A simple two-compartment Turing instability exercise

Consider the following system:

$$
\begin{aligned}
\frac{dx_L}{dt} &= x_L^2/y_L - (3/4)x_L + 1/4 \quad && + d_1\,(x_R - x_L)\\
\frac{dy_L}{dt} &= x_L^2 - (3/4)y_L + 1/2 && + d_2\,(y_R - y_L)\\
\frac{dx_R}{dt} &= x_R^2/y_R - (3/4)x_R + 1/4 && + d_1\,(x_L - x_R)\\
\frac{dy_R}{dt} &= x_R^2 - (3/4)y_R + 1/2 && + d_2\,(y_L - y_R)
\end{aligned}
$$

We think of (x_L, y_L) as the state of a "left" compartment and (x_R, y_R) as the state of a "right" compartment.

(a) Show that $(x_L, y_L, x_R, y_R) = (1, 2, 1, 2)$ is an equilibrium.

(b) Find the Jacobian J of the subsystems without diffusion, i.e.

$$
\begin{aligned}
\frac{dx}{dt} &= x^2/y - (3/4)x + 1/4\\
\frac{dy}{dt} &= x^2 - (3/4)y + 1/2\,,
\end{aligned}
$$

at the equilibrium $(1, 2)$.

(c) Is this an activator/inhibitor or a substrate-depletion system?

(d) Look at this website:

`http://www.scholarpedia.org/article/Gierer-Meinhardt_model`

Look at the Gierer-Meinhardt PDE model and at our ODE. Answer this: what do $\rho, a, h, \mu_a, \rho_a, \mu_h, \rho_h$ correspond to, in our notations? (No need for a long answer, just map these to our states and parameters. For instance, $a = x$, etc.)

Honor-system assignment: look at Figure 2 and enjoy it for a few seconds.

(e) Show that J is a stable matrix.

(f) Now let us make $d_1 = 0.1$ and $d_2 = 10$, so $D = \text{diag}(0.1, 10)$. What are the eigenvalues of $J - 2D$? Note that one of them is positive, so we might expect pattern-formation.

(g) Now simulate the 4-dimensional system, with the above diffusion coefficients, starting from a small perturbation in x_L from the equilibrium. Specifically take this initial condition: $(1.01, 2, 1, 2)$. Show a plot of the the left and right cell states, on the time interval $[0, 300]$. You should see that the level of activator is different, at steady state, among the left and right compartments. (It will about 50% higher in the left one, not a huge difference with these particular parameters.)

(h) Now suppose that dispersions are not so different. Let us say $d_1 = 0.1$ and $d_2 = 1$. What are the eigenvalues of $J - 2D$ in this case? What do you conclude about pattern formation?

Just for fun, you may want to repeat the plots for this case (no need to hand-in).