MATH 7241: Probability 1 FALL 2022 QUIZ 1

NAME: SOLUTIONS

1) A town has five hotels. Four people arrive and each person randomly and independently selects a hotel. Find the probability that at least two of the hotels are not selected by anybody.

$N =$ # hotels not selected by any one

$$\mathbb{P}(N \geq 2) = 1 - \mathbb{P}(N < 2)$$

$$= 1 - \mathbb{P}(N = 1)$$

$$= 1 - \frac{(5)(4)(3)(2)}{5^4}$$

$$= 0.81$$

2) Disease X occurs in 2% of the population. A new test for X is available. If a patient has the disease then the test returns a positive result in 90% of cases; if a patient does not have the disease then the test returns a negative result in 80% of cases. Suppose that Annie receives a positive test result. What is the probability that Annie has disease X?

$D = \{$patient has disease$\}$

$T = \{$test is positive$\}$

$$P(D|T) = \frac{P(T|D)\,P(D)}{P(T|D)\,P(D) + P(T|D^c)\,P(D^c)}$$

$$= \frac{(0.9)(0.02)}{(0.9)(0.02) + (0.2)(0.98)}$$

$$= 0.084$$

3) Let X_i, $i = 1, \ldots, 100$ be IID random variables, with mean $\mu = 1$ and variance $\sigma^2 = 4$. Use the Central Limit Theorem to estimate the probability that $X_1 + \cdots + X_{100}$ is less than 80. [Your answer should be a number; use the normal tables provided].

$$Y = X_1 + X_2 + \cdots + X_{100}$$

$$\mu = E[Y] = 1 \qquad\qquad n = 100$$

$$\sigma^2 = VAR[Y] = 4.$$

$$Z = \frac{Y - n\mu}{\sqrt{n}\,\sigma} = \frac{Y - 100}{20.}$$

$$\mathbb{P}(Y < 80) = \mathbb{P}(Z < -1)$$

$$= \mathbb{P}(Z > 1)$$

$$= 1 - \mathbb{P}(Z \leq 1)$$

$$= 1 - 0.8413$$

$$= 0.1587$$

4) X and Y are continuous random variables. X is uniformly distributed on the interval $[-1, 1]$, and it is known that

$$\mathrm{E}[Y \mid X = x] = 1 - x^2, \qquad \mathrm{E}[Y^2 \mid X = x] = 1 + x^2, \qquad \text{for } -1 \leq x \leq 1.$$

Compute the mean and variance of Y.

$$X \sim U[-1, 1].$$

$$\mathbb{E}[Y \mid X = x] = 1 - x^2$$

$$\Rightarrow \mathbb{E}[Y] = \int \mathbb{E}[Y \mid X = x] f_X(x)\, dx$$

$$= \int_{-1}^{1} (1 - x^2) \tfrac{1}{2}\, dx$$

$$= \frac{2}{3}$$

$$\mathbb{E}[Y^2 \mid X = x] = 1 + x^2.$$

$$\Rightarrow \mathbb{E}[Y^2] = \int \mathbb{E}[Y^2 \mid X = x] f_X(x)\, dx$$

$$= \int_{-1}^{1} (1 + x^2) \tfrac{1}{2}\, dx$$

$$= \frac{4}{3}.$$

$$\Rightarrow \mathrm{VAR}[Y] = \frac{4}{3} - \left(\frac{2}{3}\right)^2 = \frac{8}{9} = 0.889.$$

CHALLENGE: only attempt this if you are bored!!

5) A standard deck of 52 cards is randomly shuffled and divided into four equal piles, each with 13 cards. Find the probability that each pile contains one Ace.

4 piles, 13 cards in each pile.

Distribute 4 Aces into 52 possible slots, with 13 slots in each pile.

Total # ways $= (52)(51)(50)(49)$ (count number of ordered ways)

ways to get 1 in each pile

$= (52)(39)(26)(13)$

All outcomes equally likely

$\Rightarrow \mathbb{P}(1 \text{ Ace in each pile})$

$$= \frac{(52)(39)(26)(13)}{(52)(51)(50)(49)}$$

$$= \frac{6(13)^3}{(51)(50)(49)}$$

$$= 0.1055$$

Tables of the Normal Distribution

Probability Content from -oo to Z

Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

Far Right Tail Probabilities

Z	P{Z to oo}	Z	P{Z to oo}	Z	P{Z to oo}	Z	P{Z to oo}
2.0	0.02275	3.0	0.001350	4.0	0.00003167	5.0	2.867 E-7
2.1	0.01786	3.1	0.0009676	4.1	0.00002066	5.5	1.899 E-8
2.2	0.01390	3.2	0.0006871	4.2	0.00001335	6.0	9.866 E-10
2.3	0.01072	3.3	0.0004834	4.3	0.00000854	6.5	4.016 E-11
2.4	0.00820	3.4	0.0003369	4.4	0.000005413	7.0	1.280 E-12
2.5	0.00621	3.5	0.0002326	4.5	0.000003398	7.5	3.191 E-14
2.6	0.004661	3.6	0.0001591	4.6	0.000002112	8.0	6.221 E-16
2.7	0.003467	3.7	0.0001078	4.7	0.000001300	8.5	9.480 E-18
2.8	0.002555	3.8	0.00007235	4.8	7.933 E-7	9.0	1.129 E-19
2.9	0.001866	3.9	0.00004810	4.9	4.792 E-7	9.5	1.049 E-21