MATH 7241: Problem Set #2

Due date: Friday September 30

Reading: relevant background material for these problems can be found in the class notes, and in Ross (Chapters 2,3,5) and in Grinstead and Snell (Chapters 1,2 3, 6).

$\boxed{\textbf{Exercise 1}}$ [Ross 1-29] Suppose that $\mathbb{P}(E) = 0.6$. What can you say about $\mathbb{P}(E|F)$ when
(a) E and F are mutually exclusive?
(b) $E \subset F$?
(c) $F \subset E$?

(a) E, F mutually exclusive

$\Rightarrow \quad E \cap F = \phi \quad \Rightarrow \quad \mathbb{P}(E \cap F) = 0.$

$$\Rightarrow \quad \mathbb{P}(E | F) = \frac{\mathbb{P}(E \cap F)}{\mathbb{P}(F)} = 0.$$

(b) $E \subset F \quad \Rightarrow \quad E \cap F = E.$

$$\Rightarrow \quad \mathbb{P}(E | F) = \frac{\mathbb{P}(E \cap F)}{\mathbb{P}(F)} = \frac{\mathbb{P}(E)}{\mathbb{P}(F)}.$$

$\mathbb{P}(E) = 0.6, \quad E \subset F \quad \Rightarrow \quad 0.6 \leq \mathbb{P}(F) \leq 1.$

$$\Rightarrow \frac{0.6}{1} \leq \frac{P(E)}{P(F)} \leq \frac{0.6}{0.6}$$

$$\Rightarrow 0.6 \leq \mathbb{P}(E \mid F) \leq 1.$$

(c) $F \subset E \Rightarrow E \cap F = F.$

$$\Rightarrow \mathbb{P}(E \mid F) = \frac{\mathbb{P}(E \cap F)}{\mathbb{P}(F)} = \frac{\mathbb{P}(F)}{\mathbb{P}(F)} = 1.$$

Exercise 2 [Ross 1-42] There are three coins in a box. One is a two-headed coin, another is a fair coin, and the third is a biased coin that comes up Heads 75 percent of the time. One of the three coins is selected at random and flipped, and it comes up Heads. What is the probability that it was the two-headed coin?

2	F	B
Two-headed	Fair	Biased
$P(H) = 1$	$P(H) = \frac{1}{2}$	$P(H) = \frac{3}{4}$.

$E = \{$event that randomly selected coin comes up Heads$\}$.

$2, F, B = \{$events that selected coin is 2, F or $B\}$.

We want

$$P(2 \mid E) = \frac{P(E \mid 2)\, P(2)}{P(E \mid 2) P(2) + P(E \mid F) P(F) + P(E \mid B) P(B)}$$

$$= \frac{1 . \left(\frac{1}{3}\right)}{1\left(\frac{1}{3}\right) + \frac{1}{2}\left(\frac{1}{3}\right) + \frac{3}{4}\left(\frac{1}{3}\right)}$$

$$= \frac{4}{9}$$

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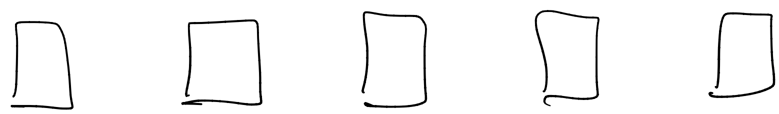
Exercise 4 In a variation of the classic Monty Hall game show, the host sets up five doors and hides prizes behind two of the doors. The contestant first guesses a door, and then the host opens one of the other four doors to show that it does not conceal a prize. The contestant is offered the opportunity to switch her guess to a different door. Should she switch or stay with her original choice? [Hint: see notes on the Monty Hall question, and try to imitate the solution provided there].



5 doors, 2 prizes

$R = \{\text{original guess was correct}\}$

$S = \{\text{correctly guesses after switching}\}$

$D = \{\text{correctly guesses without switching}\}$

$$\mathbb{P}(D) = \mathbb{P}(D \mid R)\,\mathbb{P}(R) + \mathbb{P}(D \mid R^c)\,\mathbb{P}(R^c)$$

$$= 1\left(\frac{2}{5}\right) + 0 = \frac{2}{5}$$

$$\mathbb{P}(S) = \mathbb{P}(S \mid R)\,\mathbb{P}(R) + \mathbb{P}(S \mid R^c)\,\mathbb{P}(R^c)$$

$$= \left(\frac{1}{3}\right)\left(\frac{2}{5}\right) + \left(\frac{2}{3}\right)\left(\frac{3}{5}\right)$$

$$= \frac{8}{15}$$

since $\mathbb{P}(S) > \mathbb{P}(D) \Rightarrow$ she should switch

$\boxed{\textbf{Exercise 4}}$ A fair die is rolled repeatedly. Let X_n be the result of the n^{th} roll. So X_n takes values $\{1, \ldots, 6\}$, each with probability $1/6$, and the random variables $X_1, X_2, \ldots$ are all independent. Let

$$N = \min\{n : X_n = X_{n-1} = 4,\ n \geq 2\}$$

That is, N is the first time you roll two consecutive 4's. Find $E[N]$.
[Hint: condition on the outcomes of the first two rolls, and imitate the argument we used in class for the 'rat in a maze' problem to derive a recursive equation for $\mathbb{E}[N]$.]

$R_1 = \{$ first roll is $4\}$

$R_2 = \{$ second roll is $4\}$

$N =$ number of rolls until two successive 4's.

$\mathbb{E}[N \mid R_1^c] = 1 + \mathbb{E}[N]$ "clock starts over"

$\mathbb{E}[N \mid R_1, R_2] = 2$

$\mathbb{E}[N \mid R_1, R_2^c] = 2 + \mathbb{E}[N]$ "clock starts over".

Conditioning:

$\mathbb{E}[N] = \mathbb{E}[N \mid R_1]\,\mathbb{P}(R_1) + \mathbb{E}[N \mid R_1^c]\,\mathbb{P}(R_1^c)$.

$\mathbb{E}[N \mid R_1] = \mathbb{E}[N \mid R_1, R_2]\,\mathbb{P}(R_2)$
$+ \mathbb{E}[N \mid R_1, R_2^c]\,\mathbb{P}(R_2^c)$

$$= 2\left(\frac{1}{6}\right) + \left(2 + \mathbb{E}[N]\right)\left(\frac{5}{6}\right)$$

$$\Rightarrow \mathbb{E}[N] = \left(2 + \frac{5}{6}\mathbb{E}[N]\right)\frac{1}{6}$$

$$+ \left(1 + \mathbb{E}[N]\right)\frac{5}{6}$$

$$\Rightarrow \mathbb{E}[N] = 42$$

("the answer to everything").

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Exercise Suppose X is an exponential random variable. One of the following three formulas is correct:

$$
\begin{aligned}
(a) \quad \mathbb{E}[X^2 \,|\, X > 1] &= \mathbb{E}[(X+1)^2] \\
(b) \quad \mathbb{E}[X^2 \,|\, X > 1] &= \mathbb{E}[X^2] + 1 \\
(c) \quad \mathbb{E}[X^2 \,|\, X > 1] &= (\mathbb{E}[X] + 1)^2
\end{aligned}
$$

Without doing computations, use the memoryless property of the exponential distribution to explain which answer is correct.

Given $X > 1$, memoryless property says

$$X = 1 + X'$$

where $X' \sim$ exponential, same rate as X.

$$
\begin{aligned}
\Rightarrow \quad \mathbb{E}[X^2 \,|\, X > 1] &= \mathbb{E}[(1+X')^2 \,|\, X > 1] \\
&= \mathbb{E}[(1+X')^2] \quad \text{as } X' \text{ indep. of } \{X>1\} \\
&= \mathbb{E}[(1+X)^2]
\end{aligned}
$$