14 Consider an independent trials process to be a Markov chain whose states are the possible outcomes of the individual trials. What is its fixed probability vector? Is the chain always regular? Illustrate this for Example 11.5.

15 Show that Example 11.8 is an ergodic chain, but not a regular chain. Show that its fixed probability vector $\mathbf{w}$ is a binomial distribution.

16 Show that Example 11.9 is regular and find the limiting vector.

17 Toss a fair die repeatedly. Let S_n denote the total of the outcomes through the nth toss. Show that there is a limiting value for the proportion of the first n values of S_n that are divisible by 7, and compute the value for this limit. *Hint*: The desired limit is an equilibrium probability vector for an appropriate seven state Markov chain.

18 Let $\mathbf{P}$ be the transition matrix of a regular Markov chain. Assume that there are r states and let $N(r)$ be the smallest integer n such that $\mathbf{P}$ is regular if and only if $\mathbf{P}^{N(r)}$ has no zero entries. Find a finite upper bound for $N(r)$. See if you can determine $N(3)$ exactly.

***19** Define $f(r)$ to be the smallest integer n such that for all regular Markov chains with r states, the nth power of the transition matrix has all entries positive. It has been shown,[14] that $f(r) = r^2 - 2r + 2$.

(a) Define the transition matrix of an r-state Markov chain as follows: For states s_i, with $i = 1, 2, \ldots, r-2$, $\mathbf{P}(i, i+1) = 1$, $\mathbf{P}(r-1, r) = \mathbf{P}(r-1, 1) = 1/2$, and $\mathbf{P}(r, 1) = 1$. Show that this is a regular Markov chain.

(b) For $r = 3$, verify that the fifth power is the first power that has no zeros.

(c) Show that, for general r, the smallest n such that $\mathbf{P}^n$ has all entries positive is $n = f(r)$.

20 A discrete time queueing system of capacity n consists of the person being served and those waiting to be served. The queue length x is observed each second. If $0 < x < n$, then with probability p, the queue size is increased by one by an arrival and, inependently, with probability r, it is decreased by one because the person being served finishes service. If $x = 0$, only an arrival (with probability p) is possible. If $x = n$, an arrival will depart without waiting for service, and so only the departure (with probability r) of the person being served is possible. Form a Markov chain with states given by the number of customers in the queue. Modify the program **FixedVector** so that you can input n, p, and r, and the program will construct the transition matrix and compute the fixed vector. The quantity $s = p/r$ is called the *traffic intensity*. Describe the differences in the fixed vectors according as $s < 1$, $s = 1$, or $s > 1$.

[14] E. Seneta, *Non-Negative Matrices: An Introduction to Theory and Applications*, Wiley, New York, 1973, pp. 52-54.

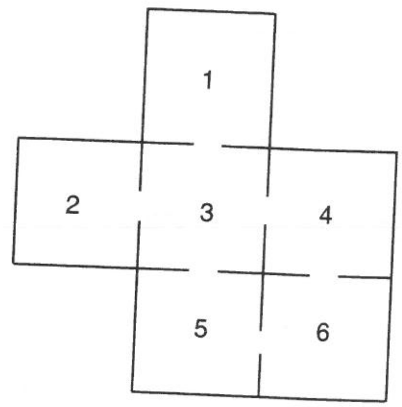

Figure 11.7: Maze for Exercise 7.

6 For the Land of Oz example (Example 11.1), make rain into an absorbing state and find the fundamental matrix **N**. Interpret the results obtained from this chain in terms of the original chain.

7 A rat runs through the maze shown in Figure 11.7. At each step it leaves the room it is in by choosing at random one of the doors out of the room.

(a) Give the transition matrix **P** for this Markov chain.

(b) Show that it is an ergodic chain but not a regular chain.

(c) Find the fixed vector.

(d) Find the expected number of steps before reaching Room 5 for the first time, starting in Room 1.

8 Modify the program **ErgodicChain** so that you can compute the basic quantities for the queueing example of Exercise 11.3.20. Interpret the mean recurrence time for state 0.

9 Consider a random walk on a circle of circumference n. The walker takes one unit step clockwise with probability p and one unit counterclockwise with probability $q = 1 - p$. Modify the program **ErgodicChain** to allow you to input n and p and compute the basic quantities for this chain.

(a) For which values of n is this chain regular? ergodic?

(b) What is the limiting vector $\mathbf{w}$?

(c) Find the mean first passage matrix for $n = 5$ and $p = .5$. Verify that $m_{ij} = d(n - d)$, where d is the clockwise distance from i to j.

10 Two players match pennies and have between them a total of 5 pennies. If at any time one player has all of the pennies, to keep the game going, he gives one back to the other player and the game will continue. Show that this game can be formulated as an ergodic chain. Study this chain using the program **ErgodicChain**.

11 Calculate the reverse transition matrix for the Land of Oz example (Example 11.1). Is this chain reversible?

12 Give an example of a three-state ergodic Markov chain that is not reversible.

13 Let $\mathbf{P}$ be the transition matrix of an ergodic Markov chain and $\mathbf{P}^*$ the reverse transition matrix. Show that they have the same fixed probability vector $\mathbf{w}$.

14 If $\mathbf{P}$ is a reversible Markov chain, is it necessarily true that the mean time to go from state i to state j is equal to the mean time to go from state j to state i? *Hint*: Try the Land of Oz example (Example 11.1).

15 Show that any ergodic Markov chain with a symmetric transition matrix (i.e., $p_{ij} = p_{ji}$) is reversible.

16 (Crowell[24]) Let $\mathbf{P}$ be the transition matrix of an ergodic Markov chain. Show that

$$(\mathbf{I}+\mathbf{P}+\cdots+\mathbf{P}^{n-1})(\mathbf{I}-\mathbf{P}+\mathbf{W}) = \mathbf{I}-\mathbf{P}^n+n\mathbf{W}\ ,$$

and from this show that

$$\frac{\mathbf{I}+\mathbf{P}+\cdots+\mathbf{P}^{n-1}}{n} \to \mathbf{W}\ ,$$

as $n \to \infty$.

17 An ergodic Markov chain is started in equilibrium (i.e., with initial probability vector $\mathbf{w}$). The mean time until the next occurrence of state s_i is $\bar{m}_i = \sum_k w_k m_{ki} + w_i r_i$. Show that $\bar{m}_i = z_{ii}/w_i$, by using the facts that $\mathbf{wZ} = \mathbf{w}$ and $m_{ki} = (z_{ii} - z_{ki})/w_i$.

18 A perpetual craps game goes on at Charley's. Jones comes into Charley's on an evening when there have already been 100 plays. He plans to play until the next time that snake eyes (a pair of ones) are rolled. Jones wonders how many times he will play. On the one hand he realizes that the average time between snake eyes is 36 so he should play about 18 times as he is equally likely to have come in on either side of the halfway point between occurrences of snake eyes. On the other hand, the dice have no memory, and so it would seem that he would have to play for 36 more times no matter what the previous outcomes have been. Which, if either, of Jones's arguments do you believe? Using the result of Exercise 17, calculate the expected to reach snake eyes, in equilibrium, and see if this resolves the apparent paradox. If you are still in doubt, simulate the experiment to decide which argument is correct. Can you give an intuitive argument which explains this result?

19 Show that, for an ergodic Markov chain (see Theorem 11.16),

$$\sum_j m_{ij} w_j = \sum_j z_{jj} - 1 = K\ .$$

[24]Private communication.