Lab 3a. Stability Problems

Task 2.1

$$\boldsymbol{A} = \frac{\mathbf{1}}{\mathbf{8}}\begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{1} \\ \mathbf{3} & \mathbf{4} & \mathbf{1} \\ \mathbf{2} & \mathbf{3} & \mathbf{4} \end{bmatrix}$$

Eigen Values are

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} \approx \begin{bmatrix} 0.8786 \\ -0.0457 \\ 0.2920 \end{bmatrix}$$

$|\lambda_i| < 1$ => System with matrix A is asymptotically stable

Task 2.2
From Matlab Code,
For s = 1,

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 1.8786 \\ -0.0569 \\ 0.1748 \end{bmatrix}$$

$|\lambda_1| > 1$ => System with matrix A is unstable

Task 2.3
Using Matlab code,
Values of "a" and "b" are,
$a = 2.9200$
$b = 0.1150$
$\therefore (-a, b) = (-2.92, 0.115)$

Lab 3 a. - Discrete Dynamical Systems and Markov Models

Problems 1.4

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 + \alpha_B - \beta_B & -\gamma_F \\ \rho_B & 1 - \beta_F \end{pmatrix}$$

Eigen Values are

$$\lambda_1 = \frac{(a+d) + \sqrt{(a-d)^2 + 4bc}}{2}$$
$$\lambda_2 = \frac{(a+d) - \sqrt{(a-d)^2 + 4bc}}{2}$$

In most cases in real world, for population dynamics, we have

$$\propto_B > \beta_B => a > 1$$
$$\gamma_F > 0 => b < 0$$
$$\rho_B > 0 => c > 0$$
$$\beta_F < 1 => d > 0$$

However,

In order to analyze different scenarios, we can mathematically try different scenarios values for a, b, c, d that may not make much sense in the real world.

1. **Both die out.**

   Let A be diagonalizable

$$A = PDP^{-1}$$
$$A^n = PD^nP^{-1}$$
$$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

   If $\lambda_1 < 0\ and\ \lambda_2 < 0$

$$\lim_{n\to\infty} A^n = \lim_{n\to\infty} PD^nP^{-1}$$

$$= \lim_{n\to\infty} P \begin{pmatrix} {\lambda_1}^n & 0 \\ 0 & {\lambda_2}^n \end{pmatrix} P^{-1}$$
$$= 0$$

   The values chosen for this scenario are as follows:

$$\propto_B = 0.5, \qquad \beta_B = 1, \qquad \rho_B = 0.01$$

$$\gamma_F = 0.5, \qquad \beta_F = 0.1$$

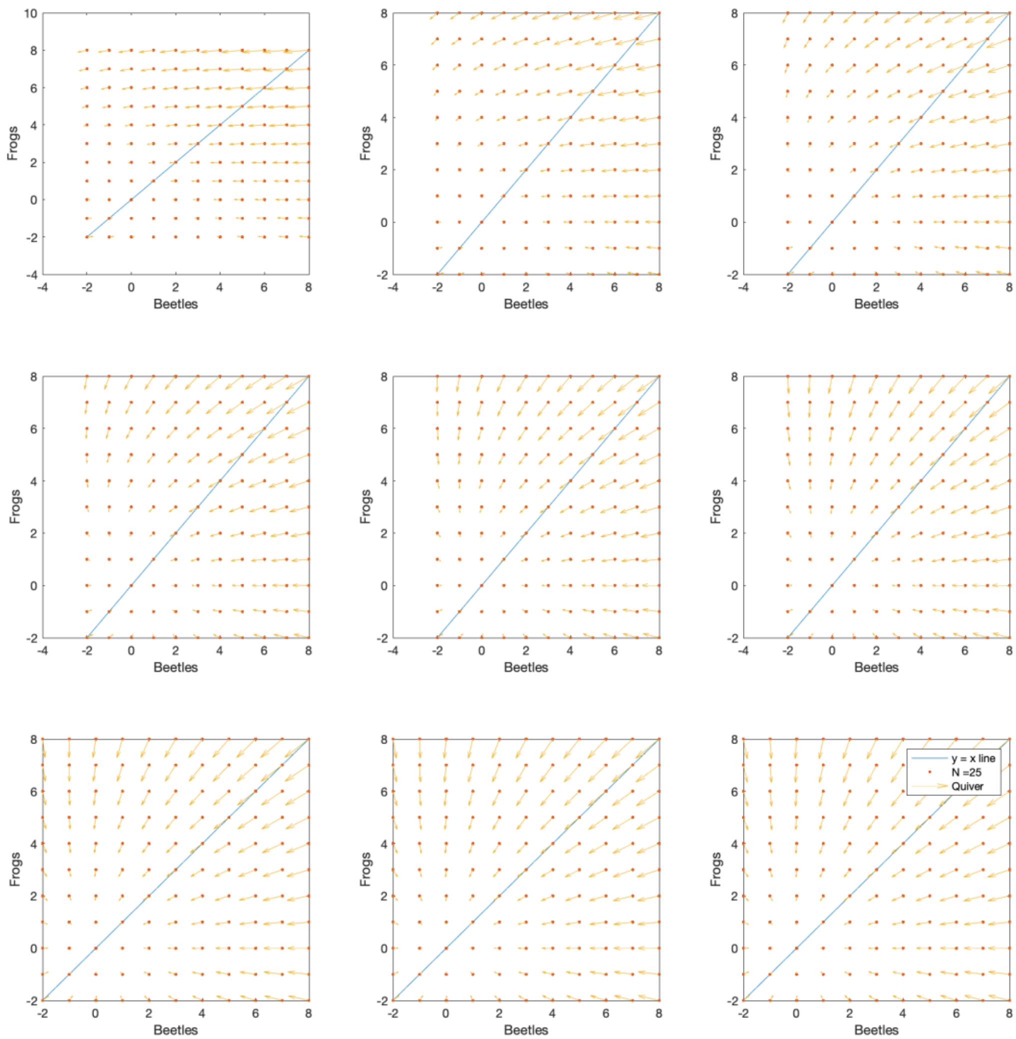
   Which yields,

$$a = 0.5, b = -0.5, c = 0.01, d = 0.9$$

The eigen values are:

$$\begin{bmatrix}\lambda_1\\ \lambda_2\end{bmatrix} = \begin{bmatrix}0.5129\\ 0.8871\end{bmatrix}$$

The graph looks like:



The initial populations chosen from range between -2 to 8 in a grid fashion.
Negative values are just for experimentation.
The analysis is done from following 9 years:
1, 4, 7, 10, 13, 16, 19, 22, 25

Population finally goes to zero.

2. Both grow exponentially

Let $\lambda_1 > 1$ and $\lambda_2 < 1$
In this case,
$\lim_{n\to\infty} A^n = \lim_{n\to\infty} PD^nP^{-1}$

$= \lim_{n\to\infty} P\begin{pmatrix} {\lambda_1}^n & 0 \\ 0 & {\lambda_2}^n \end{pmatrix} P^{-1}$

${\lambda_1}^n \to \infty$, ${\lambda_2}^n \to 0$

$\therefore Rank\ (A^n) = 1\ (\ two\ zeroes\ in\ last\ row)$

⇨ $\begin{bmatrix} B_{n+1} \\ F_{n+1} \end{bmatrix} = A^n \begin{bmatrix} B_0 \\ F_0 \end{bmatrix}$

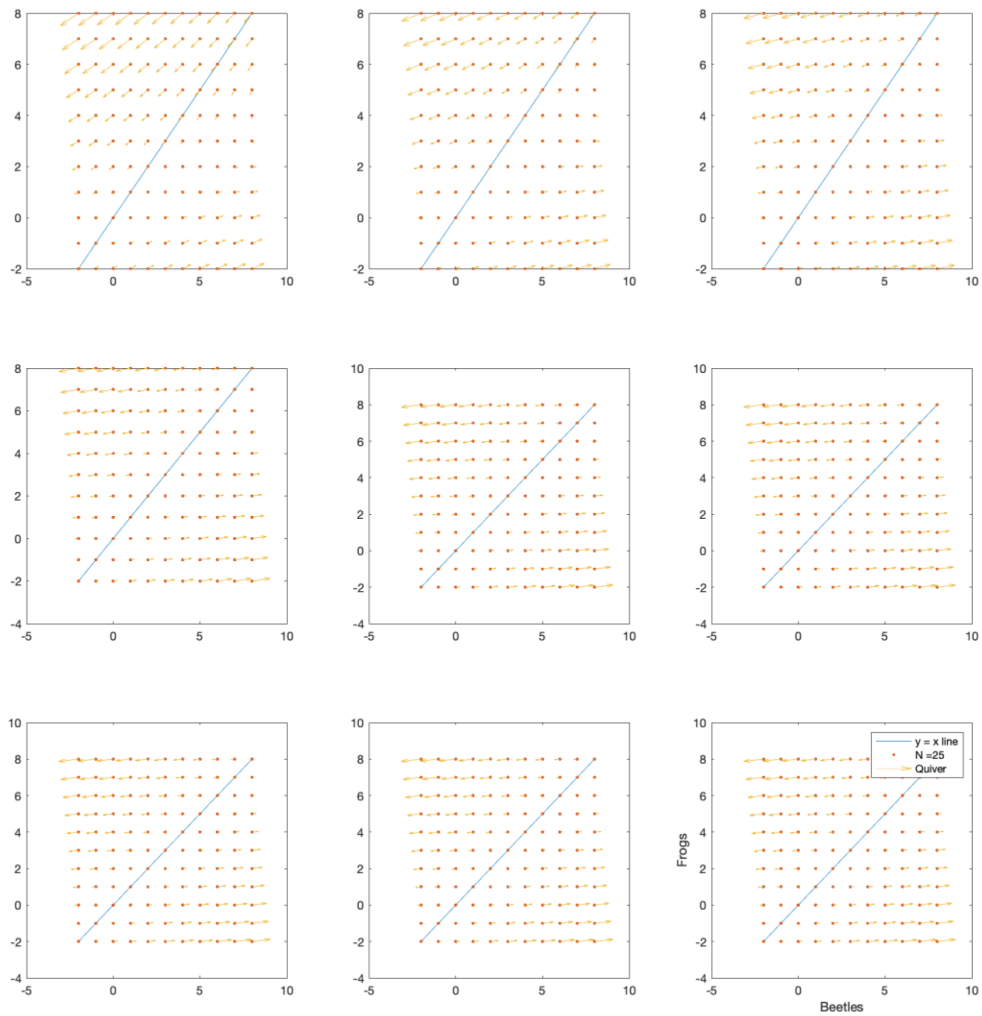
$\therefore$ Depending on initial conditions, either both increase or both decrease
Sample values for this scenario are,

$$\propto_B = 1, \qquad \beta_B = 0.5, \qquad \rho_B = 0.1$$

$$\gamma_F = 1, \qquad \beta_F = 0.5$$

$\therefore A = \begin{pmatrix} 1.5 & -1 \\ 0.1 & 0.5 \end{pmatrix}$; $\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 1.3873 \\ 0.6127 \end{bmatrix}$

The graph is shown below:



From graph, it is evident that, for higher values of initial frog population, both tend to perish which makes complete sense because if more and more frogs eat beetles, then both die as prey is only source for predator survival.

On contrary, when we have initial high beetle population, they grow exponentially, and help frigs grow exponentially. This scenario is described by points closer to x-axis.

3. **The beetles grow exponentially, but the frogs die out.**
   This looks like an impossible scenario, because if beetles grow exponentially, frogs (predator) get more and more prey (beetles) thus supporting for survival.
   But mathematically it is possible to achieve a solution which can make sense in real world, if following happens.

   Suppose beetles are infected by usage of pesticides on near by farms. Therefore, consuming more and more beetles (prey) can cause frog (predator) to die out as it is assumed that prey is only source for predator survival.

   The parameters that justify this explanation are as follows:

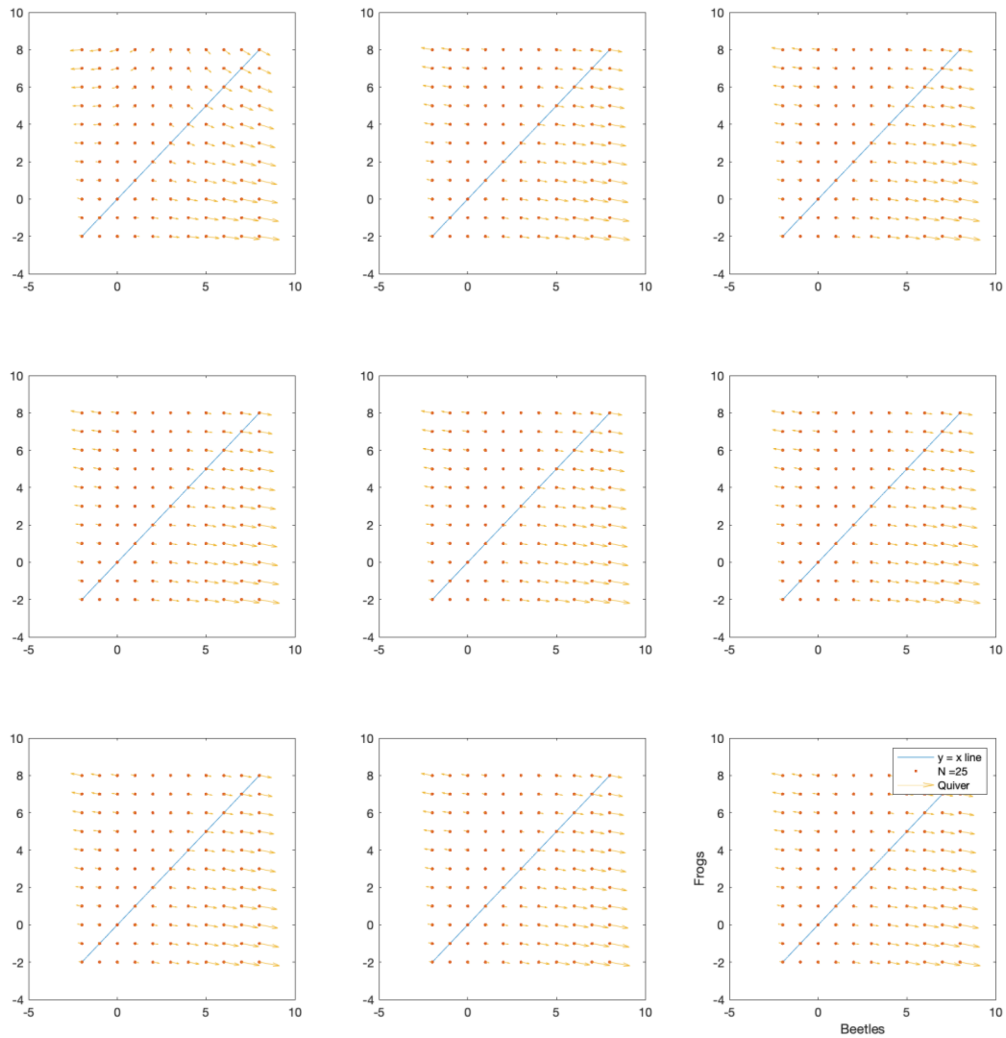
$$\propto_B = 5.5, \qquad \beta_B = 0.5, \qquad \rho_B = -1$$

$$\gamma_F = 2, \qquad \beta_F = 0.5$$

Negative value for ρ_B means eating beetle (prey) kills frog (predator)

$$\therefore A = \begin{pmatrix} 6 & -2 \\ -1 & 0.5 \end{pmatrix} ; \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 6.3423 \\ 0.1577 \end{bmatrix}$$

The graph looks like:



It can be seen that, for high initial values of beetle population, frogs eat more and more beetle and eventually perish, which is supported by direction of arrow. Once the population of frogs become zero, we need to remove it from the equation.

4. The frogs grow exponentially but the beetles die out.

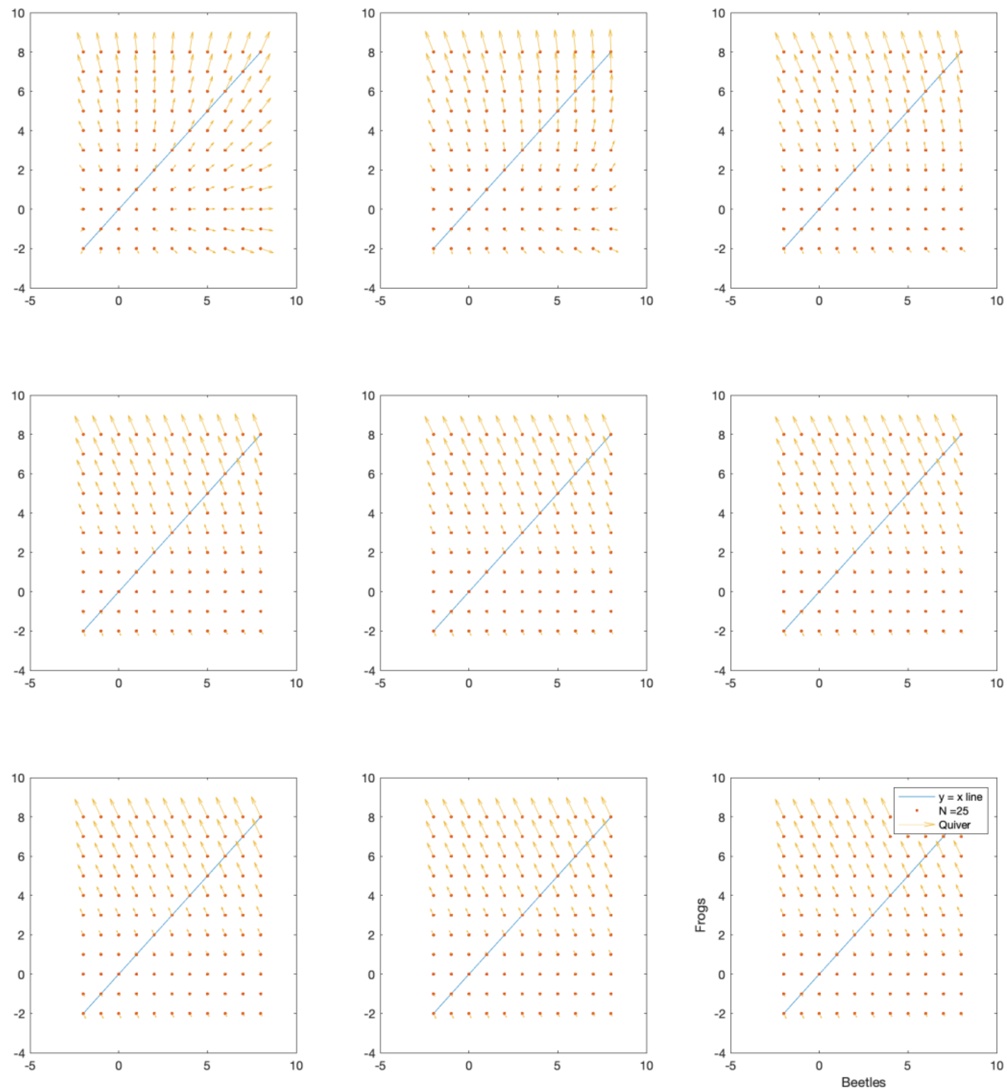
It is known that predator (frog) survives only because of prey (beetle). Therefore, if prey dies out then predator also should die. However, if $\beta_F < 0$, then mathematically we have a solution. This is not possible in real world.
The parameters for this scenario are:

$$\propto_B = 3, \qquad \beta_B = 1, \qquad \rho_B = 0.1$$

$$\gamma_F = 0.5, \qquad \beta_F = -3$$

$$\therefore A = \begin{pmatrix} 3 & -0.5 \\ 0.1 & 4 \end{pmatrix} ; \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 3.0528 \\ 3.9472 \end{bmatrix}$$

The graph looks like:



It can be seen that population of frogs increase over time and beetles die out.