Solution of ODE5 Problem 4

(4) (a) The equations analogous to the SIRS model are:

$$\begin{aligned}\frac{dS}{dt} &= -\beta SI + \gamma R - \delta S + \delta N\\ \frac{dI}{dt} &= \beta SI - \nu I - \delta I\\ \frac{dR}{dt} &= \nu I - \gamma R - \delta R\end{aligned}$$

Since $\frac{dS}{dt} + \frac{dI}{dt} + \frac{dR}{dt} = 0$, $S(t) + I(t) + R(t)$ is constant (we found a conservation law). We write $N = S(t) + I(t) + R(t)$, so that we can substitute $R(t) = N - S(t) - I(t)$, and thus we can just study the following set of ODE's:

$$\begin{aligned}\frac{dS}{dt} &= -\beta SI - (\gamma+\delta)S - \gamma I + (\gamma+\delta)N\\ \frac{dI}{dt} &= \beta SI - (\nu+\delta)I\end{aligned}$$

(b) To find the steady states, let's solve $\begin{cases}\frac{dS}{dt} = 0\\ \frac{dI}{dt} = 0\end{cases}$ for S and I.

$\frac{dI}{dt} = 0$ implies $I = 0$ or $S = \frac{\nu+\delta}{\beta}$.

Plugging $I = 0$ into $\frac{dS}{dt} = 0$, we'll obtain $S = N$. So

$$\bar{X}_1 = (S, I) = (N, 0)$$

is one steady state.

Now, by plugging $S = \frac{\nu+\delta}{\beta}$ into $\frac{dS}{dt} = 0$, we obtain $I = \frac{\gamma+\delta}{\nu+\gamma+\delta}(N - \frac{\nu+\delta}{\beta})$. So,

$$\bar{X}_2 = (S, I) = \left(\frac{\nu+\delta}{\beta}, \frac{\gamma+\delta}{\nu+\gamma+\delta}\left(N - \frac{\nu+\delta}{\beta}\right)\right)$$

is another steady state (which could have negative second component).

(c) Clearly, $\bar{X}_2$ has both coordinates positive if and only if its second coordinate $\frac{\gamma+\delta}{\nu+\gamma+\delta}(N - \frac{\nu+\delta}{\beta})$ is positive, that is, iff $N > \frac{\nu+\delta}{\beta}$. (Note that the first coordinate, $\frac{\nu+\delta}{\beta}$, is always positive.)

(d) The Jacobian matrix of the vector field

$$\begin{pmatrix} -\beta SI - (\gamma+\delta)S - \gamma I + (\gamma+\delta)N \\ \beta SI - (\nu+\delta)I \end{pmatrix}.$$

is:

$$J(S, I) = \begin{pmatrix} -\beta I - (\gamma+\delta) & -\beta S - \gamma \\ \beta I & \beta S - (\nu+\delta) \end{pmatrix}.$$

In particular, at $\bar{X}_1 = (N, 0)$ we have:

$$J_{\bar{X}_1} = \begin{pmatrix} -(\gamma+\delta) & -\beta N - \gamma \\ 0 & \beta N - (\nu+\delta) \end{pmatrix}.$$

and at $\bar{X}_2 = (\frac{\nu+\delta}{\beta}, \frac{\gamma+\delta}{\nu+\gamma+\delta}(N - \frac{\nu+\delta}{\beta}))$ we have:

$$\begin{pmatrix} -\frac{\gamma+\delta}{\nu+\gamma+\delta}(\beta N+\gamma) & -(\nu+\delta+\gamma) \\ \frac{\gamma+\delta}{\nu+\gamma+\delta}(\beta N-(\nu+\delta)) & 0 \end{pmatrix}.$$

(e) and (f) For $\bar{X}_1 = (N, 0)$ we have:

If $N > \frac{\nu+\delta}{\beta}$, then $\det(J) < 0$; which implies the steady state $(N, 0)$ is unstable.

If $N < \frac{\nu+\delta}{\beta}$, then $\det(J) > 0$ and $\operatorname{tr}(J) < 0$, so $(N, 0)$ is stable. (Note that $N < \frac{\nu+\delta}{\beta}$ implies $\beta N - (\nu+\delta) < 0$. So $\beta N - (\nu+\delta) - (\gamma+\delta) < -(\gamma+\delta) < 0$. So indeed $\operatorname{tr}(J) < 0$.)

For $\bar{X}_2$ we have: $\det(J) = -(\gamma+\delta)(\beta N - (\nu+\delta))$ and $\operatorname{tr}(J) = -(\gamma+\delta) + \beta N - (\nu+\delta)$.

The trace is always negative:

$$\operatorname{tr}(J) = -\frac{\gamma+\delta}{\nu+\gamma+\delta}(\beta N + \gamma) < 0\,.$$

So stability will be determined by the determinant. Now,

$$\det(J) = (\gamma+\delta)(\beta N - (\nu+\delta))$$

is negative if $N < \frac{\nu+\delta}{\beta}$ and is positive if $N > \frac{\nu+\delta}{\beta}$.

Thus, $(\frac{\nu+\delta}{\beta}, \frac{\gamma+\delta}{\nu+\gamma+\delta}(N - \frac{\nu+\delta}{\beta}))$ is stable if $N > \frac{\nu+\delta}{\beta}$ and is unstable if $N < \frac{\nu+\delta}{\beta}$.

In summary, if $N > \frac{\nu+\delta}{\beta}$, $(N, 0)$ is unstable and

$$\left(\frac{\nu+\delta}{\beta}, \frac{\gamma+\delta}{\nu+\gamma+\delta}\left(N - \frac{\nu+\delta}{\beta}\right)\right)$$

is stable and if

$$N < \frac{\nu+\delta}{\beta},$$

$(N, 0)$ is stable and

$$\left(\frac{\nu+\delta}{\beta}, \frac{\gamma+\delta}{\nu+\gamma+\delta}\left(N - \frac{\nu+\delta}{\beta}\right)\right)$$

is unstable.

Solution of ODE5 Problem 6

(6) (a) At $(N, 0, 0)$, $\frac{dS}{dt} = \frac{dI}{dt} = \frac{dR}{dt} = 0$, (check it!), so $(N, 0, 0)$ is a steady state.

(b) We compute the Jacobian matrix at $(\bar{N}, 0, 0)$. The Jacobian at an arbitrary point (S, I, R) is:

$$\begin{pmatrix} -\beta I - \delta & -\beta S & 0 \\ \beta I & \beta S - (\delta + \nu) & 0 \\ 0 & \nu & -\delta \end{pmatrix}.$$

So the Jacobian at $(\bar{N}, 0, 0)$ is:

$$J = \begin{pmatrix} -\delta & -\beta \bar{N} & 0 \\ 0 & \beta \bar{N} - (\delta + \nu) & 0 \\ 0 & \nu & -\delta \end{pmatrix}.$$

The eigenvalues of J are real: $\lambda_{1,2} = -\delta$ and $\lambda_3 = \beta \bar{N} - (\delta + \nu)$. (Do you see why this is true? Note the block upper-triangular form, with the 2 by 2 block itself lower triangular.)

(c) To have stability, we need all of these to be negative. The first two are, so $(\bar{N}, 0, 0)$ is stable when $\lambda_3 = \beta \bar{N} - (\delta + \nu) < 0$, or equivalently $\bar{N} < \frac{\delta + \nu}{\beta}$.

So if we choose $\bar{N}_c = \frac{\delta + \nu}{\beta}$, both the infective and the immune class eventually die out.

Solution of ODE5 Problem 10

$-(1/3)(1 + \log(3)) = 0.3005$

$R_0 = 999 * (0.003) = 2.9970, 1/999 + 1 - (1/R_0) * (1 + \log(R_0)) = 0.3011$