MATH 5110 HOMEWORK - 4 ABHILASHA JAIN

1. (f.) $ABx = \lambda x$

let $Bx = y$ $(\lambda \neq 0, \Rightarrow y \neq 0)$ — ①

Consider

$$BAy = BABx$$

$$= B\lambda x$$

$$= \lambda(Bx)$$

$\therefore$ $BAy = \lambda y$ — ②

from ① and ②

λ is a non-zero eigenvalue of BA.

Since all λ are same,

algebraic multiplicities are also same.

(2) let

$\alpha_1 \pm 0, \alpha_2 \pm 0 \ldots \alpha_p \pm 0$ are p-different non-zero eigen values of AB and BA.

$\alpha_0 = 0$ has algebraic multiplicity "k" for A.

$\therefore$ Characteristic eq^n of AB,

$\lambda^k \cdot (\lambda - \alpha_1)^{n_1} \cdot (\lambda - \alpha_2)^{n_2} \cdot \ldots \ldots \cdot (\lambda - \alpha_p)^{n_p}$ — ③

characteristic eq^n. of BA

$$\lambda^{x} \cdot (\lambda-\alpha_1)^{n_1} \cdot (\lambda-\alpha_2)^{n_2} \cdots (\lambda-\alpha_p)^{n_p} \quad — (4)$$

from (3) & (4)

$$k + n_1 + n_2 + \cdots n_p = m \quad — (5)$$
$$x + n_1 + n_2 + \cdots n_p = n \quad — (6)$$

$$(5) - (6) \Rightarrow (x - k) = n - m$$

$$\therefore \quad x = n - m + k$$

2\. (i.) Characteristic polynomial of B is

$$P(\lambda) = |B - \lambda I| \qquad B = \begin{bmatrix} 0 & -c_0 \\ 1 & -c_1 \end{bmatrix}$$

$$= \begin{bmatrix} -\lambda & -c_0 \\ 1 & -c_1 - \lambda \end{bmatrix}$$

$$= -\lambda(-c_1 - \lambda) - (-c_0).1$$
$$= \lambda c_1 + \lambda^2 + c_0$$
$$= \lambda^2 + \lambda c_1 + c_0$$

characteristic equation of C is

$$P(\lambda) = |C - \lambda I| \qquad \begin{bmatrix} 0 & 0 & -c_0 \\ 1 & 0 & -c_1 \\ 0 & 1 & -c_2 \end{bmatrix}$$

$$= \begin{vmatrix} -\lambda & 0 & -c_0 \\ 1 & -\lambda & -c_1 \\ 0 & 1 & -c_2-\lambda \end{vmatrix}$$

$$= -\lambda\left[(-\lambda)(-c_2-\lambda) - 1.(-c_1)\right] + (-c_0)$$

$$\left[1 - (-\lambda).0\right]$$

$$= -\lambda\left[\lambda c_2 + \lambda^2 + c_1\right] - c_0$$

$$\Rightarrow -\lambda^3 - \lambda^2 c_2 - \lambda c_1 - c_0$$

$$\Rightarrow \lambda^3 + \lambda^2 c_2 + \lambda c_1 + c_0.$$

(2) $f(t) = t^n + c_{n-1} t^{n-1} + \ldots\ldots + c_1 t + c_0.$

for $n = 2$ $\quad f(t) = t^2 + c_1 t + c_0$

$$= \begin{vmatrix} 0 & -c_0 \\ 1 & -c_1 \end{vmatrix} = B_2.$$

for $n=3 \Rightarrow f(t) = t^3 + c_2 t^2 + c_1 t + c_0$

$$= \begin{bmatrix} 0 & 0 & -c_0 \\ 1 & 0 & -c_1 \\ 0 & 1 & -c_2 \end{bmatrix}$$

Extending the idea
it is clear that,
every monic of the form

$$f(t) = t^n + c_{n-1} t^{n-1} + \dots c_1 t + c_0$$

is a characteristic polynomial of some matrix B.

3. Let $C = AB$

$$C^2 - tr(C) \cdot C + det(C) I_2 = 0 \quad -①$$

Given $(AB)^2 = 0 \quad - ②$

$det(C^2) = 0$

$\Rightarrow det\, C = 0. \quad — ③$

$\therefore$ from ①, ② & ③

$$tr(C) \cdot C = 0.$$

$$\therefore \quad tr(C) = 0 \quad or \quad C = 0$$

$$\Rightarrow tr(C) = 0.$$

$$\det(BA) = \det(AB) = \det(C) = 0 \quad — (4)$$

$$tr(BA) = tr(AB) = tr(C) = 0 \quad — (5)$$

$$\Rightarrow (BA)^2 - tr(BA) \times BA + \det(BA) I_2 = 0.$$

$$\Rightarrow (BA)^2 = 0.$$

Hence Proved.

4. (a) Let $\lambda_1, \lambda_2, \ldots \lambda_s$ be non-zero eigenvalues of "A" and $n_1, n_2 \ldots n_s$ be algebraic multiplicates (corresponding)

Given, $tr(A^k) = 0 \quad k = 1, 2, 3$

$$\begin{cases} n_1\lambda_1 + n_2\lambda_2 + n_s\lambda_s = 0 \\ n_1\lambda_1^2 + n_2\lambda_2^2 + n_s\lambda_s^2 = 0 \\ \vdots \\ n_1\lambda_1^s + n_2\lambda_2^s + n_s\lambda_s^s = 0 \end{cases}$$

$$\Rightarrow \begin{bmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_s \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_s^2 \\ \vdots & \vdots & & \vdots \\ \lambda_1^s & \lambda_s^s & \cdots & \lambda_s^s \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ \\ n_s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Since

$$\det \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_s \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_s^2 \\ \vdots & \vdots & & \vdots \\ \lambda_1^s & \lambda_2^s & \cdots & \lambda_s^s \end{pmatrix} = \lambda_1 \cdots \lambda_s$$

$$\Rightarrow \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_s \\ \vdots & & & \\ \lambda_1^{s-1} & \lambda_2^{s-2} & \cdots & \lambda_s^{s-1} \end{pmatrix} \neq 0$$

$$\begin{pmatrix} n_1 \\ n_2 \\ \vdots \\ n_s \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$ is the only possible eigenvalue

(b.) let $f(t) = t^n$ & $f(\lambda) = 0$

$\Rightarrow \lambda = 0$

$\therefore$ All eigen values are $\lambda_i = 0$

(multiplicity $= n$)

A annhailates $f(t)$

$\Rightarrow A^n = 0$

5. $A = \begin{bmatrix} -3 & 4 & 4 \\ -5 & 9 & 5 \\ -7 & 4 & 8 \end{bmatrix}$

$Tr(A) = -3 + 9 + 8 = 14$

$Det(A) = -3(9 \times 8 - 5 \times 4) - 4(-5 \times 8 - 5 \times (-7))$
$+ 4(-5 \times 4 - 9 \times (-7))$

$= 36$

$|A - \lambda I| = 0$.

$\Rightarrow \begin{vmatrix} -3-\lambda & 4 & 4 \\ -5 & 9-\lambda & 5 \\ -7 & 4 & 8-\lambda \end{vmatrix} = 0$

$\Rightarrow (-3-\lambda)\left[(9-\lambda)(8-\lambda) - 5 \times 4\right] - 4\left[-5(8-\lambda) - 5 \times (-7)\right]$

$+ 4\left[-5 \times 4 - (9-\lambda)(-7)\right] = 0$

$$\Rightarrow -\lambda^3 + 14\lambda^2 - 49\lambda + 36 = 0$$

$$\Rightarrow \lambda^3 - 14\lambda^2 + 49\lambda - 36 = 0.$$

Eigen values.

$$\Rightarrow (\lambda-1)(\lambda^2 - 13\lambda + 36) = 0$$

$$\Rightarrow (\lambda-1)(\lambda^2 - 9\lambda - 4\lambda + 36) = 0$$

$$\Rightarrow (\lambda-1)(\lambda(\lambda-9) - 4(\lambda-9)) = 0$$

$$\Rightarrow (\lambda-1)(\lambda-4)(\lambda-9) = 0$$

$$\Rightarrow \lambda = 1, 4, 9.$$

Eigen vectors.

$$(A - \lambda_1 I)X_1 = 0$$

$$\Rightarrow \begin{bmatrix} -4 & 4 & 4 \\ -5 & 8 & 5 \\ -7 & 4 & 7 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\Rightarrow \quad -4x_1 + 4x_2 + 4x_3 = 0 \quad - (1)$$
$$-5x_1 + 8x_2 + 5x_3 = 0 \quad - (2)$$
$$-7x_1 + 4x_2 + 7x_3 = 0 \quad - (3)$$

$$(1) \Rightarrow \quad x_1 = x_2 + x_3$$

$$(2) \Rightarrow \quad -5(x_2 + x_3) + 8x_2 + 5x_3 = 0.$$
$$\Rightarrow \quad -5x_2 - \cancel{5x_3} + 8x_2 + \cancel{5x_3} = 0.$$
$$3x_2 = 0$$
$$x_2 = 0.$$

$$(3) \Rightarrow \quad x_1 = x_3$$

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$(A - \lambda_2 I) X_2 = 0.$$

$$\Rightarrow \begin{bmatrix} -7 & 4 & 4 \\ -5 & 5 & 5 \\ -7 & 4 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0.$$

$$\Rightarrow \quad -7x_1 + 4x_2 + 4x_3 = 0. \quad — (4)$$
$$-5x_1 + 5x_2 + 5x_3 = 0 \quad — (5)$$

$$(5) \Rightarrow \quad x_1 = x_2 + x_3$$

$$-7(x_2 + x_3) + 4(x_2 + x_3) = 0.$$

$$-3(x_2 + x_3) = 0$$

$$x_2 + x_3 = 0.$$

$$x_2 = -x_3$$

$$\Rightarrow x_1 = x_2 + x_3 = -x_3 + x_3 = 0.$$

$$X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$(A - \lambda_3 I) X_3 = 0$$

$$\begin{bmatrix} -12 & 4 & 4 \\ -5 & 0 & 5 \\ -7 & 4 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -12x_1 + 4x_2 + 4x_3 = 0 \quad \text{—} (6)$$

$$-5x_1 + 5x_3 = 0 \quad \text{—} (7)$$

$$-7x_1 + 4x_2 - x_3 = 0 \quad \text{—} (8)$$

$$(7) \Rightarrow x_1 = x_3$$

$$(6) \Rightarrow x_2 + x_3 = 3x_1$$

$$2x_2 = 2x_1 \quad \Rightarrow \cancel{x_2}$$

$$X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

1. ∴

$$P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & -1 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

and it satisfies $A = PDP^{-1}$.

2. $N = P^{-1}MP \Rightarrow N^2 = (P^{-1}MP)^2$

$$= P^{-1}M^2P$$
$$= P^{-1}AP$$
$$= D$$

$\therefore ND = N.N^2 = N^3 = N^2.N = DN$

Hence D and N commutes.

3. ∵ $N^2 = D$ and N commutes with D, we have that

N is necessarily diagonal.

4. $N = \begin{bmatrix} \pm 3 & 0 & 0 \\ 0 & \pm 2 & 0 \\ 0 & 0 & \pm 1 \end{bmatrix}$ Hence there are 8 possible values of N.

$$M = PNP^{-1}$$

$$= P\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} P^{-1}$$

$$= \begin{bmatrix} 0 & 1 & 1 \\ -1 & 3 & 1 \\ -2 & 1 & 3 \end{bmatrix}$$

$\therefore$ There are 8 different M such that $M^2 = A$.