MTH 7241: Fall 2022

Practice Problems for Quiz 1

1). Four balls are chosen at random from a box which contains two Red balls and some number of Green balls. The probability that both Red balls are chosen is twice the probability that neither Red ball is chosen. How many Green balls are in the box?

$$\boxed{\begin{array}{ll} 2 & \text{Red} \\ n & \text{Green} \end{array}}$$

$$\mathbb{P}(\text{both Red}) = \binom{4}{2} \mathbb{P}(RRGG)$$

$$= 6 \frac{2}{n+2} \frac{1}{n+1}$$

$$\mathbb{P}(\text{neither Red}) = \mathbb{P}(GGGG)$$

$$= \frac{n}{n+2} \frac{n-1}{n+1} \frac{n-2}{n} \frac{n-3}{n-1}$$

$$\Rightarrow \frac{12}{(n+2)(n+1)} = 2 \frac{(n-2)(n-3)}{(n+2)(n+1)}$$

$$\Rightarrow n = 5$$

2). A maze for rats is constructed with two doors; door 1 immediately leads to the exit, door 2 leads back to the maze after 1 minute. Assume that a rat is equally likely to choose either door at all times, and that if several rats are in the maze then they choose independently.
a). A rat is put in the maze. Find the expected time until it escapes.
b). Two rats are put in the maze. Find the expected time until the first escape occurs, and find the expected time until both escape.
c). Suppose n rats are put in the maze. Find the expected time until the first escape occurs. [Hint: you may want to condition on the first choices made by all the rats].

a) T_1 = time to escape

Condition on first choice:

$$E[T_1] = 0 \cdot \tfrac{1}{2} + (1 + E[T_1])\tfrac{1}{2}$$

$$\Rightarrow E[T_1] = 1.$$

b) $T_{min}^{(2)}$ = time for first escape

$T_{max}^{(2)}$ = time for second escape

Condition on first choice of both rats:

$$E[T_{min}^{(2)}] = E[T_{min}^{(2)} \mid \text{at least one escapes on first try}] \cdot \left(\tfrac{3}{4}\right)$$

$$+ E[T_{min}^{(2)} \mid \text{neither escape first try}] \cdot \left(\tfrac{1}{4}\right)$$

$$= 0 \cdot \tfrac{3}{4} + (1 + E[T_{min}^{(2)}]) \cdot \tfrac{1}{4}$$

$$\Rightarrow E[T_{min}^{(2)}] = \tfrac{1}{3}$$

time for Rat #1

time for Rat #2

Now $T_{min}^{(2)} + T_{max}^{(2)} = T_1 + T_2$

$$\Rightarrow \quad \mathbb{E}\left[T_{min}^{(2)}\right] + \mathbb{E}\left[T_{max}^{(2)}\right] = \mathbb{E}[T_1] + \mathbb{E}[T_2]$$

$$\frac{1}{3} + \mathbb{E}\left[T_{max}^{(2)}\right] = 1 + 1$$

$$\Rightarrow \quad \mathbb{E}\left[T_{max}^{(2)}\right] = \frac{5}{3}$$

c) T_{min} = time to first escape

$$\mathbb{E}[T_{min}] = \mathbb{E}[T_{min} \mid \text{none chose door } 1] \cdot \left(\frac{1}{2}\right)^n + \mathbb{E}[T_{min} \mid \text{at least one chose door } 1] \cdot \left(1 - \left(\frac{1}{2}\right)^n\right)$$

$$= \left(1 + \mathbb{E}[T_{min}]\right) 2^{-n}$$

$$\Rightarrow \mathbb{E}[T_{min}] = \frac{1}{2^n - 1}$$

3). A biased coin has probability p of coming up Heads. The coin is tossed repeatedly. Let N_2 be the number of tosses until the first occurrence of the sequence (Heads, Tails). Use the conditional expectation method to compute $\mathrm{E}[N_2]$. [Hint: follow the methodology used in class, when we computed the expected number of tosses until the first occurrence of Heads. You will find it useful to first separately compute $\mathrm{E}[N_2|H_1]$ where H_1 is the event that the first toss comes up Heads].

$$\mathbb{E}[N_2 | H_1] = \mathbb{E}[N_2 | H_1, H_2] \, \mathbb{P}(H_2 | H_1) + \mathbb{E}[N_2 | H_1, T_2] \, \mathbb{P}(T_2 | H_1)$$

$$= (1 + \mathbb{E}[N_2 | H_1])p + 2 \cdot (1-p)$$

$$\Rightarrow \mathbb{E}[N_2 | H_1] = \frac{2-p}{1-p}$$

Now

$$\mathbb{E}[N_2] = \mathbb{E}[N_2 | H_1] \, P(H_1) + \mathbb{E}[N_2 | T_1] \, \mathbb{P}(T_1)$$

$$= \frac{2-p}{1-p} p + (1 + \mathbb{E}[N_2])(1-p)$$

$$\Rightarrow \mathbb{E}[N_2] = \frac{1}{p(1-p)}$$

4). Mary's bowl of spaghetti contains n strands. She selects two ends at random and joins them together. She does this until there are no ends left. What is the expected number of spaghetti hoops in the bowl?

Let $L_k = \begin{cases} 1 & \text{if create loop at } k^{\text{th}} \text{ step} \\ 0 & \text{else} \end{cases}$

N_n = total number of loops at end

$= L_n + L_{n-1} + L_{n-2} + \cdots + L_1$

$\Rightarrow \mathbb{E}[N_n] = \mathbb{E}[L_n] + \mathbb{E}[L_{n-1}] + \cdots + \mathbb{E}[L_1]$

Now $\mathbb{E}[L_1] = \mathbb{P}(L_1 = 1)$

$= \mathbb{P}(\text{create loop at step } 1)$

$= \dfrac{1}{2n-1}$ ← after picking first end, must pick other end of that string out of $(2n-1)$ ends

and $\mathbb{E}[L_k] = \mathbb{P}(L_k = 1)$

$= \mathbb{P}(\text{create loop at step } k)$

$= \dfrac{1}{2n-2k+1}$ ← have $2n-2k+2$ ends, pick one, then next pick must be its other end.

$\Rightarrow \mathbb{E}[N_n] = \dfrac{1}{2n-1} + \dfrac{1}{2n-3} + \cdots + \dfrac{1}{3} + 1.$

5). A patient walks in who has a fever and chills. The doctor wonders, "What is the chance that this patient has tuberculosis given the symptoms I am seeing?" Let A be the event that the patient has TB, let B be the event that the patient has fever and chills. Assume that TB is present in 0.01% of the population, whereas 3% of the population exhibits fever and chills. Assume that $P(B|A) = 0.5$. What is the answer to the doctor's question?

A: patient has TB

$$\mathbb{P}(A) = 10^{-4}$$

B: patient has fever and chills

$$\mathbb{P}(B) = 0.03$$

$$P(B|A) = 0.5$$

Doctor: what is $\mathbb{P}(A|B)$?

Use Bayes

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A)\,\mathbb{P}(A)}{\mathbb{P}(B)}$$

$$= \frac{(0.5)\,10^{-4}}{0.03}$$

$$= 1.67 \times 10^{-3}.$$

6). Let X_i, $i = 1, \ldots, 10$ be independent random variables, each being uniformly distributed over $[0,1]$. Use the Central Limit Theorem to estimate the probability that $X_1 + \cdots + X_{10}$ exceeds 7.

Let $Y = X_1 + \cdots + X_{10}$

$$\mathbb{E}[X_i] = \mu = \frac{1}{2}$$

$$\mathrm{VAR}[X_i] = \sigma^2 = \frac{1}{12}.$$

$$Z_{10} = \frac{Y - 10\mu}{\sqrt{10}\,\sigma} = \frac{Y-5}{\sqrt{10}\,\frac{1}{\sqrt{12}}}.$$

$$\mathbb{P}(Y > 7) = \mathbb{P}\left(Z_{10} > \frac{7-5}{\sqrt{10/12}}\right)$$

$$= \mathbb{P}(Z_{10} > 2\sqrt{1.2})$$

CLT

$$\approx \mathbb{P}(Z > 2\sqrt{1.2})$$

$$= 1 - \mathbb{P}(Z \leq 2.19)$$

$$= 1 - 0.9857$$

$$= 0.0143$$

Tables of the Normal Distribution

Probability Content from -oo to Z

Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

Far Right Tail Probabilities

Z	P{Z to oo}	Z	P{Z to oo}	Z	P{Z to oo}	Z	P{Z to oo}
2.0	0.02275	3.0	0.001350	4.0	0.00003167	5.0	2.867 E-7
2.1	0.01786	3.1	0.0009676	4.1	0.00002066	5.5	1.899 E-8
2.2	0.01390	3.2	0.0006871	4.2	0.00001335	6.0	9.866 E-10
2.3	0.01072	3.3	0.0004834	4.3	0.00000854	6.5	4.016 E-11
2.4	0.00820	3.4	0.0003369	4.4	0.000005413	7.0	1.280 E-12
2.5	0.00621	3.5	0.0002326	4.5	0.000003398	7.5	3.191 E-14
2.6	0.004661	3.6	0.0001591	4.6	0.000002112	8.0	6.221 E-16
2.7	0.003467	3.7	0.0001078	4.7	0.000001300	8.5	9.480 E-18
2.8	0.002555	3.8	0.00007235	4.8	7.933 E-7	9.0	1.129 E-19
2.9	0.001866	3.9	0.00004810	4.9	4.792 E-7	9.5	1.049 E-21

They are produced by APL programs written by the author,
William Knight

7). The random variable X has 10 possible values, with the pdf

Outcome	1	2	3	$\cdots$	10
Probability	p_1	p_2	p_3	$\cdots$	p_{10}

X is measured 1000 times and the frequency of each of its possible values is recorded. The goodness of fit test is used to determine if the observed data supports this pdf.

a) State the number of degrees of freedom, state the null hypothesis, and state the distribution of the test statistic under the null hypothesis. Use the tables to find the critical value of the test statistic at the 5% significance level.

b) The test statistic for the data is found to be 18.34. Decide if the observed data supports the null hypothesis at the 5% significance level.

c) Decide if the observed data supports the null hypothesis at the 1% significance level.

a) $df = 10 - 1 = 9.$

H_0: the pdf of X is given by the table.

Test statistic $D \sim \chi^2(9)$.

5% sig. level: crit. value $= \chi^2_{0.95, 9}$

$= 16.919$

b) $D = 18.34.$

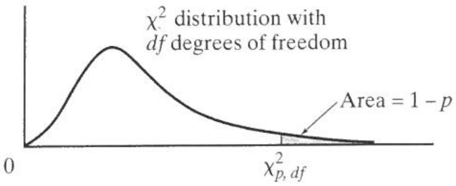
Decision rule: reject H_0 if $D > 16.919$

Conclusion: we reject H_0. Data does not support this pdf.

c) 1% sig. level: crit. value $= \chi^2_{0.99, 9} = 21.666$

Conclusion: do not reject H_0. Data supports this pdf.

Table A.3 Upper and Lower Percentiles of χ^2 Distributions

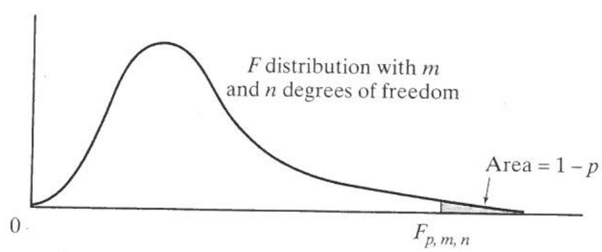


	p							
df	0.010	0.025	0.050	0.10	0.90	0.95	0.975	0.99
1	0.000157	0.000982	0.00393	0.0158	2.706	3.841	5.024	6.635
2	0.0201	0.0506	0.103	0.211	4.605	5.991	7.378	9.210
3	0.115	0.216	0.352	0.584	6.251	7.815	9.348	11.345
4	0.297	0.484	0.711	1.064	7.779	9.488	11.143	13.277
5	0.554	0.831	1.145	1.610	9.236	11.070	12.832	15.086
6	0.872	1.237	1.635	2.204	10.645	12.592	14.449	16.812
7	1.239	1.690	2.167	2.833	12.017	14.067	16.013	18.475
8	1.646	2.180	2.733	3.490	13.362	15.507	17.535	20.090
9	2.088	2.700	3.325	4.168	14.684	16.919	19.023	21.666
10	2.558	3.247	3.940	4.865	15.987	18.307	20.483	23.209
11	3.053	3.816	4.575	5.578	17.275	19.675	21.920	24.725
12	3.571	4.404	5.226	6.304	18.549	21.026	23.336	26.217
13	4.107	5.009	5.892	7.042	19.812	22.362	24.736	27.688
14	4.660	5.629	6.571	7.790	21.064	23.685	26.119	29.141
15	5.229	6.262	7.261	8.547	22.307	24.996	27.488	30.578
16	5.812	6.908	7.962	9.312	23.542	26.296	28.845	32.000
17	6.408	7.564	8.672	10.085	24.769	27.587	30.191	33.409
18	7.015	8.231	9.390	10.865	25.989	28.869	31.526	34.805
19	7.633	8.907	10.117	11.651	27.204	30.144	32.852	36.191
20	8.260	9.591	10.851	12.443	28.412	31.410	34.170	37.566
21	8.897	10.283	11.591	13.240	29.615	32.671	35.479	38.932
22	9.542	10.982	12.338	14.041	30.813	33.924	36.781	40.289
23	10.196	11.688	13.091	14.848	32.007	35.172	38.076	41.638
24	10.856	12.401	13.848	15.659	33.196	36.415	39.364	42.980
25	11.524	13.120	14.611	16.473	34.382	37.652	40.646	44.314
26	12.198	13.844	15.379	17.292	35.563	38.885	41.923	45.642
27	12.879	14.573	16.151	18.114	36.741	40.113	43.194	46.963
28	13.565	15.308	16.928	18.939	37.916	41.337	44.461	48.278
29	14.256	16.047	17.708	19.768	39.087	42.557	45.722	49.588
30	14.953	16.791	18.493	20.599	40.256	43.773	46.979	50.892
31	15.655	17.539	19.281	21.434	41.422	44.985	48.232	52.191
32	16.362	18.291	20.072	22.271	42.585	46.194	49.480	53.486
33	17.073	19.047	20.867	23.110	43.745	47.400	50.725	54.776
34	17.789	19.806	21.664	23.952	44.903	48.602	51.966	56.061

Table A.3 Upper and Lower Percentiles of χ^2 Distributions (*cont.*)

	p							
df	0.010	0.025	0.050	0.10	0.90	0.95	0.975	0.99
35	18.509	20.569	22.465	24.797	46.059	49.802	53.203	57.342
36	19.233	21.336	23.269	25.643	47.212	50.998	54.437	58.619
37	19.960	22.106	24.075	26.492	48.363	52.192	55.668	59.892
38	20.691	22.878	24.884	27.343	49.513	53.384	56.895	61.162
39	21.426	23.654	25.695	28.196	50.660	54.572	58.120	62.428
40	22.164	24.433	26.509	29.051	51.805	55.758	59.342	63.691
41	22.906	25.215	27.326	29.907	52.949	56.942	60.561	64.950
42	23.650	25.999	28.144	30.765	54.090	58.124	61.777	66.206
43	24.398	26.785	28.965	31.625	55.230	59.304	62.990	67.459
44	25.148	27.575	29.787	32.487	56.369	60.481	64.201	68.709
45	25.901	28.366	30.612	33.350	57.505	61.656	65.410	69.957
46	26.657	29.160	31.439	34.215	58.641	62.830	66.617	71.201
47	27.416	29.956	32.268	35.081	59.774	64.001	67.821	72.443
48	28.177	30.755	33.098	35.949	60.907	65.171	69.023	73.683
49	28.941	31.555	33.930	36.818	62.038	66.339	70.222	74.919
50	29.707	32.357	34.764	37.689	63.167	67.505	71.420	76.154

Source: *Scientific Tables*, 6th ed. (Basel, Switzerland: J.R. Geigy, 1962), p. 36.



The figure above illustrates the percentiles of the F distributions shown in Table A.4. Table A.4 is used with permission from Wilfrid J. Dixon and Frank J. Massey, Jr., *Introduction to Statistical Analysis* 2nd ed. (New York: McGraw-Hill, 1957), pp. 389–404.