Notes on Finite State homeogeneous Markov Chains

1 Setup

The finite state space is Ω, and the chain is

$$X = (X_0, X_1, \dots, X_n, \dots), \quad X_n \in \Omega$$

The transition matrix is

$$P = (p_{ij}), \quad i, j \in \Omega$$

where

$$p_{ij} = \mathbb{P}(X_{n+1} = j | X_n = i) = \mathbb{P}(X_1 = j | X_0 = i)$$

1.1 States

$\boxed{\text{Transient}}$ State j is *transient* if and only if $f_{jj} < 1$ if and only if $\sum_{n=1}^{\infty} p_{jj}(n) < \infty$.
$\boxed{\text{Persistent}}$ State j is *persistent* if and only if $f_{ii} = 1$ if and only if $\sum_{n=1}^{\infty} p_{jj}(n) = \infty$.
$\boxed{\text{Absorbing}}$ State j is *absorbing* if and only if $p_{jj} = 1$.

$\boxed{\text{Theorem}}$ There is at least one persistent state in Ω.

$\boxed{\text{Period}}$ The period of state j is

$$d(j) = \gcd\{n \geq 1 | p_{jj}(n) > 0\}$$

$\boxed{\text{Mean return time}}$ The mean return time of state j is

$$r(j) = \text{Expected number of steps until first return to state } j = \sum_{n=1}^{\infty} n\, f_{jj}(n)$$

If j is transient then $r(j) = \infty$.

1.2 Chains

$\boxed{\text{Accessible}}$ State j is accessible from state i if there is integer n such that $p_{ij}(n) > 0$, denoted by $i \to j$. States i and j intercommunicate if each is accessible from the other, denoted by $i \leftrightarrow j$.

$\boxed{\text{Theorem}}$ If $i \leftrightarrow j$, then either both are transient or both are persistent, and $d(i) = d(j)$.

$\boxed{\text{Irreducible}}$The chain is *irreducible* if $i \leftrightarrow j$ for all states $i, j \in \Omega$.

$\boxed{\text{Theorem}}$ For an irreducible chain, all states are persistent and all states have the same period.

$\boxed{\text{Regular}}$ The chain is *regular* if there is $n < \infty$ such that $p_{ij}(n) > 0$ for all states $i, j \in \Omega$ (including the case $i = j$). A regular chain is also irreducible.

$\boxed{\text{Theorem}}$ An irreducible chain is regular if and only if all states have period 1 (also known as an *aperiodic* chain). If $p_{jj} > 0$ for some state j and the chain is irreducible, then it is also regular.

$\boxed{\text{Theorem}}$ If the chain is non-irreducible, there is a unique decomposition of Ω into disjoint sets

$$\Omega = T \cup C_1 \cup \cdots \cup C_k$$

where T consists only of transient states, and where each set C_i is closed and contains states that all intercommunicate.

1.3 Stationary distribution

$\boxed{\text{Stationary}}$ The probability distribution $\{w_j\}$ on Ω is *stationary* if

$$w_j = \sum_{i \in \Omega} w_i \, p_{ij} \quad \text{for all } j \in \Omega$$

$\boxed{\text{Theorem}}$ If the chain is irreducible, there is a unique positive stationary distribution $\{w_j\}$, and $r(j) = 1/w_j$ for all states j.

$\boxed{\text{Theorem}}$ If the chain is regular, then

$$\mathbb{P}(X_n = j) \to w_j \quad \text{as } n \to \infty \quad \text{for all } j$$

$\boxed{\text{Reversible}}$ The chain is *reversible* if it has a stationary distribution $\{w_j\}$ that satisfies

$$w_i p_{ij} = w_j p_{ji} \quad \text{for all } i, j \in \Omega$$

$\boxed{\text{Theorem}}$ If the chain is not irreducible then the state space can be uniquely decomposed as

$$\Omega = T \cup C_1 \cup \cdots \cup C_k$$

For each set C_a there is a unique probability distribution $\{w_j^{(a)}\}$, $j \in \Omega$ satisfying $w_j^{(a)} > 0$ if and only if $j \in C_a$, $\sum_{j \in C_a} w_j^{(a)} = 1$ and

$$w_j^{(a)} = \sum_{i \in C_a} w_i^{(a)} \, p_{ij} \quad \text{for all } j \in C_a$$

For every set of numbers $x = (x_1, \ldots, x_k)$ satisfying $0 \leq x_a \leq 1$, $\sum_a x_a = 1$, there is a stationary distribution $\{w_j(x)\}$ which can be written as

$$w_j(x) = \sum_{a=1}^{k} x_a \, w_j^{(a)}$$

1.4 Absorbing chains

$\boxed{\text{Absorbing chain}}$ The chain is *absorbing* if for every state j, there is an absorbing state which is accessible from j. The state space can be decomposed as

$$\Omega = T \cup R$$

where T consists of all transient states, and where R consists of all absorbing states. With respect to this decomposition the transition matrix can be written in block form

$$P = \begin{pmatrix} Q & R \\ 0 & I \end{pmatrix}$$

$\boxed{\text{Fundamental matrix}}$ The *fundamental matrix* is $N = (I - Q)^{-1}$.

$\boxed{\text{Theorem}}$

$$\sum_{j \in T} N_{ij} = \text{Expected number of steps until absorption starting from transient state } i$$

2 Google Page Rank

A simplified version of the Google page-rank algorithm assigns a probability distribution $R(1), \dots, R(n)$ to a collection of n webpages. The webpages are connected by directed links. For each page i, let $L(i)$ be the number of outward directed links starting at i (assume that every webpage has at least one outward directed link so $L(i) \geq 1$ for all i). Then the probability distribution R is the solution of the equation

$$R(j) = \frac{(1-\delta)}{n} + \delta \sum_{i=1}^{n} R(i) p_{ij}$$

where the transition matrix P has entries

$$p_{ij} = \begin{cases} L(i)^{-1} & \text{if there is a directed link from } i \text{ to } j \\ 0 & \text{else} \end{cases}$$

and where $\delta < 1$ is a positive 'damping factor'.

As usual let $p_{ij}^{(n)}$ denote the entries of the matrix P^n. Show that R satisfies the following equation:

$$R(j) = \frac{(1-\delta)}{n} \sum_{k=0}^{\infty} \delta^k \sum_{i=1}^{n} p_{ij}^{(k)}$$