Lab 2b: Change of coordinates for circular orbit

MATH 5110: Applied Linear Algebra and Matrix Analysis
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1 Introduction

The goal of this Lab is to use a change of coordinates to compute circular orbital motion.

1.1 Uniform circular orbit

Rotation in the $x - y$ plane about the origin by angle θ is a linear map which we call R_2 (where the subscript '2' refers to two dimensions). In the standard basis we have

$$R_2(\vec{e_1}) = \cos\theta\vec{e_1} + \sin\theta\vec{e_2}, \quad R_2(\vec{e_2}) = -\sin\theta\vec{e_1} + \cos\theta\vec{e_2} \tag{1}$$

So the matrix of the rotation is

$$[R_2] = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{2}$$

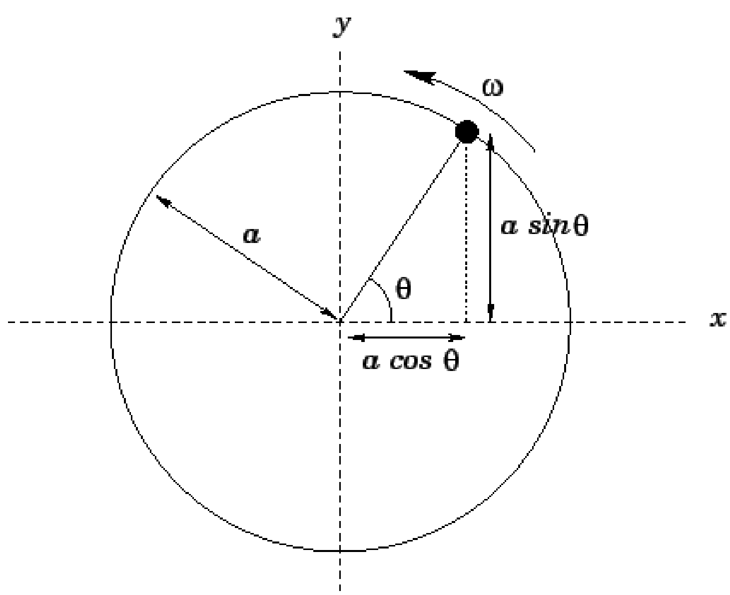
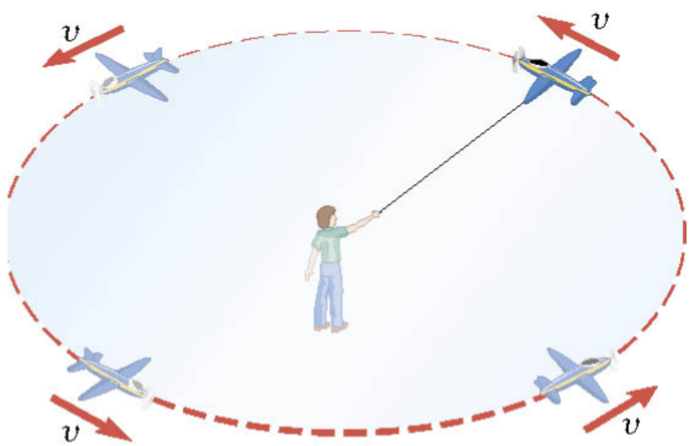


Figure 99: *Uniform circular motion.*

Now consider rotation in space by angle θ about the z-axis. We will denote this 3-dimensional linear map by R_3.



Again in the standard basis we have

$$R_3(\vec{e_1}) = \cos\theta\vec{e_1} + \sin\theta\vec{e_2}, \quad R_3(\vec{e_2}) = -\sin\theta\vec{e_1} + \cos\theta\vec{e_2}, \quad R_3(\vec{e_3}) = \vec{e_3} \tag{3}$$

so the matrix of R_3 in the standard basis is

$$[R_3]_{EE} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4}$$

1.2 Example: equatorial circular orbit

A satellite is moving in a circular orbit at the Equator at height 250 km and angular velocity $\omega = 4.22$ rad/hr. The current position of the satellite is

$$x\vec{(0)} = (0.54\, r, 0.84\, r, 0)^T \tag{5}$$

where $r = 6606$ km is the radius of the orbit. Find the position after 1 hour.

Solution: the position is

$$x\vec{(1)} = R_3(x\vec{(0)}) = [R_3]_{EE}\, x\vec{(0)} \tag{6}$$

where $[R_3]_{EE}$ is the rotation matrix (4). So we need to calculate the rotation angle θ. The formula is

$$\theta = \omega t \tag{7}$$

where ω is angular velocity and t is time. In our case $\omega = 4.22$ rad/hr and $t = 1$ hour so

$$\theta = 4.22 \text{ rad} \tag{8}$$

Hence

$$[R_3]_{EE} = \begin{pmatrix} -0.47 & 0.88 & 0 \\ -0.88 & -0.47 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{9}$$

and so

$$x(\vec{1}) = [R_3]_{EE}\, x(\vec{0}) = \begin{pmatrix} -0.47 & 0.88 & 0 \\ -0.88 & -0.47 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0.54\,r \\ 0.84\,r \\ 0 \end{pmatrix} = \begin{pmatrix} 0.49\,r \\ -0.87\,r \\ 0 \end{pmatrix} \tag{10}$$

1.3 Non-equatorial orbits

In general satellite orbits are not equatorial:

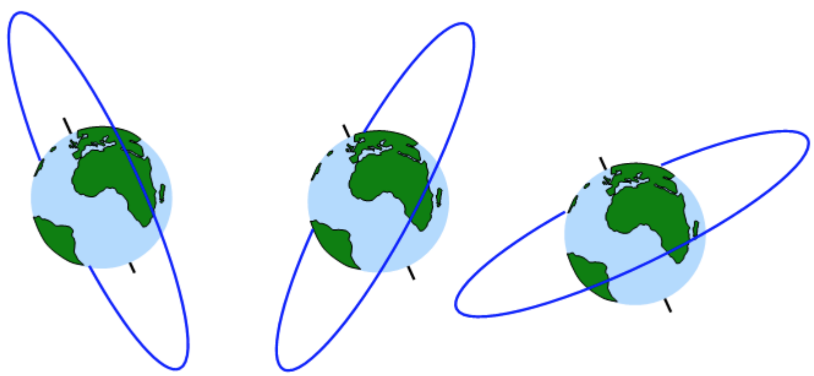


Figure 4 Figure 5 Figure 6

The orbit is specified by three orthogonal vectors $\vec{P}, \vec{Q}, \vec{W}$. The vector $\vec{W}$ is perpendicular to the orbit, and is equal to $\vec{e}_3$ for the special case of the equatorial orbit. The other two vectors $\vec{P}, \vec{Q}$ are in the plane of the orbit, and are equal to $\vec{e}_1, \vec{e}_2$ respectively for the special case of the equatorial orbit. All three vectors have unit length, and form a right-handed system, so that

$$\vec{Q} = \vec{W} \times \vec{P} \tag{11}$$

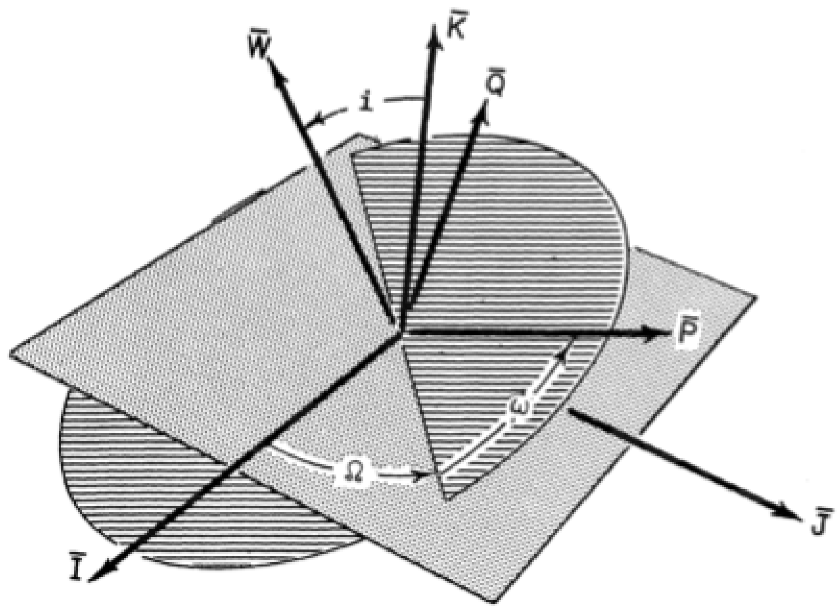


Figure 2.6-2 Relationship between **PQW** and **IJK**

As usual, E denotes the standard basis $\{\vec{e_1}, \vec{e_2}, \vec{e_3}\}$ (written $\vec{I}, \vec{J}, \vec{K}$ in the figure above) and we let U denote the basis $\{\vec{P}, \vec{Q}, \vec{W}\}$. We will consider circular motion in the plane containing the vectors $\vec{P}, \vec{Q}$. So we define a *new linear map* $\widehat{R_3}$ which is counterclockwise rotation about the $\vec{W}$ direction through angle θ. The matrix implementing $\widehat{R_3}$ in the standard basis is $[\widehat{R_3}]_{EE}$, and in the following task you will determine this 3×3 matrix for a particular non-equatorial orbit an angle θ. Note that in the special case of an equatorial orbit this matrix $[\widehat{R_3}]_{EE}$ is equal to the matrix (4). Note also that in the U-basis the matrix representation for $\widehat{R_3}$ is much simpler, namely

$$[\widehat{R_3}]_{UU} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{12}$$

which of course is the same matrix as (4) (though now with a different meaning).

1.4 Task: non-equatorial circular orbit

The task is to compute the future position of a satellite moving along a circular non-equatorial orbit, at height 600 km and angular velocity $\omega = 3.91$ rad/hr. The perpendicular vector $\vec{W}$ for the orbit is

$$\vec{W} = \frac{1}{3}\,(-1, -2, 2)^T \tag{13}$$

The current position of the satellite (measured above the ground in the standard basis) is

$$\vec{x(0)} = [\vec{x(0)}]_E = 117.67\ (4, 1, 3)^T \tag{14}$$

1.4.1 Task 1

Compute the vectors $\vec{P}, \vec{Q}$ for the orbit. Note that $\vec{P}$ should be parallel to $x(\vec{0})$, and the length of $\vec{P}$ should be 1. Then given $\vec{W}$ and $\vec{P}$ you can find $\vec{Q}$ using (11).

1.4.2 Task 2

Find the 3×3 matrix $[\mathrm{Id}]_{EU}$ which changes bases from U to E, and use it to find the inverse $[\mathrm{Id}]_{UE}$.

1.4.3 Task 3

The position of the satellite at time t is $\widehat{R_3}(x(\vec{0}))$, which in the standard basis is equal to

$$[x(\vec{t})]_E = [\widehat{R_3}]_{EE}\,[x(\vec{0})]_E \tag{15}$$

where $[\widehat{R_3}]_{EE}$ is the matrix which implements the rotation about the vector $\vec{W}$ by angle ωt. By changing to the U-basis and using the matrix (12), find the position of the satellite (in the standard basis) at times $t = 0.5, 1, 1.5$ hours.