MTH 7241 Fall 2022: Prof. C. King

Notes 6: Branching processes

Discrete time process, modeling the change in a population. Each time step produces a new generation. Each individual in generation n produces a random number of individuals in generation $n+1$. In the simplest model the number of 'offspring' has the same distribution for all individuals and does not change over time, and is independent of the current population. Let Z be the random variable representing the number of offspring, so the range of Z is $\{0, 1, 2, \dots\}$, and

$$P(Z = k) = p_k, \quad k = 0, 1, 2 \dots \tag{1}$$

The mean is

$$m = \mathbb{E}[Z] = \sum_{k \geq 0} k \, p_k \tag{2}$$

Let X_n be the size of the population in generation n then

$$X_{n+1} = \sum_{i=1}^{X_n} Z_i \tag{3}$$

where Z_i is the number of offspring of the i^{th} individual. Independence implies

$$\mathbb{E}[X_{n+1}] = \mathbb{E}[X_n]\,\mathbb{E}[Z] = m\mathbb{E}[X_n] \tag{4}$$

If the initial population size is 1, so $X_0 = 1$, then iteration of this formula hows that

$$\mathbb{E}[X_n] = m^n \tag{5}$$

So the mean population size either grows or decays exponentially depending on the mean number of offspring. If $\mathbb{E}[Z] = m = 1$ the model is said to be *critical.*

The probability of extinction is the probability that X_n is eventually zero. We define this to be

$$\rho = P(X_n = 0 \text{ eventually}) \tag{6}$$

Since $\{X_n = 0\} \Rightarrow \{X_{n+1} = 0\}$ we have

$$\{X_n = 0\} \subset \{X_{n+1} = 0\} \tag{7}$$

Therefore

$$\begin{aligned}
\{X_n = 0 \text{ eventually}\} &= \bigcup_{n\geq 0} \bigcap_{k\geq n} \{X_k = 0\} && (8)\\
&= \bigcup_{n\geq 0} \{X_n = 0\} && (9)\\
& && (10)
\end{aligned}$$

and so

$$\rho = \lim_{n\to\infty} P(X_n = 0) \tag{11}$$

Lemma 1 *Suppose that* $m = \mathbb{E}[Z] < 1$. *Then*

$$\rho = 1 \tag{12}$$

Proof:

$$\begin{aligned} \mathbb{E}[X_n] &= \sum_{k\geq 1} k\, P(X_n = k) && (13) \\ &\geq \sum_{k\geq 1} P(X_n = k) && (14) \\ &= P(X_n > 0) && (15) \end{aligned}$$

Therefore

$$P(X_n > 0) \leq m^n \to 0 \quad \text{as } n \to \infty \tag{16}$$

Hence

$$\lim_{n\to\infty} P(X_n = 0) = 1 \tag{17}$$

Formula for extinction probability

We can obtain more precise information by using a moment generating function. Define

$$\phi(s) = \mathbb{E}[s^Z] \tag{18}$$

and

$$\phi_n(s) = \mathbb{E}[s^{X_n}] \tag{19}$$

We assume $X_0 = 1$, so

$$\phi_0(s) = s, \quad \phi_1(s) = \mathbb{E}[s^Z] = \phi(s) \tag{20}$$

and we also have

$$\phi_n(0) = P(X_n = 0) \tag{21}$$

Also

$$\begin{aligned}
\phi_{n+1}(s) &= \mathbb{E}[s^{X_{n+1}}] && (22)\\
&= \mathbb{E}[\mathbb{E}[s^{Z_1+\cdots+Z_{X_n}}] \,|\, X_n] && (23)\\
&= \mathbb{E}[(\phi(s))^{X_n}] && (24)\\
&= \phi_n(\phi(s)) && (25)
\end{aligned}$$

Iterating and rearranging we get

$$\phi_{n+1}(s) = \phi(\phi_n(s)) \tag{26}$$

Now it is easy to see that

$$\phi_n(0) = P(X_n = 0) \tag{27}$$

so we know that $\{\phi_n(0)\}$ is increasing in n. Thus

$$\rho = \lim_{n\to\infty} \phi_n(0) \tag{28}$$

Furthermore taking the limit $s \to 0$ and $n \to \infty$ in the formula above we deduce

$$\rho = \phi(\rho) \tag{29}$$

So ρ is a fixed point of ϕ. Considering the graph of ϕ, and noting its convexity, we get this result.

Lemma 2 *Suppose that $P(Z \geq 2) > 0$, then*

$$\rho \begin{cases} = 1 & \text{for } m \leq 1 \\ < 1 & \text{for } m > 1 \end{cases} \tag{30}$$

$\boxed{\textbf{Example}}$ For a branching process, calculate the probability of extinction when $p_0 = 1/6$, $p_1 = 1/2$, $p_2 = 1/3$.

$\boxed{\textbf{MGF for critical case}}$

We will consider one specific critical case. Suppose that

$$P(Z = 0) = P(Z = 2) = \frac{1}{2} \tag{31}$$

Then clearly $m = \mathbb{E}[Z] = 1$ so this is critical. Therefore we know that $\rho = 1$ so eventually the process will reach 0 and the population will die out. The interesting question is the transient dynamics. Assume as usual that $X_0 = 1$ and define the mgf as before

$$\phi_n(s) = \mathbb{E}[s^{X_n}] \tag{32}$$

Then the mgf is determined by the recursion relation

$$\phi_{n+1}(s) = \frac{1}{2}\left(1 + \phi_n(s)^2\right), \quad \phi_0(s) = s \tag{33}$$

However this not very useful because is no closed form solution. Surprisingly the situation comes easier if we put in more randomness, by replacing the fixed timespan between generations by a random exponential time. Specifically, we assume that each individual waits an exponential time with rate r and then either 'dies' or produces one extra offspring, with equal probability. All waiting times are assumed to be independent. The generation index n is replaced by a time index t, so $X(t)$ denotes the population size at time t. We are interested in

$$P_k(t) = P(X(t) = k) \quad \text{for } k = 0, 1, 2 \ldots, \text{ and } t \geq 0 \tag{34}$$

This is a continuous time Markov chain, and the forward Kolmogorov equation is

$$\frac{d}{dt} P_k(t) = \begin{cases} r(k-1)\, P_{k-1}(t) + r(k+1)\, P_{k+1}(t) - 2rkP_k(t) & \text{for } k \geq 2 \\ rP_1(t) & \text{for } k = 0 \\ 2rP_2(t) - 2rP_1(t) & \text{for } k = 1 \end{cases} \tag{35}$$

The initial condition is

$$P_k(0) = \begin{cases} 1 & \text{for } k = 0 \\ 0 & \text{for } k > 0 \end{cases} \tag{36}$$

We use a generating function to solve this system. Define

$$F(s,t) = \sum_{k=0}^{\infty} s^k\, P_k(t) \tag{37}$$

Then F satisfies

$$\frac{\partial F}{\partial t} = r\,(s-1)^2\, \frac{\partial F}{\partial s}, \quad F(s,0) = s, \quad F(1,t) = 1 \tag{38}$$

Define $u = r^{-1}\,(1-s)^{-1}$ then

$$\frac{\partial F}{\partial t} = \frac{\partial F}{\partial u} \tag{39}$$

Hence

$$F = G(t+u) = G(t + \frac{1}{r(1-s)}) \tag{40}$$

for some function G. Setting $t = 0$ and using the boundary condition we find

$$G(x) = 1 - \frac{1}{x} \tag{41}$$

Substituting and expanding in powers on s we get

$$F(s,t) = \frac{rt}{1+rt} + \frac{1}{(1+rt)^2} \sum_{k=1}^{\infty} s^k \left(\frac{rt}{1+rt}\right)^{k-1} \tag{42}$$

and hence we deduce

$$P_0(t) = \frac{rt}{1+rt}, \quad P_k(t) = \frac{1}{(1+rt)^2} \left(\frac{rt}{1+rt}\right)^{k-1} \quad (k \geq 1) \tag{43}$$