Qualitative Theory of Differential Equations

Let us consider the differential equation

$$\vec{x}\,' = \vec{f}(t, \vec{x}) \qquad (1)$$

where

$$\vec{x} = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \text{ and } \vec{f}(t,\vec{x}) = \begin{pmatrix} f_1(t, x_1, \dots, x_n) \\ \vdots \\ f_n(t, x_1, \dots, x_n) \end{pmatrix}$$

Unfortunately, there are no known methods of solving Eq. (1). This, of course, is very disappointing. However, it is not necessary, in most applications, to find the solutions of (1) explicitly. For example, let $x_1(t)$ and $x_2(t)$ denote the populations, at time t, of two species competing amongst themselves for the limited food and living space in their microcosm. Suppose, moreover, that the rates of growth of $x_1(t)$ and $x_2(t)$ are governed by the differential equation (1). In this case, we are not really interested in the qualitative properties of $x_1(t)$ and $x_2(t)$. Specially, we wish to answer the following questions.

1. Do there exist value ξ_1 and ξ_2 at which the two species coexist together in a steady state? That is to say, are there numbers ξ_1, ξ_2 such that

$$x_1(t) \equiv \xi_1 \ , \ x_2(t) \equiv \xi_2$$

is a solution of (1)? Such values ξ_1, ξ_2, if they exist, are called equilibrium points of (1).

2. Suppose that the two species are coexisting in equilibrium. Suddenly, we add a few members of species 1 to the microcosm. Will $x_1(t)$ and $x_2(t)$ remain close to their equilibrium values for all future time? Or perhaps the extra few members give species 1 a large advantage and it will proceed to annihilate species 2.

3. Suppose that x_1 and x_2 arbitrary values at $t=0$. What happens as t approaches infinity? Will one species ultimately emerge victorious, or will the struggle for existence end in a draw?

More generally, we are interested in determining the following properties of solutions of (1).

1. Do there exist equilibrium values

$$\vec{x}^{\,0} = \begin{pmatrix} x_1^0 \\ x_2^0 \\ \vdots \\ x_n^0 \end{pmatrix}$$

for which $\vec{x}(t) \equiv \vec{x}^{\,0}$ is a solution of (1)?

2. Let $\vec{\varphi}(t)$ be a solution of (1). Suppose that $\vec{\psi}(t)$ is a second solution with $\vec{\psi}(0)$ very close to $\vec{\varphi}(0)$; that is, $\psi_j(0)$ is very close to $\varphi_j(0)$, $j=1,\dots,n$. Will $\vec{\psi}(t)$ remain close to $\vec{\varphi}(t)$ for all future time, or will $\vec{\psi}(t)$ diverge from $\vec{\varphi}(t)$ as t approaches infinity? This question is often referred to as the problem of stability. It is the most fundamental problem in the qualitative theory of differential equations, and has occupied the attention of many mathematicians for the past hundred years.

3. What happens to solutions $\vec{x}(t)$ of (1) as t approaches infinity? Do all solutions approach equilibrium values? If they don't approach equilibrium values, do they at least approach a periodic solution?

Remarkably, we can often give satisfactory answers to these questions, even though we cannot solve Equation (1) explicitly.

Indeed, the first question can be answered immediately. Observe that $\vec{x}'(t)$ is identically zero if $\vec{x}(t) \equiv \vec{x}^{\circ}$. Hence, $\vec{x}^{\circ}$ is an equilibrium value of (1), if, and only if,

$$\vec{f}(t, \vec{x}^{\circ}) \equiv \vec{0}. \qquad (2)$$

Example 1. Find all equilibrium values of the system of diff. equations

$$\begin{cases} \dfrac{dx_1}{dt} = 1 - x_2 \\ \dfrac{dx_2}{dt} = x_1^3 + x_2 \end{cases}$$

Solution

$$\vec{x}^{\circ} = \begin{pmatrix} x_1^{\circ} \\ x_2^{\circ} \end{pmatrix}$$

is an equilibrium value if, and only if,

$$\begin{cases} 1 - x_2^{\circ} = 0 \\ x_1^{\circ 3} + x_2^{\circ} = 0 \end{cases} \iff \begin{array}{l} x_2^{\circ} = 1 \\ x_1^{\circ} = -\sqrt[3]{x_2^{\circ}} = -\sqrt[3]{1} = -1 \end{array}$$

Hence $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ is the only equilibrium value of this system.

Example 2 Find all equilibrium solutions of the system

$$\begin{cases} \dfrac{dx}{dt} = (x-1)(y-1) \\ \dfrac{dy}{dt} = (x+1)(y+1) \end{cases}$$

Solution $\begin{cases} (x^{\circ}-1)(y^{\circ}-1) = 0 \\ (x^{\circ}+1)(y^{\circ}+1) = 0 \end{cases} \iff \left\{ \begin{pmatrix} x^{\circ} = 1 \\ y^{\circ} = -1 \end{pmatrix} \text{ or } \begin{pmatrix} y_0 = 1 \\ x_0 = -1 \end{pmatrix} \right.$

Hence, $\begin{pmatrix} x = 1 \\ y = -1 \end{pmatrix}$ and $\begin{pmatrix} x = -1 \\ y = 1 \end{pmatrix}$ are the equilibrium solutions of the system.

The question of stability is of paramount importance in all physical applications, since we can never measure initial conditions exactly. For example, consider the case of a particle of mass one slug attached to an elastic spring of force constant 1 lb/ft, which is moving in a frictionless medium. In addition, an external force $F(t) = \cos 2t$ lb is acting on the particle.

Let $y(t)$ denote the position of the particle relative to its equilibrium position. Then

$$\frac{d^2y}{dt^2} + y = \cos 2t \qquad \left[\underset{m}{[\text{inertia}]} \frac{d^2y}{dt^2} + [\text{damping}]\frac{dy}{dt} + [\text{stiffness}]\, y = F(t) \right]$$

We convert this second-order equation into a system of two first-order equations by setting

$$\begin{cases} x_1 = y \\ x_2 = y' \end{cases} \Rightarrow \begin{cases} \dfrac{dx_1}{dt} = x_2 \\ \dfrac{dx_2}{dt} = -x_1 + \cos 2t \end{cases} \qquad (3)$$

The functions $y_1(t) = \sin t$ and $y_2(t) = \cos t$ are two independent solutions of the homogeneous equation

$$y'' + y = 0 \qquad [\, r^2 + 1 = 0 \Rightarrow r_{1,2} = \pm i,\ y_i(t) = e^{r_i t} \,]$$

Moreover, $y_p = -\frac{1}{3}\cos 2t$ is a particular solution of the nonhomogeneous equation. Therefore, every solution

$$\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

of (3) is of the form

$$\vec{x}(t) = c_1 \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} + c_2 \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + \begin{pmatrix} -\frac{1}{3}\cos 2t \\ \frac{2}{3}\sin 2t \end{pmatrix} \qquad (4)$$

At time $t=0$ we measure the position and velocity of the particle and obtain

$$y(0) = 1, \quad y'(0) = 0.$$

This implies that $c_1 = 0$ and $c_2 = 4/3$. Consequently, the position and velocity of the particle for all future time are given by the equation

$$\begin{pmatrix} y(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} \frac{4}{3}\cos t - \frac{1}{3}\cos 2t \\ -\frac{4}{3}\sin t + \frac{2}{3}\sin 2t \end{pmatrix} \qquad (5)$$

However, suppose that our measurements permit an error of magnitude 10^{-4}. Will the position and velocity of the particle remain close to the values predicted by (5)?

The answer to this question had better be yes, for otherwise, Newtonian mechanics would be of no practical value to us. Fortunately, it is quite easy to show, in this case, that the position and velocity of the particle remain very close to the values predicted by (5).

Let $\hat{y}(t)$ and $\hat{y}'(t)$ denote the true values of $y(t)$ and $y'(t)$ respectively. Clearly,

$$y(t) - \hat{y}(t) = \left(\tfrac{4}{3} - c_2\right)\cos t - c_1 \sin t$$

$$y'(t) - \hat{y}'(t) = -c_1 \cos t - \left(\tfrac{4}{3} - c_2\right)\sin t$$

where c_1 and c_2 are two constants satisfying

$$-10^{-4} \le c_1 \le 10^{-4} \; ; \qquad \tfrac{4}{3} - 10^{-4} \le c_2 \le \tfrac{4}{3} + 10^{-4}$$

We can rewrite these equations in the form

$$y(t) - \hat{y}(t) = \left[c_1^2 + \left(\tfrac{4}{3} - c_2\right)^2\right]^{1/2} \cos(t - \delta_1), \quad \tan\delta_1 = \frac{c_1}{c_2 - 4/3}$$

$$y'(t) - \hat{y}'(t) = \left[c_1^2 + \left(\tfrac{4}{3} - c_2\right)^2\right]^{1/2} \cos(t - \delta_2), \quad \tan\delta_2 = \frac{4/3 - c_2}{c_1}$$

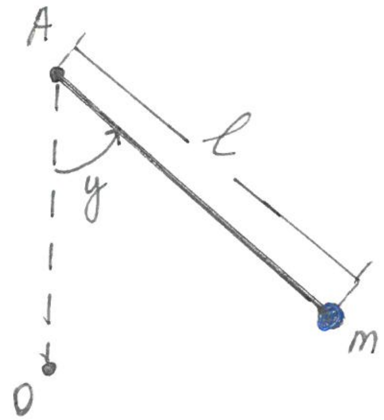
Hence, both $y(t) - \hat{y}(t)$ and $y'(t) - \hat{y}'(t)$ are bounded in absolute value by $\left[c_1^2 + \left(\tfrac{4}{3} - c_2\right)^2\right]^{1/2}$. This quantity is at most $\sqrt{2} \cdot 10^{-4}$.

Therefore, the true values of $y(t)$ and $y'(t)$ are indeed close to the values predicted by (5).

As a second example of the concept of stability, consider the case of a particle of mass m which is supported by a wire, or inelastic string, of length ℓ and of negligible mass. The wire is always straight, and the system is free to vibrate in a vertical plane. This configuration is usually referred to as a simple pendulum. The equation of motion of the pendulum is

$$\frac{d^2y}{dt^2} + \frac{g}{\ell}\sin y = 0$$

where y is the angle which the wire makes with the vertical line $\vec{AO}$ (see Figure 1).



Setting $x_1 = y$ and $x_2 = \frac{dy}{dt}$ we see that

$$\begin{cases} \dfrac{dx_1}{dt} = x_2 \\ \dfrac{dx_2}{dt} = -\dfrac{g}{\ell}\sin(x_1) \end{cases} \qquad (6)$$

The system of equations (6) has equilibrium solutions

$$\begin{pmatrix} x_1 = 0 \\ x_2 = 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_1 = \pi \\ x_2 = 0 \end{pmatrix}$$

If the pendulum is suspended in the upright position $y=\pi$ with zero velocity, then it will remain in this upright position for all future time.

These two equilibrium solutions have very different properties. If we disturb the pendulum slightly from the equilibrium position $x_1=0, x_2=0$, by either displacing it slightly, or giving it a small velocity, then it will execute small oscillations about $x_1=0$. On the other hand, if we disturb the pendulum slightly from the equilibrium position $x_1=\pi$, $x_2=0$, then it will either execute very large oscillations about $x_1=0$, or it will rotate around and around ad infinitum. Thus, the slightest disturbance causes the pendulum to deviate drastically from its equilibrium position $x_1=\pi$, $x_2=0$. Intuitively, we would say that the equilibrium value $x_1=0$, $x_2=0$ of (6) is <u>Stable</u>, while the equilibrium value $x_1=\pi$, $x_2=0$ of (6) is <u>unstable</u>.

The question of stability is usually very difficult to resolve, because we cannot solve (1) explicitly. The only case which is managable is when $\vec{f}(t,\vec{x})$ does not depend explicitly on t; that is, $\vec{f}$ is a function of $\vec{x}$ alone. Such differential equations are called <u>autonomous</u>. And even for autonomous differential equations, there are only two instances, generally, where we can completely resolve the stability question:

- The first case is when $\vec{f}(\vec{x}) = A\vec{x}$.

- The second case is when we are only interested in the stability of an equilibrium solution of

$$\vec{x}' = f(\vec{x}).$$

Question 3 is extremely important in many applications since an answer to this question is a prediction concerning the long time evolution of the system under consideration.

HW Exercises

In each of Problems 1-3, find all equilibrium values of the given system of differential equations

#1. $$\begin{cases} \dfrac{dx}{dt} = x - x^2 - 2xy \\ \dfrac{dy}{dt} = 2y - 2y^2 - 3xy \end{cases}$$

#2 $$\begin{cases} \dfrac{dx}{dt} = xy^2 - x \\ \dfrac{dy}{dt} = x \sin \pi y \end{cases}$$

#3 $$\begin{cases} \dfrac{dx}{dt} = -1 - y - e^x \\ \dfrac{dy}{dt} = x^2 + y(e^x - 1) \\ \dfrac{dz}{dt} = x + \sin z \end{cases}$$

#4. Consider the system of differential equations

$$\begin{cases} \dfrac{dx}{dt} = ax + by \\ \dfrac{dy}{dt} = cx + dy \end{cases} \qquad (*)$$

(i) Show that $x=0$, $y=0$ is the only equilibrium point of $(*)$ if $ad - bc \neq 0$.

(ii) Show that $(*)$ has a line of equilibrium points if $ad - bc = 0$.

#5. Let $x = x(t)$, $y = y(t)$ be the solution of the initial-value problem

$$\begin{cases} \dfrac{dx}{dt} = -x-y \\ \dfrac{dy}{dt} = 2x-y \end{cases} \qquad x(0)=y(0)=1.$$

Suppose that we make an error of magnitude 10^{-4} in measuring $x(0)$ and $y(0)$. What is the largest error we make in evaluating $x(t)$, $y(t)$ for $0 \le t < \infty$?