MTH 7241: Fall 2017

More Practice Problems.

1). The lifetime of a machine (in days) is an integer-valued random variable T. Assume that $1 \le T \le N$, and that all N values of T are equally likely. Given that the machine is working after k days, what is the expected value of its subsequent lifetime?
[Hint: let A_k be the event that the lifetime of the machine is at least k days. Find $E[T|A_k]$.]

Answer: $(N+k)/2$

2). Here is a hypothetical situation aboard the planned Mars probe. Its navigation system will need two working computers at all times. NASA plans to put three computers on board, so that there will be a spare in case of failure. If a computer fails, it will be immediately replaced by the extra one.

NASA asks you to estimate the expected time until the navigation system fails. You are told to assume that the lifetimes of the computers are independent exponential random variables, each with mean 1 year. Let T be the time until failure, so T is the sum of the time until the first failure and the additional time until the second failure.

Find the expected length of time until the system fails, that is find $\mathrm{E}[T]$.

Answer: λ

3). A biased coin has probability p to come up Heads. The coin is tossed 4 times and it comes up Heads 3 times. Find the value of p which maximizes the probability of this event occurring (this value is called the Maximum Likelihood Estimator for p). [Hint: compute the probability and use basic calculus].

Answer: $3/4$

4). A box contains n Red balls and m Black balls. A ball is drawn at random from the box; if it is Red, the ball is discarded; if it is Black, it is replaced in the box. Let N be the number of draws needed until all the Red balls have been taken out of the box. Find $\mathrm{E}[N]$. [Hint: write N as a sum of geometric random variables].

Answer: $\sum_{k=1}^{n}(n+m+1-k)/(n+1-k)$

5). A random number X is chosen uniformly from the interval $[0,1]$. A second random number Y is then chosen uniformly from the interval $[0,X]$.

a) Find $\mathrm{E}[X]$. [Hint: the pdf for X is $f(x)=1$ for $0 \le x \le 1$].

Answer: $1/2$

b) Calculate $\mathrm{E}[Y \mid X=x]$.

Answer: $x/2$

c) Calculate $\mathrm{E}[Y]$.

Answer: $1/4$

6). Cars pass a certain point in a highway in accordance with a Poisson process with rate $\lambda = 10$ per minute. The number of passengers in the cars are independent and identically distributed, with the following distribution: if Y is the number of passengers in a car, then $P(Y = 1) = 0.4$, $P(Y = 2) = 0.3$, $P(Y = 3) = 0.2$, $P(Y = 4) = 0.1$. A car is full if it has four passengers.
a) Find the probability that the next car that passes is full.

Answer: 0.1
b) Find the probability that two full cars pass in the next minute.

Answer: $1/2e$
c) Find the expected number of passengers that pass in the next minute.

Answer: 20
d) Find the probability that at least two passengers pass in the next ten seconds.

Answer: $1 - (5/3)e^{-5/3}$
[You may want to use the result about thinning of Poisson processes].

7). Men and women enter a bank according to independent Poisson processes at rates $\mu = 3$ and $\lambda = 2$ per minute respectively. Starting at an arbitrary time, find:
a) the probability that exactly two men enter in the next minute;

Answer: $(\mu^2 e^{-\mu})/2$
b) the probability that at least one woman enters in the next minute;

Answer: $1 - e^{-\lambda}$
c) the probability that at least one man arrives before the next woman arrives.

Answer: $\mu/(\lambda + \mu)$
d) the probability that at least two men arrive before the next two women arrive.

Answer: $\mu^2(\mu + 3\lambda)/(\mu + \lambda)^3$

8). A basic version of the Google page-rank algorithm assigns a probability distribution $R(1), \ldots, R(n)$ to a collection of n webpages. The webpages are connected by directed links. For each page i, let $L(i)$ be the number of outward directed links starting at i (assume that the graph of webpages is irreducible, and that every webpage has at least one outward directed link so $L(i) \geq 1$ for all i). Then the probability distribution R_0 is the solution of the equation

$$R_0(j) = \sum_{i=1}^{n} R(i) p_{ij}$$

where the transition matrix P has entries

$$p_{ij} = \begin{cases} L(i)^{-1} & \text{if there is a directed link from } i \text{ to } j \\ 0 & \text{else} \end{cases}$$

If all links are symmetric, so that the number of links from i to j is equal to the number of links from j to i, then this is the equation for the stationary distribution of the random walk on the graph of webpages. The constant C is determined by normalizing so that

$$\sum_{i=1}^{n} R(i) = 1$$

However this random walk may have a long mixing time, which causes a computational problem. This can be remedied by introducing a 'damping factor' $\delta < 1$ as follows:

$$R(j) = \frac{(1 - \delta)}{n} + \delta \sum_{i=1}^{n} R(i) p_{ij}$$

This shortens the mixing time, but also changes the solution of the equation. The new solution can be computed as a power series in δ. Show that the new solution R satisfies the following equation:

$$R(j) = \frac{(1-\delta)}{n} \sum_{k=0}^{\infty} \delta^k \sum_{i=1}^{n} p_{ij}^{(k)}$$

where as usual $p_{ij}^{(n)}$ denotes the entries of the matrix P^n.