**Math 5110- Applied Linear Algebra-Fall 2022**

**Instructor: He Wang**

**Test 2.**

**Student Name:** Abhilasha Jain ______ ______/50

**Rules and Instructions for Exams:**

1. Unless otherwise specified, to receive full credits you must show **all** necessary work. The grading is based on your work shown. Only a final result from computer will receive zero point.

2. You need to finish the exam yourself. Any discussions with the other people will be considered as **academic dishonesty**. **Cheating, Unauthorized Collaboration, and Facilitating Academic Dishonesty are not allowed.** You can read a description of each here http://www.northeastern.edu/osccr/academic-integrity-policy/

3. This is an open exam. You are allowed to look at textbooks, and use a computer.

4. You are **not** allowed to discuss with any other people.

5. You are **not** allowed to ask questions on any internet platform.

6. For programming questions, please following the specific instruction on the use of libraries.

Notation: $\vec{x} \in \mathbb{R}^n$ means a column vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

**1.** (8 points) Let $\mathbb{R}^5$ be the Euclidean space with dot product. Let $V$ be a subspace spanned by

$$\vec{u}_1 = \begin{bmatrix}1\\2\\1\\0\\0\end{bmatrix}, \vec{u}_2 = \begin{bmatrix}0\\0\\1\\2\\1\end{bmatrix}$$

(1) Apply the Gram-Schmidt process to find the **orthonormal** basis of $V$.

$$\vec{u}_1 = \vec{v}_1 = \begin{bmatrix}1\\2\\1\\0\\0\end{bmatrix} \qquad \vec{u}_2 = \vec{v}_2 - \frac{\vec{u}_1 \cdot \vec{v}_2}{\|\vec{u}_1\|^2} \cdot \vec{u}_1 = \begin{bmatrix}-1/6\\-1/3\\5/6\\2\\1\end{bmatrix}$$

$$\vec{e}_1 = \frac{\vec{u}_1}{\|\vec{u}_1\|} = \frac{1}{\sqrt{6}}\begin{bmatrix}1\\2\\1\\0\\0\end{bmatrix} \qquad \vec{e}_2 = \frac{\vec{u}_2}{\|\vec{u}_2\|} = \sqrt{\frac{6}{35}}\begin{bmatrix}-1/6\\-1/3\\5/6\\2\\1\end{bmatrix}$$

$\therefore$ Orthonormal basis $= \{\vec{e}, \vec{e}_2\}$

(2) Find a basis for the **orthogonal complement** of $V$.

$$A = \begin{bmatrix}1 & 2 & 1 & 0 & 0\\0 & 0 & 1 & 2 & 1\end{bmatrix}$$

$$\text{rref}(A) = \begin{bmatrix}1 & 2 & 0 & -2 & -1\\0 & 0 & 1 & 2 & 1\end{bmatrix}$$

$$\text{Basis} = \left\{\begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}2\\0\\-2\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\-1\\0\\1\end{bmatrix}\right\}$$

$$\vec{e}_3 = \frac{1}{\sqrt{5}}\begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix} \qquad \vec{e}_4 = \frac{1}{3}\begin{bmatrix}\\\\2\\1\\0\end{bmatrix} \qquad \vec{e}_5 = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\0\\-1\\0\\1\end{bmatrix}$$

Orthonormal b . of orthogonal complement of $V = \{\vec{e}_3 \; \vec{e}_4, \vec{e}_5\}$

(3) Find a formula to calculate the **shortest** distance from any point $\vec{x} \in \mathbb{R}^5$ to $V$. (You don't have to simplicify your formula.)

Shortest distance from any point $\vec{x} \in 11$ to $V$

is $\|\vec{x} - \text{proj}.\, V\|$

where

$$\text{proj}.\, V = (\vec{x}, \vec{e}_1)\,\vec{e}_1 + (\vec{x} \cdot \vec{e}_2)\,\vec{e}_2$$

**2.** (8 points) Let $M$ be a $(k+1)\times(k+1)$ matrix

$$M = \frac{1}{2k}\begin{bmatrix} k & 1 & 1 & \cdots & 1 & 1 \\ 1 & k & 1 & \cdots & 1 & 1 \\ 1 & 1 & k & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & k & 1 \\ 1 & 1 & 1 & \cdots & 1 & k \end{bmatrix}$$

All diagonal entries of $M$ are $k$ and all other entries of $M$ are 1. (Clearly write the theorem you used and the precise result. Do not use decimal numbers. )

(1) What is the largest eigenvalue $\lambda_{max}$ of $M$?

Perron-Frobenius Theorem. If A is a positive, column stochastic matrix, then:
- 1 is an eigenvalue of multiplicity one.
- 1 is the largest eigenvalue: all the other eigenvalues have absolute value smaller than 1.
- the eigenvectors corresponding to the eigenvalue 1 have either only positive entries or only negative entries. In particular, for the eigenvalue 1 there exists a unique eigenvector with the sum of its entries equal to 1

$\because$ M is column stochastic matrix ( such that the sum of column is 1)

$\therefore$ $\lambda_{max} = 1$.

(2) What is the eigenvector corresponding to $\lambda_{max}$?

Using Perron Frobenium Theorem

eigen vector corresponding to $\lambda_{max} = [x_1, x_2, x_3 \cdots\cdots x_{k+1}]$

where $x_1 = x_2 \quad x_3 = \cdots = x_{k+1}$

(3) Calculate $\lim_{n\to\infty} M^n$. Suppose A is primitive, with maximal eigenvalue $\lambda_{max}$, left eigenvector $\vec{u}$ and right eigenvector $\vec{v}$ such that, $\vec{u}\cdot\vec{v} = 1$,

then $\lim_{k\to\infty}\left(\frac{1}{\lambda_{max}}A\right)^k = \vec{u}\vec{v}^T$

Now, $\lim_{n\to\infty}\left(\frac{1}{\lambda_{max}}M\right)^n = \vec{u}\vec{v}^T$

$\Rightarrow \lim_{n\to\infty}\left(\frac{1}{1}M^n\right) = [1 \ 1 \ 1 \ 1 \ldots]_{1\times k+1}[1 \ 1 \ 1 \ \ldots]^T_{k+1}$

$\lim_{n\to\infty} M^n = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \ddots & & & \vdots \\ 1 & & & & 1 \end{bmatrix}_{(k+1)\times(k+1)}$

(4) Calculate $\lim_{n\to\infty} M^n\vec{v}$ if $\vec{v}$ is an distribution vector.

Theorem: Let A be a regular, transition n x n matrix.
1. There exists exactly one distribution vector $\vec{x}\in\mathbb{R}^n$ such that $A\vec{x}=\vec{x}$ which is called equilibrium distribution for A denoted as $\vec{x}_{equ}$
2. If $\vec{x}_0$ is any distribution vector in $\lim_{m\to\infty}(A^m\vec{x}_0) = \vec{x}_{equ}$
3. The columns of $\lim_{n\to\infty}(A^n)$ are all $\vec{x}_{equ}$ that is $\lim_{m\to\infty}(A^m) = [\vec{x}_{equ}\ \vec{x}_{equ} \ \ldots \ \vec{x}_{equ}]$

$\lim_{n\to\infty} M^n\vec{v} = \vec{v} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix}_{(k+1)\times 1}$

**3.** (8 points) Let $A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \end{bmatrix}$.

Clearly write the theorem you used. ( You can use Matlab functions eig() and rref(). You can keep 4 decimal numbers in calculation.)

(1) What is the largest eigenvalue $\lambda_{max}$ of $M$?

(Perron-Frobenius Theorem). Let A be an irreducible non-negative matrix.
- A has a positive (real) eigenvalue $\lambda_{max}$ such that all other eigenvalues of A satisfy $|\lambda| \leq \lambda_{max}$
- $\lambda_{max}$ has algebraic multiplicity 1 with a positive eigenvector $\vec{x}$ .
- Any non-negative eigenvector is a multiple of $\vec{x}$
- If A is primitive, then all other eigenvalues of A satisfy $|\lambda| \leq \lambda_{max}$

Using MATLAB, Eigen Values are: -2.2474, -0.000, 0.000, 0.000, 22.2474

$\therefore \lambda_{max} = 22.2474$

(2) What is the eigenvector corresponding to $\lambda_{max}$?

Eigen vector corresponding to $\lambda_{max}$, using MATLAB

$$\begin{bmatrix} 0.2346 \\ 0.3304 \\ 0.4262 \\ 0.5220 \\ 0.6178 \end{bmatrix}$$

(3) Calculate $\lim_{n\to\infty} \left(\frac{1}{\lambda_{max}} M\right)^n$. Suppose A is primitive, with maximal eigenvalue $\lambda_{max}$, left eigenvector $\vec{u}$ and right eigenvector $\vec{v}$ such that $\vec{u}\cdot\vec{v} = 1$, then $\lim_{n\to\infty}\left(\frac{1}{\lambda_{max}} A\right)^n = \vec{u}\vec{v}^T$

Using MATLAB, calculating left and right eign vectors.

$$\vec{u} = \begin{bmatrix} 0.2346 \\ 0.3304 \\ 0.4262 \\ 0.5220 \\ 0.6178 \end{bmatrix} = \vec{V}$$

$$\therefore \vec{u}\vec{v}^T = \begin{bmatrix} 0.0551 & 0.0775 & 0.100 & 0.1225 & 0.1449 \\ 0.0775 & 0.1092 & 0.1408 & 0.1725 & 0.2041 \\ 0.100 & 0.1408 & 0.1816 & 0.2225 & 0.2633 \\ 0.1225 & 0.1725 & 0.2225 & 0.2725 & 0.3225 \\ 0.1449 & 0.2041 & 0.2633 & 0.3225 & 0.3816 \end{bmatrix}$$

**4.** (5 points) Let $P_2(\mathbb{R})$ be the inner product space with polynomials of degree less or equal than 2, where $\langle f, g\rangle$ is defined to be $\langle f, g\rangle = \int_0^1 f(x)g(x)dx$ .

Let polynomials $f(x) = 1$ and $g(x) = 1 - x$.

(1) Find the angle $\alpha$ between $f(x)$ and $g(x)$. (Use the angle defined by the inner product.)

$$\cos\alpha = \frac{\langle f \cdot g\rangle}{\|f\|\,\|g\|} = \frac{\int_0^1 1\cdot(1-x)\,dx}{\sqrt{\int_0^1 1\cdot 1\,dx}\,\sqrt{\int_0^1 (1-x)^2 dx}} = \frac{\left[x - \frac{x^2}{2}\right]_0^1}{\sqrt{[x]_0^1}\,\sqrt{\left[x + \frac{x^3}{3} - x^2\right]_0^1}} = \frac{1/2}{1\cdot 1/\sqrt{3}} = \frac{\sqrt{3}}{2}$$

$$\alpha = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = 30^\circ$$

(2) Find $\text{proj}_f(g)$, the orthogonal projection of $g(x)$ onto $f(x)$.

$$\text{proj}_f(g) = \frac{\langle g \cdot f\rangle}{\langle f \cdot f\rangle} f(x) = \frac{1/2}{1}\cdot 1 = 1/2$$

$$\langle f, g\rangle = \langle g \cdot f\rangle = 1/2$$

$$\langle f, f\rangle = 1$$

(3) Write $g(x) = \text{proj}_f(g) + g^\perp$, where $\langle f, g^\perp\rangle = 0$. Explicitly write done $g^\perp$.

$$1 - x = \frac{1}{2} + g^\perp$$

$$\Rightarrow \quad g^\perp = \frac{1}{2} - x$$

**5.** (3 points) Let $P_2(\mathbb{R})$ be the inner product space with polynomials of degree less or equal than 2, where $\langle f, g\rangle$ is defined to be $\langle f, g\rangle = \int_0^1 f(x)g(x)dx$ .

Let $S$ be the subspace of the inner product space $P_2(\mathbb{R})$ generated by the polynomials 6 and $6x$

Find a **basis** for the **orthogonal complement** of $S$.

$$S = \text{span}\{6, 6x\}$$

$$S^\perp = \text{span}\{p \text{ such that } \langle p, s\rangle = 0 \ \forall\, s \in S\}$$

$$\Rightarrow \int_0^1 6p(x)dx = 0 \quad \& \quad \int_0^1 6x\, p(x)\, dx = 0$$

Let $p(x) = ax^2 + bx + c$

$$\Rightarrow \int_0^1 6\left[ax^2 + bx + c\right]dx = 0 \quad \& \quad \int_0^1 6x\left[ax^2 + bx + c\right]dx = 0$$

$$\Rightarrow 6\left[\frac{ax^3}{3} + \frac{bx^2}{2} + cx\right]_0^1 = 0 \quad \& \quad 6\left[\frac{ax^4}{4} + \frac{bx^3}{3} + \frac{cx^2}{2}\right]_0^1 = 0$$

$$\Rightarrow \frac{a}{3} + \frac{b}{2} + c = 0 \ — ① \quad \& \quad \frac{a}{4} + \frac{b}{3} + \frac{c}{2} = 0 \ — ②$$

Solving ① & ② $\quad a - 6c = 0$, $\quad b + 6c = 0$ $\quad \Rightarrow \begin{bmatrix} a \\ b \\ c \end{bmatrix} = c\begin{bmatrix} 6 \\ -6 \\ 1 \end{bmatrix}$

$$ax^2 + bx + c = 6cx^2 - 6cx + c$$

$= c\left[6x^2 - 6x + 1\right]$

$\therefore$ Orthogonal complement basis is

$6x^2 - 6x + 1$

**6.** (8 points) Find the **least squares approximation** to the function $f(x) = xe^x$ by a quadratic function $a + bx + cx^2$ in the interval $[0, 1]$.

(Hint: Use the distance is induced by the inner product $\langle f, g\rangle = \int_0^1 f(x)g(x)dx$. Use WolframAlpha https://www.wolframalpha.com/ to do the calculation of integrals if needed. )

Let $B = \{1, x, x^2\}$ be the basis for the vector space $P_2(x)$

$\therefore\ a(1) + b(x) + c(x^2) = xe^x$

Apply inner product on both sides using each element of basis $B = \{1, x, x^2\}$

$\langle 1, a(1) + b(x) + c(x^2)\rangle = \langle 1, xe^x\rangle$

$\Rightarrow\ a + \frac{b}{2} + \frac{c}{3} = 1 \quad — 1$

$\langle x, a(1) + b(x) + c(x^2)\rangle = \langle x, xe^x\rangle$

$\Rightarrow\ \frac{a}{2} + \frac{b}{3} + \frac{c}{4} = e - 2 = 0.7183 \quad — 2$

$\langle x^2, a(1) + b(x) + c(x^2)\rangle = \langle x^2, xe^x\rangle$

$\Rightarrow\ \frac{a}{3} \quad \frac{b}{4} + \frac{c}{5} = 6 - 2e = 0.5634 \quad — 3$

Solving 1, 2 & 3

we get $a = 0.0432$

$b = 0.5016$

$c = 2.1180$

Substituting $a, b, c$ in

$a(1) \quad b(x) + c(x^2) = xe^x$

$0.0432(1 + 0.5016(x) + 2.1180(x^2) = xe^x$

**7.** ( 4 points) Answer the following questions. Prove your the true statement and provide a counter example for the false statement.

(1) Suppose that the columns of $M \in \mathbb{R}^{n\times n}$ are orthonormal. What is the determinant of $M^2$?

Let $[a_{1j}\ a_{2j}\ \dots\ a_{ij}\ \dots a_{nj}]^T$ be the $j^{th}$ column of matrix M

$\Rightarrow$ The $j^{th}$ column is orthonormal ($\because$ all columns of M are orthonormal)

$[a_{1j}\ a_{2j} \dots a_{ij} \dots a_{nj}]^T [a_{1j}\ a_{2j} \dots a_{ij} \dots a_{nj}]\ [a_{1j}, a_{2j}, \dots, a_{ij}, \dots, a_{nj}] = \begin{cases} \pm 1 & i=j \\ 0 & i \neq j \end{cases}$

$\Rightarrow$ M is diagonal matrix with entries $\pm 1$

$\therefore$ M is orthogonal matrix

If $MM^T = I$ (true)

$\Rightarrow \det(MM^T) = 1$

$\Rightarrow \det(M)\det(M) = 1$

$\Rightarrow \det(MM) = 1 \Rightarrow \det(M^2) = 1$

(2) Suppose $A$ is any real $m \times n$ matrix and $B$ is any real $n \times m$ matrix. Is $\det(AB) = \det(BA)$?

If A is any $2\times 3$ matrix, $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ and B is $3\times 2$ matrix, $B = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

then $AB = \begin{bmatrix} 14 & 32 \\ 32 & 77 \end{bmatrix}$ $\quad BA = \begin{bmatrix} 17 & 22 & 27 \\ 22 & 29 & 36 \\ 27 & 36 & 45 \end{bmatrix}$

$\det(AB) = 54 \qquad \det(BA) = 0$

$\Rightarrow \det(AB) \neq \det(BA)$

**8.** (6 points) Suppose $N \in \mathbb{R}^{n\times n}$ is a **nilpotent** matrix. Answer the following quesitons. Explain your reason. (You can refer any result from class.)

(1) Suppose $\vec{v}$ is an eigenvector of $N$, prove that $\vec{v}$ is is an eigenvector of $N^2 + 3N + 2I_n$?

Let $\vec{v}$ be an eigenvector of N

$\Rightarrow N\vec{v} = 0\vec{v} = \vec{0}$

Then, $(N^2 + 3N + 2I_n)\vec{v} = N^2\vec{v} + 3N\vec{v} + 2I_n\vec{v} = \vec{0} + \vec{0} + 2\vec{v} = 2\vec{v}$

$\therefore \vec{v}$ is an eigenvector of $N^2 + 3N + 2I_n$

(2) What are the eigenvalues of $N^2 + 3N + 2I_n$?

If $N^2$ and $3N$ are nilpotent matrices, then $(N^2)(3N) = (3N)(N^2)$

$\therefore N^2 + 3N$ is a nilpotent matrix

$\therefore$ All eigen values of $N^2 + 3N$ equal zero.

If eigen values of A are $\lambda_1, \lambda_2 \dots \lambda_n$

Then eigen values of $A + kI_n$ are $\lambda_1 + k, \lambda_2 + k \dots \lambda_n + k$.

$\therefore$ The eigen values of $N^2 + 3N + 2I_n$ are $0+2, 0+2 \dots 0+2$.

(3) Find the determinant $\det(N^2 + 3N + 2I_n)$. $\Rightarrow$ all eigen values are 2.

Theorem: Determinant of a square matrix is the product of all eigen values then

$\det(N^2 + 3N + 2I_n) = 2^n$