times in state 0 is .0625 and the proportion of times in state 1 is .375. The astute reader will note that these numbers are the binomial distribution 1/16, 4/16, 6/16, 4/16, 1/16. We could have guessed this answer as follows: If we consider a particular ball, it simply moves randomly back and forth between the two urns. This suggests that the equilibrium state should be just as if we randomly distributed the four balls in the two urns. If we did this, the probability that there would be exactly j balls in one urn would be given by the binomial distribution $b(n, p, j)$ with $n = 4$ and $p = 1/2$. □

Exercises

1 Which of the following matrices are transition matrices for regular Markov chains?

(a) $\mathbf{P} = \begin{pmatrix} .5 & .5 \\ .5 & .5 \end{pmatrix}$.

(b) $\mathbf{P} = \begin{pmatrix} .5 & .5 \\ 1 & 0 \end{pmatrix}$.

(c) $\mathbf{P} = \begin{pmatrix} 1/3 & 0 & 2/3 \\ 0 & 1 & 0 \\ 0 & 1/5 & 4/5 \end{pmatrix}$.

(d) $\mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

(e) $\mathbf{P} = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}$.

2 Consider the Markov chain with transition matrix

$$\mathbf{P} = \begin{pmatrix} 1/2 & 1/3 & 1/6 \\ 3/4 & 0 & 1/4 \\ 0 & 1 & 0 \end{pmatrix} .$$

(a) Show that this is a regular Markov chain.

(b) The process is started in state 1; find the probability that it is in state 3 after two steps.

(c) Find the limiting probability vector $\mathbf{w}$.

3 Consider the Markov chain with general 2×2 transition matrix

$$\mathbf{P} = \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix} .$$

(a) Under what conditions is $\mathbf{P}$ absorbing?

(b) Under what conditions is $\mathbf{P}$ ergodic but not regular?

(c) Under what conditions is $\mathbf{P}$ regular?

4 Find the fixed probability vector $\mathbf{w}$ for the matrices in Exercise 3 that are ergodic.

5 Find the fixed probability vector $\mathbf{w}$ for each of the following regular matrices.

(a) $\mathbf{P} = \begin{pmatrix} .75 & .25 \\ .5 & .5 \end{pmatrix}$.

(b) $\mathbf{P} = \begin{pmatrix} .9 & .1 \\ .1 & .9 \end{pmatrix}$.

(c) $\mathbf{P} = \begin{pmatrix} 3/4 & 1/4 & 0 \\ 0 & 2/3 & 1/3 \\ 1/4 & 1/4 & 1/2 \end{pmatrix}$.

6 Consider the Markov chain with transition matrix in Exercise 3, with $a = b = 1$. Show that this chain is ergodic but not regular. Find the fixed probability vector and interpret it. Show that $\mathbf{P}^n$ does not tend to a limit, but that

$$\mathbf{A}_n = \frac{\mathbf{I} + \mathbf{P} + \mathbf{P}^2 + \cdots + \mathbf{P}^n}{n+1}$$

does.

7 Consider the Markov chain with transition matrix of Exercise 3, with $a = 0$ and $b = 1/2$. Compute directly the unique fixed probability vector, and use your result to prove that the chain is not ergodic.

8 Show that the matrix

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 0 & 1 \end{pmatrix}$$

has more than one fixed probability vector. Find the matrix that $\mathbf{P}^n$ approaches as $n \to \infty$, and verify that it is not a matrix all of whose rows are the same.

9 Prove that, if a 3-by-3 transition matrix has the property that its *column* sums are 1, then $(1/3, 1/3, 1/3)$ is a fixed probability vector. State a similar result for n-by-n transition matrices. Interpret these results for ergodic chains.

10 Is the Markov chain in Example 11.10 ergodic?

11 Is the Markov chain in Example 11.11 ergodic?

12 Consider Example 11.13 (Drunkard's Walk). Assume that if the walker reaches state 0, he turns around and returns to state 1 on the next step and, similarly, if he reaches 4 he returns on the next step to state 3. Is this new chain ergodic? Is it regular?

13 For Example 11.4 when $\mathbf{P}$ is ergodic, what is the proportion of people who are told that the President will run? Interpret the fact that this proportion is independent of the starting state.