**Math 5110 Applied Linear Algebra**

**He Wang**

he.wang@northeastern.edu

**Eigenvalues and Eigenspaces**

**Question 1.** An $n \times n$ matrix $A$ is called nilpotent if there exists an integer $k$ such that $A^k = 0$. Find all possible eigenvalues of $A$.

Suppose $\lambda$ is an eigenvalue of $A$. Then $A\vec{v} = \lambda\vec{v}$ for a nonzero $\vec{v}$. Then $A^k\vec{v} = \lambda^k\vec{v}$. On the other side, $A^k\vec{v} = 0$. So, $\lambda^k\vec{v} = 0$ for a non-zero $\vec{v}$. So, $\lambda^k = 0$. So $\lambda = 0$.

**Question 2.** Let $A \in \mathbb{R}^{2\times 2}$ defined by

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

where $a, b, c \in \mathbb{R}$. (Notice that $A$ is symmetric, that is, $A^T = A$.)

(1) Prove that $A$ has only real eigenvalues.

(2) Under what conditions on $a, b, c$ does $A$ have a multiple eigenvalue?

(1) $\det(A - \lambda I) = \begin{vmatrix} a-\lambda & b \\ b & c-\lambda \end{vmatrix} = (a-\lambda)(c-\lambda) - b^2 = \lambda^2 - (a+c)\lambda + ac - b^2$.
By quadratic polynomial, $\Delta := (a+c)^2 - 4(ac - b^2) = (a-c)^2 + b^2 \geq 0$. So, $A$ only has real eigenvalues.
(We will see that any symmetric matrix only has real eigenvalues. But the proof is much harder.)
(2) $A$ only has a multiple eigenvalue if and only if $\Delta = 0$. That is $a = c$ and $b = 0$.

**Question 3.** Let $A \in \mathbb{F}^{n\times n}$ be an invertible matrix. Show that every eigenvector of $A$ is also an eigenvector of $A^{-1}$. What is the relationship between the eigenvalues of $A$ and $A^{-1}$?

Suppose $A\vec{v} = k\vec{v}$. (Here $k \neq 0$, since $A$ is invertible and $\vec{v}$ is not zero vector. )
Then $A^{-1}A\vec{v} = A^{-1}k\vec{v}$. That is $A^{-1}\vec{v} = \frac{1}{k}\vec{v}$.
So, $\vec{v}$ is an eigenvector of $A^{-1}$ corresponding to eigenvalue $\dfrac{1}{k}$

**Question 4.** Suppose that A is a square matrix with real entries and real eigenvalues. Prove that every eigenvalue of A has an associated real eigenvector.

Let $c$ be an eigenvalue of $A$. The eigenvectors associated with $c$ are the non-trivial solutions of the linear system $(A - cI)\vec{x} = 0$. Since A and c are real, this system has non-trivial real solutions.