Module 1: Classification and properties of systems

1. What is the definition of a *system*? What are the differences between a *static* and a *dynamic* system? Explain the following properties of a system : *memory*, *causality*, *invertiblity*, *stability*, *linearity* and *time invariance.*

2. Check the following systems for memory, causality, invertiblity, stability, linearity and time invariance.

i. $y(t) = \int_t^{t+1} x(\tau - \alpha)d\tau$

ii. $y(t) = \cos(x(t-1))$

iii. $y(t) = 3x(3t+3)$

iv. $y(t) = \ln(x(t))$

v. $y(t) = e^{tx(t)}$

vi. $y(t) = 7x(t) + 6$

vii. $y(t) = \int_{-\infty}^{t} x(5\tau)d\tau$

viii. $y(t) = e^{-j\omega t} \int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}d\tau$

ix. $y(t) = \int_{t-1}^{t} x(\tau)d\tau$

x. $y(t) = x(t - t_0)$

xi. $y(t) = |x(t)|$

xii. $y(t) = \begin{cases} x(t), & x \geq 0 \\ 0, & x < 0 \end{cases}$

xiii. $y(t) = \begin{cases} -10, & x < -1 \\ 10x(t), & |x| \leq 1 \\ 10, & x > 1 \end{cases}$

xiv. $y(t) = \begin{cases} -2, & 2 < x \\ 1, & 1 < x \leq 2 \\ 0, & 0 < x \leq 1 \\ -1, & -1 < x \leq 0 \\ -2, & x \leq -1 \end{cases}$

xv. $y(t) = t\,x(t+2)$

Module 2: Singularity functions and classification of signals

1. Define Heaviside unit step, unit ramp and Dirac delta functions. Write down their different properties.

2. Prove that, $\delta(at+b) = \frac{1}{|a|}\,\delta\left(t + \frac{b}{a}\right)$.

3. Prove that, for any continuous function $f(t)$,

$$\int_{-\infty}^{\infty} f(t)\delta(at-b)\,dt = \frac{1}{|a|}\,f\left(\frac{b}{a}\right).$$

4. Find $\mathcal{L}[\delta(at+b)]$, $\mathcal{L}[\mathrm{u}(at+b)]$, and $\mathcal{L}[\mathrm{r}(at+b)]$.

5. When do you consider a signal $x(t)$ to be periodic? What is the definition of period for a periodic signal? Is the signal $x(t) = 1$ periodic? If yes, what is its period?

6. Check whether the following signals are periodic or not. If they are periodic, find their period.

i. $\sin(0.5t) + 3\cos(2t)$

ii. $\cos^2(t)$

iii. $[\sin(t)]^{p/q}$, where, $p, q \in \mathbb{N}$

iv. $\sin(t) + \sin(\pi t)$

v. $\sin\left(\dfrac{2\pi t}{T_1}\right) + \sin\left(\dfrac{2\pi t}{T_2}\right)$, where, $T_1, T_2 \in \mathbb{R}^+$

vi. $x(t) = 3\cos(4t + \pi/3)$

vii. $x(t) = e^{j(\pi t - 1)}$

viii. $x(t) = \mathcal{E}v[\sin(4\pi t)u(t)]$, where, $\mathcal{E}v[f(t)]$ represents the even part of $f(t)$

7. Define *total energy* (E_∞) and *time-averaged power* (P_∞) for continuous time signals. When do you consider a signal to be an energy signal or a power signal?

8. For the signals given below, determine E_∞ and P_∞. Then determine whether they are energy signals or power signals.

i. $u(t)$

ii. $e^{at}u(t)$

iii. $x(t) = e^{j(2t+\pi/4)}$

iv. $x(t) = \begin{cases} \dfrac{1}{2}(\cos\omega t + 1), & -\dfrac{\pi}{2} \leq t \leq \dfrac{\pi}{2} \\ 0, & \text{otherwise} \end{cases}$

v. $x(t) = \sin(\omega t)$

vi. $x(t) = t^2$

vii. $x(t) = t - \lfloor t \rfloor$

viii. $x(t) = u(t+1) - u(t-1)$

9. Prove that any signal $x(t)$ can be expressed as a sum of an odd and an even signal.

10. Check whether the following signals are even or odd. If neither, calculate their even and odd parts.

i. $x(t) = e^{-|t|}$

ii. $x(t) = 5\cos(3t)$

iii. $x(t) = \sin(2t - \pi/2)$

iv. $x(t) = u(t)$

v. $x(t) = \delta(t)$

vi. $x(t) = t$

vii. $x(t) = r(t)$

viii. $x(t) = u(t) - u(t-1)$

11. Evaluate the following integrals

i. $\displaystyle\int_{-\infty}^{\infty} \sin(2t)\,\delta(t)\,dt$

ii. $\displaystyle\int_{-\infty}^{\infty} \sin(2t)\,\delta(t-\pi/4)\,dt$

iii. $\displaystyle\int_{-\infty}^{\infty} \sin^2(t-2)\,\delta(at-b)\,dt$

Module 3: Signal representation

1. Draw the following signals

i. $x(t) = 5u(t+2) - u(t) + 3u(t-2) - 7u(t-4)$

ii. $x(t) = 4(t+2)u(t+2) - 4tu(t) - 4u(t-2) - 4(t-4)u(t-4) + 4(t-5)u(t-5)$

iii. $x(t) = r(t) - r(t-1) - r(t-2)$

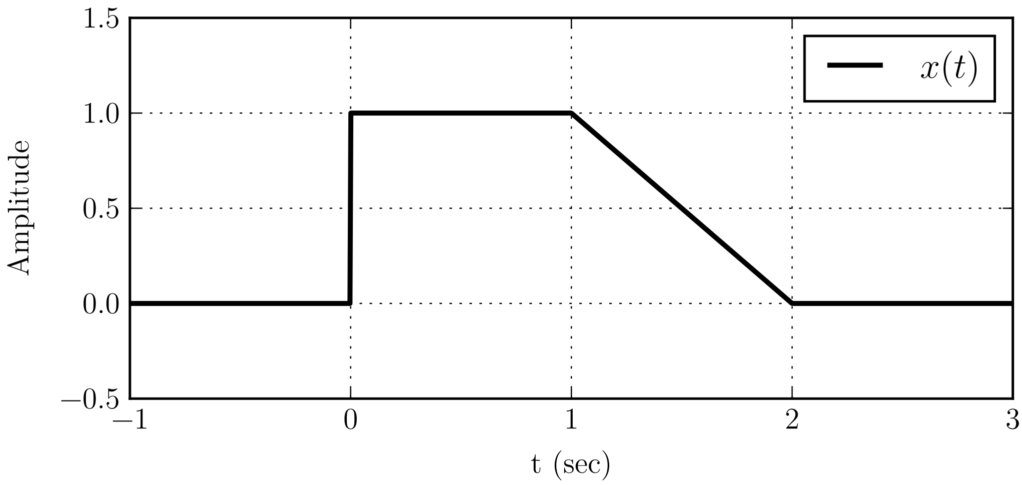
iv. $x(t) = r(t) - r(t-1) - r(t-2) + r(t-4) + r(t-5) - r(t-6)$

v. $x(t) = r(t) - r(t-1) - u(t-1)$

vi. $x(t) = \displaystyle\sum_{k=0}^{10} r(t-k) - r(t-(k+1)) - u(t-(k+1))$

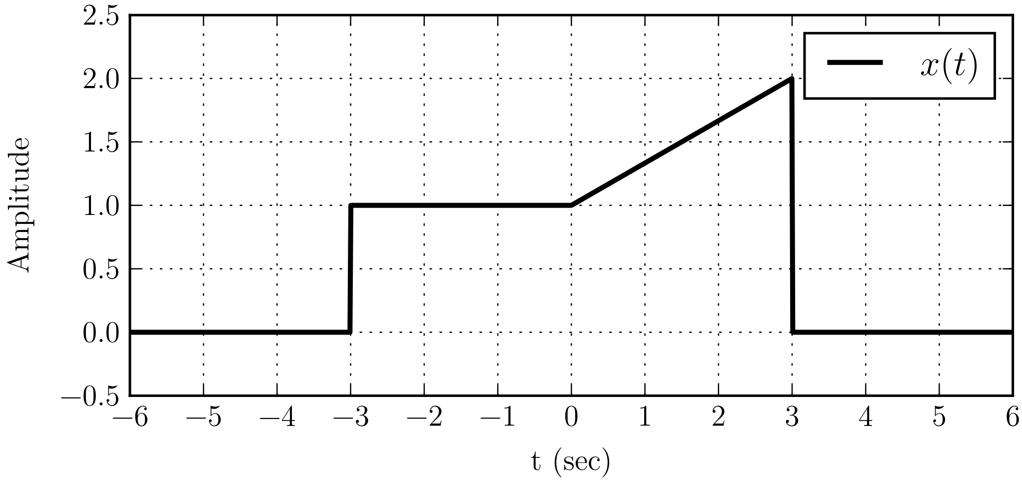
2. For the signals in **1**, draw $x(-t)$, $x(-2t+2)$, $x(\frac{t}{2}+1)$, $\mathcal{E}v[x(t)]$ and $\mathcal{O}d[x(t)]$.

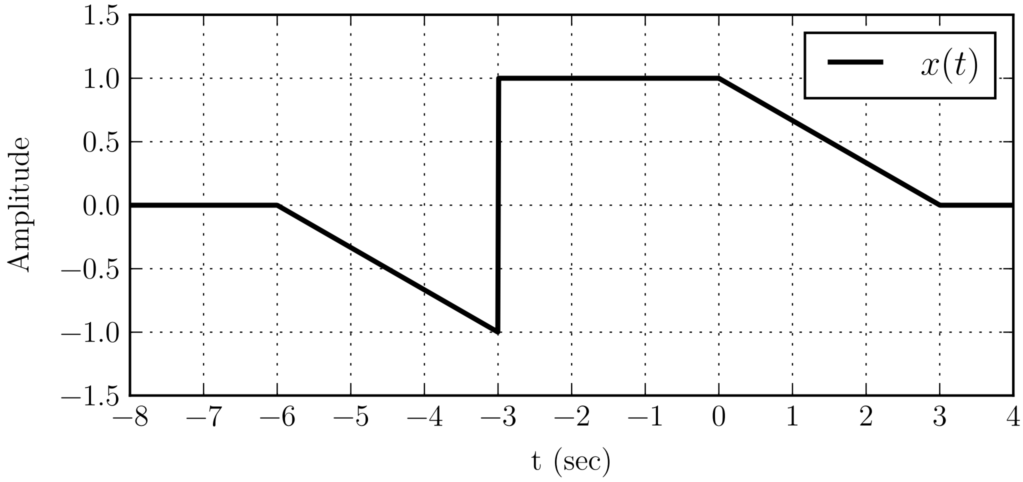
3. Describe the following signal in terms of singularity functions. Calculate its odd and even parts and draw them as well. Also, draw $x(t+0.5) + x(-t+0.5)$, $x(t+1) - x(-t-1)$ and $x(2t-1) - x(-2t-1)$.

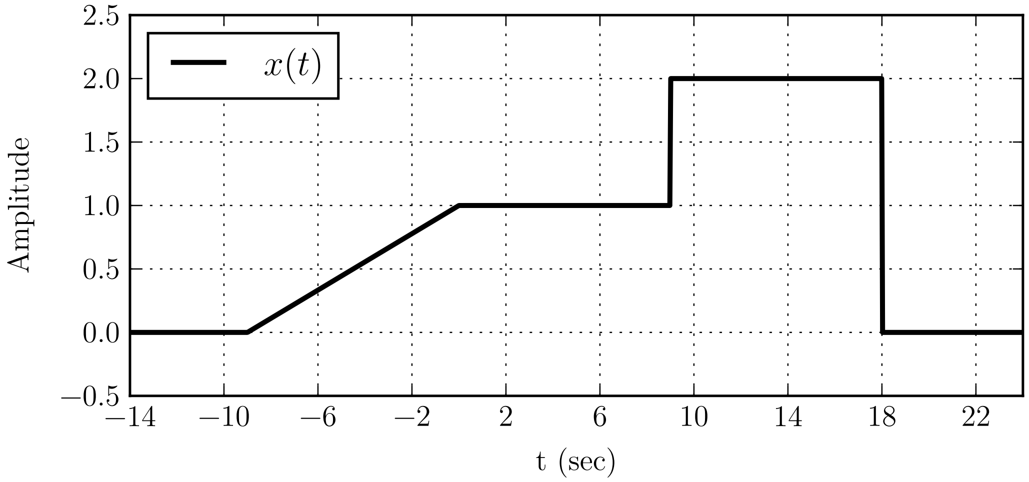


Also, for the above signal, analytically show that $x(t+1) + x(-t-1)$ is even.

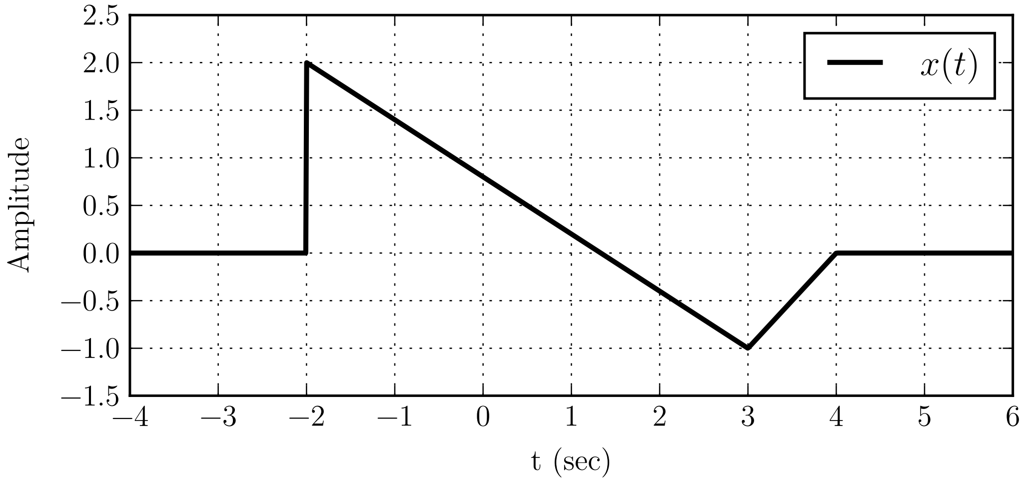
4. For the following signals, plot $x(-t/3)$, $x(-t)$, $x(3+t)$, $x(2-t)$, $4x(t) - 2$, $2x(t) + 2$, $2x(2t) + 2$ and $-4x(t) + 2$, $\mathcal{E}v[x(t)]$ and $\mathcal{O}d[x(t)]$.

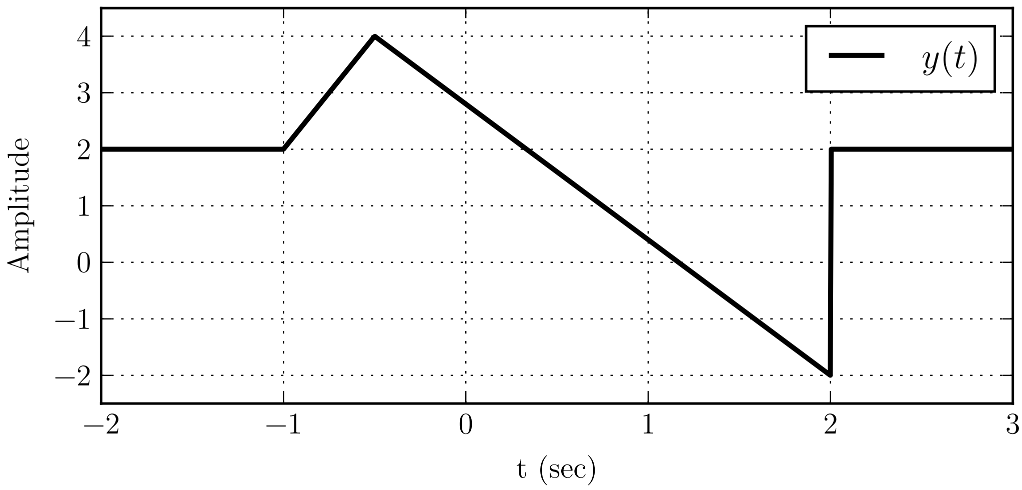




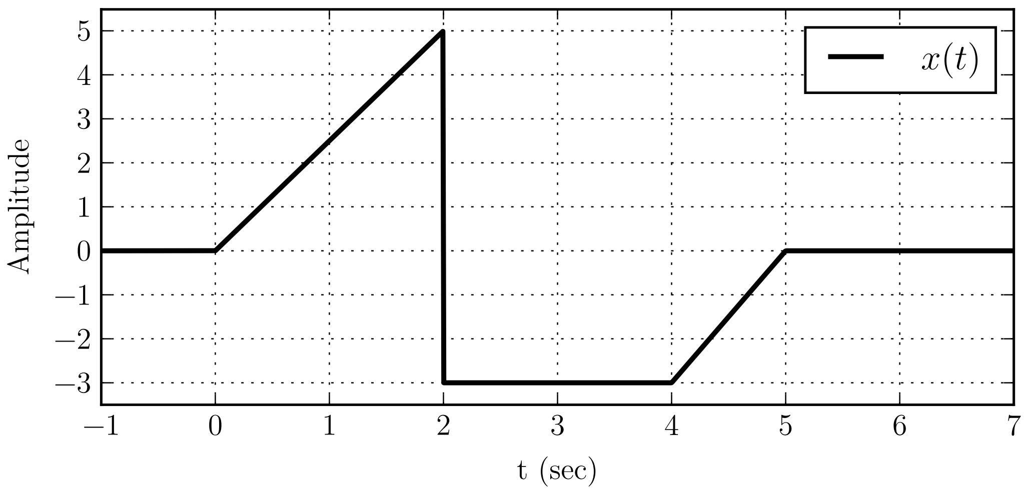


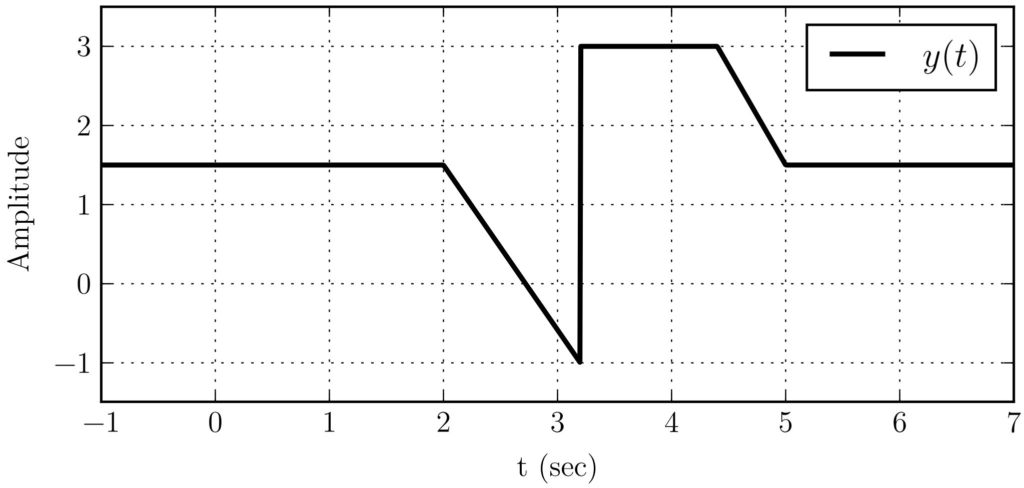
5. Given below are two signals $x(t)$ and $y(t)$. Express $y(t)$ in terms of $x(t)$ and $x(t)$ in terms of $y(t)$.





6. Given below are two signals $x(t)$ and $y(t)$. Express $y(t)$ in terms of $x(t)$ and $x(t)$ in terms of $y(t)$.





Modules 4–5: Transfer function approach for the analysis of LTI systems

1. Define the terms *transfer function*, *poles* and *zeros*. What are the restrictions of the transfer function approach used for the modelling of dynamic systems?

2. Deduce the standard transfer function models for series RLC circuit, parallel RLC circuit and a translational mechanical system with one mass one spring and one damper.

3. Define the terms undamped natural frequency, damping ratio, damped natural frequency for a standard second order system given as

$$G(s) = \frac{K\,\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

Find its step response and impulse response for $\zeta = 0$, $0 < \zeta < 1$, $\zeta = 1$, and $\zeta > 1$. What would happen if ζ is negative?

Also, derive the expressions for *rise time*, *peak time*, *peak overshoot* and *settling time* for the step response of the above system when $0 < \zeta < 1$.

4. Calculate the transient response characteristics (peak time, peak overshoot, settling time and settling time) of the step responses for the systems with transfer functions

$$G_1(s) = \frac{1}{s^2 + s + 1}, \quad G_2(s) = \frac{200}{s^2 + 20s + 225}.$$

5. If a system has a transfer function $G(S) = \dfrac{K}{\tau s + 1}$, find its step response and impulse response under zero initial conditions. Also, verify that the impulse response is nothing but the time-derivative of the step response.

6. A Mass-Spring-Damper (MKB) system has a mass of 1 kg, a spring of spring constant 100 N/m and a viscous damper with damping constant 10 N-s/m. The elements are connected as usual. The system is excited with an impulsive force of 10 N. Find the displacement (in meters) of the Mass with respect to time.

7. If the input output relationship of a dynamic system is represented in terms of an n-th order ODE with constant real coefficients as

$$a_n \frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = b_n \frac{d^n x}{dt^n} + b_{n-1}\frac{d^{n-1}x}{dt^{n-1}} + \cdots + b_1 \frac{dx}{dt} + b_0 x,$$

then show that the system is BIBO stable iff real parts of the all the roots of the equation $a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 = 0$ are strictly negative.

8. Stating the applicable conditions, prove the *initial value theorem* and *final value theorem* in relation to Laplace transforms of continuous-time functions.

9. Two systems have the following transfer functions

$$G_1(s) = \frac{1}{s^4 + s^3 + s^2 + s + 1}, \quad G_2(s) = \frac{s^2 - s + 1}{s^4 + 3s^3 + 6s^2 + 5s + 3}.$$

Find the final values of their step responses.

10. Determine the values of K and k of the system given by $G(s) = \dfrac{K}{Js^2 + Kks + K}$, so that the peak overshoot in unit-step response is 25% and the peak time is 2 sec. Assume that $J = 1\,\mathrm{kg\,m^2}$.

Modules 6–8: Frequency response analysis of signals and systems : Fourier transform

Assume that the Fourier Transform pair $f(t) \rightleftharpoons F(\omega)$ to be defined as

$$\begin{aligned} F(\omega) &= \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t} dt, \\ f(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t} d\omega. \end{aligned}$$

1. Find the Fourier Transform of $f(t) = \mathrm{rect}\left(\dfrac{t}{T}\right) = \begin{cases} 1, & -\frac{T}{2} \le t \le \frac{T}{2} \\ 0, & \text{otherwise} \end{cases}$. Then use the time scaling / shifting property of Fourier Transform to find the Fourier Transform of $g(t) = \mathrm{rect}\,(-2t + 3)$.

Hint $F(\omega) = \displaystyle\int_{-\infty}^{\infty} \mathrm{rect}\left(\frac{t}{T}\right) e^{-j\omega t} dt = \int_{-T/2}^{T/2} e^{-j\omega t} dt$. Also note that the time scaling / shifting property states : $f(at - b) \rightleftharpoons \dfrac{1}{|a|} e^{-j\omega b/a} F(\omega/a)$.

2. The 'cardinal sine' (or sinc) function is defined as $\mathrm{sinc}\,(t) = \dfrac{\sin t}{t}$. Find its Fourier Transform. Then use the scaling property of Fourier Transform to find the Fourier Transform of the normalized sinc function: $\mathrm{sinc}_\pi(t) = \dfrac{\sin \pi t}{\pi t}$. What will be the Fourier Transform of $\mathrm{sinc}_\pi(t) u(t)$. Also evaluate $\displaystyle\int_0^\infty \frac{\sin x}{x} dx$.

3. Find the Fourier Transform of an impulse train (Dirac's comb) $\Psi(t) = \displaystyle\sum_{k=-\infty}^{\infty} \delta(t - kT)$, where, T is a constant. Then find the Fourier Transform of $\xi(t) = \displaystyle\sum_{k=0}^{\infty} T^{-k}\delta(t - kT)$.

Hint Note that Dirac's comb is periodic with a period of T. Now, for the second part,

$$\begin{aligned}\xi(t) &\rightleftharpoons \Xi(\omega)\\ &= \int_{-\infty}^{\infty}\sum_{k=0}^{\infty} T^{-k}\delta(t-kT)\,e^{-j\omega t}dt\\ &= \sum_{k=0}^{\infty} T^{-k}\int_{-\infty}^{\infty}\delta(t-kT)\,e^{-j\omega t}dt\\ &= \sum_{k=0}^{\infty} T^{-k}e^{-j\omega kT},\end{aligned}$$

then sum up the GP.

4. Without using the duality property of Fourier Transform, find the Fourier Transform of $\sin(at)$.

5. Show that the Fourier Transform of a Gaussian $f(t) = \dfrac{1}{\sigma\sqrt{2\pi}}e^{-\frac{t^2}{2\sigma^2}}$ is another Gaussian. Also find its energy.

Hint The result would be $F(\omega) = \operatorname{sgn}(\sigma)\,e^{-\frac{1}{2}\sigma^2\omega^2}$. The enrgy of $f(t)$ is $\dfrac{1}{\sigma\sqrt{2\pi}}$ unit, provided that $\Re(\sigma) > 0$.

6. Find the Fourier Transform of the following

i. $\cos(at)\,\mathrm{rect}\left(\dfrac{t}{T}\right)$. **Ans** $\frac{1}{2}\,|T|\left(\mathrm{sinc}\left(\frac{1}{2}T(a-\omega)\right) + \mathrm{sinc}\left(\frac{1}{2}T(a+\omega)\right)\right)$.

ii. $u(-2t+2)$. **Ans** $\pi\delta(\omega) + \frac{je^{-j\omega}}{\omega}$.

iii. $\sin(3t)\,\delta(3t-2)$. **Ans** $\frac{1}{3}e^{-\frac{1}{3}(2j\omega)}\sin(2)$.

iv. $e^{-a|t|}$, $a > 0$. **Ans** $\frac{2a}{a^2+\omega^2}$.

v. $\sin(2\pi t)e^{-t}u(t)$. **Ans** $\frac{2\pi}{-\omega^2+2j\omega+4\pi^2+1}$.

vi. $\dfrac{1}{a^2+t^2}$. **Ans** $\frac{1}{a}\pi e^{-a|\omega|}$.

vii. $\displaystyle\sum_{k=0}^{\infty}\mathrm{rect}\left(\frac{t-2kT}{T}\right)$. **Hint** It is a periodic function.

viii. $f(t) = \begin{cases} A, & -T < t < 0\\ -A, & 0 < t < T\\ 0, & \text{otherwise}\end{cases}$.

Hint $F(\omega) = \int_{-T}^{0} Ae^{-j\omega t}dt + \int_0^T(-A)e^{-j\omega t}dt$.

7. A CT signal $x(t)$ has a Fourier Transform of $X(\omega) = \dfrac{\omega^2}{1+\omega^2}$. Find the Fourier Transform of

i. $x(1-t) + x(-1-t)$.

ii. $x(2t) + x(t/2)$.

Hint No need to find $x(t)$. Just use the time scaling / shifting property.

8. Find the energy contained in the signal $\text{sinc}_\pi(t)$.

Hint Use Parseval's relation. The energy is 1 unit.

9. A causal LTI system has a frequency response $\mathcal{H}(\omega) = \dfrac{j\omega + 4}{6 - \omega^2 + 5j\omega}$. Find the impulse response of the system. What will be the output of the system when the input is $x(t) = e^{-4t}u(t) - te^{-4t}u(t)$?

10. Find the Inverse Fourier Transform of $X(\omega) = \left(\dfrac{\sin 2\omega}{\pi\omega}\right)^2$.

Hint Find $f(t) = \mathcal{F}^{-1}\left\{\dfrac{\sin 2\omega}{\pi\omega}\right\}$. Then use convolution :

$$x(t) = f(t) \circledast f(t) = \int_{-\infty}^{\infty} f(\tau)\, f(t-\tau)d\tau.$$

The answer will be a triangular pulse.

11. Show that the LTI systems with impulse responses $h_1(t) = u(t)$, $h_2(t) = -2\delta(t) + 5e^{-2t}u(t)$ and $h_3(t) = 2te^{-t}u(t)$ all have the same response to the input $x(t) = \cos(t)$.

Hint Note that $f(t)\delta(at-b) = f(b/a)\delta(at-b)$.

12. A real and non-negative signal $x(t)$ has a Fourier Transform $X(\omega)$. Suppose we are given the following facts:

i. $\mathcal{F}^{-1}\{(1+j\omega)X(\omega)\} = Ae^{-2t}u(t)$,

ii. $\displaystyle\int_{-\infty}^{\infty} |X(\omega)|^2\, d\omega = \pi$.

Find a possible expression for $x(t)$.

Hint $X(\omega) = \dfrac{A}{1+j\omega}\mathcal{F}\left\{e^{-2t}u(t)\right\} = \dfrac{A}{(1+j\omega)(2+j\omega)}$.

13. A simple low-pass filter has a frequency response $\mathcal{H}(\omega) = \dfrac{1}{1+j\omega}$. A signal whose Power Spectral Density (PSD) $\mathcal{P}(\omega) = \pi[\delta(\omega+1) + \delta(\omega-1)]$ is fed into the system. Find the power contained in the input, the PSD of the output and the power contained in the output.

Hint $Y(\omega) = \mathcal{H}(\omega)X(\omega)$. Thus, $\underset{T\to\infty}{Lt}\dfrac{|Y_T(\omega)|^2}{T} = |\mathcal{H}(\omega)|^2 \underset{T\to\infty}{Lt}\dfrac{|X_T(\omega)|^2}{T}$.
So, we have, PSD of Output $= |\mathcal{H}(\omega)|^2\times$ PSD of Input.

14. A CT SISO LTI system is described in state space as

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 0 \\ 2 \end{bmatrix}\mathcal{U}(t), \\ y(t) &= \begin{bmatrix} 2 & 1/2 \end{bmatrix}\mathbf{x}(t).\end{aligned}$$

Find the impulse response and the frequency response of the system (under zero initial conditions). Also find the output of the system when the input is $\mathcal{U}(t) = t\,e^{-2t}u(t)$.

Module 9: Frequency response analysis of LTI systems

1. With proper labels, draw the asymptotic log-magnitude Bode plots of the systems with the following transfer functions.

i. $\dfrac{50(s+2)(s+20)}{(s+10)(s+100)}$

ii. $\dfrac{10(s+2)}{(s+5)(s+20)}$

iii. $\dfrac{(s+2)(s+50)}{(s+10)(s+20)}$

iv. $\dfrac{(s+4)(s+200)}{(s+20)(s+100)}$

v. $\dfrac{50(s+2)(s+20)}{(s+10)(s+100)}$

vi. $\dfrac{(s+5)(s+20)}{(s+1)(s+100)}$

vii. $\dfrac{(s+1)(s+100)}{(s+5)(s+20)}$

2. For the systems given below, choose any suitable value for $T_1, T_2, \cdots$, and draw their asymptotic log-magnitude Bode plots.

i. $\dfrac{1+\alpha sT_1}{1+sT_1}\dfrac{1+sT_2}{1+\alpha sT_2}$, with $\alpha = 3$ and $T_2 < T_1/6$

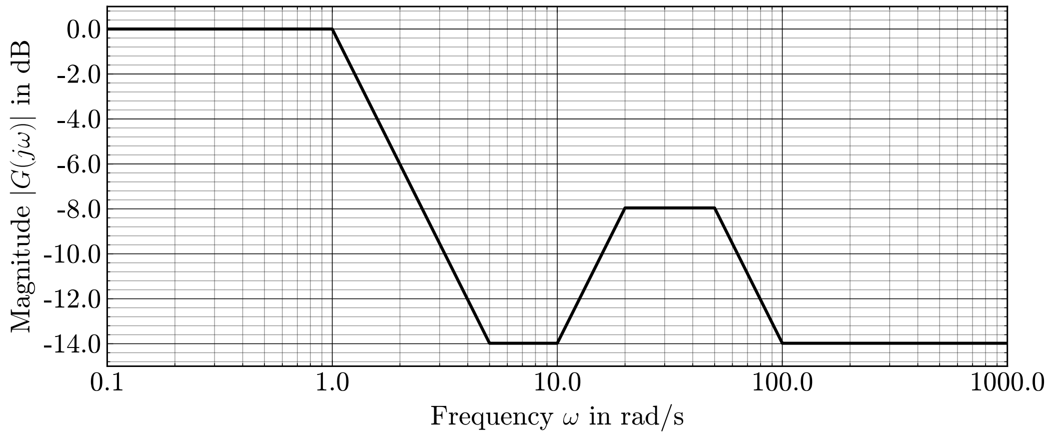
ii. $\dfrac{1+\alpha sT_1}{1+sT_1}\dfrac{1+sT_2}{1+\alpha sT_2}$, with $\alpha = 3$ and $T_1 < T_2/6$

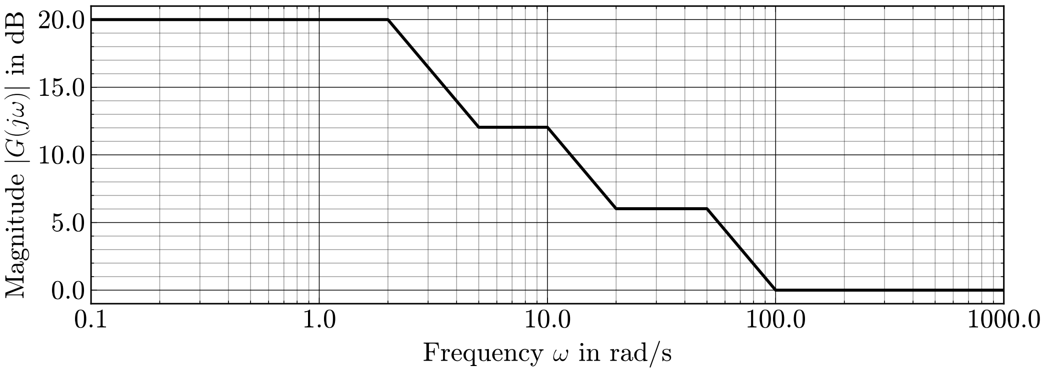
iii. $\dfrac{1+sT_1}{(1+sT_2)(1+sT_3)}$, with $\dfrac{1}{T_1} < \dfrac{1}{T_2} < \dfrac{1}{T_3}$

iv. $\dfrac{1+sT_1}{(1+sT_2)(1+sT_3)}$, with $\dfrac{1}{T_2} < \dfrac{1}{T_1} < \dfrac{1}{T_3}$

v. $\dfrac{(1+sT_1)(1+sT_2)}{(1+sT_3)(1+sT_4)}$, with $\dfrac{1}{T_3} < \dfrac{1}{T_1} < \dfrac{1}{T_2} < \dfrac{1}{T_4}$

3. The asymptotic log-magnitude Bode plots of two systems are shown below. Find out their transfer functions.





Module 10: State space

1. What are the relative advantages of state space modelling over transfer function modelling?

2. Represent the following systems in the state space

i. $G(s) = \dfrac{3}{s^3 + 2s^2 + 4s + 2}$

ii. $G(s) = \dfrac{2}{s^3 + 6s^2 + 11s + 6}$

iii. $G(s) = \dfrac{2}{6s^2 + 11s + 3}$

iv. $a_3 \dfrac{d^3y}{dt^3} + a_2 \dfrac{d^2y}{dt^2} + a_1 \dfrac{dy}{dt} + a_0 y = b_2 \dfrac{d^2x}{dt^2} + b_1 \dfrac{dx}{dt} + b_0 x$

3. An n-th order SISO LTI system is represented in the state space as

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}u, \\ y &= \mathbf{C}\mathbf{x} + Du. \end{aligned}$$

Write down the names and the dimensions of the vectors and matrices used above. Prove that the transfer function of the system can be calculated as $G(s) := \dfrac{Y(S)}{U(s)} = \mathbf{C}[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B} + D$. Also, show that the pole locations of the systems are nothing but the eigenvalues of the matrix $\mathbf{A}$.

4. Calculate the transfer functions of the following SISO LTI systems

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \begin{bmatrix} -1 & 2 \\ -8 & -6 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u(t), \\ y(t) &= \begin{bmatrix} 2 & 1/2 \end{bmatrix} \mathbf{x}(t). \end{aligned}$$

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -3 & -1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t), \\ y(t) &= \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \mathbf{x}(t). \end{aligned}$$

5. Show that the time response of a SISO LTI system represented as

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}u, \\ y &= \mathbf{C}\mathbf{x} + Du. \end{aligned}$$

can be expressed as $y(t) = \mathbf{C}\Phi(t)\mathbf{x}(0) + \mathbf{C}\int_0^t \Phi(t-\tau)\mathbf{B}u(\tau)d\tau$, where, $\Phi(t) = \mathcal{L}^{-1}\{[s\mathbf{I} - \mathbf{A}]^{-1}\}$ is the *state transition matrix*. Explain why the first part of the above expression represents the natural response and the second part represents the forced response of the system.

6. Calculate the step response of the system

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \\ y(t) &= \begin{bmatrix} 0 & 1 \end{bmatrix} \mathbf{x}(t), \end{aligned}$$

with the initial conditions $\mathbf{x}(0) = [1 \quad 1]^T$.

7. Obtain a state space model of the mechanical system shown in the following figure.