Math 5110- Applied linear algebra

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Homework 5.

Using Python or Matlab for the calculations of matrices.

Using Mathematica or https://www.wolframalpha.com/ help the calculation of integrals.

Question 1. The transition matrix $A = \begin{bmatrix} 0.1 & 0.3 \\ 0.9 & 0.7 \end{bmatrix}$.

(a) Find $\lim_{t\to\infty} A^t$.

(b) Find $\lim_{t\to\infty} A^t\vec{v}$ for a vector $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ such that $a + b = 10$. (Hint: The vector $\vec{v}$ is not a distribution vector, but $(\frac{1}{10}\vec{v})$ is a distribution vector.)

Question 2. Let $A = \begin{bmatrix} 2 & 15 & 0 \\ 1 & 2 & 1 \\ 2 & 4 & 2 \end{bmatrix}$. Find the limit $\lim_{t\to\infty}(\frac{1}{7}A)^t$

Question 3. Let $\mathbb{R}^5$ be the Euclidean space. Let V be a subspace spanned by

$$\vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

(1) Apply the Gram-Schmidt process to find the orthonormal basis of V. (2) Find the orthogonal complement of V. (3) Compute $\text{proj}_V\, \vec{y}$. (4) Write $\vec{y} = \begin{bmatrix} 2 \\ 2 \\ -1 \\ 1 \\ 1 \end{bmatrix}$ as $\vec{y} = \vec{y}_1 + \vec{y}_2$ such that $\vec{y}_1 \in V$ and $\vec{y}_2 \in V^\perp$

(5) Write a Matlab/Python function projection(y, A) to compute $\text{proj}_V\, \vec{y}$ (6) Write a Matlab/Python function OBasis(A) to compute orthogonal basis and then a function NBasis(A) to compute orthonormal basis.

Question 4. Notice that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, where

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix},$$

is an orthogonal subset of $\mathbb{R}^4$.

(1) Find a fourth vector $\vec{v}_4 \in \mathbb{R}^4$ that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is an orthogonal basis for $\mathbb{R}^4$.

(2) Find the orthogonal projection of $\vec{y} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ onto $V = \mathrm{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

Question 5. Let P be the plane in $\mathbb{R}^3$ defined by the equation $-3x + y + z = 0$.

(a) Find an orthogonal basis for P.

(b) Find the shortest distance from $(1, 1, 1)$ to the plane P.

Question 6. Show that an orthogonal transformation L from $\mathbb{R}^n$ to $\mathbb{R}^n$ preserves angles: The angle between two nonzero vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^n$ equals the angle between $L(\vec{v})$ and $L(\vec{w})$. Conversely, is any linear transformation that preserves angles orthogonal?

Question 7. Does the formula $\|\vec{x}\| := \sum_{i=1}^{n} x_i^2$ define a norm on $\mathbb{R}^n$?

Question 8. Show that the following formula defines an inner product on the vector space of all $m \times n$ matrices.

$$\langle A, B \rangle := \mathrm{trace}(A^T B).$$

Question 9. Is $C[0, 1]$ an inner product space under the following formula?

$$\langle f, g \rangle = \int_0^1 (f(x) + g(x))dx$$

Question 10. Let S be the subspace of the inner product space $P_3(\mathbb{R})$ generated by the polynomials $1 - x$ and $2 - x + x^2$ where $\langle f, g \rangle$ is defined to be $\langle f, g \rangle = \int_0^1 f(x)g(x)dx$. Find a basis for the orthogonal complement of S.