$ when $\mathbf{Ax} = \mathbf{b}$ is inconsistent.
- $\mathbf{A}^T\mathbf{Ax} = \mathbf{A}^T\mathbf{b}$ has a unique solution if and only if $rank(\mathbf{A}) = n$, in which case the unique solution is $\mathbf{x} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{b}$.
- When $\mathbf{Ax} = \mathbf{b}$ is consistent and has a unique solution, then the same is true for $\mathbf{A}^T\mathbf{Ax} = \mathbf{A}^T\mathbf{b}$, and the unique solution to both systems is given by $\mathbf{x} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{b}$.

- One more way to think about rank:

Rank and the Largest Nonsingular Submatrix

The rank of a matrix $\mathbf{A}_{m\times n}$ is precisely the order of a maximal square nonsingular submatrix of $\mathbf{A}$. In other words, to say $rank(\mathbf{A}) = r$ means that there is at least one $r \times r$ nonsingular submatrix in $\mathbf{A}$, and there are no nonsingular submatrices of larger order.

- It is impossible to increase the rank by means of matrix multiplication:$rank(\mathbf{AE}) \leq rank(\mathbf{A})$. ✓
- In a certain sense there is a dual statement for matrix addition that says that it is impossible to decrease the rank by means of a "small" matrix addition: $rank(\mathbf{A}+\mathbf{E}) \geq rank(\mathbf{A})$ whenever $\mathbf{E}$ has entries of small magnitude. ✓

Summary of Rank

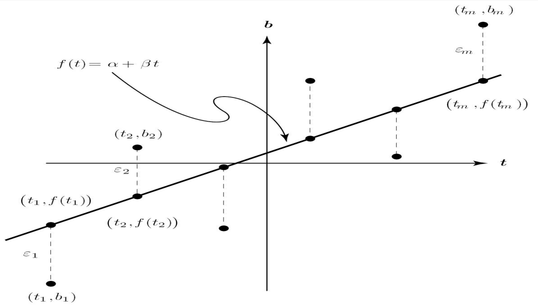
For $\mathbf{A} \in \Re^{m\times n}$, each of the following statements is true.

- $rank(\mathbf{A}) =$ The number of nonzero rows in any row echelon form that is row equivalent to $\mathbf{A}$.
- $rank(\mathbf{A}) =$ The number of pivots obtained in reducing $\mathbf{A}$ to a row echelon form with row operations.
- $rank(\mathbf{A}) =$ The number of basic columns in $\mathbf{A}$ (as well as the number of basic columns in any matrix that is row equivalent to $\mathbf{A}$).
- $rank(\mathbf{A}) =$ The number of independent columns in $\mathbf{A}$ —i.e., the size of a maximal independent set of columns from $\mathbf{A}$.
- $rank(\mathbf{A}) =$ The number of independent rows in $\mathbf{A}$ —i.e., the size of a maximal independent set of rows from $\mathbf{A}$.
- $rank(\mathbf{A}) = \dim R(\mathbf{A})$.
- $rank(\mathbf{A}) = \dim R\left(\mathbf{A}^T\right)$.
- $rank(\mathbf{A}) = n - \dim N(\mathbf{A})$.
- $rank(\mathbf{A}) = m - \dim N\left(\mathbf{A}^T\right)$.
- $rank(\mathbf{A}) =$ The size of the largest nonsingular submatrix in $\mathbf{A}$.

For $\mathbf{A} \in \mathcal{C}^{m\times n}$, replace $(\star)^T$ with $(\star)^*$.

Classical Least Squares

- At discrete points t_i (often points in time), observations b_i of some phenomenon are made, and the results are records as a set of ordered pairs:
$$\mathcal{D} = \{(t_1, b_1),\ (t_2, b_2),\ \cdots,\ (t_m, b_m)\}.$$
- **The problem is to make estimations or predictions at points $\hat{t}$ that are between or beyond the observation points t_i.**
- A standard approach is to find the equation of a curve $y = f(t)$ that closely fits the points in $\mathcal{D}$.
- The phenomenon can be estimated at any nonobservation point $\hat{t}$ with the value $\hat{y} = f(\hat{t})$.
- Let's begin by fitting a straight line to the points in $\mathcal{D}$.



- The strategy is to determine the coefficients α and β in the equation of the line $f(t) = \alpha + \beta t$ that best fits the points (t_i, b_i) in the sense that the sum of the squares of the vertical errors $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_m$ is minimal.
- The distance from (t_i, b_i) to a line $f(t) = \alpha + \beta t$ is

$$\varepsilon_i = |f(t_i) - b_i| = |\alpha + \beta t_i - b_i|$$

- The objective is to find values for α and β such that

$$\sum_{i=1}^{m} \varepsilon_i^2 = \sum_{i=1}^{m} (\alpha + \beta t_i - b_i)^2 \quad is \quad minimal.$$

- Minimization techniques tell us

$$\sum_{i=1}^{m} (\alpha + \beta t_i - b_i) = 0, \quad \sum_{i=1}^{m} (\alpha + \beta t_i - b_i) t_i = 0.$$

- It is easy to get the matrix form $\mathbf{A}^T \mathbf{A} x = \mathbf{A}^T b$, where setting

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ t_1 & t_2 & \cdots & t_m \end{pmatrix}^T, \quad \mathbf{b} = (b_1, b_2, \cdots, b_m)^T, \quad x = (\alpha, \beta)^T.$$

- This is the system of normal equations associated with the $\mathbf{A} x = b$.

- The t_i's are assumed to be distinct numbers, so $rank(\mathbf{A}) = 2$.
- The normal equations have a unique solution given by

$$
\begin{aligned}
x &= (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T b \\
&= \frac{1}{m\sum t_i^2 - (\sum t_i)^2} \begin{pmatrix} \sum t_i^2 & -\sum t_i \\ -\sum t_i & m \end{pmatrix} \begin{pmatrix} \sum b_i \\ \sum t_i b_i \end{pmatrix} \\
&= \frac{1}{m\sum t_i^2 - (\sum t_i)^2} \begin{pmatrix} \sum t_i^2 \sum b_i - \sum t_i \sum t_i b_i \\ m\sum t_i b_i - \sum t_i \sum b_i \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}
\end{aligned}
$$

- Finally, notice that the total sum of squares of the errors is given by

$$
\sum_{i=1}^{m} \varepsilon_i^2 = (\mathbf{A}x - b)^T(\mathbf{A}x - b).
$$

General Least Squares Problem

For $\mathbf{A} \in \Re^{m\times n}$ and $\mathbf{b} \in \Re^{m}$, let $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\mathbf{x}) = \mathbf{Ax} - \mathbf{b}$. The general least squares problem is to find a vector $\mathbf{x}$ that minimizes the quantity

$$\sum_{i=1}^{m} \varepsilon_i^2 = \boldsymbol{\varepsilon}^T\boldsymbol{\varepsilon} = (\mathbf{Ax} - \mathbf{b})^T(\mathbf{Ax} - \mathbf{b}).$$

Any vector that provides a minimum value for this expression is called a ***least squares solution.***

- The set of all least squares solutions is precisely the set of solutions to the system of normal equations $\mathbf{A}^T\mathbf{Ax} = \mathbf{A}^T\mathbf{b}$.
- There is a unique least squares solution if and only if $rank(\mathbf{A}) = n$, in which case it is given by $\mathbf{x} = \left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\mathbf{b}$.
- If $\mathbf{Ax} = \mathbf{b}$ is consistent, then the solution set for $\mathbf{Ax} = \mathbf{b}$ is the same as the set of least squares solutions.

- The classical least squares problem is part of a broader topic known as **linear regression**.
- It is the study of situations where attempts are made to express y as a linear combination of other variables $t_1, t_2, \cdots, t_n$.
- That means that $y = \alpha_0 + \alpha_1 t_1 + \alpha_2 t_2 + \cdots + \alpha_n t_n + \varepsilon$.
- ε is a random function whose values average out to zero in some sense.
- In other words, a linear hypothesis is the supposition that the expected (or mean) value of y at each point where the phenomenon can be observed is given by a linear equation:

$$E(y) = \alpha_0 + \alpha_1 t_1 + \alpha_2 t_2 + \cdots + \alpha_n t_n.$$

- Example: the problem of predicting the amount of weight that a pint of ice cream loses when it is stored at very low temperatures.

- There are many factors that may contribute to weight loss.
- It is reasonable to believe that storage time and temperature are the primary factors, so we will make a linear hypothesis of the form

$$y = \alpha_0 + \alpha_1 t_1 + \alpha_2 t_2 + \varepsilon.$$

- $y =$ weight loss, $t_1 =$ storage time, $t_2 =$ storage temperature.
- ε is a random function to account for all other factors.
- The expected weight loss is $E(y) = \alpha_0 + \alpha_1 t_1 + \alpha_2 t_2$.
- Suppose that we conduct an experiment as shown below.

Time (weeks)	1	1	1	2	2	2	3	3	3
Temp (^oF)	-10	-5	0	-10	-5	0	-10	-5	0
Loss (grams)	.15	.18	.20	.17	.19	.22	.20	.23	.25

If
$$\mathbf{A} = \begin{pmatrix} 1 & 1 & -10 \\ 1 & 1 & -5 \\ 1 & 1 & 0 \\ 1 & 2 & -10 \\ 1 & 2 & -5 \\ 1 & 2 & 0 \\ 1 & 3 & -10 \\ 1 & 3 & -5 \\ 1 & 3 & 0 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{pmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} .15 \\ .18 \\ .20 \\ .17 \\ .19 \\ .22 \\ .20 \\ .23 \\ .25 \end{pmatrix},$$

- If $b_i = E(y_i)$, $\mathbf{A}x = b$ is a consistent system, so we could solve for the unknown parameters α_0, α_1 and α_2.
- However, it is impossible to observe the exact value of the mean weight loss for a given storage time and temperature.
- The system $\mathbf{A}x = b$ will be inconsistent-especially when the number of observations greatly exceeds the number of parameters.
- Since we can't find exact values for α_i's, we hope for getting s set of good estimates for these parameters.
- The famous Gauss-Markov theorem states that the least squares estimates are the best way to estimate the α_i's.

Returning to our ice cream example, it can be verified that $\mathbf{b} \notin R(\mathbf{A})$, so, as expected, the system $\mathbf{Ax} = \mathbf{b}$ is not consistent, and we cannot determine exact values for α_0, α_1, and α_2. The best we can do is to determine least squares estimates for the α_i's by solving the associated normal equations $\mathbf{A}^T\mathbf{Ax} = \mathbf{A}^T\mathbf{b}$, which in this example are

$$\begin{pmatrix} 9 & 18 & -45 \\ 18 & 42 & -90 \\ -45 & -90 & 375 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 1.79 \\ 3.73 \\ -8.2 \end{pmatrix}.$$

The solution is

$$\begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} .174 \\ .025 \\ .005 \end{pmatrix},$$

and the estimating equation for mean weight loss becomes

$$\hat{y} = .174 + .025t_1 + .005t_2.$$

For example, the mean weight loss of a pint of ice cream that is stored for nine weeks at a temperature of -35^oF is estimated to be

$$\hat{y} = .174 + .025(9) + .005(-35) = .224 \text{ grams}.$$

Least Squares Curve Fitting Problem

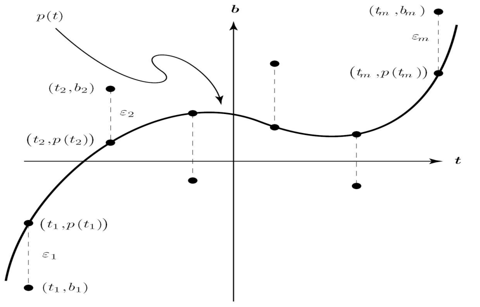
Least Squares Curve Fitting Problem: Find a polynomial

$$p(t) = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \cdots + \alpha_{n-1} t^{n-1}$$

with a specified degree that comes as close as possible in the sense of least squares to passing through a set of data points

$$\mathcal{D} = \{(t_1, b_1), \quad (t_2, b_2), \quad \ldots, \quad (t_m, b_m)\},$$

where the t_i 's are distinct numbers, and $n \leq m$.



- For the ε_i's indicated in above figure, the objective is to minimize the sum of squares

$$\sum_{i=1}^{m} \varepsilon_i^2 = \sum_{i=1}^{m} (p(t_i) - b_i)^2 = (\mathbf{Ax} - \mathbf{b})^T (\mathbf{Ax} - \mathbf{b}),$$

where

$$\mathbf{A} = \begin{pmatrix} 1 & t_1 & t_1^2 & \cdots & t_1^{n-1} \\ 1 & t_2 & t_2^2 & \cdots & t_2^{n-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & t_m & t_m^2 & \cdots & t_m^{n-1} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{n-1} \end{pmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

- The least squares polynomial of degree $n-1$ is obtained from the least squares solution associated with the system $\mathbf{A}x = b$.
- Furthermore, this least squares polynomial is unique because $\mathbf{A}$ is the Vandermonde matrix with $n \leq m$, so $rank(\mathbf{A}) = n$.
- $\mathbf{A}x = b$ has a unique least squares solution give by $(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T b$.

Exercise

1. Which of the following are spanning sets for $\mathfrak{R}^3$?
   (a) $\{(1,1,1)\}$
   (b) $\{(1,0,0),(0,0,1)\}$
   (c) $\{(1,0,0),(0,1,0),(0,0,1),(1,1,1)\}$
   (d) $\{(1,2,1),(2,0,-1),(4,4,1)\}$
2. Let $\mathcal{X}$ and $\mathcal{Y}$ be two subspace of a vector space $\mathcal{V}$. Prove that the intersection $\mathcal{X}\bigcap\mathcal{Y}$ is also a subspace of $\mathcal{V}$.
3. If $A = \begin{pmatrix} -1 & 1 & 1 & -2 & 1 \\ -1 & 0 & 3 & -4 & 2 \\ -1 & 0 & 3 & -5 & 3 \\ -1 & 0 & 3 & -6 & 4 \\ -1 & 0 & 3 & -6 & 2 \end{pmatrix}$ and $b = (-2,-5,-6,-7,-7)^T$, is $b \in R(A)$?

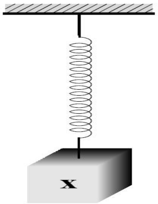
4. Which of the following sets of functions are linearly independent?
   (a) $\{\sin x,\ \cos x,\ x\sin x\}$.
   (b) $\{e^x,\ xe^x,\ x^2e^x\}$.
5. Determine the dimensions of each of the following vector spaces:
   (a) The space of polynomials having degree n or less.
   (b) The space $\mathfrak{R}^{m\times n}$of $m \times n$ matrices.
6. verify that $rank(\mathbf{A}^T\mathbf{A}) = rank(\mathbf{A}) = rank(\mathbf{A}\mathbf{A}^T)$ for

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & 1 & -4 \\ -1 & -3 & 1 & 0 \\ 2 & 6 & 2 & -8 \end{pmatrix}.$$

7. For $\mathbf{A} \in \mathfrak{R}^{m\times n}$, explain why $\mathbf{A}^T\mathbf{A} = 0$ implies $\mathbf{A} = 0$.

8. Hooke' s law says that the displacement y of an ideal spring is proportional to the force x that is appliedi.e., $y = kx$ for some constant k. Consider a spring in which k is unknown. Various masses are attached, and the resulting displacements shown in the following figure are observed. Using these observations, determine the least squares estimate for k.

x (lb)	y (in)
5	11.1
7	15.4
8	17.5
10	22.0
12	26.3



9. Using least squares techniques, fit the following data

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
y	2	7	9	12	13	14	14	13	10	8	4

with a line $y = \alpha_0 + \alpha_1 x$ and then fit the data with a quadratic $y = \alpha_0 + \alpha_1 x + \alpha_2 x^2$. Determine which of these two curves best fits the data by computing the sum of the squares of the errors in each case.