Math 5110- Applied Linear Algebra-Fall 2022

Instructor: He Wang

Test 2.

Student Name: ____________________ __________/**50**

Rules and Instructions for Exams:

1. Unless otherwise specified, to receive full credits you must show **all** necessary work. The grading is based on your work shown. Only a final result from computer will receive zero point.

2. You need to finish the exam yourself. Any discussions with the other people will be considered as **academic dishonesty**. **Cheating, Unauthorized Collaboration, and Facilitating Academic Dishonesty are not allowed.** You can read a description of each here http://www.northeastern.edu/osccr/academic-integrity-policy/

3. This is an open exam. You are allowed to look at textbooks, and use a computer.

4. You are **not** allowed to discuss with any other people.

5. You are **not** allowed to ask questions on any internet platform.

6. For programming questions, please following the specific instruction on the use of libraries.

Notation: $\vec{x} \in \mathbb{R}^n$ means a column vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

1. (8 points) Let $\mathbb{R}^5$ be the Euclidean space with dot product. Let V be a subspace spanned by

$$\vec{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

(1) Apply the Gram-Schmidt process to find the **orthonormal** basis of V.

(2) Find a basis for the **orthogonal complement** of V.

(3) Find a formula to calculate the **shortest** distance from any point $\vec{x} \in \mathbb{R}^5$ to V. (You don't have to simplicify your formula.)

2. (8 points) Let M be a $(k+1)\times(k+1)$ matrix

$$M = \frac{1}{2k}\begin{bmatrix} k & 1 & 1 & \cdots & 1 & 1 \\ 1 & k & 1 & \cdots & 1 & 1 \\ 1 & 1 & k & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & k & 1 \\ 1 & 1 & 1 & \cdots & 1 & k \end{bmatrix}$$

All diagonal entries of M are k and all other entries of M are 1. (Clearly write the theorem you used and the precise result. Do not use decimal numbers.)

(1) What is the largest eigenvalue λ_{max} of M?

(2) What is the eigenvector corresponding to λ_{max}?

(3) Calculate $\lim\limits_{n\to\infty} M^n$.

(4) Calculate $\lim\limits_{n\to\infty} M^n\vec{v}$ if $\vec{v}$ is an distribution vector.

3. (8 points) Let $A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \end{bmatrix}$.

Clearly write the theorem you used. (You can use Matlab functions eig() and rref(). You can keep 4 decimal numbers in calculation.)

(1) What is the largest eigenvalue λ_{max} of M?

(2) What is the eigenvector corresponding to λ_{max}?

(3) Calculate $\lim\limits_{n\to\infty} \left(\dfrac{1}{\lambda_{max}} M \right)^n$.

4. (5 points) Let $P_2(\mathbb{R})$ be the inner product space with polynomials of degree less or equal than 2, where $\langle f, g\rangle$ is defined to be $\langle f, g\rangle = \int_0^1 f(x)g(x)dx$.

Let polynomials $f(x) = 1$ and $g(x) = 1 - x$.

(1) Find the angle α between $f(x)$ and $g(x)$. (Use the angle defined by the inner product.)

(2) Find $\text{proj}_f(g)$, the orthogonal projection of $g(x)$ onto $f(x)$.

(3) Write $g(x) = \text{proj}_f(g) + g^{\perp}$, where $\langle f, g^{\perp}\rangle = 0$. Explicitly write done $g^{\perp}$.

5. (3 points) Let $P_2(\mathbb{R})$ be the inner product space with polynomials of degree less or equal than 2, where $\langle f, g\rangle$ is defined to be $\langle f, g\rangle = \int_0^1 f(x)g(x)dx$.

Let S be the subspace of the inner product space $P_2(\mathbb{R})$ generated by the polynomials 6 and $6x$

Find a **basis** for the **orthogonal complement** of S.

6. (8 points) Find the **least squares approximation** to the function $f(x) = xe^x$ by a quadratic function $a + bx + cx^2$ in the interval $[0, 1]$.

(Hint: Use the distance is induced by the inner product $\langle f, g \rangle = \int_0^1 f(x)g(x)dx$. Use WolframAlpha https://www.wolframalpha.com/ to do the calculation of integrals if needed.)

7. (4 points) Answer the following questions. Prove your the true statement and provide a counter example for the false statement.

(1) Suppose that the columns of $M \in \mathbb{R}^{n\times n}$ are orthonormal. What is the determinant of M^2?

(2) Suppose A is any real $m \times n$ matrix and B is any real $n \times m$ matrix. Is $\det(AB) = \det(BA)$?

8. (6 points) Suppose $N \in \mathbb{R}^{n\times n}$ is a **nilpotent** matrix. Answer the following quesitons. Explain your reason. (You can refer any result from class.)

(1) Suppose $\vec{v}$ is an eigenvector of N, prove that $\vec{v}$ is is an eigenvector of $N^2 + 3N + 2I_n$?

(2) What are the eigenvalues of $N^2 + 3N + 2I_n$?

(3) Find the determinant $\det(N^2 + 3N + 2I_n)$.