

# MATH 7241 Fall 2022: Problem Set #5

Due date: Friday October 28

## SOLUTIONS.

**Reading:** relevant background material for these problems can be found on Canvas ‘Notes 4: Finite Markov Chains’. Also Grinstead and Snell Chapter 11.

**Grinstead and Snell:** see pages 442-443 and pages 444, 467-468 on Canvas. The text is available online (free!) at

<http://www.dartmouth.edu/~chance/>

Click on the link “A GNU book”.

**Exercise 1** Grinstead & Snell, p. 442: #3.  
[Note: ergodic means irreducible].

$$P = \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix}$$

a) Absorbing if  $1-a=1$  or  $1-b=1 \Leftrightarrow ab=0$

b) if  $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow$  irreducible but not regular

so irreducible, not regular  $\Leftrightarrow ab=1$

c)  $P^2 = \begin{pmatrix} 1-a(2-a-b) & a(2-a-b) \\ b(2-a-b) & 1-b(2-a-b) \end{pmatrix}$

If  $0 < a < 1, b > 0 \Rightarrow P^2 > 0$

same for  $0 < b < 1, a > 0$ ; so regular  $\Leftrightarrow 0 < ab < 1$

**Exercise 2** Grinstead & Snell, p. 442: #5.

$$a) \quad P = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \quad w_1 = \frac{3}{4}w_1 + \frac{1}{2}w_2 \\ \Rightarrow w_1 = 2w_2 \\ \Rightarrow w = \left( \frac{2}{3}, \frac{1}{3} \right)$$

$$b) \quad P = \begin{pmatrix} \frac{9}{10} & \frac{1}{10} \\ \frac{1}{10} & \frac{9}{10} \end{pmatrix} \quad \Rightarrow \quad w = \left( \frac{1}{2}, \frac{1}{2} \right)$$

$$c) \quad P = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$$

$$w_1 = \frac{3}{4}w_1 + \frac{1}{4}w_3 \Rightarrow w_1 = w_3$$

$$w_3 = \frac{1}{3}w_2 + \frac{1}{2}w_3 \Rightarrow w_2 = \frac{3}{2}w_3$$

$$\text{so } w_3 + \frac{3}{2}w_3 + w_3 = 1 \Rightarrow w_3 = \frac{2}{7}$$

$$w = \left( \frac{2}{7}, \frac{3}{7}, \frac{2}{7} \right)$$

**Exercise 3** 'Finite Markov Chains – Problems' file: Exercise 15

Doubly stochastic  $\Rightarrow \sum_{i=1}^M p_{ij} = 1$  all  $j$ .

Claim:  $w = \left(\frac{1}{M} \quad \frac{1}{M} \quad \dots \quad \frac{1}{M}\right)$  is stationary.

Proof:  $w_j = \frac{1}{M}$

$$\sum_{i=1}^M w_i p_{ij} = \sum_{i=1}^M \frac{1}{M} p_{ij} = \frac{1}{M} \sum_{i=1}^M p_{ij} = \frac{1}{M}$$

$$\Rightarrow w_j = \sum_{i=1}^M w_i p_{ij} \text{ for all } j$$

$\Rightarrow w$  is stationary.

**Exercise 4** GS, p. 444: #17.

let  $X_n = S_n \bmod 7$  i.e.  $X_n$  is remainder when  $S_n$  is divided by 7.

$$\text{Ran}(X_n) = \{0, 1, 2, 3, 4, 5, 6\} \quad \text{by } 7, \text{ so } S_n = 7k + X_n \text{ (some } k).$$

$X_n$  is a Markov chain with transition matrix

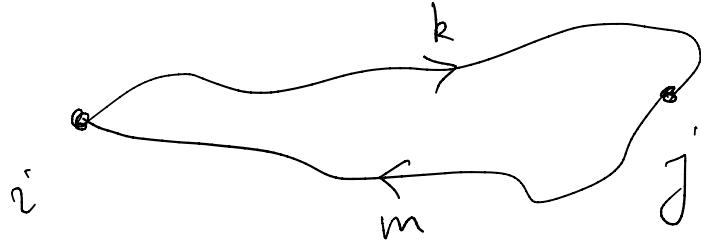
$$P = \left( \begin{array}{ccccccc|c} 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 \\ \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 1 \\ \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 2 \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 3 \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & 4 \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & 5 \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 6 \end{array} \right)$$

$P$  is doubly stochastic  $\Rightarrow$  stationary distribution  $w_i = \frac{1}{7}$   
 $P^2 > 0 \Rightarrow P$  is regular  $\Rightarrow P(X_n=0) \rightarrow \frac{1}{7}$

$\Rightarrow$  proportion of values of  $S_n$  divisible by 7  
 converges to  $\frac{1}{7}$

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**Exercise 5** For a Markov chain, suppose that state  $i$  is transient, and that state  $i$  is accessible from state  $j$  (meaning that there is some integer  $m$  such that  $p_{ji}(m) > 0$ ). Show that  $p_{ij}(n) \rightarrow 0$  as  $n \rightarrow \infty$ . [Hint: find an inequality relating  $\sum_n p_{ii}(n)$  and  $\sum_n p_{ij}(n)$ ]



$$\text{let } h = P_{j|i}(m) > 0.$$

$$\begin{aligned} \text{For all } k, \quad P_{ii}(k+m) &= P(X_{k+m} = i | X_0 = i) \\ &\geq P(X_{k+m} = i | X_k = j, X_0 = i) \\ &\quad P(X_k = j | X_0 = i) \end{aligned}$$

$$\text{Markov property} \Rightarrow P(X_{k+m} = i | X_k = j).$$

$$P(X_k = j | X_0 = i)$$

$$= P_{jk}(m) P_{ij}(k)$$

$$= h P_{ij}(k)$$

$$\Rightarrow P_{ij}(k) \leq \frac{1}{h} P_{ii}(k+m) \quad (\text{all } k \geq 1)$$

$$\text{Now } i \text{ is transient} \Rightarrow \sum_{n=1}^{\infty} P_{ii}(n) < \infty$$

$$\Rightarrow P_{ii}(n) \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

$$\Rightarrow p_{ii}(k+m) \rightarrow 0 \text{ as } k \rightarrow \infty$$

Hence from the inequality,

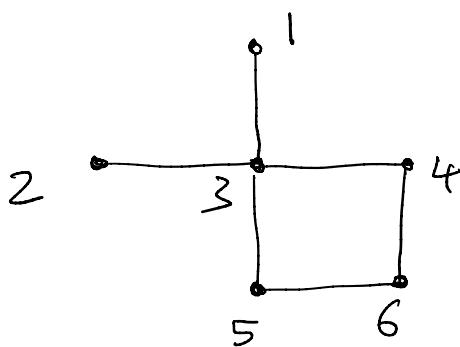
$$\lim_{k \rightarrow \infty} p_{ij}(k) \leq \frac{1}{h} \lim_{k \rightarrow \infty} p_{ii}(k+m)$$
$$= 0$$

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**Exercise** GS, p. 467: #7

[Note: ergodic means irreducible, and fixed vector means stationary distribution. For part (c), guess that the chain is reversible and solve the reversibility equations. For part (d), set up a recursion formula by conditioning on the first step].

a)



$$P = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \quad \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix}$$

- b) Can reach every room from any other room  $\Rightarrow$  it is irreducible.  
 But can return to a room only after an even number of steps.
- $\Rightarrow P_{ii}(2n+1) = 0$  all  $n$
- $\Rightarrow P$  is not regular (period = 2)

$$c) \quad w = \left( \frac{1}{12} \quad \frac{1}{12} \quad \frac{1}{3} \quad \frac{1}{6} \quad \frac{1}{6} \quad \frac{1}{6} \right)$$

$$d) \quad m_{ab} = \mathbb{E}[\# \text{ steps } a \rightarrow b \text{ for first time}]$$

$$m_{15} = 1 + m_{35}$$

$$m_{35} = 1 + \frac{1}{4}(m_{15} + m_{25} + m_{45})$$

$$m_{25} = 1 + m_{35}$$

$$m_{45} = 1 + \frac{1}{2}(m_{35} + m_{65})$$

$$m_{65} = 1 + \frac{1}{2} m_{45}$$

$$\Rightarrow m_{15} = ?.$$