Northeastern University, Department of Mathematics

MATH 5110: Applied Linear Algebra and Matrix Analysis.

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§6 Determinant

1.

Motivation

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Cofactor expansion
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 Linearity Property of the determinant function and Cramer's Rule

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1. Motivation

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Coali clefhe a number da(A) for nxn mount sty

det Af0 (A) A is highter

A is higher the.

$$A = \frac{1}{\det A} \left(\frac{1}{a_{11} a_{12}} - a_{1n} \right)$$

$$A = \begin{bmatrix} a_{11} a_{12} - a_{1n} \\ - - a_{1n} \end{bmatrix}$$

2. Cofactor expansion



Definition 1. Let A be an $n \times n$ matrix.

The first row cofactor expansion formula for the determinant of A is

Aij is the (n-1) x (n+1) makes obtained by delety i-ts now and j-th column of A

Facts about **permutation groups**.

Let [n] be the set of n integers $[n] = \{1, 2, \dots, n\}$. The **permutation group** (symmetric group) S(n) is $\{ \{ \} \} \}$

$$6 = \begin{pmatrix} 1 & 2 & -1 & N \\ 6(1) & 6(2) & -1 & 6(N) \end{pmatrix}$$

$$(N) \xrightarrow{6_1} (N) \xrightarrow{6_1} (N)$$

Ex
$$\left(\begin{pmatrix} 2 \\ 4 \end{pmatrix} \right) = (2,4)$$
 $G_1 \cdot G_2 = G_2 \cdot G_1$

A transposition is a permutation in S(n) that only switch 2 numbers.

Permutation presents only in even numbers never in odd numbers
$$\begin{pmatrix}
1 & 2 & 7 \\
2 & 3 & 1
\end{pmatrix} = (1,2)\cdot(2,7) = (1,3)\cdot(1,7)$$

$$\begin{pmatrix}
1 & 2 & 7 \\
2 & 3 & 1
\end{pmatrix} = (1,2)\cdot(2,7) = (1,3)\cdot(1,7)$$

The sign of a permutation $\sigma \in \underline{S(n)}$ is

$$\iint \frac{\operatorname{sign}(\sigma) = (-1)^{T(\sigma)}}{\blacksquare}$$

where $T(\sigma)$ is the number of transposition of σ .

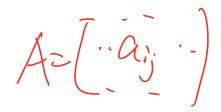
Another equivalent way to determine the sign of σ is

$$\operatorname{sign}(\sigma) = (-1)^{N(\sigma)},$$

where $N(\sigma)$ is the number of inversions of σ .

An inversion of $(\sigma(1) \ \sigma(2) \ \dots \ \sigma(n))$ is the pair of numbers $(\sigma(i) > \sigma(j))$ for i < j.

Proposition 2. If τ is obtained from σ by switch two numbers i, j, then $sign(\tau) = -sign(\sigma)$.



Theorem 3. If A is an $n \times n$ matrix, then



$$\det(A) = \sum_{\sigma \in S(n)} \operatorname{sign}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}$$

$$= \sum_{\sigma \in S(n)} \operatorname{sign}(\sigma) a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}.$$

Proof. This theorem can be proved by induction on n. For n = 1, it is true. Suppose the formula is true for n - 1, let's show that it is true for n.

$$\sum_{\sigma \in S(n)} \operatorname{sign}(\sigma) a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}$$

$$= \sum_{i=1}^{n} a_{1i} \sum_{\sigma \in S(n); \sigma(1)=i} \operatorname{sign}(\sigma) a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}$$

$$= \sum_{i=1}^{n} a_{1i} \sum_{\sigma \in S(n); \sigma(1)=i} (-1)^{1+i} \operatorname{sign}(\sigma(2) \dots \sigma(n)) a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}$$

$$= \sum_{i=1}^{n} (-1)^{1+i} a_{1i} \det A_{1i}$$

$$= \det A$$

Example 4. Let A be the 3×3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$sign(1 \ 2 \ 3) = 1$$

 $sign(1 \ 3 \ 2) = -1$
 $sign(2 \ 1 \ 3) = -1$
 $sign(2 \ 3 \ 1) = 1$
 $sign(3 \ 1 \ 2) = 1$
 $sign(3 \ 2 \ 1) = -1$

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Hence we have

$$\det(A) = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33} - a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

Example 5. Find the determinant of
$$A = \begin{bmatrix} 0 & 4 & 2 \\ 5 & 2 & 2 \\ 0 & 2 & -1 \end{bmatrix}$$
. Is A invertible?

Definition 6. Let A be an $n \times n$ matrix. Its (i, j)-th cofactor C_{ij} is

$$C_{ij} := (-1)^{i + j} det A_{ij}$$

Using cofactors, the first row cofactor expansion formula for the determinant of A is

Theorem 7. The determinant of an
$$n \times n$$
 matrix A can be computed by he i -th

row cofactor expansions

and the j-th column cofactor expansions

Example 8. Find the determinant of
$$A = \begin{bmatrix} 1 & 2 & 3 & 1.2 \\ 0 & 0 & 0 & 2 \\ 5 & 6 & 7 & \pi \\ 0 & 1 & 2 & \sqrt{2} \end{bmatrix}$$

Theorem 9. Let A be an $n \times n$ triangular matrix, the determinant



Example 10. Find out for which value of λ the matrix $A - \lambda I$ is not invertible, where

$$A = \begin{bmatrix} 2 & \sqrt{2} & 1.7 \\ 0 & 3 & 12 \\ 0 & 0 & 5 \end{bmatrix}$$

Proposition 11.

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ b_1 + c_1 & b_2 + c_2 & \cdots & b_n + c_n \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ b_1 & b_2 & \cdots & b_n \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ c_1 & c_2 & \cdots & c_n \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$



3. Row Operations and Determinant



Recall the three types of elementary row operations:

- 1. (Replacement)
- 2. (Interchange)
- 3. (Scaling)

Theorem 12 Row Operations and the Determinant). Let A be an $n \times n$ matrix.

then
$$det B = -det A$$
 $\Leftrightarrow det(E;A)$



$$det B = k det A$$



then
$$dot B = det A$$



->relA ----

Example 13. In a matrix A, if the <u>i-th</u> row equals the <u>j-th</u> row, then

Example 14. In a matrix A, if the i-th row is a scalar product of the j-th row, then det A=0

Theorem 15. An $n \times n$ matrix A is invertible if and only if $\det A \neq 0$.

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Proposition 16. Let A be an $n \times n$ matrix.

 $\det(kA) = (k^n)(\det A).$

Theorem 17 (Determinants of Products of Matrices). Let A and B be two $n \times n$ matrices.

$$\det(AB) = (\det A)(\det B).$$

Case () A) invorbbh, then A=E, E, ... Es

det (AB) = det (E1... Fr. B) = det(E1) - .. det(E3) det)

Gre (b) A is not invortible. det A dot B



Proposition 18. Let A be an $n \times n$ matrix.

$$\det(A^m) = (\det(A))^m$$



Proposition 19. Let A be an $n \times n$ invertible matrix.

$$\det(A^{-1}) = \frac{1}{\det A}.$$

Question How about $\det(A + B)$? Is it $\det(A) + \det(B)$?

Example 20. Find the determinant of $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$. Is A invertible?

Example 21. Find the determinant of
$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 9 \\ 3 & 6 & 11 & 12 \\ 4 & 8 & 15 & 16 \end{bmatrix}$$
. Is A invertible?

Definition 22 (Elementary Column Operations).

- 1. (Column Replacement) Add to one column the multiple of another column.
- 2. (Column Interchange) Interchange two columns.
- 3. (Column Scaling) Multiply all entries of a given column by a scalar.

Theorem 23 (Column Operations and the Determinant). Let A be an $n \times n$ matrix and let B be a matrix obtained from A by a single elementary row operation.

1. If B is obtained from A by a Column Replacement operation, then

$$\det B = \det A$$
.

2. If B is obtained from A by a Column Interchange operation, then

$$\det B = -\det A.$$

3. If B is obtained from A by a Column Scaling operation by a factor k, then

$$\det B = k \det A.$$

Theorem 24 (Determinant of the Transpose Matrix).

$$(\det A^T = \det A.)$$

Example 25. Vandermonde determinant

More generally, by induction on n, we can proved that

$$\det(A) = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_n \\ a_1^2 & a_2^2 & a_3^2 & \cdots & a_n^2 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ a_1^{n-1} & a_2^{n-1} & a_3^{n-1} & \cdots & a_n^{n-1} \\ a_1^n & a_2^n & a_3^n & \cdots & a_n^n \end{vmatrix} = \begin{bmatrix} & & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & \\ & & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & \\ & & & & \\ & & & \\ & & & & \\ &$$

Block Matrix.

Theorem 26. If
$$M = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$$
, then,
$$\det(M) = \det(A) \det(C).$$

Example 27. Find the determinant of
$$M = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} = \begin{bmatrix} 1 & 2 & 11 & \sqrt{3} \\ 2 & 3 & \pi & 12 \\ 0 & 0 & 3 & 9 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

4. Linearity Property of the determinant function and Cramer's Rule

Let $\underline{A} = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_n \end{bmatrix}$ be an $n \times n$ matrix

$$\begin{array}{ccc}
& & & & \\
\hline
\mathcal{L} & : & \mathbb{R}^n \longrightarrow \mathbb{R} \\
& & & & \\
\vec{\chi} & \longrightarrow \det \left[\vec{a}_1 \cdots \vec{a}_H \vec{\chi} \vec{a}_H \cdots \vec{a}_H \right]
\end{array}$$

for each i

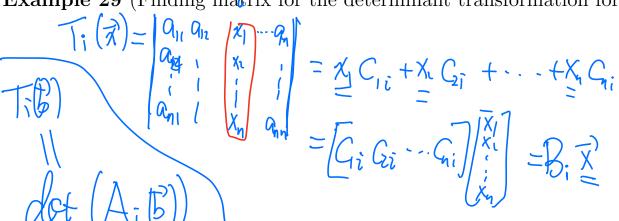
Theorem 28 (Linearity and Determinants). The transformation T defined above is a linear transformation, that is

$$0 \quad T_{r}(\vec{x}+\vec{y}) = T_{r}(\vec{x}) + T_{r}(\vec{y})$$

$$0 \quad T_{r}(\vec{x}) = c \quad T_{r}(\vec{x})$$

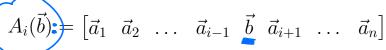
Proof. By Theorems 24, 12 and Proposition 11.

Example 29 (Finding matrix for the determinant transformation for a given A).



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Consider a matrix equation $A\vec{x} = \vec{b}$ in which A is an $n \times n$ matrix. Let





Theorem 30 (Cramer's Rule). If A is invertible, the unique solution \vec{x} of the matrix

equation $A\vec{x} = \vec{b}$ is given by

$$\chi_i = \frac{det(A_i \mathcal{E})}{det(A)} = \frac{T_i \mathcal{E}}{det A}$$

Proof. First, from cofactor expansion, $\det(A_i(b)) = \sum_{j=1}^n b_j C_{ij}$.

for any $k = 0, 1, \dots, n$. This verifies that (x_1, \dots, x_n) is a solution of $A\vec{x} = \vec{b}$.

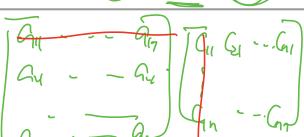
Let C be the associated $n \times n$ matrix of cofactors defined as:

$$C = \begin{bmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ C_{21} & C_{22} & \dots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \dots & C_{nn} \end{bmatrix}$$

Cij=(-1) det Al

The transpose of C is called the **adjugate matrix of** A, denoted by adjA:

Theorem 31. If A is a invertible matrix then A^{-1}



 $det(): \mathbb{R}^{n \times n} \longrightarrow \mathbb{R}$

Sit. (1)
$$det(I)=1$$

(2) Alternating: $|\vec{\alpha}_1 \cdot \vec{v} \cdot \vec{v} \cdot \vec{\alpha}_n| = 0$
(3) multiliheer:

This: there is exectly one funition spring @ 00.