MATH5210 HOMEWORK - 6 ABHILASHA JAIN

1. We consider,

$$r(t) = a + bt$$

Now, to minimize,

$$f(a,b) = (a + 24b - 47)^2 + (a + 27b - 30)^2 + (a + 22b - 35)^2 + (a + 24b - 38)^2$$

$$f(a) = 2(a + 24b - 47) + 2(a + 27b - 30) + 2(a + 22b - 35) + 2(a + 24b - 38) = 0.$$

$$8a + 194b - 300 = 0$$

$$8a + 194b = 300 \quad — ①$$

$$f(b) = 24 \times 2(a + 24b - 47) + 27 \times 2(a + 27b - 30) + 22 \times 2(a + 22b - 35) + 24 \times 2(a + 24b - 38) = 0$$

$$196a + 4730b = 7240 \quad — ②$$

① & ②

$$\Rightarrow \quad a = -\frac{1805}{23}, \quad b = \frac{110}{23}$$

$$\Rightarrow \quad r(t) = \frac{110t - 1805}{23}$$

2. Using least squares.

x^2	4	16	9	25
y^2	1	4	4	1

Here $x^2 = 4$ (least)

$y^2 = 1$ (least)

$\therefore \quad x^2 = 4y^2$

$\Rightarrow \quad x^2 = (2y)^2$

$\Rightarrow \quad x^2 - (2y)^2 = 0$

$\Rightarrow (x+2y)(x-2y) = 0$

$\Rightarrow \quad x = 2y \quad , \quad x = -2y$

$\therefore$ The above solution is the function of x that approximates y.

3. Let $W = \text{span}\{1, x\}$.
To find a least squares approximation to e^{-x} by an element in W.

$B = \{1, x\}$. Find an orthonormal basis for W by Gram - Schmidt orthonormalization process

$$f_1 = 1, \quad f_2 = x - \frac{\langle x, 1\rangle}{\langle 1, 1\rangle}.1 = x - \frac{\int_1^2 x\,dx}{\int_1^2 1\,dx}.1$$

$$= x - \frac{\left[\frac{x^2}{2}\right]_1^2}{[x]_1^2}.1$$

$$= x - \frac{(2 - 1/2)}{(2-1)}.1$$

$$= x - \frac{3}{2}.1$$

Now,

Best approximation is, $\frac{\langle e^{-x}, 1\rangle}{\langle 1, 1\rangle}.1 + \frac{\langle e^{-x}, x - 3/2\rangle}{\langle x-3/2, x-3/2\rangle}.\left(x - \frac{3}{2}\right)$

$$\langle e^{-x}, 1\rangle = \int_1^2 e^{-x}\,dx = \left[e^{-x}\right]_1^2 = -e^{-2} + e^{-1}$$

$$\langle 1, 1\rangle = \int_1^2 1\,dx = [x]_1^2 = 2-1 = 1$$

$$\langle e^{-x}, x-\tfrac{3}{2}\rangle = \int_1^2 e^{-x}\left(x-\tfrac{3}{2}\right)dx \qquad \text{(by parts.)}$$

$$= \left[\left(x-\tfrac{3}{2}\right)\frac{e^{-x}}{-1}\right]_1^2 + \int_1^2 e^{-x}\,dx$$

$$= \frac{-1}{2}e^{-2} + \left(-\frac{1}{2}\right)e^{-1} + \left[\frac{e^{-x}}{-1}\right]_1^2$$

$$= \frac{-1}{2}\left(e^{-2}+e^{-1}\right) + \left(-\left[e^{-2}-e^{-1}\right]\right)$$

$$= -\frac{3}{2}e^{-2} + \frac{1}{2}e^{-1}$$

$$\left\langle x-\tfrac{3}{2}, x-\tfrac{3}{2}\right\rangle = \int_1^2 \left(x-\tfrac{3}{2}\right)^2 dx$$

$$= \int_1^2 \left(x^2 - 3x + \tfrac{9}{4}\right)dx$$

$$= \left[\frac{x^3}{3} - \frac{3x^2}{2} + \frac{9}{4}x\right]_1^2$$

$$= \frac{8}{3} - 6 + \frac{9}{2} - \frac{1}{3} + \frac{3}{2} - \frac{9}{4}$$

$$= \frac{1}{12}$$

So, best approximation is:

$$(e^{-1}-e^{-2}).1 + \frac{\frac{1}{2}e^{-1}-\frac{3}{2}e^{-2}}{1/12}\left(x-\frac{3}{2}\right)$$

$$=\left(\frac{1}{e}-\frac{1}{e^2}\right)+6\left(\frac{1}{e}-\frac{3}{e^2}\right)\left(x-\frac{3}{2}\right)$$

$$=\frac{1}{e}-\frac{1}{e^2}+\left(\frac{6}{e}-\frac{18}{e^2}\right)x-9\left(\frac{1}{e}-\frac{3}{e^2}\right)$$

$$=\left(\frac{26}{e^2}-\frac{8}{e}\right)+\left(\frac{6}{e}-\frac{18}{e^2}\right)x$$

4. Considering the basis for the vector space $P_2(x)$.

$$B=\{1, x, x^2\}$$

Suppose $\sin x \in P_2(x)$

$\therefore \quad a(1)+b(x)+c(x^2)=\sin x$

Applying inner product on both sides using each element of Basis $B=\{1,x,x^2\}$

$$\langle 1, a(1)+b(x)+c(x^2)\rangle = \langle 1, \sin x\rangle$$

Applying linearity property

$$a\langle 1,1\rangle + b\langle 1,x\rangle + c\langle 1,x^2\rangle = \langle 1, \sin x\rangle$$

$$\Rightarrow a\int_0^{\pi}(1).1\,dx + b\int_0^{\pi} 1.(x)\,dx + c\int_0^{\pi}(1)\,x^2\,dx = \int_0^{\pi}(1.)\sin x\,dx$$

$$\Rightarrow a[x]_0^{\pi} + b\left[\frac{x^2}{2}\right]_0^{\pi} + c\left[\frac{x^3}{3}\right]_0^{\pi} = -[\cos x]_0^{\pi}$$

$$\Rightarrow a\pi + b\left[\frac{\pi^2}{2}\right] + c\left[\frac{\pi^3}{3}\right] = \cos 0 - \cos\pi.$$

$$\Rightarrow a\pi + b\frac{\pi^2}{2} + c\frac{\pi^3}{3} = 2 \quad \text{— } (1)$$

$$\langle x, a(1)+b(x)+c(x^2)\rangle = \langle x, \sin x\rangle$$

Applying linearity property.

$$a\langle x,1\rangle + b\langle x,x\rangle + c\langle x,x^2\rangle = \langle x, \sin x\rangle$$

$$\Rightarrow a\int_0^{\pi}(x)1\,dx + b\int_0^{\pi}(x)\,x\,dx + c\int_0^{\pi}(x)\,x^2\,dx = \int_0^{\pi}(x)\sin x\,dx$$

$$\Rightarrow a\int_0^{\pi}x\,dx + b\int_0^{\pi}x^2\,dx + c\int_0^{\pi}x^3\,dx = \int_0^{\pi}x.\sin x\,dx.$$

$$\Rightarrow a\left[\frac{x^2}{2}\right]_0^{\pi} + b\left[\frac{x^3}{3}\right]_0^{\pi} + c\left[\frac{x^4}{4}\right]_0^{\pi} = \left[-x\cos x + \sin x\right]_0^{\pi}$$

$$\Rightarrow a\left[\frac{\pi^2}{2}\right] + b\left[\frac{\pi^3}{3}\right] + c\left[\frac{\pi^4}{4}\right] = \left[-\pi\cos\pi + \sin\pi - (-(0)\cos 0 + \sin 0)\right]$$

$$\Rightarrow a\frac{\pi^2}{2} + b\frac{\pi^3}{3} + c\frac{\pi^4}{4} = \pi. \quad \text{--- } ②$$

$$\langle x^2, a(1) + b(x) + c(x^2)\rangle = \langle x^2, \sin x\rangle$$

Applying linearity property

$$a\langle x^2, 1\rangle + b\langle x^2, x\rangle + c\langle x^2, x^2\rangle = \langle x^2, \sin x\rangle$$

$$\Rightarrow a\int_0^{\pi}(x^2).1dx + b\int_0^{\pi}(x^2)x\,dx + c\int_0^{\pi}(x^2)x^2\,dx = \int_0^{\pi} x^2. \sin x\,dx.$$

$$\Rightarrow a\int_0^{\pi}x^2dx + b\int_0^{\pi}x^3dx + c\int_0^{\pi}x^4dx = \int_0^{\pi}x^2\sin x\,dx.$$

$$\Rightarrow a\left[\frac{x^3}{3}\right]_0^{\pi} + b\left[\frac{x^4}{4}\right]_0^{\pi} + c\left[\frac{x^5}{5}\right]_0^{\pi} = \left[-(-2+x^2)\cos x + 2x\sin x\right]_0^{\pi}$$

$$\Rightarrow a\left[\frac{\pi^3}{3}\right] + b\left[\frac{\pi^4}{4}\right] + c\left[\frac{\pi^5}{5}\right] = \left[-(-2+\pi^2)\cos\pi + 2\pi\sin\pi + (-2+0^2)\cos 0 - 2(0)\sin 0\right]$$

$$\Rightarrow a\frac{\pi^3}{3} + b\frac{\pi^4}{4} + c\frac{\pi^5}{5} = -2 + \pi^2 - 2$$

$$\Rightarrow \quad a\frac{\pi^3}{3} + b\frac{\pi^4}{4} + c\frac{\pi^5}{5} = -4 + \pi^2 \quad -③$$

from ①, ② and ③ solving for a, b, c

we get,

$$a = \frac{12(-10+\pi^2)}{\pi^3}$$

$$b = -\frac{60(-12+\pi^2)}{\pi^4}$$

$$c = \frac{60(-12+\pi^2)}{\pi^5}$$

Substituting a, b, c in

$$a(1) + b(x) + c(x^2) = \sin x$$

$$\frac{12(-10+\pi^2)}{\pi^3}(1) - \frac{60(-12+\pi^2)}{\pi^4}(x) + \frac{60(-12+\pi^2)}{\pi^5}(x^2) = \sin x.$$