Irreducible and regular

Notation: for a matrix T write $T \geq 0$ if $T_{ij} \geq 0$ for all i, j and $T > 0$ if $T_{ij} > 0$ for all i, j.

Definition 2 *Let P be the transition matrix of a Markov chain.*

(1) The Markov chain is irreducible *if for all states i, j there is an integer $n(i,j)$ such that $p_{ij}(n(i,j)) > 0$.*

(2) The Markov chain is regular *if there is an integer n such that $P^n > 0$.*

Example 4 *Recall the balls in boxes model:*

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 1 & 0 \end{pmatrix} \tag{14}$$

Since

$$P^2 = \begin{pmatrix} 1/4 & 1/2 & 1/4 \\ 1/8 & 3/4 & 1/8 \\ 1/4 & 1/2 & 1/4 \end{pmatrix} \tag{15}$$

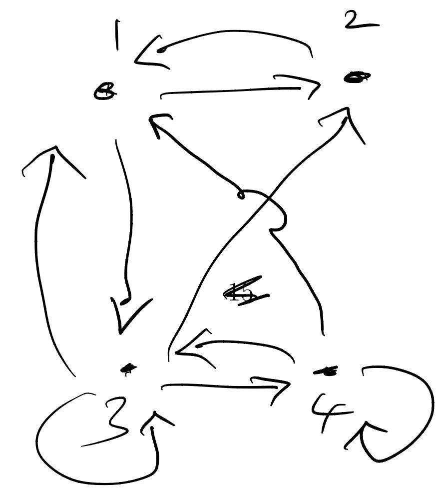
it follows that P is regular.

Ex 4 state Markov chain

$$P = \begin{pmatrix} 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ 1 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{6} & 0 & \frac{1}{6} & \frac{2}{3} \end{pmatrix}$$

Irreducible?

Yes.



Regular?

$$P^4 = \begin{pmatrix} 0.3345 & 0.1794 & \cdots & - \\ 0.2301 & 0.2083 & & \\ \vdots & & \ddots & \\ & & & 0.3222 \end{pmatrix}$$

Numerical: all entries in P^4 are positive

$\Rightarrow$ **YES**

Stationary distribution:

$$\text{regular} \underset{\text{P-F.}}{\Rightarrow} \quad P^k \xrightarrow[k \to \infty]{} \begin{pmatrix} w_1 & w_2 & w_3 & w_4 \\ w_1 & w_2 & w_3 & w_4 \\ & & & \\ & \text{---} & 4 \text{---} & \\ & & & \end{pmatrix}$$

Compute P^{30}:

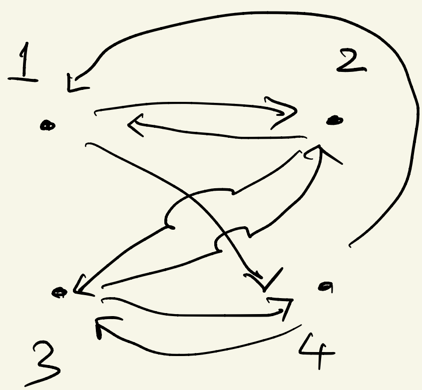
$$P^{20} = \begin{pmatrix} 0.2885 & 0.1731 & 0.3077 & 0.2308 \\ & & \| & \\ & & & \end{pmatrix}$$

(all rows are equal).

Best method to compute (w_1, w_2, w_3, w_4).

<u>Example</u>

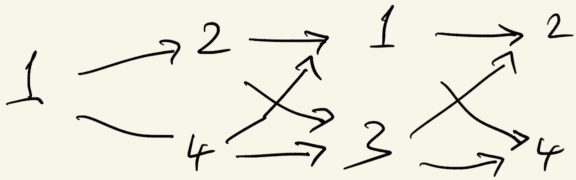
$$P = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & 0 & \frac{3}{4} & 0 \\ 0 & \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{pmatrix}$$



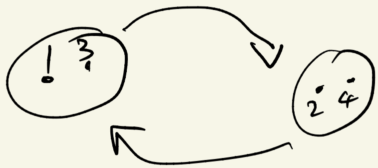
Irreducible?

<u>Yes</u>.

Regular?



$$\{1, 3\} \longrightarrow \{2, 4\} \longrightarrow \{1, 3\} \rightarrow \{2, 4\} \rightarrow$$



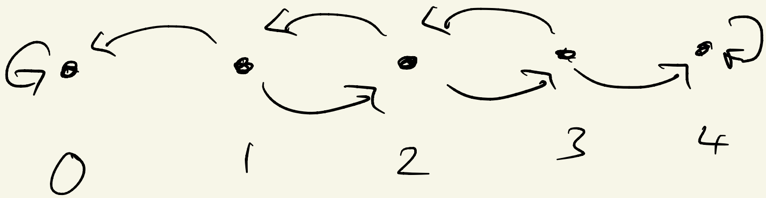
Compute:

$$P^4 = \begin{pmatrix} 0.4 & 0 & 0.6 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0.4 & 0 & 0.6 & 0 \\ 0 & 0.4 & 0 & 0.6 \end{pmatrix}$$

Not regular.

Example. Drunkard's Walk.

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$



$\{0\}$ $\{4\}$ absorbing states, persistent

$\{1,2,3\}$ transient.

Stationary distribution.

$$w_j = \sum_i P_{ij} w_i$$

$j=0$: $w_0 = w_0 P_{00} + w_1 P_{10} + w_2 P_{20}$

$$+ w_3 P_{30} + w_4 P_{40}$$

$$w_0 = w_0 + \frac{1}{2} w_1$$

$j=1$: $w_1 = w_0 P_{01} + w_1 P_{11} + w_2 P_{21}$

$$+ w_3 P_{31} + w_4 P_{41}$$

$$w_1 = w_2 \left(\frac{1}{2}\right)$$

$j=2$: $w_2 = w_0 P_{02} + w_1 P_{12} + w_2 P_{22}$

$$+ w_3 P_{32} + w_4 P_{42}$$

$$w_2 = \frac{1}{2} w_1 + \frac{1}{2} w_3$$

$j=3$: $w_3 = \frac{1}{2} w_2$

$j=4$: $w_4 = w_4 + \frac{1}{2} w_3$

Solution: $w_1 = w_2 = w_3 = 0$

(makes sense b/c these are transient states).

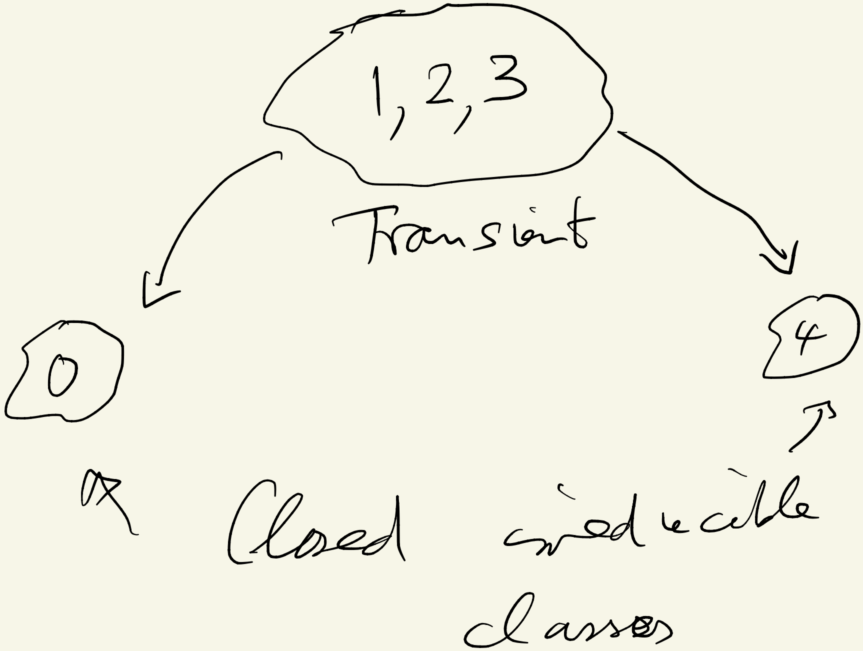
$$w_0 + w_4 = 1$$

Infinite number of solution:

$w_0 = x$ $\quad\quad$ $0 \le x \le 1$

$w_4 = 1 - x$

$$(w_0, w_1, w_2, w_3, w_4) = (x, 0, 0, 0, 1-x)$$



$$(w_0, w_1, w_2, w_3, w_4) = x(1,0,0,0,0) + (1-x)(0,0,0,0,1).$$

stat. dist. for state 0 by itself

stat. dist. for state 4 by itself.

combination of stationary states for closed irreducible classes.

Ex. 4

$$Y_n = (X_n, X_{n-1}) \qquad i, j, k, \dots$$

$$Y_{n+1} = (X_{n+1}, X_n)$$

$$\mathbb{P}\left(Y_{n+1} = (i,j) \;\middle|\; Y_n = (k,l)\right)$$

$$= \mathbb{P}\left(X_{n+1} = i,\ X_n = j \;\middle|\; X_n = k,\ X_{n-1} = l\right)$$

$$\lim_{n\to\infty} \mathbb{P}\left(Y_n = (i,j)\right)$$

$$\lim_{n\to\infty} \mathbb{P}(X_n = k) = w_k.$$

Example 5 *Define the two-state swapping chain:*

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{16}$$

Then $P^2 = I$ *is the identity, hence for all* $n \geq 1$

$$P^{2n} = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad P^{2n+1} = P \tag{17}$$

So P *is irreducible but not regular.*

0.1 Perron-Frobenius Theorem

Let e denote the vector in $\mathbb{R}^n$ with all entries 1, so

$$e = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & \cdots & 1 \end{pmatrix}^T \tag{18}$$

Theorem 2 [Perron-Frobenius] *Suppose P is a regular $n \times n$ transition matrix. Then there is a unique strictly positive vector $w \in \mathbb{R}^n$ such that*

$$w^T P = w^T \tag{19}$$

and such that

$$P^k \to e\, w^T \quad \textit{as } k \to \infty \tag{20}$$

P is regular $n \times n$ transition matrix.

Assume $P_{ij} > 0$ all i, j.

$$y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

column vector

$$m_0 = \min_i y_i$$

$$M_0 = \max_i y_i$$

$$Py = \underbrace{\begin{pmatrix} & & \\ & P_{ij} & \\ & & \end{pmatrix}}_{n \times n} \underbrace{\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}}_{n \times 1} = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}$$

$$m_1 = \min_i z_i$$

$$M_1 = \max_i z_i$$

$$z_i = (Py)_i = \sum_{j=1}^{n} P_{ij} y_j$$

$$= P_{ik} m_0 + \sum_{j \neq k} P_{ij} y_j$$

(smallest term)

$$\leq P_{ik} m_0 + \sum_{j \neq k} P_{ij} M_0$$

$$= P_{ik} m_0 + M_0 (1 - P_{ik})$$

$$z_i \leq M_0 - P_{ik} (M_0 - m_0)$$

choose $z_i = M_1$ (largest in z)

$$\Rightarrow \boxed{M_1 < M_0}$$

$M_0 = \max y_i$

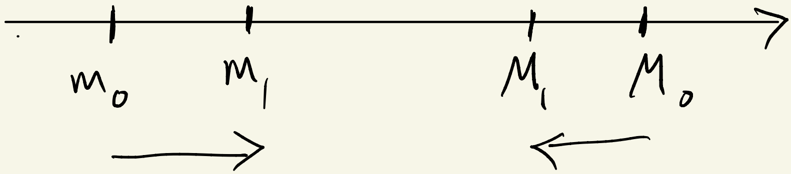
$M_1 = \max z_i$

Similar argument

$$\boxed{m_1 > m_0}$$

$m_0 = \min y_i$

$m_1 = \min z_i$

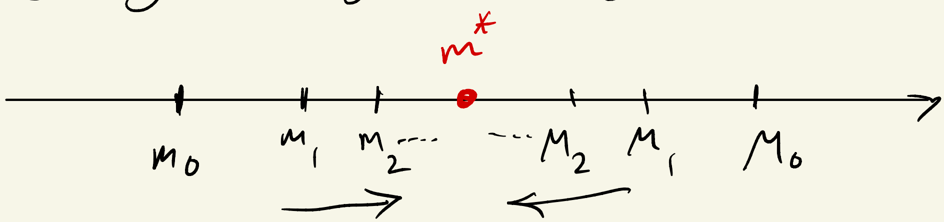


$M_0 - m_0 =$ range of y

$M_1 - m_1 =$ range of $z = Py$.

So P contracts the range of y.

Multiply by P again: $P^2 y$



$P^k y$ contracts k times.

$$\Rightarrow \quad M_k \to m^*, \qquad m_k \to m^*.$$

as $k \to \infty$.

$$\text{So} \quad P^k y \longrightarrow \begin{pmatrix} m^* \\ m^* \\ \vdots \\ m^* \end{pmatrix} \quad \text{as } k \to \infty$$

all equal values.

m^* depends on y.

choose $y = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = e_1$.

then $P^k e_1 \to \begin{pmatrix} w_1 \\ w_1 \\ \vdots \\ w_1 \end{pmatrix}$ define this to be w_1

choose $y = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = e_2$; $P^k e_2 \to \begin{pmatrix} w_2 \\ w_2 \\ \vdots \\ w_2 \end{pmatrix}$

$\vdots$

choose $y = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} = e_n$; $P^k e_n \to \begin{pmatrix} w_n \\ w_n \\ \vdots \\ w_n \end{pmatrix}$.

$$P^k e_j \to \begin{pmatrix} w_j \\ w_j \\ \vdots \\ w_j \end{pmatrix} \quad j = 1 \ldots n$$

$$\sum_{j=1}^{n} P^k e_j = P^k \cdot \mathbb{1}$$ (vector with all 1's)

$$= \mathbb{1}.$$

$$= \begin{pmatrix} \sum_j w_j \\ \sum_j w_j \\ \vdots \\ \sum_j w_j \end{pmatrix}$$

$\Rightarrow \sum_j w_j = 1 \Rightarrow$ stationary dist.

Proof of Perron-Frobenius: We show that for all vectors $y \in \mathbb{R}^n$,

$$P^k y \to e\, w^T y \tag{21}$$

which is a positive multiple of the constant vector e. This implies the result.

Suppose first that $P > 0$ so that $p_{ij} > 0$ for all $i, j \in S$. Let $d > 0$ be the smallest entry in P (so $d \leq 1/2$). For any $y \in \mathbb{R}^n$ define

$$m_0 = \min_j \, y_j, \quad M_0 = \max_j \, y_j \tag{22}$$

and

$$m_1 = \min_j \,(Py)_j, \quad M_1 = \max_j \,(Py)_j \tag{23}$$

Consider $(Py)_i = \sum_j p_{ij} y_j$. This is maximized by pairing the smallest entry m_0 of y with the smallest entry d of p_{ij}, and then taking all other entries of y to be M_0. For any i,

$$\begin{aligned}
(Py)_i &= \sum_j p_{ij} y_j \\
&= p_{ik} m_0 + \sum_{j \neq k} p_{ij} y_j && (24) \\
&\leq p_{ik} m_0 + (1 - p_{ik}) M_0 && (25) \\
&\leq d m_0 + (1-d) M_0 && (26)
\end{aligned}$$

and therefore

$$M_1 \leq d m_0 + (1-d) M_0$$

By similar reasoning,

$$m_1 = \min_i (Py)_i \geq (1-d) m_0 + d M_0 \tag{27}$$

Subtracting these bounds gives

$$M_1 - m_1 \leq (1 - 2d)(M_0 - m_0) \tag{28}$$

Now we iterate to give

$$M_k - m_k \leq (1 - 2d)^k \,(M_0 - m_0) \tag{29}$$

where again

$$M_k = \max_i (P^k y)_i, \quad m_k = \min_i (P^k y)_i \tag{30}$$

Furthermore the sequence $\{M_k\}$ is decreasing since

$$M_{k+1} = \max_i (PP^k y)_i = \max_i \sum_j p_{ij} (P^k y)_j \leq M_k \tag{31}$$

and the sequence $\{m_k\}$ is increasing for similar reasons. Therefore both sequences converge as $k \to \infty$, and the difference between them also converges to zero. Hence we conclude that the components of the vector $P^k y$ converge to a constant value, meaning that

$$P^k y \to m\, e \tag{32}$$

for some m.

We can pick out the value of m with the inner product

$$m(e^T e) = e^T \lim_{k \to \infty} P^k y = \lim_{k \to \infty} e^T P^k y \tag{33}$$

Note that for $k \geq 1$,

$$e^T P^k y \geq m_k (e^T e) \geq m_1 (e^T e) = \min_i (Py)_i (e^T e)$$

Since P is assumed positive, if $y_i \geq 0$ for all i it follows that $(Py)_i > 0$ for all i, and hence $m > 0$.

Now define

$$w_j = \lim_{k\to\infty} P^k e_j/(e^T e) \tag{34}$$

where e_j is the vector with entry 1 in the j^{th} component, and zero elsewhere. It follows that $w_j > 0$ so w is strictly positive, and

$$P^k \to ew^T \tag{35}$$

By continuity this implies

$$\lim_{k\to\infty} P^k P = ew^T P \tag{36}$$

and hence $w^T P = w^T$. This proves the result in the case where $P > 0$.

Now turn to the case where P is regular. Since P is regular, there exists integer N such that

$$P^N > 0 \tag{37}$$

Hence by the previous result there is a strictly positive $w \in \mathbb{R}^n$ such that

$$P^{kN} \to ew^T \tag{38}$$

as $k \to \infty$, satisfying $w^T P^N = w^T$.

It follows that $P^{N+1} > 0$, and hence there is also a vector v such that

$$P^{k(N+1)} \to ev^T \tag{39}$$

as $k \to \infty$, and $v^T P^{N+1} = v^T$. Considering convergence along the subsequence $kN(N+1)$ it follows that $w = v$, and hence

$$w^T P^{N+1} = v^T P^{N+1} = v^T = w^T = w^T P^N \tag{40}$$

and so

$$w^T P = w^T \tag{41}$$

The subsequence $P^{kN}y$ converges to $ew^T y$ for every y, and we want to show that the full sequence $P^m y$ does the same. For any $\epsilon > 0$ there is $K < \infty$ such that for all $k \geq K$ and all probability vectors y

$$\|(P^{kN} - ew^T)y\| \leq \epsilon \tag{42}$$

Let $m = kN + j$ where $j < N$, then for any probability vector y

$$\|(P^m - ew^T)y\| = \|(P^{kN+j} - ew^T)y\| = \|(P^{kN} - ew^T)P^j y\| \leq \epsilon \tag{43}$$

which proves convergence along the full sequence.

$\boxed{\text{QED}}$

$\boxed{\textbf{Corollary of Perron-Frobenius Theorem}}$

Note that as a corollary of the Theorem we deduce that the vector w is the unique (up to scalar multiples) solution of the equation

$$w^T P = w^T \tag{44}$$

Also since $v^T e = \sum v_i = 1$ for a probability vector v, it follows that

$$v^T P^n \to w^T \tag{45}$$

for any probability vector v.

Example 6 *Recall the balls in boxes model:*

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 1 & 0 \end{pmatrix} \tag{46}$$

We saw that P is regular. Solving the equation $w^T P = w^T$ yields the solution

$$w^T = (1/6, 2/3, 1/6) \tag{47}$$

Furthermore we can compute

$$P^{10} = \begin{pmatrix} 0.167 & 0.666 & 0.167 \\ 0.1665 & 0.667 & 0.1665 \\ 0.167 & 0.666 & 0.167 \end{pmatrix} \tag{48}$$

showing the rate of convergence.

Meaning of Perron-Frobenius

Concerning the interpretation of the result. Suppose that the distribution of X_0 is

$$P(X_0 = i) = \alpha_i \tag{49}$$

for all $i \in S$. Then

$$\begin{aligned} P(X_k = j) &= \sum_i P(X_k = j | X_0 = i)\, P(X_0 = i) \\ &= \sum_i (P^k)_{ij}\, \alpha_i \\ &= (\alpha^T P^k)_j \end{aligned}$$

where α is the vector with entries α_i. Using our Theorem we deduce that

$$P(X_k = j) \to w_j \tag{50}$$

as $k \to \infty$ for any initial distribution α. Furthermore if $\alpha = w$ then $\alpha^T P^k = w^T P^k = w^T$ and therefore

$$P(X_k = j) = w_j \tag{51}$$

for all k. So w is called the *equilibrium* or *stationary* distribution of the chain. The Theorem says that the state of the chain rapidly forgets its initial distribution and converges to the stationary value.

0.2 Finite and irreducible

Now suppose the chain is irreducible but not regular. Then we get a similar but weaker result.

Theorem 3 *Let P be the transition matrix of an irreducible Markov chain. Then there is a unique strictly positive probability vector w such that*

$$w^T P = w^T \tag{52}$$

Furthermore

$$\frac{1}{n+1}\Big(I + P + P^2 + \cdots + P^n\Big) \to e w^T \tag{53}$$

as $n \to \infty$.

This Theorem allows the following interpretation: for an irreducible chain, w_j is the long-run fraction of time the chain spends in state j.

Proof for finite state irreducible: define

$$Q = \frac{1}{2}I + \frac{1}{2}P \tag{54}$$

Then Q is a transition matrix. Also

$$2^n Q^n = \sum_{k=0}^{n} \binom{n}{k} \; P^k \tag{55}$$

Because the chain is irreducible, for all pairs of states i, j there is an integer $n(i,j)$ such that $(P^{n(i,j)})_{ij} > 0$. Let $n = \max n(i,j)$, then for all i, j we have

$$2^n (Q^n)_{ij} = \sum_{k=0}^{n} \binom{n}{k} \; (P^k)_{ij} \geq \binom{n}{n(i,j)} \; (P^{n(i,j)})_{ij} > 0 \tag{56}$$

and hence Q is regular. Let w be the unique stationary vector for Q then

$$w^T Q = w^T \leftrightarrow w^T P = w^T \tag{57}$$

which shows existence and uniqueness for P.

Let $W = e w^T$ then a calculation shows that for all n

$$\Big(I + P + P^2 + \cdots + P^{n-1}\Big)(I - P + W) = I - P^n + nW \tag{58}$$

Note that $I - P + W$ is invertible: indeed if $y^T(I - P + W) = 0$ then

$$y^T - y^T P + (y^T e) w = 0 \tag{59}$$

Multiply by e on the right and use $Pe = e$ to deduce

$$y^T e - y^T P e + (y^T e)(w^T e) = (y^T e)(w^T e) = 0 \tag{60}$$

Since $w^T e = 1 > 0$ it follows that $y^T e = 0$ and so $y^T - y^T P = 0$. By uniqueness this means that y is a multiple of w, but then $y^T e = 0$ means that $y = 0$. Therefore $I - P + W$ is invertible, and so

$$I + P + P^2 + \cdots + P^{n-1} = (I - P^n + nW)(I - P + W)^{-1} \tag{61}$$

Now $WP = W = W^2$ hence

$$W(I - P + W) = W \implies W = W(I - P + W)^{-1} \tag{62}$$

Therefore

$$I + P + P^2 + \cdots + P^{n-1} = (I - P^n)(I - P + W)^{-1} + nW \tag{63}$$

and so

$$\frac{1}{n}\Big(I + P + P^2 + \cdots + P^{n-1}\Big) = W + \frac{1}{n}(I - P^n)(I - P + W)^{-1} \tag{64}$$

It remains to show that the norm of the matrix $(I - P^n)(I - P + W)^{-1}$ is bounded as $n \to \infty$, or equivalently that $\|(I - P^n)\|$ is uniformly bounded. This follows from the bound

$$\|P^n z\| \leq \sum_{ij} (P^n)_{ij} |z_j| = \sum_j |z_j| \tag{65}$$

Therefore $\frac{1}{n}(I - P^n)(I - P + W)^{-1} \to 0$ and the result follows,

$\boxed{\text{QED}}$

0.3 Sojourn times

Definition 3 *Consider an irreducible Markov chain.*

(1) *starting in state i, m_{ij} is the expected number of steps to visit state j for the first time (by convention $m_{ii} = 0$)* for $i \neq j$.

(2) *starting in state i, r_i is the expected number of steps for the first return to state i*

(3) *the fundamental matrix is $Z = (I - P + W)^{-1}$* (*)

Theorem 4 *Let w be the stationary distribution of an irreducible finite state Markov chain. Then for all states $i, j \in S$,*

$$r_i = \frac{1}{w_i}, \quad m_{ij} = \frac{z_{jj} - z_{ij}}{w_j} \tag{66}$$

where z_{ij} is the (i, j) entry of the fundamental matrix Z.

$w_i =$ stat. prob for state i

$=$ long run fraction of time spent in state i.

$r_i =$ mean time between successive visits to state i.

$$r_i = \frac{1}{w_i}.$$

(*) $$W = \begin{pmatrix} w_1 & w_2 & \cdots & w_n \\ w_1 & w_2 & \cdots & w_n \\ \vdots & & & \vdots \\ w_1 & w_2 & \cdots & w_n \end{pmatrix}$$

<u>Ex.</u> <u>Balls in boxes.</u>

$m_{02} =$ mean time $(0 \mapsto 2)$

$m_{02} \geq 2.$

Condition on first step:

$$m_{02} = \underset{\text{first step}}{1} + \underset{\text{remaining steps.}}{m_{12}}$$

$$m_{12} = \text{mean time } (1 \to 2)$$

condition on first step

$$= 1 \cdot P_{12} + (1 + m_{12}) \cdot P_{11} + (1 + m_{02}) \cdot P_{10}$$

solve for m_{02}, m_{12}:

$$m_{12} = 5, \quad m_{02} = 6.$$

$$\Gamma_2 = \text{mean time } (2 \to 2)$$

$$= \frac{1}{w_2} = 6.$$

recall stat. dist. is

$$\left(\tfrac{1}{6}, \tfrac{2}{3}, \tfrac{1}{6}\right).$$

Proof: let M be the matrix with entries $M_{ij} = m_{ij}$, let E be the matrix with entries $E_{ij} = 1$, and let D be the diagonal matrix with diagonal entries $D_{ii} = r_i$. For all $i \neq j$,

$$m_{ij} = p_{ij} + \sum_{k \neq j} p_{ik}(m_{kj} + 1) = 1 + \sum_{k \neq j} p_{ik} m_{kj} \tag{67}$$

For all i,

$$r_i = \sum_k p_{ik}(m_{ki} + 1) = 1 + \sum_k p_{ik} m_{ki} \tag{68}$$

Thus for all i, j,

$$M_{ij} = 1 + \sum_{k \neq j} p_{ik} M_{kj} - D_{ij} \tag{69}$$

which can be written as the matrix equation

$$M = E + PM - D \tag{70}$$

Multiplying on the left by w^T and noting that $w^T = w^T P$ gives

$$0 = w^T E - w^T D \tag{71}$$

The i^{th} component of the right side is $1 - w_i r_i$, hence this implies that for all i

$$r_i = \frac{1}{w_i} \tag{72}$$

Recall the definition of the matrix $Z = (I - P + W)^{-1}$, and vector $e = (1, 1, \ldots, 1)^T$. Since $Pe = We = e$ it follows that $(I - P + W)e = e$ and hence $Ze = e$ and $ZE = E = ee^T$. Furthermore $w^T P = w^T W = w^T$ and so similarly $w^T Z = w^T$ and $W = WZ$. Therefore from (70),

$$Z(I - P)M = ZE - ZD = E - ZD \tag{73}$$

Since $Z(I - P) = I - ZW = I - W$ this yields

$$M = E - ZD + WM \tag{74}$$

The (i, j) component of this equation is

$$m_{ij} = 1 - z_{ij} r_j + (w^T M)_j \tag{75}$$

Setting $i = j$ gives $0 = 1 - z_{jj} r_j + (w^T M)_j$, hence

$$m_{ij} = (z_{jj} - z_{ij}) r_j = \frac{z_{jj} - z_{ij}}{w_j} \tag{76}$$

$\boxed{\text{QED}}$

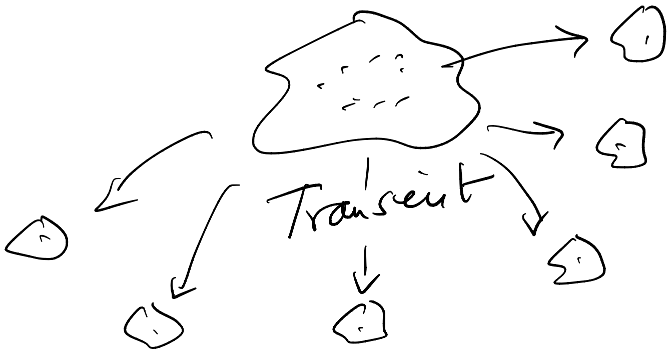
0.4 Absorbing chains

Definition 4 *A state i is* absorbing *if $p_{ii} = 1$. A chain is* absorbing *if for every state i there is an absorbing state which is accessible from i. A non-absorbing state in an absorbing chain is called a* transient *state.*

Consider an absorbing chain with r absorbing states and t transient states. Re-order the states so that the transient states come first, then the absorbing states. The transition matrix then has the form

$$P = \begin{pmatrix} Q & R \\ 0 & I \end{pmatrix} \tag{77}$$

where I is the $r \times r$ identity matrix.



$t = \#$ transient

$r = \#$ absorbing

Order the states so transient are first, then absorbing.

$$P = \begin{array}{c} \left(\begin{array}{c|c} Q & R \\ \hline 0 & I \end{array}\right) \\ \leftarrow \text{transient} \rightarrow \quad \leftarrow \text{abs} \rightarrow \end{array} \quad \begin{array}{c} \uparrow \text{transient} \downarrow \\ \uparrow \text{absorbing} \downarrow \end{array}$$

zero matrix $(r \times t)$

identity matrix $(r \times r)$

Q : $t \times t$ between transient states

R : $t \times r$ from transient to absorbing.

$$P = \begin{pmatrix} Q & R \\ 0 & I \end{pmatrix}$$

$$P^2 = \begin{pmatrix} Q & R \\ 0 & I \end{pmatrix}\begin{pmatrix} Q & R \\ 0 & I \end{pmatrix}$$

$$= \begin{pmatrix} Q^2 & QR+R \\ 0 & I \end{pmatrix}$$

$$P^k = \begin{pmatrix} Q^k & R_k \\ 0 & I \end{pmatrix}$$

mess

$$(Q^k)_{ij} = \mathbb{P}(X_k = j \mid X_0 = i)$$

where i, j are both transient.

Since j is transient, we get

$$(Q^k)_{ij} \to 0 \quad \text{as} \quad k \to \infty.$$

<u>Consequence</u>: $I - Q$ is invertible.

Define

$$N = (I - Q)^{-1}.$$

Fundamental matrix of absorbing chain.