0.4 Absorbing chains

Definition 4 *A state i is* absorbing *if $p_{ii} = 1$. A chain is* absorbing *if for every state i there is an absorbing state which is accessible from i. A non-absorbing state in an absorbing chain is called a* transient *state.*

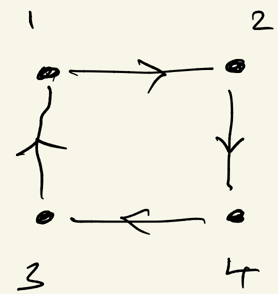
Consider an absorbing chain with r absorbing states and t transient states. Re-order the states so that the transient states come first, then the absorbing states. The transition matrix then has the form

$$P = \begin{pmatrix} Q & R \\ 0 & I \end{pmatrix} \tag{77}$$

where I is the $r \times r$ identity matrix.

Types of Markov Chains.

- Irreducible $\Leftrightarrow$ all states intercommunicate



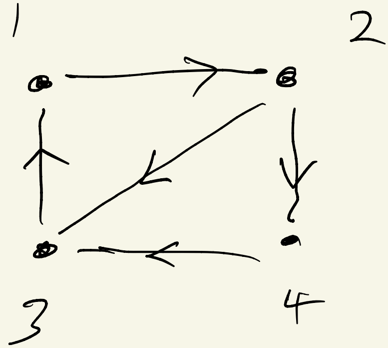
- Regular ? $\Leftrightarrow$ $P^4 > 0$?

  $\Leftrightarrow$ period of chain $= 1$

  NO: period of this chain is 4,

  therefore not regular.

- Add one jump:



$\{3, 4, 6, 7, 8, 9, 10, 11, \dots\}$

$\Rightarrow$ period $= 1$.

$\Rightarrow$ regular.

<u>State i:</u> d_i = period of state i

$$= \gcd\{n : P_{ii}(n) > 0\}$$

gcd = greatest common divisor.

e.g. $S_1 = \{3, 6, 12, 15, 21, 30, 33, \dots\}$

$\gcd(S_1) = 3$.

$S_2 = \{6, 12, 18, 19, 24, 30, \dots\}$

$\gcd(S_2) = 1$.

If $d_i = 1 \Rightarrow$ state i is aperiodic.

<u>Theorem</u>

If states i and j intercommunicate

Then $d_i = d_j$, i.e. same period.

<u>Corollary</u>

If chain is irreducible then all states have the same period, hence this is the period of the chain.

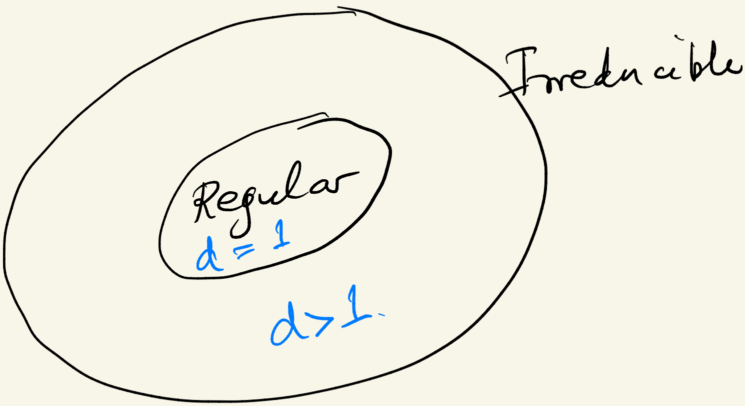
<u>Note</u>: if chain is irreducible and aperiodic then it is regular.

In this case there is a unique stationary distribution w, and

$$\mathbb{P}(X_n = j \mid X_0 = i) \longrightarrow w_j$$

as $n \to \infty$.

(<u>Perron-Frobenius</u>).

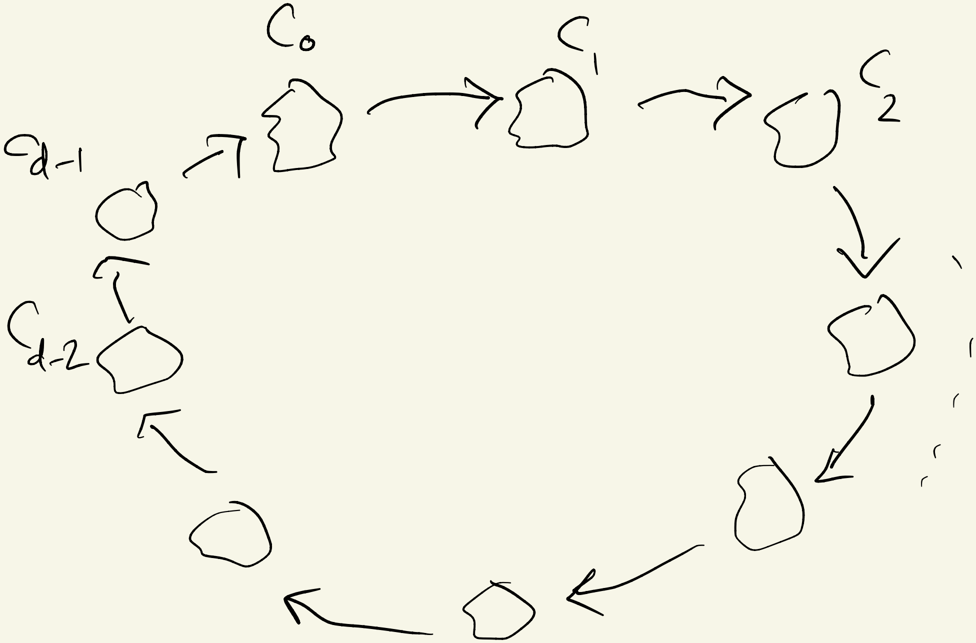


- Chain is irreducible but not regular, so period $d > 1$.

Theorem

Let P be irreducible with period d. Then can divide the states into d disjoint subsets $C_0, C_1, \ldots, C_{d-1}$, re-order the numbering of states according to $C_0, C_1, \ldots, C_{d-1}$. Then P has block matrix form.

$$P = \begin{array}{cc}
\left(\begin{array}{cccccc}
0 & Q_{01} & 0 & \cdots & 0 & 0 \\
0 & 0 & Q_{12} & \cdots & 0 & 0 \\
0 & 0 & 0 & \ddots & & 0 \\
\vdots & & & & \ddots & \vdots \\
\vdots & & & & & Q_{d-2,d-1} \\
Q_{d-1,0} & 0 & \cdots & & 0 & 0
\end{array}\right) &
\begin{array}{c} C_0 \\ C_1 \\ C_2 \\ \vdots \\ \vdots \\ C_{d-1} \end{array} \\
\begin{array}{cccccc} C_0 & C_1 & C_2 & \cdots & & C_{d-1} \end{array} &
\end{array}$$



The periodic structure becomes apparent here.

$$P^d = \begin{array}{cc} \begin{pmatrix} T_0 & 0 & 0 & \cdots & 0 \\ 0 & T_1 & 0 & \cdots & 0 \\ 0 & 0 & T_2 & & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \cdots & T_{d-1} \end{pmatrix} & \begin{array}{c} C_0 \\ C_1 \\ C_2 \\ \vdots \\ C_{d-1} \end{array} \end{array}$$

where

$$T_0 = Q_{01}\, Q_{12}\, Q_{23} \cdots Q_{d-1,0}$$

$$T_1 = Q_{12}\, Q_{23} \cdots Q_{d-1,0}\, Q_{01}$$

$\vdots$

etc.

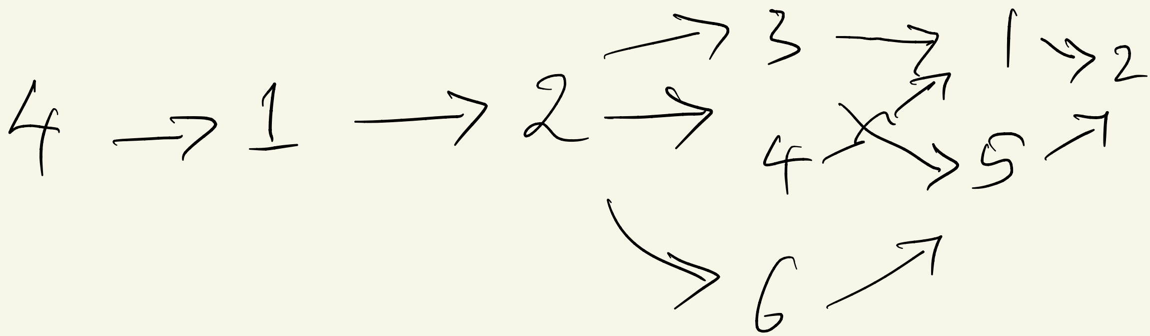
T_i is regular with stationary dist. $w^{(i)}$ (dimension is $|C_i|$ = # states in C_i).

The stationary distribution of the full chain is

$$w = \frac{1}{d}\left(w^{(0)}, w^{(1)}, w^{(2)}, \ldots, w^{(d-1)}\right).$$

<u>Example</u> 6×6 transition matrix

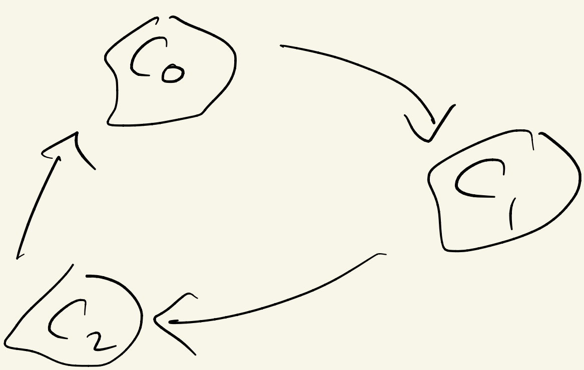
$$P = \begin{array}{cc} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \\ \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} & \end{array}$$



$$C_0 = \{2\}$$

$$C_1 = \{3, 4, 6\}$$

$$C_2 = \{1, 5\}$$



$d = 3$

irréducible

Re-order states 2, 3, 4, 6, 1, 5

$$
\hat{P} = \begin{array}{c c}
\left(\begin{array}{c|ccc|cc}
0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\
\hline
 & & & & \frac{1}{2} & \frac{1}{2} \\
0 & & 0 & & 1 & 0 \\
 & & & & 0 & 1 \\
\hline
1 & & & & & \\
1 & & 0 & & 0 &
\end{array}\right) & \begin{array}{c} 2 \\ 3 \\ 4 \\ 6 \\ 1 \\ 5 \end{array} \\
\begin{array}{cccccc} 2 & 3 & 4 & 6 & 1 & 5 \end{array} &
\end{array}
$$

$$
Q_{01} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}
$$

$$Q_{12} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$Q_{20} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

<u>Stationary distribution:</u>

$$P^3 = \begin{array}{cc} \left(\begin{array}{c|ccc|cc} 1 & & 0 & & 0 & \\ \hline & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & & \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \\ & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & & \\ \hline 0 & & 0 & & \frac{1}{2} & \frac{1}{2} \\ & & & & \frac{1}{2} & \frac{1}{2} \end{array}\right) & \begin{array}{c} 2 \\ 3 \\ 4 \\ 6 \\ 1 \\ 5 \end{array} \end{array}$$

$$T_0 = \begin{pmatrix} 1 \end{pmatrix} \qquad w^{(0)} = \begin{pmatrix} 1 \end{pmatrix}$$

$$T_1 = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \qquad w^{(1)} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

$$T_2 = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad w^{(2)} = \left(\frac{1}{2}, \frac{1}{2}\right)$$

$\Rightarrow$ stationary distribution is

$$w = \frac{1}{3}\left(1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{2}, \frac{1}{2}\right)$$

<u>Meaning</u>: since has period $d = 3$

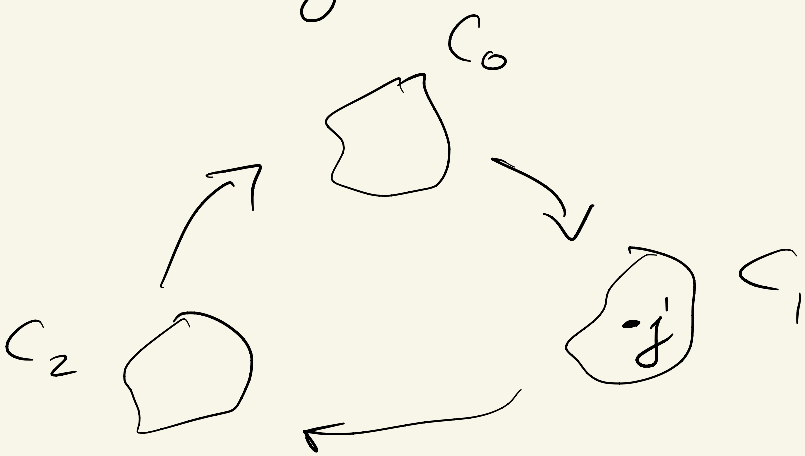
it is <u>not</u> regular,

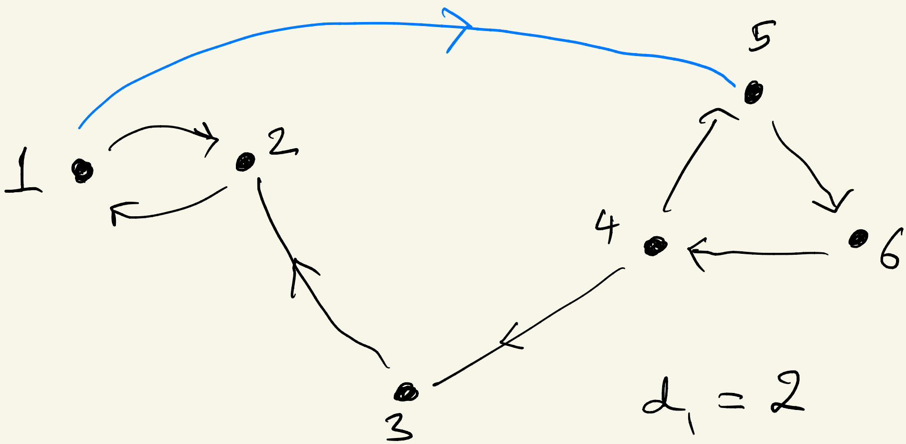
So Perron–Frobenius does not apply.

But we can say:

w_j = stat. dist. in state j

= long-run fraction of time that the chain spends in state j.





$d_1 = 2$

$d_3 = \gcd(\phi)$ N/A

$d_5 = 3$

$d_1 = 1$

$d_3 = 1$

$d_5 = \gcd\{3, 6, 8, 9, \dots\} = 1$

irreducible $\Rightarrow$ all same!

Example 7 *For the drunkard's walk, show that*

$$Q = \begin{pmatrix} 0 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1/2 & 0 \end{pmatrix}, \quad R = \begin{pmatrix} 1/2 & 0 \\ 0 & 0 \\ 0 & 1/2 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{78}$$

Absorbing Chains

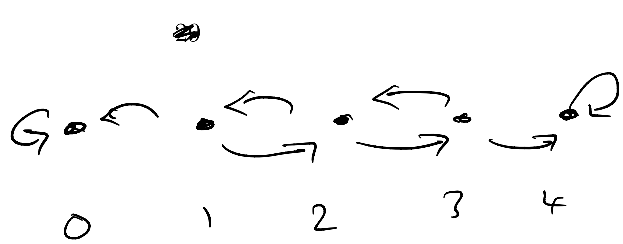
A chain is absorbing if

- it has at least one absorbing state ($P_{ii} = 1$)
- starting from any state, there is nonzero probability to reach an absorbing state.

$$\text{States} = \underset{\text{transient}}{\{T\}} \cup \underset{\text{absorbing}}{\{A\}}$$

$$P = \begin{array}{c} \\ \left(\begin{array}{c:c} Q & R \\ \hdashline O & I \end{array}\right) \end{array} \begin{array}{l} \text{transient} \\ \text{absorbing} \end{array}$$

Drunkard's Walk:



$$P = \begin{array}{c} \left(\begin{array}{ccc|cc} 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \hline & & & 1 & 0 \\ & O & & 0 & 1 \end{array}\right) \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 0 \\ 4 \end{array}$$

$$Q = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \qquad R = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

$$Q = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$Q^2 = \left(\frac{1}{2}\right)^2 \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$Q^3 = \frac{1}{8}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \frac{1}{8}\begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} = \frac{1}{2}Q$$

$$Q^{2n+1} = \left(\tfrac{1}{2}\right)^n Q.$$

So $\quad Q^n \to 0 \quad$ as $\quad n \to \infty.$

<u>Why?</u> $\quad (Q^n)_{ij} = \mathbb{P}(X_n = j \mid X_0 = i)$

where both i, j are transient states.

But the chain has smaller and smaller prob. to visit transient states.

So it makes sense that $Q^n \to 0.$

<u>Consequence:</u>

suppose there is a vector x such that

$$Qx = x.$$

$$\Rightarrow Q^2 x = Qx = x.$$

$$Q^3 x = Q^2 x = Qx = x.$$

$$\Rightarrow Q^n x = x \qquad \text{all } n.$$

But $Q^n \to 0$ as $n \to \infty$.

$\Rightarrow \quad x = 0.$

$$Qx = x \iff \underbrace{(Q - I)x = 0}$$

$\Rightarrow$ the kernel of $Q - I$ is the zero vector

$\Rightarrow (Q-I)^{-1}$ exists.

Define the fundamental matrix

$$N = (I - Q)^{-1}.$$

Simple calculations show that for all $n \geq 1$

$$P^n = \begin{pmatrix} Q^n & R_n \\ 0 & I \end{pmatrix} \tag{79}$$

where R_n is a complicated matrix depending on Q and R.

Lemma 2 *As $n \to \infty$,*

$$(Q^n)_{ij} \to 0$$

for all absorbing states i, j.

Proof: for a transient state i, there is an absorbing state k, an integer n_i and $\delta_i > 0$ such that

$$p_{ik}(n_i) = \delta_i > 0 \tag{80}$$

Let $n = \max n_i$, and $\delta = \min \delta_i$, then for any $i \in T$, there is a state $k \in R$ such that

$$p_{ik}(n) \geq \delta \tag{81}$$

Hence for any $i \in T$,

$$\sum_{j \in T} Q_{ij}^n = 1 - \sum_{k \in R} P_{ik}^n = 1 - \sum_{k \in R} p_{ik}(n) \leq 1 - \delta \tag{82}$$

In particular this means that $Q_{ij}^n \leq 1 - \delta$ for all $i, j \in T$. So for all $i \in T$ we get

$$\sum_{j \in T} Q_{ij}^{2n} = \sum_{k \in T} Q_{ik}^n \sum_{j \in T} Q_{kj}^n \leq (1 - \delta) \sum_{k \in T} Q_{ik}^n \leq (1 - \delta)^2 \tag{83}$$

This iterates to give

$$\sum_{j \in T} Q_{ij}^{kn} \to 0 \quad \text{as } k \to \infty \tag{84}$$

for all $i \in T$.

It remains to notice that

$$\sum_{j \in T} Q_{ij}^{m+1} = \sum_{k \in T} Q_{ik}^m \sum_{j \in T} Q_{kj} \leq \sum_{k \in T} Q_{ik}^m \tag{85}$$

and hence the sequence $\{\sum_{k \in T} Q_{ik}^m\}$ is monotone decreasing in m. Therefore

$$\sum_{j \in T} Q_{ij}^k \to 0 \quad \text{as } k \to \infty \tag{86}$$

for all $i \in T$, which proves the result.

$\boxed{\text{QED}}$

Notice what the result says: the probability of remaining in the transient states goes to zero, so eventually the chain must transition to the absorbing states. So the quantities of interest are related to the time (=number of steps) needed until the chain exits the transient states and enters the absorbing states, and the number of visits to other transient states.

Consider the equation

$$x = Qx \tag{87}$$

Applying Q to both sides we deduce that

$$x = Q^2 x \tag{88}$$

and iterating this leads to

$$x = Q^n x \tag{89}$$

for all n. Since $Q^n \to 0$ it follows that $x = 0$. Hence there is no nonzero solution of the equation $x = Qx$ and therefore the matrix $I - Q$ is non-singular and so invertible.

Define the fundamental matrix

$$N = (I - Q)^{-1} \tag{90}$$

Note that

$$(I + Q + Q^2 + \cdots + Q^n)(I - Q) = I - Q^{n+1} \tag{91}$$

and letting $n \to \infty$ we deduce that

$$N = I + Q + Q^2 + \cdots \tag{92}$$

Theorem 5 *Let i, j be transient states. Then*

(1) N_{ij} is the expected number of visits to state j starting from state i (counting initial state if $i = j$).

(2) $\sum_j N_{ij}$ is the expected number of steps of the chain, starting in state i, until it is absorbed.

(3) define the $t \times r$ matrix $B = NR$. Then B_{ik} is the probability that the chain is absorbed in state k, given that it started in state i.