You can compute eigenvalues or you can use the trace/determinant plane. Here, I'll use eigenvalues for some problems, but the tr/det plane will give the same results.

Solution of ODE2 Problem 1c

(1c)

By definition, a steady state is the root of the algebraic equation

$$F(N,P) = 0,$$

where

$$F(N,P) = \begin{pmatrix} \frac{dN}{dt} \\ \frac{dP}{dt} \end{pmatrix}.$$

In this problem

$$F(N,P) = \begin{pmatrix} N(1-P) \\ \alpha P(N-1) \end{pmatrix}.$$

So a steady state is the same thing as a solution (N,P) of the two simultaneous equations

$$N(1-P) = 0$$

$$\alpha P(N-1) = 0$$

From the first equation $P = 1$ or $N = 0$. So, for a steady state $(\bar{N},\bar{P})$ either $\bar{N} = 0$ or $\bar{P} = 1$. If $\bar{N} = 0$, since also

$$\alpha \bar{P}(\bar{N}-1) = 0$$

must hold, so $\bar{P} = 0$. Hence $(\bar{N},\bar{P}) = (0,0)$ is a steady state. Now if $\bar{P} = 1$ then from second equation $\bar{N} = 1$. Hence $(\bar{N},\bar{P}) = (1,1)$ is another steady state.

To determine the stability of the steady states, we linearize the system:

At any point (N,P) the Jacobian F' of F is:

$$\begin{pmatrix} 1-P & -N \\ \alpha P & \alpha(N-1) \end{pmatrix}$$

In particular in $(\bar{N},\bar{P}) = (0,0)$ we have

$$\begin{pmatrix} 1 & 0 \\ 0 & -\alpha \end{pmatrix}$$

Since the eigenvalues of the Jacobian at $(0,0)$, $\lambda_1 = 1$ and $\lambda_2 = -\alpha$, have opposite signs, $(0,0)$ is a saddle.

Also the Jacobian of F at $(\bar{N},\bar{P}) = (1,1)$ is:

$$\begin{pmatrix} 0 & -1 \\ \alpha & 0 \end{pmatrix}$$

with eigenvalues λ given by

$$\begin{vmatrix} -\lambda & -1 \\ \alpha & -\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad \lambda_1, \lambda_2 = \pm i\sqrt{\alpha}$$

Thus $(\bar{N},\bar{P}) = (1,1)$ is a center, since the eigenvalues are purely imaginary.

Solution of ODE2 Problem 4bce

(4b)

The steady states are the solutions of $f_1(N_1, N_2) = f_2(N_1, N_2) = 0$ which in general would be:

$$(N_1, N_2) = (0, 0)$$
$$(N_1, N_2) = (1, 0)$$
$$(N_1, N_2) = (0, 1)$$

and we also have a fourth possible steady state that satisfies

$$(1 - a_{12}a_{21})N_1 = 1 - a_{12}, \quad (1 - a_{12}a_{21})N_2 = 1 - a_{21}.$$

When a_{12} and a_{21} equal one, these can be solved by any (N_1, N_2). If they do not equal 1, but $a_{12}a_{21} = 1$, then there is no solution. If $a_{12}a_{21} \neq 1$, there is also this fourth solution:

$$(N_1, N_2) = \left(\frac{1 - a_{12}}{1 - a_{12}a_{21}}, \frac{1 - a_{21}}{1 - a_{12}a_{21}}\right)$$

To determine the stability of the steady states, we linearize the system:

At any point (N_1, N_2), the Jacobian $F' = J$ of

$$F = \begin{pmatrix} f_1(N_1, N_2) \\ f_2(N_1, N_2) \end{pmatrix}$$

is:

$$J(N_1, N_2) = \begin{pmatrix} 1 - 2N_1 - a_{12}N_2 & -a_{12}N_1 \\ -\alpha a_{21}N_2 & \alpha(1 - 2N_2 - a_{21}N_1) \end{pmatrix}.$$

In order to determine the stability of the steady states, we look at the real part of the eigenvalues of $J(N_1, N_2)$. If both the eigenvalues have negative real part, the steady state, (N_1, N_2), is stable; otherwise it is unstable.

The first steady state, $(0, 0)$, is unstable since the eigenvalues of $J(0, 0)$ given by

$$J(0, 0) = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$$

$$|J(0, 0) - \lambda I| = \begin{vmatrix} 1 - \lambda & 0 \\ 0 & \alpha - \lambda \end{vmatrix} = 0 \quad \Rightarrow \quad \lambda_1 = 1, \ \lambda_2 = \alpha$$

are positive.

The eigenvalues of the Jacobian at the second steady state, namely, $(1, 0)$, are given by:

$$|J(1, 0) - \lambda I| = \begin{vmatrix} -1 - \lambda & -a_{12} \\ 0 & \alpha(1 - a_{21}) - \lambda \end{vmatrix} = 0 \quad \Rightarrow \quad \lambda_1 = -1, \ \lambda_2 = \alpha(1 - a_{21}).$$

If $\lambda_2 < 0$, then since $\lambda_1 = -1 < 0$, the steady state $(1,0)$ is stable:

$$\lambda_2 = \alpha(1 - a_{21}) < 0 \quad \text{iff} \quad a_{21} > 1 \quad (\text{or equivalently} \quad \frac{b_{21}k_1}{k_2} > 1).$$

On the other hand, if $\lambda_2 > 0$, (i.e. $a_{21} < 1$ or equivalently $\dfrac{b_{21}k_1}{k_2} < 1$), then $(1,0)$ is unstable.

Similarly for the third steady state, $(0,1)$, the eigenvalues are

$$|J(0,1) - \lambda I| = \begin{vmatrix} 1 - a_{12} - \lambda & 0 \\ -\alpha a_{21} & -\alpha - \lambda \end{vmatrix} = 0 \quad \Rightarrow \quad \lambda_1 = -\alpha, \ \lambda_2 = 1 - a_{12},$$

and so $(0,1)$ is stable if $a_{12} > 1$, and is unstable if $a_{12} < 1$.

(4c)

Arguing as in part (b), $(0,0)$ is always an unstable steady state; $(1,0)$ is unstable since $a_{21} < 1$ and $(1,0)$ is also unstable since $a_{12} < 1$. Finally since $a_{12}a_{21} = \frac{1}{4} \neq 1$, the last steady state is positive and it is $(\frac{2}{3}, \frac{2}{3})$.

Also the Jacobian matrix J for this last steady state is:

$$J = (1 - a_{12}a_{21})^{-1} \begin{pmatrix} a_{12} - 1 & a_{12}(a_{12} - 1) \\ \alpha a_{21}(a_{21} - 1) & \alpha(a_{21} - 1) \end{pmatrix}$$

with $a_{12} = a_{21} = \frac{1}{2}$, i.e.

$$J = \frac{4}{3} \begin{pmatrix} -1/2 & -1/4 \\ -\alpha/4 & -\alpha/2 \end{pmatrix}$$

Since $\operatorname{tr}(J) < 0$ and $\det(J) = 3\alpha/16 > 0$, $(\frac{2}{3}, \frac{2}{3})$ is a *stable* steady state.

(4e)

Arguing as in part *b*, $(0,0)$ is always an unstable steady state; $(1,0)$ is stable since $a_{21} > 1$ and $(1,0)$ is also stable since $a_{12} > 1$. Finally since $a_{12}a_{21} = 4 \neq 1$, the last steady state is positive and it is $(\frac{1}{3}, \frac{1}{3})$. To determine the stability of $(\frac{1}{3}, \frac{1}{3})$, we compute the Jacobian:

$$J = \frac{-1}{3} \begin{pmatrix} -1/3 & -2/3 \\ -2\alpha/3 & -\alpha/3 \end{pmatrix}$$

Since $\operatorname{tr}(J) < 0$ and $\det(J) < 0$, $(\frac{1}{3}, \frac{1}{3})$ is a *saddle*.

Solution of ODE2 Problem 8d

(8d)

To find the steady states, let $\frac{dN_1(t)}{dt} = \frac{dN_2(t)}{dt} = 0$, i.e.

$N_1(t)[\alpha - (N_1(t) + N_2(t))] = 0$ and $N_2(t)[\beta - (N_1(t) + N_2(t))] = 0$.

We get:

$(N_1, N_2) = (0, 0)$; $(N_1, N_2) = (0, \beta)$; $(N_1, N_2) = (\alpha, 0)$.

The last two steady states are positive when $\alpha, \beta > 0$.

To determine the stability of each steady state, we first calculate the Jacobian matrix of $\begin{pmatrix} \frac{dN_1}{dt} \\ \frac{dN_2}{dt} \end{pmatrix}$ at a generic point (N_1, N_2) which is:

$$J = \begin{pmatrix} \alpha - 2N_1 - N_2 & -N_1 \\ -N_2 & \beta - 2N_2 - N_1 \end{pmatrix}.$$

The Jacobian at $(0, 0)$ is:

$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$$

The eigenvalues of J at $(0, 0)$ are α and β which both of them are positive. Hence $(0, 0)$ is unstable.

The Jacobian at $(0, \beta)$ is:

$$\begin{pmatrix} \alpha - \beta & 0 \\ -\beta & -\beta \end{pmatrix}$$

The eigenvalues of J at $(0, \beta)$ are $\alpha - \beta$ and $-\beta$. $\alpha - \beta = (\frac{a}{b} - 1) - (\frac{a}{b} - s) = s - 1 < 0$ and also $-\beta < 0$, hence $(0, \beta)$ is a stable point.

The Jacobian at $(\alpha, 0)$ is:

$$\begin{pmatrix} -\alpha & -\alpha \\ 0 & \beta - \alpha \end{pmatrix}$$

The eigenvalues of J at $(\alpha, 0)$ are $\beta - \alpha$ and $-\alpha$.

$\beta - \alpha = (\frac{a}{b} - s) - (\frac{a}{b} - 1) = 1 - s > 0$ and $-\alpha < 0$, so $(\alpha, 0)$ is a saddle.

Solution of ODE3 Problem 6d

Recall that in problem 6c, you found this state with nonzero N:

$$(N_2, C_2) = (\frac{\alpha_1\alpha_2}{\alpha_3} - \frac{\alpha_1}{\alpha_1 - \alpha_3}, \frac{\alpha_3}{\alpha_1 - \alpha_3})$$

(6d) To determine the stability of (N_2, C_2), we calculate the Jacobian at this point, which is:

$$\begin{pmatrix} 0 & \frac{\alpha_1 N_2}{(1+C_2)^2} \\ -\frac{\alpha_3}{\alpha_1} & -\frac{N_2}{(1+C_2)^2} - 1 \end{pmatrix}.$$

Since $N_2 > 0$, $\det(J) > 0$ and $\mathrm{tr}(J) < 0$, and hence (N_2, C_2) is stable.