Northeastern University, Department of Mathematics

MATH 5110: Applied Linear Algebra and Matrix Analysis. (Fall 2021)

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§11 Least Squares and Data Fitting

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Review:

1. Inner Product Space (vector space V with an inner product $\langle\,,\,\rangle$)

$\Downarrow$ Geometry $\to$ norm, $\to$ angle

Example: 1. $\mathbb{R}^n$ (weighted) dot product

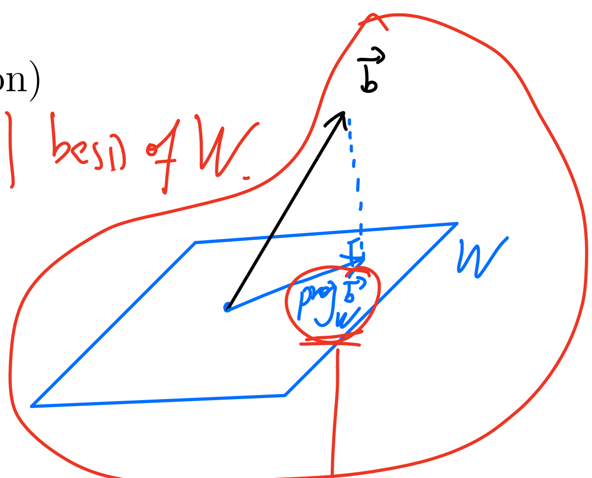
2. $C[a,b]$. $\langle f, g\rangle := \int_a^b f(x)g(x)\,dx$

2. Use orthogonal basis (to find orthogonal projection)

$B=\{\vec{v}_1 \cdots \vec{v}_s\}$ of $W \subset V$

$\{\vec{b}_1 \cdots \vec{b}_s\}$ basis of W.

$$\operatorname{proj}_W \vec{b} = \frac{\langle \vec{b}, \vec{v}_1\rangle}{\langle \vec{v}_1, \vec{v}_1\rangle}\vec{v}_1 + \cdots + \frac{\langle \vec{b}, \vec{v}_s\rangle}{\langle \vec{v}_s, \vec{v}_s\rangle}\vec{v}_s = c_1\vec{b}_1 + \cdots + c_s\vec{b}_s$$



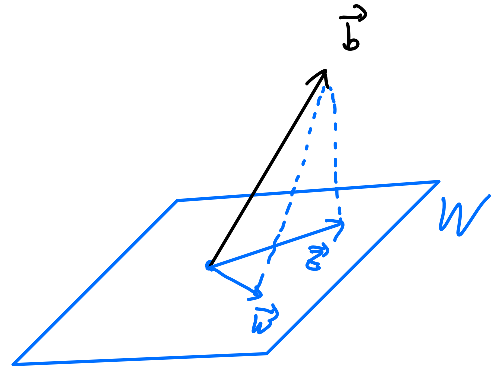
3. Find orthogonal basis (Gram-Schmidt process)

A basis $\{\vec{b}_1 \cdots \vec{b}_s\}$ of $W \longmapsto$ orthogonal basis $\{\vec{v}_1 \cdots \vec{v}_s\}$ of W.

Orthogonal: $(\vec{b} - \operatorname{proj}_W \vec{b}) \perp W$

projection: $\operatorname{proj}_W \vec{b} \in W$

General Least Squares Problem



Set up: • V : inner product space.

• $W \subset V$ subspace of V.

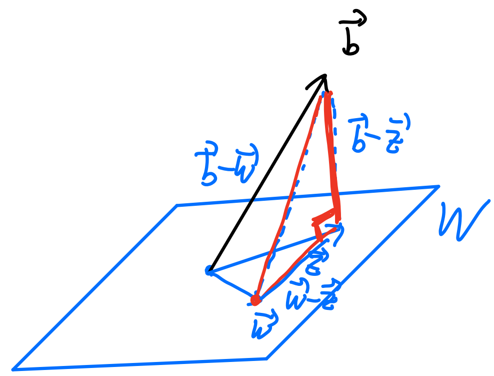
• $\vec b \in V$. $\vec b \notin W$

Question: What is the "closest" vector $\vec z \in W$ to $\vec b$?

$\Longleftrightarrow$ Find $\vec z \in W$ s.t. $\|\vec b - \vec z\| \le \|\vec b - \vec w\|$ for any $\vec w \in W$.

norm

Answer: inner product Thm: $\boxed{\vec z = \mathrm{proj}_W \vec b}$ is the answer



Proof: $(\vec b - \vec w) = (\vec b - \vec z) + (\vec z - \vec w)$

$\vec b^{\perp} = \vec b - \vec z \perp W$, $\vec z - \vec w \in W$ $\Longrightarrow$ $\vec b - \vec z \perp \vec z - \vec w$

By pythagorean Thm $\Longrightarrow$ $\|\vec b - \vec w\|^2 = \|\vec b - \vec z\|^2 + \|\vec z - \vec w\|^2$ ≥ 0

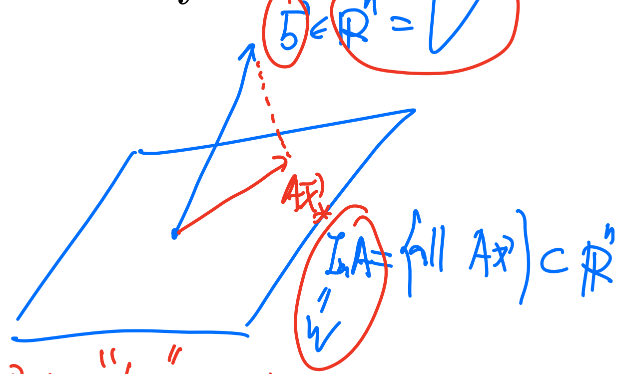
$\Longrightarrow \|\vec b - \vec w\|^2 \ge \|\vec b - \vec z\|^2$

Calculation: $B = \{\vec v_1 \cdots \vec v_s\}$ orthogonal of $W \subset V$

Method (1) $\mathrm{proj}_W \vec b := \dfrac{\langle \vec b, \vec v_1\rangle}{\langle \vec v_1, \vec v_1\rangle}\vec v_1 + \cdots + \dfrac{\langle \vec b, \vec v_s\rangle}{\langle \vec v_s, \vec v_s\rangle}\vec v_s$

Method (2) $\{\vec b_1 \dots \vec b_s\}$ basis of W. $\mathrm{proj}_W \vec b = c_1 \vec b_1 + \cdots + c_s \vec b_s$

$$\boxed{\langle c_1 \vec b_1 + \cdots + c_s \vec b_s - \vec b, \ \vec b_i \rangle = 0 \text{ for } i = 1, \dots, s}$$



Set up:

Let A be an $n \times m$ matrix.

Let $\vec{b} \in \mathbb{R}^n$.

$\vec{b} \notin \text{im}A$

Suppose $A\vec{x} \overset{\approx}{=} \vec{b}$ has no solution.

Q: Find $\vec{x}_*$ s.t. $A\vec{x}_*$ is "closest" to $\vec{b}$

Consider $\mathbb{R}^n$ with any inner product.

☆

[Least-Squares Problem/Solution for $A\vec{x} = \vec{b}$]

Problem: Find the vector(s) $\vec{x}_* \in \mathbb{R}^m$ such that for all $x \in \mathbb{R}^m$,

$$||\underbrace{A\vec{x}_*}_{\vec{z}} - \vec{b}|| \leq ||\underbrace{A\vec{x}}_{\vec{w}} - \vec{b}||$$

Solutions:

$$A\vec{x}_* = \text{proj}_{\text{im}A}\vec{b}$$

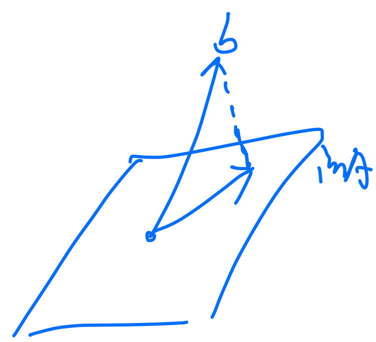
Example 1. Find the least-squares solutions for $A\vec{x} = \vec{b}$, where $A = \begin{bmatrix} -1 & 4 \\ 1 & 8 \\ -1 & 4 \end{bmatrix}$ and $\begin{bmatrix} 14 \\ -4 \\ 0 \end{bmatrix}$

dot product on $\mathbb{R}^3$

Step 0: Orthogonal basis $\{\vec{a}_1, \vec{a}_2\}$ since $\vec{a}_1 \cdot \vec{a}_2 = 0$

Step 1: $\vec{z} = \text{proj}_{\text{im}A}\vec{b} = \dfrac{\vec{b}\cdot\vec{a}_1}{\vec{a}_1\cdot\vec{a}_1}\vec{a}_1 + \dfrac{\vec{b}\cdot\vec{a}_2}{\vec{a}_2\cdot\vec{a}_2}\vec{a}_2$

$= -6\vec{a}_1 + \frac{1}{4}\vec{a}_2 = \begin{bmatrix} 7 \\ -4 \\ 7 \end{bmatrix}$



Step 2: Solve $A\vec{x} = \vec{z}$

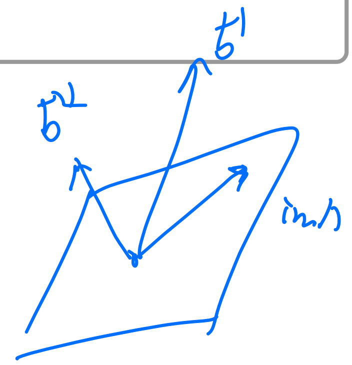
$$\left[\begin{array}{cc|c} -1 & 4 & 7 \\ 1 & 8 & -4 \\ -1 & 4 & 7 \end{array}\right] \to \cdots \to \text{rref} = \left[\begin{array}{cc|c} 1 & 0 & -6 \\ 0 & 1 & 1/4 \\ 0 & 0 & 0 \end{array}\right]$$

So $\vec{x}_* = \begin{bmatrix} -6 \\ 1/4 \end{bmatrix}$ is the least squares solution of $A\vec{x} = \vec{b}$.

In particular, if we consider $\boxed{\textbf{dot product}}$ on $\mathbb{R}^n$, we have the following formula.

Theorem 2. *(Normal Equation) The Least-Square solutions of $A\vec{x} = \vec{b}$ coincide with the solutions of of* ***normal equations***

$$(A^T A)\vec{x} = A^T\vec{b}.$$



- $\vec{x}_*$ is a least-square soln. of $A\vec{x}=\vec{b}$

$\iff A\vec{x}_* = \text{proj}_{\text{im}A}\vec{b}$

$\iff \vec{b}-A\vec{x}_* = \vec{b}^\perp \in \boxed{(\text{im}A)^\perp = \ker(A^T)}$

$\iff A^T(\vec{b}-A\vec{x}_*) = \vec{0}$

$\iff A^T\vec{b} - A^TA\vec{x}_* = \vec{0}$

$\iff A^TA\vec{x}_* = A^T\vec{b}$ easy

If $W=I_n$, then $\langle \vec{u},\vec{v}\rangle_W = \vec{u}^T\vec{v} = \vec{u}\cdot\vec{v}$

More generally, we can also consider $\boxed{\textbf{weighted dot product}}$ on $\mathbb{R}^n$,

$$\langle \vec{u}, \vec{v}\rangle_W := \vec{u}^T W \vec{v}$$

where W is a positive-definite symmetric matrix.

e.g. $W = \begin{bmatrix} c_1 & & & \\ & c_2 & & \\ & & \ddots & \\ & & & c_n \end{bmatrix}$

$c_i > 0$

- Find $\vec{x}_*$ s.t. $\|A\vec{x}_*-\vec{b}\|_W \leq \|A\vec{x}-\vec{b}\|_W$.

- Answer: $A\vec{x}_* = \text{proj}^W_{\text{im}A}\vec{b}$,

$(\text{im}A)^{\perp_W} = \ker\boxed{(WA)^T}$

$= \ker A^TW^T$

$= \ker A^TW$

$\iff A^TWA\vec{x}_* = A^TW\vec{b}$

Example 3. Find the least-squares solutions for $A\vec{x} = \vec{b}$, where $A = \begin{bmatrix} -1 & 4 \\ 1 & 8 \\ -1 & 4 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 14 \\ -4 \\ 0 \end{bmatrix}$

$$A^T A = \begin{bmatrix} -1 & 1 & -1 \\ 4 & 8 & 4 \end{bmatrix} \begin{bmatrix} -1 & 4 \\ 1 & 8 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 96 \end{bmatrix}$$

$$A^T \vec{b} = \begin{bmatrix} -1 & 1 & -1 \\ 4 & 8 & 4 \end{bmatrix} \begin{bmatrix} 14 \\ -4 \\ 0 \end{bmatrix} = \begin{bmatrix} -18 \\ 24 \end{bmatrix}$$

Solve the normal equation $A^T A \vec{x} = A^T \vec{b}$

$$\left[\begin{array}{cc|c} 3 & 0 & -18 \\ 0 & 96 & 24 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & -6 \\ 0 & 1 & 4 \end{array}\right]$$

So $\vec{x} = \begin{bmatrix} -6 \\ 4 \end{bmatrix}$ is the least-squares solution

(2) The image $\text{im}(A)$ is a plane in $\mathbb{R}^3$ passing the origin. Find the distance from the vector $\vec{b}$ (or the point $(14, -4, 0)$) to the plane $\text{im}(A)$.

The distance is given be the norm of $\vec{b}^{\perp} = \vec{b} - \text{proj}_{\text{im}(A)} \vec{b}$.

We know that $\text{proj}_{\text{im}(A)} \vec{b} = Ax_* = \begin{bmatrix} -1 & 4 \\ 1 & 8 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} -6 \\ 1/4 \end{bmatrix} = \begin{bmatrix} 7 \\ -4 \\ 7 \end{bmatrix}$.

So, $\vec{b}^{\perp} = \begin{bmatrix} 14 \\ -4 \\ 0 \end{bmatrix} - \begin{bmatrix} 7 \\ -4 \\ 7 \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \\ -7 \end{bmatrix}$. So the distance is $||\vec{b}^{\perp}|| = 7\sqrt{2}$.

Example 4. Find the least-squares solutions for the system $A\vec{x} = \vec{b}$, where $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 4 \end{bmatrix}$

Step 1. Construct the normal equation $A^TA\vec{x} = A^T\vec{b}$

$$A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

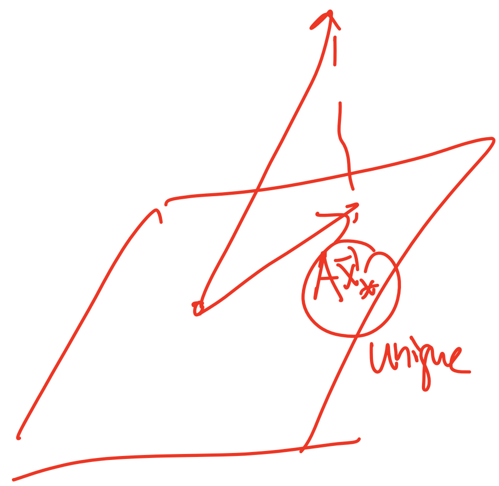
$$A^T\vec{b} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 10 \\ 4 \\ 6 \end{bmatrix}$$

Solve the normal equation

$$\left[\begin{array}{ccc|c} 4 & 2 & 2 & 10 \\ 2 & 2 & 0 & 4 \\ 2 & 0 & 2 & 6 \end{array}\right] \to \cdots \to \text{rref} = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 = 3 - x_3$
$x_2 = -1 + x_3$
x_3 free

$$\vec{x}_* = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 - x_3 \\ -1 + x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$



$$A\vec{x}_* = A\begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}$$

A technical property:

Proposition 5. *Let A be an $n \times m$ matrix.*

$$\ker(A) = \ker(A^T A)$$

$\operatorname{rank} A = \operatorname{rank}(A^T A)$

$\underset{m\times n}{A^T}\,\underset{n\times m}{A}$ ($m\times m$)

A^TA is invertible $\Leftrightarrow$ $\operatorname{rank} A^TA = m$

Corollary 6. *If* rank $A = m$*, the normal equation* $(A^T A)\vec{x} = A^T\vec{b}$ *has a unique solution:*

$$\boxed{\vec{x} = (A^T A)^{-1} A^T \vec{b}}$$

QR factorization method Suppose A is $n \times m$ matrix with full column rank. Solve the least squares solution using QR factorization $A = QR$ where Q is an orthogonal matrix $n \times m$ and R is an $m \times m$ upper triangular matrix with rank m.

rank $A = m$

$Q^TQ = I_m$

$\vec{x} = R^{-1}Q^T\vec{b}$

$\operatorname{proj}_{\operatorname{Im} A}\vec{b} = QQ^T\vec{b}$

$A\vec{x}_* = \cancel{Q}Q^T\vec{b}$

$\cancel{Q}R\vec{x}_* =$

3. **Data Fitting**

Problem: Fitting a function of a certain type of data. We use the following three example to illustrate this application.

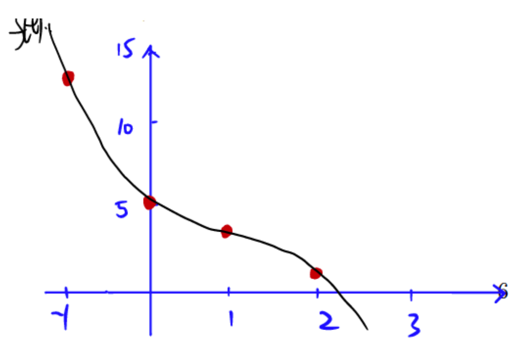
Example 7. Find a cubic polynomial $f(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3$ whose graph passes through the points $(0, 5)$, $(1, 3)$, $(-1, 13)$, $(2, 1)$

Solution:

We need to solve the linear system $\begin{cases} c_0 & = 5 \\ c_0 + c_1 + c_2 + c_3 & = 3 \\ c_0 - c_1 + c_2 - c_3 & = 13 \\ c_0 + 2c_1 + 4c_2 + 8c_3 & = 1 \end{cases}$

$$[A|\vec{b}] = \begin{bmatrix} 1 & 0 & 0 & 0 & 5 \\ 1 & 1 & 1 & 1 & 3 \\ 1 & -1 & 1 & -1 & 13 \\ 1 & 2 & 4 & 8 & 1 \end{bmatrix} \to \cdots \to \mathbf{rref}[A|\vec{b}] = \begin{bmatrix} 1 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

So, the linear system has the unique solution $\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \\ 3 \\ -1 \end{bmatrix}$ So, the cubic polynomial is $f(t) = 5 - 4t + 3t^2 - t^3$.



perfect fit, but calculation is hard.

Example 8. Fit a quadratic function $g(t) = c_0 + c_1 t + c_2 t^2$ to the four data points $(0, 5)$, $(1, 3)$, $(-1, 13)$, $(2, 1)$

t_0	t	t^2	y
1	0	$(0)^2$	5
1	1	$(1)^2$	3
1	-1	$(-1)^2$	13
1	2	$(2)^2$	1

We need to solve the linear system

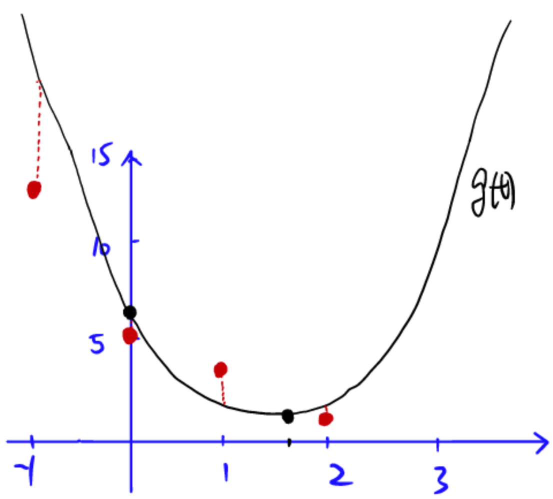
$$\begin{cases} c_0 & = 5 \\ c_0 + c_1 + c_2 & = 3 \\ c_0 - c_1 + c_2 & = 13 \\ c_0 + 2c_1 + 4c_2 & = 1 \end{cases}$$

As matrix equation $A\vec{x} = \vec{b}$, where $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 2 & 4 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 5 \\ 3 \\ 13 \\ 1 \end{bmatrix}$

$A^TA = \begin{bmatrix} 4 & 2 & 6 \\ 2 & 6 & 8 \\ 6 & 8 & 18 \end{bmatrix}$ $\quad A^T\vec{b} = \begin{bmatrix} 22 \\ -8 \\ 20 \end{bmatrix}$

Solve the normal equation $A^TA\vec{x} = A^T\vec{b}$ $\quad \vec{x} = \begin{bmatrix} 5.9 \\ -5.3 \\ 1.5 \end{bmatrix} = \vec{c}^*$

So, the quadratic function $g(t) = 5.9 - 5.3t + 1.5t^2$



$A\vec{c}^* = \begin{bmatrix} g(a_1) \\ g(a_2) \\ g(a_3) \\ g(a_4) \end{bmatrix}$ $\quad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$

$$\|\vec{b} - A\vec{c}^*\|^2 = (b_1 - g(a_1))^2 + (b_2 - g(a_2))^2 + (b_3 - g(a_3))^2 + (b_4 - g(a_4))^2$$

The sum of the vertical distances between graph and data points is minimal.

Example 9. Fit a linear function $h(t) = c_0 + c_1 t$ to the four data points $(0, 5)$, $(1, 3)$, $(-1, 13)$, $(2, 1)$

t_0	t_1	y
1	0	5
1	1	3
1	-1	13
1	2	1

$\vec{y}$

We need to solve the linear system

$$\vec{x} = \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} \qquad \begin{cases} c_0 & = 5 \\ c_0 + c_1 & = 3 \\ c_0 - c_1 & = 13 \\ c_0 + 2c_1 & = 1 \end{cases}$$

As matrix equation $A\vec{x} = \vec{b}$, where $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & -1 \\ 1 & 2 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 5 \\ 3 \\ 13 \\ 1 \end{bmatrix}$

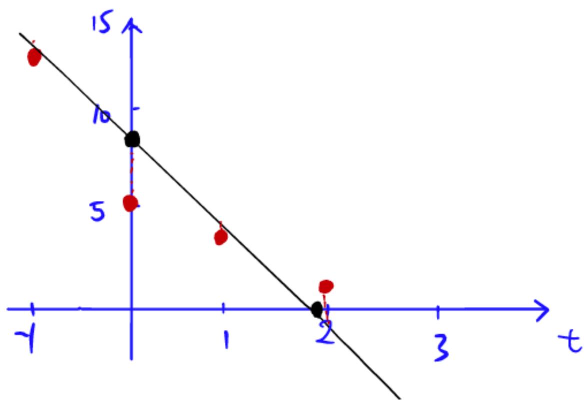
$$\left\| \begin{bmatrix} h(0) \\ h(1) \\ h(-1) \\ h(2) \end{bmatrix} - \begin{bmatrix} 5 \\ 3 \\ 13 \\ 1 \end{bmatrix} \right\|$$

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix}$$

$$A^T \vec{b} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 13 \\ 1 \end{bmatrix} = \begin{bmatrix} 22 \\ -8 \end{bmatrix}$$

Solve the normal equation $A^T A \vec{x} = A^T \vec{b}$. $\quad \vec{x} = \begin{bmatrix} 7.4 \\ -3.8 \end{bmatrix}$

So the linear function is $h(t) = 7.4 - 3.8t$

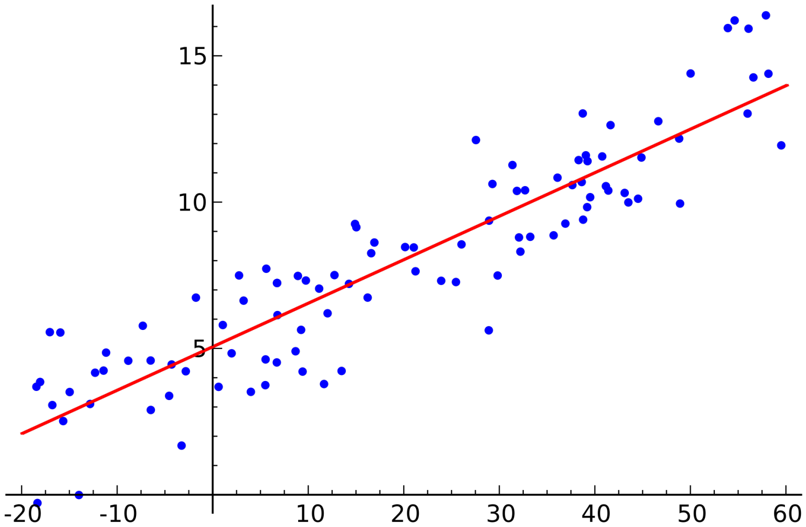


Remark: More generally, we can consider n-points $(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)$.

- Find a linear function $h(t) = c_0 + c_1 t$ fits the data by the least squares.

More generally, the following question is very standard in statistics.

Example 10. Consider the data with n points (a_1, b_1), (a_2, b_2), $\ldots$, (a_n, b_n). Find a linear function $h(t) = c_0 + c_1 t$ fits the data by the least squares. (Suppose $a_1 \neq a_2$)



$\operatorname{rank} A = 2$ $\quad \vec{x} = (A^TA)^{-1}A^T\vec{b}$

$A^TA\vec{x} = A^T\vec{b}$

$A\vec{x} = \operatorname{proj}_{\operatorname{im}A}\vec{b}$

We need to solve the least-squares problem for $A\vec{x} = \vec{b}$, for $A = \begin{bmatrix} 1 & a_1 \\ 1 & a_2 \\ \vdots & \\ 1 & a_n \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$

$$A^TA = \begin{bmatrix} 1 & a_1 \\ 1 & a_2 \\ \vdots & \\ 1 & a_n \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ a_1 & a_2 & \cdots & a_n \end{bmatrix} = \begin{bmatrix} n & \sum_{i=1}^n a_i \\ \sum_{i=1}^n a_i & \sum_{i=1}^n a_i^2 \end{bmatrix}$$

$$A^Tb = \begin{bmatrix} 1 & a_1 \\ 1 & a_2 \\ \vdots & \\ 1 & a_n \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^n b_i \\ \sum_{i=1}^n a_i a_i \end{bmatrix}$$

Since $a_1 \neq a_2$, we know that rank $A = 2$.
The normal equation $A^TA\vec{x} = A^T\vec{b}$ has a unique solution

$$\vec{x}_* = (A^TA)^{-1}A^T\vec{b} = \frac{1}{n\sum_{i=1}^n a_i^2 - (\sum_{i=1}^n a_i)^2} \begin{bmatrix} \sum_{i=1}^n a_i^2 & -\sum_{i=1}^n a_i \\ -\sum_{i=1}^n a_i & n \end{bmatrix} \begin{bmatrix} \sum_{i=1}^n b_i \\ \sum_{i=1}^n a_i a_i \end{bmatrix}$$

$$= \frac{1}{n\sum_{i=1}^n a_i^2 - (\sum_{i=1}^n a_i)^2} \begin{bmatrix} (\sum_{i=1}^n a_i^2)(\sum_{i=1}^n b_i) - (\sum_{i=1}^n a_i)(\sum_{i=1}^n a_i a_i) \\ -(\sum_{i=1}^n a_i)(\sum_{i=1}^n b_i) + n\sum_{i=1}^n a_i a_i \end{bmatrix}$$

Example 11. Consider the data with m inputs and 1 output:

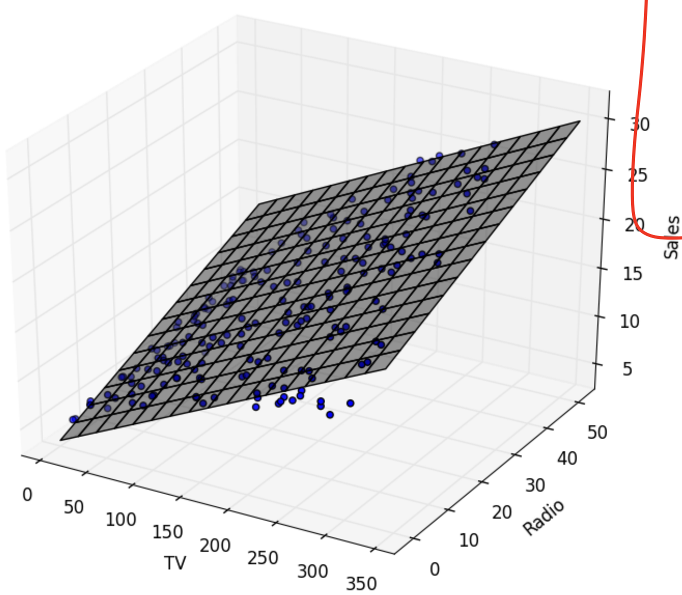
$$(a_{11}, a_{12}, ..., a_{1m}, b_1),\ (a_{21}, a_{22}, ..., a_{2m}, b_2),\ \ldots,\ (a_{n1}, a_{n2}, ..., a_{nm}, b_n).$$

Find a linear function $h(t_1, t_2, ..., t_n) = \boxed{c_0} + c_1t_1 + c_2t_2 + \cdots + c_nt_n$ fits the data by the least squares.

$1 \quad a_{11} \quad a_{12} \quad \cdots \quad b_1$

$A \qquad b$

For example, when $m = 2$,



We need to solve the least-squares problem for $A\vec{x} = \vec{b}$, for $A = \begin{bmatrix} 1 & a_{11} & a_{12} & \ldots & a_{1m} \\ 1 & a_{21} & a_{22} & \ldots & a_{2m} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & a_{n1} & a_{n2} & \ldots & a_{nm} \end{bmatrix}$

and $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$

Example 12. Consider the data with m inputs and s outputs:

$$(a_{11}, a_{12}, ..., a_{1m}, b_{11}, ..., b_{1s}),\ (a_{21}, a_{22}, ..., a_{2m}, b_{21}, ..., b_{2s}),\ \ldots,\ (a_{n1}, a_{n2}, ..., a_{nm}, b_{n1}, ..., b_{ns}).$$

Find a linear function $H(\vec{t}) = \vec{c}_0 + C\vec{t}$ fits the data by the least squares.

4. Best Approximation for Functions

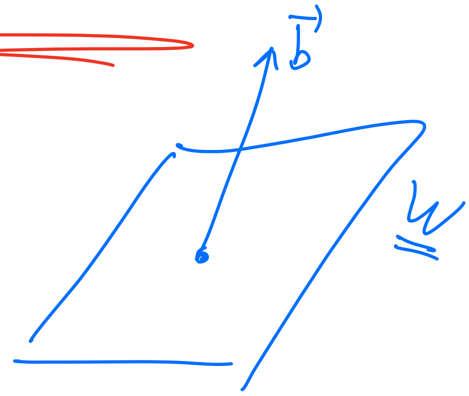
Set up:

Let V be the vector space of continuous functions. e.g. $f(x) = e^x \in V$

Consider the inner product on V: $\langle f(x), g(x) \rangle = \int_a^b f(x) \cdot g(x) dx$

$= \{a_0 + a_1 x + \cdots + a_n x^n\}$

Let W be the subspace of polynomials of degree $\leq n$.



Consider a function $\vec{b} = f(x) \in V$. (e.g., $f(x) = e^x) \notin W$

Question:

Find the "best" $=\vec{z}$ degree n polynomial approximation of $\vec{b} = f(x)$.

$\|\vec{b} - \vec{z}\| \leq \|\vec{b} - \vec{w}\|$ for any $\vec{w} \in W$.

<u>Answer</u> $\vec{z} = \text{proj}_W \vec{b}$

Example: