Northeastern University, Department of Mathematics

MATH 5110: Applied Linear Algebra and Matrix Analysis.

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§7 Diagonalization; Eigenvalues; Eigenvectors;

Contents

$A = PCP^{-1} \overset{\text{def}}{\Longleftrightarrow} A$ and C are "similar".

1. Diagonalization

Let D be an diagonal matrix. The power D^k is easy to calculate. For example,

$$D^k = \begin{bmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & d_4 \end{bmatrix}^k = \begin{bmatrix} (d_1)^k & 0 & 0 & 0 \\ 0 & (d_2)^k & 0 & 0 \\ 0 & 0 & (d_3)^k & 0 \\ 0 & 0 & 0 & (d_4)^k \end{bmatrix}$$

Definition 1. An $n \times n$ matrix A is said to be **diagonalizable** if

$$A = PDP^{-1}$$

Application of diagonalization:

$$A^k = PD\cancel{P^{-1}}\cancel{P}D\cancel{P^{-1}} \cdots \cancel{P}DP^{-1} = PD^kP^{-1}$$

Question:

1. Are all $n \times n$ matrices A diagonalizable?

$A = PDP^{-1}$

2. If a matrix A is diagonalizable, how to find the invertible matrix P and the diagonal matrix D? The answer for this question is called **diagonalize** matrix A.

Solve $AP = PD$

$\Leftrightarrow$ Solve $A\vec{x} = c\vec{x}$

and get solution $(\vec{b}_1, c_1)$, $(\vec{b}_2, c_2)$, $\vdots$ $(\vec{b}_n, c_n)$

independent.

$T_A : \mathbb{R}^n \to \mathbb{R}^n$

$T(\vec{b}_1) = c_1\vec{b}_1$

$T(k\vec{b}_1) = kT(\vec{b}_1) = kc_1\vec{b}_1 = c_1 k\vec{b}_1$

$\boxed{P = [\vec{b}_1 \dots \vec{b}_n]}$

invertible

$D = \begin{bmatrix} c_1 & & \\ & \ddots & \\ & & c_n \end{bmatrix}$

$A = PDP^{-1}$

2. Eigenvalues and Eigenvectors.

Consider a linear transformation $T : \mathbb{F}^n \to \mathbb{F}^n$ by matrix $T\vec{x} = A\vec{x}$. ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.)

Definition 2. If there exist a nonzero vector $\vec{x} \in \mathbb{F}^n$ and a number $\lambda \in \mathbb{F}$ such that

$$A\vec{x} = \lambda\vec{x}$$

then, the vector $\vec{x}$ is **an eigenvector corresponding to the eigenvalue** λ.

Definition 3. A basis $\vec{b}_1, \ldots, \vec{b}_n$ of $\mathbb{F}^n$ is called an **eigenbasis** for A if the vectors $\vec{b}_1, \ldots, \vec{b}_n$ are eigenvectors of A.

$A\vec{v} = \lambda\vec{v}$

Example 4. If $\vec{v}$ is an eigenvector of A corresponding to λ, is $\vec{v}$ an eigenvector of A^k? Is λ an eigenvalue of A^k?

NO

$A^k\vec{v} = A\cdots A\vec{v} = \lambda \overbrace{A\cdots A}^{k-1}\vec{v} = (\lambda^k)\vec{v}$

$A = PDP^{-1}$

Theorem 5. *A is diagonalizable if and only if it has n linearly independent eigenvectors $\vec{b}_1, \ldots, \vec{b}_n$ (eigenbasis).*
In this case $A = PDP^{-1}$ where the columns of P are eigenvectors of A; the diagonal entries of D are the eigenvalues of A corresponding to the eigenvectors given by the columns of P.

$P = [\vec{b}_1 \cdots \vec{b}_n]$ $\quad D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \end{bmatrix}$

Proof. We already verified that system of equations $A\vec{b}_1 = \lambda_1\vec{b}_1$, $A\vec{b}_2 = \lambda_2\vec{b}_2$, $\dots$, $A\vec{b}_n = \lambda_n\vec{b}_n$. is equivalent to matrix equation

$$AP = PD$$

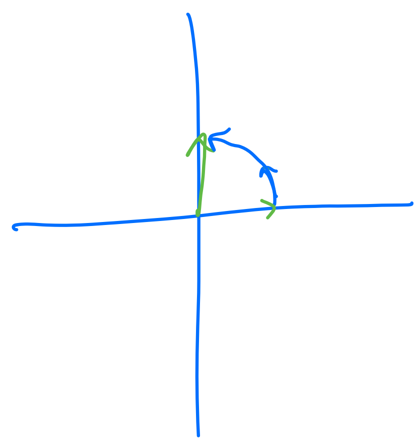
where $P = [\vec{b}_1 \ \dots \ \vec{b}_n]$ and $D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$.

P is invertible if and only if $\{\vec{b}_1, \dots, \vec{b}_n\}$ is a basis of $\mathbb{R}^n$. In this case, $A = PDP^{-1}$ and A is diagonalizable.

□

Example 6. Let T be the projection transformation onto a line $L = \text{Span}\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\}$ $\mathbb{R}^3$.

Find a basis $\mathscr{B} = [\vec{b}_1 \ \vec{b}_2 \ \vec{b}_3]$ for $\mathbb{R}^3$ such that the $\mathscr{B}$-matrix of the T is the diagonal matrix $D = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$.



Example 7. Let T be the rotation through an angle of $\pi/2$ in the counterclock direction. So the matrix of T is $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Find all eigenvalues and eigenvectors of A. Is A diagonalizable?

$[T(\vec{e}_1) \ T(\vec{e}_2)] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

NO

Example 8. Which matrix has 0 as an eigenvalue?

$\boxed{A\vec{x} = \vec{0}\vec{x}}$ has non-zero soln. $\vec{x} \neq \vec{0}$

$\Longleftrightarrow$ A is not invertible

2. Eigenvalues and Characteristic Polynomials

Let A be an $n \times n$ matrix.

$A\vec{x} = \lambda \vec{x}$

$A\vec{x} - \lambda\vec{x} = \vec{0}$

$(A - \lambda I)\vec{x} = \vec{0}$ has non-zero soln $\iff |A - \lambda I| = 0$

Theorem 9 (**The Characteristic Equation of** A)**.**

λ is an eigenvalue of $A \iff |A - \lambda I| = 0$

Example 10. Finding Eigenvalues for the following matrices:

$$A = \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} i & 1 \\ 2 & 3 \end{bmatrix}$$

$$eig(A) = \begin{pmatrix} -1 \\ 7 \end{pmatrix}$$

$$eig(B) = \begin{pmatrix} \frac{3}{2} - \frac{\sqrt{16-6\,\mathrm{i}}}{2} + \frac{1}{2}\,\mathrm{i} \\ \frac{\sqrt{16-6\,\mathrm{i}}}{2} + \frac{3}{2} + \frac{1}{2}\,\mathrm{i} \end{pmatrix} \approx \begin{pmatrix} -0.5337 + 0.8688i \\ 3.5337 + 0.1312i \end{pmatrix}$$

Theorem 11. *The eigenvalues of a triangular $n \times n$ matrix A equal the diagonal entries of A.*

Proof. Suppose A is an upper triangular matrix.

$$\det(A - \lambda I) = \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{nn} - \lambda \end{vmatrix} = (a_{11} - \lambda)(a_{22} - \lambda)\cdots(a_{nn} - \lambda) = 0$$

Hence, the eigenvalues of A are a_{ii} for $i = 1, \ldots, n$. □

Practice: Finding Eigenvalues for the following matrices:

$$A = \begin{bmatrix} 2 & 5 & \sqrt{2} \\ 3 & 4 & 7 \\ 0 & 0 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 4 & 0 \\ 3 & 5 & 7 \end{bmatrix}$$

In general for a $n \times n$ matrix A,

$$\det(A - \lambda I) = \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix}$$

$$= (a_{11} - \lambda)(a_{22} - \lambda) \cdots (a_{nn} - \lambda) + \sum(\text{ terms of degree} \leq (n-2))$$

$$= (-\lambda)^n - \underbrace{(a_{11} + a_{22} + \cdots + a_{nn})}_{= tr(A)}(-\lambda)^{n-1} + \sum(\text{ terms of degree} \leq (n-2))$$

Definition 12 (Characteristic Polynomial). If A is an $n \times n$ matrix, the **characteristic polynomial** of A is

$$f_A(\lambda) := \det(A - \lambda I)$$

Example 13. Find the characteristic polynomial for a 2×2 arbitrary matrix.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = \lambda^2 - \underbrace{(a+d)}_{= Tr(A)}\lambda + \underbrace{ad-bc}_{= \det A}$$

Definition 14. Th sum of the diagonal entries of a square matrix is called the **trace** of A,

$$tr(A) := a_{11} + a_{22} + \cdots + a_{nn}$$

The characteristic polynomial for a 2×2 matrix A:

- $tr(A^T) = tr A$
- $tr(A+B) = tr(A) + tr(B)$
- $tr(kA) = k\, tr A$
- $tr(AB) = tr(BA)$ ✓
- $tr(AB) \neq (trA)(trB)$

More generally,

Q: Are there any $n \times n$ matrices A, B such that $AB - BA = I_n$?

tr

$$0 = tr(AB) - tr(BA) = tr(I_n) = n$$

$$= (-\lambda)^n + \underline{(\lambda_1 + \cdots + \lambda_n)}(-\lambda)^{n-1} + \cdots + \underline{\lambda_1\lambda_2\cdots\lambda_n}$$

$$= (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda)$$

Theorem 15. *Let A be an $n \times n$ matrix. Then the characteristic polynomial of A is*

$$f_A(\lambda) = (-\lambda)^n + \operatorname{tr} A (-\lambda)^{n-1} + \cdots + \det A$$

$$f_A(\lambda) = |A - \lambda I| \qquad f_A(0) = |A|$$

More properties on Characteristic Polynomials

Definition 16 (Algebraic Multiplicity).
An eigenvalue λ_0 of A is said to have **algebraic multiplicity** k if it has multiplicity k as a root of the characteristic polynomial $f_A(t)$. Equivalently,

$$f_A(\lambda) = (\lambda_0 - \lambda)^k g(\lambda)$$

such that $g(\lambda_0) \neq 0$.

Theorem 17. *An $n \times n$ matrix has at most n eigenvalues, even counted with algebraic multiplicities.*

has n complex eigenvalues

Example 18. Find all eigenvalues and their algebraic multiplicities of $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Example 19. Find the characteristic polynomial of $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$. Which of the following numbers $1, -1, 4$ are eigenvalues of A?

Example 20. Find the characteristic polynomial of $A = \begin{bmatrix} 2 & 1 & 2 \\ -1 & 4 & 2 \\ -1 & 1 & 5 \end{bmatrix}$. Verify that 3 and 5 are eigenvalues.

eig( )

Theorem 21. *Let A be an $n \times n$ matrix. Suppose A has n eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, (listed with algebraic multiplicities.) Then*

$$\operatorname{tr} A = \lambda_1 + \cdots + \lambda_n$$

$$\det A = \lambda_1 \cdots \lambda_n$$

$A = PCP^{-1} \Leftrightarrow A, C$ similar

$A = PBP^{-1}$

Theorem 22 (On Eigenvalues of Similar Matrices). *If A and B are similar,*

then $f_A(\lambda) = f_B(\lambda)$

then A, B have the same eigenvalues.

$$f_A(\lambda) = |A - \lambda I| = |PBP^{-1} - \lambda I| = |PBP^{-1} - PIP^{-1}|$$

$$= |P(B - \lambda I)P^{-1}|$$

$$= |B - \lambda I| = f_B(\lambda)$$

Suppose $B = PAP^{-1}$

then $B = PAP^{-1} = PIP^{-1} = PP^{-1} = I$ contradictory

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

The converse of the preceding theorem is not generally true. There do exist matrices with the same characteristic polynomial that are not similar matrices. See example later.

$A = PBP^{-1}$

Proposition 23. *If A and B are similar, then*

$\operatorname{tr} A = \operatorname{tr} B$

$\det A = \det B$

$A = PBP^{-1}$

Proposition 24. *If A and B are similar, then*

$\operatorname{rank} A = \operatorname{rank} B$

Example 25. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ are not similar, but they have the same rank, the same determinant, the same trace, the same eigenvalues.

No!

Example 26. Are the following two matrices similar to each other? $A = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 5 \\ 2 & 3 \end{bmatrix}$

$\operatorname{tr} B = 6$ $\neq$ $\operatorname{tr} A = 5$

Ex: If $A\vec{v} = \lambda\vec{v}$, then $P^{-1}\vec{v}$ is eigenvector for B

Warning: Similar matrices may have **different** eigenvectors.

$A = PBP^{-1}$

$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}$

3. Eigenvectors and Eigenspaces

Theorem 27. *Let A be an $n \times n$ matrix. A scalar λ is an eigenvalue for A if and only if the matrix equation*

$$(A - \lambda I_n)\vec{x} = \vec{0}$$

has a nontrivial *solution $\vec{x}$.*
Said differently, λ is an eigenvalue for A if and only if

$$\mathrm{Nul}(A - \lambda I_n) \neq \{\vec{0}\}.$$

Definition 28. Let A be an $n \times n$ matrix and λ be a eigenvalue of A. The set of all eigenvectors of A corresponding to λ together with the zero vector, is called the **eigenspace** of A corresponding to λ, and it equals the subspace

$$\mathrm{Nul}(A - \lambda I_n).$$

The dimension of the eigenspace $\mathrm{Nul}(A-\lambda I_n)$ is called the **geometric multiplicity** of λ.

Proposition 29.

$$1 \leq \boxed{\textit{Geometric multiplicity of } \lambda} \leq \textit{Algebraic multiplicity of } \lambda \leq n.$$

$A = PDP^{-1}$

Example 30. Let T be the projection transformation onto a line $L = \mathrm{Span}\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\}$ $\mathbb{R}^3$.

Explain the geometric meaning of the eigenvalues and eigenspaces.

Lemma 31. *Let A be an $n \times n$ matrix and let $\vec{v}_1, \ldots, \vec{v}_p$ be eigenvectors of A that correspond to* distinct *eigenvalues $\lambda_1, \ldots, \lambda_p$ respectively. Then $\{\vec{v}_1, \ldots, \vec{v}_p\}$ is a linearly independent set of vectors.*

Proof. We prove this by induction on p. If $p = 1$, it is clear. Suppose this is true for $p - 1$ vectors. $\square$

Lemma 32. *Let A be an $n \times n$ matrix and let $\lambda_1, \ldots, \lambda_p$ be distinct eigenvalues with corresponding independent set of eigenvectors $V_1, \ldots, V_p$. Then $V_1 \cup \cdots \cup V_p$ is a linearly independent set of vectors.*

Recall that an $n \times n$ matrix A is diagonalizable if and only if it has n linearly independent eigenvectors.

$$A = PDP^{-1}$$

n independent eigenvectors

eigenvalues

Proposition 33 (Case of Distinct Eigenvalues)**.** *If an $n \times n$ matrix A has n **distinct** eigenvalues, then its corresponding eigenvectors are linearly independent and A is diagonalizable.*

$E_k = \ker(A - \lambda_k I)$

Theorem 34. *Let $\lambda_1, \ldots, \lambda_p$ be **distinct** eigenvalues of A such that*

$$f_A(\lambda) = \det(A - \lambda I) = (\lambda_1 - \lambda)^{k_1}(\lambda_2 - \lambda)^{k_2} \cdots (\lambda_p - \lambda)^{k_p}.$$

Suppose $k_1 + k_2 + \cdots + k_p = n$. Let E_k be the eigenspace of λ_k. with basis B_k

A is "diagonalizable" $\iff$ $B_1 \cup B_2 \cdots \cup B_p$ is a basis for $\mathbb{R}^n$

$A = PDP^{-1}$

$\iff \dim E_i = k_i$ for any $i = 1, \ldots, p$

$\iff \dim E_1 + \cdots + \dim E_p = n$

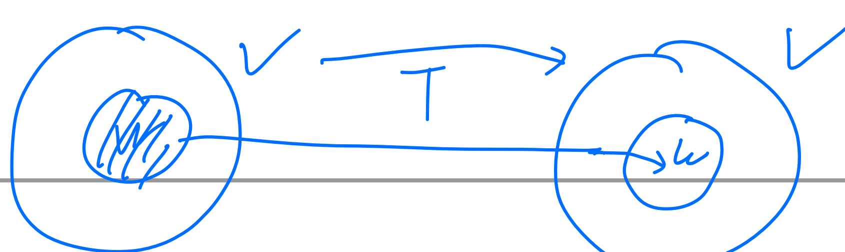
~~*Proof.* A is~~ diagonalizable if and only if it has n linearly independent eigenvectors. ☐

• Ex: $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ not diagonalizable!

$\lambda = 1$ a.m.$(\lambda=1) = 2$

g.m$(\lambda=1) = 1$

$\ker(A - \lambda I) = \ker\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$



Another point of view of the eigenspaces is the invariant subspace. $\mathbb{R}^n \xrightarrow{T_A} \mathbb{R}^n$

"eigenspace"!

Definition 35. Let $T : V \to V$ be a linear transformation on a vector space V. A subspace $W \subseteq V$ is said to be **invariant** under T if

$$T(W) \subseteq W.$$

$\dim(S)=1$ $\quad S=\text{span}\{\vec v\}$ $\quad T(S)\subseteq S$

Proposition 36. *A one-dimensional subspace is invariant under the linear transformation T_A if and only if it is an eigenspace spanned by an eigenvector of A.*

$T(\vec v)\in S$

$\vec v$ is an eigenvector of A.

$T_A(\vec v)=\lambda\vec v$

$P=[\vec b_1 \cdots \vec b_n]$ $\quad D=\begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix}$

$\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$ $\quad A=PDP^{-1} \iff A$ is diagonalizable

Theorem 37. *An $n \times n$ matrix A is similar to a diagonal matrix D, (i.e., $A = PDP^{-1}$) if and only if there exists a decomposition of* $A\vec b_i=\lambda_i\vec b_i$

$$\mathbb{F}^n = V_1 \oplus V_2 \oplus \cdots \oplus V_n$$

such that each V_i is one dimensional and invariant under T_A.

$V_i = \text{Span}\{\vec b_i\}$

$A\vec b_i = \lambda_i \vec b_i$

In Matlab, $[P, D] = \text{eig}(A)$ returns diagonal matrix D of eigenvalues and matrix P whose columns are the corresponding right eigenvectors, such that $AP = PD$.

Example 38. Diagonalizing Matrices

$$A = \begin{bmatrix} 4 & -1 & 0 \\ 2 & 1 & 0 \\ 2 & -1 & 2 \end{bmatrix}. \quad B = \begin{bmatrix} 5 & 1 & 0 \\ 2 & 4 & 0 \\ 2 & 1 & 3 \end{bmatrix}. \quad C = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}. \quad M = \begin{bmatrix} 2 & 1 & 2 \\ -1 & 4 & 2 \\ -1 & 1 & 5 \end{bmatrix}.$$

Remark[Non Diagonalizing Result] For any $n > 1$ there exist examples of $n \times n$ matrices that are not diagonalizable.

$A=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}=PDP^{-1}$ $\quad |A-\lambda I| = \lambda^2+1=0$

$\lambda = \pm i$

Real Matrices Acting on $\mathbb{C}^n$

Let A be a real $n \times n$ matrix and λ be an eigenvalue of A.

- If λ is a real number, then there exist real eigenvectors associate to λ, as well as complex eigenvector.
- If λ is a complex (non-real) eigenvalue of A, then every eigenvector $\vec{x}$ associated to λ is a complex (non-real) vector.

Suppose A is an $n \times n$ matrix with real number entries so that $\overline{A} = A$. Let λ be a complex eigenvalue of A with associated eigenvector $\vec{x}$. Then

$$\overline{A \cdot \vec{x}} = \overline{A} \cdot \overline{\vec{x}} = A \cdot \overline{\vec{x}}$$
$$\overline{A \cdot \vec{x}} = \overline{\lambda \cdot \vec{x}} = \overline{\lambda} \cdot \overline{\vec{x}}$$

Combining the two we obtain

$$A \cdot \overline{\vec{x}} = \overline{\lambda} \cdot \overline{\vec{x}}.$$

Theorem 39. *Let A be an $n \times n$ matrix with real number entries and let λ be an eigenvalue of A with associated eigenvector $\vec{x}$. Then $\overline{\lambda}$ is also an eigenvalue of A with associated eigenvector $\overline{\vec{x}}$.*