Classification of states

Define

$$f_{ij}(n) = P(X_1 \neq j, X_2 \neq j, \ldots, X_{n-1} \neq j, X_n = j \mid X_0 = i) \tag{9}$$

to be the probability that starting in state i the chain first visits state j after n steps. Define

$$\begin{aligned} f_{ij} &= \sum_{n=1}^{\infty} f_{ij}(n) \\ &= P(X_n = j \text{ for some } n \geq 1 \mid X_0 = i) \end{aligned}$$

This is the probability that the chain eventually visits state j starting in state i.

Definition 1 *The state j is* persistent *if $f_{jj} = 1$. The state j is* transient *if $f_{jj} < 1$.*

So the state j is persistent if the chain must eventually return to state j, given that it started at state j. Since the clock starts over at each return to state j, we conclude that if the state j is persistent then the chain must return to j infinitely often.

Similarly if j is transient, then the chain will return to state j only finitely often. Eventually it will never return to the state.

One particular type of persistent state is the class of *absorbing states*: these are the states for which $p_{ii} = 1$. So once the chain enters an absorbing state it can never leave. The end states of the drunkard's walk are absorbing states.

Deciding about persistence

There is a useful way to decide if a state is persistent or transient.

With the conventions $p_{ij}(0) = \delta_{ij}$ and $f_{ij}(0) = 0$, we have for each n

$$p_{ij}(n) = \delta_{ij}\,\delta_{n,0} + \sum_{k=1}^{n} f_{ij}(k)\,p_{jj}(n-k)$$

where we sum over k, the time of first visit to state j. This is a convolution formula so we can solve it by a transform. Define the generating functions

$$P_{ij}(s) = \sum_{n=0}^{\infty} s^n p_{ij}(n), \quad F_{ij}(s) = \sum_{n=0}^{\infty} s^n f_{ij}(n)$$

Then the convolution formula implies

$$P_{ij}(s) = \delta_{ij} + F_{ij}(s)\,P_{jj}(s)$$

For the case $i = j$ this yields

$$P_{ii}(s) = \frac{1}{1 - F_{ii}(s)}$$

Now consider what happens in the limit $s \to 1$. Recall Abel's theorem: if $a_n \geq 0$ for all n and $\sum_n a_n s^n$ is finite for all $|s| < 1$, then

$$\lim_{s \uparrow 1} \sum_{n=0}^{\infty} a_n s^n = \sum_{n=0}^{\infty} a_n$$

We apply this with

$$\lim_{s \uparrow 1} F_{ii}(s) = f_{ii}$$

We conclude that the state i is persistent ($F_{ii}(1) = 1$) if and only if

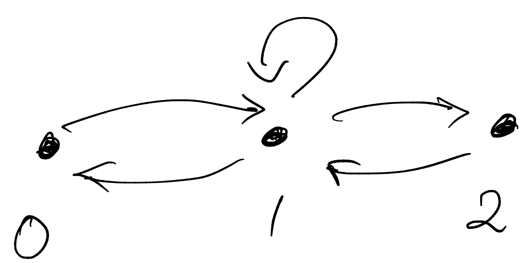
$$P_{ii}(1) = \sum_n p_{ii}(n) = \infty$$

So this is our useful test: the state i is persistent if and only if

$$\sum_{n=1}^{\infty} P(X_n = i \,|\, X_0 = i) = \infty$$

Ex Balls in boxes.

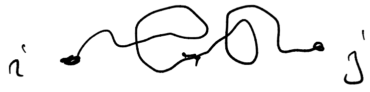
X_n = # Black balls in Box 1



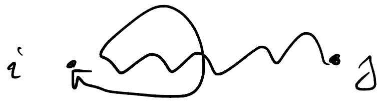
Deduced that state 1 is persistent.

i.e $f_{11} = 1$.

Are states 0,2 also persistent?



$P_{ij}(n) = \mathbb{P}(X_n = j \mid X_0 = i) > 0$



$P_{ji}(m) = \mathbb{P}(X_m = i \mid X_0 = j) > 0$

Classification of Markov chains

We say that the states i and j *intercommunicate* if there are integers n, m such that $p_{ij}(n) > 0$ and $p_{ji}(m) > 0$. In other words it is possible to go from each state to the other after a finite number of steps.

Theorem 1 *Suppose i, j intercommunicate, then they are either both transient or both persistent.*

Proof: Since i, j intercommunicate there are integers n, m such that

$$h = p_{ij}(n)p_{ji}(m) > 0 \tag{10}$$

Hence for any r,

$$(*) \qquad p_{ii}(n+m+r) \geq p_{ij}(n)p_{jj}(r)p_{ji}(m) = h\, p_{jj}(r) \tag{11}$$

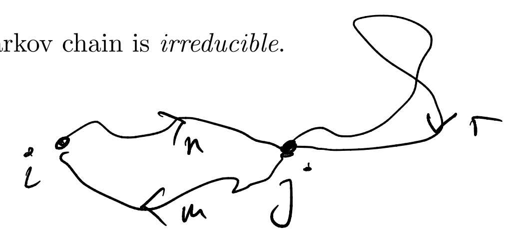
Sum over r to deduce

$$\sum_k p_{ii}(k) \geq \sum_r p_{ii}(n+m+r) \geq h \sum_r p_{jj}(r) \tag{12}$$

Therefore either both sums are finite or both are infinite, hence either both states are transient or both are persistent.

If all states intercommunicate we say that the Markov chain is *irreducible*.

$(*)$ $P_{ii}(n+m+r)$



$$\geq \underbrace{P_{ij}(n)}\, P_{jj}(r)\, \underbrace{P_{ji}(m)} = h\, P_{jj}(r).$$

$h > 0$

all integers $r \geq 0$.

Sum over r:

$$\sum_{r=0}^{\infty} P_{ii}(n+m+r) \geq h \sum_{r=0}^{\infty} P_{jj}(r)$$

Notice that

$$\sum_{r=0}^{\infty} P_{ii}(n+m+r) \leq \sum_{\ell=0}^{\infty} P_{ii}(\ell)$$

$$\Rightarrow \sum_{l=0}^{\infty} p_{ii}(l) \geq h \sum_{r=0}^{\infty} p_{jj}(r)$$

suppose state j is persistent

$\Rightarrow$ RHS $= \infty$.

$\Rightarrow$ LHS $= \infty$

$\Rightarrow$ state i is also persistent.

Ex. Balls in boxes

States $0, 1$: intercommunicate ?

$$\left.\begin{array}{l} p_{01} > 0 \\ p_{10} > 0 \end{array}\right\} \checkmark \text{ yes.}$$

States 0, 2 intercommunicate?

$$\left.\begin{aligned} P_{02}(2) &> 0 \quad \text{(two steps)} \\ P_{20}(2) &> 0 \quad \text{(two steps)} \end{aligned}\right\} \checkmark \text{ yes.}$$

<u>Notation</u>: when all states in chain intercommunicate, we say the chain is <u>irreducible</u>.

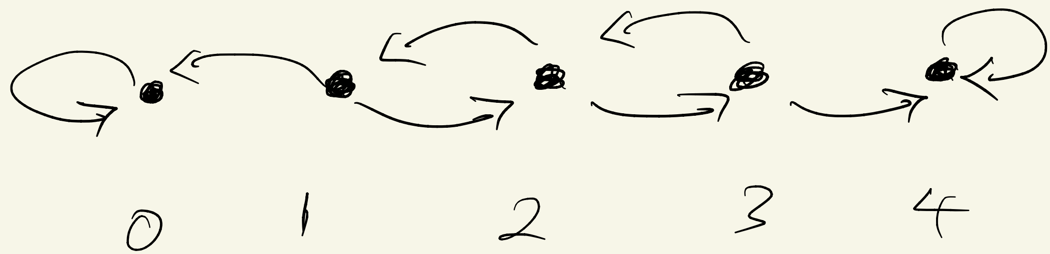
<u>Therefore</u>:

know state 1 is persistent

$\Rightarrow$ states 0, 2 are persistent.

Ex

Drunkard's Walk



States 0, 4 are persistent:

$$P_{00}(n) = \mathbb{P}(X_n = 0 \mid X_0 = 0) = 1$$

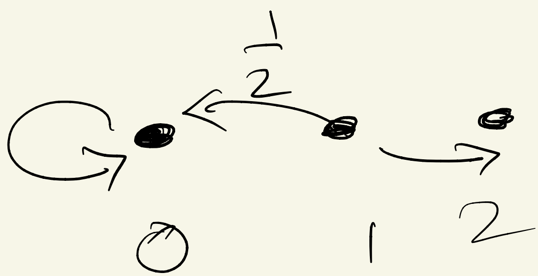
$$\Rightarrow \sum_{n=0}^{\infty} P_{00}(n) = \sum_{n=0}^{\infty} 1 = \infty.$$

Absorbing state is persistent.

What about states 1, 2, 3?

They intercommunicate, so either all persistent or all transient.

State 1:



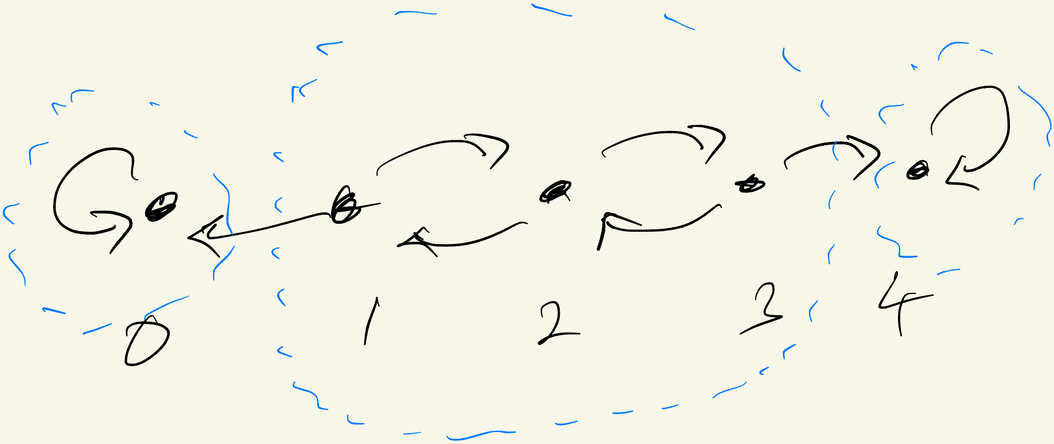
$$\mathbb{P}(X_1 = 0 \mid X_0 = 1) = \frac{1}{2}.$$

$$f_{00}(n) = \mathbb{P}(X_n = 0, X_{n-1} \neq 0, \dots, X_1 \neq 0 \mid X_0 = 0)$$

$$\leq \frac{1}{2}$$

$$\boxed{\begin{gathered}\mathbb{P}(X_n = 1 \text{ some } n \geq 1 \mid X_0 = 1) \\ \leq \frac{1}{2}\end{gathered}}$$

$f_{11} \leq \frac{1}{2} \Rightarrow$ state 1 is transient.



Three equivalence classes of states:

$\{0\}$ persistent class (closed)

$\{4\}$ persistent class (closed)

$\{1, 2, 3\}$ transient. class.

Irreducible $\Leftrightarrow$ one equiv. class.

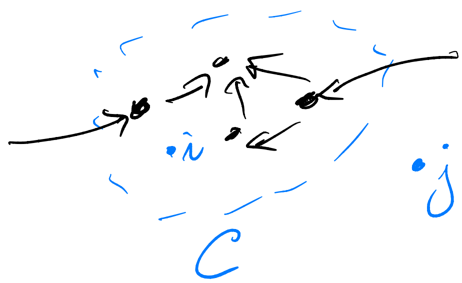
A class of states C in Ω is called *closed* if $p_{ij} = 0$ whenever $i \in C$ and $j \notin C$.

There is a fairly obvious decomposition of the state space.
The state space Ω can be partitioned uniquely as

$$\Omega = T \cup C_1 \cup C_2 \cup \cdots \tag{13}$$

where T is the set of all transient states, and each class C_i is closed and irreducible.

If the chain starts with $X_0 \in C_i$ then it stays in C_i forever. If it starts with $X_0 \in T$ then eventually it enters one of the classes C_i and stays there forever. So the irreducible classes determine the *long-time behavior*.

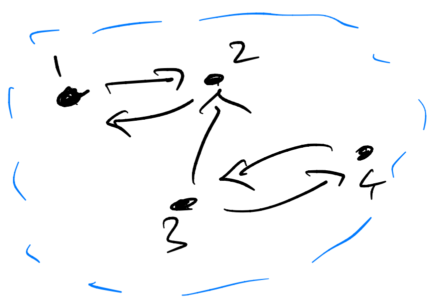


$i \in C$

$j \notin C$

No jumps out from C

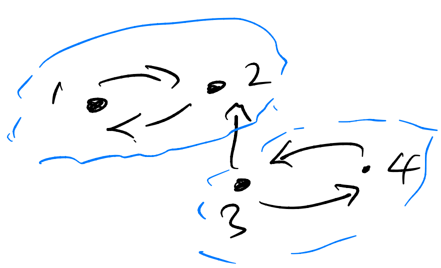
$\Rightarrow$ C is closed.



closed

not irreducible

$\{1,2\}$ closed, irreduc.



$\{3,4\}$ transient.