Differential Equations lab 5

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Task1 Consider following system of differential equation $y' = Ay + f(t)$, where A - constant $3x3$ matrix, y, f - 3-dimensional vectors:

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 0 & -1 \\ 2 & -1 & 2 \end{pmatrix}, f(t) = \begin{pmatrix} 24 \\ -e^{2t} \\ 2e^{2t} \end{pmatrix}$$

Task 1.1 Solve homogeneous system of diff. eqs. $y' = Ay$

$$\begin{vmatrix} 3-\lambda & 1 & 1 \\ 1 & -\lambda & -1 \\ 2 & -1 & 2-\lambda \end{vmatrix} = 0;$$

$$-8 - 2\lambda + 5\lambda^2 - \lambda^3 = 0 \lambda_1 = 4, v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \lambda_2 = 2, v_2 = \begin{pmatrix} -1 \\ -2 \\ 3 \end{pmatrix} \lambda_3 = -1, v_3 = \begin{pmatrix} -2 \\ 5 \\ 3 \end{pmatrix}$$

hence

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = C_1 e^{4t} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} -1 \\ -2 \\ 3 \end{pmatrix} + C_3 e^{-t} \begin{pmatrix} -2 \\ 5 \\ 3 \end{pmatrix}$$

Task 1.2 Calculate e^{At}, $det(e^{-A})$

$$\lambda_1 = 4, \lambda_2 = 2, \lambda_3 = -1 \implies J_A = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$A = H^{-1} J_A H$ where H - transition matrix to Jordan bases, hence $AH = J_A H$

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \begin{pmatrix} 3 & 1 & 1 \\ 1 & 0 & -1 \\ 2 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

$$\begin{cases} 3a_1 + a_2 + 2a_3 & = 4a_1 \\ a_1 - a_3 & = 4a_2 \\ a_1 - a_2 + 2_a 3 & = 4a_3 \end{cases} \implies a_1 = 7, a_2 = 1, a_3 = 3$$

$$\begin{cases} 3b_1 + b_2 + 2b_3 & = 2b_1 \\ b_1 - b_3 & = 2b_2 \\ b_1 - b_2 + 2_b 3 & = 2b_3 \end{cases} \implies b_1 = 1, b_2 = 1, b_3 = -1$$

$$\begin{cases} 3c_1 + c_2 + 2c_3 & = -c_1 \\ c_1 - c_3 & = -c_2 \\ c_1 - c_2 + 2_c 3 & = -c_3 \end{cases} \implies c_1 = 1, c_2 = -1, c_3 = 2$$

In total:

$$H = \begin{pmatrix} 7 & 1 & 3 \\ 1 & 1 & -1 \\ 1 & -1 & 2 \end{pmatrix}; H^{-1} = 0.5 \begin{pmatrix} -1 & 5 & 4 \\ 3 & -11 & 10 \\ 2 & -8 & -6 \end{pmatrix}$$

$$e^{At} = H^{-1} e^{J_A} H = \begin{pmatrix} 6e^{-t} + 3e^{2t} - 7e^{4t} & -24e^{-t} - 11e^{2t} + 35e^{4t} & -18e^{-t} - 10e^{2t} + 28e^{4t} \\ 2e^{-t} + 3e^{2t} - e^{4t} & -8e^{-t} - 11e^{2t} + 5e^{4t} & -6e^{-t} - 10e^{2t} + 4e^{4t} \\ 4e^{-t} - 3e^{2t} - e^{4t} & -16e^{-t} + 11e^{2t} + 5e^{4t} & -12e^{-t} + 10e^{2t} + 4e^{4t} \end{pmatrix}$$

Task 1.3 Solve non-homogeneous diff. eq. using undefined coefficients method

$f_1 = 24 = e^{\sigma t} P_m(t) \implies \sigma = 0, m = 0, s = 0$

$$\begin{cases} x_{part,1} &= A \\ y_{part,1} &= B \\ z_{part,1} &= C \end{cases}$$

substituting into equation infer:

$$\begin{cases} 3A + B + C + 24 &= 0 \\ A - C &= 0 \\ 2A - B + 2C &= 0 \end{cases} \implies A = -3, B = 0, C = 3$$

$f_2 = -e^{\sigma 2t} \implies \sigma = 2, m = 0, s = 1$

$$\begin{cases} x_{part,2} &= (A_1 t + A_2)e^{2t} \\ y_{part,2} &= (B_1 t + B_2)e^{2t} \\ z_{part,2} &= (C_1 t + C_2)e^{2t} \end{cases}$$

substituting into equation after canselling out all exponents we get:

$$\begin{cases} 2A_1 t + 2A_2 + A_1 = 3(A_1 t + A_2) + (B_1 t + B_2) + (C_1 t + C_2) \\ 2B_1 t + 2B_2 + B_1 = (A_1 t + A_2) - (C_1 t + C_2) - 1 \\ 2C_1 t + 2C_2 + C_1 = 2(A_1 t + A_2) - (B_1 t + B_2) + 2(C_1 t + C_2) \end{cases} \implies$$

$$\implies A_1 = 0, B_1 = 0, C_1 = 0, A_2 = 0, B_2 = -1, C_2 = 1;$$

hence

$$\begin{cases} x_{part,2} &= 0 \\ y_{part,2} &= -e^{2t} \\ z_{part,2} &= e^{2t} \end{cases}$$

$f_3 = 2e^t \implies \sigma = 2, m = 0, s = 1$

$$\begin{cases} x_{part,3} &= (A_1 t + A_2)e^{2t} \\ y_{part,3} &= (B_1 t + B_2)e^{2t} \\ z_{part,3} &= (C_1 t + C_2)e^{2t} \end{cases}$$

substituting into equation after canselling out all exponents we get:

$$\begin{cases} 2A_1 t + 2A_2 + A_1 = 3(A_1 t + A_2) + (B_1 t + B_2) + (C_1 t + C_2) \\ 2B_1 t + 2B_2 + B_1 = (A_1 t + A_2) - (C_1 t + C_2) \\ 2C_1 t + 2C_2 + C_1 = 2(A_1 t + A_2) - (B_1 t + B_2) + 2(C_1 t + C_2) + 2 \end{cases} \implies$$

$$\implies A_1 = 0, B_1 = 0, C_1 = 0, A_2 = -\frac{2}{3}, B_2 = -\frac{-4}{3}, C_2 = 2;$$

hence

$$\begin{cases} x_{part,3} &= -\frac{2}{3}e^{2t} \\ y_{part,3} &= -\frac{-4}{3}e^{2t} \\ z_{part,3} &= 2e^{2t} \end{cases}$$

totally:

$$\begin{cases} x_{total} = C_1e^{4t} - C_2e^{2t} - 2e^{-t} - \frac{2}{3}e^{2t} - 3 \\ y_{total} = 2C_2e^{2t} + 5e^{-t} - e^{2t} - \frac{4}{3}e^{2t} \\ z_{total} = -C_1e^{4t} + 3C_2e^{2t} + 3C_3e^{-t} + e^{2t} + 2e^{2t} + 3 \end{cases}$$

Task 2 Solve given system of non-homogeneous differential equations using Lagrange's method

$$\begin{cases} \dot{x} = -4x + y - e^{-t}tg2t \\ \dot{y} = -3x + 2y \end{cases}$$

corresponding homogeneous system:

$$\begin{cases} \dot{x} = -4x + y \\ \dot{y} = -3x + 2y \end{cases} ; \quad A = \begin{pmatrix} -4 & 1 \\ -3 & 2 \end{pmatrix}; \quad f_1 = -e^{-t}tg(2t), \quad f_2 = 0;$$

$$\lambda^2 - 2\lambda - 5 = 0 \quad \lambda_{1,2} = -1 \pm \sqrt{6} \quad v_{1,2} = \begin{pmatrix} \frac{1}{3}(3 \pm \sqrt{6}) \\ 1 \end{pmatrix}$$

From this point I'd like to make expressions less obfuscated by making the following substitution:

$$\alpha_1 = \frac{1}{3}(3 + \sqrt{6})\alpha_2 = \frac{1}{3}(3 + \sqrt{6})$$

$$\begin{cases} x = \alpha_1C_1e^{\left(-1-\sqrt{6}\right)t} + \alpha_2C_2e^{\left(-1+\sqrt{6}\right)t} \\ y = C_1e^{\left(-1-\sqrt{6}\right)t} + C_2e^{\left(-1+\sqrt{6}\right)t} \end{cases}$$

Applying Lagranges method we get the following:

$$\begin{cases} \alpha_1C_1'e^{\left(-1-\sqrt{6}\right)t} + \alpha_2C_2'e^{\left(-1+\sqrt{6}\right)t} = -e^{-t}tg(2t) \\ C_1'e^{\left(-1-\sqrt{6}\right)t} + C_2'e^{\left(-1+\sqrt{6}\right)t} = 0 \end{cases}$$

$$C_2' = \frac{e^{-t}tg(2t)}{e^{\left(-1+\sqrt{6}\right)t}\left(\alpha_2 - \alpha_1\right)}C_1' = -e^{2\sqrt{6}t}\frac{e^{-t}tg(2t)}{e^{\left(-1+\sqrt{6}\right)t}\left(\alpha_2 - \alpha_1\right)}$$

$$C_1 = \int\left(- e^{2\sqrt{6}t}\frac{e^{-t}tg(2t)}{e^{\left(-1+\sqrt{6}\right)t}\left(\alpha_2 - \alpha_1\right)}\right)dt + \tilde{C_1}C_2 = \int\left(\frac{e^{-t}tg(2t)}{e^{\left(-1+\sqrt{6}\right)t}\left(\alpha_2 - \alpha_1\right)}\right)dt + \tilde{C_2}$$

totally:

$$\begin{cases} x = \alpha_1\int\left(- e^{2\sqrt{6}t}\frac{e^{-t}tg(2t)}{e^{\left(-1+\sqrt{6}\right)t}\left(\alpha_2-\alpha_1\right)}\right)dt + \tilde{C_1}e^{\left(-1-\sqrt{6}\right)t} + \alpha_2C_2 = \int\left(\frac{e^{-t}tg(2t)}{e^{\left(-1+\sqrt{6}\right)t}\left(\alpha_2-\alpha_1\right)}\right)dt + \tilde{C_2}e^{\left(-1+\sqrt{6}\right)t} \\ y = \int\left(- e^{2\sqrt{6}t}\frac{e^{-t}tg(2t)}{e^{\left(-1+\sqrt{6}\right)t}\left(\alpha_2-\alpha_1\right)}\right)dt + \tilde{C_1}e^{\left(-1-\sqrt{6}\right)t} + C_2 = \int\left(\frac{e^{-t}tg(2t)}{e^{\left(-1+\sqrt{6}\right)t}\left(\alpha_2-\alpha_1\right)}\right)dt + \tilde{C_2}e^{\left(-1+\sqrt{6}\right)t} \end{cases}$$

applly inverse substitution by yourself