SOLUTION NOTES

MTH 7241: Fall 2022

First Practice Problems for Test 1

1). 3 balls are distributed in 3 boxes. At each step, one of the balls is selected at random, taken out of whichever box it is in, and moved at random to one of the other boxes. Let X_n be the number of balls in the first box, after n steps.

a). Find the transition matrix of the chain $X_0, X_1, \ldots$.

b). Find the stationary distribution of the chain.

$$P = \begin{array}{cc} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & \frac{2}{3} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 1 & 0 \end{pmatrix} & \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \end{matrix} \\ \begin{matrix} 0 & \quad 1 & \quad 2 & \quad 3 \end{matrix} & \end{array}$$

2). Consider the following transition probability matrix for a Markov chain on 4 states:

$$P = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Number the states $\{1, 2, 3, 4\}$ in the order presented.

a). Find and classify the equivalence classes of the states (irreducible and transient).

b). Find a stationary distribution for the chain.

Irreducible, period $= 3$

$w = \frac{1}{6}(1, 1, 3, 2)$

3). Suppose that coin 1 has probability 0.7 of coming up Heads, and coin 2 has probability 0.4 of coming up Heads. If the coin tossed today comes up Heads, then we select coin 1 to toss tomorrow, and if it comes up Tails, then we select coin 2 to toss tomorrow. If the coin initially tossed is equally likely to be coin 1 or coin 2, then what is the probability that the coin tossed on the third day after the initial toss is coin 1?

$$P = \begin{array}{cc} \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix} & \begin{array}{c} H \\ T \end{array} \\ \begin{array}{cc} H & \quad T. \end{array} & \end{array}$$

$$\mathbb{P}(\text{coin } 1 \text{ on Day } 3) = \mathbb{P}(\text{Heads on Day } 2)$$

$$= \mathbb{P}(X_2 = H).$$

$$= \mathbb{P}(X_2 = H \mid \text{initial coin } 1) \cdot \tfrac{1}{2} + \mathbb{P}(X_2 = H \mid \text{initial coin } 2) \cdot \tfrac{1}{2}$$

$$\mathbb{P}(X_2 = H \mid \text{initial coin } 1) = \mathbb{P}(X_2 = H \mid X_0 = H) \cdot \mathbb{P}(X_0 = H \mid \text{initial coin } 1) + \mathbb{P}(X_2 = H \mid X_0 = T) \cdot \mathbb{P}(X_0 = T \mid \text{initial coin } 1)$$

$$= (0.61)(0.7) + (0.52)(0.3).$$

Similarly

$$\mathbb{P}(X_2 = H \mid \text{initial coin } 2) = (0.61)(0.4) + (0.52)(0.6)$$

4) Four balls are shared between box #1 and box #2. At each step a biased coin is tossed which comes up Heads with probability p. If the coin comes up Heads and box #1 is not empty, a ball is removed from box #1 and placed in box #2. If the coin comes up Heads and box #1 is empty, no balls are moved. If the coin comes up Tails and box #2 is not empty, a ball is removed from box #2 and placed in box #1. If the coin comes up Tails and box #2 is empty, no balls are moved. Let X_n be the number of balls in box #1 after n steps.

Find the transition matrix for the Markov chain $\{X_n\}$ (your answer will depend on p).

$$P = \begin{array}{cc} \begin{pmatrix} p & 1-p & 0 & 0 & 0 \\ p & 0 & 1-p & 0 & 0 \\ 0 & p & 0 & 1-p & 0 \\ 0 & 0 & p & 0 & 1-p \\ 0 & 0 & 0 & p & 1-p \end{pmatrix} & \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \\ \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} & \end{array}$$

5) Consider the following transition probability matrix for a Markov chain on 5 states:

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Number the states $\{1, 2, 3, 4, 5\}$ in the order presented.

Given that the chain starts in state 1, find the expected number of steps until the first return to state 1.

Irreducible, period $= 2$.

$$\mu_1 = \frac{1}{w_1} = 8.$$

6) Let $\{X_n\}$ be a Markov chain, and suppose that for state i we have

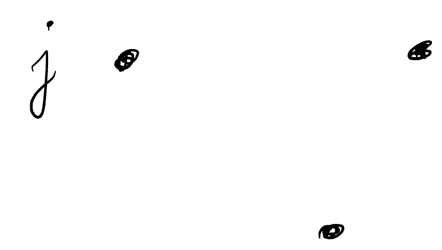
$$\sum_{k=1}^{n} p_{ii}(k) = \sum_{k=1}^{n} \mathrm{P}(X_k = i \,|\, X_0 = i) = 3 - \frac{9}{\sqrt{n+8}} \quad \text{for all } n \geq 1.$$

Determine whether state i is transient or persistent (explain your reasoning).

$$\lim_{n\to\infty} \sum_{k=1}^{n} p_{ii}(k) = 3 < \infty$$

$\Rightarrow$ transient.

7) Consider an irreducible chain on 3 states. **Either** prove that $p_{jj}(6) > 0$ for every state j, **or** give an example where $p_{jj}(6) = 0$ for some state j.



Irreducible $\Rightarrow$ $P_{jj}(n) > 0$ some n.

3 states $\Rightarrow$ $n \leq 3$.

$\Rightarrow$ $n \in \{1, 2, 3\}$.

all factors of 6

$\Rightarrow$ $P_{jj}(6) > 0$