MATH 5110

1). Write down bases for the column spaces of the matrices A, B, C:

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 0 \end{pmatrix}$$

2). Consider the following matrix together with its row reduced echelon form:

$$A = \begin{pmatrix} 1 & 4 & 2 & 5 \\ 2 & 8 & 0 & -1 \\ -1 & 2 & 0 & 3 \end{pmatrix}, \quad \operatorname{rref}(A) = \begin{pmatrix} 1 & 0 & 0 & -2.17 \\ 0 & 1 & 0 & 0.42 \\ 0 & 0 & 1 & 2.75 \end{pmatrix}$$

a) Find the dimensions of the columnspace, the nullspace and the rowspace of A.
b) Find bases for the columnspace, the nullspace and the rowspace of A.

3). Find all solutions of the equation $A\vec{x} = \vec{b}$ where

$$A = \begin{pmatrix} 3 & 5 & 25 \\ 7 & 9 & 53 \\ -4 & 5 & -10 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 53 \\ 105 \\ 11 \end{pmatrix}, \quad \operatorname{rref}(A\ \vec{b}) = \begin{pmatrix} 1 & 0 & 5 & 6 \\ 0 & 1 & 2 & 7 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

4). Suppose that A is a 3×5 matrix whose rank is 3. Write down $\operatorname{rref}(A^T)$.

5). In each case, either prove the statement or give a counterexample to show that it is false:

- $n \geq 3$, $\vec{u}, \vec{v}, \vec{w}$ are linearly independent vectors in $\mathbb{R}^n$, and

  $$S = \{\vec{x} \in \mathbb{R}^n \ : \ \vec{u} - \vec{x} \in \operatorname{Span}(\vec{v},\ \vec{w})\}$$

  Then S is a subspace.

- A is a matrix whose first two column vectors are linearly independent; then the first two columns of $\operatorname{rref}(A)$ are pivot columns.

- $\vec{u}, \vec{v}, \vec{w}$ are independent vectors in $\mathbb{R}^n$, $S_1 = \text{Span}\,(\vec{u}, \vec{v}, \vec{w})$, $S_2 = \text{Span}\,(\vec{u} - \vec{v},\ \vec{v} - \vec{w},\ \vec{w} - \vec{u})$; then $S_1 = S_2$.
- A is a 3×5 matrix whose row vectors are $\vec{v_1}, \vec{v_2}, \vec{v_3} \in \mathbb{R}^5$. Also

$$\text{rref}(A) = \begin{pmatrix} 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

  The vectors $\vec{v_1}, \vec{v_2}, \vec{v_3}$ are linearly independent.

6). Consider $A = (\vec{u_1}\ \vec{u_2}\ \vec{u_3})$ and let $\vec{b_1} = (2,\ 1,\ 4)^T$ and $\vec{b_2} = (0,\ -1,\ 2)^T$. Also

$$\text{rref}(A\ \vec{b_1}\ \vec{b_2}) = \begin{pmatrix} 1 & 0 & 0 & 22/3 & 22/3 \\ 0 & 1 & 0 & 7/3 & 7/3 \\ 0 & 0 & 1 & -7/3 & -1/3 \end{pmatrix}$$

Find scalars $c_1,\ c_2$ such that $\vec{x} = (0,\ 0,\ 1)^T$ is the solution of $A\vec{x} = c_1\vec{b_1} + c_2\vec{b_2}$.

7). Suppose that A is a 3×5 matrix whose columns are $\vec{v_1}, \ldots, \vec{v_5}$. The rref form is

$$\text{rref}(A) = \begin{pmatrix} 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Write $\vec{v_2}$ and $\vec{v_4}$ as linear combinations of $\vec{v_1}, \vec{v_3}, \vec{v_5}$.

8). Write down the standard matrix representation for the linear map $L : \mathbb{R}^2 \to \mathbb{R}^2$ which shrinks every vector's length by 1/2.

9). The kernel of a linear map $L : \mathbb{R}^4 \to \mathbb{R}^3$ is spanned by the vectors $(1, 2, 3, -1), (0, 1, 0, 2)$. Suppose that $\vec{x} = (1, 1, 2, -1)$ is a solution of the equation $L(\vec{x}) = \vec{b}$ for some vector $\vec{b}$. Find a vector $\vec{y} = (*, 0, *, 0)$ which also solves $L(\vec{y}) = \vec{b}$.

10). Let $\vec{u}, \vec{v}$ be nonzero vectors in $\mathbb{R}^n$. Find the rank and nullity of the matrix $A = \vec{u}\,\vec{v}^T$.

11). [Challenge problem!] Let $\vec{u}, \vec{v}$ be linearly independent vectors in $\mathbb{R}^n$. Find the rank and nullity of the matrix $A = \vec{u}\,\vec{v}^T + \vec{v}\,\vec{u}^T$.