MATH 5131 Fall 2021 **Take-Home Test 2**

Print your NAME: **Score:** **/100**

In order to receive full credit, SHOW ALL YOUR WORK. **Box your answer.**

You may work in groups.

1. (20 pts) Consider the spreading of a highly communicable disease on an isolated island with population size N. A portion of the population travels abroad and returns to the island infected with the disease. You would like to predict the number of people X who will be infected by some time t. Consider the following model:

$$\frac{dX}{dt} = kX(N - X)$$

a. List two major assumptions implicit in the preceding model. How reasonable are the assumptions you list?

b. Graph dX/dt versus X.

c. Graph X versus t if the initial number of infections is $X_1 < N/2$. Graph X versus t if the initial number of infections is $X_2 > N/2$.

d. Solve the model for X as a function of t.

e. From Part d, find the limit of X as t approaches infinity.

f. Consider an island with a population of 5000. At various times during the epidemic the number of people infected was recorded as follows:

t (days)	2	6	10
X (people infected)	1887	4087	4853
$\ln(X/(N - X))$	-0.5	1.5	3.5

Do the data collected support the given model?

g. Use the results in Part f to estimate the constants in the model, and predict the number of people who will be infected by $t = 12$ days.

2. (20 pts) The Department of Fish and Game in a certain state is planning to issue deer hunting permits. It is known that if the deer population falls below a certain level m, then the deer will become extinct. It is also known that if the deer population goes above the maximum carrying capacity M, the population will decrease to M.

a. Discuss the reasonableness of the following model for the growth rate of the deer population as a function of time

$$\frac{dP}{dt} = kP(M - P)(P - m)$$

where P is the population of the deer and k is a constant of proportionality. Include a graph of dP/dt versus P as a part of your discussion.

b. Explain how this growth rate model differs from the logistic model $dP/dt = kP(M - P)$. Is it better or worse than the logistic model?

c. Show that if $P > M$ for all t, then the limit of $P(t)$ as $t \to \infty$ is M.

d. Discuss what happens if $P < m$ for all t.

e. Assuming that $m < P < M$ for all t, solve the differential equation of the model.

f. Graphically discuss the solutions to the differential equations. What are the equilibrium points of the model? Explain the dependence of the equilibrium level of P on the initial conditions. How many deer hunting permits should be issued?

3. (20 pts) Consider the competitive hunter model defined by

$$\frac{dx}{dt} = a(1 - x/k_1)x - bxy$$

$$\frac{dy}{dt} = m(1 - y/k_2)y - nxy$$

where x represents the trout population and y the bass population.

a. What are assumptions are implicitly being made about the growth of trout and bass in the absence of competition?

b. Interpret the constants a, b, m, n, k_1, k_2 in terms of the physical problem.

c. Perform a graphical analysis and answer the following questions:

i. What are the possible equilibrium levels?

ii. Is coexistence possible?

iii. Pick several typical starting points and sketch typical trajectories in the phase plane.

iv. Interpret the outcomes predicted by your graphical analysis in terms of the constants a, b, m, n, k_1, and k_2.

Note: When you get to Step i, you should realize that at least five cases exists. You will need to analyze all five cases. One case is when the lines are coincident.

4. (20 pts) Consider two species whose survival depends upon their mutual cooperation. An example would be species of bee that feeds primarily on the nectar of one plant species and simultaneously pollinates that plant. One simple model of this mutualism is given by the autonomous system:

$$\frac{dx}{dt} = -ax + bxy$$

$$\frac{dy}{dt} = -my + nxy$$

a. What assumptions are implicitly being made about the growth of each species in the absence of cooperation?

b. Interpret the constants a, b, m, and n in terms of the physical problem?

c. What are the equilibrium levels?

d. Perform a graphical analysis and indicate the trajectory directions in the phase plane.

e. Find the analytic solution and sketch typical trajectories in the phase plane.

f. Interpret the outcomes predicted by your graphical analysis. Do you believe the model is realistic?

5. (20 pts) In many instances, it is the adult members of the prey who are chiefly attacked by the predators, while the young members are better protected, either by their smaller size, or by their living in a different station. Let x_1 be the number of adult prey, x_2 the number of young prey, and y the number of predators. Then,

$$\frac{dx_1}{dt} = -a_1x_1 + a_2x_2 - bx_1y$$

$$\frac{dx_2}{dt} = nx_1 - (a_1 + a_2)x_2$$

$$\frac{dy}{dt} = -cy + dx_1y$$

where a_2x_2 represents the number of young (per unit time) growing into adults, and n represents the birth rate proportional to the number of adults. Find all equilibrium solutions of the system.