MATH G5110: Worksheet 1 SOLUTIONS

1. A: basis $\{(1,0,0)^T\}$

   B: basis $\{(1,0,0)^T, (0,2,0)^T\}$

   C: basis $\{(1,2,0)^T\}$

2. a) dim col space = # pivots = 3

   dim null space = # free = 1

   dim row space = rank = 3

   b) Col space basis: $\{(1,2,-1), (4,8,2), (2,0,0)\}$

   null space basis: $\{(2.17, -0.42, -2.75, 1)\}$.

   rows space basis: $\{(1,4,2,5), (2,8,0,-1), (-1,2,0,3)\}$

3. $$\begin{aligned} x + 5z &= 6 \\ y + 2z &= 7 \end{aligned} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 7 \\ 0 \end{pmatrix} + z \begin{pmatrix} -5 \\ -2 \\ 1 \end{pmatrix}, \quad -\infty < z < \infty.$$

4. rank = 3, $\Rightarrow \operatorname{rref}(A^T) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$

(5) (a) FALSE: check if $\vec{0} \in S$:

$\Rightarrow$ need $\vec{u} = a\vec{v} + b\vec{w}$, some scalars a, b.

But independent $\Rightarrow a = b = 0 \Rightarrow$ impossible

$\Rightarrow$ not a subspace.

(b) TRUE: independent cols give pivot cols in rref.

(c) FALSE; for example, $\vec{u} \notin S_2$, since

$\vec{u} \stackrel{?}{=} a(\vec{u}-\vec{v}) + b(\vec{v}-\vec{w}) + c(\vec{w}-\vec{u}) = (a-c)\vec{u} + (b-a)\vec{v} + (c-b)\vec{w}$

$\Leftrightarrow a = b = c,\ a - c = 1$ impossible, so $S_1 \not\subset S_2$

(but note $S_2 \subset S_1$). uses independence of $\vec{u}, \vec{v}, \vec{w}$ to deduce this

(d) ~~For~~ TRUE; rank $= 3 =$ dim rowspace.

(6) $$\text{Nullspace}(A\ \vec{b}_1\ \vec{b}_2) := \text{Span}\left(\begin{pmatrix} -22/3 \\ -7/3 \\ 7/3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -22/3 \\ -7/3 \\ 1/3 \\ 0 \\ 1 \end{pmatrix}\right) = \text{Span}(\vec{v}_1, \vec{v}_2)$$

Want $\begin{pmatrix} 0 \\ 0 \\ 1 \\ -c_1 \\ -c_2 \end{pmatrix}$ in nullspace $\Rightarrow$ $= a\begin{pmatrix} -22/3 \\ -7/3 \\ 7/3 \\ 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} -22/3 \\ -7/3 \\ 1/3 \\ 0 \\ 1 \end{pmatrix}$

$\Rightarrow a+b=0, \quad a\frac{7}{3} + b\frac{1}{3} = 1 \Rightarrow a = \frac{1}{2}, \; b = -\frac{1}{2}$

$\Rightarrow c_1 = -\frac{1}{2}, \; c_2 = \frac{1}{2},$

7. $\vec{v}_2 = 2\vec{v}_1, \quad \vec{v}_4 = -\vec{v}_1 + 3\vec{v}_3$

8. $[L] = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \frac{1}{2} I_2$

9. $L(\vec{x}) = \vec{b}, \quad \vec{x} = (1,1,2,-1).$

$\vec{y} = \vec{x} + c_1(1,2,3,-1) + c_2(0,1,0,2) = (*, 0, *, 0)$

$\Rightarrow 1 + 2c_1 + c_2 = 0, \; -1 - c_1 + 2c_2 = 0 \Rightarrow c_1 = -\frac{3}{5}, \; c_2 = \frac{1}{5}$

$\Rightarrow \vec{y} = \frac{1}{5}(2, 0, 1, 0).$

10. All columns are multiples of $\vec{u}$

$\Rightarrow$ columnspace $= \mathrm{Span}(\vec{u}) \Rightarrow$ rank $= 1$, nullity $= n-1$
$(\vec{u} \neq \vec{0})$

11. Each column is $a\vec{u} + b\vec{v}$ some scalars a, b.

$\Rightarrow$ colspace $\subset$ span$(\vec{u}, \vec{v})$. (b/c $\vec{u}, \vec{v}$ independent)

Let $B = \begin{pmatrix} \vec{u}^T \\ \vec{v}^T \end{pmatrix}$ ($2 \times n$ matrix), rank B = dim rowspace $= 2$.

$\Rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in$ colspace$(B) \Rightarrow$ there is vector $\vec{x} \in \mathbb{R}^n$ s.t.

$$B\vec{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Now $B\vec{x} = \begin{pmatrix} \vec{u}^T\vec{x} \\ \vec{v}^T\vec{x} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\Rightarrow \vec{u}^T\vec{x} = 1, \quad \vec{v}^T\vec{x} = 0.$

$\Rightarrow A\vec{x} = (\vec{u}\,\vec{v}^T\vec{x} + \vec{v}\,\vec{u}^T\vec{x}) = \vec{v}.$

$\Rightarrow \vec{v} \in$ colspace(A).

Similarly there is $\vec{y} \in \mathbb{R}^n$ s.t. $B\vec{y} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$,

so $\vec{u} \in$ col space(A)

$\Rightarrow$ span$(\vec{u}, \vec{v}) \subset$ colspace(A)

$\Rightarrow$ span$(\vec{u}, \vec{v}) =$ colspace(A).

$\Rightarrow$ rank$(A) = 2$, nullity$(A) = n-2$