MATHS110

Homework 2.

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1. V is a vector space over $\mathbb{R}$
when we multiply any scaler 'c' with $\vec{v}$
we get $c\vec{v}$ which is not an element
in $\{0, \vec{v}\}$. Therefore $\{0, \vec{v}\}$ is not
closure on multiplication and hence not
a subspace.

2. 1. $S = \left\{ \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 \;\middle|\; x_1 x_2 = 0 \right\}$.

for $x_1 x_2 = 0$, either x_1 or x_2 has to
be 0.

for vectors $\begin{bmatrix} x_1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ x_2 \end{bmatrix}$

$$\begin{bmatrix} x_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \notin S$$

$\therefore$ S is not closure on addition
$\therefore$ S is not a subspace of $\mathbb{R}^2$

2. $T = \{ \vec{x} \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 1 \}$

for
$\vec{x} = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}$

$c\vec{x} \notin T$ $\therefore$ T is not closure on
scaler multiplication. $\therefore$ T is not subspace.

3. 1. $U_{3\times3} = \left\{ \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ 0 & x_{22} & x_{23} \\ 0 & 0 & x_{33} \end{bmatrix} \middle| \; x_{ij} \in \mathbb{R},\ i,j = 1,2,3 \right\}$

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$

$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & 0 & b_{33} \end{bmatrix}$

$A + B = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ 0 & a_{22}+b_{22} & a_{23}+b_{23} \\ 0 & 0 & a_{33}+b_{33} \end{bmatrix}$

$\in U_{3\times3}$.

$\therefore$ $U_{3\times3}$ is closure on addition

Let c be some scaler

$cA = \begin{bmatrix} ca_{11} & ca_{12} & a_{13} \\ 0 & ca_{22} & ca_{23} \\ 0 & 0 & ca_{33} \end{bmatrix} \in U_{3\times3}$

$\therefore$ $U_{3\times3}$ is closure on scaler multiplication.

if $E = 0$ space. in $R^{3\times3}$.

$$E \times A = \{0\} \quad \in U_{3\times3}.$$

$\therefore$ $U_{3\times3}$ contain zero vector.

$\therefore$ ~~Subset~~ $U_{3\times3}$ of all 3×3 upper triangle matrices with real entries contains a zero vector, is closure on addition and scaler multiplication.

$\therefore$ $U_{3\times3}$ is as subspace of $\mathbb{R}^{3\times3}$.

2. $$T_{3\times3} = \left\{ \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ 0 & x_{22} & x_{23} \\ 0 & 0 & x_{33} \end{bmatrix} \text{ or } \begin{bmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & x_{23} & x_{33} \end{bmatrix} \middle| \; x_{ij} \in R, i,j = 1,2,3 \right\}$$

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$ $B = \begin{bmatrix} b_{11} & 0 & 0 \\ b_{21} & b_{22} & 0 \\ b_{31} & b_{23} & b_{33} \end{bmatrix}$

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} & a_{13} \\ b_{21} & a_{22} + b_{22} & a_{23} \\ b_{31} & b_{23} & a_{33} + b_{33} \end{bmatrix} \notin T$$

T is not closure on addition

$\therefore$ T is not a subspace of $R^{3\times 3}$.

3. 3. $W = \{ (t + at^2) \mid a \in \mathbb{R} \}$

for $a = 0$

$W = \{ t \}$

$\therefore$ W doesnot contain $\{0\}$

$\therefore$ W is not a subspace of P.

4. $\cancel{c}$. $S = \text{Span}\left\{ \vec{b_1} = \begin{bmatrix} -1 \\ -2 \\ 4 \\ -2 \end{bmatrix} \quad \vec{b_2} = \begin{bmatrix} 0 \\ 1 \\ -5 \\ 4 \end{bmatrix} \right\}$

1. $\vec{v} = \begin{bmatrix} -1 \\ 0 \\ -6 \\ 6 \end{bmatrix}$

$c_1 \vec{b_1} + c_2 \vec{b_2} = \vec{v}$

$\Rightarrow \left[\begin{array}{cc|c} -1 & 0 & -1 \\ -2 & 1 & 0 \\ 4 & -5 & -6 \\ -2 & 4 & 6 \end{array}\right]$

rref $= \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow c_1 = 1$

$c_2 = 2$.

$\therefore \vec{v}$ ~~is~~ belongs to S.

2 $\vec{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

$$c_1 \vec{b}_1 + c_2 \vec{b}_2 = \vec{w}$$

$$\Rightarrow \left[\begin{array}{cc|c} -1 & 0 & 1 \\ -2 & 1 & 1 \\ 4 & -5 & 1 \\ -2 & 4 & 1 \end{array}\right]$$

$$\text{rref} = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \quad c_1 = 1$$
$$c_2 = 1$$

but $1 \begin{bmatrix} -1 \\ -2 \\ 4 \\ -2 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \\ -5 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \\ 2 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

$\therefore \vec{w}$ doesnot belong to S.

5. $S = \text{Span}\left\{ \vec{b_1} = \begin{bmatrix} -1 & -2 \\ 4 & -2 \end{bmatrix}, \vec{b_2} = \begin{bmatrix} 0 & 1 \\ -5 & 4 \end{bmatrix} \right\}$

1. $\vec{v} = \begin{bmatrix} -1 & 0 \\ -6 & 6 \end{bmatrix}$

$c_1 \vec{b_1} + c_2 \vec{b_2} = \vec{v}$

$-1c_1 + 0 \cdot c_2 = -1 \Rightarrow c_1 = 1$

$-2 \cdot c_1 + 1 \cdot c_2 = 0 \Rightarrow c_2 = 2$

$4c_1 - 5c_2 = -6 \Rightarrow -6 = -6$

$-2c_1 + 4c_2 = 6 \Rightarrow 6 = 6$

$\therefore$ for $c_1 = 1$ and $c_2 = 2$
$\vec{v}$ belongs to S

2. $\vec{w} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$c_1 \vec{b_1} + c_2 \vec{b_2} = \vec{w}$

$-1 c_1 + 0 \cdot c_2 = 1 \Rightarrow c_1 = -1$

$-2 \cdot c_1 + 1 \cdot c_2 = 1 \Rightarrow c_2 = -2$

$4 c_1 - 5 c_2 = 1 \Rightarrow 6 \neq 1$

$-2 c_1 + 4 c_2 = 1 \Rightarrow -6 \neq 1$

$\therefore \vec{w}$ doesnot belong to S.

6. $S = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$

$T = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

We can see that $S \subset T$.

To show

vector $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \in S$

Let $\vec{b_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{b_2} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

$c_1 \vec{b_1} + c_2 \vec{b_2} = \vec{v}$

$\Rightarrow \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{array}\right]$

$\text{rref} = \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$ $\Rightarrow c_1 = 2$, $c_2 = 2$

$\therefore \vec{v} \in S$

Hence S and $T \subset \mathbb{R}^3$

7 (1) If ~~UUV~~ $\vec{u}, \vec{v} \in U \cup V$

such that $\vec{u} \in U$ and $\vec{v} \in V$

but $\vec{u} + \vec{v} \notin U \cup V$

Ex:

Consider x-axis and y-axis.

A is a point on x-axis $(5, 0)$

B is a point on y-axis $(0, 8)$

$A + B = (5, 8) \notin$ ~~the~~ $x \cup y$.

$\therefore$ Union of subspaces doesn't belong to the subspace

(2) Let $\vec{0} \in U$ and $\vec{0} \in V$, $\therefore \vec{0} \in U \cap V$

If $\vec{u}, \vec{v} \in U \cap V$, then

$\vec{u} + \vec{v} \in U$

$\vec{u} + \vec{v} \in V$

$\vec{u} + \vec{v} \in$ ~~UU~~ $U \cap V$

$\therefore$ an intersection is closure on addition

If $\vec{u} \in U \cap V$, then

$c\vec{u} \in U$ and $c\vec{v} \in V$

for and $c \in \mathbb{R}$

$c\vec{u} \in U \cap V$. $\therefore$ an intersection is closure on scalar product.

∴ This intersection of subspace is a subspace.

8. Let $U = \{u_1, u_2 \ldots u_n\}$

$V = \{v_1, v_2 \ldots v_m\}$

$W = \{u_1, v_1, w_1 \ldots w_p\}$.

$L.H.S = (U+V) \cap W$

$= \{u_1 \ldots, u_n, v_1 \ldots v_m\} \cap \{u_1, v_1, w_1 \ldots w_p\}$

$= \{u_1, v_1\}$

$R.H.S = (U \cap W) + (V \cap W)$

$= \{u_1\} + \{v_1\}$

$= \{u_1, v_1\}$

∴ $L.H.S. = R.H.S.$

9. Since

$$U_1 \cap U_2 = 0$$
$$U_2 \cap U_3 = 0$$
$$U_1 \cap U_3 = 0$$

$$U_1 \cap U_2 \cap U_3 = 0$$

We know: $V = U \oplus W$ if and only if
$V = U + W$ and $U \cap W = \{\vec{0}\}$

~~We know~~ Since $V = U_1 + U_2 + U_3$
and $U_1 \cap U_2 \cap U_3 = 0$

$$\therefore V = U_1 \oplus U_2 \oplus U_3$$

10. Given, V be a vector space with dimension n over the field of Z_2 of two elements.

i.e. $Z_2 = \{0, 1\}$

V is n dimensional space.

So let $\{v_1, v_2, \ldots v_n\}$ be basis of V.

Then,

every element of V is uniquely expressed in the linear combination of basis.

i.e. $\{a_1v_1 + a_2v_2 + \ldots + a_nv_n\}$ $\forall a_i \in Z_2$.

Z_2 has two elements.

$\therefore$ a_i has two choices (0 or 1)

$\therefore$ we have 2^n choices ~~alt~~ altogether.

$\therefore$ V has 2^n elements.

11. Given $\{\vec{u}, \vec{v}\}$, $\{\vec{v}, \vec{w}\}$ and $\{\vec{w}, \vec{u}\}$ are linearly independant.

$$\Rightarrow \quad x_1\vec{u} + x_2\vec{v} = 0 \quad \text{--- } (1)$$
$$x_3\vec{v} + x_4\vec{w} = 0 \quad \text{--- } (2)$$
$$x_5\vec{w} + x_6\vec{u} = 0 \quad \text{--- } (3)$$

~~Eq~~ (1) + (2) + (3)

$$\Rightarrow \quad x_1\vec{u} + x_2\vec{v} + x_3\vec{v} + x_4\vec{w} + x_5\vec{w} + x_6\vec{u} = 0$$
$$\Rightarrow \quad (x_1 + x_6)\vec{u} + (x_2 + x_3)\vec{v} + (x_4 + x_5)\vec{w} = 0.$$
$$\Rightarrow \quad a\vec{u} + b\vec{v} + c\vec{w} = 0$$

where $a = x_1 + x_6$

$b = x_2 + x_3$

$c = x_4 + x_5$

$\therefore$ $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly independent.

12. $V = \left\{ \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \\ 13 \end{bmatrix} \right\} \in \mathbb{R}^3$

$c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 = \vec{0}$

$-1 \cdot c_1 + 1 \cdot c_2 - 3c_3 = 0 \Rightarrow c_1 = c_2 - 3c_3$

$1 \cdot c_1 - 1 c_2 + 3c_3 = 0 \Rightarrow c_1 = c_2 - 3c_3$

$3c_1 - 2c_2 + 13c_3 = 0.$

$$3(c_2 - 3c_3) - 2c_2 + 13c_3 = 0$$

$$\Rightarrow 3c_2 - 9c_3 - 2c_2 + 13c_3 = 0$$

$$\Rightarrow c_2 + 4c_3 = 0$$

$$\Rightarrow c_2 = -4c_3$$

$$c_1 = c_2 - 3c_3$$

$$= -4c_3 - 3c_3$$

$$= -7c_3$$

for $c_3 = 1$

$c_2 = -4$

$c_1 = -7$

$\therefore$

$$\vec{v}_3 = -7\vec{v}_1 - 4\vec{v}_2$$

Q13. $\text{rank}(A) \leq m < n$

$\therefore$ Nullity$(A) = n - \text{rank}(A)$

$= n - m > 0$

So, the null space has dimension ≥1

$A\vec{x} = \vec{0}$ has a non-trivial solution

for $\forall\, x \in R^n$.

Q14. 1. $\vec{u_1} = \begin{bmatrix} 1 \\ 4 \\ 0 \\ -5 \\ 1 \end{bmatrix}$ $\vec{u_2} = \begin{bmatrix} 1 \\ 3 \\ 0 \\ -4 \\ 0 \end{bmatrix}$ $\vec{u_3} = \begin{bmatrix} 0 \\ 4 \\ 1 \\ 1 \\ 4 \end{bmatrix} \in \mathbb{R}^5$

let $A = \{ \vec{u_1}, \vec{u_2}, \vec{u_3} \}$

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ $\vec{u_1}$, $\vec{u_2}$ and $\vec{u_3}$ are Independent.

2.
Lets adjoint the vectors $\vec{v_1} = (1\ 0\ 0\ 0\ 0)$, $\vec{v_2} = (0\ 1\ 0\ 0\ 0)$ to the basis A

Now $B = \{ \vec{u_1}, \vec{u_2}, \vec{u_3}, \vec{v_1}, \vec{v_2} \} \in \mathbb{R}^5$

To show that B is a basis for $\mathbb{R}^5$, we need to prove B is linearly indepent

Suppose $a_1, a_2, a_3, a_4, a_5 \in \mathbb{R}^5$ satisfy

$a_1 \vec{u_1} + a_2 \vec{u_2} + a_3 \vec{u_3} + a_4 \vec{v_1} + a_5 \vec{v_2} = \vec{0}$

$$\Rightarrow a_1 \begin{bmatrix} 1 \\ 4 \\ 0 \\ -5 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 3 \\ 0 \\ -4 \\ 0 \end{bmatrix} + a_3 \begin{bmatrix} 0 \\ 4 \\ 1 \\ 1 \\ 4 \end{bmatrix} + a_4 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + a_5 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow a_1 + a_2 + a_4 = 0$ —— (1)

$4a_1 + 3a_2 + 4a_3 + a_5 = 0$ —— (2)

$a_3 = 0$ —— (3)

$-5a_1 - 4a_2 + a_3 = 0$ —— (4)

$a_1 + 4a_3 = 0$ —— (5)

eq (3) and eq (5)

$a_1 = 0$ —— (6)

from eq^n (3) (4) and (6)

$a_2 = 0$ —— (7)

from eq^n (1), (6), (7)

$a_4 = 0$ —— (8)

from eq^n (2) (3), (6), (7) $a_5 = 0$ —— (9)

$\Rightarrow a_1 = a_2 = a_3 = a_4 = a_5 = 0.$

$\therefore B = \{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{v}_1, \vec{v}_2\}$ is linearly independent in $\mathbb{R}^5$

Hence B is the basis for $\mathbb{R}^5$

15. Let $U = \begin{bmatrix} -1 & 3 & 1 & 0 \\ 1 & 0 & 1 & -1 \\ 2 & 2 & 1 & -3 \end{bmatrix}$

$\text{rref}(U) = \begin{bmatrix} 1 & 0 & 0 & -8/7 \\ 0 & 1 & 0 & -3/3 \\ 0 & 0 & 1 & 1/7 \end{bmatrix}$

$\dim(U) = 3.$

$W = \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & 3 & 1 & -1 \\ 2 & -2 & -1 & -1 \\ 2 & 2 & 1 & -1 \end{bmatrix}$

$\text{rref}(W) = \begin{bmatrix} 1 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & -1/2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\dim(W) = 3.$

$$\text{rref}(U+W) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\dim(U+W) = 4.$

$\dim(U+W) = \dim(U) + \dim(W) - \dim(U \cap W)$

$\Rightarrow \dim(U \cap W) = 3 + 3 - 4 = 2$

$$\frac{\mathbb{R}^4}{U \oplus U} = \mathbb{R}^4$$

$\dim \mathbb{R}^4/U = 1.$

S''ly. $\dim \mathbb{R}^4/W = 1.$

16. (a.) Since $\vec{v}_1, \vec{v}_2 \ldots \vec{v}_n$ be n linearly dependent vectors, there exist non-trivial solution $\vec{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ such that $\sum_{j=1}^{n} c_j \vec{v}_j = \vec{0}$.

Claim: All c_j are non zero.
Suppose one $c_j = 0$ such that $c_i = 0$.
then
$\vec{v}_1 \ldots \hat{\vec{v}}_i \ldots \vec{v}_n$ is dependent
which contradict the assumption that any $n-1$ of the vectors $\vec{v}_1 \ldots \vec{v}_n$ are linearly independent.

(b.) Suppose $\beta_1 = \gamma \alpha_1$, then

$$\gamma \sum_{j=1}^{n} \alpha_j \vec{v}_j - \sum_{j=1}^{n} \beta_j \vec{v}_j = \vec{0}$$

Hence

$$\sum_{j=1}^{n} (\gamma \alpha_j - \beta_j) \vec{v}_j = \sum_{j=2}^{n} (\gamma \alpha_j - \beta_j) \vec{v}_j = \vec{0}$$

Since $n-1$ of the vectors $\vec{v}_2 \ldots \vec{v}_n$ are linearly independent, we have $\gamma \alpha_j - \beta_j = 0$ for $j = 2 \ldots n$

17) $M = \begin{bmatrix} 3 & 3 & 2 & 8 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 3 & 5 \\ -2 & 4 & 6 & 8 \end{bmatrix}$

$U = \begin{bmatrix} 3 & 3 & 2 & 8 \\ 1 & 1 & -1 & 1 \end{bmatrix}$

$W = \begin{bmatrix} 1 & 1 & 3 & 5 \\ -2 & 4 & 6 & 8 \end{bmatrix}$

$rref(U) = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

$dim(U) = 2$

$rref(W) = \begin{bmatrix} 1 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$dim(w) = 1$

$U + W = M$

$rref(M) = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\dim(U+W) = 2$

$\dim(U+W) = \dim(U) + \dim(W) - \dim(U\cap W)$

$\Rightarrow \dim(U\cap W) = 2+1-2$

$= 1$

18. Let $S = \{1, x, x^2, x^3\}$ be standard basis of $P_4(\mathbb{R})$

Consider the vectors f_1, f_2, g_1 and g_2 within a matrix.

Let $A = \begin{bmatrix} 1 & 0 & 2 & 1 \\ -2 & 1 & 2 & -1 \\ 0 & 1 & -4 & 1 \\ 1 & -1 & 1 & 0 \end{bmatrix}$

$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Hence basis for $U+V = \{f_1, f_2, g_1\}$

basis for $U\cap V = \{g_2\}$