大数据分析

Spectral Graph Theory: Basics

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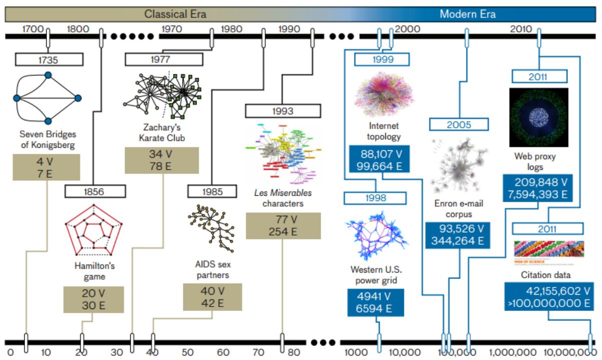
Overview

谱图理论核心问题

- **The central issue in spectral graph theory is**
  - **understanding, estimating, and finding** 理解、估计、寻找
  - **eigenvectors and eigenvalues of graphs.** 图的特征向量、特征值
- Amazement at learning that combinatorial properties of graphs could be revealed by an examination of the **eigenvalues and eigenvectors** of their associated matrices. 令人惊讶的是，通过检查图的关联矩阵的特征值和特征向量，可以揭示图的组合属性。
- Many of the graph problems that ***linear algebra*** apply to, give ***almost linear*** *time* algorithms

许多线性代数应用的图问题，给出了几乎是线性的时间算法

Overview of Graph Research



Nadya T. Bliss and Matthew C. Schmidt: Confronting the Challenges of Graphs and Networks, LINCOLN LABORATORY JOURNAL, VOL 20, No. 1, 2013

Outline

- Graphs and related matrix
- **Graph Laplacian**
- **Spectral Graph Theory**
- **Eigen Value and Vectors: connected components**
- **Spectral Sparsification**

谱稀疏化

Mathematically defined graphs

- **Path**
  - 顶点 边
  - **the vertices are {1, ..., *n*}. The edges are (*i*, *i* + 1) for 1 ≤ *i* < *n***
- **Ring**
  - **the vertices are {1, ..., *n*}. The edges are in the path, plus the edge (1, n)**
- **Hypercube** 超立方体
  - **The vertices are elements of *{0, 1}^k*. Edges exist between vertices that differ in only one coordinate**

顶点是{0，1}^k的元素，边在顶点之间，各个顶点只有一个坐标不同

Matrices for Graphs

- **A spreadsheet**
  - **if we wish to draw a matrix as a table, then we need to decide which vertex corresponds to which row and column**
  - **The first row of a matrix has no special importance**
  - **adjacency matrix**

$$A_G(u,v) = \begin{cases} 1 & \text{if } (u,v) \in E \\ 0 & \text{otherwise.} \end{cases}$$

- **An operator**
  - **diffusion operator**
  - **the stuff at a vertex will be uniformly distributed to its neighbors**

扩散算子
一个顶点上的物质会均匀分布到它的邻居

$$W_G = D_G^{-1} A_G.$$

- **A quadratic form**二次型
  - **Laplacian matrix**

拉普拉斯矩阵

$$L_G \stackrel{\text{def}}{=} D_G - A_G. \qquad x^T L_G x = \sum_{(u,v)\in E} (x(u) - x(v))^2.$$

The Adjacency Matrix

$$A(i,j) = \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

λ is eigenvalue and v is eigenvector if

$$Av = \lambda v$$

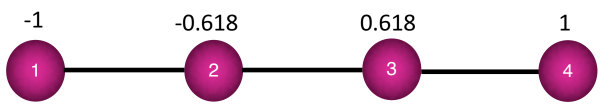
Think of $v \in \mathbb{R}^V$. even better $v : V \to \mathbb{R}$

Symmetric -> n real eigenvalues and
real eigenvectors form orthonormal basis

对称阵：n个实特征值
实特征向量形成标准正交基

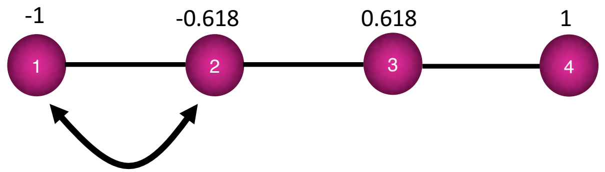
Example

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ -0.618 \\ 0.618 \\ 1 \end{pmatrix} = 0.618 \begin{pmatrix} -1 \\ -0.618 \\ 0.618 \\ 1 \end{pmatrix}$$



Example: invariant under re-labeling 不变性

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ -0.618 \\ 0.618 \\ 1 \end{pmatrix} = 0.618 \begin{pmatrix} -1 \\ -0.618 \\ 0.618 \\ 1 \end{pmatrix}$$



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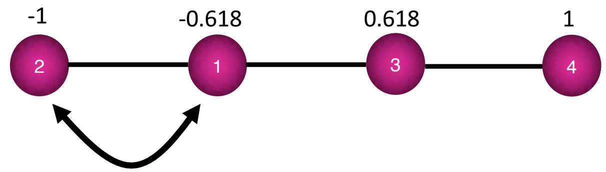

Invariance under permutations

置换不变性

Let $\mathbf{\Pi}$ be a permutation matrix. That is, there is a permutation $\pi : V \to V$ so that

$$\mathbf{\Pi}(u, v) = \begin{cases} 1 & \text{if } u = \pi(v), \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Prove that if

$$\boldsymbol{A}\psi = \lambda\psi,$$

then

$$\left(\mathbf{\Pi} \boldsymbol{A} \mathbf{\Pi}^T\right)(\mathbf{\Pi}\psi) = \lambda(\mathbf{\Pi}\psi).$$

Operators and Quadratic Forms

View of A as an operator:

$$y = Ax \qquad\qquad y(i) = \sum_{j:(i,j)\in E} x(j)$$

View of A as quadratic form:

$$x^T A x = \sum_{(i,j)\in E} x(i)x(j)$$

if $Ax = \lambda x$ and $\|x\| = 1$ then $x^T A x = \lambda$

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- **Graph Laplacian**
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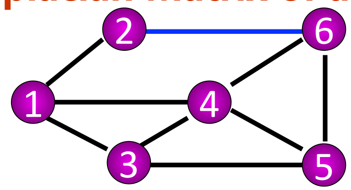
The Laplacian Matrix of a Graph



$$\begin{pmatrix} 3 & -1 & -1 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & -1 \\ -1 & 0 & 3 & -1 & -1 & 0 \\ -1 & 0 & -1 & 4 & -1 & -1 \\ 0 & 0 & -1 & -1 & 3 & -1 \\ 0 & -1 & 0 & -1 & -1 & 3 \end{pmatrix}$$

Symmetric

Non-positive off-diagonals

Diagonally dominant

对称的
非正的
非对角的
对角占优

拉普拉斯:图上的自然二次型

Laplacian: natural quadratic form on graphs

$$x^T L x = \sum_{(i,j)\in E} (x(i) - x(j))^2$$

D其中为图的度矩阵(对角)，A为图的邻接矩阵

$L = D - A$ where D is diagonal matrix of degrees

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

Embedding graph in line (Hall ' 70)

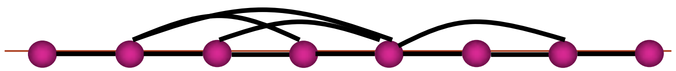
map $V \longrightarrow \mathbb{R}$

minimize $\sum_{(i,j)\in E} (x(i) - x(j))^2 = x^T L x$

trivial solution: $x = \mathbf{1}$ So, require $x \perp \mathbf{1}$

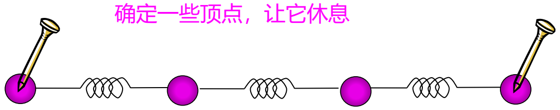
Solution $x = v_2$

Atkins, Boman, Hendrickson '97:
Gives correct embedding for graphs like



Spring Networks

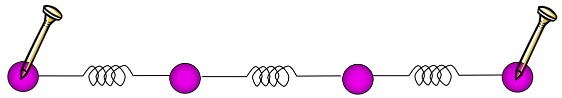
- View edges as rubber bands or ideal linear springs 视边缘为橡皮筋或理想的线性弹簧
- Nail down some vertices, let rest settle
  确定一些顶点，让它休息



- In equilibrium, nodes are averages of neighbors. 在平衡状态下，节点是邻居的平均值

Spring Networks

- View edges as rubber bands or ideal linear springs
- Nail down some vertices, let rest settle

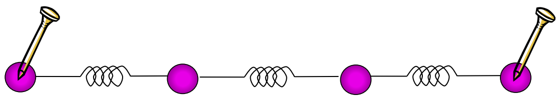


When stretched to length ℓ

potential energy is $\ell^2/2$

Spring Networks

- Nail down some vertices, let rest settle

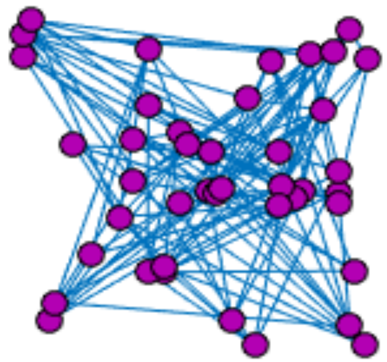


Physics: position minimizes total potential energy

$$\frac{1}{2}\sum_{(i,j)\in E}(x(i)-x(j))^2$$

subject to boundary constraints (nails)

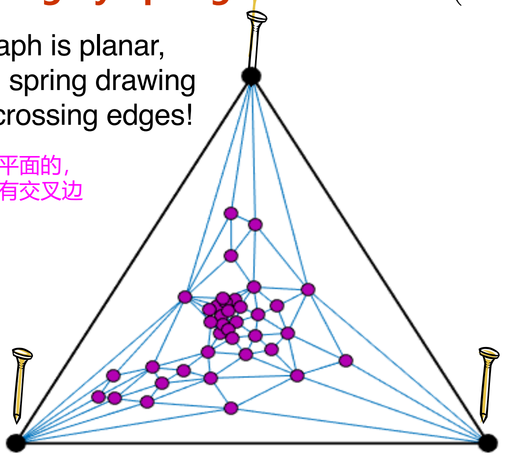
Drawing by Spring Networks (Tutte '63)



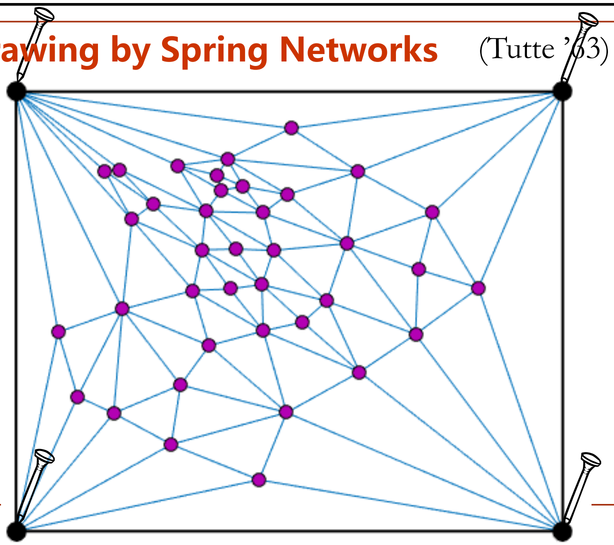
Drawing by Spring Networks (Tutte '63)

If the graph is planar,
then the spring drawing
has no crossing edges!

如果图形是平面的，
则弹簧图没有交叉边



Drawing by Spring Networks (Tutte '63)



Graph Laplacian

对于任何使能量最小的x，能量对x(u)的偏导数必须是零

For any x that minimizes the energy, the partial derivative of the energy with respect to $x(u)$ must be zero.

$$\frac{1}{2}\sum_{v:(u,v)\in E} w_{u,v}2(\boldsymbol{x}(u)-\boldsymbol{x}(v)) = \sum_{v:(u,v)\in E} w_{u,v}(\boldsymbol{x}(u)-\boldsymbol{x}(v)).$$

方程等于0，得

Setting this to zero gives the equations we previously derived:

$$d(u)\boldsymbol{x}(u) - \sum_{v:(u,v)\in E} w_{u,v}\boldsymbol{x}(v) = 0,$$

$$x(u) = \sum_{v:(u,v)\in E} \frac{w_{u,v}}{d(u)} x(v)$$

note: random walk iteration

It corresponds to the row of the Laplacian matrix for vertex u.

它对应于顶点u的拉普拉斯矩阵的行

Laplacian operator and graph Laplacian

The Laplace operator is a second order differential operator in the n-dimensional Euclidean space, defined as the divergence($\nabla\cdot$) of the gradient(∇f).

$$\Delta f = \nabla^2 f = \nabla\cdot\nabla f$$

拉普拉斯算子是一个二阶微分算子在n维欧几里得空间,定义为梯度(∇f)的散度(∇·)。

where the latter notations derive from formally writing

$$\nabla = \left(\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_n}\right).$$

the Laplacian of f is the sum of all the *unmixed* second partial derivatives in the Cartesian coordinates x_i:

$$\Delta f = \sum_{i=1}^{n}\frac{\partial^2 f}{\partial x_i^2}$$

f的拉普拉斯变换是笛卡尔坐标系xi中所有非混合二阶偏导的和:

Laplacian operator

- **Consider one-variable case:**

$$\triangle f = \frac{\partial^2 f}{\partial x^2} = \left(\frac{f(x+\delta x) - f(x)}{\delta x}\right)_x = \frac{f(x+\delta x)_x - f(x)_x}{\delta x}$$
$$= \left(\frac{f(x+2\delta x) - f(x+\delta x)}{\delta x} - \frac{f(x+\delta x) - f(x)}{\delta x}\right) / \delta x$$

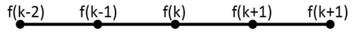
$$\triangle f = \left(\frac{f(x+\delta x) - f(x)}{\delta x} - \frac{f(x) - f(x-\delta x)}{\delta x}\right) / \delta x$$
$$= \frac{f(x+\delta x) + f(x-\delta x) - 2f(x)}{\delta x^2}$$

$$f(x+\delta x) = f(x) + \delta x f'(x) + \frac{\delta x^2}{2} f''(x) + \ldots = f(x) + \delta x f'(x) + \frac{\delta x^2}{2} f''(\bar{x})$$
$$f(x-\delta x) = f(x) - \delta x f'(x) + \frac{\delta x^2}{2} f''(x) + \ldots = f(x) - \delta x f'(x) + \frac{\delta x^2}{2} f''(\bar{\bar{x}})$$
$$f(x+\delta x) + f(x-\delta x) = 2f(x) + \delta x^2 f''(x) + \ldots = 2f(x) + \frac{\delta x^2}{2}(f''(\bar{x}) + f''(\bar{\bar{x}}))$$
$$\triangle f = \frac{\partial^2 f}{\partial x^2} = \frac{(f''(\bar{x}) + f''(\bar{\bar{x}}))}{2}$$

- **averages the derivatives of function**

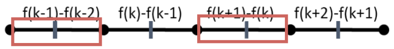
求函数导数的平均值

Graph Laplacian



1D discretizing derivatives 一维离散导数

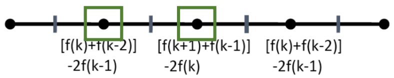
Function f: $\mathbb{R} \to \mathbb{R}$, discretize at the points $(\ldots, k-2, k-1, k, k+1, k+2, \ldots)$



On edges $\nabla_{P_n} f$

1[st] derivative at the line midpoints $\frac{df}{dx}$

一阶导是点的中点



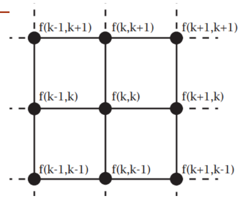
$-L_{P_n} f$

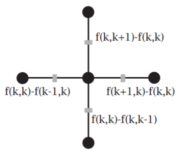
2[nd] derivative at the original points, $\frac{d^2 f}{dx^2}$

二阶导是点

Graph Laplacian

2D discretizing derivatives

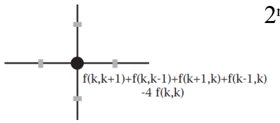




1[st] on horizontal: $\frac{df}{dx}$, vertical : $\frac{df}{dy}$

On edges → The discretization of the gradient $\nabla_{G_{n,m}} f$

2[nd] derivative



The discretized Laplacian in 2d of f is $-L_{G_{n,m}} f$

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Spectral Graph Theory

这儿要记住要考

A n-by-n symmetric matrix has n real eigenvalues $\lambda_1 \le \lambda_2 \cdots \le \lambda_n$ and eigenvectors $v_1, \ldots, v_n$ such that

$$L v_i = \lambda_i v_i$$

These eigenvalues and eigenvectors tell us a lot about a graph!

(excluding $\lambda_1 = 0, v_1 = \mathbb{1}$)

排除

Rayleigh quotient

瑞利商

Definition 1.6.2. *The Rayleigh quotient of a vector $\boldsymbol{x}$ with respect to a matrix $\boldsymbol{M}$ is the ratio*

$$\frac{\boldsymbol{x}^T \boldsymbol{M} \boldsymbol{x}}{\boldsymbol{x}^T \boldsymbol{x}}.$$

Observe that if ψ is an eigenvector of $\boldsymbol{M}$ of eigenvalue λ, then

$$\frac{\psi^T \boldsymbol{M} \psi}{\psi^T \psi} = \frac{\psi^T \lambda \psi}{\psi^T \psi} = \frac{\lambda \psi^T \psi}{\psi^T \psi} = \lambda.$$

Theorem 1.6.3. *Let $\boldsymbol{M}$ be a symmetric matrix and let $\boldsymbol{x}$ be a non-zero vector that maximizes the Rayleigh quotient with respect to $\boldsymbol{M}$. Then, $\boldsymbol{x}$ is an eigenvector of $\boldsymbol{M}$ with eigenvalue equal to the Rayleigh quotient. Moreover, this eigenvalue is the largest eigenvalue of $\boldsymbol{M}$.*

令M是对称阵，x是非零向量，最大化M对应得瑞利商，
然后x是M的特征向量，
且瑞利商等于特征值。另外这个特征值是M的最大特征值。

2019/10/29

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Courant-Fischer definition of eigvals/vecs

$$\lambda_1 = \min_{x \neq 0} \frac{x^T L x}{x^T x} \qquad v_1 = \arg\min_{x \neq 0} \frac{x^T L x}{x^T x}$$

$$\lambda_2 = \min_{x \perp v_1} \frac{x^T L x}{x^T x} \qquad v_2 = \arg\min_{x \perp v_1} \frac{x^T L x}{x^T x}$$

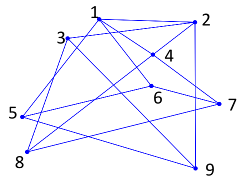
here $v_1 = \mathbf{1}$

$$\lambda_k = \min_{S \text{ of dim } k} \max_{x \in S} \frac{x^T L x}{x^T x}$$

$$v_k = \arg \min_{x \perp v_1, \ldots, v_{k-1}} \frac{x^T L x}{x^T x}$$

Spectral Graph Drawing

(Hall '70)

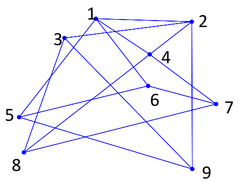


Original Drawing

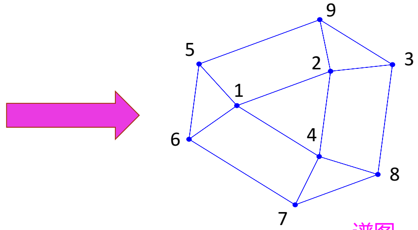
Spectral Graph Drawing

(Hall '70)

Plot vertex a at $(v_2(a), v_3(a))$
draw edges as straight lines



Original Drawing 原图



谱图

Spectral Drawing

Spectral Graph Drawing

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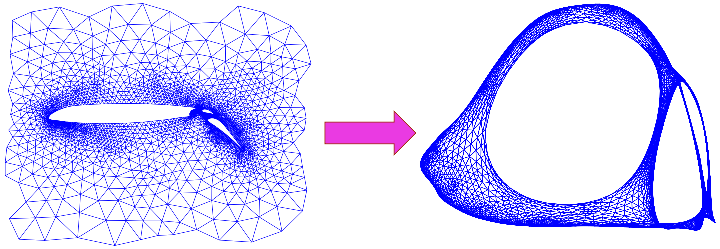
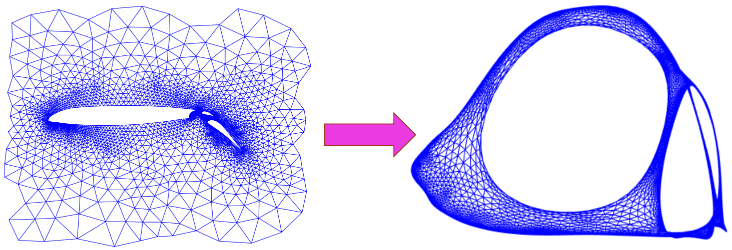


Original Drawing

Spectral Drawing

Spectral Graph Drawing

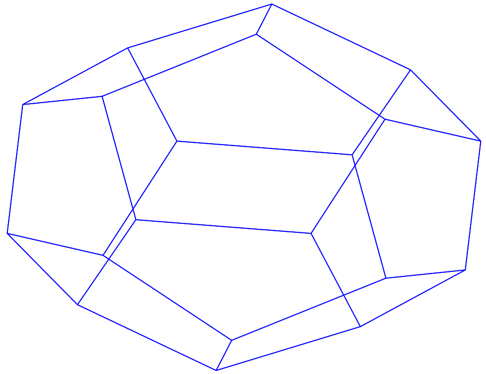
(Hall '70)



Original Drawing

Spectral Drawing

Dodecahedron



Best embedded by first three eigenvectors 最好由前三个特征向量嵌入

When there is a "nice" drawing

Most edges are short
Vertices are spread out and don't clump too much

大多数的边都是短的
顶点被分散开来，不会聚集太多

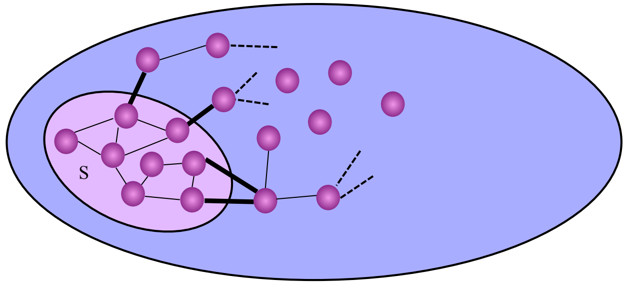
$\Rightarrow$ λ_2 is close to 0

When λ_2 is big, say $> 10/|V|^{1/2}$
there is no nice picture of the graph

Measuring boundaries of sets

集的测量边界

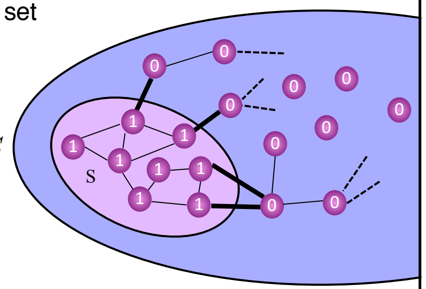
Boundary: edges leaving a set



Measuring boundaries of sets

Boundary: edges leaving a set
Characteristic Vector of S:

$$x(a) = \begin{cases} 1 & a \text{ in } S \\ 0 & a \text{ not in } S \end{cases}$$



Measuring boundaries of sets

Boundary: edges leaving a set
Characteristic Vector of S: S特征向量

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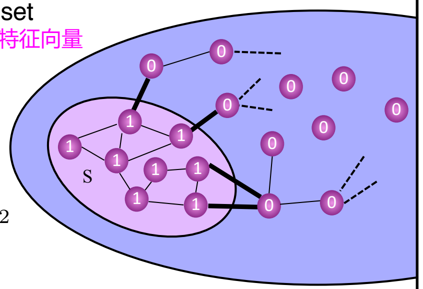


$$\sum_{(a,b)\in E} (x(a) - x(b))^2$$

$$= |\text{boundary}(S)|$$

这里S不等于全集

Spectral Clustering and Partitioning

找到小边界的大集合

Find large sets of small boundary

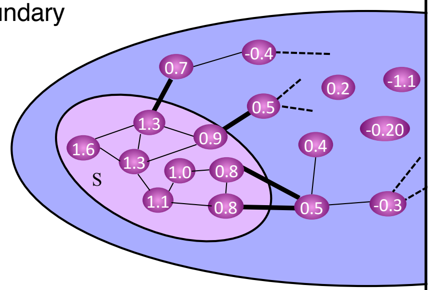
Heuristic to find 启发式寻找

$\boxed{x \text{ with } x^T L_G x \text{ small}}$

Compute eigenvector

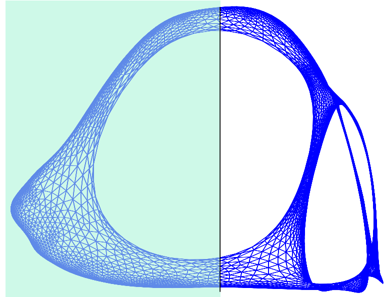
$$L_G v_2 = \lambda_2 v_2$$

Consider the level sets



Spectral Partitioning

(Donath-Hoffman '72, Barnes '82, Hagen-Kahng '92)

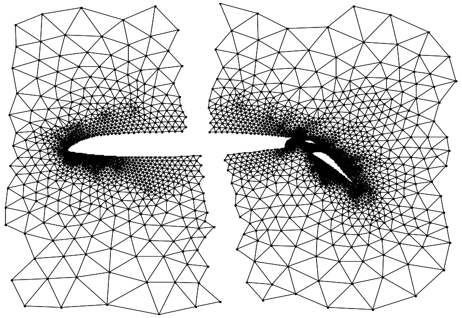


$S = \{a : v_2(a) \le t\}$ for some t

Cheeger's inequality implies good approximation

Spectral Partitioning

(Donath-Hoffman '72, Barnes '82, Hagen-Kahng '92)



$S = \{a : v_2(a) \le t\}$ for some t

Cheeger's inequality implies good approximation

Graph Laplacian

- **Complete graph** 完全图：Kn，每对不同顶点之间恰有一条边相连

***Kn*, which has edge set $\{(u,v): u \neq v\}$.**

完全图拉普拉斯矩阵有一重0特征值，n-1重特征值n

Lemma 2.5.1. *The Laplacian of K_n has eigenvalue 0 with multiplicity 1 and n with multiplicity $n-1$.*

当第二个特征值很大（接近n）的时候可以认为接近完全图

- **Star graph**

***Sn*, which has edge set $\{(1,u): 2 \le u \le n\}$.**

星图：Sn，一个节点与其他所有点相连

Lemma 2.5.3. *The graph S_n has eigenvalue 0 with multiplicity 1, eigenvalue 1 with multiplicity $n-2$, and eigenvalue n with multiplicity 1.*

星状图Sn有一重特征值0和n，
n-2重特征值1

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Graph Laplacian

超立方体

具有顶点集{0,1}^d的图，顶点之间的边的名称仅相差一位。

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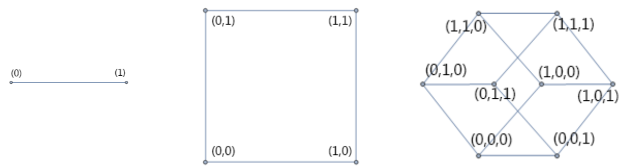
- **Hypercube**

Graph with vertex set $\{0,1\}^d$, with edges between vertices whose names differ in exactly one bit.



The eigenvalues of its Laplacian matrix the numbers (0, 2, ..., 2n), kth eigenvalue has multiplicity $\binom{n}{k}$. 特征值有{0,2,4...,2n}，每个特征值（^n _K）重

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Triangle: Computations

[Tsourakakis ICDM 2008]

计算三角形

Triangles are expensive to compute

(3-way join; several approx. algos)

三角形计算昂(3路连接;几个近似算法)

Q: Can we do that quickly?

A: Yes!

#triangles = 1/6 Sum (λ_i^3)

(and, because of skewness (S2) ,

we only need the top few eigenvalues!

因为偏态（S2），只需要最大的几个特征值

15-826 (c) C. Faloutsos, 2017 46

Outline

- **Graphs and related matrix**
- **Graph Laplacian**
- **Spectral Graph Theory**
- **Eigen Value and Vectors: connected components**连通分支
- **Spectral Sparsification**

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Eigenvalue Interlacing

特征值交叉

A是n*n对称阵，B是A的n-1维主子矩阵

Theorem 7.5.2 (Eigenvalue Interlacing). *Let A be an n-by-n symmetric matrix and let B be a principal submatrix of A of dimension $n-1$ (that is, B is obtained by deleting the same row and column from A). Then,*

$$\alpha_1 \geq \beta_1 \geq \alpha_2 \geq \beta_2 \geq \cdots \geq \alpha_{n-1} \geq \beta_{n-1} \geq \alpha_n,$$

where $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$ and $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_{n-1}$ are the eigenvalues of A and B, respectively.

α和β分别是A和B的特征值

If S is the subgraph of G, then

$$\lambda_{max}(A_G) \geq \lambda_{max}(A_S) \geq \lambda_{min}(A_S) \geq \lambda_{min}(A_G)$$

note: A_S has fewer eigenvalues than A_G, e.g. $\lambda_{n-1}(A_S) = \lambda_{min}(A_S)$

recall: subset of edges $F \subseteq E$ then $L_H \preccurlyeq L_G$ then $\lambda_k(L_H) \leq \lambda_k(L_G)$

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Perron-Frobenius Theorem

佩龙一弗罗宾尼斯定理

对称阵。G是连通带权图，A是邻接矩阵，μ1>=μ2>=,,,,,,μn，是特征值，则

Theorem 3.5.1. *[Perron-Frobenius, Symmetric Case] Let G be a connected weighted graph, let $\boldsymbol{A}$ be its adjacency matrix, and let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ be its eigenvalues. Then*

a. $\mu_1 \geq -\mu_n$, *and*

b. $\mu_1 > \mu_2$,

c. The eigenvalue μ_1 has a strictly positive eigenvector. μ1有严格正值的特征向量

Corollary 7.5.1. *Let M be a matrix with non-positive off-diagonal entries, such that the graph of the non-zero off-diagonally entries is connected. Let λ_1 be the smallest eigenvalue of M and let $\boldsymbol{v}_1$ be the corresponding eigenvector. Then $\boldsymbol{v}_1$ may be taken to be strictly positive, and λ_1 has multiplicity 1.*

拉普拉斯的v1是1向量

M非对角元素非正的矩阵，使得非零非对角元素的图是连通的。令λ1是M最小的特征值，v1是对应特征向量。则，v1可能严格正的，λ1是1重特征值

Proof. Consider the matrix $A = \sigma I - M$, for some large σ. For σ sufficiently large, this matrix will be non-negative, and the graph of its non-zero entries is connected. So, we may apply the Perron-Frobenius theory to A to conclude that its largest eigenvalue α_1 has multiplicity 1, and the corresponding eigenvector $\boldsymbol{v}_1$ may be assumed to be strictly positive. We then have $\lambda_1 = \sigma - \alpha_1$, and $\boldsymbol{v}_1$ is an eigenvector of λ_1. □

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Fiedler's Nodal Domain Theorem

费得勒节域定理

V2是什么

Theorem 7.6.1 ([Fie75]). *Let $G = (V, E, w)$ be a weighted connected graph, and let L_G be its Laplacian matrix. Let $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n$ be the eigenvalues of L_G and let $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ be the corresponding eigenvectors. For any $k \geq 2$, let*

$$W_k = \{i \in V : \boldsymbol{v}_k(i) \geq 0\}.$$

Then, the graph induced by G on W_k has at most $k-1$ connected components.

G是带权连通图，LG是拉普拉斯矩阵，
特征值0=λ1<...<λn，v1,...,vn是特征向量，
对于任意k>=2，使得Wk={}，则
G在Wk上包含的图至多k-1个连通分量。

recall that $v_1 = \mathbf{1}$ and v_k is orthogonal to v_1

So v_k has both positive and negative entries.

回想一下v1=1 vk正交于v1
所以vk有正的和负的分量

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Fiedler's Nodal Domain Theorem

Proof. cont.

Assume $G(W_k)$ (Graph induced by W_k)has t connected components.

Re-order vertices let vertices in one connected component of $G(W_k)$ appear first. $(x_i \geq 0\,, y < 0)$

$$L_G v_k = \lambda_k v_k$$

$$\begin{bmatrix} B_1 & \mathbf{0} & \mathbf{0} & \cdots & C_1 \\ \mathbf{0} & B_2 & \mathbf{0} & \cdots & C_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & B_t & C_t \\ C_1^T & C_2^T & \cdots & C_t^T & \boxed{D} \end{bmatrix} \begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \vdots \\ \boldsymbol{x}_t \\ \boldsymbol{y} \end{pmatrix} = \lambda_k \begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \vdots \\ \boldsymbol{x}_t \\ \boldsymbol{y} \end{pmatrix}$$

每个Bi中的非零项的图是连通的，而每个Ci是非正的，并且至少有一个非零项(否则图G将断开连接)

Graph of non-zero entries in each B_i is connected, and that each C_i is non-positive, and has at least one non-zero entry (otherwise the graph G would be disconnected).

y是特征向量的负值部分
D是LG中与vk相应的剩下的元素

note: y is the part of negative values in eigenvector v_k
D is the remaining elements of L_G corresponding to v_k

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Fiedler's Nodal Domain Theorem

Proof. cont.

As each entry in C_i is non-positive and $\boldsymbol{y}$ is strictly negative, each entry of $C_i\boldsymbol{y}$ is non-negative and some entry of $C_i\boldsymbol{y}$ is positive. Thus, $\boldsymbol{x}_i$ cannot be all zeros,

$$B_i\boldsymbol{x}_i = \lambda_k\boldsymbol{x}_i - C_i\boldsymbol{y} \leq \lambda_k\boldsymbol{x}_i$$

and

$$\boldsymbol{x}_i^T B_i \boldsymbol{x}_i \leq \lambda_k \boldsymbol{x}_i^T \boldsymbol{x}_i.$$

Goal: to prove $\lambda_{min}(B_i) < \lambda_k$

1) x_i all positive, then $x_i C_i y > 0$,and

$$\boldsymbol{x}_i^T B_i \boldsymbol{x}_i = \lambda_k \boldsymbol{x}_i^T \boldsymbol{x}_i - \boldsymbol{x}_i^T C_i \boldsymbol{y} < \lambda_k \boldsymbol{x}_i^T \boldsymbol{x}_i.$$

xi有0项，Perron-Frobenius定理告诉我们xi不能是最小特征值的特征向量，所以Bi的最小特征值小于λk

2) x_i has zero entry, Perron-Frobenius theorem tells us that x_i cannot be an eigenvector of smallest eigenvalue, and so the smallest eigenvalue of B_i is less than λ_k.

Note: refer to Corollary 7.5.1, B_i has non-positive off-diagonal entries, and its non-zero off-diagonally entries are connected as the i-th connected component.

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参照推论7.5.1,B_i具有非正的非对角分量，其非零的非对角分量作为第i个连通分量被连通。

Fiedler’s Nodal Domain Theorem

Proof. cont.

Thus, the matrix

$$B = \begin{bmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_t \end{bmatrix}$$

has at least t eigenvalues less than λ_k. By the eigenvalue interlacing theorem, this implies that L_G has at least t eigenvalues less than λ_k. We may conclude that t, the number of connected components of $G(W_k)$, is at most $k-1$. □

B至少有t特征值比λk小。
通过特征值交叉理论，LG至少有t个特征值小于λk。
我们得到结论，G(Wk)的连通分量数t，最多为k-1。

interlacing eigenvalue e.g. 特征值交错

$$\lambda_1 \le \lambda_1(B) \le \lambda_2 \le \lambda_2(B) \le \cdots \le \lambda_t(B) < \lambda_k \le \cdots \le \lambda_n$$

So

$$t \le k-1$$

Outline

- Graphs and related matrix
- Graph Laplacian
- Spectral Graph Theory
- Eigen Value and Vectors: connected components
- Spectral Sparsification

谱稀疏化

Spectral Sparsification

- **Every graph can be approximated by a sparse graph with a similar Laplacian**

每个图都可以用具有相似拉普拉斯算子的
稀疏图来近似

Approximating Graphs

A graph H is an ϵ-approximation of G if

for all x $$\frac{1}{1+\epsilon} \le \frac{x^T L_H x}{x^T L_G x} \le 1+\epsilon$$

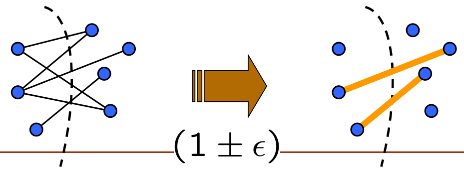
$$L_H \approx_\epsilon L_G$$

Approximating Graphs 近似图

A graph H is an ϵ-approximation of G if

for all x $\frac{1}{1+\epsilon} \le \frac{x^T L_H x}{x^T L_G x} \le 1+\epsilon$

Preserves boundaries of every set



Approximating Graphs

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Solutions to linear equations are similar 线性方程的解是相似的

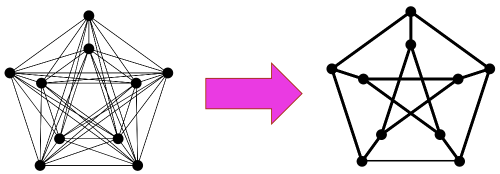
$$L_H \approx_\epsilon L_G \iff L_H^{-1} \approx_\epsilon L_G^{-1}$$

As are effective resistances 有效电阻也是如此

Expanders Sparsify Complete Graphs 扩展器使完全图稀疏化

Every set of vertices has large boundary

λ_2 is large



Random regular graphs are usually expanders

Sparsification by Random Sampling 随机抽样稀疏化

Assign a probability $p_{a,b}$ to each edge (a,b)

Include edge (a,b) in H with probability $p_{a,b}$.

If include edge (a,b), give it weight $w_{a,b}/p_{a,b}$

$$\mathbb{E}[\ L_H\ ] = \sum_{(a,b)\in E} p_{a,b}(w_{a,b}/p_{a,b})L_{a,b} = L_G$$

Sparsification by Random Sampling

Pab为W乘ab之间有效电阻

Choose $p_{a,b}$ to be $w_{a,b}$ times the effective resistance between a and b.

Low resistance between a and b means there are many alternate routes for current to flow and that the edge is not critical.

意味着电流有许多可供选择的流动路线，而且边缘不是临界的。

Proof by random matrix concentration bounds (Rudelson, Ahlswede-Winter, Tropp, etc.)

通过随机矩阵浓度界证明

Only need $O(n \log n/\epsilon^2)$ edges

(S, Srivastava '08)

Optimal Graph Sparsification?

最优图稀疏表示

For every $G = (V, E, w)$, there is a $H = (V, F, z)$ s.t.

$$L_G \approx_\epsilon L_H \quad \text{and} \quad |F| \le (2+\epsilon)^2 n/\epsilon^2$$

(Batson, S, Srivastava '09)

Take away

考试的形式

- $L_G = D - A$
- $L_G = \sum_e L_e$
- $L_G = \sum_{(u,v)\in E} w_{u,v}(\delta_u - \delta_v)(\delta_u - \delta_v)^T$
  $= \sum_{(u,v)\in E} w_{u,v} L_{G_{u,v}}$
- $x^T L_G x = \sum_{(u,v)\in E} w_{u,v}\big(x(u) - x(v)\big)^2$

$if\ \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$
- $\lambda_2 > 0$. **if and only if graph is connected**
- $\lambda_1 = 0, it's\ eigen\ vector\ \phi_1 = \mathbf{1}$
- **Normalize Laplace:** $\mathrm{D}^{-1/2}\mathrm{L}\mathrm{D}^{-1/2} = I - \mathrm{D}^{-1/2}\mathrm{A}\mathrm{D}^{-1/2}$

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Reference

- **Gary Miller (CMU)**
  - http://www.cs.cmu.edu/afs/cs/academic/class/15859n-f16/schedule.html
- **Dan Spielman (Yale)**
  - http://www.cs.yale.edu/homes/spielman/561/

Questions?