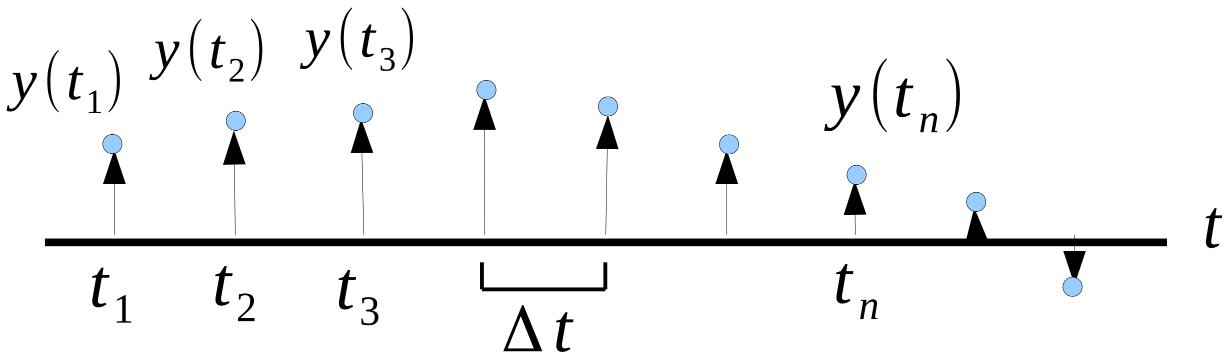
Sampled data

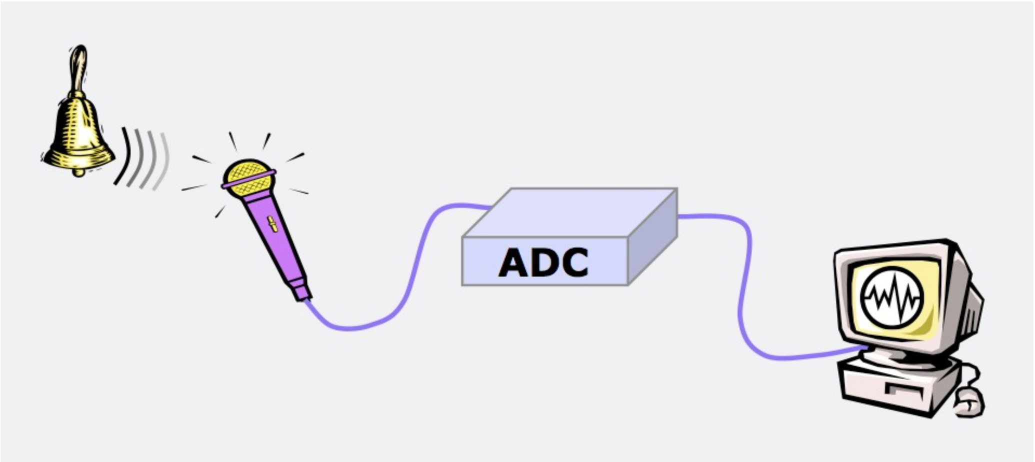


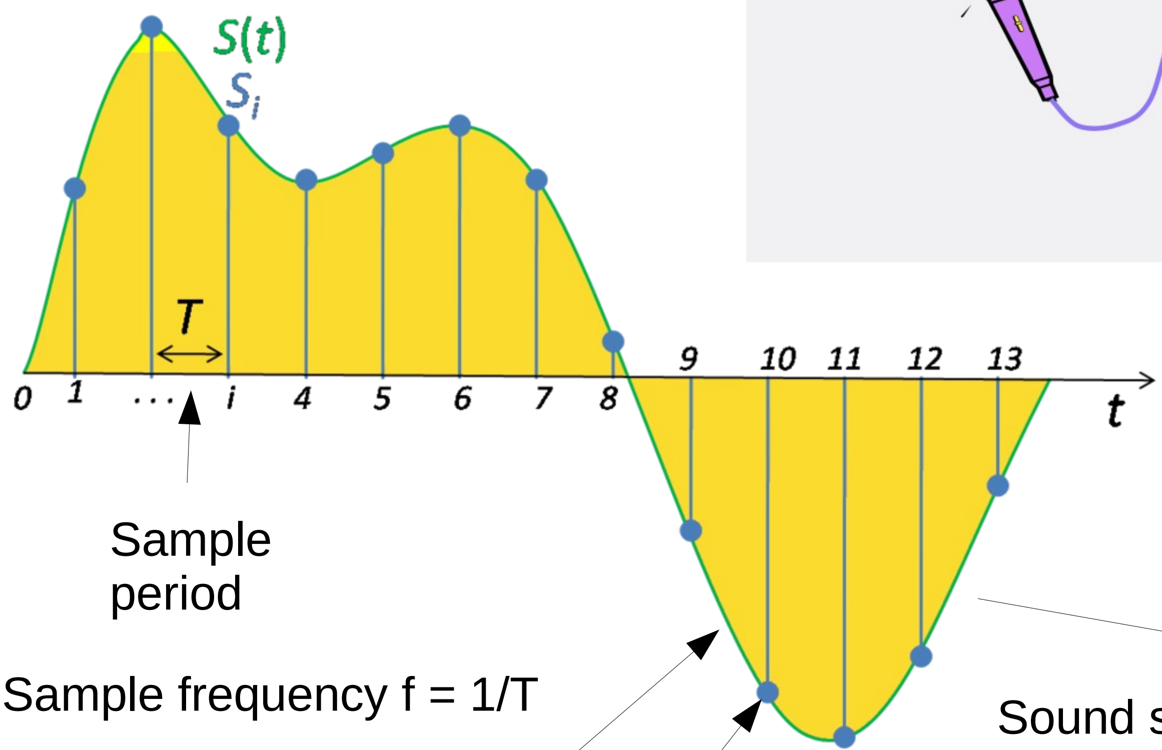
- Consider continuous time function $y(t)$.
- Sample the function at regular intervals.
- Sample points t_n
- The result is a vector of values of y:

  $[y_1, y_2, y_3, \cdots, y_n, \cdots]$

Example: Digital audio

Sound wave captured by computer





Sample period

Sample frequency f = 1/T

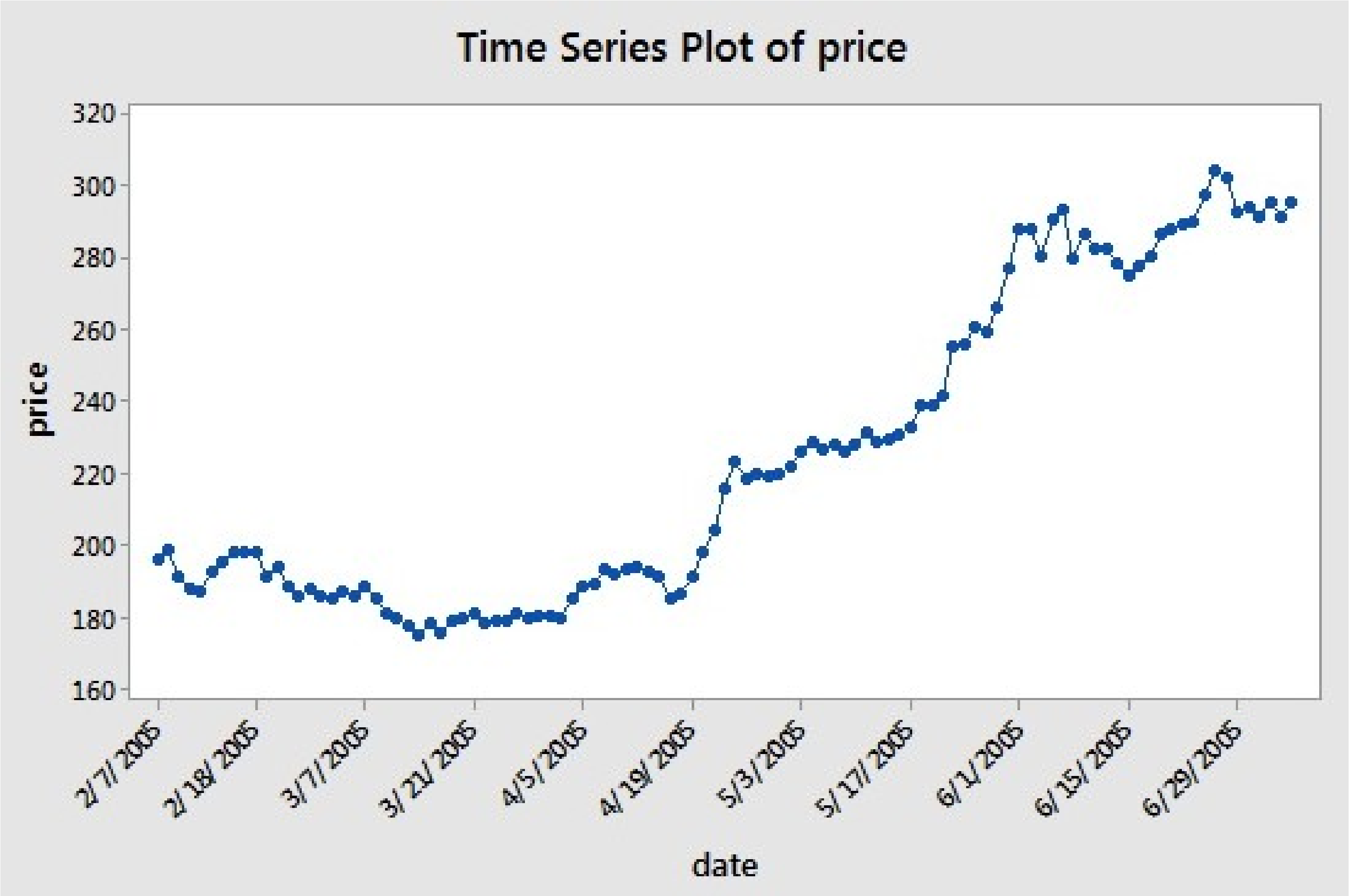
Sound signal

Sound samples

Sound samples represented as vector of numbers

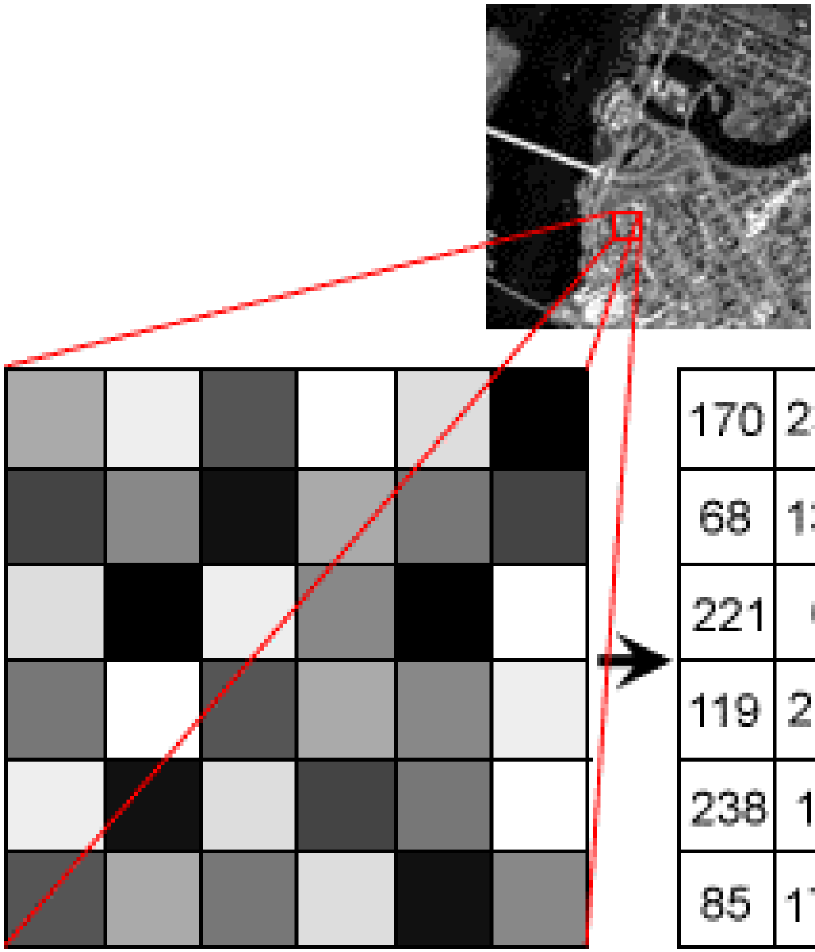
0.3717
0.6901
0.9096
0.9989
0.9450
0.7557
0.4582
0.0951
-0.2817
-0.6182
-0.8660
-0.9898
-0.9718
-0.8146
-0.5406
-0.1893
0.1893

Example: stock prices vs. time



The idea is to treat the data vector as a function which varies in time.

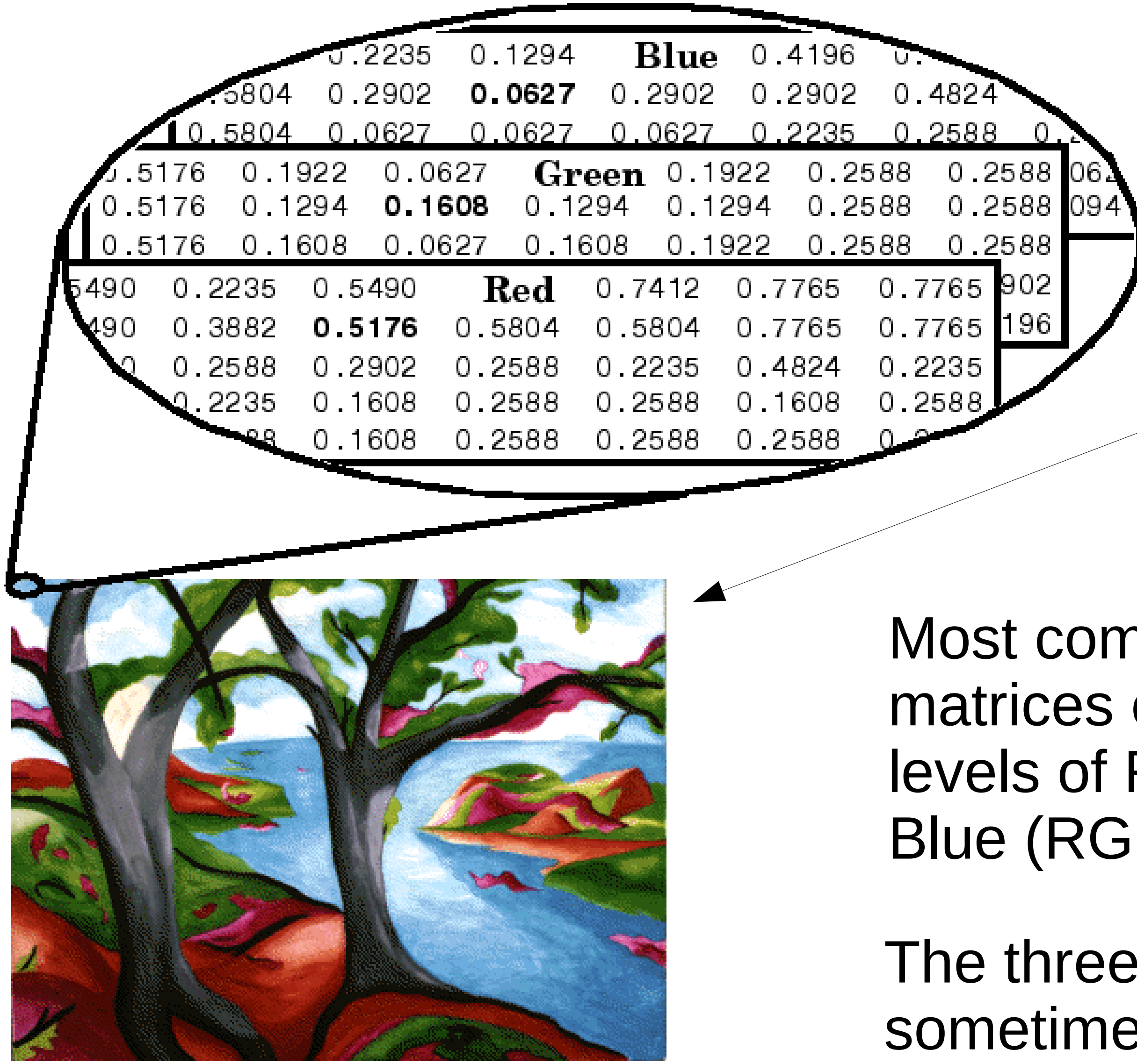
Example: Digital images (2D)



170	238	85	255	221	0
68	136	17	170	119	68
221	0	238	136	0	255
119	255	85	170	136	238
238	17	221	68	119	255
85	170	119	221	17	136

B&W image stored as matrix of “pixels” -- numbers signifying black/white level at each point.

Color images



Color image stored as three matrices of “pixels” -- numbers signifying intensity level at each point.

Most commonly, the three matrices correspond to levels of Red, Green, and Blue (RGB).

The three matrices are sometimes called “color planes”

Callable function vs. Sampled data function

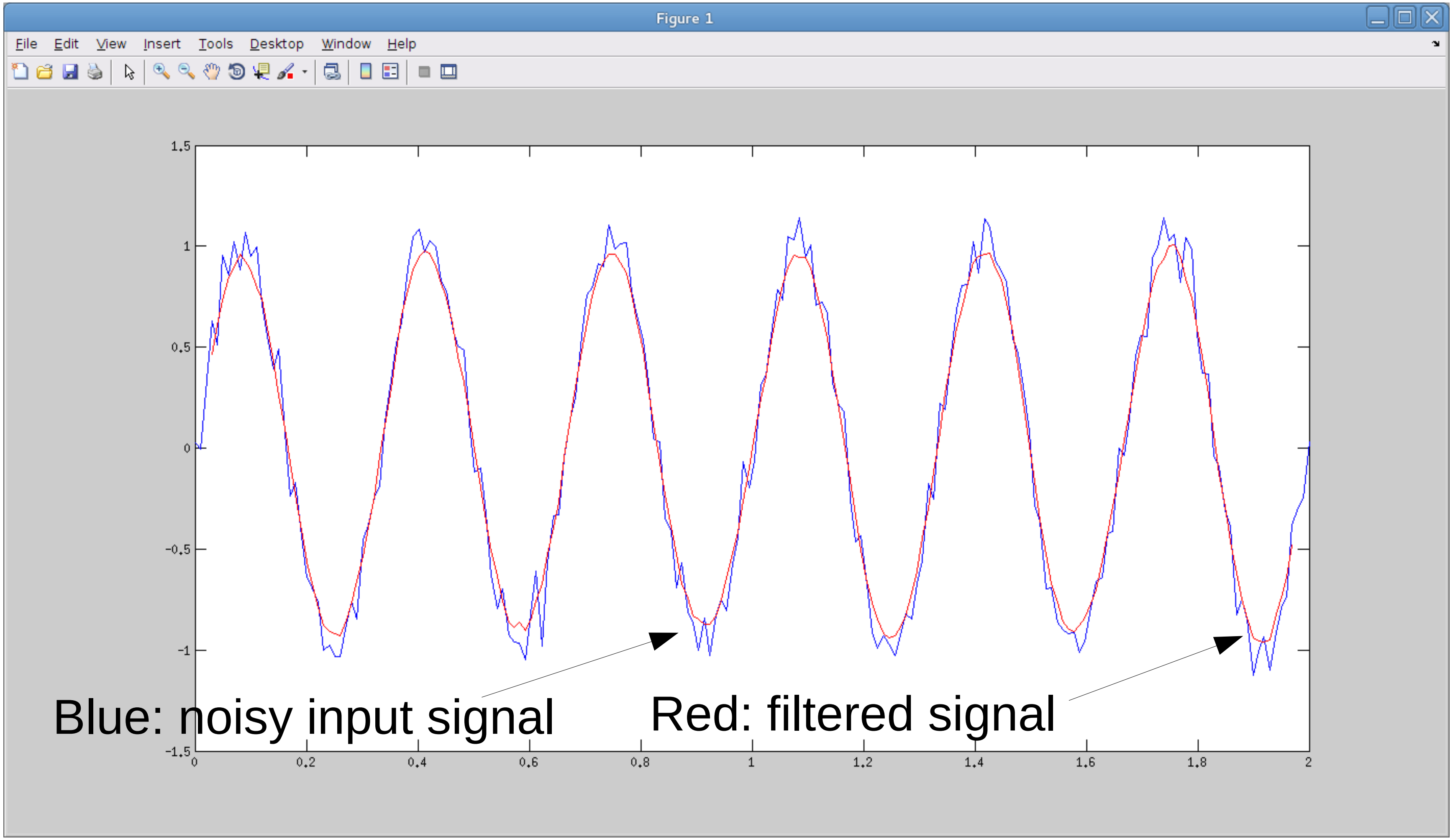
- We usually think of a function as callable.
- Now we need to think of a vector of data as another representation of a function.
- In general, sampling of real data is done using fixed (constant) period.
- For signal vector $f_n = f(t_n)$, here is always an implicit time vector tn lurking behind the scenes.
- Concept of streaming vs. batch processing.

Application: Filtering of data

- Input: Noisy data

Imagine an audio stream (music)

- Desired output: Data with noise removed.

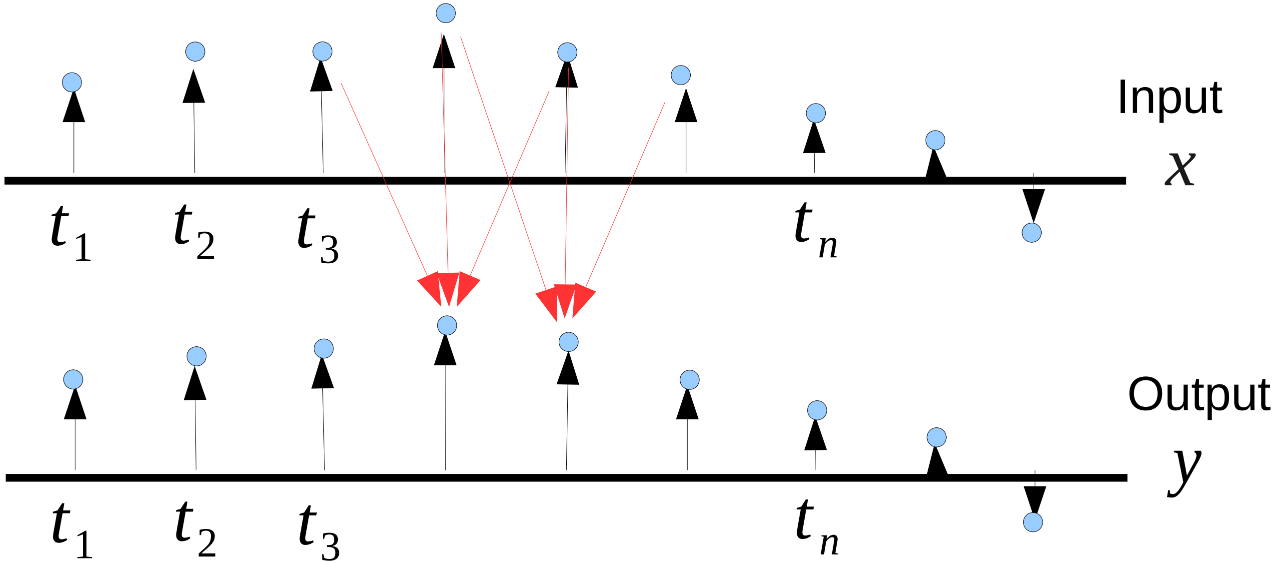


Blue: noisy input signal

Red: filtered signal

Simplest filter: moving box average

Three-point centered average.



- Idea: Take average of surrounding samples. Do this for each sample.

$$y_n = \frac{x_{n-1} + x_n + x_{n+1}}{3}$$

- The hardest part is getting the index arithmetic right....

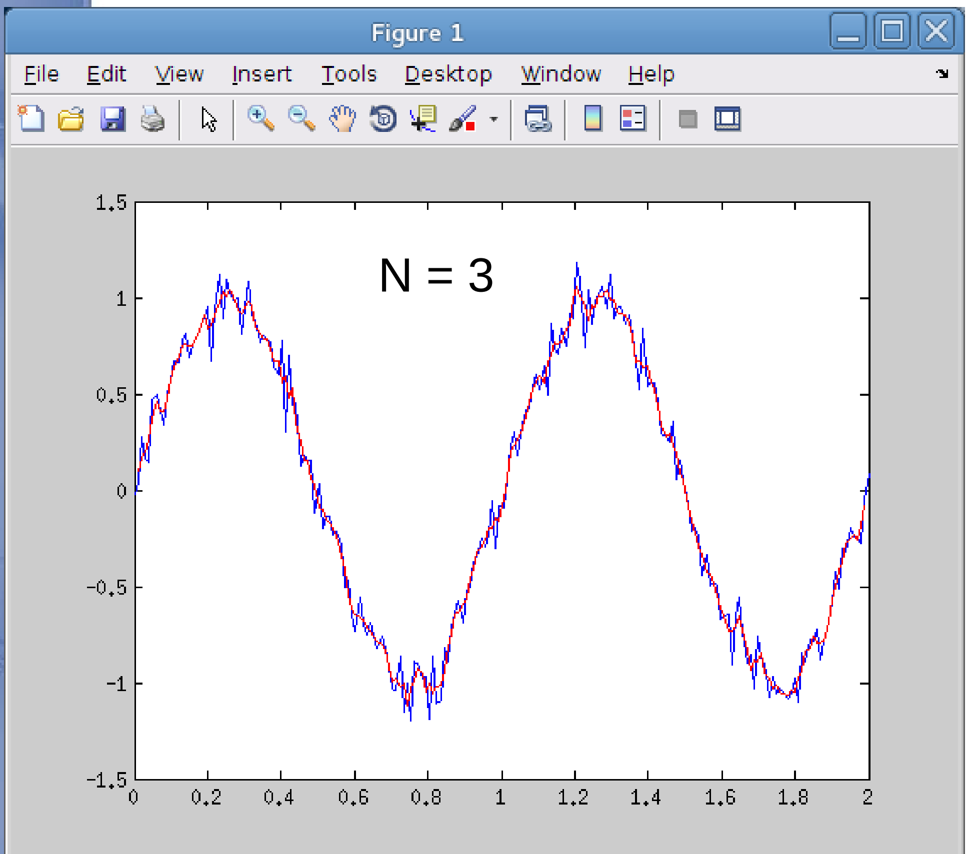
```
function [tf, yf] = box_filter_centered(t, x, Npts)
  % Performs centered box average over Npts points.

  % Check that Npts is an odd number
  if (mod(Npts, 2) == 0)
    error('Npts must be an odd number!')
  end

  % Compute number of points to the left & right
  Noffset = (Npts-1)/2;

  Nx = length(x)
  yf = zeros(Nx-Npts+1, 1);
  tf = zeros(Nx-Npts+1, 1);

  % Loop over input pts, compute box average
  for n = (1+Noffset):(Nx-Noffset)
    idx = (n-Noffset):(n+Noffset);
    tf(n-Noffset) = t(n);
    yf(n-Noffset) = sum(x(idx))/Npts;
  end

end
```

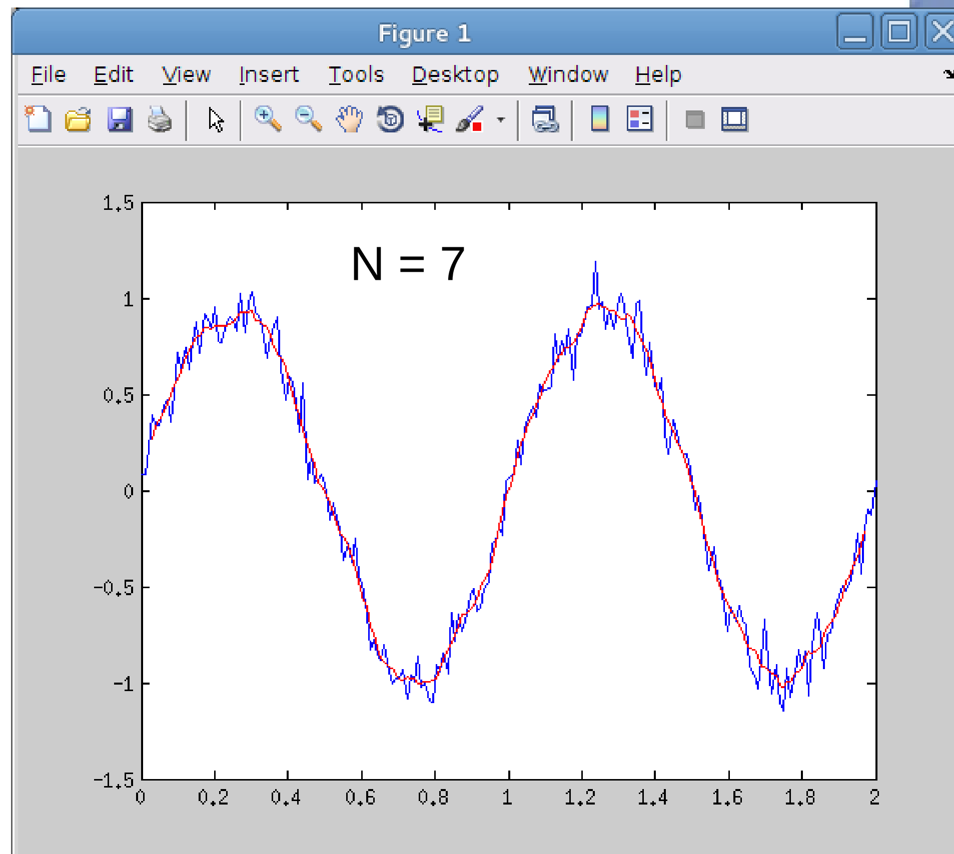
Note that I create a new time series vector along with the signal vector.

Here's where we compute the average of the input signal

$$y_n = \frac{x_{n-1} + x_n + x_{n+1}}{3}$$

Effect of different Npts





- Averaging more points together -> less noise in signal.

Some remarks

- The general computation is

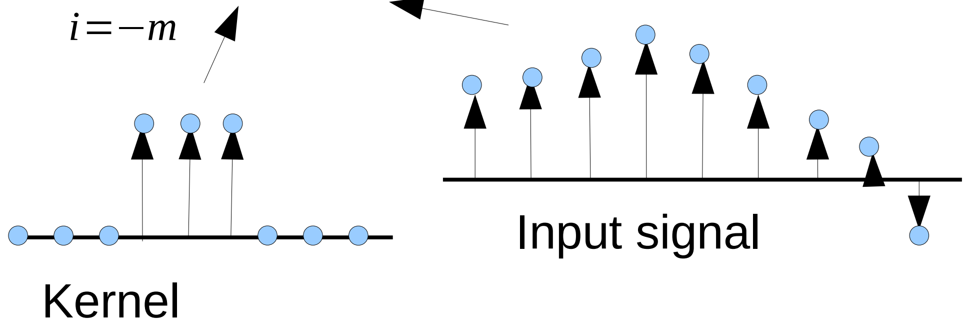
$$y_n = \sum_{i=-m}^{m} w_{n-i} x_i$$

Convolution – Remember this expression!

- Coefficients w_n must sum to 1 (to preserve "energy" in signal).
- How to deal with points at end?
- Concept: causal vs. non-causal filters
  - Centered average filter is non-causal.

Filter kernels

$$y_n = \sum_{i=-m}^{m} w_{n-i} x_i$$



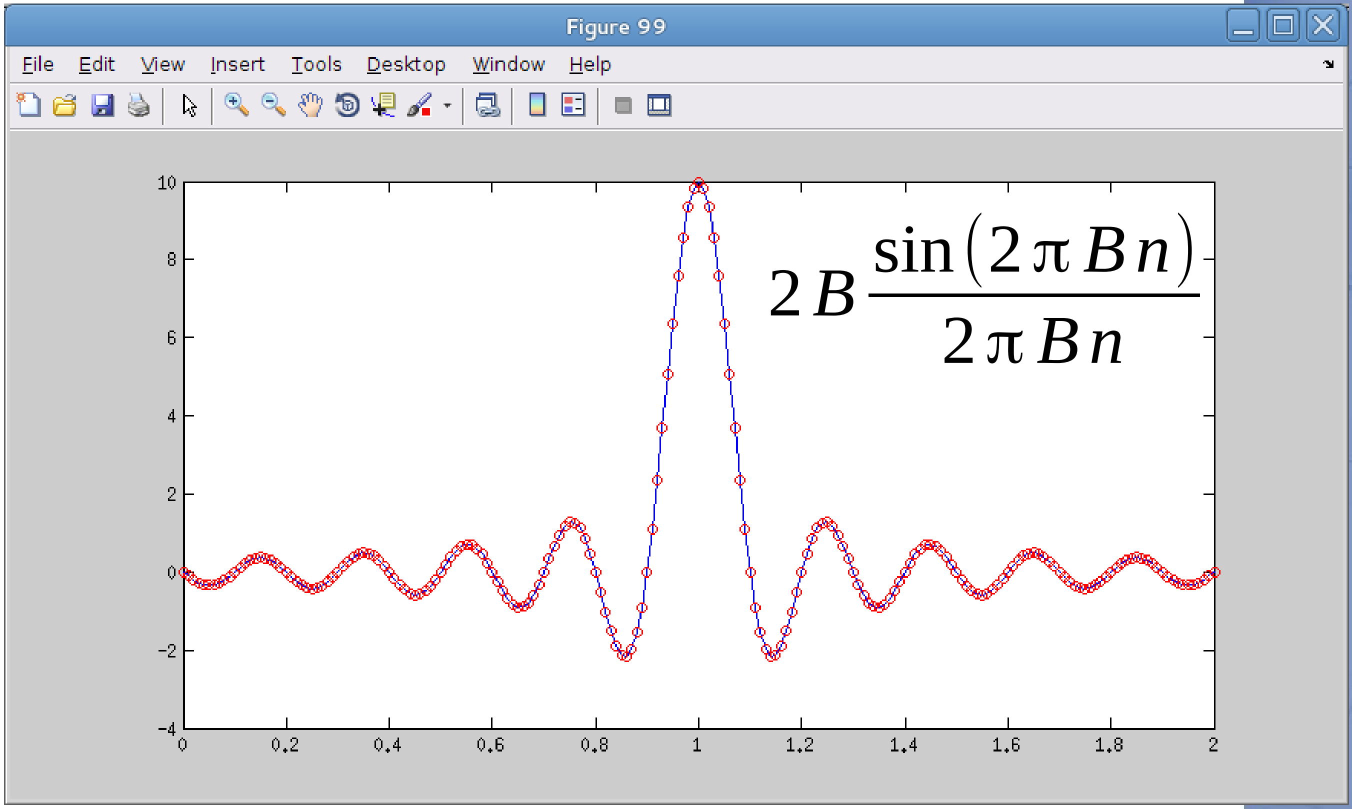
- Regard w coefficients as a function.
  - That function is called the “kernel”.
- Different kernels give different filter characteristics. (Recall effect of different Npts.)

Example: Sinc kernel

- Consider function sinc(x) = sin(x)/x
- Use as kernel in filter:

$$y_n = \sum_{i=-m}^{m} w_{n-i} x_i$$

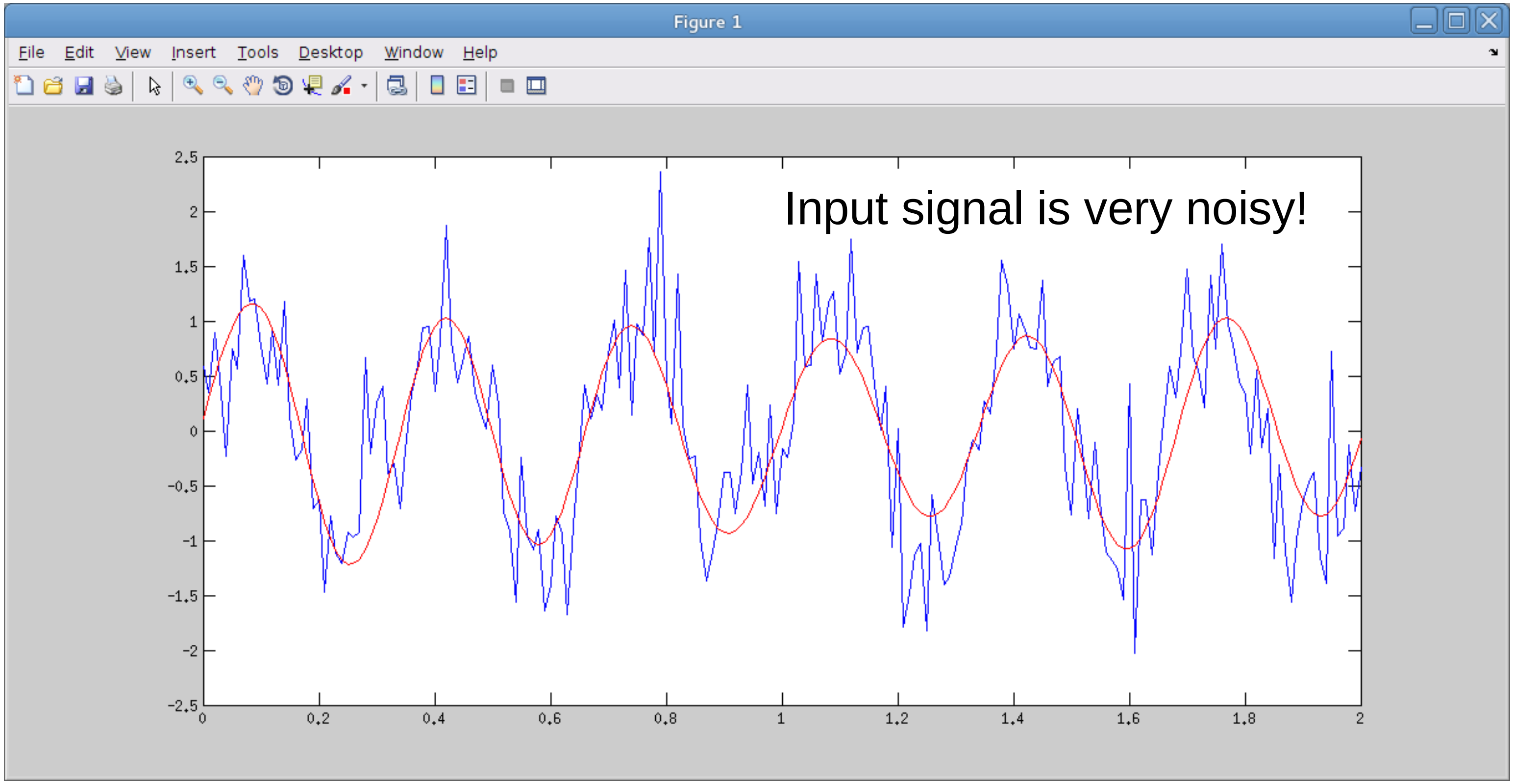
- Why use this crazy function?



To be revealed in session 2

```matlab
function yf = sinc_filter_centered(t, x, B)
  % Filters x using sinc kernel.  The desired filter bandwidth
  % (cut-off freq) is B.  We apply the filter to a cyclic
  % version of the input signal.  That is, we assume the input
  % x(t) is periodic, and the input vector contains one period of x.

  % For everything to work, we require length(t) to be odd.
  N = length(t);
  if (mod(N, 2) == 0)
    error('length(t) must be odd!')
  end

  % Create filter kernel
  Tmax = t(end);
  w = 2*B*sinc(2*B*(t-Tmax/2));
  % Now shift it 1/2 around
  w = circshift(w, [0, (N-1)/2]);

  figure(99)
  plot(t, w);
  hold on
  plot(t, w, 'ro');

  % Create index used in computation
  idx = 1:N;

  % Loop over input pts, compute filtered signal
  for i = 1:N
    j = circshift(idx, [0, i-1]);
    yf(i) = dot(x(idx), w(j))/(N/2);
  end

end
```

Filtered signal

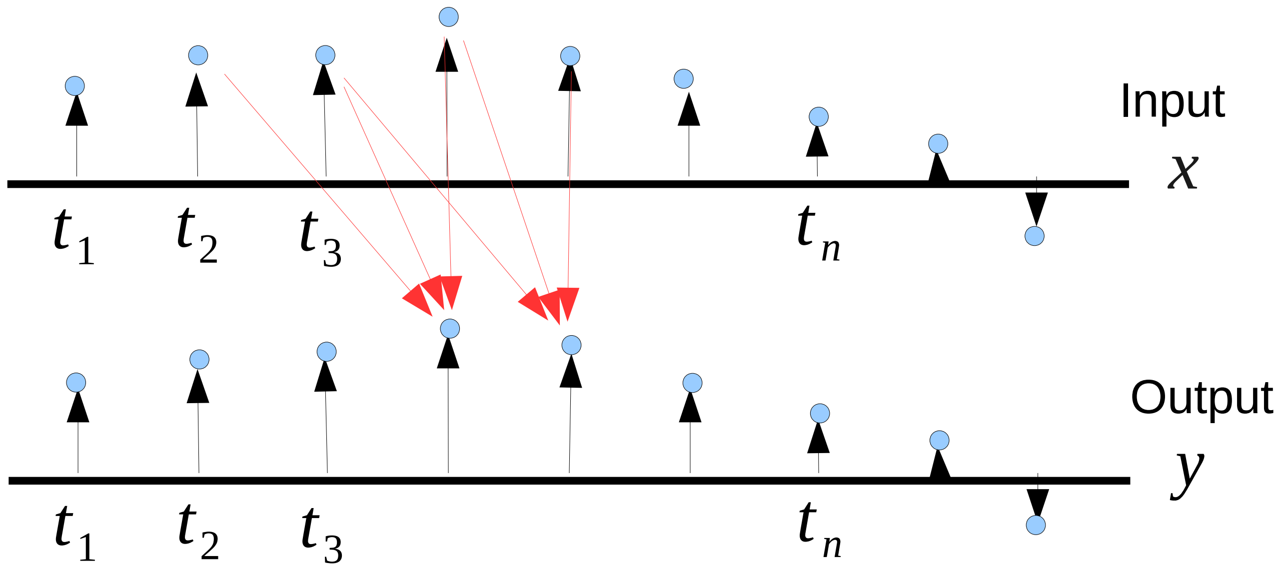


$f_0 = 3\,Hz \qquad B = 4\,Hz \qquad A_n = 0.5$

- Sinc() is applied to cyclic copies of input signal to deal with question of signal ends.
- Noise is very successfully reduced.

Causal filter (trailing box average)

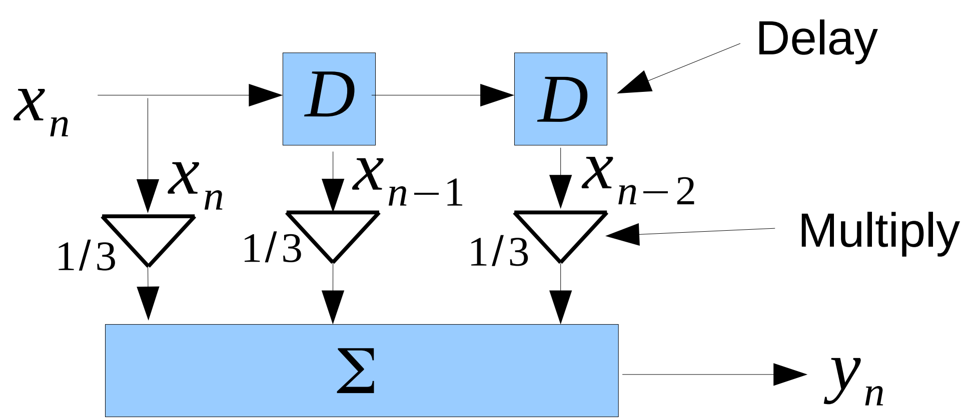
Three-point centered average.



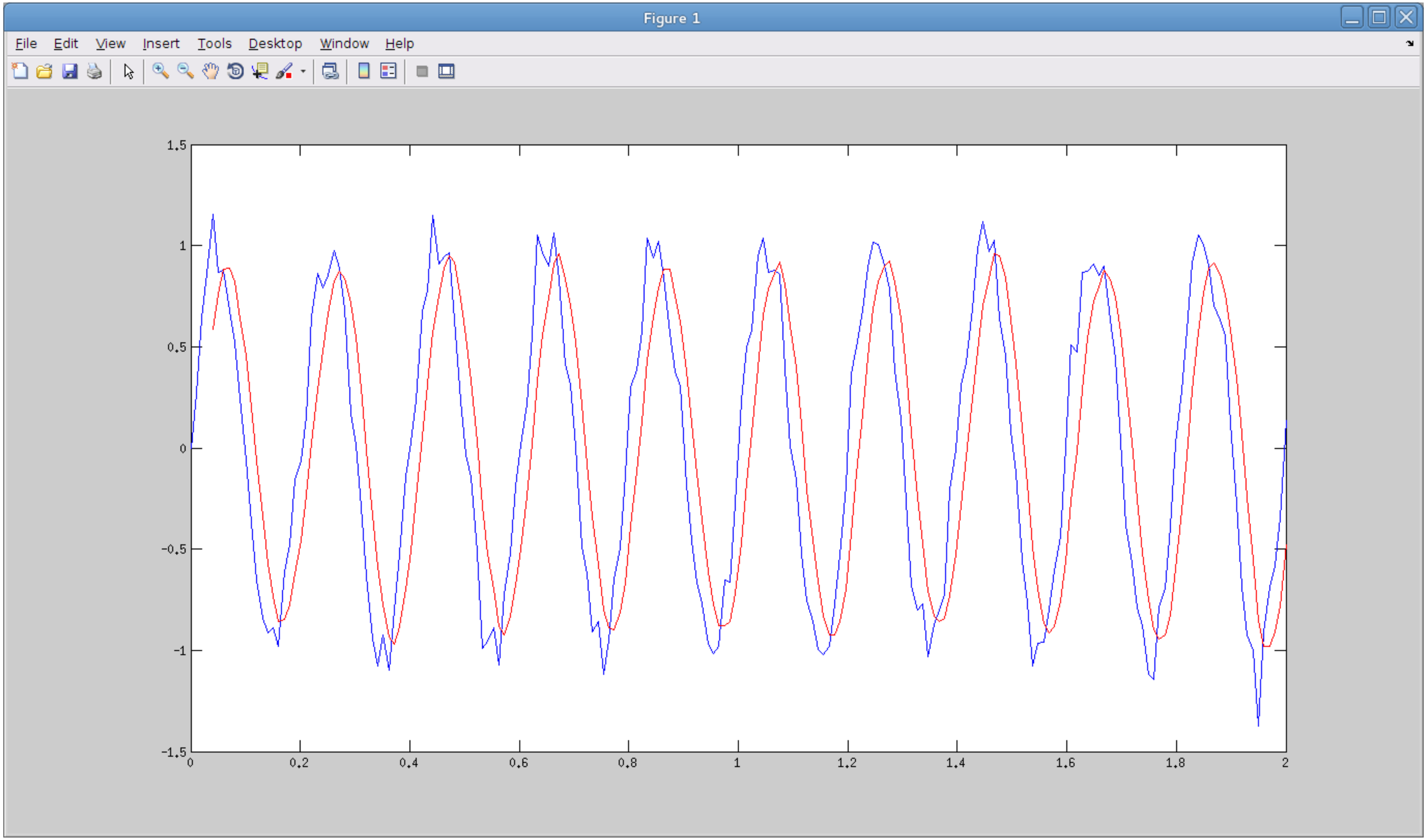
$$y_n = \frac{x_{n-2} + x_{n-1} + x_n}{3}$$

Note that this filter depends only upon present and past values of x (not upon future values).

Dataflow diagram



Filtering with trailing box average



- Note output signal has been delayed

Simple moving average

- Note SMA is delayed

Takeaway points

- You can filter a signal using different kernels.
  - Box
  - Sinc
  - Gaussian (homework)
- The weights themselves can be considered to be a function.
  - Filtering = convolution of kernel with input signal.
  - In DSP these are referred to as “FIR” filters.
- Different kernels have different properties.
- What kernel to use depends upon your specific signal and your specific goals.

Fourier series and Fourier transforms

Joseph Fourier

1768 -- 1830

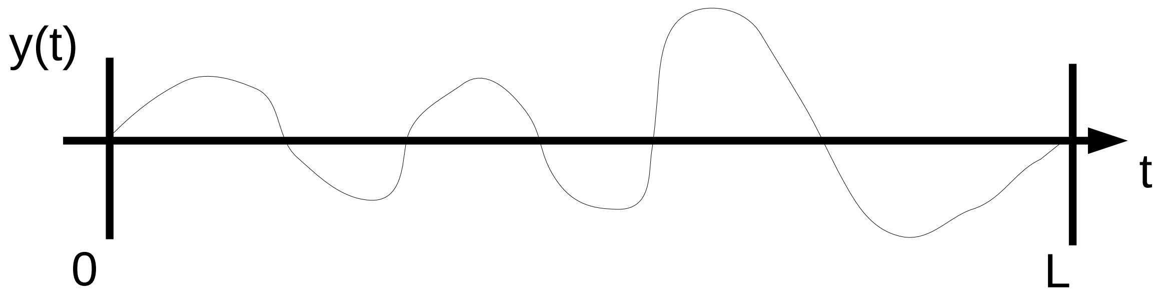
Goal: Expanding a function

- It's all about writing an expansion for a given function *y(t)*.
- Recall Taylor's series expansion around a point:

$$y(t-t_0)=a_0+a_1(t-t_0)+a_2(t-t_0)^2+a_3(t-t_0)^3+\cdots$$

- Properties:
  - Provides good approximation to $y(t-t_0)$ in neighborhood $t\approx t_0$
  - Usually only need a few terms for good approximation.

Consider function on finite interval



- Taylor expansion not very good here – polynomial order required is too high.
- Can I do a different expansion which works over entire interval?
- Note this example fcn is zero at boundaries.
- I claim yes: Fourier sin series:

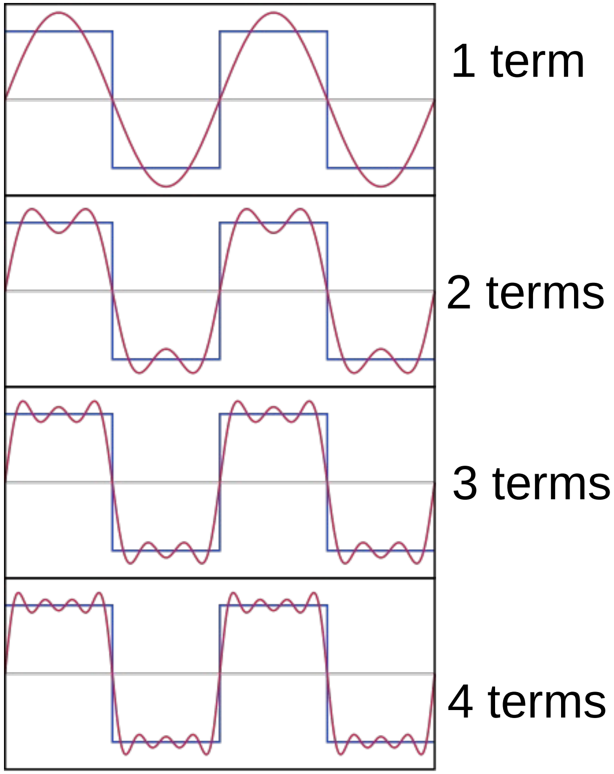
$$y(t) = a_1 \sin\left(\frac{\pi t}{L}\right) + a_2 \sin\left(\frac{2\pi t}{L}\right) + a_3 \sin\left(\frac{3\pi t}{L}\right) + \cdots$$

Fourier sin series

- Important theorem: I can expand any bounded, continuous function which is zero at the boundaries as a sum of sin functions (Fourier, 18th Century).

$$y(t)=\sum_{n=1}^{\infty} a_n \sin(n\pi t/L)$$

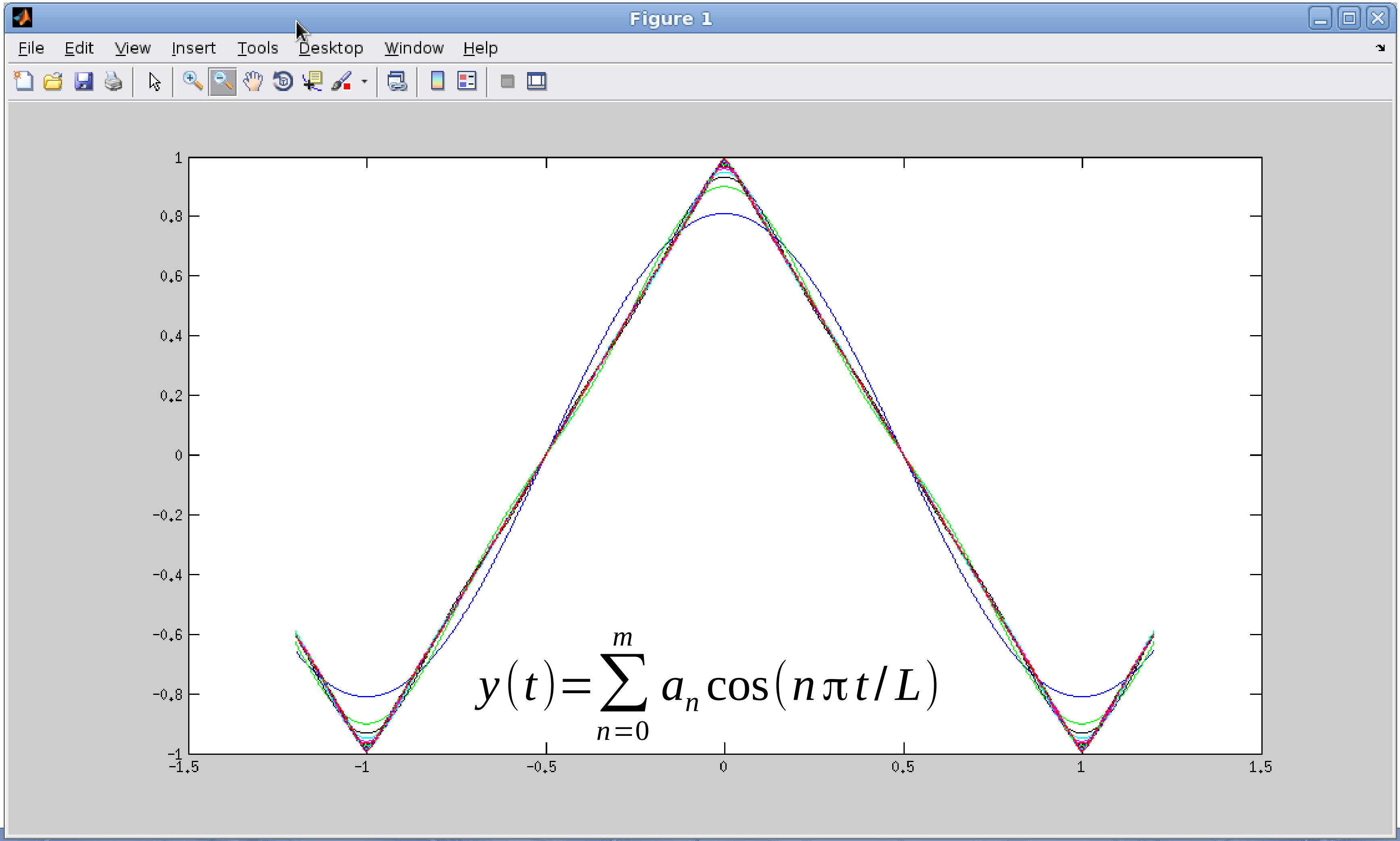
- I might need an infinite number of terms.
- Series converges over entire interval.
- Each term in expansion has coefficient a_n
- But how to get coefficients?



There is a similar theorem involving cos(t)

Fourier series is remarkable

- Says you can expand even functions with discontinuities using sin/cos functions.



MIT Mathlets

- Demo: http://mathlets.org/mathlets/fourier-coefficients/

- To generate square waves, coefficients are:

$$a_n = \frac{4}{n\pi} \quad \text{Odd n}$$

$$= 0 \quad \text{Even n}$$

```
>> n = 1:2:11

n =

    1     3     5     7     9    11

>> an = (4./(n*pi))'

an =

   1.273239544735163
   0.424413181578388
   0.254647908947033
   0.181891363533595
   0.141471060526129
   0.115749049521378
```

How to get coefficients?

- Consider integrating the product of two sin functions:

$$\int_0^L dt \sin\left(\frac{n\pi t}{L}\right)\sin\left(\frac{m\pi t}{L}\right)$$

m, n are integers

- Recall trig identity:

$$\sin(x)\sin(y)=\frac{1}{2}(\cos(x-y)-\cos(x+y))$$

- So: $$\int_0^L dt \sin\left(\frac{n\pi t}{L}\right)\sin\left(\frac{m\pi t}{L}\right)$$

$$=\frac{1}{2}\int_0^L dt\left[\cos\left(\frac{(n-m)\pi t}{L}\right)-\cos\left(\frac{(n+m)\pi t}{L}\right)\right]$$

Deriving coefficients.....

- Consider expression

$$\frac{1}{2}\int_0^L dt\left[\cos\left(\frac{(n-m)\pi t}{L}\right)-\cos\left(\frac{(n+m)\pi t}{L}\right)\right]$$

- When $n \neq m$ we have two terms like:

$$\int_0^L dt\cos\left(\frac{p\pi t}{L}\right)$$

p is integer

$$=\frac{L}{p\pi}\sin\left(\frac{p\pi t}{L}\right)\Bigg|_0^L$$

Draw picture on
blackboard to show why
this integrates to zero

0

$$=\frac{L}{p\pi}\left(\sin p\pi-0\right)=0$$

When n = m...

$$\frac{1}{2}\int_0^L dt\left[\cos\left(\frac{(n-m)\pi t}{L}\right)-\underbrace{\cancel{\cos\left(\frac{(n+m)\pi t}{L}\right)}}_{=0}\right]$$

$$=\frac{1}{2}\int_0^L dt\cos\left(\frac{0\pi t}{L}\right)$$

$$=\frac{L}{2}$$

- Conclusion: integral is non-zero only when n = m.

Orthogonal functions

- Sin functions are orthogonal over interval [0, L]

$$\int_0^L dt \sin\left(\frac{n\pi t}{L}\right)\sin\left(\frac{m\pi t}{L}\right) \begin{cases} =0 \text{ for } n \neq m \\ =\dfrac{L}{2} \text{ for } n=m \end{cases}$$

- Similar to orthogonality of vectors:

$$\vec{u}\cdot\vec{v} \quad \begin{cases} =0 \text{ for } \vec{u}\neq\vec{v} \\ =C \text{ for } \vec{u}=\vec{v} \end{cases}$$

Consider what this means for Fourier expansion

- Start with

$$y(t)=\sum_{n=1}^{\infty} a_n \sin(n\pi t/L)$$

- Multiply through both sides and integrate:

$$\int_0^L dt\, y(t)\sin\left(\frac{m\pi t}{L}\right)=\sum_{n=1}^{\infty} a_n \int_0^L dt \sin\left(\frac{m\pi t}{L}\right)\sin\left(\frac{n\pi t}{L}\right)$$

- Use orthogonality:

Method to get coefficients

$$\boxed{a_m=\frac{2}{L}\int_0^L dt\, y(t)\sin\left(\frac{m\pi t}{L}\right)}$$

Therefore, we can go in two directions

- Fourier series expansion:

$$y(t) \Leftrightarrow \sum_{n=1}^{\infty} a_n \sin(n\pi t/L)$$

- You can go back and forth:

$$y(t) = \sum_{n=1}^{\infty} a_n \sin(n\pi t/L)$$

Get function from coefficients

$$a_m = \frac{2}{L}\int_0^L dt\, y(t) \sin\left(\frac{m\pi t}{L}\right)$$

Get coefficients from function

Generalize to any function defined on an interval

- You can expand any function, regardless of values on boundary using:

Real *y(t)*

$$y(t)=\frac{b_0}{2}+\sum_{n=1}^{\infty} b_n \cos(\omega_n t)+\sum_{n=1}^{\infty} a_n \sin(\omega_n t)$$

a, b are real

Complex *y(t)*

$$y(t)=\sum_{n=-\infty}^{\infty} c_n e^{-i\omega_n t}$$

c is complex

Full definition – use this

- Valid for arbitrary periodic function on interval [-L, L]

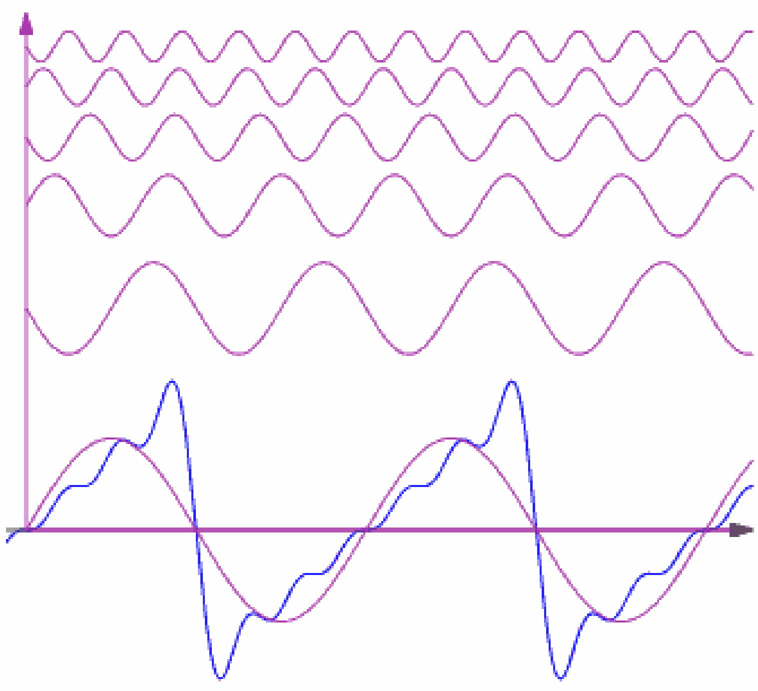
$$y(t)=\frac{b_0}{2}+\sum_{n=1}^{\infty} b_n \cos\left(\frac{n\pi t}{L}\right)+\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi t}{L}\right)$$

$$b_0=\frac{1}{L}\int_{-L}^{L} y(t)\,dt \qquad b_n=\frac{1}{L}\int_{-L}^{L} y(t)\cos\left(\frac{n\pi t}{L}\right)dt$$

$$a_n=\frac{1}{L}\int_{-L}^{L} y(t)\sin\left(\frac{n\pi t}{L}\right)dt$$

Consider meaning of a_n coefficients

- Define $\omega_n = n\pi/L$
- Then Fourier series is $y(t) = \sum_{n=1}^{\infty} a_n \sin(\omega_n t)$
- Interpret ω_n as a frequency
- Height of a_n determines amplitude of that frequency component in signal *y(t)*.
- **Key point:** Any signal can be viewed as composed of a sum of sin/cos waves.



Complex Fourier series

- Fourier series expansion:

$$y(t) = \sum_{n=-\infty}^{\infty} a_n e^{-int}$$

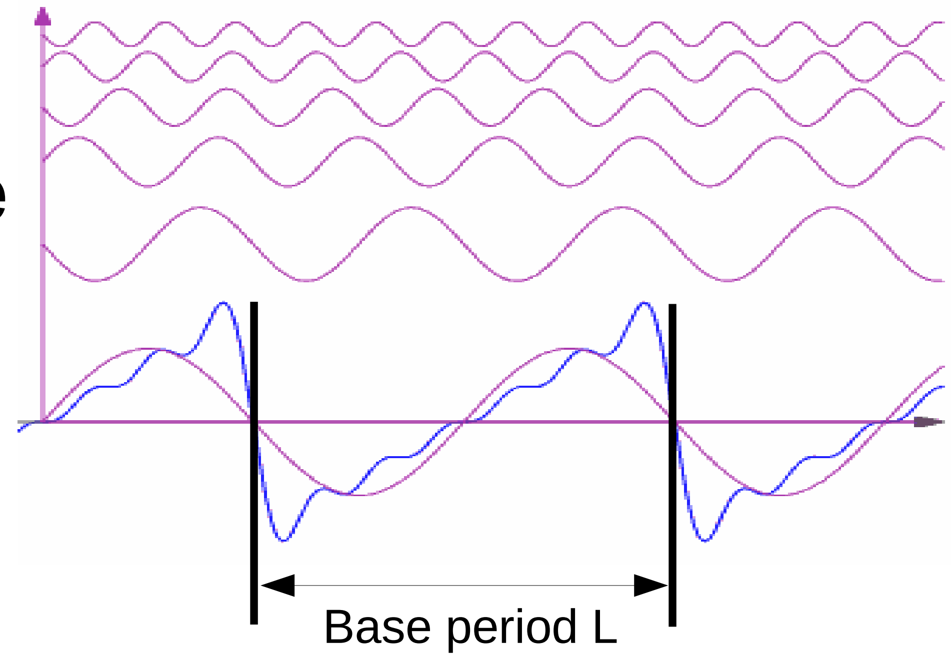
- Given coefficients, compute function:

$$a_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} dt\, y(t) e^{int}$$

- Note different limits of summation and integration.

What happens outside interval [0, L]?

- Basis functions sin, cos mean expansion extends to infinity, and is periodic.
- Base period L
- Therefore, you can use Fourier series to expand:
  - Any periodic function
  - Any function defined on a finite interval



Fourier transform

- Fourier series defined for signal on finite interval or periodic.
- What if signal is infinite (i.e. extends to $t=\pm\infty$)?
- Fourier transform pair:

Transform to frequency domain

$$Y(\omega)=\int_{-\infty}^{\infty} dt\, y(t)\, e^{-i\omega t}$$

Transform to time domain

$$y(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, Y(\omega)\, e^{i\omega t}$$

Fourier transform vs. series

Fourier series

$$a_m = \frac{2}{L}\int_0^L dt\, y(t) \sin\left(\frac{m\pi t}{L}\right)$$

$$y(t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi t}{L}\right)$$

- Valid for:
  - Periodic function
  - Function on interval
- Continuous function, discrete spectrum

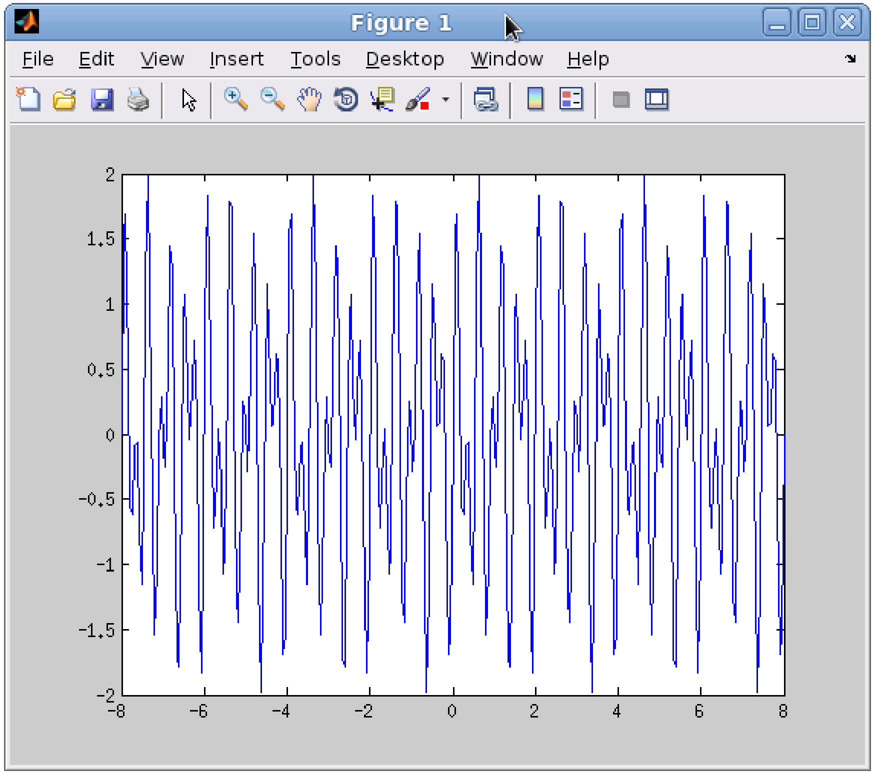
Fourier transform

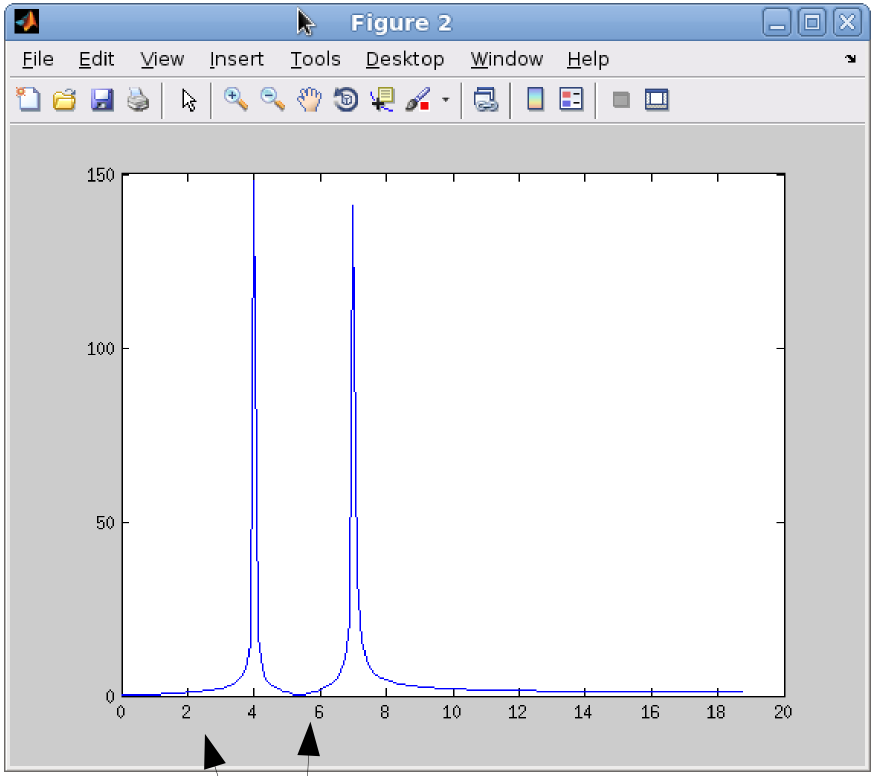
$$Y(\omega) = \int_{-\infty}^{\infty} dt\, y(t)\, e^{-i\omega t}$$

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, Y(\omega)\, e^{i\omega t}$$

- Valid for any function
- Function and spectrum continuous.

Fourier transform converts time-varying signal into frequency spectrum






```
f1 = 4;
f2 = 7;
w = -t.*t + 64;
y = w.*(sin(2*pi*f1*t) + sin(2*pi*f2*t));
```

Create signal with two frequencies: 4Hz and 7Hz.

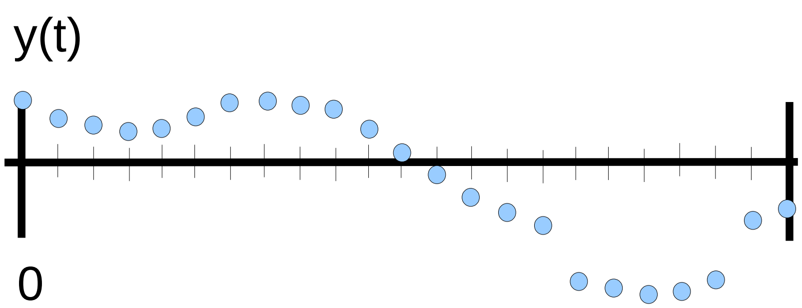
Take Fourier transform. Observe two delta functions at 4 and 7 Hz.

Time and frequency are duals

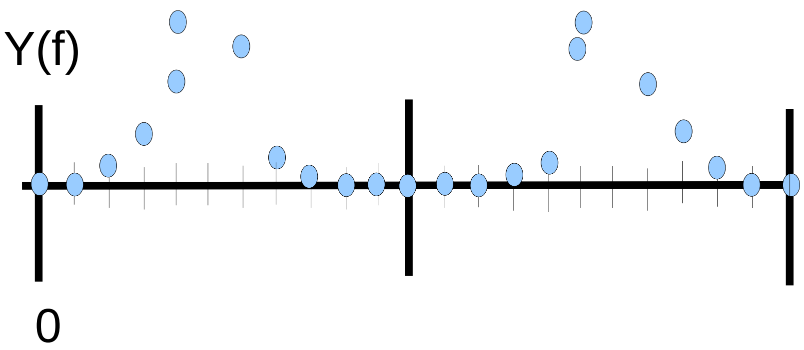
- Fourier transform pair: you can go back and forth from time domain to frequency domain.

$$y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \, Y(\omega) \, e^{i\omega t}$$

$$Y(\omega) = \int_{-\infty}^{\infty} dt \, y(t) \, e^{-i\omega t}$$

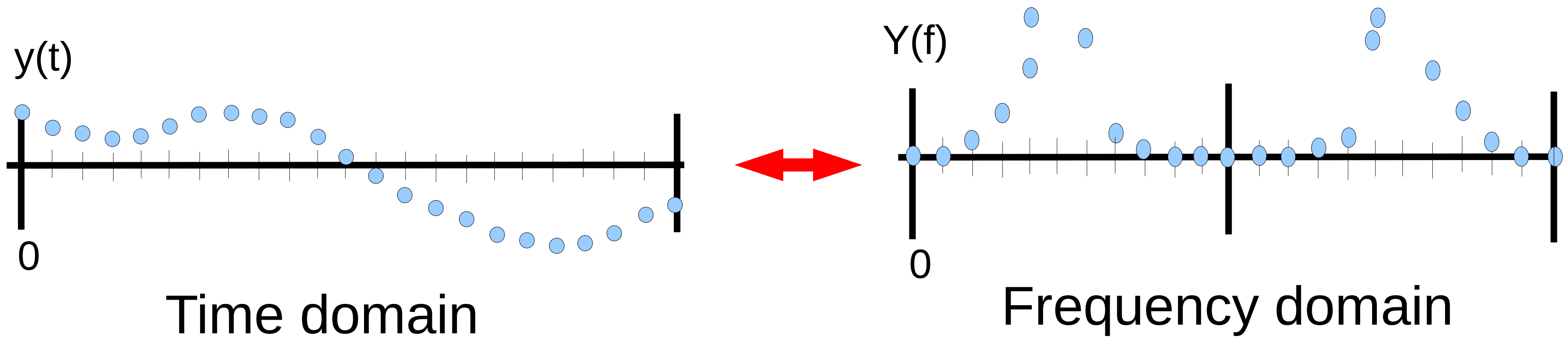


Time domain

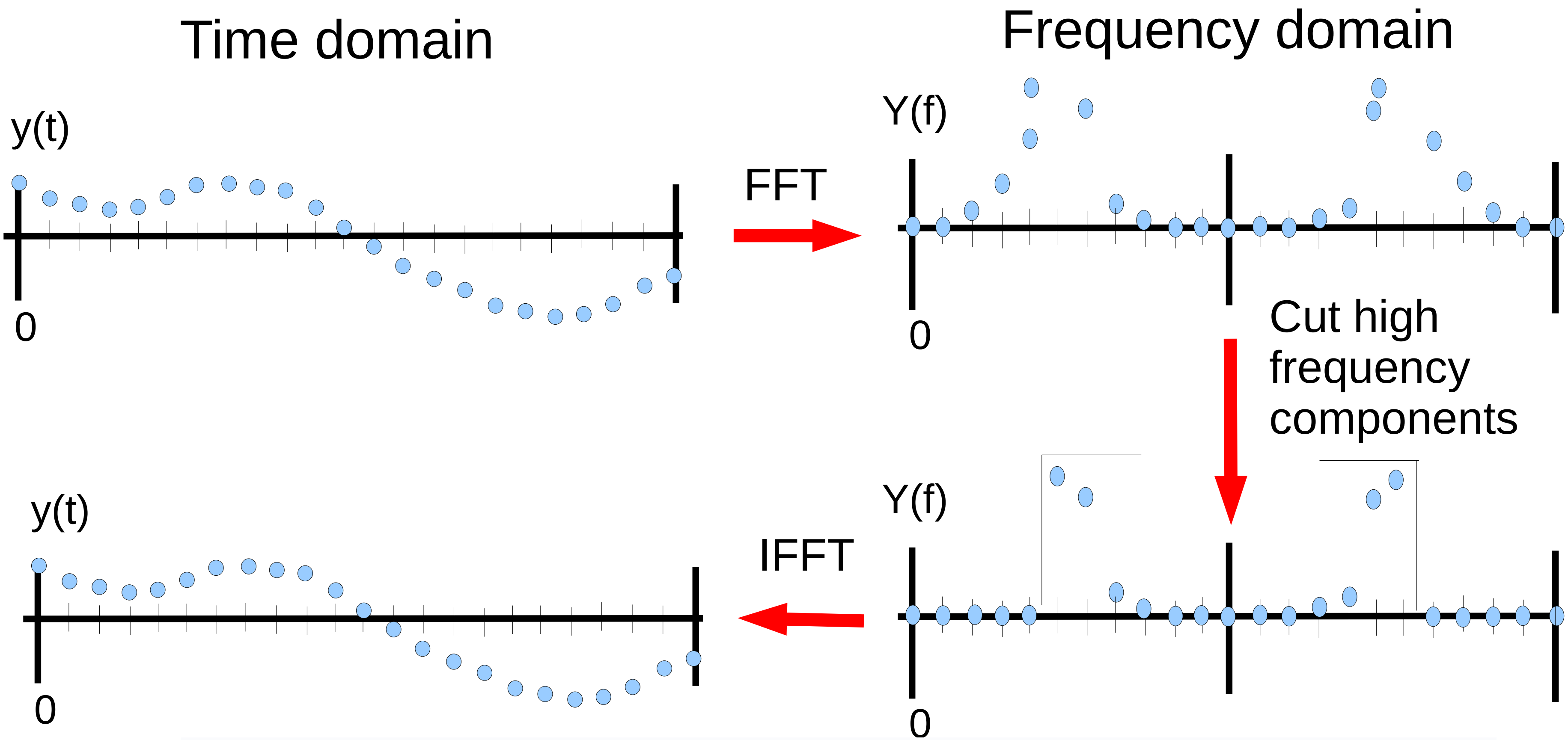


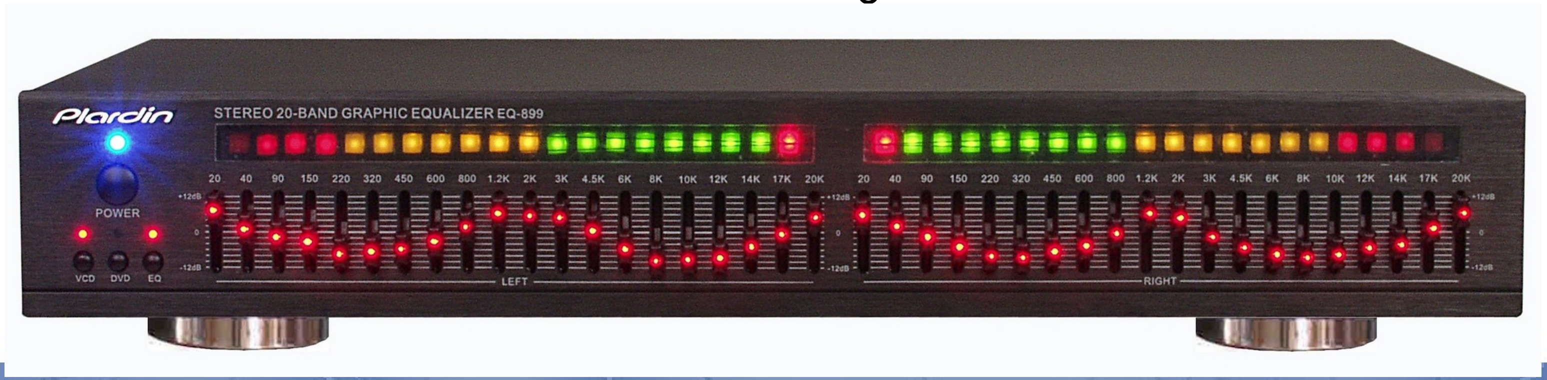
Frequency domain

Time domain and frequency domain



Application: filtering





Stereo equalizer

Session summary

- Sampled data
- Simple filters
- Fourier series
- Fourier transform