Fundamental Matrix Solutions; e^{At}

If $\vec{x}^1(t), \dots, \vec{x}^n(t)$ are n linearly independent solutions of the differential equation

$$\vec{x}' = A\vec{x} \qquad (1)$$

then every solution $\vec{x}(t)$ can be written in the form

$$\vec{x}(t) = c_1\vec{x}^1(t) + c_2\vec{x}^2(t) + \dots + c_n\vec{x}^n(t). \qquad (2)$$

Let $X(t)$ be the matrix whose columns are $\vec{x}^1(t), \dots, \vec{x}^n(t)$. Then, Equation (2) can be written in the concise form

$$\vec{x}(t) = X(t)\vec{c}, \text{ where } \vec{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

<u>Definition</u> A matrix $X(t)$ is called a <u>fundamental matrix solution</u> of (1) if its columns form a set of n linearly independent solutions of (1).

We will show that the matrix e^{At} can be computed directly from any fundamental matrix solution of (1). This is rather remarkable since it does not appear possible to sum the infinite series

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \dots + \frac{A^nt^n}{n!} + \dots$$

exactly, for an arbitrary matrix A. Specifically, we have the following theorem.

Theorem 1 Let $X(t)$ be a fundamental matrix solution of the differential equation

$$\vec{x}' = A\vec{x}.$$

Then,

$$e^{At} = X(t)X^{-1}(0). \qquad (3)$$

In other words, the product of any fundamental matrix solution of (1) with its inverse at $t=0$ must yield e^{At}.

We prove Theorem 1 in three steps. First, we establish a simple test to determine whether a matrix-valued function is a fundamental matrix solution of (1). Then, we use this test to show that e^{At} is a fundamental matrix solution of (1). Finally, we establish a connection between any two fundamental matrix solutions of (1).

Lemma 1 A matrix $X(t)$ is a fundamental matrix solution of (1) if, and only if,

$$X'(t) = AX(t) \quad \text{and} \quad \det X(0) \neq 0.$$

[The derivative of a matrix-valued function $X(t)$ is the matrix whose components are the derivatives of the corresponding components of $X(t)$.].

Proof Let $\vec{x}^1(t), \dots, \vec{x}^n(t)$ denote the n columns of $X(t)$. Observe that

$$X'(t) = (\vec{x}^{1\prime}(t), \dots, \vec{x}^{n\prime}(t)) \text{ and } AX(t) = (A\vec{x}^1(t), \dots, A\vec{x}^n(t)).$$

Hence, the n vector equations $\vec{x}^{1\prime}(t) = A\vec{x}^1(t), \dots, \vec{x}^{n\prime}(t) = A\vec{x}^n(t)$ are equivalent to the single matrix equation $X'(t) = AX(t)$. Moreover, n solutions $\vec{x}^1(t), \dots, \vec{x}^n(t)$ of (1) are linearly independent if, and only if, $\vec{x}^1(0), \dots, \vec{x}^n(0)$ are linearly independent vectors in $\mathbb{R}^n$. These vectors, in turn, are linearly independent if, and only if, $\det X(0) \neq 0$. Consequently, $X(t)$ is a fundamental matrix solution of (1) if, and only if, $X'(t) = AX(t)$ and $\det X(0) \neq 0$.

Lemma 2 The matrix-valued function

$$e^{At} \equiv I + At + \frac{A^2t^2}{2!} + \cdots$$

is a fundamental matrix solution of (1).

Proof

$$\frac{d}{dt}e^{At} = \frac{d}{dt}\left(I + At + \frac{A^2t^2}{2!} + \cdots + \frac{A^nt^n}{n!} + \cdots\right)$$

$$= 0 + A + \frac{2A^2t}{2!} + \cdots + \frac{nA^nt^{n-1}}{n!} + \frac{(n+1)A^{n+1}t^n}{(n+1)!} + \cdots$$

$$= A\left(I + At + \cdots + \frac{A^{n-1}t^{n-1}}{(n-1)!} + \frac{A^nt^n}{n!} + \cdots\right)$$

$$= A(e^{At})$$

Hence e^{At} is a solution of the matrix differential equation

$$X'(t) = AX(t)$$

Moreover, its determinant, evaluated at $t=0$, is one since $e^{A\cdot 0} = I$. Therefore, by Lemma 1, e^{At} is a fundamental matrix solution of (1).

Lemma 3 Let $X(t)$ and $Y(t)$ be two fundamental matrix solutions of (1). Then there exists a constant matrix C such that

$$Y(t) = X(t)C.$$

Proof By definition, the columns $\vec{x}^1(t), \ldots, \vec{x}^n(t)$ of $X(t)$ and $\vec{y}^1(t), \ldots, \vec{y}^n(t)$ of $Y(t)$ are linearly independent sets of solutions of (1). In particular, therefore, each column of $Y(t)$ can be written as a linear combination of the columns of $X(t)$; i.e., there exist constants $c_1^j, \ldots, c_n^j$ such that

$$\vec{y}^j(t) = c_1^j\vec{x}^1(t) + c_2^j\vec{x}^2(t) + \cdots + c_n^j\vec{x}^n(t), \quad j = 1, \ldots, n. \tag{4}$$

Let C be the matrix $(\vec{c}^1, \vec{c}^2, \dots, \vec{c}^n)$ where

$$\vec{c}^j = \begin{pmatrix} c_1^j \\ \vdots \\ c_n^j \end{pmatrix}.$$

Then, the n equations (4) are equivalent to the single matrix equation

$$Y(t) = X(t)C.$$

We prove Theorem 1.

Proof of Theorem 1 Let $X(t)$ be a fundamental matrix solution of (1). Then, by Lemma 2 and 3 there exists a constant matrix C such that $e^{At} = X(t)C$ (5)

Setting $t=0$ in (5) gives $I = X(0)C$, which implies that $C = X^{-1}(0)$.

Hence, $e^{At} = X(t)X^{-1}(0)$.

Example 1 Find e^{At} if

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 5 \end{pmatrix}$$

Solution Our first step is to find 3 linearly independent solutions of the differential equation:

$$\vec{x}' = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 5 \end{pmatrix} \vec{x}.$$

$$p(\lambda) = \det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & 1 & 1 \\ 0 & 3-\lambda & 2 \\ 0 & 0 & 5-\lambda \end{pmatrix} = (1-\lambda)(3-\lambda)(5-\lambda) \Rightarrow$$

$\lambda_1 = 1,\ \lambda_2 = 3,\ \lambda_3 = 5.$

i) $\lambda_1 = 1$:

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} v_2 + v_3 = 0 \\ 2v_2 + 2v_3 = 0 \\ 4v_3 = 0 \end{cases} \Rightarrow \begin{pmatrix} v_1 \text{ is any} \\ v_2 = -v_3 = 0 \\ v_3 = 0 \end{pmatrix} \Rightarrow$$

$$\vec{v}^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

and $\vec{x}^1(t) = e^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is one solution of $\vec{x}' = A\vec{x}$.

ii) $\lambda_2 = 3$

$$(A-3I)\vec{v} = \begin{pmatrix} -2 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} -2v_1 + v_2 + v_3 = 0 \\ 2v_3 = 0 \\ 2v_3 = 0 \end{cases} \Leftrightarrow \begin{cases} 2v_1 = v_2 \\ v_3 = 0 \end{cases}$$

Hence $\vec{v}^2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \Rightarrow \vec{x}^2(t) = e^{3t}\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ is a second solution.

iii) $\lambda_3 = 5$

$$(A-5I)\vec{v} = \begin{pmatrix} -4 & 1 & 1 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} -4v_1 + v_2 + v_3 = 0 \\ -2v_2 + 2v_3 = 0 \\ 0 = 0 \end{cases} \Leftrightarrow \begin{cases} -4v_1 + 2v_2 = 0 \\ v_2 = v_3 \end{cases}$$

$$\begin{cases} v_2 = 2v_1 \\ v_3 = v_2 \end{cases} \Rightarrow \vec{v}^3 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \Rightarrow \vec{x}^3(t) = e^{5t}\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$ is a third solution.

These solutions are linearly independent. Therefore,

$$X(t) = \begin{pmatrix} e^t & e^{3t} & e^{5t} \\ 0 & 2e^{3t} & 2e^{5t} \\ 0 & 0 & 2e^{5t} \end{pmatrix}$$

is a fundamental matrix solution. We compute

$$X^{-1}(0) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{pmatrix}^{-1} = \frac{1}{4}\begin{bmatrix} 4 & -2 & 0 \\ 0 & 2 & -2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

Therefore,

$$\exp\left(\begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 5 \end{bmatrix} t\right) = \begin{pmatrix} e^t & e^{3t} & e^{5t} \\ 0 & 2e^{3t} & 2e^{5t} \\ 0 & 0 & 2e^{5t} \end{pmatrix}\begin{pmatrix} 1 & -1/2 & 0 \\ 0 & 1/2 & -1/2 \\ 0 & 0 & 1/2 \end{pmatrix}$$

$$= \begin{pmatrix} e^t & \frac{1}{2}(e^{3t} - e^t) & \frac{1}{2}(e^{5t} - e^{3t}) \\ 0 & e^{3t} & e^{5t} - e^{3t} \\ 0 & 0 & e^{5t} \end{pmatrix}$$

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The nonhomogeneous equation; variation of parameters.

Consider now the nonhomogeneous equation $\vec{x}' = A\vec{x} + \vec{f}(t)$

In this case, we can use our knowledge of the solutions of the homogeneous equation

$$\vec{x}' = A\vec{x} \qquad (1)$$

to help us find the solution of the initial-value problem

$$\vec{x}' = A\vec{x} + f(t), \quad \vec{x}(t_0) = \vec{x}^0. \qquad (2)$$

Let $\vec{x}^1(t), \dots, \vec{x}^n$ be n linearly independent solutions of the homogeneous equation (1). Since the general solution of (1) is $c_1\vec{x}^1(t) + \dots + c_n\vec{x}^n(t)$, it is natural to seek a solution of (2) of the form

$$\vec{x}(t) = u_1(t)\vec{x}^1(t) + u_2(t)\vec{x}^2(t) + \dots + u_n(t)\vec{x}^n(t) \qquad (3)$$

This equation can be written concisely in the form

$$\vec{x}(t) = X(t)\vec{u}(t), \text{ where } X(t) = (\vec{x}^1(t), \dots, \vec{x}^n(t))$$

and

$$\vec{u}(t) = \begin{pmatrix} u_1(t) \\ \vdots \\ u_n(t) \end{pmatrix}$$

Plugging this expression into the differential equation (2) gives

$$X'(t)\vec{u}(t) + X(t)\vec{u}'(t) = AX(t)\vec{u}(t) + f(t) \qquad (4)$$

The matrix $X(t)$ is a fundamental matrix solution of (1). Hence, $X'(t) = AX(t)$, and equation (4) reduces to

$$X(t)\vec{u}'(t) = f(t) \qquad (5)$$

Recall that the columns of $X(t)$ are linearly independent vectors of $\mathbb{R}^n$ at every time t. Hence $X^{-1}(t)$ exists, and

$$\vec{u}\,'(t) = X^{-1}(t)\vec{f}(t). \qquad (6)$$

Integrating this expression between t_0 and t gives

$$\vec{u}(t) = \vec{u}(t_0) + \int_{t_0}^{t} X^{-1}(s)\vec{f}(s)\,ds$$

$$= X^{-1}(t_0)\vec{x}^{\,0} + \int_{t_0}^{t} X^{-1}(s)\vec{f}(s)\,ds.$$

Consequently,

$$\vec{x}(t) = X(t)X^{-1}(t_0)\vec{x}^{\,0} + X(t)\int_{t_0}^{t} X^{-1}(s)\vec{f}(s)\,ds \qquad (7)$$

If $X(t)$ is the fundamental matrix solution e^{At}, then Equation (7) simplifies considerably. To wit, if

$$X(t) = e^{At} \Rightarrow X^{-1}(s) = e^{-As} \Rightarrow$$

$$\vec{x}(t) = e^{At}e^{-At_0}\vec{x}^{\,0} + e^{At}\int_{t_0}^{t} e^{-As}\vec{f}(s)\,ds$$

$$= e^{A(t-t_0)}\vec{x}^{\,0} + \int_{t_0}^{t} e^{A(t-s)}\vec{f}(s)\,ds.$$

<u>Example 1</u>. Solve the initial-value problem

$$\vec{x}\,' = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{pmatrix}\vec{x} + \begin{pmatrix} 0 \\ 0 \\ e^{t}\cos 2t \end{pmatrix}, \quad \vec{x}(0) = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

<u>Solution</u> We first find e^{At}

$$\det(A - \lambda I) = \begin{pmatrix} 1-\lambda & 0 & 0 \\ 2 & 1-\lambda & -2 \\ 3 & 2 & 1-\lambda \end{pmatrix} = (1-\lambda)^3 + 4(1-\lambda) = (1-\lambda)\left((1-\lambda)^2+4\right)$$

$$(1-\lambda)=0 \Rightarrow \lambda_1 = 1$$
$$(1-\lambda)^2=-4 \Rightarrow 1-\lambda = 2i \Rightarrow \lambda_2 = 1-2i$$
$$1-\lambda = -2i \Rightarrow \lambda_3 = 1+2i$$

The eigenvalues of A are $\lambda_1 = 1$, $\lambda_{2,3} = 1 \pm 2i$

i) $\lambda_1 = 1. \Rightarrow (A-I)\vec{v} = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & -2 \\ 3 & 2 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} 0=0 \\ 2v_1 - 2v_3 = 0 \\ 3v_1 + 2v_2 = 0 \end{cases} \Rightarrow$

$\begin{cases} v_1 = v_3 \\ v_2 = -\frac{3}{2} v_1 \end{cases} \Rightarrow \vec{v} = \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} \Rightarrow \vec{x}^1(t) = e^t \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}$

ii) $\lambda_2 = 1+2i \qquad (A-(1+2i)I)\vec{v} = \begin{pmatrix} -2i & 0 & 0 \\ 2 & -2i & -2 \\ 3 & 2 & -2i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$

$\begin{cases} -2iv_1 = 0 \\ 2v_1 - 2iv_2 - 2v_3 = 0 \\ 3v_1 + 2v_2 - 2iv_3 = 0 \end{cases} \Rightarrow \begin{cases} v_1 = 0 \\ v_3 = -iv_2 \\ v_2 = iv_3 \end{cases} \Rightarrow \begin{cases} v_1 = 0 \\ v_3 = -iv_2 \end{cases} \Rightarrow \vec{v} \begin{pmatrix} 0 \\ 1 \\ -i \end{pmatrix} \Rightarrow$

$$\vec{x}(t) = e^{(1+2i)t}\begin{pmatrix} 0 \\ 1 \\ -i \end{pmatrix} = e^t(\cos 2t + i\sin 2t)\left[\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + i\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}\right]$$
$$= e^t\cos 2t\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - e^t\sin 2t\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} + ie^t\left(\sin 2t\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \cos 2t\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right)$$

Consequently,

$$\vec{x}^2(t) = e^t\begin{pmatrix} 0 \\ \cos 2t \\ \sin 2t \end{pmatrix} \quad \text{and} \quad \vec{x}^3(t) = e^t\begin{pmatrix} 0 \\ \sin 2t \\ -\cos 2t \end{pmatrix}$$

Therefore,

$$X(t) = \begin{pmatrix} 2e^t & 0 & 0 \\ -3e^t & e^t\cos 2t & e^t\sin 2t \\ 2e^t & e^t\sin 2t & -e^t\cos 2t \end{pmatrix}$$

is a fundamental matrix solution of $\vec{x}' = A\vec{x}$.

$$X^{-1}(0) = \begin{pmatrix} 2 & 0 & 0 \\ -3 & 1 & 0 \\ 2 & 0 & -1 \end{pmatrix}^{-1} = -\frac{1}{2}\begin{pmatrix} -1 & 0 & 0 \\ -3 & -2 & 0 \\ -2 & 0 & 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ \frac{3}{2} & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

$$e^{At} = X(t)X^{-1}(0) = \begin{pmatrix} 2e^t & 0 & 0 \\ -3e^t & e^t\cos 2t & e^t\sin 2t \\ 2e^t & e^t\sin 2t & -e^t\cos 2t \end{pmatrix}\begin{pmatrix} \frac{1}{2} & 0 & 0 \\ \frac{3}{2} & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

$$= e^t\begin{pmatrix} 1 & 0 & 0 \\ -3/2 + 3/2\cos 2t + \sin 2t & \cos 2t & -\sin 2t \\ 1 + 3/2\sin 2t - \cos 2t & \sin 2t & \cos 2t \end{pmatrix}.$$

$\Downarrow$

$$\vec{x}(t) = e^{A(t-0)}\vec{x}^0 + \int_0^t e^{A(t-s)}\vec{f}(s)\,ds$$

$$\vec{x}(t) = e^{At}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + e^{At}\int_0^t e^{-s}\begin{pmatrix} 1 & 0 & 0 \\ -3/2 + 3/2\cos 2s - \sin 2s & \cos 2s & \sin 2s \\ 1 - 3/2\sin 2s - \cos 2s & -\sin 2s & \cos 2s \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ e^s\cos 2s \end{pmatrix}ds$$

$$= e^t\begin{pmatrix} 0 \\ \cos 2t - \sin 2t \\ \sin 2t + \cos 2t \end{pmatrix} + e^{At}\int_0^t \begin{pmatrix} 0 \\ \cos 2s\sin 2s \\ \cos^2 2s \end{pmatrix}ds$$

$$\int_0^t \sin 2s\cos 2s\,ds = \frac{1}{2\cdot 2}\sin^2 2s\Big|_0^t = \frac{1}{4}\sin^2 2t = \frac{1}{4}\left(\frac{1-\cos 4t}{2}\right)$$

$$\int_0^t \cos^2 2s\,ds = \int_0^t \frac{1+\cos 4s}{2}ds = \frac{1}{2}s\Big|_0^t + \frac{\sin 4s}{8}\Big|_0^t$$

$$= e^t\begin{pmatrix} 0 \\ \cos 2t - \sin 2t \\ \sin 2t + \cos 2t \end{pmatrix} + e^{At}\begin{pmatrix} 0 \\ (1-\cos 4t)/8 \\ \frac{1}{2}t + \frac{\sin 4t}{8} \end{pmatrix}$$

$$= e^t\begin{pmatrix} 0 \\ \cos 2t - \sin 2t \\ \sin 2t + \cos 2t \end{pmatrix} + e^t\begin{pmatrix} 0 \\ \frac{-t\sin 2t}{2} + \frac{\cos 2t - \cos 4t\cos 2t - \sin 4t\sin 2t}{8} \\ \frac{t\cos 2t}{2} + \frac{\sin 4t\cos 2t - \sin 2t\cos 4t + \sin 2t}{8} \end{pmatrix}$$

$$= e^t\begin{pmatrix} 0 \\ \cos 2t - (1+\frac{1}{2}t)\sin 2t \\ (1+\frac{1}{2}t)\cos 2t + \frac{5}{4}\sin 2t \end{pmatrix}$$

(Quite tedious and laborious!)