Northeastern University, Department of Mathematics

MATH 5110: Applied Linear Algebra and Matrix Analysis.

• Instructor: **He Wang**   Email: **he.wang@northeastern.edu**

§12 Spectral Theorem and quadratic forms

Contents

$a_{ij} = dist(\vec{x}_i, \vec{x}_j)$

$\vec{x}_i$ $\vec{x}_j$

$A = \begin{bmatrix} & a_{ij} & \end{bmatrix}$

$A = A^T$

$M \in \mathbb{R}^{m\times n}$

MM^T or M^TM symmetric $\Rightarrow$ S.V.D

$\Downarrow$

PCA.

1. Spectral Theorem

In this section, we deal real matrix.

An $n \times n$ matrix A is called **symmetric** if $\boxed{A^T = A,}$ i.e.,

$$a_{ij} = a_{ji} \qquad \text{for all } i, j \in \{1, 2, ..., n\}$$

Example 1 (Diagonalizing a Symmetric Matrix)**.**

$$A = \begin{bmatrix} 10 & 6 \\ 6 & 1 \end{bmatrix}. \qquad B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}. \qquad C = \begin{bmatrix} 1 & 1 & 7 \\ 1 & 7 & 1 \\ 7 & 1 & 1 \end{bmatrix}.$$

$A = PDP^{-1}$

$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $\lambda = i, -i$

$a+bi$
$\|$
$\lambda \in \mathbb{C}$

$\lambda = \bar{\lambda} \Rightarrow \lambda \in \mathbb{R}$

real

Proposition 2. *A symmetric $n \times n$ matrix A has n real eigenvalues if they are counted with their algebraic multiplicities.*

- Suppose $A\vec{x} = \lambda\vec{x}$. for $\lambda \in \mathbb{C}$ $\vec{x} \neq \vec{0} \in \mathbb{C}^n$

- $\langle A\vec{x}, \vec{x}\rangle_{\mathbb{C}} = \langle \lambda\vec{x}, \vec{x}\rangle_{\mathbb{C}} = \lambda\langle \vec{x}, \vec{x}\rangle_{\mathbb{C}}$

$\|$ adjoint

$\langle \vec{x}, A^*\vec{x}\rangle_{\mathbb{C}} = \langle \vec{x}, A\vec{x}\rangle_{\mathbb{C}} = \langle \vec{x}, \lambda\vec{x}\rangle_{\mathbb{C}} = \bar{\lambda}\langle \vec{x}, \vec{x}\rangle$

$\Rightarrow \lambda = \bar{\lambda}$

Proposition 3. *Let A be a* real *symmetric matrix and let λ, μ* $\in \mathbb{R}$ *be two distinct eigenvalues of A with associated eigenvectors $\vec{v}, \vec{w}$. Then*

$$\vec{v} \cdot \vec{w} = 0.$$

$\lambda \neq \mu$

$A\vec{v} = \lambda \vec{v}$

$A\vec{w} = \mu \vec{w}$

$\lambda \neq \mu$

$A = A^T$

$$\langle A\vec{v}, \vec{w} \rangle = \lambda \langle \vec{v}, \vec{w} \rangle$$

$$\|$$

$$\langle \vec{v}, A^{*}\vec{w} \rangle$$

$$\|$$

$$\langle \vec{v}, A\vec{w} \rangle = \bar{\mu} \langle \vec{v}, \vec{w} \rangle = \mu \langle \vec{v}, \vec{w} \rangle$$

$(\lambda - \mu)\langle \vec{v}, \vec{w} \rangle = 0$

$\Rightarrow \langle \vec{v}, \vec{w} \rangle = 0$

E_λ is orthogonal to E_μ for distinct eigenvalues λ, μ (in that $\vec{v} \cdot \vec{w} = 0$ for all $\vec{v} \in E_\lambda$ and $\vec{w} \in E_\mu$).

Definition 4 (Orthogonal Diagonalization)**.** An $n \times n$ matrix is **orthogonally diagonalizable** if there exist diagonal matrix D and orthogonal matrix P such that

$P^{-1} = P^T$

$$A = PDP^{-1} = PDP^T.$$

$A = PDP^T$

Theorem 5 (On Orhtogonal Diagonalizability)**.** *An $n \times n$ matrix A is orthogonally diagnonalizable if and only if A is a symmetric matrix.*

$A = A^T$

"$\Rightarrow$" $A = PDP^T$ $\quad A^T = (PDP^T)^T = (P^T)^T D^T P^T = PDP^T$

"$\Leftarrow$" "by induction" $\quad n = 1$ ✓

· suppose it is true for $(n-1) \times (n-1)$ matrix.

suppose $A\vec{v}_1 = \lambda \vec{v}_1$, $\|\vec{v}_1\| = 1$. Choose an orthonormal basis for $\mathbb{R}^n$. $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$

$AP = P[\quad]$

$$A = P \begin{bmatrix} \lambda_1 & 0 \cdots 0 \\ 0 & \\ \vdots & B \\ 0 & \end{bmatrix} P^{-1} = P[\quad] P^T$$

$P = [\vec{v}_1 \cdots \vec{v}_n]$

$P^{-1} = P^T$

$A^T = A \implies B^T = B$

$[A\vec{v}_1 \cdots A\vec{v}_n] = [\vec{v}_1 \cdots \vec{v}_n]\begin{bmatrix} \lambda & \cdots \\ 0 & \\ \vdots & \\ 0 & \end{bmatrix}$, $A\vec{v}_1 = \lambda\vec{v}_1$

Need $\vec{v}_1^T A = \lambda \vec{v}_1^T$ left eigenvector

$(A\vec{v}_1)^T = (\lambda\vec{v}_1)^T$

$\vec{v}_1^T A^{T} = \lambda \vec{v}_1^T$

By induction $B = QDQ^T$ $Q^T = Q^{-1}$

$A = P\begin{bmatrix} 1 & 0 \\ 0 & Q \end{bmatrix}\begin{bmatrix} \lambda & 0 \\ 0 & D \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & Q^T \end{bmatrix}P^T$

$= U\begin{bmatrix} \lambda & 0 \\ 0 & D \end{bmatrix}U^T$

$P = [\vec{u}_1\ \vec{u}_2 \cdots \vec{u}_n]$ $D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$

3. The Spectral Decomposition

Let A be an $n \times n$ symmetric matrix and let D and P be a diagonal and orthogonal matrix with $A = PDP^{-1}$. $= PDP^T$

Theorem 6 (Spectral Decomposition for Symmetric Matrices)**.**

$$A = \lambda_1 \vec{u}_1\vec{u}_1^T + \lambda_2 \vec{u}_2\vec{u}_2^T + \cdots + \lambda_n \vec{u}_n\vec{u}_n^T$$

$$A = PDP^T = [\vec{u}_1 \cdots \vec{u}_n]\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}\begin{bmatrix} \vec{u}_1^T \\ \vdots \\ \vec{u}_n^T \end{bmatrix}$$

$$= [\lambda_1\vec{u}_1\ \ \lambda_2\vec{u}_2 \cdots \lambda_n\vec{u}_n]\begin{bmatrix} \vec{u}_1^T \\ \vdots \\ \vec{u}_n^T \end{bmatrix}$$

Denote

$L_i := \text{Span}\{\vec{u}_i\}$ $\text{proj}_{L_i}\vec{y} = (\vec{u}_i\vec{u}_i^T)\vec{y}$

- Algorithm: ① Find eigenvalues $\lambda_1 \cdots \lambda_k$ $\Longrightarrow$ $D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$

② Find basis B_i for each E_{λ_i}

③ Apply G-S to B_i to get orthonormal basis $\{\ \}$ for E_{λ_i}

$P = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix}$

- T/F?

Ex 1. A, B diagonalizable, then A+B diagonalizable. (F)

$A = P_1 D_1 P_1^{-1}$ $B = P_2 D_2 P_2^{-1}$ $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Ex 2. A, B orthogonally diagonalizable, then A+B orthogonally diagonalizable. True.

$\Updownarrow$ $\Updownarrow$

A, B symm. $\Longrightarrow$ A+B symm.

Ex: A, B orthogonally diagonalizable, then AB orthogonally diagonalizable. True only if $AB = BA$

$\Updownarrow$ $\Updownarrow$

A, B symm. $\overset{?}{\Longrightarrow}$ AB symm.

$A^T = A$ $B^T = B$ $\overset{?}{\Longrightarrow}$ $(AB)^T = AB$

$(AB)^T = B^T A^T = BA$

2. **Quadratic forms and positive definite**

Definition 7. A function $\boxed{p(x_1, ..., x_n)}$ from $\mathbb{R}^n$ to $\mathbb{R}$ is call a **quadratic form**, if it is a linear combination of forms $x_i x_j$.

So, a quadratic form can be written as

$$p(x_1, ..., x_n) = \sum_{i,j} c_{ij} x_i x_j \quad = \sum_{i=1}^{3} \sum_{j=1}^{3} a_{ij} x_i x_j$$

Another way to write quadratic form is using symmetric matrices

$$p(x_1, ..., x_n) = \vec{x} \cdot A\vec{x} = \vec{x}^T A\vec{x} \quad = [x_1 \ x_2 \ x_3] \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The unique symmetric matrix A is called the matrix for the quadratic form.

Example 8. Consider $p(x_1, ..., x_3) = 3x_1^2 + 4x_2^2 - 5x_3^2 - 2x_1x_2 + 4x_1x_3 + 6x_2x_3$

$$= [x_1 \ x_2 \ x_3] \begin{bmatrix} 3 & -1 & 2 \\ -1 & 4 & 3 \\ 2 & 3 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \vec{x}^T A \vec{x}$$

$\vec{x} \in \mathbb{R}^3$

① $P(\vec{x}) = 3x_1^2 + 2x_2^2 + x_3^2 > 0$ for any $\vec{x} \neq \vec{0}$ $A = \begin{bmatrix} 3 & & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

② $P(\vec{x}) = x_1^2 + 3x_2^2 \geq 0$ $A = \begin{bmatrix} 1 & & \\ & 3 & \\ & & 0 \end{bmatrix}$

③ $P(\vec{x}) = x_1^2 - x_3^2$

Definition 9. An real symmetric matrix A is called **positive definite** if the quadratic form

$$\vec{x}^T A \vec{x} > 0$$

for all nonzero $\vec{x} \in \mathbb{R}^n$.
The matrix A is called **positive semidefinite** if the quadratic form

$$\vec{x}^T A \vec{x} \geq 0$$

for all $\vec{x} \in \mathbb{R}^n$.

Theorem 10. *(1) An real symmetric matrix A is positive definite if and only if all eigenvalues of A are positive.* $\Leftrightarrow$
(2) An real symmetric matrix A is positive semidefinite if and only if all eigenvalues of A are non-negative.

$A = A^T \Leftrightarrow A = PDP^T = PDP^T$ $D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$

$q(x_1 \cdots x_n) = \vec{x}^T A \vec{x} = \vec{x}^T P D P^T \vec{x}$

Denote $\vec{y} = P^T \vec{x}$ invertible.

$= \vec{y}^T D \vec{y}$

$= \lambda_1 y_1^2 + \cdots + \lambda_n y_n^2 \geq 0$

Ex $A = \begin{bmatrix} 1 & 1 & 7 \\ 1 & 7 & 1 \\ 7 & 1 & 1 \end{bmatrix} = PDP^T$ $D = \begin{bmatrix} 9 & & \\ & 6 & \\ & & -6 \end{bmatrix}$

$$P = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \vec{y} = P^T \vec{x} = \begin{bmatrix} \frac{1}{\sqrt{3}}x_1 + \frac{1}{\sqrt{3}}x_2 + \frac{1}{\sqrt{3}}x_3 \\ \text{\_\_\_\_\_\_} \\ \text{\_\_\_\_\_\_} \end{bmatrix}$$

$$q(\vec{x}) = \vec{x}^T A \vec{x} = x_1^2 + 7x_2^2 + x_3^2 + 2x_1x_2 + 14x_1x_3 + 2x_2x_3$$

$$= 9(\quad y_1 \quad)^2 + 6(\quad y_2 \quad)^2 + 6(\quad y_3 \quad)^2$$

$$M = \left(\quad k(\vec{x}_i, \vec{x}_j) \quad \right)$$

$$k(\vec{x}, \vec{y}) := \langle \phi(\vec{x}), \phi(\vec{y}) \rangle \overset{e.g.}{=} (\vec{x}^T\vec{y} + a)^n$$

$$\phi : \mathbb{R}^d \longrightarrow \mathbb{R}^D$$

$$\vec{x} \longrightarrow \phi(\vec{x})$$

$D \gg d$

Theorem 11. *Let V be an "inner product space" over $\mathbb{R}$ and let $\{\vec{b}_1, ..., \vec{b}_n\}$ be a basis of V. Then the Gram matrix G is positive definite.*

Here the Gram matrix G is defined by $G_{ij} = \langle \vec{b}_j, \vec{b}_i \rangle = \langle \vec{b}_i, \vec{b}_j \rangle$

$G = \begin{bmatrix} & G_{ij} & \end{bmatrix}$

$q(x_1 \cdots x_n) = \vec{x}^T G \vec{x} = \sum_{i,j} G_{ij} x_i x_j$

$= \sum_{i,j} \langle \vec{b}_i, \vec{b}_j \rangle x_i x_j$

$= \langle x_1 \vec{b}_1 + \cdots x_n \vec{b}_n, x_1 \vec{b}_1 - - x_n \vec{b}_n \rangle$

$= \| x_1 \vec{b}_1 + \cdots + x_n \vec{b}_n \|^2 \geq 0$

Ex: If $V = \mathbb{R}^n$, dot prod, $\{\vec{b}_1 - - \vec{b}_n\}$ is basis for $\mathbb{R}^n$, $B = [\vec{b}_1 - - \vec{b}_n]$

$G = B^T B$

Proposition 12. *Let A be an $m \times n$ real matrix. Then $A^T A$ is positive semidefinite. Further more, if* $\text{rank}(A) = n$, *then $A^T A$ is positive definite.*

$\vec{x}^T A^T A \vec{x} = \|A\vec{x}\|^2 \geq 0$

$\text{rank}(A^T A) = \text{rank } A$

Thm: $A \in \mathbb{R}^{n \times n}$ is symmetric, positive-semi define $\Leftrightarrow A = XX^T$

$\Leftarrow$ ✓

$\Rightarrow A = PDP^{-1} = PDP^T$

$A = P\sqrt{D}\sqrt{D}P^T = XX^T$ here $X = P\sqrt{D}$

$D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$ $\lambda_i \geq 0$ $\sqrt{D} = \begin{bmatrix} \sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_n} \end{bmatrix}$

Thm (Cholesky decompn)

A is symm. positive-defn $\iff$ $A = LL^T$

"L is lower triangular"

$\Rightarrow A = XX^T$ $\quad X^T = QR$

$= (QR)^T QR$

$= R^T Q^T Q R$

$= R^T R = LL^T \qquad L = R^T$

any $X \in \mathbb{R}^{m \times n}$

$X^T X$ or XX^T symmetric

Positive Definite Complex Hermitian Matricies. $A^* = A$

$A \in \mathbb{C}^{n \times n}$