MATH 5131 Fall 2021 Take-Home Test 1

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1. (16 pts) Solve the initial value problem explicitly for y using separation of variables

$$x^2\frac{dy}{dx} = \frac{4x^2-x-2}{(x+1)(y+1)}, \quad y(1)=1$$

$$(y+1)\,dy = \frac{(4x^2-x-2)}{(x+1)x^2}\,dx \Rightarrow \int (y+1)\,dy = \int\left(\frac{3}{x+1}+\frac{1}{x}-\frac{2}{x^2}\right)dx$$

Partial Fraction Decomposition Method:

$$\frac{4x^2-x-2}{(x+1)x^2} = \frac{A}{x+1}+\frac{B}{x}+\frac{C}{x^2} = \frac{Ax^2+Bx(x+1)+C(x+1)}{x^2(x+1)} \Rightarrow$$

$$4x^2-x-2 = Ax^2+Bx^2+Bx+Cx+C = x^2(A+B)+x(B+C)+C$$

x^2: $4=A+B$ $\qquad C=-2$

x^1: $-1=B+C$ $\quad\Leftrightarrow\quad$ $B=-1-C=-1-(-2)=1$

x^0: $-2=C$ $\qquad A=4-B=4-1=3$

$$\frac{(y+1)^2}{2} = 3\ln|x+1|+\ln|x|+\frac{2}{x}+C_1$$

$$(y+1)^2 = 6\ln|x+1|+2\ln|x|+\frac{4}{x}+C_1$$

$$(y+1)^2 = \ln(x+1)^6+\ln(x^2)+\frac{4}{x}+C_1$$

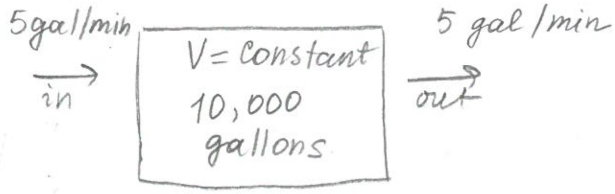
I.C. $(1+1)^2 = \ln(1+1)^6+\underbrace{\ln(1)^2}_{0}+\frac{4}{1}+C_1$

$$\cancel{4} = \ln(2^6)+\cancel{4}+C_1 \Rightarrow C_1=-\ln(2^6)=-6\ln(2)$$

$$y+1=\sqrt{\ln(x+1)^6+\ln(x^2)+\frac{4}{x}-6\ln 2}$$

Finally, $\boxed{y(x)=\sqrt{6\ln(x+1)+2\ln x+\frac{4}{x}-6\ln 2}-1}$

2. (18 pts) A swimming pool whose volume is 10,000 gallons contains water that is 0.01% chlorine. Starting at $t = 0$, city water containing 0.001% chlorine is pumped into the pool at a rate of 5 gal/min. The pool water flows out at the same rate. What is the percentage of chlorine in the pool after 1 hour? When will the pool water be 0.002% chlorine?



$x(t)$ = volume of chlorine at time t

$$\frac{dx}{dt} = \text{input rate} - \text{output rate} = 5\,\frac{gal}{min} \cdot \frac{0.001\%}{100\%} - 5\,\frac{gal}{min}\,\frac{x(t)\ gal}{10{,}000\ gal}$$

$$\frac{dx}{dt} = 5\cdot 10^{-5} - 5\cdot 10^{-4}x \Rightarrow \frac{dx}{dt} + 5\cdot 10^{-4}x = 5\cdot 10^{-5} \quad (\text{linear} + \text{integrating factor})$$

$$I(t) = e^{\int 5\cdot 10^{-4}dt} = e^{5\cdot 10^{-4}t}$$

$$\Downarrow$$

$$(e^{5\cdot 10^{-4}t}x) = \int e^{5\cdot 10^{-4}t}\cdot 5\cdot 10^{-5}dt = 5\cdot 10^{-5}\frac{e^{5\cdot 10^{-4}t}}{5\cdot 10^{-4}} + C_1 = 0.1e^{5\cdot 10^{-4}t} + C_1 \Rightarrow$$

$$x(t) = C_1 e^{-5\cdot 10^{-4}t} + 0.1$$

$$x(0) = C_1 + 0.1 = \frac{0.01\%}{100\%} \times 10{,}000 = 1 \Rightarrow C_1 = 1 - 0.1 = 0.9$$

$$\boxed{x(t) = 0.9e^{-5\cdot 10^{-4}t} + 0.1}$$

$$x(60\,min) = 0.9e^{-5\cdot 10^{-4}\cdot 60} + 0.1 = 0.9e^{-0.03} + 0.1 = 0.9734 \approx 0.97$$

$$\boxed{\frac{x(60)}{100\%} = \frac{0.97}{100} = 0.0097\%}\ \text{after 1 hour}$$

$$\frac{x(t^*)}{10{,}000}\cdot 100\% = 0.002\%$$

$$\frac{0.9e^{-5\cdot 10^{-4}t^*} + 0.1}{100} = 2\cdot 10^{-3} \iff 0.9e^{-5\cdot 10^{-4}t^*} + 0.1 = 0.2 \Rightarrow$$

$$t^* = \frac{1}{-5\cdot 10^{-4}}\ln\left(\frac{0.1}{0.9}\right) = -0.2\cdot 10^4 \ln(1/9) = 0.2\cdot 10^4 \ln(9) = 2000\ln(9)$$

$$\boxed{t^* \approx 4394.45\ min \quad \text{or} \quad 73.24\ hours}$$

3. (16 pts) Carbon dating is often used to determine the age of a fossil. For example, a humanoid skull was found in a cave in South Africa along with the remains of a campfire. Archaeologists believe the age of the skull to be the same age as the campfire. It is determined that only 2 % of the original amount of carbon-14 remains in the burnt wood of the campfire. Estimate the age of the skull if the half-life of carbon-14 is about 5568 years.

half-life of carbon-14 = 5568 years

Let R_0 be the amount of carbon-14 in the burnt wood when it was produced, at $t=0$

let $R(t^*)$ be the amount of carbon-14 in the found sample, then

$R(t^*) = 0.02 R_0$

Model for the radioactive decay is $R(t) = R_0 e^{-kt}$ and

$k = \frac{\ln 2}{5568}$

$R(t^*) = R_0 e^{-kt^*} = 0.02 R_0 \Rightarrow e^{-kt^*} = 0.02 \Rightarrow t^* = -\frac{1}{k} \ln(0.02)$

$t^* = -\frac{1}{\ln 2/5568} \ln(0.02) = -\frac{5568 \ln(0.02)}{\ln 2} \approx \boxed{31,425 \text{ years}}$

4. (15 pts) The separation of equation $\frac{dy}{dx} = p(y)q(x)$ requires division by $p(y)$, and this may disguise the fact that the roots of the equation $p(y) = 0$ are actually constant solutions to the differential equation.

(a) To explore this further, separate the equation

$$\frac{dy}{dx} = (x-3)(y+1)^{2/3},$$

to derive the solution,

$$y = -1 + (x^2/6 - x + C)^3.$$

(b) Show that $y \equiv -1$ satisfies the original equation $\frac{dy}{dx} = (x-3)(y+1)^{2/3}$.

(c) Show that there is no choice of the constant C that will make the solution in part (a) yield the solution $y \equiv -1$. Thus, we lost the solution $y \equiv -1$ when we divide by $(y+1)^{2/3}$.

(a) $\frac{dy}{(y+1)^{2/3}} = (x-3)\,dx$

$$\int (y+1)^{-2/3}\, d(y+1) = \int (x-3)\, d(x-3)$$

$$\Downarrow$$

$$\frac{(y+1)^{-2/3+1}}{-2/3+1} = \frac{(x-3)^2}{2} + C$$

$$3(y+1)^{1/3} = \frac{1}{2}(x-3)^2 + C$$

$$(y+1)^{1/3} = \frac{1}{6}(x-3)^2 + C = \frac{1}{6}(x^2-6x+9) + C = \frac{1}{6}x^2 - x + \underbrace{\frac{3}{2} + C}_{\text{combine}}$$

$$\uparrow 3 \mid (y+1)^{1/3} = \frac{1}{6}x^2 - x + C \Rightarrow \boxed{y = -1 + (x^2/6 - x + C)^3}$$

(b) $y \equiv -1$; LHS $\frac{d}{dx}(-1) = 0$; RHS $= (x-3)(-1+1)^{2/3} = 0 \Rightarrow$

$y \equiv 1$ indeed satisfies the equation

(c) From $y = -1 + (\frac{x^2}{6} - x + C)^3$ we try to determine the constant C

$$-1 = -1 + (\frac{x^2}{6} - x + C)^3$$

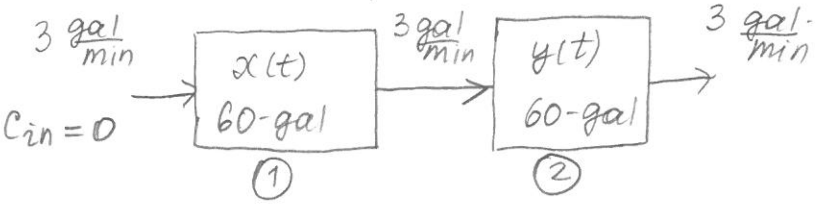
$$\Downarrow$$

$$0 = (\frac{x^2}{6} - x + C)^3 \Rightarrow \frac{x^2}{6} - x + C = 0$$

We can not find a C (one constant) that makes the quadratic function be zero for all x. Impossible! Therefore the solution set $y = -1 + (\frac{x^2}{6} - x + C)^3$ does not include the solution $y \equiv -1$.

5. (18 pts) Beginning at time $t = 0$, fresh water is pumped at the rate of 3 gal/min into a 60-gal tank initially filled with brine. The resulting less-and-less salty mixture overflows at the same rate into a second 60-gal tank that initially contained only pure water, and from there it eventually spills onto the ground. Assuming perfect mixing in both tanks, when will the water in the second tank taste saltiest? And exactly how salty will it then be, compared with the original brine?

A diagram (cascade)



Let $x(t)$ be the amount of salt in tank 1 and $y(t)$ be the amount of salt in tank 2, in kilograms

$C_1(t) = \frac{x(t)}{60}$, the concentration of salt in tank 1

$C_2(t) = \frac{y(t)}{60}$, — " —— " —— " —— " —— " tank 2

$C_1(0) = C_{10}$, $C_2(0) = 0$

Model: ① $\frac{dx}{dt} = 3\frac{gal}{min} \cdot 0 \frac{kg}{gal} - 3\frac{gal}{min}\,\frac{x}{60}\,\frac{kg}{gal} = -\frac{3}{60}x \left[\frac{kg}{min}\right]$ $x(0) = x_0$

② $\frac{dy}{dt} = 3\frac{gal}{min}\,\frac{x(t)}{60}\,\frac{kg}{min} - 3\frac{gal}{min}\,\frac{y(t)}{60}\,\frac{kg}{gal} = \frac{1}{20}x - \frac{1}{20}y \left[\frac{kg}{min}\right]$

$y(0) = 0$

Solving ①: $\begin{cases} x' = -\frac{1}{20}x \\ x(0) = x_0 \end{cases} \Rightarrow$ $x(t) = Ce^{-\frac{1}{20}t}$ - general solution

$x(t) = x_0 e^{-\frac{1}{20}t}$ - particular solution

Solving ② $\begin{cases} y' + \frac{1}{20}y = \frac{1}{20}x_0 e^{-\frac{1}{20}t} \\ y(0) = 0 \end{cases}$, $I(t) = e^{\int \frac{1}{20}dt} = e^{\frac{1}{20}t}$

$\frac{d}{dt}\left(e^{\frac{1}{20}t}y\right) = \frac{1}{20}x_0 \Rightarrow e^{\frac{1}{20}t}y = \frac{1}{20}x_0 t + C \Rightarrow y(t) = \frac{1}{20}x_0 t e^{-\frac{1}{20}t} + Ce^{-\frac{1}{20}t}$

$y(0) = 0 + C = 0 \Rightarrow \boxed{y(t) = \frac{x_0 t}{20}e^{-\frac{1}{20}t}}$

- Maximum occurs when $y'(t) = 0 = \frac{x_0}{20}\left(e^{-\frac{1}{20}t} - \frac{t}{20}e^{-\frac{1}{20}t}\right) = \frac{x_0 e^{-\frac{1}{20}t}}{20}\left(1 - \frac{t}{20}\right) \Rightarrow$

$1 - \frac{t}{20} = 0 \Rightarrow \boxed{t = 20 \text{ min}}$

- salinity: $\frac{y(20)}{x_0} = \frac{x_0[(20)/20]e^{-\frac{1}{20}\cdot 20}}{x_0} = \boxed{e^{-1}} \approx 0.367879 \approx 0.37$

about 37% of the original brine.

6. (17 pts) Use the method discussed under *Bernoulli Equations* to solve the problem

$$\frac{dy}{dx} = \frac{y^2 + 2yx}{x^2}$$

$$\frac{dy}{dx} = \frac{y^2}{x^2} + \frac{2}{x}y$$

$$\frac{dy}{dx} - \frac{2}{x}y = \frac{1}{x^2}y^2 \qquad (n=2)$$

Substitution $w = y^{1-2} = \frac{1}{y}$ $(y \neq 0)$ (Checking later)

$$\frac{dw}{dx} + (1-2)\left(-\frac{2}{x}\right)w = \frac{1}{x^2}(1-2) \iff \frac{dw}{dx} + \frac{2}{x}w = -\frac{1}{x^2}$$

$$I(x) = e^{\int \frac{2}{x}dx} = e^{\ln|x^2|} = x^2$$

$$\frac{d}{dx}(x^2 w) = -1 \Rightarrow x^2 w = -x + C \Rightarrow w = -\frac{1}{x} + \frac{C}{x^2}$$

Returning to the old variable y:

$$y = \frac{1}{w} = \frac{1}{-\frac{1}{x} + \frac{C}{x^2}} = \frac{1}{\frac{-x+C}{x^2}} = \frac{x^2}{C-x}$$

Checking $y \equiv 0$: LHS: $\frac{d(0)}{dx} = 0$ RHS: $\frac{0^2}{x^2} + \frac{2}{x} \cdot 0 = 0$.

Therefore the general solution is

$$\boxed{\begin{array}{l} y = \dfrac{x^2}{C-x} \\ \\ y = 0 \end{array}}$$

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