

Mixing Problems

A problem for which the one-compartment system provides a useful representation is the mixing of fluids in a tank. Let $x(t)$ represent the amount of a substance in a tank (compartment) at time t . To use the compartmental analysis model, we must be able to determine the rates at which this substance enters and leaves the tank. In mixing problems one is often given the rate at which a fluid containing the substance flows into the tank, along with the concentration of the substance in that fluid. Hence, multiplying the flow rate (volume/time) by the concentration (amount/volume) yields the input rate (amount/time):

$$\begin{aligned} \text{input rate} &= \text{flow rate} \left[\frac{\text{volume}}{\text{time}} \right] \times \text{concentration} \left[\frac{\text{amount}}{\text{volume}} \right] \\ &\qquad\qquad\qquad \underbrace{\qquad\qquad}_{R_{in}(t)} \\ &= \left[\frac{\text{amount}}{\text{time}} \right] \end{aligned}$$

The output rate of substance is usually more difficult to determine. If we are given the exit rate of the mixture of fluids in the tank, then how do we determine the concentration of the substance in the mixture? One simplifying assumption that we might make is that the concentration is kept uniform in the mixture. Then we can compute the concentration of the substance in the mixture by dividing the amount $x(t)$ by the volume of the mixture in the tank at time t . Multiplying this concentration by the exit rate of the mixture then gives the desired output rate of the substance.

$$\begin{aligned} \text{output rate} &= \text{exit rate} \left[\frac{\text{volume}}{\text{time}} \right] \times \text{concentration in tank} \left[\frac{x(t)}{\text{volume}} \right] = \left[\frac{x(t)}{\text{time}} \right]. \\ &\qquad\qquad\qquad \underbrace{\qquad\qquad}_{R_{out}(t)} \qquad\qquad\qquad \underbrace{\qquad\qquad}_{C_{out}(t)} \end{aligned}$$

Mathematical Model:

$$\frac{dx}{dt} = R_{in} \cdot C_{in} - R_{out} \cdot C_{out}$$

↑ constants ↑ variable
typically constants

$$[x' = a - bx]$$

a first-order linear equation

A Lake Pollution Model

Pollution in our lakes and rivers has become a major problem, particularly over the past 50 years. In order to improve this situation in the future it is necessary to gain a good understanding of the processes involved. We need to be able to predict how pollutant amounts or concentrations vary over time and under different management strategies.



The Great Lakes — Superior, Michigan, Huron, Erie, and Ontario — are the largest system of fresh surface water on earth, by area. They contain approximately 18% of the world's fresh water supply.

As usual, we need to make some assumptions while developing the model, and we assume that the lake has a constant volume V , and that it is continuously well mixed so that the pollution is uniform throughout.

Let $C(t)$ be the concentration of the pollutant in the lake at time t . Let F be the rate at which water flows out of the lake in m^3/day . Since the volume is constant, we have

$$\left\{ \begin{array}{l} \text{flow of} \\ \text{mixture} \\ \text{into lake} \end{array} \right\} = \left\{ \begin{array}{l} \text{flow of} \\ \text{mixture} \\ \text{out of lake} \end{array} \right\} = F$$

Applying the balance law to the mass of the pollutant $M(t)$ we can describe the process in words,

$$\left\{ \begin{array}{l} \text{rate of} \\ \text{change of} \\ \text{mass of} \\ \text{pollutant} \\ \text{in lake} \end{array} \right\} = \left\{ \begin{array}{l} \text{rate at} \\ \text{which the} \\ \text{pollutant} \\ \text{enters the} \\ \text{lake} \end{array} \right\} - \left\{ \begin{array}{l} \text{rate at} \\ \text{which the} \\ \text{pollutant} \\ \text{leaves the} \\ \text{lake} \end{array} \right\}$$

This translates into the differential equation for the changing mass

$$(1) \quad M'(t) = F \cdot c_{in} - F \frac{M(t)}{V}, \quad \left[\begin{array}{l} \text{balance law; relationship between} \\ \text{introduced variables} \end{array} \right]$$

where c_{in} is the concentration (in units of mass per unit of volume, such as g/m^3) of the pollutant in the flow entering the lake.

Now, since $M(t) = C(t)V$ we have $M'(t) = C'(t)V$ (since V is constant) and hence that $C'(t) = M'(t)/V$, we transform (1) to a differential equation for the concentration of the pollutant in the lake,

$$(2) \quad \boxed{\frac{dC}{dt} = \frac{F}{V} c_{in} - \frac{F}{V} C}$$

If the flow rate F is constant with time, then we can use separation of variables to solve the equation.

Example 1 Solve the initial-value problem

$$\frac{dC}{dt} = \frac{F}{V} c_{in} - \frac{F}{V} C, \quad C(0) = c_0. \quad [c_{in} = \text{constant}].$$

$(c_{in} - c \neq 0)$

Solution $\frac{dC}{dt} = \frac{F}{V} (c_{in} - C) \Rightarrow \frac{dC}{c_{in} - C} = \frac{F}{V} dt \Rightarrow \int \frac{dC}{c_{in} - C} = \int \frac{F}{V} dt \Rightarrow$

$$-\ln|c_{in} - C| = \frac{F}{V} t + K \Rightarrow \ln|c_{in} - C| = K - \frac{F}{V} t \Rightarrow$$

[constant of integration]

$$e^{\ln|c_{in} - C|} = e^{K - \frac{F}{V} t} \Rightarrow |c_{in} - C| = e^K e^{-\frac{F}{V} t} \Rightarrow c_{in} - C = K e^{-\frac{F}{V} t} \Rightarrow$$

$$C(t) = c_{in} - K e^{-\frac{F}{V} t}$$

$$C(0) = c_0 = c_{in} - K \Rightarrow K = c_{in} - c_0$$

$$\boxed{C(t) = c_{in} - (c_{in} - c_0) e^{-\frac{F}{V} t}}$$

The solution can be divided into two parts:

$$C(t) = \underbrace{c_{in} \left[1 - e^{-\frac{F}{V} t} \right]}_{\substack{\text{contribution from} \\ \text{the pollution inflow}}} + \underbrace{c_0 e^{-\frac{F}{V} t}}_{\substack{\text{contribution from the initial data}}}$$

Also note that

$$\lim_{t \rightarrow \infty} C(t) = \lim_{t \rightarrow \infty} (C_{in} - (C_{in} - C_0) e^{-\frac{F}{V}t}) = C_{in}$$

and concentration in the lake increases/decreases steadily to C_{in} .

Example 2 Case when input concentration C_{in} varies with time, that is $C_{in} = C_{in}(t)$, solve the initial-value problem:

$$\begin{cases} \frac{dC}{dt} = \frac{F}{V} C_{in}(t) - \frac{F}{V} C \\ C(0) = C_0. \end{cases}$$

Solution

$$\frac{dC}{dt} + \frac{F}{V} C = \frac{F}{V} C_{in}(t)$$

integrating factor: $\rho(t) = \frac{F}{V} \Rightarrow u(t) = e^{\int \frac{F}{V} dt} = e^{\frac{F}{V} t}$

We rewrite the equation as follows:

$$\frac{d}{dt} (e^{\frac{F}{V}t} C) = \frac{F}{V} C_{in}(t) e^{\frac{F}{V}t} \quad \left[\begin{array}{l} \text{Using the F.T.C.:} \\ \int_a^b f(x) dx = F(b) - F(a), \quad F' = f \end{array} \right]$$

$$\int_0^t \frac{d}{ds} (e^{\frac{F}{V}s} C) ds = \int_0^t \frac{F}{V} C_{in}(s) e^{\frac{F}{V}s} ds$$

$$e^{\frac{F}{V}s} C(s) \Big|_0^t = \int_0^t \frac{F}{V} e^{-\frac{F}{V}s} C_{in}(s) ds$$

$$e^{\frac{F}{V}t} C(t) - e^{\frac{F}{V}0} \underbrace{C(0)}_{C_0} = \int_0^t \frac{F}{V} e^{-\frac{F}{V}s} C_{in}(s) ds.$$

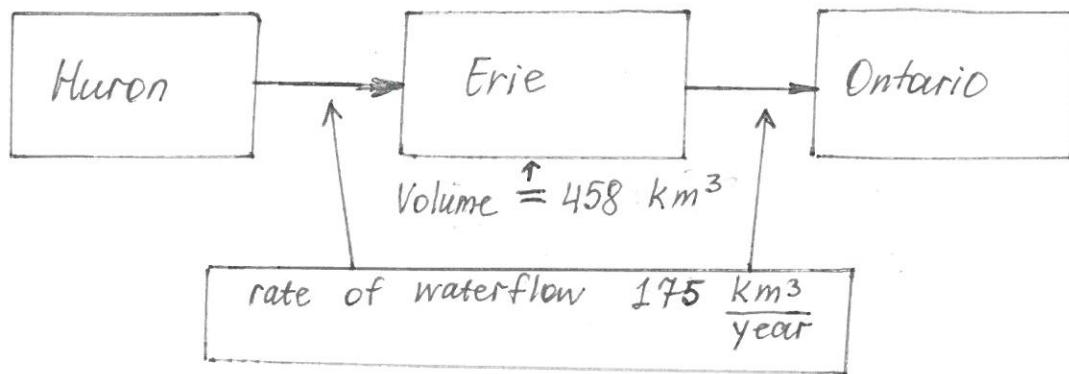
$$e^{\frac{F}{V}t} C(t) = C_0 + \frac{F}{V} \int_0^t e^{-\frac{F}{V}s} C_{in}(s) ds$$

$$C(t) = \underbrace{C_0 e^{-\frac{F}{V}t}}_{\text{the response corresponds to the initial data}} + \underbrace{\frac{F}{V} e^{-\frac{F}{V}t} \int_0^t e^{-\frac{F}{V}s} C_{in}(s) ds}_{\text{the response corresponds to the input.}}$$

the response corresponds to the initial data

the response corresponds to the input.

Example 2 Cleaning up Lake Erie



Problem: - The present concentration of pollutants in Erie is $5 \times \underline{\text{concentration}}_c$ of pollutants in Huron;
 - Pollution of Erie is ordered to stop

Question: - When will the pollution concentration in Erie reach $2 \times \underline{\text{concentration}}_c$ in Huron?

Solution Using the differential equation describing the situation:

$$M' = F c_{in} - \frac{F}{V} M \quad [M = C(t) \cdot V]$$

with $M(0) = 5C \cdot 458$, we need to find t^* so that $M(t^*) = 2C \cdot 458$. It implies

$$M' = 175 \cdot C - \frac{175}{458} M \Leftrightarrow M' + \frac{175}{458} M = 175C \Rightarrow$$

the integrating factor:

$$U(t) = e^{\int \frac{175}{458} dt} = e^{\frac{175}{458} t} \Rightarrow \frac{d}{dt} \left(e^{\frac{175}{458} t} M(t) \right) = e^{\frac{175}{458} t} \cdot 175C \Rightarrow$$

$$M(t) = e^{-\frac{175}{458} t} \left[\int 175C e^{\frac{175}{458} t} dt + K \right] \Rightarrow$$

$$M(t) = e^{-\frac{175}{458} t} \left[\frac{175C}{175/458} e^{\frac{175}{458} t} + K \right] = 458C + ke^{-\frac{175}{458} t}$$

$$M(0) = 458C + K = 458 \cdot 5 \cdot C \Rightarrow K = 4 \cdot 458C = 1832C \Rightarrow$$

$$M(t) = 458C + 1832C e^{-\frac{175}{458} t} \quad - \text{the particular solution}$$

$$M(t^*) = 458C + 1832C e^{-\frac{175}{458} t^*} = 2C \cdot 458 = 916C \Rightarrow$$

$$t^* = \ln \left(\frac{916C - 458C}{1832C} \right) \cdot \left(-\frac{458}{175} \right) = \ln \left(\frac{1}{4} \right) \cdot \left(-\frac{458}{175} \right) \approx \boxed{3.6 \text{ years}}$$

Example 3 Lake Burley Griffin

Lake Burley Griffin in Canberra, the capital city of Australia, was created artificially in 1962 for both recreational and aesthetic purposes. In 1974 the public health authorities indicated that pollution standards set down for safe recreational use were being violated and this was attributed to the sewage works in Queanbeyan upstream (or rather the discharge of untreated sewage into the lake's feeder river).

It was established that, while the sewage plants certainly exacerbated the problem, there were significant contribution from rural and urban runoff as well, particularly during summer rainstorms.

In 1974 the mean concentration of the bacteria faecal coliform count was approximately 10^7 bacteria per m^3 at the point where the river feeds into the lake. The safety threshold for this faecal coliform count in the water is such that for contact recreational sports no more than 10% of total samples over a 30-day period should exceed 4×10^6 bacteria per m^3 .

Given that the lake was polluted, it is of interest to examine how, if sewage management were improved, the lake would flush out and if and when the pollution level would drop below the safety threshold.

The system can be modeled, very simple, under a few assumptions. Flow (F) into the lake is assumed equal flow out of the lake, and the volume (V) of the lake will be considered constant and is approximately $28 \times 10^6 m^3$. Further, the lake can be considered as well mixed in the sense that the pollution concentration throughout will be taken as constant. Under these assumptions a suitable differential equation model for the pollutant concentration is

$$\left\{ \begin{array}{l} \frac{dc}{dt} = \frac{F}{V} c_{in} - \frac{E}{V} c \quad (\text{D.E.}) \\ c(0) = c_0 \quad \text{I.C.} \end{array} \right.$$

where c_{in} is the concentration of the pollutant entering the lake, c_0 is the initial concentration of the pollutant.

The solution is

$$C(t) = C_{in} - (C_{in} - C_0) e^{-\frac{F}{V}t}$$

If $C_{in} = 0$ (only fresh water entering the lake)

$$F = 4 \times 10^6 \frac{m^3}{month} \quad (\text{a mean monthly flow}), V = 28 \times 10^6 m^3$$

$C_0 = 10^7 \frac{\text{bacteria}}{m^3}$ (the initial faecal coliform count as was measured in 1974)

How long does it take for the pollution level to drop below the safety threshold?

Solution Find t^* such that $C(t^*) = 4 \times 10^6 = C_*$

$$C(t^*) = C_0 e^{-\frac{F}{V}t^*} \Rightarrow C_* = C_0 e^{-\frac{F}{V}t^*} \Rightarrow t^* = -\frac{V}{F} \ln\left(\frac{C_*}{C_0}\right) \text{ or}$$

$$t^* = \frac{V}{F} \ln\left(\frac{C_0}{C_*}\right) = \frac{28 \times 10^6}{4 \times 10^6} \ln\left(\frac{10^7}{4 \times 10^6}\right) = 4 \ln\left(\frac{5}{2}\right) \approx 6.4 \text{ months}$$

It takes approximately 6 months for the pollution level to drop below the safety threshold. However, pure water entering the lake is not a very realistic scenario with three sewage plants and much farmland upstream, and so including the entrance of polluted river water into the lake model is essential.

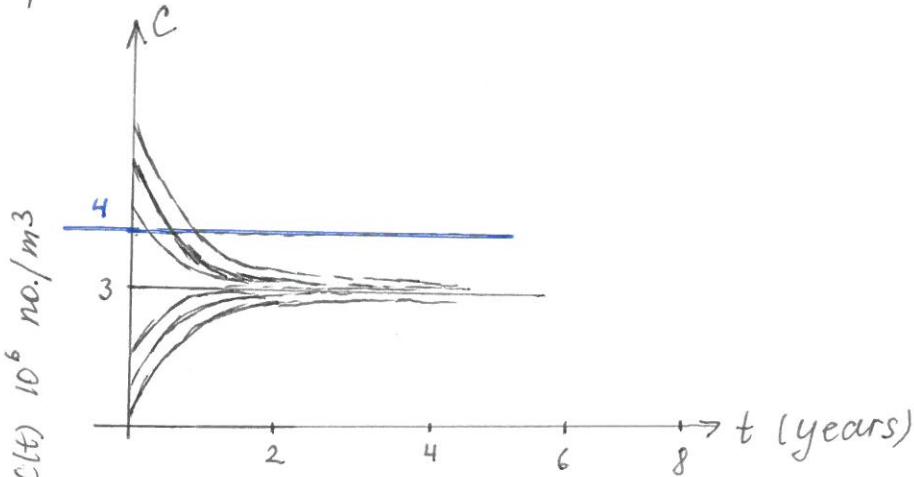


Figure 1: Pollution levels in Lake Burley Griffin over time with constant rates of flow and pollution. A range of initial concentrations has been used. The blue line is the pollution threshold level.

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> restart : with(plots) : cin := 3; V := 28; F := 4 * 12; threshold := 4; init_c := 10; de1 :=
  diff(C(t), t) = F/V * (cin - C(t)); soln := c0 → dsolve({de1, C(0) = c0}, C(t), numeric) :
  plot1 := c0 → odeplot(soln(c0), [t, C(t)], 0 .. 8) : list1 := seq(plot1(i/2), i = 1 .. 12) :
  line1 := plot([[0, threshold], [8, threshold]]) : display(list1, line1);

$$\frac{cin := 3}{V := 28}$$

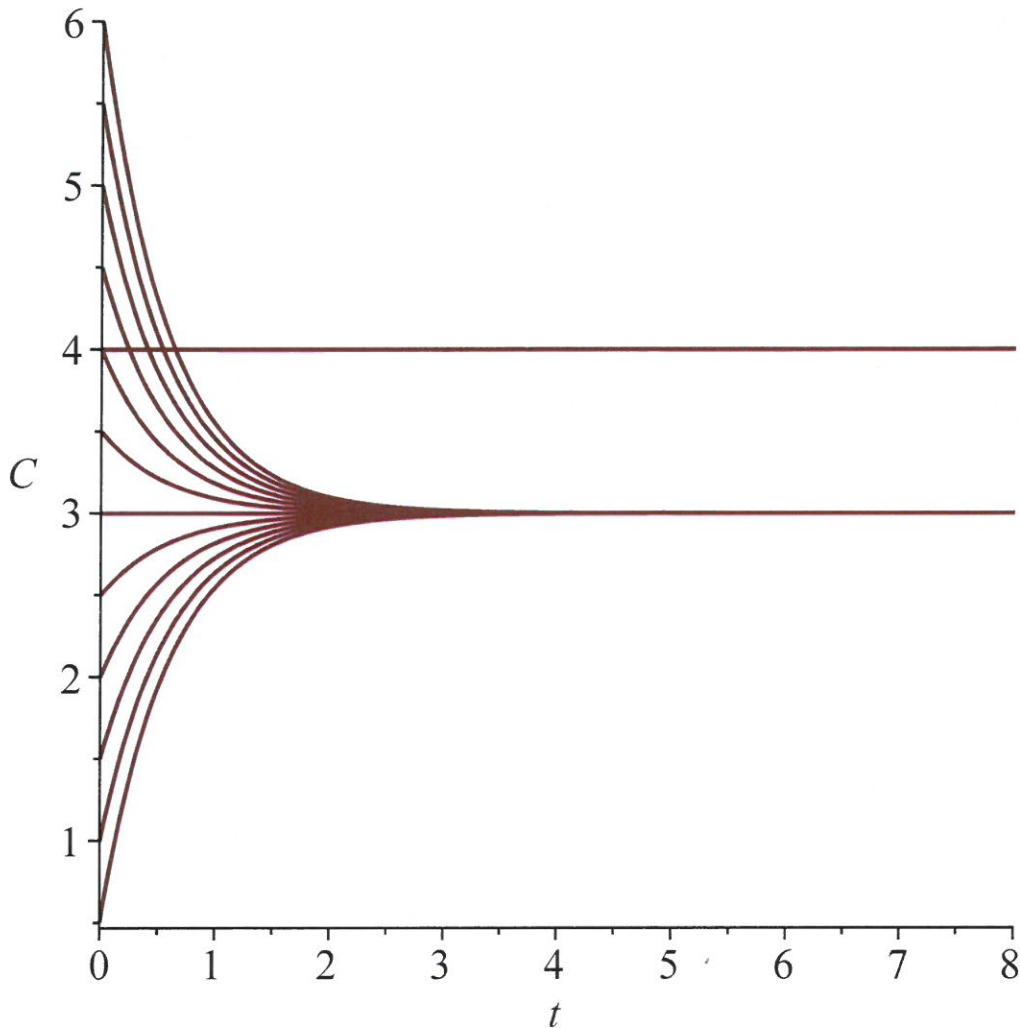

$$F := 48$$


$$threshold := 4$$


$$init\_c := 10$$


$$de1 := \frac{d}{dt} C(t) = \frac{36}{7} - \frac{12 C(t)}{7}$$


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However, the model is still simplistic. Flow rates change over the year with on average a seasonal pattern, while the amount of pollution reaching the lake will itself be seasonal.

Assuming a sinusoidal pattern over the year, a rough estimate of the inflow concentration from the available data in the 1970s is

$$C_{in}(t) = 10^6 (10 + 10 \cos(2\pi t)) \frac{\text{bacteria}}{\text{m}^3}$$

and for the flow rate,

$$F(t) = 10^6 (10 + 6 \sin(2\pi t)) \frac{\text{m}^3}{\text{year}}$$

Note that the concentration of the pollutant increases when the flow is low

We need to re-solve the differential equation (2) with F no longer constant. See below the numerical solutions derived by using Maple.

Example 4 Drug assimilation into the blood.

We investigate two simple models of cold pill assimilation into the bloodstream. In the first model we consider a single cold pill and in the second a course of cold pills. The basic idea of using a compartmental diagram to build the models is used.

Background The drug dissolves in the gastrointestinal tract (GI-tract) and each ingredient is diffused into the bloodstream. They are carried to the locations in which they act and are removed from the blood by the kidneys and the liver. The assimilation and removal may occur at different rates for the different ingredients of the same pill.

General Compartment Model

We can consider this problem as a compartmental model with two compartments, corresponding to the GI-tract and the bloodstream.



Input-output compartmental diagram for drug assimilation.

```
> restart : with(plots) : cin := (10 + 10·cos(2·Pi·t)) ; V := 28 ; F := (10 + 6·sin(2·Pi·t)) ;
threshold := 4; init_c := 10; de1 := diff(C(t), t) = (F/V) · (cin - C(t)) ; soln := c0
→ dsolve( {de1, C(0) = c0}, C(t), numeric) : plot1 := c0 → odeplot(soln(c0), [t, C(t)], 0
..8) : list1 := seq(plot1(i/2), i=1..20) : line1 := plot([ [0, threshold], [8, threshold]]) :
display(list1, line1);
```

$$cin := 10 + 10 \cos(2 \pi t)$$

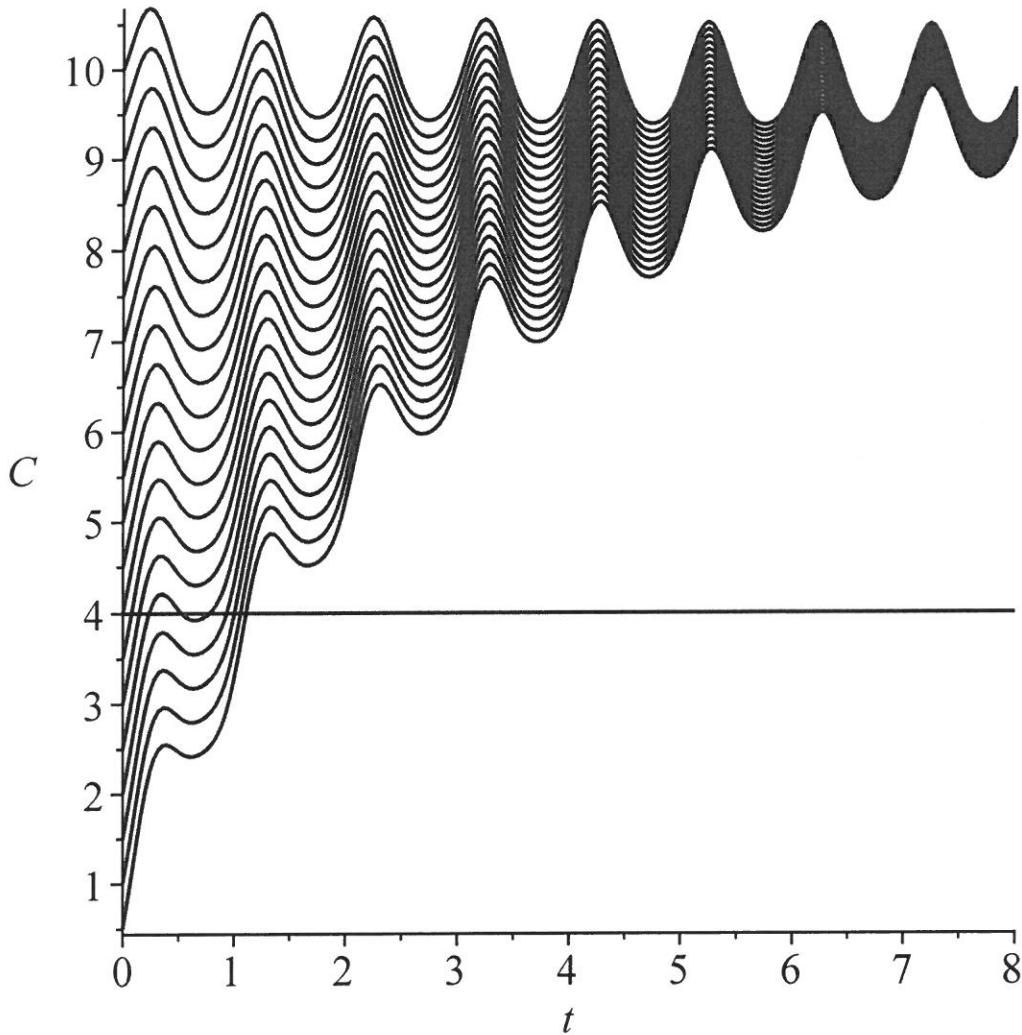
$$V := 28$$

$$F := 10 + 6 \sin(2 \pi t)$$

$$threshold := 4$$

$$init_c := 10$$

$$de1 := \frac{d}{dt} C(t) = \frac{(10 + 6 \sin(2 \pi t)) (10 + 10 \cos(2 \pi t) - C(t))}{28}$$



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=>
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GI-tract compartment has a single input and output and the blood stream compartment has a single input and output.

We have

$$\left\{ \begin{array}{l} \text{rate of change} \\ \text{of drug} \\ \text{in GI tract} \end{array} \right\} = \left\{ \begin{array}{l} \text{rate of} \\ \text{drug} \\ \text{intake} \end{array} \right\} - \left\{ \begin{array}{l} \text{rate} \\ \text{drug} \\ \text{leaves GI-tract} \end{array} \right\}$$

$$\left\{ \begin{array}{l} \text{rate of change} \\ \text{of drug} \\ \text{in blood} \end{array} \right\} = \left\{ \begin{array}{l} \text{rate} \\ \text{drug} \\ \text{enters blood} \end{array} \right\} - \left\{ \begin{array}{l} \text{rate} \\ \text{drug} \\ \text{leaves blood} \end{array} \right\}$$

We let $x(t)$ be the amount of a drug in the GI-tract at time t and $y(t)$ the amount in the bloodstream at time t . We consider two models: a single cold pill where there is no ingestion of the drug except that which occurs initially, and a course of cold pills where the drug intake is assumed to occur continuously.

The common cold remains without a cure. However, there are pills that can be taken to relieve some of the congestion and symptoms, such as watering eyes and a running nose, through the action of a decongestant and an antihistamine. The cold pills we consider in the following two models consist of these two drugs.

Model I: A single fast-dissolving cold pill

In the GI-tract we consider the pill to have been swallowed, and so after this event (over subsequent time) we have nothing more entering the GI-tract. The pill quickly dissolves and the drug begins to enter the bloodstream from the GI-tract. So, for the GI-tract there is only an output term.

Assuming the output rate is proportional to the GI-tract drug concentration, which is therefore proportional to the amount of drug in the bloodstream, then

$$(3) \quad \left\{ \begin{array}{l} \frac{dx}{dt} = -k_1 x, \\ x(0) = x_0, \end{array} \right.$$

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where x_0 is the amount of a drug in the pill, and k_1 is a positive coefficient of proportionality (the rate constant).

We assume that the instant the pill enters the GI-tract at $t=0$ it dissolves instantaneously so $x(0)=x_0$.

In the bloodstream the initial amount of the drug is zero, so $y(0)=0$. The level in the bloodstream increases as the drug diffuses from the GI-tract and decreases as the kidneys and liver remove it. Thus,

$$(4) \quad \begin{cases} \frac{dy}{dt} = k_1 x - k_2 y, \\ y(0) = 0, \end{cases}$$

with k_2 another positive constant of proportionality. The cold pill is made up of a decongestant and an antihistamine, and the coefficients of proportionality, k_1 and k_2 , are different for the different component in the pill. The coefficients also depend on the age and health of the person involved, and the concentration of a drug may also depend on the person's body mass, which means that for some people the doses may peak higher and/or faster than for the average person, and of course this can be potentially dangerous.

As expected, as t increases both x and y approaches zero although the rate at which this occurs depends on the coefficients k_1 and k_2 associated with each drug.

Exact solution for Model I.

The differential equations (3) and (4) are not coupled b/c the first equation is totally independent of y .

$$\begin{cases} \frac{dx}{dt} = -k_1 x \\ x(0) = x_0 \end{cases} \Rightarrow x(t) = x_0 e^{-k_1 t}$$

Using this solution, the second I.V.P. (4) becomes

$$\begin{cases} \frac{dy}{dt} + k_2 y = k_1 x_0 e^{-k_1 t} \\ y(0) = 0 \end{cases}$$

Solution Integrating factor $e^{\int k_2 dt} = e^{k_2 t}$

$$\frac{d}{dt} (e^{k_2 t} y) = k_1 x_0 e^{-k_1 t} e^{k_2 t} = k_1 x_0 e^{(k_2 - k_1)t} \Rightarrow$$

$$y(t) = e^{-k_2 t} \int_0^t k_1 x_0 e^{(k_2 - k_1)s} ds \Rightarrow \boxed{\text{if } k_1 = k_2 \text{ then}}$$

$$y(t) = e^{-k_2 t} \left[k_1 x_0 \frac{e^{(k_2 - k_1)s}}{k_2 - k_1} \Big|_0^t \right]$$

$$= e^{-k_2 t} \left[\frac{k_1 x_0}{k_2 - k_1} e^{(k_2 - k_1)t} - \frac{k_1 x_0}{k_2 - k_1} \right]$$

$$\boxed{y(t) = \frac{k_1 x_0}{k_2 - k_1} \left[e^{-k_1 t} - e^{-k_2 t} \right]}$$

Model II : A course of slowly dissolving cold pills

In reality, particularly for a cold, we take a course of pills rather than just one. To model this let us assume that the drug is delivered to the GI-tract continuously, which is reasonable for pills that dissolve slowly in the GI-tract. Thus we assume a constant rate of drug input, I (e.g. ml of drug per hour). Also, since the pill dissolves slowly, we assume that initially there is no drug in the GI-tract.

$$\begin{cases} \frac{dx}{dt} = I - k_1 x, & x(0) = 0, \\ \frac{dy}{dt} = k_1 x - k_2 y, & y(0) = 0, \end{cases}$$

where I is a positive constant representing the rate of ingestion of the drug (in grams per unit of time).

Once again we have a sequence (cascade) of linear equations that are not coupled and thus can be exactly solved sequentially:

$$\begin{cases} \frac{dx}{dt} + k_1 x = I \\ x(0) = 0 \end{cases} \Rightarrow u(t) = e^{\int k_1 dt} = e^{k_1 t} \Rightarrow$$

$$x(t) = e^{-k_1 t} \int_0^t I e^{k_1 s} ds = e^{-k_1 t} \left[I \frac{e^{k_1 s}}{k_1} \right]_0^t$$

$$= e^{-k_1 t} \left[\frac{I}{k_1} e^{k_1 t} - \frac{I}{k_1} e^0 \right]$$

$$x(t) = \frac{I}{k_1} (1 - e^{-k_1 t})$$

Then solving for $y(t)$:

$$\frac{dy}{dt} + k_2 y = k_1 \frac{I}{k_1} (1 - e^{-k_1 t}) \Rightarrow$$

$$\begin{cases} \frac{dy}{dt} + k_2 y = I (1 - e^{-k_1 t}) \\ y(0) = 0 \end{cases} \quad e^{\int k_2 dt} = e^{k_2 t} \text{ --- integrating factor}$$

$$\frac{d}{dt} (e^{k_2 t} y) = I e^{k_2 t} (1 - e^{-k_1 t}) = I e^{k_2 t} - I e^{(k_2 - k_1)t} \Rightarrow$$

$$y(t) = e^{-k_2 t} \int_0^t \{ I e^{k_2 s} - I e^{(k_2 - k_1)s} \} ds$$

$$= e^{-k_2 t} \left[\frac{I}{k_2} (e^{k_2 t} - 1) - \frac{I}{k_2 - k_1} (e^{(k_2 - k_1)t} - 1) \right]$$

$$= \frac{I}{k_2} (1 - e^{-k_2 t}) - \frac{I}{k_2 - k_1} (e^{-k_1 t} - e^{-k_2 t}) \Rightarrow$$

$$y(t) = \frac{I}{k_2} \left[1 - e^{-k_2 t} - \frac{k_2}{k_2 - k_1} (e^{-k_1 t} - e^{-k_2 t}) \right]$$

$$\begin{aligned}
 y(t) &= \frac{I}{k_2} \left[1 - \underbrace{e^{-k_2 t}}_{\sim} - \frac{k_2}{k_2 - k_1} e^{-k_1 t} + \frac{k_2}{k_2 - k_1} e^{-k_2 t} \right] \\
 &= \frac{I}{k_2} \left[1 - \frac{k_2}{k_2 - k_1} e^{-k_1 t} - \left[e^{-k_2 t} \cdot \frac{(k_2 - k_1)}{(k_2 - k_1)} - \frac{k_2}{k_2 - k_1} e^{-k_2 t} \right] \right] \\
 &= \frac{I}{k_2} \left[1 - \frac{k_2}{k_2 - k_1} e^{-k_1 t} - e^{-k_2 t} \left[\frac{k_2 - k_1 - k_2}{k_2 - k_1} \right] \right] \\
 &= \frac{I}{k_2} \left[1 - \frac{k_2}{k_2 - k_1} e^{-k_1 t} + \frac{k_1}{k_2 - k_1} e^{-k_2 t} \right] \Rightarrow \\
 y(t) &= \frac{I}{k_2} \left[1 - \frac{1}{k_2 - k_1} \left[k_2 e^{-k_1 t} - k_1 e^{-k_2 t} \right] \right] \quad (k_1 \neq k_2)
 \end{aligned}$$

As usual we are interested in what happens over time, in this case, as $t \rightarrow \infty$, then

$$\lim_{t \rightarrow \infty} x(t) = \lim_{t \rightarrow \infty} \frac{I}{k_1} \left(1 - \underbrace{e^{-k_1 t}}_{\rightarrow 0} \right) = \frac{I}{k_1}$$

$$\lim_{t \rightarrow \infty} y(t) = \lim_{t \rightarrow \infty} \frac{I}{k_2} \left(1 - \frac{1}{k_2 - k_1} \left(k_2 \underbrace{e^{-k_1 t}}_{\rightarrow 0} - k_1 \underbrace{e^{-k_2 t}}_{\rightarrow 0} \right) \right) = \frac{I}{k_2}$$

Our assumption of I constant holds in the case where drugs are embedded in resins and dissolve at constant rates allowing the drug to be released slowly and evenly over a period of hours. In reality some pills dissolve quickly and thus $I(t)$ should be a pulsing function, possibly a sinusoidal function, representing repeated doses, or some other function of t providing an initial and substantial boost to the drug level and then very little during the remaining time period before the next dose is taken.

HW #2 Ex. 2.8(a), 2.10, 2.11