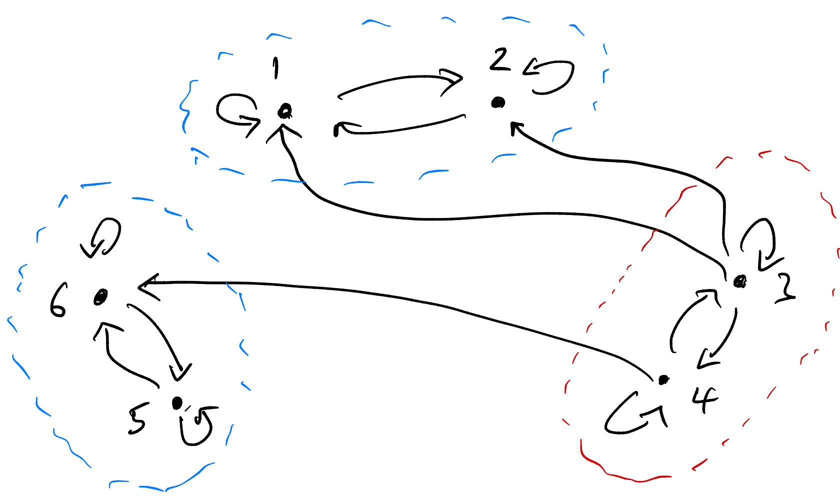
MATH 7241 Fall 2022: Problem Set #4

Due date: Friday October 21

Reading: relevant background material for these problems can be found on Canvas 'Notes 4: Finite Markov Chains'. Also Grinstead and Snell Chapter 11.

$\boxed{\textbf{Exercise 1}}$ 'Finite Markov Chains - Problems', Exercise 1.

Hint: draw a graph with 6 nodes to represent the states of the chain, and draw a directed edge between each pair of nodes (i, j) for which the transition matrix entry p_{ij} is positive. You can identify the set of transient states as the group of nodes from which edges exit, but into which there are no entering edges. Once you find the transient states, the remaining states are all irreducible, and break up into subsets which intercommunicate.



$\{3,4\}$ transient

$\{1,2\}$, $\{5,6\}$ both closed, intercommunicating

&

$\Rightarrow$ persistent

$\boxed{\textbf{Exercise 2}}$ 'Finite Markov Chains - Problems', Exercises 2 a), 2 b).

a) $\{3,4\}$ transient

$\{1,2,5\}$ closed, irreducible, persistent.

b) Stationary distribution:

$$W = (W_1, W_2, W_3, W_4, W_5)$$

$$= \left(\frac{1}{3}, \frac{1}{3}, 0, 0, \frac{1}{3}\right)$$ <u>unique solution</u>.

Exercise 3 'Finite Markov Chains - Problems', Exercise 3.

Hint: to represent this by a Markov chain you must use the result of two successive trials as your state. So there are four states: SS, SF, FS, FF where S is success and F is failure, and SF means success on trial n and failure on trial $n+1$. Then if your current state is SF, your next state must be either FS or FF.

X_n = result of n^{th} trial, $X_n \in \{S, F\}$

$Y_n = (X_n, X_{n-1})$ results of n^{th}, $(n-1)^{\text{st}}$ trial

Y_n is a Markov chain w/ transition matrix

$$P = \begin{array}{c} \\ \\ \\ \\ \\ \end{array}\begin{array}{cc} \begin{pmatrix} 0.8 & 0.2 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 \end{pmatrix} & \left.\begin{array}{c} SS \\ SF \\ FS \\ FF \end{array}\right\} Y_0 = \text{input} \\ \underbrace{\begin{array}{cccc} SS & SF & FS & FF \end{array}}_{Y_1 = \text{output}} & \end{array}$$

Stationary distribution: $W = \left(\frac{5}{11}, \frac{2}{11}, \frac{2}{11}, \frac{2}{11} \right)$.

Long run proportion of successful trials is

$$\lim_{n\to\infty} \mathbb{P}(X_n = S) = \lim_{n\to\infty} \left(\mathbb{P}(Y_n = SS) + \mathbb{P}(Y_n = SF) \right)$$

$$= W_{SS} + W_{SF} = \frac{7}{11}$$

Exercise 4 'Finite Markov Chains – Problems' file: Exercise ~~3~~ 4

Claim 1: $\{Y_n\}$ is Markov.

Proof: $\mathbb{P}(Y_{n+1}=(j_1,j_2) \mid Y_n=(i_1,i_2), Y_{n-1}=(k_1,k_2),\dots)$

$$= \mathbb{P}(X_n=j_1, X_{n+1}=j_2 \mid X_{n-1}=i_1, X_n=i_2, X_{n-2}=k_1, X_{n-1}=k_2, \dots)$$

Only need to check cases where this event has nonzero prob.

$\Rightarrow i_1=k_2, \dots$ (all consistent)

$$= \mathbb{P}(X_{n+1}=j_2 \mid X_n=j_1, X_{n-1}=i_1, X_n=i_2, X_{n-2}=k_1,\dots).$$
$$\mathbb{P}(X_n=j_1 \mid X_{n-1}=i_1, X_n=i_2, X_{n-2}=k_1,\dots)$$

nonzero only if $j_1=i_2$ in which case it equals 1

$$= \mathbb{P}(X_{n+1}=j_2 \mid X_n=i_2, X_{n-1}=i_1,\dots)\cdot \delta_{j_1,i_2}$$

$$= P_{i_2,j_2}\cdot \delta_{j_1,i_2} \quad \text{by Markov property for } X$$

$$= \mathbb{P}(Y_{n+1}=(j_1,j_2) \mid Y_n=(i_1,i_2))$$

$\Rightarrow Y_n$ is Markov chain.

By assumption,

$$\lim_{n\to\infty} \mathbb{P}(X_n = j) = w_j \quad \text{all } j.$$

$$\Rightarrow \lim_{n\to\infty} \mathbb{P}(Y_n = (i,j))$$

$$= \lim_{n\to\infty} \mathbb{P}(X_{n-1} = i, X_n = j)$$

$$= \lim_{n\to\infty} \mathbb{P}(X_n = j \mid X_{n-1} = i)\, \mathbb{P}(X_{n-1} = i)$$

$$= P_{ij} \lim_{n\to\infty} \mathbb{P}(X_{n-1} = i)$$

$$= w_i P_{ij}$$

Exercise ~~4~~ 5 Grinstead & Snell, p. 442: #2.

Note: see pages 442-443 from Grinstead and Snell on Canvas. The text is available online (free!) at

http://www.dartmouth.edu/∼chance/

Click on the link "A GNU book".

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ \frac{3}{4} & 0 & \frac{1}{4} \\ 0 & 1 & 0 \end{pmatrix}$$

a) $$P^2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ \frac{3}{8} & \frac{1}{2} & \frac{1}{8} \\ \frac{3}{4} & 0 & \frac{1}{4} \end{pmatrix} \qquad P^3 = \begin{pmatrix} & & \\ & * & \\ & & \end{pmatrix} \text{ all non-zero}$$

$\Rightarrow$ P is regular.

b) $$\mathbb{P}(X_2 = 3 \mid X_0 = 1) = (P^2)_{13} = \frac{1}{6}$$

c) $$w_j = \sum_{i=1}^{3} w_i P_{ij} \quad \text{for } j = 1, 2, 3$$

$$\Rightarrow w = \left(\frac{1}{2}, \frac{1}{3}, \frac{1}{6}\right)$$