Northeastern University, Department of Mathematics

MATH 5110: Applied Linear Algebra and Matrix Analysis.

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§4. Bases

Contents

Ex: $A = \left[\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \ \vec{v}_2 = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} \right]$

Ex: $\{ \vec{v}_1 = \sin t, \ \vec{v}_2 = e^t \}$ independence

1. Linear Independence

$x_1 \sin t + x_2 e^t = 0 \Rightarrow x_1 = x_2 = 0$

$x_1 \vec{v}_1 + x_2 \vec{v}_2 = \vec{0}$

$\Leftrightarrow$ $A\vec{x} = \vec{0}$

Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_t$ be vectors in a vector space V.

Definition 1. • The set of vectors $\vec{v}_1, \ldots, \vec{v}_p$ in V is said to be **(linearly) independent** if

(*) $x_1 \vec{v}_1 + \cdots + x_p \vec{v}_p = \vec{0}$ only has trivial solution

$x_1 = \cdots = x_p = 0$

• The set $\{\vec{v}_1, \ldots, \vec{v}_p\}$ is said to be **(linearly) dependent** if

(*) has nontrivial solu.

An infinite subset W of a vector space V is said to be *linearly independent* if all finite subsets of W are linearly independent.

$\{1, t, t^2, t^3, \ldots, t^n, \ldots\}$

We say a vector $\vec{v}_i$ (for $i \geq 2$) is **redundant** if it is a linear combination of the preceding vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{i-1}\}$.

$\vec{v}_i = a_1 \vec{v}_1 + \cdots + a_{i-1} \vec{v}_{i-1}$

delete it

Proposition 2. *Suppose $\vec{v}_i$ is redundant in $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$. Then*

$$\operatorname{Span}\{\vec{v}_1, \ldots, \vec{v}_p\} = \operatorname{Span}\{\vec{v}_1 \ldots \hat{\vec{v}}_i \ldots \vec{v}_p\}$$

$\vec{x} = a_1\vec{v}_1 + \cdots a_i\vec{v}_i + \cdots + a_p\vec{v}_p$ $\subseteq$

$\supseteq$ ✓

Proposition 3. • *Suppose $\vec{v}_1 \neq \vec{0}$. The set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ is independent if and only if none of them is redundant.* $\Longleftrightarrow$

• *If the set $\{\vec{v}_1, \ldots, \vec{v}_p\}$ of vectors contains the zero vector $\vec{0}$, then it is linearly dependent.*

• *If a subset of the set $\{\vec{v}_1, \ldots, \vec{v}_p\}$ is linearly dependent, then $\{\vec{v}_1, \ldots, \vec{v}_p\}$ is dependent.*

Example 4. (1) A set $\{\vec{v}\}$ is linearly dependent if and only if $\vec{v} = \vec{0}$

(2) A set $\{\vec{u}, \vec{v}\}$ is linearly dependent if and only if $\vec{u} = c\vec{v}$ or $\vec{v} = c\vec{u}$

In $\mathbb{F}^n$, $(*) \iff A\vec{x} = \vec{0} \iff [A \mid \vec{0}]$

$A = [\vec{v}_1 \cdots \vec{v}_p]$

Proposition 5. *The set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\} \subset \mathbb{F}^n$ is independent if and only if*

$\iff$ rank $A = p$

Proposition 6. *If $p > n$, then a set $\{\vec{v}_1, \ldots, \vec{v}_p\}$ of vectors in $\mathbb{F}^n$ is linearly dependent.*

dim V

V

Span $\{$ $\}$

2. Basis of a vector space

Let V be vector space over $\mathbb{F}$.

Definition 7. A subset $B = \{\vec{b}_1, \ldots, \vec{b}_n\}$ of V is called a **basis** for V if

① $V = \text{Span}\{\vec{b}_1 \cdots \vec{b}_n\}$

② $\{\vec{b}_1, \ldots, \vec{b}_n\}$ is independent.

Example 8. Standard basis for $\mathbb{R}^n$.

Example 9. Find a basis for the vector space M_2 of all 2×2 matrices.

$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Example 10. Find a basis for the vector space P_2 of all polynomials of degree ≤ 2.

$\{1, t, t^2\}$ $\{a_0 + a_1 t + a_2 t^2\}$

$V = \mathbb{R}^p$

> **Theorem 11.** *If $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is independent, and $V = \text{Span}\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m\}$, then*
>
> $n \leq m$

$$\vec{v}_1 = a_{11}\vec{w}_1 + a_{12}\vec{w}_2 + \cdots + a_{1m}\vec{w}_n$$
$$\vec{v}_2 = a_{21} \quad - \quad - \quad - \quad a_{2m}$$
$$\vec{v}_n = a_{n1} \quad \text{———} \quad a_{nm}$$

$p \times n \qquad p \times m \qquad m \times n$

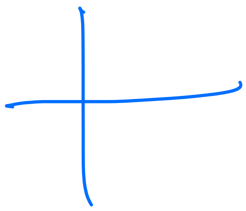
$$[\vec{v}_1 \cdots \vec{v}_n] = [\vec{w}_1 \cdots \vec{w}_m] A$$

$$V = WA$$

$$n = \text{rank}\, V = \text{rank}\, WA \leq \text{rank}\, A \leq m$$

> **Theorem 12** (Spanning Set Theorem). *Let V be a vector space and let $S = \{\vec{v}_1, \ldots, \vec{v}_p\}$ be a subset of V with $\text{Span}\{\vec{v}_1, \ldots, \vec{v}_p\} = H$.*
>
> - *If one of the vectors in S, say $\vec{v}_k$, is a linear combination of the remaining vectors in S, then the set $S - \{\vec{v}_k\}$ still spans H,*
>
> $$H = \text{Span}\{\vec{v}_1, \ldots, \vec{v}_{k-1}, \vec{v}_{k+1}, \ldots, \vec{v}_p\}$$
>
> - *If $H \neq \{\vec{0}\}$ then some subset of S is a basis for H.*

Proposition 13. *(1) Every spanning set of a finite-dimensional vector space can be reduced to a basis.*
(2) Any finite-dimensional vector space has a basis.
(3) Any independent set in a finite-dimensional vector space can be extended to a basis.



$\mathbb{R}^2$

$\{\vec{e}_1, \vec{e}_2\}$

$\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix}\right\}$

3. The Dimension of a Subspace

For a finite-dimensional vector space V, it has many different bases. However, they contain some common properties.

Theorem 14. *If $\mathscr{B} = \{\vec{b}_1, \dots, \vec{b}_p\}$ and $\mathscr{D} = \{\vec{d}_1, \dots, \vec{d}_m\}$ are two bases for V, then $p = m$.*

$p \le m$

$m \le p$

Definition 15 (The Dimension of a Vector Space)**.** The **dimension** of a vector space V is defined as

$\dim V =$ cardinality of a basis of V.

#

If $V = \{\vec{0}\}$

$\dim(V) = 0$

Lemma 16. *Suppose $\mathscr{B} = \{\vec{b}_1, \dots, \vec{b}_p\}$ is a basis for V.*
(1) Any set of more than p vectors is linearly dependent.
(2) Any set of less than p vectors can not span V.

#(independent set) $\le p$

$\begin{bmatrix}1 & 2 & 4\\0 & 3 & 1\\0 & 0 & 1\end{bmatrix}$

$P^2 = \{a_0 + a_1 t + a_2 t^2\}$

$\{1, t, t^2\}$ is a basis for P^2.

$x_1(1) + x_2(2+3t) + x_3(4+t+t^2) = 0$

Q: Is $\{1, 2+3t, 4+t+t^2\}$ a basis for P^2.

$\dim P^2 = 3$

Theorem 17 (The Basis Theorem). *Let V be a vector space with $\dim(V) = p \geq 1$.*

- any independent set $\{\vec{v}_1 \cdots \vec{v}_p\} \subset V$ is a basis of V.
- If $V = \operatorname{Span}\{\vec{w}_1 \cdots \vec{w}_p\}$, then $\{\vec{w}_1 \cdots \vec{w}_p\}$ is a basis of V.

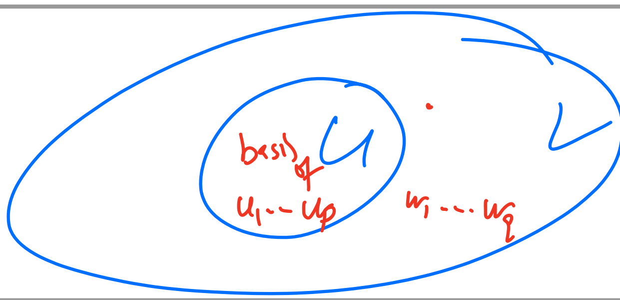
Ex: $\mathbb{R}^2$ $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ is a basis of $\mathbb{R}^2$ $\qquad \mathbb{R}^2 = \operatorname{Span}\{\vec{v}_1, \vec{v}_2\}$

$\dim \mathbb{R}^2 = 2$ $\qquad \subseteq \qquad \supseteq$

Theorem 18. *Let U be a subspace of a finite-dimensional space V. There is a subspace W such that $V = U \oplus W$.*

W is not unique.

$W = \operatorname{Span}\{w_1 \cdots w_q\}$



$\dim(U \oplus V) = \dim U + \dim V$

Theorem 19. *Let U and V be subspaces of a finite-dimensional space. Then*

$$\dim(U + V) = \dim U + \dim V - \dim(U \cap V)$$

$p+q+n \qquad p+q \qquad p+n \qquad p$

basis of $U \cap V$ $\quad w_1 \cdots w_p$ $\to$ basis of U $\quad w_1 \cdots w_p \; u_1 \cdots u_q$

$\to$ basis of V $\quad w_1 \cdots w_p \; v_1 \cdots v_n$

prop

$\boxed{V = U + W \text{ and } U \cap W = \{\vec{0}\} \iff V = U \oplus W}$

Corollary 20. *Let U and W be subspaces of an n-dimensional space V. Suppose $\dim U + \dim W = \dim V$ and $U \cap W = \{\vec{0}\}$, then*

$$V = U \oplus W$$

$\dim(U+W) = n$

$U + W \subseteq V$

Theorem 21. *Suppose V is a finite dimensional and $U_1, \ldots, U_p$ are subspaces of V such that $V = U_1 + \cdots + U_p$ and $\dim V = \dim U_1 + \cdots + \dim U_p$. Then $V = U_1 \oplus \cdots \oplus U_p$.*

Suppose $p=2$ $\quad V = U_1 + U_2$ and $\dim V = \dim U_1 + \dim U_2 \implies V = U_1 \oplus U_2$

ker T $\quad$ V $\quad$ im T $\qquad \Rightarrow U_1 \cap U_2 = \{\vec{0}\}$

4. Basis of Null space and range

$$\mathbb{R}^m \xrightarrow{T_A} \mathbb{R}^n, \quad \vec{x} \to A\vec{x}$$

Let $T : V \to W$ be a linear transformation.

- $\operatorname{rank}(T) := \dim(\operatorname{im} T)$
- $\operatorname{nullity}(T) := \dim(\ker T)$

Rank-Nullity Thm.

Theorem 22. *Let $T : V \to W$ be a linear transformation. Then*

$$\underbrace{\dim V}_{p+m} = \underbrace{\dim(\ker T)}_{p} + \dim(\operatorname{im} T)$$

- Suppose $\{\vec{u}_1 \cdots \vec{u}_p\}$ is a basis for $\ker T$, extend it to be a basis for V $\{\vec{u}_1 \ldots \vec{u}_p\ \vec{b}_1 \ldots \vec{b}_m\}$
- Claim: $\{T(\vec{b}_1), \ldots, T(\vec{b}_m)\}$ is a basis for $(\operatorname{im} T)$ (by def of basis)

Let A be an $m \times n$ matrix. The linear transformation defined by A is

$$T_A : \mathbb{F}^n \to \mathbb{F}^m$$
$$\vec{x} \to A\vec{x}$$

Theorem 23 (Basis for $\operatorname{im}(A)$). *A basis for the image* $\operatorname{im}(A)$ *is given by the pivot columns of A. In particular,* $\dim(\operatorname{im} A) = \operatorname{rank} A$.

$$A = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 1 & 6 & 2 & 2 \\ 2 & 12 & 2 & 3 \end{bmatrix} \longrightarrow \operatorname{rref} A = \begin{bmatrix} 1 & 6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Theorem 24 (Basis for $\ker(A)$). *Let A be an $m \times n$ matrix. Solve the matrix equation $A\vec{x} = \vec{0}$. Write $\vec{x}$ as a linear combination of vectors $\vec{v}_1, \ldots, \vec{v}_p$ with the weights corresponding to the free variables.*
Then $\{\vec{v}_1, \ldots, \vec{v}_p\}$ is a basis for $\ker(A)$. $= \{\text{all soln of } A\vec{x} = \vec{0}\}$

$$\vec{x} = x_2 \begin{bmatrix} -6 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \\ 3 \\ ? \end{bmatrix}$$

$T: \mathbb{R}^n \to \mathbb{R}^m$

$T_A : \mathbb{R} \to \mathbb{R}$

Proposition 25 (The Dimensions of ker(A) and im(A)). *Let A be an $m \times n$ matrix. Then,*

$$\dim(\ker(A)) + \underbrace{\dim(\operatorname{im}(A))}_{\operatorname{rank} A} = n = \dim(\mathbb{R}^n)$$

Proposition 26. *Let A be an $n \times n$ square matrix. A is* ***invertible****, if and only if*

• Q: • Find basis for $U+V$

• Find basis for $U \cap V$

• Ex: A $n \times n$ $T_A : \mathbb{R}^n \to \mathbb{R}^n$

• If $\ker A \cap \operatorname{im} A = \{\vec{0}\}$, then $\mathbb{R}^n = \ker A + \operatorname{im} A$

$\ker A + \operatorname{im} A = n$

Ex: $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\ker A = \operatorname{im} A = \operatorname{Span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$