MATH 7241 Fall 2022: Problem Set #6

Due date: Friday November 4

Reading: relevant background material for these problems can be found on Canvas 'Notes 4: Finite Markov Chains', 'FSHMC'. Also Grinstead and Snell Chapter 11.

$\boxed{\textbf{Exercise 1}}$ In each case below, determine whether or not the chain is reversible (note: the condition for reversibility is $w_i p_{ij} = w_j\, p_{ji}$ for all states i, j).

$$(a) \qquad P = \begin{pmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{pmatrix}$$

$$(b) \qquad P = \begin{pmatrix} 3/4 & 1/4 & 0 \\ 0 & 2/3 & 1/3 \\ 1/4 & 1/4 & 1/2 \end{pmatrix}$$

$\boxed{\textbf{Exercise 2}}$ A box contains N balls, some red and some blue. At each step, a coin is flipped with probability p of coming up Heads, and probability $1-p$ of coming up Tails. If the coin comes up Heads, a ball is chosen at random from the box and is replaced by a red ball; if the coin comes up Tails then a ball is chosen randomly from the box and replaced by a blue ball. Let X_n denote the number of red balls in the box after n steps. Find the transition matrix for the chain $\{X_n\}$, and find the stationary distribution. [Hint: is the chain reversible?] Compute $\lim_{n\to\infty} \mathrm{E}[X_n]$, and explain why you could have guessed your answer without doing the calculation.

$\boxed{\textbf{Exercise 3}}$ A knight moves randomly on a standard 8×8 chessboard. At each step it chooses at random one of the possible legal moves available. Given that the knight starts in a corner of the chessboard, find the expected number of steps until its first return to its initial position. [Hint: model the knight's position using a Markov chain, and try to show that the chain is reversible]

Exercise 4 Grinstead and Snell p.423, #7.

Exercise 5 Grinstead and Snell p.423, #9.

Exercise 6 Grinstead and Snell p.427, #24.

Exercise 7 Recall the Gambler's Ruin Problem: a random walk on the integers $\{0, 1 \ldots, N\}$ with probability p to jump right and $q = 1 - p$ to jump left at every step, and absorbing states at 0 and N. Starting at $X_0 = k$, the probability to reach N before reaching 0 is

$$P_k = \frac{1 - (q/p)^k}{1 - (q/p)^N} \quad \text{for } p \neq \frac{1}{2}, \qquad\qquad P_k = \frac{k}{N} \quad \text{for } p = \frac{1}{2}.$$

Starting at $X_0 = k$, let R_k be the probability to reach state N without returning to state k. Use the Gambler's Ruin result to compute R_k for all $k = 0, \ldots, N$, and for all $0 < p < 1$. [Hint: condition on the first step and use the formula given above].

$\boxed{\textbf{Exercise 8}}$ The Markov chain $X = \{X_n\}$ is defined on the state space $S = \{0, 1, 2, \dots\}$. The chain is irreducible, aperiodic and positive persistent, with stationary distribution $\{w_k\}$ $(k = 0, 1, 2, \dots)$. Let $Y = \{Y_n\}$ be an independent copy of X, and define $Z = (X, Y)$.

a). Write down the transition matrix for Z, and compute its stationary distribution (your answer will depend on w).

b). Given that the chain Z starts at the state (k, k) (so that $X_0 = Y_0 = k$), find an expression for the expected number of steps until the first return to (k, k).