Conditional expectation

If X and Y are discrete r.v.'s then we can compute conditional probabilities as above:

$$\mathbb{P}(X = x|Y = y) = \frac{\mathbb{P}(X = x, Y = y)}{\mathbb{P}(Y = y)}$$

There is also the formula for total probability

$$\mathbb{P}(X = x) = \sum_y \mathbb{P}(X = x|Y = y)\, \mathbb{P}(Y = y)$$

where the sum runs over all possible values of Y. Conditioning is a very useful method for solving problems in probability, because it is often much easier to compute conditional probabilities and then sum over the result to find the 'unconditioned' probability.

Example 3 *Best prize: n distinct prizes arrive in sequence, all have different values, and one is the best. You must pick a prize or else move on to the next one (no going back to earlier ones). Your knowledge consists of the values of the previous prizes. You want to use a strategy that will maximize the probability of selecting the best prize. The prizes are randomly arranged in sequence.*

Strategy: reject the first k prizes, then select the first one which is better than all of these previous ones. Let X be the position of the best prize. Use

$$P_k(best) = \sum_{i=1}^{n} P_k(best|X = i)\, P(X = i)$$

to deduce

$$P_k(best) \simeq \frac{k}{n} \log \frac{n}{k}$$

Find value of k to maximize this.

We define the conditional expectation of X conditioned on the value $Y = y$ as

$$\mathbb{E}[X|Y=y] = \sum_x x\,\mathbb{P}(X=x|Y=y)$$

This number is defined for each possible value of Y. Putting these all together we get the r.v. $\mathbb{E}[X|Y]$ as a function of Y. You should think of $\mathbb{E}[X|Y]$ as a random variable which is determined by the random variable Y, like Y^2 or e^{tY}: if you know the value of Y, then you know the value of $\mathbb{E}[X|Y]$. There is a very useful relation between the conditional expectation $\mathbb{E}[X|Y]$ and the 'unconditioned' expectation $\mathbb{E}[X]$.

Theorem 1

$$\mathbb{E}\Big[\mathbb{E}[X|Y]\Big] = \mathbb{E}[X]$$

Note that on the left side we are first averaging over X, with Y fixed, and then we average over Y. On the right side we do it all in just one step.

$$\text{LHS} = \mathbb{E}_Y\Big[\mathbb{E}_X[X|Y]\Big]$$

average over X with Y fixed.

average over Y

$$\text{RHS} = \mathbb{E}[X]$$

average over X

Why are these equal?

$$\text{LHS} = \sum_y \mathbb{E}[X|Y=y]\,\mathbb{P}(Y=y)$$

$$= \sum_y \Big\{ \sum_x x\,\mathbb{P}(X=x|Y=y) \Big\}\,\mathbb{P}(Y=y).$$

switch order

$$= \sum_x x \sum_y \mathbb{P}(X=x|Y=y)\,\mathbb{P}(Y=y)$$

$$= \sum_x x \sum_y \frac{\mathbb{P}(X=x, Y=y)}{\cancel{\mathbb{P}(Y=y)}} \cancel{\mathbb{P}(Y=y)}$$

$$= \sum_x x \left\{ \sum_y \mathbb{P}(X=x, Y=y) \right\}$$

Sum over y gives the marginal pdf for X

$$= \sum_x x \, \mathbb{P}(X=x).$$

$$= \mathbb{E}[X].$$

Example 4 *Let $N, X_1, X_2, \ldots$ be independent, where X_i are IID. Define*

$$Y = \sum_{i=1}^{N} X_i$$

Y = total amount of claims in a month.

For example, N is the number of insurance claims in a month, and X_i is the size of the i^{th} claim. Then

$$\mathbb{E}[Y] = \mathbb{E}[X]\,\mathbb{E}[N]$$

Find
$$\mathbb{E}[Y] = \mathbb{E}\left[\sum_{i=1}^{N} X_i\right]$$
$$= \mathbb{E}[X_1 + X_2 + X_3 + \cdots + X_N]$$
$$\overset{?}{\cancel{=}} \mathbb{E}[X_1] + \mathbb{E}[X_2] + \cdots + \mathbb{E}[X_N].$$

NO! b/c N is random

Condition on N, this fixes the number of terms:

$$\mathbb{E}[Y \mid N = n] = \mathbb{E}\left[\sum_{i=1}^{N} X_i \,\middle|\, N = n\right]$$
$$= \mathbb{E}\left[\sum_{i=1}^{n} X_i \,\middle|\, N = n\right]$$
$$= \mathbb{E}\left[\sum_{i=1}^{n} X_i\right]$$

b/c X_i are indep. of N

$$= \mathbb{E}[X_1 + X_2 + \cdots + X_n]$$
$$= \mathbb{E}[X_1] + \mathbb{E}[X_2] + \cdots + \mathbb{E}[X_n]$$

Let $\mu = \mathbb{E}[X_i]$ (all i, all the same)

$$\mathbb{E}[Y \mid N=n] = \mu + \mu + \cdots + \mu$$
$$= n\mu.$$

Use the conditioning theorem

<u>under the conditioning</u>:

$$\mathbb{E}[Y] = \mathbb{E}\big[\mathbb{E}[Y \mid N]\big]$$
$$= \sum_n \mathbb{E}[Y \mid N=n]\, \mathbb{P}(N=n)$$
$$= \sum_n n\mu\, \mathbb{P}(N=n)$$

$$= \mu \sum_n n \, \mathbb{P}(N=n)$$

$$= \mu \, \mathbb{E}[N].$$

$$\boxed{\mathbb{E}[Y] = \mathbb{E}[X] \, \mathbb{E}[N]}$$

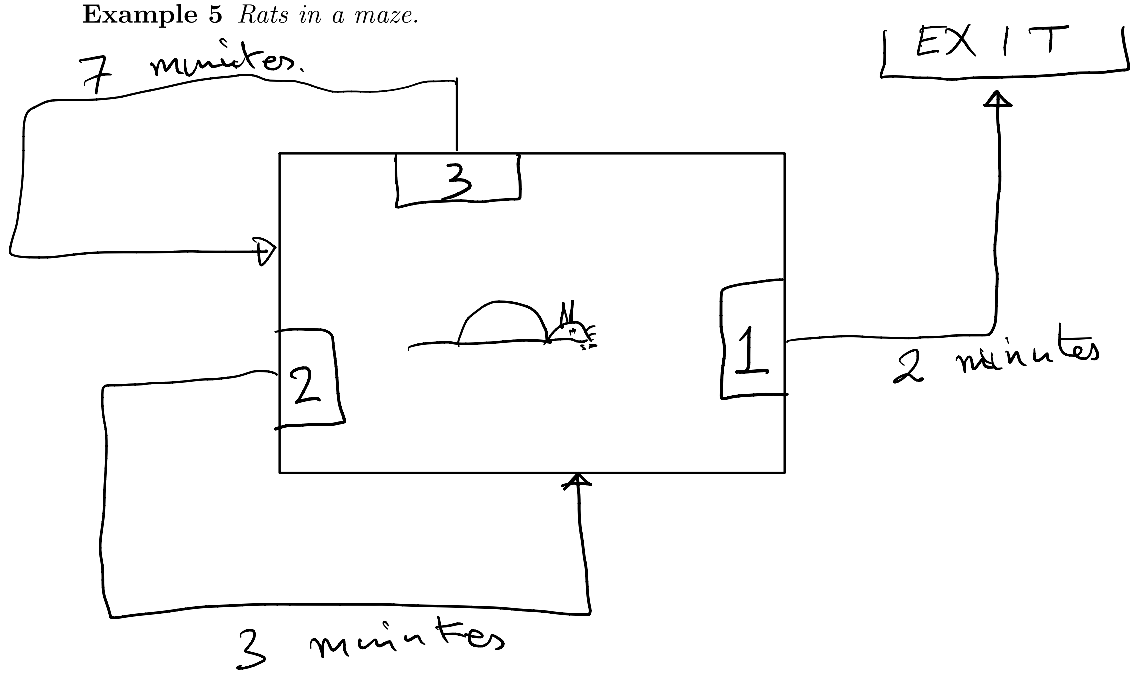
$$Y = \sum_{i=1}^{N} X_i$$ — random sum of random variables.

mean of total

= (mean of each claim)

× (mean number of claims).

Example 5 *Rats in a maze.*



When rat enters the room, it randomly chooses one of the doors.

Rat has no memory!

T = time until rat escapes.

Find $E[T]$.

<u>Use conditioning:</u>

conditioning on first door chosen.

$$\mathbb{E}[T \mid 1] = 2$$

$$\mathbb{E}[T \mid 2] = 3 + \mathbb{E}[T]$$

clock starts over when it returns to the room

$$\mathbb{E}[T \mid 3] = 7 + \mathbb{E}[T].$$

$$\Rightarrow \mathbb{E}[T] = \mathbb{E}[T \mid 1]\, \mathbb{P}(\text{chose } 1) + \mathbb{E}[T \mid 2]\, \mathbb{P}(\text{chose } 2) + \mathbb{E}[T \mid 3]\, \mathbb{P}(\text{chose } 3)$$

$$\mathbb{E}[T] = 2\left(\frac{1}{3}\right) + \left(3 + \mathbb{E}[T]\right)\frac{1}{3} + \left(7 + \mathbb{E}[T]\right)\frac{1}{3}$$

Solve for $\mathbb{E}[T] = 12$

Conditioning with respect to a continuous random variable

Although we will not define conditioning with respect to a continuous random variable in full detail, it is a very useful notion. Let X be a continuous random variable, then for any event A we have

$$\mathbb{P}(A) = \int_{-\infty}^{\infty} \mathbb{P}(A \mid X = x)\, f_X(x) dx$$

pdf of X

It is often convenient to use a shorthand and write this as

average over X

$$\mathbb{P}(A) = \mathbb{E}[\mathbb{P}(A \mid X)]$$

where it is understood that the quantity $\mathbb{P}(A \mid X)$ is a random variable which is a function of X. Many interesting examples arise when the event A involves another random variable.

Example 6 *Suppose that X, Y are independent exponentials with mean 1 and we want $\mathbb{P}(X+Y \geq z)$ where $z \geq 0$. Now*

$$\mathbb{P}(X+Y \geq z \,|\, X=x) = \mathbb{P}(Y \geq z-x \,|\, X=x) = \mathbb{P}(Y \geq z-x)$$

because they are independent. Thus

$$\mathbb{P}(Y \geq z-x) = \begin{cases} e^{-(z-x)} & \textit{for } z-x \geq 0 \\ 1 & \textit{for } z-x < 0 \end{cases}$$

and hence

$$\begin{aligned}
\mathbb{P}(X+Y \geq z) &= \int_0^\infty \mathbb{P}(X+Y \geq z \,|\, X=x) e^{-x} dx \\
&= \int_0^\infty \mathbb{P}(Y \geq z-x) e^{-x} dx \\
&= \int_0^z e^{-z} dx + \int_z^\infty e^{-x} dx \\
&= ze^{-z} + e^{-z}
\end{aligned}$$

X, Y are independent.

Both are exponential r.v.'s with mean 1.

pdf of X: $f_X(x) = e^{-x}$ $(x \geq 0)$

pdf of Y: $f_Y(x) = e^{-x}$ $(x \geq 0)$

Want $\mathbb{P}(X+Y \geq z)$.

Calculate this by conditioning on Y:

$$\mathbb{P}(X+Y \geq z \,|\, Y=y)$$

$$= \mathbb{P}(X+y \geq z \,|\, Y=y)$$

b/c X, Y independent

$$= \mathbb{P}(X+y \geq z)$$

$$= \mathbb{P}(X \geq z-y)$$

Recall: for any $t > 0$,

$$\mathbb{P}(X \geq t) = e^{-t}$$ (special property of exp. r.v.)

for any $t \leq 0$,

$$\mathbb{P}(X \geq t) = 1$$ (b/c X is positive).

$$\Rightarrow \mathbb{P}(X \geq z-y) = \begin{cases} e^{-(z-y)} & \text{if } z-y \geq 0 \\ 1 & \text{if } z-y < 0. \end{cases}$$

Therefore

$$\mathbb{P}(X+Y \geq z) = \int_0^\infty P(X+Y \geq z \mid Y=y) \cdot e^{-y} \, dy$$

pdf of Y

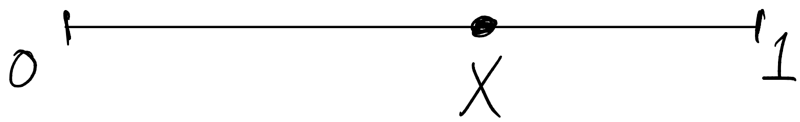
$$= \int_0^z e^{-(z-y)} e^{-y} \, dy$$

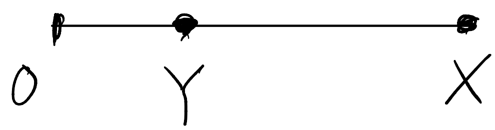
$$+ \int_z^\infty 1 \cdot e^{-y} \, dy$$

$$= z e^{-z} + e^{-z}.$$

The same technique can be applied even when the random variables are dependent.

Example 7 *Suppose X is uniform on $[0,1]$ and Y is uniform on $[0,X]$. Calculate $\mathbb{E}[Y]$.*





Know $\mathbb{E}[X] = \frac{1}{2}$

$$\mathbb{E}[Y] = \ ?$$

Condition on X:

$$\mathbb{E}[Y \mid X = x] = \frac{x}{2}$$

Undo conditioning:

$$\mathbb{E}[Y] = \int_0^1 \mathbb{E}[Y \mid X = x] \, f_X(x) \, dx$$

$$= \int_0^1 \frac{x}{2} \cdot 1 \cdot dx$$

$$= \frac{1}{4}$$

Abstract: $\mathbb{E}[Y \mid X = x] = \frac{x}{2}$

$$\mathbb{E}[Y|X] = \frac{X}{2}$$

Advantage:

$$\mathbb{E}[Y] = \mathbb{E}\big[\mathbb{E}[Y|X]\big]$$

$$= \mathbb{E}\left[\frac{X}{2}\right]$$

$$= \frac{1}{2}\,\mathbb{E}[X]$$

$$= \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}.$$

Memoryless property of exponential r.v.'s

Conditioning can have quite unexpected effects on the distributions of random variables. One well-known example is the memoryless property of the exponential random variable. Suppose that X is exponential with rate λ, so that its pdf is

$$f_X(t) = \lambda e^{-\lambda t} \quad \text{for } t \geq 0$$

$X \geq 0$

$\lambda = \text{rate} > 0.$

$\frac{1}{\lambda} = \text{mean} = \mathbb{E}[X].$

Then an easy calculation shows that

$$\mathbb{P}(X > t) = e^{-\lambda t}$$

for all $t > 0$. If we condition on this event we find that

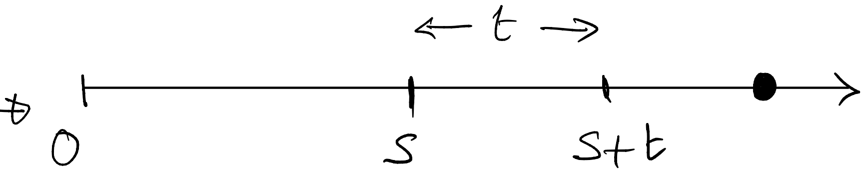
$$\mathbb{P}(X > t + s \,|\, X > s) = e^{-\lambda t}$$

This can be interpreted as a memoryless property by viewing X as the time to failure of a device. Conditioning on the event $\{X > s\}$ means that we condition on the device not having failed up to time s. The result above says that given this event, the subsequent lifetime of the device has the same distribution as a fresh lifetime.

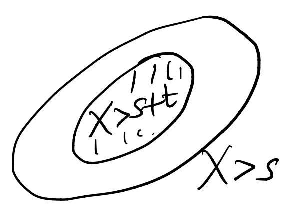
$t > 0$:

$$P(X > t) = \int_t^\infty \lambda e^{-\lambda u}\, du$$

$$= e^{-\lambda t}$$



Condition on $\{X > s\}$:



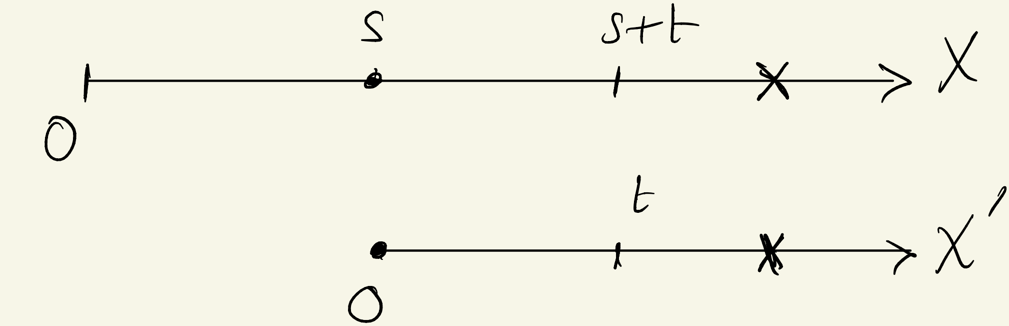
$$P(X > s+t \mid X > s)$$

$$= \frac{P(X > s+t,\ X > s)}{P(X > s)}$$

$$= \frac{P(X > s+t)}{P(X > s)}$$

$$= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} \quad \text{(b/c } X \text{ is expon.)}$$

$$= e^{-\lambda t}$$



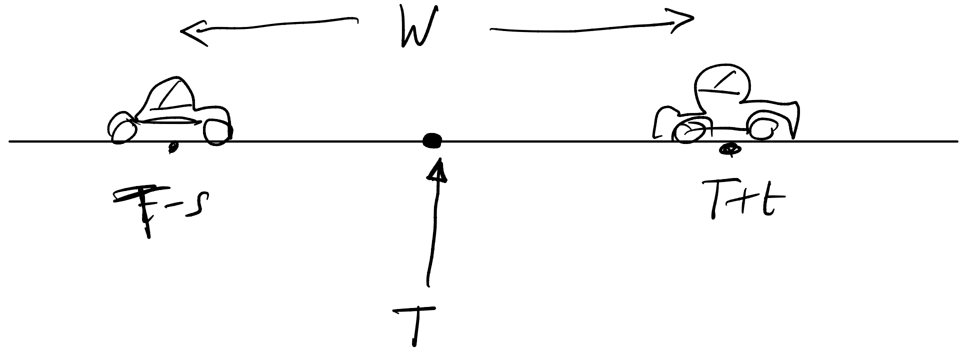
$$\mathbb{P}(X > s+t \mid X > s) = \mathbb{P}(X' > t) = e^{-\lambda t}$$

X' is a new copy of the exponential r.v. that starts at time s.

Given that X is at least as big

as s, then the remaining time for X is distributed as an exponential with the same rate, i.e. it starts over again at time s.

Example 8 *Cars pass a point on a highway. The times between successive cars are independent exponential random variables with the same mean m. Suppose at a random time you stand at the point on the highway. What is the mean time until the next car passes?*



T = time we stand by highway.

W = time between successive cars.

$W \sim$ exp. rate λ.

S = time since previous car passed.

previous car at time $T-s$.

$\mathbb{P}$(next car passes at $T+t$).

$$= \mathbb{P}(W > t+s \mid W > s)$$

$$= \mathbb{P}(W > t)$$

$$= e^{-\lambda t}$$ by memoryless property

Mean time until next car:

$$\mathbb{E}[W \mid W > s]$$

$$= s + \mathbb{E}[\text{additional time}]$$

$$= s + \frac{1}{\lambda}$$ ($\frac{1}{\lambda}$ = mean time between cars).

Say $\frac{1}{\lambda} = 30$ sec = mean time between cars

Then if you arrive at random time, the mean time until next cars is 30 sec.