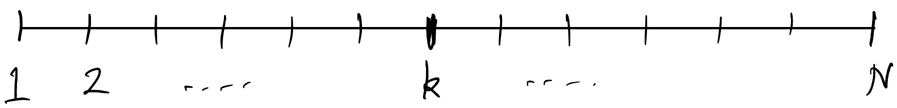
MTH 7241: Fall 2022

More Practice Problems

1). The lifetime of a machine (in days) is an integer-valued random variable T. Assume that $1 \le T \le N$, and that all N values of T are equally likely. Given that the machine is working after k days, what is the expected value of its subsequent lifetime?
[Hint: let A_k be the event that the lifetime of the machine is at least k days. Find $E[T|A_k]$.]

Answer:



$$A_k = \{ T > k \}$$

$$P(A_k) = P(T > k) = \frac{N-k}{N} \quad \text{(uniform pdf).}$$

$$\Rightarrow \text{need} \quad E[T \mid A_k] = \frac{E[T \, \mathbb{1}_{A_k}]}{P(A_k)}$$

where $\mathbb{1}_{A_k} = \begin{cases} 1 & \text{if } T > k \\ 0 & \text{if } T \le k. \end{cases}$

$$\Rightarrow E[T \mathbb{1}_{A_k}] = \sum_{j=k+1}^{N} j \frac{1}{N} = \frac{1}{N} \frac{(N-k)(N+k+1)}{2}$$

$$\Rightarrow E[T \mid A_k] = \frac{N+k+1}{2}.$$

2). Here is a hypothetical situation aboard the Space Station. Its navigation system will need two working computers at all times. NASA plans to put three computers on board, so that there will be a spare in case of failure. If a computer fails, it will be immediately replaced by the extra one.

NASA asks you to estimate the expected time until the navigation system fails. You are told to assume that the lifetimes of the computers are independent exponential random variables, each with mean 1 year. Let T be the time until failure. Find the expected length of time until the system fails, that is find $\mathrm{E}[T]$.

Answer: 1 *year*

T_1, T_2, T_3 are lifetimes of computers

IID exponential, mean $= 1$.

Say we T_1, T_2 fail first.

$\Rightarrow L =$ lifetime $= \min(T_1, T_2) + \min(T_1', T_3)$

where $T_1' =$ remaining lifetime $\sim$ exp, mean 1

by memoryless property.

$$\Rightarrow \mathbb{E}[L] = \mathbb{E}[\min(T_1, T_2)] + \mathbb{E}[\min(T_1', T_3)]$$

$$= \frac{1}{2} + \frac{1}{2} = 1.$$

3). A box contains n Red balls and m Black balls. A ball is drawn at random from the box; if it is Red, the ball is discarded; if it is Black, it is replaced in the box. Let N be the number of draws needed until all the Red balls have been taken out of the box. Find E[N]. [Hint: write N as a sum of geometric random variables].

Answer: $\sum_{k=1}^{n}(n+m+1-k)/(n+1-k)$

Let T_1 = # draws until first Red ball

T_2 = # additional draws until second Red ball

$\vdots$

T_n = # additional draws until n^{th} Red ball

$$N = T_1 + T_2 + \cdots + T_n$$

$$\Rightarrow \mathbb{E}[N] = \mathbb{E}[T_1] + \mathbb{E}[T_2] + \cdots + \mathbb{E}[T_n].$$

$T_1 \sim$ geometric, $p = \frac{n}{n+m}$ (number of coin tosses until first Heads).

$$\Rightarrow \mathbb{E}[T_1] = \frac{1}{p} = \frac{n+m}{n}.$$

$T_2 \sim$ geometric, $p = \frac{n-1}{n-1+m}$

$$\Rightarrow \mathbb{E}[T_2] = \frac{n-1+m}{n-1}$$

$\vdots$

$$\mathbb{E}[T_n] = \frac{1+m}{1}.$$

$$\Rightarrow \mathbb{E}[N] = m+1 + \frac{m+2}{2} + \frac{m+3}{3} + \cdots + \frac{m+n}{n}$$

$$= \sum_{k=1}^{n} \frac{m+1+k}{k}.$$

4). A random number X is chosen uniformly from the interval $[0,1]$. A second random number Y is then chosen uniformly from the interval $[0,X]$.

a) Find E$[X]$. [Hint: the pdf for X is $f(x)=1$ for $0 \le x \le 1$].

Answer: 1/2

b) Calculate E$[Y \mid X = x]$.

Answer: $x/2$

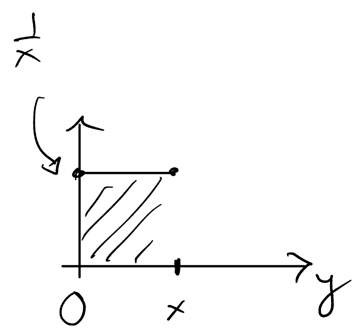
c) Calculate E$[Y]$.

Answer: 1/4

a) $\mathbb{E}[X] = \frac{1}{2}$

b) $\mathbb{E}[Y \mid X = x]$

$= \frac{x}{2}$



c) $\mathbb{E}[Y] = \mathbb{E}[\,\mathbb{E}[Y \mid X]\,]$

$= \int_0^1 \mathbb{E}[Y \mid X = x] \, f(x)\, dx$

$= \int_0^1 \frac{x}{2}\, dx = \frac{1}{4}.$

5). A fair die is repeatedly rolled. Find the expected number of rolls needed until the numbers $\{1, 2\}$ occur consecutively, in either order. [Hint: condition on the first occurrence of either number 1 or number 2].

Answer: 21

N_{12} = # rolls until first occurrence of (1,2) or (2,1).

Condition on X_1 = first roll.

$$\mathbb{E}[N_{12} \mid X_1 = 1] = \underbrace{\tfrac{1}{6}(2)}_{\text{second roll} = 2} + \underbrace{\tfrac{1}{6}\left(1 + \mathbb{E}[N_{12} \mid X_1 = 1]\right)}_{\text{second roll} = 1} + \underbrace{\tfrac{4}{6}\left(2 + \mathbb{E}[N_{12}]\right)}_{\text{second roll} = 3,4,5,6}$$

$$\Rightarrow \tfrac{5}{6}\,\mathbb{E}[N_{12} \mid X_1 = 1] = \tfrac{11}{6} + \tfrac{4}{6}\,\mathbb{E}[N_{12}].$$

$$\Rightarrow \mathbb{E}[N_{12} \mid X_1 = 1] = \tfrac{11}{5} + \tfrac{4}{5}\,\mathbb{E}[N_{12}].$$

Similarly $\mathbb{E}[N_{12} \mid X_1 = 2] = \tfrac{11}{5} + \tfrac{4}{5}\,\mathbb{E}[N_{12}]$.

Finally $\mathbb{E}[N_{12} \mid X_1 \neq \{1,2\}] = 1 + \mathbb{E}[N_{12}]$.

$$\Rightarrow \mathbb{E}[N_{12}] = \tfrac{1}{6}\,\mathbb{E}[N_{12} \mid X_1 = 1] + \tfrac{1}{6}\,\mathbb{E}[N_{12} \mid X_1 = 2] + \tfrac{4}{6}\,\mathbb{E}[N_{12} \mid X_1 \neq \{1,2\}]$$

$$= \tfrac{2}{6}\left(\tfrac{11}{5} + \tfrac{4}{5}\,\mathbb{E}[N_{12}]\right) + \tfrac{4}{6}\left(1 + \mathbb{E}[N_{12}]\right)$$

$$\Rightarrow \tfrac{2}{30}\,\mathbb{E}[N_{12}] = \tfrac{22}{30} + \tfrac{4}{6}$$

$$\Rightarrow \mathbb{E}[N_{12}] = 11 + 10 = 21$$

6). Cars pass a certain point on a highway in accordance with a Poisson process with rate $\lambda = 10$ per minute. The number of passengers in the cars are independent and identically distributed, with the following distribution: if Y is the number of passengers in a car, then $P(Y = 1) = 0.4$, $P(Y = 2) = 0.3$, $P(Y = 3) = 0.2$, $P(Y = 4) = 0.1$. A car is full if it has four passengers.

a) Find the expected number of passengers that pass in the next minute.

Answer: 20

b) Find the probability that no full cars pass in the next two minutes.

Answer: [redacted]

c) Suppose that 4 cars pass in the next 30 seconds. Find the probability that 6 cars pass in the following 30 seconds.

$N(t) = \#$ cars in $[0, t]$. $\sim$ P.P. $\lambda = 10 \text{ min}^{-1}$

a) $\mathbb{E}[\#\text{passengers in next minute} \mid N(1) = n]$

$$= n\,\mathbb{E}[Y] = n\,(0.4 + 2(0.3) + 3(0.2) + 4(0.1))$$

$$= 2n$$

$$\Rightarrow \mathbb{E}[\#\text{passengers in next minute}] = 2\,\mathbb{E}[N(1)]$$

$$= 2(10) = 20.$$

b) $N_4(t) = \#$ full cars in $[0, t]$. $\sim$ P.P. $\lambda_4 = \lambda(0.1)$

$= 1 \text{ min}^{-1}$

Thinning result.

$$\Rightarrow \mathbb{P}(N_4(2) = 0) = e^{-2\lambda_4} = e^{-2}$$

c) Independent increments

$$\Rightarrow \mathbb{P}(N(\tfrac{1}{2}) = 6) = \frac{(\lambda(\frac{1}{2}))^6}{6!} e^{-\lambda(\frac{1}{2})} = \frac{5^6}{6!} e^{-5}$$

7). Men and women enter a bank according to independent Poisson processes at rates $\mu = 3$ and $\lambda = 2$ per minute respectively. Starting at an arbitrary time, find:
a) the probability that exactly two men enter in the next minute;
Answer: $(\mu^2 e^{-\mu})/2$
b) the probability that at least one woman enters in the next minute;
Answer: $1 - e^{-\lambda}$
c) the probability that at least one man arrives before the next woman arrives.
Answer: $\mu/(\lambda + \mu)$

a) $\mathbb{P}(N_{men}(1) = 2) = \frac{\mu^2}{2!} e^{-\mu} \qquad (\mu = 3)$

b) $\mathbb{P}(N_{women}(1) \geq 1) = 1 - \mathbb{P}(N_{women}(1) = 0)$

$= 1 - e^{-\lambda} \qquad (\lambda = 2)$

c) $\mathbb{P}(\text{first arrival is Man}) = \frac{\mu}{\lambda + \mu}.$

8). Suppose that A, B and C are events in a sample space. Show that

$$\mathbb{P}(A) + \mathbb{P}(B) \leq 1 + \mathbb{P}(C) + \mathbb{P}(A \cap B \cap C^c)$$

$$\mathbb{P}(A^c \cup B^c \cup C) \leq \mathbb{P}(A^c) + \mathbb{P}(B^c) + \mathbb{P}(C)$$

$$\Leftrightarrow \quad 1 - \mathbb{P}(A \cap B \cap C^c) \leq 1 - \mathbb{P}(A) + 1 - \mathbb{P}(B) + \mathbb{P}(C)$$

$$\Leftrightarrow \quad \mathbb{P}(A) + \mathbb{P}(B) \leq 1 + \mathbb{P}(C) + \mathbb{P}(A \cap B \cap C^c).$$

9). A standard deck of 52 cards is shuffled and 5 cards are drawn at random. Let X be the number of aces remaining in the pack after this drawing. Find $E[X]$.

Answer: 47/13

Let $A_1 = \begin{cases} 1 & \text{if first card is Ace} \\ 0 & \text{else} \end{cases}$

$A_2 = \begin{cases} 1 & \text{if second card is Ace} \\ 0 & \text{else} \end{cases}$

$\vdots$

$A_5 = \begin{cases} 1 & \text{if fifth card is Ace} \\ 0 & \text{else.} \end{cases}$

$$\Rightarrow X = 4 - A_1 - A_2 - A_3 - A_4 - A_5$$

$$\Rightarrow \mathbb{E}[X] = 4 - 5\,\mathbb{E}[A_1]$$

$$= 4 - 5 \cdot \mathbb{P}(\text{first card is Ace})$$

$$= 4 - 5 \cdot \frac{1}{13}$$

$$= \frac{47}{13}$$

10). A fair die is rolled repeatedly. Let X_n be the result of the n^{th} roll. So X_n takes values $\{1, \ldots, 6\}$, each with probability $1/6$, and the random variables $X_1, X_2, \ldots$ are all independent.

Let

$$N = \min\{n : X_n = X_{n-1},\ n \geq 2\}$$

That is, N is the first roll where the result is equal to the previous roll. [e.g., if you roll the sequence $2, 3, 1, 4, 4, 6, \ldots$ then $N = 5$.] Find $E[N]$.

Answer: 7

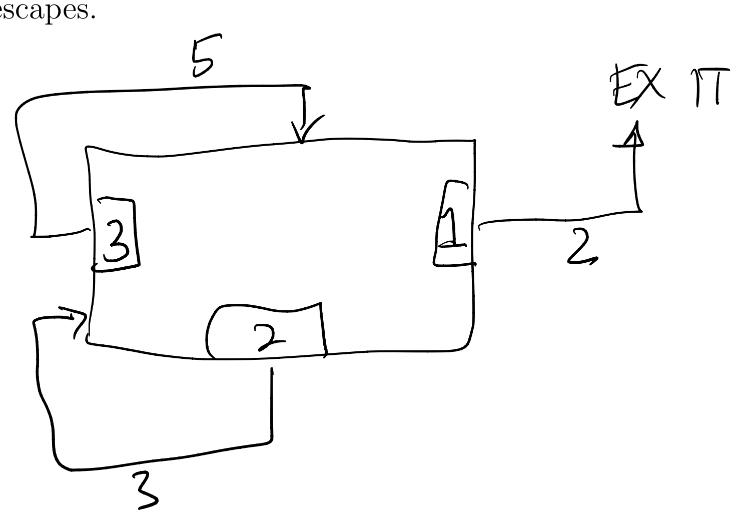
11). A maze for rats is constructed with three doors; door 1 leads to the exit after 2 minutes, door 2 leads back to the maze after 3 minutes, and door 3 leads back to the maze after 5 minutes.

a). A rat is put in the maze. Assume that the rat is equally likely to choose any door at all times. Find the expected time until the rat escapes.

Answer: 10

b). A rat is put in the maze. Assume that this rat remembers the last door that it chose, but does not remember any doors before its last choice. So if it went through door 2 or door 3 on the previous step, then it will not go through the same door again on this step. At every time it randomly chooses between the exit and the available doors, not including any doors that it chose at the last step. Find the expected time until the rat escapes.

Answer: 22/3



a) T = time to escape.

Condition on first door chosen.

$$E[T] = E[T|1]\cdot\tfrac{1}{3} + E[T|2]\,\tfrac{1}{3} + E[T|3]\,\tfrac{1}{3}$$

$$= 2\left(\tfrac{1}{3}\right) + \left(3 + E[T]\right)\tfrac{1}{3} + \left(5 + E[T]\right)\tfrac{1}{3}$$

$$\Rightarrow E[T] = 10$$

b) Let D be previous door chosen.

$$\mathbb{E}[T \mid D=2] = \tfrac{1}{2}(2) + \tfrac{1}{2}\left(5 + \mathbb{E}[T \mid D=3]\right)$$

chosen door 1   chose door 3

$$\mathbb{E}[T \mid D=3] = \tfrac{1}{2}(2) + \tfrac{1}{2}\left(3 + \mathbb{E}[T \mid D=2]\right)$$

Together these give

$$\mathbb{E}[T \mid D=2] = \frac{19}{3}$$

$$\mathbb{E}[T \mid D=3] = \frac{17}{3}$$

Now condition on first door chosen:

$$\mathbb{E}[T] = \tfrac{1}{3}(2) + \tfrac{1}{3}\left(3 + \mathbb{E}[T \mid D=2]\right)$$

$$+\frac{1}{3}\left(5+\mathbb{E}[T \mid D=3]\right)$$

$$=\frac{10}{3}+\frac{1}{3}\cdot\frac{19}{3}+\frac{1}{3}\cdot\frac{17}{3}$$

$$=\frac{66}{9}$$

$$=\frac{22}{3}$$

12). A biased coin has probability p of coming up Heads. The coin is tossed n times, and the number of changeovers is counted. Call this number N_n. (Recall that a changeover occurs when Heads is followed by Tails, or vice versa). Find $E[N_n]$.

Answer: $(n-1)2p(1-p)$

$N_n = \#$ changeovers.

e.g. HTTTHT $\Rightarrow$ $n=5$, $N_5 = 3$.

let $X_k = \begin{cases} 1 & \text{if changeover at } k^{th} \text{ toss} \\ 0 & \text{else} \end{cases}$

where $k = 2, \ldots, n$.

$\Rightarrow N_n = X_2 + X_3 + \cdots + X_n$.

$\Rightarrow \mathbb{E}[N_n] = \mathbb{E}[X_2] + \mathbb{E}[X_3] + \cdots + \mathbb{E}[X_n]$

$= (n-1)\,\mathbb{E}[X]$.

Now $\mathbb{E}[X] = \mathbb{P}(\text{changeover})$

$= \mathbb{P}(HT) + \mathbb{P}(TH)$

$= 2p(1-p)$.

$\Rightarrow \mathbb{E}[N_n] = 2p(1-p)(n-1)$.

13). Consider the following transition probability matrix for a Markov chain on 5 states:

$$P = \begin{pmatrix} 0.5 & 0.3 & 0 & 0 & 0.2 \\ 0 & 0.5 & 0 & 0 & 0.5 \\ 0 & 0.4 & 0.4 & 0.2 & 0 \\ 0.3 & 0 & 0.2 & 0 & 0.5 \\ 0.5 & 0.2 & 0 & 0 & 0.3 \end{pmatrix}$$

Number the states $\{1, 2, 3, 4, 5\}$ in the order presented.

a). Find and classify the equivalence classes of the states.

b). Find a stationary distribution for the chain. Is it unique?

c). Compute the expected number of steps needed to first return to state 1, conditioned on starting in state 1.

d). Compute the expected number of steps needed to first reach any of the states $\{1, 2, 5\}$, conditioned on starting in state 3.

a) $\{1,2,5\}$ persistent class, irreducible

$\{3,4\}$ transient.

b) $w = (w_1, w_2, 0, 0, w_5)$ unique.

c) $\tau_1 = \frac{1}{w_1}$

d) Make $\{1,2,5\}$ an absorbing state. $(*)$

$$\Rightarrow \hat{P} = \begin{array}{cc} \begin{pmatrix} 0.4 & 0.2 & 0.4 \\ 0.2 & 0 & 0.8 \\ 0 & 0 & 1 \end{pmatrix} & \begin{array}{c} 3 \\ 4 \\ (*) \end{array} \\ \begin{array}{ccc} 3 & 4 & (*) \end{array} & \end{array}$$

$$\Rightarrow Q = \begin{pmatrix} 0.4 & 0.2 \\ 0.2 & 0 \end{pmatrix}$$

$$N = (I - Q)^{-1} = \begin{pmatrix} 0.6 & -0.2 \\ -0.2 & 1 \end{pmatrix}^{-1}$$

$$= \frac{1}{0.6 - 0.04} \begin{pmatrix} 1 & 0.2 \\ 0.2 & 0.6 \end{pmatrix}.$$

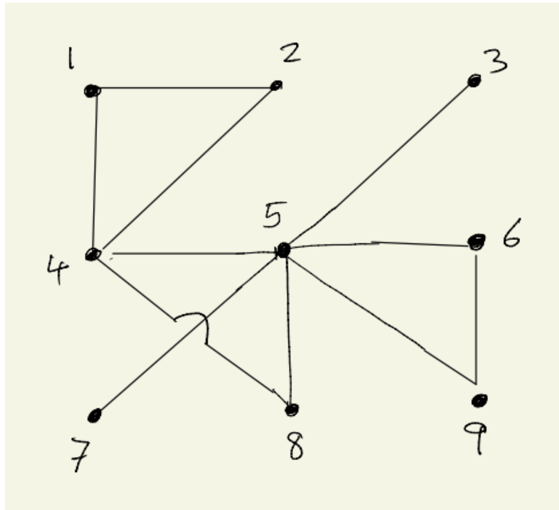
$$= \frac{1}{0.56} \begin{pmatrix} 1 & 0.2 \\ 0.2 & 0.6 \end{pmatrix}$$

$\Rightarrow$ expected # steps to reach (X)

starting in 3 is

$$\frac{1}{0.56}(1 + 0.2) = \frac{1.2}{0.56} = \frac{30}{14}.$$

14)



Consider a random walk on the graph shown above. At each step the walker randomly jumps along an edge to a neighboring vertex. Let $d(i)$ denote the number of edges at vertex i, then the transition matrix is

$$p_{i,j} = \begin{cases} \dfrac{1}{d(i)} & \text{if there is an edge from } i \text{ to } j \\ \\ 0 & \text{otherwise} \end{cases}$$

a). Find the stationary distribution of the chain. [Hint: use the same method as the chessboard example].

b). Find the mean return time to vertex number 1.

c). You can add one extra edge from vertex number 5 to any other vertex (double edges are allowed). Which choice of new edge will cause the largest reduction in the mean return time to vertex 1?

a) $w_i = \dfrac{d_i}{\sum_j d_j}$

b) $r_1 = \dfrac{1}{w_1}$

c) Add edge between 1 and 5.

15) Consider a random walk $\{X_n\}$ on the set $\{0, 1, 2, 3, \ldots, N\}$ with the following transition probabilities:

$$\begin{aligned} p_{k,k+1} &= \frac{2}{5} \quad \text{for } k = 0, 1, 2, 3, \ldots, N-1 \\ p_{k,k-1} &= \frac{3}{5} \quad \text{for } k = 1, 2, 3, \ldots, N \\ p_{0,0} &= \frac{3}{5}, \qquad p_{N,N} = \frac{2}{5} \end{aligned}$$

a). Write down the reversible equations for the stationary distribution.

b). Is the chain reversible? Explain your reasoning.

c). The mean first return time to state 0 is

$$r_0 = 3\left(1 - \left(\frac{2}{3}\right)^{N+1}\right)$$

Let $w = (w_0, w_1, \ldots, w_N)$ be the stationary distribution of the chain. Find an expression for w_k in terms of r_0 and k, for all $k = 0, 1, \ldots, N$.

a) $\frac{2}{5} w_k = \frac{3}{5} w_{k+1}$.

b) Yes, can solve reversible equations.

c) $w_0 = \frac{1}{r_0} = \frac{1}{3}\left(1-\left(\frac{2}{3}\right)^{N+1}\right)^{-1}$.

$w_{k+1} = \frac{2}{3} w_k \Rightarrow w_k = \left(\frac{2}{3}\right)^k w_0$

$= \left(\frac{2}{3}\right)^k \frac{1}{3}\left(1-\left(\frac{2}{3}\right)^{N+1}\right)^{-1}$.

16). The number of bacteria in an experiment is described by a branching process with the following distribution for number of offspring:

$$p_0 = \mathbb{P}(Z = 0) = 1/6, \quad p_1 = \mathbb{P}(Z = 1) = 5/12, \quad p_2 = \mathbb{P}(Z = 2) = 5/12.$$

a). Find the mean number of offspring.

b). Calculate the probability of extinction.

c). Three independent copies of this experiment are run. Find the probability that at least one of the populations does not become extinct.

a) $m = 0\left(\frac{1}{6}\right) + 1\left(\frac{5}{12}\right) + 2\left(\frac{5}{12}\right) = \frac{15}{12}$

b) solve $s = \phi(s) = \frac{1}{6} + s\left(\frac{5}{12}\right) + s^2\left(\frac{5}{12}\right).$

$\Leftrightarrow \quad 5s^2 + 5s + 2 = 12s$

$\Leftrightarrow \quad 5s^2 - 7s + 2 = 0$

$\Leftrightarrow \quad (s-1)(5s-2) = 0.$

$\Rightarrow \quad \rho = \frac{2}{5}$

c) $X = \#$ not extinct $\sim Bin(n, p)$ $\quad n = 3$, $p = 1 - \rho = \frac{3}{5}.$

so $\mathbb{P}(X \geq 1) = 1 - \mathbb{P}(X = 0) = 1 - (1-p)^3 = 1 - \left(\frac{2}{5}\right)^3$

17). For a special species of bacteria it is known that the number of offspring of each individual can be either 0, 1 or 3 (no other values are possible). Let p_0, p_1, p_3 be the probabilities of $0, 1, 3$ offspring. It is also known that the probability of extinction is $\rho = 1/4$. Assume that the growth of the population is described by a branching process.

a). The probabilities satisfy the linear equation $p_0 + p_1 + p_3 = 1$. Use the value of ρ to find another linear equation satisfied by these probabilities.

b). By eliminating p_3 you can use the two equations from part (a) to find a linear equation satisfied by p_0 and p_1. Use this equation to find the largest possible value of p_0.

a) $\rho = \phi(\rho) = p_0 + \rho p_1 + \rho^3 p_3$

$\Leftrightarrow \frac{1}{4} = p_0 + \frac{1}{4} p_1 + \frac{1}{64} p_3.$

b) $p_3 = 1 - p_0 - p_1$

$\Rightarrow \frac{1}{4} = p_0 + \frac{1}{4} p_1 + \frac{1}{64}(1 - p_0 - p_1)$

$\Leftrightarrow 16 = 64 p_0 + 16 p_1 + 1 - p_0 - p_1$

$\Leftrightarrow 15 = 63 p_0 + 15 p_1$

$\Leftrightarrow 5 = 21 p_0 + 5 p_1.$

Need $p_1 \geq 0 \Rightarrow \boxed{p_0 \leq \frac{5}{21}.}$

18) Recall the Gambler's Ruin Problem: a random walk on the integers $\{0, 1 \ldots, N\}$ with probability p to jump right and $q = 1 - p$ to jump left at every step, and absorbing states at 0 and N. Starting at $X_0 = k$, the probability to reach N before reaching 0 is

$$P_k = \frac{1 - (q/p)^k}{1 - (q/p)^N} \quad \text{for } p \neq \frac{1}{2}, \qquad\qquad P_k = \frac{k}{N} \quad \text{for } p = \frac{1}{2}.$$

Consider integers m, k satisfying $0 < m < k < N$. Starting at $X_0 = k$, compute the probability to visit state m exactly once before being absorbed at state N.

19) A new strain of COVID-19 is composed of three types, with proportions $\theta = (\theta_1, \theta_2, \theta_3)$, where $\theta_1 + \theta_2 + \theta_3 = 1$. A random sample of 30 sick people is tested and the following frequencies are found for each type:

Type	1	2	3
Number	15	8	7

a) Based on previous testing the prior distribution $f_0(\theta)$ is assumed to be the Dirichlet distribution $\pi_\alpha(\theta)$ where $\alpha = (10, 6, 9)$. The posterior distribution for θ is also a Dirichlet distribution $\pi_{\alpha'}(\theta)$. Find the parameters α'.

b) Write down the means and maximum likelihood estimators for $(\theta_1, \theta_2, \theta_3)$ using the posterior distribution.

a) $\alpha' = \alpha + (15, 8, 7)$

$= (25, 14, 16)$

b) Means: $\dfrac{\alpha_i'}{\sum_j \alpha_j'}$

MLE: $\dfrac{\alpha_i' - 1}{\sum_j (\alpha_j' - 1)}$