Northeastern University, Department of Mathematics

MATH 5110: Applied Linear Algebra and Matrix Analysis

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§3. Linear Spaces over Fields

Contents

$\begin{pmatrix}0\\1\end{pmatrix}$

~~Subspaces in $\mathbb{R}^n$~~

1. **Linear Spaces**

$\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ or $\mathbb{Z}_p$

$0 \in$

Definition 1. Let $\mathbb{F}$ be a field. A **vector space** over $\mathbb{F}$ is any nonempty set V with two **closed** operations,

- **Sum.** $\vec{u} + \vec{v} \in V$
- **Scalar product.** $c \cdot \vec{u} \in V$

$+ : V \times V \to V$

$\mathbb{F} \times V \to V$

subject to **axioms**:

1.) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$.
2.) $(\vec{u} + \vec{v}) + \vec{w} = \vec{v} + (\vec{u} + \vec{w})$.
3.) $\exists$ a zero vector $\vec{0} \in V$ s.t. $\vec{u} + \vec{0} = \vec{u}$.
4.) $\forall\ \vec{u} \in V$, $\exists$ a vector $-\vec{u} \in V$ s.t. $\vec{u} + (-\vec{u}) = \vec{0}$.
5.) $c \cdot (\vec{u} + \vec{v}) = c \cdot \vec{u} + c \cdot \vec{v}$.
6.) $(c + d) \cdot \vec{u} = c \cdot \vec{u} + d \cdot \vec{v}$.
7.) $c(d \cdot \vec{u}) = (cd)\vec{u}$.
8.) $1 \cdot \vec{u} = \vec{u}$.

e_1

(V, sum) abelian group

$(V, \text{sum}, \text{scalar prod.})$

Proposition 2. *(1) Zero vector is unique.*
(2) For any $\vec{u}$, the inverse vector $-\vec{u}$ is unique.

Proposition 3. $0 \cdot \vec{u} = \vec{0}$

$$\vec{0} = \vec{u} + (-\vec{u}) = 1 \cdot \vec{u} + (-\vec{u}) = (0+1)\cdot\vec{u} + (-\vec{u}) = (0\cdot\vec{u} + 1\cdot\vec{u}) + (-\vec{u})$$
$$= (0\cdot\vec{u} + \vec{u}) + (-\vec{u})$$
$$= 0\cdot\vec{u} + (\vec{u} + (-\vec{u}))$$
$$= 0\cdot\vec{u} + \vec{0}$$
$$= 0\cdot\vec{u}$$

Proposition 4. $c \cdot \vec{0} = \vec{0}$

Proposition 5. $-\vec{u} = (-1)\vec{u}$.

$1 \in \mathbb{F}$ $\quad (-1) + 1 = 0 \quad \mathbb{F}$

$\mathbb{R}$

Example 6. Vectors over $\mathbb{F}$:

1. $\mathbb{F}^n$ is a vector space over $\mathbb{F}$. $\vec{u} \in \mathbb{F}^n$ $\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$ $u_i \in \mathbb{F}$

2. The set of all $m \times n$ matrices (denoted as $\mathbb{F}^{m\times n}$ or $M_{m\times n}$) with matrix sum and scalar product is a vector space with entries in $\mathbb{F}$. $= \vec{u} \in \mathbb{F}^{m\times n}$

3. The set of all functions $f : \mathbb{F} \to \mathbb{F}$ with sum and scalar product is a vector space. $V = \{$ e.g. e^x, $\sin x$, ... $\}$ $f(x)+g(x)$ $kf(x)$

4. The set of all continuous (or smooth) functions $f : \mathbb{R} \to \mathbb{R}$ with sum and scalar product is a vector space. infinite dimension

5. The set of all maps $\mathbb{F}^m \to \mathbb{F}^n$ is a vector space. $V = \{ \}$

2. Subspaces

$V \times V \to V$

Definition 7. A **subspace** of the vector space V is a subset H of V that satisfies the following three properties:
(1). $\vec{0} \in H$.
(2). If $\vec{u}, \vec{v} \in H$ then $\vec{u} + \vec{v} \in H$. $H \times H \to H$ "closed for sum"
(3). If $\vec{u}, \vec{v} \in H$ and $c \in \mathbb{F}$, then $c\vec{u} \in H$. $\mathbb{F} \times H \to H$ "closed for scalar prod"
any any

Theorem 8. *A subspace H of a linear space V is a linear space.*

Proposition 9. $\{\vec{0}\}$ *is a subspace of linear space V, called zero space.*

Example 10. Determine which of the following set is a subspace (vector space). $P_2 = \{a_0 + a_1 t + a_2 t^2 \mid a_0, a_1, a_2 \in \mathbb{R}\}$

1. Let P_n be the set of all polynomials of degree $\leq n$ (over $\mathbb{R}$).

> *Proof.* The set of polynomials of degree $\leq n$, $P_n = \{a_0 + a_1 t + \cdots a_n t^n | a_i \in F\}$ is a subset of the linear space of all functions $F \to F$.
> First, $0 \in P_n$.
> Second, For any two such polynomials $a_0 + a_1 t + \cdots a_n t^n$ and $b_0 + b_1 t + \cdots b_n t^n \in P_n$, the sum $a_0 + a_1 t + \cdots a_n t^n + b_0 + b_1 t + \cdots b_n t^n = (a_0 + b_0) + (a_1 + b_2)t + \cdots (a_n + b_n)t^n$ is in P_n.
> Third, for any polynomials $a_0 + a_1 t + \cdots a_n t^n \in P_n$ and any $c \in F$, the scalar product $c(a_0 + a_1 t + \cdots a_n t^n) = ca_0 + ca_1 t + \cdots ca_n t^n$ is in P_n.
> So P_n is a subspace of the linear space of all functions $F \to F$. □

2. Let P be the set of all polynomials. vector space

3. Let H be the set of all polynomials of degree exactly 3. $\vec{0} \notin H$
Not vector space

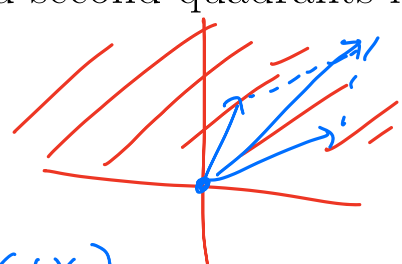
Fixon
4. The set $D_{n\times n}$ of all $n \times n$ diagonal matrices with real entries. Yes.

5. The set of all $n \times n$ "invertible" matrices with real entries. $\vec{0} \notin$
No

6. The union of the first and second quadrants in the xy-plane:

① $\vec{0} \in W$

$$W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| \; y \geq 0 \right\}$$



$\mathbb{R}^2$

No

② If $y_1 \geq 0$ $y_2 \geq 0$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix} \in W$$

③
$$(-2)\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \end{bmatrix} \notin W$$

Ex: $\vec{v_1} = e^x$ $\quad \vec{v_2} = \sin x$

$\mathrm{Span}\{\vec{v_1}, \vec{v_2}\} = \{ a_1 e^x + a_2 \sin x \mid \text{all } a_i \in \mathbb{R}\}$ vector space

Definition 11. A ***linear combination*** of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ in V is a vector in V defined as

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m$$

The **span** of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ is the set of all linear combinations

$$\mathrm{Span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m) := \left\{ c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m \;\middle|\; c_1 \cdots c_m \in \mathbb{R} \right\}$$

Theorem 12. *Then* $\mathrm{Span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m)$ *is a subspace of* V.

Proof. We prove the theorem by verifying the definition.
1. Choose all $c_i = 0$ so $\vec{0} \in \mathrm{Span}(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_m)$
2. For any two vectors $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m$ and $d_1\vec{v}_1 + d_2\vec{v}_2 + \cdots + d_m\vec{v}_m$ in $\mathrm{Span}(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_m)$, the sum

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m + d_1\vec{v}_1 + d_2\vec{v}_2 + \cdots + d_m\vec{v}_m =$$
$$(c_1 + d_1)\vec{v}_1 + (c_2 + d_2)\vec{v}_2 + \cdots + (c_m + d_m)\vec{v}_m$$

is an element in $\mathrm{Span}(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_m)$.
3. For any vector $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m$ in $\mathrm{Span}(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_m)$ and any $k \in \mathbb{F}$, the scalar product

$$k(c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m) = kc_1\vec{v}_1 + kc_2\vec{v}_2 + \cdots + kc_m\vec{v}_m$$

is an element in $\mathrm{Span}(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_m)$. □

Ex: $V = \mathbb{R}^2$ $\quad U = \{(0, y)\}$

Proposition 13. *Any subspace* U $\{u_i\}$ *of* V *can be written as span of "some" vectors in* V.

$U = \mathrm{Span}\{u_i\} = \mathrm{Span}\{\text{smallest set}\}$

If a vector space V can be written as a span of finite number of vectors in V, then V is called a **finite-dimensional** vector space.

$\| \quad \mathrm{Span}\{V_1, \cdots, V_n\}$

3. Linear transformations

set (sum, scalar prod)

Let V and W be vector spaces over a field $\mathbb{F}$. A **transformation** T from V to W is a rule

map

function

$$T \colon V \to W$$

Definition 14. A transformation $T \colon V \to W$ is called ***linear*** if

$T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$

and $T(c\vec{u}) = cT(\vec{u})$ for any $\vec{u}, \vec{v} \in V$, $c \in F$

Proposition 15. *If $T \colon V \to W$ is a linear transformation, then*

① $T(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1T(\vec{v}_1) + c_2T(\vec{v}_2)$

② $T(\vec{0}_V) = \vec{0}_W$

Example 16. 1. Zero map is linear transformation.

2. Identity map $\mathrm{id} : V \to V$ is a linear transformation.

Example 17. Is $T : \mathbb{R}^3 \to \mathbb{R}^2$, defined as $T(\vec{x}) = \begin{bmatrix} x_1 + 2x_2 + 1 \\ x_2 - x_3 \end{bmatrix}$ a linear transformation?

$S \circ T\big(S(\vec{u}+\vec{v}) - S(\vec{u}) - S(\vec{v})\big) = 0 \implies S(\vec{u}+\vec{v}) - S(\vec{u}) - S(\vec{v}) = 0$

Proposition 18. *If $T \colon X \to Y$ is linear and has a inverse S, then S is linear.*

$X \xleftarrow{S} Y$

$T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$

$T \circ S = \mathrm{id}_Y$

$S \circ T = \mathrm{id}_X$

check $\boxed{S(\vec{u}+\vec{v}) = S(\vec{u}) + S(\vec{v})}$

$S(c\vec{u}) = cS(\vec{u})$

Definition 19. Two vector spaces V and W are called **isomorphic**, denoted as

$$V \cong W$$

if there is an invertible linear transformation $T : V \to W$.

S

bijective

Let V and W be vector spaces. Denote $\mathscr{L}(V,W)$ the set of all linear transformations from V to W. Thus $T \in \mathscr{L}(V,W)$ if and only if

(i) $T : V \to W$,

(ii) $T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2)$ for $\vec{v}_1, \vec{v}_2 \in V$,

(iii) $T(a\vec{v}) = aT(\vec{v})$ for $\vec{v} \in V$, $a \in \mathbb{F}$.

$T_1, T_2 \in \mathcal{L}(V,W)$

$(T_1 + T_2)(\vec{v}) := T_1(\vec{v}) + T_2(\vec{v})$

$(cT)(\vec{v}) := c\cdot(T(\vec{v}))$

Theorem 20. *$\mathscr{L}(V,W)$ is a vector space.*

Theorem 21. *$\mathscr{L}(\mathbb{F}^n, \mathbb{F}^m)$ is isomorphic to the matrix space $\mathbb{F}^{m\times n}$.*

$\mathbb{R}^n \xrightarrow{T_A} \mathbb{R}^m$ $\quad T_A \quad f \quad A$

$\vec{x} \longrightarrow T_A(\vec{x}) := A\vec{x}$

$T \longrightarrow [T(e_1)\ T(e_2) \cdots T(e_n)]$

$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \quad \cdots$

Definition 22. The vector space $V^* := \mathscr{L}(V, \mathbb{F})$ is called the **dual space** of the vector space V.

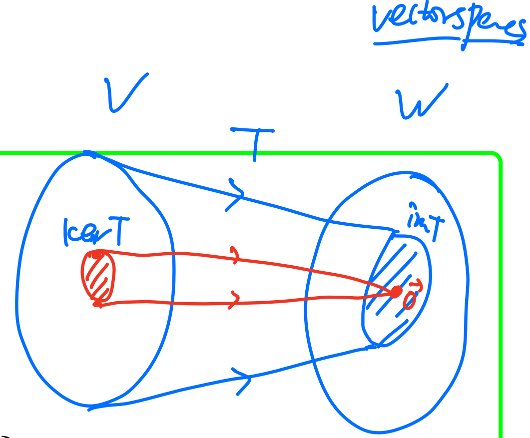
Ex : $V = \mathbb{R}^2$

$\boxed{V^* := \mathcal{L}(\mathbb{R}^2, \mathbb{R})} \rightleftarrows \mathbb{R}^{1\times 2}$

T_A

$\begin{bmatrix} x \\ y \end{bmatrix} \longrightarrow [x\ \ y] = A$

4. Kernel and Image



Definition 23. Consider a transformation/map

vector spaces

$$T : V \to W.$$

The ***image*** of T if defined as

$$\operatorname{im}(T) := \{T(\vec{x}) \mid \text{ all } \vec{x} \in V\}$$

The **kernel** of T is defined as

$$\ker(T) := \{\vec{x} \in V \mid T(\vec{x}) = \vec{0}\}$$

$T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$

$T(c\vec{u}) = cT(\vec{u})$

linear

$\Leftarrow$ $T(\vec{x}) = \vec{b}$ has at most one sln for any $\vec{b}$.

? $\Leftrightarrow$

$T(\vec{x}) = \vec{0} \Rightarrow \vec{x} = \vec{0}$

Proposition 24. *$T : V \to W$ is injective if and only if* $\ker(T) = \{\vec{0}\}$.
$T : V \to W$ is surjective if and only if $\operatorname{im}(T) = W$.

"$\Leftarrow$"

Suppose $T(\vec{u}) = \vec{b}$, $T(\vec{v}) = \vec{b}$, then $T(\vec{u}) = T(\vec{v}) \Leftrightarrow T(\vec{u}) - T(\vec{v}) = \vec{0}$

$\Leftrightarrow T(\vec{u}-\vec{v}) = \vec{0} \Rightarrow \vec{u}-\vec{v} = \vec{0}$

Theorem 25. *Let $T : V \to W$ be a "linear" transformation. Then* $\operatorname{im}(T)$ *is a subspace of W and* $\ker(T)$ *is a subspace of V.*

$\begin{bmatrix}0\\b\end{bmatrix} + N = \left\{ \begin{bmatrix}0\\b\end{bmatrix} + \begin{bmatrix}x\\0\end{bmatrix} \;\middle|\; \begin{bmatrix}x\\0\end{bmatrix} \in N \right\}$

ring $V = \mathbb{Z}$ $N = 2\mathbb{Z}$

$a \sim b \Leftrightarrow a - b \in N$

$V/N = \left\{ \left[\begin{bmatrix}0\\y\end{bmatrix} \right] \;\middle|\; y \in \mathbb{R} \right\} \cong \mathbb{R}$

y

$[a] := a + 2\mathbb{Z}$

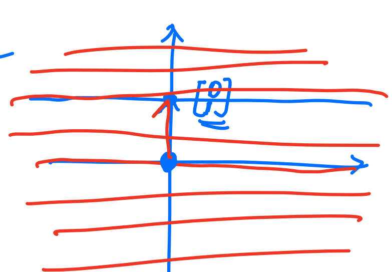
$= \{a + 2n \mid n \in \mathbb{Z}\}$

$\mathbb{Z}/2\mathbb{Z} = \{[0], [1]\}$

$[a] + [b] := [a+b]$

$[a][b] := [ab]$

Ex: $V = \mathbb{R}^2$



N = x-axis

$= \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} \in \mathbb{R}^2 \right\}$

5. **Quotient spaces.**

An **equivalent relation** $\sim$ on a set V is a binary relation such that for any $\vec{u}, \vec{v}, \vec{w} \in V$,

- $\vec{v} \sim \vec{v}$.
- If $\vec{v} \sim \vec{w}$, then $\vec{w} \sim \vec{v}$.
- If $\vec{u} \sim \vec{v}$ and $\vec{v} \sim \vec{w}$, then $\vec{u} \sim \vec{w}$.

Let V be a vector space over a field $\mathbb{F}$. Let N be a subspace of V. We can define an equivalence relation on V by defining that

$$\boxed{\vec{v} \sim \vec{w}} \text{ if and only if } \vec{v} - \vec{w} \in N$$

The **equivalence class** (or, called the **coset**) of $\vec{v}$ is defined

$$[\vec{v}] := \vec{v} + N = \{\vec{v} + \vec{a} \mid \vec{a} \in N\}$$

Definition 26. The **quotient space** V/N is a the set of all cosets. Sum and scalar product are defined as

- $[\vec{v}] + [\vec{w}] := [\vec{v} + \vec{w}]$.
- $c[\vec{v}] := [c\vec{v}]$.

Proposition 27. *Quotient space V/N is a vector space.*

Remark:

There is a natural epimorphism from $p : V \to V/N$ defined by $p(\vec{v}) = [\vec{v}]$. The kernel is $\ker p = N$. There exists a **short exact sequence**

$$0 \to N \hookrightarrow V \twoheadrightarrow V/N \to 0$$

$$\vec{v} \to [\vec{v}]$$

$V = \mathbb{R}^2$

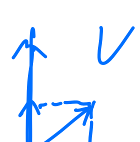
$N =$ x-axis

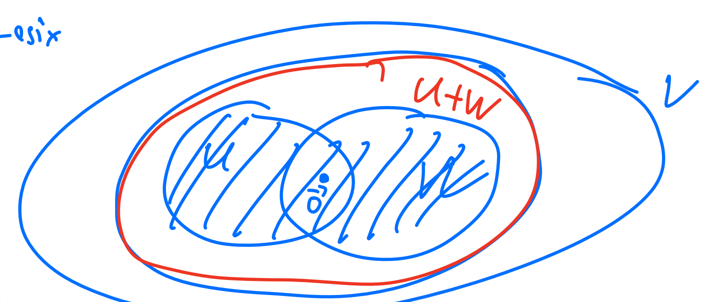
Here, exact means ker=im at each connecting place.

Remark: The idea of quotient is used in almost all mathematics, e.g., quotient group, quotient ring, quotient field, quotient module, quotient algebra, quotient space in topology, etc.

$V = \mathbb{R}^2$ $\quad$ $V =$ y-axis

$U =$ x-axis





- $U \cap W$ is a subspace of V
- $U \cup W$ is not a vector

Space

Not subspace of V

6. Sum and direct sum of subspaces

Let U and W be subspaces of a vector space V.

Definition 28. The **sum** of U and W is defined as

$$U + W := \mathrm{Span}\{\vec{x} \mid \vec{x} \in U \cup W\} = \mathrm{Span}(U \cup W)$$
$$:= \{\vec{u}+\vec{w} \mid \vec{u} \in U,\ \vec{w} \in W\}$$

$\vec{v}$ "

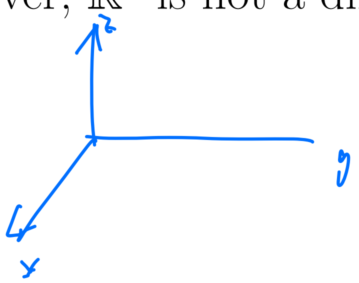
Proposition 29. *$U + W$ is a subspace of V.*

Definition 30. A sum $V = U + W$ is called the **direct sum** of U and W, denoted by

$$V = U \oplus W$$

if each $v \in V$ can be "**uniquely**" written as $\vec{v} = \vec{u} + \vec{w}$. $\vec{u} \in U$, $\vec{w} \in W$

Example 31. $U = \{\cancel{x, y, 0}\} \subset \mathbb{R}^3$ $\begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$ = x-y-plane and $V = \{\cancel{0, y, z}\} \subset \mathbb{R}^3$ $\begin{bmatrix} 0 \\ y \\ z \end{bmatrix}$ = yz-plane, then $\boxed{U + V \overset{\subseteq}{\underset{\supseteq}{=}} \mathbb{R}^3}$. However, $\mathbb{R}^3$ is not a direct sum of U and V.



$\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y/2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ y/2 \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix}$$

Example 32. $\mathbb{R}^3$ is a direct sum of $U = \{\cancel{x, 0}, 0\} \subset \mathbb{R}^3$ and $V = \{\cancel{0, y}, z\} \subset \mathbb{R}^3$.

U = x-axis $\begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix}$, V = yz plane $\begin{bmatrix} 0 \\ y \\ z \end{bmatrix}$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \vec{u} + \vec{v} = \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ y \\ z \end{bmatrix}$$

$= \vec{0} + \vec{0}$ ← $\vec{u} \in U$ $\vec{w} \in W$

Proposition 33. *Let $V = U + W$. If $\vec{0} = \vec{u} + \vec{w}$ implies $\vec{u} = \vec{w} = \vec{0}$, then*

$$V = U \oplus W$$

- For each $\vec{v} \in V$, suppose $\vec{v} = \vec{u} + \vec{w} = \vec{u}' + \vec{w}'$ $\quad \vec{u}, \vec{u}' \in U$, $\vec{w}, \vec{w}' \in W$

$\Rightarrow \underbrace{(\vec{u} - \vec{u}')}_{\in U} + \underbrace{(\vec{w} - \vec{w}')}_{\in W} = \vec{0} \Rightarrow \vec{u} - \vec{u}' = \vec{0}$ and $\vec{w} - \vec{w}' = \vec{0}$

Proposition 34. *$V = U \oplus W$ if and only if $V = U + W$ and $U \cap W = \{\vec{0}\}$.*

"$\Rightarrow$" Suppose $\vec{v} \in U \cap W$, then $-\vec{v} \in U \cap W$. $\underbrace{\vec{0} = \vec{v} + (-\vec{v})}$ $\in W$ $\in U$

then $\vec{v} = \vec{0}$

"$\Leftarrow$" Suppose $\vec{0} = \vec{u} + \vec{w}$ and $\vec{u} \in U$, $\vec{w} \in W$, then $\vec{u} = -\vec{w} \in W \Rightarrow \vec{u} \in U \cap W$

$\vec{u} = \vec{0}$

7. Linear transformations and matrices

Theorem 35. *Given an $m \times n$ matrix A. There is a linear transformation $T : \mathbb{F}^n \to \mathbb{F}^m$ defined as*

$\mathbb{R}$ $\mathbb{R}$

$\vec{x} \to A\vec{x}$

$$T(\vec{x}) := A\vec{x}$$

Denote $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_m$ be the column vectors of the identity matrix I_m. We call them the standard vectors in $\mathbb{F}^m$.

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \cdots \quad \vec{e}_m = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

The next theorem is very effective for finding the matrix for a given linear transformation.

Theorem 36 (Transformation matrix)**.** *Let $T : \mathbb{F}^n \to \mathbb{F}^m$ be a linear transformation.*
There exists an $m \times n$ matrix A such that $T(\vec{x}) = A\vec{x}$.
Further more, the matrix of T is given by

$$A = [T(\vec{e}_1) \quad T(\vec{e}_2) \quad \cdots \quad T(\vec{e}_n)].$$

Theorem 37. *An $n \times n$ matrix A is invertible if and only if the linear transformation T_A is injective; if and only if T_A is surjective.*

$\Longleftrightarrow$ T_A bijective

$$\begin{tikzcd} \mathbb{R}^p \arrow[r, "T_B"] \arrow[rr, bend left] & \mathbb{R}^n \arrow[r, "T_A"] & \mathbb{R}^m \end{tikzcd}$$

$$\vec{x} \longrightarrow B\vec{x} \longrightarrow \underline{A(B\vec{x})} = (AB)\vec{x}$$

$$T_A \circ T_B(\vec{x}) = \underline{AB}\,\vec{x}$$

$$T_B(\vec{x}) = B\vec{x}$$

Theorem 38. *Let A be an $m \times n$ matrix and B be a $n \times p$ matrix. Then the product AB is the matrix of the transformation composition $T_A \circ T_B$.*

Corollary 39. *An $n \times n$ matrix A is invertible if and only if T_A is invertible. Moreover, $(T_A)^{-1} = T_{A^{-1}}$.*

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "T_A"] & \mathbb{R}^n \arrow[l, bend left, "S"] \end{tikzcd}$$

$$T_A \circ S = id$$

$$S \circ T_A = id$$

8. Tensor product of spaces and Kronecker product of matrices