Find the fundamental matrix **N**, and also **Nc** and **NR**. Interpret the results.

7 In Example 11.8, make states 0 and 4 into absorbing states. Find the fundamental matrix **N**, and also **Nc** and **NR**, for the resulting absorbing chain. Interpret the results.

8 In Example 11.13 (Drunkard's Walk) of this section, assume that the probability of a step to the right is 2/3, and a step to the left is 1/3. Find **N**, **Nc**, and **NR**. Compare these with the results of Example 11.15.

9 A process moves on the integers 1, 2, 3, 4, and 5. It starts at 1 and, on each successive step, moves to an integer greater than its present position, moving with equal probability to each of the remaining larger integers. State five is an absorbing state. Find the expected number of steps to reach state five.

10 Using the result of Exercise 9, make a conjecture for the form of the fundamental matrix if the process moves as in that exercise, except that it now moves on the integers from 1 to n. Test your conjecture for several different values of n. Can you conjecture an estimate for the expected number of steps to reach state n, for large n? (See Exercise 11 for a method of determining this expected number of steps.)

***11** Let b_k denote the expected number of steps to reach n from $n-k$, in the process described in Exercise 9.

(a) Define $b_0 = 0$. Show that for $k > 0$, we have

$$b_k = 1 + \frac{1}{k}\big(b_{k-1} + b_{k-2} + \cdots + b_0\big) \ .$$

(b) Let

$$f(x) = b_0 + b_1 x + b_2 x^2 + \cdots \ .$$

Using the recursion in part (a), show that $f(x)$ satisfies the differential equation

$$(1-x)^2 y' - (1-x)y - 1 = 0 \ .$$

(c) Show that the general solution of the differential equation in part (b) is

$$y = \frac{-\log(1-x)}{1-x} + \frac{c}{1-x} \ ,$$

where c is a constant.

(d) Use part (c) to show that

$$b_k = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} \ .$$

12 Three tanks fight a three-way duel. Tank A has probability 1/2 of destroying the tank at which it fires, tank B has probability 1/3 of destroying the tank at which it fires, and tank C has probability 1/6 of destroying the tank at which *problem* is the problem of finding the probability w_x of winning an amount T before losing everything, starting with state x. Show that this problem may be considered to be an absorbing Markov chain with states 0, 1, 2, ..., T with 0 and T absorbing states. Suppose that a gambler has probability $p = .48$ of winning on each play. Suppose, in addition, that the gambler starts with 50 dollars and that $T = 100$ dollars. Simulate this game 100 times and see how often the gambler is ruined. This estimates w_{50}.

24 Show that w_x of Exercise 23 satisfies the following conditions:

(a) $w_x = pw_{x+1} + qw_{x-1}$ for $x = 1, 2, \ldots, T-1$.

(b) $w_0 = 0$.

(c) $w_T = 1$.

Show that these conditions determine w_x. Show that, if $p = q = 1/2$, then

$$w_x = \frac{x}{T}$$

satisfies (a), (b), and (c) and hence is the solution. If $p \neq q$, show that

$$w_x = \frac{(q/p)^x - 1}{(q/p)^T - 1}$$

satisfies these conditions and hence gives the probability of the gambler winning.

25 Write a program to compute the probability w_x of Exercise 24 for given values of x, p, and T. Study the probability that the gambler will ruin the bank in a game that is only slightly unfavorable, say $p = .49$, if the bank has significantly more money than the gambler.

***26** We considered the two examples of the Drunkard's Walk corresponding to the cases $n = 4$ and $n = 5$ blocks (see Example 11.13). Verify that in these two examples the expected time to absorption, starting at x, is equal to $x(n - x)$. See if you can prove that this is true in general. *Hint*: Show that if $f(x)$ is the expected time to absorption then $f(0) = f(n) = 0$ and

$$f(x) = (1/2)f(x-1) + (1/2)f(x+1) + 1$$

for $0 < x < n$. Show that if $f_1(x)$ and $f_2(x)$ are two solutions, then their difference $g(x)$ is a solution of the equation

$$g(x) = (1/2)g(x-1) + (1/2)g(x+1)\ .$$

Also, $g(0) = g(n) = 0$. Show that it is not possible for $g(x)$ to have a strict maximum or a strict minimum at the point i, where $1 \le i \le n-1$. Use this to show that $g(i) = 0$ for all i. This shows that there is at most one solution. Then verify that the function $f(x) = x(n - x)$ is a solution.