MATH 5131 Lecture 8

Introduction to Systems and Phase Plane Analysis

We develop some powerful theory, which often allows us to predict the dynamics of a system in general terms. It provides the means by which we can establish the phase-plane behavior of a system and predict the outcome for any possible parameter combination. The theory is developed for both linear and nonlinear systems.

Introduction to the Phase Plane

We study systems of two first-order equations of the form

$$\begin{cases} \dfrac{dx}{dt} = f(x,y) \\ \\ \dfrac{dy}{dt} = g(x,y) \end{cases} \qquad (1)$$

The independent variable t does not appear in the right-hand terms $f(x,y)$ and $g(x,y)$; such systems are called <u>autonomous</u>.

We note that the solutions to autonomous systems have a "time-shif immunity" in the sense that if the pair $x(t)$, $y(t)$ solves (1), so does the <u>time-shifted</u> pair $x(t+c)$, $y(t+c)$ for any constant c.

Specifically, if we let $X(t) := x(t+c)$ and $Y(t) := y(t+c)$, then by the chain rule

$$\frac{dX}{dt}(t) = \frac{dx}{dt}(t+c) = f(x(t+c), y(t+c)) = f(X(t), Y(t)),$$

$$\frac{dY}{dt}(t) = \frac{dy}{dt}(t+c) = g(x(t+c), y(t+c)) = g(X(t), Y(t)),$$

proving that $X(t)$, $Y(t)$ is also a solution to (1).

Since t does not appear explicitly in the system (1), it is certainly tempting to divide the two equations, invoke the chain rule

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

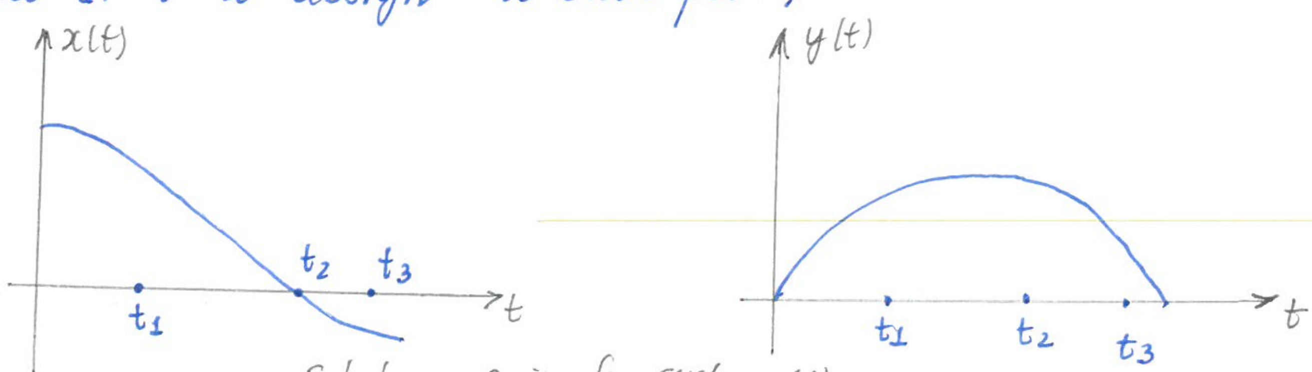
and consider the single first-order differential equation

$$\frac{dy}{dx} = \frac{g(x,y)}{f(x,y)} \qquad (2)$$

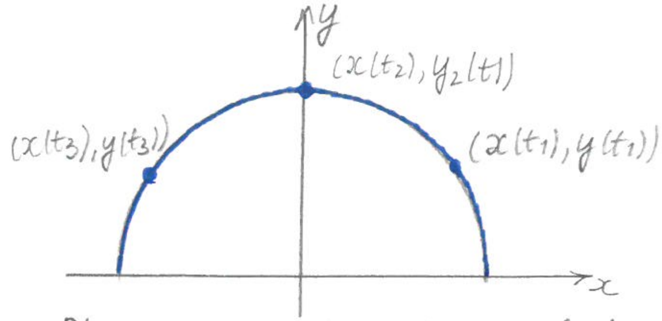
We will refer to (2) as the phase plane equation. Earlier we mastered several appoaches to equations like (2): the use of direction fields to visualize the solution graphs, and the analytic techniques for the cases of separability, linearity.

So the form (2) certainly has advantages over (1), but it is important to maintain our perspective by noting these distinctions:

(i) A solution to the original problem (1) is a pair of functions of t — $x(t), y(t)$ — that satisfies (1) for all t in some interval I. These functions can be visualized as a pair of graphs. If, in the xy-plane, we plot the points $(x(t), y(t))$ as t varies over I, the resulting curve is known as the trajectory of the solution pair $x(t), y(t)$, and the xy-plane is called the phase plane. Note, however, that the trajectory in this plane contains less information than the original graphs, because the t-dependence has been suppressed. In principle we can construct, point by point, the trajectory from the solution graphs, but we cannot reconstruct the solution graphs from the phase plane trajectory alone (because we would not know what value of t to assign to each point).



Solution pair for system (1)



Phase plane trajectory of the solution pair for system (1)

ii) Nonetheless, the slope dy/dx of a trajectory in the phase plane is given by the right-hand side of (2). So, in solving equation (2) we are indeed locating the trajectories of the system (1) in the phase plane.

iii) In the context of the system (1) x and y are both dependent variables on an equal footing, and t is the independent variable

Except for the very special case of linear systems with constant coefficients finding all solutions to the system (1) is generally an impossible task. But it is relatively easy to find <u>constant solutions</u>; if $f(x_0, y_0) = 0$ and $g(x_0, y_0) = 0$, then the constant functions $x(t) \equiv x_0$, $y(t) \equiv y_0$ solve (1).

Critical Points and Equilibrium Solutions

<u>Def</u>. A point (x_0, y_0) where $f(x_0, y_0) = 0$ and $g(x_0, y_0) = 0$ is called a <u>critical point</u>, or <u>equilibrium point</u>, of the system

$$\begin{cases} \frac{dx}{dt} = f(x,y) \\ \frac{dy}{dt} = g(x,y) \end{cases}$$

and the corresponding constant solution $x(t) \equiv x_0$, $y(t) \equiv y_0$ is called an <u>equilibrium solution</u>. The set of all critical points is called the <u>critical point set</u>.

Notice that trajectories of equilibrium solutions consists of just single points (the equilibrium points). But what can be said about the other trajectories? Can we predict any of their features from closer examination of the equilibrium points?

Linear Theory

We develop techniques that allow us to predict, for each equilibrium point, the behavior of the trajectories close to that point. From this we can establish a complete pichere of the system phase-plane. Initially, we consider the linear case (a pair of coupled linear equations in two unknowns) and then show how this can be extended to the nonlinear case. (While we restrict our analysis to two equations in two unknowns, the theory is applicable to larger systems with many unknowns.)

The general linear system

Consider a pair of coupled linear equations:

$$X' = a_1 X + b_1 Y$$

$$Y' = a_2 X + b_2 Y,$$

where differentiation is with respect to time t and a_1, a_2, b_1 and b_2 are constant.

We denote an equilibrium point (critical point or steady point) for the system by (x_e, y_e). Thus

$$\begin{cases} a_1 x_e + b_1 y_e = 0 \\ a_2 x_e + b_2 y_e = 0 \end{cases}$$

Linear algebra notation

The system above can be written in terms of matricies and vectors in the following way. Let

$$\vec{x}(t) = \begin{bmatrix} X(t) \\ Y(t) \end{bmatrix} \quad \text{and} \quad \vec{x}' = \begin{bmatrix} X'(t) \\ Y'(t) \end{bmatrix},$$

where $\vec{x}$ is a vector function of time. Let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

where A is a matrix.

The above system of two equations can be written as

$$\vec{x}' = A\vec{x}$$

This means that

$$\begin{bmatrix} X' \\ Y' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} a_{11}X + a_{12}Y \\ a_{21}X + a_{22}Y \end{bmatrix}$$

using normal matrix multiplication.

<u>Example 1</u> Carry out the multiplication $A\vec{x}_1$ and $A\vec{x}_2$, where

$$A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}, \quad \vec{x}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

<u>Solution</u>

$$A\vec{x}_1 = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3\cdot(-1) + (-2)(1) \\ 1\cdot(-1) + 0\cdot 1 \end{bmatrix} = \begin{bmatrix} -3-2 \\ -1 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$$

$$A\vec{x}_2 = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3\cdot(2) + (-2)(1) \\ 1\cdot 2 + 0\cdot 1 \end{bmatrix} = \begin{bmatrix} 6-2 \\ 2+0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

The effect of multiplying vectors $\vec{x}_1$ and $\vec{x}_2$ by a matrix A.

So multiplication by a matrix A maps a vector onto another vector. In the case of $\vec{x}_2$, we have that

$$A\vec{x}_2 = 2\vec{x}_2,$$

So the effect of multiplying by A is the same as multiplying by a scalar or a number (which is 2 in this case). We use this notion of the "equivalence" of multiplication by a matrix and a scalar in the process of finding eigenvalues and eigenvectors.

These values turn out to be essential in predicting the behavior of trajectories in the phase-plane associated with the system.

The eigenvalue-eigenvector method of finding solutions

Consider the first-order linear homogeneous differential equation

$$\dot{\vec{x}} = A\vec{x}, \quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \qquad (1)$$

Our goal is to find n linearly independent solutions

$$\vec{x}^1(t), \vec{x}^2(t), \ldots, \vec{x}^n(t).$$

Recall that both the first-order and second-order linear homogeneous scalar equations have exponential functions as solutions. This suggests that we try

$$\vec{x}(t) = e^{\lambda t}\vec{v},$$

where $\vec{v}$ is a constant vector, as a solution of (1). Observe that

$$\frac{d}{dt} e^{\lambda t}\vec{v} = \lambda e^{\lambda t}\vec{v}$$

and

$$A(e^{\lambda t}\vec{v}) = e^{\lambda t}A\vec{v}.$$

Hence, $\vec{x}(t) = e^{\lambda t}\vec{v}$ is a solution of (1) if, and only if,

$$\lambda e^{\lambda t}\vec{v} = e^{\lambda t}A\vec{v}.$$

Dividing both sides of this equation by $e^{\lambda t}$ gives

$$A\vec{v} = \lambda\vec{v}. \qquad (2)$$

Thus, $\vec{x}(t) = e^{\lambda t}\vec{v}$ is a solution of (1) if, and only if, λ and $\vec{v}$ satisfy (2).

<u>Definition</u> A nonzero vector $\vec{v}$ satisfying (2) is called an <u>eigenvector</u> of A with <u>eigenvalue</u> λ.

<u>Remark</u> The vector $\vec{v} = \vec{0}$ is excluded because it is uninteresting. Obviously, $A\vec{0} = \lambda \cdot \vec{0}$ for any number λ.

An eigenvector of a matrix A is a rather special vector: under the linear transformation $\vec{x} \to A\vec{x}$, it goes into a multiple λ of itself. Vectors which are transformed into multiples of themselves play an important role in many applications. To find such vectors, we rewrite Equation (2) in the form

$$\vec{0} = A\vec{v} - \lambda\vec{v} = (A - \lambda I)\vec{v}. \qquad (3)$$

Equation (3) has a nonzero solution $\vec{v}$ only if $\det(A - \lambda I) = 0$. Hence the eigenvalues λ of A are the roots of the equation

$$0 = \det(A - \lambda I) = \det \begin{pmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{pmatrix}$$

and the eigenvectors of A are then the nonzero solutions of the equations

$$(A - \lambda I)\vec{v} = \vec{0},$$

for these values of λ.

The determinant of the matrix $A - \lambda I$ is clearly a polynomial in λ of degree n, with leading term $(-1)^n \lambda^n$. It is customary to call this polynomial the <u>characteristic polynomial</u> of A and to denote it by $p(\lambda)$.

For each root λ_j of $p(\lambda)$, that is, for each number λ_j such that $p(\lambda_j)=0$, there exists at least one nonzero vector $\vec{v}^j$ such that

$$A\vec{v}^j = \lambda_j \vec{v}^j.$$

Every polynomial of degree $n \geq 1$ has at least one (possibly complex) root. Therefore, every matrix has at least one eigenvalue, and consequently, at least one eigenvector. On the other hand, $p(\lambda)$ has at most n distinct roots. Therefore, every $n \times n$ matrix has at most n eigenvalues. Finally, observe that every $n \times n$ matrix has at most n linearly independent eigenvectors, since the space of all vectors

$$\vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

has dimension n.

<u>Remark</u>. Let $\vec{v}$ be an eigenvector of A with eigenvalues λ. Observe that

$$A(c\vec{v}) = cA\vec{v} = c\lambda\vec{v} = \lambda(c\vec{v})$$

for any constant c. Hence, any constant multiple $(c \neq 0)$ of an eigenvector of A is again an eigenvector of A, with the same eigenvalue.

For each eigenvector $\vec{v}^j$ of A with eigenvalue λ_j, we have a solution $\vec{x}^j(t) = e^{\lambda_j t}\vec{v}^j$ of (1). If A has n linearly independent eigenvectors $\vec{v}^1, \ldots, \vec{v}^n$ with eigenvalues $\lambda_1, \ldots, \lambda_n$ respectively ($\lambda_1, \ldots, \lambda_n$ need not be distinct), then

$$\vec{x}^j(t) = e^{\lambda_j t}\vec{v}^j, \quad j = 1, \ldots, n$$

are n linearly independent solutions of (1).

Then every solution $\vec{x}(t)$ of (1) is of the form

$$\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}^1 + c_2 e^{\lambda_2 t}\vec{v}^2 + \dots + c_n e^{\lambda_n t}\vec{v}^n, \qquad (4)$$

which is called the <u>general</u> <u>solution</u> of (1).

The situation is simplest when A has n distinct real eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ with eigenvectors $\vec{v}^1, \vec{v}_2, \dots, \vec{v}_n$ respectively, for in this case we are guaranteed that $\vec{v}^1, \vec{v}^2, \dots, \vec{v}^n$ are linearly independent.

<u>Example 1</u> Solve the initial-value problem

$$\vec{x}' = \begin{pmatrix} 1 & 12 \\ 3 & 1 \end{pmatrix}\vec{x}, \quad \vec{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

<u>Solution</u>

$$p(\lambda) = \det\begin{pmatrix} 1-\lambda & 12 \\ 3 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 - 36 = 1 - 2\lambda + \lambda^2 - 36 = \lambda^2 - 2\lambda - 35$$

$$= (\lambda+5)(\lambda-7)$$

$\lambda_1 = -5, \quad \lambda_2 = 7$

i) $\underline{\lambda_1 = -5}$: $(A-(-5)I)\vec{v} = \begin{pmatrix} 1-(-5) & 12 \\ 3 & 1-(-5) \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 6 & 12 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow$

$6v_1 + 12v_2 = 0 \Leftrightarrow v_1 + 2v_2 = 0 \Rightarrow v_1 = -2v_2 \Rightarrow \vec{v} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \Rightarrow$

$$\vec{x}^1(t) = e^{-5t}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

ii) $\underline{\lambda_2 = 7}$: $(A-7I)\vec{v} = \begin{pmatrix} -6 & 12 \\ 3 & -6 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow 3v_1 - 6v_2 = 0 \Leftrightarrow v_1 = 2v_2 \Rightarrow$

$\vec{v}^2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \Rightarrow \vec{x}^2(t) = e^{7t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} \Rightarrow$

$$\vec{x}(t) = c_1\vec{x}^1(t) + c_2\vec{x}^2(t)$$

From the initial conditions: $\vec{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = c_1\begin{pmatrix} -2 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 2 \\ 1 \end{pmatrix} \Rightarrow \begin{cases} c_1 - c_2 = 0 \\ c_1 + c_2 = 1 \end{cases} \Rightarrow$

$c_1 = \frac{1}{2}, \; c_2 = \frac{1}{2} \Rightarrow \boxed{\vec{x}(t) = \frac{1}{2}e^{-5t}\begin{pmatrix} -2 \\ 1 \end{pmatrix} + \frac{1}{2}e^{7t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}}$ or $\vec{x}(t) = \begin{pmatrix} e^{7t} - e^{-5t} \\ \frac{1}{2}e^{7t} + \frac{1}{2}e^{-5t} \end{pmatrix}$

Complex roots

If $\lambda = \alpha + i\beta$ is a complex eigenvalue of A with eigenvector $\vec{v} = \vec{v}^1 + i\vec{v}^2$, then $\vec{x}(t) = e^{\lambda t}\vec{v}$ is a complex-valued solution of the differential equation (1). This complex-valued solution gives rise to two real-valued solutions.

<u>Lemma</u> Let $\vec{x}(t) = \vec{y}(t) + i\vec{z}(t)$ is a complex-valued solution of (1). Then, both $\vec{y}(t)$ and $\vec{z}(t)$ are real-valued solutions of (1).

The complex-valued function $\vec{x}(t) = e^{(\alpha+i\beta)t}(\vec{v}^1 + i\vec{v}^2)$ can be written in the form

$$\vec{x}(t) = e^{\alpha t}(\cos\beta t + i\sin\beta t)(\vec{v}^1 + i\vec{v}^2)$$
$$= e^{\alpha t}\left[(\vec{v}^1\cos\beta t - \vec{v}^2\sin\beta t) + i(\vec{v}^1\sin\beta t + \vec{v}^2\cos\beta t)\right]$$

Hence, if $\lambda = \alpha + i\beta$ is an eigenvalue of A with eigenvector $\vec{v} = \vec{v}^1 + i\vec{v}^2$, then

$$\vec{y}(t) = e^{\alpha t}(\vec{v}^1\cos\beta t - \vec{v}^2\sin\beta t)$$

and

$$\vec{z}(t) = e^{\alpha t}(\vec{v}^1\sin\beta t + \vec{v}^2\cos\beta t)$$

are two real-valued solutions of (1). Moreover, these two solutions must be linearly independent.

<u>Example 2</u> Solve the initial-value problem

$$\vec{x}' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}\vec{x}, \quad \vec{x}(0) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Solution

$$p(\lambda) = \det(A-\lambda I) = \det\begin{pmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & -1 \\ 0 & 1 & 1-\lambda \end{pmatrix} = (1-\lambda)^3 + (1-\lambda) = (1-\lambda)\left((1-\lambda)^2+1\right)$$

$$p(\lambda) = (1-\lambda)(1-2\lambda+\lambda^2+1) = (1-\lambda)(\lambda^2-2\lambda+2)$$

$$\lambda_1 = 1, \quad \lambda_{2,3} = \frac{2 \pm \sqrt{4-4\cdot 1\cdot 2}}{2} = \frac{2\pm\sqrt{-4}}{2} = \frac{2\pm i2}{2} = 1\pm i$$

(i) $\underline{\lambda_1 = 1}$:
$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} v_1 \text{ is any} \\ -v_3 = 0 \\ v_2 = 0 \end{cases} \Rightarrow \vec{v}^{\,1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$$

$$\vec{x}^{\,1}(t) = e^t\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

(ii) $\lambda_2 = 1+i$:
$$\begin{pmatrix} 1-(1+i) & 0 & 0 \\ 0 & 1-(1+i) & -1 \\ 0 & 1 & 1-(1+i) \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow$$

$$\begin{pmatrix} -i & 0 & 0 \\ 0 & -i & -1 \\ 0 & 1 & -i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -iv_1 = 0 \\ -iv_2 - v_3 = 0 \\ v_2 - iv_3 = 0 \end{cases} \Leftrightarrow \begin{cases} v_1 = 0 \\ iv_2 + v_3 = 0 \quad \cdot(-i) \\ v_2 - iv_3 = 0 \end{cases} \Leftrightarrow \begin{cases} v_1 = 0 \\ v_2 - iv_3 = 0 \end{cases} \Rightarrow$$

$$\vec{v} = \begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix} \Rightarrow \vec{x}^{\,2}(t) = e^{(1+i)t}\begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix} = e^t(\cos t + i\sin t)\left[\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + i\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right] \Rightarrow$$

$$\vec{x}^{\,2}(t) = e^t\cos t\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - e^t\sin t\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + ie^t\cos t\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + ie^t\sin t\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= e^t\begin{pmatrix} 0 \\ -\sin t \\ \cos t \end{pmatrix} + ie^t\begin{pmatrix} 0 \\ \cos t \\ \sin t \end{pmatrix} \Rightarrow \text{By Lemma } \vec{x}^{\,2}(t) = e^t\begin{pmatrix} 0 \\ -\sin t \\ \cos t \end{pmatrix}, \ \vec{x}^{\,3}(t) = e^t\begin{pmatrix} 0 \\ \cos t \\ \sin t \end{pmatrix}$$

The three solutions $\vec{x}^{\,1}(t)$, $\vec{x}^{\,2}(t)$, and $\vec{x}^{\,3}(t)$ are linearly independent since their initial values

$$\vec{x}^{\,1}(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{x}^{\,2}(0) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \vec{x}^{\,3}(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

are linearly independent vectors in $\mathbb{R}^3$.

Therefor, the solution $\vec{x}(t)$ of the initial-value problem must have the form

$$\vec{x}(t) = c_1 e^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^t \begin{pmatrix} 0 \\ -\sin t \\ \cos t \end{pmatrix} + c_3 e^t \begin{pmatrix} 0 \\ \cos t \\ \sin t \end{pmatrix}.$$

Setting $t=0$, $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \Rightarrow c_1 = c_2 = c_3 = 1 \Rightarrow$

$$\vec{x}(t) = e^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + e^t \begin{pmatrix} 0 \\ -\sin t \\ \cos t \end{pmatrix} + e^t \begin{pmatrix} 0 \\ \cos t \\ \sin t \end{pmatrix} = e^t \begin{pmatrix} 1 \\ \cos t - \sin t \\ \cos t + \sin t \end{pmatrix}.$$

Equal roots

If the characteristic polynomial of A does not have n distinct roots, then A may not have n linearly independent eigenvectors.

Suppose that the $n \times n$ matrix A has only $k < n$ linearly independent eigenvectors. Then, the differential equation $\vec{x}' = A\vec{x}$ has only k linearly independent solutions of the form $e^{\lambda t}\vec{v}$. Our problem is to find an additional $n-k$ linearly independent solutions.

Recall that $x(t) = e^{at}c$ is a solution of the scalar differential equation $x' = ax$, for every constant c. Analogously, we would like to say that $\vec{x}(t) = e^{At}\vec{v}$ is a solution of the vector differential equation

$$\vec{x}' = A\vec{x}$$

for every constant vector $\vec{v}$. However, e^{At} is not defined if A is an $n \times n$ matrix. This is not a serious difficulty, though. There is a very natural way of defining e^{At} so that it resembles the scalar exponential e^{at}; simply set

$$e^{At} \equiv I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \dots + \frac{A^nt^n}{n!} + \dots \qquad (5)$$

! It can be shown that the infinite series converges for all t, and can be differentiable term by term.

In particular

$$\frac{d}{dt}e^{At} = A + A^2 t + \cdots \frac{A^{n+1}}{n!}t^n + \cdots = A\left[I + At + \cdots + \frac{A^n t^n}{n!} + \cdots\right] = Ae^{tA}$$

This implies that $e^{At}\vec{v}$ is a solution of (1) for every constant vector $\vec{v}$, since

$$\frac{d}{dt}\left(e^{At}\vec{v}\right) = Ae^{At}\vec{v} = A\left(e^{At}\vec{v}\right).$$

Remark. The matrix exponential e^{At} and the scalar exponential e^{at} satisfy many similar properties. For example,

$$\left(e^{At}\right)^{-1} = e^{-At} \quad \text{and} \quad e^{A(t+s)} = e^{At}e^{As}.$$

However, $e^{At+Bt} = e^{At}\cdot e^{Bt}$ only if $AB = BA$.

There are several classes of matricies A for which the infinite series (5) can be summed exactly. In general, though, it does not seem possible to express e^{At} in closed form. Yet, the remarkable fact is that we can always find n linearly independent vectors $\vec{v}$ for which the infinite series $e^{At}\vec{v}$ can be summed exactly. Moreover, once we know n linearly independent solutions of (1), we can even compute e^{At} exactly.

We now show how to find n linearly independent vectors $\vec{v}$ for which the infinite series $e^{At}\vec{v}$ can be summed exactly. Observe that $e^{At}\vec{v} = e^{(A-\lambda I)t}e^{\lambda I t}\vec{v}$ for any constant λ, since $(A-\lambda I)(\lambda I) = (\lambda I)(A - \lambda I)$. (see the above Remark)

Moreover,

$$e^{\lambda I t}\vec{v} = \left[I + \lambda I t + \frac{\lambda^2 I^2 t^2}{2!} + \cdots\right]\vec{v} = \left[1 + \lambda t + \frac{\lambda^2 t^2}{2!} + \cdots\right]\vec{v} = e^{\lambda t}\vec{v}.$$

Hence, $e^{At}\vec{v} = e^{\lambda t} e^{(A-\lambda I)t}\vec{v}$

Next, we make the crucial observation that if $\vec{v}$ satisfies $(A-\lambda I)^m \vec{v} = \vec{0}$ for some integer m, then the infinite series $e^{(A-\lambda I)t}\vec{v}$ terminates after m terms. If $(A-\lambda I)^m\vec{v} = \vec{0}$, then $(A-\lambda I)^{m+\ell}\vec{v} = 0$ for every positive integer ℓ, since

$$(A-\lambda I)^{m+\ell}\vec{v} = (A-\lambda I)^{\ell}\left[(A-\lambda I)^m \vec{v}\right] = \vec{0}$$

Consequently,

$$e^{(A-\lambda I)t}\vec{v} = \vec{v} + t(A-\lambda I)\vec{v} + \cdots + \frac{t^{m-1}}{(m-1)!}(A-\lambda I)^{m-1}\vec{v}$$

and

$$\begin{aligned} e^{At}\vec{v} &= e^{\lambda t} e^{(A-\lambda I)t}\vec{v} \\ &= e^{\lambda t}\left[\vec{v} + t(A-\lambda I)\vec{v} + \cdots + \frac{t^{m-1}}{(m-1)!}(A-\lambda I)^{m-1}\vec{v}\right]. \end{aligned}$$

This suggests the following algorithm for finding n linearly independent solutions of (1).

(I) Find all the eigenvalues and eigenvectors of A. If A has n linearly independent eigenvectors, then the differential equation $\vec{x}' = A\vec{x}$ has n linearly independent solutions of the form $e^{\lambda t}\vec{v}$. (Observe that the infinite series $e^{(A-\lambda I)t}\vec{v}$ terminates after one term if $\vec{v}$ is an eigenvector of A with eigenvalue λ.).

(II) Suppose that A has only $k<n$ linearly independent eigenvectors. Then, we have only k linearly independent solutions of the form $e^{\lambda t}\vec{v}$. To find additional solutions we pick an eigenvalue λ of A and find all vectors $\vec{v}$ for which $(A-\lambda I)^2\vec{v}=\vec{0}$, but $(A-\lambda I)\vec{v}\neq\vec{0}$. For each such vector $\vec{v}$

$$\boxed{e^{At}\vec{v} = e^{\lambda t} e^{(A-\lambda I)t}\vec{v} = e^{\lambda t}[\vec{v} + t(A-\lambda I)\vec{v}]}$$

is an additional solution of $\vec{x}' = A\vec{x}$. We do this for all the eigenvalues λ of A.

(III) If we still do not have enough solutions, then we find all vectors $\vec{v}$ for which $(A-\lambda I)^3\vec{v}=\vec{0}$, but $(A-\lambda I)^2\vec{v}\neq\vec{0}$. For each such vector $\vec{v}$,

$$\boxed{e^{At}\vec{v} = e^{\lambda t}\left[\vec{v} + t(A-\lambda I)\vec{v} + \frac{t^2}{2!}(A-\lambda I)^2\vec{v}\right]}$$

is an additional solution of $\vec{x}' = A\vec{x}$

(IV) We keep proceeding in this manner until, hopefully, we obtain n linearly independent solutions.

The following lemma from linear algebra, which we accept without proof, guarantees that this algorithm always works. Moreover, it puts an upper bound on the number of steps we have to perform in this algorithm.

Lemma 1 Let the characteristic polynomial of A have k distinct roots $\lambda_1, \lambda_2, \ldots, \lambda_k$ with multiplicity $n_1, n_2, \ldots, n_k$ respectively. (This means that $p(\lambda)$ can be factored into the form $(\lambda_1-\lambda)^{n_1}(\lambda_2-\lambda)^{n_2}\cdots(\lambda_k-\lambda)^{n_k}$) Suppose that A has only $\nu_j < n_j$ linearly independent eigenvectors with eigenvalue λ_j. Then the equation $(A-\lambda I)^2\vec{v}=\vec{0}$ has at least ν_j+1 independent solutions. More generally, if the equation $(A-\lambda_j I)^m\vec{v}=\vec{0}$ has only $m_j < n_j$ independent solutions, then the equation $(A-\lambda_j I)^{m+1}\vec{v}=\vec{0}$ has at least m_j+1 independent solutions.

Lemma 1 clearly implies that there exists an integer d_j with $d_j \le n_j$, such that the equation $(A-\lambda_j I)^{d_j}\vec{v}=\vec{0}$ has at least n_j linearly independent solutions. Thus, for each eigenvalue λ_j of A, we can compute n_j linearly independent solutions of $\vec{x}\,' = A\vec{x}$. All these solutions have the form

$$\boxed{\vec{x}(t) = e^{\lambda_j t}\left[\vec{v} + t(A-\lambda_j I)\vec{v} + \cdots + \frac{t^{d_j-1}}{(d_j-1)!}(A-\lambda_j I)^{d_j-1}\vec{v}\right]}$$

In addition, it can be shown that the set of $n_1+\ldots+n_k = n$ solutions thus obtained must be linearly independent.

Example 3 Find the general solution of of the differential equation

$$\vec{x}\,' = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}\vec{x}.$$

Solution

$$p(\lambda) = \det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 1 & 0 & 1-\lambda & 1 \\ 0 & 1-\lambda & 0 & 0 & 1-\lambda \\ 0 & 0 & 2-\lambda & 0 & 0 \end{vmatrix}$$

$$= (1-\lambda)(1-\lambda)(2-\lambda) + \cancel{1\cdot 0\cdot 0}^{\,0} + \cancel{0\cdot 0\cdot 0}^{\,0} - \cancel{0\cdot(1-\lambda)\cdot 0}^{\,0} - \cancel{0\cdot 0\cdot(1-\lambda)}^{\,0} - \cancel{(2-\lambda)\cdot 0\cdot 1}^{\,0}$$

$$\boxed{p(\lambda) = (1-\lambda)^2(2-\lambda)}$$

(I) $\underline{\lambda=1}$. We seek all nonzero vectors $\vec{v}$ such that

$$(A-1\cdot I)\vec{v} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} v_2 = 0 \\ 0 = 0 \\ v_3 = 0 \end{cases} \Leftrightarrow \begin{pmatrix} v_1 \text{ is any} \\ v_2 = 0 \\ v_3 = 0 \end{pmatrix}$$

Consequently, $\vec{x}^{\,1} = e^t\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

is one solution of $\vec{x}\,' = A\vec{x}$. Since A has only one linearly independent eigenvector with eigenvalue 1, we look for all solutions of the equation

$$(A-1\cdot I)^2\vec{v} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\vec{v} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0=0 \\ 0=0 \\ v_3=0 \end{pmatrix} \Rightarrow$$

$$\begin{pmatrix} v_1 \text{ is any} \\ v_2 \text{ is any} \\ v_3 = 0 \end{pmatrix}$$

The vector $\vec{v} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ satisfies $(A-I)^2\vec{v} = \vec{0}$, but

$$(A-I)\vec{v} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \neq \vec{0}.$$

We could just as well choose any $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ 0 \end{pmatrix}$ for which $v_2 \neq 0$.

Hence,

$$\vec{x}^2 = e^{At}\begin{pmatrix}0\\1\\0\end{pmatrix} = e^t e^{(A-I)t}\begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$= e^t[I + t(A-I)]\begin{pmatrix}0\\1\\0\end{pmatrix} = e^t\left[\begin{pmatrix}0\\1\\0\end{pmatrix} + t\begin{pmatrix}0&1&0\\0&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}\right]$$

$$= e^t\begin{pmatrix}0\\1\\0\end{pmatrix} + te^t\begin{pmatrix}1\\0\\0\end{pmatrix} = e^t\begin{pmatrix}t\\1\\0\end{pmatrix}$$

is a second linearly independent solution.

(II) $\lambda = 2$: $(A-2I)\vec{v} = \begin{pmatrix}-1&1&0\\0&-1&0\\0&0&0\end{pmatrix}\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \Leftrightarrow \begin{cases}-v_1+v_2=0\\-v_2=0\\0=0\end{cases} \Rightarrow \begin{pmatrix}v_1=0\\v_2=0\\v_3 \text{ is any}\end{pmatrix}$

Hence $\vec{x}^3(t) = e^{2t}\begin{pmatrix}0\\0\\1\end{pmatrix}$, a third linearly independent solution.

Therefore, the general solution is

$$\vec{x}(t) = C_1\vec{x}^1 + C_2\vec{x}^2 + C_3\vec{x}^3 = C_1e^t\begin{pmatrix}1\\0\\0\end{pmatrix} + C_2e^t\begin{pmatrix}t\\1\\0\end{pmatrix} + C_3e^{2t}\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\boxed{\vec{x}(t) = \begin{pmatrix}C_1e^t + C_2te^t\\C_2e^t\\C_3e^{2t}\end{pmatrix}}$$