0.2 The Poisson process

This is the following special case, given in the second description: the process is $\{N(t)\}$ on $\Omega = \{0, 1, 2, \dots\}$, with $N(0) = 0$, $\lambda_i = \lambda$ and

$$p_{ij} = \begin{cases} 1 & \text{for } j = i+1 \\ 0 & \text{else} \end{cases}$$

So the chain starts at 0, and increases by jumps which occur after holding times which are IID exponential with rate λ. There are several immediate consequences.

Theorem 3

- *For every t, $N(t)$ has the Poisson distribution with parameter λt, that is*

$$P(N(t) = j) = \frac{(\lambda t)^j}{j!} e^{-\lambda t}$$

- *The arrival times $0 = T_0 < T_1 < T_2 < \dots$ have gamma distribution.*
- *If $s < t$ then $N(t) - N(s)$ is Poisson with rate $\lambda(t-s)$, and is independent of the arrival times during the interval $[0, s]$.*

The Poisson process is widely used as the model for an arrival process, both because it does a good job in many cases, and also because it allows exact computations in many ways.

Inter arrival times are IID exponential.

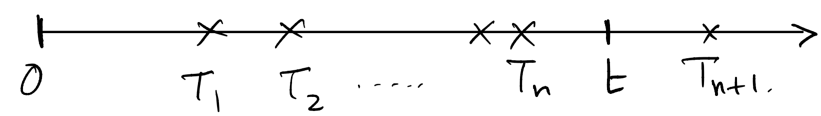
The Poisson process has many remarkable properties not shared by other CTMC's, here are some examples.

Theorem 4 (The superposition theorem) *Let $N_1(t)$ and $N_2(t)$ be independent Poisson processes with rates λ_1 and λ_2 respectively. Then $N_1 + N_2$ is a Poisson process with rate $\lambda_1 + \lambda_2$.*

There is also a converse.

Theorem 5 (The thinning theorem) *Suppose that $N(t)$ is a Poisson process with rate λ, and each arrival is independently classified as one of two types: Type 1 with probability p, or Type 2 with probability $1-p$. Let $N_1(t)$ and $N_2(t)$ be the number of arrivals of type 1 and 2 respectively. Then N_1 and N_2 are independent Poisson processes with rates λp and $\lambda(1-p)$ respectively.*

Conditional Arrival Times



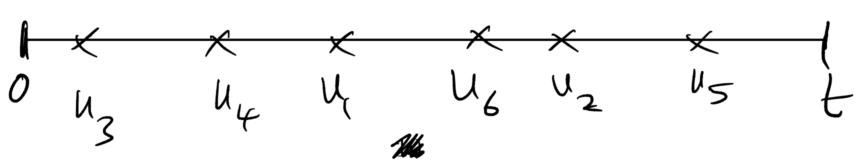
Suppose we know $N(t) = n$.

$\Rightarrow 0 < T_1 < T_2 < \cdots < T_n \leq t < T_{n+1}$

Condition on the event $\{N(t) = n\}$.

What is the joint conditional distribution of $\{T_1, \ldots, T_n\}$?

Let $U_1, U_2, \ldots, U_n$ be IID uniform on $[0, t]$.



pdf: $f_U(x) = \begin{cases} \frac{1}{t} & \text{for } 0 \leq x \leq t \\ 0 & \text{else.} \end{cases}$

Main result:

Joint pdf of $\{T_1, \ldots, T_n\}$ conditioned on $\{N(t) = n\}$ is the same as the joint pdf of $\{U_1, \ldots, U_n\}$.

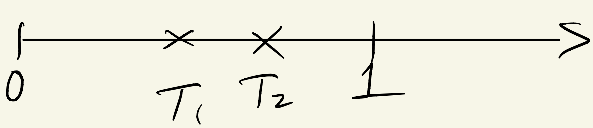
Meaning: given that $N(t) = n$, the first n arrival times $T_1, \ldots, T_n$ are distributed uniformly and independently on $[0, t]$.

Example 1

Suppose $N \sim$ P.P. rate $\lambda = 5 \text{ min}^{-1}$.

Given $N(1) = 2$, find

$$\mathbb{P}\left(T_2 \leq \tfrac{1}{2} \,\middle|\, N(1) = 2\right).$$



Use Theorem: $U_1, U_2 \sim$ IID uniform $[0,1]$.

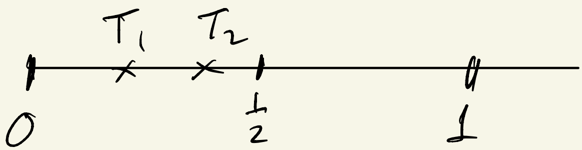
$$P(T_2 \le \tfrac{1}{2} \mid N(1) = 2)$$

$$= P(T_1 < T_2 \le \tfrac{1}{2} \mid N(1) = 2)$$

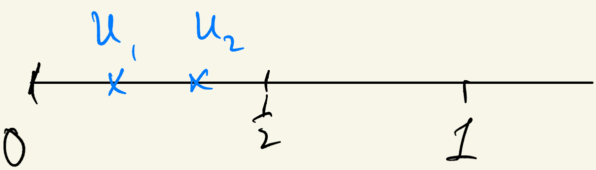
same as (U_1, U_2)

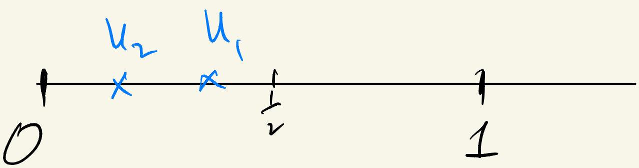
$$= P(U_1 \le \tfrac{1}{2},\ U_2 \le \tfrac{1}{2})$$

$$= P(U_1 \le \tfrac{1}{2})\, P(U_2 \le \tfrac{1}{2})$$



$$= \left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right) = \tfrac{1}{4}.$$





Define the order statistics

$$U_{(1)} = \min(U_1, U_2, \dots, U_n)$$

$$U_{(2)} = \min\{(U_1, U_2, \dots, U_n) \setminus U_{(1)}\}$$

$$U_{(3)} = \min\{(U_1, U_2, \dots, U_n) \setminus (U_{(1)}, U_{(2)})\}.$$

$$\vdots$$

$$U_{(n)} = \max(U_1, U_2, \dots, U_n).$$

not independent : $U_{(1)} \le U_{(2)} \le \dots \le U_{(n)}$

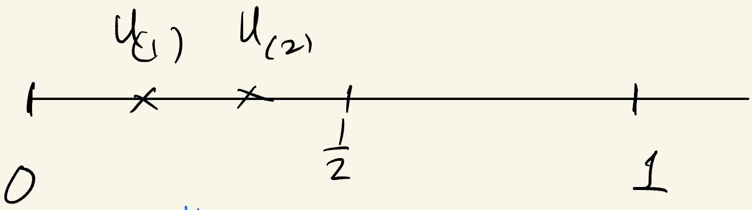
$$\begin{array}{cccc} \downarrow & \downarrow & \dots & \downarrow \\ T_1 & T_2 & & T_n \end{array}$$

<u>Example again:</u>

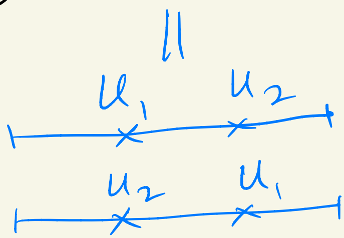
$$\mathbb{P}(T_1 < T_2 \le \tfrac{1}{2} \mid N(1) = 2)$$

$$= \mathbb{P}\left(U_{(1)} < U_{(2)} \leq \tfrac{1}{2} \right)$$

$U_{(1)} = \min(U_1, U_2)$



$U_{(2)} = \max(U_1, U_2)$



$$= \mathbb{P}\left(U_1 < U_2 \leq \tfrac{1}{2} \right) + \mathbb{P}\left(U_2 < U_1 \leq \tfrac{1}{2} \right)$$

$$= \mathbb{P}\left(U_1 \leq \tfrac{1}{2} \quad U_2 \leq \tfrac{1}{2} \right)$$

$$= \mathbb{P}\left(U_1 \leq \tfrac{1}{2} \right) \cdot \mathbb{P}\left(U_2 \leq \tfrac{1}{2} \right)$$

Takeaway

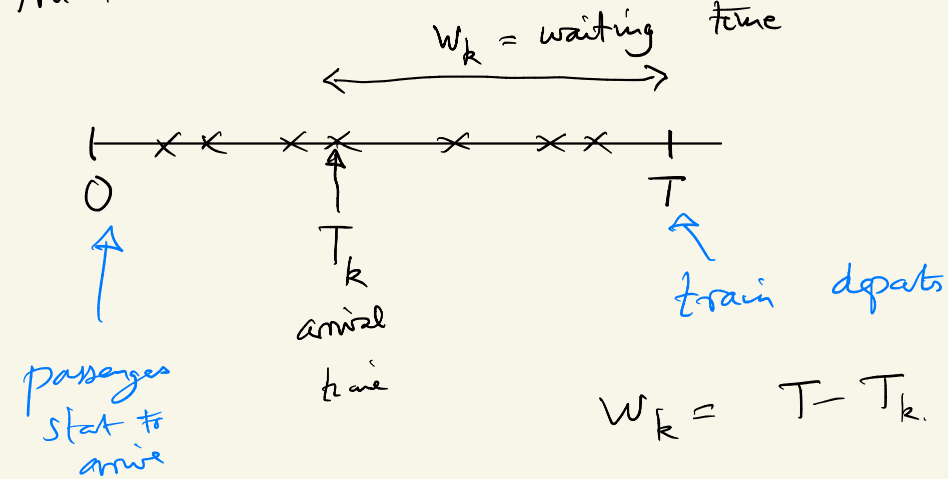
Given $N(t) = n$, the arrival times are uniform and random.

Example 2 (P.S. #7, Ex. 7).

Passengers waiting for train.

Passengers arrive as P.P. rate λ.

Train leaves at time t.



$$W_k = T - T_k.$$

$N(T) =$ number of passengers who depart on the train.

Average waiting time

$$\overline{W} = \frac{1}{N(T)} \left(W_1 + W_2 + \cdots + W_{N(T)} \right).$$

Question: find $\mathbb{E}[\overline{W}]$.

Condition on $T = t$.

Condition on $N(t) = n$.

$$\mathbb{E}[\overline{W} \mid T = t, N(t) = n]$$

$$= \frac{1}{n} \mathbb{E}[W_1 + W_2 + \cdots + W_n \mid T = t, N(t) = n]$$

$$= \frac{1}{n} \mathbb{E}\left[t - T_1 + t - T_2 + \cdots + t - T_n \,\middle|\, T = t, N(t) = n \right]$$

$$= t - \frac{1}{n} \mathbb{E}\left[T_1 + T_2 + \cdots + T_n \,\middle|\, T = t \;\; N(t) = n \right]$$

$\downarrow$ Theorem

$$= t - \frac{1}{n} \mathbb{E}\left[U_{(1)} + U_{(2)} + \cdots + U_{(n)} \right]$$

$\downarrow$ order stats. of $U_1, U_2, \ldots, U_n$ (IID uniform on $[0, t]$)

$\downarrow$ b/c $(n=2)$ $\underbrace{\min(U_1, U_2) + \max(U_1, U_2)}_{= U_1 + U_2}$

$$= t - \frac{1}{n} \mathbb{E}\left[U_1 + U_2 + \cdots + U_n \right]$$

0.3 The birth-death process

This can be viewed as a generalization of the Poisson process where there are both arrivals and departures. For each state j there is a departure rate μ_j and an arrival rate λ_j, and the holding time at state j is exponential with rate $\lambda_j + \mu_j$. The transition matrix is

$$p_{ij} = \begin{cases} \frac{\lambda_i}{\lambda_i+\mu_i} & \text{for } j = i+1 \\ \frac{\mu_i}{\lambda_i+\mu_i} & \text{for } j = i-1 \\ 0 & \text{else} \end{cases}$$

where $\mu_0 = 0$. So the generator is

$$G_{ij} = \begin{cases} -(\lambda_i + \mu_i) & \text{for } i = j \\ \lambda_i & \text{for } j = i+1 \\ \mu_i & \text{for } j = i-1 \\ 0 & \text{else} \end{cases}$$

This is used as a model for population growth where arrivals are births and departures are deaths. It also arises in queuing systems, as we will see shortly. For the moment we just determine the stationary distribution. This satisfies

$$\sum_i \pi_i \, G_{ij} = 0$$

for all j. The equations are

$$\begin{aligned} \lambda_0\pi_0 + \mu_1\pi_1 &= 0 \\ \lambda_{n-1}\pi_{n-1} - (\lambda_n + \mu_n)\pi_n + \mu_{n+1}\pi_{n+1} &= 0 \end{aligned}$$

Rather than solving directly, we note that similarly to the discrete Markov chain, the condition for time reversibility is

$$\pi_i p_{ij}(t) = \pi_j p_{ji}(t)$$

for all $t > 0$. Taking the limit $t \to 0$ this gives

$$\pi_i G_{ij} = \pi_j G_{ji}$$

Applying this to the birth-death model we get the equation

$$\pi_i \lambda_i = \pi_{i+1} \mu_{i+1}$$

which immediately gives the recursion relation

$$\pi_{i+1} = \frac{\lambda_i}{\mu_{i+1}} \pi_i$$

Hence we get the solution for $n \geq 1$

$$\pi_n = \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} \pi_0$$

So the condition for existence of the stationary distribution is

$$\sum_n \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} < \infty$$

Example 2 *The Yule process is a pure birth process in which each individual gives birth at rate λ, independent of all others. Also known as the preferential attachment process. So each starting individual $N_i(0) = 1$ gives rise to a family $N_i(t)$, where $N_i(t)$ is a CTMC with*

$$\lambda_n = n\lambda, \quad n \geq 1$$

The total population at time t is $\sum_i N_i(t)$.

The process $N_i(t)$ has arrival times $T_1 < T_2 < \cdots$, so let $X_1, X_2, \ldots$ be the interarrival times. These are independent exponential r.v.'s with increasing rates $\lambda, 2\lambda, \ldots$.

Now

$$N_i(t) > n \Leftrightarrow T_n < t \Leftrightarrow X_1 + \cdots + X_n < t$$

Thus

$$P(N_i(t) > n) = P(X_1 + \cdots + X_n < t)$$

This probability is given by

$$P(X_1 + \cdots + X_n < t) = \left(1 - e^{-\lambda t}\right)^n$$

This can be proved by induction on n. Thus

$$\begin{aligned} P(N_i(t) > n) &= P(X_1 + \cdots + X_n < t) \\ &= \left(1 - e^{-\lambda t}\right)^n \end{aligned}$$

and so

$$P(N_i(t) = n) = \left(1 - e^{-\lambda t}\right)^{n-1} e^{-\lambda t}$$

This has a useful interpretation; let $p = e^{-\lambda t}$, and consider a biased coin which comes up Heads with probability p. Then $N_i(t)$ is the number of tosses needed until the first Heads appears. Thus $\sum_{i=1}^m N_i(t)$ is the number of tosses needed until we get m Heads. This has a negative binomial distribution:

$$P\left(\sum_{i=1}^m N_i(t) = k\right) = \binom{k-1}{m-1} e^{-m\lambda t} \left(1 - e^{-\lambda t}\right)^{k-1}$$

Finally note that the relation between $P(t)$ and G can also be expressed as a differential equation: the forward equations

$$P(t)'_{ij} = \sum_k P(t)_{ik} G_{kj}$$

and the backward equations

$$P(t)'_{ij} = \sum_k G_{ik} P(t)_{kj}$$