Stability and Bifurcation

We begin with the simplest equation

$$\frac{dy}{dt} = ay \quad (a = \text{constant}) \qquad (1)$$

with the variable t interpreted as the time. It is easy to obtain the general solution:

$$y = Ce^{at} \qquad (2)$$

where C is an arbitrary constant determined from the initial condition

$$y(t_0) = y_0$$

Substituting into (2), we get

$$y_0 = Ce^{at_0}, \text{ or } C = y_0 e^{-at_0},$$

and finally

$$y = y_0 e^{a(t-t_0)} \qquad (3)$$

In particular, when $y_0 = 0$ we get the zero solution $y \equiv 0$. Now suppose that the initial value y_0, which we regarded as zero, is actually different from zero, albeit just slightly. Then how will the solution behave with time, that is, as t increases? Will such a perturbed solution approach the unperturbed zero solution or will it recede from it?

The answer to these questions depends essentially on the sign of the coefficient a. If $a<0$, then (3) shows immediately that as t increases the solutions approach zero, so that for large t they practically vanish. In such a situation, the unperturbed solution is said to be asymptotically stable relative to the change (perturbation) of the initial condition.

The picture will be quite different for $a>0$. Here, when $y_0 \neq 0$ and t is increasing, the solution increases in absolute value without bound, that is to say; it becomes significant even if y_0 was arbitrarily small. Here the unperturbed solution is said to be unstable.

For $a > 0$ we have equation (1), for example, when considering the growth of bacteria in a nutrient medium with y denoting the mass of bacteria per unit volume and a, the intensity of growth. It is clear that if at the initial time there were no bacteria at all, then of course none will appear in the course of time. But this picture is unstable in the sense that a purposeful or accidental introduction of any arbitrarily small amount of bacteria into the medium will, in time, lead to extreme growth and pollution of the medium with bacteria.

The intermediate case $a = 0$ is also interesting. Here the solutions will merely be constant and for this reason for a small initial deviation of the perturbed solution from the unperturbed solution the former will be close to the latter even when t increases, although the approach will not be asymptotic (as $t \to \infty$). Such a picture is called nonasymptotic stability of an unperturbed solution.

Now let us consider the more general equation

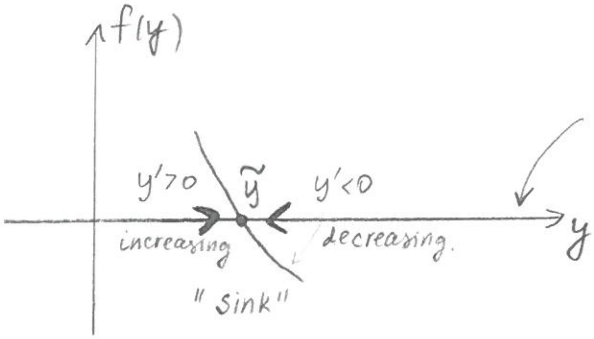
(autonomous equation) $$\frac{dy}{dt} = f(y) \qquad (4)$$

It is easy to find all the stationary solutions, that is, solutions of the form $y =$ constant. To do this, in (4) set $y = \tilde{y} =$ constant to get

$$f(\tilde{y}) = 0 \qquad (5)$$

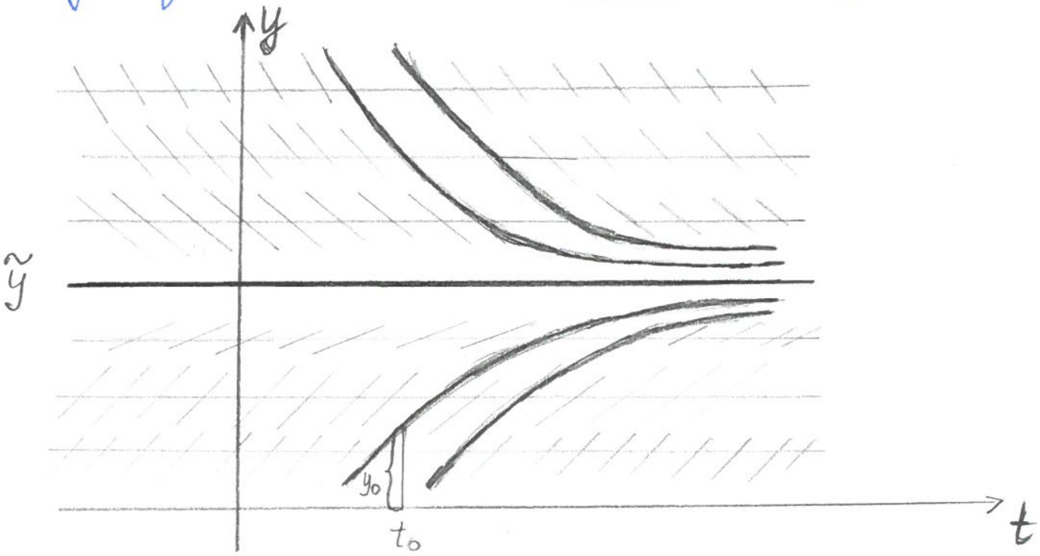
Thus the stationary solutions of (4) are the zeros of the function $f(y)$ in the right-hand side. Let us examine one such solution $y = \tilde{y}$ and determine whether it is stable or not.

First assume that the function $f(y)$ is decreasing, at least in a certain neighborhood of the value $y = \tilde{y}$; then if y passes through the value $\tilde{y}$, it follows that $f(y)$ passes from positive values to negative values.



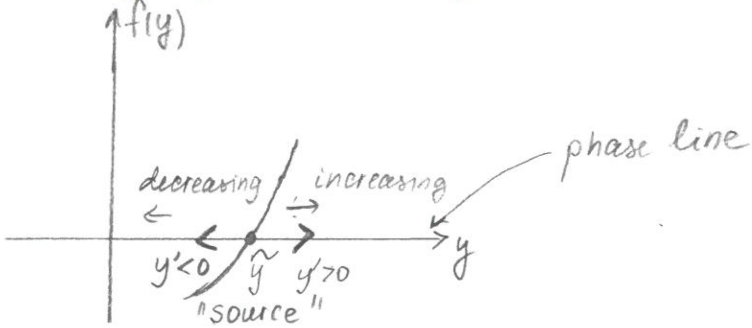
[The arrows on the y-axis indicate the direction of movement of y]

In this case the approximate picture of the direction field defined by equation (5) is shown below:

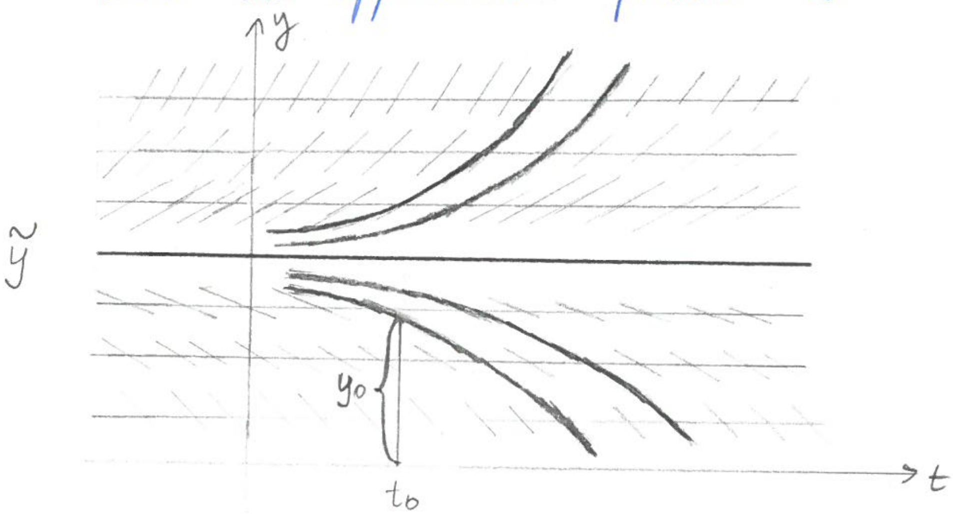


In constructing this field, bear in mind that the isoclines for (5) have the form $y = \text{constant}$ (b/c $f(y) = \text{constant}$, no t explicitly), that is, they are straight lines parallel to the t-axis. The heavy lines (curves) indicate the integral straight line $y = \tilde{y}$, which depicts an unperturbed stationary solution, and several other integral curves that result from changes in the initial condition. It is clear that if y_0 differs slightly from $\tilde{y}$ (say, within the limits of the drawing), then the perturbed solution does not differ considerably from the unperturbed one even when t increases, and when $t \to \infty$ it asymptotically approaches the unperturbed solution. Thus, in this case the unperturbed solution is asymptotically stable.

Now let $f(y)$ increase from negative values to positive ones when y passes through the value $\tilde{y}$.



Then the approximate picture is



It is clear that no matter how close y_0 is to $\tilde{y}$ (but not equal to $\tilde{y}$!), the approximate solution $y(t)$ will recede from the unperturbed solution to a finite but substantial distance as t increases. This means that in the case at hand the unperturbed stationary solution is unstable.

The criteria obtained here can be derived differently, without resorting to a geometric picture. Expand the right side of (4) in a power series about the value $y = \tilde{y}$; then we get

$$\frac{dy}{dt} = \underbrace{\cancelto{0}{f(\tilde{y})}}_{\text{(b/c of (5))}} + f'(\tilde{y})(y-\tilde{y}) + \frac{f''}{2!}(\tilde{y})(y-\tilde{y})^2 + \dots$$

That is,

$$\frac{d(y-\tilde{y})}{dt} = f'(\tilde{y})(y-\tilde{y}) + \dots \qquad (6)$$

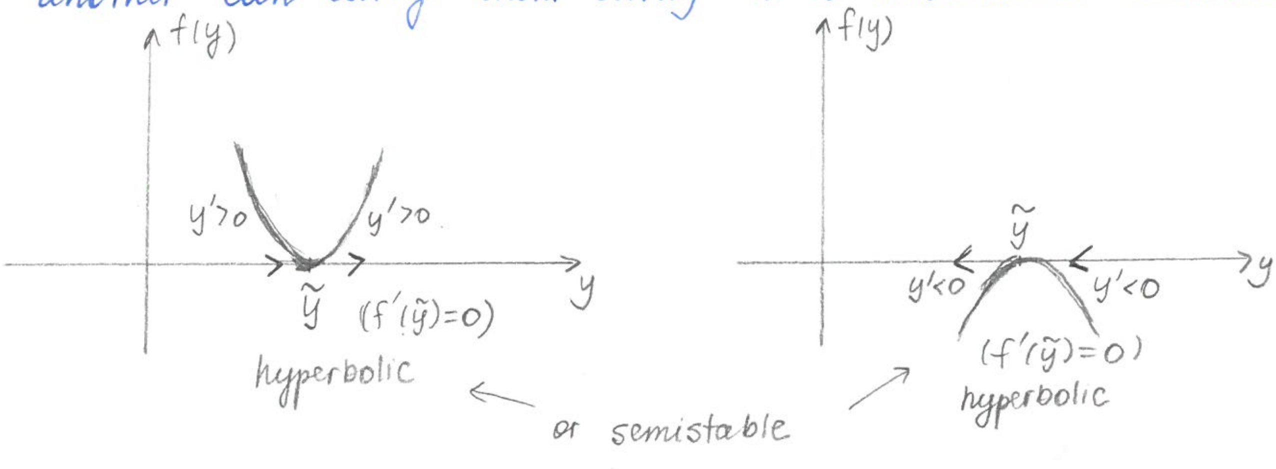
where the dots stand for higher-order terms.

It must be stressed that when determining stability, we study the behavior of perturbed solutions that differ but slightly from the unperturbed solution, that is to say, we consider only small values of $y-\tilde{y}$. For this reason, the main role in the right-hand side of (6) is played by the linear term that is written out. Discarding higher-order terms, we obtain an equation of the form (1) in which

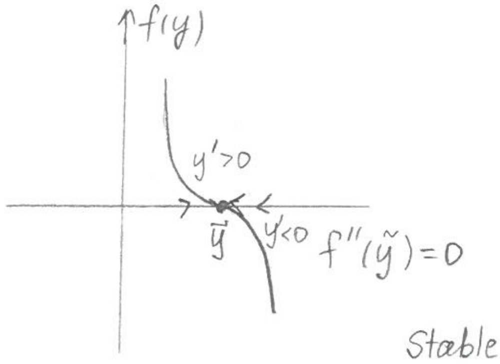
$$a = f'(\tilde{y}).$$

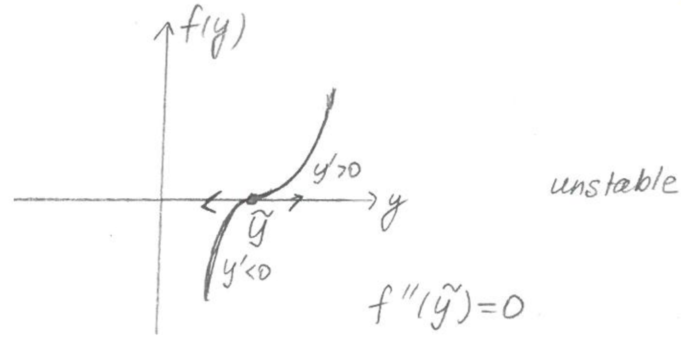
Applying the results obtained above for equation (1), we see that for $f'(\tilde{y})<0$ the solution $y-\tilde{y}=0$, or $y=\tilde{y}$, is asymptotically stable; but if $f'(\tilde{y})>0$, then the solution $y=\tilde{y}$ is unstable. But in the first case the function $f(y)$ decreases (at least about the value $y=\tilde{y}$), while in the latter case, it increases, so that we arrive at the same conclusions that were obtained via geometrical reasoning.

In a particular case where $f'(y)=a=0$ for equation (1) we have nonasymptotic stability, that is, the solutions close to the unperturbed solution do not tend to it and do not recede from it; then in the complete equation (6) a substantial role is played by higher-order terms, which in one instance can direct the perturbed solutions to the unperturbed solution, while in another can carry them away to a substantial distance.

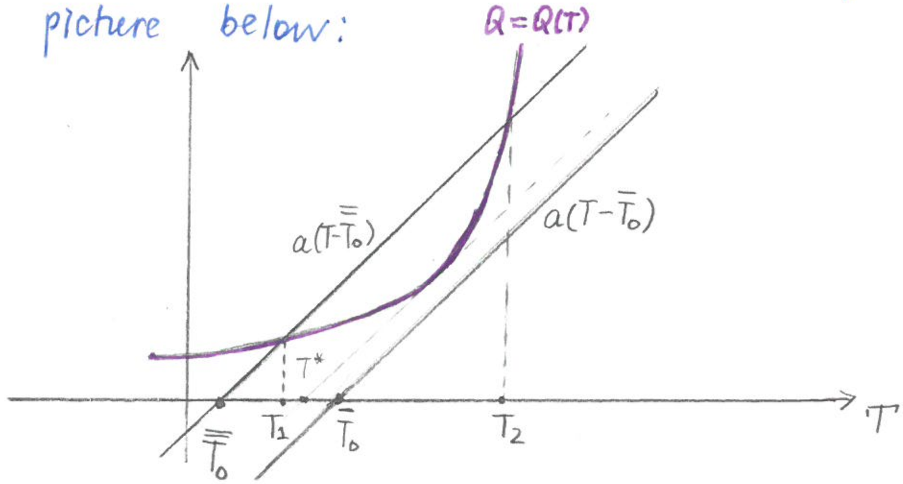


$f''(\tilde{y}) \neq 0$





To illustrate, let us examine the thermal regime in a certain volume where we have a chemical reaction associated with release of heat, which is carried off into the ambient space. Since the rate of reaction depends on the temperature T in the given volume (we consider the mean temperature at a given time t), the rate Q of heat release in the reaction depends on T, $Q = Q(T)$. We assume the relationship to be as shown in the picture below:



We also suppose the rate of heat dissipation into the ambient space to be equal to $a(T-T_0)$, where a is the proportionality constant and T_0 is the temperature of the ambient medium. Then, for a constant heat capacity c of the volume at hand, the differential equation of the process becomes

$$\frac{d}{dt}(cT) \equiv c\,\frac{dT}{dt} = Q(T) - a(T-T_0) \qquad (7)$$

[the temperature balance equation]

By virtue of the foregoing, a stationary state (in which the temperature remains constant during the reaction) is possible for those T for which the right side vanishes, which means that the graph of $Q(T)$ crosses the graph of $a(T-T_0)$.

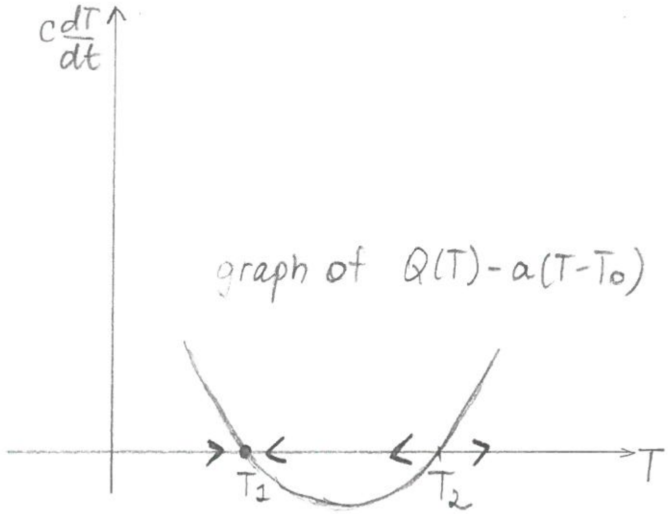
We see that if the ambient temperature T_0 is sufficiently great (when $T_0 = \overline{T}_0$ in the picture), a stationary state is impossible: the supply of heat will all the time be greater than dissipation and the volume will constantly heat up.

If the temperature is low (when $T_0 = \overline{\overline{T}}_0$ in the picture), two stationary states having temperature T_1 and T_2 are feasible. Near the value T_1, the right-hand side of (7) passes from positive values to negative values, which means it decreases. We have already seen that such a state is a stable state. This is evidedent from the picture, for if the temperahere T falls below T_1, then more heat will be released in the reaction than is dissipated, which means the volume will heat up, and if T rises above T_1, then more heat will be dissipated than released and the volume will cool off.

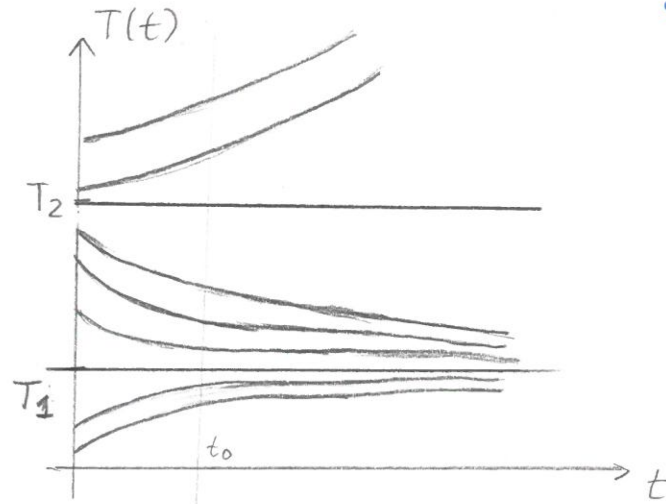
In the similar fashion we can verify that the stationary temperature T_2 will be unstable. Thus, when $T_0 = \overline{\overline{T}}_0$ the development of the process depends on the initial temperahere as follows: if it was less than T_2, then in time the temperature tends to the stationary value T_1; if the initial temperature was greater than T_2, then the temperature builds up catastrophically. Such was the reasoning that served as the basis for the theory of thermal explosion.

Key to the qualitative analysis is to determine the equilibria of an equation and the dynamics near the equilibria.

Let us summarize what we got from the considered example.

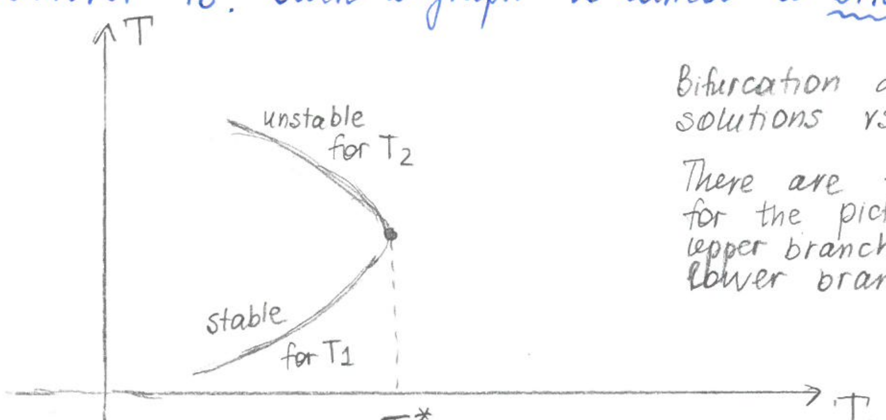


Generic phase line plot.
The T axis is interpreted as a one-dimensional state, or phase, space on which the solution $T = T(t)$ moves in time. The arrows indicate the direction of movement



Time series plots for various initial conditions of the equation whose phase line plot is shown in the phase line diagram

We can sketch the locus $Q(T)-a(T-T_0)=0$ on T_0T-coordinate system (with T_0 the independent variable) to graphically indicate the dependence of the equilibrium solutions on the parameter T_0. Such a graph is called a bifurcation diagram.



Bifurcation diagram showing equilibrium solutions vs. the parameter T_0.

There are two branches at least for the picture we were given, the upper branch is unstable, and the lower branch is stable.

In this context, the ambient temperature T_0 is called a bifurcation parameter. We observed that if T_0 is too great ($T_0 > T^*$), then no stationary state exists and the volume will heat up. If $T_0 < T^*$, then there are two equilibrium states.

Example 1.

Consider the model

$$u' = (\lambda - b)u - au^3,$$

where a and b are fixed positive constants and λ is a parameter that varies.

(a) If $\lambda < b$ show that there is a single equilibrium and that it is asymptotically stable.

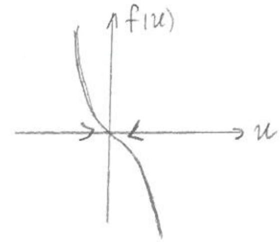
Stationary (equilibrium) points are the zeros of $f(u) = (\lambda - b)u - au^3$:

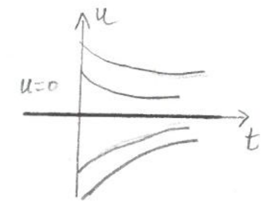
$(\lambda - b)u - au^3 = 0$

$u(\lambda - b - au^2) = 0$

$u = 0$, $\quad au^2 = \lambda - b$

$u^2 = \frac{\lambda - b}{a} < 0$ (no solution)





There is one equilibrium point $u = 0$

Stability: $f'(u) = (\lambda - b) - 3au^2$

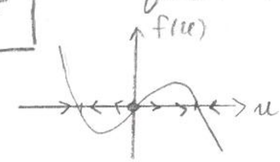
$f'(0) = \lambda - b < 0 \Rightarrow f(u)$ at $u = 0$ is decreasing, it implies that $u = 0$ is asymptotically stable

(b) If $\lambda > b$ find all equilibria and determine their stability.

Equilibria $f(u) = 0$

$u(\lambda - b - au^2) = 0$

$\boxed{u = 0}$, $\quad u^2 = \frac{\lambda - b}{a} > 0 \Rightarrow \boxed{u = \sqrt{\frac{\lambda - b}{a}}}$, $\boxed{u = -\sqrt{\frac{\lambda - b}{a}}}$ – Equilibria



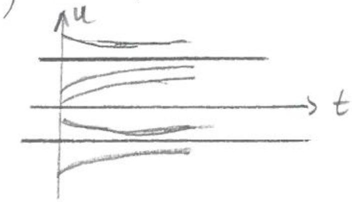
Stability: $f'(u) = \lambda - b - 3au^2$

$f'(0) = \lambda - b - 3a \cdot 0^2 = \lambda - b > 0$, f at $u = 0$ is increasing, therefore it is unstable stationary point

$f'\left(\sqrt{\frac{\lambda - b}{a}}\right) = \lambda - b - 3a\left(\sqrt{\frac{\lambda - b}{a}}\right)^2 = \lambda - b - 3\cancel{a}\frac{(\lambda - b)}{\cancel{a}} = \lambda - b - 3(\lambda - b) =$

$= -2(\lambda - b) < 0$, $\quad u = \sqrt{\frac{\lambda - b}{a}}$ is stable

$f'\left(-\sqrt{\frac{\lambda - b}{a}}\right) = \lambda - b - 3a\left(-\sqrt{\frac{\lambda - b}{a}}\right)^2 = -2(\lambda - b) < 0$, $\quad u = -\sqrt{\frac{\lambda - b}{a}}$ is stable



(c) Sketch the bifurcation diagram showing how equilibra vary with λ. Label each branch of the curves shown in the bifurcation diagram as stable or unstable.