Northeastern University, Department of Mathematics

MATH 5110: Applied Linear Algebra and Matrix Analysis.

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§5. Coordinate and matrix of a transformation

Contents

$V = \mathbb{P}_2 = \{a_0 + a_1 t + a_2 t^2\}$ $\mathcal{B} = \{1, t, t^2\}$

1. Coordinates

Theorem 1 (Unique Representation Theorem). *Let $\mathscr{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ be a basis for an $\mathbb{F}$-vector space V. Then,*

span independent

for any $\vec{v} \in V$, there exits unique scalars $c_1 \cdots c_n \in \mathbb{R}$, s.t.

$$\vec{v} = c_1 \vec{b}_1 + c_2 \vec{b}_2 + \dots + c_n \vec{b}_n$$

- $V = \text{Span}\{\vec{b}_1 \dots \vec{b}_n\} \Longrightarrow$ existence.
- Suppose $\vec{v} = d_1 \vec{b}_1 + d_2 \vec{b}_2 + \dots + d_n \vec{b}_n$

$$(c_1 - d_1)\vec{b}_1 + \dots + (c_n - d_n)\vec{b}_n = \vec{0}$$

by independence $\Longrightarrow c_1 = d_1, \dots, c_n = d_n$

$V = \mathbb{P}_2 = \{a_0 + a_1 t + a_2 t^2\}$ $\mathcal{B} = \{1, t, t^2\}$

$\vec{v} = 3 + 4t + 5t^2 \longleftrightarrow (\vec{v})_{\mathcal{B}} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} \in \mathbb{R}^3$

$$\boxed{\vec{v} = c_1 \vec{b}_1 + \dots + c_n \vec{b}_n}$$

Definition 2 (Coordinates Relative to a Basis)**.** Let $\mathscr{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$ be a basis for an $\mathbb{F}$-vector space V. Then, The **coordinates of** $\vec{v} \in V$ **relative to** $\mathscr{B}$ are

$\mathbb{R}$

$$[\vec{v}]_B := \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^n$$

Example 3. [The standard basis for $\mathbb{R}^n$]

$\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \vec{v}$

$\vec{v} = (2)\vec{e}_1 + (3)\vec{e}_2 + (4)\vec{e}_3$

$[\vec{v}]_E = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \vec{v}$

fix

Definition 4. Let $\mathscr{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$ be a basis for an $\mathbb{F}$-vector space V. **The coordinate map** is

$\mathbb{R}$

$$V \longrightarrow \mathbb{R}^n$$

$$\vec{x} \longrightarrow [\vec{x}]_B$$

Theorem 5. *For any choice of basis $\mathscr{B}$ of the vector space V, the associated coordinate map is an isomorphism from V to $\mathbb{F}^n$.*

$\mathbb{R}$

- linear
- inj.
- surj.

$[\vec{x}+\vec{y}]_B = [\vec{x}]_B + [\vec{y}]_B$

Answered Classification of all finite-dim Vector Spaces

Example 6. The standard basis for the vector space $\mathbb{R}^{2\times 2}$ of all 2×2 matrices is

$$\left\{ E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, E_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, E_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, E_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

$$T : \mathbb{R}^{2\times 2} \to \mathbb{R}^4$$

$$aE_1 + bE_2 + cE_3 + dE_4 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \to \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

Example 7. Let V be the vector space of all polynomials of degree ≤ 1.

P_1

$\mathcal{B} = \{ \vec{b}_1 = 1 + t, \ \vec{b}_2 = 1 + 2t \}$

$\vec{v} = c_1 \vec{b}_1 + c_2 \vec{b}_2$

$\vec{v} = -5 + t$. Find $[\vec{v}]_{\mathcal{B}}$

Example 8. Consider a basis $\mathscr{B} = \{\vec{b}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}\}$ for $\mathbb{R}^2$.

(1) Suppose $\vec{x} = \begin{bmatrix} -5 \\ 1 \end{bmatrix}$. Find the coordinate vector $[\vec{x}]_{\mathscr{B}}$. $= \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} -5 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} -5 \\ 1 \end{bmatrix} = \vec{x} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} \qquad \left[\begin{array}{cc|c} 1 & -1 & -5 \\ 1 & 2 & 1 \end{array}\right]$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

(2) Suppose $\vec{x} \in \mathbb{R}^2$ has the coordinate vector $[\vec{x}]_{\mathscr{B}} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$. Find $\vec{x}$. $= -3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$= [\vec{b}_1 \ \vec{b}_2] [\vec{x}]_{\mathcal{B}}$

$P_{\mathcal{B}} = [\vec{b}_1 \cdots \vec{b}_n]$

Thm: $\boxed{\vec{x} = P_{\mathcal{B}} [\vec{x}]_{\mathcal{B}}}$

$\vec{x} = c_1 \vec{b}_1 + \cdots + c_n \vec{b}_n$

$= [\vec{b}_1 \cdots \vec{b}_n] \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

$[\vec{x}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1} \vec{x}$

$\mathbb{R}^m \xrightarrow{T} \mathbb{R}^n$ linear

$\vec{x} \to T(\vec{x}) = A\vec{x}$

$A = [T(\vec{e}_1) \cdots T(\vec{e}_n)]$

2. The matrix of a transformation

e.g.: $V \to W$

$f(x) \to f'(x) + 2f''(x)$

Definition 9. Let $T : V \to W$ be a linear transformation between vector spaces. Suppose $\mathscr{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$ is a basis for V and $\mathscr{C} = \{\vec{w}_1, \ldots, \vec{w}_m\}$ is basis for W.

The **matrix C of the transformation corresponding to bases $\mathscr{B}$ and $\mathscr{C}$**, (or the $\mathscr{B}$-$\mathscr{C}$-matrix of T) is

$$[T]_{\mathcal{B}\mathcal{C}} = C = \begin{bmatrix} [T(\vec{b}_1)]_{\mathcal{C}} & [T(\vec{b}_2)]_{\mathcal{C}} & \cdots & [T(\vec{b}_n)]_{\mathcal{C}} \end{bmatrix}$$

$m \times n$

$$\begin{tikzcd}
\vec{x} \in V \arrow[r, "T"] \arrow[d, shift right, "{[-]_{\mathcal{B}}}"', "\cong"] & W \arrow[d, shift left, "{[-]_{\mathcal{C}}}", "\cong"'] & T(\vec{x}) \arrow[d] \\
\mathbb{R}^n \arrow[r, dashed, "T_{\mathcal{C}}"] \arrow[u, shift right] & \mathbb{R}^m \arrow[u, shift left] & \\
{[\vec{x}]_{\mathcal{B}}} \quad \vec{v} \arrow[r] & C \cdot \vec{v} = & {[T(\vec{x})]_{\mathcal{C}}}
\end{tikzcd}$$

$$T_{\mathcal{C}} = [-]_{\mathcal{C}} \circ T \circ [-]_{\mathcal{B}}^{-1}$$

Theorem 10. *With assumptions in Definition 9, for any $\vec{x} \in V$,*

$$[T(\vec{x})]_{\mathscr{C}} = C \cdot [\vec{x}]_{\mathscr{B}}$$

$\vec{x} = x_1\vec{b}_1 + x_2\vec{b}_2 + \cdots + x_n\vec{b}_n$

$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$$[T(\vec{x})]_{\mathcal{C}} = \left[T\left(x_1\vec{b}_1 + x_2\vec{b}_2 + \cdots + x_n\vec{b}_n\right)\right]_{\mathcal{C}}$$

$$\overset{T \text{ linear}}{=} \left[x_1 T(\vec{b}_1) + x_2 T(\vec{b}_2) + \cdots + x_n T(\vec{b}_n)\right]_{\mathcal{C}}$$

$$\overset{[-]_{\mathcal{C}} \text{ linear}}{=} x_1[T(\vec{b}_1)]_{\mathcal{C}} + x_2[T(\vec{b}_2)]_{\mathcal{C}} + \cdots + x_n[T(\vec{b}_n)]_{\mathcal{C}}$$

$$= \begin{bmatrix} [T(\vec{b}_1)]_{\mathcal{C}} & \cdots & [T(\vec{b}_n)]_{\mathcal{C}} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = C[\vec{x}]_{\mathcal{B}}$$

Definition 11. The **rank of the linear transformation** T is defined to be the rank of the $\mathscr{B}$-$\mathscr{C}$-matrix of T.

$$V \xrightarrow{T} W$$

$$\operatorname{rank} T := \dim(\operatorname{im} T) \qquad \dim(\operatorname{im}) + \dim(\ker) = \dim V$$

3. Change of coordinates

Applications of Theorem ~~??~~ in $\mathbb{F}^n$.

(I) When $V = \mathbb{F}^n$ and $W = \mathbb{F}^m$ with standard bases.

$$\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix} = \vec{e}_1 \cdots \vec{e}_n \qquad \begin{bmatrix}0\\\vdots\\1\\\vdots\\0\end{bmatrix} = \vec{e}_1, \cdots, \vec{e}_n$$

$$\begin{tikzcd}
\vec{x} \quad \mathbb{R}^n \arrow[r, "T"] \arrow[d, "\mathrm{id}"'] & \mathbb{R}^m \quad T(\vec{x}) \arrow[d, "\mathrm{id}"] \\
\vec{x} \quad \mathbb{R}^n \arrow[r, "T"] & \mathbb{R}^m \quad T(\vec{x}) = C(\vec{x})
\end{tikzcd}$$

$$[T]_{EE} = C = [T(\vec{e}_1) \cdots T(\vec{e}_n)]$$

(II) When $V = W = \mathbb{F}^n$ with $T : V \to W$ the identity map. Use basis $\mathscr{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$ for $V = \mathbb{F}^n$ and use standard basis E for $W = \mathbb{F}^n$.

$$B = \{\vec{b}_1 \cdots \vec{b}_n\} \qquad E = \{\vec{e}_1 \cdots \vec{e}_n\}$$

$$\begin{tikzcd}
\vec{x} \quad \mathbb{R}^n \arrow[r, "T = \mathrm{id}"] \arrow[d, "{[\ ]_B}"'] & \mathbb{R}^n \arrow[d, "{[\ ]_E = \mathrm{id}}"] \\
\mathbb{R}^n \arrow[r, "T_C"] & \mathbb{R}^n
\end{tikzcd}$$

$$[\vec{x}]_B = C^{-1}\vec{x}$$

$$(\mathrm{id})_{BE} = C = [\,[T(\vec{b}_1)]_E \cdots [T(\vec{b}_n)]_E\,] = [\vec{b}_1 \cdots \vec{b}_n]$$

is also the matrix for $[\ ]_B^{-1}$

C^{-1} is the matrix for $[\ ]_B$

Proposition 12. *Let $\mathscr{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$ be a basis for $\mathbb{F}^n$. Let $P_{\mathscr{B}}$ be the $n \times n$ matrix*

$$C = P_{\mathscr{B}} = [\vec{b}_1 \ \ \ldots \ \ \vec{b}_n] = [\mathrm{id}]_{BE}$$

Then,

$$[\vec{x}]_B = C^{-1}[\vec{x}]_E$$

or

$$[\vec{x}]_E = C \cdot [\vec{x}]_B$$

$[id]_{BE} = C :=$

Definition 13. The matrix $P_{\mathscr{B}}$ from the previous theorem is called the **change-of-coordinates matrix** from the basis $\mathscr{B}$ to the standard basis $\mathscr{E} = \{\vec{e}_1, \ldots, \vec{e}_n\}$.

Proposition 14. *The change-of-coordinates matrix $P_{\mathscr{B}}$ is always* ***invertible****, and*

$$[\vec{x}]_{\mathscr{B}} = P_{\mathscr{B}}^{-1} \cdot \vec{x}.$$

(III) The third particular case of Theorem ?? 10 is when $V = W = \mathbb{R}^n$ with basis $\mathscr{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$.

$P = [\vec{b}_1 \cdots \vec{b}_n]$

Proposition 15. *Let $\mathscr{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$ be a basis for $\mathbb{R}^n$. Let T be a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$. There is an $n \times n$ matrix (called $\mathscr{B}$-matrix) C such that*

$$[T(\vec{x})]_{\mathscr{B}} = C[\vec{x}]_{\mathscr{B}}$$

$$C = \left[[T(b_1)]_B \ \cdots \ [T(b_n)]_B \right] = [T]_{BB}$$

$$\begin{tikzcd}
B \quad \vec{x} \in \mathbb{R}^n \arrow[r, "T", "A"'] \arrow[d, "{[\ ]_B}"', "P^{-1}"] & \mathbb{R}^n \quad B \arrow[d, "P^{-1}", "{[\ ]_B}"'] \\
\mathbb{R}^n \arrow[r, "T_C", "C"'] & \mathbb{R}^n
\end{tikzcd}$$

$$\begin{tikzcd}
\vec{x} \arrow[r] \arrow[d] & T(\vec{x}) = A\vec{x} \arrow[d] \\
{[\vec{x}]_B} \arrow[r] & C[\vec{x}]_B = [T(\vec{x})]_B
\end{tikzcd}$$

$P^{-1}A\vec{x} = CP^{-1}\vec{x}$

$P^{-1}A = CP^{-1}$

$A = PCP^{-1}$

or $C = P^{-1}AP$

Theorem 16. *Let $\mathscr{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$ be a basis for $\mathbb{R}^n$ and denote $P = [\vec{b}_1\ \vec{b}_2 \ldots \vec{b}_n]$. Let $T_A : \mathbb{R}^n \to \mathbb{R}^n$ be the transformation defined by an $n \times n$ matrix A. Let C be the $\mathscr{B}$-matrix of T_A.*

$$A = PCP^{-1}$$

$$[T]_{EE} = A = [id]_{BE}\,[T]_{BB}\,[id]_{EB}$$

Example 17. Consider a basis $\mathscr{B} = \{\vec{b}_1, \vec{b}_2\}$ for $\mathbb{R}^2$ where $\vec{b}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{b}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. Suppose a transformation T is defined by matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. What is the matrix C of the transformation T respect to basis $\mathscr{B}$? (Find the $\mathscr{B}$-matrix of T.)

$$T(\vec{x}) = A\vec{x} \qquad C = P^{-1}AP$$

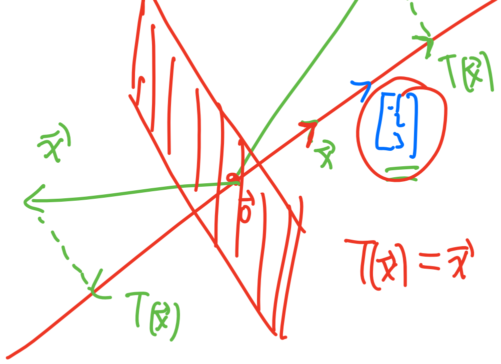
$$C = \left[[T(b_1)]_{\mathcal{B}} \quad [T(b_2)]_{\mathcal{B}} \right]$$

Example 18. Let T be the projection transformation onto a line $L = \text{Span}\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\}$ $\mathbb{R}^3$.

$\{\vec{b}_1\ \vec{b}_2\ \vec{b}_3\}$

C

Find a basis $\mathscr{B}$ for $\mathbb{R}^3$ such that the $\mathscr{B}$-matrix of the T is diagonal.



$$\begin{tikzcd}
\mathcal{B} \quad \mathbb{R}^3 \arrow[r, "T=\mathrm{proj}", "A"'] \arrow[d, "{[\ ]_{\mathcal{B}}}"'] & \mathbb{R}^3 \quad \mathcal{B} \arrow[d, "{[\ ]_{\mathcal{B}}}"] \\
\mathbb{R}^3 \arrow[r, "C"', "\parallel"] & \mathbb{R}^3
\end{tikzcd}$$

$$C = \begin{bmatrix} c_1 & & \\ & c_2 & \\ & & c_3 \end{bmatrix}$$

$$A = PCP^{-1}$$

$$AP = PC \qquad P = [\vec{b}_1 \ \vec{b}_2 \ \vec{b}_3]$$

$$A[\vec{b}_1 \ \vec{b}_2 \ \vec{b}_3] = [\vec{b}_1 \ \vec{b}_2 \ \vec{b}_3]\begin{bmatrix} c_1 & & \\ & c_2 & \\ & & c_3 \end{bmatrix}$$

$$[A\vec{b}_1 \ \ A\vec{b}_2 \ \ A\vec{b}_3] = [c_1 b_1 \ \ c_2 b_2 \ \ c_3 b_3]$$

$$\begin{cases} A\vec{b}_1 = c_1 \vec{b}_1 \\ A\vec{b}_2 = c_2 \vec{b}_2 \\ A\vec{b}_3 = c_3 \vec{b}_3 \end{cases} \qquad \underline{A\vec{x} = \lambda \vec{x}}$$

$$T\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right) = 1\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad \vec{b}_1, \vec{b}_2, \vec{b}_3$$

$$T\left(\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}\right) = \vec{0} = 0\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & & \\ & 0 & \\ & & 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}\right) = \vec{0} = 0\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$