**Exercise 1. [Ross 1-29] Suppose that *P(E)* = 0.6. What can you say about *P(E|F)* when**

**(a) *E* and *F* are mutually exclusive?**

**(b) $E \subset F$?**

**(c) $F \subset E$?**

**Solution 1.** Given that, *P(E)* = 0.6

(a) The events '*E*' and '*F*' are mutually exclusive then *P(EF)* = 0

$$\therefore P(E|F) = \frac{P(EF)}{P(F)} = 0$$

(b) $E \subset F$, then $P\,(E \cap F)$ *or* $P(EF) = P(E)$

$$\therefore P(E|F) = \frac{P(EF)}{P(F)}$$

$$= \frac{P(E)}{P(F)}$$

$\because E \subset F$, we have $P(E) < P(F)$

$=> 0.6 < P(F) \leq 1$
$\therefore P(E|F)$ is observed to be greater than or equal to 0.6 and less than 1.

(c) $F \subset E$, then $P(EF) = P(F)$

$$\therefore P(E|F) = \frac{P(EF)}{P(F)}$$

$$= \frac{P(F)}{P(F)}$$

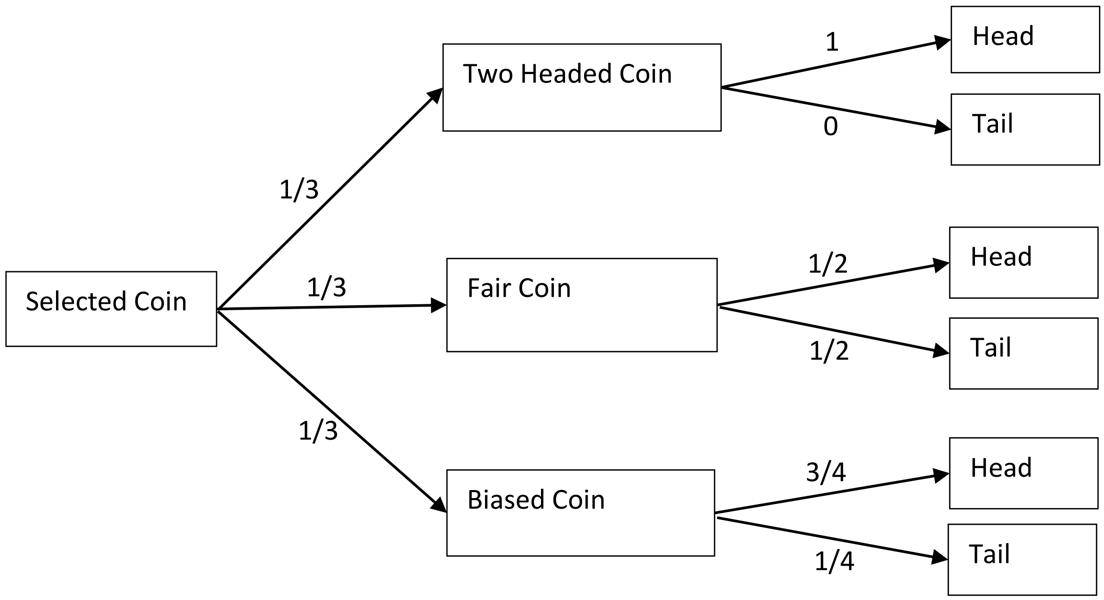
$$= 1$$

**Exercise 2. [Ross 1-42] There are three coins in a box. One is a two-headed coin, another is a fair coin, and the third is a biased coin that comes up Heads 75 percent of the time. One of the three coins is selected at random and flipped, and it comes up Heads. What is the probability that it was the two-headed coin?**

**Solution 2.** The probability that the flipped coin shows head is obtained below:

Let event T is denoted as two-headed, event F denoted as coin is fair, event B denoted as coin is biased and event H is denoted as flipped coin shows head.

From the information given, the tree diagram is drawn as below:




Probability of getting a head is:

$$P(H) = [\{P(T)P(H|T)\} + \{P(F)P(H|F)\} + \{P(B)P(H|B)\}]$$

$$P(H) = \left[\left(\frac{1}{3} \times 1\right) + \left(\frac{1}{3} \times \frac{1}{2}\right) + \left(\frac{1}{3} \times \frac{3}{4}\right)\right]$$

$$P(H) = \left(\frac{1}{3} + \frac{1}{6} + \frac{1}{4}\right)$$

$$P(H) = \frac{9}{12} = \frac{3}{4}$$

The probability that it was the two-headed coin, given that the flipped coin shows head is:

$$P(T|H) = \frac{P(T)P(H|T)}{P(H)}$$

$$P(T|H) = \frac{\frac{1}{3} \times 1}{\frac{3}{4}}$$

$$P(T|H) = \frac{4}{9} = 0.4444$$

**Exercise 3. In a variation of the classic Monty Hall game show, the host sets up five doors and hides prizes behind two of the doors. The contestant first guesses a door, and then the host opens one of the other four doors to show that it does not conceal a prize. The contestant is offered the opportunity to switch her guess to a different door. Should she switch or stay with her original choice?**

**[Hint: see notes on the Monty Hall question and try to imitate the solution provided there].**

**Solution 3.** There are 5 doors and 2 prizes.

Let O = {Originally guessed correct answer}.

S = {Correct guess after switching}.

W = {Correct guess without switching}

$$P(W) = P(W|O)P(O) + P(W|O^c)P(O^c)$$
$$P(W) = \left(1 \times \frac{2}{5}\right) + 0 = \frac{2}{5} = 0.4$$
$$P(S) = P(S|O)P(O) + P(S|O^c)P(O^c)$$
$$P(S) = \left(\frac{1}{3} \times \frac{2}{5}\right) + \left(\frac{2}{3} \times \frac{3}{5}\right) = \frac{8}{15} = 0.53$$

$\because P(S) > P(W)$ , the contestant should switch.

**Exercise 4. A fair die is rolled repeatedly. Let $X_n$ be the result of the $n^{th}$ roll. So $X_n$ takes values {1, . . . , 6}, each with probability 1/6, and the random variables X1, X2, . . . are all independent. Let**

$$\boldsymbol{N = min\{n : X_n = X_{n-1} = 4, n \geq 2\}}$$

**That is, N is the first time you roll two consecutive 4's. Find E[N].**

**[Hint: condition on the outcomes of the first two rolls and imitate the argument we used in class for the 'rat in a maze' problem to derive a recursive equation for E[N].]**

**Solution 4.** $E\ [\ N \mid X_1 = a\ ]$ is same for all a = 1, 2, 3, 4, 5, 6

$$=> E[N] = \sum_{a=1}^{6} E[\ N \mid X_1 = a]\ P(X_1 = a) = E[\ N \mid X_1 = 4]$$
$$E[\ N \mid X_1 = 4] = E[\ N \mid X_1 = 4, X_2 = 4]\ P(\ X_2 = 4 \mid X_1 = 4)$$
$$E[\ N \mid X_1 = 4] = \frac{1}{6}\ E[\ N \mid X_1 = 4, X_2 = 4]$$
$$E[\ N \mid X_1 = 4] = \frac{1}{6}\ (2)$$

$E[\, N \mid X_1 = 4] = \dfrac{1}{3}$

$=> E[N] = E[\, N| \; X_1 = 4] = \dfrac{1}{3}$

**Exercise 5. Suppose X is an exponential random variable. One of the following three formulas is**

**correct:**

**(a)** $\boldsymbol{E[X^2| \, X > 1] = E[(X + 1)^2]}$

**(b)** $\boldsymbol{E[X^2| \, X > 1] = E[X^2] + 1}$

**(c)** $\boldsymbol{E[X^2| \, X > 1] = (E[X] + 1)^2}$

**Without doing computations, use the memoryless property of the exponential distribution to explain which answer is correct.**

**Solution 5.** Give X > 1, memoryless property says

$$X = 1 + X'$$

Where, X' ~ exponential, same rate at X.

$$=> E[X^2 \,|X > 1] = E[(1 + X')^2 \,|X > 1]$$

$= E[(1 + X')^2]$ (X' is independent of {X > 1})

$$= E[(1 + X)^2]$$

$$= E[(X + 1)^2]$$