Northeastern University, Department of Mathematics

MATH 5110: Applied Linear Algebra and Matrix Analysis.

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§7 Diagonalization; Eigenvalues; Eigenvectors;

Contents

1. Diagonalization

Let D be an diagonal matrix. The power D^k is easy to calculate. For example,

$$D^k = \begin{bmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & d_4 \end{bmatrix}^k = \begin{bmatrix} (d_1)^k & 0 & 0 & 0 \\ 0 & (d_2)^k & 0 & 0 \\ 0 & 0 & (d_3)^k & 0 \\ 0 & 0 & 0 & (d_4)^k \end{bmatrix}$$

Definition 1. An $n \times n$ matrix A is said to be **diagonalizable** if

Application of diagonalization:

Question:

1. Are all $n \times n$ matrices A diagonalizable?

2. If a matrix A is diagonalizable, how to find the invertible matrix P and the diagonal matrix D? The answer for this question is called **diagonalize** matrix A.

2. Eigenvalues and Eigenvectors.

Consider a linear transformation $T : \mathbb{F}^n \to \mathbb{F}^n$ by matrix $T\vec{x} = A\vec{x}$. ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.)

Definition 2. If there exist a nonzero vector $\vec{x} \in \mathbb{F}^n$ and a number $\lambda \in F$ such that

$$A\vec{x} = \lambda\vec{x}$$

then, the vector $\vec{x}$ is **an eigenvector corresponding to the eigenvalue** λ.

Definition 3. A basis $\vec{b}_1, \ldots, \vec{b}_n$ of $\mathbb{F}^n$ is called an **eigenbasis** for A if the vectors $\vec{b}_1, \ldots, \vec{b}_n$ are eigenvectors of A.

Example 4. If $\vec{v}$ is an eigenvector of A corresponding to λ, is $\vec{v}$ an eigenvector of A^k? Is λ an eigenvalue of A^k?

Theorem 5. *A is diagonalizable if and only if it has n linearly independent eigenvectors $\vec{b}_1, \ldots, \vec{b}_n$ (eigenbasis).*
In this case $A = PDP^{-1}$ where the columns of P are eigenvectors of A; the diagonal entries of D are the eigenvalues of A corresponding to the eigenvectors given by the columns of P.

Proof. We already verified that system of equations $A\vec{b}_1 = \lambda_1\vec{b}_1$, $A\vec{b}_2 = \lambda_2\vec{b}_2$, $\ldots$, $A\vec{b}_n = \lambda_n\vec{b}_n$. is equivalent to matrix equation

$$AP = PD$$

where $P = [\vec{b}_1 \ \ldots \ \vec{b}_n]$ and $D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$.

P is invertible if and only if $\{\vec{b}_1, \ldots, \vec{b}_n\}$ is a basis of $\mathbb{R}^n$. In this case, $A = PDP^{-1}$ and A is diagonalizable.

□

Example 6. Let T be the projection transformation onto a line $L = \text{Span}\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\}$ $\mathbb{R}^3$.

Find a basis $\mathscr{B} = [\vec{b}_1 \ \vec{b}_2 \ \vec{b}_3]$ for $\mathbb{R}^3$ such that the $\mathscr{B}$-matrix of the T is the diagonal matrix $D = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$.

Example 7. Let T be the rotation through an angle of $\pi/2$ in the counterclock direction. So the matrix of T is $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Find all eigenvalues and eigenvectors of A. Is A diagonalizable?

Example 8. Which matrix has 0 as an eigenvalue?

2. **Eigenvalues and Characteristic Polynomials**

Let A be an $n \times n$ matrix.

Theorem 9 (**The Characteristic Equation of** A).

Example 10. Finding Eigenvalues for the following matrices:

$$A = \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix} \qquad\qquad B = \begin{bmatrix} i & 1 \\ 2 & 3 \end{bmatrix}$$

$$eig(A) = \begin{pmatrix} -1 \\ 7 \end{pmatrix}$$

$$eig(B) = \begin{pmatrix} \frac{3}{2} - \frac{\sqrt{16-6\,\mathrm{i}}}{2} + \frac{1}{2}\,\mathrm{i} \\ \frac{\sqrt{16-6\,\mathrm{i}}}{2} + \frac{3}{2} + \frac{1}{2}\,\mathrm{i} \end{pmatrix} \approx \begin{pmatrix} -0.5337 + 0.8688i \\ 3.5337 + 0.1312i \end{pmatrix}$$

Theorem 11. *The eigenvalues of a triangular $n \times n$ matrix A equal the diagonal entries of A.*

Proof. Suppose A is an upper triangular matrix.

$$\det(A - \lambda I) = \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{nn} - \lambda \end{vmatrix} = (a_{11} - \lambda)(a_{22} - \lambda) \cdots (a_{nn} - \lambda) = 0$$

Hence, the eigenvalues of A are a_{ii} for $i = 1, ..., n$. $\square$

Practice: Finding Eigenvalues for the following matrices:

$$A = \begin{bmatrix} 2 & 5 & \sqrt{2} \\ 3 & 4 & 7 \\ 0 & 0 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 4 & 0 \\ 3 & 5 & 7 \end{bmatrix}$$

In general for a $n \times n$ matrix A,

$$\det(A - \lambda I) = \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix}$$
$$= (a_{11} - \lambda)(a_{22} - \lambda) \cdots (a_{nn} - \lambda) + \sum (\text{ terms of degree} \leq (n-2))$$
$$= (-\lambda)^n - (a_{11} + a_{22} + \cdots + a_{nn})(-\lambda)^{n-1} + \sum (\text{ terms of degree} \leq (n-2))$$

Definition 12 (Characteristic Polynomial). If A is an $n \times n$ matrix, the **characteristic polynomial** of A is

Example 13. Find the characteristic polynomial for a 2×2 arbitrary matrix.

Definition 14. Th sum of the diagonal entries of a square matrix is called the **trace** of A,

The characteristic polynomial for a 2×2 matrix A:

More generally,

Theorem 15. *Let A be an $n \times n$ matrix. Then the characteristic polynomial of A is*

More properties on Characteristic Polynomials

Definition 16 (Algebraic Multiplicity)**.**
An eigenvalue λ_0 of A is said to have **algebraic multiplicity** k if it has multiplicity k as a root of the characteristic polynomial $f_A(t)$. Equivalently,

$$f_A(\lambda) = (\lambda_0 - \lambda)^k g(\lambda)$$

such that $g(\lambda_0) \neq 0$.

Theorem 17. *An $n \times n$ matrix has at most n eigenvalues, even counted with algebraic multiplicities.*

Example 18. Find all eigenvalues and their algebraic multiplicities of $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Example 19. Find the characteristic polynomial of $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$. Which of the following numbers 1, -1, 4 are eigenvalues of A?

Example 20. Find the characteristic polynomial of $A = \begin{bmatrix} 2 & 1 & 2 \\ -1 & 4 & 2 \\ -1 & 1 & 5 \end{bmatrix}$. Verify that 3 and 5 are eigenvalues.

Theorem 21. *Let A be an $n \times n$ matrix. Suppose A has n eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, (listed with algebraic multiplicities.) Then*

Theorem 22 (On Eigenvalues of Similar Matrices)**.** *If A and B are similar,*

The converse of the preceding theorem is not generally true. There do exist matrices with the same characteristic polynomial that are not similar matrices. See example later.

Proposition 23. *If A and B are similar, then*

Proposition 24. *If A and B are similar, then*

Example 25. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ are not similar, but they have the same rank, the same determinant, the same trace, the same eigenvalues.

Example 26. Are the following two matrices similar to each other? $A = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 5 \\ 2 & 3 \end{bmatrix}$

Warning: Similar matrices may have **different** eigenvectors.

3. Eigenvectors and Eigenspaces

Theorem 27. *Let A be an $n \times n$ matrix. A scalar λ is an eigenvalue for A if and only if the matrix equation*

$$(A - \lambda I_n)\vec{x} = \vec{0}$$

has a nontrivial *solution $\vec{x}$.*
Said differently, λ is an eigenvalue for A if and only if

$$\mathrm{Nul}(A - \lambda I_n) \neq \{\vec{0}\}.$$

Definition 28. Let A be an $n \times n$ matrix and λ be a eigenvalue of A. The set of all eigenvectors of A corresponding to λ together with the zero vector, is called the **eigenspace** of A corresponding to λ, and it equals the subspace

$$\mathrm{Nul}(A - \lambda I_n).$$

The dimension of the eigenspace $\mathrm{Nul}(A - \lambda I_n)$ is called the **geometric multiplicity** of λ.

Proposition 29.

1$\leq$ Geometric multiplicity of $\lambda \leq$ Algebraic multiplicity of $\lambda \leq n$.

Example 30. Let T be the projection transformation onto a line $L = \mathrm{Span}\{\begin{bmatrix}1\\2\\3\end{bmatrix}\}$ $\mathbb{R}^3$.

Explain the geometric meaning of the eigenvalues and eigenspaces.

Lemma 31. *Let A be an $n \times n$ matrix and let $\vec{v}_1, \ldots, \vec{v}_p$ be eigenvectors of A that correspond to* distinct *eigenvalues $\lambda_1, \ldots, \lambda_p$ respectively. Then $\{\vec{v}_1, \ldots, \vec{v}_p\}$ is a linearly independent set of vectors.*

Proof. We prove this by induction on p. If $p = 1$, it is clear. Suppose this is true for $p - 1$ vectors.

□

Lemma 32. *Let A be an $n \times n$ matrix and let $\lambda_1, \ldots, \lambda_p$ be distinct eigenvalues with corresponding independent set of eigenvectors $V_1, \ldots, V_p$. Then $V_1 \cup \cdots \cup V_p$ is a linearly independent set of vectors.*

Recall that an $n \times n$ matrix A is diagonalizable if and only if it has n linearly independent eigenvectors.

Proposition 33 (Case of Distinct Eigenvalues)**.** *If an $n \times n$ matrix A has n **distinct** eigenvalues, then its corresponding eigenvectors are linearly independent and A is diagonalizable.*

Theorem 34. *Let $\lambda_1, \ldots, \lambda_p$ be **distinct** eigenvalues of A such that*

$$f_A(\lambda) = \det(A - \lambda I) = (\lambda_1 - \lambda)^{k_1}(\lambda_2 - \lambda)^{k_2} \cdots (\lambda_p - \lambda)^{k_p}.$$

Suppose $k_1 + k_2 + \cdots + k_p = n$. Let E_k be the eigenspace of λ_k.

Proof. A is diagonalizable if and only if it has n linearly independent eigenvectors.

□

Another point of view of the eigenspaces is the invariant subspace.

Definition 35. Let $T : V \to V$ be a linear transformation on a vector space V. A subspace $W \subseteq V$ is said to be **invariant** under T if

Proposition 36. *A one-dimensional subspace is invariant under the linear transformation T_A if and only if it is an eigenspace spanned by an eigenvector of A.*

Theorem 37. *An $n \times n$ matrix A is similar to a diagonal matrix D, (i.e., $A = PDP^{-1}$) if and only if there exists a decomposition of*

$$\mathbb{F}^n = V_1 \oplus V_2 \oplus \cdots \oplus V_n$$

such that each V_i is one dimensional and invariant under T_A.

In Matlab, $[P, D] = \text{eig}(A)$ returns diagonal matrix D of eigenvalues and matrix P whose columns are the corresponding right eigenvectors, such that $AP = PD$.

Example 38. Diagonalizing Matrices

$$A = \begin{bmatrix} 4 & -1 & 0 \\ 2 & 1 & 0 \\ 2 & -1 & 2 \end{bmatrix}. \quad B = \begin{bmatrix} 5 & 1 & 0 \\ 2 & 4 & 0 \\ 2 & 1 & 3 \end{bmatrix}. \quad C = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}. \quad M = \begin{bmatrix} 2 & 1 & 2 \\ -1 & 4 & 2 \\ -1 & 1 & 5 \end{bmatrix}.$$

Remark[Non Diagonalizing Result] For any $n > 1$ there exist examples of $n \times n$ matrices that are not diagonalizable.

Real Matrices Acting on $\mathbb{C}^n$

Let A be a real $n \times n$ matrix and λ be an eigenvalue of A.

- lf λ is a real number, then there exist real eigenvectors associate to λ, as well as complex eigenvector.
- If λ is a complex (non-real) eigenvalue of A, then every eigenvector $\vec{x}$ associated to λ is a complex (non-real) vector.

Suppose A is an $n \times n$ matrix with real number entries so that $\overline{A} = A$. Let λ be a complex eigenvalue of A with associated eigenvector $\vec{x}$. Then

$$\overline{A \cdot \vec{x}} = \overline{A} \cdot \overline{\vec{x}} = A \cdot \overline{\vec{x}}$$
$$\overline{A \cdot \vec{x}} = \overline{\lambda \cdot \vec{x}} = \overline{\lambda} \cdot \overline{\vec{x}}$$

Combining the two we obtain

$$A \cdot \overline{\vec{x}} = \overline{\lambda} \cdot \overline{\vec{x}}.$$

Theorem 39. *Let A be an $n \times n$ matrix with real number entries and let λ be an eigenvalue of A with associated eigenvector $\vec{x}$. Then $\overline{\lambda}$ is also an eigenvalue of A with associated eigenvector $\overline{\vec{x}}$.*