Northeastern University, Department of Mathematics

MATH 5110: Applied Linear Algebra and Matrix Analysis.

• Instructor: **He Wang** Email: **he.wang@northeastern.edu**

§6 Determinant

Contents

1. Motivation

Definition 1. Let A be an $n \times n$ matrix.
The **first row cofactor expansion** formula for the **determinant** of A is

Facts about **permutation groups**.

Let $[n]$ be the set of n integers $[n] = \{1, 2, \ldots, n\}$.

The **permutation group** (symmetric group) $S(n)$ is

A **transposition** is a permutation in $S(n)$ that only switch 2 numbers.

The **sign** of a permutation $\sigma \in S(n)$ is

$$\text{sign}(\sigma) = (-1)^{T(\sigma)}$$

where $T(\sigma)$ is the number of transposition of σ.

Another equivalent way to determine the sign of σ is

$$\text{sign}(\sigma) = (-1)^{N(\sigma)},$$

where $N(\sigma)$ is the number of inversions of σ.

An inversion of $(\sigma(1)\ \sigma(2)\ \ldots\ \sigma(n))$ is the pair of numbers $(\sigma(i) > \sigma(j))$ for $i < j$.

Proposition 2. *If τ is obtained from σ by switch two numbers i, j, then* $\operatorname{sign}(\tau) = -\operatorname{sign}(\sigma)$.

Theorem 3. *If A is an $n \times n$ matrix, then*

$$\begin{aligned}\det(A) &= \sum_{\sigma \in S(n)} \operatorname{sign}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)} \\ &= \sum_{\sigma \in S(n)} \operatorname{sign}(\sigma) a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}.\end{aligned}$$

Proof. This theorem can be proved by induction on n. For $n = 1$, it is true. Suppose the formula is true for $n-1$, let's show that it is true for n.

$$\begin{aligned}&\sum_{\sigma \in S(n)} \operatorname{sign}(\sigma) a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)} \\ &= \sum_{i=1}^{n} a_{1i} \sum_{\sigma \in S(n); \sigma(1)=i} \operatorname{sign}(\sigma) a_{2,\sigma(2)} \cdots a_{n,\sigma(n)} \\ &= \sum_{i=1}^{n} a_{1i} \sum_{\sigma \in S(n); \sigma(1)=i} (-1)^{1+i} \operatorname{sign}(\sigma(2) \ \dots \ \sigma(n)) a_{2,\sigma(2)} \cdots a_{n,\sigma(n)} \\ &= \sum_{i=1}^{n} (-1)^{1+i} a_{1i} \det A_{1i} \\ &= \det A\end{aligned}$$

$\square$

Example 4. Let A be the 3×3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{aligned}\operatorname{sign}(1 \ 2 \ 3) &= 1 \\ \operatorname{sign}(1 \ 3 \ 2) &= -1 \\ \operatorname{sign}(2 \ 1 \ 3) &= -1 \\ \operatorname{sign}(2 \ 3 \ 1) &= 1 \\ \operatorname{sign}(3 \ 1 \ 2) &= 1 \\ \operatorname{sign}(3 \ 2 \ 1) &= -1\end{aligned}$$

Hence we have

$$\det(A) = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33} - a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

Example 5. Find the determinant of $A = \begin{bmatrix} 0 & 4 & 2 \\ 5 & 2 & 2 \\ 0 & 2 & -1 \end{bmatrix}$. Is A invertible?

Definition 6. Let A be an $n \times n$ matrix. Its (i, j)**-th cofactor** C_{ij} is

Using cofactors, the first row cofactor expansion formula for the determinant of A is

Theorem 7. *The determinant of an $n \times n$ matrix A can be computed by he i-**th row** cofactor expansions*

*and the j-**th column** cofactor expansions*

Example 8. Find the determinant of $A = \begin{bmatrix} 1 & 2 & 3 & 1.2 \\ 0 & 0 & 0 & 2 \\ 5 & 6 & 7 & \pi \\ 0 & 1 & 2 & \sqrt{2} \end{bmatrix}$

Theorem 9. *Let A be an $n \times n$ triangular matrix, the determinant*

Example 10. Find out for which value of λ the matrix $A - \lambda I$ is not invertible, where $A = \begin{bmatrix} 2 & \sqrt{2} & 1.7 \\ 0 & 3 & 12 \\ 0 & 0 & 5 \end{bmatrix}$

Proposition 11.

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ b_1 + c_1 & b_2 + c_2 & \cdots & b_n + c_n \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ b_1 & b_2 & \cdots & b_n \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ c_1 & c_2 & \cdots & c_n \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

3. Row Operations and Determinant

Recall the three types of *elementary row operations*:
1. (Replacement)
2. (Interchange)
3. (Scaling)

Theorem 12 (Row Operations and the Determinant)**.** *Let A be an $n \times n$ matrix.*

Example 13. In a matrix A, if the i-th row equals the j-th row, then

Example 14. In a matrix A, if the i-th row is a scalar product of the j-th row, then

Theorem 15. *An $n \times n$ matrix A is invertible if and only if* $\det A \neq 0$.

Proposition 16. *Let A be an $n \times n$ matrix.*

$$\det(kA) = (k^n)(\det A).$$

Theorem 17 (Determinants of Products of Matrices)**.** *Let A and B be two $n \times n$ matrices.*

$$\det(AB) = (\det A)(\det B).$$

Proposition 18. *Let A be an $n \times n$ matrix.*

$$\det(A^m) = (\det(A))^m$$

Proposition 19. *Let A be an $n \times n$ invertible matrix.*

$$\det(A^{-1}) = \frac{1}{\det A}.$$

Question: How about $\det(A+B)$? Is it $\det(A) + \det(B)$?

Example 20. Find the determinant of $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$. Is A invertible?

Example 21. Find the determinant of $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 9 \\ 3 & 6 & 11 & 12 \\ 4 & 8 & 15 & 16 \end{bmatrix}$. Is A invertible?

Definition 22 (Elementary Column Operations)**.**
1. (Column Replacement) Add to one column the multiple of another column.
2. (Column Interchange) Interchange two columns.
3. (Column Scaling) Multiply all entries of a given column by a scalar.

Theorem 23 (Column Operations and the Determinant)**.** *Let A be an $n \times n$ matrix and let B be a matrix obtained from A by a single elementary row operation.*
1. If B is obtained from A by a Column Replacement operation, then

$$\det B = \det A.$$

2. If B is obtained from A by a Column Interchange operation, then

$$\det B = -\det A.$$

3. If B is obtained from A by a Column Scaling operation by a factor k, then

$$\det B = k \det A.$$

Theorem 24 (Determinant of the Transpose Matrix)**.**

$$\det A^T = \det A.$$

Example 25. Vandermonde determinant

$$\det(A) = \begin{vmatrix} 1 & 1 & 1 \\ a_1 & a_2 & a_3 \\ a_1^2 & a_2^2 & a_3^2 \end{vmatrix} =$$

More generally, by induction on n, we can proved that

$$\det(A) = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_n \\ a_1^2 & a_2^2 & a_3^2 & \cdots & a_n^2 \\ \vdots & \vdots & \cdots & \vdots & \\ a_1^{n-1} & a_2^{n-1} & a_3^{n-1} & \cdots & a_n^{n-1} \\ a_1^n & a_2^n & a_3^n & \cdots & a_n^n \end{vmatrix} =$$

Block Matrix.

Theorem 26. *If* $M = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$, *then,*

$$\det(M) = \det(A)\det(C).$$

Example 27. Find the determinant of $M = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} = \begin{bmatrix} 1 & 2 & 11 & \sqrt{3} \\ 2 & 3 & \pi & 12 \\ 0 & 0 & 3 & 9 \\ 0 & 0 & 1 & 4 \end{bmatrix}$

4. Linearity Property of the determinant function and Cramer's Rule

Let $A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_n \end{bmatrix}$ be an $n \times n$ matrix

Theorem 28 (Linearity and Determinants)**.** *The transformation T defined above is a linear transformation, that is*

Proof. By Theorems 24, 12 and Proposition 11. □

Example 29 (Finding matrix for the determinant transformation for a given A)**.**

Consider a matrix equation $A\vec{x} = \vec{b}$ in which A is an $n \times n$ matrix. Let

$$A_i(\vec{b}) = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_{i-1} & \vec{b} & \vec{a}_{i+1} & \dots & \vec{a}_n \end{bmatrix}$$

Theorem 30 (Cramer's Rule)**.** *If A is invertible, the unique solution $\vec{x}$ of the matrix equation $A\vec{x} = \vec{b}$ is given by*

Proof. First, from cofactor expansion, $\det(A_i(\vec{b})) = \sum_{j=1}^{n} b_j C_{ij}$.

$$\begin{aligned} a_{k1}x_1 + a_{k2}x_2 + \cdots a_{kn}x_n &= \frac{1}{\det(A)}(\sum_{i=1}^{n} a_{ki} \sum_{j=1}^{n} b_j C_{ij}) \\ &= \frac{1}{\det(A)}(\sum_{j=1}^{n} b_j \sum_{i=1}^{n} a_{ki} C_{ij}) \\ &= \frac{1}{\det(A)}(b_k \det(A)) \\ &= b_k \end{aligned}$$

for any $k = 0, 1, \cdots, n$. This verifies that $(x_1, \dots, x_n)$ is a solution of $A\vec{x} = \vec{b}$. □

Let C be the associated $n \times n$ matrix of cofactors defined as:

$$C = \begin{bmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ C_{21} & C_{22} & \dots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \dots & C_{nn} \end{bmatrix}$$

The transpose of C is called the **adjugate matrix of** A, denoted by $\text{adj}A$:

$$\text{adj}A = C^T = \begin{bmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \dots & C_{nn} \end{bmatrix}$$

Theorem 31. *If A is a invertible matrix then $A^{-1} = \dfrac{1}{\det A} \cdot adjA$*