Solutions of ODE2 Problems 1b and 4a

(1b) We start by writing:

Let

$$N = N^*\hat{N}, \quad P = P^*\hat{P} \ \ and \ \ t = t^*\hat{t},$$

where stars indicate new variables and the hats are constants to be chosen. Proceeding purely formally, we substitute these into the differential equations:

$$\frac{d(N^*\hat{N})}{d(t^*\hat{t})} = N^*\hat{N}(a - bP^*\hat{P}) \Rightarrow \frac{\hat{N}dN^*}{\hat{t}dt^*} = N^*\hat{N}b\hat{P}\left(\frac{a}{b\hat{P}} - P^*\right) \Rightarrow \frac{dN^*}{dt^*} = b\hat{t}\hat{P}N^*\left(\frac{a}{b\hat{P}} - P^*\right)$$

$$\frac{dP^*\hat{P}}{dt^*\hat{t}} = P^*\hat{P}(cN^*\hat{N} - d) \Rightarrow \frac{\hat{P}dP^*}{\hat{t}dt^*} = P^*\hat{P}c\hat{N}\left(N^* - \frac{d}{c\hat{N}}\right) \Rightarrow \frac{dP^*}{dt^*} = c\hat{t}\hat{N}P^*\left(N^* - \frac{d}{c\hat{N}}\right)$$

Let's look at the last equations: we would like to have $\frac{a}{b\hat{P}} = 1$ and $\frac{d}{c\hat{N}} = 1$, so we pick $\hat{P} = \frac{a}{b}$ and $\hat{N} = \frac{d}{c}$.

Also, we choose $\hat{t} = \frac{1}{a}$ and $\alpha = \frac{d}{a}$ and after dropping the stars we obtain:

$$\tfrac{dN}{dt} = N(1 - P)$$

$$\tfrac{dP}{dt} = \alpha P(N - 1)$$

(4a) We start by writing:

$$N_1 = N_1^*\hat{N}_1, \quad N_2 = N_2^*\hat{N}_2 \ \ and \ \ t = t^*\hat{t}$$

where stars indicate new variables and the hats are constants to be chosen. Proceeding purely formally, we substitute these into the differential equations:

$$\frac{d(N_1^*\hat{N}_1)}{d(t^*\hat{t})} = r_1 N_1^*\hat{N}_1\left[1 - \frac{N_1^*\hat{N}_1}{k_1} - b_{12}\frac{N_2^*\hat{N}_2}{k_1}\right] \ \Rightarrow \ \frac{dN_1^*}{dt^*} = r_1\hat{t}N_1^*\left[1 - \frac{N_1^*\hat{N}_1}{k_1} - b_{12}\frac{N_2^*\hat{N}_2}{k_1}\right]$$

$$\frac{d(N_2^*\hat{N}_2)}{d(t^*\hat{t})} = r_2 N_2^*\hat{N}_2\left[1 - \frac{N_2^*\hat{N}_2}{k_2} - b_{21}\frac{N_1^*\hat{N}_1}{k_2}\right] \ \Rightarrow \ \frac{dN_2^*}{dt^*} = r_2\hat{t}N_2^*\left[1 - \frac{N_2^*\hat{N}_2}{k_2} - b_{21}\frac{N_1^*\hat{N}_1}{k_2}\right]$$

By taking $\hat{N}_1 = k_1$, $\hat{N}_2 = k_2$, $a_{12} = b_{12}\frac{k_2}{k_1}$, $a_{21} = b_{21}\frac{k_1}{k_2}$, $\hat{t} = \frac{1}{r_1}$ and $\alpha = \frac{r_2}{r_1}$ and dropping the stars, we have:

$$\tfrac{dN_1}{dt} = N_1(1 - N_1 - a_{12}N_2) = f_1(N_1, N_2)$$

$$\tfrac{dN_2}{dt} = \alpha N_2(1 - N_2 - a_{21}N_1) = f_2(N_1, N_2)$$

Solution of ODE3 Problem 4a

(4a) To find a change of variables, we start by writing:

$$N = N^*\hat{N}, \;\; C = C^*\hat{C}, \;\; t = t^*\hat{t}.$$

We substitute these into the differential equations:

$$\tfrac{d(N^*\hat{N})}{d(t^*\hat{t})} = kC^*\hat{C}N^*\hat{N} - \tfrac{F}{V}N^*\hat{N}$$

$$\Rightarrow \; \tfrac{dN^*}{dt^*} = k\hat{C}\hat{t}C^*N^* - \tfrac{F}{V}\hat{t}N^*$$

$$\tfrac{d(C^*\hat{C})}{d(t^*\hat{t})} = -\alpha kC^*\hat{C}N^*\hat{N} - \tfrac{F}{V}C^*\hat{C} + \tfrac{F}{V}C_0$$

$$\Rightarrow \; \tfrac{dC^*}{dt^*} = -\alpha k\hat{t}\hat{N}C^*N^* - \tfrac{F\hat{t}}{V}C^* + \tfrac{\hat{t}}{\hat{C}}\tfrac{F}{V}C_0$$

Choose $\hat{t}$, $\hat{C}$ and $\hat{N}$ such that $\frac{F}{V}\hat{t} = 1$, $k\hat{C}\hat{t} = 1$ and $\alpha k\hat{t}\hat{N} = 1$; i.e $\hat{t} = \frac{V}{F}$, $\hat{C} = \frac{F}{Vk}$, $\hat{N} = \frac{F}{\alpha kV}$ and let $a = \frac{C_0Vk}{F}$; dropping the stars, we have:

$$\tfrac{dN}{dt} = CN - N$$

$$\tfrac{dC}{dt} = -CN - C + a.$$

Solution of ODE3 Problems 6abc

(6a)

$$\frac{dN}{dt} = \frac{K_{max}C}{k_n+C}N - \mu N$$

$$\frac{dC}{dt} = -\alpha\frac{K_{max}C}{k_n+C}N - \frac{CF}{V} + \frac{C_0F}{V}$$

(6b) We start by writing: $N = N^*\hat{N}$, $C = C^*\hat{C}$, $t = t^*\hat{t}$, where stars indicate new variables and the hats are constants to be chosen. Proceeding purely formally, we substitute these into the differential equations:

$$\frac{d(N^*\hat{N})}{d(t^*\hat{t})} = \frac{k_{max}\hat{C}\hat{N}}{k_n+\hat{C}C^*}C^*N^* - \mu\hat{N}N^*$$

$$\Rightarrow \frac{dN^*}{dt^*} = \frac{\hat{t}k_{max}}{\frac{k_n}{\hat{C}}+C^*}C^*N^* - \mu\hat{t}N^*$$

$$\frac{d(C^*\hat{C})}{d(t^*\hat{t})} = -\alpha\frac{k_{max}\hat{C}\hat{N}}{k_n+\hat{C}C^*}C^*N^* - \frac{\hat{C}C^*F}{V} + \frac{C_0F}{V}$$

$$\Rightarrow \frac{dC^*}{dt^*} = -\alpha\frac{k_{max}\hat{t}\hat{N}}{\hat{C}}\frac{C^*N^*}{\frac{k_n}{\hat{C}}+C^*} - \frac{\hat{t}F}{V}C^* + \frac{C_0F\hat{t}}{V\hat{C}}$$

Taking $\hat{t} = \frac{V}{F}$, $\hat{C} = k_n$ and $\hat{N} = \frac{\hat{C}}{\alpha k_{max}\hat{t}} = \frac{k_n}{\alpha k_{max}\frac{V}{F}}$ and dropping the stars, we have:

$$\frac{dN}{dt} = \alpha_1\frac{C}{1+C}N - \alpha_3 N$$

$$\frac{dC}{dt} = -\frac{C}{1+C}N - C + \alpha_2,$$

where $\alpha_1 = \frac{k_{max}V}{F}$, $\alpha_2 = \frac{C_0}{k_n}$ and $\alpha_3 = \frac{\mu V}{F}$.

(6c)

To find the steady states, let $\frac{dN}{dt} = 0$ and $\frac{dC}{dt}$ and solve the equations for N and C:

$$\frac{dN}{dt} = 0 \Rightarrow \alpha_1\frac{C}{1+C}N - \alpha_3 N = 0$$

$$\Rightarrow N = 0, C = \frac{\alpha_3}{\alpha_1-\alpha_3}$$

$$\frac{dC}{dt} = 0 \Rightarrow -\frac{C}{1+C}N - C + \alpha_2 = 0$$

$$\Rightarrow N = \frac{\alpha_1\alpha_2}{\alpha_3} - \frac{\alpha_1}{\alpha_1-\alpha_3}, C = \alpha_2.$$

So $(N_1, C_1) = (0, \alpha_2)$ and $(N_2, C_2) = (\frac{\alpha_1\alpha_2}{\alpha_3} - \frac{\alpha_1}{\alpha_1-\alpha_3}, \frac{\alpha_3}{\alpha_1-\alpha_3})$ are the steady states of the system.

N_2 and C_2 are positive if:

(i) $\frac{\alpha_1\alpha_2}{\alpha_3} - \frac{\alpha_1}{\alpha_1-\alpha_3} = \alpha_1(\frac{\alpha_2}{\alpha_3} - \frac{1}{\alpha_1-\alpha_3}) > 0$, i.e $\frac{\alpha_2}{\alpha_3} - \frac{1}{\alpha_1-\alpha_3} > 0$ (because $\alpha_1 > 0$), i.e $\alpha_2 > \frac{\alpha_3}{\alpha_1-\alpha_3}$ and

(ii) Since $\alpha_3 > 0$, to have a positive amount for C_2 we should have $\alpha_1 - \alpha_3 > 0$, i.e $\alpha_1 > \alpha_3$.

Solution of ODE3 Problems 13

(a) The matrix A has a first column of all ones, and the second column has the inverses of the C_i's.

(b) This code works:

```
load('data_for_fitting_michaelis_menten.mat')
C = DATA(:,1)
observe = DATA(:,2)
howmany = length(C)
Cinv = 1./C
observeinv = 1./observe
A = [ones(howmany,1) Cinv]
b = observeinv
fit = lsqr(A,b)
kmax = 1/fit(1)
kn = kmax * fit(2)
% this should give the answers kmax = 4.9416 and kn = 1.6005
plot(C,observe,'linewidth',3,'color','blue')
hold on
plot(C,kmax*C ./ (kn + C),'linewidth',3,'color','red')
hold off
```

(c)