Math 4570 Matrix methods for DA and ML

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Homework 0- linear algebra prerequisite.

Here is a quick test about your linear algebra knowledge. We also need some basic knowledge about probability and partial derivatives in calculus 3. My teaching page on 2321, 2331, 3081 contains all the materials. See Canvas for the details.

1. Let $A = \begin{bmatrix} 1 & 3 & 6 & 2 \\ 1 & 5 & 8 & 6 \\ 3 & 6 & 15 & 0 \end{bmatrix}$.

(1). Determine the reduced row echelon form **rref** of the matrix A. (Write all details)

$$A = \begin{bmatrix} 1 & 3 & 6 & 2 \\ 1 & 5 & 8 & 6 \\ 3 & 6 & 15 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 6 & 2 \\ 0 & 2 & 2 & 4 \\ 0 & -3 & -3 & -6 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 6 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & -3 & -3 & -6 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 6 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to$$

$$\to \begin{bmatrix} 1 & 0 & 3 & -4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \textbf{rref}(A)$$

(2) Determine all solutions $\vec{x}$ of $A\vec{x} = \vec{0}$ in **parametric vector form.**

From $\textbf{rref}(A)$, $\begin{cases} x_1 + 3x_3 - 4x_4 & = 0 \\ x_2 + x_3 + 2x_4 & = 0 \end{cases}$

So the solutions in vector form is $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3x_3 + 4x_4 \\ -x_3 - 2x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 4 \\ -2 \\ 0 \\ 1 \end{bmatrix}$

2. Let M be a matrix with row echelon form $\textbf{ref}(M) = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 0 & 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

(i) What is the **rank** of the matrix M? Answer: _________

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(ii) Is $M\vec{x} = \vec{b}$ has a solution for **every** $\vec{b} \in \mathbb{R}^3$?

(A) Yes. **(B) No.** (C) Can not be determined.

(iii) How many **free variables** in the solution set of $M\vec{x} = \vec{0}$ for $\vec{0} \in \mathbb{R}^3$? Answer: _________

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3. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 2 & 4 & 7 \end{bmatrix}$. Answer the following 4 questions.

(i) Use Elementary Row Operations to find the **inverse** of matrix A. (Write down all your work.)

$$[A|I_3] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 6 & 0 & 1 & 0 \\ 2 & 4 & 7 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3-2R_1]{R_2-2R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right] \xrightarrow{R_1-3R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 7 & 0 & -3 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right] \xrightarrow{R_1-2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 11 & -2 & -3 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right] = [I_3|A^{-1}]$$

$$A^{-1} = \begin{bmatrix} 11 & -2 & -3 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

(ii) Find all solutions for $A\vec{x} = \vec{b}$ for $\vec{b} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ using the result in (i).

Unique Solution $\vec{x} = A^{-1}\vec{b} = \begin{bmatrix} 11 & -2 & -3 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -8 \\ 1 \\ 2 \end{bmatrix}$

(iii) Does $A^3\vec{x} = \vec{b}$ have a unique solution for any $\vec{b} \in \mathbb{R}^3$? (Explain the reason.)

Yes. Since A is invertible, so A^3 is invertible. Hence $\vec{x} = (A^3)^{-1}\vec{b}$ is the unique solution.

(iv) Find $(A^T)^{-1}$.

$(A^T)^{-1} = (A^{-1})^T = \begin{bmatrix} 11 & -2 & -2 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$

Cont.

4. Let $A = \begin{bmatrix} 1 & 4 & 0 & 2 & 5 \\ 2 & 8 & 1 & -3 & 4 \\ 4 & 16 & 1 & 1 & 14 \end{bmatrix}$. Suppose $\textbf{rref}(A) = \begin{bmatrix} 1 & 4 & 0 & 2 & 5 \\ 0 & 0 & 1 & -7 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Answer the following questions.

(1)(4 points) Find a **basis** for the **kernel** of A.

$$\begin{cases} x_1 + 4x_2 + 2x_4 + 6x_5 = 0 \\ x_3 - 7x_4 - 6x_5 = 0 \end{cases} \implies \begin{cases} x_1 = -4x_2 - 2x_4 - 5x_5 \\ x_3 = 7x_4 + 6x_5 \\ x_2, x_4, x_5 \text{ are free variables} \end{cases}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -4x_2 - 2x_4 - 5x_5 \\ x_2 \\ 7x_4 + 6x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ 7 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -5 \\ 0 \\ 6 \\ 0 \\ 1 \end{bmatrix}$$

A basis for $\ker(A)$ is $\left\{ \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 7 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 6 \\ 0 \\ 1 \end{bmatrix} \right\}$

(2)(2 points) Find a **basis** for the **image** of A.

A basis for the image $\mathrm{im}A$ is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

(3) (3 points) $\dim((\ker A)^{\perp}) =$ _____ $\quad \dim(\mathrm{im}(A^T)) =$ _____ $\quad \dim(\mathrm{im}(A)^{\perp}) =$ _____

$\dim((\ker A)^{\perp}) = 2$ $\quad \dim(\mathrm{im}(A^T)) = 2$ $\quad \dim(\mathrm{im}(A)^{\perp}) = 1$

(4)(1 point) Is $\mathrm{im}(A) = \mathrm{im}(\textbf{rref}(A))$? Explain your reason.

No. $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ is not $\mathrm{im}(\textbf{rref}(A))$

Cont.

5. Suppose $\vec{x}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}$, $\vec{x}_2 = \begin{bmatrix}3\\3\\0\end{bmatrix}$ and $\vec{x}_3 = \begin{bmatrix}4\\2\\0\end{bmatrix}$ form a basis $\mathscr{B}$ for the vector space $\mathbb{R}^3$.

(1). (7 points) Using the *Gram-Schmidt process* to $\mathscr{B}$, find an **orthogonal** basis for the vector space $\mathbb{R}^3$.

$$\vec{v}_1 = \vec{b}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$\vec{v}_2 = \vec{v}_2 - \left(\frac{\vec{b}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\right)\vec{v}_1 = \begin{bmatrix}3\\3\\0\end{bmatrix} - \frac{6}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}1\\1\\-2\end{bmatrix}$$

$$\vec{v}_3 = \vec{b}_3 - \left(\frac{\vec{b}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\right)\vec{v}_1 - \left(\frac{\vec{b}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\right)\vec{v}_2 = \begin{bmatrix}4\\2\\0\end{bmatrix} - \frac{6}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} - \frac{6}{6}\begin{bmatrix}1\\1\\-2\end{bmatrix} = \begin{bmatrix}1\\-1\\0\end{bmatrix}$$

(2). (3 points) Normalize the result in (2), find an **orthonormal** basis for the vector space $\mathbb{R}^3$.

$$\vec{u}_1 = \frac{\vec{v}_1}{||\vec{v}_1||} = \begin{bmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{bmatrix};\ \vec{u}_2 = \frac{\vec{v}_2}{||\vec{v}_2||} = \begin{bmatrix}1/\sqrt{6}\\1/\sqrt{6}\\-2/\sqrt{6}\end{bmatrix};\ \vec{u}_3 = \frac{\vec{v}_3}{||\vec{v}_3||} = \begin{bmatrix}1/\sqrt{2}\\-1/\sqrt{2}\\0\end{bmatrix};$$

(3). (5 points) Find the QR-factorization of $A = \begin{bmatrix}1 & 3 & 4\\1 & 3 & 2\\1 & 0 & 0\end{bmatrix}$.

$A = QR$. The matrix $Q = \begin{bmatrix}1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2}\\1/\sqrt{3} & 1/\sqrt{6} & -1/\sqrt{2}\\1/\sqrt{3} & -2/\sqrt{6} & 0\end{bmatrix}$

Recall that R is the upper triangular matrix with $r_{ii} = ||\vec{v}_i||$ and $r_{ij} = \vec{u}_i \cdot \vec{b}_j$. So,

The matrix $R = \begin{bmatrix}\sqrt{3} & 2\sqrt{3} & 2\sqrt{3}\\0 & \sqrt{6} & \sqrt{6}\\0 & 0 & \sqrt{2}\end{bmatrix}$

(Or calculate $R = Q^T A$.)

Cont.

6. Vectors $\vec{v}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} -1/2 \\ -1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \vec{v}_4 = \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$ form an **orthonormal basis** for $\mathbb{R}^4$.
Using these vectors to answer the following questions.

(1) Let $V = \text{Span}\{\vec{v}_1, \vec{v}_2\}$ be a subspace of $\mathbb{R}^4$.

Write down an orthonormal basis for V and an orthonormal basis for $V^{\perp}$?

$\{\vec{v}_1, \vec{v}_2\}$ is an orthonormal basis for V.
$\{\vec{v}_3, \vec{v}_4\}$ is an orthonormal basis for $V^{\perp}$.
No reason is ok for correct answers. Partial credits can be given for reasons.

(2) Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be the transformation defined by the orthogonal projection onto $V = \text{Span}\{\vec{v}_1, \vec{v}_2\}$. Find the matrix of the linear transformation T.

Let $U = [\vec{v}_1\ \vec{v}_2\ \vec{v}_3]$. The matrix for T is

$$UU^T = \begin{bmatrix} 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \end{bmatrix}$$

(3). Suppose $\vec{y} \in \mathbb{R}^4$ satisfies $\vec{y} \cdot \vec{v}_1 = \sqrt{2}, \quad \vec{y} \cdot \vec{v}_2 = 2\sqrt{2}, \quad \vec{y} \cdot \vec{v}_3 = 4$, and $\vec{y} \cdot \vec{v}_4 = 2$.

(i) Find $\text{proj}_V \vec{y}$, the orthogonal projection of $\vec{y}$ onto $V = \text{Span}\{\vec{v}_1, \vec{v}_2\}$.

(ii) Find the vector $\vec{y}$.

(i) $\text{proj}_V \vec{y} = \left(\dfrac{\vec{y} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \right) \vec{v}_1 + \left(\dfrac{\vec{y} \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \right) \vec{v}_2 = \sqrt{2}\vec{v}_1 + 2\sqrt{2}\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}$

(ii) $\vec{y} = \sqrt{2}\vec{v}_1 + 2\sqrt{2}\vec{v}_2 + 4\vec{v}_3 + 2\vec{v}_4 = \begin{bmatrix} 0 \\ -1 \\ 2 \\ 5 \end{bmatrix}$

Cont.

7. Using the least-squares method, find the **line** $f(t) = c_0 + c_1 t$ of best fit through the points $(-1, 1)$, $(-2, 0)$, $(3, 5)$.

Solve the linear system $A\vec{x} = \vec{b}$, where $A = \begin{bmatrix} 1 & -1 \\ 1 & -2 \\ 1 & 3 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}$

the normal equation $A^T A\vec{x} = A^T\vec{b}$.

$A^T A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & -2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 14 \end{bmatrix}$

$A^T\vec{b} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \end{bmatrix}$

Solve the normal equation by the augmented matrix $\begin{bmatrix} 3 & 0 & 6 \\ 0 & 14 & 14 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$

So, $\vec{x}^* = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

So the line is $f(t) = 2 + t$

Cont.

8. Let A be the matrix $A = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 0 & x & 9 & 2 \\ 0 & 2 & 3 & 2 \\ 4 & 3 & 5 & 1 \end{bmatrix}$. Answer the following questions.

(1) Compute the **determinant** of A. Write down all steps. The final answer is a formula with x.

$$\det(A) = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 0 & x & 9 & 2 \\ 0 & 2 & 3 & 2 \\ 0 & 0 & 3 & 0 \end{bmatrix} = 4 \begin{vmatrix} x & 9 & 2 \\ 2 & 3 & 2 \\ 0 & 3 & 0 \end{vmatrix} = 4(3)(-1)^{3+2} \begin{vmatrix} x & 2 \\ 2 & 2 \end{vmatrix} = -12(2x-4)$$

(2) For which value of x is the matrix A **not** invertible?

A is not invertible if and only if $\det(A) = 0$.
So $x = 2$.

9. (4 points) Suppose M and N are 3×3 matrices with determinants $\det M = 8$ and $\det N = -1$. Find the determinant of the matrix $2M^{-1}N^3M^T$.

$\det\left(2M^{-1}N^3M^T\right) = 2^3 \det(M^{-1})(\det N)^3 \det M = 2^3(1/8)(-1)^3(8) = -8$

10. Suppose A is a 5×9 matrix such that $\operatorname{rank}(A) = 3$. Answer the following questions:

$\dim(\ker A) =$ _______________; $\dim(\operatorname{im} A) =$ _______________;

$\dim((\ker A)^\perp) =$ _______________ $\dim((\operatorname{im} A^T)^\perp) =$ _______________;

$\dim(\ker(A)) = 6$; $\dim(\operatorname{im}(A)) = 3$;
$\dim(\ker(A)^\perp) = 3$; $\dim((\operatorname{im} A^T)^\perp) = 6$;

11. Consider the matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix}$. Suppose $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ is an eigenvector of A with the corresponding eigenvalue $\lambda = 6$.

(1) Find a basis for the eigenspace E_λ with eigenvalue $\lambda = 0$.

For $\lambda = 0$. $A - 0I = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. The vector form of the solutions is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_2 - 1x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. So the eigenspace E_λ has a basis by $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

(2) Diagonalize the matrix A. That is, find an **invertible** matrix P and a **diagonal** matrix D such that $A = PDP^{-1}$.

$$P = \begin{bmatrix} 1/\sqrt{6} & -2/\sqrt{5} & -1/\sqrt{2} \\ 2/\sqrt{6} & 1/\sqrt{5} & 0 \\ 1/\sqrt{6} & 0 & 1/\sqrt{2} \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Normalization is not required.

(3) (2 points) Determine whether the quadratic form $q(x_1, x_2, x_3) = \vec{x}^T A \vec{x}$ is positive semi-definite, or positive definite, or neither. (Reason)

The quadratic form positive semi-definite since the eigenvalues are non negative.

Cont.

12. Suppose A is a 3×3 matrix with eigenvalues $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 2$. Answer the following questions.
(1) What are the **eigenvalues** of $A^2 - 2A$?

Eigenvalues of $A^2 - 2A$ are calculated by $\lambda^2 - 2\lambda$, which are 0, -1 and 0.

(2) Is the matrix $A^2 - 2A$ invertible? (Reason)

No.
Since zero is an eigenvalue of $A^2 - 2A$. Or determinant of $A^2 - 2A$ is 0.

(3) Is A diagonalizable? Answer: (a) Yes (b) No (c) Can not be determined

(a)

(4) Is $A^2 - 2A$ diagonalizable? Answer: (a) Yes (b) No (c) Can not be determined

(a)

13. Let $A = \begin{bmatrix} 4 & 2 & 3 \\ 2 & 1 & x \\ 0 & 0 & 5 \end{bmatrix}$.

(1) Find all **eigenvalues** for the matrix A. (Write all details of calculation.)

$$\det(A - \lambda I) = \begin{vmatrix} 4-\lambda & 2 & 3 \\ 2 & 1-\lambda & x \\ 0 & 0 & 5-\lambda \end{vmatrix} = (5-\lambda)[(4-\lambda)(1-\lambda) - 4] = (5-\lambda)[\lambda^2 - 5\lambda] = (5-\lambda)^2\lambda = 0$$

So the eigenvalues of A are 0, and 5, 5.

(2) For which values of x is the matrix A diagonalizable? (Hint: Consider geometric multiplicity.)

$$A - 5I = \begin{bmatrix} -1 & 2 & 3 \\ 2 & -4 & x \\ 0 & 0 & 0 \end{bmatrix} \to \ldots \to \begin{bmatrix} 1 & -2 & -3 \\ 0 & 0 & x+6 \\ 0 & 0 & 0 \end{bmatrix}$$

So, the geometric multiplicity of $\lambda = 5$ is 2 if and only if $x + 6 = 0$.
That is $x = -6$.

Cont.

14. Consider the matrix $A = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 0 & -1 \end{bmatrix}$ and $B = A^T A = \begin{bmatrix} 2 & 3 & 0 \\ 3 & 9 & 3 \\ 0 & 3 & 2 \end{bmatrix}$.

(1) Verify that the vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 3 \\ -2 \\ 3 \end{bmatrix}$ are eigenvectors of $B = A^T A$. What are their corresponding eigenvalues?

$B\vec{v}_1 = 11\vec{v}_1$, so $\vec{v}_1$ with eigenvalue 11.
$B\vec{v}_2 = 2\vec{v}_2$, so $\vec{v}_2$ with eigenvalue 2.
$B\vec{v}_3 = 0\vec{v}_3$ so $\vec{v}_3$ with eigenvalue 0.

(2) What are the singular values of A?

$\sigma_1 = \sqrt{11}$, $\sigma_2 = \sqrt{2}$, $\sigma_3 = 0$

(3) Calculate vectors $A\vec{v}_1$, $A\vec{v}_2$ and $A\vec{v}_3$.

$$A\vec{v}_1 = \begin{bmatrix} 11 \\ 0 \end{bmatrix}$$
$$A\vec{v}_2 = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$
$$A\vec{v}_3 = \vec{0}$$

(4) Write a singular value decomposition $A = U\Sigma V^T$. (Explicitly write down all three matrices U, Σ, and V)

$$\Sigma = \begin{bmatrix} \sqrt{11} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix}$$
$$V = \begin{bmatrix} 1/\sqrt{11} & -1/\sqrt{2} & -3/\sqrt{22} \\ 3/\sqrt{11} & 0 & -2/\sqrt{22} \\ 1/\sqrt{11} & 1/\sqrt{2} & 3/\sqrt{22} \end{bmatrix}$$
$$U = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

The End.