Northeastern University, Department of Mathematics

MATH 5110: Applied Linear Algebra and Matrix Analysis.

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§12 Spectral Theorem and quadratic forms

Contents

1. Spectral Theorem

In this section, we deal real matrix.

An $n \times n$ matrix A is called **symmetric** if $A^T = A$, i.e.,

$$a_{ij} = a_{ji} \qquad \text{for all } i, j \in \{1, 2, ..., n\}$$

Example 1 (Diagonalizing a Symmetric Matrix)**.**

$$A = \begin{bmatrix} 10 & 6 \\ 6 & 1 \end{bmatrix}. \qquad B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}. \qquad C = \begin{bmatrix} 1 & 1 & 7 \\ 1 & 7 & 1 \\ 7 & 1 & 1 \end{bmatrix}.$$

Proposition 2. *A symmetric $n \times n$ matrix A has n real eigenvalues if they are counted with their algebraic multiplicities.*

Proposition 3. *Let A be a symmetric matrix and let λ, μ be two distinct eigenvalues of A with associated eigenvectors $\vec{v}, \vec{w}$. Then*

$$\vec{v} \cdot \vec{w} = 0.$$

E_λ is orthogonal to E_μ for distinct eigenvalues λ, μ (in that $\vec{v} \cdot \vec{w} = 0$ for all $\vec{v} \in E_\lambda$ and $\vec{w} \in E_\mu$).

Definition 4 (Orthogonal Diagonalization)**.** An $n \times n$ matrix is **orthogonally diagonalizable** if there exist diagonal matrix D and orthogonal matrix P such that

$$A = PDP^{-1} = PDP^T.$$

Theorem 5 (On Orhtogonal Diagonalizability)**.** *An $n \times n$ matrix A is orthogonally diagnonalizable if and only if A is a symmetric matrix.*

3. The Spectral Decomposition

Let A be an $n \times n$ matrix and let D and P be a diagonal and orthogonal matrix with $A = PDP^{-1}$.

Theorem 6 (Spectral Decomposition for Symmetric Matrices)**.**

2. **Quadratic forms and positive definite**

Definition 7. A function $p(x_1, ..., x_n)$ from $\mathbb{R}^n$ to $\mathbb{R}$ is call a **quadratic form**, if it is a linear combination of forms $x_i x_j$.

So, a quadratic form can be written as

$$p(x_1, ..., x_n) = \sum_{i,j} c_{ij} x_i x_j$$

Another way to write quadratic form is using symmetric matrices

$$p(x_1, ..., x_n) = \vec{x} \cdot A\vec{x} = \vec{x}^T A\vec{x}$$

The unique symmetric matrix A is called the matrix for the quadratic form.

Example 8. Consider $p(x_1, ..., x_3) = 3x_1^2 + 4x_2^2 - 5x_3^2 - 2x_1x_2 + 4x_1x_3 + 6x_2x_3$

Definition 9. An real symmetric matrix A is called **positive definite** if the quadratic form

$$\vec{x}^T A\vec{x} > 0$$

for all nonzero $\vec{x} \in \mathbb{R}^n$.
The matrix A is called **positive semidefinite** if the quadratic form

$$\vec{x}^T A\vec{x} \geq 0$$

for all $\vec{x} \in \mathbb{R}^n$.

Theorem 10. *(1) An real symmetric matrix A is positive definite if and only if all eigenvalues of A are positive.*
(2) An real symmetric matrix A is positive semidefinite if and only if all eigenvalues of A are non-negative.

Theorem 11. *Let V be an inner product space over $\mathbb{R}$ and let $\{\vec{b}_1, ..., \vec{b}_n\}$ be a basis of V. Then the Gram matrix G is positive definite.*

Here the Gram matrix G is defined by $G_{ij} = \langle \vec{b}_j, \vec{b}_i \rangle$.

Proposition 12. *Let A be an $m \times n$ real matrix. Then $A^T A$ is positive semidefinite. Further more, if $\operatorname{rank}(A) = n$, then $A^T A$ is positive definite.*