

Classification of states

Define

$$f_{ij}(n) = P(X_1 \neq j, X_2 \neq j, \dots, X_{n-1} \neq j, X_n = j | X_0 = i) \quad (9)$$

to be the probability that starting in state i the chain first visits state j after n steps. Define

$$\begin{aligned} f_{ij} &= \sum_{n=1}^{\infty} f_{ij}(n) \\ &= P(X_n = j \text{ for some } n \geq 1 | X_0 = i) \end{aligned}$$

*persistent
= recurrent*

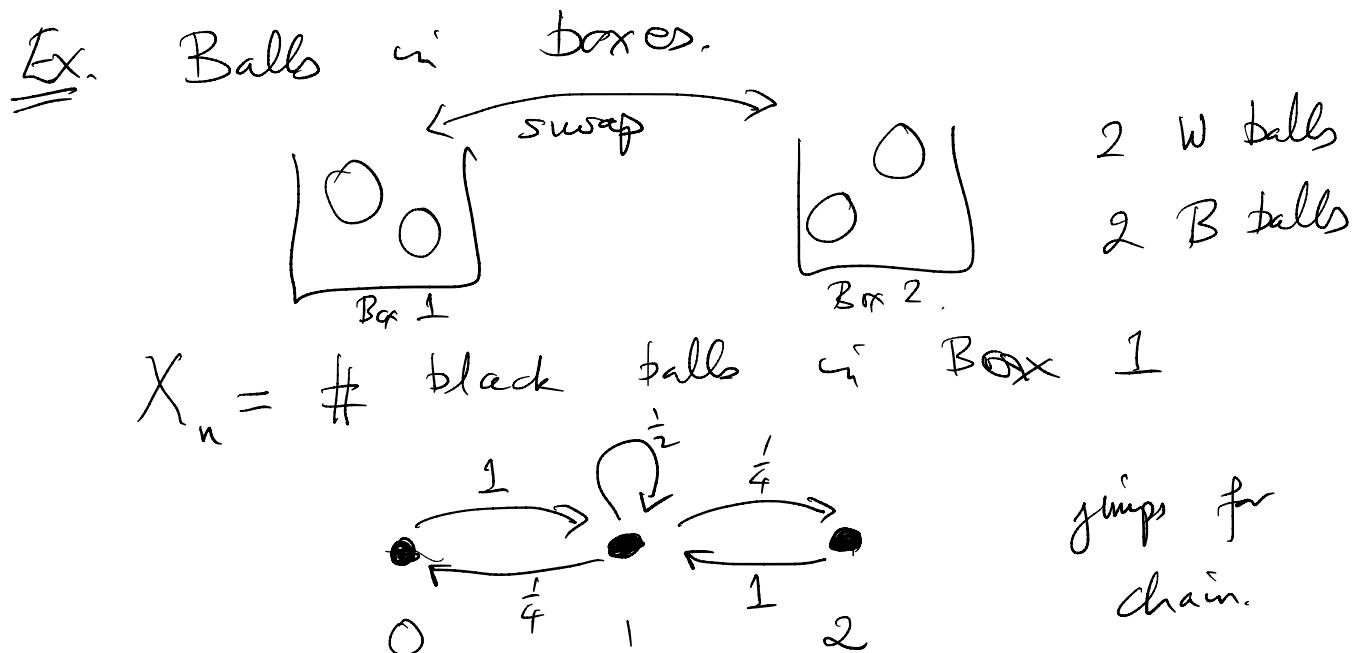
This is the probability that the chain eventually visits state j starting in state i .

Definition 1 The state j is persistent if $f_{jj} = 1$. The state j is transient if $f_{jj} < 1$.

So the state j is persistent if the chain must eventually return to state j , given that it started at state j . Since the clock starts over at each return to state j , we conclude that if the state j is persistent then the chain must return to j infinitely often.

Similarly if j is transient, then the chain will return to state j only finitely often. Eventually it will never return to the state.

One particular type of persistent state is the class of *absorbing states*: these are the states for which $p_{ii} = 1$. So once the chain enters an absorbing state it can never leave. The end states of the drunkard's walk are absorbing states.



Is state 1 persistent?

$$\begin{aligned} f_{11} &= P(X_n = 1 \text{ some } n \geq 1 | X_0 = 1) \\ &= \sum_{n=1}^{\infty} f_{11}(n) \end{aligned}$$

*prob. first return
after n steps.*

$$f_{11}(1) = \mathbb{P}(X_1=1, \text{ first return} | X_0=1)$$

$$= \frac{1}{2}$$

$$f_{11}(2) = \mathbb{P}(X_2=1, \text{ first return} | X_0=1)$$

$$= \mathbb{P}(X_2=1, X_1 \neq 1 | X_0=1)$$

$$= \mathbb{P}(X_2=1, X_1=0 | X_0=1)$$

$$+ \mathbb{P}(X_2=1, X_1=2 | X_0=1)$$

$$= \left(\frac{1}{4}\right)(1) + \left(\frac{1}{4}\right)(1)$$

$$= \frac{1}{2}.$$

$$f_{11}(3) = \mathbb{P}(X_3=1, X_2 \neq 1, X_1 \neq 1 | X_0=1)$$

$$= 0$$
$$f_{11}(n) = 0 \quad \text{all } n \geq 3.$$

$$\Rightarrow f_{11} = f_{11}(1) + f_{11}(2) + f_{11}(3) + \dots$$
$$= \frac{1}{2} + \frac{1}{2} + 0 + 0 + \dots$$
$$= 1$$

\Rightarrow state 1 is persistent.

Deciding about persistence

There is a useful way to decide if a state is persistent or transient.

With the conventions $p_{ij}(0) = \delta_{ij}$ and $f_{ij}(0) = 0$, we have for each n

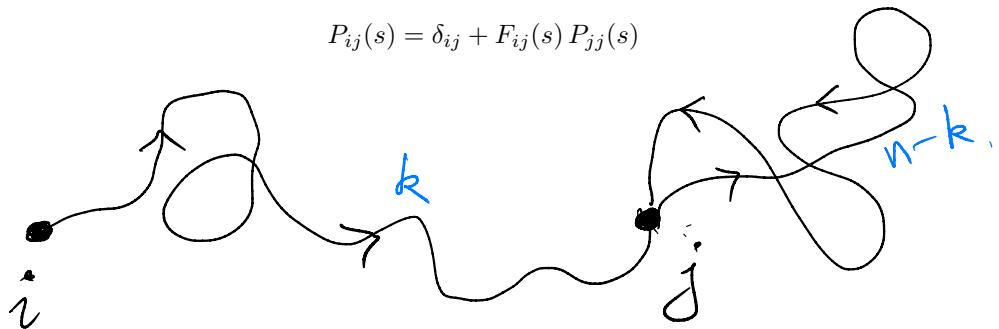
$$p_{ij}(n) = \delta_{ij} \delta_{n,0} + \sum_{k=1}^n f_{ij}(k) p_{jj}(n-k)$$

where we sum over k , the time of first visit to state j . This is a convolution formula so we can solve it by a transform. Define the generating functions

$$P_{ij}(s) = \sum_{n=0}^{\infty} s^n p_{ij}(n), \quad F_{ij}(s) = \sum_{n=0}^{\infty} s^n f_{ij}(n)$$

Then the convolution formula implies

$$P_{ij}(s) = \delta_{ij} + F_{ij}(s) P_{jj}(s)$$



$$P_{ij}(n) = P(X_n = j | X_0 = i)$$

- say $X_k = j$ is the first visit to state j after k steps

Possible values of k :

$k = 1, 2, 3, \dots, n$ - disjoint events.

$$\Rightarrow P_{ij}(n) = \sum_{k=1}^n P(X_n = j, X_k = j, X_{k-1} \neq j, X_{k-2} \neq j, \dots, X_1 \neq j | X_0 = i)$$

$$= \sum_{k=1}^n P(X_n=j \mid X_k=j).$$

$$P(X_k=j \mid X_{k-1}=j, \dots, X_1=j \mid X_0=i)$$

$$P_{ij}(n) = \sum_{k=1}^n P_{jj}(n-k) f_{ij}(k) \quad (i \neq j) \\ (n \geq 1)$$

Define

$$P_{ij}(0) = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

$$P_{ij}(n) = \delta_{ij} \delta_{n0} + \sum_{k=1}^n f_{ij}(k) P_{jj}(n-k) \quad (\text{X})$$

$$\delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$$

$$\delta_{n0} = \begin{cases} 1 & n=0 \\ 0 & n>0 \end{cases}$$

- holds for all i, j and all $n \geq 0$.

Define

$$\hat{P}_{ij}(s) = \sum_{n=0}^{\infty} s^n P_{ij}^{(n)}$$

$$\hat{f}_{ij}(s) = \sum_{n=0}^{\infty} s^n f_{ij}^{(n)}$$

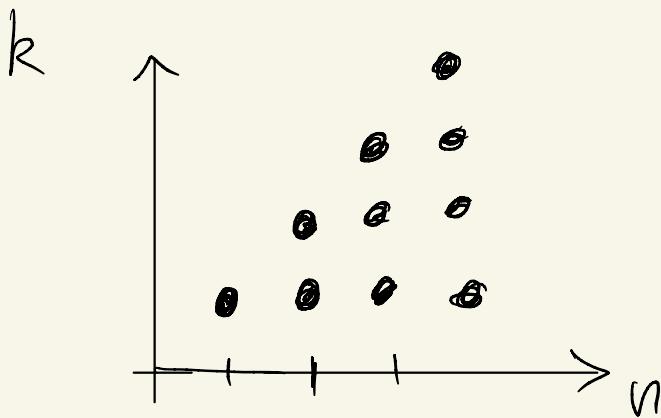
where define $f_{ij}^{(0)} = 0$ all i, j .

(*) $s^n P_{ij}^{(n)} = s^n \delta_{ij} \delta_{n0}$

$$+ \sum_{k=1}^n s^k f_{ij}^{(k)}.$$
$$\times s^{n-k} P_{jj}^{(n-k)}$$

Sum over n :

$$\sum_{n=0}^{\infty} s^n p_{ij}(n) = \delta_{ij}^{ir} + \sum_{n=0}^{\infty} \sum_{k=1}^n s^k f_{ij}^r(k) s^{n-k} p_{jj}(n-k)$$



$$\sum_{n=0}^{\infty} \sum_{k=1}^n = \sum_{k=1}^{\infty} \sum_{n=k}^{\infty}$$

$$\Rightarrow \hat{P}_{ij}(s) = \delta_{ij} + \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} s^k f_{ij}^n(k),$$

$$s^{n-k} P_{jj}^{(n-k)}$$

let $m = n - k$.

$\Rightarrow m = 0, 1, 2, \dots$

$$= \delta_{ij} + \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} s^k f_{ij}^m(k),$$

$$s^m P_{jj}^{(m)}$$

$$= \delta_{ij} + \left(\sum_{k=1}^{\infty} s^k f_{ij}(k) \right) \cdot \left(\sum_{m=0}^{\infty} s^m p_{jj}(m) \right)$$

$\hat{f}_{ij}(s)$
 $\hat{p}_{jj}(s)$

⇒

$$\hat{p}_{ij}(s) = \delta_{ij} + \hat{f}_{ij}(s) \hat{p}_{jj}(s)$$

S=1

$$f_{ij}(1) = \sum_{n=0}^{\infty} (1)^n f_{ij}(n) = \sum_{n=1}^{\infty} f_{ij}(n)$$

$$= f_{ij}$$

let $i=j$, put $s=1$:

$$\hat{P}_{jj}(1) = 1 + \hat{f}_{jj}(1) \hat{P}_{jj}(1)$$

$$\Rightarrow \hat{P}_{jj}(1) = \frac{1}{1 - \hat{f}_{jj}(1)}.$$

Note $\hat{f}_{jj}(1) = f_{jj}$

$$\hat{P}_{jj}(1) = \sum_{n=0}^{\infty} P_{jj}(n).$$

Suppose $j \hookrightarrow$ persistent

$$\Leftrightarrow f_{jj} = 1 \Leftrightarrow \sum_{n=0}^{\infty} P_{jj}(n) = \infty$$

Suppose j is transient.

$$\Leftrightarrow f_{jj} < 1 \Leftrightarrow \sum_{n=0}^{\infty} P_{jj}(n) < \infty$$

For the case $i = j$ this yields

$$P_{ii}(s) = \frac{1}{1 - F_{ii}(s)}$$

Now consider what happens in the limit $s \rightarrow 1$. Recall Abel's theorem: if $a_n \geq 0$ for all n and $\sum_n a_n s^n$ is finite for all $|s| < 1$, then

$$\lim_{s \uparrow 1} \sum_{n=0}^{\infty} a_n s^n = \sum_{n=0}^{\infty} a_n$$

We apply this with

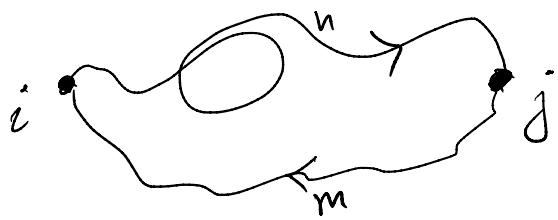
$$\lim_{s \uparrow 1} F_{ii}(s) = f_{ii}$$

We conclude that the state i is persistent ($F_{ii}(1) = 1$) if and only if

$$P_{ii}(1) = \sum_n p_{ii}(n) = \infty$$

So this is our useful test: the state i is persistent if and only if

$$\sum_{n=1}^{\infty} P(X_n = i \mid X_0 = i) = \infty$$



$$p_{ij}(n) > 0.$$

$$p_{ji}(m) > 0.$$

Classification of Markov chains

We say that the states i and j *intercommunicate* if there are integers n, m such that $p_{ij}(n) > 0$ and $p_{ji}(m) > 0$. In other words it is possible to go from each state to the other after a finite number of steps.

Theorem 1 Suppose i, j intercommunicate, then they are either both transient or both persistent.

Proof: Since i, j intercommunicate there are integers n, m such that

$$h = p_{ij}(n)p_{ji}(m) > 0 \quad (10)$$

Hence for any r ,

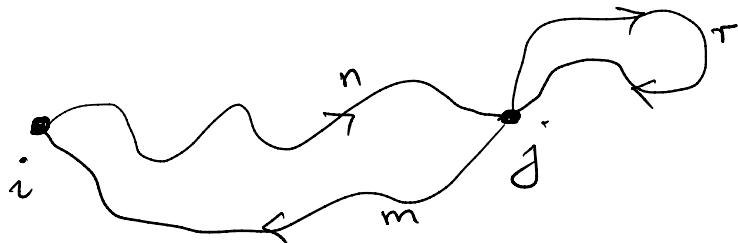
$$\text{(X)} \quad p_{ii}(n+m+r) \geq p_{ij}(n)p_{jj}(r)p_{ji}(m) = h p_{jj}(r) \quad (11)$$

Sum over r to deduce

$$\sum_k p_{ii}(k) \geq \sum_r p_{ii}(n+m+r) \geq h \sum_r p_{jj}(r) \quad (12)$$

Therefore either both sums are finite or both are infinite, hence either both states are transient or both are persistent.

If all states intercommunicate we say that the Markov chain is *irreducible*.



$$\text{(X)} \quad P_{ij}(n) P_{jj}(r) P_{ji}(m) = P(X_n=j | X_0=i) P(X_{n+r}=j | X_{n+r}=j) \\ = P(X_{n+r}=j | X_0=i) \\ = P(X_{n+r}=j | X_{n+r}=i, X_{n+r}=j | X_n=j) \\ = P(X_{n+r+m}=i, X_{n+r}=j, X_n=j | X_0=i)$$

Markov
Prop

$$P_{ij}^{ij}(n+r+m) = P(X_{n+r+m} = i | X_0 = i) \\ \geq p_{ij}^i(n) p_{jj}^j(r) p_{ji}^i(m).$$