Differential Equations lab 6

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Task 1. Examine following system for stable/asymptotically stable/unstable solutions depending on parameters a, b

$$\begin{cases} \dot{x} &= y \\ \dot{y} &= x - z \\ \dot{z} &= x + ay - bz \end{cases}$$

Coefficients matrix:

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & a & -b \end{pmatrix}$$

Characteristic polynomial:

$$det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & -1 \\ 1 & a & -b-\lambda \end{vmatrix} = -\lambda \begin{vmatrix} -\lambda & -1 \\ a & -b-\lambda \end{vmatrix} - \begin{vmatrix} 1 & -1 \\ 1 & -b-\lambda \end{vmatrix} = 0$$

$$-\lambda(\lambda(b+\lambda)+a) + (b+\lambda-1) = 0$$
$$\lambda^3 + b\lambda^2 + (a+1)\lambda - (b-1) = 0$$

Corresponding Hurwitz matrix:

$$\begin{pmatrix} b & 1 & 0 \\ 1-b & a+1 & b \\ 0 & 0 & 1-b \end{pmatrix},$$

$$\begin{cases} \Delta_1 = b > 0 \\ \Delta_2 = b(a+1) - (1-b) > 0 \\ \Delta_3 = (1-b)\big(b(a+1) - (1-b)\big) \end{cases} \implies \begin{cases} b > 0 \\ a > \frac{1}{b} - 2 \end{cases}$$

Hence, for any $b > 0$, $a > \frac{1}{b} - 2$ solutions of given differential equation will be asymtotiaclly stable.

For solution stability we need to find a, b such that $Re\lambda \leq 0$, but we already derived bounds for a, b to achieve $Re\lambda < 0$, so now we only need to find roots of characteristic polynomial and derive new boundaries for a, b such that $Re\lambda = 0$

Our characteristic polynomial – is a cubic, thus we'd like to apply Kardano's formula. First we would like to get rid of $b\lambda^2$ term. We can achieve this by applying next substitution:

$$\lambda = y - \frac{b}{3}$$

After denoting p, q as follows

$$p = \frac{1}{3}(3(a+1) - b^2)$$
$$q = \frac{1}{27a^3}(2b^3 - 9b(a+1) + 27(1-b))$$

We will get:

$$y^3 + py + q = 0$$

Now we are ready to apply Kardano's formula:

$$
\begin{aligned}
Q &= \left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2 \\
\alpha &= \sqrt[3]{-\frac{q}{3} + \sqrt{Q}} \\
\beta &= \sqrt[3]{-\frac{q}{3} - \sqrt{Q}}
\end{aligned}
$$

In given notation roots will take next form:

$$
\begin{aligned}
y_1 &= \alpha + \beta \\
y_{2,3} &= -\frac{\alpha+\beta}{2} \pm i\frac{\alpha-\beta}{2}\sqrt{3}
\end{aligned}
$$

After inverse substitution we got equations to define real parts of eigenvalues:

$$
\begin{aligned}
\lambda = y - \frac{b}{3} \implies \lambda_1 &= \alpha + \beta - \frac{b}{3} \\
\lambda_{2,3} &= -\frac{\alpha+\beta}{2} - \frac{b}{3} \pm i\frac{\alpha-\beta}{2}\sqrt{3}
\end{aligned}
$$

Applying eigenvalues real parts constraits we come up with following system:

$$
\begin{cases}
\alpha + \beta - \frac{b}{3} = 0 \\
-\frac{\alpha+\beta}{2} - \frac{b}{3} = 0
\end{cases}
\implies -\frac{b}{6} - \frac{b}{3} = 0 \implies b = 0
$$

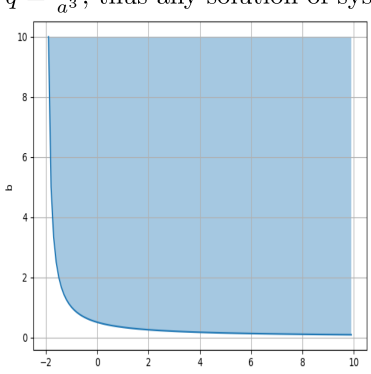
Now we are able to simplify coefficients p, q by plugging in $b = 0$:

$$
p = \frac{3(a+1)}{3} = (a+1); q = \frac{1}{a^3}
$$

After expanding first equation from our system:

$$
\left.
\begin{aligned}
\alpha + \beta &= 0 \\
\sqrt[3]{-\frac{q}{3} - \sqrt{Q}} + \sqrt[3]{-\frac{q}{3} + \sqrt{Q}} &= 0 \\
-\frac{q}{3} + \sqrt{Q} &= -\left(-\frac{q}{3} + \sqrt{Q}\right) \\
-\frac{q}{2} &= \frac{q}{2}
\end{aligned}
\right. \implies q = 0
$$

but $q = \frac{1}{a^3}$, thus any solution of system, which satisfies parameter constraits, will be asymptotically stable.



Task 2 Find all equilibrium points of differential equations system

$$\begin{cases} \dot{x} = 4x^2 - y^2 \\ \dot{y} = 2xy - 4x - 8 \end{cases}$$

$$\begin{cases} 4x^2 - y^2 = 0 \\ 2xy - 4x - 8 = 0 \end{cases}$$

Assuming that we only allow real-valued solutions, we got:

$$\begin{aligned} x_1 = 2, y_1 &= 4 \\ x_2 = -1, y_2 &= -2 \end{aligned}$$

Applying first approximation to tayor series of first and second function we get

$$\begin{pmatrix} 8x & -2y \\ 2y-4 & 2x \end{pmatrix}$$

Substituting founded equlibrium points and after calculating eigenvalues we got:

$$\begin{aligned} &(x,y) = (2,4): \\ \lambda_{1,2} = \frac{3 \pm 2i}{2} &\implies \text{unstable solve} \\ &(x,y) = (-1,-2): \\ \lambda_1 = 3, \lambda_2 = -1 &\implies \text{unstable solve} \end{aligned}$$

Task 3 Build Lyapunov function for given system of differential equations and examine trivial solution for stability.

$$\begin{cases} \dot{x} = 4y - x^5 \\ \dot{y} = -3x - y^3 \end{cases}$$

After several attempts to pick a suitable Lyapunov function One will come up with following form:

$$V(x,y) = A(x) + B(y)$$

Now we'd like to apply Lyapunov stability theorem:

$$\begin{aligned} \frac{dV}{dt} &= \frac{dV}{dx}\frac{dx}{dt} + \frac{dV}{dy}\frac{dy}{dt} = \frac{dV}{dx}\dot{x} + \frac{dV}{dy}\dot{y} \\ \frac{dV}{dt} &= A'(x)(4y - x^5) - B'(y)(ex + y^3) \\ &= 4A'(x)y - A'(x)x^5 - 3B'(y)(3x + y^3) \end{aligned}$$

Now, if we plug $A'(x) = Ax, B'(y) = Bx, 4A - 3B = 0$ we will get the fillowing:

$$\begin{aligned} \frac{dV}{dt} &= 4Axy - Ax^6 - 3Bxy - By^4 (4A = 3B) \\ &= -Ax^6 - -By^4 < 0 \end{aligned}$$

By letting $A = 3, B = 4$ we get:

$$\begin{aligned} V(x,y) &= \frac{A}{2}x^2 + \frac{B}{2}y^2, \\ V(0,0) &= 0 \end{aligned} \implies \text{Trivial solution is stable}$$