Northeastern University, Department of Mathematics

MATH 5110: Applied Linear Algebra and Matrix Analysis.

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§5. Coordinate and matrix of a transformation

Contents

Topics: 1. Coordinates; 2. matrix of linear transformations; 3. Change of coordinates;

1. Coordinates

Theorem 1 (Unique Representation Theorem). *Let V be a vector space and let $\mathscr{B} = \{\vec{b}_1, \dots, \vec{b}_p\}$ be a basis for V. Then* ***each*** *vector $\vec{v}$ in V can be written as a linear combination*

$$\vec{v} = c_1 \cdot \vec{b}_1 + \cdots + c_p \cdot \vec{b}_p,$$

for a **unique** *set of scalars $c_1, \dots, c_p$.*

Proof. Since $\mathscr{B} = \{\vec{b}_1, \dots, \vec{b}_p\}$ is a basis, $V = \text{Span}\mathscr{B}$. So, any $\vec{v} \in V$ can be written as a linear combination of $\vec{b}_1, \dots, \vec{b}_p$.
Suppose that

$$\vec{v} = c_1 \cdot \vec{b}_1 + \cdots + c_p \cdot \vec{b}_p, \text{ and } \vec{v} = d_1 \cdot \vec{b}_1 + \cdots + d_p \cdot \vec{b}_p$$

Then $c_1 \cdot \vec{b}_1 + \cdots + c_p \cdot \vec{b}_p = d_1 \cdot \vec{b}_1 + \cdots + d_p \cdot \vec{b}_p$. Hence $(c_1 - d_1) \cdot \vec{b}_1 + \cdots + (c_p - c_p) \cdot \vec{b}_p = \vec{0}$.
Then $c_1 = d_1$, $c_2 = d_2$, $\dots$, $c_p = d_p$, since $\{\vec{b}_1, \dots, \vec{b}_p\}$ is independent. $\square$

Definition 2 (Coordinates Relative to a Basis). Let V be a vector space and let $\mathscr{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ be a basis for V. The **coordinates of** $\vec{v} \in V$ relative to $\mathscr{B}$ are the unique weights $c_1, \dots, c_n \in \mathbb{F}$ for which

$$\vec{v} = c_1 \cdot \vec{b}_1 + \cdots + c_p \cdot \vec{b}_n,$$

In this case, we write

$$[\vec{v}]_{\mathscr{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

Example 3. [The standard basis for $\mathbb{R}^n$] The **standard basis** for $\mathbb{R}^n$ is the set $E = \{\vec{e}_1, \ldots, \vec{e}_n\}$. The associated E-coordinates are called the **standard coordinates** of a vector in $\mathbb{R}^n$, and $[\vec{x}]_E = \vec{x}$.

Definition 4. Let $\mathscr{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$ be a basis for a vector space V. The map

$$T\colon V \to \mathbb{F}^n, \text{ given by } T(\vec{x}) = [\vec{x}]_{\mathscr{B}}$$

is called the **coordinate map** from V to $\mathbb{F}^n$ with respect to $\mathscr{B}$.

The coordinate mapping allows us to view vectors $\vec{x}$ in the abstract vector space V by means of coordinates of vectors in the concrete and familiar vector space $\mathbb{R}^n$.

Theorem 5. *For any choice of basis $\mathscr{B}$ of the vector space V, the associated coordinate map $T(\vec{x}) = [\vec{x}]_{\mathscr{B}}$ is a one-to-one linear transformation from V onto $\mathbb{F}^n$. That is V is **isomorphic** to $\mathbb{F}^n$, i.e., $V \cong \mathbb{F}^n$.*

Proof. It is easy to verify that T is a linear transformation, i.e., $T(\vec{x}+\vec{y}) = T(\vec{x}) + T(\vec{y})$ and $T(k\vec{x}) = kT(\vec{x})$ for any $\vec{x}, \vec{y} \in V$ and any $k \in \mathbb{F}$.

For any vector $\vec{v} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{F}^n$, let $\vec{x} = c_1\vec{b}_1 + \cdots c_n\vec{b}_n \in V$, then $T(\vec{x}) = \vec{v}$. So, the coordinate map T is surjective.

Suppose there are $\vec{x}$ and $\vec{y} \in V$ such that $T(\vec{x}) = T(\vec{y})$. Since $T(\vec{x}) = [\vec{x}]_{\mathscr{B}} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and $T(\vec{y}) = [\vec{y}]_{\mathscr{B}} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$. So, $[\vec{x}]_{\mathscr{B}} = [\vec{y}]_{\mathscr{B}}$ and $\vec{x} = \vec{y}$. Hence T is injective. $\square$

Example 6. The standard basis for the vector space M_2 of all 2×2 matrices is

$$\left\{ E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, E_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, E_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, E_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

The coordinate map $T : M_2 \to \mathbb{R}^4$ sends a matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2$ to its coordinate $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \mathbb{R}^4$.

Example 7. [The Coordinate Mapping] Let V be the vector space of all polynomials of degree ≤ 2.

Example 8. [Coordinates Relative to a Basis] Consider a basis $\mathscr{B} = \{\vec{b}_1, \vec{b}_2\}$ for $\mathbb{R}^2$ where $\vec{b}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{b}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. Suppose $\vec{x} \in \mathbb{R}^2$ has the coordinate vector $[\vec{x}]_{\mathscr{B}} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$. Find $\vec{x}$.

2. The matrix of a transformation

Definition 9. Let $T : V \to W$ be a linear transformation between vector spaces over $\mathbb{F}$. Suppose $\mathscr{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$ is a basis for V and $\mathscr{C} = \{\vec{w}_1, \ldots, \vec{w}_m\}$ is basis for W.
The **matrix** C **of the transformation** corresponding to bases $\mathscr{B}$ and $\mathscr{C}$, (or the $\mathscr{B}$-$\mathscr{C}$-matrix of T) is a $m \times n$ matrix defined as

$$C = \big[[T(\vec{b}_1)]_{\mathscr{C}}\ [T(\vec{b}_2)]_{\mathscr{C}}\ \ldots\ [T(\vec{b}_n)]_{\mathscr{C}}\big]$$

Definition 10. The **rank of the linear transformation** T is defined to be the rank of matrix C.

The next theorem is easy but very important in applications.

Theorem 11. *With assumptions in Definition 9, for any $\vec{x} \in V$,*

$$[T(\vec{x})]_{\mathscr{C}} = C \cdot [\vec{x}]_{\mathscr{B}}$$

Proof. Suppose $\vec{x} = x_1\vec{b}_1 + \cdots + x_n\vec{b}_n \in V$, then,

$$\begin{aligned}
[T(\vec{x})]_{\mathscr{C}} &= [T(x_1\vec{b}_1 + \cdots + x_n\vec{b}_n)]_{\mathscr{C}} \\
&= [x_1T(\vec{b}_1) + \cdots + x_nT(\vec{b}_n)]_{\mathscr{C}} \\
&= x_1[T(\vec{b}_1)]_{\mathscr{C}} + \cdots + x_n[T(\vec{b}_n)]_{\mathscr{C}} \\
&= \big[[T(\vec{b}_1)]_{\mathscr{C}}\ [T(\vec{b}_2)]_{\mathscr{C}}\ \ldots\ [T(\vec{b}_n)]_{\mathscr{C}}\big] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \\
&= C \cdot [\vec{x}]_{\mathscr{B}}
\end{aligned}$$

□

Remark 12. The matrix C of a linear transformation $T : V \to W$ depends on the bases for both vector spaces V and W.

Remark 13. The **advantage** of dealing with transformation is that it is not depending on bases.

We can consider the matrix C in the diagram

$$\begin{tikzcd}
V \arrow[r, "T"] \arrow[d, "{[\ ]_{\mathscr{B}}}"'] & W \arrow[d, "{[\ ]_{\mathscr{C}}}"] \\
\mathbb{F}^n \arrow[r, "T_C"] & \mathbb{F}^m
\end{tikzcd}$$

The equation $[T(\vec{x})]_{\mathscr{C}} = C \cdot [\vec{x}]_{\mathscr{B}}$ means $[\ ]_{\mathscr{C}} \circ T = [\ ]_{\mathscr{B}} \circ T_C$, which means the diagram is commutative.

3. Change of coordinate

Now let's look at a particular cases of Theorem 11 in $\mathbb{F}^n$.

Theorem in §3 is a particular case when $V = \mathbb{F}^n$ and $W = \mathbb{F}^m$ with standard bases.

Another particular case is when $V = W = \mathbb{F}^n$ with $T : V \to W$ the identity map, i.e. $T(\vec{x}) = \vec{x}$. We use basis $\mathscr{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$ for $V = \mathbb{F}^n$ and use standard basis for $W = \mathbb{F}^n$.

$$\begin{tikzcd} V = \mathbb{F}^n \arrow[r, "\mathrm{id}"] \arrow[d, "{[\ ]_{\mathscr{B}}}"'] & W = \mathbb{F}^n \arrow[d, "{[\ ]_E}"] \\ \mathbb{F}^n \arrow[r, "T_C"] & \mathbb{F}^n \end{tikzcd}$$

The matrix $C = [T(\vec{b}_1)\ \ldots\ T(\vec{b}_n)] = [\vec{b}_1\ \ldots \vec{b}_n]$.

Proposition 14. *Let $\mathscr{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$ be a basis for $\mathbb{F}^n$ and let $\vec{x} \in \mathbb{F}^n$ be any vector. Let $P_{\mathscr{B}}$ be the $n \times n$ matrix whose columns are $\vec{b}_1, \cdots, \vec{b}_n$ written in the standard basis for $\mathbb{F}^n$*

$$P_{\mathscr{B}} = \begin{bmatrix} \vec{b}_1 & \ldots & \vec{b}_n \end{bmatrix}$$

Then the standard coordinates of $\vec{x} \in \mathbb{F}$ can be calculated from the $\mathscr{B}$-coordinates $[\vec{x}]_{\mathscr{B}}$ of $\vec{x}$ as

$$\vec{x} = P_{\mathscr{B}} \cdot [\vec{x}]_{\mathscr{B}}.$$

Definition 15 (Change-of-coordinates Matrix)**.** The matrix $P_{\mathscr{B}}$ from the previous theorem is called the **change-of-coordinates matrix** from the basis $\mathscr{B}$ to the standard basis $\mathscr{E} = \{e_1, \ldots, e_n\}$.

Proposition 16. *The change-of-coordinates matrix $P_{\mathscr{B}}$ is always **invertible**, and equation $\vec{x} = P_{\mathscr{B}} \cdot [\vec{x}]_{\mathscr{B}}$ can be used to find the $\mathscr{B}$-coordinates of $\vec{x}$ in terms of the standard coordinates of $\vec{x}$ as*

$$[\vec{x}]_{\mathscr{B}} = P_{\mathscr{B}}^{-1} \cdot \vec{x}.$$

This means that the matrix for the coordinate map $[\ ]_{\mathscr{B}} : \mathbb{F}^n \to \mathbb{F}^n$ is given by $P_{\mathscr{B}}^{-1}$.

Example 17. [The Change of Coordinates Matrix] Let $\vec{x} = \begin{bmatrix} 2 \\ 8 \end{bmatrix}$. Find the coordinate vector $[\vec{x}]_{\mathscr{B}}$ of $\vec{x}$ relative to the basis $\mathscr{B}$ for $\mathbb{R}^2$ as in the above example.

The third particular case of Theorem 11 is when $V = W = \mathbb{F}^n$ with basis $\mathscr{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$.

Proposition 18 (The matrix of a linear transformation)**.** *Let $\mathscr{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$ be a basis for $\mathbb{F}^n$. Let T be a linear transformation from $\mathbb{F}^n$ to $\mathbb{F}^n$. There is an $n \times n$ matrix C such that*

$$[T(\vec{x})]_{\mathscr{B}} = C[\vec{x}]_{\mathscr{B}}$$

The matrix C can by calculated by

$$C = \Big[[T(\vec{b}_1)]_{\mathscr{B}}\ [T(\vec{b}_2)]_{\mathscr{B}}\ \cdots\ [T(\vec{b}_n)]_{\mathscr{B}} \Big]$$

The matrix C is called the matrix of T respect to basis $\mathscr{B}$, or $\mathscr{B}$-matrix.

Theorem 19. *Let $\mathscr{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ be a basis for $\mathbb{F}^n$ and denote matrix $P = [\vec{b}_1\ \vec{b}_2 \dots \vec{b}_n]$. Let $T_A : \mathbb{F}^n \to \mathbb{F}^n$ be the transformation defined by an $n \times n$ matrix A, i.e., $T_A(\vec{x}) = A\vec{x}$. Let C be the $\mathscr{B}$-matrix of T_A. Then,*

$$A = PCP^{-1}$$

or equivalently,

$$C = P^{-1}AP$$

Proof. By Proposition 18, for any $\vec{x} \in \mathbb{F}^n$,

$$[T_A(\vec{x})]_{\mathscr{B}} = C[\vec{x}]_{\mathscr{B}}$$

So,

$$[A(\vec{x})]_{\mathscr{B}} = C[\vec{x}]_{\mathscr{B}}$$

By Proposition 16,

$$P^{-1}A(\vec{x}) = CP^{-1}(\vec{x}) \quad \text{for any } \vec{x} \in \mathbb{F}^n.$$

So $P^{-1}A = CP^{-1}$. □

We can consider this theorem as the commutative diagram

$$\begin{tikzcd}
\mathbb{F}^n \arrow[r, "T_A"] \arrow[d, "{[\ ]_{\mathscr{B}}}"'] & \mathbb{F}^n \arrow[d, "{[\ ]_{\mathscr{B}}}"] \\
\mathbb{F}^n \arrow[r, "T_C"] & \mathbb{F}^n
\end{tikzcd}
\qquad
\begin{tikzcd}
\vec{x} \in \mathbb{F}^n \arrow[r, "A"] \arrow[d, "P^{-1}"'] & \mathbb{F}^n \arrow[d, "P^{-1}"] \\
\mathbb{F}^n \arrow[r, "C"] & \mathbb{F}^n
\end{tikzcd}$$

So, $P^{-1}A\vec{x} = CP^{-1}\vec{x}$ for any $\vec{x} \in \mathbb{F}^n$ on the top left corner.

Example 20. Consider a basis $\mathscr{B} = \{\vec{b}_1, \vec{b}_2\}$ for $\mathbb{R}^2$ where $\vec{b}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{b}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. Suppose a transformation T is defined by matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. What is the matrix C of the transformation T respect to basis $\mathscr{B}$?

Example 21. Let T be the projection transformation onto a line $L = \text{Span}\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\}\ \mathbb{R}^3$. The matrix of T is $A = \frac{1}{14}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$

(1)Find a basis $\mathscr{B} = \{\vec{b}_1\ \vec{b}_2\ \vec{b}_3\}$ for $\mathbb{R}^3$ such that the $\mathscr{B}$-matrix of the T is the diagonal matrix $D = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$. (2) Equivalently, find a matrix $B = [\vec{b}_1\ \vec{b}_2\ \vec{b}_3]$ and D such that $A = PDP^{-1}$.

Step 1. Compare the columns of D. It is equivalent to find independent vectors $\vec{b}_1, \vec{b}_2, \vec{b}_3$ and numbers d_1, d_2, d_3 such that

$$T(\vec{b}_1) = d_1(\vec{b}_1), \quad T(\vec{b}_2) = d_2(\vec{b}_2), \quad T(\vec{b}_2) = d_2(\vec{b}_2)$$

Step 2. Use the geometric properties of the transformation to find those vectors and numbers. (We will develop algebraic method to solve this systematically.)
We need to find vectors $\vec{b}_1, \vec{b}_2, \vec{b}_3$ such that the projection $\text{proj}_L \vec{b}_i$ is the scalar product of $\vec{b}_i$.

Let $\vec{b}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Then, $A\vec{b}_1 = 1\vec{b}_1$. So, $d_1 = 1$.

Let $\vec{b}_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$. Then, $A\vec{b}_2 = \vec{0} = 0\vec{b}_2$. So, $d_2 = 0$.

Let $\vec{b}_1 = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$. Then, $A\vec{b}_3 = \vec{0} = 0\vec{b}_3$. So, $d_3 = 0$.

The key is to solve $T(\vec{x}) = \lambda\vec{x}$ or equivalently $A\vec{x} = \lambda\vec{x}$.