Numerical Analysis 1 – Class 3

Friday, January 27th, 2023

Subjects covered

- Dense and sparse matrices.
- Matrix multiplication algorithms and complexity.
- Matrix norm, condition number.
- Introducing the SVD (singular value decomposition).
- Visualizing a matrix.

Reading

- Kutz, Chapters 2.1 and 15.1 – 15.2.
- C. Moler, "Numerical Computing with MATLAB", Chapter 2, "Linear Equations" (linked on Canvas).
- "The Extraordinary SVD", by C. Martin and M. Porter (linked on Canvas).
- "Visualizing a Matrix", S. Brorson (linked on Canvas).

Problems

Most of the following problems require you to write a program. For each program you write, please make sure you also write a test which validates your program. E-mail your answers to our TA: Hiu Ying Man, man.h@northeastern.edu.

Problem 1

The goal of this problem is to explore the relationship between matrix condition number and errors incurred when solving the linear system $Ax = b$ for x.

A family of matrices which appears frequently in applications are called the "Vandermonde matrices". The general form of a square Vandermonde matrix is shown below. (Note that a Vandermonde matrix may be rectangular, but for the purposes of this problem we will specialize on a square matrix.)

$$V_N = \begin{pmatrix} x_1^0 & x_1^1 & x_1^2 & x_1^3 & \cdots & x_1^{N-1} \\ x_2^0 & x_2^1 & x_2^2 & x_2^3 & \cdots & x_2^{N-1} \\ x_3^0 & x_3^1 & x_3^2 & x_3^3 & \cdots & x_3^{N-1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ x_N^0 & x_N^1 & x_N^2 & x_N^3 & \cdots & x_3^{N-1} \end{pmatrix}$$

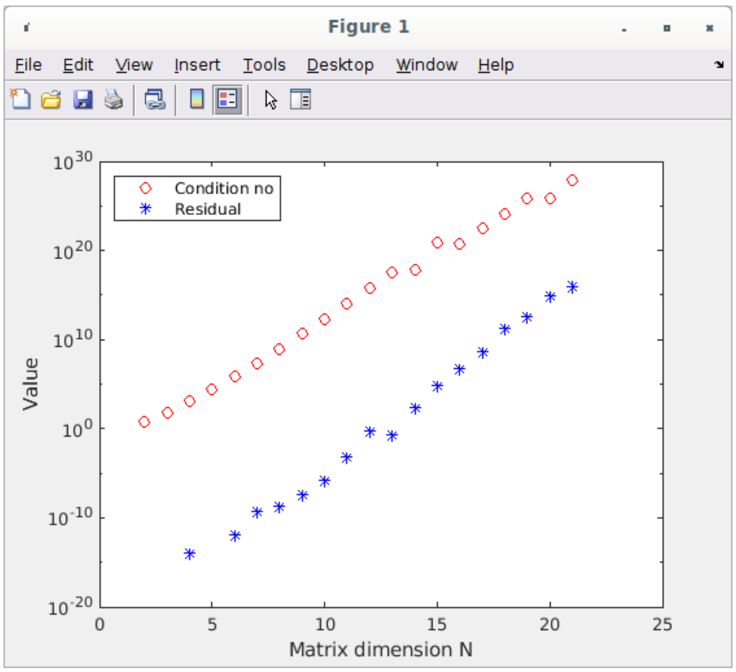
Consider the linear system $V_N\, x = b$. Write a program which does the following:

1. Start with a known vector $x_0 = [1, 2, 3, ..., N]^T$ where x_0 is of length N.
2. Create the Nth degree Vandermonde matrix V_N. (This is best done as a sub-function.) Matlab

has a function “vander” which also returns a Vandermonde matrix (using a different ordering of the elements). Please write your own program, but feel free to check your program’s return against Matlab’s “vander”.

3. Compute the matrix condition number $c_n = \mathrm{cond}(V_n)$.
4. Compute the matrix vector product $b = V_n x_0$.
5. Now perform the linear solve operation $x_c = V_n \setminus b$.
6. Compute the norm of the difference between the computed and the starting x_0: $r_n = \|x_0 - x_c\|$
   In theory, what should be the value of r_n (i.e. assuming perfect computations)?
7. Loop through values of n = 2, 3, …., 21 and make a plot of r_n and c_n vs. matrix order n. My plot is shown below.

Now answer the questions: How many orders of magnitude separate the value of the residual and the value of the condition number? Why is this the case? Can you change something in your program to demonstrate why this is true? You can turn in your answers in a .txt file.



Problem 2

The goal of this problem is to get some additional experience with thinking about symmetric matrices and quadratic forms. Consider the 2x2 matrix A where the parameter a takes values $a \in [0,2]$.

$$A = \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix}$$

Please do the following:

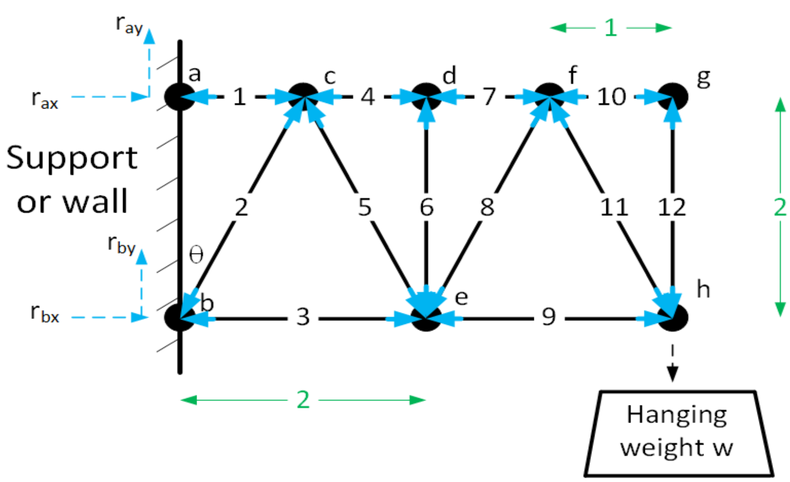
1. Compute the eigenvalues of A as functions of parameter a.
2. As mentioned in class, the singular values and eigenvalues of a *symmetric* matrix are related by $\sigma_i = |\lambda_i|$. Knowing this, please write down an expression for the condition number of matrix A as a function of the parameter a. What value of a causes A to become singular?

3. Consider the quadratic form $f(u)=u^T A u$ where *u* is the two-element vector describing a point in the [*x*, *y*] plane. That is, $u=[x,y]^T$. Write a program which computes and makes plots of $f(u)$ while varying the parameter a.
4. What happens to the quadratic form when the matrix is singular?

Problem 3

A truss is a architectural structure frequently used to create a rigid framework capable of bearing great loads. You will recognize many trusses in the real world acting as bridges, roof braces, and power-line supports amongst other structures. An example truss crane is shown on the right. Analysis of a truss involves computing the forces on all of its constituent beams to verify that none of the beams are near their breaking limit.

Consider the simple truss crane shown in the figure below. It is inspired by two sections of the arm of the crane in the photo. Its joints and beams are lettered/numbered for analysis. Joints a and b are considered fixed to a support wall and can't move either horizontally or vertically. This is entered into the analysis by employing reaction forces $r_{ax}, r_{ay}, r_{bx},$ and r_{by} coming from the wall and acting on joints a and b. The other joints (lettered) are connected by beams (numbered). The crane is used to lift a weight *w* which hangs vertically from joint h. The dimensions of the truss are annotated on the figure. To start this problem, take the weight to be *w* = 15 tons.



A simple analysis of this truss is performed by considering the force balance equations. They may be expressed as a matrix which relates the unknown internal forces on all beams to the known externally applied forces (i.e. the weight). The reaction forces from the wall are regarded as internal forces since they are unknown. We write this relation as

$$A f_i = -f_e$$

where f_i is a vector (list) of the internal forces on each beam, and f_e is the vector of known external forces. The matrix *A* holds the information detailing how the forces interact, i.e. the structure of the truss. The goal is to find the internal forces f_i .

To find the matrix *A*, we use the fact that the sum of all forces at each joint is zero in equilibrium. (Otherwise, the joint will move, which means the truss is collapsing.) The forces at each joint are resolved into *x* and *y* (horizontal and vertical) components. For each force component, we write an equation in which the forces sum to zero.

For the crane truss shown above the first few equations are:

- Joint a, x: $f_1 - r_{ax} = 0$
- Joint a, y: $r_{ay} = 0$
- Joint b, x: $f_2 \sin(\theta) + f_3 - r_{bx} = 0$
- Joint b, y: $f_2 \cos(\theta) + r_{by} = 0$
- Joint c, x: $-f_1 - f_2 \sin(\theta) + f_4 + f_5 \sin(\theta) = 0$
- Joint c, y: $-f_2 \cos(\theta) - f_5 \cos(\theta) = 0$
- etc.

The forces at each joint are depicted by the blue arrows in the figure; the direction of the arrow shows the direction of the force. We use the convention that force components pointing up or to the right are positive, down or to the left are negative. Your goal is to compute the internal forces on the beams and verify none of them exceed the limit of the beam. We assume the truss members have the following limits:

- Compression < 25 tons. If compression is larger than this limit, the beam will bend and break.
- Tension < 35 tons. If tension is larger than this limit, the beam will snap apart.
- The wall is capable of supporting any force, whether compression or tension.

Note that as drawn, compressive forces on the beams are negative, and tension forces on the beams are positive. (Compression and tension follow the blue arrows in the figure.)

Please do the following:

1. Complete the above analysis and write down the entire matrix equation describing the forces at each joint. Don't forget to include the reaction forces in the matrix.
2. Write down the external force vector, f_e .
3. Solve the system (using Matlab, or your preferred solver), and find the vector of member forces, $f_1, f_2, f_3,$ etc.
4. Will the crane collapse if it is used to lift 15 tons?
5. What happens if you try to lift 20 tons with this crane?

The goal of this problem is to create and use a sparse matrix in a in real-world application. If you need help with this problem, I have placed an analysis of a similar problem onto Canvas. It shows the general approach of how to create the equilibrium matrix for a truss, solve it, and then interpret the solution.