Math 5110 Applied Linear Algebra

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Homework 2.

1. Reading: [Gockenbach], 2.2-2.7. 3.1-3.7. The rest of the two chapters as extra reading.

2. Questions:

Question 1. Let V be a vector space over $\mathbb{R}$ and let $\vec{v} \in V$ be a nonzero vector. Is the subset $\{0, \vec{v}\}$ is a subspace of V? Prove your result.

Question 2. Determine whether or not the following set a subspace of $\mathbb{R}^2$. Prove your result.

(1) $S = \{\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 \mid x_1 x_2 = 0\}$.

(2) $T = \{\vec{x} \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 1\}$ the unit disc in $\mathbb{R}^2$.

Question 3. (1) Let $U_{3\times3}$ be the set of all 3×3 upper triangular matrices with real entries. Is $U_{3\times3}$ a subspace of $\mathbb{R}^{3\times3}$? Prove your result.

(2) Let $T_{3\times3}$ be the set of all 3×3 triangular matrices with real entries. Is $T_{3\times3}$ a subspace of $\mathbb{R}^{3\times3}$?

(3) Let W be the set of all polynomials in the form $\{t + at^2\}$ where a is any real number. Is W a subspace of P the vector space of all polynomials.

Question 4. (Allow to use Matlab for **rref**) Let S be the following subspace of $\mathbb{R}^4$:

$$S = \operatorname{Span}\left\{ \vec{b}_1 = \begin{bmatrix} -1 \\ -2 \\ 4 \\ -2 \end{bmatrix}, \ \vec{b}_2 = \begin{bmatrix} 0 \\ 1 \\ -5 \\ 4 \end{bmatrix} \right\}.$$

Determine if each vector belongs to S:

(1.) $\vec{v} = \begin{bmatrix} -1 \\ 0 \\ -6 \\ 6 \end{bmatrix}$; (2.) $\vec{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

Question 5. Let S be the following subspace of $\mathbb{R}^{2\times2}$:

$$S = \operatorname{Span}\left\{ \vec{b}_1 = \begin{bmatrix} -1 & -2 \\ 4 & -2 \end{bmatrix}, \ \vec{b}_2 = \begin{bmatrix} 0 & 1 \\ -5 & 4 \end{bmatrix} \right\}.$$

Determine if each vector belongs to S:

(1.) $\vec{v} = \begin{bmatrix} -1 & 0 \\ -6 & 6 \end{bmatrix}$; (2.) $\vec{w} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

Question 6. Show that $S = \operatorname{Span}\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\}$ and $T = \operatorname{Span}\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\}$ are the same subspace of $\mathbb{R}^3$.

Question 7. Suppose U and V are two subspaces of a vector space W.

(1) Is the union of two subspace $U \cup V$ a subspace?

(2) Is the intersection $U \cap V$ is a subspace?

Question 8. Prove or disprove the following statement: if U, V, W are subspaces of a vector space, then $(U + V) \cap W = (U \cap W) + (V \cap W)$.

Question 9. Let U_1, U_2, U_3 be subspaces of a vector space such that $U_i \cap U_j = 0$ for $i \neq j$. Is it true that the subspace $U_1 + U_2 + U_3$ equals $U_1 \oplus U_2 \oplus U_3$? Justify your answer.

Question 10. If V is a vector space with dimension n over the field $\mathbb{Z}_2$ of two elements, how many elements does V contain? Prove your result.

Question 11. If $\{\vec{u}, \vec{v}\}, \{\vec{v}, \vec{w}\}$ and $\{\vec{w}, \vec{u}\}$ are linearly independent subsets, is the subset $\{\vec{u}, \vec{v}, \vec{w}\}$ linearly independent?

Question 12. Show that $\{\begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \\ 13 \end{bmatrix}\} \in \mathbb{R}^3$ is linearly dependent by writing one of the vectors as a linear combination of the others.

Question 13. Let A be an $m \times n$ matrix with real entries, and suppose $n > m$. Prove the linear transformation defined by A is not injective. (That is, $A\vec{x} = \vec{0}$ has a nontrivial solution $x \in \mathbb{R}^n$.)

Question 14. Let $\vec{u}_1 = \begin{bmatrix} 1 \\ 4 \\ 0 \\ -5 \\ 1 \end{bmatrix}; \vec{u}_2 = \begin{bmatrix} 1 \\ 3 \\ 0 \\ -4 \\ 0 \end{bmatrix}; \vec{u}_3 = \begin{bmatrix} 0 \\ 4 \\ 1 \\ 1 \\ 4 \end{bmatrix}$ be vectors in $\mathbb{R}^5$.

(1) Show that $\vec{u}_1, \vec{u}_2, \vec{u}_3$ is linearly independent.

(2) Extend $\vec{u}_1, \vec{u}_2, \vec{u}_3$ to a basis for $\mathbb{R}^5$.

Question 15. Consider the linear subspaces U and W of $\mathbb{R}^4$ spanned by $\vec{u}_1 := \begin{bmatrix} -1 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \vec{u}_2 := \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \vec{u}_3 := \begin{bmatrix} 2 \\ 2 \\ 1 \\ -3 \end{bmatrix}$ and $\vec{w}_1 := \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{w}_2 := \begin{bmatrix} 1 \\ 3 \\ 1 \\ -1 \end{bmatrix}, \vec{w}_3 := \begin{bmatrix} 2 \\ -2 \\ -1 \\ -1 \end{bmatrix}, \vec{w}_4 := \begin{bmatrix} 2 \\ 2 \\ 1 \\ -1 \end{bmatrix}$ respectively.

Find the **dimensions** of the sum $U + W$, the intersection $U \cap W$, and the quotient spaces $\mathbb{R}^4/U$ and $\mathbb{R}^4/W$.

Question 16. Let V be a vector space over a field $\mathbb{K}$, and let $\vec{v}_1, ..., \vec{v}_n$ be n linearly dependent vectors of V such that any $n - 1$ of the vectors $\vec{v}_1, ..., \vec{v}_n$ are linearly independent. Show:

(a) There exist scalars $\alpha_1, \dots, \alpha_n$ in $\mathbb{K}$, **all** nonzero, such that $\sum_{j=1}^{n} \alpha_j \vec{v}_j = \vec{0}$.

(b) If $\alpha_1, \dots, \alpha_n$ and $\beta_1, \dots, \beta_n$ are two sets of nonzero scalars in $\mathbb{K}$ such both $\sum_{j=1}^{n} \alpha_j \vec{v}_j = 0$ and $\sum_{j=1}^{n} \beta_j \vec{v}_j = 0$ then there exists a nonzero scalar γ in K such that $\beta_j = \gamma \alpha_j$ for each $j = 1, ..., n$.

Question 17. Let M be the matrix $M = \begin{bmatrix} 3 & 3 & 2 & 8 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 3 & 5 \\ -2 & 4 & 6 & 8 \end{bmatrix}$, and let U and W be the subspaces of $\mathbb{R}^4$ generated by rows 1 and 2 of M, and by rows 3 and 4 of M respectively. Find the dimensions of the subspaces $U + W$ and $U \cap W$.

Question 18. Define polynomials $f_1 = 1 - 2x + x^3$, $f_2 = x + x^2 - x^3$ and also $g_1 = 2 + 2x - 4x^2 + x^3$, $g_2 = 1 - x + x^2$, $g_3 = 2 + 3x - x^2$. Let $U = \mathrm{Span}(f_1, f_2)$ and $V = \mathrm{Span}(g_1, g_2, g_3)$ be subspaces of $P_4(\mathbb{R})$, polynomials of degree smaller than 4. Find a basis for $U + V$ and a basis for $U \cap V$.