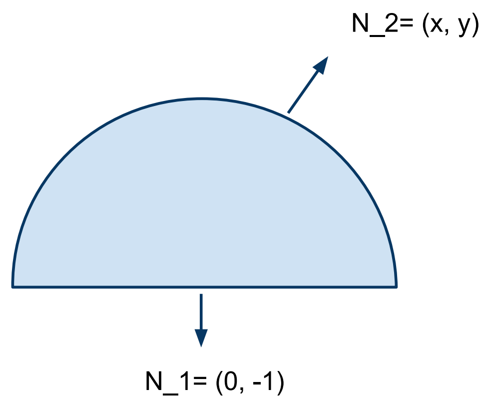
Solution of ODE8 Problem 5



Let D be the region formed by intersecting the unit disk $x^2 + y^2 \leq 1$ with the upper half-plane. There are two types of outward normal vectors:

$N_1 = (0, -1)$ when $y = 0$ and $|x| \leq 1$ (this vector is the same everywhere in the bottom segment);

$N_2 = (x, y)$ when $x^2 + y^2 \leq 1$ and $y \geq 0$ (this vector depends on where in the semicircle we are).

a) $(f, g) \cdot N_1 = (-x, -y^2) \cdot (0, -1) = y^2 = 0$,

which means that the vector field is tangent at the bottom of the semidisk.

On N_2, we have $(f, g) \cdot N_1 = (-x, -y^2) \cdot (x, y) = -x^2 - y^3 \leq 0$ (because $y \geq 0$ on the semidisk).

So D is a trapping region.

b) $(f, g) \cdot N_1 = (y, -x) \cdot (0, -1) = x$.

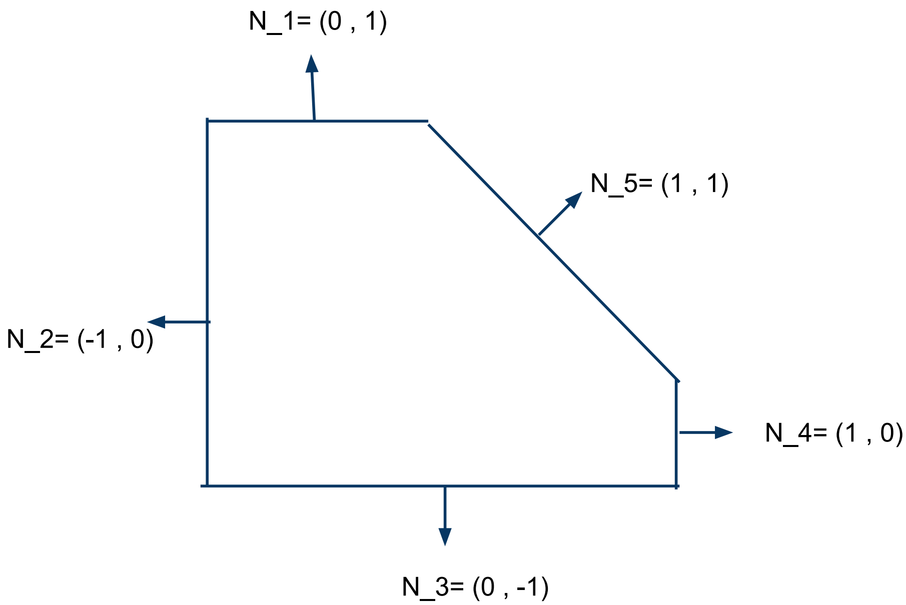
This expression is positive when x is positive, and is negative when x is negative. So the region D cannot be a trapping region, because trajectories exit it in the first quadrant. (In fact, this is a harmonic oscillator, and the solutions are clockwise-oriented circles.)

c) $(f, g) \cdot N_1 = (-xy^2 - x, x^2y) \cdot (0, -1) = -x^2y = 0$, which means that the vector field is tangent at the bottom of the semidisk.

On N_2, we have $(f, g) \cdot N_2 = (-xy^2 - x, x^2y) \cdot (x, y) = -x^2y^2 - x^2 + x^2y^2 = -x^2 \leq 0$.

So D is a trapping region.

Solution of ODE8 Problem 8



a) The outward normal vector of each segment is shown in the picture.

In segment 1,

$$(\dot{x}, \dot{y}) \cdot N_1 = (-x + ay + x^2y, b - ay - x^2y) \cdot (0, 1) = b - ay - x^2y = -\frac{b}{a}x^2 \leq 0,$$

since $y = \frac{b}{a}$.

In segment 2,

$$(\dot{x}, \dot{y}) \cdot N_2 = (-x + ay + x^2y, b - ay - x^2y) \cdot (-1, 0) = x - ay - x^2y = -ay \leq 0,$$

since $x = 0$.

In segment 3,

$$(\dot{x}, \dot{y}) \cdot N_3 = (-x + ay + x^2y, b - ay - x^2y) \cdot (0, -1) = -b + ay + x^2y = -b < 0,$$

since $y = 0$.

In segment 4,

$$(\dot{x}, \dot{y}) \cdot N_4 = (-x + ay + x^2y, b - ay - x^2y) \cdot (1, 0) = -x + ay + x^2y \leq 0,$$

since $y \leq \frac{x}{a+x^2}$.

In segment 5,

$$(\dot{x}, \dot{y}) \cdot N_5 = (-x + ay + x^2y, b - ay - x^2y) \cdot (1, 1) = -x + ay + x^2y + b - ay - x^2y = -x + b \leq 0,$$

a since $x \geq b$.

So the arrows on the boundary are really as shown, and the region is a trapping one.

b) Since at $(b, \frac{b}{a+b^2})$, we have

$$(\dot{x}, \dot{y}) = (-b + a\frac{b}{a+b^2} + b^2\frac{b}{a+b^2}, b - a\frac{b}{a+b^2} - b^2\frac{b}{a+b^2}) = (0,0),$$

$(b, \frac{b}{a+b^2})$ is a steady state.

c) In this part, we'll show that $(b, \frac{b}{a+b^2})$ is a repelling node.

To show this, we need to compute the Jacobian matrix of

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -x + ay + x^2y \\ b - ay - x^2y \end{pmatrix}$$

at $(b, \frac{b}{a+b^2})$. This is:

$$J = \begin{pmatrix} -1 + 2b\frac{b}{a+b^2} & a + b^2 \\ -2b\frac{b}{a+b^2} & -(a+b^2) \end{pmatrix}.$$

We see that

$$\operatorname{tr}(J) = -1 + 2b\frac{b}{a+b^2} - (a+b^2) = -\frac{b^4 + b^2(2a-1) + a + a^2}{a+b^2} > 0$$

since $b^4 + b^2(2a-1) + a + a^2 < 0$. Hence the steady state is a repelling node.

We have a bounded trapping region and a repelling node inside the region, so by the Poincaré Bendixson theorem, there exists a periodic orbit in the region.

d) $a = .01$ and $b = 0.1$ satisfy $b^4 + b^2(2a-1) + a + a^2 < 0$.