MATH 7241 Fall 2022: Problem Set #6

Due date: Friday November 4

SOLUTIONS

Reading: relevant background material for these problems can be found on Canvas 'Notes 4: Finite Markov Chains', 'FSHMC'. Also Grinstead and Snell Chapter 11.

Exercise 1 In each case below, determine whether or not the chain is reversible (note: the condition for reversibility is $w_i p_{ij} = w_j p_{ji}$ for all states i, j).

$$(a) \qquad P = \begin{pmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{pmatrix}$$

$$(b) \qquad P = \begin{pmatrix} 3/4 & 1/4 & 0 \\ 0 & 2/3 & 1/3 \\ 1/4 & 1/4 & 1/2 \end{pmatrix}$$

a) Find stationary distribution:

$$w = \left(\frac{2}{3}, \frac{1}{3}\right)$$

Check reversible equation:

$$w_1 p_{12} \stackrel{?}{=} w_2 p_{21}$$

$$\frac{2}{3} \cdot \frac{1}{4} \stackrel{?}{=} \frac{1}{3} \cdot \frac{1}{2} \qquad \text{YES} \Rightarrow \text{reversible}$$

b) $p_{12} = \frac{1}{4}$, $p_{21} = 0 \Rightarrow$ impossible to satisfy $w_1 p_{12} = w_2 p_{21}$

$\Rightarrow$ not reversible

$\boxed{\textbf{Exercise 2}}$ A box contains N balls, some red and some blue. At each step, a coin is flipped with probability p of coming up Heads, and probability $1-p$ of coming up Tails. If the coin comes up Heads, a ball is chosen at random from the box and is replaced by a red ball; if the coin comes up Tails then a ball is chosen randomly from the box and replaced by a blue ball. Let X_n denote the number of red balls in the box after n steps. Find the transition matrix for the chain $\{X_n\}$, and find the stationary distribution. [Hint: is the chain reversible?] Compute $\lim_{n\to\infty} \mathrm{E}[X_n]$, and explain why you could have guessed your answer without doing the calculation.

Ex. 2 X_n = # red balls in Box 1.

State space $\{0, 1, 2, \dots, N-1\}$.

Transition matrix:

$$P_{ij} = \mathbb{P}(X_{n+1} = j \mid X_n = i) = \begin{cases} p\,\dfrac{N-i}{N} & \text{if } j = i+1 \\ (1-p)\,\dfrac{i}{N} & \text{if } j = i-1 \\ 1 - p\,\dfrac{N-i}{N} - (1-p)\,\dfrac{i}{N} & \text{if } j = i \end{cases}$$

Note: if $j = i+1$, then must toss Heads (prob $= p$) and must select a blue ball (prob $= \frac{N-i}{N}$). etc.

Try to solve reversible equations:

$$W_i P_{ij} = W_j P_{ji}.$$

In this case we only have the equations

$$W_i P_{i,i+1} = W_{i+1} P_{i+1,i}$$

$$\Rightarrow \quad W_{i+1} = W_i \frac{P_{i,i+1}}{P_{i+1,i}}$$

$$= W_i \frac{p}{1-p} \frac{N-i}{i+1}$$

$$= W_{i-1} \left(\frac{p}{1-p}\right)^2 \frac{N-i}{i+1} \frac{N-i+1}{i}$$

$$= \cdots\cdots$$

$$= W_0 \left(\frac{p}{1-p}\right)^{i+1} \frac{(N-i)(N-i+1)\cdots(N)}{(i+1)(i)\cdots(1)}$$

$$= W_0 \left(\frac{p}{1-p}\right)^{i+1} \binom{N}{i+1}$$

$$\Rightarrow \quad W_i = W_0 \left(\frac{p}{1-p}\right)^{i} \binom{N}{i} \qquad (i = 0, 1, 2, \cdots, N)$$

Normalize to find w_0:

$$\sum_{i=0}^{N} w_i = w_0 \sum_{i=0}^{N} \left(\frac{p}{1-p}\right)^i \binom{N}{i}$$

$$= w_0 \left(1 + \frac{p}{1-p}\right)^N$$

$$= w_0 (1-p)^{-N}.$$

$$\Rightarrow \quad w_0 = (1-p)^N.$$

$$\Rightarrow \quad w_i = p^i (1-p)^{N-i} \binom{N}{i}$$

This is the binomial dist. w/ prob. p.

$$\Rightarrow \quad \lim_{N\to\infty} \mathbb{E}[X_n] = \sum_{i=0}^{N} i\, w_i$$

$$= Np. \quad \text{b/c binomial.}$$

We could have expected this; as $N \to \infty$ the balls are randomly mixed so that the fraction of Red balls is p, and the fraction of Blue balls is $1-p$.

Exercise 3 A knight moves randomly on a standard 8×8 chessboard. At each step it chooses at random one of the possible legal moves available. Given that the knight starts in a corner of the chessboard, find the expected number of steps until its first return to its initial position. [Hint: model the knight's position using a Markov chain, and try to show that the chain is reversible]

$E[\#\text{steps first return to state } i] = w_i^{-1}$

Random walk on graph is reversible:

$$W_i = \frac{d_i}{\sum_j d_j}$$ $d_i = \#$ nearest neighbors of node i.

8x8 chess board

2	3	4	4	4	4	3	2
3	4	6	6	6	6	4	3
4	6	8	8	8	8	6	4
4	6	8	8	8	8	6	4
4	6	8	8	8	8	6	4
4	6	8	8	8	8	6	4
3	4	6	6	6	6	4	3
2	3	4	4	4	4	3	2

d_i for each square

$$\sum_j d_j = 336$$

$$\Rightarrow W_{corner} = \frac{2}{336} = \frac{1}{168}$$

$\Rightarrow$ mean return time $= 168$ steps

$\boxed{\textbf{Exercise 4}}$ Grinstead and Snell p.423, #7.

Make 0,4 into absorbing states

$$\Rightarrow P = \begin{array}{c} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & 0 & \frac{3}{4} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{3}{4} & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array}$$

Reorder states 1,2,3, 0,4:

$$P = \begin{pmatrix} Q & R \\ 0 & I \end{pmatrix} := \begin{pmatrix} 0 & \frac{3}{4} & 0 & \frac{1}{4} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{3}{4} & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$Q = \begin{pmatrix} 0 & \frac{3}{4} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{3}{4} & 0 \end{pmatrix} \Rightarrow N = (I - Q)^{-1} = \begin{pmatrix} \frac{5}{2} & 3 & \frac{3}{2} \\ 2 & 4 & 2 \\ \frac{3}{2} & 3 & 5/2 \end{pmatrix}$$

$$NR = \begin{pmatrix} \frac{5}{2} & 3 & \frac{3}{2} \\ 2 & 4 & 2 \\ \frac{3}{2} & 3 & 5/2 \end{pmatrix} \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & 0 \\ 0 & \frac{1}{4} \end{pmatrix} = \begin{pmatrix} 5/8 & 3/8 \\ 1/2 & 1/2 \\ 3/8 & 5/8 \end{pmatrix}$$

$\boxed{\textbf{Exercise 5}}$ Grinstead and Snell p.423, #9.

$$P = \begin{array}{cc} \begin{pmatrix} 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \end{array}$$

$\{1,2,3,4\}$ transient

$$\Rightarrow \quad Q = \begin{pmatrix} 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \quad N = (I - Q)^{-1} = \begin{array}{cc} \begin{pmatrix} 1 & \frac{1}{4} & \frac{1}{3} & \frac{1}{2} \\ 0 & 1 & \frac{1}{3} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \end{array}$$

$$\text{Mean \# steps } (1 \to 5) = N_{11} + N_{12} + N_{13} + N_{14}$$

$$= 1 + \frac{1}{4} + \frac{1}{3} + \frac{1}{2} = \frac{25}{12}$$

Exercise 6 Grinstead and Snell p.427, #24.

Case $p=q$ done in class.

$\underline{p \neq q}$: try

$$w_x = \frac{(q/p)^x - 1}{(q/p)^T - 1} = A(q/p)^x + B$$

$$\Rightarrow \quad p w_{x+1} + q w_{x-1} = p A (q/p)^{x+1} + pB + q A (q/p)^{x-1} + qB$$

$$= q A (q/p)^x + p A (q/p)^x + A$$

$$= w_x \quad \checkmark$$

Boundary conditions: $w_0 = 0 \Rightarrow A + B = 0$

$w_T = 1 \Rightarrow A\left((q/p)^T - 1\right) = 1$.

Exercise 7 Recall the Gambler's Ruin Problem: a random walk on the integers $\{0, 1 \ldots, N\}$ with probability p to jump right and $q = 1 - p$ to jump left at every step, and absorbing states at 0 and N. Starting at $X_0 = k$, the probability to reach N before reaching 0 is

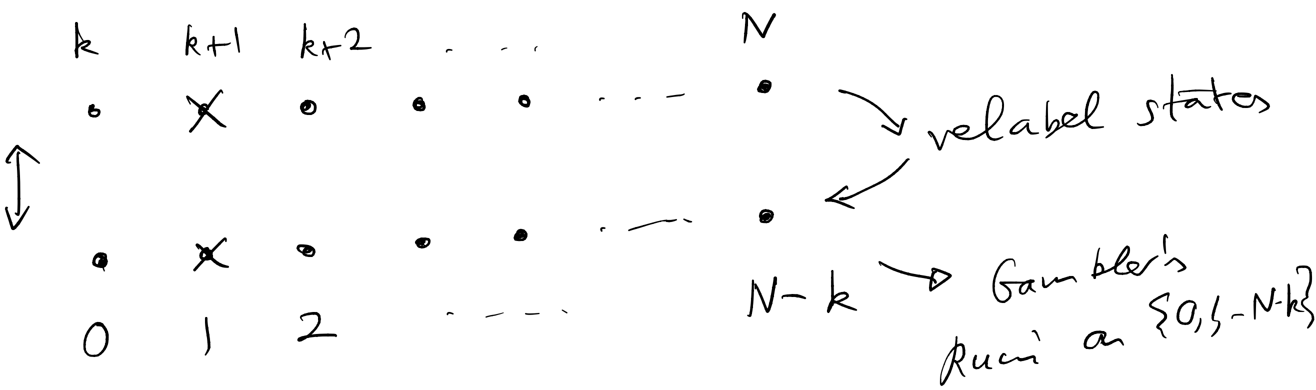
$$P_k = \frac{1-(q/p)^k}{1-(q/p)^N} \quad \text{for } p \neq \frac{1}{2}, \qquad P_k = \frac{k}{N} \quad \text{for } p = \frac{1}{2}.$$

Starting at $X_0 = k$, let R_k be the probability to reach state N without returning to state k. Use the Gambler's Ruin result to compute R_k for all $k = 0, \ldots, N$, and for all $0 < p < 1$. [Hint: condition on the first step and use the formula given above].

Condition on Step 1:

$$R_k = \mathbb{P}(\text{reach } N \text{ before } k \mid X_1 = k-1)\cdot q \quad \longrightarrow 0$$

$$+ \mathbb{P}(\text{reach } N \text{ before } k \mid X_1 = k+1)\cdot p$$



$$\Rightarrow \mathbb{P}(\text{reach } N \text{ before } k \mid X_1 = k+1) \quad \longleftarrow \text{original problem}$$

$$= \mathbb{P}(\text{reach } N-k \text{ before } 0 \mid X_0 = 1) \quad \longleftarrow \text{Gambler's Ruin}$$

$$\Rightarrow R_k = p \cdot \mathbb{P}(\text{reach } N-k \text{ before } 0 \mid X_0 = 1)$$

Use formula above for P_k:

$$\Rightarrow R_k = \begin{cases} p \dfrac{1 - q/p}{1-(q/p)^{N-k}} & p \neq \frac{1}{2} \\ \frac{1}{2} \dfrac{1}{N-k} & p = \frac{1}{2} \end{cases}$$

Exercise 8 The Markov chain $X = \{X_n\}$ is defined on the state space $S = \{0, 1, 2, \dots\}$. The chain is irreducible, aperiodic and positive persistent, with stationary distribution $\{w_k\}$ ($k = 0, 1, 2, \dots$). Let $Y = \{Y_n\}$ be an independent copy of X, and define $Z = (X, Y)$.

a). Write down the transition matrix for Z, and compute its stationary distribution (your answer will depend on w).

b). Given that the chain Z starts at the state (k, k) (so that $X_0 = Y_0 = k$), find an expression for the expected number of steps until the first return to (k, k).

a) $\mathbb{P}(Z_1 = (k,l) \mid Z_0 = (i,j))$ $\qquad Z = (X, Y)$

$$= \mathbb{P}(X_1 = k \mid X_0 = i)\ \mathbb{P}(Y_1 = l \mid Y_0 = j)$$

$$= P_{ik} P_{jl}$$

Stationary: $W_{(k,l)} = W_k W_l$.

Check:
$$\sum_{i,j} W_{(i,j)} P_{(i,j),(k,l)} = \sum_{i,j} W_i W_j P_{ik} P_{jl}$$
$$= \left(\sum_i w_i P_{ik}\right)\left(\sum_j w_j P_{jl}\right)$$
$$= W_k W_l$$
$$= W_{(k,l)}$$

b) Mean return time to (k,k) is

$$\frac{1}{W_{(k,k)}} = \frac{1}{W_k^2}$$