Northeastern University, Department of Mathematics

MATH 5110: Applied Linear Algebra and Matrix Analysis.

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§3. Topics: 1. Linear Spaces over fields; 2. Subspaces; 3. linear transformation.

1. **Linear Spaces**

We already met real vector spaces $\mathbb{R}^n$, or complex vector spaces $\mathbb{C}^n$. Now, let us study a more general definition of abstract vector spaces.

Definition 1 ((abstract) vector space, or linear spaces)**.** Let $\mathbb{F}$ be a field. A **vector space** over $\mathbb{F}$ is any nonempty set V of objects, called *vectors*, on which there are defined two **closed** operations,
• **vector addition (sum)** $\vec{u}+\vec{v}\in V$, and
• **multiplication by a scalar** $c\in\mathbb{F}$**,** $c\vec{u}\in V$ **(scalar product)**,
subject to the rules below, called **axioms of a vector space**:
1.) $\vec{u}+\vec{v}=\vec{v}+\vec{u}$.
2.) $(\vec{u}+\vec{v})+\vec{w}=\vec{v}+(\vec{u}+\vec{w})$.
3.) There is a zero vector $\vec{0}\in V$ such that $\vec{u}+\vec{0}=\vec{u}$.
4.) For each $\vec{u}\in V$, there is a vector $-\vec{u}\in V$ such that $\vec{u}+(-\vec{u})=\vec{0}$.
5.) $c\cdot(\vec{u}+\vec{v})=c\cdot\vec{u}+c\cdot\vec{v}$.
6.) $(c+d)\cdot\vec{u}=c\cdot\vec{u}+d\cdot\vec{v}$.
7.) $c(d\cdot\vec{u})=(cd)\vec{u}$.
8.) $1\cdot\vec{u}=\vec{u}$.
These must hold for all $\vec{u},\vec{v},\vec{w}\in V$ and all $c,d\in\mathbb{F}$.

Axioms 1-4 implies that $(V,+)$ is a abelian group.

Some nice properties seems obvious in $\mathbb{R}^n$, but need a proof in general starting from the axioms.

Proposition 2. *(1) Zero vector is unique.*
(2) For any $\vec{u}$, the inverse vector $-\vec{u}$ is unique.

Proof. Since $(V,+)$ is an abelian group, we have proved this in §1. □

Proposition 3. $0\cdot\vec{u}=\vec{0}$,

Proof. $\vec{0}=\vec{u}+(-\vec{u})=1\cdot\vec{u}+(-\vec{u})=(0+1)\cdot\vec{u}+(-\vec{u})=0\cdot\vec{u}+1\cdot\vec{u}+(-\vec{u})=0\cdot\vec{u}+\vec{0}=0\cdot\vec{u}$
In the proof, we used axioms 4, 8, 6, 2, 3. □

Proposition 4. $c\cdot\vec{0}=\vec{0}$

Proof. $c \cdot \vec{0} = c \cdot (0 \cdot \vec{u}) = (c0) \cdot \vec{u} = 0 \cdot \vec{u} = \vec{0}$ □

Proposition 5. $-\vec{u} = (-1)\vec{u}$.

Proof. $\vec{0} = 0 \cdot \vec{u} = (1-1) \cdot \vec{u} = 1\vec{u} + (-1)\vec{u} = \vec{u} + (-1)\vec{u}$. So $-\vec{u} = (-1)\vec{u}$ since inverse is unique. □

We have the following examples of vector spaces with the corresponding sum and scalar product.

1. $\mathbb{F}^n$ is a vector space.

2. The set of all $m \times n$ matrices (denoted as $\mathbb{F}^{m\times n}$) with matrix sum and scalar product is a vector space with entries in $\mathbb{F}$.

3. The set of all functions $f : \mathbb{F} \to \mathbb{F}$ with sum and scalar product is a vector space. (Infinite dimensional)

4. The set of all continuous functions $f : \mathbb{R} \to \mathbb{R}$ with sum and scalar product is a vector space. (Infinite dimensional)

4. The set of all functions(transformations) $\mathbb{F}^m \to \mathbb{F}^n$ is a vector space.

It is not hard but we need to verify that those two operations are closed and all those eight axioms are satisfied.

Once we already have some vector spaces, we will consider subsets of the vector space with the same sum and scalar product operations. It will be easier to check using the following definition.

2. **Subspaces**

Definition 6. (Subspace) A **subspace** of the vector space V is a subset H of V that satisfies the following three properties.
(1). $\vec{0} \in H$.
(2). If $\vec{u}, \vec{v} \in H$ then $\vec{u} + \vec{v} \in H$. (Closed under addition)
(3). If $\vec{u}, \vec{v} \in H$ and $c \in \mathbb{F}$, then $c\vec{u} \in H$. (Closed under scalar product)

Theorem 7. *A subspace H of a linear space V is a linear space.*

Remark: In the definition, we can put non-empty subset and omit (1).

Proposition 8. $\{\vec{0}\}$ *is a subspace of linear space V, called zero space.*

Example 9. Determine which of the following set is a subspace (vector space).

1. Let P be the set of all polynomials.

2. Let P_n be the set of all polynomials of degree $\leq n$.

Proof. The set of polynomials of degree $\leq n$, $P_n = \{a_0 + a_1 t + \cdots a_n t^n | a_i \in F\}$ is a vector space. First, $0 \in P_n$.
Second, For any two such polynomials $a_0 + a_1 t + \cdots a_n t^n$ and $b_0 + b_1 t + \cdots b_n t^n \in P_n$, the sum $a_0 + a_1 t + \cdots a_n t^n + b_0 + b_1 t + \cdots b_n t^n = (a_0 + b_0) + (a_1 + b_2)t + \cdots (a_n + b_n)t^n$ is in P_n.
Third, for any polynomials $a_0 + a_1 t + \cdots a_n t^n \in P_n$ and any $c \in F$, the scalar product $c(a_0 + a_1 t + \cdots a_n t^n) = ca_0 + ca_1 t + \cdots ca_n t^n$ is in P_n. So P_n is a subspace of the linear space of all functions $F \to F$. $\square$

3. Let H be the set of all polynomials of degree exactly 3, with real coefficients.

4. Let $H = \{ax^4 + b \mid a, b \in \mathbb{R}\}$. Is H a subspace of P_4?

5. Let $H = \{x^2 + a \mid a \in \mathbb{R}\}$. Is H a subspace of P?

6. The set $U_{n\times n}$ of all $n \times n$ upper triangular matrices with real entries.

7. The set $L_{n\times n}$ of all $n \times n$ lower triangular matrices with real entries.

8. The set $D_{n\times n}$ of all $n \times n$ diagonal matrices with real entries.

9. The set $T_{m\times n}$ of all $n \times n$ triangular matrices with real entries.

Yes: 1, 2, 4,6, 7, 8,
No: 3, 5, 9

Example 10. Determine which of the following set is a subspace (vector space).

1. Let L be the set of vectors on the line $2x_1 - x_2 = 0$.

2. Let L be the set of vectors on the line $2x_1 - x_2 = 1$.

3. Let H be the set of vectors on the plane $3x_1 - 5x_2 + x_3 = 0$.

4. Let $V = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0 \right\}$.

5. The union of the first and second quadrants in the xy-plane: $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| \; y \geq 0 \right\}$

6. Any line or plane passing zero in $\mathbb{R}^n$.

Yes: 1, 3, 6
No: 2, 4,5

Definition 11. A ***linear combination*** of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ in V is a vector in V defined as

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m$$

where $c_1, \ldots, c_m \in \mathbb{F}$. The set of all linear combinations of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ in V is called the **span** of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$, denoted as

$$\mathrm{Span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m) := \{c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m |\ \text{all } c_i \in \mathbb{F}\}$$

Theorem 12. *Let $\vec{u}_1, \vec{v}_2, \ldots, \vec{v}_m$ be vectors in a linear space V. Then $\mathrm{Span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m)$ is a subspace of V.*

Proof. We prove the theorem by verifying the definition.
1. Choose all $c_i = 0$ so $\vec{0} \in \mathrm{Span}(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_m)$
2. For any two vectors $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m$ and $d_1\vec{v}_1 + d_2\vec{v}_2 + \cdots + d_m\vec{v}_m$ in $\mathrm{Span}(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_m)$, the sum $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m + d_1\vec{v}_1 + d_2\vec{v}_2 + \cdots + d_m\vec{v}_m = (c_1+d_1)\vec{v}_1 + (c_2+d_2)\vec{v}_2 + \cdots + (c_m+d_m)\vec{v}_m$ is an element in $\mathrm{Span}(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_m)$.
3. For any vector $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m$ in $\mathrm{Span}(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_m)$ and any $k \in \mathbb{F}$, the scalar product $k(c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m) = kc_1\vec{v}_1 + kc_2\vec{v}_2 + \cdots + kc_m\vec{v}_m$ is an element in $\mathrm{Span}(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_m)$. $\square$

The other direction is also correct:

Proposition 13. *Any subspace U of V can be written as span of some vectors in V.*

The choice of span is not unique. The largest is $U = \mathrm{Span}(U)$.

If a vector space V can be written as a span of finite number of vectors in V, then V is called a **finite-dimensional** vector space. In our class, we will focus on finite-dimensional vector spaces.

3. Sum and direct sum of subspaces

Let U and W be subspaces of a vector space V.

Definition 14. The **sum** of U and W is defined to be the set of all possible sums of elements in U and W,

$$U + W = \{u + w | u \in U; w \in W\}$$

Proposition 15. *$U + W$ is a subspace of V.*

Definition 16. If $V = U + W$ and each $v \in V$ can be uniquely written as $v = u + w$, then, V is called the **direct sum** of U and W, denoted by

$$V = U \oplus W$$

Example 17. $U = \{x, y, 0\} \subset \mathbb{R}^3$ and $V = \{0, y, z\} \subset \mathbb{R}^3$, then $U + V = \mathbb{R}^3$. However, $\mathbb{R}^3$ is not a direct sum of U and V.

In order to verify direct sum, we only need to verify that $\vec{0}$ only have a unique decomposition.

Proposition 18. *Let $V = U + W$. If $\vec{0} = \vec{u} + \vec{w}$ implies $\vec{u} = \vec{w} = \vec{0}$, then*
$$V = U \oplus W$$

Proof. Suppose $v \in V$ can be written as two decompositions $\vec{v} = \vec{u}_1 + \vec{w}_1 = \vec{u}_2 + \vec{w}_2$, then $(\vec{u}_1 - \vec{u}_2) + (\vec{w}_1 - \vec{w}_2) = \vec{0}$. Then $\vec{u}_1 = \vec{u}_2$ and $\vec{w}_1 = \vec{w}_2$. Hence, $V = U \oplus W$. □

Proposition 19. *$V = U \oplus W$ if and only if $V = U + W$ and $V \cap W = \{\vec{0}\}$.*

Proof. $\Rightarrow$: Suppose $V = U \oplus W$, then $V = U + W$. If $\vec{v} \in V \cap W$, then $\vec{0} = \vec{v} + (-\vec{v})$. We also have $\vec{0} = \vec{0} + \vec{0}$. Hence $\vec{v} = \vec{0}$.
$\Leftarrow$: Suppose $\vec{v} = \vec{u}_1 + \vec{w}_1 = \vec{u}_2 + \vec{w}_2$, then $(\vec{u}_1 - \vec{u}_2) = -(\vec{w}_1 - \vec{w}_2) \in U \cap W$. So $\vec{u}_1 - \vec{u}_2 = -(\vec{w}_1 - \vec{w}_2) = \vec{0}$. Hence $\vec{u}_1 = \vec{u}_2$ and $\vec{w}_1 = \vec{w}_2$. □

Example 20. $\mathbb{R}^3$ is a direct sum of $U = \{x, 0, 0\} \subset \mathbb{R}^3$ and $V = \{0, y, z\} \subset \mathbb{R}^3$.

Example 21. Suppose U and V are two subspaces of a vector space W.

(1) Is the union of two subspace $U \cup V$ a subspace?

(2) Is the intersection $U \cap V$ is a subspace?

4. **Linear transformations**

Let V and W be two vector spaces over a field $\mathbb{F}$. A **transformation** (or function or map) T from V to W is a rule
$$T\colon V \to W$$
of assigning to each vector $\vec{v} \in V$ a vector $T(\vec{v}) = \vec{w} \in W$. This is only in set level.

Definition 22. A transformation $T\colon V \to W$ is called ***linear*** if
(i) $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ for all $\vec{u}, \vec{v} \in V$.
(ii) $T(c \cdot \vec{u}) = c \cdot T(\vec{u})$ for all $\vec{u} \in V$ and all $c \in \mathbb{F}$.

From definition, we can easily get the following properties for linear transformation.

Proposition 23. *If $T\colon V \to W$ is a linear transformation, then*
(i) $T(\vec{0}) = \vec{0}$;
(ii) $T(c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p) = c_1T(\vec{v}_1) + c_2T(\vec{v}_2) + \cdots + c_pT(\vec{v}_p)$ for all scalars $c_1, \ldots, c_p \in \mathbb{F}$ and all vectors $\vec{v}_1, \ldots, \vec{v}_p \in V$.

Example 24. 1. Zero map is linear transformation.

2. Identity map $\mathrm{id} : V \to V$ is a linear transformation.

Definition 25. Consider a transformation $T\colon V \to W$. We call V the **domain** of T.
The set of all vectors $T(\vec{x})$ in W, for all $\vec{x} \in V$, is called the ***image*** of T, denoted as $\mathrm{im}(T)$, that is

$$\mathrm{im}(T) := \{T(\vec{x}) \mid \text{ all } \vec{x} \in V\} \subset W.$$

The **kernel** of T is defined as

$$\ker(T) := \{\vec{x} \in V \mid T(\vec{x}) = \vec{0}\} \subset V.$$

Recall that a transformation $T\colon V \to W$ is called **surjective** (or onto) if for any vectors $\vec{b} \in V$, $T(\vec{x}) = \vec{b}$ has at least a solution. T is called **injective** (or one-to-one) if $T(\vec{a}) = T(\vec{b})$ implies $\vec{a} = \vec{b}$ for any vectors $\vec{a}, \vec{b} \in V$.

We can use kernel and image to describe injective and surjective.

Proposition 26. *A linear transformation $T\colon V \to W$ is injective if and only if $\ker(T) = \{\vec{0}\}$, (i.e., $T(\vec{v}) = \vec{0}$ implies $\vec{v} = \vec{0}$.)*
$T\colon V \to W$ is surjective if and only if $\mathrm{im}(T) = W$.

Proof. The forward direction ($\Rightarrow$) is clear. Let's show the backward direction ($\Leftarrow$). For any vectors $\vec{a}, \vec{b} \in \mathbb{R}^n$, if $T(\vec{a}) = T(\vec{b})$, then $T(\vec{a} - \vec{b}) = \vec{0}$. So, $\vec{a} - \vec{b} = \vec{0}$, that is $\vec{a} = \vec{b}$. Hence T is injective. $\square$

Theorem 27. *Let $T\colon V \to W$ be a linear transformation. Then $\mathrm{im}(T)$ is a subspace of W and $\ker(T)$ is a subspace of V.*

(Vector space $\ker(T)$ is also called **Null space** and $\mathrm{im}(T)$ is called ***range*** of T)

Proof. (1) First, $\vec{0} \in \mathrm{im}(T)$ since $T(\vec{0}) = \vec{0}$. Second, if $\vec{a}$ and $\vec{b} \in \mathrm{im}(T)$, then there exists $\vec{u}, \vec{v} \in V$ such that $T(\vec{u}) = \vec{a}$ and $T(\vec{v}) = \vec{b}$. So, $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v}) = \vec{a} + \vec{b}$. Hence $\vec{a} + \vec{b} \in \mathrm{im}(T)$. Third, if $\vec{a} \in \mathrm{im}(T)$ and $c \in F$, then there exists $\vec{u} \in V$ such that $T(\vec{u}) = \vec{a}$. So, $T(c\vec{u}) = cT(\vec{u}) = c\vec{a}$. Hence, $c\vec{a} \in \mathrm{im}(T)$. Hence, $\mathrm{im}(T)$ is a subspace of W.
(2) First, $\vec{0} \in \ker(T)$ since $T(\vec{0}) = \vec{0}$. Second, if $\vec{a}$ and $\vec{b} \in \ker(T)$, then $T(\vec{a}) = T(\vec{b}) = \vec{0}$. So, $T(\vec{a} + \vec{b}) = T(\vec{a}) + T(\vec{b}) = \vec{0}$. Hence $\vec{a} + \vec{b} \in \ker(T)$. Third, if $\vec{a} \in \ker(T)$ and $c \in F$, then $T(\vec{a}) = \vec{0}$. So, $T(c\vec{a}) = cT(\vec{a}) = \vec{0}$. Hence, $c\vec{a} \in \ker(T)$. Hence, $\ker(T)$ is a subspace of V. $\square$

A linear transformation $T\colon X \to Y$ is said to be **invertible** (or **bijective**) if the equation $T(\vec{x}) = \vec{y}$ has a unique solution $\vec{x}$ for any $\vec{y} \in Y$. Equivalently, there is a transformation $S\colon Y \to X$ such that

$$S(T(\vec{x})) = \vec{x}, \text{ for all } \vec{x} \in X;$$

$$T(S(\vec{y})) = \vec{y}, \text{ for all } \vec{y} \in Y.$$

Remark. Such a transformation $S\colon Y \to X$, if it exists is unique and linear, is called the **inverse transformation** of T. We shall use the notion $S = T^{-1}$ to denote the inverse transformation of T.

Definition 28. Two vector spaces V and W are called **isomorphic**, denoted as $V \cong W$, if there is an invertible linear transformation $T : V \to W$.

Let V and W be vector spaces. Denote $\mathscr{L}(V, W)$ the set of all linear transformations from V to W. Thus $T \in \mathscr{L}(V, W)$ if and only if

(i) $T : V \to W$,
(ii) $T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2)$ for $\vec{v}_1, \vec{v}_2 \in V$,
(iii) $T(a\vec{v}) = aT(\vec{v})$ for $\vec{v} \in V$, $a \in \mathbb{F}$.

Theorem 29. *$\mathscr{L}(V, W)$ is a vector space.*

Proof. All transformations from V to W form a vector space under the sum and scalar products. We only need to show $\mathscr{L}(V, W)$ is a subspace by verifying the three conditions. Zero transformation is a linear transformation. Sum and scalar product preserve linearity. (verify). $\square$

6. **Quotient spaces.**

Let V be a vector space over a field $\mathbb{F}$.

An **equivalent relation** $\sim$ on V is a binary relation such that reflexive, symmetric and transitive. That is, for any $\vec{u}$, $\vec{v}$, $\vec{w} \in V$,

- $\vec{v} \sim \vec{v}$.
- If $\vec{v} \sim \vec{w}$, then $\vec{w} \sim \vec{v}$.
- If $\vec{u} \sim \vec{v}$ and $\vec{v} \sim \vec{w}$, then $\vec{u} \sim \vec{w}$.

Let N be a subspace of V. We can define an equivalence relation on V by defining that

$$\vec{v} \sim \vec{w} \text{ if } \vec{v} - \vec{w} \in N$$

The **equivalence class** (or, called the **coset**) of $\vec{v}$ is defined $[\vec{v}] := \vec{v} + N = \{\vec{v} + \vec{a} \mid \vec{a} \in N\}$.

The **quotient space** V/N is a the set of all cosets by the equivalent relation. Addition and scalar product is defined as

- $[\vec{v}] + [\vec{w}] = [\vec{v} + \vec{w}]$.
- $c[\vec{v}] = [c\vec{v}]$.

Check that this definition is not depending on the choice of representative.

Quotient space V/N is a vector space.

There is a natural epimorphism from $p : V \to V/N$ defined by $p(\vec{v}) = [\vec{v}]$. The kernel is $\ker p = N$. There exists a **short exact sequence**

$$0 \to N \to V \to V/N \to 0$$

Here, exact means ker=im at each connecting place.

On the level of dimension, $\dim(V/N) = \dim(V) - \dim(N)$.

The idea of quotient is used in almost all mathematics, e.g., quotient group, quotient ring, quotient field, quotient module, quotient algebra, quotient space in topology, etc.

7. **Linear transformations and matrices**

Theorem 30. *Given an $m \times n$ matrix A. There is a linear transformation $T\colon \mathbb{F}^n \to \mathbb{F}^m$ defined as*

$$T(\vec{x}) = A \cdot \vec{x}$$

Proof. If A is an $m \times n$ matrix, $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^m$ and c is a scalar, then $A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v}$ and $A(c\vec{u}) = c(A\vec{u})$. So, T is a linear transformation. □

Denote $\vec{e}_1$, $\vec{e}_2$, ... , $\vec{e}_m$ be the column vectors of the identity matrix I_m. We call them the standard vectors in $\mathbb{F}^m$.

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \cdots \qquad \vec{e}_m = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

The next theorem is very effective for finding the matrix for a given linear transformation.

Theorem 31 (Transformation matrix)**.** *Let $T\colon \mathbb{F}^n \to \mathbb{F}^m$ be a linear transformation. There exists an $m \times n$ matrix A such that $T(\vec{x}) = A \cdot \vec{x}$. Further more, the matrix of T is given by*

$$A = [T(\vec{e}_1) \quad T(\vec{e}_2) \quad \cdots \quad T(\vec{e}_n)].$$

Proof. For any $\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_n\vec{e}_n \in \mathbb{F}^n$,

$$\begin{aligned} T(\vec{x}) &= T(x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_n\vec{e}_n) \\ &= x_1T(\vec{e}_1) + x_2T(\vec{e}_2) + \cdots + x_nT(\vec{e}_n) \\ &= [T(\vec{e}_1)\ T(\vec{e}_2)\ \cdots\ T(\vec{e}_n)]\vec{x} \\ &= A\vec{x} \end{aligned}$$

□

Theorem 32. *$\mathscr{L}(\mathbb{F}^n, \mathbb{F}^m)$ is isomorphic to the matrix space $\mathbb{F}^{m\times n}$.*

Proof. Define a map $f : \mathbb{F}^{m\times n} \to \mathscr{L}(F^n, F^m)$ send matrix A to the linear transformation T_A.
f is linear, since $f(A + B) = f(A) + f(B)$ and $f(cA) = cf(A)$.
$\ker(f) = \{0\}$ is the zero matrix.
By the above theorem, f is surjective. So, f is a isomrophism. □

Definition 33. The vector space $V^* := \mathscr{L}(V, \mathbb{F})$ is called the **dual space** of the vector space V.

In particular,

Theorem 34. *An $n \times n$ matrix A is invertible if and only if the linear transformation T_A is injective; if and only if the linear transformation T_A is surjective.*

Proof. T_A is injective if and only if the linear system $A\vec{x} = \vec{0}$ has only the trivial solution. T_A is surjective if and only if the $T_A(\vec{x}) = \vec{b}$ has at least a solution for any $\vec{b} \in \mathbb{F}^n$. □

Theorem 35 (Matrix product and composition of transformations)**.** *Let A be an $m \times n$ matrix and B be a $n \times p$ matrix. Then the product AB is the matrix of the transformation composition $T_A \circ T_B$.*

Proof. $(T_A \circ T_B)(\vec{x}) = T_A(B\vec{x}) = A(B\vec{x}) = AB\vec{x}$. □

Corollary 36. *An $n \times n$ matrix A is invertible if and only if T_A is invertible. Moreover, $(T_A)^{-1} = T_{A^{-1}}$.*

We know that any linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is defined by an $m \times n$ matrix $A = [\vec{a}_1\ \vec{a}_2\ \cdots\ \vec{a}_n]$. So, the image of T is

$$\begin{aligned}\operatorname{im}(T) &= \{A(\vec{x}) \mid \text{ all } \vec{x} \in \mathbb{R}^n\} \\ &= \{x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n \mid \text{ all real numbers } x_i\} \\ &= \{\text{all linear combinations of } \vec{a}_1, \vec{a}_2, \cdots, \vec{a}_n\} \\ &= \operatorname{Span}(\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_n)\end{aligned}$$

The kernel of T is the solution set

$$\ker(T) = \{\vec{x} \in \mathbb{R}^n \mid A(\vec{x}) = \vec{0}\} = \{\text{all solutions of } A(\vec{x}) = \vec{0}\}$$

Proposition 37. *For an $m \times n$ matrix A,*

(1.) $\ker(A) = \{\vec{0}\}$ *if and only if $A\vec{x} = \vec{0}$ only has zero solution; if and only if* $\operatorname{rank}(A) = n$ *(i.e., no free variable).*

(2.) *If* $\ker(A) = \{\vec{0}\}$ *(T_A is injective), then $m \geq n$ (i.e., more equations than variables).*

(3) *If* $\operatorname{im}(T_A)$ *is surjective, then $n \geq m$.*

Proof. Proof of (1) and (2) is clear. For proof of (3), $\operatorname{im}(A) = \operatorname{Span}(\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_n) = \mathbb{R}^m$. We will finish the argument in next section using basis and dimension. □

Example 38. Find a transformation which has kernel as a plane

$$2x - y + 3z = 0.$$

How many such transformation can we find?

Example 39. What is the geometry of image and kernel of the transformation defined by $T(\vec{x}) = \vec{v} \cdot \vec{x}$ for $\vec{v} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$?

Example 40. Find the image and kernel of the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ defined by matrix $A = \begin{bmatrix} 0 & 0 & 2 & 8 \\ 1 & 5 & 2 & -5 \\ 2 & 10 & 6 & -2 \end{bmatrix} = [\vec{a}_1 \ \vec{a}_2 \ \vec{a}_3 \ \vec{a}_4]$. Answer: The **image** of T is $\text{im}(T) = \text{Span}(\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4)$.

We can calculated that $\textbf{rref}(A) = \begin{bmatrix} 1 & 5 & 0 & -13 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The solutions of $A(\vec{x}) = \vec{0}$ can be described as vector form:

$$\vec{x} = x_2 \begin{bmatrix} -5 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 13 \\ 0 \\ -4 \\ 1 \end{bmatrix} = x_2\vec{v}_1 + x_4\vec{v}_2$$

So the **kernel** is $\ker(T) = \text{Span}\{v_1, v_2\}$.

Consider an $m \times n$ matrix. We know that $\ker(T)$ is the set of all solutions of $A(\vec{x}) = \vec{0}$.

8. Tensor product of spaces and Kronecker product of matrices