Fast Fourier Transforms

- Complex eigenvalues
- Inner Products on functions
- Fourier series
- Fourier Transform
- Discrete Fourier Transform
- Fast Fourier Transform

➢ Fourier Transform

Data

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\exp(i\omega t)\,d\omega$$

$$F(\omega) := \int_{-\infty}^{\infty} f(t)\exp(-i\omega t)\,dt$$

"frequency"

Joseph Fourier (1768-1830) was obsessed with the physics of heat and developed the Fourier series and transform to model heat-flow problems.

➢ Fourier series

Complex

Recall that the Hermitian inner product for functions $f(x)$ and $g(x)$ defined for x on a domain $x \in [a, b]$:

$$\langle f(x), g(x) \rangle = \int_{a}^{b} f(x)\bar{g}(x)\, dx$$

π, $-\pi$

where $\bar{g}$ denotes the complex conjugate,

Data $\quad f(x+T)=f(x)$

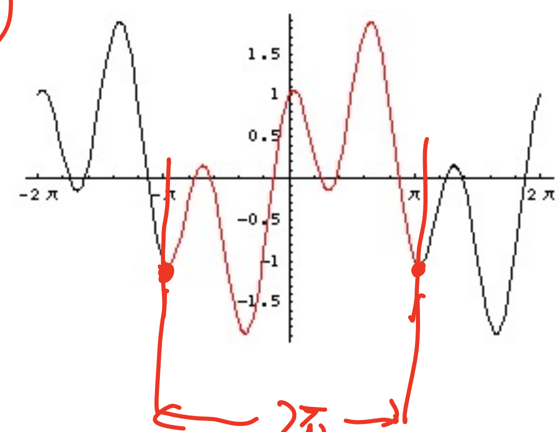
A fundamental result in Fourier analysis is that if $f(x)$ is T **periodic** and **piecewise smooth**, then it can be written in terms of a Fourier series, which is an infinite sum of cosines and sines of increasing frequency.

If $f(x)$ is 2π- **periodic** $\quad x \in \mathbb{R}$

frequency

$\mathbb{R}$ $\quad \mathbb{R}$

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos(kx) + b_k \sin(kx) \right).$$



$f(x) \in S \quad \mathrm{proj}_S f$

amplitude

$S = \text{Span}$ $\cos(0)=1$ $\cancel{\sin(0)=0}$

$\sin kx$ is $\frac{2\pi}{k}$-periodic

Consider the set $\{\cos(kx), \sin(kx)\}$ for $k = 0, 1, 2, \dots$

$\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0$

- The set is **orthogonal.** ! $\Longrightarrow$ independence

$\int_{-\pi}^{\pi} \sin mx \sin nx = \begin{cases} 0 & m \neq n \\ \pi & m = n \end{cases}$

The coordinate coefficients a_k and b_k are given by

$$a_k = \frac{1}{\|\cos(kx)\|^2} \langle f(x), \cos(kx) \rangle$$
$$b_k = \frac{1}{\|\sin(kx)\|^2} \langle f(x), \sin(kx) \rangle,$$

$\|1\|^2 = \int_{-\pi}^{\pi} 1^2 \, dx = 2\pi$

$\|\sin kx\|^2 = \pi$

$= \int_{-\pi}^{\pi} \sin kx \sin kx \, dx = \pi.$

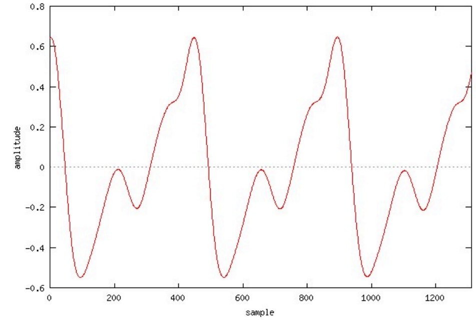
That is

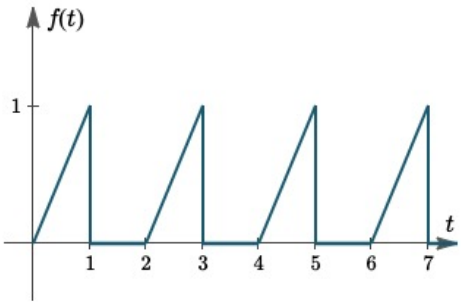
$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx) dx$$
$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx) dx,$$

$$\frac{a_0}{2} = \frac{\int_{-\pi}^{\pi} f(x) \cdot 1 \, dx}{2\pi}$$

➢ The Fourier series for an ***L*-periodic** function

***L*-periodic** functions:





The Fourier series for an ***L*-periodic** function on $[0, L)$ is given by:

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos\left(\frac{2\pi k x}{L}\right) + b_k \sin\left(\frac{2\pi k x}{L}\right) \right)$$

$\sin\left(\frac{2\pi x}{L}\right)$ is L-periodic

with coordinate coefficients a_k and b_k are given by

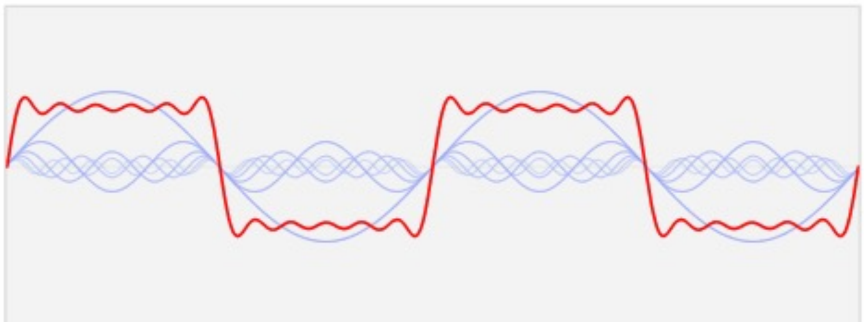
$$a_k = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{2\pi k x}{L}\right) dx$$

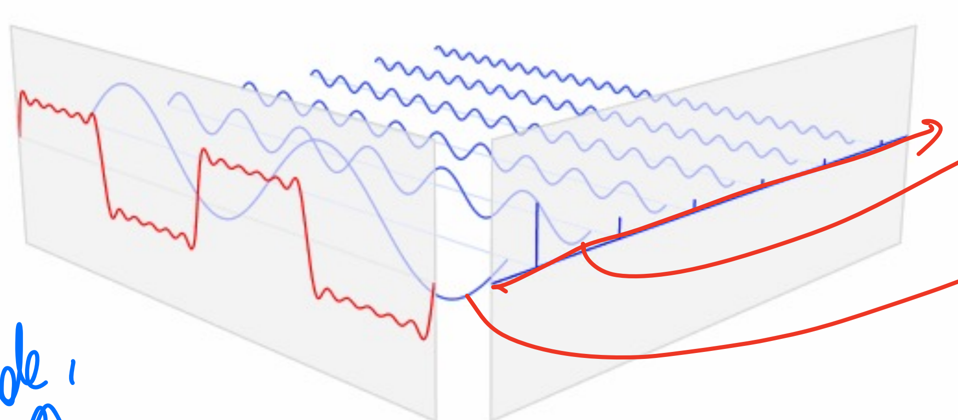
$$b_k = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{2\pi k x}{L}\right) dx$$

$\|1\|^2 = L$

f(t)

$\to t$ time domain

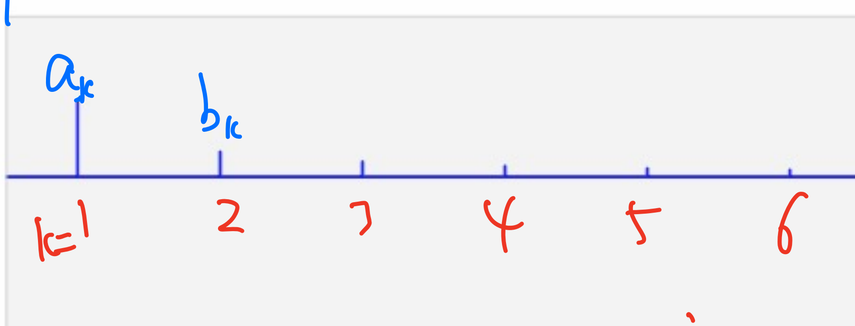




$a_2 \cos\left(\frac{2\pi 2t}{L}\right)$

$a_1 \sin\left(\frac{2\pi t}{L}\right)$

amplitude

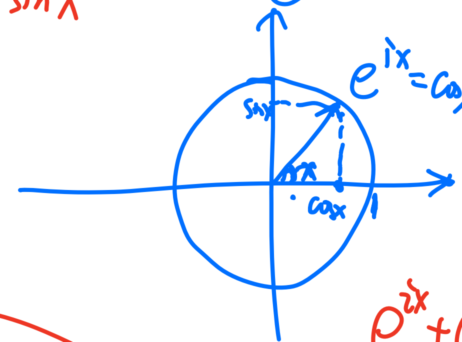


k frequency domain

$1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \cdots$ Taylor series

$e^{-ix} = \cos x - i\sin x$

2π-periodic



- **Complex functions:**

Euler's formula: $e^{ix} = \cos(x) + i\,\sin(x)$ with $i := \sqrt{-1}$

$x \in \mathbb{R}$

$\cos x = \frac{e^{ix} + e^{-ix}}{2}$

$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$

If $f(x)$ is **periodic** and **piecewise smooth** complex valued function,

New Notation

$e^{ikx} = (\cos x + i\sin x)^k = \cos kx + i\sin kx$

$\mathrm{proj}_S f(x)$

$$
\begin{aligned}
f(x) &= \sum_{k=-\infty}^{\infty} c_k e^{ikx} \\
&= \sum_{k=-\infty}^{\infty} (\alpha_k + i\beta_k)(\cos(kx) + i\sin(kx)) \\
&= (\alpha_0 + \underbrace{i\beta_0}_{=0}) + \sum_{k=1}^{\infty}\left[(\alpha_{-k} + \alpha_k)\cos(kx) + (\beta_{-k} - \beta_k)\sin(kx)\right] \\
&\quad + i\sum_{k=1}^{\infty}\left[\underbrace{(\beta_{-k} + \beta_k)}_{=0}\cos(kx) - \underbrace{(\alpha_{-k} - \alpha_k)}_{=0}\sin(kx)\right]
\end{aligned}
$$

$\in \mathbb{R}$

If $f(x)$ is real-valued, then $\alpha_{-k} = \alpha_k$ and $\beta_{-k} = -\beta_k$, so that $c_{-k} = c_k$.

➢ The **Fourier series** of complex function:

$\cos kx + i \sin kx$

$S = \mathrm{Span}\{$ Consider the periodic complex functions $\psi_k := e^{ikx}$ for $k \in \mathbb{Z}$. $\}$

These functions are **orthogonal**

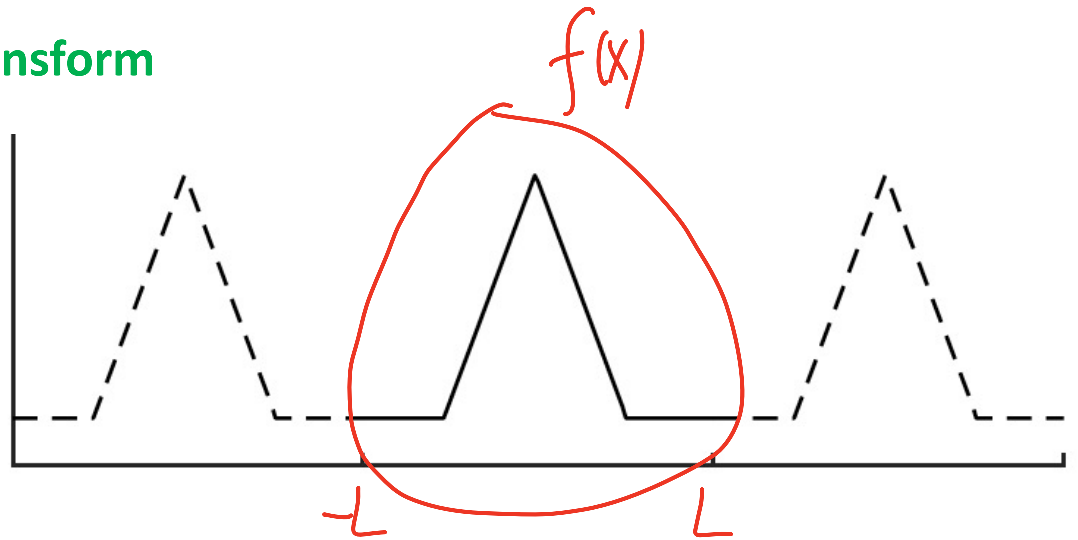
$$\langle \psi_j, \psi_k \rangle = \int_{-\pi}^{\pi} e^{ijx} e^{-ikx} dx = \int_{-\pi}^{\pi} e^{i(j-k)x} dx = \left[\frac{e^{i(j-k)x}}{i(j-k)} \right]_{-\pi}^{\pi} = \begin{cases} 0 & \text{if } j \neq k \\ 2\pi & \text{if } j = k \end{cases}$$

The **Fourier series** of $f(x)$ is the orthogonal projection:

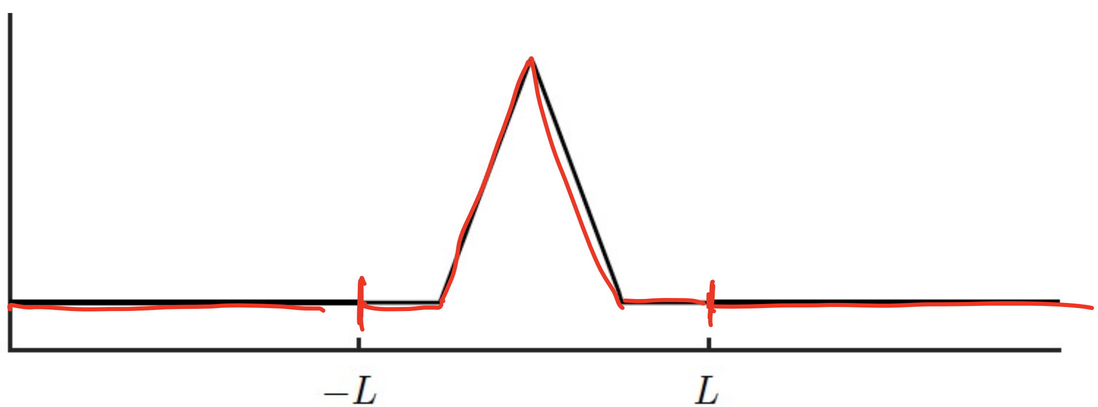
$$f(x) = \sum_{k=-\infty}^{\infty} c_k \psi_k(x) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \langle f(x), \psi_k(x) \rangle \psi_k(x)$$

$\mathrm{proj}_S f(x)$

➢ Fourier Transform



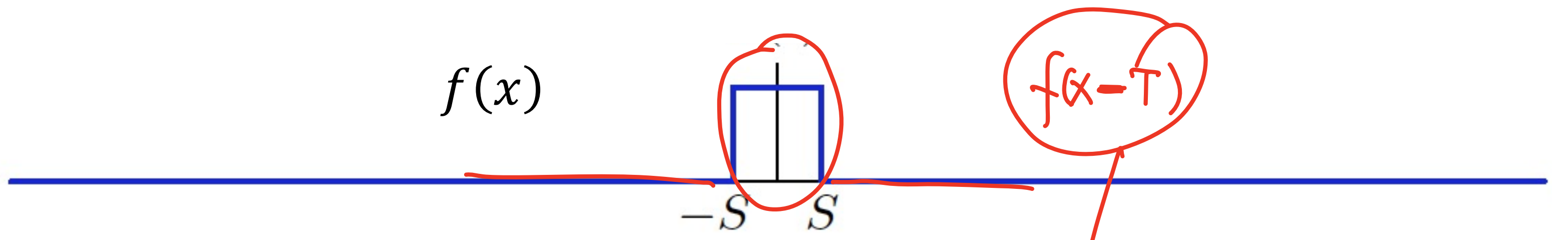
Fourier series is only valid for a function that is **periodic** on the domain $[-L;\ L)$.



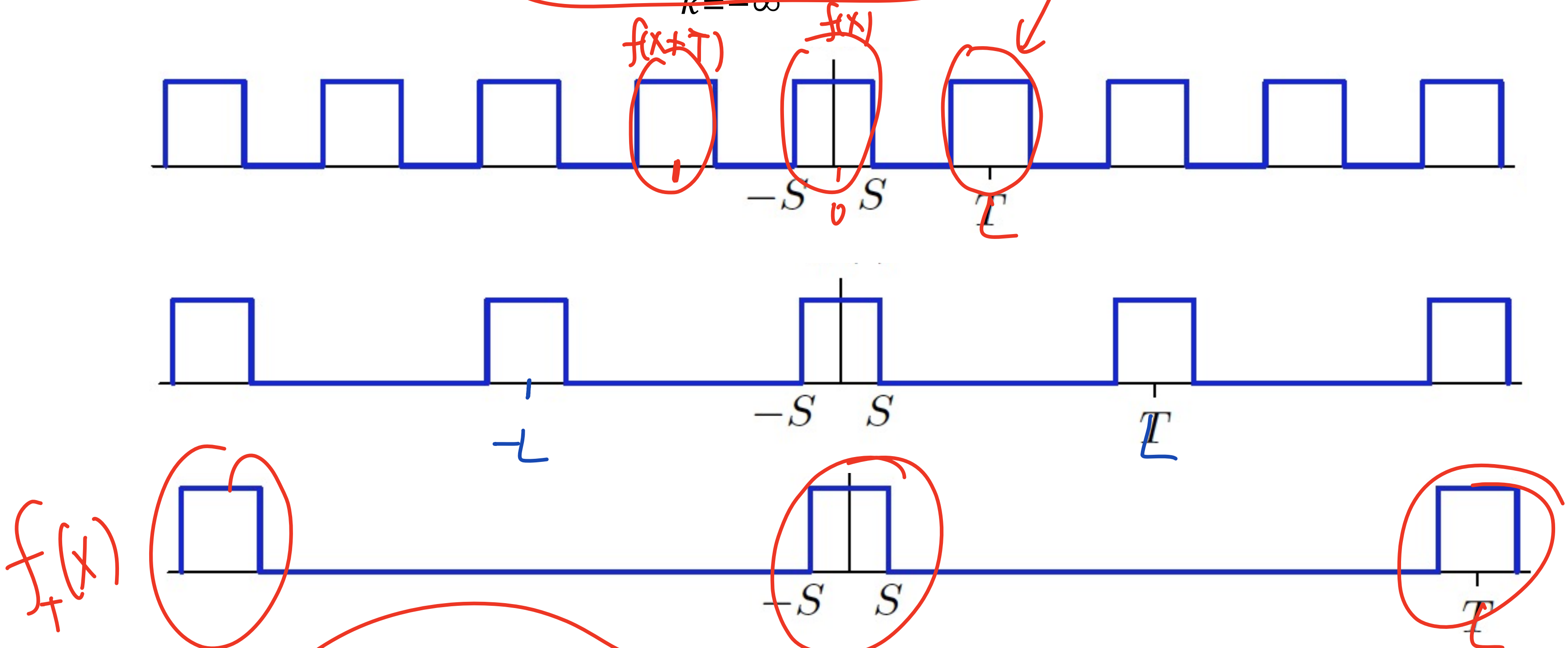
The Fourier transform is valid for generic **nonperiodic** functions.

The Fourier transform integral is essentially the limit of a Fourier series as the length of the domain goes to infinity.

Example: Let $f(x)$ represent an aperiodic signal.



Periodic extension: $f_T(x) = \sum_{k=-\infty}^{\infty} f(x + kT)$



$$f(x) = \lim_{T\to\infty} f_T(x)$$ 2T - periodic.

T=L

Consider the Fourier series on a domain $[-L, L)$ and then let $L \longrightarrow \infty$
On this domain, the Fourier series is:

2L-periodic

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}\left[a_k \cos\left(\frac{k\pi x}{L}\right) + b_k \sin\left(\frac{k\pi x}{L}\right)\right] = \boxed{\sum_{k=-\infty}^{\infty} c_k e^{ik\pi x/L}}$$

$e^{i\frac{\pi x}{L}}$ is $\frac{2\pi}{\pi/L} = 2L$ periodic.

with the coefficients given by:

$$c_k = \frac{1}{2L}\langle f(x), \psi_k\rangle = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-ik\pi x/L}\,dx$$

$\langle \psi_k, \psi_k \rangle$

$f(x)$ is now represented by a sum of sines and cosines with a discrete set of frequencies ~~$\omega/2\pi$~~ given by $\omega = \frac{k\pi}{L}$

$\frac{2\pi}{\omega} = \frac{2L}{k}$ periodic

let $L \longrightarrow \infty$, these discrete frequencies become a continuous range of frequencies

$\sin(kx)$ $\frac{2\pi}{k}$ periodic

$\Delta\omega = \frac{\pi}{L}$

frequency

Fourier transform pair (Joseph Fourier 1822)

$$f(x) = \lim_{\Delta\omega \to 0} \sum_{k=-\infty}^{\infty} \frac{\Delta\omega}{2\pi} \underbrace{\int_{-\pi/\Delta\omega}^{\pi/\Delta\omega} f(\xi) e^{-ik\Delta\omega\xi}\, d\xi}_{\langle f(x), \psi_k(x)\rangle} e^{ik\Delta\omega x}$$

where $\Delta\,\omega = \frac{\pi}{L}$

$\Delta\omega \to 0$

Define the **Fourier transform** of $f(x)$ as

$$\hat{f}(\omega) = \mathcal{F}\left(f(x)\right) := \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx$$

On the other side, we have the **inverse Fourier transform**

$$f(x) = \mathcal{F}^{-1}\left(\hat{f}(\omega)\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x}\, d\omega$$

Another format of the Fourier transform pairs:

The question is where to put the 2π: as a factor in front or in the exponential.

Fourier transform

$$\hat{f}(s) = \int_{-\infty}^{\infty} e^{-2\pi i s t} f(t)\, dt$$

inverse Fourier transform

$$f(t) = \int_{-\infty}^{\infty} \hat{f}(s) e^{2\pi i s t}\, ds$$

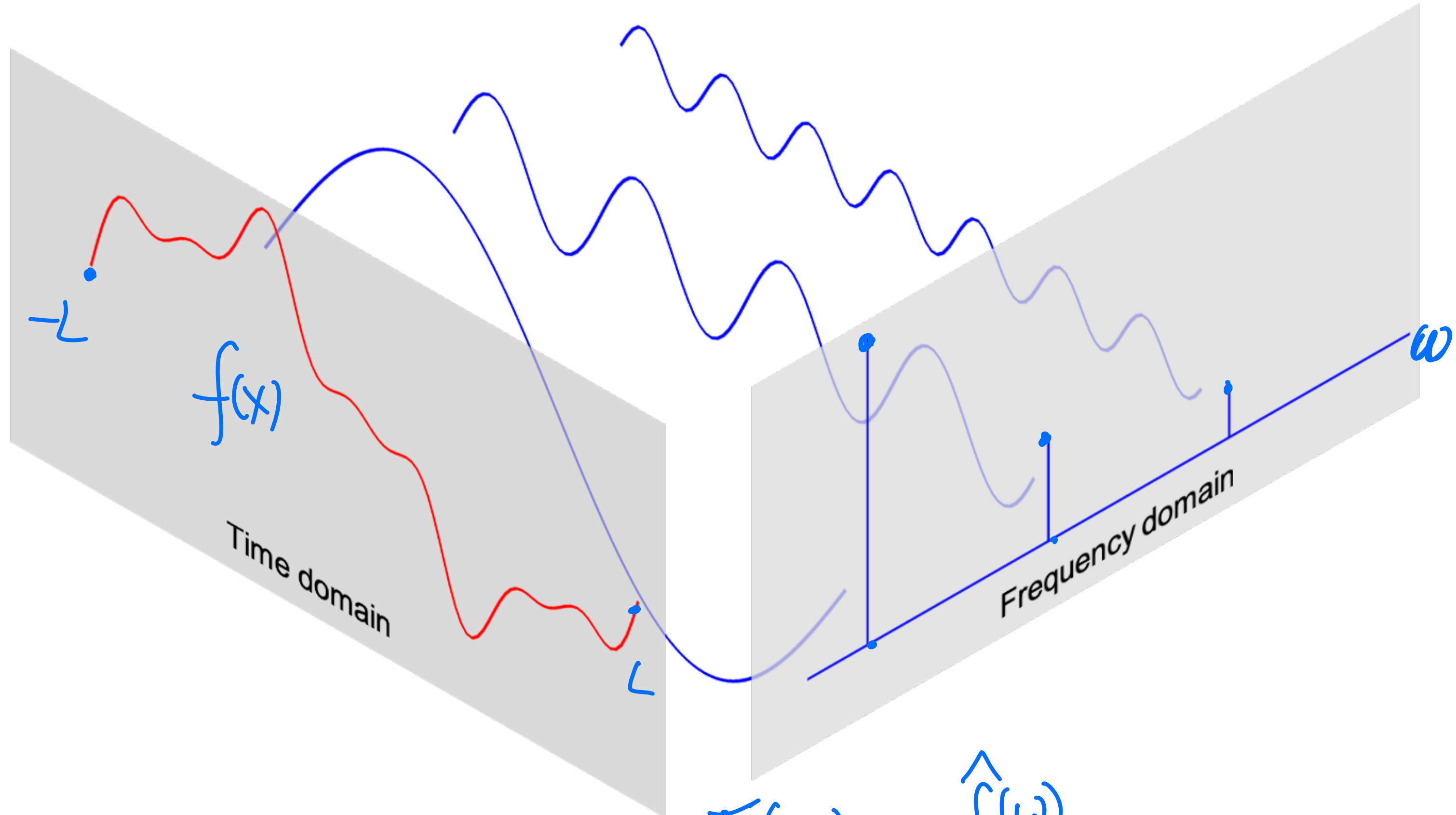
The relation is to change the variable by a **scaling** $\omega = 2\pi s$. Equivalently, $s = \frac{k}{2L}$

There are even more formats in practice:

$$\mathcal{F}f(s) = \frac{1}{A}\int_{-\infty}^{\infty} e^{iBst} f(t)\, dt$$

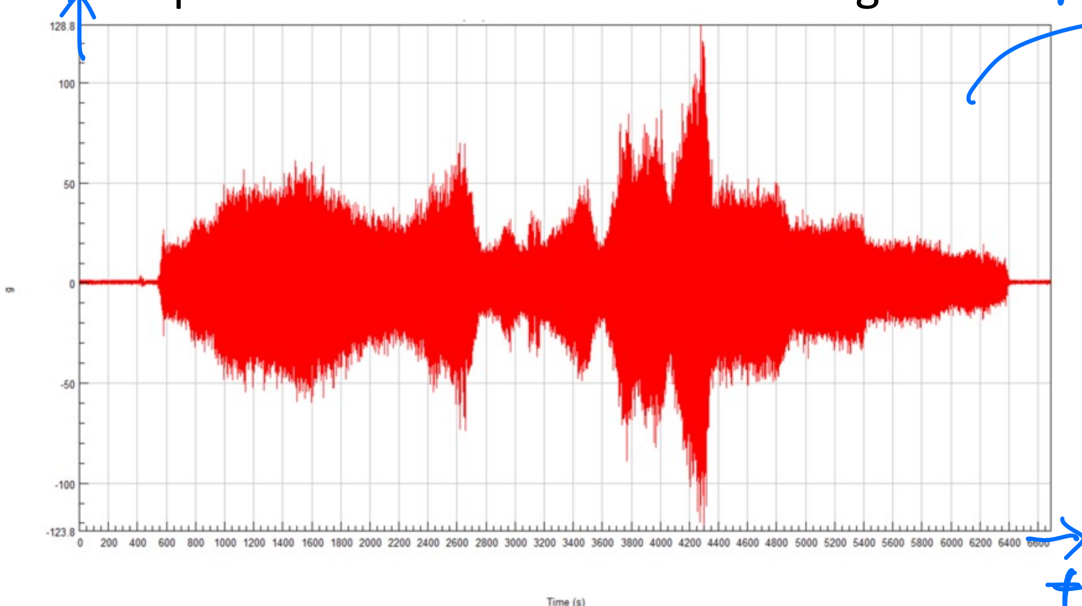
with

$$\begin{array}{ll} A = \sqrt{2\pi} & B = \pm 1 \\ A = 1 & B = \pm 2\pi \\ A = 1 & B = \pm 1 \end{array}$$

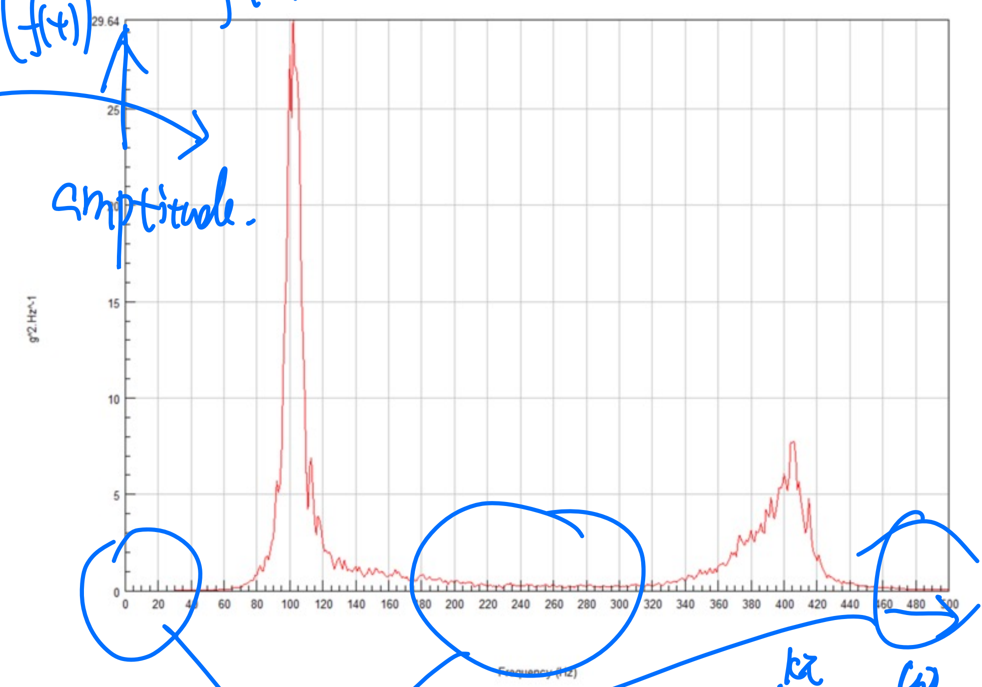


$f(t)$

Sample Accelerometer Data from Flight Test

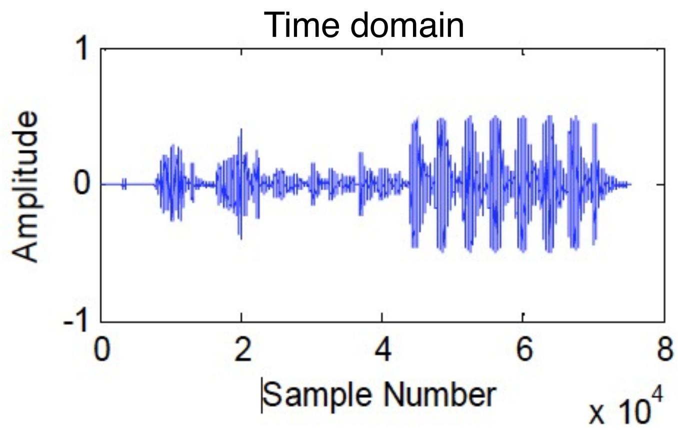


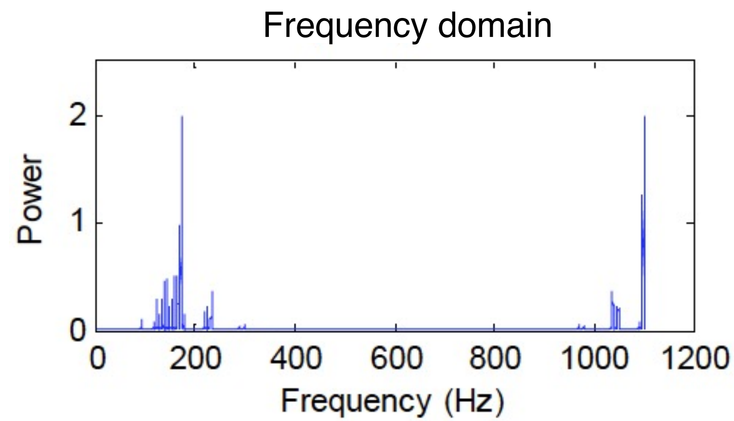
$F(f(x))$ $\hat{f}(\omega)$



noise.

A signal has one or more frequencies in it, and can be viewed fromtwo different standpoints: **Time domain** $(\boldsymbol{t})$ and **Frequency domain** $(\boldsymbol{\omega})$





- Time-domain figure: how a signal changes over time
- Frequency-domain figure: how much of the signal lies within each given frequency band over a range of frequencies.

- To decompose a complex signal into simpler parts to facilitate analysis.
- Differential and difference equations and convolution operations in the time domain become algebraic operations in the frequency domain.
- Fast Algorithm (FFT)

Applications:

- The Fourier transform of the derivative of a function is given by:

$$\begin{aligned}
\mathcal{F}\left(\frac{d}{dx}f(x)\right) &= \int_{-\infty}^{\infty} \overbrace{f'(x)}^{dv}\,\overbrace{e^{-i\omega x}}^{u}\,dx \\
&= \Big[\underbrace{f(x)e^{-i\omega x}}_{uv}\Big]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \underbrace{f(x)}_{v}\Big[\underbrace{-i\omega e^{-i\omega x}}_{du}\Big]\,dx \\
&= i\omega \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx \\
&= i\omega\mathcal{F}(f(x)).
\end{aligned}$$

Fourier transform can turn PDEs into ODEs

$$u_{tt} = cu_{xx} \quad \xrightarrow{\mathcal{F}\ \mathcal{F}} \quad \hat{u}_{tt} = -c\omega^2\hat{u}$$

- The Fourier transform is a **linear operator**

$$\mathcal{F}(\alpha f(x) + \beta g(x)) = \alpha \mathcal{F}(f) + \beta \mathcal{F}(g)$$

$$\mathcal{F}^{-1}(\alpha \hat{f}(\omega) + \beta \hat{g}(\omega)) = \alpha \mathcal{F}^{-1}(\hat{f}) + \beta \mathcal{F}^{-1}(\hat{g})$$

- **Parseval's theorem**

$$\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 \, d\omega = 2\pi \int_{-\infty}^{\infty} |f(x)|^2 \, dx$$

Fourier transform preserves the L2 norm, up to a constant.

$$\|f(x)\|^2 = \int_{-\infty}^{\infty} f(x) f(x) \, dx$$

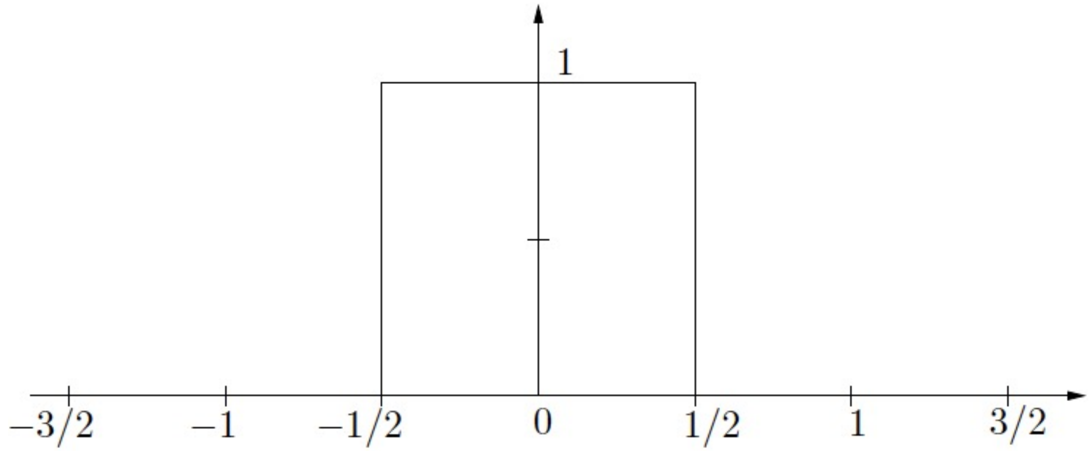
- More applications to Convolutions, Noise filtering, Image processing, etc.

➢ Example

$$f(x) = \sum_{k=-\infty}^{\infty} c_k \psi_k(x) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \langle f(x), \psi_k(x) \rangle \psi_k(x)$$

Let's take a specific, simple, and important example.

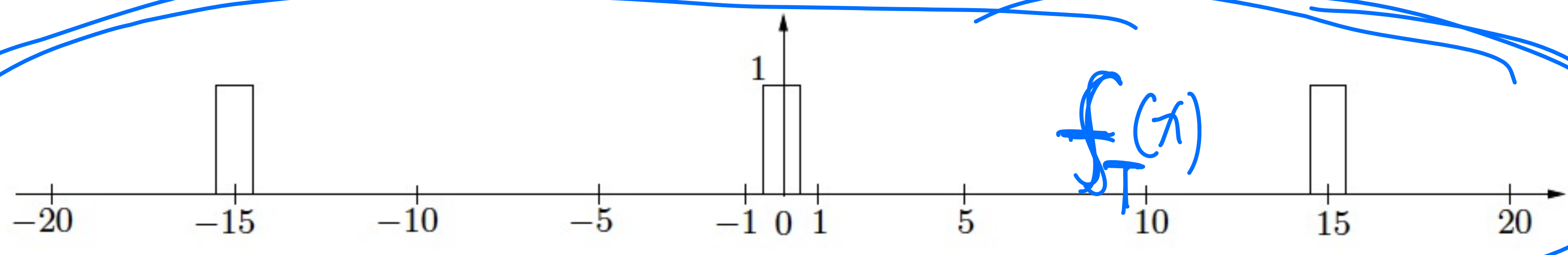
$$f(t) = \begin{cases} 1 & |t| < 1/2 \\ 0 & |t| \geq 1/2 \end{cases}$$



Box function/rectangle function/top hat function/ indicator function/characteristic function

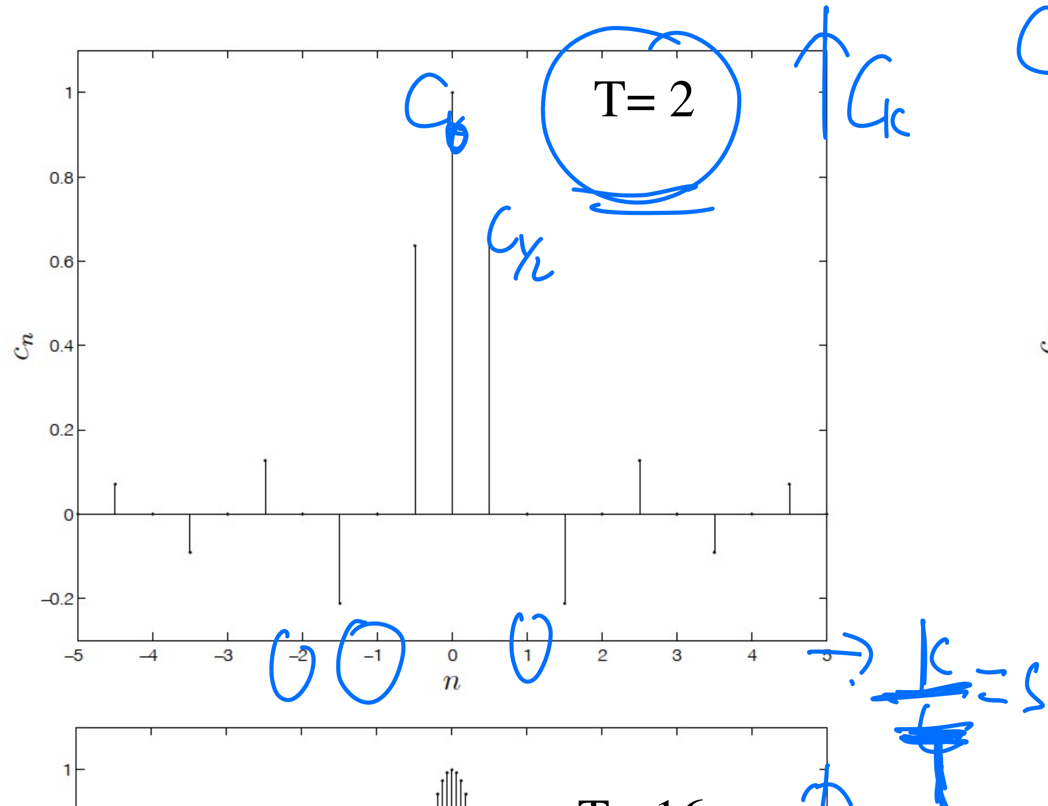
$f(t)$ is not periodic. It doesn't have a Fourier series. An aperiodic signal can be thought of as periodic with infinite period.

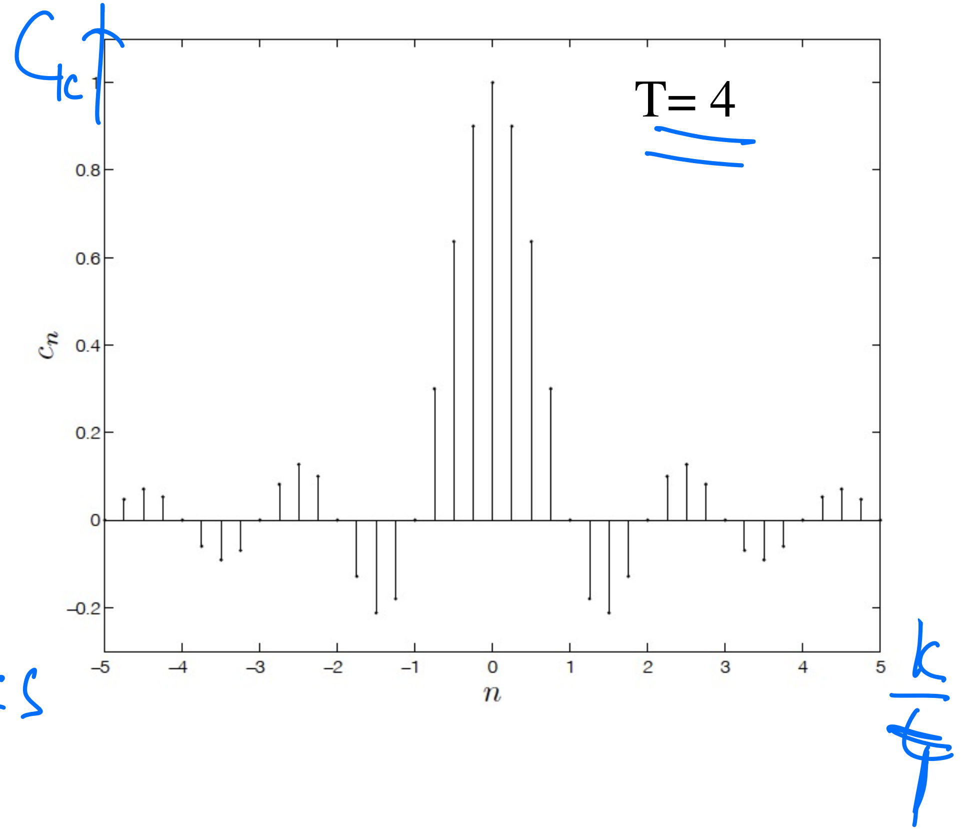
Here's a plot of $f(t)$ periodized to have period 15.

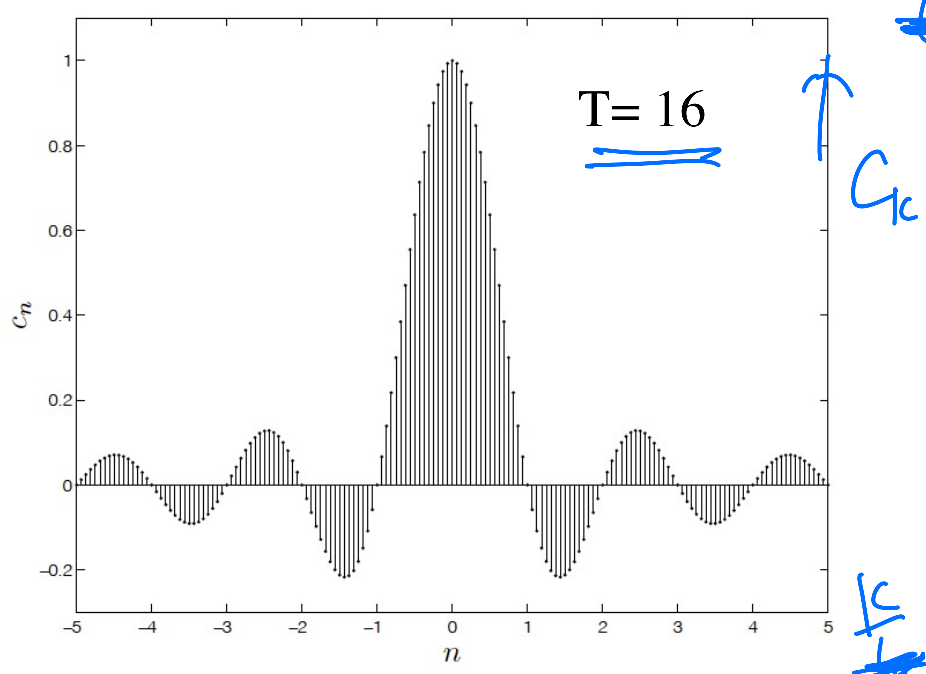


$\int_T f(x)$

➢ **Example continue1.** Fourier coefficients of periodized rectangle functions with period T







We see that as the period increases the frequencies are getting closer and closer together and it looks as though the coefficients are tracking some definite curve.

- **Example continue2.**

A general function $f(t)$ of period T the Fourier series has the form

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n t/T}$$

the frequencies are $0, \pm\frac{1}{T},\ \pm\frac{2}{T}, \ldots.$

Points in the spectrum are spaced $1/T$ apart and, indeed, in the pictures above the spectrum is getting more tightly packed as the period T increases. The n-th Fourier coefficient is given by

$$c_n = \frac{1}{T}\int_0^T e^{-2\pi i n t/T} f(t)\,dt = \frac{1}{T}\int_{-T/2}^{T/2} e^{-2\pi i n t/T} f(t)\,dt$$

T

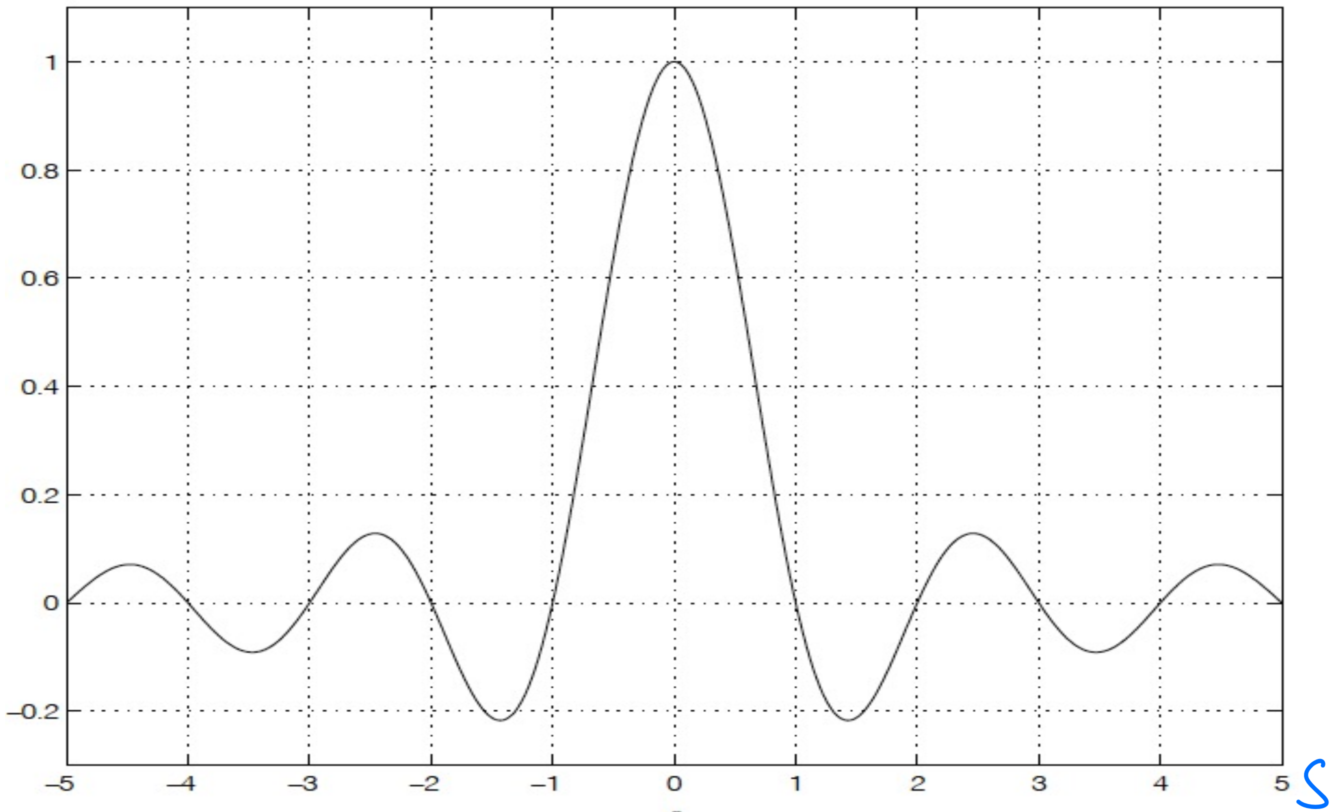
We can calculate this Fourier coefficient $\displaystyle = \frac{1}{T}\int_{-1/2}^{1/2} e^{-2\pi i n t/T}\cdot 1\,dt$

$$= \frac{1}{T}\left[\frac{1}{-2\pi i n/T} e^{-2\pi i n t/T}\right]_{t=-1/2}^{t=1/2} = \frac{1}{2\pi i n}\left(e^{\pi i n/T} - e^{-\pi i n/T}\right) = \frac{1}{\pi n}\sin\left(\frac{\pi n}{T}\right)$$

Example continue3.

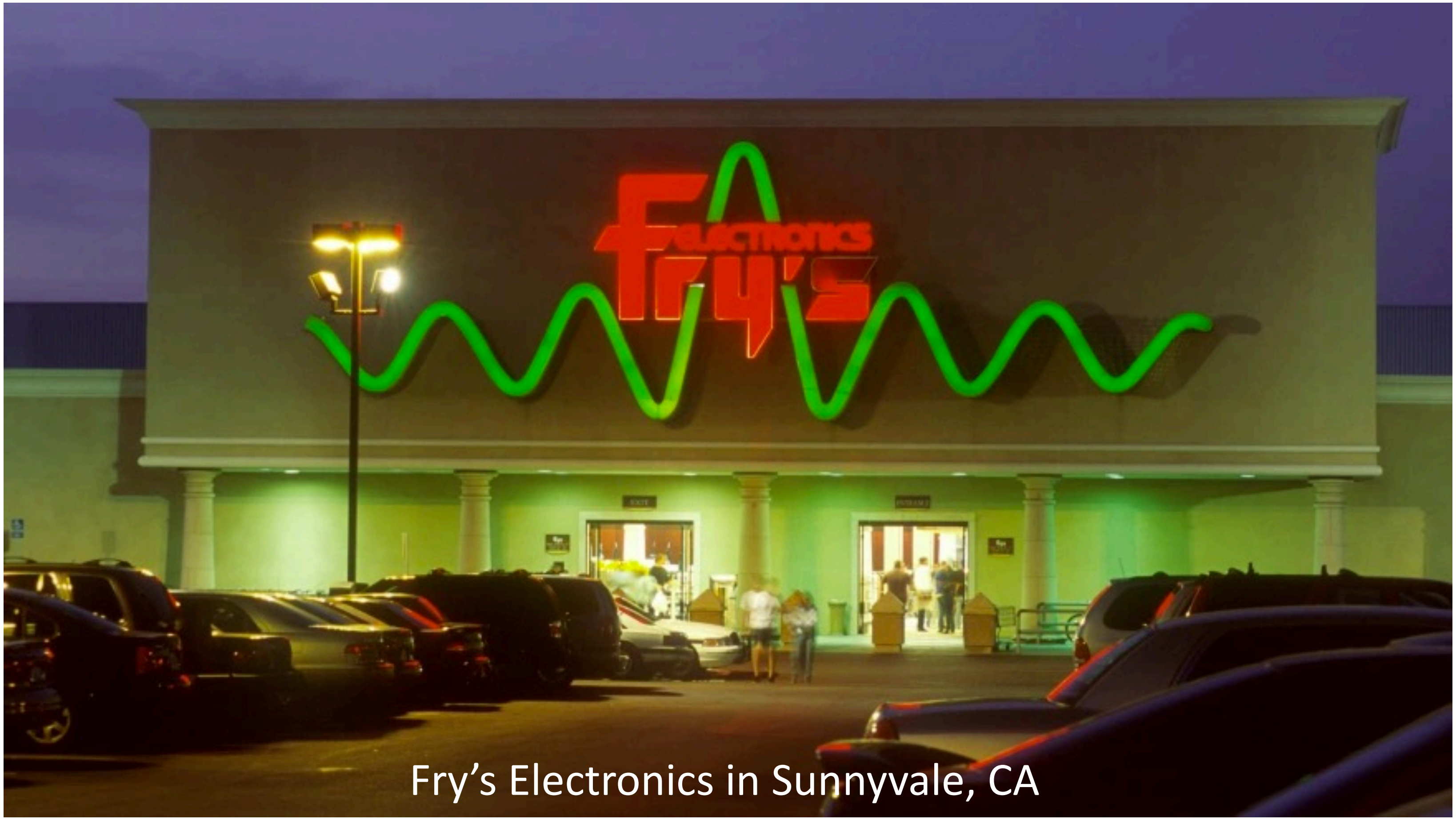
If T is large then we can think of replacing the closely packed discrete points n/T by a continuous variable, $s = n/T$ we would then write

$$\hat{f}(s) = \frac{\sin \pi s}{\pi s}$$



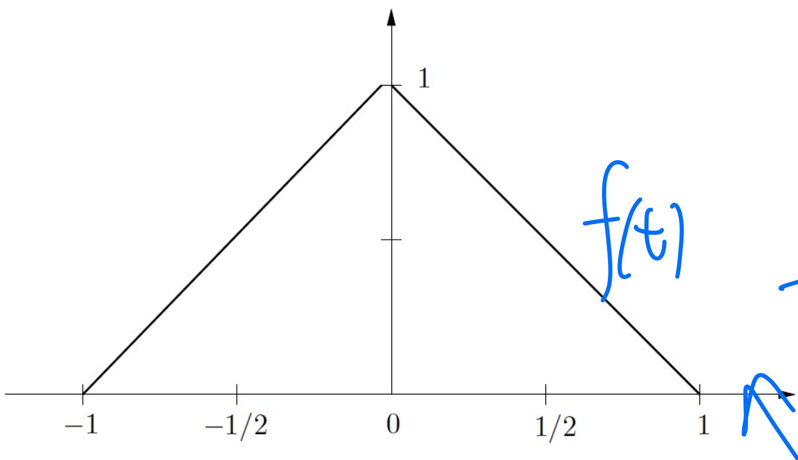
$$\operatorname{sinc} x = \frac{\sin \pi x}{\pi x}$$

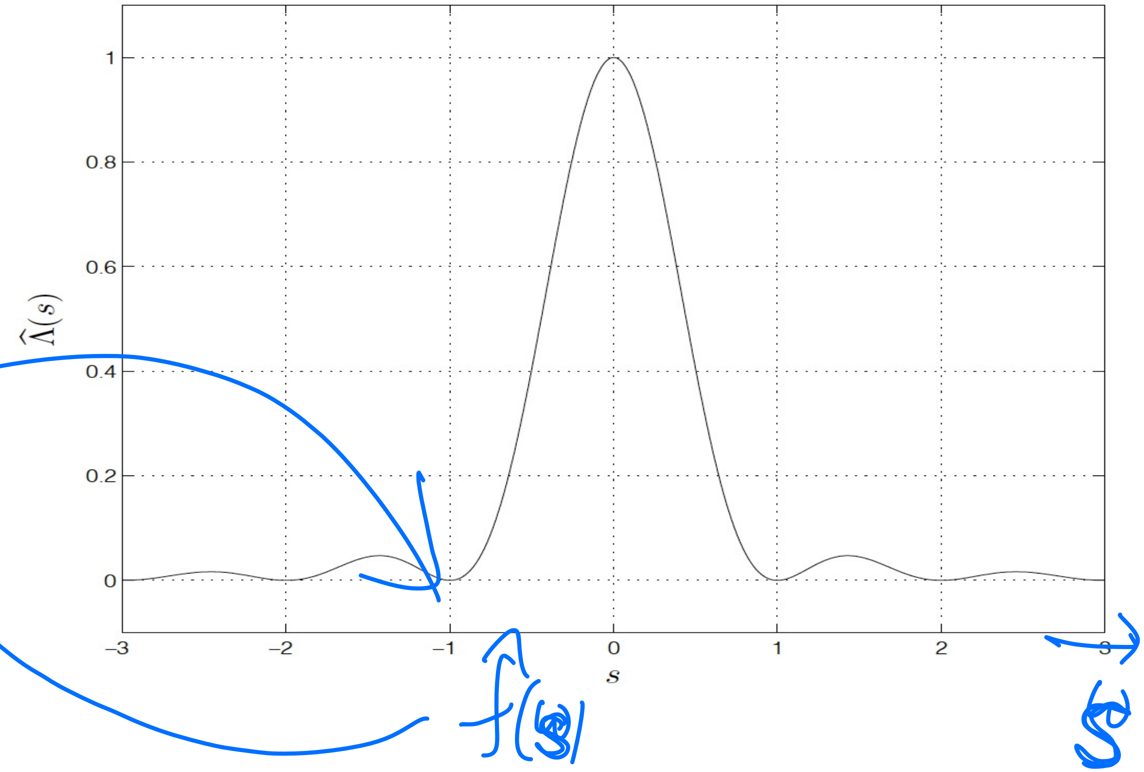
- **Example end.**



Fry’s Electronics in Sunnyvale, CA

Exercise: (Rectangle function)





$$\Lambda(x) = \begin{cases} 1 - |x| & |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

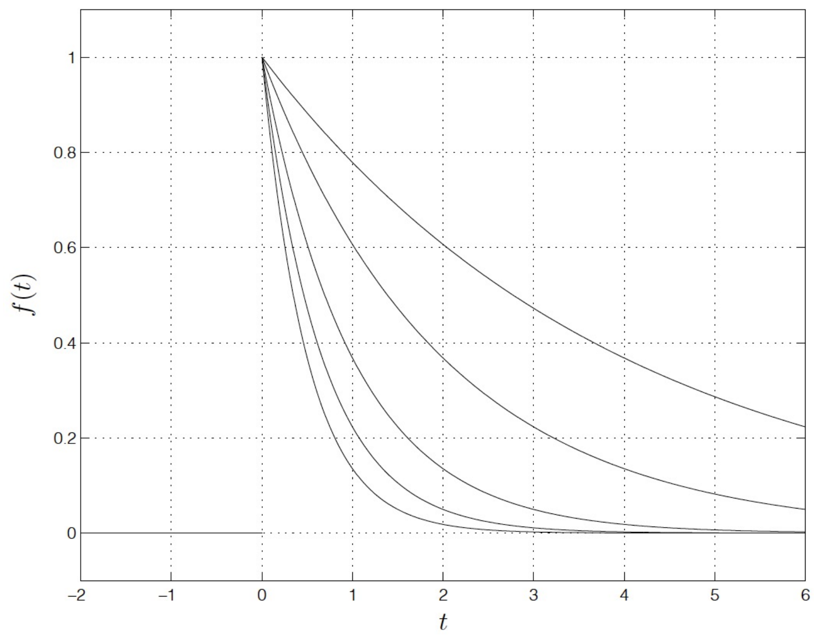
$$\begin{aligned} \mathcal{F}\Lambda(s) &= \int_{-\infty}^{\infty} \Lambda(x) e^{-2\pi i s x}\, dx \\ &= \int_{-1}^{0} (1+x) e^{-2\pi i s x}\, dx + \int_{0}^{1} (1-x) e^{-2\pi i s x}\, dx \\ &= \left(\frac{\sin \pi s}{\pi s}\right)^2 = \operatorname{sinc}^2 s. \end{aligned}$$

Exercise: (The exponential decay)

$$f(t) = \begin{cases} 0 & t \leq 0 \\ e^{-at} & t > 0 \end{cases}$$

where a is a positive constant

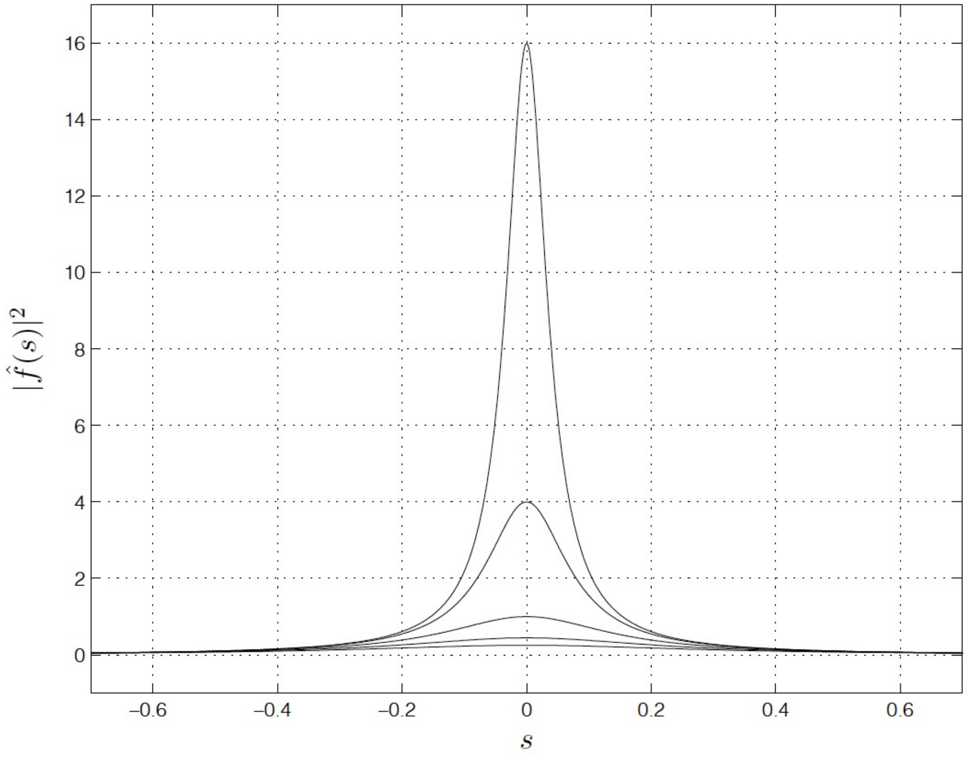
This function models a signal that is zero, switched on, and then decays exponentially. Here are graphs for $a = 2, 1.5, 1.0, 0.5, 0.25$.



$$\mathcal{F}f(s) = \int_0^\infty e^{-2\pi i s t} e^{-at}\, dt = \frac{1}{2\pi i s + a}$$

The **power spectrum (energy spectrum)** of the exponential decay is

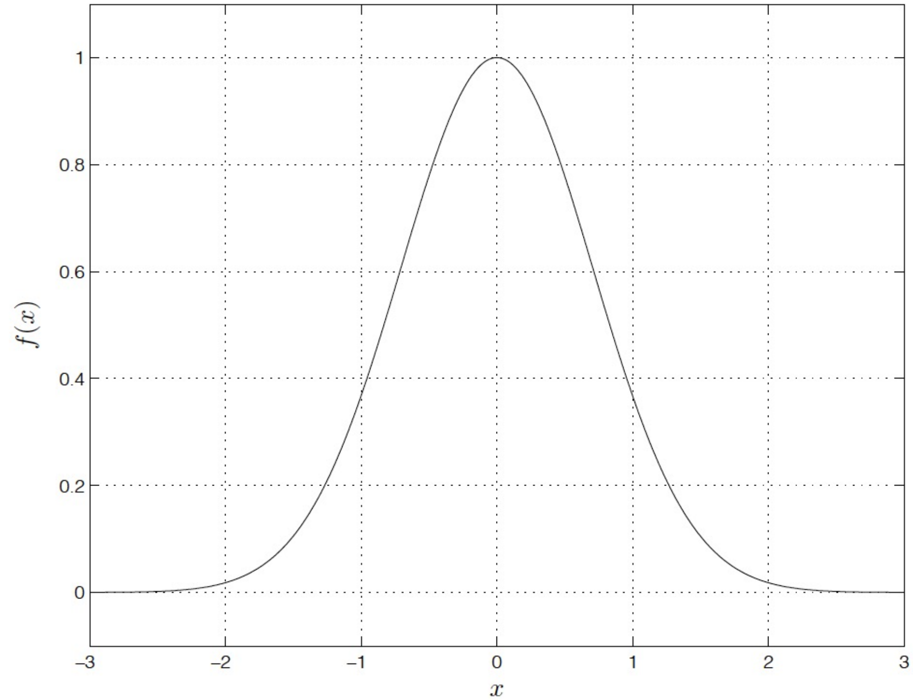
$$|\mathcal{F}f(s)|^2 = \frac{1}{|2\pi i s + a|^2} = \frac{1}{a^2 + 4\pi^2 s^2}$$



Exercise: (Gaussian)

The Gaussian function:

$$f(x) = e^{-\pi x^2}$$



The Fourier transform of the Gaussian $f(x)$

$$\mathcal{F}f(s) = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i s x}\, dx = e^{-\pi s^2}$$

Exercise: (Gaussian) continue

Details of calculation of Fourier transform of the Gaussian

Differentiate with respect to s:

$$\frac{d}{ds}\mathcal{F}f(s) = \int_{-\infty}^{\infty} e^{-\pi x^2}(-2\pi i x)e^{-2\pi i s x}\,dx$$

$$\begin{aligned} &= -\int_{-\infty}^{\infty} i e^{-\pi x^2}(-2\pi i s)e^{-2\pi i s x}\,dx \\ &= -2\pi s\int_{-\infty}^{\infty} e^{-\pi x^2}e^{-2\pi i s x}\,dx \\ &= -2\pi s\mathcal{F}f(s) \end{aligned}$$

(integration by parts)

$$\begin{aligned} u &= e^{-2\pi i s x} \\ dv &= -2\pi i x e^{-\pi x^2}\,dx \end{aligned}$$

Solve the differential equation with initial condition $\mathcal{F}f(0) = \int_{-\infty}^{\infty} e^{-\pi x^2}\,dx = 1$

Hence, $\mathcal{F}f(s) = \mathcal{F}f(0)e^{-\pi s^2} = e^{-\pi s^2}$

❖ **Take a breath!**

We have considered the Fourier series and Fourier transform for continuous functions.

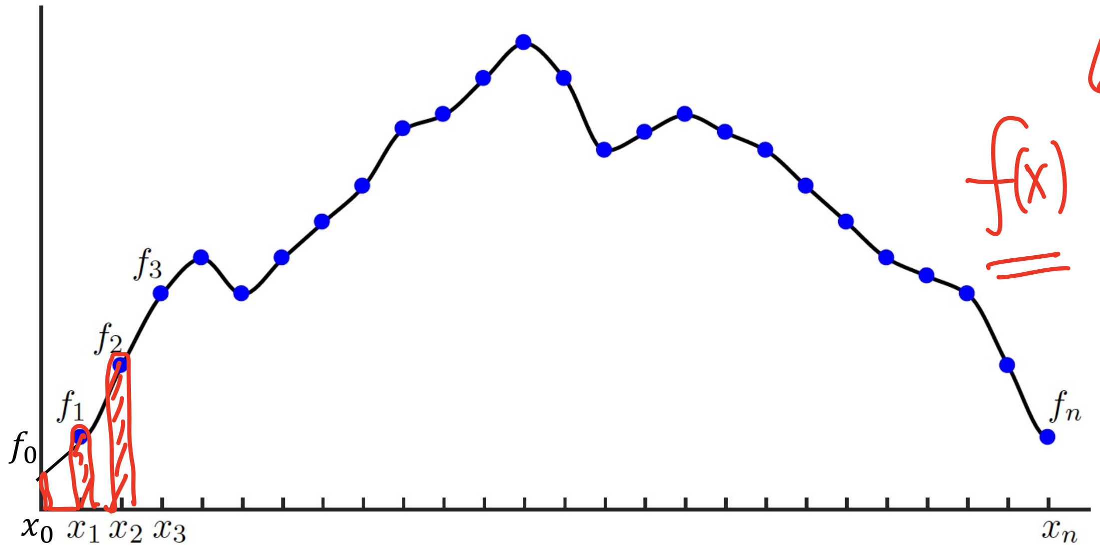
Fast
Computer
Computation

Approximate the Fourier transform on discrete vectors of data.

- **Discrete Fourier Transform (DFT)**
- **Fast Fourier Transform (FFT)**

Discrete Fourier Transform (DFT)

The DFT is tremendously useful for numerical approximation and computation.



$f(x) \rightsquigarrow \hat{f}(x)$

$n = 10000$

Convert $f(x)$ into a digital signal vector in $\mathbb{C}^n$ $\mathbb{R}^n$

$$\mathbf{f} = \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_{n-1} \end{bmatrix} \in \mathbb{C}^n \quad \mathbb{R}^n$$

$$\hat{f} = \begin{bmatrix} \hat{f}_0 \\ \hat{f}_1 \\ \vdots \\ \hat{f}_{n-1} \end{bmatrix} \in \mathbb{C}^n$$

$\cos\frac{-2\pi}{n} + i\sin\left(\frac{-2\pi}{n}\right)$

$\parallel$

- **The discrete Fourier transform (DFT)**

$$\hat{f}_k = \sum_{j=0}^{n-1} f_j\, e^{-2\pi i jk/n} = \sum_{j=0}^{n-1} f_j\, {\omega_n}^{jk}$$

Denote $\omega_n := e^{-2\pi i/n}$

$$i = \sqrt{-1}$$

$$e^{ix} = \cos(x) + i\sin(x)$$

- **The inverse discrete Fourier transform**

$$f_k = \frac{1}{n}\sum_{j=0}^{n-1} \hat{f}_j\, e^{i2\pi jk/n} = \frac{1}{n}\sum_{j=0}^{n-1} \hat{f}_j\, {\omega_n}^{-jk}$$

$n = 4$

$$\hat{f}_0 = f_0 + f_1 + f_2 + f_3$$

$$\hat{f}_1 = f_0 + w f_1 + w^2 f_2 + w^3 f_3$$

$$\hat{f}_2 = f_0 + w^2 f_1 + w^4 f_2 + w^6 f_3$$

$$\hat{f}_3 = f_0 + w^3 f_1 + w^6 f_2 + w^9 f_3$$

$$F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & w & w^2 & w^3 \\ 1 & w^2 & w^4 & w^6 \\ 1 & w^3 & w^6 & w^9 \end{bmatrix}$$

$$\begin{tikzcd} \mathbb{C}^4 \arrow[r, "F_4"] & \mathbb{C}^4 \arrow[l, bend left, "F_4^{-1}"] \end{tikzcd}$$

- **discrete Fourier transform** is a **linear** operator maps the data points to the frequency domain.

$$\mathbf{f} = \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_{n-1} \end{bmatrix} \xrightarrow{\textbf{DFT}} \hat{\mathbf{f}} = \begin{bmatrix} \hat{f}_0 \\ \hat{f}_1 \\ \hat{f}_2 \\ \vdots \\ \hat{f}_{n-1} \end{bmatrix}$$

- The DFT is computed by matrix multiplication

$\omega_n := e^{-2\pi i/n}$

$$\begin{bmatrix} \hat{f}_0 \\ \hat{f}_1 \\ \hat{f}_2 \\ \vdots \\ \hat{f}_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_n & \omega_n^2 & \cdots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^4 & \cdots & \omega_n^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \cdots & \omega_n^{(n-1)^2} \end{bmatrix}_{n\times n} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_{n-1} \end{bmatrix}_{n\times 1}$$

Here $\omega_n := e^{-2\pi i/n}$

$n \gg 0$

- The **discrete Fourier transform (DFT)** of any vector $\boldsymbol{f}$ in $\mathbb{C}^n$ is

$$\hat{\boldsymbol{f}} = F_n \boldsymbol{f}$$

- The matrix F_n of the DFT is called the **Fourier matrix.**

Convention: Some authors use the conjugate of F_n for Fourier matrix!

➢ Code for DFT matrix

Generate discrete Fourier transform matrix F_n (MATLAB or any other)

```
clear all, close all, clc
n = 256;
w = exp(-i*2*pi/n);

% Slow
for i=1:n
    for j=1:n
        DFT(i,j) = w^((i-1)*(j-1));
    end
end

% Fast
[I,J] = meshgrid(1:n,1:n);
DFT = w.^((I-1).*(J-1));
imagesc(real(DFT))
```

In Matlab, the built-in DFT matrix function: a=dftmtx(n)

Computation Complexity

- However, a direct calculation of the Discrete Fourier transform $\hat{f} = F_n f$ involves multiplication by a dense $n \times n$ matrix, requiring $O(n^2)$ operations.

  E.g., if $N = 10^3$, then $O(n^2) = 10^6$, i.e., a million.

- This makes the straightforward method slow and impractical, even for a moderately long sequence.

➢ Fast Fourier Transform (FFT)

$$\hat{\mathrm{f}} = F_n\,\mathrm{f}$$

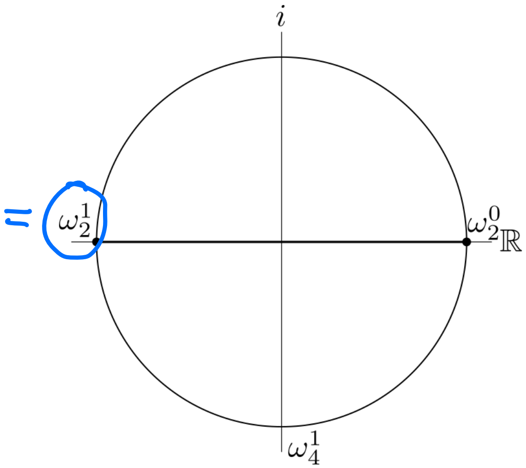
The **Fourier matrix** F_n is a Vandermonde matrix

$$F_n = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_n & \omega_n^2 & \cdots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^4 & \cdots & \omega_n^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \cdots & \omega_n^{(n-1)^2} \end{bmatrix}$$

$$\omega_n := e^{-2\pi i/n} = \cos\left(\frac{-2\pi}{n}\right) + i\sin\left(\frac{-2\pi}{n}\right)$$

➢ Roots of Unity and the Fourier Matrix

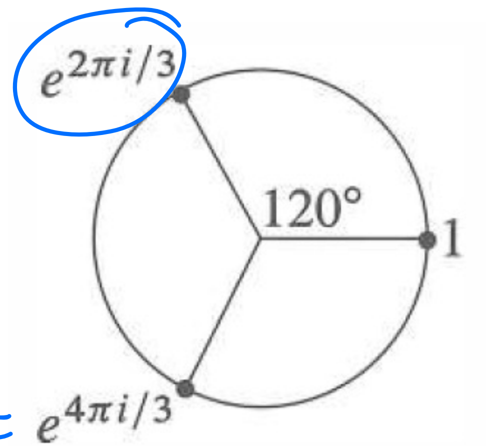
The solutions of $z^n = 1$ are the "n-th roots of unity."



The solution is the complex number

$$w_n := e^{2\pi i/n} = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$$

$\overline{\omega_3} = w_3 =$

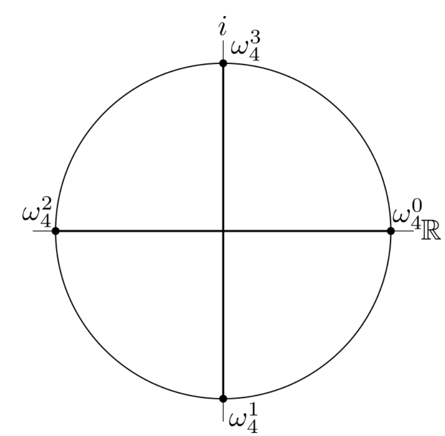


$\overline{\omega_3^2} = w_3^2 =$

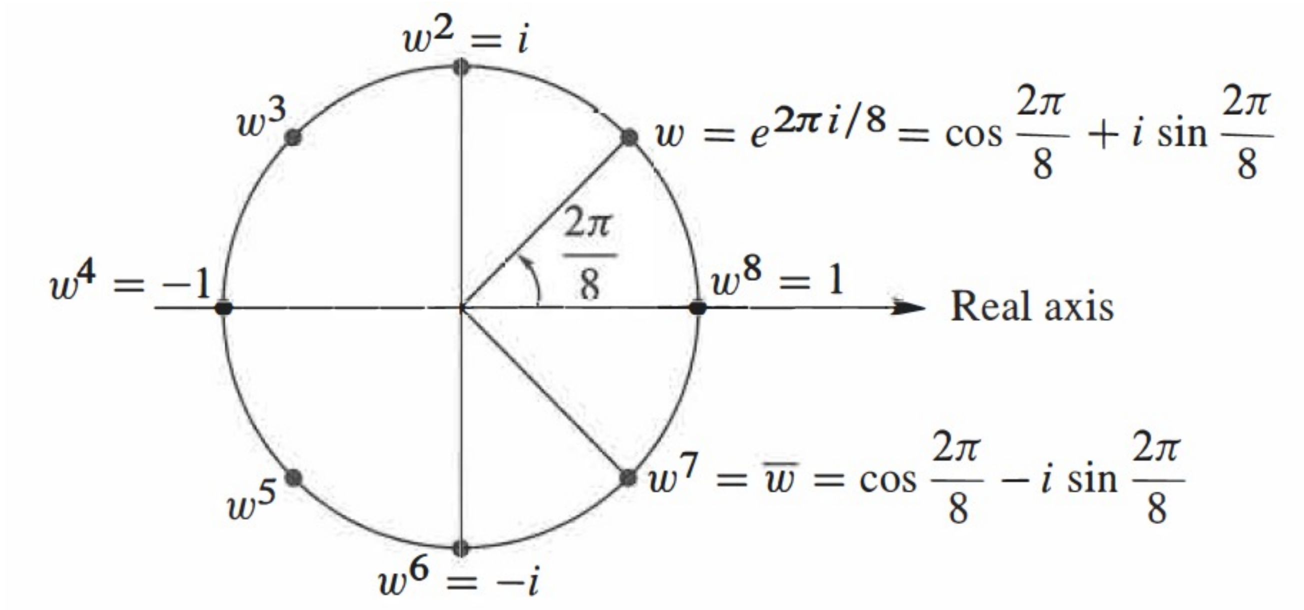
Relation with $\omega_n := e^{-2\pi i/n}$,

$$w_n = \overline{\omega_n}$$

These roots are n evenly spaced points around the unit circle in the complex plane.

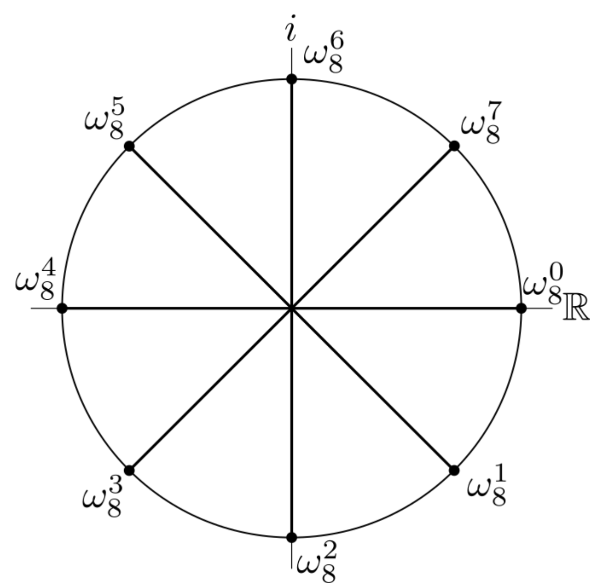


n=8



The eight solutions to $z^8 = 1$ are $1, w, w^2, \ldots, w^7$ with $w = (1+i)/\sqrt{2}$.

Use $\omega_n := e^{-2\pi i/n}$



➢ Properties of Fourier matrix

Proposition:

Define: $$E_k = \begin{bmatrix} 1 \\ w^k \\ w^{2k} \\ \vdots \\ w^{(n-1)k} \end{bmatrix}$$

1) $E_k = E_{k+n}$
2) $\{E_0, E_1, E_2, \dots, E_{n-1}\}$ *is orthogonal.*
   That is $\langle E_i, \ E_j \rangle = 0$ if $0 < |i-j| < n$
3) $\langle E_i, \ E_i \rangle = n$

Here $\langle \boldsymbol{u}, \boldsymbol{v} \rangle := u^* \, v$ the standard inner product in $\mathbb{C}^n$

$A^T = A^{-1}$

$\overline{U}^T = U^{-1}$

The **Fourier matrix** F_n is a Vandermonde matrix

$$F_n = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_n & \omega_n^2 & \cdots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^4 & \cdots & \omega_n^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \cdots & \omega_n^{(n-1)^2} \end{bmatrix}$$

$F_n^* = \overline{F_n}^T$

- The columns of the Fourier matrix F_n are the conjugate of $E_0, E_1, \dots, E_{n-1}$

- $F_n \underbrace{F_n^*}_{n} = \cancel{n}\, I_n$ $\qquad F_n^{-1} = \frac{1}{n} \underline{\underline{F_n^*}}$

- The determinant of F_n is $\underline{\sqrt{n}}$

- $\frac{1}{\sqrt{n}} F_n$ is a **unitary** matrix. That is, $(\frac{1}{\sqrt{n}} F_n)\left(\frac{1}{\sqrt{n}} F_n^*\right) = I_n$

- $F_n^{-1} = \frac{1}{n} F_n^* = \frac{1}{n} \overline{F_n}$

Recall that the DFT of $\boldsymbol{f} \in \mathbb{C}^n$ is the linear transformation $\hat{\boldsymbol{f}} = F_n\, \boldsymbol{f}$

The inverse discrete Fourier transform

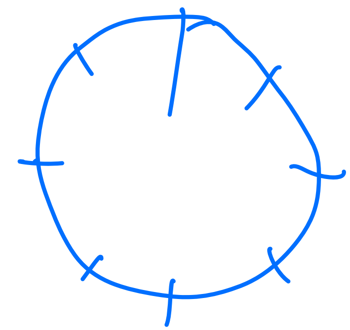
$$\boldsymbol{f} = F_n^{-1} \hat{\boldsymbol{f}} = \frac{1}{n} F_n^* \hat{\boldsymbol{f}} = \frac{1}{n} \overline{F_n} \hat{\boldsymbol{f}}$$

Examples:

$$F_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\omega_n := e^{-2\pi i/n} = \cos\frac{2\pi}{n} - i\sin\frac{2\pi}{n}$$

$$F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -\mathrm{i} & (-\mathrm{i})^2 & (-\mathrm{i})^3 \\ 1 & (-\mathrm{i})^2 & (\mathrm{i})^4 & (-\mathrm{i})^6 \\ 1 & (-\mathrm{i})^3 & (-\mathrm{i})^6 & (-\mathrm{i})^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -\mathrm{i} & -1 & \mathrm{i} \\ 1 & -1 & 1 & -1 \\ 1 & \mathrm{i} & -1 & -\mathrm{i} \end{bmatrix}$$



$$F_8 = \begin{bmatrix} \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 \\ \omega^0 & \omega^1 & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 & \omega^7 \\ \omega^0 & \omega^2 & \omega^4 & \omega^6 & \omega^8 & \omega^{10} & \omega^{12} & \omega^{14} \\ \omega^0 & \omega^3 & \omega^6 & \omega^9 & \omega^{12} & \omega^{15} & \omega^{18} & \omega^{21} \\ \omega^0 & \omega^4 & \omega^8 & \omega^{12} & \omega^{16} & \omega^{20} & \omega^{24} & \omega^{28} \\ \omega^0 & \omega^5 & \omega^{10} & \omega^{15} & \omega^{20} & \omega^{25} & \omega^{30} & \omega^{35} \\ \omega^0 & \omega^6 & \omega^{12} & \omega^{18} & \omega^{24} & \omega^{30} & \omega^{36} & \omega^{42} \\ \omega^0 & \omega^7 & \omega^{14} & \omega^{21} & \omega^{28} & \omega^{35} & \omega^{42} & \omega^{49} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & -i & -i\omega & -1 & -\omega & i & i\omega \\ 1 & -i & -1 & i & 1 & -i & -1 & i \\ 1 & -i\omega & i & \omega & -1 & i\omega & -i & -\omega \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -\omega & -i & i\omega & -1 & \omega & i & -i\omega \\ 1 & i & -1 & -i & 1 & i & -1 & -i \\ 1 & i\omega & i & -\omega & -1 & -i\omega & -i & \omega \end{bmatrix}$$

where, $\omega = e^{-\frac{2\pi i}{8}} = \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$

Example. Suppose $N = 4$ and $\mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}$.

Compute the DFT of y.
Compare the norms ||y|| and $||F_4\, y||$

Inverse of F_4.

$F_4^{-1} = \frac{1}{4}\overline{F_4}$

$$F_4\mathbf{y} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -\mathrm{i} & -1 & \mathrm{i} \\ 1 & -1 & 1 & -1 \\ 1 & \mathrm{i} & -1 & -\mathrm{i} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2-2\mathrm{i} \\ -2 \\ 2+2\mathrm{i} \end{bmatrix}$$

$$\|\mathbf{y}\|^2 = 1 + 2^2 + (-1)^2 = 6$$

$$\frac{1}{4}||F_4\, y|| = \frac{1}{4}[2^2 + (2-2\mathrm{i})(2+2\mathrm{i}) + (-2)^2 + (2+2\mathrm{i})(2-2\mathrm{i})] = \frac{1}{4}[4+8+4+8] = 6$$

Exercise: The signal corresponding to a single impulse at time zero is (roughly) described

$$e_1 = y = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$$

Find the Fourier transform of this signal we compute.

Fourier basis

Proposition:

Let $\{\boldsymbol{e_1}, \ldots, \boldsymbol{e_n}\}$ be the standard basis for $\mathbb{C}^n$.

- The Fourier transform $F_n E_k = \boldsymbol{e_{k+1}}$
- Set $\boldsymbol{u_j} = \frac{1}{\sqrt{n}} \boldsymbol{F_n e_j}$. Then $\{\boldsymbol{u_1}, \ldots, \boldsymbol{u_n}\}$ is an orthonormal basis for $\mathbb{C}^n$, called the **Fourier basis**.

Example.

$$\begin{bmatrix} \hat{f}_0 \\ \hat{f}_1 \\ \hat{f}_2 \\ \hat{f}_3 \end{bmatrix} = F_4 \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} f_0+f_2 + f_1+f_3 \\ f_0-f_2 - i(f_1-f_3) \\ f_0+f_2 - (f_1+f_3) \\ f_0-f_2 + i(f_1-f_3) \end{bmatrix}$$

Recall the Fourier matrix:

$$F_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}$$

It can be decomposed as

$$F_4 = \left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -i \\ \hline 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & i \end{array}\right)\left(\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \left[\begin{array}{c|c} I & D_2 \\ \hline I & -D_2 \end{array}\right]\left[\begin{array}{c|c} F_2 & 0 \\ \hline 0 & F_2 \end{array}\right] P$$

$$F_8 = \begin{bmatrix} \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 \\ \omega^0 & \omega^1 & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 & \omega^7 \\ \omega^0 & \omega^2 & \omega^4 & \omega^6 & \omega^8 & \omega^{10} & \omega^{12} & \omega^{14} \\ \omega^0 & \omega^3 & \omega^6 & \omega^9 & \omega^{12} & \omega^{15} & \omega^{18} & \omega^{21} \\ \omega^0 & \omega^4 & \omega^8 & \omega^{12} & \omega^{16} & \omega^{20} & \omega^{24} & \omega^{28} \\ \omega^0 & \omega^5 & \omega^{10} & \omega^{15} & \omega^{20} & \omega^{25} & \omega^{30} & \omega^{35} \\ \omega^0 & \omega^6 & \omega^{12} & \omega^{18} & \omega^{24} & \omega^{30} & \omega^{36} & \omega^{42} \\ \omega^0 & \omega^7 & \omega^{14} & \omega^{21} & \omega^{28} & \omega^{35} & \omega^{42} & \omega^{49} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & -i & -i\omega & -1 & -\omega & i & i\omega \\ 1 & -i & -1 & i & 1 & -i & -1 & i \\ 1 & -i\omega & i & \omega & -1 & i\omega & -i & -\omega \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -\omega & -i & i\omega & -1 & \omega & i & -i\omega \\ 1 & i & -1 & -i & 1 & i & -1 & -i \\ 1 & i\omega & i & -\omega & -1 & -i\omega & -i & \omega \end{bmatrix}$$

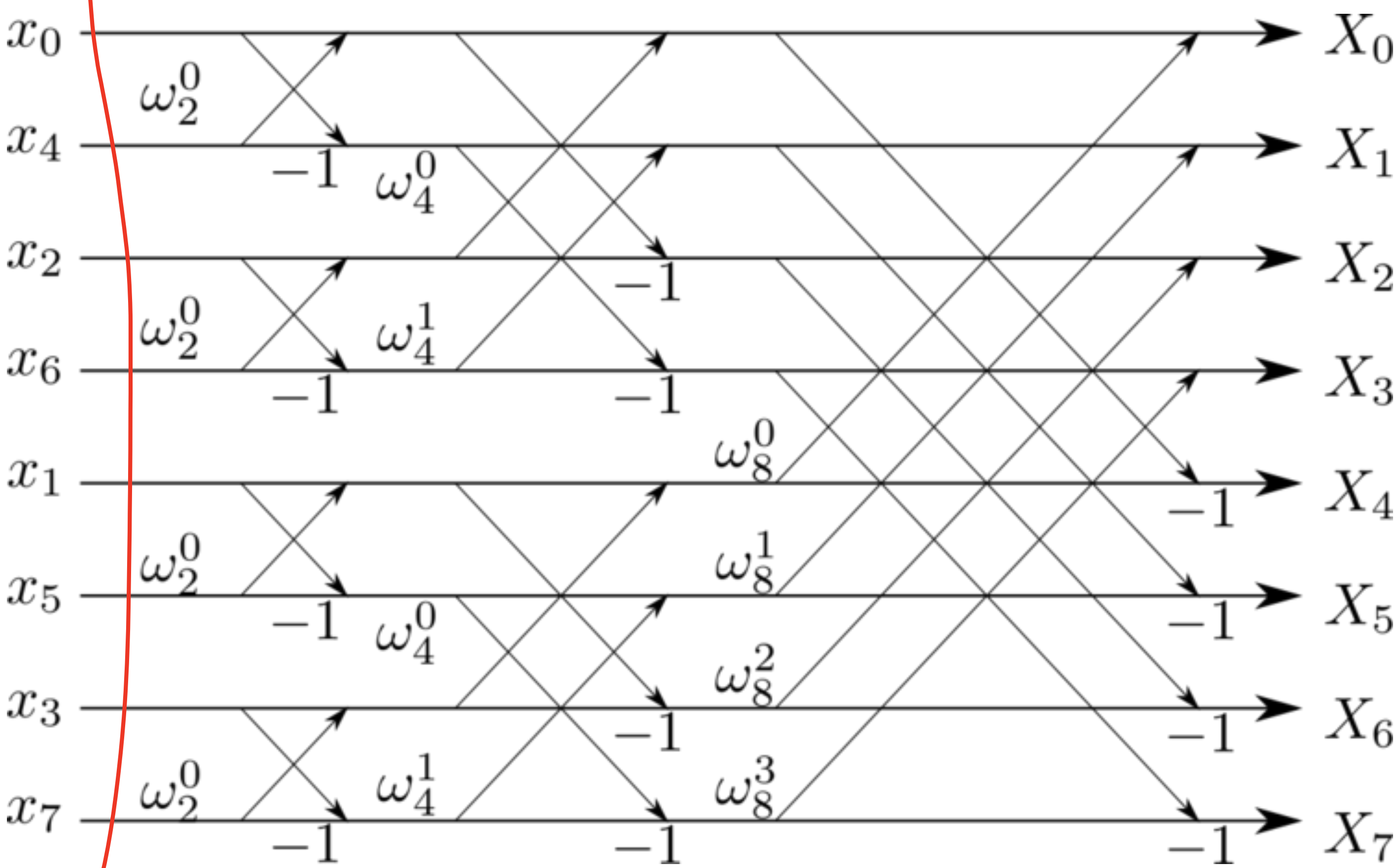
$$
\mathbf{F}_8\mathbf{P}_8 = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & \omega & -i & \omega^3 & -1 & -\omega & i & -\omega^3 \\
1 & \omega^2 & -1 & -\omega^2 & 1 & \omega^2 & -1 & -\omega^2 \\
1 & \omega^3 & i & \omega & -1 & -\omega^3 & -i & -\omega \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & -\omega & -i & -\omega^3 & -1 & \omega & i & \omega^3 \\
1 & -\omega^2 & -1 & \omega^2 & 1 & -\omega^2 & -1 & \omega^2 \\
1 & -\omega^3 & i & -\omega & -1 & \omega^3 & -i & \omega
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

$$
\begin{pmatrix}
\begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & -i & -1 & i \\
1 & -1 & 1 & -1 \\
1 & i & -1 & -i
\end{pmatrix} &
\begin{pmatrix}
1 & 1 & 1 & 1 \\
\omega & \omega^3 & -\omega & -\omega^3 \\
\omega^2 & -\omega^2 & \omega^2 & -\omega^2 \\
\omega^3 & \omega & -\omega^3 & -\omega
\end{pmatrix} \\
\begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & -i & -1 & i \\
1 & -1 & 1 & -1 \\
1 & i & -1 & -i
\end{pmatrix} &
\begin{pmatrix}
-1 & -1 & -1 & -1 \\
-\omega & -\omega^3 & \omega & \omega^3 \\
-\omega^2 & \omega^2 & -\omega^2 & \omega^2 \\
-\omega^3 & -\omega & \omega^3 & \omega
\end{pmatrix}
\end{pmatrix}
$$

$$
\begin{pmatrix}
1 & 1 & 1 & 1 \\
\omega & \omega^3 & -\omega & -\omega^3 \\
\omega^2 & -\omega^2 & \omega^2 & -\omega^2 \\
\omega^3 & \omega & -\omega^3 & -\omega
\end{pmatrix} =
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \omega & 0 & 0 \\
0 & 0 & \omega^2 & 0 \\
0 & 0 & 0 & \omega^3
\end{pmatrix}
\begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & -i & -1 & i \\
1 & -1 & 1 & -1 \\
1 & i & -1 & -i
\end{pmatrix} = \mathbf{D}_8\mathbf{F}_4
$$

$$
\begin{pmatrix}
-1 & -1 & -1 & -1 \\
-\omega & -\omega^3 & \omega & \omega^3 \\
-\omega^2 & \omega^2 & -\omega^2 & \omega^2 \\
-\omega^3 & -\omega & \omega^3 & \omega
\end{pmatrix} = -
\begin{pmatrix}
1 & 1 & 1 & 1 \\
\omega & \omega^3 & -\omega & -\omega^3 \\
\omega^2 & -\omega^2 & \omega^2 & -\omega^2 \\
\omega^3 & \omega & -\omega^3 & -\omega
\end{pmatrix} = -\mathbf{D}_8\mathbf{F}_4
$$

f


x_0
x_4
x_2
x_6
x_1
x_5
x_3
x_7
ω_2^0
ω_2^0
ω_2^0
ω_2^0
ω_4^0
ω_4^1
ω_4^0
ω_4^1
ω_8^0
ω_8^1
ω_8^2
ω_8^3
-1
X_0
X_1
X_2
X_3
X_4
X_5
X_6
X_7

$$\hat{\mathbf{f}} = \mathbf{F}_{1024}\mathbf{f} = \begin{bmatrix} \mathbf{I}_{512} & -\mathbf{D}_{512} \\ \mathbf{I}_{512} & -\mathbf{D}_{512} \end{bmatrix} \begin{bmatrix} \mathbf{F}_{512} & \mathbf{0} \\ \mathbf{0} & \mathbf{F}_{512} \end{bmatrix} \begin{bmatrix} \mathbf{f}_{\text{even}} \\ \mathbf{f}_{\text{odd}} \end{bmatrix}$$

$$\mathbf{D}_{512} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \omega & 0 & \cdots & 0 \\ 0 & 0 & \omega^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \omega^{511} \end{bmatrix}$$

$n = 2^s$

Fourier matrices can be broken down into chunks with lots of zero entries; Fourier probably didn't notice this. Gauss did, but didn't realize how significant a discovery this was.

The fact that $\omega_n^2 = \omega_n$ reveals a nice relationship between F_n and F_{2n}

$$F_{2n} = \begin{bmatrix} I_n & D_n \\ I_n & -D_n \end{bmatrix} \begin{bmatrix} F_n & 0 \\ 0 & F_n \end{bmatrix} P.$$

P is a $2n \times 2n$ permutation matrix.

$n \times n$

E.g., when $n = 1024 = 2^{10}$.
Directly multiplying by F_n requires over 10^6 calculations.
The fast Fourier transform can be completed with only $\frac{1}{2} n \log_2 n = 5*1024$ calculations. This is 200 times faster!

In 1965, James W. Cooley (IBM) and John W. Tukey (Princeton) developed the revolutionary fast Fourier transform (FFT) algorithm which needs $O(n \log(n))$ operations. (The algorithm was formulated by Gauss in 1805.)

"the most important numerical algorithm of our lifetime"
--Gilbert Strang 1994

This is only possible because Fourier matrices are special matrices with orthogonal columns. In the next lecture we'll return to dealing exclusively with real numbers and will learn about positive definite matrices, which are the matrices most often seen in applications