SOLUTIONS

MATH 7241 Fall 2022: Problem Set #7

Due date: Wednesday December 7

Reading: relevant background material for these problems can be found on Canvas 'Notes 7: Bayesian Inference, 'Notes 9: Continuous time Markov chains'.

Exercise 1 [Use Bayesian inference to solve this problem] A population of bacteria is composed of four types, with unknown proportions $\theta = (\theta_1, \theta_2, \theta_3, \theta_4)$. A random sample is measured and the following numbers are found for each type:

Type	1	2	3	4
Number	25	8	17	13

a) Find the posterior distribution for θ (Note: you should assume a uniform prior distribution for θ on the simplex Δ_4, and your result should be a Dirichlet distribution. You do not need to evaluate the evidence).

b) Write down the means and maximum likelihood estimators for $(\theta_1, \theta_2, \theta_3, \theta_4)$ using the posterior distribution.

a) Prior: $f_0(\theta) = \pi_{\alpha_0}(\theta)$ $\qquad \alpha_0 = (1,1,1,1)$

$$= N(\alpha_0)\, \theta_1^{\alpha_1 - 1}\, \theta_2^{\alpha_2 - 1}\, \theta_3^{\alpha_3 - 1}\, \theta_4^{\alpha_4 - 1}$$

$$= N(\alpha_0).$$

Likelihood: $P(\mathcal{D} \mid \theta) = C\, \theta_1^{25}\, \theta_2^{8}\, \theta_3^{17}\, \theta_4^{13}$

constant which counts number of different orderings

Posterior: $f_1(\theta \mid \mathcal{D}) = \dfrac{P(\mathcal{D} \mid \theta)\, f_0(\theta)}{Z}.$

Just focus on the θ-dependence:

$$f_1(\theta|\mathcal{D}) \propto \theta_1^{25}\,\theta_2^{8}\,\theta_3^{17}\,\theta_4^{13}$$

$$\propto \theta_1^{\alpha_1'-1}\,\theta_2^{\alpha_2'-1}\,\theta_3^{\alpha_3'-1}\,\theta_4^{\alpha_4'-1}.$$

So the dependence is the same as for

$\pi_{\alpha'}(\theta)$ with $\alpha' = (\alpha_1', \alpha_2', \alpha_3', \alpha_4')$

$$= (26, 9, 18, 14).$$

Since $f_1(\theta|\mathcal{D})$ is a pdf, the constant of proportionality must be the same as in $\pi_{\alpha'}(\theta)$, that is $N(\alpha')$.

[so no need to evaluate the evidence Z].

$$\Rightarrow \quad f_1(\theta|\mathcal{D}) = \pi_{\alpha'}(\theta).$$

<u>Dirichlet</u>

b) Means of Dirichlet:

$$\theta_{1,\text{mean}} = \frac{\alpha_1'}{\sum \alpha_j'} = \frac{26}{67}$$

$$\theta_{2,\text{mean}} = \frac{\alpha_2'}{\sum \alpha_j'} = \frac{9}{67}$$

$$\theta_{3,\text{mean}} = \frac{\alpha_3'}{\sum \alpha_j'} = \frac{18}{67}$$

$$\theta_{4,\text{mean}} = \frac{\alpha_4'}{\sum \alpha_j'} = \frac{14}{67}$$

$$\theta_{1,MLE} = \frac{\alpha_1' - 1}{\sum \alpha_j' - 1} = \frac{25}{63}$$

$$\theta_{2,MLE} = \frac{8}{63}$$

$$\theta_{3,MLE} = \frac{17}{63}$$

$$\theta_{4,MLE} = \frac{13}{63}$$

Exercise 2 A continuous random variable X is uniformly distributed over the interval $[0, \theta]$. However the parameter θ is unknown, and initially it is modeled as the value of a continuous random variable which is uniformly distributed on the interval $[0, 1]$.

a). Write down the prior distribution f_0 for θ.

b). Write down the likelihood function $f(x \mid \theta)$ (this is the pdf for X conditioned on the value θ). Be careful to specify the intervals where $f(x \mid \theta)$ is zero and where it is nonzero.

c). The random variable X is measured and gives the value $x_1 \in [0, 1]$. Let $D = \{x_1\}$ denote the data from this trial. The posterior pdf of θ is given by the Bayesian update rule

$$f_1(\theta \mid D) = \frac{f(x_1 \mid \theta)\, f_0(\theta)}{Z}$$

where the evidence is

$$Z = \int_0^1 f(x_1 \mid \theta)\, f_0(\theta)\, d\theta$$

Sketch a graph of $f(x_1 \mid \theta)\, f_0(\theta)$ as a function of θ. Again be careful to account for the intervals where $f(x_1 \mid \theta)$ is zero and where it is nonzero.

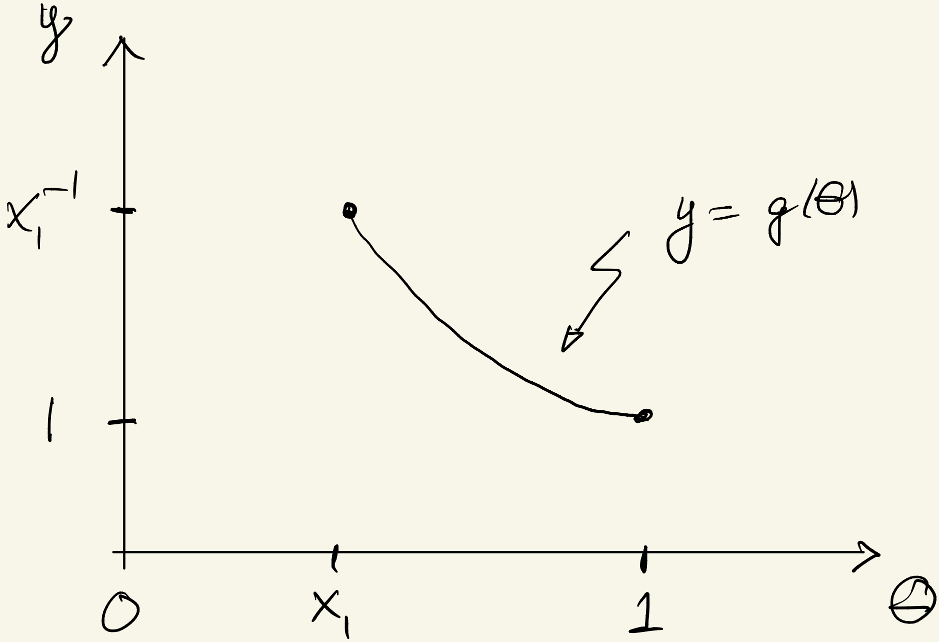
d). Evaluate the evidence Z and hence determine the posterior pdf f_1.

a) Prior: $f_0(\theta) = \begin{cases} 1 & 0 \le \theta \le 1 \\ 0 & \text{else} \end{cases}$

b) Likelihood: $f(x \mid \theta) = \begin{cases} \frac{1}{\theta} & 0 \le x \le \theta \\ 0 & x < 0 \text{ or } x > \theta \end{cases}$

c) let $g(\theta) = f(x_1 \mid \theta)\, f_0(\theta)$

$$= \begin{cases} \frac{1}{\theta} & x_1 \le \theta \le 1 \\ 0 & \theta < x_1 \text{ or } \theta > 1 \end{cases}$$



d) $$Z = \int_0^1 f(x_1|\theta)\, f_0(\theta)\, d\theta$$

$$= \int_0^1 g(\theta)\, d\theta$$

$$= \int_{x_1}^1 \frac{1}{\theta}\, d\theta$$

$$= \ln\theta \Big|_{x_1}^1$$

$$= -\ln x_1$$

$$= \ln\left(\frac{1}{x_1}\right)$$

$$\Rightarrow \quad f_1(\theta|\mathcal{D}) = \begin{cases} \dfrac{1}{\theta \ln\left(\frac{1}{x_1}\right)} & x_1 \leq \theta \leq 1 \\ 0 & \theta < x_1 \text{ or } \theta > 1. \end{cases}$$

$\boxed{\textbf{Exercise 3}}$ Events occur according to a Poisson process with rate $\lambda = 1.5$ per hour.
(a) What is the probability that no event occurs between 8pm and 10pm?
(b) Starting at noon, what is the expected time at which the fifth event occurs?
(c) What is the probability that two or more events occur between 6pm and 8pm?

a) $\mathbb{P}(N(10) - N(8) = 0)$

$= \mathbb{P}(N(2) = 0)$ (memoryless)

$= e^{-\lambda(2)} = e^{-3}$

b) Memoryless $\Rightarrow$ $\mathbb{E}[T_5] = 5\,\mathbb{E}[T_1]$

$= \frac{5}{\lambda} = \frac{5}{1.5} = 3{:}20$ pm

c) $\mathbb{P}(N(8) - N(6) \geq 2) = \mathbb{P}(N(2) \geq 2)$ (memoryless)

$= 1 - \mathbb{P}(N(2) = 0) - \mathbb{P}(N(2) = 1)$

$= 1 - e^{-2\lambda} - 2\lambda e^{-2\lambda}$

$= 1 - 4e^{-3}$

$\boxed{\textbf{Exercise 4}}$ Cars pass a certain point in a highway in accordance with a Poisson process with rate $\lambda = 10$ per minute. The number of passengers in the cars are independent and identically distributed, with the following distribution: if Y is the number of passengers in a car, then $P(Y = 1) = 0.5$, $P(Y = 2) = 0.3$, $P(Y = 3) = 0.1$, $P(Y = 4) = 0.1$. A car is full if it has four passengers.
a) Find the probability that the next car that passes is full.
b) Find the probability that two full cars pass in the next minute.
c) Find the expected number of passengers that pass in the next minute.
d) Find the probability that at least two passengers pass in the next ten seconds.
[You may want to use the result about thinning of Poisson processes].

a) # passengers is IID in each car

$\Rightarrow \mathbb{P}(\text{next car is full}) = \mathbb{P}(Y=4) = 0.1$

b) $N_4(t)$ = # full cars in $[0,t]$

Thinning result $\Rightarrow$ $N_4 \sim$ P.P. rate $\lambda(0.1) = 1 = \lambda_4$

$\Rightarrow \mathbb{P}(N_4(1) = 2) = \frac{(\lambda_4 t)^2}{2!} e^{-\lambda_4 t} = \frac{1}{2} e^{-1} = \frac{1}{2e}$

c) $\mathbb{E}[\text{\# passengers in car}] = \mathbb{E}[Y]$

$= 0.5 + 2(0.3) + 3(0.1) + 4(0.1)$

$= 1.8$

$\Rightarrow \mathbb{E}[\text{\# passengers in next minute}]$

$= \mathbb{E}[Y]\, \mathbb{E}[\text{\# cars in next minute}]$

$= (1.8)\,\lambda = 18.$

d) $\mathbb{P}$(at least 2 passengers in 10 sec.)

$= 1 - \mathbb{P}$(0 passenger in 10 sec)

$\quad - \mathbb{P}$(1 passenger in 10 sec)

$$= 1 - \mathbb{P}\left(N\left(\tfrac{1}{6}\right) = 0\right) - \mathbb{P}(Y=1)\,\mathbb{P}\left(N\left(\tfrac{1}{6}\right)=1\right)$$

$$= 1 - e^{-\lambda\left(\frac{1}{6}\right)} - (0.5)\,\lambda\left(\tfrac{1}{6}\right) e^{-\lambda\left(\frac{1}{6}\right)}$$

$$= 1 - e^{-5/3} - (0.5)\left(\tfrac{10}{6}\right) e^{-5/3}$$

$$= 1 - \tfrac{11}{6} e^{-5/3}$$

Exercise 5 Men and women enter a bank according to independent Poisson processes at rates $\mu = 4$ and $\lambda = 5$ per minute respectively. Starting at an arbitrary time, find the probability that at least two men arrive before the next two women arrive. [You may want to use the result about superposition of Poisson processes].

Men & Women are independent P.P.'s

$\Rightarrow$ can merge into one P.P., where M, W are chosen independently.

$\Rightarrow$ $\mathbb{P}$(at least 2 Men before next 2 Women)

$$= \mathbb{P}(MM) + \mathbb{P}(MWM) + \mathbb{P}(WMM)$$

$$\mathbb{P}(M) = \frac{\mu}{\mu+\lambda} = \frac{4}{9}$$

$$\mathbb{P}(W) = \frac{\lambda}{\mu+\lambda} = \frac{5}{9}$$

$\Rightarrow$ $\mathbb{P}$(at least 2M before next 2W)

$$= \left(\frac{4}{9}\right)^2 + \left(\frac{4}{9}\right)^2\left(\frac{5}{9}\right) + \left(\frac{4}{9}\right)^2\left(\frac{5}{9}\right)$$

$$= \left(\frac{4}{9}\right)^2\left(\frac{19}{9}\right) = \frac{304}{729} = 0.42$$

Exercise 6 Customers arrive at a bank according to a Poisson process with rate λ. Suppose two customers arrived during the first hour. What is the probability that (a) both arrived during the first 20 minutes? (b) at least one arrived during the first 20 minutes? [You may want to use the result about conditional distribution of arrival times].

Given $N(1) = 2$

$\Rightarrow$ arrival times are $\left.\begin{array}{l} T_1 = U_{(1)} \\ T_2 = U_{(2)} \end{array}\right\}$ order stats of $U_1, U_2 \sim IID$ unif on a $[0,1]$.

a) $\mathbb{P}($ both arrived in first 20 mins $\mid N(1) = 2)$

$= \mathbb{P}(T_1 \leq \frac{1}{3}, T_2 \leq \frac{1}{3} \mid N(1) = 2)$

$= \mathbb{P}(U_1 \leq \frac{1}{3}, U_2 \leq \frac{1}{3})$ (independent)

$= \mathbb{P}(U \leq \frac{1}{3})^2 = \frac{1}{9}$

b) $\mathbb{P}($ at least one arrived in first 20 mins $\mid N(1) = 2)$

$= 1 - \mathbb{P}($ none arrived in first 20 min $\mid N(1) = 2)$

$= 1 - \mathbb{P}(T_1 > \frac{1}{3}, T_2 > \frac{1}{3} \mid N(1) = 2)$

$= 1 - \mathbb{P}(U > \frac{1}{3})^2 = 1 - \left(\frac{2}{3}\right)^2 = \frac{5}{9}$

Exercise 7 Starting at 9 : 00am, assengers arrive at a train station as a Poisson process with rate $\lambda = 2$ per minute. The next train will depart at time T, where T is uniformly distributed between 9:10 am and 9:15 am. Compute the mean waiting time for a passenger at the station before the train departs.

$N(t)$ = # passenges who arrive between 9am and time t.

$\sim$ PP rate $\lambda = 2$ per minute.

T_k = arrival time of k^{th} passenger.

T = departure time of train

$\sim$ uniform between 9:10 & 9:15 am

$T \sim U[10, 15]$ (units are minutes).

$\Rightarrow N(T)$ = # passenges who depart on train

W_k = waiting time of k^{th} passenger

$= T - T_k$ if $T_k < T$

$= T - T_k$ if $k \leq N(T)$.

$\overline{W}$ = mean waiting time for passengers

$$= \frac{1}{N(T)} \sum_{k=1}^{N(T)} W_k$$

$N(T)$ = # passengers who depart on train

$$\Rightarrow \overline{W} = \frac{1}{N(T)} \sum_{k=1}^{N(T)} (T - T_k)$$

$$= T - \frac{1}{N(T)} \sum_{k=1}^{N(T)} T_k.$$

$$\Rightarrow \mathbb{E}[\overline{W}] = \mathbb{E}[T] - \mathbb{E}\left[\frac{1}{N(T)} \sum_{k=1}^{N(T)} T_k\right]$$

$$= \frac{10+15}{2} - \mathbb{E}\left[\frac{1}{N(T)} \sum_{k=1}^{N(T)} T_k\right]$$

$$= \frac{25}{2} - \mathbb{E}\left[\frac{1}{N(T)} \sum_{k=1}^{N(T)} T_k\right]$$

Condition on T:

$$\mathbb{E}\left[\overline{W} \mid T=t\right] = \frac{25}{2} - \mathbb{E}\left[\frac{1}{N(t)} \sum_{k=1}^{N(t)} T_k\right]$$

Condition on $N(t)$:

$$\mathbb{E}\left[\overline{W} \mid T=t, N(t)=n\right]$$

$$= \frac{25}{2} - \mathbb{E}\left[\frac{1}{n} \sum_{k=1}^{n} T_k \,\middle|\, N(t)=n\right]$$

$$= \frac{25}{2} - \frac{1}{n} \mathbb{E}\left[\sum_{k=1}^{n} U_{(k)} \,\middle|\, N(t)=n\right]$$

order stats

$$= \frac{25}{2} - \frac{1}{n} \mathbb{E}\left[\sum_{k=1}^{n} U_k \,\middle|\, N(t)=n\right]$$

$$= \frac{25}{2} - \frac{1}{n} n \mathbb{E}[U]$$

uniform on $[0,t]$

$$= \frac{25}{2} - \frac{t}{2}.$$

$$\Rightarrow \mathbb{E}[\bar{W}] = \frac{25}{2} - \frac{1}{2}\mathbb{E}[T]$$

$$= \frac{25}{4}$$

$$= 6 \text{ mins. } 15 \text{ sec.}$$

Exercise 8 Suppose that the distribution of offspring Z for a branching process is geometric, so that

$$p_k = \mathbb{P}(Z = k) = x^k (1 - x) \quad \text{for} \quad k = 0, 1, 2, \ldots$$

where $0 < x < 1$.

a). Find the largest value of x for which extinction is guaranteed. Call this value x_m.

b). For $x > x_m$ compute the probability of extinction.

a) Extinction is guaranteed if $m \leq 1$.

$$m = \mathbb{E}[Z] = \sum_{k=0}^{\infty} k\, x^k (1-x)$$

$$= x(1-x) \sum_{k=1}^{\infty} k\, x^{k-1}$$

$$= x(1-x) \frac{1}{(1-x)^2}$$

$$= \frac{x}{1-x}$$

so $m \leq 1 \iff \frac{x}{1-x} \leq 1 \iff x \leq \frac{1}{2}$

$\Rightarrow$ $x_m = \frac{1}{2}$ is the largest value of x when extinction is guaranteed.

f) Let $x > x_m \Rightarrow m > 1$

let ρ = prob. of extinction

solve
$$s = \phi(s)$$
$$= \mathbb{E}[s^Z]$$
$$= \sum_{k=0}^{\infty} s^k x^k (1-x)$$
$$= \frac{1-x}{1-sx}$$

$$\Leftrightarrow \quad s(1-sx) = 1-x$$

$$\Leftrightarrow \quad s^2 x - s + 1 - x = 0$$

$$\Leftrightarrow \quad (s-1)(sx-1+x) = 0$$

ρ = smallest positive root $= \frac{1-x}{x}$