Lab
Fourier analysis

MATH 5110: Applied Linear Algebra and Matrix Analysis

Northeastern University
Author: Chris King

1 Vector space of functions

Let $PC[a, b]$ denote the set of all piecewise continuous real-valued functions whose domain of definition is the interval $[a, b]$. It is a fact that $PC[a, b]$ is a vector space over the real numbers (the proof involves checking that the axioms for a vector space hold; the most important properties are that for every $f, g \in PC[a, b]$ and $s \in \mathbb{R}$, both sf and $f + g$ are in $PC[a, b]$). Clearly the dimension of $PC[a, b]$ is greater than n for every integer n, so we say that $PC[a, b]$ is an infinite-dimensional vector space.

[Technical aside for those interested: A function f is said to have a *jump discontinuity* at c if both one-sided limits $\lim_{x\to c-} f(x)$ and $\lim_{x\to c+} f(x)$ exist and are not equal. A function f is *piecewise continuous* if f is continuous except possibly at a finite set of points where it has jump discontinuities.]

We will be interested in the space $PC[-\pi, \pi]$, consisting of piecewise continuous functions on the interval $[-\pi, \pi]$. Examples are $\sin(t)$, $\cos(t)$, t, $|t|$ (all continuous), also

$$f(t) = \begin{cases} -1 & -\pi \le t \le 0 \\ 1 & 0 < t \le \pi \end{cases}$$

For any two functions f, g in $PC[-\pi, \pi]$ we define

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\, g(t)\, dt \tag{1}$$

It is a fact that $\langle \cdot, \cdot \rangle$ is an inner product on $PC[-\pi, \pi]$. This means that it satisfies the following properties:

- $\langle f, g \rangle = \langle g, f \rangle$
- $\langle f + g, h \rangle = \langle f, h \rangle + \langle g, h \rangle$
- $\langle cf, g \rangle = c \langle f, g \rangle$ for all scalars c
- $\langle f, f \rangle > 0$ for all nonzero f in $PC[-\pi, \pi]$

The definition (1) should be viewed as the infinite-dimensional extension of the usual notion of the inner product on $\mathbb{R}^n$. Using the inner product we define the length or norm of a function as

$$\|f\| = \sqrt{\langle f, f\rangle}$$

and the angle between functions f, g as

$$\theta = \cos^{-1} \frac{\langle f, g\rangle}{\|f\| \, \|g\|}$$

In particular f, g are orthogonal if $\langle f, g\rangle = 0$. We will often work with finite-dimensional subspaces of $PC[-\pi, \pi]$, in which case the usual notions of basis and orthogonal projection will apply.

2 Subspaces of trigonometric functions

For each integer n we define the subspace $T_n \subset PC[-\pi, \pi]$ to be the span of the functions $\{1, \sin t, \cos t, \sin(2t), \cos(2t), \ldots, \sin(nt), \cos(nt)\}$. So the space T_n consists of all functions of the form

$$f(t) = a + b_1 \sin t + c_1 \cos t + \cdots + b_n \sin(nt) + c_n \cos(nt)$$

These are called the trigonometric polynomials of order less than or equal to n. Recall the following integration formulas:

$$\begin{aligned}
\int_{-\pi}^{\pi} \sin(kt) \cos(mt)\, dt &= 0 \quad \text{for integers } k, m \\
\int_{-\pi}^{\pi} \sin(kt) \sin(mt)\, dt &= 0 \quad \text{for distinct integers } k, m \\
\int_{-\pi}^{\pi} \cos(kt) \cos(mt)\, dt &= 0 \quad \text{for distinct integers } k, m
\end{aligned}$$

These formulas imply that the functions $\{1, \sin t, \cos t, \sin(2t), \cos(2t), \ldots, \sin(nt), \cos(nt)\}$ are all orthogonal and hence linearly independent. Thus the dimension of T_n is $2n + 1$. Furthermore for all positive integers m,

$$\int_{-\pi}^{\pi} \sin^2(mt)\, dt = \int_{-\pi}^{\pi} \cos^2(mt)\, dt = \pi$$

and hence the functions $\sin t, \cos t, \sin(2t), \cos(2t), \ldots, \sin(nt), \cos(nt)$ are unit vectors in $PC[-\pi, \pi]$. The norm of the constant function 1 is $\sqrt{2}$, so we conclude that the functions

$$\left\{ \frac{1}{\sqrt{2}}, \sin t, \cos t, \sin(2t), \cos(2t), \ldots, \sin(nt), \cos(nt) \right\}$$

form an orthonormal basis of T_n.

3 Fourier coefficients

If f is a piecewise continuous function on $[-\pi, \pi]$ then its projection onto T_n is

$$\operatorname{proj}_{T_n} f(t) = a_0 \frac{1}{\sqrt{2}} + b_1 \sin t + c_1 \cos t + \cdots + b_n \sin(nt) + c_n \cos(nt)$$

where

$$
\begin{aligned}
b_k &= \langle f(t), \sin(kt) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\ \sin(kt)\, dt \\
c_k &= \langle f(t), \cos(kt) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\ \cos(kt)\, dt \\
a_0 &= \langle f(t), \frac{1}{\sqrt{2}} \rangle = \frac{1}{\sqrt{2}\pi} \int_{-\pi}^{\pi} f(t)\, dt
\end{aligned}
$$

The numbers $\{a_0, b_k, c_k\}$ are called the Fourier coefficients of the function f. The function

$$f_n(t) = a_0 \frac{1}{\sqrt{2}} + b_1\ \sin t + c_1\ \cos t + \cdots + b_n\ \sin(nt) + c_n\ \cos(nt)$$

is called the nth order Fourier approximation of f. The function $b_k\ \sin(kt) + c_k\ \cos(kt)$ is called the kth harmonic of f. This can be rewritten as

$$b_k\ \sin(kt) + c_k\ \cos(kt) = A_k\ \sin(k(t + \delta_k))$$

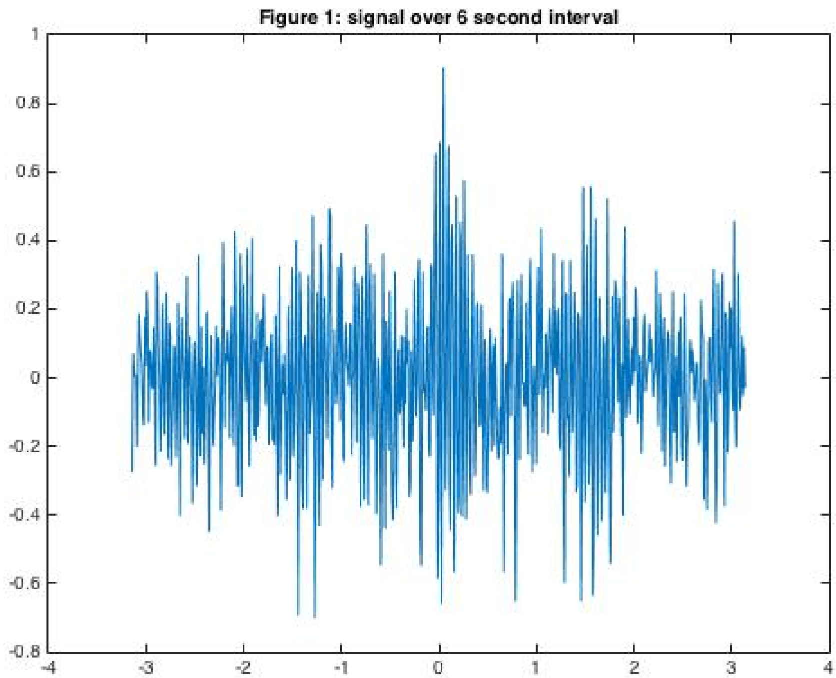
where $A_k = \sqrt{b_k^2 + c_k^2}$ is the amplitude and δ_k is the phase shift.

3.1 Task 1

The file HDN_spectrum.xlsx contains the 48 largest terms in the spectral decomposition of the famous opening guitar chord in the Beatles song ‘A hard day’s night’. The first column is the frequency (in Hz) and the second column is the amplitude (arbitrary units). Let f_i be the frequency and A_i the amplitude in row i. For simplicity we assume that the phase shifts are zero. Then the signal is

$$S_1(t) = \sum_{i=1}^{48} A_i\ \sin(f_i\, t), \quad -\pi \le t \le \pi$$

Plot the function $S_1(t)$ over the interval $[-\pi, \pi]$. You should get something like the plot in Figure 1.

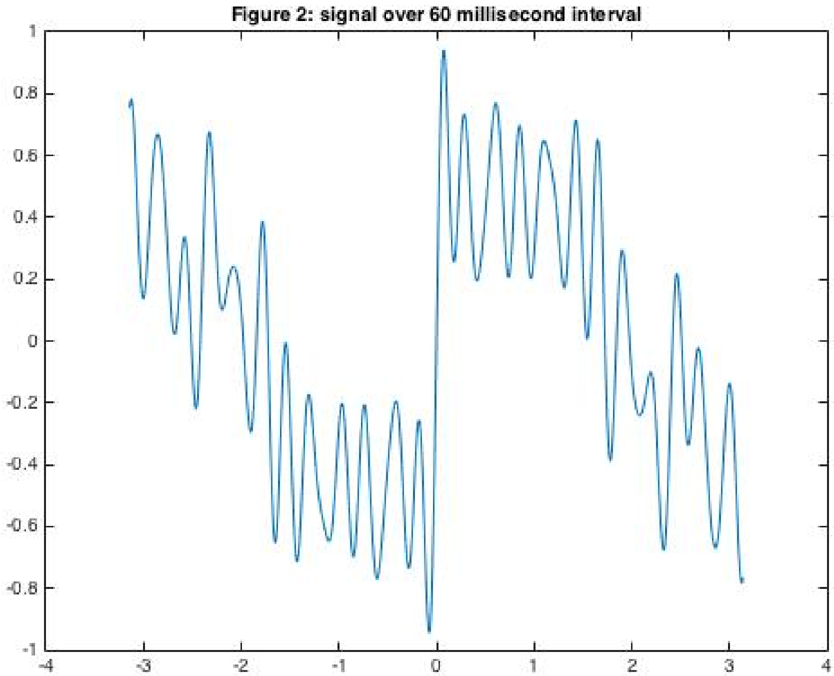


3.2 Task 2

Now rescale time by a factor 0.01 to get the function over a 60ms interval: define

$$S_2(t) = \sum_{i=1}^{48} A_i \sin(f_i\, t/100), \quad -\pi \le t \le \pi$$

Plot the function $S_2(t)$ over the interval $[-\pi, \pi]$, you should get something like the plot in Figure 2.



3.3 Task 3

Let $S_2^{(n)}(t)$ denote the nth order Fourier approximation as described above. Compute and plot the functions $S_2^{(n)}(t)$, for $n = 10, 20, 30, 40, 50, 60$. Note: in Matlab the function trapz can be used to compute integrals when the integrand is given as a numerical array.

3.4 Task 4

The error of the nth order Fourier approximation is computed as

$$E_n = \frac{1}{\pi} \int_{-\pi}^{\pi} |S_2(t) - S_2^{(n)}(t)|^2 \, dt$$

Find the smallest integer n such that

$$E_n \leq 0.005$$