Northeastern University, Department of Mathematics

MATH 5110: Applied Linear Algebra and Matrix Analysis.

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§12 Spectral Theorem and quadratic forms

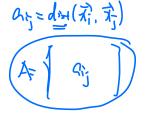
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1. Spectral Theorem

In this section, we deal real matrix.

An $n \times n$ matrix A is called **(symmetric)** if $A^T = A$, i.e., for all $i, j \in \{1, 2, ..., n\}$

Example 1 (Diagonalizing a Symmetric Matrix).

$$A = \begin{bmatrix} 10 & 6 \\ 6 & 1 \end{bmatrix}. \qquad B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}. \qquad C = \begin{bmatrix} 1 & 1 & 7 \\ 1 & 7 & 1 \\ 7 & 1 & 1 \end{bmatrix}.$$

$$A = PDP^{-1}$$

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \lambda = 1, -i$$

Proposition 2. A symmetric $n \times n$ matrix \underline{A} has \underline{n} real eigenvalues if they are counted with their algebraic multiplicities.

Suppose
$$A\vec{x} = \lambda \vec{x}$$
. for $\lambda \in \mathbb{C}$ $\vec{x} \neq \vec{0} \in \mathbb{C}^n$

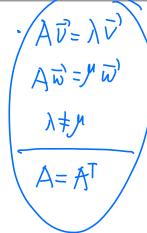
$$\frac{\langle A\vec{x}, \vec{x} \rangle_c}{||} = \langle \lambda \vec{x}, \vec{x} \rangle_c = |\lambda \langle \vec{x}, \vec{x} \rangle_c$$

$$\frac{||}{||} \text{ adjusht}$$

$$\frac{\langle \vec{x}, \vec{x} \vec{y} \rangle_c}{||} = \langle \vec{x}, A\vec{x} \rangle_c = \langle \vec{x}, \lambda \vec{x} \rangle_c = |\vec{\lambda} \langle \vec{x}, \vec{x} \rangle$$

1

Proposition 3. Let A be a symmetric matrix and let λ , μ be two distinct eigenvalues of A with associated eigenvectors \vec{v} , \vec{w} . Then $\vec{v} \cdot \vec{w} = 0.$



$$\langle A\vec{v}, \vec{w} \rangle = \lambda \langle \vec{v}, \vec{v} \rangle$$

$$(\lambda - \mu) \langle \vec{v}, \vec{w} \rangle = 0$$

$$\langle \vec{v}, A\vec{w} \rangle = \mu \langle \vec{v}, \vec{v} \rangle = \mu \langle \vec{v}, \vec{v} \rangle$$

$$\langle \vec{v}, A\vec{w} \rangle = \mu \langle \vec{v}, \vec{v} \rangle = \mu \langle \vec{v}, \vec{v} \rangle$$

 E_{λ} is orthogonal to E_{μ} for distinct eigenvalues λ, μ (in that $\vec{v} \cdot \vec{w} = 0$ for all $\vec{v} \in E_{\lambda}$ and $\vec{w} \in E_{\mu}$).

Definition 4 (Orthogonal Diagonalization). An $n \times n$ matrix is **orthogonally diagonalizable** if there exist diagonal matrix D and orthogonal matrix P such that

 $(A) = PDP^{-1} = PDP^{T}.$

Theorem 5 (On Orhtogonal Diagonalizability). $An(n \times n)$ matrix A is orthogonally diagnonalizable if and only if A is a symmetric matrix.

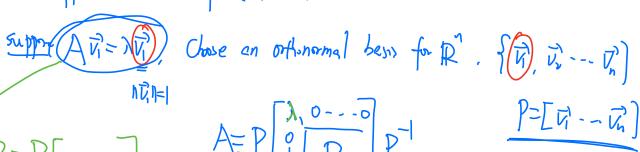
$$A=PDP^{T}$$

$$A^{T}=(PDP^{T})^{T}=(PT)^{T}D^{T}P^{T}=PDP^{T}$$

$$A=PDP^{T}$$

$$A=P$$

· suppose it i) true for (n-1) X (n-1) mother.



AP=P[

$$A^{T}=A$$

$$\Rightarrow B^{T}=B$$

$$A^{T}=A$$

$$\Rightarrow A^{T}=A$$

$$\Rightarrow A^{T}=$$

By hahrton
$$B = QDQ^T$$
 $Q^T = Q^T =$

$$P=[\vec{U}_1\vec{U}_1\cdots\vec{U}_n]$$

$$\mathbb{D}=\begin{bmatrix}\lambda_1\\ \lambda_h\end{bmatrix}$$

3. The Spectral Decomposition

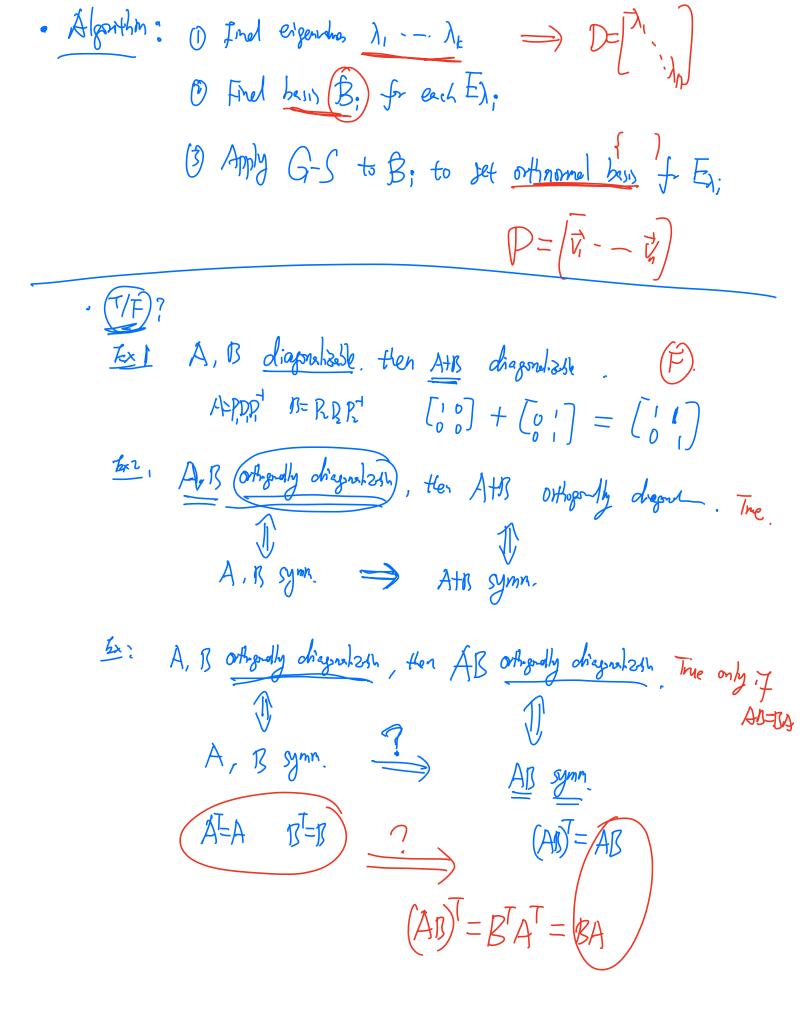
Let A be an $n \times n$ matrix and let D and P be a diagonal and orthogonal matrix with $A = PDP^{-1}$

Theorem 6 (Spectral Decomposition for Symmetric Matrices).

$$A = PDP = \begin{bmatrix} \vec{\alpha}_1 & \cdots & \vec{\alpha}_n \end{bmatrix} \begin{bmatrix} \vec{\lambda}_1 & \cdots & \vec{\lambda}_n \end{bmatrix} \begin{bmatrix} \vec{\alpha}_1^T & \cdots & \vec{\lambda}_n \end{bmatrix} \begin{bmatrix} \vec{\alpha}_1^T & \cdots & \vec{\lambda}_n \end{bmatrix} \begin{bmatrix} \vec{\alpha}_1^T & \cdots & \vec{\lambda}_n \vec{\alpha}_n^T \end{bmatrix} \begin{bmatrix} \vec{\alpha}_1^T & \cdots & \vec{\lambda}_n^T \end{bmatrix} \begin{bmatrix} \vec{\alpha}_1^$$

Dente

$$\operatorname{Proj}_{i} \vec{y} = (\vec{u}_{i} \vec{u}_{i}^{T}) \vec{y}$$



2. Quadratic forms and positive definite

Definition 7. A function $p(x_1,...,x_n)$ from \mathbb{R}^n to \mathbb{R} is call a **quadratic form**, if it is a linear combination of forms $x_i x_j$.

So, a quadratic form can be written as

$$\underline{p(x_1, ..., x_n)} = \sum_{i,j} c_{ij} x_i x_j \qquad = \sum_{i=1}^{3} \sum_{j=1}^{3} \alpha_{ij} \gamma_j \gamma_j$$

Another way to write quadratic form is using symmetric matrices
$$p(x_1,...,x_n) = \vec{x} \cdot A\vec{x} = (\vec{x}^T A \vec{x}) = (x_1 \times x_2) (x_1 \times x_3) (x_2 \times x_4) (x_3 \times x_5) (x_4 \times x_5) (x_4 \times x_5) (x_5 \times x_5$$

The unique symmetric matrix A is called the matrix for the quadratic for

Example 8. Consider
$$p(x_1, ..., x_3) = 3x_1^2 + 4x_2^2 - 5x_3^2 - 2x_1x_2 + 4x_1x_3 + 6x_2x_3$$

$$= \sqrt[3]{x_1} + \sqrt[4]{x_2} - 5x_3^2 - 2x_1x_2 + 4x_1x_3 + 6x_2x_3$$

$$= \sqrt[7]{x_1} + \sqrt[7]{x_1} + \sqrt[7]{x_1} + \sqrt[7]{x_1} + \sqrt[7]{x_1} + \sqrt[7]{x_2} + \sqrt[7]{x_1} + \sqrt[7]{x_2} + \sqrt[7]{x_1} + \sqrt[7]{x_2} + \sqrt[7]{x_2} + \sqrt[7]{x_1} + \sqrt[7]{x_2} + \sqrt[7]{x_$$

(1)
$$P(x) = 3x_1^2 + 2x_1^2 + x_1^2 > 0$$
 for any $x \neq 0$ $A = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 2 & 0 \end{bmatrix}$
(1) $P(x') = x_1^2 + 3x_1^2 > 0$
(2) $P(x') = x_1^2 - x_1^2 = x_1^2 - x_1^2 = x_1^2$

Definition 9. An real symmetric matrix \underline{A} is called **positive definite** if the quadratic form

$$\vec{x}^T A \vec{x} > 0$$

for all nonzero $\vec{x} \in \mathbb{R}^n$.

The matrix A is called **positive semidefinite** if the quadratic form

$$\vec{x}^T A \vec{x} > 0$$

for all $\vec{x} \in \mathbb{R}^n$.

Theorem 10. (1) An real symmetric matrix A is positive definite if and only if all eigenvalues of A are positive.

(2) An real symmetric matrix A is positive semidefinite if and only if all eigenvalues of A are non-negative.

$$A = A^{T} \iff A = PDP^{T} = PDP^{T}$$

$$\frac{g(x_{1} \cdots x_{n})}{g(x_{1} \cdots x_{n})} = x^{T}Ax = x^{T}PDP^{T}x$$

$$= x^{T}Dx$$

$$\frac{1}{4}$$
 $A = \begin{bmatrix} 1 & 1 & 7 \\ 1 & 1 & 1 \end{bmatrix} = p p p^T$ $P = \begin{bmatrix} 8 & 6 & 7 \\ 7 & 1 & 1 \end{bmatrix}$

$$\mathcal{N} = \left(\begin{array}{c} k(\overline{x_1}, x_2) \\ \end{array} \right)$$

$$k(\vec{x}, \vec{y}) := \langle \phi \vec{x} \rangle, \phi(\vec{y}) \rangle = (\vec{x}'\vec{y} + \vec{a})^n$$

$$\phi: \mathbb{R}^{\underline{d}} \longrightarrow \mathbb{R}^{\underline{d}}$$

$$\vec{\chi}' \longrightarrow \phi_{(\underline{\chi}')}$$

Theorem 11. Let V be an inner product space over \mathbb{R} and let $\{\vec{b}_1, ..., \vec{b}_n\}$ be a basis of V. Then the Gram matrix G is positive definite.

Here the Gram matrix
$$G$$
 is defined by $G_{ij} = \langle \vec{b}_j, \vec{b}_i \rangle = \langle \vec{b}_j, \vec{b}_i \rangle = \langle \vec{b}_j, \vec{b}_j \rangle$

$$\begin{cases}
\gamma(A_{1} \cdots X_{n}) = \overline{X}^{7} G \overline{X} = \sum_{i,j} G_{i,j} A_{i} Y_{j} \\
= \sum_{i,j} \langle \overline{J}_{i}, \overline{J}_{i,j} \rangle A_{i} X_{j} \\
= \langle x_{1} \overline{J}_{i} + \cdots \times_{n} \overline{J}_{n} \rangle, \quad x_{1} \overline{J}_{i} - x_{n} \overline{J}_{n} \rangle$$

$$= ||A_{1} \overline{J}_{i} + \cdots + X_{n} \overline{J}_{n}||^{2} \geq 0$$



Proposition 12. Let A be an $m \times n$ real matrix. Then $(A^T A)$ is positive semidefinite. Further more, if rank(A) = n, then A^TA is positive definite.

$$X_{\perp} Y_{\perp} \forall \underline{x}, = || \forall \underline{x}, ||_{\mathcal{I}} \geq \overline{0}$$

This: A is symmether, postther-semi obtaine
$$A = XX^T$$
 $A = PDDDDD^T = XX^T$
 $A = PDDDD^T = XX^T$
 $A = PDDDD^T$
 $A = PDDD^T$
 $A = PDD^T$
 $A = PDD^T$

Thm (Cholesky deampon) A is symm. positie-defin I is lower totaryour. $\Rightarrow A = XX^{T}$ X=QR $=(QR)^TQR$ $=R^{T}Q^{T}QR$ L=RT $= R^T R = L L^T$

any $X \in \mathbb{R}^{m \times n}$

Ty or XXT symmetin

Positive Definite Complex Hermitian Matricies. $A \in \mathbb{C}^{n \times n}$

