MATH 7241: Problems #1

Due date: Wednesday September 21

Reading: relevant background material for these problems can be found in the class notes, and in Ross (Sections 2.1 2.2, 2.3, 2.4) and in Grinstead and Snell (Chapters 1,2 3, 6).

Exercise 1 Let A and B be events such that $P(A) = 0.7$ and $P(B) = 0.9$. Find the largest and smallest possible values of $P(A \cup B) - P(A \cap B)$ (note: the event $A \cup B$ means either A or B or both are true, the event $A \cap B$ means both A and B are true).

Exercise 2 Each of the following random variables is a well-known type. Identify each by name:
a) an airplane has four engines, and each engine may independently fail (very small probability!). X is the number of engines that fail.
b) flies are randomly landing on my pizza at a steady average rate. X is the number of flies that land on my pizza in the next five minutes.
c) a spammer sends a fake email to a new address every second. X is the number of attempts until somebody responds.
d) a farm raises several hundred thousand chickens. X is the weight of a randomly selected chicken.

Exercise 3 A town has five hotels; three people arrive and each randomly and independently selects a hotel. Find the probability that exactly two of them stay in the same hotel.

Exercise 4 Find the mean of X, where the pdf is:

$$P(X = n) = (1 - p)^2 \, n \, p^{n-1}, \quad n = 1, 2, \ldots$$

[**Hint**: note that $\sum_{n=0}^{\infty} n(n-1) \, p^{n-2} = \frac{d^2}{dp^2} \sum_{n=0}^{\infty} p^n$].

Exercise 5 Randomly distribute r balls in n boxes. Find the probability that the first box is empty. Find the probability that the first two boxes are both empty.

$\boxed{\textbf{Exercise 6}}$ We start with a stick of length 1, and break it in two pieces at a randomly chosen position (chosen uniformly over its length). Find the mean length of the longer end of the broken stick.

$\boxed{\textbf{Exercise 7}}$ The current in a resistor is a random variable X. The pdf of X is $f(x) = e^{-(x-1)}$ for $x \geq 1$. The power dissipated in the resistor is $Y = X^2$. Find the pdf of Y.

$\boxed{\textbf{Exercise 8}}$ Derive the formula

$$\mathrm{VAR}[X_1 + X_2 + \cdots + X_n] = \sum_{k=1}^{n} \mathrm{VAR}[X_k] + 2 \sum_{i<j} \mathrm{COV}(X_i, X_j) \tag{1}$$

$\boxed{\textbf{Exercise 9}}$ Find a random number generator that generates uniformly on $[0, 1]$ (for example the command *rand* in Matlab). Using this generator, estimate the volume of the region under the surface

$$z = \frac{1}{3} \cosh \sqrt{x^2 + y^2}$$

and above the unit square $0 \leq x \leq 1$, $0 \leq y \leq 1$. [Note: generate three independent uniform random variables for each run, corresponding to the three coordinates of a random point in the unit cube. Do enough runs to be confident that you have an accurate estimate of the first two decimal places].

$\boxed{\textbf{Exercise 10}}$ In class we considered this problem: "An urn contains n Red balls and m Black balls. Suppose that k balls are withdrawn from the urn, and let X be the number of Red balls among these. Find $\mathbb{E}[X]$ assuming (i) replacement, and (ii) no replacement." Using the same reasoning as in class, compute $\mathrm{VAR}[X]$ assuming (i) replacement, and (ii) no replacement. [Hint: use the formula from Exercise 8 above. The answers will be different for the two cases].

$\boxed{\textbf{Exercise 11}}$ A typing firm has three typists A,B and C. The number of errors per 100 pages made by typist A is a Poisson random variable with mean 2.6; the number of errors per 100 pages made by typist B is a Poisson random variable with mean 3; the number of errors per 100 pages made by typist C is a Poisson random variable with mean 3.4. A manuscript of 300 pages is sent to the firm. Let X denote the number of errors in the typed manuscript.

a) Assume that one typist is randomly selected to do all the work. Find the mean and variance of X.

b) Assume instead that the work is divided into three equal parts which are given to the three typists. Find the mean and variance of X in this case.