Northeastern University, Department of Mathematics

MATH 5110: Applied Linear Algebra and Matrix Analysis.

• Instructor: **He Wang** Email: **he.wang@northeastern.edu**

§6 Determinant

Contents

1. Motivation

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$ad-bc \neq 0$

$ad-bc = \det(A)$

Goal: define a number $\det(A)$ for $n \times n$ matrix s.t.

$\det A \neq 0 \iff A$ is invertible

• $A^{-1} = \dfrac{1}{\det A}\begin{bmatrix} \vdots & - & - & \vdots \\ \vdots & & & \\ - & & & \end{bmatrix}$

$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & - & - & \end{bmatrix}$

2. Cofactor expansion

Definition 1. Let A be an $n \times n$ matrix.
The **first row cofactor expansion** formula for the **determinant** of A is

$$\det(A) := a_{11}\det A_{11} - a_{12}\det A_{12} + \cdots + (-1)^{1+n}\det A_{1n}$$

Here A_{ij} is the $(n-1)\times(n-1)$ matrix obtained by deleting i-th row and j-th column of A

Facts about **permutation groups**.

Let $[n]$ be the set of n integers $[n] = \{1, 2, \ldots, n\}$.

The **permutation group** (symmetric group) $S(n)$ is $= \{\text{all bijections } \sigma : [n] \to [n]\}$

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix}$$

$$: [n] \xrightarrow{\sigma_1} [n] \xrightarrow{\sigma_2} [n]$$

Ex) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 3 & 2 & 5 \end{pmatrix} = (2,4)$

$\sigma_1 \cdot \sigma_2 \neq \sigma_2 \circ \sigma_1$

A **transposition** is a permutation in $S(n)$ that only switch 2 numbers.

Permutation presents only in even numbers never in odd numbers

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (1,2)\circ(2,3) = (1,3)\circ(1,2)$$

$T(\sigma) = 2$

The **sign** of a permutation $\sigma \in S(n)$ is

$$\text{sign}(\sigma) = (-1)^{T(\sigma)}$$

where $T(\sigma)$ is the number of transposition of σ.

Another equivalent way to determine the sign of σ is

$$\text{sign}(\sigma) = (-1)^{N(\sigma)},$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

where $N(\sigma)$ is the number of inversions of σ.

An inversion of $(\sigma(1)\ \sigma(2)\ \ldots\ \sigma(n))$ is the pair of numbers $(\sigma(i) > \sigma(j))$ for $i < j$.

Proposition 2. *If τ is obtained from σ by switch two numbers i, j, then* $\operatorname{sign}(\tau) = -\operatorname{sign}(\sigma)$.

$A = [\cdots a_{ij} \cdots]$

Theorem 3. *If A is an $n \times n$ matrix, then*

$$\det(A) = \sum_{\sigma \in S(n)} \operatorname{sign}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}$$
$$= \sum_{\sigma \in S(n)} \operatorname{sign}(\sigma) a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}.$$

Proof. This theorem can be proved by induction on n. For $n = 1$, it is true. Suppose the formula is true for $n-1$, let's show that it is true for n.

$$\sum_{\sigma \in S(n)} \operatorname{sign}(\sigma) a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}$$
$$= \sum_{i=1}^{n} a_{1i} \sum_{\sigma \in S(n); \sigma(1)=i} \operatorname{sign}(\sigma) a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}$$
$$= \sum_{i=1}^{n} a_{1i} \sum_{\sigma \in S(n); \sigma(1)=i} (-1)^{1+i} \operatorname{sign}(\sigma(2) \ \ldots \ \sigma(n)) a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}$$
$$= \sum_{i=1}^{n} (-1)^{1+i} a_{1i} \det A_{1i}$$
$$= \det A$$

$\square$

Example 4. Let A be the 3×3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{aligned}
\operatorname{sign}(1 \ 2 \ 3) &= 1 \\
\operatorname{sign}(1 \ 3 \ 2) &= -1 \\
\operatorname{sign}(2 \ 1 \ 3) &= -1 \\
\operatorname{sign}(2 \ 3 \ 1) &= 1 \\
\operatorname{sign}(3 \ 1 \ 2) &= 1 \\
\operatorname{sign}(3 \ 2 \ 1) &= -1
\end{aligned}$$

Hence we have

$$\det(A) = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33} - a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

Example 5. Find the determinant of $A = \begin{bmatrix} 0 & 4 & 2 \\ 5 & 2 & 2 \\ 0 & 2 & -1 \end{bmatrix}$. Is A invertible?

Definition 6. Let A be an $n \times n$ matrix. Its (i,j)**-th cofactor** C_{ij} is

$$C_{ij} := \underline{(-1)^{i+j}}\ \underline{\det}\ A_{ij}$$

Using cofactors, the first row cofactor expansion formula for the determinant of A is

$$\det A = a_{11}C_{11} + a_{12}C_{12} \cdots + C_{1n}$$

Theorem 7. *The determinant of an $n \times n$ matrix A can be computed by he i-**th row** cofactor expansions*

$$\det A = a_{i1}C_{i1} + \cdots + a_{in}C_{in}$$

*and the j-**th column** cofactor expansions*

$$\det A = a_{1j}C_{1j} + \cdots + a_{nj}C_{nj}$$

Example 8. Find the determinant of $A = \begin{bmatrix} 1 & 2 & 3 & 1.2 \\ 0 & 0 & 0 & 2 \\ 5 & 6 & 7 & \pi \\ 0 & 1 & 2 & \sqrt{2} \end{bmatrix}$

Theorem 9. *Let A be an $n \times n$ triangular matrix, the determinant*

$$\det A = a_{11} a_{22} \cdots a_{nn}$$

Example 10. Find out for which value of λ the matrix $A - \lambda I$ is not invertible, where
$A = \begin{bmatrix} 2 & \sqrt{2} & 1.7 \\ 0 & 3 & 12 \\ 0 & 0 & 5 \end{bmatrix}$

Proposition 11.

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ b_1 + c_1 & b_2 + c_2 & \cdots & b_n + c_n \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ b_1 & b_2 & \cdots & b_n \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ c_1 & c_2 & \cdots & c_n \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$\begin{bmatrix} 1 & & \\ & d & \\ & & 1 \end{bmatrix}$ $\quad \det(E_i(k)) = k \quad \det(E_{ij}) = -1$

$\begin{bmatrix} 1 & & \\ & \ddots & \\ & & k \end{bmatrix}$ $\quad \det(E_{ij}(d)) = 1$

3. Row Operations and Determinant

Recall the three types of *elementary row operations*: $\Longleftrightarrow$ Elementary matrix

1. (Replacement) — $E_{ij}(d)$
2. (Interchange) — E_{ij}
3. (Scaling) — $E_i(k)$

Theorem 12 (Row Operations and the Determinant)**.** *Let A be an $n \times n$ matrix.*

① $A \xrightarrow{R_i \leftrightarrow R_j} B = E_{ij}A$ then $\det B = -\det A$ $\Longleftrightarrow$ $\det(E_{ij}A) = -\det A$

② $A \xrightarrow{kR_i} B$ then $\det B = k \det A$ $\Longleftrightarrow$ $\det(E_i(k)A) = k \det A$

③ $A \xrightarrow{R_i + dR_j} B$ then $\det B = \det A$ $\Longleftrightarrow$ $\det(E_{ij}(d)A) = \det A$

$A \to - - - - - - \to \boxed{\text{ref} A} \to - - - - \to \text{rref} A$

Example 13. In a matrix A, if the i-th row equals the j-th row, then

$\det A = 0$

Example 14. In a matrix A, if the i-th row is a scalar product of the j-th row, then

$\det A = 0$

Theorem 15. *An $n \times n$ matrix A is invertible if and only if* $\det A \neq 0$.

$A \to \cdots \to \operatorname{rref} A$

$\operatorname{rank}(\operatorname{rref} A) = n \iff \det(\operatorname{rref} A) = 1$

$\iff \det A \neq 0$

Proposition 16. *Let A be an $n \times n$ matrix.*

$$\det(kA) = (k^n)(\det A).$$

Theorem 17 (Determinants of Products of Matrices). *Let A and B be two $n \times n$ matrices.*

$$\det(AB) = (\det A)(\det B).$$

Case ① A is invertible, then $A = E_1 E_1 \cdots E_s$

$\det(AB) = \det(E_1 \cdots E_s B) = \det(E_1) \cdots \det(E_s) \det B$

$= \det A \det B$

Case ② A is not invertible. $\det A = 0$.

$\operatorname{rank}(AB) \le \operatorname{rank} A < n$

$\Rightarrow \det(AB) = 0$

Proposition 18. *Let A be an $n \times n$ matrix.*

$$\det(A^m) = (\det(A))^m$$

$A A^{-1} = I_n$

Proposition 19. *Let A be an $n \times n$ invertible matrix.*

$$\det(A^{-1}) = \frac{1}{\det A}.$$

Question: How about $\det(A + B)$? Is it $\det(A) + \det(B)$?

Example 20. Find the determinant of $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$. Is A invertible?

Example 21. Find the determinant of $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 9 \\ 3 & 6 & 11 & 12 \\ 4 & 8 & 15 & 16 \end{bmatrix}$. Is A invertible?

Definition 22 (Elementary Column Operations).
1. (Column Replacement) Add to one column the multiple of another column.
2. (Column Interchange) Interchange two columns.
3. (Column Scaling) Multiply all entries of a given column by a scalar.

Theorem 23 (Column Operations and the Determinant). *Let A be an $n \times n$ matrix and let B be a matrix obtained from A by a single elementary row operation.*
1. If B is obtained from A by a Column Replacement operation, then

$$\det B = \det A.$$

2. If B is obtained from A by a Column Interchange operation, then

$$\det B = -\det A.$$

3. If B is obtained from A by a Column Scaling operation by a factor k, then

$$\det B = k \det A.$$

Theorem 24 (Determinant of the Transpose Matrix).

$$\det A^T = \det A.$$

Example 25. Vandermonde determinant

$$\det(A) = \begin{vmatrix} 1 & 1 & 1 \\ a_1 & a_2 & a_3 \\ a_1^2 & a_2^2 & a_3^2 \end{vmatrix} = (a_2-a_1)(a_3-a_1)(a_3-a_2)$$

More generally, by induction on n, we can proved that

$$\det(A) = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_n \\ a_1^2 & a_2^2 & a_3^2 & \cdots & a_n^2 \\ \vdots & \vdots & \cdots & \vdots \\ a_1^{n-1} & a_2^{n-1} & a_3^{n-1} & \cdots & a_n^{n-1} \\ a_1^n & a_2^n & a_3^n & \cdots & a_n^n \end{vmatrix} = \prod_{1 \le j < i \le n} (a_i - a_j)$$

Block Matrix.

Theorem 26. *If* $M = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$, *then,*

$$\det(M) = \det(A)\det(C).$$

Example 27. Find the determinant of $M = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} = \begin{bmatrix} 1 & 2 & 11 & \sqrt{3} \\ 2 & 3 & \pi & 12 \\ 0 & 0 & 3 & 9 \\ 0 & 0 & 1 & 4 \end{bmatrix}$

4. Linearity Property of the determinant function and Cramer's Rule

Let $A = [\vec{a}_1 \ \ \vec{a}_2 \ \ \dots \ \ \vec{a}_n]$ be an $n \times n$ matrix

Def: $T_i : \mathbb{R}^n \to \mathbb{R}$ for each i

$\vec{x} \to \det[\vec{a}_1 \cdots \vec{a}_{i-1}\, \vec{x}\, \vec{a}_{i+1} \cdots \vec{a}_n]$

Theorem 28 (Linearity and Determinants). *The transformation T defined above is a linear transformation, that is*

① $T_i(\vec{x}+\vec{y}) = T_i(\vec{x}) + T_i(\vec{y})$

② $T_i(c\vec{x}) = c\,T_i(\vec{x})$

Proof. By Theorems 24, 12 and Proposition 11. ☐

$C_{ij} = (-1)^{i+j} \det A_{ij}$

Example 29 (Finding matrix for the determinant transformation for a given A).

$$T_i(\vec{x}) = \begin{vmatrix} a_{11} & a_{12} & x_1 & \cdots & a_{1n} \\ a_{21} & \vdots & x_2 & & \vdots \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \vdots & x_n & & a_{nn} \end{vmatrix} = x_1 C_{1i} + x_2 C_{2i} + \cdots + x_n C_{ni}$$

$$= [C_{1i}\ C_{2i} \cdots C_{ni}] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = B_i \vec{x}$$

$T_i(\vec{b}) = \det(A_i(\vec{b}))$

Consider a matrix equation $A\vec{x} = \vec{b}$ in which A is an $n \times n$ matrix. Let

$A\vec{x} = \vec{b}$

$\vec{x} = A^{-1}\vec{b}$

$$A_i(\vec{b}) := [\vec{a}_1 \quad \vec{a}_2 \quad \dots \quad \vec{a}_{i-1} \quad \vec{b} \quad \vec{a}_{i+1} \quad \dots \quad \vec{a}_n]$$

Theorem 30 (Cramer's Rule)**.** *If A is invertible, the unique solution $\vec{x}$ of the matrix equation $A\vec{x} = \vec{b}$ is given by*

$$x_i = \frac{\det(A_i(\vec{b}))}{\det(A)} = \frac{T_i(\vec{b})}{\det A}$$

Check $A\vec{x} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$

Check: $b_1 = a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n$

Proof. First, from cofactor expansion, $\det(A_i(\vec{b})) = \sum_{j=1}^{n} b_j C_{ij}$.

For each k

$$\boxed{a_{k1}x_1 + a_{k2}x_2 + \cdots a_{kn}x_n} = \frac{1}{\det(A)}\left(\sum_{i=1}^{n} a_{ki} \sum_{j=1}^{n} b_j C_{ij}\right)$$
$$= \frac{1}{\det(A)}\left(\sum_{j=1}^{n} b_j \boxed{\sum_{i=1}^{n} a_{ki} C_{ij}}\right)$$
$$= \frac{1}{\det(A)}(b_k \det(A))$$
$$= b_k$$

$\det A$ if $j=k$

0 if $j \neq k$

for any $k = 0, 1, \cdots, n$. This verifies that $(x_1, \dots, x_n)$ is a solution of $A\vec{x} = \vec{b}$. □

Let C be the associated $n \times n$ matrix of cofactors defined as:

$$C = \begin{bmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ C_{21} & C_{22} & \dots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \dots & C_{nn} \end{bmatrix}$$

$C_{ij} = (-1)^{i+j} \det A_{ij}$

The transpose of C is called the **adjugate matrix of** A, denoted by $\mathrm{adj}A$:

$$\mathrm{adj}A = C^T = \begin{bmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \dots & C_{nn} \end{bmatrix}$$

Theorem 31. *If A is a invertible matrix then $A^{-1} = \dfrac{1}{\det A} \cdot \mathrm{adj}A$*

Check

$$\begin{bmatrix} a_{11} & \dots & a_{1n} \\ a_{21} & \dots & a_{2n} \\ \vdots & & \\ a_{n1} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ \vdots & & & \\ C_{1n} & \dots & & C_{nn} \end{bmatrix} = \det A \cdot I$$

$$= \begin{bmatrix} \det A & & 0 \\ & \ddots & \\ 0 & & \det A \end{bmatrix}$$

Def: $\det(\ \ ) : \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}$

s.t.

① $\det(I) = 1$

② Alternating: $|\vec{a}_1 \cdots \vec{v} \cdots \vec{v} \cdots \vec{a}_n| = 0$

③ multilinear:

$$|\vec{a}_1 \cdots \vec{a}_{j-1}\ \boxed{r\vec{a}_j + \vec{b}}\ \vec{a}_{j+1} \cdots \vec{a}_n| = r|\vec{a}_1 \cdots \vec{a}_n| + |\vec{a}_1 \cdots \vec{a}_{j-1}\ \vec{b}\ \vec{a}_{j+1} \cdots \vec{a}_n|$$

Thm: There is exactly one function satisfy ① ② ③.