Math 5110 Applied Linear Algebra

He Wang

he.wang@northeastern.edu

Homework 2.

1. Reading: [Gockenbach], Chapter 1 and Chapter 2.

2. Questions:

Questions 1-10 are about vector spaces.

Question 1. Let V be a vector space over $\mathbb{R}$ and let $\vec{v} \in V$ be a nonzero vector. Is the subset $\{0, \vec{v}\}$ is a subspace of V? Prove your result.

No. $2\vec{v}$ is not in the subset. So the set $\{0, \vec{v}\}$ is not closed under scalar product.

Question 2. Determine whether or not the following set a subspace of $\mathbb{R}^2$. Prove your result.

(1) $S = \{\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 \mid x_1 x_2 = 0\}$.

(2) $T = \{\vec{x} \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 1\}$ the unit disc in $\mathbb{R}^2$.

(1) No, the set S is not closed under sum. For example, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are in S, but their sum is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ which is not in S.
(2) No. the set T is not closed under scalar product. For example. $3\begin{bmatrix} 0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 0 \end{bmatrix}$ which is not in T.

Question 3. (1) Let $U_{3\times3}$ be the set of all 3×3 upper triangular matrices with real entries. Is $U_{3\times3}$ a subspace of $\mathbb{R}^{3\times3}$? Prove your result.

(2) Let $T_{3\times3}$ be the set of all 3×3 triangular matrices with real entries. Is $T_{3\times3}$ a subspace of $\mathbb{R}^{3\times3}$?

(3) Let W be the set of all polynomials in the form $\{t + at^2\}$ where a is any real number. Is W a subspace of P the vector space of all polynomials.

(1) Yes. Verify three conditions.
(2) No. Sum is not closed.
(3) No. Not include zero.

Question 4. (Allow to use Matlab for **rref**) Let S be the following subspace of $\mathbb{R}^4$:

$$S = \text{Span}\left\{\vec{b}_1 = \begin{bmatrix} -1 \\ -2 \\ 4 \\ -2 \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} 0 \\ 1 \\ -5 \\ 4 \end{bmatrix}\right\}.$$

Determine if each vector belongs to S:

(1.) $\vec{v} = \begin{bmatrix} -1 \\ 0 \\ -6 \\ 6 \end{bmatrix}$; (2.) $\vec{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

It is the same question as whether or not $x_1\vec{b}_1 + x_2\vec{b}_2 = \vec{v}$ or $x_1\vec{b}_1 + x_2\vec{b}_2 = \vec{w}$ has a solution.
Set up augmented matrix $[\vec{b}_1\ \vec{b}_2|\vec{v}]$ and $[\vec{b}_1\ \vec{b}_2|\vec{w}]$ and find their **rref**.
(1) Yes. (2) No.

Question 5. Let S be the following subspace of $\mathbb{R}^{2\times 2}$:

$$S = \text{Span}\left\{\vec{b}_1 = \begin{bmatrix} -1 & -2 \\ 4 & -2 \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} 0 & 1 \\ -5 & 4 \end{bmatrix}\right\}.$$

Determine if each vector belongs to S:

(1.) $\vec{v} = \begin{bmatrix} -1 & 0 \\ -6 & 6 \end{bmatrix}$; (2.) $\vec{w} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

See above question.
It is the same question as whether or not $x_1\vec{b}_1 + x_2\vec{b}_2 = \vec{v}$ or $x_1\vec{b}_1 + x_2\vec{b}_2 = \vec{w}$ has a solution.
(1) Yes. (2) No.

Question 6. Show that $S = \text{Span}\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\}$ and $T = \text{Span}\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\}$ are the same subspace of $\mathbb{R}^3$.

It is clear that $S \subset T$.

We only need to show that $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \in S$. That is show that $\vec{v}_3 = x_1\vec{u}_1 + x_2\vec{u}_2$ has a solution.

Question 7. Suppose U and V are two subspaces of a vector space W.

(1) Is the union of two subspace $U \cup V$ a subspace?

(2) Is the intersection $U \cap V$ is a subspace?

(1) No. Sum is not closed.
(2) Yes. Verify three conditions:
1. $\vec{0} \in U$ and $\vec{0} \in V$, so $\vec{0} \in U \cap V$
2. If $\vec{u}, \vec{v} \in U \cap V$, then $\vec{u} + \vec{v} \in U$ and $\vec{u} + \vec{v} \in V$. So, $\vec{u} + \vec{v} \in U \cap V$.
2. If $\vec{u} \in U \cap V$, then $k\vec{u} \in U$ and $k\vec{v} \in V$ for any $k \in F$. So, $k\vec{u} \in U \cap V$.

Question 8. Prove or disprove the following statement: if U, V, W are subspaces of a vector space, then $(U+V)\cap W = (U\cap W)+(V\cap W)$.

The statement is false in general.

For example, consider the three subspaces $U = \mathrm{Span}(\begin{bmatrix}1\\0\end{bmatrix})$, $V = \mathrm{Span}(\begin{bmatrix}0\\1\end{bmatrix})$ and $W = \mathrm{Span}(\begin{bmatrix}1\\1\end{bmatrix})$

$U+V=\mathbb{R}^2$, $U\cap W = V\cap W = 0$. So, $(U+V)\cap W = W$ but $(U\cap W)+(V\cap W) = 0$

Question 9. Let U_1, U_2, U_3 be subspaces of a vector space such that $U_i \cap U_j = 0$ for $i \neq j$. Is it true that the subspace $U_1 + U_2 + U_3$ equals $U_1 \oplus U_2 \oplus U_3$? Justify your answer.

No.

For example, consider the three subspaces $U = \mathrm{Span}(\begin{bmatrix}1\\0\end{bmatrix})$, $V = \mathrm{Span}(\begin{bmatrix}0\\1\end{bmatrix})$ and $W = \mathrm{Span}(\begin{bmatrix}1\\1\end{bmatrix})$

Question 10. If V is a vector space with dimension n over the field $\mathbb{Z}_2$ of two elements, how many elements does V contain? Prove your result.

Solution. If $n = 0$, then $V = 0$ and so V has only one vector, the zero vector.
Let $n > 0$, and choose a basis of V , say $\{v_1, ..., v_n\}$. Then each vector $\vec{v}$ of V can be uniquely written in the form $\vec{v} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$ where $c = 0$ or 1. Consequently there are 2^n possibilities for $\vec{v}$.

Question 11. If $\{\vec{u},\vec{v}\}, \{\vec{v},\vec{w}\}$ and $\{\vec{w},\vec{u}\}$ are linearly independent subsets, is the subset $\{\vec{u},\vec{v},\vec{w}\}$ linearly independent?

Not in general. For example, $\vec{u} = \begin{bmatrix}1\\0\end{bmatrix}$ $\vec{v} = \begin{bmatrix}0\\1\end{bmatrix}$ and $\vec{v} = \begin{bmatrix}1\\1\end{bmatrix}$

Question 12. Show that $\{\begin{bmatrix}-1\\1\\3\end{bmatrix}, \begin{bmatrix}1\\-1\\-2\end{bmatrix}, \begin{bmatrix}-3\\3\\13\end{bmatrix}\} \in \mathbb{R}^3$ is linearly dependent by writing one of the vectors as a linear combination of the others.

Solve $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{0}$
We can get a solution $x_1 = -7; x_2 = -4; x_3 = 1$.
So, $\vec{v}_3 = 7\vec{v}_1 + 4\vec{v}_2$

Question 13. Let A be an $m\times n$ matrix with real entries, and suppose $n > m$. Prove the linear transformation defined by A is not injective. (That is, $A\vec{x} = \vec{0}$ has a nontrivial solution $x \in \mathbb{R}^n$.)

There are several different arguments for this question.
$\mathrm{rank}(A) \leq m < n$. So $Nullity(A) = n - \mathrm{rank}(A) = n - m > 0$. So the null space has dimension ≥ 1. So, $A\vec{x} = \vec{0}$ has a nontrivial solution $x \in \mathbb{R}^n$.

Question 14. Let $\vec{u}_1 = \begin{bmatrix} 1 \\ 4 \\ 0 \\ -5 \\ 1 \end{bmatrix}; \vec{u}_2 = \begin{bmatrix} 1 \\ 3 \\ 0 \\ -4 \\ 0 \end{bmatrix}; \vec{u}_3 = \begin{bmatrix} 0 \\ 4 \\ 1 \\ 1 \\ 4 \end{bmatrix}$ be vectors in $\mathbb{R}^5$.

(1) Show that $\vec{u}_1, \vec{u}_2, \vec{u}_3$ is linearly independent.

(2) Extend $\vec{u}_1, \vec{u}_2, \vec{u}_3$ to a basis for $\mathbb{R}^5$.

(1) Let $A = [\vec{u}_1\ \vec{u}_2\ \vec{u}_3]$. Then $\mathbf{rref}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. So. $\vec{u}_1, \vec{u}_2, \vec{u}_3$ is independent.

(2) You can try to add $\vec{e}_4$ and $\vec{e}_5$, but we need to check that $\mathbf{rref}([\vec{u}_1\ \vec{u}_2\ \vec{u}_3\ \vec{e}_4\ \vec{e}_5]) = I_5$. An more general method is to use decomposition

$$\mathbb{R}^5 = \mathrm{Row}(A^T) \oplus \ker A^T$$

where

$$A^T = \begin{bmatrix} 1 & 4 & 0 & -5 & 1 \\ 1 & 3 & 0 & -4 & 0 \\ 0 & 4 & 1 & 1 & 4 \end{bmatrix}$$

Question 15. Consider the linear subspaces U and W of $\mathbb{R}^4$ spanned by $\vec{u}_1 := \begin{bmatrix} -1 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \vec{u}_2 := \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \vec{u}_3 := \begin{bmatrix} 2 \\ 2 \\ 1 \\ -3 \end{bmatrix}$ and $\vec{w}_1 := \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{w}_2 := \begin{bmatrix} 1 \\ 3 \\ 1 \\ -1 \end{bmatrix}, \vec{w}_3 := \begin{bmatrix} 2 \\ -2 \\ -1 \\ -1 \end{bmatrix}, \vec{w}_4 := \begin{bmatrix} 2 \\ 2 \\ 1 \\ -1 \end{bmatrix}$ respectively.

Find the **dimensions** of the sum $U + W$, the intersection $U \cap W$, and the quotient spaces $\mathbb{R}^4/U$ and $\mathbb{R}^4/W$.

Let $A = [\vec{u}_1\ \vec{u}_2\ \vec{u}_3]$, $B = [\vec{w}_1\ \vec{w}_2\ \vec{w}_3\ \vec{w}_4]$, $C = [\vec{u}_1\ \vec{u}_2\ \vec{u}_3\ \vec{w}_1\ \vec{w}_2\ \vec{w}_3\ \vec{w}_4]$
Calculate $\mathrm{rank}(A) = 3$, $\mathrm{rank}(A) = 3$, $\mathrm{rank}(A) = 4$. So, $\dim U = 3$, $\dim W = 3$ and $\dim(U + W) = 4$.
By $\dim(U + W) = \dim U + \dim W - \dim(U \cap W)$. $\dim U \cap W = 2$.
$(\mathbb{R}^4/U) \oplus U = \mathbb{R}^4$. So, $\dim \mathbb{R}^4/U = 1$.
Similarly, $\dim \mathbb{R}^4/W = 1$.

Question 16. Let V be a vector space over a field $\mathbb{K}$, and let $\vec{v}_1, ..., \vec{v}_n$ be n linearly dependent vectors of V such that any $n - 1$ of the vectors $\vec{v}_1, ..., \vec{v}_n$ are linearly independent. Show:

(a) There exist scalars $\alpha_1, \ldots, \alpha_n$ in $\mathbb{K}$, **all** nonzero, such that $\sum_{j=1}^n \alpha_j \vec{v}_j = \vec{0}$.

(b) If $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ are two sets of nonzero scalars in $\mathbb{K}$ such both $\sum_{j=1}^n \alpha_j \vec{v}_j = 0$ and $\sum_{j=1}^n \beta_j \vec{v}_j = 0$ then there exists a nonzero scalar γ in K such that $\beta_j = \gamma \alpha_j$ for each $j = 1, ..., n$.

(a) Since $\vec{v}_1, ..., \vec{v}_n$ be n linearly dependent vectors, there exist non-trivial solution $\vec{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$ such that $\sum_{j=1}^{n} c_j \vec{v}_j = \vec{0}$. Claim: all c_j are non-zero. Suppose one of c_j is zero, for example $c_i \neq 0$, then $\vec{v}_1, ..., \widehat{\vec{v}_i}, ..., \vec{v}_n$ is dependent, which contradict the assumption that any $n-1$ of the vectors $\vec{v}_1, ..., \vec{v}_n$ are linearly independent.

(b) Suppose $\beta_1 = \gamma\alpha_1$. Then

$$\gamma \sum_{j=1}^{n} \alpha_j \vec{v}_j - \sum_{j=1}^{n} \beta_j \vec{v}_j = \vec{0}$$

Hence $\sum_{j=1}^{n}(\gamma\alpha_j - \beta_j)\vec{v}_j = \sum_{j=2}^{n}(\gamma\alpha_j - \beta_j)\vec{v}_j = \vec{0}$. Since $n-1$ of the vectors $\vec{v}_2, \vec{v}_3, ..., \vec{v}_n$ are linearly independent, we have $\gamma\alpha_j - \beta_j = 0$ for $j = 2, 3, ..., n$.

Question 17. Let M be the matrix $M = \begin{bmatrix} 3 & 3 & 2 & 8 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 3 & 5 \\ -2 & 4 & 6 & 8 \end{bmatrix}$, and let U and W be the subspaces of $\mathbb{R}^4$ generated by rows 1 and 2 of M, and by rows 3 and 4 of M respectively. Find the dimensions of the subspaces $U + W$ and $U \cap W$.

Note that $U + W$ is just the row space of M, and so $\dim(U + W)$ equals the rank of M. Putting M in reduced row echelon form, we find that the rank is 3. Thus $\dim(U + W) = 3$.
Next $\dim(U) = 2$ since rows 1 and 2 of M are clearly linearly independent; similarly $dim(W) = 2$. Hence $\dim(U \cap W) = \dim(U) + \dim(W) - \dim(U + W) = 1$.

Question 18. Define polynomials $f_1 = 1 - 2x + x^3$, $f_2 = x + x^2 - x^3$ and also $g_1 = 2 + 2x - 4x^2 + x^3$, $g_2 = 1 - x + x^2$, $g_3 = 2 + 3x - x^2$. Let $U = \mathrm{Span}(f_1, f_2)$ and $V = \mathrm{Span}(g_1, g_2, g_3)$ be subspaces of $P_4(\mathbb{R})$, polynomials of degree smaller than 4. Find a basis for $U + V$ and a basis for $U \cap V$.

Use the ordered basis $1, x, x^2, x^3$, and write down the matrix whose columns are the coordinate vectors of f_1, f_2, g_1, g_2, g_3.

$$M = \begin{bmatrix} 1 & 0 & 2 & 1 & 2 \\ -2 & 1 & 2 & -1 & 3 \\ 0 & 1 & -4 & 1 & -1 \\ 1 & -1 & 1 & 0 & 0 \end{bmatrix}$$

To find a basis of $U + W$, put M in reduced column echelon form, and delete all zero columns. The remaining columns will be the coordinate vectors of a set of elements in a basis of $U + V$. The reduced column echelon form of M is

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Hence a basis for $U + W$ is $1, x, x^2, x^3$, which means that $U + W = P_4(\mathbb{R})$.

To find a basis for $U \cap W$, first find a basis for the null space of M (kernel of M). This turns out to be

$$\begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

The first two entries lead to a basis of $U \cap W$. Thus a basis for $U \cap W$ is $(-1)f_1 + (-1)f_2 = -1 + x - x^2$. (Reason?) It can also be calculated as $0g_1 + 1(g_2) + 0g_3$.