Math 5110 Applied Linear Algebra -Fall 2021.

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Homework 3.

Questions: (You can use Matlab if needed.)

The following questions are about matrix of linear transformation and coordinate.

Question 1. Suppose $L : \mathbb{R}^3 \to \mathbb{R}^3$ is linear, $\vec{b} \in \mathbb{R}^3$ is given, and $\vec{u} = (1, 0, 1), \vec{v} = (1, 1, -1)$ are two solutions to $L(x) = b$. Find two more solutions to $L(\vec{x}) = \vec{b}$.

Question 2. Find matrix of each linear operator: (Hint: using theorem on matrix of linear transformation. The matrix is $[T(\vec{e}_1)\ T(\vec{e}_2)\ ...\ T(\vec{e}_n)]$)

(1.) Let $R : \mathbb{R}^2 \to \mathbb{R}^2$ be the **rotation** of angle θ about the origin (a positive θ indicates a counterclockwise rotation). Find the matrix A such that $R(x) = Ax$ for all $x \in \mathbb{R}^2$.

(2.) Consider the linear operator mapping $\mathbb{R}^2$ into itself that sends each vector $\begin{bmatrix} x \\ y \end{bmatrix}$ to its **projection** onto the x-axis, namely, $\begin{bmatrix} x \\ 0 \end{bmatrix}$. Find the matrix representing this linear operator.

(3.) A (horizontal) **shear** acting on the plane maps a **point** $\begin{bmatrix} x \\ y \end{bmatrix}$ to the point $\begin{bmatrix} x + ry \\ y \end{bmatrix}$, where r is a real number. Find the matrix representing this operator.

(4.) A linear operator $L : \mathbb{R}^n \to \mathbb{R}^n$ defined by $L(x) = r\vec{x}$ is called a **dilation** if $r > 1$ and a **contraction** if $0 < r < 1$. What is the matrix of L?

Question 3. Consider the following geometrically defined linear maps of $\mathbb{R}^3$ to itself. Describe each of them by a matrix with respect to the canonical basis of $\mathbb{R}^3$. (Hint: using theorem on matrix of linear transformation.)

(a) Orthogonal projection onto the xz-plane.

(b) Counterclockwise rotation by 45°about the x-axis.

(c) The map (rotation) of part (b) then followed by the map(projection) of part (a).

(d) Rotation by 120° about the main diagonal in space (spanned by the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, taken counterclockwise as you look towards the origin.

Question 4. Let $x \in R^N$ be denoted as $x = (x_1, x_2, ..., x_N)$. Given $\vec{x}, \vec{y} \in \mathbb{R}^N$, the **convolution** of $\vec{x}$ and $\vec{y}$ is the vector $\vec{x} * \vec{y} \in \mathbb{R}^N$ defined by

$$(\vec{x} * \vec{y})_n = \sum_{m=1}^{N} x_m y_{n-m}, \quad \text{for } n = 1, 2, ..., N.$$

In this formula, $\vec{y}$ is regarded as defining a periodic vector of period N; therefore, if $n - m \leq 0$, we take $y_{n-m} = y_{N+n-m}$. For instance, $y_0 = y_N, y_{-1} = y_{N-1}, y_{-2} = y_{N-2}$, and so forth.

(1) Prove that if $y \in \mathbb{R}^N$ is fixed, then the mapping

$$L : \vec{x} \to \vec{x} * \vec{y}$$

is linear. (2) Find the matrix representing this operator L.

Question 5. Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by the following conditions:

(a) L is linear; (b) $L(\vec{e}_1) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$; (c) $L(\vec{e}_2) = \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}$; (d) $L(\vec{e}_3) = \begin{bmatrix} 7 \\ -3 \\ 9 \end{bmatrix}$;

Here $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ is the standard basis for $\mathbb{R}^3$. Prove that there is a 3×3 matrix A such that $L(x) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^3$. What is the matrix A?

Question 6. Let $\mathscr{B} = \{\vec{b}_1, \vec{b}_2\} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ be a basis for $V = \mathrm{Span}\{\vec{b}_1, \vec{b}_2\}$.

(1). Find the coordinate of $\vec{x} = \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}$ relative to $\mathscr{B}$.

(2). Suppose the coordinate of $\vec{y} \in V$ is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Find the vector $\vec{y}$.

Question 7. Let $V = \{a_1 t + a_2 t^2 + a_3 t^3 \mid a_1, a_2, a_3 \in \mathbb{R}\}$ with basis $\mathscr{B} = \{t, t^2, t^3\}$. Let $P_2 = \{a_0 + a_1 t + a_2 t^2 \mid a_0, a_1, a_2 \in \mathbb{R}\}$ with basis $\mathscr{C} = \{1, t, t^2\}$. Let $T : V \to P_2$ be a transformation defined by derivatives $T(p) = 2f' - f''$.

(1) Prove that T is a linear transformation. (using properties of derivative.)

(2) Find the matrix $[T]_{\mathscr{B}\mathscr{C}}$ of the transformation T respective to the bases $\mathscr{B}$ and $\mathscr{C}$.

(3) Is T an isomorphism?

Question 8. Let V be a subspace of $\mathbb{R}^n$. Suppose $\mathscr{B} = \{\vec{b}_1, \ldots, \vec{b}_s\}$ and $\mathscr{C} = \{\vec{v}_1, \ldots, \vec{v}_s\}$ are two bases of V.

(1) Find the $\mathscr{B} - \mathscr{C}$-matrix $S = [\mathrm{id}]_{\mathscr{B}\mathscr{C}}$ of the identity map from V to V. This matrix is also called **change of coordinate matrix** from $\mathscr{B}$ to $\mathscr{C}$.

(2) Show that $[\vec{b}_1 \ \ldots \ \vec{b}_s] = [\vec{v}_1 \ \ldots \ \vec{v}_s]S$.

Question 9. Let V be a subspace of $\mathbb{R}^3$. Suppose $\mathscr{B} = \{\vec{b}_1, \vec{b}_2\} = \left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ -3 \end{bmatrix} \right\}$ and $\mathscr{C} = \{\vec{v}_1, \vec{v}_2\} = \left\{ \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$ are two bases of V. (Example for the above question.)

(1) Find change of coordinate matrix S from $\mathscr{B}$ to $\mathscr{C}$.

(2) Verify that $[\vec{b}_1 \ \vec{b}_2] = [\vec{v}_1 \ \vec{v}_2]S$.

Question 10. A function $T : P_4(\mathbb{R}) \to P_4(\mathbb{R})$ is defined by the rule $T(f) = xf" - 2xf' - f$. Show that T is a linear operator, and find the matrix that represents T with respect to the standard basis of $P_4(\mathbb{R})$.

The following questions are about determinant

Question 11. Consider the real $n \times n$ matrix $A_n = (a_{ij})_{i,j=1,\dots,n}$ which has 2s on the main diagonal, -1s on the two diagonals next to the main diagonal, and 0s elsewhere. For example $A_2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, $A_3 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$,

$$A_4 = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}.$$

Compute $\det(A_n)$ in terms of n.

Question 12. Compute the area of the hexagon with vertices (3, 1), (12, 8), (10, 7), (-1,-1), (-10,-8) and (-8,-7). Compute by hand (using determinant) and verify by Matlab use polyshape.

Question 13. True or False: (Briefly explain the reason.)

(1) $\det(A + B) = \det(A) + \det(B)$ for all 5×5 matrices A and B.
(2) The equation $\det(-A) = \det(A)$ holds for all 6×6 matrices.
(3) If all the entries of a 7×7 matrix A are 7, then $\det(A)$ must be 7^7
(4) An 8×8 matrix fails to be invertible if (and only if) its determinant is nonzero.
(5) If B is obtained be multiplying a column of A by 9, then the equation $\det(B) = 9\det(A)$ must hold.
(6) If A is any $n \times n$ matrix, then $\det(AA^T) = \det(A^T A)$
(7) There is an invertible matrix of the form $\begin{bmatrix} a & e & f & j \\ b & 0 & g & 0 \\ c & 0 & h & 0 \\ d & 0 & i & 0 \end{bmatrix}$
(8) If A is an invertible $n \times n$ matrix, then $\det(A^T)\det(A^{-1}) = 1$.
(9) $\det(4A) = 4\det(A)$ for all 4×4 matrices A.
(10) There is a nonzero 4×4 matrix A such that $\det(4A) = 4\det(A)$.
(11) $\det(AB) = \det(BA)$ for all $n \times n$ matrices A and B.

Question 14. Is there a 3×3 matrix such that $A^2 + I = \mathbf{0}$? Answer the question in real numbers and complex numbers. Show your reason.

Question 15. Let A be the matrix $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 8 & 9 \\ 2 & 4 & 6 & 10 \\ 1 & 5 & 10 & 9 \end{bmatrix}$. Compute by hand the **determinant** of A. Write down all steps you are using. (Hint: using row operations together with cofactor expansion)

Question 16. Let $A = \begin{bmatrix} 0 & 0 & \dots & 0 & a_1 \\ 0 & 0 & \dots & a_2 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & a_{n-1} & \dots & 0 & 0 \\ a_n & 0 & \dots & 0 & 0 \end{bmatrix}$. Find the determinant of A and prove your result..

Question 17. Find a 5×5 permutation matrix P such that $P[x_1\ x_2\ x_3\ x_4\ x_5]^T = [x_3\ x_1\ x_4\ x_5\ x_2]^T$

Question 18. (Vandermonde determinants.) Consider distinct real numbers $a_0, a_1, ..., a_n$. We define the $(n+1)\times(n+1)$ matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ a_0 & a_1 & a_2 & \cdots & a_n \\ a_0^2 & a_1^2 & a_2^2 & \cdots & a_n^2 \\ \vdots & \vdots & \cdots & \vdots & \\ a_0^{n-1} & a_1^{n-1} & a_2^{n-1} & \cdots & a_n^{n-1} \\ a_0^n & a_1^n & a_2^n & \cdots & a_n^n \end{pmatrix}$$

Vandermonde showed that

$$\det(A) = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ a_0 & a_1 & a_2 & \cdots & a_n \\ a_0^2 & a_1^2 & a_2^2 & \cdots & a_n^2 \\ \vdots & \vdots & \cdots & \vdots & \\ a_0^{n-1} & a_1^{n-1} & a_2^{n-1} & \cdots & a_n^{n-1} \\ a_0^n & a_1^n & a_2^n & \cdots & a_n^n \end{vmatrix} = \prod_{0\le j<i\le n} (a_i - a_j)$$

(a) Verify Vandermonde's formula for the case $n = 1$.

(b) Suppose the Vandermonde formula holds for $n1$. You are asked to demonstrate it for n. Consider the function

$$f(t) = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ a_0 & a_1 & a_2 & \cdots & a_{n-1} & t \\ a_0^2 & a_1^2 & a_2^2 & \cdots & a_{n-1}^2 & t^2 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \\ a_0^{n-1} & a_1^{n-1} & a_2^{n-1} & \cdots & a_{n-1}^{n-1} & t^{n-1} \\ a_0^n & a_1^n & a_2^n & \cdots & a_{n-1}^n & t^n \end{vmatrix}$$

Explain why $f(t)$ is a polynomial of n-th degree. Find the coefficient k of t^n using Vandermondes formula for $a_0, \dots, a_{n-1}$. Explain why $f(a_0) = f(a_1) = \ = f(a_{n-1}) = 0$. Conclude that $f(t) = k(t-a_0)(t-a_1)(t-a_{n-1})$ for the scalar k you found above. Substitute $t = a_n$ to demonstrate Vandermondes formula.