MTH 7241: Fall 2022

Second Practice Problems for Test 1

1). Consider the following transition probability matrix for a Markov chain on 5 states:

$$P = \begin{pmatrix} 1/4 & 3/4 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1/3 & 2/3 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Number the states $\{1, 2, 3, 4, 5\}$ in the order presented.

a). Write down two different stationary distributions for the chain.

b). Starting from state 4, find the expected number of steps until first reaching state 3.

c). Starting from state 5, find the expected number of steps until first reaching state 2.

2). A particle moves between 12 points which are spaced around a circle. At each step the particle is equally likely to move one point clockwise or one point counterclockwise. Find the mean number of steps for the particle to return to its starting position. [Hint: make use of the solution of the Gambler's Ruin problem as derived in class].

3) A stack of m cards is shuffled as follows: at each step a random number X is chosen from $\{1, 2, \dots, m\}$, then the card which sits at position X is removed and placed at the top of the deck. By modeling this process as a Markov chain, show that in the long-run the deck becomes randomly shuffled, so that all $m!$ orderings are equally likely.

4) Consider the following transition probability matrix for a Markov chain on 3 states:

$$P = \begin{pmatrix} 1/2 & 1/3 & 1/6 \\ 0 & 1/3 & 2/3 \\ 1/2 & 0 & 1/2 \end{pmatrix}$$

Number the states $\{1, 2, 3\}$ in the order presented.

Find the long-run probability that the chain jumps in the following sequence of states: $1 \to 2 \to 3 \to 1 \to 2$.

5) Let $\{X_n\}$ be a Markov chain, and suppose that for state i we have

$$\sum_{k=1}^{n} p_{ii}(k) = \sum_{k=1}^{n} \mathrm{P}(X_k = i \,|\, X_0 = i) \geq \frac{9}{\sqrt{n+8}} \quad \text{for all } n \geq 1.$$

Determine whether state i is transient or persistent (explain your reasoning).

6) Consider an irreducible chain on 3 states. **Either** prove that $p_{jj}(3) > 0$ for every state j, **or** give an example where $p_{jj}(3) = 0$ for some state j.

7) For a branching process calculate the probability of extinction when the offspring probabilities are $p_0 = 1/4$, $p_1 = 1/2$, $p_2 = 1/8$, $p_3 = 1/8$.

8) Either give an example of a finite Markov chain with no transient states and two different stationary distributions, or show that that this cannot happen.

9) Either give an example of a finite Markov chain with a unique stationary distribution and one transient state, or show that that this cannot happen.

10) Let X_n be an irreducible regular finite Markov chain with a unique stationary distribution $\{w_i\}$. Compute

$$\lim_{n\to\infty} \mathrm{P}\left(X_{n+1} = X_n\right)$$