Math 5110 Applied Linear Algebra -Fall 2021.

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Homework 3 .

2. Questions: (You can use Matlab if needed.)

The following questions are about matrix of linear transformation and coordinate.

Question 1. Suppose $L : \mathbb{R}^3 \to \mathbb{R}^3$ is linear, $\vec{b} \in \mathbb{R}^3$ is given, and $\vec{u} = (1, 0, 1), \vec{v} = (1, 1, -1)$ are two solutions to $L(x) = b$. Find two more solutions to $L(\vec{x}) = \vec{b}$.

Since L is linear, $L(\vec{u}) = \vec{b}$ and $L(\vec{v}) = \vec{b}$, hence, $L(2\vec{u} - \vec{v}) = 2L(\vec{u}) - L(\vec{v}) = 2\vec{b} - \vec{b} = \vec{b}$.
So, $2\vec{u} - \vec{v} = (1, -1, 1)$ is another solution.
Similar, we can find infinitely many solutions $\vec{u} + t(\vec{u} - \vec{v})$.

Question 2. Find matrix of each linear operator: (Hint: using theorem on matrix of linear transformation.)

(1.) Let $R : \mathbb{R}^2 \to \mathbb{R}^2$ be the **rotation** of angle θ about the origin (a positive θ indicates a counterclockwise rotation). Find the matrix A such that $R(x) = Ax$ for all $x \in \mathbb{R}^2$.

(2.) Consider the linear operator mapping $\mathbb{R}^2$ into itself that sends each vector $\begin{bmatrix} x \\ y \end{bmatrix}$ to its **projection** onto the x-axis, namely, $\begin{bmatrix} x \\ 0 \end{bmatrix}$. Find the matrix representing this linear operator.

(3.) A (horizontal) **shear** acting on the plane maps a **point** $\begin{bmatrix} x \\ y \end{bmatrix}$ to the point $\begin{bmatrix} x + ry \\ y \end{bmatrix}$, where r is a real number. Find the matrix representing this operator.

(4.) A linear operator $L : \mathbb{R}^n \to \mathbb{R}^n$ defined by $L(x) = r\vec{x}$ is called a **dilation** if $r > 1$ and a **contraction** if $0 < r < 1$. What is the matrix of L?

The matrix of a linear transformation $L : \mathbb{R}^n \to \mathbb{R}^m$ is given by $[L(\vec{e}_1)\ L(\vec{e}_2)\ ...\ L(\vec{e}_n)]$

(1) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

(2) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

(3) $\begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix}$

(4) $\begin{bmatrix} r & 0 & \cdots & 0 \\ 0 & r & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r \end{bmatrix}$

Question 3. Consider the following geometrically defined linear maps of $\mathbb{R}^3$ to itself. Describe each of them by a matrix with respect to the canonical basis of $\mathbb{R}^3$. (Hint: using theorem on matrix of linear transformation.)

(a) Orthogonal projection onto the xz-plane.

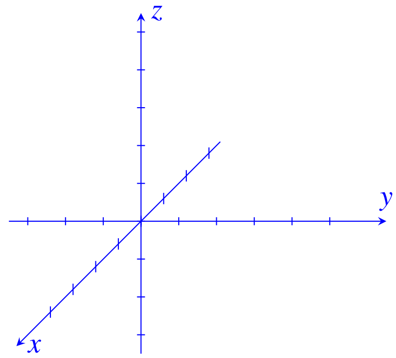
(b) Counterclockwise rotation by 45°about the x-axis.

(c) The map (rotation) of part (b) then followed by the map(projection) of part (a).

(d) Rotation by 120° about the main diagonal in space (spanned by the vector $\begin{bmatrix}1\\1\\1\end{bmatrix}$, taken counterclockwise as you look towards the origin.

(d*) In question (d), if the rotation is an angle θ, what is the matrix? (Optional. It is the same as lab2.)

The matrix of a linear transformation $L : \mathbb{R}^3 \to \mathbb{R}^3$ is given by $[L(\vec{e}_1)\ L(\vec{e}_2)\ L(\vec{e}_3)]$, where $\vec{e}_1 = (1,0,0)$, $\vec{e}_2 = (0,1,0)$, $\vec{e}_3 = (0,0,1)$.



(a). $L(\vec{e}_1) = \begin{bmatrix}1\\0\\0\end{bmatrix}$ $L(\vec{e}_2) = \begin{bmatrix}0\\0\\0\end{bmatrix}$ $L(\vec{e}_3) = \begin{bmatrix}0\\0\\1\end{bmatrix}$. So, the matrix is $A = \begin{bmatrix}1 & 0 & 0\\0 & 0 & 0\\0 & 0 & 1\end{bmatrix}$

(b) $L(\vec{e}_1) = \begin{bmatrix}1\\0\\0\end{bmatrix}$ $L(\vec{e}_2) = \begin{bmatrix}0\\\cos(\theta)\\\sin(\theta)\end{bmatrix}$ $L(\vec{e}_3) = \begin{bmatrix}0\\-\sin(\theta)\\\cos(\theta)\end{bmatrix}$. So, the matrix is $B = \begin{bmatrix}1 & 0 & 0\\0 & \sqrt{2}/2 & -\sqrt{2}/2\\0 & \sqrt{2}/2 & \sqrt{2}/2\end{bmatrix}$

(c) The matrix is $C = AB$.

(d) The plane perpendicular to the vector $\begin{bmatrix}1\\1\\1\end{bmatrix}$ and passing vector $\vec{e}_1$ is $x + y + z = 1$. In fact, this plane also pass $\vec{e}_2$ and $\vec{e}_3$. Draw the triangle and from the geometry, we can see that, $L(\vec{e}_1) = e_2$, $L(\vec{e}_2) = e_3$, $L(\vec{e}_3) = e_1$. So, the matrix is $D = \begin{bmatrix}0 & 0 & 1\\1 & 0 & 0\\0 & 1 & 0\end{bmatrix}$

Question 4. Let $x \in R^N$ be denoted as $x = (x_1, x_2, ..., x_N)$. Given $\vec{x}, \vec{y} \in \mathbb{R}^N$, the **convolution** of $\vec{x}$ and $\vec{y}$ is the vector $\vec{x} * \vec{y} \in \mathbb{R}^N$ defined by

$$(\vec{x} * \vec{y})_n = \sum_{m=1}^{N} x_m y_{n-m}, \quad \text{for } n = 1, 2, ..., N.$$

In this formula, $\vec{y}$ is regarded as defining a periodic vector of period N; therefore, if $n - m \leq 0$, we take $y_{n-m} = y_{N+n-m}$. For instance, $y_0 = y_N$, $y_{-1} = y_{N-1}$, $y_{-2} = y_{N-2}$, and so forth.

(1) Prove that if $y \in \mathbb{R}^N$ is fixed, then the mapping

$$L : \vec{x} \longrightarrow \vec{x} * \vec{y}$$

is linear. (2) Find the matrix representing this operator L.

(1) (i) Check $L(c\vec{x}) = cL(\vec{x})$.

$$[L(c\vec{x})]_n = [c\vec{x} * \vec{y}]_n = \sum_{m=1}^{N} c x_m y_{n-m} = c \sum_{m=1}^{N} x_m y_{n-m} = [cL(\vec{x})]_n$$

(ii) Check $L(\vec{x} + \vec{z}) = L(\vec{x}) + L(\vec{z})$.

$$[L(\vec{x}+\vec{y})]_n = [(\vec{x}+\vec{z})*\vec{y}]_n = \sum_{m=1}^{N}(x_m+z_m)y_{n-m} = \sum_{m=1}^{N} x_m y_{n-m}+z_m y_{n-m} = \sum_{m=1}^{N} x_m y_{n-m}+\sum_{m=1}^{N} z_m y_{n-m} = L(\vec{x})+L(\vec{z})$$

(2) The matrix of the linear transformation $L : \mathbb{R}^N \to \mathbb{R}^N$ is given by $[L(\vec{e}_1)\ L(\vec{e}_2)\ ...\ L(\vec{e}_N)]$

$$[L(\vec{e}_1)]_n = [\vec{e}_1 * \vec{y}]_n = \sum_{m=1}^{N}(\vec{e}_1)_m y_{n-m} = (\vec{e}_1)_1 y_{n-1} = y_{n-1}$$

So, $L(\vec{e}_1) = \begin{bmatrix} y_N \\ y_1 \\ y_2 \\ \vdots \\ y_{N-1} \end{bmatrix}$

$$[L(\vec{e}_2)]_n = [\vec{e}_2 * \vec{y}]_n = \sum_{m=1}^{N}(\vec{e}_2)_m y_{n-m} = (\vec{e}_2)_2 y_{n-1} = y_{n-2}$$

So, $L(\vec{e}_2) = \begin{bmatrix} y_{N-1} \\ y_N \\ y_1 \\ \vdots \\ y_{N-2} \end{bmatrix}$. Similarly, $L(\vec{e}_3) = \begin{bmatrix} y_{N-2} \\ y_{N-1} \\ y_N \\ \vdots \\ y_{N-3} \end{bmatrix}$... and ..., $L(\vec{e}_N) = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_N \end{bmatrix}$ So, the matrix of the transformation is

$$\begin{bmatrix} y_N & y_{N-1} & y_{N-2} & \cdots & y_1 \\ y_1 & y_N & y_{N-1} & \cdots & y_2 \\ y_2 & y_1 & y_N & \cdots & y_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ y_{N-1} & y_{N-2} & y_{N-3} & \cdots & y_N \end{bmatrix}$$

Question 5. Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by the following conditions:

(a) L is linear; (b) $L(\vec{e}_1) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$; (c) $L(\vec{e}_2) = \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}$; (d) $L(\vec{e}_3) = \begin{bmatrix} 7 \\ -3 \\ 9 \end{bmatrix}$;

Here $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ is the standard basis for $\mathbb{R}^3$. Prove that there is a 3×3 matrix A such that $L(x) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^3$. What is the matrix A?

Proof in lecture notes. The matrix of a linear transformation $L : \mathbb{R}^3 \to \mathbb{R}^3$ is given by $A = [L(\vec{e}_1)\ L(\vec{e}_2)\ L(\vec{e}_3)]$. So, $A = \begin{bmatrix} 1 & 5 & 7 \\ 2 & 2 & -3 \\ 3 & 1 & 9 \end{bmatrix}$

Question 6. Let $\mathscr{B} = \{\vec{b}_1, \vec{b}_2\} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ be a basis for $V = \text{Span}\{\vec{b}_1, \vec{b}_2\}$.

(1). Find the coordinate of $\vec{x} = \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}$ relative to $\mathscr{B}$.

(2). Suppose the coordinate of $\vec{y} \in V$ is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Find the vector $\vec{y}$.

(1) To find to coordinate $[\vec{x}]_{\mathscr{C}}$, we need to solve $x_1\vec{b}_1 + x_2\vec{b}_2 = \vec{x}$. We get $x_1 = 2$ and $x_2 = 1$. Hence, $[\vec{x}]_{\mathscr{C}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

(2) $\vec{y} = [\vec{b}_1\ \vec{b}_2][\vec{x}]_{\mathscr{C}} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$

Question 7. Let $V = \{a_1 t + a_2 t^2 + a_3 t^3 \mid a_1, a_2, a_3 \in \mathbb{R}\}$ with basis $\mathscr{B} = \{t, t^2, t^3\}$. Let $P_2 = \{a_0 + a_1 t + a_2 t^2 \mid a_0, a_1, a_2 \in \mathbb{R}\}$ with basis $\mathscr{C} = \{1, t, t^2\}$. Let $T : V \to P_2$ be a transformation defined by derivatives $T(p) = 2f' - f''$.

(1) Prove that T is a linear transformation. (using properties of derivative.)

(2) Find the matrix $[T]_{\mathscr{B}\mathscr{C}}$ of the transformation T respective to the bases $\mathscr{B}$ and $\mathscr{C}$.

(3) Is T an isomorphism?

(1) The i-th derivative satisfies $(f(t)+g(t))^{(i)} = (f(t))^{(i)} + (g(t))^{(i)}$ and $(cf(t))^{(i)} = c(f(t))^{(i)}$ Hence, we can verify that $T(f+g) = T(f) + T(g)$ and $T(cf) = cT(f)$ for any polynomials f, g in V and any real number c.
(2) $[T]_{\mathscr{B}\mathscr{C}} = [[T(t)]_{\mathscr{C}}\ [T(t^2)]_{\mathscr{C}}\ [T(t^3)]_{\mathscr{C}}]$
$T(t) = 2$, $T(t^2) = 4t - 2$, $T(t^3) = 6t^2 - 6t$
$[T(t)]_{\mathscr{C}} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$, $[T(t^2)]_{\mathscr{C}} = \begin{bmatrix} -2 \\ 4 \\ 0 \end{bmatrix}$, $[T(t^3)]_{\mathscr{C}} = \begin{bmatrix} 0 \\ -6 \\ 6 \end{bmatrix}$
Hence $[T]_{\mathscr{B}\mathscr{C}} = \begin{bmatrix} 2 & -2 & 0 \\ 0 & 4 & -6 \\ 0 & 0 & 6 \end{bmatrix}$
(2) T is an isomorphism, since T is linear and the rank is 3.

Question 8. Let V be a subspace of $\mathbb{R}^n$. Suppose $\mathscr{B} = \{\vec{b}_1, \dots, \vec{b}_s\}$ and $\mathscr{C} = \{\vec{v}_1, \dots, \vec{v}_s\}$ are two bases of V.

(1) Find the $\mathscr{B} - \mathscr{C}$-matrix $S = [\mathrm{id}]_{\mathscr{B}\mathscr{C}}$ of the identity map from V to V. This matrix is also called **change of coordinate matrix** from $\mathscr{B}$ to $\mathscr{C}$.

The $\mathscr{B} - \mathscr{C}$-matrix $S = [\mathrm{id}]_{\mathscr{B}\mathscr{C}}$ is given by $[[\vec{b}_1]_{\mathscr{C}}\ [\vec{b}_2]_{\mathscr{C}}\ \cdots\ [\vec{b}_s]_{\mathscr{C}}]$

(2) Show that $[\vec{b}_1\ \dots\ \vec{b}_s] = [\vec{v}_1\ \dots\ \vec{v}_s]S$.

By definition of coordinate $[\vec{v}_1\ \dots\ \vec{v}_s][\vec{x}]_{\mathscr{C}} = \vec{x}$ for any $\vec{x} \in V$. Hence, each $\vec{b}_i = [\vec{v}_1, \dots, \vec{v}_s][\vec{b}_i]_{\mathscr{C}}$. So, $[\vec{b}_1, \dots, \vec{b}_s] = [\vec{v}_1, \dots, \vec{v}_s]S$.

Question 9. Let V be a subspace of $\mathbb{R}^3$. Suppose $\mathscr{B} = \{\vec{b}_1, \vec{b}_2\} = \left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ -3 \end{bmatrix} \right\}$ and $\mathscr{C} = \{\vec{v}_1, \vec{v}_2\} = \left\{ \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$ are two bases of V. (Example for the above question.)

(1) Find change of coordinate matrix S from $\mathscr{B}$ to $\mathscr{C}$.

The change of coordinate matrix S from $\mathscr{B}$ to $\mathscr{C}$ is given by $[[\vec{b}_1]_{\mathscr{C}}\ [\vec{b}_2]_{\mathscr{C}}]$
To find $[\vec{b}_1]_{\mathscr{C}}$, we need to solve $\vec{b}_1 = x_1\vec{v}_1 + x_2\vec{v}_2$. We get $x_1 = 2$ and $x_2 = 1$.
To find $[\vec{b}_2]_{\mathscr{C}}$, we need to solve $\vec{b}_2 = x_1\vec{v}_1 + x_2\vec{v}_2$. We get $x_1 = -1$ and $x_2 = 4$.
Hence, the change of coordinate matrix S from $\mathscr{B}$ to $\mathscr{C}$ is $S = \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix}$

(2) Verify that $[\vec{b}_1\ \vec{b}_2] = [\vec{v}_1\ \vec{v}_2]S$.

Calculate $[\vec{v}_1\ \vec{v}_2]S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & -1 \\ -3 & -3 \end{bmatrix} = [\vec{b}_1\ \vec{b}_2]$.

Question 10. A function $T : P_4(\mathbb{R}) \to P_4(\mathbb{R})$ is defined by the rule $T(f) = xf" - 2xf' - f$. Show that T is a linear operator, and find the matrix that represents T with respect to the standard basis of $P_4(\mathbb{R})$.

To show that T is a linear operator we simply verify that $T(f_1 + f_2) = T(f_1) + T(f_2)$ and $T(cf_1) = cT(f_1)$ where c is a constant.
To find the matrix representing T, compute $T(1) = -1$, $T(x) = -x$, $T(x^2) = 2x - 5x^2$ and $T(x^3) = 6x^2 - 4x^3$.
The columns of the required matrix are the coordinate vectors of $T(1), T(x), T(x^2), T(x^3)$ respect to the ordered basis $1, x, x^2, x^3$. Hence, the matrix is

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -5 & 6 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

The following questions are about determinant

Question 11. Consider the real $n \times n$ matrix $A_n = (a_{ij})_{i,j=1,\dots,n}$ which has 2's on the main diagonal, -1's on the two diagonals next to the main diagonal, and 0's elsewhere. For example $A_2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, $A_3 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$, $A_4 = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$.

Compute $\det(A_n)$ in terms of n.

We can calculate that $\det(A_1) = 2$, $\det(A_2) = 3$, $\det(A_3) = 2\det(A_2) - \det(A_1) = 4$.
In general, by cofactor expansion,

$$\det(A_n) = 2\det(A_{n-1}) - \det(A_{n-2})$$

for $n \geq 3$.
So, $\det(A_n) - \det(A_{n-1}) = \det(A_{n-1}) - \det(A_{n-2}) = \det(A_2) - \det(A_1) = 1$.
So $\det(A_n)$ is an arithmetic sequence with $d = 1$. That is $\det(A_n) = n + 1$.

Question 12. Compute the area of the hexagon with vertices (3, 1), (12, 8), (10, 7), (-1,-1), (-10,-8) and (-8,-7). Compute by hand (using determinant) and verify by Matlab use polyshape.

List the vertices in counter clock-wise order. A(3, 1), B(12, 8), C(10, 7), D(-1,-1), E(-10,-8) and F(-8,-7)
Then, we can divide the hexagon into 4 triangles ABC, ACD, ADE, AEF.
The area of ABC is is the determinant of $[\vec{AB}\ \vec{AC}] = \begin{bmatrix} 9 & 7 \\ 7 & 6 \end{bmatrix}$ which is 6
The total area is 15.
We can sketch the hexagon or using Matlab draw the hexagon.

Matlab Input

```
pgon = polyshape([3 12 10 -1 -10 -8],[1 8 7 -1 -8 -7])

plot(pgon)

a=area(pgon)
```

Output

```
 pgon =
  polyshape with properties:

      Vertices: [6  2 double]
    NumRegions: 1
      NumHoles: 0
```

Question 13. True or False: (Briefly explain the reason.)

(1) $\det(A + B) = \det(A) + \det(B)$ for all 5×5 matrices A and B.
(2) The equation $\det(-A) = \det(A)$ holds for all 6×6 matrices.
(3) If all the entries of a 7×7 matrix A are 7, then $\det(A)$ must be 7^7
(4) An 8×8 matrix fails to be invertible if (and only if) its determinant is nonzero.
(5) If B is obtained be multiplying a column of A by 9, then the equation $\det(B) = 9\det(A)$ must hold.
(6) If A is any $n \times n$ matrix, then $\det(AA^T) = \det(A^TA)$
(7) There is an invertible matrix of the form $\begin{bmatrix} a & e & f & j \\ b & 0 & g & 0 \\ c & 0 & h & 0 \\ d & 0 & i & 0 \end{bmatrix}$
(8) If A is an invertible $n \times n$ matrix, then $\det(A^T)\det(A^{-1}) = 1$.
(9) $\det(4A) = 4\det(A)$ for all 4×4 matrices A.
(10) There is a nonzero 4×4 matrix A such that $\det(4A) = 4\det(A)$.
(11) $\det(AB) = \det(BA)$ for all $n \times n$ matrices A and B.

(1) No. (2) Yes. (3) No. $\det(A) = 0$. (4) No. (5) Yes. (6) Yes. (7) No. (8) Yes. (9) No. (10) Yes. Choose $\det(A) = 0$. (11) Yes.

Question 14. Is there a 3×3 matrix such that $A^2 + I = \mathbf{0}$? Answer the question in real numbers and complex numbers. Show your reason.

Over real numbers, No. If $A^2 = -I_3$, then $\det(A^2) = [\det(A)]^2 = \det(-I_3) = -1$, which is impossible over real numbers.

Over complex numbers, it is possible, for example, $\begin{bmatrix} 1 & & \\ & i & \\ & & i \end{bmatrix}$

Question 15. Let A be the matrix $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 8 & 9 \\ 2 & 4 & 6 & 10 \\ 1 & 5 & 10 & 9 \end{bmatrix}$. Compute by hand the **determinant** of A. Write down all steps you are using. (Hint: using row operations together with cofactor expansion)

$$\det(A) = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 3 & 6 & 5 \end{vmatrix} = 1 \begin{vmatrix} 1 & 2 & 1 \\ 0 & 0 & 2 \\ 3 & 7 & 5 \end{vmatrix} = 2(-1)^{2+3} \begin{vmatrix} 1 & 2 \\ 3 & 7 \end{vmatrix} = -2(7-6) = -2$$

Question 16. Let $A = \begin{bmatrix} 0 & 0 & \dots & 0 & a_1 \\ 0 & 0 & \dots & a_2 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & a_{n-1} & \dots & 0 & 0 \\ a_n & 0 & \dots & 0 & 0 \end{bmatrix}$. Find the determinant of A and prove your result..

Let $D_n = \det(A)$. By cofactor expansion of the first column, we have $D_n = (-1)^{n-1} a_n D_{n-1}$.
Using recurrence formula, we have $D_n = (-1)^{n-1+n-2+\cdots+2+1} a_n a_{n-1} \cdots a_2 a_1 = (-1)^{n(n-1)/2} a_1 a_2 \cdots a_n$

Question 17. Find a 5×5 permutation matrix P such that $P[x_1\ x_2\ x_3\ x_4\ x_5]^T = [x_3\ x_1\ x_4\ x_5\ x_2]^T$

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Question 18. (Vandermonde determinants.) Consider distinct real numbers $a_0, a_1, ..., a_n$. We define the $(n+1) \times (n+1)$ matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ a_0 & a_1 & a_2 & \cdots & a_n \\ a_0^2 & a_1^2 & a_2^2 & \cdots & a_n^2 \\ \vdots & \vdots & \dots & \vdots & \\ a_0^{n-1} & a_1^{n-1} & a_2^{n-1} & \cdots & a_n^{n-1} \\ a_0^n & a_1^n & a_2^n & \cdots & a_n^n \end{pmatrix}$$

Vandermonde showed that

$$\det(A) = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ a_0 & a_1 & a_2 & \cdots & a_n \\ a_0^2 & a_1^2 & a_2^2 & \cdots & a_n^2 \\ \vdots & \vdots & \cdots & \vdots & \\ a_0^{n-1} & a_1^{n-1} & a_2^{n-1} & \cdots & a_n^{n-1} \\ a_0^n & a_1^n & a_2^n & \cdots & a_n^n \end{vmatrix} = \prod_{0\le j<i\le n} (a_i - a_j)$$

(a) Verify Vandermonde's formula for the case $n = 1$.

(b) Suppose the Vandermonde formula holds for $n-1$. You are asked to demonstrate it for n. Consider the function

$$f(t) = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ a_0 & a_1 & a_2 & \cdots & a_{n-1} & t \\ a_0^2 & a_1^2 & a_2^2 & \cdots & a_{n-1}^2 & t^2 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \\ a_0^{n-1} & a_1^{n-1} & a_2^{n-1} & \cdots & a_{n-1}^{n-1} & t^{n-1} \\ a_0^n & a_1^n & a_2^n & \cdots & a_{n-1}^n & t^n \end{vmatrix}$$

Explain why $f(t)$ is a polynomial of n-th degree. Find the coefficient k of t^n using Vandermonde's formula for $a_0, \ldots, a_{n-1}$. Explain why $f(a_0) = f(a_1) = \ldots = f(a_{n-1}) = 0$. Conclude that $f(t) = k(t-a_0)(t-a_1)\ldots(t-a_{n-1})$ for the scalar k you found above. Substitute $t = a_n$ to demonstrate Vandermonde's formula.

a If $n = 1$, then $A = \begin{bmatrix} 1 & 1 \\ a_0 & a_1 \end{bmatrix}$, so $\det(A) = a_1 - a_0$ (and the product formula holds).

b Expanding the given determinant down the right-most column, we see that the coefficient k of t^n is the $n-1$ Vandermonde determinant which we assume is

$$\prod_{n-1\ge i>j} (a_i - a_j).$$

Now $f(a_0) = f(a_1) = \cdots = f(a_{n-1}) = 0$, since in each case the given matrix has two identical columns, hence its determinant equals zero. Therefore

$$f(t) = \left(\prod_{n-1\ge i>j} (a_i - a_j) \right) (t-a_0)(t-a_1)\cdots(t-a_{n-1})$$

and

$$\det(A) = f(a_n) = \prod_{n\ge i>j} (a_i - a_j),$$

as required.