MATH 5131 Lecture 2

- **Basic first-order differential equations.**

$$\frac{dy}{dx} = f(x,y), \quad \text{unknown}: \ y = y(x)$$

Some types of 1st ode's

Separable

$$\frac{dy}{dx} = p(y)q(x)$$

$$\Downarrow$$

$$\frac{1}{p(y)}\,dy = q(x)\,dx \qquad [\,p(y) \not\equiv 0\,].$$

$$\Downarrow$$

to solve $\boxed{\int \frac{1}{p(y)}\,dy = \int q(x)\,dx}$ is the general solution, usually implicitly

Linear 1st-order equations (integrating factor method)

$$y' + p(x)y = q(x)$$

$$\Downarrow$$

$$\underbrace{u(x)y' + u(x)p(x)y}_{\frac{d}{dx}(uy) = ?} = u(x)q(x) \qquad (u(x) \not\equiv 0) \quad u(x) = ?$$

$$\Downarrow$$

$$\underbrace{uy' + upy} = \frac{d}{dx}(uy) \overset{\text{Product Rule}}{=} \underbrace{uy' + u'y}$$

compare

Should be the same

$$\Downarrow$$

$$\cancel{uy'} + upy = \cancel{uy'} + u'y \Rightarrow upy = u'y \Rightarrow$$

$$(up - u')y = 0 \Rightarrow (y \not\equiv 0) \quad up - u' = 0 \Rightarrow u' = up \Rightarrow$$

$$\frac{du}{dx} = up \Rightarrow \frac{du}{u} = p\,dx \Rightarrow \boxed{u = e^{\int p(x)\,dx}} \text{ - our integrating factor}$$

do not include a constant of integration

Once we get an integrating factor:

$$\underbrace{uy' + upy}_{\Downarrow} = uq$$

$$\frac{d}{dx}(uy) = uq \Rightarrow \int d(uy) = \int uq\,dx \Rightarrow$$

$$uy = \int uq\,dx \Rightarrow y = \frac{1}{u}\left[\int uq\,dx\right] \quad \text{or}$$

$$y = e^{-\int p\,dx}\left[\underbrace{\int e^{\int p\,dx} q\,dx}\right]$$

the arbitrary constant appears when we complete the integration

Example 1 Solve $y' = 3x^2(y+1)$ for $y(x)$.

Solution 1 (Separable)

$$\frac{dy}{dx} = 3x^2(y+1) \qquad (y+1 \neq 0)$$

$$\Downarrow$$

$$\frac{1}{y+1}\,dy = 3x^2\,dx$$

$$\Downarrow$$

$$\int \frac{1}{y+1}\,dy = \int 3x^2\,dx$$

$$\Downarrow$$

$$\ln|y+1| = x^3 + C$$

$$\Downarrow$$

$$e^{\ln|y+1|} = e^{x^3+C} \quad [e^c \cdot e^{x^3}]$$

$$\Downarrow$$

$$(y+1) = Ce^{x^3}$$

$$\Downarrow$$

$$\boxed{y = Ce^{x^3} - 1}$$ – general solution

Check whether $y=-1$ is a solution
$\frac{d}{dx}(-1) = 0 \Rightarrow 0 = 3x^2(-1+1)$ ✓
but it is included in $(Ce^{x^3} - 1)$ when $C = 0$.

Solution 2 (1st-order linear, integrating factor)

$$y' - 3x^2 y = 3x^2, \quad p(x) = -3x^2$$

$$u(x) = e^{\int -3x^2\,dx} = e^{-x^3}$$

$$\Downarrow$$

$$y(x) = \frac{1}{e^{-x^3}}\left[\int e^{-x^3} \cdot \underbrace{3x^2\,dx}\right]$$

Substitution
$v = x^3$
$dv = 3x^2\,dx$

$$= e^{x^3}\left[-e^{-x^3} + C\right]$$

$$= -1 + Ce^{x^3}$$

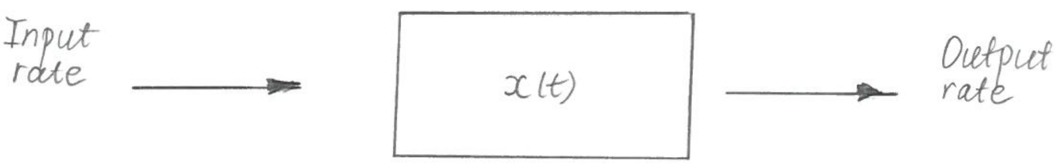
$$\Downarrow$$

$$\boxed{y(x) = Ce^{x^3} - 1}$$ the general solution.

Compartmental Analysis

Many complicated processes can be broken down into distinct stages and the entire system modeled by describing the interactions between the various stages. Such systems are called compartmental and are graphically depicted by block diagrams.

The basic one-compartment system consists of a function $x(t)$ that represents the amount of a substance in the compartment at time t, an input rate at which the substance enters the compartment, and an output rate at which the substance leaves the compartment:



Because the derivative of x with respect to t can be interpreted as the rate of change in the amount of the substance in the compartment with respect to time, the one-compartment system suggests

$$\boxed{\frac{dx}{dt} = \text{input rate} - \text{output rate}}$$

[Balance Law]

as a mathematical model for the process.

In this lecture we concentrate on examples of models that involve first-order differential equations. In studying these and in building your own models the following broad outline of the process may be helpful.

Radioactive Decay

The physicist Rutherford and his colleagues showed that the atoms of certain "radioactive" elements are unstable and that within a given time period a fixed proportion of the atoms spontaneously disintegrates to form atoms of a new element. Because radioactivity is a property of the atom, Rutherford showed that the radioactivity of a substance is directly proportional to the number of atoms of the substance present.

Thus, if $N(t)$ denotes the number of atoms present at time t, then dN/dt, the number of atoms that disintegrate per unit time is proportional to N, that is,

$$\boxed{\frac{dN}{dt} = -kN} \qquad (1)$$

The constant k which is positive, is known as the decay constant of the substance. The larger k is, of course, the faster the substance decays.

One measure of the rate of disintegration of a substance is its <u>half-life</u> which is defined as the time required for half of a given quantity of radioactive atoms to decay. To compute the half-life of a substance in terms of k, assume that at time t_0, $N(t_0) = N_0$. Then the solution of the initial-value problem

$$\begin{cases} \dfrac{dN}{dt} = -kN \\ N(t_0) = N_0 \end{cases} \Rightarrow \quad N = Ce^{-kt}$$

$$N(t_0) = \underbrace{Ce^{-kt_0} = N_0}, \Rightarrow C = N_0 e^{kt_0}$$

is $$\boxed{N(t) = N_0 e^{-k(t-t_0)}}$$

or $\dfrac{N}{N_0} = e^{-k(t-t_0)}$. Taking logarithms of both sides we obtain that

$$-k(t-t_0) = \ln\left(\frac{N}{N_0}\right). \qquad (2)$$

Now if $\dfrac{N}{N_0} = \dfrac{1}{2}$ then $-k(t-t_0) = \ln\left(\frac{1}{2}\right)$ so that

$$(t-t_0) = -\frac{1}{k}\ln(2^{-1}) = \frac{\ln 2}{k} = \frac{0.6931}{k}. \qquad (3)$$

Thus, the half-life of a substance is $\ln 2$ divided by the decay constant k. The dimension of k is reciprocal time. If t is measured in years then k has the dimension of reciprocal years, and if t is measured in minutes, then k has the dimension of reciprocal minutes. The half-lives of many substances have been determined and recorded. For example, the half-life of carbon-14 (^{14}C) is 5568 years and the half-life of uranium-238 (^{238}U) is 4.5 billion years.

Radiocarbon Dating

One of the most accurate ways of dating archaeological finds is the method of carbon-14 (^{14}C) dating discovered by Willard Libby around 1949. The basis of this method is delightfully simple: The atmosphere of the earth is continuously bombarded by cosmic rays. These cosmic rays produce neutrons in the earth's atmosphere, and these neutrons combine with nitrogen to produce ^{14}C, which is usually called radiocarbon, since it decays radioactively. Now, this radiocarbon is incorporated in carbon dioxide (CO_2) and thus moves through the atmosphere to be absorbed by plants. In living tissue, the rate of ingestion of ^{14}C exactly balances the rate of disintegration of ^{14}C. When an organism dies, though, it ceases to ingest carbon-14 and thus its ^{14}C concentration begins to decrease through disintegration of the ^{14}C present. Now, it is a fundamental assumption of physics that the rate of bombardment of the earth's atmosphere by cosmic rays has always been constant. This implies that the original rate of disintegration of the ^{14}C in a sample such as charcoal is the same as the rate measured today. This assumption enables us to determine the age of a sample of charcoal.

Let $N(t)$ denote the amount of carbon-14 present in a sample at time t, and N_0 the amount present at time $t=0$ when the sample was formed. If k denotes the decay constant of ^{14}C (the half-life of carbon-14 is 5568 years) then $dN/dt = -kN(t)$, $N(0) = N_0$. Consequently, $N(t) = N_0 e^{-kt}$. Now the present rate $R(t)$ of disintegration of ^{14}C in the sample is given by

$$R(t) = kN(t) = kN_0 e^{-kt}$$

and the original rate of disintegration is

$$R(0) = k N_0.$$

Thus, $$\frac{R(t)}{R(0)} = \frac{k N_0 e^{-kt}}{k N_0} = e^{-kt}$$

so that $$t = -\frac{1}{k} \ln\left(\frac{R(t)}{R(0)}\right) = \frac{1}{k} \ln\left(\frac{R(0)}{R(t)}\right)$$

Hence if we measure $R(t)$, the present rate of disintegration of the ^{14}C in the charcoal, and observe that $R(0)$ must equal the rate of disintegration of the ^{14}C in a comparable amount of living wood, then we can compute the age t of the charcoal. The following example is real life illustration of this method.

Example 2 In the Cave of Lascaux in France there are some ancient wall paintings, believed to be prehistoric. Using a Geiger counter, the current decay rate of ^{14}C in charcoal fragments collected from the cave was measured as approximately 1.69 disintegrations per minute per gram of carbon. In comparison, for living tissue in 1950 the measurement was 13.5 disintegrations per minute per gram of carbon.
How long ago was the radioactive carbon formed and the Lascaux Cave paintings painted?

Solution (The present rate of disintegration in the sample) = 1.69 [$R(t)$].

(The original rate of disintegration when the sample was formed like in the modern living wood) = 13,5 [$R(0)$].

Recall, $$k = \frac{\ln 2}{\text{half-life}} = \frac{\ln 2}{5568}$$

so that $$t = \frac{1}{k} \ln\left(\frac{R(0)}{R(t)}\right) = \frac{5568}{\ln 2} \ln\left(\frac{13.5}{1.69}\right) = 16,692 \text{ years.}$$

Detecting Art Forgeries (Reading: Sec. 2.3 in the textbook)

The key to the dating of paintings and other materials such as rocks and fossils lies in the phenomenon of radioactivity.

Now the basis of "radioactive dating" is essentially the following. From Equation (2):

$$-k(t-t_0) = \ln\left(\frac{N}{N_0}\right)$$

we can solve for $(t-t_0) = \frac{1}{k}\ln\left(\frac{N_0}{N}\right)$.

If t_0 is the time the substance was initially formed or manufactered, then the age of the substance is $\frac{1}{k}\ln\left(\frac{N_0}{N}\right)$.

The decay constant k is known or can be computed, in most instances. Moreover, we can usually evaluate N quite easily. Thus, if we knew N_0 we could determine the age of the substance. But this is the real difficulty of course, since we usually do not know N_0. In some instances though, we can either determine N_0 indirectly, or else determine certain suitable ranges for N_0, and such is the case for the forgeries of Van Meegeren.

We begin with following well-known facts of elementary chemistry. Almost all rocks in the earth's crust contain a small amount of uranium. The uranium in the rock decays to another radioactive element, and that one decays to another and another, and so forth (see Figure 1) in a series of elements that results in lead, which is not radioactive. The uranium (whose half-life is over four billion years) keeps feeding the elements following it in the series, so that as fast as they decay, they are replaced by the elements before them.

Now, all paintings contain a small amount of the radioactive element lead-210 (^{210}Pb), and an even smaller amount of radium-226 (^{226}Ra), since these elements are contained in white lead (lead oxide), which is a pigment that artists have used for over 2000 years. For the analysis which follows, it is important to note that white lead is made from lead metal, which, in turn, is extracted from a rock called lead ore, in a process called smelting.

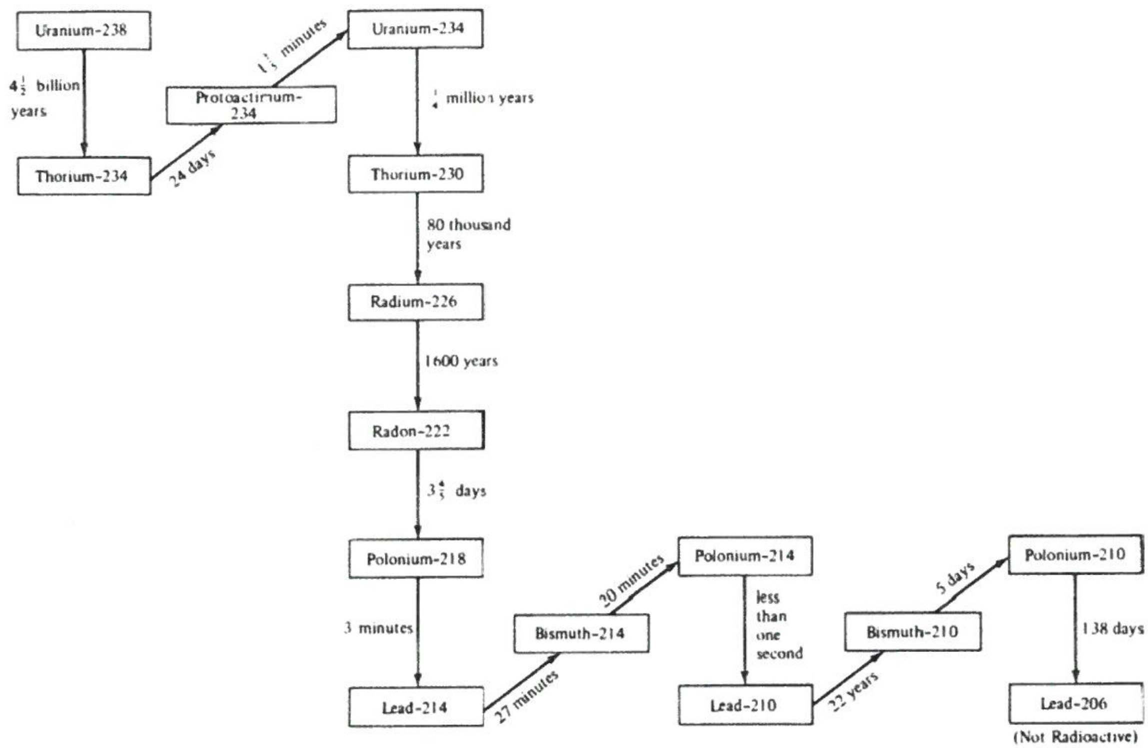


Figure 1. The Uranium series. (The times shown on the arrows are the half-lives of each step.)

In this process, the lead-210 in the ore goes along with the lead metal. However, 90-95% of the radium and its descendants are removed with other waste products in a material called slag. Thus, most of the supply of lead-210 is cut off and it begins to decay very rapidly, with a half-life of 22 years. This process continues until the lead-210 in the white lead is once more in radioactive equilibrium with the small amount of radium present, i.e. the disintegration of the lead-210 is exactly balanced by the disintegration of the radium.

Let us now use this information to compute the amount of lead-210 present in a sample in terms of the amount originally present at the time of manufacture.

Let $y(t)$ be the amount of lead-210 per gram of white lead at time t, y_0 the amount of lead-210 per gram of white lead present at the time of manufacture t_0, and $r(t)$ the number of disintegrations of radium-226 per minute per gram of white lead, at time t. If k is the decay constant for lead-210, then

$$\begin{cases} \dfrac{dy}{dt} = -ky + r(t) \\ y(t_0) = y_0 \end{cases} \qquad (4)$$

Since we are only interested in a time period of at most 300 years we may assume that the radium-226, whose half-life is 1600 years, remains constant, so that $r(t)$ is a constant r. Multiplying both sides of the differential equation by the integrating factor

$$u(t) = e^{kt}$$

we obtain that

$$\frac{d}{dt}\left(e^{kt} y\right) = r e^{kt}.$$

Hence

$$e^{kt} y(t) - e^{kt_0} y_0 = \frac{r}{k}\left(e^{kt} - e^{kt_0}\right)$$

or

$$y(t) = \frac{r}{k}\left(1 - e^{-k(t-t_0)}\right) + y_0 e^{-k(t-t_0)} \qquad (5)$$

Now $y(t)$ and r can be easily measured. Thus, if we knew y_0 we could use equation (5) to compute $(t-t_0)$ and consequently, we could determine the age of the painting. As we pointed out, though, we cannot measure y_0 directly. One possible way out of this difficulty is to use the fact that the original quantity of lead-210 was in radioactive equilibrium with the larger amount of radium-226 in the ore from which the metal was extracted. Let us, therefore, take samples of different ores and count the number of disintegrations of the radium-226 in the ores. This was done for a variety of ores and the results are given in Table 1 below. These numbers vary from 0.18 to 140. Consequently, the number of disintegrations of the lead-210 per minute per gram of white lead at the time of manufacture will vary from 0.18 to 140. This implies that y_0 will also vary over a very large interval, since the number of disintegrations of lead-210 is proportional to the amount present. Thus, we cannot use Equation (5) to obtain an accurate, or even a crude estimate, of the age of a painting.

Table 1. Ore and ore concentrate samples. All disintegration rates are per gram of white lead.

Description and Source		Disintegrations per minute of ^{226}Ra
Ore concentrate	(Oklahoma–Kansas)	4.5
Crushed raw ore	(S.E. Missouri)	2.4
Ore concentrate	(S.E. Missouri)	0.7
Ore concentrate	(Idaho)	2.2
Ore concentrate	(Idaho)	0.18
Ore concentrate	(Washington)	140.0
Ore concentrate	(British Columbia)	1.9
Ore concentrate	(British Columbia)	0.4
Ore concentrate	(Bolivia)	1.6
Ore concentrate	(Australia)	1.1

However, we can still use equation (5) to distinguish between a 17th century painting and a modern forgery. The basis for this statement is the simple observation that if the paint is very old compared to the 22 year half-life of lead, then the amount of radioactivity from the lead-210 in the paint will be nearly equal to the amount of radioactivity from the radium in the paint. On the other hand, if the painting is modern (approximately 20 years old, or so) then the amount of radioactivity from the lead-210 will be much greater than the amount of radioactivity from the radium.

We make this agrument precise in the following manner. Let us assume that the painting in question is either very very new or about 300 years old. Set

$$t - t_0 = 300 \text{ years}$$

in (5). Then, after some algebra, we see that

$$k y_0 = k y(t) e^{300k} - r(e^{300k} - 1) \qquad (6)$$

If the painting is indeed a modern forgery, then ky_0 will be absurdly large. To determine what is an absurdly high disintegration rate we observe that if the lead-210 decayed originally (at the time of manufacture) at the rate of 100 disintegrations per minute per gram of white lead, then the ore from which it was extracted had an uranium content of approximately 0.014 per cent. This is a fairly high concentration of uranium since the average amount of uranium in rocks of the earth's crust is about 2.7 parts per million. On the other hand, there are some very rare ores in the Western Hemisphere whose uranium content is 2-3 per cent. To be on the safe side, we will say that a disintegration rate of lead-210 is certainly absurd if it exceeds 30,000 disintegrations per minute per gram of white lead.

To evaluate ky_0, we must evaluate the present disintegration rate, $ky(t)$, of the lead-210, the disintegration rate r of the radium-226, and e^{300k}. Since the disintegration rate of polonium-210 (^{210}Po) equals that of lead-210 after several years, and since it is easier to measure the disintegration rate of polonium-210, we substitute these values for those of lead-210. To compute e^{300k}, we observe from (3) that

$$k = \frac{\ln(2)}{22}.$$

Hence

$$e^{300k} = e^{(300/22)\ln 2} = 2^{(150/11)}.$$

The disintegration rates of polonium-210 and radium-226 were measured for the "Disciples at Emmaus" and various other alleged forgeries and are given in Table 2 below.

Table 2. Paintings of questioned authorship. All disintegration rates are per minute, per gram of white lead.

Description	^{210}Po disintegration	^{226}Ra disintegration
"Disciples at Emmaus"	8.5	0.8
"Washing of Feet"	12.6	0.26
"Woman Reading Music"	10.3	0.3
"Woman Playing Mandolin"	8.2	0.17
"Lace Maker"	1.5	1.4
"Laughing Girl"	5.2	6.0

If we now evaluate ky_0 from (6) for the white lead in the painting "Disciples at Emmaus" we obtain that

$$ky_0 = (8.5)2^{150/11} - 0.8\left(2^{150/11} - 1\right) = 98,050$$

which is unacceptably large. Thus, this painting must be a modern forgery.

HW 2.5, 2.7, 2.12