Math 5110 Applied Linear Algebra

He Wang

he.wang@northeastern.edu

Homework -Least Squares

Using Python or Matlab for the calculations of matrices.

Using Mathematica or https://www.wolframalpha.com/ help the calculation of integrals.

Question 1. The following data were collected for the mean annual temperature t and rainfall r in a certain region; use the Method of Least Squares to find a linear approximation for r in terms of t.

t	24	27	22	24
r	47	30	35	38

Let $r = at + b$ be the linear relation that best fits the data. Then we need to find the least squares solution to

$$\begin{cases} 24a + b = 47 \\ 27a + b = 30 \\ 22a + b = 35 \\ 24a + b = 38 \end{cases}$$

Solve the linear system using normal equation

$$\begin{bmatrix} 2365 & 97 \\ 97 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3620 \\ 150 \end{bmatrix}$$

which has a unique solution $a = -70/51, b = 3610/51$.
Hence, the linear relation is $r == (-70/51)t + 3610/51$

Question 2. In a tropical rain forest the following data were collected for the numbers x and y (per square kilometer) of a prey species and a predator species over a number of years. Use least squares to find a quadratic function of x that approximates y.

x	2	4	3	5
y	1	2	2	1

Let $r = a + bx + cx^2$ be the quadratic relation that best fits the data. Then we need to find the least squares solution to

$$\begin{cases} a + 2b + 4c = 1 \\ a + 4b + 16c = 2 \\ a + 3b + 9c = 2 \\ a + 5b + 25c = 1 \end{cases}$$

Solve the linear system using normal equation

$$\begin{bmatrix} 4 & 14 & 54 \\ 14 & 54 & 224 \\ 54 & 225 & 978 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 6 \\ 21 \\ 79 \end{bmatrix}$$

which has a unique solution $a = -4, b = 7/2, c = -1/2$.
Hence, the quadratic relation is $y = -4 + \dfrac{7}{2}x - \dfrac{1}{2}x^2$

Question 3. Find a least squares approximation to the function e^{-x} by a linear function in the interval $[1, 2]$. Use the inner product $\langle f, g \rangle = \int_1^2 f(x)g(x)dx$

Let S be the subspace of $C[1, 2]$ which consists of all linear functions.
Thus S has the basis $1, x$. The least squares approximation to e^{-x} by a linear function is the projection of e^{-x} on S. So we have to find a and b such that $e^{-x} - a - bx$ belongs to $S^\perp$, that is, it must be orthogonal to 1 and x.
The condition for this are $\langle e^{-x} - a - bx, 1 \rangle = 0$ and $\langle e^{-x} - a - bx, x \rangle = 0$. Perform the integration we have

$$\begin{cases} a + \frac{3}{2}b = e^{-1} - e^{-2} \\ \frac{3}{2}a + \frac{7}{3}b = 2e^{-1} - 3e^{-2} \end{cases}$$

The solution is

$$\begin{cases} a = -8e^{-1} + 26e^{-2} \\ b = 6e^{-1} - 18e^{-2} \end{cases}$$

So the linear approximation to e^{-x} is $a + bx$ with a, b above.

Question 4. Find a least square approximation for the function $\sin x$ as a quadratic function of x in the interval $[0, \pi]$. Use the inner product $\langle f, g \rangle = \int_0^\pi f(x)g(x)dx$

Let S be the subspace of $C[1,2]$ which consists of all quadratic functions. (degree ≤ 2)
Thus S has the basis $1, x, x^2$. The least squares approximation to $\sin x$ by a linear function is the projection of $\sin x$ on S. So we have to find a, b and c such that $\sin x - a - bx - cx^2$ belongs to $S^\perp$, that is, it must be orthogonal to 1, x and x^2.
The condition for this are $\langle \sin x - a - bx - cx^2, 1\rangle = 0$; $\langle \sin x - a - bx - cx^2, x\rangle = 0$ and $\langle \sin x - a - bx - cx^2, x^2\rangle = 0$. Perform the integration we have

$$\begin{cases} 2 - \pi a - \frac{\pi^2}{2} b - \frac{\pi^3}{3} c = 0 \\ \pi - \frac{\pi^2}{2} a - \frac{\pi^3}{3} b - \frac{\pi^4}{4} c = 0 \\ \pi^2 - 4 - \frac{\pi^3}{3} a - \frac{\pi^4}{4} b - \frac{\pi^5}{5} c = 0 \end{cases}$$

The solution is

$$\begin{cases} a = 12(\pi^2 - 10)/\pi^3, \\ b = 60(-\pi^2 + 12)/\pi^4 \\ c = 60(\pi^2 - 12)/\pi^5 \end{cases}$$

So the linear approximation to $\sin x$ is $a + bx + cx^2$ with a, b, c as above.