Problem 1.

Derivation of two first-order equations:

To rewrite the second-order differential equation (1) as two first-order equations, we introduce a new variable $y = \dot{x}$, which represents the velocity of the reed. Then, we can write the system as follows:

$$m\,\dot{x} = y$$

$$m\,\dot{y} = -k\,x + a\,y - b(y^3)$$

This is a system of two first-order differential equations. To see this, we can rewrite it in vector form as:

$$dx/dt = [\dot{x}, y]$$

$$dy/dt = [-k/m\,x + a/m\,y - b/m\,(y^3), -x/m]$$

Problem 2.

Derivation of Stability Condition:

For Heun's method, the local truncation error is O(h^3), and the global error is O(h^2). The method is said to be stable if the error does not grow exponentially with each step. To ensure stability, we need to derive the stability condition for Heun's method.

Let us consider the test equation $y' = \lambda y$, where λ is a complex number.

Using Heun's method, we get:

$$y_{n+1} = y_n + h/2\,(f(t_n, y_n) + f(t_n + 1, y_n + 1))$$

where,

$$y_{n+1} = y_n + h\lambda\,y_n = (1 + h\lambda)\,y_n$$

Substituting y_{n+1} and $y_n + 1$ into the test equation, we get:

$$y_{n+1} = y_n + h/2(\lambda y_n + \lambda(1 + h\lambda)y_n)$$

Simplifying the above equation, we get:

$$y_{n+1} = (1 + h\lambda + (h\lambda)^2/2)\,y_n$$

We can rewrite the above equation in terms of the amplification factor A:

$$y_{n+1} = A\,y_n$$

where $A = 1 + h\lambda + (h\lambda)^2/2$

For stability, we need the absolute value of A to be less than or equal to 1.

$$|A| = |1 + h\lambda + (h\lambda)^2/2| \leq 1$$

Expanding the absolute value, we get:

$$-1 \leq 1 + h\lambda + (h\lambda)^2/2 \leq 1$$

Simplifying the above equation, we get:

$$-2 \leq h\lambda + (h\lambda)^2 \leq 0$$

We can solve the above quadratic inequality to get the stability interval for Heun's method:

$$-2 \leq h\lambda \leq 0$$

Therefore, Heun's method is stable over the interval $(-2, 0)$ for real $h\lambda$.

Problem 3.

Using the Butcher tableau, the actual set of equations used to implement the explicit midpoint method is,

$k_1 = h * f(t_n, y_n)$

$k_2 = h * f(t_n + h/2, y_n + k_1/2)$

$y_{n+1} = y_n + k_2$