Math 5110 Applied Linear Algebra -Fall 2020.

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Homework 4.

1. Reading: [Gockenbach], Chapters 4 .

2. Questions: (You can use Matlab if needed, e.g. eigenvalues by eig(A))

The following questions are about eigenvalues and eigenspaces.

Question 1. An $n \times n$ matrix A is called nilpotent if there exists an integer k such that $A^k = 0$. Find all possible eigenvalues of A.

Suppose λ is an eigenvalue of A. Then $A\vec{v} = \lambda\vec{v}$ for a nonzero $\vec{v}$. Then $A^k\vec{v} = \lambda^k\vec{v}$. On the other side, $A^k\vec{v} = 0$. So, $\lambda^k\vec{v} = 0$ for a non-zero $\vec{v}$. So, $\lambda^k = 0$. So $\lambda = 0$.

Question 2. Let $A \in \mathbb{R}^{2\times 2}$ defined by

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

where $a, b, c \in \mathbb{R}$. (Notice that A is symmetric, that is, $A^T = A$.)

(1) Prove that A has only real eigenvalues.

(2) Under what conditions on a, b, c does A have a multiple eigenvalue?

(1) $\det(A - \lambda I) = \begin{vmatrix} a-\lambda & b \\ b & c-\lambda \end{vmatrix} = (a-\lambda)(c-\lambda) - b^2 = \lambda^2 - (a+c)\lambda + ac - b^2.$
By quadratic polynomial, $\Delta := (a+c)^2 - 4(ac-b^2) = (a-c)^2 + b^2 \geq 0$. So, A only has real eigenvalues.
(We will see that any symmetric matrix only has real eigenvalues. But the proof is much harder.)
(2) A only has a multiple eigenvalue if and only if $\Delta = 0$. That is $a = c$ and $b = 0$.

Question 3. Let $A \in \mathbb{F}^{n\times n}$ be an invertible matrix. Show that every eigenvector of A is also an eigenvector of A^{-1}. What is the relationship between the eigenvalues of A and A^{-1}?

Suppose $A\vec{v} = k\vec{v}$. (Here $k \neq 0$, since A is invertible and $\vec{v}$ is not zero vector.)
Then $A^{-1}A\vec{v} = A^{-1}k\vec{v}$. That is $A^{-1}\vec{v} = \frac{1}{k}\vec{v}$.
So, $\vec{v}$ is an eigenvector of A^{-1} corresponding to eigenvalue $\dfrac{1}{k}$

Question 4. Suppose that A is a square matrix with real entries and real eigenvalues. Prove that every eigenvalue of A has an associated real eigenvector.

Let c be an eigenvalue of A. The eigenvectors associated with c are the non-trivial solutions of the linear system $(A - cI)\vec{x} = 0$. Since A and c are real, this system has non-trivial real solutions.

Question 5. For each of the following real matrices, find the eigenvalues and a basis for each eigenspace. (Use Matlab)

(1) $A = \begin{bmatrix} -15 & 0 & 8 \\ 0 & 1 & 0 \\ -28 & 0 & 15 \end{bmatrix}$

(2) $B = \begin{bmatrix} -4 & -4 & -5 \\ -6 & -2 & -5 \\ 11 & 7 & 11 \end{bmatrix}$

(3) $C = \begin{bmatrix} 6 & -1 & 1 \\ 4 & 1 & 1 \\ -12 & 3 & -1 \end{bmatrix}$

(4) $D = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$

(5) $E = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

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A=[-15 0 8; 0 1 0; -28 0 15]
B=[-4 -4 -5; -6 -2 -5; 11 7 11]
C=[6 -1 1; 4 1 1; -12 3 -1]
D=[1 2; -2 1]
E=[0 1 1; 1 0 1; 1 1 0]
A=sym(A)
B=sym(B)
C=sym(C)
D=sym(D)
E=sym(E)
[VA,DA]=eig(A)
[VB,DB]=eig(B)
[VC,DC]=eig(C)
[VD,DD]=eig(D)
[VE,DE]=eig(E)
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Output: (The first matrix contains eigenvectors, the second matrix contains eigenvalues.)

$$\begin{pmatrix} \frac{4}{7} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} -\frac{5}{9} & -\frac{1}{2} \\ -\frac{5}{9} & -\frac{1}{2} \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{4} & -\frac{1}{4} \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\begin{pmatrix} \mathrm{i} & -\mathrm{i} \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1-2\,\mathrm{i} & 0 \\ 0 & 1+2\,\mathrm{i} \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Question 6. Which matrices in the above question are diagonalizable? If it is diagonalizable, find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

Matrices A, D, E are diagonalizable.
The first matrix is P the second matrix is D for each example.

Question 7. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$T(\vec{x}) = \begin{bmatrix} x_2 + x_3 \\ x_1 + x_3 \\ x_1 + x_2 \end{bmatrix}.$$

Find a basis $\mathscr{B}$ for $\mathbb{R}^3$ such that $[T]_{\mathscr{B},\mathscr{B}}$ is diagonal. What is the matrix $[T]_{\mathscr{B},\mathscr{B}}$?

The matrix A of the transformation T is $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. That is $T(\vec{x}) = A\vec{x}$.

From (5) of the above question, the basis $\mathscr{B}$ is the eigenbasis given by $\mathscr{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

$[T]_{\mathscr{B},\mathscr{B}} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

Question 8. Suppose $A \in \mathbb{F}^{m \times n}$ and $B \in \mathbb{F}^{n \times m}$ and $n \geq m$.

(1) Show that AB and BA has the same non-zero eigenvalues with the same algebraic multiplicities.

(2) If 0 is an eigenvalue of AB with algebraic multiplicity k, what is the algebraic multiplicity of 0 as eigenvalue of BA.

(1) Suppose $AB\vec{v} = \lambda\vec{v}$ such that $\lambda \neq 0$ and $\vec{v} \neq \vec{0}$. Then $BAB\vec{v} = B\lambda\vec{v} = \lambda B\vec{v}$. Notice that $B\vec{v} \neq 0$. So, $B\vec{v}$ is an eigenvector of AB with eigenvalue λ.
Similar, any nonzero eigenvalue of BA is an eigenvalue of of AB.
(2) The rest are zero eigenvalues. So, $k + n - m$.

Question 9. (1) Find the characteristic polynomial of $B = \begin{bmatrix} 0 & -c_0 \\ 1 & -c_1 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 0 & -c_0 \\ 1 & 0 & -c_1 \\ 0 & 1 & -c_2 \end{bmatrix}$.

(2) Shows that every monic polynomial

$$f(t) = t^n + c_{n-1}t^{n-1} + \cdots + c_1 t + c_0$$

is the characteristic polynomial of some matrix B. (Hint: look at (1))

Let $B = \begin{bmatrix} 0 & 0 & \ldots & 0 & -c_0 \\ 1 & 0 & \ldots & 0 & -c_1 \\ 0 & 1 & \ldots & 0 & -c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & -c_{n-1} \end{bmatrix}$. Then $\det(B - tI) = f(t)$

The following two questions are about Cayley-Hamilton Theorem and Jordan normal forms.

Question 10. Let A and B be 2×2 matrices such that $(AB)^2 = \mathbf{0}$. Prove that $(BA)^2 = \mathbf{0}$.

The characteristic polynomial of AB is $f_{AB}(t) = t^2 - (tr(AB))t + \det(AB)$.
The characteristic polynomial of BA is $f_{BA}(t) = t^2 - (tr(BA))t + \det(BA)$.
We know that $tr(AB) = tr(BA)$ and $\det(AB) = \det(BA)$. Since $(AB)^2 = \mathbf{0}$. We have $\det(AB) = \det(BA) = 0$.
By Cayley-Hamilton Theorem, $(AB)^2 - (tr(AB))AB + \det(AB) = \mathbf{0}$. So $(tr(AB))AB = \mathbf{0}$. So $(tr(AB))$ must equal to 0.
By Cayley-Hamilton Theorem, $(BA)^2 - (tr(BA))BA + \det(BA) = \mathbf{0}$. So $(BA)^2 = (tr(BA))BA = \mathbf{0}$

Question 11. (1) Let A be a 3×3 matrix such that the traces $tr(A^k)$=0 for $k = 1, 2, 3$. Show that all eigenvalues of A are zeros.

(2) Is there a 3×3 nilpotent matrix such that $A^3 \neq \mathbf{0}$?

(1)
Suppose the eigenvalues of A are λ_1, λ_2, and λ_3.
The matrix A has Jordan canonical form by $A = PJP^{-1}$, where $J = \begin{bmatrix} \lambda_1 & * & 0 \\ 0 & \lambda_2 & * \\ 0 & 0 & \lambda_3 \end{bmatrix}$ with $* = 0$ or 1.
$tr(A^k)$=0 for $k = 1, 2, 3$.
So, $\begin{cases} \lambda_1 + \lambda_2 + \lambda_3 = 0 \\ \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 0 \\ \lambda_1^3 + \lambda_2^3 + \lambda_3^3 = 0 \end{cases}$ This means that, $\begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \\ \lambda_1^3 & \lambda_2^3 & \lambda_3^3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. This means that the $A\vec{x} = \vec{0}$ has non-trivial solutions. So, $\det(A) = \begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{vmatrix} = 0$. By Vandermonde matrix, $\det(A) = \lambda_1\lambda_2\lambda_3(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3) = 0$. So, it is impossible that they all distinct and non-zero.
Similarly, if there is a zero eigenvalue, we can consider the rest use the same method. If two eigenvalues are the same, we can group them together as $2\lambda_i^k$ in the linear system. In each situation, the size of the matrix will smaller than 3×3.
We can show that it is impossible to have distinct and non-zero eigenvalues. Keep reduce the size of the matrix we can show that all eigenvalues must be zeros.
(2) If A is nilpotent, then A only has zero eigenvalues. So, the characteristic polynomial of A is $f_A(t) = -t^3$. So, by Cayley-Hamilton Theorem $-A^3 = -\mathbf{0}$.

(Remark for (1): Actually, after we analyzed the problem, we can prove the problem for an $n\times n$ matrix. When we start the writing, we can consider all non-zero, distinct eigenvalues λ_1, ..., λ_s with algebraic multiplicity $k_1, ..., k_s \geq 1$ and show that this is impossible. The writing will be clear.)

Question 12. Consider the matrix

$$A = \begin{bmatrix} -3 & 4 & 4 \\ -5 & 9 & 5 \\ -7 & 4 & 8 \end{bmatrix}$$

The aim is to find a matrix $M \in \mathbb{R}^{3\times3}$ such that $M^2 = A$ (a "square root" of A).

(1) Find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

The characteristic polynomial of A is $p(\lambda) = -\lambda^3 + 14\lambda^2 - 49\lambda + 36$. An obvious root of this polynomial is 1, and we can factorize $p(\lambda) = -(\lambda - 1)(\lambda - 4)(\lambda - 9)$, which gives us the eigenvalues $1, 4, 9$.

We use Gaussian elimination to compute eigenspace $E_1 = \ker(A - 1I)$, and we get $E_1 = \operatorname{Span}\begin{bmatrix}1\\0\\1\end{bmatrix}$.

Similarly, we get $E_4 = \operatorname{Span}\begin{bmatrix}0\\1\\-1\end{bmatrix}$ and $E_9 = \operatorname{Span}\begin{bmatrix}1\\2\\1\end{bmatrix}$. We then define the invertible matrix P and the diagonal matrix D as

$$P = \begin{bmatrix}1 & 0 & 1\\0 & 1 & 2\\1 & -1 & 1\end{bmatrix}, \qquad D = \begin{bmatrix}1 & 0 & 0\\0 & 4 & 0\\0 & 0 & 9\end{bmatrix}$$

so that $A = PDP^{-1}$.

(2) Let M be in $\mathbb{R}^{3\times 3}$ and let us assume that $M^2 = A$. Let us consider $N = P^{-1}MP$. Show that $N^2 = D$. Then prove that N commutes with D, i.e., $ND = DN$.

Exploiting the associativity of matrix multiplication, we obtain

$$N^2 = (P^{-1}MP)(P^{-1}MP) = P^{-1}M(PP^{-1})MP = P^{-1}M^2P = P^{-1}AP = D$$

and, therefore,

$$ND = N(N^2) = N^3 = (N^2)N = DN.$$

(3) Explain that N is thus necessarily diagonal.
Hint: Note that all the diagonal values of D are distinct.

Intuitively, as D is diagonal, the product ND multiplies the columns of N while DN multiplies the rows of N. But as $ND = DN$, and D has different values on the diagonal, then N has to be diagonal. Let us prove this result formally.

Let us denote by $n_{i,j}$ the coefficient of matrix N at row i and column j and let d_i denote the i^{th} coefficient on the diagonal of D. Note that in our example, i and j will be ranged in $\{1, 2, 3\}$, but this result extends to matrices of arbitrary size. Let i and j be in $\{1, 2, 3\}$. The coefficient of ND at row i and column j is equal to $n_{i,j}d_j$, while that of DN is equal to $d_i n_{i,j}$. The matrix equality $ND = DN$ yields

$$\forall i, j \in \{1, 2, 3\}\colon\ n_{i,j}d_j = n_{i,j}d_i,$$

i.e.,

$$\forall i, j \in \{1, 2, 3\}\colon\ n_{i,j}(d_j - d_i) = 0. \tag{1}$$

In general, a product is null if and only if at least one of its factors is null. But as all the values on the diagonal of D are different, (1) is equivalent to

$$\forall i, j \in \{1, 2, 3\}\colon\ (i \neq j) \implies (n_{i,j} = 0),$$

which ensures that N is diagonal. Note that if two values on the diagonal of D were equal, N would not necessarily be diagonal and we would have infinitely many candidates for N, and thus as many for M.

(4) What can you say about N's possible values? Compute a matrix M, whose square is equal to A. How many different such matrices are there?

We can write N as $N = \mathrm{diag}(n_1, n_2, n_3)$ and $N^2 = D$ requires that $n_1^2 = 1, n_2^2 = 4$ and $n_3^2 = 9$. As all diagonal values are positive, we have exactly two distinct square roots for each one. Therefore, we have 8 possible values for N that we gather in the following set:

$$\{\mathrm{diag}(n_1, n_2, n_3) \mid n_1 \in \{-1, +1\}, n_2 \in \{-2, +2\}, n_3 \in \{-3, +3\}\}.$$

Now, let us set $N = \mathrm{diag}(1, 2, 3)$ and compute the product $M = PNP^{-1}$. First, Gaussian elimination gives us

$$P^{-1} = \frac{1}{2}\begin{bmatrix} 3 & -1 & -1 \\ 1 & 0 & -1 \\ -1 & 1 & 1 \end{bmatrix},$$

and we find one square root of A as

$$M = PNP^{-1} = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 3 & 1 \\ -2 & 1 & 3 \end{bmatrix}.$$

We can check that M^2 indeed equals A. We can choose amongst the 8 different possible values of N to find a new square root of A. Hence, there are equally many different such matrices M.