MTH 7241: Fall 2022

SOLUTIONS

Second Practice Problems for Test 1

1). Consider the following transition probability matrix for a Markov chain on 5 states:

$$P = \begin{pmatrix} 1/4 & 3/4 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1/3 & 2/3 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

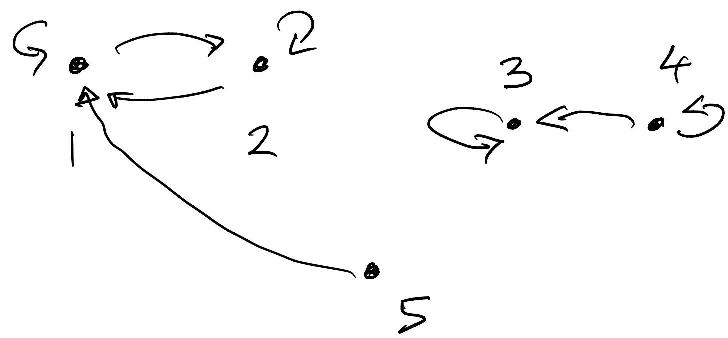
Number the states $\{1, 2, 3, 4, 5\}$ in the order presented.

a). Write down two different stationary distributions for the chain.

b). Starting from state 4, find the expected number of steps until first reaching state 3.

c). Starting from state 5, find the expected number of steps until first reaching state 2.

a)



$\{1,2\}$ closed, irreducible

$\{3\}$ absorbing

$\{4,5\}$ transient.

Different stationary distribution for each closed, irreducible class.

$\underline{\{1,2\}}$: $P = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

$$\Rightarrow \quad w_1 = \frac{1}{4} w_1 + \frac{1}{2} w_2$$

$$\Rightarrow \quad w_1 = \frac{2}{3} w_2$$

$$\Rightarrow \quad w = \left(\frac{2}{5}, \frac{3}{5}, 0, 0, 0 \right)$$

$$\{5\} : \quad w = (0, 0, 1, 0, 0)$$

b) $\{3, 4\}$:

$$P = \begin{array}{cc} \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ 0 & 1 \end{pmatrix} & \begin{array}{c} 4 \\ 3 \end{array} \\ \begin{array}{cc} 4 \quad & \quad 3 \end{array} & \end{array}$$

so $\quad Q = \left(\frac{2}{3} \right)$

$$\Rightarrow \quad N = (I - Q)^{-1} = \left(\frac{1}{3} \right)^{-1} = (3)$$

$\Rightarrow$ Expected number of steps $(4 \to 3) = 3$.

c) since $5 \to 1$.

$\Rightarrow$ Expected # steps $(5 \to 2)$

$= 1 +$ Expected # steps $(1 \to 2)$.

$\{1, 2\}$:

$$P = \begin{array}{cc} \begin{pmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} & \begin{array}{c} 1 \\ 2 \end{array} \end{array}$$

m_{12} = Expected # steps $(1 \to 2)$.

Condition on first step.

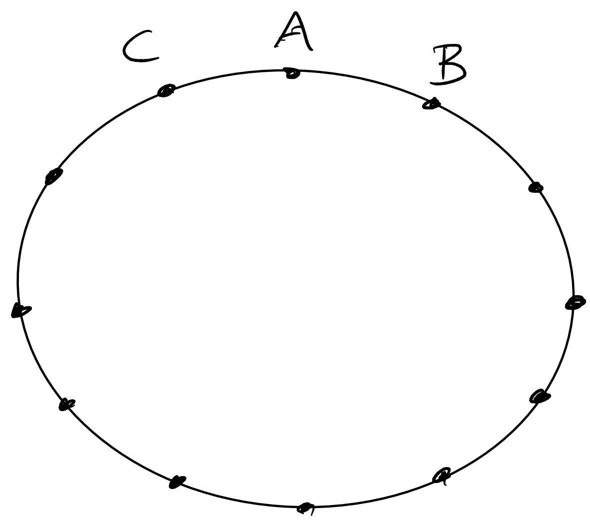
$$m_{12} = \frac{1}{4}(1 + m_{12}) + \frac{3}{4}(1)$$

$$= 1 + \frac{1}{4} m_{12}$$

$$\Rightarrow m_{12} = \frac{4}{3}.$$

$$\Rightarrow m_{52} = 1 + \frac{4}{3} = \frac{7}{3}$$

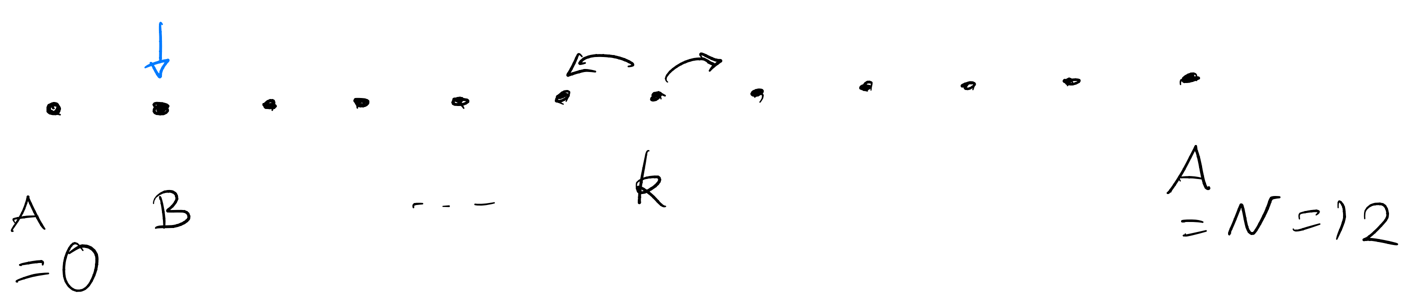
2). A particle moves between 12 points which are spaced around a circle. At each step the particle is equally likely to move one point clockwise or one point counterclockwise. Find the mean number of steps for the particle to return to its starting position. [Hint: make use of the solution of the Gambler's Ruin problem as derived in class].



Start at A:

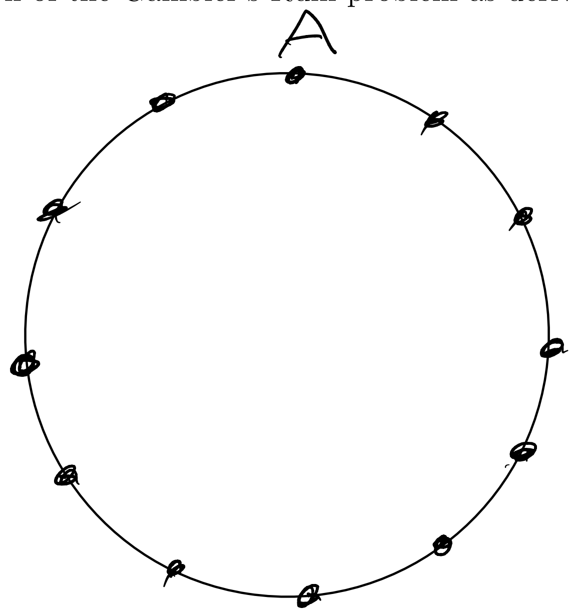
first step goes to B or C, with prob. $\frac{1}{2}$.

Suppose it jumps to B:



Let m_k = expected number of steps to reach either endpoint starting at k.

2). A particle moves between 12 points which are spaced around a circle. At each step the particle is equally likely to move one point clockwise or one point counterclockwise. Find the mean number of steps for the particle to return to its starting position. [Hint: make use of the solution of the Gambler's Ruin problem as derived in class].



This is a random walk on graph; every vertex has degree $d = 2$.

So the Markov chain is reversible and has stationary distribution

$$W_i = \frac{d_i}{\sum_j d_j} = \frac{2}{12(2)} = \frac{1}{12}$$

Therefore the mean first return time is

$$\frac{1}{W_i} = 12.$$

Condition on first step:

$$m_k = \frac{1}{2}(1+m_{k+1}) + \frac{1}{2}(1+m_{k-1}).$$

$$\Rightarrow m_k = 1 + \frac{1}{2} m_{k+1} + \frac{1}{2} m_{k-1}.$$

Boundary conditions:

$$m_0 = m_N = 0.$$

<u>Solution:</u> guess

$$m_k = a + bk + ck^2.$$

(some constants a, b, c)

$$\Rightarrow a + bk + ck^2 = 1 + \frac{1}{2}\left(a + b(k+1) + c(k+1)^2\right)$$

$$+\frac{1}{2}\left(a+b(k-1)+c(k-1)^2\right)$$

$$= 1 + a + bk$$

$$+ ck^2 + c$$

$$\Rightarrow \boxed{c = -1}$$

B.C. $m_0 = a = 0$ $\boxed{a = 0}$

$$m_N = a + bN + cN^2 = 0.$$

$$\Rightarrow bN = N^2 \Rightarrow \boxed{b = N}$$

$$\Rightarrow m_k = Nk - k^2$$

$$= k(N-k).$$

So for our problem:

$N = 12, \quad k = 1$

$\Rightarrow$ expected number of steps to return to A starting from B

$= 1(N-1) = 11$

Since the same holds if we start from C, we deduce

Expected # steps for return to A

$= 1 + 11 = 12$

3) A stack of m cards is shuffled as follows: at each step a random number X is chosen from $\{1, 2, \ldots, m\}$, then the card which sits at position X is removed and placed at the top of the deck. By modeling this process as a Markov chain, show that in the long-run the deck becomes randomly shuffled, so that all $m!$ orderings are equally likely.

State space = { all $m!$ orderings }

state $s = (k_1, k_2, \ldots, k_m)$

where k_i are distinct integers from $\{1, 2, \ldots, m\}$.

Transition matrix:

$$s \to s'$$

$$(k_1, \ldots, k_m) \longrightarrow (k_j, k_1, k_2, \ldots, k_{j-1}, k_{j+1}, \ldots, k_m)$$

$X = j \Rightarrow$ move card k_j to front

and $P_{ss'} = \dfrac{1}{m}$ (all equally likely).

The shuffling evolves as a Markov chain with this transition matrix. Clearly it is irreducible and aperiodic, so it converges to its stationary distribution. What is the stationary distribution?

Check that the transition matrix is doubly stochastic:

given s', find $s \to s'$:

say $s' = (k_1, k_2, \ldots, k_m)$

then
$$
\left\{
\begin{array}{l}
s = (k_1, k_2, k_3, \ldots, k_m) \\
s = (k_2, k_1, k_3, \ldots, k_m) \\
s = (k_2, k_3, k_1, \ldots, k_m) \\
\vdots \\
s = (k_2, k_3, \ldots, k_m, k_1)
\end{array}
\right.
$$

m states for $s \to s'$

all have probability $= \frac{1}{m}$

$\Rightarrow$ matrix is doubly stochastic

$\Rightarrow$ stationary distribution is <u>uniform</u>.

$$\Rightarrow \quad W_s = \frac{1}{m!}$$

$\Leftrightarrow$ all orderings equally likely.

4) Consider the following transition probability matrix for a Markov chain on 3 states:

$$P = \begin{array}{cc} \begin{pmatrix} 1/2 & 1/3 & 1/6 \\ 0 & 1/3 & 2/3 \\ 1/2 & 0 & 1/2 \end{pmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \\ \begin{array}{ccc} 1 & \quad 2 & \quad 3 \end{array} & \end{array}$$

Number the states $\{1, 2, 3\}$ in the order presented.

Find the long-run probability that the chain jumps in the following sequence of states: $1 \to 2 \to 3 \to 1 \to 2$.

$$\lim_{n\to\infty} \mathbb{P}(X_n = 1, X_{n+1} = 2, X_{n+2} = 3, X_{n+3} = 1, X_{n+4} = 2)$$

$$= P_{12} P_{23} P_{31} P_{12} \lim_{n\to\infty} \mathbb{P}(X_n = 1)$$

$$= P_{12} P_{23} P_{31} P_{12} w_1.$$

Now $w = (w_1, w_2, w_3) = \left(\frac{2}{5}, \frac{1}{5}, \frac{2}{5}\right)$

$\Rightarrow$ get $\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)\left(\frac{2}{5}\right)$

$$= \frac{2}{135}$$

5) Let $\{X_n\}$ be a Markov chain, and suppose that for state i we have

$$\sum_{k=1}^{n} p_{ii}(k) = \sum_{k=1}^{n} \mathrm{P}(X_k = i \mid X_0 = i) \geq \frac{9}{\sqrt{n+8}} \quad \text{for all } n \geq 1.$$

Determine whether state i is transient or persistent (explain your reasoning).

Correction: it should be

$$P_{ii}(n) \geq \frac{9}{\sqrt{n+8}} \quad \text{for all } n \geq 1.$$

Then

$$\sum_{n=1}^{\infty} P_{ii}(n) \geq \sum_{n=1}^{\infty} \frac{9}{\sqrt{n+8}}$$

$$\geq 9 \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+8}}$$

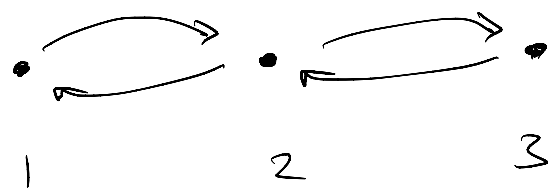
divergent harmonic series

$$= \infty$$

$\Rightarrow$ state i is persistent

6) Consider an irreducible chain on 3 states. **Either** prove that $p_{jj}(3) > 0$ for every state j, **or** give an example where $p_{jj}(3) = 0$ for some state j.

<u>False</u>: here is example



In this case $p_{jj}(3) = 0$ for every j.

7) For a branching process calculate the probability of extinction when the offspring probabilities are $p_0 = 1/4$, $p_1 = 1/2$, $p_2 = 1/8$, $p_3 = 1/8$.

Mean offspring:

$$m = 0\left(\tfrac{1}{4}\right) + 1\left(\tfrac{1}{2}\right) + 2\left(\tfrac{1}{8}\right) + 3\left(\tfrac{1}{8}\right)$$

$$= \frac{9}{8}$$

Since $m > 1 \Rightarrow \rho < 1$ (prob. of extinction)

$$\phi(s) = \frac{1}{4} + \frac{1}{2}s + \frac{1}{8}s^2 + \frac{1}{8}s^3.$$

solve

$$\phi(s) = s$$

$$\Leftrightarrow \frac{1}{8}s^3 + \frac{1}{8}s^2 - \frac{1}{2}s + \frac{1}{4} = 0.$$

$$\Leftrightarrow s^3 + s^2 - 4s + 2 = 0.$$

$$(s-1)(s^2 + 2s - 2) = 0$$

$$\Downarrow$$

$$s = \frac{-2 \pm \sqrt{4+8}}{2} = -1 \pm \sqrt{3}.$$

<u>Solution</u>: ρ = smallest positive solution

$$\Downarrow$$

$$\Rightarrow \rho = -1 + \sqrt{3}.$$

8) Either give an example of a finite Markov chain with no transient states and two different stationary distributions, or show that that this cannot happen.

Example:

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Stationary distributions:

$$w = \left(\tfrac{1}{2}, \tfrac{1}{2}, 0, 0\right)$$

$$w = \left(0, 0, \tfrac{1}{2}, \tfrac{1}{2}\right)$$

9) Either give an example of a finite Markov chain with a unique stationary distribution and one transient state, or show that that this cannot happen.

<u>Example</u>.

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$\{3\}$ transient.

$$W = \left(\tfrac{1}{2}, \tfrac{1}{2}, 0\right)$$

unique.

10) Let X_n be an irreducible regular finite Markov chain with a unique stationary distribution $\{w_i\}$. Compute

$$\lim_{n\to\infty} \mathrm{P}\left(X_{n+1} = X_n\right)$$

$$\mathbb{P}(X_{n+1} = X_n)$$

$$= \sum_i \mathbb{P}(X_{n+1} = X_n = i \mid X_n = i)\, \mathbb{P}(X_n = i)$$

$$= \sum_i P_{ii}\, \mathbb{P}(X_n = i)$$

$$\Rightarrow \lim_{n\to\infty} \mathbb{P}(X_{n+1} = X_n) = \sum_i P_{ii}\, w_i$$