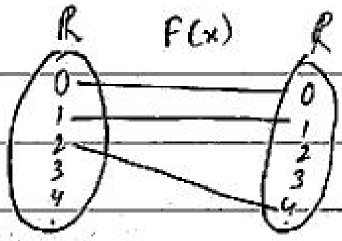
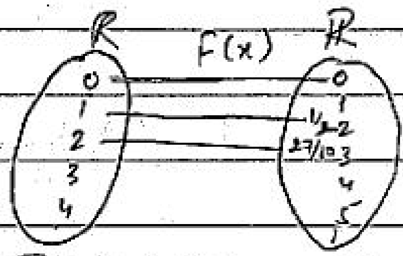
MATH 5110 HOMEWORK - 1. ABHILASHA JAIN

Q1. (a) $F(x) = x^2$



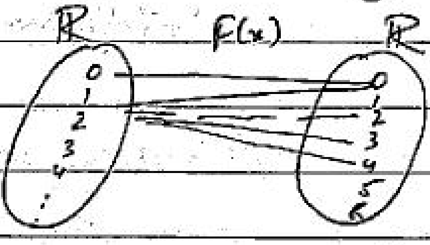
Surjective

(b) $F(x) = x^3/(x^2+1)$



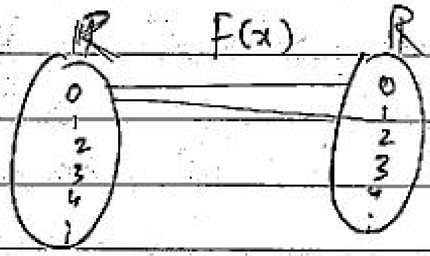
Injective

(c) $F(x) = x(x-1)(x-2)$



Surjective

(d) $F(x) = e^x + 2$



Bijective

Q2. 1. Given $\mathbb{Q}[\sqrt{2}] = \{ a + b\sqrt{2} : a, b \in \mathbb{Q} \}$

(a) Closure Property : Let $a_1, b_1, a_2, b_2 \in \mathbb{Q}[\sqrt{2}]$

then, $x = a_1 + b_1\sqrt{2} \in \mathbb{Q}[\sqrt{2}]$

and $y = a_2 + b_2\sqrt{2} \in \mathbb{Q}[\sqrt{2}]$

Now,

$$x + y = (a_1 + b_1\sqrt{2}) + (a_2 + b_2\sqrt{2})$$
$$= (a_1 + a_2) + (b_1 + b_2)\sqrt{2} \in \mathbb{Q}[\sqrt{2}]$$

$\because$ $a_1, a_2 \in \mathbb{Q}[\sqrt{2}]$ then $a_1 + a_2 \in \mathbb{Q}[\sqrt{2}]$

& $b_1, b_2 \in \mathbb{Q}[\sqrt{2}]$ then $b_1 + b_2 \in \mathbb{Q}[\sqrt{2}]$

Hence, $x, y \in \mathbb{Q}[\sqrt{2}]$ then $x + y \in \mathbb{Q}[\sqrt{2}]$

Now, $x \cdot y = (a_1 + b_1\sqrt{2}) \times (a_2 + b_2\sqrt{2})$

$= (a_1 \times a_2 + 2 b_1 \times b_2) + (a_1 \times b_2 + b_1 \times a_2)\sqrt{2} \in \mathbb{Q}[\sqrt{2}]$

So, both $+$ and $\times$ are closed on $\mathbb{Q}[\sqrt{2}]$

Field $(\mathbb{Q}[\sqrt{2}], +, \times)$ is a subfield of $(\mathbb{R}, +, \times)$

(b) Associativity: As addition and multiplication are associative on $\mathbb{R}$
Hence they are also associative on $\mathbb{Q}[\sqrt{2}]$

(c) Commutativity: As addition and multiplication are commutative on $\mathbb{R}$, $\therefore$ they are also commutative on $\mathbb{Q}[\sqrt{2}]$

(d) Identity: We have,

$$(a + b\sqrt{2}) + (0 + 0\sqrt{2}) = (a+0) + (b+0)\sqrt{2}$$
$$= a + b\sqrt{2}$$

and similarly for $(0 + 0\sqrt{2}) + (a + b\sqrt{2})$

So $(0 + 0\sqrt{2})$ is the identity for $+$ on $\mathbb{Q}[\sqrt{2}]$

Then:

$$(a + b\sqrt{2}) \times (1 + 0\sqrt{2}) = (a \times 1 + 2 \times b \times 0) + (b \times 1 + a \times 0)\sqrt{2}$$
$$= a + b\sqrt{2}$$

and similarly for $(1 + 0\sqrt{2}) \times (a + b\sqrt{2})$

So, $(1 + 0\sqrt{2})$ is the identity for $\times$ on $\mathbb{Q}[2]$

(e) Distributivity: As multiplication distributes over addition on $\mathbb{R}$, hence, $\times$ is distributive over $+$ on $\mathbb{Q}[\sqrt{2}]$

(f.) Inverses: We have,

$$(a+b\sqrt{2}) + (-a+(-b)\sqrt{2}) = (a-a) + (b-b)\sqrt{2}$$
$$= 0 + 0\sqrt{2}$$

and similarly for $(-a+(-b)\sqrt{2}) + (a+b\sqrt{2})$

So, $(-a+(-b)\sqrt{2})$ is the inverse of $(a+b\sqrt{2})$ for $+$ on $\mathbb{Q}[\sqrt{2}]$

We have,

$$(a+b\sqrt{2})(a-b\sqrt{2}) = a^2 - 2b^2$$
$$\Rightarrow (a+b\sqrt{2})\left(\frac{a-b\sqrt{2}}{a^2-2b^2}\right) = 1 = 1 + 0\sqrt{2}$$

The product inverse of $a+b\sqrt{2}$ is $\frac{a}{a^2-2b^2} - \frac{b\sqrt{2}}{a^2-2b^2}$

As a & b are rational, $\therefore \frac{a}{a^2-2b^2}$ and $\frac{b}{a^2-2b^2}$

So, the product inverse of $(a+b\sqrt{2})$ is an element of $\mathbb{Q}[\sqrt{2}]$.

Q2. 2. $C = \{a + b\sqrt{-1} : a, b \in \mathbb{R}\}$. $\{\because \sqrt{-1} = i$
$= a + bi$

a. Closure: Let $a_1, b_1, a_2, b_2 \in C$

then $x = a_1 + ib_1 \in C$

$y = a_2 + ib_2 \in C$

Now, $x+y = (a_1+ib_1) + (a_2+ib_2)$
$= (a_1+a_2) + i(b_1+b_2) \in C.$

$\because a_1, a_2 \in C$ then $a_1 + a_2 \in C$

& $b_1, b_2 \in C$ then $b_1 + b_2 \in C$

$\therefore x, y \in C$ then $x+y \in C.$

Now $x \cdot y = (a_1 + ib_1)(a_2 + ib_2)$

$= ((a_1 \cdot a_2) + \cancel{i} (b_1 \cdot b_2)) + (a_1 b_2 + b_1 a_2) i \in C$

So both $+$ & $\times$ are closed on C.

b. Identity: We have,

$$(a + ib) + (0 + 0i) = (a+0) + (b+0)i$$
$$= a + ib$$

& similarly for $(0 + 0i) + (a + ib)$

So, $(0 + 0i)$ is the identity for $+$ on C.

Then,

$$(a + ib) \times (1 + 0i) = (a \times 1 + \cancel{i} b \times 0) + (b \times 1 + a \times 0) i$$
$$= a + ib$$

& similarly for $(1 + i0) \times (a + ib)$

So, $(1 + 0i)$ is the identity for $\times$ on C.

c. Inverse: We have

$$(a + ib) + (-a + i(-b)) = (a - a) + (b - b)i$$
$$= 0 + 0i$$

& similarly for $(-a + i(-b)) + (a + ib)$

So, $(-a + (-b)i)$ is the inverse of $(a + bi)$ for $+$ on C

We have

$$(a + bi)(a - bi) = a^2 - b^2$$
$$\Rightarrow (a + bi)\left(\frac{a - bi}{a^2 - b^2}\right) = 1 = 1 + 0i$$

$\therefore$ The product of inverse of $a + bi$ is $\frac{a}{a^2 - b^2}$ $\frac{bi}{a^2 - b^2}$

d. Commutativity: $x = (a_1 + b_1 i)$

$y = (a_2 + b_2 i)$

$$x+y = (a_1 + b_1 i) + (a_2 + b_2 i)$$
$$= (a_1 + a_2) + (b_1 + b_2) i$$
$$= (a_2 + a_1) + (b_2 + b_1) i$$
$$= (a_2 + b_2 i) + (a_1 + b_1 i)$$
$$= y + x.$$

$\therefore$ + is commutative on C

$$x.y = (a_1 + b_1 i).(a_2 + b_2 i)$$
$$= a_1 a_2 + a_1 b_2 i + a_2 b_1 i + b_1 b_2$$
$$= (a_1 a_2 + b_1 b_2) + (a_1 b_2 + b_1 a_2) i$$

also,

$$y.x = (a_2 + b_2 i)(a_1 + b_1 i)$$
$$= a_1 a_2 + b_1 a_2 i + a_1 b_2 i + b_1 b_2$$
$$= (a_1 a_2 + b_1 b_2) + (a_1 b_2 + b_1 a_2) i$$

$\therefore x.y = y.x$

$\therefore$ x is commutative on C

e. Associativity. $x = (a_1 + b_1 i)$; $y = (a_2 + b_2 i)$; $z = (a_3 + b_3 i)$

$$x + (y+z) = (a_1 + b_1 i) + [(a_2 + b_2 i) + (a_3 + b_3 i)]$$
$$= (a_1 + b_1 i) + [(a_2 + a_3) + (b_2 + b_3) i]$$
$$= (a_1 + (a_2 + a_3)) + (b_1 + (b_2 + b_3)) i$$
$$= [(a_1 + a_2) + (b_1 + b_2) i] + (a_3 + b_3 i)$$
$$= [(a_1 + b_1 i) + (a_2 + b_2 i)] + (a_3 + b_3 i)$$
$$= (x + y) + z.$$

$\therefore$ + is ~~associative~~ ~~on~~ associative on C.

$$x(yz) = x[(a_2+b_2i)(a_3+b_3i)]$$

$$= (a_1+b_1i)[(a_2a_3+b_2b_3)+(a_2b_3+b_2a_3)i]$$

$$= [(a_1a_2a_3+a_1b_2b_3)+(b_1a_2b_3+b_1b_2a_3)]+$$
$$[(a_1a_2b_3+a_1b_2a_3)+(b_1a_2a_3+b_1b_2b_3)]i$$

$$(x\cdot y)z = [(a_1+b_1i)(a_2+b_2i)](a_3+b_3i)$$

$$= [(a_1a_2+b_1b_2)+(a_1b_2+b_1a_2)i](a_3+b_3i)$$

$$= [(a_1a_2a_3+b_1b_2a_3)+(a_1b_2b_3+b_1a_2b_3)]+$$
$$[(a_1a_2b_3+b_1b_2b_3)+(a_1b_2a_3+b_1a_2a_3)]i$$

$\therefore x(y\cdot z) = (x\cdot y)z$.

$\therefore$ X is associative on C.

$\therefore$ C is a field. (Unordered field)

Q3. The set of all $n\times n$ matrices over $\mathbb{R}$ is not a field

let A be an $n\times n$ matrix

then its inverse is given by

$$A^{-1} = \frac{1}{|A|}\,\text{adj}.A. \quad \text{where } |A| \text{ is } \det A.$$

if $|A| = 0 \Rightarrow A^{-1}$ doesn't exist.

&, there exist non-zero $n\times n$ matrices whose determinant is zero.

$\Rightarrow \exists$ $n\times n$ matrix whose inverse doesn't exist.

Hence, set of all $n\times n$ matrix is not field. [$\because$ for a field we need inverse of each element]

Q4.

+	[0]	[1]	[2]
[0]	[0]	[1]	[2]
[1]	[1]	[2]	[0]
[2]	[2]	[0]	[1]

X	[0]	[1]	[2]
[0]	[0]	[0]	[0]
[1]	[0]	[1]	[2]
[2]	[0]	[2]	[1]

Q5. we have $x = a_1 + b_1 i \in C$

$y = a_2 + b_2 i \in C$

$$x + y = (a_1 + b_1 i) + (a_2 + b_2 i)$$
$$= (a_1 + a_2) + (b_1 + b_2) i \quad \in C$$
$$= (a_2 + b_2 i) + (a_1 + b_1 i)$$
$$= y + x.$$

$$x \cdot y = (a_1 + b_1 i) \cdot (a_2 + b_2 i)$$
$$= (a_1 a_2 - b_1 b_2) + (a_1 b_2 + b_1 a_2) i \quad \in C$$
$$y \cdot x = (a_2 + b_2 i) \cdot (a_1 + b_1 i)$$
$$= (a_1 a_2 - b_1 b_2) + (a_1 b_2 + b_1 a_2) i$$

$\therefore x \cdot y = y \cdot x.$

$\therefore$ + and X are commutative over C. and + and X are also closure over C.

$(0 + 0i) + (a + bi) = (0 + a) + (0 + b) i = a + bi$

$\therefore$ C has an additive identity.

$$(a + ib) \cdot (1 + 0i) = (a \times 1 + b \times 0) + (b \times 1 + a \times 0) i$$
$$= a + bi$$

$\therefore$ C has an multiplicative identity.

$(a+bi) + (-a+i(-b)) = (a-a) + (b-b)i = 0+0i$

$\therefore (-a+(-b)i)$ is the inverse of $(a+bi)$ for $+$ on C.

$(a+bi) \cdot (a-bi) = a^2 - b^2$

$(a+bi) \left(\frac{a-bi}{a^2-b^2}\right) = 1 = 1+0 \cdot i$

$\therefore \frac{a}{a^2-b^2} - \frac{b}{a^2-b^2} i$ is the product inverse of $a+bi$

$\therefore C$ is a field.

Q6.

(a) $[A|b] = \left[\begin{array}{cc|c} 1 & h & 4 \\ 3 & 6 & 8 \end{array}\right]$

$\sim \left[\begin{array}{cc|c} 1 & h & 4 \\ 1.5 & 3 & 4 \end{array}\right] \quad R_2 \to R_2/2$

$\sim \left[\begin{array}{cc|c} 1 & h & 4 \\ 0 & 1.5h-3 & 2 \end{array}\right] \quad R_2 \to 1.5R_1 - R_2$

Case I.

$1.5h - 3 = 0$

$\Rightarrow h = 2.$

At $h = 2$, the rank of A is 1

but rank of $[A|b]$ is 2.

$\because$ both rank are not equal

Hence, the system is inconsistent.

Case 2.

$$1.5h - 3 \neq 0$$

which implies $h \neq 2$.

For each $h \in \mathbb{R} - \{2\}$, the given augmented matrix is consistent.

Also, the rank of A and rank of [A|b] are equal and equal to no. of unknowns.

Therefore,

For each $h \in \mathbb{R} - \{2\}$, [A|b] is consistent and has the unique solution.

Q6.(b.) $[A|b] = \left[\begin{array}{cc|c} -4 & 12 & h \\ 2 & -6 & -3 \end{array}\right]$

$$\sim \left[\begin{array}{cc|c} -4 & 12 & h \\ 0 & 0 & -6+h \end{array}\right] \quad R_2 \to 2R_2 + R_1$$

$$h - 6 = 0$$

$\Rightarrow h = 6$.

for $h = 6$ rank of A is 1 and rank of [A|b] is also 1.

$\therefore$ [A|b] is consistent for $h = 6$.

Q7. B and D.

Q8. $Z_2 = \{[0], [1]\}$ where $[0] = \{0, \pm 2, \pm 4 \ldots\}$

$[1] = \{1 \pm 2, 1 \pm 4 \ldots\}$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$A^2 = A \cdot A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Q9. When determinent of a matrix is zero, it has no inverses.

$\therefore$ $|A| = 0$ has no inverse.

$$\begin{vmatrix} 6 & -1 & 1 \\ t & 0 & 1 \\ 0 & 1 & t \end{vmatrix} = 0$$

$\Rightarrow$ $6(0-1) - (-1)(t^2 - 0) + 1(t - 0) = 0$

$\Rightarrow$ $-6 + t^2 + t = 0$

$\Rightarrow$ $t^2 + t - 6 = 0$

$\Rightarrow$ $t^2 + 3t - 2t - 6 = 0$

$\Rightarrow$ $t(t+3) - 2(t+3) = 0$

$\Rightarrow (t+3)(t-2) = 0.$

$\therefore t = -3, 2$

$\therefore$ for $t = -3$ or 2, A doesn't have inverse.

Q10. (1.) 3x2 rref:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & * \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

(2.) 2x3 rref:

$$\begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \end{bmatrix}, \begin{bmatrix} 1 & * & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & * & * \\ 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(3.) $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ 4x1 rref.

Q12. a : any real number. ; $b = 0$
$c = 1$; $d = 0$; $e = 0$

Q12.

(1.)

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & -4 & -5 & -6 \end{bmatrix} \qquad \begin{bmatrix} R_3 \to R_3 - 2R_1 \\ R_2 \to R_1 - R_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 7 & 2 \end{bmatrix} \qquad \left[R_3 \to R_3 + 4R_2 \right]$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 2/7 \end{bmatrix} \qquad \left[R_3 \to R_3 / 7 \right]$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 0 & 22/7 \\ 0 & 1 & 0 & 8/7 \\ 0 & 0 & 1 & 2/7 \end{bmatrix} \qquad \begin{bmatrix} R_2 \to R_2 - 3R_3 \\ R_1 \to R_1 - 3R_3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 6/7 \\ 0 & 1 & 0 & 8/7 \\ 0 & 0 & 1 & 2/7 \end{bmatrix} \qquad \left[R_1 \to R_1 - 2R_2 \right]$$

$$= \text{rref}(A)$$

(2) $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 2 \end{bmatrix}$

$= \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 3 & 2 & 1 \end{bmatrix}$ $\begin{bmatrix} R_3 \to R_3 - 2R_1 \\ R_2 \to R_2 - R_1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ $\begin{bmatrix} R_3 \to R_3 - 3R_2 \end{bmatrix}$

$= \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\begin{bmatrix} R_3 \to R_3/2 \end{bmatrix}$

$= \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\begin{bmatrix} R_2 \to R_2 - 2R_3 \\ R_1 \to R_1 - 4R_3 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\begin{bmatrix} R_1 \to R_1 - 2R_2 \end{bmatrix}$

$= rref(A)$

```
A = [ 1 2 3 4; 1 1 0 2; 2 0 1 2]
S = sym(A)
rref(S)
rref(A)
Ar2 = rrefgf(A,2)
Ar3 = rrefgf(A,3)
Ar5 = rrefgf(A,5)
Ar7 = rrefgf(A,7)
```

```
Ar2 =

     1     0     0     0
     0     1     0     0
     0     0     1     0


Ar3 =

     1     0     0     0
     0     1     0     2
     0     0     1     2


Ar5 =

     1     0     0     3
     0     1     0     4
     0     0     1     1


Ar7 =

     1     0     4     0
     0     1     3     0
     0     0     0     1
```

(4.) Comparing Ar 2, Ar 3, Ar 5 and Ar 7, we can see that the Ar2 ~~is~~ ~~Ar3~~ and Ar 7 has different first three column.

So, A has different rank over Z_2 and Z_7.

Q13. Given $A = \begin{bmatrix} 3 & 1 & 4 \\ 5 & 2 & 6 \\ 0 & 5 & 2 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}$

1. $[A \mid \vec{b}] = \left[\begin{array}{ccc|c} 3 & 1 & 4 & 1 \\ 5 & 2 & 6 & 5 \\ 0 & 5 & 2 & 1 \end{array}\right]$

$= \left[\begin{array}{ccc|c} 3 & 1 & 4 & 1 \\ 0 & 1 & 5 & 3 \\ 0 & 5 & 2 & 1 \end{array}\right]$ $R_2 \to 3R_2 - 5R_1$

$= \left[\begin{array}{ccc|c} 3 & 1 & 4 & 1 \\ 0 & 1 & 5 & 3 \\ 0 & 0 & 2 & 0 \end{array}\right]$ $R_3 \to 5R_2 - R_3$

$=$ rref (A).

2. $A\vec{x} = \vec{b} \mod 7$

$$3x_1 + x_2 + 4x_3 = 1 \quad \text{— (1)}$$

$$x_2 + 5x_3 = 3 \quad \text{— (2)}$$

$$2x_3 = 0 \quad \Rightarrow \quad x_3 = 0.$$

By (2) $\longrightarrow$ $x_2 = 3$

By (1) $\Rightarrow$ $3x_1 + 3 + 0 = 1$

$3x_1 = (-2) \bmod 7.$

$3x_1 = 5$

$x_1 = 5/3$

Hence the system of linear eq^n $A\vec{x} = \vec{b}$ with field Z_7 is

$x_1 = 5/3$, $x_2 = 3$ and $x_3 = 0$.

Q14. $$\text{rref}(A) = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

No solution because the system is inconsistent.

```
A = [ 3 11 19 -2; 7 23 39 10; -4 -3 -2 6];
S = sym(A);
rref(S)
syms x y z
eqn1 = 3*x + 11*y + 19*z == -2;
eqn2 = 7*x + 23*y +39*z == 10;
eqn3 = -4*x - 3*y - 2*z == 6;

[A,B] = equationsToMatrix([eqn1, eqn2, eqn3],[x, y, z])

X = linsolve(A,B)
```

```
ans =

[1, 0, -1, 0]
[0, 1,  2, 0]
[0, 0,  0, 1]


A =

[ 3, 11, 19]
[ 7, 23, 39]
[-4, -3, -2]


B =

-2
10
 6

Warning: Solution does not exist because the system is inconsistent.
> In symengine
In sym/privBinaryOp (line 1136)
In sym/linsolve (line 63)
In HW1_Solve_Linear_Systems (line 11)

X =

Inf
Inf
Inf
```

Q15 $\quad \text{rref}(A) = \begin{bmatrix} 1 & 2 & 3 & 0 & 5 & 6 \\ 0 & 0 & 0 & 1 & 2 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

So, $x_1 = 6 - 2x_2 - 3x_3 - 5x_5$

$\quad\; x_4 = 7 - 2x_5$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 6 - 2x_2 - 3x_3 - 5x_5 \\ x_2 \\ x_3 \\ 7 - 2x_5 \\ x_5 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 0 \\ 7 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -5 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

where x_2, x_3, x_5 are any real numbers.

```
A = [ 3 6 9 5 25 53; 7 14 21 9 53 105; -4 -8 -12 5 -10 11];
S = sym(A);
rref(S)
syms v w x y z
eqn1 = 3*v + 6*w + 9*x + 5*y + 25*z == 53;
eqn2 = 7*v + 14*w + 21*x + 9*y + 53*z == 105;
eqn3 = -4*v - 8*w - 12*x + 5*y - 10*z == 11;

[A,B] = equationsToMatrix([eqn1, eqn2, eqn3],[v, w, x, y, z])

X = linsolve(A,B)
```

```
[1, 2, 3, 0, 5, 6]
[0, 0, 0, 1, 2, 7]
[0, 0, 0, 0, 0, 0]


A =

[ 3,   6,    9, 5,   25]
[ 7,  14,   21, 9,   53]
[-4,  -8,  -12, 5,  -10]


B =

 53
105
 11

Warning: Solution is not unique because the system is rank-deficient.
> In symengine
In sym/privBinaryOp (line 1136)
In sym/linsolve (line 63)
In HW1_Solve_Linear_Systems (line 11)

X =

6
0
0
7
0
```

Q16.

$$\mathrm{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -8221/4340 \\ 0 & 1 & 0 & 0 & 0 & 8591/8680 \\ 0 & 0 & 1 & 0 & 0 & 4695/434 \\ 0 & 0 & 0 & 1 & 0 & -459/434 \\ 0 & 0 & 0 & 0 & 1 & 699/434 \end{bmatrix}$$

$x_1 = -8221/4340 = -1.89$

$x_2 = 8591/8680 = 0.99$

$x_3 = 4695/434 = 10.82$

$x_4 = -459/434 = -1.06$

$x_5 = 699/434 = 1.61$

```
A = [ 2 4 3 5 6 37; 4 8 7 5 2 74; -2 -4 3 4 -5 20; 1 2 2 -1 2 26; 5 -10 4 6 4 24];
S = sym(A);
rref(S)
syms v w x y z
eqn1 = 2*v + 4*w + 3*x + 5*y + 6*z == 37;
eqn2 = 4*v + 8*w + 7*x + 5*y + 2*z == 74;
eqn3 = -2*v - 4*w + 3*x + 4*y - 5*z == 20;
eqn4 = v + 2*w + 2*x - y + 2*z == 26;
eqn5 = 5*v - 10*w + 4*x + 6*y + 4*z == 24;

[A,B] = equationsToMatrix([eqn1, eqn2, eqn3, eqn4, eqn5],[v, w, x, y, z])

X = linsolve(A,B)
```

```
[1, 0, 0, 0, 0, -8221/4340]
[0, 1, 0, 0, 0,  8591/8680]
[0, 0, 1, 0, 0,   4695/434]
[0, 0, 0, 1, 0,   -459/434]
[0, 0, 0, 0, 1,    699/434]


A =


[ 2,   4, 3,  5,  6]
[ 4,   8, 7,  5,  2]
[-2,  -4, 3,  4, -5]
[ 1,   2, 2, -1,  2]
[ 5, -10, 4,  6,  4]


B =


37
74
20
26
24


X =


-8221/4340
 8591/8680
  4695/434
  -459/434
   699/434
```

Q17. 1. By invertible theorem, A and C are invertible and $A^{-1} = BC$ and $C^{-1} = AB$.

Then $AB = C^{-1}$ then $CAB = I$

So B is invertible and $B^{-1} = CA$.

2. If AB is invertible, then there exist a matrix C such that $ABC = I$ then, by (1) each matrix is invertible.

Q18. $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ & $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$(AB)^2 = \begin{bmatrix} 9 & 12 \\ 0 & 0 \end{bmatrix}$

$[A^2B^2] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Q19. We know.

$$A^{-1} = \frac{1}{|A|} \text{ adj. } A.$$

Suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$A^{-1} = \frac{\begin{vmatrix} d & -b \\ -c & a \end{vmatrix}}{ad - bc} \qquad A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

To get $A^{-1} = A^T$

We set $ad - bc = 1$.
and $a = d$ and $b = -c$

So,
$A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ such that $a^2 + b^2 = 1$.
is the kind of matrix whose $A^{-1} = A^T$

eg: $\begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}$

Q20. 1. Symmetric Matrix.

2X2

$$\begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix}$$

3x3

$$\begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & 5 \\ 1 & 5 & 3 \end{bmatrix}$$

4x4

$$\begin{bmatrix} 1 & 3 & 7 & 9 \\ 3 & 4 & 5 & 2 \\ 7 & 5 & 6 & 12 \\ 9 & 2 & 12 & 8 \end{bmatrix}$$

Skew symmetric Matrix

2X2

$$\begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}$$

3x3

$$\begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & 5 \\ -2 & -5 & 0 \end{bmatrix}$$

4x4

$$\begin{bmatrix} 0 & 2 & -4 & 6 \\ -2 & 0 & -7 & 1 \\ 4 & 7 & 0 & -11 \\ -6 & -1 & 11 & 0 \end{bmatrix}$$

2. All zeroes.

3. Zero matrix.

4. $(A + A^T)^T = A^T + (A^T)^T = A^T + A$

$(AA^T)^T = A^T (A^T)^T = AA^T$

$(A^T A)^T = (A^T)^T (A^T) = A^T A^T$

$\therefore$ Above all are symmetric.

$(A - A^T)^T = A^T - (A^T)^T = A^T - A = -(A - A^T)$

$\therefore (A - A^T)^T$ is skew symmetric.

5. $\frac{1}{2}\left[(A + A^T) + (A - A^T)\right] = A.$

Q 21. $H_n = I_n - 2\vec{u}\,\vec{u}^T$ — ①

where I_n is identity matrix of n^{th} order and $\vec{u}$ is unit vector in $\mathbb{R}^n$

1. $H_n^T = (I_n - 2\vec{u}\vec{u}^T)^T$

We know

$I_n^T = I_n$

$(AB)^T = B^T A^T$

$(A^T)^T = A.$

$= I_n^T - 2(\vec{u}\vec{u}^T)^T$

$= I_n^T - 2(\vec{u}^T)^T\vec{u}^T$

$= I_n - 2\vec{u}\vec{u}^T$

$= H_n$

$\therefore$ H_n is symmetric.

2. $H_n^T \cdot H_n = H_n H_n$

$= (I_n - 2\vec{u}\vec{u}^T)(I_n - 2\vec{u}\vec{u}^T)$

$= I_n I_n - 2\vec{u}\vec{u}^T I_n - 2\vec{u}\vec{u}^T I_n + 4\vec{u}\vec{u}^T\vec{u}\vec{u}^T$

$= I_n - 4\vec{u}\vec{u}^T + 4\vec{u}(\vec{u}^T\vec{u})\vec{u}^T$ — ③

we know that $\vec{u}$ is a unit vector in $\mathbb{R}^n$

$\because \vec{u} \in \mathbb{R}^n \Rightarrow \vec{u}^T \in \mathbb{R}^n$

$\vec{u}$ is a column vector.

$\vec{u}_{n\times 1}$ i.e. $\vec{u}^T_{1\times n}$ is a row vector.

So $\vec{u}^T\vec{u} = 1$ — ④

$\{ \vec{u}^T_{1\times n}\vec{u}_{n\times 1} = 1_{1\times 1} = 1$ as $\vec{u}$ is a unit vector.

In eq^n ③

$H_n^T H_n = I_n - 4\vec{u}\vec{u}^T + 4\vec{u}(1)\vec{u}^T$

$H_n^T H_n = I_n - 4\vec{u}\vec{u}^T + 4\vec{u}\vec{u}^T$

$H_n^T H_n = I_n$

$\therefore$ H_n is an orthogonal matrix.

3. $H_n^2 = H_n \cdot H_n$

$= H_n^T H_n$

$= I_n$

4. $H_n\vec{u} = \cancel{\#} \left(I_n - 2\vec{u}\,\vec{u}^T\right)\vec{u}$

$= I_n\vec{u} - 2\vec{u}\,\vec{u}^T\vec{u}$

$= \vec{u} - 2\vec{u}\,(\vec{u}^T\vec{u})$ — (4)

$= \vec{u} - 2\vec{u}\ (1)$

$\Rightarrow -\vec{u}$

$H_n\vec{u} = -\vec{u}$

5. $\vec{u} = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \Rightarrow \vec{u}^T = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}$

if $n = 3$

$H_3 = I_3 - 2\frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{2}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1/3 & -2/3 & -2/3 \\ -2/3 & 1/3 & -2/3 \\ -2/3 & -2/3 & 1/3 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix}$

H_3 ~~is~~ symmetric and orthogonal.

$$H_4 = I_4 - 2\vec{u}\,\vec{u}^T$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - 2\frac{1}{\sqrt{4}}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\frac{1}{\sqrt{4}}\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 & -1/2 & -1/2 & -1/2 \\ -1/2 & 1/2 & -1/2 & -1/2 \\ -1/2 & -1/2 & 1/2 & -1/2 \\ -1/2 & -1/2 & -1/2 & 1/2 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}$$

H_4 is symmetrical and orthagonal.

Q22. $A = \begin{bmatrix} 4 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix}$

$LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix}$

$R_2 \to R_2 - \frac{1}{4} R_1 \; ; \; R_3 \to R_3 - 0.R_1$

$R_4 \to R_4 - 0.R_1$

$LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/4 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 1 & 0 & 0 \\ 0 & 15/4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix}$

$R_3 \to R_3 - \frac{4}{15} R_2 \; ; \; R_4 \to R_4 - 0.R_2$

$LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/4 & 1 & 0 & 0 \\ 0 & 4/15 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 1 & 0 & 0 \\ 0 & 15/4 & 1 & 0 \\ 0 & 0 & 56/15 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix}$

$R_4 \to R_4 - \frac{15}{56} R_3$

$$LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/4 & 1 & 0 & 0 \\ 0 & 4/15 & 1 & 0 \\ 0 & 0 & 15/56 & 1 \end{bmatrix} \begin{bmatrix} 4 & 1 & 0 & 0 \\ 0 & 15/4 & 1 & 0 \\ 0 & 0 & 56/15 & 1 \\ 0 & 0 & 0 & 209/56 \end{bmatrix}$$

Q2.3. $q_1 = d_1$

$q_i = l_i r_{i-1} + d_i$

$p_i = l_i d_{i-1}$

$u_i = r_i$

Q24. $$L.U. = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1/4 & 1 & \ddots & & \vdots \\ 0 & 4/15 & 1 & \ddots & \vdots \\ \vdots & \ddots & 15/56 & 1 & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \ddots & 1 \end{bmatrix} \begin{bmatrix} 4 & 1 & 0 & \cdots & 0 \\ 0 & 15/4 & 1 & \ddots & \vdots \\ 0 & \ddots & 56/15 & 1 & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & \end{bmatrix}$$