MATH5110

HOMEWORK - 5

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1. a. $A = \begin{bmatrix} 0.1 & 0.3 \\ 0.9 & 0.7 \end{bmatrix}$

eigenvalues $\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} -0.2 \\ 1 \end{bmatrix}$ eigenvectors $V_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$,

$V_2 = \begin{bmatrix} 1/3 \\ 1 \end{bmatrix}$

A as PDP^{-1}

$$A = \begin{bmatrix} 0.1 & 0.3 \\ 0.9 & 0.7 \end{bmatrix} = \begin{bmatrix} 1/3 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -0.2 \end{bmatrix} \begin{bmatrix} 0.75 & 0.75 \\ -0.75 & 0.25 \end{bmatrix}$$

$$\Rightarrow A^t = \begin{bmatrix} 1/3 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -0.2 \end{bmatrix}^t \begin{bmatrix} 0.75 & 0.75 \\ -0.75 & 0.25 \end{bmatrix}$$

$$\Rightarrow \lim_{t \to \infty} A^t = \begin{bmatrix} 1/3 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -0.2 \end{bmatrix}^t \begin{bmatrix} 0.75 & 0.75 \\ -0.75 & 0.25 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0.75 & 0.75 \\ -0.75 & 0.25 \end{bmatrix}$$

$$= \begin{bmatrix} 0.25 & 0.25 \\ 0.75 & 0.75 \end{bmatrix}$$

(b.) $$\lim_{t \to \infty} A^t \vec{v} = \begin{bmatrix} 0.25 & 0.25 \\ 0.75 & 0.75 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$= \begin{bmatrix} 0.25(a+b) \\ 0.75(a+b) \end{bmatrix}$$

$$= \begin{bmatrix} 0.25 \\ 0.75 \end{bmatrix}$$

$a + b = 10.$

2. $A = \begin{bmatrix} 2 & 15 & 0 \\ 1 & 2 & 1 \\ 2 & 4 & 2 \end{bmatrix}$

$$A = 2(4-4) - 15(2-2) + 0(4-4)$$
$$= 0$$

$$\lim_{t \to \infty} \left(\frac{A}{7}\right)^t = \lim_{t \to \infty} \left(\frac{0}{7}\right)^t$$

$$= \lim_{t \to \infty} 0$$

$$\therefore \lim_{t \to \infty} \left(\frac{A}{7}\right)^t = 0.$$

3.1. $V = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

$$\vec{u}_1 = \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{e}_1 = \frac{\vec{u}_1}{\|\vec{u}_1\|} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{u}_2 = \vec{v}_2 - \frac{\vec{u}_1 \cdot \vec{v}_2}{\|\vec{u}_1\|^2} \vec{u}_1 = \begin{bmatrix} -2/3 \\ 1/3 \\ 1/3 \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{e}_2 = \frac{\vec{u}_2}{\|\vec{u}_2\|} = \sqrt{\frac{3}{5}} \begin{bmatrix} -2/3 \\ 1/3 \\ 1/3 \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{u}_3 = \vec{v}_3 - \frac{\vec{u}_1 \cdot \vec{v}_3}{\|\vec{u}_1\|^2} \cdot \vec{u}_1 - \frac{\vec{u}_2 \cdot \vec{v}_3}{\|\vec{u}_2\|^2} \vec{u}_2$$

$$= \begin{bmatrix} 1/5 \\ -3/5 \\ 2/5 \\ 1/5 \\ 1 \end{bmatrix}$$

$$\vec{e}_3 = \frac{\vec{u}_3}{\|\vec{u}_3\|} = \sqrt{\frac{5}{8}} \begin{bmatrix} 1/5 \\ -3/5 \\ 2/5 \\ 1/5 \\ 1 \end{bmatrix}$$

$\therefore$ Orthonormal basis $= \{\vec{e_1}, \vec{e_2}, \vec{e_3}\}$

2. $$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Basis or orthogonal complement $= \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

$$\vec{u_4} = \vec{v_4} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \quad ; \quad \vec{e_4} = \frac{\vec{u_4}}{\|\vec{u_4}\|} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{u_5} = \vec{v_5} - \frac{(\vec{u_4} \cdot \vec{v_5})}{\|\vec{u_4}\|^2} \cdot \vec{u_4}$$

$$= \vec{v_5} - \frac{1}{3}\vec{u_4} = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/3 \\ 1 \\ -2/3 \\ -1/3 \\ 1 \end{bmatrix}$$

$$\therefore \vec{e}_5 = \sqrt{\frac{3}{8}} \begin{bmatrix} -1/3 \\ 1 \\ -2/3 \\ -1/3 \\ 1 \end{bmatrix}$$

Orthonormal basis of orthogonal complement of V $= \{ \vec{e}_4 , \vec{e}_5 \}$

3. $\vec{e}_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{e}_2 = \sqrt{\frac{3}{5}} \begin{bmatrix} -2/3 \\ 1/3 \\ 1/3 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{e}_3 = \sqrt{\frac{5}{8}} \begin{bmatrix} 1/5 \\ -3/5 \\ 2/5 \\ 1/5 \\ 1 \end{bmatrix}$

$$\text{proj}_V \vec{y} = (\vec{e}_1 \cdot \vec{y})\vec{e}_1 + (\vec{e}_2 \cdot \vec{y})\vec{e}_2 + (\vec{e}_3 \cdot \vec{y})\vec{e}_3$$

$$= \frac{1}{3} \times 3 \times \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \frac{3}{5} \times 0 + \frac{5}{8} \times 0$$

$$= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

4. $\vec{y}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ (calculated in part 3)

$$\vec{e}_4 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{e}_5 = \sqrt{\frac{3}{8}} \begin{bmatrix} -1/3 \\ 1 \\ -2/3 \\ -1/3 \\ 1 \end{bmatrix}$$

$$y_2 = \text{Proj}_{V^\perp} \vec{y} = \underline{(\vec{e}_4 \cdot \vec{y})\,\vec{e}_4} + (\vec{e}_5 \cdot \vec{y})\,\vec{e}_5$$

$$= \frac{1}{3} \times 4 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + \frac{3}{8} \times \frac{8}{3} \begin{bmatrix} -1/3 \\ 1 \\ -2/3 \\ -1/3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \\ 1 \end{bmatrix}$$

$$\therefore y = y_1 + y_2$$

$$= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$

4. 1. $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & -1 & 1 \\ -1 & 2 & 1 & 0 \end{bmatrix}$$

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & 1/3 \\ 0 & 0 & 1 & -1/3 \end{bmatrix}$$

$$\therefore \vec{v}_4 = \begin{bmatrix} -1/3 \\ -1/3 \\ 1/3 \\ 1 \end{bmatrix}$$

2. $$\text{Proj}_V \vec{y} = \frac{(\vec{v}_1 \cdot \vec{y})}{\|\vec{v}_1\|^2} \vec{v}_1 + \frac{(\vec{v}_2 \cdot \vec{y})}{\|\vec{v}_2\|^2} \vec{v}_2 + \frac{(\vec{v}_3 \cdot \vec{y})}{\|\vec{v}_3\|^2} \vec{v}_3$$

$$= \frac{1}{2}(2)\vec{v}_1 + \frac{1}{4}(2)\vec{v}_2 + \frac{1}{6}(2)\vec{v}_3$$

$$= \frac{1}{6}\begin{bmatrix} 7 \\ 7 \\ 5 \\ 3 \end{bmatrix}$$

5. (a) $-3x + y + z = 0.$

$$z = 3x - y$$

One possible basis is, $\left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$

$\vec{v}_1 \quad \vec{v}_2$

$$\vec{u}_1 = \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

$$\vec{u}_2 = \vec{v}_2 - \frac{(\vec{v}_2 \cdot \vec{u}_1)}{\|\vec{u}_1\|^2} (\vec{u}_1)$$

$$= \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + \frac{3}{10} \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3/10 \\ 1 \\ -1/10 \end{bmatrix}$$

Orthogonal basis $= \left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3/10 \\ 1 \\ -1/10 \end{bmatrix} \right\}$

b. $\text{proj}_V \vec{y} = \frac{(\vec{y}\cdot\vec{u}_1)}{\|\vec{u}_1\|^2}\vec{u}_1 + \frac{(\vec{y}\cdot\vec{u}_2)}{\|\vec{u}_2\|}\vec{u}_2$

$$= \begin{bmatrix} 8/11 \\ 12/11 \\ 12/11 \end{bmatrix}$$

$$\text{proj}_{V^\perp}\vec{y} = \vec{y} - \text{proj}_V \vec{y}$$

$$= \begin{bmatrix} 3/11 \\ -1/11 \\ -1/11 \end{bmatrix}$$

$\therefore$ Shortest distance $= \|\text{proj}_{V^\perp}\vec{y}\| = \frac{\sqrt{11}}{11} = \frac{1}{\sqrt{11}}$

6. $L(\vec{u}) = A\vec{u}$

"L" is orthogonal in $\mathbb{R}^n \to \mathbb{R}^n \Rightarrow AA^T = I$

$\therefore \langle L(\vec{v}), L(\vec{w})\rangle = L(\vec{v}) \cdot L(\vec{w})$

$= A\vec{v} \cdot A\vec{w}$

$= (A\vec{v})^T \cdot A\vec{w}$

$= \vec{v}^T \underbrace{A^T \cdot A}_{I_n} \vec{w}$

$= \vec{v}^T \vec{w}$

$$= \langle \vec{v}, \vec{w} \rangle$$

$$\| L(\vec{v}) \|^2 = L(\vec{v})^T L(\vec{v})$$

$$= (A\vec{v})^T A\vec{v}$$

$$= \vec{v}^T A^T A \vec{v}$$

$$= \vec{v}^T \vec{v}$$

$$= \langle \vec{v}, \vec{v} \rangle$$

$$= \|\vec{v}\|^2$$

$\because$ Linear transformations preserve products and angles.

$$\therefore \theta(\vec{v}, \vec{w}) = \frac{\langle \vec{v}, \vec{w} \rangle}{\|\vec{v}\| \times \|\vec{w}\|}$$

$\therefore$ angle is also preserved.

Let, $L: R^2 \to R^2$, such that $L(\vec{u}) = 3\vec{u}$

$$\theta_{(L(\vec{v}), L(\vec{w}))} = \frac{3\vec{v} \cdot 3\vec{w}}{3\|\vec{v}\| \cdot 3\|\vec{w}\|} = \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \|\vec{w}\|}$$

$$= \theta_{(\vec{v}, \vec{w})}$$

$\therefore$ angle is preserved but "L" is not orthogonal.

7. $$\|\vec{x}\| := \sum_{i=1}^{n} x_i^2$$

$$\|\alpha \vec{x}\| := \sum_{i=1}^{n} \alpha^2 x_i^2$$

$$:= \alpha^2 \sum_{i=1}^{n} x_i^2$$

$$:= \alpha^2 \|\vec{x}\| \neq |\alpha| \, \|\vec{x}\|$$

$\therefore \|\vec{x}\|$ does not define a norm.

8. 1. Linearity Check

$$\langle xA + yC, B \rangle = tr\left((x_A + y_C)^T B \right)$$

$$= tr\left(xA^T B + yC^T B \right)$$

$$= x\, tr(A^T B) + y\, tr(C^T B)$$

$$= x \langle A, B \rangle + y \langle C, B \rangle \checkmark$$

2. Symmetry Check:

$$\langle A, B\rangle = tr(A^T B)$$

$$= tr((A^T B)^T)$$

$$= tr(B^T A)$$

$$= \langle B, A\rangle$$ ✓

3. Positive Definite Check:

$$\langle A, A\rangle = tr(A^T A)$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{n} a_{ij}^2$$

$$\geq 0 \quad \left[= 0 \Rightarrow a_{ij} = 0 \ \forall i,j \right]$$

✓

9. $C[0,1] = \{$ set of all continuous functⁿ from $[0,1]$ to $\mathbb{R}\}$

Let $f(x) = -1$ for $x \in [0,1]$

then $f(x)$ is continuous $\Rightarrow f(x) \in C[0,1]$

Now $\langle f,f \rangle = \int_0^1 2f(x)\,dx$

$= \int_0^1 -2\,dx$

$= [-2x]_0^1 = -2 \not> 0$

~~is if fails~~

$\therefore$ it fails to provide positive definite property

$\therefore C[0,1]$ is not an inner product.

10. Let $p(x) = ax^3 + bx^2 + cx + d$ be in the orthogonal complement.

$\int_0^1 (1-x)\,p(x) = 0 \Rightarrow 3a + 5b + 10c + 30d = 0$

and

$$\int_0^1 (2-x+x^2)\cdot p(x) = 0 \Rightarrow 28a + 37b + 55c + 110d = 0$$

$3a + 5b = -10c - 30d$

$28a + 37b = -55c - 110d$

$$= \begin{bmatrix} 3 & 5 \\ 28 & 37 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -10c - 30d \\ -55c - 110d \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{29} \begin{bmatrix} 95c + 560d \\ -115c - 510d \end{bmatrix}$$

Thus $p(x) = c\left[\frac{95}{25}x^3 - \frac{115}{29}x^2 + x\right]$

$$+ d\left[\frac{560}{29}x^3 - \frac{510}{29}x^2 + 1\right]$$

Hence, basis for orthogonal complement of S

$$= \left\{\frac{95}{29}x^3 - \frac{115}{29}x^2 + x,\ \frac{560}{29}x^3 - \frac{510}{29}x^2 + 1\right\}$$