Math 4570 Matrix methods for DA and ML

He Wang

Homework 0- linear algebra prerequisite.

Here is a quick test about your linear algebra knowledge. (No submission needed. Solutions are provided)

We also need some basic knowledge about probability and partial derivatives in calculus 3. My teaching page on 2321, 2331, 3081 contains all the materials. See Canvas for the details.

1. Let $A = \begin{bmatrix} 1 & 3 & 6 & 2 \\ 1 & 5 & 8 & 6 \\ 3 & 6 & 15 & 0 \end{bmatrix}$.

(1). Determine the reduced row echelon form **rref** of the matrix A. (Write all details)

(2) Determine all solutions $\vec{x}$ of $A\vec{x} = \vec{0}$ in **parametric vector form.**

2. Let M be a matrix with row echelon form $\mathbf{ref}(M) = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 0 & 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

(i) What is the **rank** of the matrix M? Answer: ________

(ii) Is $M\vec{x} = \vec{b}$ has a solution for **every** $\vec{b} \in \mathbb{R}^3$?

(A) Yes. (B) No. (C) Can not be determined.

(iii) How many **free variables** in the solution set of $M\vec{x} = \vec{0}$ for $\vec{0} \in \mathbb{R}^3$? Answer: ________

3. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 2 & 4 & 7 \end{bmatrix}$. Answer the following 4 questions.

(i) Use Elementary Row Operations to find the **inverse** of matrix A. (Write down all your work.)

(ii) Find all solutions for $A\vec{x} = \vec{b}$ for $\vec{b} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ using the result in (i).

(iii) Does $A^3\vec{x} = \vec{b}$ have a unique solution for any $\vec{b} \in \mathbb{R}^3$? (Explain the reason.)

(iv) Find $(A^T)^{-1}$.

4. Let $A = \begin{bmatrix} 1 & 4 & 0 & 2 & 5 \\ 2 & 8 & 1 & -3 & 4 \\ 4 & 16 & 1 & 1 & 14 \end{bmatrix}$. Suppose $\mathbf{rref}(A) = \begin{bmatrix} 1 & 4 & 0 & 2 & 5 \\ 0 & 0 & 1 & -7 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Answer the following questions.

(1)(4 points) Find a **basis** for the **kernel** of A.

(2)(2 points) Find a **basis** for the **image** of A.

(3) (3 points) $\dim((\ker A)^{\perp}) =$ _____ $\dim(\mathrm{im}(A^T)) =$ _____ $\dim(\mathrm{im}(A)^{\perp}) =$ _____

(4)(1 point) Is $\mathrm{im}(A) = \mathrm{im}(\mathbf{rref}(A))$? Explain your reason.

5. Suppose $\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{x}_2 = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix}$ and $\vec{x}_3 = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}$ form a basis $\mathscr{B}$ for the vector space $\mathbb{R}^3$.

(1). (7 points) Using the *Gram-Schmidt process* to $\mathscr{B}$, find an **orthogonal** basis for the vector space $\mathbb{R}^3$.

(2). (3 points) Normalize the result in (2), find an **orthonormal** basis for the vector space $\mathbb{R}^3$.

(3). (5 points) Find the QR-factorization of $A = \begin{bmatrix} 1 & 3 & 4 \\ 1 & 3 & 2 \\ 1 & 0 & 0 \end{bmatrix}$.

6. Vectors $\vec{v}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} -1/2 \\ -1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$, $\vec{v}_4 = \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$ form an **orthonormal basis** for $\mathbb{R}^4$.
Using these vectors to answer the following questions.

(1) Let $V = \text{Span}\{\vec{v}_1, \vec{v}_2\}$ be a subspace of $\mathbb{R}^4$.

Write down an orthonormal basis for V and an orthonormal basis for $V^{\perp}$?

(2) Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be the transformation defined by the orthogonal projection onto $V = \text{Span}\{\vec{v}_1, \vec{v}_2\}$. Find the matrix of the linear transformation T.

(3). Suppose $\vec{y} \in \mathbb{R}^4$ satisfies $\vec{y} \cdot \vec{v}_1 = \sqrt{2}$, $\quad \vec{y} \cdot \vec{v}_2 = 2\sqrt{2}$, $\quad \vec{y} \cdot \vec{v}_3 = 4$, and $\vec{y} \cdot \vec{v}_4 = 2$.

(i) Find $\text{proj}_V \, \vec{y}$, the orthogonal projection of $\vec{y}$ onto $V = \text{Span}\{\vec{v}_1, \vec{v}_2\}$.

(ii) Find the vector $\vec{y}$.

7. Using the least-squares method, find the **line** $f(t) = c_0 + c_1 t$ of best fit through the points $(-1, 1)$, $(-2, 0)$, $(3, 5)$.

8. Let A be the matrix $A = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 0 & x & 9 & 2 \\ 0 & 2 & 3 & 2 \\ 4 & 3 & 5 & 1 \end{bmatrix}$. Answer the following questions.

(1) Compute the **determinant** of A. Write down all steps. The final answer is a formula with x.

(2) For which value of x is the matrix A **not** invertible?

9. (4 points) Suppose M and N are 3×3 matrices with determinants $\det M = 8$ and $\det N = -1$. Find the determinant of the matrix $2M^{-1}N^3M^T$.

Cont.

10. Suppose A is a 5×9 matrix such that $\text{rank}(A) = 3$. Answer the following questions:

$\dim(\ker A) =$ _______________; $\dim(\operatorname{im} A) =$ _______________;

$\dim((\ker A)^{\perp}) =$ _______________ $\dim((\operatorname{im} A^T)^{\perp}) =$ _______________;

11. Consider the matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix}$. Suppose $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ is an eigenvector of A with the corresponding eigenvalue $\lambda = 6$.

(1) Find a basis for the eigenspace E_λ with eigenvalue $\lambda = 0$.

(2) Diagonalize the matrix A. That is, find an **invertible** matrix P and a **diagonal** matrix D such that $A = PDP^{-1}$.

(3) (2 points) Determine whether the quadratic form $q(x_1, x_2, x_3) = \vec{x}^T A \vec{x}$ is positive semi-definite, or positive definite, or neither. (Reason)

12. Suppose A is a 3×3 matrix with eigenvalues $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 2$. Answer the following questions.

(1) What are the **eigenvalues** of $A^2 - 2A$?

(2) Is the matrix $A^2 - 2A$ invertible? (Reason)

(3) Is A diagonalizable? Answer: (a) Yes (b) No (c) Can not be determined

(4) Is $A^2 - 2A$ diagonalizable? Answer: (a) Yes (b) No (c) Can not be determined

13. Let $A = \begin{bmatrix} 4 & 2 & 3 \\ 2 & 1 & x \\ 0 & 0 & 5 \end{bmatrix}$.

(1) Find all **eigenvalues** for the matrix A. (Write all details of calculation.)

(2) For which values of x is the matrix A diagonalizable? (Hint: Consider geometric multiplicity.)

14. Consider the matrix $A = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 0 & -1 \end{bmatrix}$ and $B = A^T A = \begin{bmatrix} 2 & 3 & 0 \\ 3 & 9 & 3 \\ 0 & 3 & 2 \end{bmatrix}$.

(1) Verify that the vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 3 \\ -2 \\ 3 \end{bmatrix}$ are eigenvectors of $B = A^T A$. What are their corresponding eigenvalues?

(2) What are the singular values of A?

(3) Calculate vectors $A\vec{v}_1$, $A\vec{v}_2$ and $A\vec{v}_3$.

(4) Write a singular value decomposition $A = U\Sigma V^T$. (Explicitly write down all three matrices U, Σ, and V)

The End.