Math 5110 - Applied Linear Algebra -Fall 2022

Instructor: He Wang

Test 1. (In class)

Student Name: ____________________

Rules and Instructions for Exams:

1. Unless otherwise specified, to receive full credits you must show **all** necessary work. The grading is based on your work shown. Only a final result from calculator will receive zero point.

2. You need to finish the exam yourself. Any discussions with the other people will be considered as **academic dishonesty**. **Cheating, Unauthorized Collaboration, and Facilitating Academic Dishonesty are not allowed.** You can read a description of each here http://www.northeastern.edu/osccr/academic-integrity-policy/

3. You are allowed to bring one lecture notes or a textbook.

4. However, you are **not** allowed to bring the homework or practice questions.

5. However, you are **not** allowed to use any electronic devices.

Notation: $\vec{x} \in \mathbb{R}^n$ means a column vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

Student Name: ____________________ __________/**50**

1. (2 points) Consider the finite ring $\mathbb{Z}_p$. Answer the following questions:
(1) Which of the following is the **additive inverse** of $[5] \in \mathbb{Z}_9$? Answer: __________

A. [1] B. [2] C. [3] D. [4] E. [5] F. [0] G. not exist

(2) Which of the following is the **multiplicative inverse** of $[5] \in \mathbb{Z}_9$? Answer: __________

A. [1] B. [2] C. [3] D. [4] E. [5] F. [0] G. not exist

(1) D (2) B

2. (2 points) Let $\mathbb{F}$ be the field contains all numbers of the form $a+b\sqrt{5}$ where a and b are rational numbers, with the usual addition and multiplication of arithmetic

$$(a + b\sqrt{5}) + (c + d\sqrt{5}) := a + c + (b + d)\sqrt{5}$$

$$(a + b\sqrt{5}) \times (c + d\sqrt{5}) := ac + 5bd + (ad + bc)\sqrt{5}$$

(1) What is the **additive inverse** of $2 - \sqrt{5}$? Answer: __________

A. $-2 + \sqrt{5}$ B. $-2 - \sqrt{5}$ C. $1 + \sqrt{5}$ D. $-1 - \sqrt{5}$ E. $-2 - \sqrt{5}$ F. not exist

(2) What is the **multiplicative inverse** of $2 - \sqrt{5}$? Answer: __________

A. $-2 + \sqrt{5}$ B. $2 - \sqrt{5}$ C. $1 + \sqrt{5}$ D. $-1 - \sqrt{5}$ E. $-2 - \sqrt{5}$ F. not exist

(1) A (2) E

3. (4 points) Determine whether or not the following set S a **subspace** of V. Explain your reason. If you use a theorem, write down the theorem.
(1) Let $V = \mathbb{R}^3$ and $S = \{\vec{x} \in \mathbb{R}^3 \mid x_1 + x_2 = 1\}$.

No. Since S does not contains zero vector

(2) Let $V :=$ { all funcitons $f(x) : \mathbb{R} \to \mathbb{R}$}be the vector space of functions.

Let $S = \text{Span}\{\sin x, x^3 + 1, e^x\}$ be the subset of V.

Yes. Since the Span of any subset of V is always a subspace.

4. Let $\mathbb{R}^{3\times 3}$ be the vector space of all 3×3 matrices.

Let U be the subspace contains all 3×3 **upper** triangular matrices.

Let L be the subspace contains all 3×3 **lower** triangular matrices.

Recall that a 3×3 upper triangular matrix A satisfies $a_{21} = a_{31} = a_{32} = 0$.

(1) (2 points) Is any 3×3 matrix $M \in \mathbb{R}^{3\times 3}$ can be written as the sum of a upper triangular matrix and a lower triangular matrix? Explain the reason.

Yes. $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ a_{21} & 0 & 0 \\ a_{31} & a_{32} & 0 \end{bmatrix}$

(2) (1 points) Write down a **basis** for the intersection $U \cap L$?

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(3) (1 point) Is $\mathbb{R}^{3\times 3} = U + L$? Answer: ____

Yes

(4) (1 point) Is $\mathbb{R}^{3\times 3} = U \oplus L$? Answer: ____

No.

(5) (1 point) Is $\mathbb{R}^{3\times 3} = U \cup L$? Answer: _____

No

5. (4 points) Let $\vec{a}_1,\ \vec{a}_2,\ \vec{a}_3$ be vectors in a vector space V. Answer the following questions and explain the reason.

(1) Suppose $\vec{a}_3$ is a linear combination of $\vec{a}_1$ and $\vec{a}_2$. Is $\vec{a}_1,\ \vec{a}_2,\ \vec{a}_3$ dependent?

(2) Suppose $\vec{a}_1,\ \vec{a}_2,\ \vec{a}_3$ is dependent. Is $\vec{a}_3$ always a linear combination of $\vec{a}_1$ and $\vec{a}_2$?

(1) Yes, since a_3 is redundent.
(2) No. For example, suppose $\vec{a}_1 = \vec{a}_2$, then $\vec{a}_1,\ \vec{a}_2,\ \vec{a}_3$ is dependent.
But $\vec{a}_3$ can be any vector. Use a concrete example in $\mathbb{R}^2$ is even better.

6. Let $A = [\vec{a}_1\ \vec{a}_2\ \vec{a}_3\ \vec{a}_4]$ with $\mathbf{rref}(A) = \begin{bmatrix} \mathbf{1} & 0 & -7 & -8 \\ 0 & \mathbf{1} & -2 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Answer the following questions.

(1) (4 points) Find a **basis** for the kernel (null) space $\ker A$.

So, $\begin{cases} x_1 - 7x_3 - 8x_4 = 0 \\ x_2 - 2x_3 - 3x_4 \\ x_3, x_4 \text{ is a free variable} \end{cases}$

The **vector form** is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 7x_3 + 8x_4 \\ 2x_3 + 3x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 7 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 8 \\ 3 \\ 0 \\ 1 \end{bmatrix}$

So, a basis for $\ker A$ is $\{ \begin{bmatrix} 7 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 8 \\ 3 \\ 0 \\ 1 \end{bmatrix}$

(2) (2 points) Is $\vec{a}_4$ a linear combination of $\vec{a}_1$, $\vec{a}_2$, $\vec{a}_3$? If Yes, write it done. If No, explain the reason.

Yes. Solve augmented matrix $[\vec{a}_1\ \vec{a}_2\ \vec{a}_3 \mid \vec{a}_4]$, we find a solution $(-8, -3, 0, 1)$. So $\vec{a}_4 = -8\vec{a}_1 + -3\vec{a}_2$.

7. (4 Points) Suppose that A is a 3×3 matrix with column vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3$ and the determinant $Det(A) = Det(\vec{a}_1|\vec{a}_2|\vec{a}_3) = 2$. Suppose B is a 3×3 invertible matrix. Compute the following determinants. (Write your)

(1). $Det(\vec{a}_1 - \vec{a}_2|\vec{a}_2 - \vec{a}_1|\vec{a}_3) =$

(2). $Det(\vec{a}_2|\vec{a}_2 - \vec{a}_1|2\vec{a}_3) =$

(3). $Det(BA^2B^{-1}A^T) =$

(1). $Det(\vec{a}_1 - \vec{a}_2|\vec{a}_2 - \vec{a}_1|\vec{a}_3) = 0$
(2). $Det(\vec{a}_2|\vec{a}_2 - \vec{a}_1|2\vec{a}_3) = Det(\vec{a}_2| - \vec{a}_1|2\vec{a}_3) = Det(A) = 2$
(3). $Det(BA^2B^{-1}A^T) = Det(A)^3 = 8$

8. Let $A = [\vec{a}_1\ \vec{a}_2\ \vec{a}_3\ \vec{a}_4\ \vec{a}_5\ \vec{a}_6]$. Let $U = \text{Span}\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ and $V = \text{Span}\{\vec{a}_4, \vec{a}_5, \vec{a}_6\}$.

Suppose $\textbf{rref}(A) = \begin{bmatrix} \mathbf{1} & 0 & 0 & \mathbf{2} & 0 & 0 \\ 0 & \mathbf{1} & 0 & \mathbf{3} & 0 & 0 \\ 0 & 0 & \mathbf{1} & \mathbf{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} \end{bmatrix}$ and $\textbf{rref}([\vec{a}_4\ \vec{a}_5\ \vec{a}_6]) = \begin{bmatrix} \mathbf{1} & 0 & 0 \\ 0 & \mathbf{1} & 0 \\ 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 \end{bmatrix}$

(1) (5 points) What are the **dimensions** of U, V, $U + V$, $\mathbb{R}^5/U$ and $U \cap V$?

$\dim(U) = 3$
$\dim(V) = 3$
$\dim(U + V) = 5$
$\dim(\mathbb{R}^5/U) = 5 - \dim U = 2$
From $\dim(U + V) = \dim U + \dim V - \dim U \cap V$, we have $\dim U \cap V = 1$

(2)(2 points) Find a **basis** for $U \cap V$? write the basis vector as a linear combinations of $\vec{a}_1$, $\vec{a}_2$, $\vec{a}_3$.

Step 1. From $\textbf{rref}(A)$ we can find solutions.
Step 2. Then, a basis is given by $\{2\vec{a}_1 + 3\vec{a}_2 + 4\vec{a}_3\}$

9. (4 points) Let V be a vector space over a field. Suppose U, S, W are any subspaces of V. Answer the following quesitons. If it is true, prove it. If it is false, provide a counter-example.

(1) Is $(U \cap S) + W \subseteq (U + W) \cap (S + W)$?

(2) Is $(U \cap S) + W = (U + W) \cap (S + W)$?

(1) True. Since $U \cap S \subseteq U$ and $U \cap S \subseteq S$, we have $(U \cap S) + W \subseteq U + W)$ and $(U \cap S) + W \subseteq S + W$.
Hence $(U \cap S) + W \subseteq (U + W) \cap (S + W)$.
(2) The statement is false in general.
For example, consider the three subspaces $U = \text{Span}(\begin{bmatrix} 1 \\ 0 \end{bmatrix})$, $V = \text{Span}(\begin{bmatrix} 0 \\ 1 \end{bmatrix})$ and $W = \text{Span}(\begin{bmatrix} 1 \\ 1 \end{bmatrix})$
Then, $(U \cap S) + W = W$ and $(U + W) \cap (S + W) = \mathbb{R}^2$

10. Let $P_4(\mathbb{R}) = \text{Span}\{1, x, x^2, x^3\}$. A function $T : P_4(\mathbb{R}) \to P_4(\mathbb{R})$ is defined by the rule

$$T(f) = f'' + f.$$

(1) (2 points) Show that T is a **linear** transformation.

To show that T is a linear operator we simply verify that $T(f_1 + f_2) = T(f_1) + T(f_2)$ and $T(cf_1) = cT(f_1)$ where c is a constant.

(2) (3 points) Find the **matrix** M of T with respect to the standard basis $\{1, x, x^2, x^3\}$ of $P_4(\mathbb{R})$.

To find the matrix representing T, compute $T(1) = 1$, $T(x) = x$, $T(x^2) = 2 + x^2$ and $T(x^3) = 6x + x^3$. The columns of the required matrix are the coordinate vectors of $T(1), T(x), T(x^2), T(x^3)$ respect to the ordered basis $1, x, x^2, x^3$. Hence, the matrix is

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(3) (2 points) Find the **dimension** of $\ker(T)$ and the **dimension** of $\text{im}(T)$.

$\dim(\ker(T)) = 0$ and $\dim \text{im}(T) = 4$.

11. (4 points) In each case, if the statement is true, prove it. If it is false, provide a counter-example.

1. Does there exist real invertible 3×3 matrices A and S such that $S^{-1}AS = 2A$.

No. Since $\det(S^{-1}AS) = \det(A)$, if such matrices exist, we have $\det(A) = 2^3 \det(A)$, that is $1 = 2^3$ which is impossible.

2. Does there exist real invertible 3×3 matrices A such that $AB - BA + A - A^T = I_3$.

No. Use Trace. $trace(AB - BA + A - A^T) = 0$ but $trace(I_3) = 3$