Math 5110 Applied Linear Algebra -Fall 2021.

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Homework 4.

(You can use Matlab if needed, e.g. eigenvalues by eig(A))

Question 1. Suppose $A \in \mathbb{F}^{m\times n}$ and $B \in \mathbb{F}^{n\times m}$ and $n \geq m$.

(1) Show that AB and BA has the same non-zero eigenvalues with the same algebraic multiplicities.

(2) If 0 is an eigenvalue of AB with algebraic multiplicity k, what is the algebraic multiplicity of 0 as eigenvalue of BA.

Question 2. (1) Find the characteristic polynomial of $B = \begin{bmatrix} 0 & -c_0 \\ 1 & -c_1 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 0 & -c_0 \\ 1 & 0 & -c_1 \\ 0 & 1 & -c_2 \end{bmatrix}$.

(2) Shows that every monic polynomial

$$f(t) = t^n + c_{n-1}t^{n-1} + \cdots + c_1 t + c_0$$

is the characteristic polynomial of some matrix B. (Hint: look at (1))

The following two questions are about Cayley-Hamilton Theorem and Jordan normal forms.

Question 3. Let A and B be 2×2 matrices such that $(AB)^2 = \mathbf{0}$. Prove that $(BA)^2 = \mathbf{0}$.

Question 4. (1) Let A be a 3×3 matrix such that the traces $tr(A^k)$=0 for $k = 1, 2, 3$. Show that all eigenvalues of A are zeros.

(2) Is there a 3×3 nilpotent matrix such that $A^3 \neq \mathbf{0}$?

(Remark for (1): Actually, after we analyzed the problem, we can prove the problem for an $n\times n$ matrix. When we start the writing, we can consider all non-zero, distinct eigenvalues λ_1, ..., λ_s with algebraic multiplicity $k_1, ..., k_s \geq 1$ and show that this is impossible. The writing will be clear.)

Question 5. Consider the matrix

$$A = \begin{bmatrix} -3 & 4 & 4 \\ -5 & 9 & 5 \\ -7 & 4 & 8 \end{bmatrix}$$

The aim is to find a matrix $M \in \mathbb{R}^{3\times 3}$ such that $M^2 = A$ (a "square root" of A).

(1) Find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. (Use Matlab)

(2) Let M be in $\mathbb{R}^{3\times 3}$ and let us assume that $M^2 = A$. Let us consider $N = P^{-1}MP$. Show that $N^2 = D$. Then prove that N commutes with D, i.e., $ND = DN$.

(3) Explain that N is thus necessarily diagonal.
Hint: Note that all the diagonal values of D are distinct.

(4) What can you say about N's possible values? Compute a matrix M, whose square is equal to A. How many different such matrices are there?