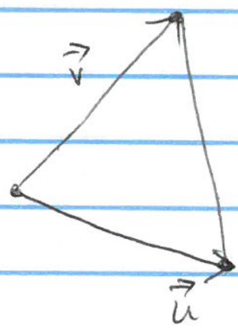
MATH 5110 Worksheet 2: SOLUTIONS

(1). $\det\left(3\vec{c}_1 \;\; \vec{c}_2 \;\; -4\vec{c}_3\right) = (3)(-4)\det\left(\vec{c}_1 \;\; \vec{c}_2 \;\; \vec{c}_3\right) = -24$

$$\det\left(\vec{c}_3 \;\; \vec{c}_2 \;\; \vec{c}_1\right) = -\det\left(\vec{c}_2 \;\; \vec{c}_3 \;\; \vec{c}_1\right) = \det\left(\vec{c}_2 \;\; \vec{c}_1 \;\; \vec{c}_3\right)$$
$$= -\det\left(\vec{c}_1 \, \vec{c}_2 \, \vec{c}_3\right) = -2$$

$$\det\left(\vec{c}_1 - 2\vec{c}_2 + \vec{c}_3 \quad \vec{c}_2 - \vec{c}_1 \quad 5\vec{c}_3\right) = 5\det\left(\vec{c}_1 - 2\vec{c}_2 \quad \vec{c}_2 - \vec{c}_1 \quad \vec{c}_3\right)$$
$$= 5\det\left(\vec{c}_1 \;\; \vec{c}_2 \;\; \vec{c}_3\right) - 10\det\left(\vec{c}_2 \;\; -\vec{c}_1 \;\; \vec{c}_3\right)$$
$$= 10 - 10(2) = -10$$

(2)



$$\vec{u} = (3,-2) - (1,1) = (2,-3)$$
$$\vec{v} = (2,6) - (1,1) = (1,5)$$
$$\text{Area} = \frac{1}{2}\left|\det\left(\vec{u} \;\; \vec{v}\right)\right| = \frac{1}{2}\det\begin{pmatrix} 2 & 1 \\ -3 & 5 \end{pmatrix} = \frac{13}{2}$$

(3) $$A = \begin{pmatrix} 0 & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & 0 \end{pmatrix} \qquad \det A = -\det\begin{pmatrix} 0 & & 1 \\ & 1 & \\ 1 & & 0 \end{pmatrix} = -\det\begin{pmatrix} 0 & & 1 \\ & 1 & \\ 1 & & 0 \end{pmatrix}$$
$$= \det\begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix} = \det\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -$$

(4) $\det(AB) = \det A \det B = \det BA^2 = \det B\,(\det A)^2$

B non-singular $\Rightarrow \det B \neq 0 \Rightarrow \det A = (\det A)^2 \Rightarrow \det A = \begin{cases} 0 \\ 1. \end{cases}$

(5) $\det(S) = \det(S^T) = \det(-S) = (-1)^3 \det S = -\det S$

$$\Rightarrow \det S = 0$$

6. $\det A = \det(SDS^{-1}) = \det S \det D \, (\det S)^{-1}$

$= \det D = (2)(3)(-1)(-1) = 6$

7. $\lambda I - A = \begin{pmatrix} \lambda-1 & 0 & 0 \\ 0 & \lambda-1 & -3 \\ 0 & 0 & \lambda-2 \end{pmatrix} \qquad p(\lambda) = (\lambda-1)^2(\lambda-2)$

$\Rightarrow$ eigenvalues $\{1, 2\}$

$\underline{\underline{\lambda=1}}$: algebraic mult. $= 2$,

eigenspace $= \text{null}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -3 \\ 0 & 0 & -1 \end{pmatrix} = \text{Span}\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}$

$\Rightarrow$ geom. mult. $= 2$.

$\underline{\underline{\lambda=2}}$: alg. mult. $= 1 \Rightarrow$ geom. mult. $= 1$.

8. $S = (\vec{V}_1 \ \vec{V}_2) = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \quad ; \quad D = \text{diag}(\lambda_1, \lambda_2) = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$

9. $A = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ -1 & \frac{3}{2} \end{pmatrix} \qquad \lambda I - A = \begin{pmatrix} \lambda+\frac{1}{2} & -\frac{1}{2} \\ 1 & \lambda-\frac{3}{2} \end{pmatrix}$

$p(\lambda) = (\lambda+\tfrac{1}{2})(\lambda-\tfrac{3}{2}) + \tfrac{1}{2} = \lambda^2 - \lambda - \tfrac{1}{4} = 0.$

$\lambda = \frac{1 \pm \sqrt{1+1}}{2} = \frac{1 \pm \sqrt{2}}{2},$

$\lambda = \frac{1 \pm \sqrt{2}}{2}$: $\begin{pmatrix} \lambda+\frac{1}{2} & -\frac{1}{2} \\ 1 & \lambda-\frac{3}{2} \end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \Rightarrow (\lambda+\tfrac{1}{2})x - \tfrac{1}{2}y = 0$

$y = (2\lambda+1)x$

$\lambda_1 = \frac{1+\sqrt{2}}{2}$; $y = (2+\sqrt{2})x \Rightarrow \vec{v}_1 = k\begin{pmatrix} 1 \\ 2+\sqrt{2} \end{pmatrix}$