MATH 5131 Lecture 9

Phase portraits of linear systems.

We present a complete picture of all trajectories of the linear differential equation

$$\vec{x}' = A\vec{x}, \quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \qquad (1)$$

This picture is called a phase portrait, and it depends almost completely on the eigenvalues of the matrix A. It also changes drastically as the eigenvalues of A change sign or become imaginary.

Let λ_1 and λ_2 denote the two eigenvalues of A. We distinguish the following cases.

1. $\lambda_2 < \lambda_1 < 0$. Let $\vec{v}^1$ and $\vec{v}^2$ be eigenvectors of A with eigenvalues λ_1 and λ_2 respectively.
   In the x_1x_2-plane we draw the four half-lines l_1, l_1', l_2, and l_2'. The rays l_1 and l_2 are parallel to $\vec{v}^1$ and $\vec{v}^2$ while the rays l_1' and l_2' are parallel to $-\vec{v}^1$ and $-\vec{v}^2$.

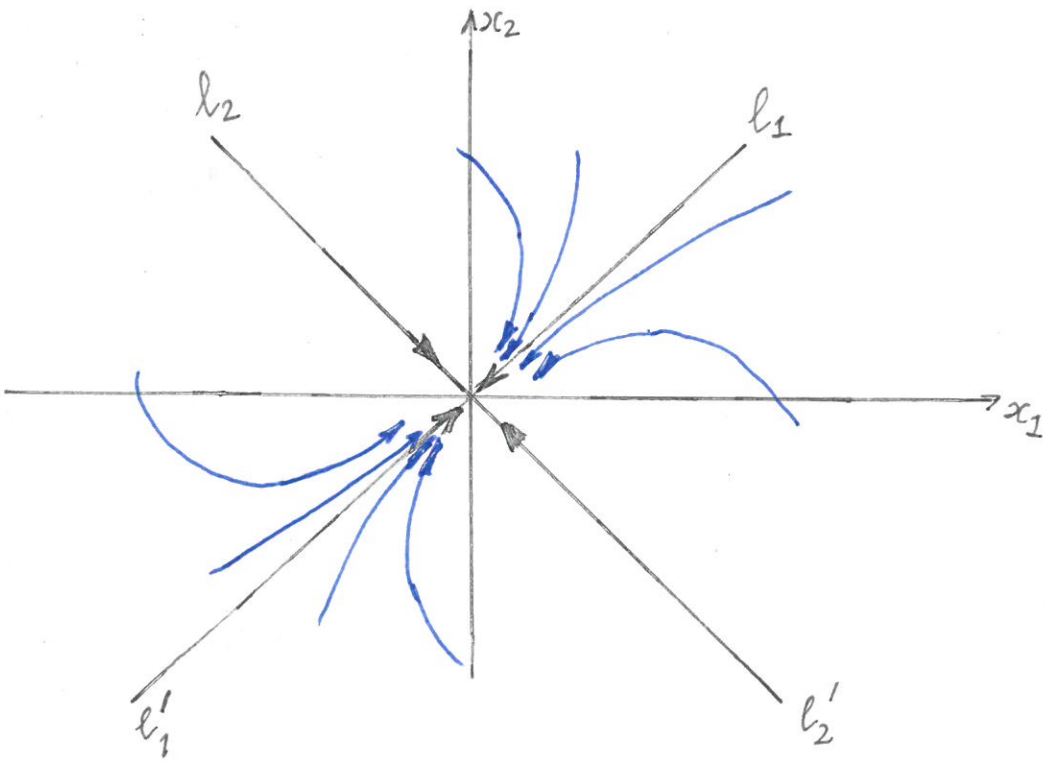


Figure 1: Phase portrait of a stable node

Observe first that

$$\vec{x}(t) = c e^{\lambda_1 t} \vec{v}^1$$

is a solution of (1) for any constant c. This solution is always proportional to $\vec{v}^1$, and the constant of proportionality, $ce^{\lambda_1 t}$, runs from $\pm\infty$ to 0, depending as to whether c is positive or negative. Hence, the trajectory of this solution is the half-line l_1 for $c>0$, and the half-line l_1' for $c<0$.

Similarly, the trajectory of the solution

$$\vec{x}(t) = c e^{\lambda_2 t} \vec{v}^2$$

is the half-line l_2 for $c>0$, and the half-line l_2' for $c<0$. The arrows on these four lines in Figure 1 indicate in what direction $\vec{x}(t)$ moves along its trajectory.

Next, recall that every solution $\vec{x}(t)$ of (1) can be written in the form

$$\vec{x}(t) = c_1 e^{\lambda_1 t} \vec{v}^1 + c_2 e^{\lambda_2 t} \vec{v}^2 \qquad (2)$$

for some choice of constants c_1 and c_2. Obviously, every solution $\vec{x}(t)$ of (1) approaches $\vec{0}$ as t approaches infinity. Hence, every trajectory of (1) approaches the origin

$$x_1 = x_2 = 0 \quad \text{as } t \to \infty.$$

We can make an even stronger statement by observing that $e^{\lambda_2 t}\vec{v}^2$ is very small compared to $e^{\lambda_1 t}\vec{v}^1$ when t is very large. Therefore, $\vec{x}(t)$, for $c_1 \neq 0$, comes closer and closer to $c_1 e^{\lambda_1 t}\vec{v}^1$ as t approaches infinity. This implies that the tangent to the trajectory of $\vec{x}(t)$ approaches l_1 if c_1 is positive, and l_1' if c_1 is negative. Thus, the phase portrait of (1) has the form described in Figure 1.

The distinguishing feature of this phase portrait is that every trajectory, with the exception of a single line, approaches the origin in a fixed direction (if we consider the directions $\vec{v}^1$ and $-\vec{v}^1$ equivalent). In this case we say that the equilibrium solution $\vec{x}=\vec{0}$ of (1) is a <u>stable node</u>.

<u>Remark</u> The trajectory of every solution $\vec{x}(t)$ of (1) approaches the origin $x_1=x_2=0$ as $t\to\infty$. However, this point does not belong to the trajectory of any nontrivial solution $\vec{x}(t)$.

1'. $0<\lambda_1<\lambda_2$. The phase portrait of (1) in this case is exactly the same as Figure 1, except that the direction of the arrows is reversed. Hence, the equilibrium solution $\vec{x}(t)=\vec{0}$ of (1) is an <u>unstable node</u> if both eigenvalues of A are positive.

2. $\lambda_1=\lambda_2<0$. In this case, the phase portrait of (1) depends on whether A has one or two linearly independent eigenvectors.

(a) Suppose that A has two linearly independent eigenvectors $\vec{v}^1$ and $\vec{v}^2$ with eigenvalue $\lambda<0$. In this case, every solution $\vec{x}(t)$ of (1) can be written in the form

$$\vec{x}(t)=c_1e^{\lambda t}\vec{v}^1+c_2e^{\lambda t}\vec{v}^2=e^{\lambda t}(c_1\vec{v}^1+c_2\vec{v}^2) \quad (3)$$

for some choice of constants c_1 and c_2. Now, the vector

$e^{\lambda t}(c_1\vec{v}^1+c_2\vec{v}^2)$ is parallel to $c_1\vec{v}^1+c_2\vec{v}^2$ for all t.

Hence, the trajectory of every solution $\vec{x}(t)$ of (1) is a half-line. Moreover, the set of vectors

$$\{c_1\vec{v}^1+c_2\vec{v}^2\},$$

for all choices of c_1 and c_2, cover every direction in the x_1x_2-plane, since $\vec{v}^1$ and $\vec{v}^2$ are linearly independent. Hence, the phase portrate of (1) has the form described in Figure 2.

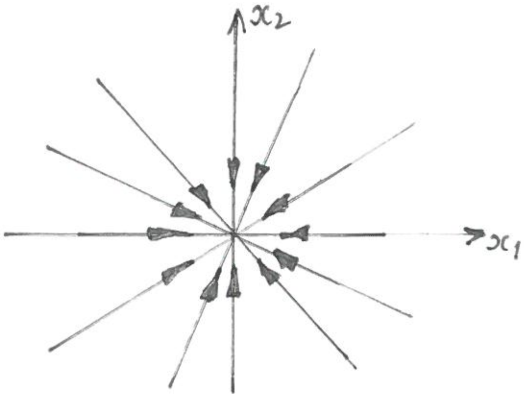


Figure 2.

(b) Suppose that A has only one linearly independent eigenvector $\vec{v}$, with eigenvalue λ. In this case,

$$\vec{x}^1(t) = e^{\lambda t}\vec{v}$$

is one solution of (1). To find a second solution of (1) which is independent of $\vec{x}^1$, we observe that

$$(A-\lambda I)^2\vec{u} = \vec{0} \quad \text{for every vector } \vec{u}.$$

[Let $(A-\lambda I)\vec{u} = \vec{v}$ | Apply $(A-\lambda I)$

$$(A-\lambda I)(A-\lambda I)\vec{u} = \underbrace{(A-\lambda I)\vec{v}}_{\vec{v} \text{ is an eigenvector}} = \vec{0}$$

$$\Downarrow$$

$$(A-\lambda I)^2\vec{u} = \vec{0}$$]

Hence,

$$\vec{x}(t) = e^{At}\vec{u} = e^{At-\lambda It+\lambda It}\vec{u} = e^{(A-\lambda I)t}e^{\lambda It}\vec{u}$$

$$= e^{(A-\lambda I)t}e^{\lambda t}\vec{u} = e^{\lambda t}e^{(A-\lambda I)t}\vec{u}$$

$$= e^{\lambda t}\left[I + (A-\lambda I)t + \frac{1}{2!}(A-\lambda I)^2t^2 + 0 + \dots\right]\vec{u}$$

$$= e^{\lambda t}\left[I + (A-\lambda I)t\right]\vec{u}\,] = e^{\lambda t}\left[\vec{u} + t(A-\lambda I)\vec{u}\right]$$

$$\vec{x}(t) = e^{\lambda t}\left[\vec{u} + t\underbrace{(A-\lambda I)\vec{u}}_{\text{multiple of } \vec{v}}\right] \qquad (4)$$

is a solution of (1) for any choice of $\vec{u}$.

Equation (4) can be simplified by observing that $(A-\lambda I)\vec{u}$ must be a multiple k of $\vec{v}$. This follows immediately from the equation $(A-\lambda I)[(A-\lambda I)\vec{u}]=\vec{0}$, and the fact that A has only one linearly independent eigenvector $\vec{v}$. Choosing $\vec{u}$ independent of $\vec{v}$, we see that every solution $\vec{x}(t)$ of (1) can be written in the form

$$\vec{x}(t) = c_1 e^{\lambda t}\vec{v} + c_2 e^{\lambda t}(\vec{u} + kt\vec{v}) = e^{\lambda t}(c_1\vec{v} + c_2\vec{u} + c_2 kt\vec{v}),$$

for some choice of constants c_1 and c_2. Obviously, every solution $\vec{x}(t)$ of (1) approaches $\vec{0}$ as $t\to\infty$. In addition, observe that $c_1\vec{v}+c\vec{u}$ is very small compared to $c_2 kt\vec{v}$ if c_2 is unequal to zero and t is very large. Hence, the tangent to the trajectory of $\vec{x}(t)$ approaches $\pm\vec{v}$ (depending on the sign of c_2) as $t\to\infty$, and the phase portrait of (1) has the form described in Figure 3.

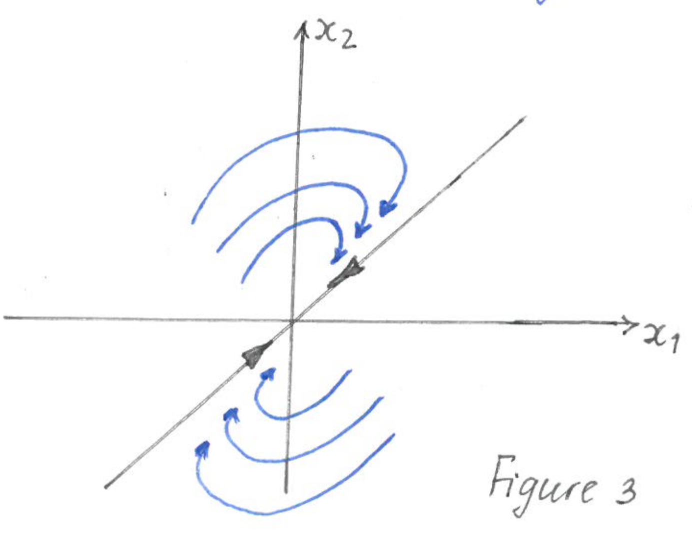


Figure 3

2.' $\lambda_1=\lambda_2>0$. The phase portraits of (1) in the cases (2a)' and (2b)' are exactly the same as Figure 2 and 3, except that the direction of the arrow is reversed.

3. $\lambda_1 < 0 < \lambda_2$. Let $\vec{v}^1$ and $\vec{v}^2$ be eigenvectors of A with eigenvalues λ_1 and λ_2 respectively. In the x_1x_2-plane we draw the four half-lines l_1, l_1', l_2, and l_2'; the half-lines l_1 and l_2 are parallel to $\vec{v}^1$ and $\vec{v}^2$, while the half-lines l_1' and l_2' are parallel to $-\vec{v}^1$ and $-\vec{v}^2$. Observe first that every solution $\vec{x}(t)$ of (1) is of the form

$$\vec{x}(t) = c_1 e^{\lambda_1 t} \vec{v}^1 + c_2 e^{\lambda_2 t} \vec{v}^2 \qquad (5)$$

for some choice of constants c_1 and c_2. The trajectory of the solution $\vec{x}(t) = c_1 e^{\lambda_1 t} \vec{v}^1$ is l_1 for $c_1 > 0$ and l_1' for $c_1 < 0$, while the trajectory of the solution $\vec{x}(t) = c_2 e^{\lambda_2 t} \vec{v}^2$ is l_2 for $c_2 > 0$ and l_2' for $c_2 < 0$, Note, too, the direction of the arrows on l_1, l_1', l_2, and l_2'; the solution $\vec{x}(t) = c_1 e^{\lambda_1 t} \vec{v}_1 \to \vec{0}$ as $t \to \infty$; whereas the solution $\vec{x}(t) = c_2 e^{\lambda_2 t} \vec{v}^2$ becomes unbounded (for $c_2 \neq 0$) as $t \to \infty$. Next, observe that $e^{\lambda_1 t} \vec{v}^1$ is very small compared to $e^{\lambda_2 t} \vec{v}^2$ when t is very large. Hence, every solution $\vec{x}(t)$ of (1) with $c_2 \neq 0$ becomes unbounded as $t \to \infty$, and its trajectory approaches either l_2 or l_2'. Finally, observe that $e^{\lambda_2 t} \vec{v}_2$ is very small compared to $e^{\lambda_1 t} \vec{v}^1$ when t is very large negative. Hence, the trajectory of any solution $\vec{x}(t)$ of (1), with $c_1 \neq 0$, approaches either l_1 or l_1' as $t \to -\infty$. Consequently, the phase portrait of (1) has the form described in Figure 4.

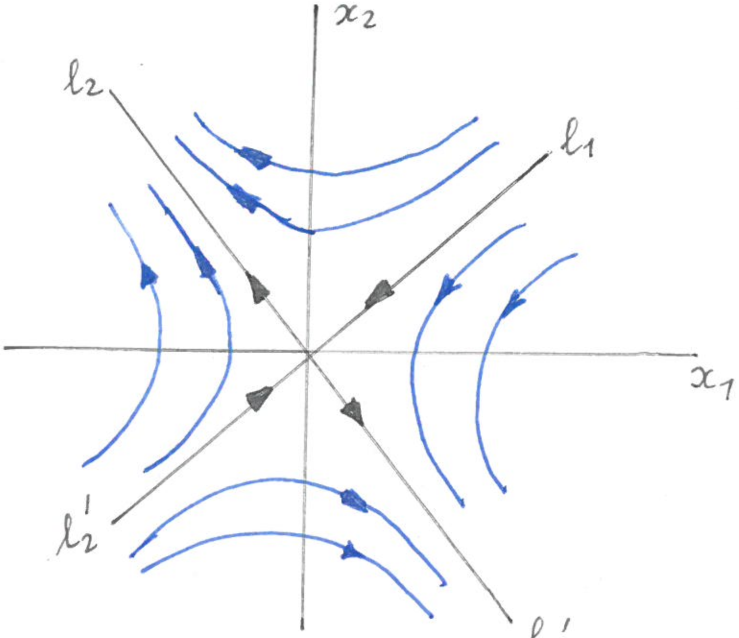


Figure 4

This phase portrait resembles a "saddle" near $x_1 = x_2 = 0$. For this reason, we say that the equilibrium solution $\vec{x}(t) = \vec{0}$ of (1) is a <u>saddle point</u> if the eigenvalues of A have opposite sign.

4. $\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$, $\beta \neq 0$. Our first step in deriving the phase portrait of (1) is to find the general solution of (1). Let $\vec{z} = \vec{u} + i\vec{v}$ be an eigenvector of A with the eigenvalue $\alpha + i\beta$. Then,

$$\vec{x}(t) = e^{(\alpha+i\beta)t}(\vec{u} + i\vec{v}) = e^{\alpha t}(\cos\beta t + i\sin\beta t)(\vec{u} + i\vec{v})$$
$$= e^{\alpha t}[\vec{u}\cos\beta t - \vec{v}\sin\beta t] + i e^{\alpha t}[\vec{u}\sin\beta t + \vec{v}\cos\beta t].$$

is a complex-valued solution of (1). Therefore,

$$\vec{x}^1(t) = e^{\alpha t}[\vec{u}\cos\beta t - \vec{v}\sin\beta t]$$

and $\quad \vec{x}^2(t) = e^{\alpha t}[\vec{u}\sin\beta t + \vec{v}\cos\beta t]$

are two real-valued linearly independent solutions of (1), and every solution $\vec{x}$ of (1) is of the form

$$\vec{x}(t) = c_1\vec{x}^1(t) + c_2\vec{x}^2(t).$$

This expression can be written in the form

$$\vec{x}(t) = e^{\alpha t} \begin{pmatrix} R_1 \cos(\beta t - \delta_1) \\ R_2 \cos(\beta t - \delta_2) \end{pmatrix} \qquad (6)$$

for some choice of constants $R_1 \geq 0$, $R_2 \geq 0$, δ_1 and δ_2. We distinguish the following cases.

(a) $\alpha = 0$: Observe that both

$$x_1(t) = R_1 \cos(\beta t - \delta_1) \quad \text{and} \quad x_2(t) = R_2 \cos(\beta t - \delta_2)$$

are periodic functions of time with period $\frac{2\pi}{\beta}$. The function $x_1(t)$ varies between $-R_1$ and $+R_1$, while $x_2(t)$ varies between $-R_2$ and R_2. Consequentely, the trajectory of any solution $\vec{x}(t)$ of (1) is a closed curve surrounding the origin $x_1 = x_2 = 0$, and the phase portrait of (1) has the form described in Figure 5a. For this reason, we say that the equilibrium solution $\vec{x}(t) = \vec{0}$ of (1) is a <u>center</u> when the eigenvalues of A are pure imaginary.

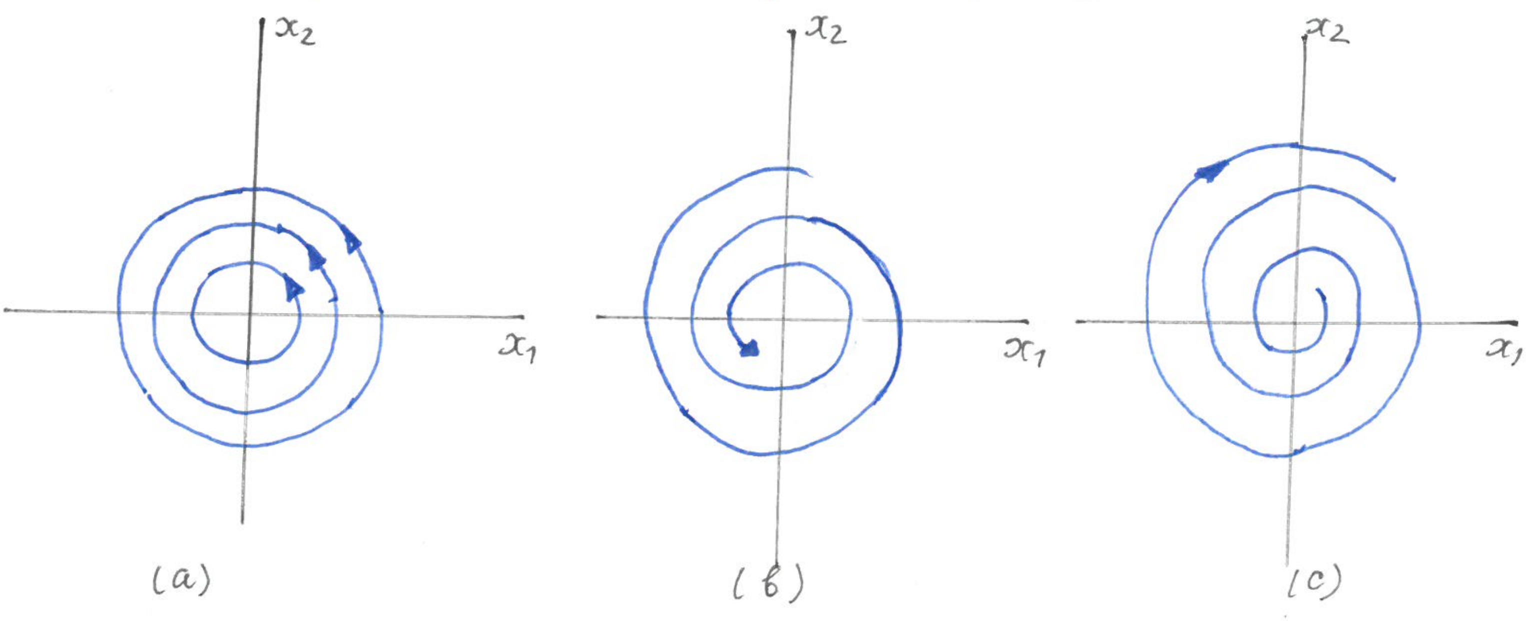


Figure 5.

The direction of the arrows in Figure 5a must be determined from the differential equation (1). The simplest way of doing this is to check the sign of x_2' when $x_2 = 0$. If x_2' is greater than zero for $x_2 = 0$ and $x_1 > 0$, then all solutions $\vec{x}(t)$ of (1) move in the counterclockwise direction; if x_2' is less than zero for $x_2 = 0$ and $x_1 > 0$, then all solutions $\vec{x}(t)$ of (1) move in the clockwise direction.

(b) $\alpha < 0$: In this case, the effect of the factor $e^{\alpha t}$ in Equation (6) is to change the simple closed curves of Figure 5a into the spirals of Figure 5b. This is because the point $\vec{x}\left(\frac{2\pi}{\beta}\right) = e^{\frac{2\pi\alpha}{\beta}} \vec{x}(0)$ is closer to the origin than $\vec{x}(0)$. Again, the direction of the arrows in Figure 5b must be determined directly from the differential eq. (1). In this case, we say that the equilibrium solution $\vec{x}(0) = \vec{0}$ of (1) is a <u>stable focus</u>.

(c) $\alpha > 0$: In this case, all trajectories of (1) spiral away from the origin as $t \to \infty$ (see Figure 5c), and the equilibrium solution $\vec{x}(t) = \vec{0}$ of (1) is called an unstable focus.

Example 1. Draw the phase portrait of the linear equation

$$\vec{x}\,' = A\vec{x} = \begin{pmatrix} -2 & -1 \\ 4 & -7 \end{pmatrix}\vec{x}$$

Solution

$$\begin{vmatrix} -2-\lambda & -1 \\ 4 & -7-\lambda \end{vmatrix} = (-2-\lambda)(-7-\lambda) - 4(-1) = (\lambda+2)(\lambda+7)+4 = 0$$

$\lambda^2 + 9\lambda + 14 + 4 = 0$

$\lambda^2 + 9\lambda + 18 = 0$

$D = 9^2 - 4\cdot(18) = 81 - 72 = 9 > 0$

$\lambda_1 = \dfrac{-9-3}{2} = \dfrac{-12}{2} = -6$

$\lambda_2 = \dfrac{-9+3}{2} = \dfrac{-6}{2} = -3.$

$\Rightarrow$ Eigenvalues: $\lambda_1 = -6$, $\lambda_2 = -3$.

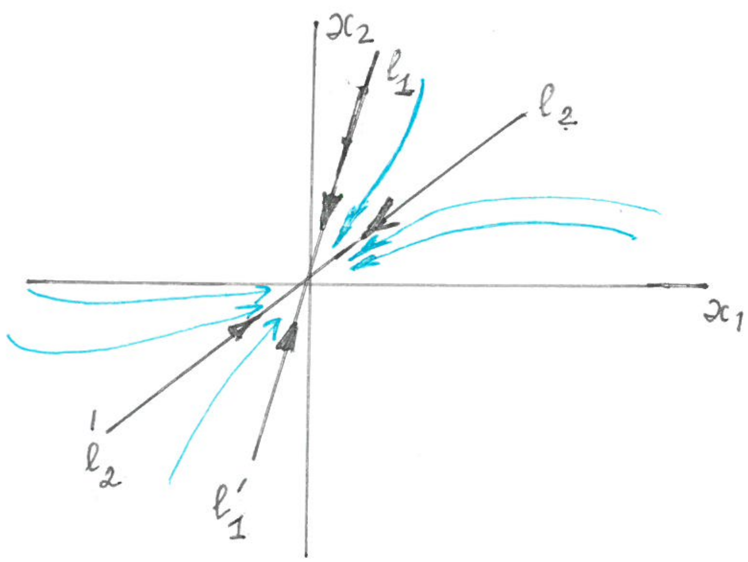
Eigenvectors: • $\lambda_1 = -6$: $\begin{pmatrix} -2-(-6) & -1 \\ 4 & -7-(-6) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow$

$\begin{pmatrix} 4 & -1 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \quad 4x_1 - x_2 = 0 \Rightarrow x_2 = 4x_1 = \vec{v}^{\,1} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

• $\lambda_2 = -3$: $\begin{pmatrix} -2-(-3) & -1 \\ 4 & -7-(-3) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & -1 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow$

$x_1 - x_2 = 0 \Rightarrow \quad x_1 = x_2 \quad \Rightarrow \vec{v}^{\,2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Since $\lambda_1 < \lambda_2 < 0 \Rightarrow \vec{x} = \vec{0}$ is a stable node.



Example 2 Draw the phase portrait of the linear equation

$$\vec{x}\,' = A\vec{x} = \begin{pmatrix} 1 & -3 \\ -3 & 1 \end{pmatrix}\vec{x}$$

Solution

$$\begin{vmatrix} 1-\lambda & -3 \\ -3 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - (-3)^2 = (1-\lambda)^2 - 9 = 1 - 2\lambda + \lambda^2 - 9 = \lambda^2 - 2\lambda - 8 = 0$$

$\lambda^2 - 2\lambda - 8 = 0$

$(\lambda - 4)(\lambda + 2) = 0$

$\lambda_1 = 4, \quad \lambda_2 = -2 \quad \Rightarrow$ Eigenvalues $\lambda_1 = 4$, $\lambda_2 = -2$ (saddle).

Eigenvectors: $\lambda_1 = 4$: $\begin{pmatrix} 1-4 & -3 \\ -3 & 1-4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -3 & -3 \\ -3 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$-3x_1 - 3x_2 = 0$

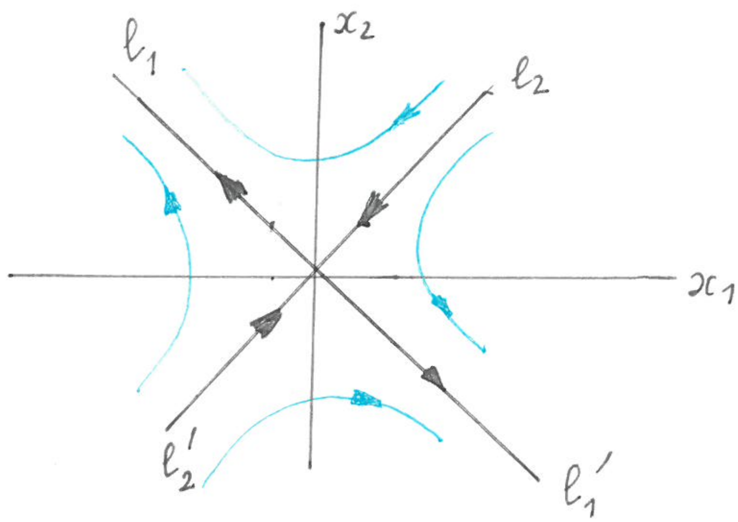
$x_1 + x_2 = 0$

$x_1 = -x_2 \quad \Rightarrow \quad \vec{v}^{\,1} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\lambda_2 = -2$: $\begin{pmatrix} 1-(-2) & -3 \\ -3 & 1-(-2) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 3 & -3 \\ -3 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow$

$3x_1 - 3x_2 = 0$

$x_1 = x_2 \quad \Rightarrow \quad \vec{v}^{\,2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$



Example 3 Draw the phase portrait of the linear equation

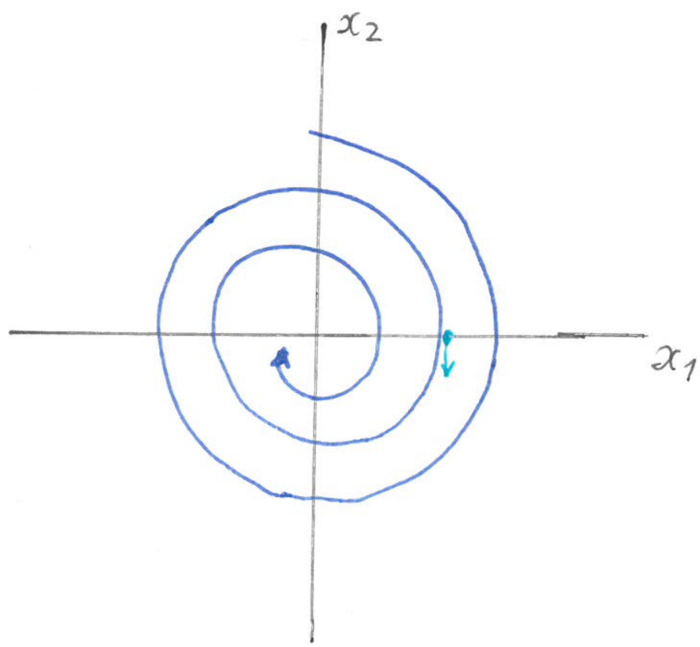
$$\vec{x}' = A\vec{x} = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix} \vec{x}.$$

Solution

$$\begin{vmatrix} -1-\lambda & 1 \\ -1 & -1-\lambda \end{vmatrix} = (-1-\lambda)^2 - (-1)(1) = (1+\lambda)^2 + 1 = \lambda^2 + 2\lambda + 2 = 0.$$

$$\lambda_{1,2} = \frac{-2 \pm \sqrt{4-4\cdot 1\cdot 2}}{2} = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm i2}{2} = -1 \pm i$$

Hence, $\vec{x} = \vec{0}$ is a stable focus of the system and every nontrivial trajectory spirals into the origin as $t \to \infty$. To determine the direction of rotation of the spiral, we observe that $x_2' = -x_1$ when $x_2 = 0$. Thus $x_2' < 0$ for $x_1 > 0$ and $x_2 = 0$. Consequently, all nontrivial trajectories spiral into the origin in the clockwise direction.



Practice Problems on Systems

Given $\begin{pmatrix} x' \\ y' \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$

1) Find the general solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$

2) Sketch some of the solution curves

(a) $A = \begin{pmatrix} 2 & 14 \\ 4 & 3 \end{pmatrix}$ (b) $A = \begin{pmatrix} 1 & 2 \\ -5 & -1 \end{pmatrix}$

(c) $A = \begin{pmatrix} -7 & -3 \\ 3 & -17 \end{pmatrix}$ (d) $A = \begin{pmatrix} 3 & \sqrt{2} \\ \sqrt{2} & -2 \end{pmatrix}$

e) $A = \begin{pmatrix} -1/2 & 1 \\ -1 & -1/2 \end{pmatrix}$