MTH 7241 (Prof. King): FALL 2015: FINAL December, 2015

You may use your notes and the texts and handouts for the course. You may not use other resources, either online or offline. Do not discuss the problems with anybody until you have handed back the exam. Please contact me by email if you have any questions. Return the exam to me either by email or in person by 4:00pm on Monday December 14. Good luck!

1). Consider the following transition probability matrix for a Markov chain on 6 states:

$$P = \begin{pmatrix} 1/3 & 2/3 & 0 & 0 & 0 & 0 \\ 2/3 & 1/3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/4 & 3/4 & 0 & 0 \\ 0 & 0 & 1/5 & 4/5 & 0 & 0 \\ 1/4 & 0 & 1/4 & 0 & 1/4 & 1/4 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \end{pmatrix}$$

a). Label the states $\{1, 2, 3, 4, 5, 6\}$ in the order given in the matrix. Classify the states as persistent or transient.

b). Find a stationary distribution for each persistent class in the chain.

c). Compute the expected number of steps needed to first return to state 1, conditioned on starting in state 1.

d). Compute the expected number of steps needed to first reach state 2, conditioned on starting in state 1.

e). Suppose the chain starts in state 5. Find the probability that it eventually reaches state 2.

2). a). Suppose that n independent trials are to be performed, with probability of success equal to x on each trial. Find the mean of the number of successes that occur in these n trials. [Your answer will depend on x].

b). In a related experiment the probability of success is a random variable U which is uniform on [0,1], so its pdf is

$$f_U(x) = \begin{cases} 1 & \text{for } 0 \le x \le 1 \\ 0 & \text{else} \end{cases}$$

Suppose again that n trials are to be performed and that conditional on $\{U = x\}$ these trials will be independent with probability of success equal to x. Find the mean of the number of successes that occur in these n trials. [Hint: condition on U, use your answer to part (a), and then undo the conditioning on U]

c). For the same experiment as in part (b), find the variance of the number of successes that occur in these n trials.

3). Coin 1 has $P(\text{Heads}) = 0.6$, and Coin 2 has $P(\text{Heads}) = 0.5$. Coin 1 is flipped until the first time Tails appears. Then Coin 2 is flipped until the first time Tails appears. This process is continued, flipping and switching each time Tails appears.

a). The process starts with flipping Coin 1. Find the expected number of flips until the first switch to Coin 2 occurs.

b). What is the probability that the third flip is with Coin 2?

c). What proportion of time, on average, is Coin 1 flipped? [Hint: set up the problem as a two-state Markov chain, and find the stationary distribution]

4). A Poisson process $N(t)$ has rate $\lambda = 1$. Suppose there are exactly 2 arrivals in the interval $[0,1]$, so $N(1) = 2$.

a). Find the probability that both of these arrivals occur in the interval $[0, 1/3]$.

b). Let T_1 and T_2 be the first and second arrival times. Find the mean and variance of the sum $T_1 + T_2$, given that $N(1) = 2$. [Hint: use the Theorem concerning conditional distribution of the arrival times in an interval $[0, s]$, conditioned on the number of arrivals in this interval. This Theorem allows you to express T_1, T_2 in terms of two independent uniform random variables U_1 and U_2].

c). Let $X_2 = T_2 - T_1$ be the time between the first and second arrivals of the process. Find the mean of X_2, given that $N(1) = 2$. [Hint: again use the Theorem from the last class concerning conditional distribution of the arrival times, and express X_2 in terms of two independent uniform random variables U_1 and U_2].

5). Starting from time 0, buses arrive at a bus stop according to a Poisson process of rate λ. Passengers arrive at the bus stop according to an independent Poisson process of rate μ. When a bus arrives, all waiting passengers instantly board the bus and subsequent passengers wait for the next bus.

a). Let X_m denote the interarrival time between the $(m-1)^{st}$ bus and the m^{th} bus. Find the conditional expectation for the number of passengers who board the m^{th} bus, given that $X_m = x$.

b). Find the expected number of passengers who board the m^{th} bus. [Hint: use your result from part a) and undo the conditioning over X_m]

c). Find the pmf for the number of passengers who board the m^{th} bus. [Hint: for each n, find the conditional probability that n passengers board the bus, conditioned on $X_m = x$, and then undo the conditioning].

d). Suppose $\lambda = 2$ hour^{-1} and $\mu = 12$ hour^{-1}. Given that a bus arrives at 10:30am and that no bus arrives between 10:30am and 11:00am, find the mean and variance for the number of passengers who board the next bus.