MATH 5131 Lecture 4

First-order Differential Equations

Here we introduce some general concepts referring to differential equations and will discuss certain classes of equations that are not hard to solve.

Direction Field / Geometric meaning of a first-order diff. equation

We wish to emphasize the geometric significance of a solution of a first-order differential equation. In many practical problems, a rough geometrical approximation to a solution may be all that is needed.

A first-order differential equation of the form

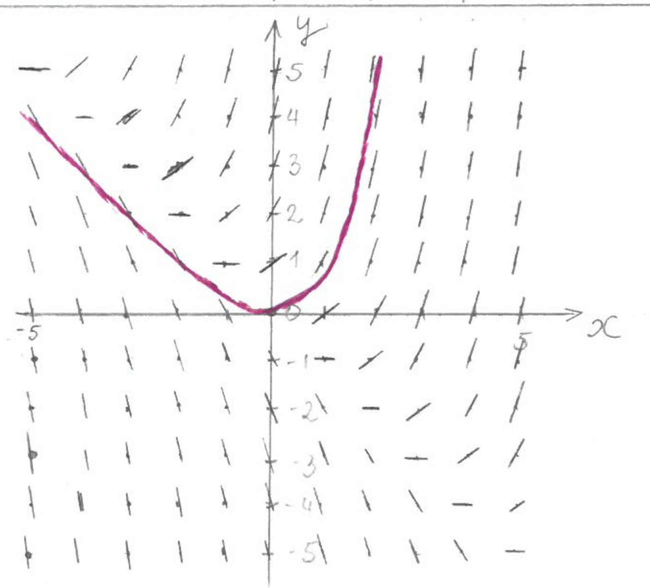
$$\frac{dy}{dx} = f(x,y) \qquad (1)$$

defines a slope field (or direction field) in the xy-plane: the value $f(x,y)$ is the slope of a tiny line segment at the point (x,y). It has geometrical significance for solution curves (also called integral curves) which are simply graphs of solutions $y(x)$ of the equation: at each point (x_0, y_0) on a solution curve, $f(x_0,y_0)$ is the slope of the tangent line to the curve. If we sketch the slope field, and then try to draw curves which are tangent to this field at each point, we get a good idea how the various solutions behave. In particular, if we pick (x_0, y_0) and try to draw a curve passing through (x_0, y_0) which is everywhere tangent to the slope field, we should get the graph of the solution for the intial-value problem

$$\begin{cases} \dfrac{dy}{dx} = f(x,y) \\ y(x_0) = y_0 \end{cases}$$

Let us illustrate this with a simple example: $\frac{dy}{dx} = x+y$. We first take a very low-tech approach and simply calculate the slope at various values of x and y:

y \ x	-5	-4	-3	-2	-1	0	1	2	3	4	5
-5	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0
-4	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1
-3	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2
-2	-7	-6	-5	-4	-3	-2	-1	0	1	2	3
-1	-6	-5	-4	-3	-2	-1	0	1	2	3	4
0	-5	-4	-3	-2	-1	0	1	2	3	4	5
1	-4	-3	-2	-1	0	1	2	3	4	5	6
2	-3	-2	-1	0	1	2	3	4	5	6	7
3	-2	-1	0	1	2	3	4	5	6	7	8
4	-1	0	1	2	3	4	5	6	7	8	9
5	0	1	2	3	4	5	6	7	8	9	10



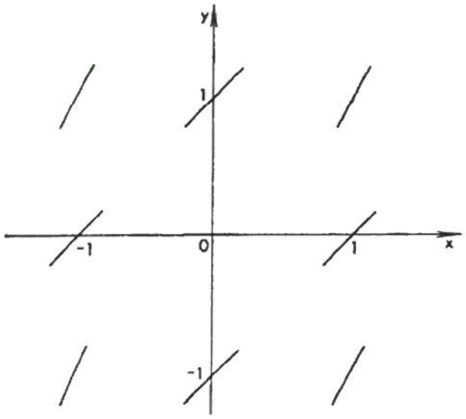
$$\begin{cases} \dfrac{dy}{dx} = x+y \\ y(0) = 0 \end{cases}$$

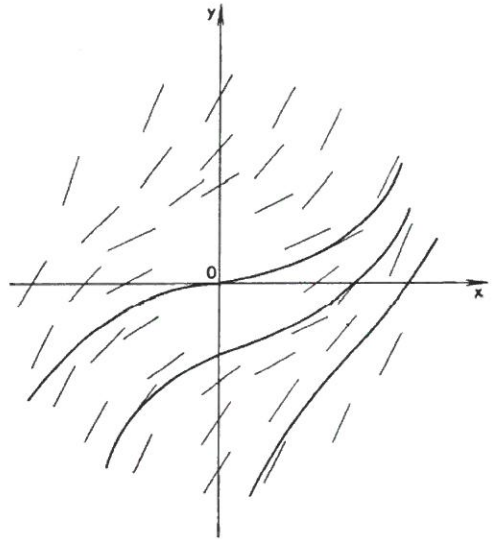
$$\boxed{y = e^x - x - 1}$$

$$\begin{cases} \underline{\text{check}}:\ y' - y \overset{?}{=} x \\ y' = e^x - 1 \\ y' - y = e^x - 1 - (e^x - x - 1) = x \checkmark \\ y(0) = e^0 - 0 - 1 = 1 - 1 = 0 \checkmark \end{cases}$$

The construction of line segments is unquestionably a tedious job. Further, if a sufficient number of them is not constructed in close proximity, it may be difficult or impossible to choose the correct line element for the particular integral curve we wish to find. If such doubt exists in a certain neighborhood, it then becomes necessary to construct additional line elements in this area until the doubt is resolved. Using the slope field, we can then sketch solution curves: the solution curve starts at (0,0) and it is everywhere tangent to the slope curve.

Low-tech :

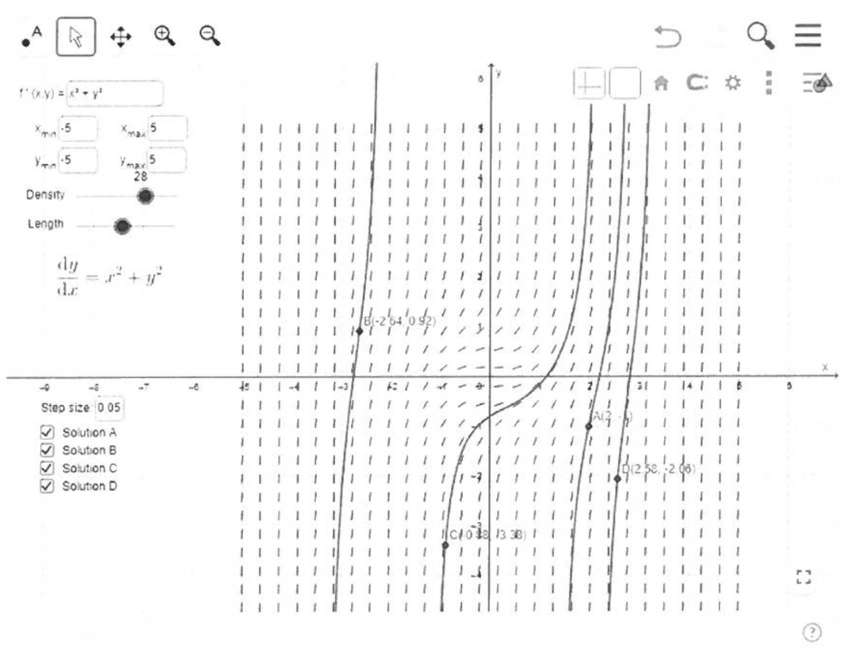




$$\frac{dy}{dx} = x^2 + y^2$$

High-tech :

A direction field (or slope field / vector field) is a picture of the general solution to a first order differential equation with the form $\frac{dy}{dx} = f(x, y)$



- Edit the gradient function $f'(x, y)$ in the input box at the top. The function you input will be shown in blue underneath as $dy/dx = f(x, y)$

The Method of Isoclines

An <u>isocline</u> for the differential equation

$$y' = f(x,y)$$

is a set of points in the xy-plane where all the solutions have the same slope dy/dx; thus, it is a level curve for the function $f(x,y)$. For example, if

$$y' = f(x,y) = x+y,$$

the isoclines are simply the curves (straight lines)

$$x+y=c \quad \text{or} \quad y=-x+c.$$

Here c is an arbitrary constant. But c can be interpreted as the numerical value of the slope dy/dx of every solution curve as it crosses the isocline. (Note that c is not the slope of the isocline itself; the latter is -1. Figure 1 depicts the isoclines for $y' = x+y$.

To implement the method of isoclines for sketching direction fields, we draw hash marks with slope c along the isocline $f(x,y)=c$ for a few selected values of c. If we then erase the underlying isocline curves, the hash marks constitute a part of the direction field for the differential equation. Figure 2 depicts this process for the isoclines shown in Figure 1, and Figure 3 displays some solution curves.

<u>Remark</u> The isoclines themselves are not always straight lines. For equation

$$y' = x^2 - y$$

they are parabolas $x^2-y=c$. When the isocline curves are complicated, this method is not practical.

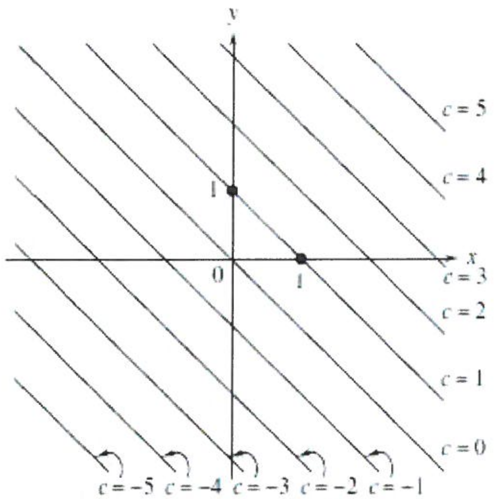


Figure 1 : Isoclines for $y'=x+y$

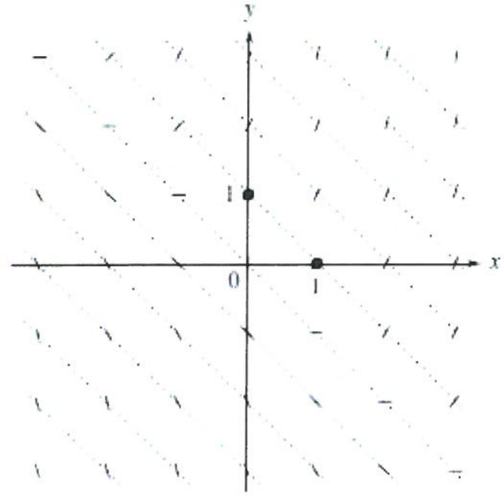


Figure 2 : Direction field for $y'=x+y$

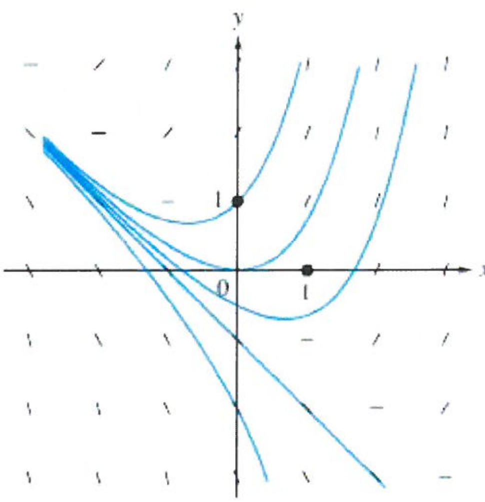


Figure 3 : Solutions to $y'=x+y$

Remark. For our illustration, we chose an $f(x,y)$ which, when we set equal to c, could be solved expeicitly for y. We were therefore able to find the isoclines of the direction field without much trouble, but in many practical cases, $f(x,y)=c$ may be more difficult to solve than the given differential equation itself. In such cases, we must resort to other means to find a solution.

An integral curve which has been drawn by means of a direction field may be looked upon as if it were formed by a particle moving in such way that is tangent to each of its line elements. Therefore the path of this particle (which is an integral curve) is sometimes referred to as a streamline of the field moving in the direction of the field. Every student of physics has witnessed the formation of a direction field when he/she has gently tapped a glass, covered with iron filings, which had been placed over a bar magnet. Each iron filing assumes the direction of a line element, and the imaginary curve which has the proper line elements as tangents is a streamline.

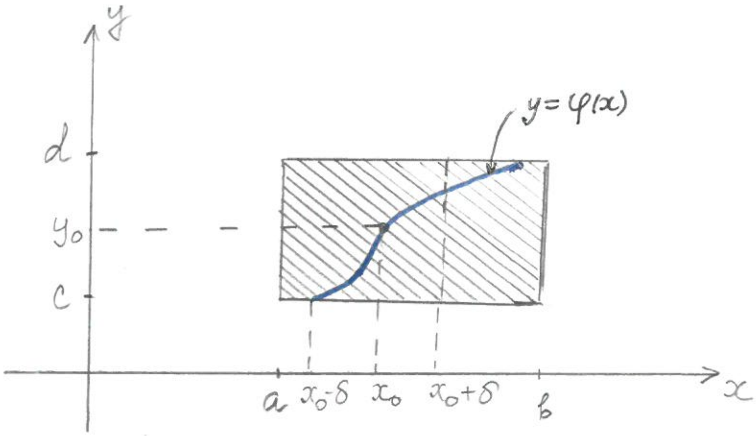
Existence and Uniqueness of Solutions

We now state an existence and uniqueness theorem for first-order initial value problems. We presume the differential equation has been cast into the format

$$\frac{dy}{dx} = f(x,y).$$

Of course, the right-hand side, $f(x,y)$, must be well defined at the starting value x_0 for x and the stipulated initial value $y_0 = y(x_0)$ for y.

The hypotheses of the theorem, moreover, require continuity of both f and $\partial f/\partial y$ for x in some interval $a < x < b$ containing x_0, and for y in some interval $c < y < d$ containing y_0. Notice that the set of points in the xy-plane that satisfy $a < x < b$ and $c < y < d$ constitutes a rectangle. The figure below depicts this "rectangle of continuity" with the initial point (x_0, y_0) in its interior and a sketch of a portion of the solution curve contained therein.



Theorem 1 Existence and Uniqueness of Solution

Given the initial value problem

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0,$$

assume that f and $\partial f/\partial y$ are continuous functions in a rectangle $R = \{(x,y) : a < x < b,\ c < y < d\}$ that contains the point (x_0, y_0). Then the initial value problem has a unique solution $\varphi(x)$ in some interval $x_0 - \delta < x < x_0 + \delta$, where δ is a positive number.

This theorem tells us two things.

- First, when an equation satisfies the hypotheses of Theorem 1, we are assured that a solution to the initial value problem exists. Naturally, it is desirable to know whether the equation we are trying to solve actually has a solution before we spend too much time trying to solve it.
- Second, when the hypotheses are satisfied, there is a unique solution to the initial value problem. This uniqueness tells us that if we can find a solution, then it is the only solution for the initial value problem.
- Graphically, the theorem says that there is only one solution curve that passes through the point (x_0, y_0). In other words, for this first-order equation, two solutions cannot cross anywhere in the rectangle.
- Notice that the existence and uniqueness of the solution holds only in some neighborhood $(x_0-\delta, x_0+\delta)$. Unfortunately, the theorem does not tell us the span (2δ) of this neighborhood (merely that it is not zero).
- When initial value problems are used to model physical phenomena, many practitioners tacitly presume the conclusions of Theorem 1 to be valid. Indeed, for the initial value problem to be reasonable model, we certainly expect it to have a solution, since physically "something does happen". Moreover, the solution should be unique in those cases when repetition of the experiment under identical conditions yields the same result.
- The proof of Theorem 1 involves converting the initial value problem into an integral equation

$$\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases} \Rightarrow \int_{y_0}^{y} dy = \int_{x_0}^{x} f(t, y(t))\,dt \Rightarrow y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\,dt$$

and then using Picard's method to generate a sequence of successive approximations that converge to the solution:

$$y_0(x) = y_0$$

$$y_1(x) = y_0 + \int_{x_0}^{x} f(t, y_0(t))\,dt$$

$$y_2(x) = y_0 + \int_{x_0}^{x} f(t, y_1(t))\,dt$$

$$y_3(x) = y_0 + \int_{x_0}^{x} f(t, y_2(t))\,dt, \ldots, y_n(x) = y_0 + \int_{x_0}^{x} f(t, y_{n-1}(t))\,dt$$

Example 1 For the initial value problem

$$\frac{dy}{dx} = x^2 - xy^3 \quad , \; y(1)=6,$$

does Theorem 1 imply the existence of a unique solution?

Solution $f(x,y) = x^2 - xy^3$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 - xy^3) = 0 - x \cdot 3y^2 = -3xy^2$$

Both of these functions are continuous in any rectangle containing the point $(x_0, y_0) = (1,6)$, so the hypotheses of Theorem 1 are satisfied. It then follows from theorem 1 that the initial value problem has a unique solution in an interval about $x=1$ of the form $(1-\delta, 1+\delta)$, where δ is some positive number.

Example 2 For the initial value problem

$$\frac{dy}{dx} = 3y^{2/3}, \quad y(2)=0,$$

does Theorem 1 imply the existence of a unique solution?

Solution $f(x,y) = 3y^{2/3}$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(3y^{2/3}) = 3 \cdot \frac{2}{3} \cdot y^{2/3-1} = 2y^{-1/3} = \frac{2}{\sqrt[3]{y}}$$

Unfortunately $\frac{\partial f}{\partial y}$ is not continuous or even defined when $y=0$.

Consequently, there is no rectangle containing $(x_0, y_0) = (2,0)$ in which both f and $\partial f/\partial y$ are continuous. Because the hypotheses of Theorem 1 don't hold, we cannot use Theorem 1 to determine whether the initial value problem does or does not have a unique solution. It turns out that this initial value problem has more than one solution: $y_1(x) = (x-2)^3$ & $y_2(x) \equiv 0$.

Suppose the initial condition is changed to $y(2)=1$. Then, since f and $\partial f/\partial y$ are continuous in any rectangle that contains the point $(2,1)$ but does not intersect the x-axis – say, $R = \{(x,y) : 0 < x < 10, \; 0 < y < 5\}$ it follows from theorem 1 that this new initial value problem has a unique solution in some interval about $x=2$.

Example 3 Find the first four Picard approximations for

$$\begin{cases} y' = xy \\ y(0) = 1 \end{cases}$$

Solution $f(x,y) = xy$, $x_0 = 0$, $y_0 = 1$

Our first approximation is $y_0(x) = 1$. The succeeding approximations are :

$$y_1(x) = 1 + \int_0^x f(t, y_0)\,dt = 1 + \int_0^x t\,dt = 1 + \frac{t^2}{2}\Big|_0^x = 1 + \frac{x^2}{2},$$

$$y_2(x) = 1 + \int_0^x f(t, y_1)\,dt = 1 + \int_0^x t\left(1 + \frac{t^2}{2}\right)dt$$

$$= 1 + \int_0^x \left(t + \frac{t^3}{2}\right)dt = 1 + \left(\frac{t^2}{2} + \frac{t^4}{8}\right)\Big|_0^x = 1 + \frac{x^2}{2} + \frac{x^4}{8},$$

$$y_3(x) = 1 + \int_0^x f(t, y_2)\,dt = 1 + \int_0^x t\left(1 + \frac{t^2}{2} + \frac{t^4}{8}\right)dt$$

$$= 1 + \int_0^x t + \frac{t^3}{2} + \frac{t^5}{8}\,dt = 1 + \left(\frac{t^2}{2} + \frac{t^4}{8} + \frac{t^6}{48}\right)\Big|_0^x$$

$$= 1 + \frac{x^2}{2} + \frac{x^4}{8} + \frac{x^6}{48}.$$

Note, if we solve the differential equation by the method of separation of variables, we obtain :

$$\frac{dy}{dx} = xy \Leftrightarrow \frac{dy}{y} = x\,dx \Leftrightarrow \ln|y| = \frac{x^2}{2} + C \Leftrightarrow y = Ce^{\frac{x^2}{2}}$$

$$y(0) = C = 1 \Rightarrow \boxed{y(x) = e^{x^2/2}}$$

The particular solution $y = e^{x^2/2}$ whose series expansion is

$$e^{\frac{x^2}{2}} = 1 + \left(\frac{x^2}{2}\right)^1 + \frac{1}{2}\left(\frac{x^2}{2}\right)^2 + \frac{1}{3!}\left(\frac{x^2}{2}\right)^3 + \dots + \frac{1}{n!}\left(\frac{x^2}{2}\right)^n + \dots$$

$$= 1 + \frac{x^2}{2} + \frac{x^2}{8} + \frac{x^6}{48} + \cdots$$

Note that each succeeding function in the sequence $y_0, y_1, y_2, y_3, \ldots$ is a closer approximation to the actual solution than is the previous one.

Remark. For different permissible starting approximations $y_0(x)$, different sequences $y_0(x), y_1(x), \ldots, y_n(x)$ will result. However, each will have the property that for an x in an interval about x_0,

$$\lim_{n\to\infty} y_n(x) = y(x),$$

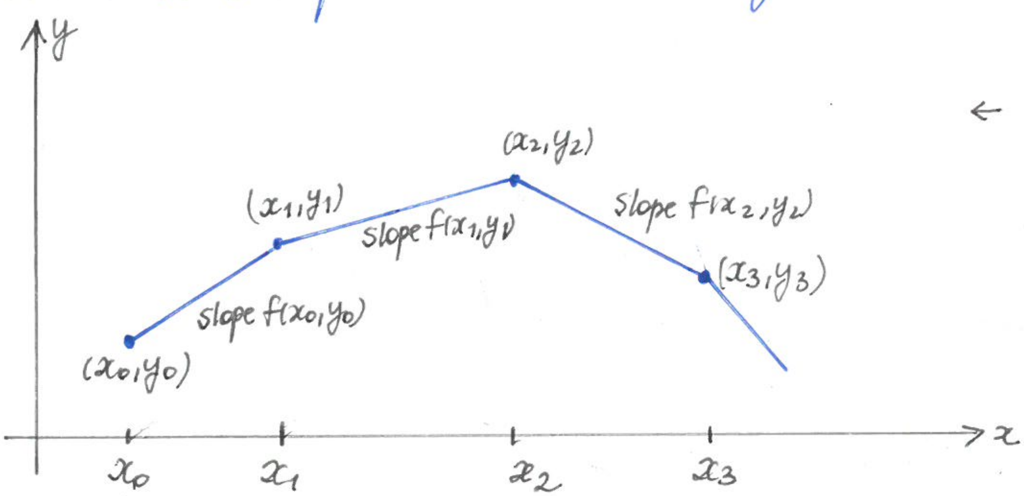
where $y(x)$ is the solution of $dy/dx = f(x,y)$ satisfying $y(x_0)=y_0$. The rapidity with which the sequence of approximations will converge to the solution $y(x)$ will depend on how closely the starting solution $y_0(x)$ approximates the actual solution $y(x)$; the closer the approximation, the quicker the convergence.

The Approximation Method of Euler

Euler's method (or the tangent-line method) is a procedure for constructing approximate solutions to an initial-value problem for a first-order differential equation

$$y' = f(x,y) \quad , \quad y(x_0) = y_0. \qquad (2)$$

It could be described as a "mechanical" or "computerized" implementation of the informal procedure for hand sketching the solution curve from a picture of the direction field. As such, we will see that it remains subject to the failing that it may skip across solution curves. However, under fairly general conditions, iterations of the procedure do converge to true solutions.



← Polygonal-line approximation given by Euler's method

D E S C R I P T I O N

Method: Starting at the initial point (x_0, y_0), we follow the straight line with slope $f(x_0, y_0)$, the tangent line, for some distance to the point (x_1, y_1). Then we reset the slope to the value $f(x_1, y_1)$ and follow this line to (x_2, y_2). In this way we construct polygonal (broken line) approximations to the solution. As we take smaller spacings between points (and thus employ more points), we may expect to converge to the true solution.

- Assume that the initial-value problem (2) has a unique solution $\varphi(x)$ in some interval centered at x_0. Let h be a fixed positive number (called the <u>step size</u>) and consider the equally spaced points

$$x_n := x_0 + nh, \quad n = 0, 1, 2, \ldots$$ (":=" means "is defined to be")

The construction of values y_n that approximate the solution values $\varphi(x_n)$ proceeds as follows.

At the point (x_0, y_0), the slope of the solution to (2) is given by

$$\left.\frac{dy}{dx}\right|_{\substack{x = x_0 \\ y = y_0}} = f(x_0, y_0)$$

Hence, the tangent line to the solution curve at the initial point (x_0, y_0) is

$$y = y_0 + (x - x_0) f(x_0, y_0)$$

Using this tangent line to approximate $\varphi(x)$, we find that for the point $x_1 = x_0 + h$

$$\varphi(x_1) \approx y_1 := y_0 + (x_1 - x_0) f(x_0, y_0) = y_0 + h f(x_0, y_0).$$

Next, starting at the point (x_1, y_1), we construct the line with slope given by the direction field at the point (x_1, y_1) – that is, with slope equal to $f(x_1, y_1)$. If we follow this line [namely,

$$y = y_1 + (x - x_1) f(x_1, y_1)]$$

in stepping from x_1 to $x_2 = x_1 + h$, we arrive at the approximation

$$\varphi(x_2) \approx y_2 := y_1 + (x_2 - x_1) f(x_1, y_1) = y_1 + h f(x_1, y_1).$$

Repeating the process we get

$$\varphi(x_3) \approx y_3 := y_2 + (x_3 - x_2) f(x_2, y_2) = y_2 + h f(x_2, y_2)$$

$$\varphi(x_4) \approx y_4 := y_3 + (x_4 - x_3) f(x_3, y_3) = y_3 + h f(x_3, y_3), \text{ etc.}$$

This simple procedure is <u>Euler's method</u> and can be summarized by the recursive formulas:

$$\boxed{\begin{aligned} x_{n+1} &= x_n + h \\ y_{n+1} &= y_n + h f(x_n, y_n), \quad n = 0, 1, 2, \dots \end{aligned}}$$

(3)

(4)

<u>Example 4</u> Use Euler's method to find approximations to the solution of the initial-value problem

$$y' = y, \quad y(0) = 1 \qquad (5)$$

at $x = 1$, taking 1, 2, 4, 8 and 16 steps.

<u>Remark</u>. Observe that the solution to (5) is just $\varphi(x) = e^x$, so Euler's method will generate algebraic approximations to the transcendental number $e = 2.71828\dots$

<u>Solution</u> Here $f(x, y) = y$, $x_0 = 0$, and $y_0 = 1$. The recursive formula for Euler's method is

$$y_{n+1} = y_n + h y_n = (1 + h) y_n.$$

To obtain approximations at $x = 1$ with N steps, we take the step size

$$h = \frac{1}{N}.$$

For $N = 1$, we have

$$\varphi(1) \approx y_1 = (1+1)(1) = 2.$$

For $N=2$, $\varphi(x_2)=\varphi_1 \approx y_2$. In this case we get

$$y_1 = (1+0.5)(1) = 1.5$$

$$\varphi(1) \approx y_2 = (1+0.5)(1.5) = 2.25.$$

For $N=4$, $\varphi(x_4) = \varphi(1) \approx y_4$, where

$$y_1 = (1+0.25)(1) = 1.25,$$

$$y_2 = (1+0.25)(1.25) = 1.5625$$

$$y_3 = (1+0.25)(1.5625) = 1.95313$$

$$\varphi(1) \approx y_4 = (1+0.25)(1.95313) = 2.44141$$

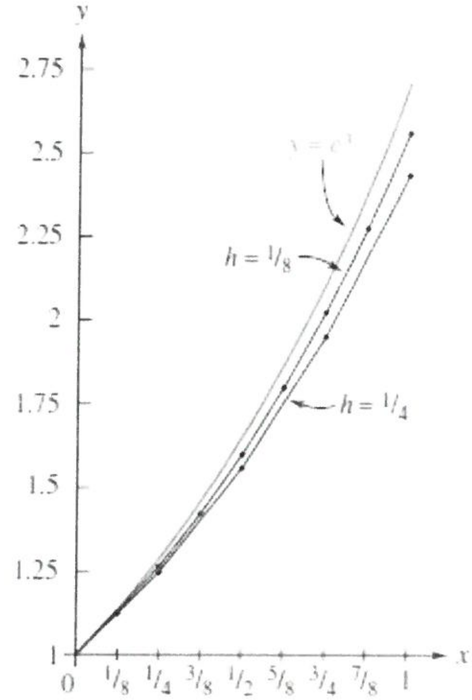
(In the above computations, we have rounded to five decimal places.) Similarly, taking $N=8$ and 16, we obtain even better estimates for $\varphi(1)$.

Euler's Method for $y'=y$, $y(0)=1$		
N	h	Approximation for $\varphi(1)=e$
1	1.0	2.0
2	0.5	2.25
4	0.25	2.44141
8	0.125	2.56578
16	0.0625	2.63793

Convergence: $\varphi(x) - y(x;h) = O(h)$ (for a fixed x)

$$\left[\lim_{h\to 0} y(x;h) = \varphi(x)\right]$$

The error $\varphi(x) - y(x;h)$ tends to zero like a constant times h.
[Euler's method is of order 1].



Approximations of e^x using Euler's method with $h = 1/4$ and $1/8$

Integrable types of first-order equations

Let us list several types of differential equations that can be solved without difficulty.

1. The simplest differential equation is the pure time equation

$$u' = g(t), \qquad (6)$$

where g is a given continuous function. By the Fundamental Theorem of Calculus, all antiderivatives are given by

$$u(t) = \int_a^t g(s)\,ds + C, \qquad (7)$$

where C is an arbitrary constant of integration and a is any lower limit of integration. Thus (7) gives all solutions of the pure time equation (6).

2. Separable Equations. These a equations of the form

$$\frac{du}{dt} = h(u)\,g(t).$$

Here the right-hand side is a product of two functions, one of which depends on u alone and the other on t alone. We rewrite the equation as following

$$\frac{du}{h(u)} = g(t)\,dt$$

Integrating the right and left sides, we get

$$\int \frac{du}{h(u)} = \int g(t)\,dt + C$$

[Caution: Constant functions $u \equiv k$ such that $h(k) \equiv 0$ are also solutions to the equation, which may or may not be included in the general solution.

(we write only one arbitrary constant because both constants that appear when evaluating the integrals can be combine into one.) When the integrals are resolved, it gives the solution u implicitly as a function of t. One may, or may not, be able to solve for u and find an explicit form for the solutions.

An autonomous equations, when the right side of the differential equation does not depend on time t,

$$u' = f(u),$$

has implicit solution $\int \frac{1}{f(u)}\,du = t + C$.

3. Linear Equations A first-order linear equation has the form

$$u' + p(t)u = q(t)$$

Method: integrating factor $I(t) = e^{\int p(t)dt}$

$$Iu' + p(t)Iu = \frac{d}{dt}(Iu) = Iq(t) \Rightarrow Iu = \int Iq(t)\,dt \Rightarrow$$

$$u = \frac{1}{I}\int Iq\,dt = e^{-\int p(t)\,dt}\left[\int I(t)q(t)\,dt\right] \Rightarrow$$

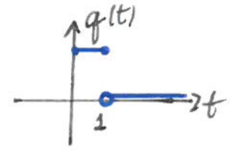
$$\boxed{u = e^{-\int p(t)dt}\left[\int I(t)q(t)\,dt + C\right]}$$

Example 5 ($q(t)$ is discontinuous $\Rightarrow u(t)$ is not differentiable)

Solve the initial-value problem

$$u' + 2u = q(t)\,, \quad u(0)=2,$$

where $q(t)$ is the function $q(t) = \begin{cases} 2, & \text{if } 0 \le t \le 1 \\ 0, & \text{if } t > 1 \end{cases}$



Solution $p(t) = 2 \Rightarrow I(t) = e^{\int 2\,dt} = e^{2t}$

$$\underbrace{e^{2t}u' + 2e^{2t}u}_{\frac{d}{dt}(e^{2t}u)} = e^{2t}q(t) \quad \Rightarrow \quad \boxed{e^{2t}u = \int q(t)e^{2t}\,dt}$$

This shows that the solution is continuous for $t \ge 0$, but we need to evaluate the integral separately for $0 \le t \le 1$ and $t > 1$.

For $0 \le t \le 1$ we have $q(t) = 2$, and integration yields

$$\int 2e^{2t}\,dt = e^{2t} + C_0, \quad \text{where } C_0 \text{ is an arbitrary constant.}$$

Solving for u: $\quad u(t) = e^{-2t}(e^{2t} + C_0) = 1 + C_0 e^{-2t}$

$u(0) = 1 + C_0 = 2 \Rightarrow C_0 = 1 \Rightarrow \boxed{u(t) = 1 + e^{-2t}, \quad 0 \le t \le 1}$

For $t > 1$, we need to integrate with $q(t) = 0$:

$$e^{2t}u = \int 0\,dt = C_1 \Rightarrow u(t) = C_1 e^{-2t}.$$

To find the constant C_1 we need to know the value of $u(t)$ at some $t > 1$. We do not have this information, but if the solution $u(t)$ is to be continuous for $t \ge 0$, the limiting values of $u(t)$ as $t \to 1^-$ and $t \to 1^+$ must agree. In other words,

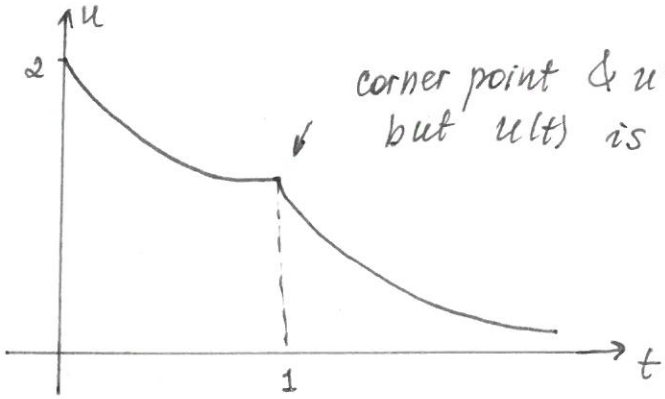
$$\lim_{\substack{t \to 1^- \\ (0 \le t \le 1)}} u(t) = u(1) = 1 + e^{-2\cdot 1} = \lim_{\substack{t \to 1^+ \\ (t > 1)}} u(t) = u(1) = C_1 e^{-2\cdot 1} \Rightarrow$$

$$1 + e^{-2} = C_1 e^{-2} \Rightarrow C_1 = \frac{1 + e^{-2}}{e^{-2}} = e^2(1 + e^{-2}) = e^2 + 1$$

So $\boxed{u(t) = (1 + e^2)e^{-2t}, \quad t > 1}$

Putting these formulae together, we conclude that the solution is given by

$$u(t)=\begin{cases}1+e^{-2t}, & \text{if } 0\le t\le 1\\ (1+e^{2})e^{-2t}, & \text{if } t>1\end{cases} \qquad (t=1 \text{ is a corner point}).$$



corner point & $u'(t)$ has a finite jump of discontinuity, but $u(t)$ is continuous for $t\ge 0$.

4. A nonlinear equation reducible to linear form (Bernoulli's Eq.)

The equation of the type

$$\frac{du}{dt}+p(t)u=q(t)u^{n}, \qquad (n\neq 0,\ n\neq 1)$$

in which n may be regarded different from zero and unity, can be reduced to linear form by substitution

$$w=u^{1-n}$$

Then, $\dfrac{dw}{dt}=(1-n)u^{-n}\dfrac{du}{dt} \Rightarrow \dfrac{du}{dt}=\dfrac{1}{(1-n)}u^{n}\dfrac{dw}{dt}$

and the original equation becomes

$$\frac{1}{(1-n)}u^{n}\frac{dw}{dt}+p(t)u=q(t)u^{n}$$

$$u^{n}\Big[\frac{1}{1-n}\frac{dw}{dt}+p(t)\underbrace{u^{1-n}}_{w}\Big]=q(t)u^{n} \quad \Big|\div u^{n}\neq 0$$

$$\frac{1}{1-n}\frac{dw}{dt}+p(t)w=q(t) \Rightarrow \underbrace{\boxed{\frac{dw}{dt}+(1-n)p(t)w=(1-n)q(t)}}_{\text{linear.}}$$

Example 6 Find the general solution

$$u' - tu = t^2u^2$$

Solution $n=2 \Rightarrow w = u^{1-2} = u^{-1}$, $g(t) = t^2$, $p(t) = -t$

$$\frac{dw}{dt} + (1-2)(-t)\,w = (1-2)t^2$$

$$\Downarrow$$

$$\frac{dw}{dt} + tw = -t^2 \Rightarrow I(t) = e^{\int t\,dt} = e^{\frac{t^2}{2}}$$

$$w(t) = e^{-\frac{t^2}{2}}\left[-\int t^2 e^{\frac{t^2}{2}}\,dt\right] = e^{-\frac{t^2}{2}}\left[-\left\{te^{t^2/2} - \int e^{t^2/2}\,dt\right\}\right]$$

$$\begin{bmatrix} u=t & dv = te^{\frac{t^2}{2}}dt \\ du=dt & v = e^{\frac{t^2}{2}} \end{bmatrix}$$

$$w(t) = e^{-\frac{t^2}{2}}\left\{-te^{t^2/2} + \int_0^t e^{s^2/2}\,ds\right\} = -t + e^{-\frac{t^2}{2}}\int_0^t e^{s^2/2}\,ds + Ce^{-\frac{t^2}{2}}$$

$$u = \frac{1}{w} = \frac{1}{Ce^{-t^2/2} - t + e^{-t^2/2}\int_0^t e^{s^2/2}\,ds} \cdot \frac{e^{t^2/2}}{e^{t^2/2}} = \frac{e^{t^2/2}}{C - te^{t^2/2} + \int_0^t e^{s^2/2}\,ds}$$

$$\boxed{u(t) = \frac{e^{t^2/2}}{C - te^{t^2/2} + \int_0^t e^{s^2/2}\,ds}}$$ general solution