Northeastern University, Department of Mathematics

MATH 5110: Applied Linear Algebra and Matrix Analysis.

• Instructor: **He Wang** Email: **he.wang@northeastern.edu**

§9 Dynamical Systems

Contents

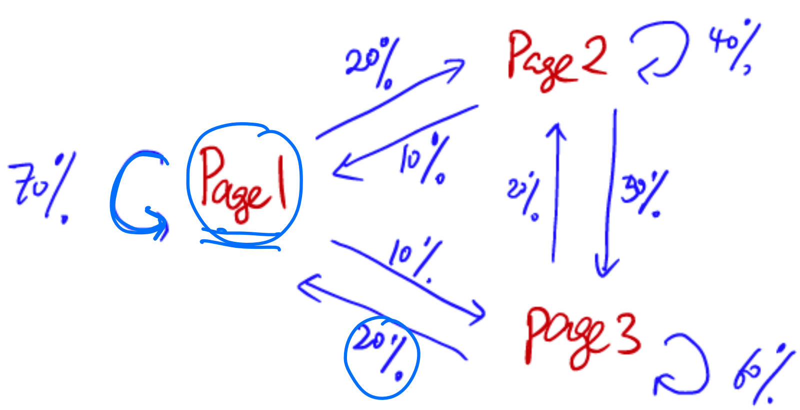
1. Dynamical Systems and Eigenvectors.

Google's $\boxed{\textbf{PageRank}}$ **Algorithm** (Larry Page and Sergey Brin, 1996)

Consider a mini-web with only three pages: Page1, Page2, Page3. Initially, there is an equal number of surfers on each page. The initial probability distribution vector is

$$\begin{bmatrix}100\\100\\100\end{bmatrix} = 300\begin{bmatrix}1/3\\1/3\\1/3\end{bmatrix} \qquad \vec{x}_0 = \begin{bmatrix}1/3\\1/3\\1/3\end{bmatrix} \text{ or } \begin{bmatrix}1/2\\1/3\\1/6\end{bmatrix} = \begin{bmatrix}P_1\\P_2\\P_3\end{bmatrix}$$

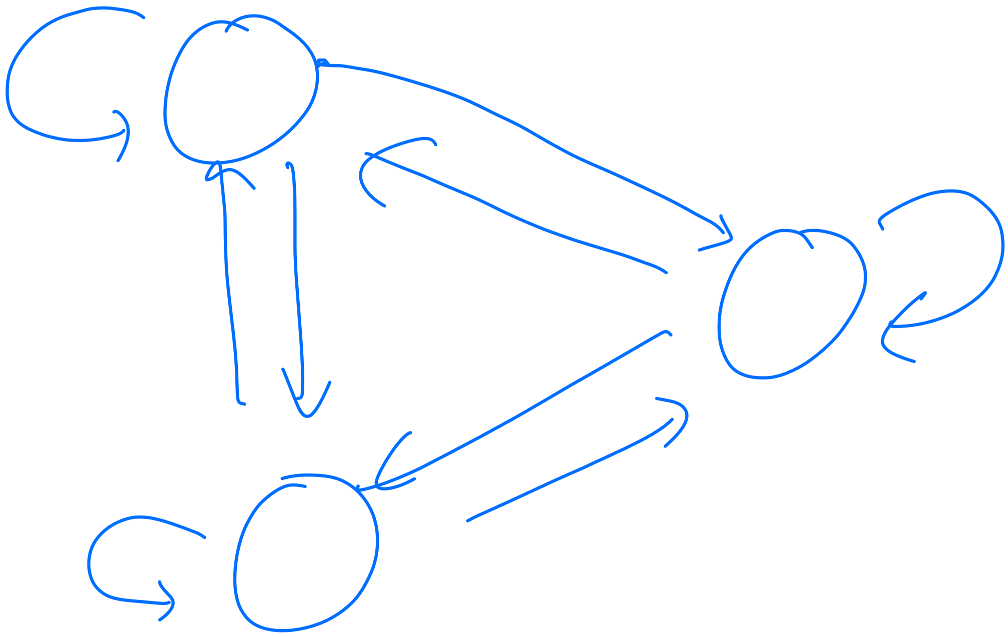
After 1 minute, some people will move onto different pages with a probability distribution vector $\vec{x}_1$, as in the following diagram



$$\vec{x}_1 = \begin{bmatrix}0.7P_1 + 0.1P_2 + 0.2P_3\\ 0.2P_1 + 0.4P_2 + 0.2P_3\\ 0.1P_1 + 0.5P_2 + 0.6P_3\end{bmatrix} = \begin{bmatrix}0.7 & 0.1 & 0.2\\ 0.2 & 0.4 & 0.2\\ 0.1 & 0.5 & 0.6\end{bmatrix}\vec{x}_0 = A\vec{x}_0$$

Example 1. There is a bicycle sharing company in MA. Records indicate that, on average, 10% of the customers taking a bicycle in downtown go to Cambridge and 30% go to suburbs. Customers boarding in Cambridge have a 30% chance of going to downtown and a 30% chance of going to the suburbs, while suburban customers choose downtown 40% of the time and Cambridge 30% of the time. The owner of the bicycle sharing company is interested in knowing where the bicycle will end up, on average.

$$A = \begin{bmatrix} 0.6 & 0.3 & 0.4 \\ 0.1 & 0.4 & 0.3 \\ 0.3 & 0.3 & 0.3 \end{bmatrix}$$



- $\vec{x}_0 \xrightarrow{A} \vec{x}_1 \xrightarrow{A} \vec{x}_2 \quad \cdots\cdots \to \quad \vec{x}_\infty$

- discrete Dynamical System with initial value $\vec{x}_0$

$$\vec{x}_{t+1} = A\vec{x}_t \quad \text{for} \quad t = 0, 1, \ldots$$

- $\vec{x}_{t+1} = A^{t+1}\vec{x}_0$

Calculate A^{t+1}

A^∞

If

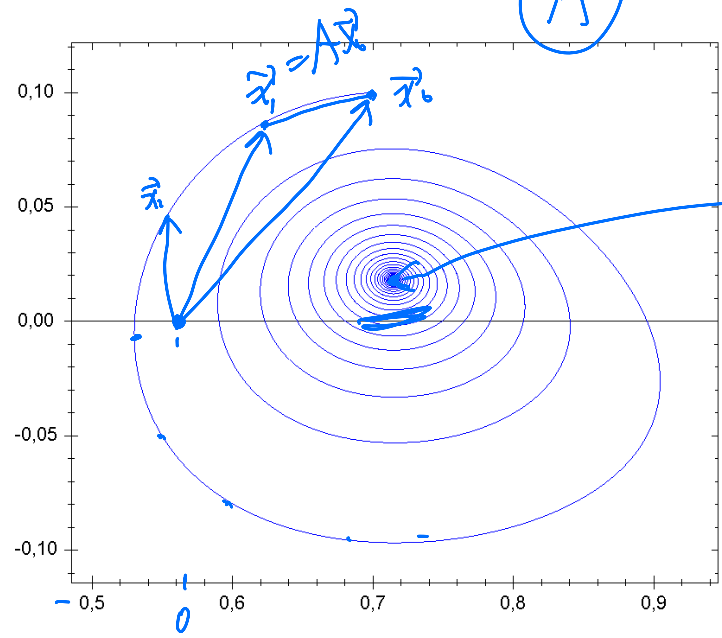
$A = PDP^{-1}$

$A = PJP^{-1}$

Remark: Let A be a 2×2 matrix The endpoints of state vectors $\vec{x}(0)$, $\vec{x}(1)$, $\cdots$, $\vec{x}(t)$, $\ldots$, form the discrete **trajectory** of the system. A **phase portrait** of the dynamical system shows trajectories for various initial states.

A

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$



$A^{\infty}\vec{x}_0$

PageRank Example:

sum=1 sum=1

$$A = \begin{bmatrix} 0.7 & 0.1 & 0.2 \\ 0.2 & 0.4 & 0.2 \\ 0.1 & 0.5 & 0.6 \end{bmatrix} \quad \text{and} \quad \vec{x}_0 = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

$A = PDP^{-1}$

$A^t = PD^tP^{-1}$

Example 2. Find explicit formulas for A^t.

Example 3. Find explicit formulas for $A^t\vec{x}_0$

Example 4. Find $\lim\limits_{t\to\infty} A^t$

Example 5. Find $\lim\limits_{t\to\infty} A^t\vec{x}_0$

1. Find all eigenvalues by $\det(A-\lambda I)=0$

$\lambda_1=1$, $\lambda_2=0.5$, $\lambda_3=0.2$

2. Find an eigenbasis for A

A basis for eigenspace E_{λ_1} is $\left\{\begin{bmatrix}7\\5\\8\end{bmatrix}\right\}$

A basis for eigenspace E_{λ_2} is $\left\{\begin{bmatrix}1\\0\\-1\end{bmatrix}\right\}$

A basis for eigenspace E_{λ_3} = $\left\{\begin{bmatrix}-1\\-3\\4\end{bmatrix}\right\}$

3. $A=PDP^{-1}$ where $P=\begin{bmatrix}7&1&-1\\5&0&-3\\8&-1&4\end{bmatrix}$ $D=\begin{bmatrix}1&&\\&0.5&\\&&0.2\end{bmatrix}$

rank | matrix

4. $\boxed{A^t=PD^tP^{-1}}=\begin{bmatrix}7&1&-1\\5&0&-3\\8&-1&4\end{bmatrix}\begin{bmatrix}1&&\\&0.5^t\to 0&\\&&0.2^t\to 0\end{bmatrix}\frac{1}{60}\begin{bmatrix}3&3&3\\44&-36&-16\\5&-15&5\end{bmatrix}$

$t\to\infty$

$=\left[\vec{b}_1\quad 0.5^t\vec{b}_2\quad 0.2^t\vec{b}_3\right]P^{-1}$

$[\vec{x}_0]_B=\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\begin{bmatrix}1/20\\-2/45\\-1/36\end{bmatrix}$ $\quad\vec{x}_0=c_1\vec{b}_1+c_2\vec{b}_2+c_3\vec{b}_3$ $\quad\vec{x}_0=P\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}$

$\boxed{A^t\vec{x}_0}=c_1\lambda_1^t\vec{b}_1+c_2\lambda_2^t\vec{b}_2+c_3\lambda_3^t\vec{b}_3 = PD^tP^{-1}\vec{x}_0=[B_1\ B_2\ B_3]\begin{bmatrix}1^t&&\\&0.5^t&\\&&0.2^t\end{bmatrix}P^{-1}\vec{x}_0$

$=\frac{1}{20}\cdot\vec{b}_1-\frac{2}{45}(0.5^t)\vec{b}_2-\frac{1}{36}(0.2^t)\vec{b}_3$

$\lim_{t\to\infty}A^t=[\vec{b}_1\ \vec{0}\ \vec{0}]P^{-1}=\begin{bmatrix}7&0&0\\5&0&0\\8&0&0\end{bmatrix}\frac{1}{60}\begin{bmatrix}3&3&3\\44&-36&-16\\5&-15&5\end{bmatrix}$

$=\frac{1}{20}\begin{bmatrix}7&7&7\\5&5&5\\8&8&8\end{bmatrix}$

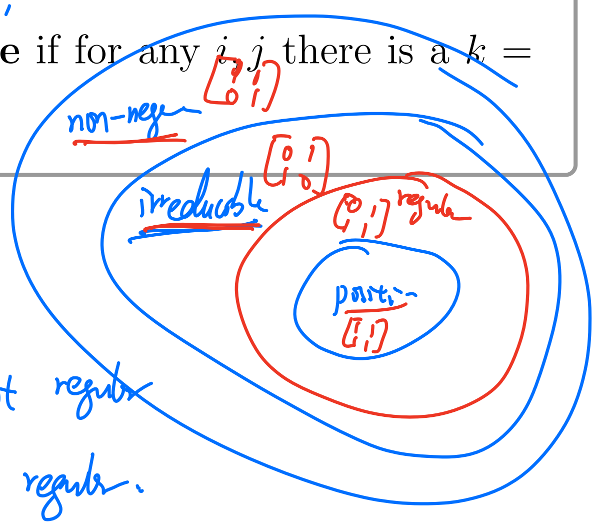
$\lim_{t\to\infty}A^t\vec{x}_0=\frac{1}{20}\vec{b}_1=\frac{1}{20}\begin{bmatrix}7\\5\\8\end{bmatrix}=\begin{bmatrix}7/20\\5/20\\8/20\end{bmatrix}$

Equilibria for regular transition matrices:

Let us start with some terminologies:

Definition 6.

- A matrix A is said to be **non-negative** if each entry of matrix A is not negative.
- A matrix A is said to be **positive** if each entry of matrix A is positive.
- A non-negative matrix A is said to be "**regular**" (or "**primitive**", or eventually positive) if the matrix A^m is positive for some integer $m > 0$.
- A non-negative matrix A is called "**irreducible**" if for any i, j there is a $k = k(i,j)$ such that $(A^k)_{ij} > 0$.



Ex: $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ positive

Ex: $A=\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $B=\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ non-negative

$A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$ $\quad B^2 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

A not regular

B is regular.

Ex: $C=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad C^2=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad C^3=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ $\quad \vec{v}$ $\quad \frac{1}{a+b+c}\begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$\begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix}$ $\quad p_i \geq 0$ $\quad p_1 + \cdots + p_n = 1$

Definition 7.

- A vector $\vec{x} \in \mathbb{R}^n$ is said to be a **distribution** vector if its entries are non-negative and the sum is 1.
- A square matrix A is said to be a **transition matrix** (or "column" stochastic matrix) if all its columns are distributions vectors.

Lemma 8. *If A is a transition matrix and $\vec{x}$ a distribution vector, then $A\vec{x}$ is a distribution vector.*

$$A\vec{x} = \begin{bmatrix} x_1 a_{11} + \dots + x_n a_{1n} \\ x_1 a_{12} + \dots \\ \vdots \\ x_1 a_{1n} \dots \end{bmatrix}$$

Ex!

Lemma 9. *A and A^T have the same characteristic polynomial.*

$\det(A - \lambda I)$ - - - - -

Lemma 10. *Suppose A is an $n \times n$ positive matrix such that the sum of each row is 1. Then,*

(1) *$\lambda = 1$ is an eigenvalue of A with algebraic multiplicity 1.* $\lambda_{max} = 1$

(2) *Consider an eigenvector $\vec{v}$ of A with positive entries. Show that the associated eigenvalue is less than or equal to 1.*

(3) *Show that absolute value of the eigenvalue is less than or equal to 1.*

(4) *-1 is not an eigenvalue of A.* all $|\lambda| < 1$

(1) $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$ $A\vec{u} = \vec{u}$

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ & \vdots & \\ a_{n1} & \dots & a_{nn} \end{bmatrix}$$

(2) $A\vec{v} = \lambda \vec{v}$

$$\begin{bmatrix} 0 \\ 1/2 \\ 1/2 \end{bmatrix}$$

By the above two lemmas, we have proved the special case of Perron-Frobenius Theorem.

Theorem 11 (Perron-Frobenius Theorem (special case for transition matrix))**.** *If A is a positive column stochastic matrix, then:*

- *1 is an eigenvalue of multiplicity one.* $\lambda_{max}=1$
- *1 is the largest eigenvalue: all the other eigenvalues have absolute value smaller than 1.* $|\lambda|<1$ $\lambda=-1$ not eigenval.
- *the eigenvectors corresponding to the eigenvalue 1 have either only positive entries or only negative entries. In particular, for the eigenvalue 1 there exists a unique eigenvector with the sum of its entries equal to 1.*

Theorem 12. *Let A be a "regular" transition $n \times n$ matrix.*
1. There exists exactly one distribution vector $\vec{x} \in \mathbb{R}^n$ such that

$$A\vec{x} = \vec{x}$$

*which is called **equilibrium** distribution for A denoted as $\vec{x}_{equ}$.*
2. If $\vec{x}_0$ is "any" distribution vector in $\mathbb{R}^n$, then

$$\lim_{m\to\infty} (A^m\vec{x}_0) = \vec{x}_{equ}$$

3. The columns of $\lim_{n\to\infty}(A^n)$ are all $\vec{x}_{equ}$, that is

$$\lim_{m\to\infty} (A^m) = [\vec{x}_{equ}\ \vec{x}_{equ} \ldots \vec{x}_{equ}]$$

(1) A regular $\Longrightarrow$ A^m positive.

λ is eigenval of $A \Longrightarrow \lambda^m$ is eigenval of A^m.

$\lambda^m = 1$ is the largest eigval of A^m.

$\lambda = 1$ is the largest eigenval of A

(2) Suppose $A = PDP^{-1}$ $\quad D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$ $\quad 1 = \lambda_1 > \lambda_2 \geq \cdots \geq \lambda_n > -1$

$A^m \vec{x}_0 = A^m (c_1 \vec{v}_1 + \cdots + c_n \vec{v}_n)$ $\quad P = [\vec{v}_1 \cdots \vec{v}_n]$ $\quad \vec{x}_0 = c_1 \vec{v}_1 + \cdots + c_n \vec{v}_n$

$A\vec{v}_1 = \vec{v}_1$

$= c_1 \lambda_1^m \vec{v}_1 + \cdots + c_n \lambda_n^m \vec{v}_n$

$= c_1 \vec{v}_1 + \boxed{c_2 \lambda_2^m + \cdots + c_n \lambda_n^m \vec{v}_n} \longrightarrow c_1 \vec{v}_1$

e.g. $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $\quad c_1 \vec{v}_1 = \begin{bmatrix} 1/6 \\ 2/6 \\ 3/6 \end{bmatrix}$

In general, using Jordan decomposition $A = PJP^{-1}$, where $P = [\vec{b}_1, ..., \vec{b}_n]$ and

$$J = \begin{bmatrix} 1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & J_{n_2}(\lambda_2) & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & J_{n_m}(\lambda_m) \end{bmatrix}$$

Hence, $A^m P = PJ^m$.

$$[J_{n_i}(\lambda_i)]^k = \begin{bmatrix} \lambda_i^k & \binom{k}{1}\lambda_i^{k-1} & \binom{k}{2}\lambda_i^{k-2} & \cdots & \binom{k}{n_i-1}\lambda_i^{k-n_i+1} \\ 0 & \lambda_i^k & \binom{k}{1}\lambda_i^{k-1} & \cdots & \binom{k}{n_i-2}\lambda_i^{k-n_i+2} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_i^k & \binom{k}{1}\lambda_i^{k-1} \\ 0 & 0 & \cdots & 0 & \lambda_i^k \end{bmatrix},$$

Claim: If $|\lambda_i| < 0$, then $\lim_{m \to \infty} ([J_{n_i}(\lambda_i)]^m) = \mathbf{0}$.

Hence,

$$\lim_{m \to \infty} (J^m) = \begin{bmatrix} 1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix}$$

Markov Chains (1906) can be used to study real word questions like PageRank of a webpage as used by Google, automatic speech recognition systems, probabilistic forecasting, cruise control systems in motor vehicles, queues or lines of customers arriving at an airport/train station/..., currency exchange rates, animal population dynamics, music, etc.

Convention in Probability: all vectors are transposed if you read some probability books about Markov chains. A stochastic matrix P comes from a stochastic process $\{X_0, ..., X_n\}$ with values in $\{1, ..., n\}$.

$$p_{ij} = P(X_{t+1} = i \mid X_t = j)$$

3. Perron-Frobenius Theorem

Example 13. (Ranking of Players) The results of a round tournament be represented by the following matrix.

$$A = \begin{bmatrix} 0.5 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0.5 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0.5 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0.5 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0.5 \end{bmatrix}$$

Here $a_{i,j} = 1$ represents player i win v.s. player j; and $a_{i,j} = 0$ represents player i loss v.s. player j.

Question: How to rank those 6 players from the results?

Suppose before the game, all ranked 1, represented by ranking vector $\vec{r_0} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ After the tournament, the ranking is $\vec{r_1} = A\vec{r_0} = \begin{bmatrix} 4.5000 \\ 2.5000 \\ 3.5000 \\ 1.5000 \\ 3.5000 \\ 2.5000 \end{bmatrix}$.

The rank is $P_1 > P_5 = P_3 > P_2 = P_6 > P_4$.

Consider the strength of the opponents, we calculate $\vec{r_2} = A\vec{r_1} = \begin{bmatrix} 14.2500 \\ 6.2500 \\ 8.2500 \\ 5.2500 \\ 9.2500 \\ 5.2500 \end{bmatrix}$, and $\vec{r_3} = A\vec{r_2} = \begin{bmatrix} 36.1250 \\ 17.6250 \\ 20.8750 \\ 16.8750 \\ 23.3750 \\ 14.1250 \end{bmatrix}$. Now we can see the rank: $P_1 > P_5 > P_3 > P_2 > P_4 > P_6$.

The eigenvalues of A are $2.7261; 0.0028; 0.1303 \pm 1.3750i; 0.0052 \pm 1.0451i$.

$\lambda = 2.7261$ is the largest eigenvalue with eigenvector $\begin{bmatrix} 0.2721 \\ 0.1372 \\ 0.1689 \\ 0.1222 \\ 0.1831 \\ 0.1165 \end{bmatrix}$. This vector is almost the same as $\vec{r}_{\geq 10}$ divided by the sum of the entries.

Let A be a real matrix.

Proposition 14. *If A is irreducible, then $I + A$ is primitive.*

Theorem 15 (**Perron-Frobenius Theorem**). *Let A be an irreducible non-negative matrix.*

- *A has a positive (real) eigenvalue $\lambda_{\max}$ such that all other eigenvalues of A satisfy $|\lambda| \leq \lambda_{\max}$*
- *$\lambda_{\max}$ has algebraic multiplicity 1 with a positive eigenvector $\vec{x}$.*
- *Any non-negative eigenvector is a multiple of $\vec{x}$.*
- *If A is primitive, then all other eigenvalues of A satisfy $|\lambda| < \lambda_{\max}$*

This theorem was first proved for positive matrices by Oskar Perron in 1907 and extended by Ferdinand Georg Frobenius to non-negative irreducible matrices in 1912.

The **spectrum** of a square matrix A, denoted by $\sigma(A)$, is the set of all eigenvalues of A. The **spectral radius** of A, denoted by $\rho(A)$, is the maximum eigenvalue of A in absolute value.

Theorem 16. *Suppose A is a primitive matrix, with spectral radius λ. Then λ is a simple root of the characteristic polynomial which is strictly greater than the absolute value of any other root, and λ has strictly positive eigenvectors.*

4. More Applications

4.1. Powers of a primitive matrix.

Let A be a primitive matrix. By the Perron-Frobenius theorem, let $\lambda_{\max}$ be its maximal eigenvalue.

Let $\vec{u}$ be a (right-handed) positive eigenvector of A with eigenvalue $\lambda_{\max}$, so $A\vec{u} = \lambda_{\max}\vec{u}$.

Let $\vec{v}$ be the left-handed eigenvector vector such that $\vec{v}^T A = \lambda_{\max}\vec{v}$ and $\vec{v} \cdot \vec{u} = 1$.

Theorem 17. *Suppose A is primitive, with maximal eigenvalue $\lambda_{\max}$, left eigenvector $\vec{u}$ and right eigenvector $\vec{v}$ such that $\vec{v} \cdot \vec{u} = 1$, then*

$$\lim_{k\to\infty} \left(\frac{1}{\lambda_{\max}} A\right)^k = \vec{u}\vec{v}^T$$

4.2. **Graphs and Non-negative matrices.**

A **directed graph** is a pair (V, E) consisting of a vertex set V and a subset edge set $E \subset V \times V$. The directed edge (v_i, v_j) goes form v_i to v_j. For example,

$$v_1 \to v_2 \to \cdots \to v_n$$

The graph associated to the non-negative square $n \times n$ matrix A has vertex set $V = v_1, ..., v_n$ and edge set

$$E = \{(v_j, v_i) \mid a_{ij} \neq 0)\}$$

The **adjacency matrix** A of the graph (V, E) is the $n \times n$ matrix B with

$$b_{ij} = \begin{cases} 1 & \text{if } (v_j, v_i) \in E \\ 0 & \text{otherwise.} \end{cases}$$

A **path** is a sequence of edges connecting v and w. The number of edges in the path is called the **length** of the path.

If A is the adjacency matrix of the graph, then $(A^2)_{ij}$ gives the number of paths of length two joining v_j to v_i, and, more generally, $(A^m)_{ij}$ gives the number of paths of length m joining v_j to v_i.

Theorem 18. *A is irreducible if and only if its associated graph is strongly connected, i.e., for any two vertices v_i and v_j there is a path (of some length) joining v_i to v_j .*

A **cycle** is a path starting and ending at the same vertex.

If M is primitive, then there are (at least) two cycles whose lengths are relatively prime.

Theorem 19. *If the graph associated to M is strongly connected and has two cycles of relatively prime lengths, then M is primitive.*

4.3. **Population model (The Leslie Model).**

1. (The Fibonacci Model) Simple Population model.

A_t : the number of adult pairs of rabbits at the end of month t.

Y_t : the number of youth pairs of rabbits at the end of month t.

Start with one pair of youth rabbits (1 month old). Each youth pair takes two months to mature into adulthood.

In this simple model, both adults and youth give birth to a pair at the end of every month, but once a youth pair matures to adulthood and reproduces, it then becomes extinct.

$A_0 = 0$, $Y_0 = 1$; $A_1 = 1$, $Y_1 = 1$; $A_2 = Y_1$, $Y_2 = A_1 + Y_1$; ... ; $A_t = Y_{t-1}$, $Y_t = A_{t-1} + Y_{t-1}$; ... Hence,

$$\begin{bmatrix} Y_{t+1} \\ A_{t+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} Y_t \\ A_t \end{bmatrix}$$

Here, the sequence Y_t give us the Fibonacci numbers.

2. In the simple model, the 1's in the first row represent the number of offspring produced so we can replace these 1's with birth rates b_1 and b_2. Since the lower 1 in our matrix represents a youth surviving into adulthood we will replace it by $0 < s \leq 1$, which is called the survival rate.

$$\vec{f}(t+1) = \begin{bmatrix} Y_{t+1} \\ A_{t+1} \end{bmatrix} = \begin{bmatrix} b_1 & b_2 \\ s & 0 \end{bmatrix} \begin{bmatrix} Y_t \\ A_t \end{bmatrix}$$

3. Lesli Model. More generally, if we consider k age classes other than 2 age classes, we have the Lesli Model (1945). The population to consider consists of the females of a species, and the stratification is by age group.

So the population is described by a vector $\vec{f}(t) = \begin{bmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{bmatrix}$, where the i-th entry $f_i(t)$ is the number of females in the i-th age group.

Let b_i be the female birth rate in the i-th age group and s_i the survival rate of females in the i-th age group

The transition after one time unit is given by the Leslie matrix

$$L = \begin{bmatrix} b_1 & b_2 & \dots & b_{n-1} & b_n \\ s_1 & 0 & 0 & 0 & 0 \\ 0 & s_2 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & s_{n-1} & 0 \end{bmatrix}$$

Proposition 20. *L is irreducible.*

The graph associated to L consists of n vertices

$$\begin{tikzcd} v_1 \arrow[r] & v_2 \arrow[r] & v_3 \arrow[r] \arrow[ll, bend right] & \cdots \arrow[r] \arrow[lll, bend right] & v_n \end{tikzcd}$$

(and possibly others when $b_i \neq 0$) connected to v_1 and so is strongly connected.

Proposition 21. *If there are two relative prime numbers i and j such that $b_i > 0$ and $b_j > 0$, to one another then L is primitive.*

4.4. **Economic growth.** Consider an economy, with activity level $x_i \geq 0$ in sector i for $i = 1, ..., n$.

Given activity level $\vec{x}_t$ in period t, in period $t+1$ we have $\vec{x}_{t+1} = A\vec{x}_t$, with A non-negative.

$a_{ij} \geq 0$ means activity in sector j does not decrease activity in sector i, i.e., the activities are mutually non-inhibitory.

4.5. **SVD analysis(in the last section).**

Further reading about the PageRank:

Other lectures:

http://pi.math.cornell.edu/~mec/Winter2009/RalucaRemus/Lecture3/lecture3.html

A little more professional:

https://www.rose-hulman.edu/~bryan/googleFinalVersionFixed.pdf

https://www.math.purdue.edu/~ttm/google.pdf

http://www.ams.org/publicoutreach/feature-column/fcarc-pagerank

Original paper:

Sergey Brin and Lawrence Page http://infolab.stanford.edu/~backrub/google.html