Math 5110- Applied linear algebra

He Wang

he.wang@northeastern.edu

Homework 5.

Using Python or Matlab for the calculations of matrices.

Using Mathematica or https://www.wolframalpha.com/ help the calculation of integrals.

Question 1. The transition matrix $A = \begin{bmatrix} 0.1 & 0.3 \\ 0.9 & 0.7 \end{bmatrix}$.

(a) Find $\lim\limits_{t\to\infty} A^t$.

A has eigenvalues 1 and -0.2, and corresponding eigenvectors $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

The equilibrium distribution for A is the distribution vector $\vec{x}_{equ} = \frac{1}{4}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0.25 \\ 0.75 \end{bmatrix}$

So, $\lim\limits_{t\to\infty} A^t = [\vec{x}_{equ}\ \vec{x}_{equ}] = \begin{bmatrix} 0.25 & 0.25 \\ 0.75 & 0.75 \end{bmatrix}$

(b) Find $\lim\limits_{t\to\infty} A^t\vec{v}$ for a vector $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ such that $a + b = 10$. (Hint: The vector $\vec{v}$ is not a distribution vector, but $(\frac{1}{10}\vec{v})$ is a distribution vector.)

$(\frac{1}{10}\vec{v})$ is a distribution vector since $a + b = 10$.

$\lim\limits_{t\to\infty} A^t\vec{v} = 10 \lim\limits_{t\to\infty} A^t(\frac{1}{10}\vec{v}) = 10\begin{bmatrix} 0.25 \\ 0.75 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 7.5 \end{bmatrix}$

Question 2. Let $A = \begin{bmatrix} 2 & 15 & 0 \\ 1 & 2 & 1 \\ 2 & 4 & 2 \end{bmatrix}$. Find the limit $\lim\limits_{t\to\infty}(\frac{1}{7}A)^t$

A is primitive since A^2 is positive.
The eigenvalues of A are $7, -1, 0$
The largest eigenvalue of A is 7.

The right eigenvector with eigenvalue 7 of A is $\vec{u} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$

The left eigenvector with eigenvalue 7 of A is the eigenvalue of A^T is $\vec{v} = \begin{bmatrix} 7 \\ 5 \\ 5 \end{bmatrix}$

The dot product $\vec{v} \cdot \vec{u} = 21 + 5 + 10 = 36$
So the limit is $\lim_{t \to \infty} (\frac{1}{7}A)^t = \frac{1}{36}\vec{u}\vec{v}^T$

Question 3. Let $\mathbb{R}^5$ be the Euclidean space. Let V be a subspace spanned by

$$\vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

(1) Apply the Gram-Schmidt process to find the orthonormal basis of V. (2) Find the orthogonal complement of V. (3) Compute $\text{proj}_V \vec{y}$. (4) Write $\vec{y} = \begin{bmatrix} 2 \\ 2 \\ -1 \\ 1 \\ 1 \end{bmatrix}$ as $\vec{y} = \vec{y}_1 + \vec{y}_2$ such that $\vec{y}_1 \in V$ and $\vec{y}_2 \in V^\perp$

(5) Write a Matlab/Python function projection(y, A) to compute $\text{proj}_V \vec{y}$ (6) Write a Matlab/Python function OBasis(A) to compute orthogonal basis and then a function NBasis(A) to compute orthonormal basis.

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$
Matlab QR decomposition.
$\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$ form a basis for $V^\perp = \ker A^T$

Question 4. Notice that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, where

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix},$$

is an orthogonal subset of $\mathbb{R}^4$.

(1) Find a fourth vector $\vec{v}_4 \in \mathbb{R}^4$ that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is an orthogonal basis for $\mathbb{R}^4$.

(2) Find the orthogonal projection of $\vec{y} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ onto $V = \mathrm{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

(1) $\vec{v}_4 \in V^\perp$ where $V = \mathrm{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

$V^\perp = \ker A$ where $A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & -1 & 1 \\ -1 & 2 & 1 & 0 \end{bmatrix}$

So $V^\perp = \mathrm{Span}\{\begin{bmatrix} -1/3 \\ -1/3 \\ 1/3 \\ 1 \end{bmatrix}\}$

(3) One way is by a direct calculation.
The other faster way is project onto $V^\perp$ first.

$$\vec{y}_2 = \mathrm{proj}_{V^\perp} \vec{y} = \frac{\vec{y} \cdot \vec{v}_4}{\vec{v}_4 \cdot \vec{v}_4} \vec{v}_4 = \frac{-2}{12} \begin{bmatrix} 1 \\ 1 \\ -1 \\ -3 \end{bmatrix}$$

So, $\vec{y}_1 = \vec{y} - \vec{y}_2 = \begin{bmatrix} 7/6 \\ 7/6 \\ 5/6 \\ 1/2 \end{bmatrix}$

Question 5. Let P be the plane in $\mathbb{R}^3$ defined by the equation $-3x + y + z = 0$.

(a) Find an orthogonal basis for P.

(b) Find the shortest distance from $(1, 1, 1)$ to the plane P.

The normal vector of the plane is $\vec{n} = \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$. So the plane $P = \vec{n}^\perp$

An orthogonal basis for P is $\{\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 10 \end{bmatrix}\}$

(b) $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\mathrm{proj}_P \vec{v} = \dfrac{4}{10} \vec{u}_1 + \dfrac{2}{110} \vec{u}_2$

The shortest distance from $(1, 1, 1)$ to the plane P is $\|\vec{v}^\perp\| = 1/\sqrt{11}$.

Question 6. Show that an orthogonal transformation L from $\mathbb{R}^n$ to $\mathbb{R}^n$ preserves angles: The angle between two nonzero vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^n$ equals the angle between $L(\vec{v})$ and $L(\vec{w})$. Conversely, is any linear transformation that preserves angles orthogonal?

Suppose L is orthogonal. Then L preserves dot products and norm.

$$\cos(\angle(L(\vec{v}), L(\vec{w}))) = \frac{L(\vec{v}) \cdot L(\vec{w})}{\|L(\vec{v})\|\|L(\vec{w})\|} = \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\|\|\vec{w}\|} = \cos(\angle(\vec{v}, \vec{w}))$$

So, L preserves angle.
In general, no. For example, projection to a line will map any angle to zero.

Question 7. Does the formula $\|\vec{x}\| := \sum_{i=1}^{n} x_i^2$ define a norm on $\mathbb{R}^n$?

No. Since $\|c\vec{x}\| \neq c\|\vec{x}\|$.

Question 8. Show that the following formula defines an inner product on the vector space of all $m \times n$ matrices.

$$\langle A, B \rangle := \text{trace}(A^T B).$$

Verify all the axioms.

Question 9. Is $C[0, 1]$ an inner product space under the following formula?

$$\langle f, g \rangle = \int_0^1 (f(x) + g(x))dx$$

No. $\langle f_1 + f_2, g \rangle$ does not equal $\langle f_1, g \rangle + \langle f_2, g \rangle$. Or use the reason $\langle f, f \rangle$ might be negative.

Question 10. Let S be the subspace of the inner product space $P_3(\mathbb{R})$ generated by the polynomials $1 - x$ and $2 - x + x^2$ where $\langle f, g \rangle$ is defined to be $\langle f, g \rangle = \int_0^1 f(x)g(x)dx$. Find a basis for the orthogonal complement of S.

Let $f(x) = a + bx + cx^2$ be in $P_3(\mathbb{R})$. The condition for f to be in $S^\perp$ are

$$\begin{cases} \int_0^1 (a + bx + cx^2)(1 - x)dx = 0 \\ \int_0^1 (a + bx + cx^2)(2 - x + x^2)dx = 0 \end{cases}$$

Perform the integration

$$\begin{cases} 40a + 15b + 8c = 0 \\ 110a + 55b + 37c = 0 \end{cases}$$

The general solution of this system is $a = 23t, b = -120t, c = 110t$ with t arbitrary. Hence the polynomial $23 - 120x + 110x$ forms a basis for $S^\perp$.