Northeastern University, Department of Mathematics

MATH G5110: Applied Linear Algebra and Matrix Analysis. (Fall 2020)

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Reading for §7: Review of Polynomials

Definition 1. • A **polynomial** with coefficients in field $\mathbb{F}$ is a function $p : \mathbb{F} \to \mathbb{F}$ of the form

$$p(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n.$$

- If $a_m \neq 0$, we say that the polynomial $p(t)$ has **degree** n
- A number λ is called a **root** of the polynomial $p(t)$ if $p(\lambda) = 0$.

Proposition 2. *λ is root of a degree n polynomial $p(t)$ if and only if there is a degree $n-1$ polynomial $q(t)$ such that*

$$p(t) = (t - \lambda)q(t)$$

Proof. Backward direction"⇐" is obvious. Let's show forward direction "⇒"
Since λ is root, we have $a_0 + a_1\lambda + a_2\lambda^2 + \cdots + a_n\lambda^n = 0$.
So,

$$\begin{aligned} p(t) &= p(t) - a_0 + a_1\lambda + a_2\lambda^2 + \cdots + a_n\lambda^n \\ &= a_1(t-\lambda) + a_2(t^2 - \lambda^2) + \cdots + a_n(t^n - \lambda^n) \\ &= (t-\lambda)\big[a_1 + a_2(t+\lambda) + \cdots + a_n(t^{n-1} + t^{n-2}\lambda + \cdots + \lambda^{n-1})\big] \\ &= (t-\lambda)q(t) \end{aligned}$$

Here $q(t)$ has degree $n-1$ since $a_n \neq 0$. □

Proposition 3. *A degree n polynomial has at most n (distinct) roots in $\mathbb{F}$.*

Proof. From the above theorem by induction. □

Proposition 4. *If $a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n = 0$ for all $t \in \mathbb{F}$, then $a_0 = a_1 = \cdots = a_n = 0$.*

Proof. Only zero polynomial $p = 0$ has infinitely many solutions. □

This means that $\{1, t, t^2, \ldots, t^n\}$ is independent in polynomial vector space P.

Proposition 5 (Division Algorithm). *Suppose $p(t)$ and $q(t)$ are non-zero polynomials. There exists polynomials $r(t)$ and $s(t)$ such that*

$$p(t) = s(t)q(t) + r(t)$$

and $\deg(r) < \deg(q)$.

Similar as integers, we can think this as divide $p(t)$ by $q(t)$ and the remainder is $r(t)$.

Theorem 6 (Fundamental Theorem of Algebra). *Every polynomial $p(t)$ of degree $n \geq 1$ with complex coefficient has n roots. That is*

$$p(t) = a_n(t - z_1)(t - z_2) \cdots (t - z_n)$$

The above factorization is unique if we do not count the order.

Proposition 7. *Suppose $p(t)$ is a polynomial with real coefficients. If $z \in \mathbb{C}$ is a root of $p(t)$, then the conjugate of z is also a root.*

Proof. If $p(z) = 0$, then take the conjugate of both sides, we have $\overline{p(z)} = 0$ and hence $p(\bar{z}) = 0$ by properties of conjugate. $\square$

Theorem 8 (Real roots). *Every polynomial $p(t)$ of degree $n \geq 1$ with real coefficient can be factorized as*

$$p(t) = a_n(t - c_1)(t - c_2) \cdots (t - c_p)(t^2 + a_1 t + b_1)(t^2 + a_2 t + b_2) \cdots (t^2 + a_m t + b_m)$$

where all numbers in the factorization are real numbers and $a_i^2 < 4b_i$ for $i = 1, 2, ..., m$

Proof. First $p(t) = a_n(t - z_1)(t - z_2) \cdots (t - z_n)$ has been factored as complex roots. Since complex roots come in pairs for real polynomials. Suppose $z = a + bi$ is a root, then $p(t)$ contains a real polynomial factor $(t - z)(t - \bar{z}) = t^2 - 2at + |z|^2$. $\square$

Proposition 9 (Rational roots). *Let $p(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$ be a polynomial of degree $n \geq 1$ with integer coefficient. Suppose rational number $\frac{p}{q}$ is a root of $p(t)$ such that $(p, q) = 1$, then $p|a_0$ and $q|a_n$.*

Complex vectors

We list some basic knowledge of complex numbers.

- Just as $\mathbb{R}$ denotes the set of real numbers, we will use $\mathbb{C}$ to denote the set of complex numbers $z = a + ib$. Here $i = \sqrt{-1}$, and a and b are real numbers called/denoted

$$a = Re(z) = \textbf{real part of } z$$
$$b = Im(z) = \textbf{imaginary part of } z$$

- The **complex conjugate** of $z = a + bi \in \mathbb{C}$ is $\bar{z} := a - bi$
- The **absolute value** of z is $|z| = \sqrt{a^2 + b^2}$.
- $z\bar{z} = |z|^2$
- Complex numbers $\mathbb{C}$ can be viewed as a 2-dimensional $\mathbb{R}$-vector space $\mathbb{R}^2$. Furthermore, there is a product operation on it.

Similarly to $\mathbb{R}^n$ denoting n-dimensional real vectors (that is $n \times 1$ matrices with real number entries), so $\mathbb{C}^n$ shall denote n-dimensional complex vectors, that is $n \times 1$ matrices with complex number entries.

If A is an $m \times n$ matrix and $\vec{x} \in \mathbb{C}^n$ an n-dimensional complex vector, then $A\vec{x}$ is defined in exactly the same way as it is in the case of a real n-dimensional vector $\vec{x}$.

Definition 10 (Real and Imaginary Parts of Vectors)**.** Let $\vec{x} \in \mathbb{C}^n$ be a complex n-dimensional vector.

- The **complex conjugate vector** $\overline{\vec{x}}$ of $\vec{x}$ is the vector made up from the complex conjugate entries of $\vec{x}$.
- The **real part** of $\vec{x}$, denoted $Re(\vec{x})$ is the (real) vector consisting of the real parts of the entries of $\vec{x}$.
- The **imaginary part** of $\vec{x}$, denoted $Im(\vec{x})$ is the (real) vector consisting of the imaginary parts of the entries of $\vec{x}$.

Note that

$$\vec{x} = Re(\vec{x}) + i \cdot Im(\vec{x}) \quad \text{and} \quad \overline{\vec{x}} = Re(\vec{x}) - i \cdot Im(\vec{x}).$$

Remark 11. Replacing the complex vector $\vec{x}$ from the previous definition by a complex $m \times n$ matrix A, leads to the

- Complex conjugate matrix $\overline{A}$.
- Real part $Re(A)$ of A.
- Imaginary part $Im(A)$ of A.

The analogues of above equations apply, in addition to

$$\overline{\lambda \cdot \vec{x}} = \overline{\lambda} \cdot \overline{\vec{x}}, \qquad \overline{A \cdot \vec{x}} = \overline{A} \cdot \overline{\vec{x}}, \qquad \overline{A \cdot B} = \overline{A} \cdot \overline{B}.$$