Stability of Linear Systems

We consider the stability question for solutions of autonomous differential equations.

Let $\vec{x} = \vec{\varphi}(t)$ be a solution of the differential equation

$$\vec{x}' = f(\vec{x}) \qquad (1)$$

We are interested in determining whether $\vec{\varphi}(t)$ is stable or unstable. That is to say, we seek to determine whether every solution $\vec{\psi}(t)$ of (1) which starts sufficiently close to $\vec{\varphi}(t)$ at $t=0$ must remain close to $\vec{\varphi}(t)$ for all future time $t>0$. We begin with the following formal definition of stability.

<u>Definition</u> The solution $\vec{x}(t) = \vec{\varphi}(t)$ of (1) is <u>stable</u> if every solution $\vec{\psi}(t)$ of (1) which starts sufficiently close to $\vec{\varphi}(t)$ at $t=0$ must remain close to $\vec{\varphi}(t)$ for all future time t. The solution $\vec{\varphi}(t)$ is <u>unstable</u> if there exists at least one solution $\vec{\psi}(t)$ of (1) which starts near $\vec{\varphi}(t)$ at $t=0$ but which does not remain close to $\vec{\varphi}(t)$ for all future time. More precisely, the solution $\vec{\varphi}(t)$ is stable if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon)$ such that

$$|\psi_j(t) - \varphi_j(t)| < \varepsilon \text{ if } |\psi_j(0) - \varphi_j(0)| < \delta(\varepsilon), \quad j = 1, \dots, n$$

for every solution $\vec{\psi}(t)$ of (1).

The stability question can be completely resolved for each solution of the linear differential equation

$$\vec{x}' = A\vec{x} \qquad (2)$$

This is not surprising, of course, since we can solve Eq. (2) exactly.

We have the following important theorem.

Theorem 1 (Stability)

(a) Every solution $\vec{x} = \vec{\varphi}(t)$ of (2) is stable if all the eigenvalues of A have negative real part.

(b) Every solution $\vec{x} = \vec{\varphi}(t)$ of (2) is unstable if at least one eigenvalue of A has positive real part.

(c) Suppose that all the eigenvalues of A have real part ≤ 0 and $\lambda_1 = i\sigma_1, \ldots, \lambda_\ell = i\sigma_\ell$ have zero real part. Let $\lambda_j = i\sigma_j$ have multiplicity k_j. This means that the characteristic polynomial of A can be factored into the form

$$p(\lambda) = (\lambda - i\sigma_1)^{k_1} \cdots (\lambda - i\sigma_\ell)^{k_\ell} q(\lambda)$$

where all the roots of $q(\lambda)$ have negative real part. Then, every solution $\vec{x} = \vec{\varphi}(t)$ of (1) is stable if A has k_j linearly independent eigenvectors for each eigenvalue $\lambda_j = i\sigma_j$. Otherwise, every solution $\vec{\varphi}(t)$ is unstable.

Our first step in proving Theorem 1 is to show that every solution $\vec{\varphi}(t)$ is stable if the equilibrium solution $\vec{x}(t) \equiv \vec{0}$ is stable, and every solution $\vec{\varphi}(t)$ is unstable if $\vec{x}(t) \equiv \vec{0}$ is unstable.

Let $\vec{\psi}(t)$ be any solution of (2). Observe that $\vec{z}(t) = \vec{\varphi}(t) - \vec{\psi}(t)$ is again a solution of (2). Therefore, if the equilibrium solution $\vec{x}(t) \equiv \vec{0}$ is stable, then $\vec{z}(t) = \vec{\varphi}(t) - \vec{\psi}(t)$ will always remain small if $\vec{z}(0) = \vec{\varphi}(0) - \vec{\psi}(0)$ is sufficiently small. Consequently, every solution $\vec{\varphi}(t)$ of (2) is stable.

On the other hand suppose that $\vec{x}(t) \equiv \vec{0}$ is unstable. Then, there exists a solution $\vec{x} = \vec{h}(t)$ which is very small initially, but which becomes large as t approaches infinity. The function $\vec{\psi}(t) = \vec{\varphi}(t) + \vec{h}(t)$ is clearly a solution of (2). Moreover, $\vec{\psi}(t)$ is close to $\vec{\varphi}(t)$ initially, but diverges from $\vec{\varphi}(t)$ as t increases. Therefore, every solution $\vec{x}(t) = \vec{\varphi}(t)$ of (2) is unstable.

Our next step in proving Theorem 1 is to reduce the problem of showing that n quantities $\psi_j(t)$, $j = 1, \ldots, n$ are small to the much simpler problem of showing that only one quantity is small.

This is accomplished by introducing the concept of length, or magnitude, of a vector.

<u>Definition</u> Let

$$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

be a vector with n components. The numbers $x_1, \dots, x_n$ may be real or complex. We define the length of $\vec{x}$, denoted by $\|\vec{x}\|$ as

$$\|\vec{x}\| \equiv \max\{|x_1|, |x_2|, \dots, |x_n|\}.$$

For example, if

$$\vec{x} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}, \text{ then } \|\vec{x}\| = 3$$

and if

$$\vec{x} = \begin{pmatrix} 1+2i \\ 2 \\ -1 \end{pmatrix}, \text{ then } \|\vec{x}\| = \sqrt{5}.$$

The concept of the length, or magnitude of a vector corresponds to the concept of the length, or magnitude of a number. Observe that $\|\vec{x}\| \geq 0$ for any vector $\vec{x}$ and $\|\vec{x}\| = 0$ only if $\vec{x} = \vec{0}$. Second, observe that

$$\|\lambda\vec{x}\| = \max\{|\lambda x_1|, |\lambda x_2|, \dots, |\lambda x_n|\} = |\lambda| \max\{|x_1|, \dots, |x_n|\} = |\lambda| \|\vec{x}\|$$

Finally, observe that

$$\|\vec{x} + \vec{y}\| = \max\{|x_1 + y_1|, \dots, |x_n + y_n|\} \leq \max\{|x_1| + |y_1|, \dots, |x_n| + |y_n|\}$$

$$\leq \max\{|x_1|, \dots, |x_n|\} + \max\{|y_1|, \dots, |y_n|\} = \|\vec{x}\| + \|\vec{y}\|.$$

Thus, our definition really captures the meaning of length.

<u>Proof of Theorem 1</u>

(a) Every solution $\vec{x} = \vec{\psi}(t)$ of $\vec{x}' = A\vec{x}$ is of the form

$$\vec{\psi}(t) = e^{At}\vec{\psi}(0)$$

Let $\varphi_{ij}(t)$ be the ij element of the matrix e^{At}, and let $\psi_1^\circ, \dots \psi_n^\circ$ be the components of $\vec{\psi}(0)$. Then, the ith component of $\vec{\psi}(t)$ is

$$\psi_i(t) = \varphi_{i1}(t)\psi_1^\circ + \dots + \varphi_{in}(t)\psi_n^\circ \equiv \sum_{j=1}^{n} \varphi_{ij}(t)\psi_j^\circ .$$

Suppose that all the eigenvalues of A have negative real part. Let α_1 be the largest of the real parts of the eigenvalues of A. It is a simple matter to show that for every number $-\alpha$, with $-\alpha_1 < -\alpha < 0$, we can find a number K such that

$$|\varphi_{ij}(t)| \le K e^{-\alpha t}, \quad t \ge 0. \Rightarrow$$

$$|\psi_i(t)| \le \sum_{j=1}^{n} K e^{-\alpha t} |\psi_j^\circ| = K e^{-\alpha t} \sum_{j=1}^{n} |\psi_j^\circ|$$

for some positive constants K and α. Now, $|\psi_j^\circ| \le \|\vec{\psi}(0)\|$. Hence,

$$\|\vec{\psi}(t)\| = \max\{|\psi_1(t)|, \dots, |\psi_n(t)|\} \le nKe^{-\alpha t}\|\vec{\psi}(0)\|.$$

Let $\varepsilon > 0$ be given. Choose $\delta(\varepsilon) = \dfrac{\varepsilon}{nK}$. Then

$$\|\vec{\psi}(t)\| < \varepsilon \quad \text{if } \|\vec{\psi}(0)\| < \delta(\varepsilon) \text{ and } t \ge 0, \text{ since}$$

$$\|\vec{\psi}(t)\| \le nKe^{-\alpha t}\|\vec{\psi}(0)\| < nK\varepsilon/nK = \varepsilon.$$

Consequently, the equilibrium solution $\vec{x}(t) \equiv 0$ is stable.

(b) Let λ be an eigenvalue of A with positive real part and let $\vec{v}$ be an eigenvector of A with eigenvalue λ. Then,

$$\vec{\psi}(t) = ce^{\lambda t}\vec{v}$$

is a solution of $\vec{x}' = A\vec{x}$ for any constant c.

If λ is real then $\vec{v}$ is also real and

$$\|\vec{\varphi}(t)\| = |c|\, e^{\lambda t}\, \|\vec{v}\|.$$

Clearly, $\|\vec{\varphi}(t)\|$ approaches infinity as $t \to \infty$, for any choice of $c \neq 0$, no matter how small. Therefore, $\vec{x}(t) \equiv \vec{0}$ is unstable.

If $\lambda = \alpha + i\beta$ is complex, then $\vec{v} = \vec{v}^1 + i\vec{v}^2$ is also complex. In this case

$$e^{(\alpha+i\beta)t}(\vec{v}^1 + i\vec{v}^2) = e^{\alpha t}(\cos\beta t + i\sin\beta t)(\vec{v}^1 + i\vec{v}^2)$$

$$= e^{\alpha t}[\vec{v}^1\cos\beta t - \vec{v}^2\sin\beta t] + i[\vec{v}^1\sin\beta t + \vec{v}^2\cos\beta t]$$

is a complex-valued solution of (2). Therefore

$$\vec{\varphi}^1(t) = c e^{\alpha t}(\vec{v}^1\cos\beta t - \vec{v}^2\sin\beta t)$$

is a real-valued solution of (2), for any choice of constant c. Clearly, $\|\vec{\varphi}^1\|$ is unbounded as $t \to \infty$ if c and either $\vec{v}^1$ or $\vec{v}^2$ is nonzero. Thus, $\vec{x}(t) \equiv 0$ is unstable.

(c) If A has k_j linearly independent eigenvectors for each eigenvalue $\lambda_j = i\sigma_j$ of multiplicity k_j, then we can find a constant K such that

$$|(e^{At})_{ij}| \leq K.$$

There, $\|\vec{\varphi}(t)\| \leq nK\|\vec{\varphi}(0)\|$ for every solution $\vec{\varphi}(t)$ of (2). It now follows immediately from the proof of (a) that $\vec{x}(t) \equiv \vec{0}$ is stable.

On the other hand, if A has fewer than k_j linearly independent eigenvectors with eigenvalue $\lambda_j = i\sigma_j$, then $\vec{x}\,' = A\vec{x}$ has solutions $\vec{\varphi}(t)$ of the form

$$\vec{\varphi}(t) = c e^{i\sigma_j t}[\vec{v} + t(A - i\sigma_j I)\vec{v}]$$

where $(A - i\sigma_j I)\vec{v} \neq \vec{0}$. If $\sigma_j = 0$, then $\vec{\varphi}(t) = c(\vec{v} + tA\vec{v})$ is real-valued.

Moreover, $\|\vec{\psi}(t)\|$ is unbounded as $t \to \infty$ for any choice of $C \neq 0$. Similarly, both the real and imaginary parts of $\vec{\psi}(t)$ are unbounded in magnitude for arbitrary small $\vec{\psi}(0) \neq \vec{0}$ if $\sigma_j \neq 0$. Therefore, the equilibrium solution $\vec{x}(t) \equiv \vec{0}$ is unstable. //

If all the eigenvalues of A have negative real parts, then every solution $\vec{x}(t)$ of $\vec{x}' = A\vec{x}$ approaches zero as $t \to \infty$. This follows immediately from the estimate

$$\|\vec{x}\| \le K e^{-\alpha t} \|\vec{x}(0)\|$$

which we derived in the proof of part (a) of Theorem 1. Thus, not only is the equilibrium solution $\vec{x}(t) \equiv \vec{0}$ stable, but every solution $\vec{\psi}(t)$ of (2) approaches it as t approaches infinity. This very strong type of stability is known as <u>asymptotic</u> stability.

<u>Definition</u> A solution $\vec{x} = \vec{\varphi}(t)$ of (1) is asymptotically stable if it is stable, and if every solution $\vec{\psi}(t)$ which starts sufficiently close to $\vec{\varphi}(t)$ must approach $\vec{\varphi}(t)$ as t approaches infinity. In particular, an equilibrium solution $\vec{x}(t) = \vec{x}^0$ of (1) is asymptotically stable if every solution $\vec{x} = \vec{\psi}(t)$ of (1) which starts sufficiently close to $\vec{x}^0$ at time $t=0$ not only remains close to $\vec{x}^0$ for all future time, but ultimately approaches $\vec{x}^0$ as t approaches infinity.

<u>Remark</u> The asymptotic stability of any solution $\vec{x} = \vec{\varphi}(t)$ of (2) is clearly equivalent to the asymptotic stability of the equilibrium solution $\vec{x}(t) \equiv \vec{0}$.

<u>Example 1</u>. Determine whether each solution $\vec{x}(t)$ of the differential equation

$$\vec{x}' = \begin{pmatrix} -1 & 0 & 0 \\ -2 & -1 & 2 \\ -3 & -2 & -1 \end{pmatrix} \vec{x}$$

is stable, asymptotically stable, or unstable.

Solution The characteristic polynomial of A is

$$p(\lambda) = \det(A-\lambda I) = \det\begin{pmatrix} -1-\lambda & 0 & 0 \\ -2 & -1-\lambda & 2 \\ -3 & -2 & -1-\lambda \end{pmatrix} = -(1+\lambda)^3 + 4(-1-\lambda)$$

$$= -(1+\lambda)^3 - 4(1+\lambda) = -(1+\lambda)\big(\underbrace{(1+\lambda)^2+4}_{\lambda+1=\pm 2i}\big) = -(1+\lambda)(\lambda^2+2\lambda+1+4)$$

$$= -(1+\lambda)(\lambda^2+2\lambda+5)$$

$\lambda_1 = -1$, $\lambda_{2,3} = -1 \pm 2i$

Since all three eigenvalues have negative real part, we conclude that every solution of the differential equation $\vec{x}' = A\vec{x}$ is asymptotically stable.

Example 2 Prove that every solution of the differential equation

$$\vec{x}' = \begin{pmatrix} 1 & 5 \\ 5 & 1 \end{pmatrix}\vec{x}$$

is unstable.

Solution $p(\lambda) = \det(A-\lambda I) = \det\begin{pmatrix} 1-\lambda & 5 \\ 5 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 - 25$

Hence $\lambda_1 = 6$ and $\lambda_2 = -4$ are the eigenvalues of A. Since one eigenvalue of A is positive, we conclude that every solution $\vec{x} = \vec{\varphi}(t)$ of $\vec{x}' = A\vec{x}$ is unstable.

Example 3 Show that every solution of the differential equation

$$\vec{x}' = \begin{pmatrix} 0 & -3 \\ 2 & 0 \end{pmatrix}\vec{x}$$

is stable, but not asymptotically stable.

Solution $p(\lambda) = \det(A-\lambda I) = \det\begin{pmatrix} -\lambda & -3 \\ 2 & -\lambda \end{pmatrix} = \lambda^2 + 6$

$\lambda_{1,2} = \pm i\sqrt{6}$. Therefore, by part (c) of Theorem 1, every solution $\vec{x} = \vec{\varphi}(t)$ of $\vec{x}' = A\vec{x}$ is stable. However, no solution is asymptotototically stable.

This follows immediately from the fact that the general solution of $\vec{x}' = A\vec{x}$ is

$$\vec{x}(t) = c_1 \begin{pmatrix} -\sqrt{6}\,\sin\sqrt{6}t \\ 2\cos\sqrt{6}t \end{pmatrix} + c_2 \begin{pmatrix} \sqrt{6}\cos\sqrt{6}t \\ 2\sin\sqrt{6}t \end{pmatrix}.$$

Hence, every solution $\vec{x}(t)$ is periodic, with period $\frac{2\pi}{\sqrt{6}}$, and no solution $\vec{x}(t)$ (except $\vec{x}(t) \equiv \vec{0}$) approaches 0 as t approaches infinity.

Example 4 Show that every solution of the differential equation

$$\vec{x}' = \begin{pmatrix} 2 & -3 & 0 \\ 0 & -6 & -2 \\ -6 & 0 & -3 \end{pmatrix} \vec{x}$$

is unstable.

Solution

$$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & -3 & 0 \\ 0 & -6-\lambda & -2 \\ -6 & 0 & -3-\lambda \end{vmatrix}$$

$$= (2-\lambda)(-6-\lambda)(-3-\lambda) + 36 = (2-\lambda)(6+\lambda)(3+\lambda) + 36$$
$$= (12 - 4\lambda - \lambda^2)(3+\lambda) + 36 = \cancel{36} + \cancel{12\lambda} - \cancel{12\lambda} - 4\lambda^2 - 3\lambda^2 - \lambda^3 - \cancel{36}$$
$$= -7\lambda^2 - \lambda^3 = -\lambda^2(\lambda + 7).$$

Hence, the eigenvalues of A are $\lambda_1 = -7$, and $\lambda_{2,3} = 0$. Every eigenvector $\vec{v}$ of A with eigenvalue 0 must satisfy the equation

$$A\vec{v} = \begin{pmatrix} 2 & -3 & 0 \\ 0 & -6 & -2 \\ -6 & 0 & -3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} 2v_1 = 3v_2 \\ -6v_2 = 2v_3 \end{matrix} \Rightarrow \vec{v} = c \begin{pmatrix} 3 \\ 2 \\ -6 \end{pmatrix}$$

Consequently, every solution $\vec{x} = \vec{\varphi}(t)$ of $\vec{x}' = A\vec{x}$ is unstable, since $\lambda = 0$ is an eigenvalue of multiplicity two and A has only one linearly independent eigenvector with eigenvalue 0.

Stability of Equilibrium Solutions

The next simplest equation is

$$\vec{x}\,' = A\vec{x} + \vec{g}(\vec{x}), \text{ where } \vec{g}(\vec{x}) = \begin{pmatrix} g_1(\vec{x}) \\ \vdots \\ g_n(\vec{x}) \end{pmatrix} \quad (1)$$

is very small compared to $\vec{x}$. Specifically we assume that

$$\frac{g_1(\vec{x})}{\max\{|x_1|,\dots,|x_n|\}}, \dots, \frac{g_n(\vec{x})}{\max\{|x_1|,\dots,|x_n|\}}$$

are continuous functions of $x_1,\dots,x_n$ which vanish for $x_1 = \dots = x_n = 0$. This is always the case if each component of $\vec{g}(\vec{x})$ is a polynomial in $x_1,\dots,x_n$ which begins with terms of order 2 or higher. For example, if

$$\vec{g}(\vec{x}) = \begin{pmatrix} x_1 x_2^2 \\ x_1 x_2 \end{pmatrix},$$

then both $\dfrac{x_1 x_2^2}{\max\{|x_1|,|x_2|\}}$ and $\dfrac{x_1 x_2}{\max\{|x_1|,|x_2|\}}$ are continuous functions

of x_1, x_2 which vanish for $x_1 = x_2 = 0$.

If $\vec{g}(\vec{0}) = \vec{0}$ then $\vec{x}(t) \equiv \vec{0}$ is an equilibrium solution of (1). We would like to determine whether it is stable or unstable. At first glance this would seem impossible to do, since we cannot solve Eq. (1) explicitly. However, if $\vec{x}$ is very small, then $\vec{g}(\vec{x})$ is very small compared to $A\vec{x}$. Therefore, it seems plausible that the stability of the equilibrium solution $\vec{x}(t) \equiv \vec{0}$ of (1) should be determined by the stability of the "approximate" equation $\vec{x}\,' = A\vec{x}$. This is almost the case as the following theorem indicates.

Theorem 2 Suppose that the vector-valued function

$$\frac{\vec{g}(\vec{x})}{\|\vec{x}\|} \equiv \frac{\vec{g}(\vec{x})}{\max\{|x_1|,\dots,|x_n|\}}$$

is a continuous function of $x_1,\dots,x_n$ which vanishes for $\vec{x} \equiv \vec{0}$.

Then,

(a) The equilibrium solution $\vec{x}(t) \equiv \vec{0}$ of (1) is asymptotically stable if the equilibrium solution $\vec{x}(t) \equiv \vec{0}$ of the linearized equation $\vec{x}' = A\vec{x}$ is asymptotically stable.
Equivalently, the solution $\vec{x}(t) \equiv \vec{0}$ of (1) is asymptotically stable if all the eigenvalues of A have negative real part.

(b) The equilibrium solution $\vec{x}(t) \equiv 0$ of (1) is unstable if at least one eigenvalue of A has positive real part.

(c) The stability of the equilibrium solution $\vec{x}(t) \equiv \vec{0}$ of (1) cannot be determined from the stability of the equilibrium solution $\vec{x}(t) \equiv \vec{0}$ of $\vec{x}' = A\vec{x}$ if all the eigenvalues of A have real part ≤ 0 but at least one eigenvalue of A has zero real part.

Proof (a) The key step in many stability proofs is to use the variation of parameters formula. This formula implies that any solution $\vec{x}(t)$ of (1) can be written in the form

$$\vec{x}(t) = e^{At}\vec{x}(0) + \int_0^t e^{A(t-s)} g(\vec{x}(s))\,ds. \qquad (2)$$

We wish to show that $\|\vec{x}(t)\|$ approaches 0 as $t \to \infty$. Recall that if all the eigenvalues of A have negative real part, then we can find positive constants K and α such that

$$\|e^{At}\vec{x}(0)\| \leq Ke^{-\alpha t}\|\vec{x}(0)\|$$

and $$\|e^{A(t-s)}\vec{g}(\vec{x}(s))\| \leq Ke^{-\alpha(t-s)}\|g(\vec{x}(s))\|.$$

Moreover, we can find a positive constant σ such that

$$\|\vec{g}(\vec{x})\| \leq \frac{\alpha}{2K}\|\vec{x}\| \quad \text{if } \|\vec{x}\| \leq \sigma.$$

This follows immediately from our assumption that $\frac{\vec{g}(\vec{x})}{\|\vec{x}\|}$ is continuous and vanishes at $\vec{x} = \vec{0}$.

Consequently, Equation (2) implies that

$$\|\vec{x}(t)\| \le \|e^{At}\vec{x}(0)\| + \int_0^t \|e^{A(t-s)}\vec{g}(\vec{x}(s))\|\, ds$$

$$\le Ke^{-\alpha t}\|\vec{x}(0)\| + \frac{\alpha}{2}\int_0^t e^{-\alpha(t-s)}\|\vec{x}(s)\|\, ds$$

as long as $\|\vec{x}(s)\| \le \sigma$, $0 \le s \le t$. Multiplying both sides of this inequality by $e^{\alpha t}$ gives

$$e^{\alpha t}\|\vec{x}(t)\| \le K\|\vec{x}(0)\| + \frac{\alpha}{2}\int_0^t e^{\alpha s}\|x(s)\|\, ds. \qquad (3)$$

This inequality (3) can be simplified by setting $z(t) = e^{\alpha t}\|\vec{x}(t)\|$, for then

$$z(t) \le K\|\vec{x}(0)\| + \frac{\alpha}{2}\int_0^t z(s)\, ds. \qquad (4)$$

We would like to differentiate both sides of (4) with respect to t. However, we cannot, in general, differentiate both sides of an inequality and still preserve the sense of the inequality. We circumvent this difficulty by the clever trick of setting

$$U(t) = \frac{\alpha}{2}\int_0^t z(s)\, ds.$$

Then $$\frac{dU}{dt} = \frac{\alpha}{2}z(t) \le \frac{\alpha}{2}K\|\vec{x}(0)\| + \frac{\alpha}{2}U(t).$$

or $$\frac{dU}{dt} - \frac{\alpha}{2}U(t) \le \frac{\alpha K}{2}\|\vec{x}(0)\|.$$

Multiplying both sides of this inequality by the integrating factor $e^{-\alpha t/2}$ gives

$$\frac{d}{dt}e^{-\frac{\alpha t}{2}}U \le \frac{\alpha K}{2}\|\vec{x}(0)\|e^{-\frac{\alpha t}{2}},$$

or

$$\frac{d}{dt}e^{-\frac{\alpha t}{2}}\left[U(t) + K\|\vec{x}(0)\|\right] \le 0$$

Consequently,

$$e^{-\frac{\alpha t}{2}}\left[U(t) + K\|\vec{x}(0)\|\right] \le U(0) + K\|\vec{x}(0)\| = K\|\vec{x}(0)\|,$$

so that $\quad U(t) \le -K\|\vec{x}(0)\| + K\|\vec{x}(0)\|e^{\frac{\alpha t}{2}}$.

Returning to the inequality (4), we see that

$$\|\vec{x}(t)\| = e^{-\alpha t} z(t) \le e^{-\alpha t}\left[K\|\vec{x}(0)\| + U(t)\right] \le K\|\vec{x}(0)\|e^{-\frac{\alpha t}{2}} \qquad (5)$$

as long as $\|\vec{x}(s)\| \le \sigma$, $0 \le s \le t$. Now, if $\|\vec{x}(0)\| \le \frac{\sigma}{K}$, then the inequality (5) guarantees that $\|\vec{x}(t)\| \le \sigma$ for all future timet. Consequently, the inequality (5) is true for all $t \ge 0$ if $\|\vec{x}(0)\| \le \frac{\sigma}{K}$. Finally, observe from (5) that $\|\vec{x}(t)\| \le K\|\vec{x}(0)\|$ and $\|\vec{x}(t)\|$ approaches 0 as $t \to \infty$. Therefore, the equilibrium solution $\vec{x}(t) \equiv 0$ of (1) is asymptotically stable.

(b) The proof of (b) is too difficult to present here.

(c) We will present two differential equations of the form (1) where the nonlinear term $\vec{g}(\vec{x})$ determines the stability of the equilibrium solution $\vec{x}(t) \equiv \vec{0}$. Consider first the system of differential equations

$$\begin{cases} \dfrac{dx_1}{dt} = x_2 - x_1(x_1^2 + x_2^2) \\[2ex] \dfrac{dx_2}{dt} = -x_1 - x_2(x_1^2 + x_2^2) \end{cases} \qquad (6)$$

The linearized equation is

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

and the eigenvalues of the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{are} \quad \pm i.$$

To analyze the behavior of the nonlinear system (6) we multiply the first equation by x_1, the second equation by x_2 and add; this gives

$$x_1 \frac{dx_1}{dt} + x_2 \frac{dx_2}{dt} = -x_1^2(x_1^2+x_2^2) - x_2^2(x_1^2+x_2^2)$$
$$= -(x_1^2+x_2^2)^2.$$

But $\quad x_1 \dfrac{dx_1}{dt} + x_2 \dfrac{dx_2}{dt} = \dfrac{1}{2}\dfrac{d}{dt}(x_1^2+x_2^2).$

Hence, $\quad \dfrac{d}{dt}(x_1^2+x_2^2) = -2(x_1^2+x_2^2)^2.$

This implies that $\quad x_1^2(t) + x_2^2(t) = \dfrac{c}{1+2ct}$

where $\quad c = x_1^2(0) + x_2^2(0)$

Thus, $x_1^2(t)+x_2^2(t)$ approaches zero as t approaches infinity for any solution $x_1(t), x_2(t)$ of (6). Moreover, the value of $x_1^2+x_2^2$ at any time t is always less than its value at $t=0$. We conclude, therefore, that $x_1(t)\equiv 0$, $x_2(t)\equiv 0$ is asymptotically stable.

On the other hand, consider the system of equations

$$\begin{cases} \dfrac{dx_1}{dt} = x_2 + x_1(x_1^2+x_2^2) \\[2ex] \dfrac{dx_2}{dt} = -x_1 + x_2(x_1^2+x_2^2) \end{cases} \qquad (7)$$

Here too, the linearized system is

$$\vec{x}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\vec{x}.$$

In this case, though,

$$\frac{d}{dt}\left(x_1^2 + x_2^2\right) = 2(x_1^2 + x_2^2)^2.$$

This implies that

$$x_1^2(t) + x_2^2(t) = \frac{c}{1-2ct}, \quad c = x_1^2(0) + x_2^2(0).$$

Notice that every solution $x_1(t), x_2(t)$ of (7) with $x_1^2(0) + x_2^2(0) \neq 0$ approaches infinity in finite time. We conclude, therefore, that the equilibrium solution $x_1(t) \equiv 0$, $x_2(t) \equiv 0$ is unstable.

Example 1 Consider the system of diff. equations

$$\begin{cases} \dfrac{dx_1}{dt} = -2x_1 + x_1 + 3x_3 + 9x_2^3 \\[2ex] \dfrac{dx_2}{dt} = -6x_2 - 5x_3 + 7x_3^5 \\[2ex] \dfrac{dx_3}{dt} = -x_3 + x_1^2 + x_2^2 \end{cases}$$

Determine, if possible, whether the equilibrium solution $x_1(t) \equiv 0$, $x_2(t) \equiv 0$, $x_3(t) \equiv 0$ is stable or unstable.

Solution We rewrite this system in the form $\vec{x}' = A\vec{x} + \vec{g}(\vec{x})$ where
$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad A = \begin{pmatrix} -2 & 1 & 3 \\ 0 & -6 & -5 \\ 0 & 0 & -1 \end{pmatrix} \text{ and } \vec{g}(\vec{x}) = \begin{pmatrix} 9x_2^3 \\ 7x_3^5 \\ x_1^2 + x_2^2 \end{pmatrix}.$$

The function $\vec{g}(\vec{x})$ satisfies the hypotheses of Theorem 2, and the eigenvalues of A: $\lambda_1 = -2$, $\lambda_2 = -6$, $\lambda_3 = -1$. Hence, the equilibrium solution $\vec{x}(t) \equiv \vec{0}$ is asymptotically stable.

Theorem 2 can also be used to determine the stability of equilibrium solutions of arbitrary autonomous differential equations.

Let $\vec{x}^0$ be an equilibrium value of the differential equation

$$\vec{x}' = \vec{f}(\vec{x}) \qquad (8)$$

and set $\vec{z}(t) = \vec{x}(t) - \vec{x}^0$. Then

$$\vec{z}'(t) = \vec{x}' = \vec{f}(\vec{x}^0 + \vec{z}) \qquad (9)$$

Clearly, $\vec{z}(t) \equiv \vec{0}$ is an equilibrium solution of (9) and the stability of

$$\vec{x}(t) \equiv \vec{x}^0$$

is equivalent to the stability of $\vec{z}(t) \equiv \vec{0}$.

Next, we show that $\vec{f}(\vec{x}^0 + \vec{z}(t))$ can be written in the form

$$\vec{f}(\vec{x}^0 + \vec{z}) = A\vec{z} + \vec{g}(\vec{z}),$$

where $\vec{g}(\vec{z})$ is small compared to $\vec{z}$.

Lemma 1 Let $\vec{f}(\vec{x})$ have two continuous partial derivatives with respect to each of its variables $x_1, \dots, x_n$. Then $\vec{f}(\vec{x}^0 + \vec{z})$ can be written in the form

$$\vec{f}(\vec{x}^0 + \vec{z}) = \vec{f}(\vec{x}^0) + A\vec{z} + \vec{g}(\vec{z}) \qquad (10)$$

where $\vec{g}(z) / \max\{|z_1|, \dots |z_n|\}$ is a continuous function of $\vec{z}$ which vanishes for $\vec{z} = \vec{0}$.

Proof Equation (10) is an immediate consequence of Taylor's Theorem which states that each component $f_j(\vec{x}^0 + \vec{z})$ of $\vec{f}(\vec{x}^0 + \vec{z})$ can be written in the form

$$f_j(\vec{x}^0 + \vec{z}) = f_j(\vec{x}^0) + \frac{\partial f_j}{\partial x_1}(\vec{x}^0) z_1 + \dots + \frac{\partial f_j}{\partial x_n}(\vec{x}^0) z_n + g_j(\vec{z})$$

where $\dfrac{g_j(\vec{z})}{\max\{|z_1|, \dots, |z_n|\}}$ is a continuous function of $\vec{z}$ which vanishes for $\vec{z} = \vec{0}$.

Hence,
$$\vec{f}(\vec{x}^\circ + \vec{z}) = \vec{f}(\vec{x}^\circ) + A\vec{z} + \vec{g}(\vec{z})$$

where
$$A = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\vec{x}^\circ) & \cdots & \frac{\partial f_1}{\partial x_n}(\vec{x}^\circ) \\ \vdots & & \\ \frac{\partial f_n}{\partial x_1}(\vec{x}^\circ) & \cdots & \frac{\partial f_n}{\partial x_n}(\vec{x}^\circ) \end{pmatrix}$$

Theorem 2 and Lemma 1 provide us with the following algorithm for determining whether the equilibrium solution $\vec{x}(t) \equiv \vec{x}^\circ$ of $\vec{x}' = \vec{f}(\vec{x})$ is stable or unstable:

1. Set $\vec{z} = \vec{x} - \vec{x}^\circ$

2. Write $\vec{f}(\vec{x}^\circ + \vec{z})$ in the form $A\vec{z} + \vec{g}(\vec{z})$ where $\vec{g}(\vec{z})$ is a vector-valued polynomial in $z_1, \dots, z_n$ beginning with terms of order two or more.

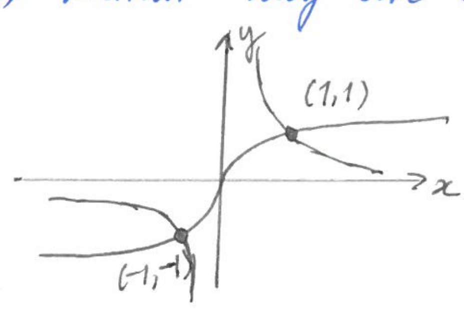
3. Compute the eigenvalues of A. If all the eigenvalues of A have negative real part, then $\vec{x}(t) \equiv \vec{x}^\circ$ is asymptotically stable. If one eigenvalue of A has positive real part, then $\vec{x}(t) \equiv \vec{x}^\circ$ is unstable.

Example 2 Find all equilibrium solutions of the system of differential equations

$$\begin{cases} \frac{dx}{dt} = 1 - xy \\ \frac{dy}{dt} = x - y^3 \end{cases} \qquad (11)$$

and determine (if possible) whether they are stable or unstable.

Solution $\begin{cases} 1 - xy = 0 \\ x - y^3 = 0 \end{cases}$



Hence, $\begin{pmatrix} x(t) \equiv 1 \\ y(t) \equiv 1 \end{pmatrix}$ and $\begin{pmatrix} x(t) \equiv -1 \\ y(t) \equiv -1 \end{pmatrix}$ are the only equilibrium solutions of (11).

(i) $x(t) = 1,\ y(t) = 1$: Set $u = x-1,\ v = y-1$. Then

$$\frac{du}{dt} = \frac{dx}{dt} = 1-(u+1)(v+1) = 1-(uv+u+v+1) = -u-v-uv$$

$$\frac{dv}{dt} = \frac{dy}{dt} = (u+1)-(v+1)^3 = u+1-(v^3+3v^2+3v+1) = u-3v-3v^2-v^3.$$

We rewrite this system in the form

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} uv \\ -3v^2-v^3 \end{pmatrix}$$

The matrix $\begin{pmatrix} -1 & -1 \\ 1 & -3 \end{pmatrix}$ has a single eigenvalue $\lambda = -2$ since

$$\det\begin{pmatrix} -1-\lambda & -1 \\ 1 & -3-\lambda \end{pmatrix} = (-1-\lambda)(-3-\lambda)+1 = (1+\lambda)(3+\lambda)+1 = 3+4\lambda+\lambda^2+1 = (\lambda+2)^2$$

Hence, the equilibrium solution $x(t) \equiv 1,\ y(t) \equiv 1$ of (11) is asymptotically stable.

(ii) $x(t) \equiv -1,\ y(t) \equiv -1$: Set $u = x+1,\ v = y+1$. Then

$$\frac{du}{dt} = \frac{dx}{dt} = 1-(u-1)(v-1) = u+v-uv$$

$$\frac{dv}{dt} = \frac{dy}{dt} = u-1-(v-1)^3 = u-\cancel{1}-(v^3-3v^2+3v-\cancel{1}) = u-3v+3v^2-v^3$$

We rewrite this system in the form

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} -uv \\ 3v^2-v^3 \end{pmatrix}$$

The eigenvalues of the matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}$$

are: $\det\begin{pmatrix} 1-\lambda & 1 \\ 1 & -3-\lambda \end{pmatrix} = (1-\lambda)(-3-\lambda) - 1 = -3 - \lambda + 3\lambda + \lambda^2 - 1 =$

$$= \lambda^2 + 2\lambda - 4 = 0$$

$$\lambda_{1,2} = \frac{-2 \pm \sqrt{2^2 - 4\cdot(1)(-4)}}{2} = \frac{-2 \pm 2\sqrt{5}}{2}.$$

$\lambda_1 = -1-\sqrt{5} < 0$, and $\lambda_2 = -1+\sqrt{5} > 0$. Therefore, the equilibrium solution $x(t) \equiv -1$, $y(t) \equiv -1$ of (11) is unstable.

Example 3 Find all equilibrium solutions of the system of differential equations

$$\begin{cases} \dfrac{dx}{dt} = \sin(x+y) \\ \dfrac{dy}{dt} = e^x - 1 \end{cases} \qquad (12)$$

and determine whether they are stable or unstable.

Solution $\begin{matrix} \sin(x+y) = 0 \\ e^x - 1 = 0 \end{matrix} \iff \begin{cases} \sin(x+y) = \sin(\pi n) \\ \quad x+y = \pi n \Rightarrow y = -x + \pi n \\ x = 0 \end{cases}$

Consequently, $x(t) \equiv 0$, $y(t) \equiv \pi n$, $n = 0, \pm 1, \pm 2, \ldots$ are equilibrium solutions of (12).

Setting $u = x$, $v = y - \pi n$, gives.

$$\frac{du}{dt} = \sin(u + v + \pi n), \quad \frac{dv}{dt} = e^u - 1.$$

Now, $\sin(u+v+\pi n) = \cos(\pi n)\sin(u+v) = (-1)^n \sin(u+v)$. Therefore,

$$\frac{du}{dt} = (-1)^n \sin(u+v), \quad \frac{dv}{dt} = e^u - 1.$$

Next, $\sin(u+v) = u+v - \frac{(u+v)^3}{3!} + \dots$, $e^u - 1 = u + \frac{u^2}{2!} + \dots$

Hence $\frac{du}{dt} = (-1)^n \left[u+v - \frac{(u+v)^3}{3!} + \dots \right]$, $\frac{dv}{dt} = u + \frac{u^2}{2!} + \dots$

We rewrite the system in the form

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} (-1)^n & (-1)^n \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \text{terms of order 2 or higher in } u \text{ and } v.$$

The eigenvalues of the matrix

$$\begin{pmatrix} (-1)^n & (-1)^n \\ 1 & 0 \end{pmatrix} \quad \text{are} \quad \lambda_1 = \frac{(-1)^n - \sqrt{1+4(-1)^n}}{2}$$

$$((-1)^n - \lambda)\cdot(-\lambda) - (-1)^n = 0. \qquad \lambda_2 = \frac{(-1)^n + \sqrt{1+4(-1)^n}}{2}$$

$$\lambda^2 - (-1)^n \lambda - (-1)^n = 0.$$

When n is even, $\lambda_1 = \frac{1-\sqrt{5}}{2} < 0$ and $\lambda_2 = \frac{1+\sqrt{5}}{2} > 0$. Hence, $x(t) \equiv 0$, $y(t) \equiv n\pi$ is unstable if n is even. When n is odd, both $\lambda_1 = \frac{(-1-\sqrt{3}i)}{2}$ and $\lambda_2 = \frac{(-1+\sqrt{3}i)}{2}$ have negative real part. Therefore, the equilibrium solution $x(t) \equiv 0$, $y(t) = \pi n$ is asymptotically stable if n is odd.