MTH 7241: Fall 2022

First Practice Problems for Test 1

1). 3 balls are distributed in 3 boxes. At each step, one of the balls is selected at random, taken out of whichever box it is in, and moved at random to one of the other boxes. Let X_n be the number of balls in the first box, after n steps.

a). Find the transition matrix of the chain $X_0, X_1, \ldots$.

b). Find the stationary distribution of the chain.

2). Consider the following transition probability matrix for a Markov chain on 4 states:

$$P = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Number the states $\{1, 2, 3, 4\}$ in the order presented.

a). Find and classify the equivalence classes of the states (irreducible and transient).

b). Find a stationary distribution for the chain.

3). Suppose that coin 1 has probability 0.7 of coming up Heads, and coin 2 has probability 0.4 of coming up Heads. If the coin tossed today comes up Heads, then we select coin 1 to toss tomorrow, and if it comes up Tails, then we select coin 2 to toss tomorrow. If the coin initially tossed is equally likely to be coin 1 or coin 2, then what is the probability that the coin tossed on the third day after the initial toss is coin 1?

4) Four balls are shared between box #1 and box #2. At each step a biased coin is tossed which comes up Heads with probability p. If the coin comes up Heads and box #1 is not empty, a ball is removed from box #1 and placed in box #2. If the coin comes up Heads and box #1 is empty, no balls are moved. If the coin comes up Tails and box #2 is not empty, a ball is removed from box #2 and placed in box #1. If the coin comes up Tails and box #2 is empty, no balls are moved. Let X_n be the number of balls in box #1 after n steps.

Find the transition matrix for the Markov chain $\{X_n\}$ (your answer will depend on p).

5) Consider the following transition probability matrix for a Markov chain on 5 states:

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Number the states $\{1, 2, 3, 4, 5\}$ in the order presented.

Given that the chain starts in state 1, find the expected number of steps until the first return to state 1.

6) Let $\{X_n\}$ be a Markov chain, and suppose that for state i we have

$$\sum_{k=1}^{n} p_{ii}(k) = \sum_{k=1}^{n} \mathrm{P}(X_k = i \,|\, X_0 = i) = 3 - \frac{9}{\sqrt{n+8}} \quad \text{for all } n \geq 1.$$

Determine whether state i is transient or persistent (explain your reasoning).

7) Consider an irreducible chain on 3 states. **Either** prove that $p_{jj}(6) > 0$ for every state j, **or** give an example where $p_{jj}(6) = 0$ for some state j.