Lab - Stability Problems

MATH 5110: Applied Linear Algebra and Matrix Analysis

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1 Background

Let A be a real $n \times n$ matrix, and consider the n-dimensional linear dynamical system described by the update equation

$$\vec{x}(1) = A\,\vec{x}(0)$$

By iterating this equation we obtain the sequence of vectors $\vec{x}(0), \vec{x}(1), \vec{x}(2), \ldots$, where

$$\vec{x}(k) = A^k\,\vec{x}(0), \quad k = 1, 2, 3, \ldots$$

The system is said to be *asymptotically stable* if for every initial vector $\vec{x}(0)$ we have

$$\vec{x}(k) = A^k\,\vec{x}(0) \to \vec{0} \quad \text{as } k \to \infty$$

For example, suppose $n = 2$ and

$$A = \begin{pmatrix} 0.5 & 0 \\ 0 & -0.2 \end{pmatrix}, \quad \vec{x}(0) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Then for every integer k we have

$$\vec{x}(k) = A^k\,\vec{x}(0) = \begin{pmatrix} (0.5)^k\,x_1 \\ (-0.2)^k\,x_2 \end{pmatrix}$$

Since $r^k \to 0$ for every $|r| < 1$ this shows that $(0.5)^k\,x_1 \to 0$ and $(-0.2)^k\,x_2 \to 0$, hence the system is asymptotically stable.

Now suppose that A is diagonalizable and consider the factorization

$$A = S\,D\,S^{-1}$$

where $D = \text{diag}(\lambda_1, \ldots, \lambda_n)$. Then

$$A^k = S\,D^k\,S^{-1}, \quad k = 1, 2, 3, \ldots$$

where $D^k = \text{diag}(\lambda_1^k, \ldots, \lambda_n^k)$. We see that $\lambda_1^k \to 0$ if $|\lambda_1| < 1$, and similarly for the other eigenvalues. Thus the necessary and sufficient condition for asymptotic stability is that all eigenvalues of A have absolute value less than 1.

2 Computation

Consider the 3-dimensional dynamical system with matrix

$$A = \frac{1}{8} \begin{pmatrix} 1 & 2 & 1 \\ 3 & 4 & 1 \\ 2 & 3 & 4 \end{pmatrix}$$

2.1 Task 1

Show that the system with matrix A is asymptotically stable.

2.2 Task 2

Let $\vec{u}, \vec{w}$ be vectors given by

$$\vec{u} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad \vec{w} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

and define the matrix

$$B = \frac{1}{5} \vec{u} \, \vec{w}^T$$

We suppose that the original system with matrix A is perturbed by the addition of matrix B: the perturbed system has matrix

$$A(s) = A + s\,B$$

where s is a real number. So for example $A(0) = A$, and $A(1) = A + B$. From Task 1 we know that the matrix $A(s)$ is stable at $s = 0$. Use Matlab to show that the system is unstable at $s = 1$. Use Matlab to find the smallest integer m such that the matrix $A(s)$ is unstable at $s = -m$.

2.3 Task 3

The matrix $A(s)$ is stable for all s in an open interval $(-a, b)$ where a, b are real positive numbers, and it is unstable for s outside this interval. Use Matlab to compute the numbers a and b to 2 decimal places.