MATH 5120 HOMEWORK - 3 ABHILASHA JAIN

1. $L: R^3 \to R^3$

$\vec{u} = (1,0,1) \qquad \vec{v} = (1,1,-1)$

$\vec{u} - \vec{v} = (0, -1, +2)$

If $L(\vec{u} - \vec{u}) = 0$

$\therefore L(\vec{v} + \alpha(\vec{u} - \vec{v})) = L(\vec{u}) + \alpha(0) = b \quad \forall \alpha \in R$

$\Rightarrow \vec{v} + 2(\vec{u} - \vec{v}) = (1, -1, 3)$

$\vec{v} + 5(\vec{u} - \vec{v}) = (1, -4, 9)$

2.(a.) $[T(\vec{e}_1) \; T(\vec{e}_2) \; - - - \; T(\vec{e}_n)]$

Two standard vectors $e_1 = (1,0)$ & $e_2 = (0,1)$

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

(b.) $\vec{b} = \begin{bmatrix} x \\ y \end{bmatrix}$

Then $\vec{e}_1 \to \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ & $\vec{e}_2 \to \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\Rightarrow A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

(c.) $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) \to \begin{pmatrix} x + ry \\ y \end{pmatrix}$

Then $e_1 \to \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ & $e_2 \to \begin{bmatrix} r \\ 1 \end{bmatrix}$

$$\Rightarrow A = \begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix}$$

(d.) $L: R^n \to R^n$

$L = r\vec{x}$, $r > 0$

$L(e_i) = re_i$

$$A = \begin{bmatrix} r & 0 & 0 & \cdots & 0 \\ 0 & r & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & r \end{bmatrix}$$

$$L = A\vec{x} = r\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

3.(a) The eq^n of xz-plane, where $y = 0$

$\{(1,0,0), (0,0,1)\}$ is orthonormal basis to $y=0$

$$p(x,y,z) = \langle (x,y,z), (1,0,0) \rangle (1,0,0) + \langle (x,y,z), (0,0,1) \rangle (0,0,1)$$

$$= (x,0,0) + (0,0,z)$$

$$= (x,0,z)$$

$$\Rightarrow M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) A counter clockwise around the x-axis

$$(x', y', z') = (x, y(\cos 45) - z(\sin 45)\}, y(\sin 45) + z(\cos 45))$$

$$= \left(x, \frac{y-z}{\sqrt{2}}, \frac{y+z}{\sqrt{2}}\right)$$

$$\Rightarrow M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

(c.) $P(x', y', z') = \left(x, 0, \frac{y+z}{\sqrt{2}}\right)$

$$\Rightarrow M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} \\ 0 & 0 & 1/\sqrt{2} \end{bmatrix}$$

(d.) $R = R_z R_y R_x$

$$R_z = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_y = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is inclined when $\alpha = \sin^{-1}(1/\sqrt{3})$

A 120° w.r.t. origin results in $\theta = 120 + \sin^{-1}(1/\sqrt{3})$

$\therefore R = R_z R_y R_x$ where $\theta = 120^\circ + \sin^{-1}(1/\sqrt{3})$

4.(a.) $(\vec{x} - \vec{y})_n = \sum_{n=1}^{n} x_n y_{nm}$

Let $z \in R$

$$((n+z)y)_n = \sum_{m=1}^{N} (x+z)_m y_{n-m} = \sum_{m=1}^{N} x_m y_{n-m} + \sum_{m=1}^{N} z_m y_{n-m}$$

$$= (x * y)_m + (z * y)_m.$$

Let $\alpha \in R$

$$((\alpha n) x y)_n = \sum_{m=1}^{N} (\alpha x)_m y_{n-m} = \alpha \sum_{m=1}^{N} (x_m) y_{n-m} = \alpha (xy)_n$$

4(b) $L : \vec{x} \to \vec{x} * \vec{y}$

$F(e_k)_n = y_{n-k}$, $n = 0, 1 \ldots \ldots N$

$\Rightarrow F(e_0) = (y_0, y_1, \ldots y_N)$

$F(e_1) = (y_{N-1}, y_0, \ldots y_{N-2})$

$$A = \begin{bmatrix} y_0 & y_{N-1} & \cdots & y_1 \\ y_1 & y_0 & \cdots & y_2 \\ \vdots & \vdots & & \vdots \\ y_{N-1} & y_{N-2} & \cdots & y_0 \end{bmatrix}$$

5. $L(x) = x_1 L(e_1) + x_2 L(e_2) + x_3 L(e_3)$ $\rightarrow$ L is Linear.

$$L(x) = \begin{bmatrix} x_1 + 5x_2 + 7x_3 \\ 2x_1 + 2x_2 - 3x_3 \\ 3x_1 + x_2 + 9x_3 \end{bmatrix} = \begin{bmatrix} 1 & 5 & 7 \\ 2 & 2 & -3 \\ 3 & 1 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} 1 & 5 & 7 \\ 2 & 2 & -3 \\ 3 & 1 & 9 \end{bmatrix}$$

6.(a) $A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 1 & 5 \\ 3 & 1 & 7 \end{bmatrix}$ $\quad rref(A) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$\vec{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

6.(b) $\vec{y} = 1\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$

7(a.) $T(P) = 2p' + p''$

$\vec{0} \in T(P)$.

$T(x+y) = 2(x+y)' + (x+y)'' = (2x' + x'') + (2y' + y'') = T(x) + T(y)$

$T(\alpha x) = 2(\alpha x)' + (\alpha x)'' = \alpha(2x' + x'') = \alpha T(x)$

$\Rightarrow T(P)$ is a linear transformation.

7(b.) $\mathcal{B} = \{ t, t^2, t^3 \}$ $\qquad$ $\mathcal{b} = \{1, t, t^2\}$

$T(t) = 2(1) + 0 = 2 - 2(1) + 0(t) + 0(t^2)$

$\therefore \vec{T_1} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$

$T(t^2) = 4t + 2 = 2(1) + 4(t) + 0(t^2)$

$\therefore \vec{T_2} = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}$

$T(t^3) = 2(3t^2) + 6t = 0(1) + 6(t) + 6(t)$

$\therefore \vec{T_3} = \begin{bmatrix} 0 \\ 6 \\ 6 \end{bmatrix}$

$$\therefore [T]_{\mathcal{B}\mathcal{b}} = \begin{bmatrix} 2 & 2 & 0 \\ 0 & 4 & 6 \\ 0 & 0 & 6 \end{bmatrix}$$

7(c.) $$rref(T) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow \ker T = \{0\}$

T is an isomorphism.

8(a.) Let $S = [\vec{S}_1, \vec{S}_2 \; \dots \; \vec{S}_s]$, $B = [\vec{b}_1 \; \vec{b}_2 \; \dots \; \vec{b}_s]$,

$C = [\vec{v}_1 \; \vec{v}_2 \dots . \vec{v}_s]$

$C(\vec{S}_1) = \vec{b}_1 \Rightarrow \vec{S}_1 = C^{-1} \vec{b}_1$

$\therefore S_i = C^{-1} \vec{b}_i$

Then, $S = [C^{-1}\vec{b}_1, C^{-1}\vec{b}_2 \; \dots . \; C^{-1}\vec{b}_s]$

$S = C^{-1}B$

$\Rightarrow S = [id]_{\mathcal{B}\mathcal{b}} = C^{-1}B$

8(b.) $S = C^{-1}B \Rightarrow B = CS$

$\Rightarrow [\vec{b}_1, \vec{b}_2 \; \dots . \; \vec{b}_s] = [\vec{v}_1, \vec{v}_2 \; \dots \; \vec{v}_s] S.$

9.(a) $\begin{bmatrix}1\\2\\-3\end{bmatrix} = c_1\begin{bmatrix}0\\1\\-1\end{bmatrix} + c_2\begin{bmatrix}1\\0\\-1\end{bmatrix}$

$$\text{rref}\left(\left[\begin{array}{cc|c}0&1&1\\1&0&2\\-1&-1&-3\end{array}\right]\right)$$

$$\Rightarrow \vec{S}_1 = \begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix}$$

$$\text{rref}\left(\left[\begin{array}{cc|c}0&1&4\\1&0&-1\\-1&-1&-3\end{array}\right]\right)$$

$$\Rightarrow \vec{S}_2 = \begin{bmatrix}-1\\4\end{bmatrix}$$

$$\therefore S = \begin{bmatrix}2&-1\\1&4\end{bmatrix}$$

9(b.) $[\vec{v}_1\ \ \vec{v}_2]\,S = \begin{bmatrix}0&1\\1&0\\-1&-1\end{bmatrix}\begin{bmatrix}2&-1\\1&4\end{bmatrix} = \begin{bmatrix}1&4\\2&-1\\-3&-3\end{bmatrix} = [\vec{b}_1\ \ \vec{b}_2]$

10. $T(F) = xF'' - 2xF' - F$

$$\Rightarrow T(af(x)+g(x)) = x[af(x)+g(x)]'' - 2x[af(x)+g(x)]' - [af(x)+g(x)]$$

$$= axf''(x) + xg''(x) - 2axf'(x) - 2xg'(x) - af(x) + g(x)$$

$$= a[xf''(x) - 2xf'(x) - f(x)] + xg''(x) - 2xg'(x) + g(x)$$

$$= aT(f(x)) + T(g(x))$$

11. $$A_n = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \ddots & \vdots \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & \cdots & & 2 \end{bmatrix} \to \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ 0 & 3/2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \ddots & \vdots \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & \cdots & & 2 \end{bmatrix} \to$$

$$\to \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ 0 & 3/2 & -1 & \cdots & 0 \\ 0 & 0 & 4/3 & \ddots & \vdots \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & & & 2 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & 0 & 0 & \cdots \\ 0 & 3/2 & 0 & 0 & \\ 0 & 0 & 4/3 & 0 & \\ 0 & 0 & 0 & 5/4 & \\ \vdots & & & & \ddots \\ 0 & & & & \frac{n+1}{n} \end{bmatrix}$$

$$\Rightarrow \det(A_n) = 2\left(\frac{3}{2}\right)\left(\frac{4}{3}\right) \cdots \left(\frac{n+1}{n}\right) = (n+1)$$

12. $$A = \frac{1}{2}\left(\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} + \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} + \cdots + \begin{vmatrix} x_n & x_1 \\ y_n & y_1 \end{vmatrix} \right)$$

In counter clockwise order: $(3,1)\ (12,8)\ (10,7)\ (-1,-1)\ (-10,-8)\ (-8,-7)$

$$\Rightarrow A = \frac{1}{2}\left(\begin{vmatrix} 3 & 12 \\ 1 & 8 \end{vmatrix} + \begin{vmatrix} 12 & 10 \\ 8 & 7 \end{vmatrix} + \begin{vmatrix} 10 & -1 \\ 7 & -1 \end{vmatrix} + \begin{vmatrix} -1 & -10 \\ -1 & -8 \end{vmatrix} + \begin{vmatrix} -10 & -8 \\ -8 & -7 \end{vmatrix} + \begin{vmatrix} -8 & 3 \\ -7 & 1 \end{vmatrix} \right)$$

$$= \frac{1}{2}\left(12 + 4 - 3 - 2 + 6 + 13\right) = 15$$

13.(a.) Let $A = B = I$.

$\det(A+B) = 2^5 = 32$

$\det(A) + \det(B) = 1+1 = 2$

False.

(b.) $\det(A) = -1^n \det(A)$

where $n = 6$

$\det(-A) = \det(A)$

True.

(c.) If any row is a linear combination of any other row(s) $\Rightarrow \det(A) = 0$

False

(d.) If $\det(A) \neq 0$ then $A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$

then A is invertible

False.

(e.) If calculated determinant with the column $\times 9$ then take 9 as a common term for all $\det(B) = 9(\det(A))$

True

(f.) $\det(A \times A^T) = \det A \times \det A^T$

$= \det(A^T \times A)$

True

(g.) 2 and 4 are Linearly dependent.

False

(h.) $\det(A^T) = \det(A)$

$\det(A^T)\det(A^{-1}) = \det(A)\frac{1}{\det(A)} = 1$

True.

(i) $\det(4A) = 4^4 \det(A)$

False.

(j) $\det(4A) = 4^4 \det(A) = 256 \det(A)$

If $\det(A) = 0$

then $256 \det(A) = 4\det(A)$

Infinite matrices except [0]

True

(k.) $\det(AB) = \det(A) \times \det(B)$

$= \det(B) \times \det(A)$

$= \det(BA)$

True

14. $A^2 = -[id]$

$\Rightarrow [\det(A)]^2 = (-1)^3 \Rightarrow \det(A) = \pm i$

$\Rightarrow$ $\det(A)$ is complex and A has complex entries.

A has two eigen values $+i$ and $-i$

A is similar to $B = \begin{pmatrix} iI_m & 0 \\ 0 & -iI_n \end{pmatrix}$ st. $m+n=3$

15. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 8 & 9 \\ 2 & 4 & 6 & 10 \\ 1 & 5 & 10 & 9 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 4 & 6 & 8 \\ 2 & 5 & 8 & 9 \\ 2 & 4 & 6 & 10 \\ 1 & 5 & 10 & 9 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 3 & 7 & 5 \end{bmatrix} \rightarrow$

$\rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 1 \\ 0 & 3 & 7 & 5 \\ 0 & 0 & 0 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

$\Rightarrow A = -(1 \times 1 \times 1 \times 2) = -2$

16. $\begin{bmatrix} 0 & a_1 \\ a_2 & 0 \end{bmatrix}$

$E_{12} = \begin{bmatrix} a_2 & 0 \\ 0 & a_1 \end{bmatrix}$

$\det = \det(E_{12}) \det \begin{pmatrix} a_2 & 0 \\ 0 & a_1 \end{pmatrix} = -1 \prod_{i=1}^{2} a_i$

When n is even, $n/2$ row swaps

$\Rightarrow \det A = (-1)^{n/2} \prod a_i$

When n is odd, $(-1)^{(n-1)/2} \prod a_i$ swaps.

17. $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} \to \begin{bmatrix} 3 \\ 2 \\ 1 \\ 4 \\ 5 \end{bmatrix} \to \begin{bmatrix} 3 \\ 2 \\ 4 \\ 1 \\ 5 \end{bmatrix} \to \begin{bmatrix} 3 \\ 1 \\ 4 \\ 2 \\ 5 \end{bmatrix} \to \begin{bmatrix} 3 \\ 1 \\ 4 \\ 5 \\ 2 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 4 \\ 5 \\ 2 \end{bmatrix}$

18.(a) When $n=1$ then $\begin{bmatrix} 1 & 1 \\ a_0 & a_1 \end{bmatrix}$

$\det(A) = 1(a_1) - 1(a_0) = \prod_{0 \le j \le i \le n} (a_i - a_j)$

$\therefore$ Vandermonde holds base case

18 (b) Assume true for $(n-1)$

(i) Take transpose

(ii) Swap top and bottom rows

(iii) Result in n-degree polynomial of t

The coefficient for t^n is the $\det A_{n-1}$

If t is equal to any a then the last column is equal to a previous column $\Rightarrow \det = 0$.

$\therefore$ $t = kt^n + c_1 t^{n-1} + \ldots c_{n-1} t + c_n$ (the n roots are $a_0 \ldots a_{n-1}$)

$$= k\left(t^n + \frac{c_1}{k} t^{n-1} + \ldots \frac{c_{n-1}}{k} t + \frac{c_n}{k}\right)$$

$\Rightarrow k = 0 \leq j \leq i \leq n-1 \; (a_i - a_j)$ and when $t = a_n$ multiplied by k

the det comes as $0 \leq j \leq i \leq n \; (a_i - a_j)$