Problem 1.

(1) Verifying the solution of (1):

We have the following differential equation:

$$\frac{dp}{dt} = Kp^2$$

We can rewrite this equation as:

$$\frac{dp}{p^2} = K\,dt$$

Integrating both sides:

$$\int \frac{dp}{p^2} = \int K\,dt$$

Solving the left-hand integral:

$$-\frac{1}{p} = Kt + C_1$$

Multiplying both sides by -p:

$$1 = -Ktp + C_1 p$$

Solving for p:

$$p(t) = 1\,/\,(C_1 - Kt)$$

Using the initial condition $p(0) = 500\ million$, we get:

$$C_1 = \frac{1}{500}$$

Therefore, the solution is given by:

$$p(t) = \frac{1}{(1/500 - Kt)}$$

Comparing this solution with the given solution, we can see that they are equivalent.

Writing down the expression for y vs. t:

We have:

$$y = 1/p = 1/500 - Kt$$

Rearranging this expression:

$$Kt = 1/500 - y$$

$$t = (1/K)(1/500 - y)$$

Problem 2.

7 points fit a polynomial of degree 2,

$$y = a_0 + a_1(x - x_0) + a_2(x - x_0)^2$$

Vandermonde System to solve for *a* coefficient:

$$\begin{pmatrix} 1 & (x_{-3} - x_0) & (x_{-3} - x_0)^2 \\ 1 & (x_{-2} - x_0) & (x_{-2} - x_0)^2 \\ 1 & (x_{-1} - x_0) & (x_{-1} - x_0)^2 \\ 1 & 0 & 0 \\ 1 & (x_1 - x_0) & (x_1 - x_0)^2 \\ 1 & (x_2 - x_0) & (x_2 - x_0)^2 \\ 1 & (x_3 - x_0) & (x_3 - x_0)^2 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} y_{-3} \\ y_{-2} \\ y_{-1} \\ y_0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

Sample points are equally spaced:

$(x_{-3} - x_0) = -3h;\ (x_3 - x_0) = 3h$

$(x_{-2} - x_0) = -2h;\ (x_2 - x_0) = 2h$

$(x_{-1} - x_0) = -h;\ (x_1 - x_0) = h$

$$\therefore \begin{pmatrix} 1 & -3 & 9 \\ 1 & -2 & 4 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} y_{-3} \\ y_{-2} \\ y_{-1} \\ y_0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

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X =

    1   -3    9

    1   -2    4

    1   -1    1

    1    0    0

    1    1    1

    1    2    4

    1    3    9

>> 21*inv(X'*X)*X'

ans =
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-2.0000	3.0000	6.0000	7.0000	6.0000	3.0000	-2.0000
-2.2500	-1.5000	-0.7500	0	0.7500	1.5000	2.2500
1.2500	0	-0.7500	-1.0000	-0.7500	0	1.2500

Computation for a coefficients:

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \frac{1}{21}\begin{pmatrix} -2 & 3 & 6 & 7 & 6 & 3 & -2 \\ -2.25 & -1.5 & -0.75 & 0 & 0.75 & 1.5 & 2.25 \\ 1.25 & 0 & -0.75 & -1 & -0.75 & 0 & 1.25 \end{pmatrix}\begin{pmatrix} y_{-3} \\ y_{-2} \\ y_{-1} \\ y_0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

Recall Savitzky-Golay filter identifies constant term with filtered value:

$$\tilde{y}_n = a_0 = \frac{1}{21}(-2y_{-3} + 3y_{-2} + 6y_{-1} + 7y_0 + 6y_1 + 3y_2 - 2y_3)$$

7-point quadratic SG filter kernel:

$$w = \frac{1}{21}(-2, 3, 6, 7, 6, 3, -2)$$