0.2 The Poisson process

This is the following special case, given in the second description: the process is $\{N(t)\}$ on $\Omega = \{0, 1, 2, \dots\}$, with $N(0) = 0$, $\lambda_i = \lambda$ and

$$p_{ij} = \begin{cases} 1 & \text{for } j = i + 1 \\ 0 & \text{else} \end{cases}$$

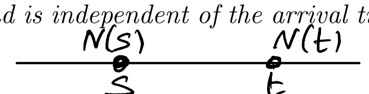
So the chain starts at 0, and increases by jumps which occur after holding times which are IID exponential with rate λ. There are several immediate consequences.

Theorem 3 • *For every t, $N(t)$ has the Poisson distribution with parameter λt, that is*

$$P(N(t) = j) = \frac{(\lambda t)^j}{j!} e^{-\lambda t} \qquad \Rightarrow \mathbb{E}[N(t)] = \lambda t.$$

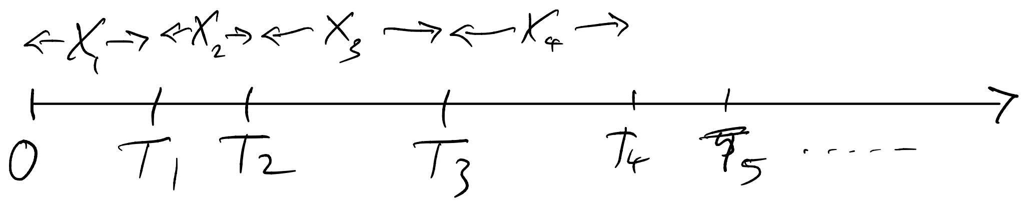
- *The arrival times $0 = T_0 < T_1 < T_2 < \dots$ have gamma distribution.*
- *If $s < t$ then $N(t) - N(s)$ is Poisson with rate $\lambda(t - s)$, and is independent of the arrival times during the interval $[0, s]$.*

Independent increments



$N(t) \geq N(s)$

The Poisson process is widely used as the model for an arrival process, both because it does a good job in many cases, and also because it allows exact computations in many ways.



Interarrival times $X_1, X_2, X_3, \dots$ are IID, with pdf

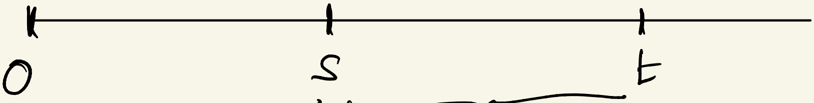
$$f_X(t) = \lambda e^{-\lambda t} \quad (t \geq 0)$$

$$T_n = X_1 + X_2 + \dots + X_n \quad \Rightarrow \mathbb{E}[T_n] = n \frac{1}{\lambda}$$

pdf: $$f_{T_n}(t) = \lambda \frac{(\lambda t)^{n-1}}{(n-1)!} e^{-\lambda t} \quad (t \geq 0)$$

(gamma distribution).

$$\{T_n \leq t\} \Leftrightarrow \{N(t) \geq n\}$$ same event.



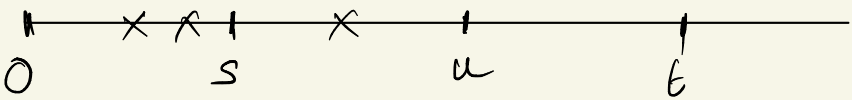
increment $\qquad$ increment

$N(s) - N(0) = N(s)$ $\qquad$ $N(t) - N(s)$

independent

cf. Markov property.

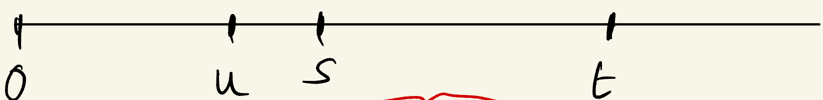
$0 < s < u < t$



increment $= N(s)$ $\qquad$ increment $N(u) - N(s)$ $\qquad$ increment $N(t) - N(u)$

independent

$0 < u < s < t$

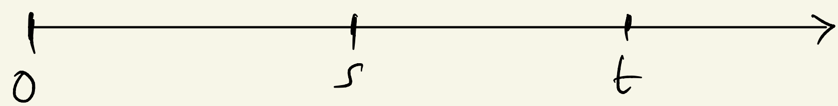


increment $N(s)$ $\qquad$ increment $N(t) - N(u)$

not independent

Summary: increments of the Poisson process on <u>disjoint</u> intervals are independent.

Application: calculate $\mathrm{COV}[N(s), N(t)]$ where $0 < s < t$.



$$\mathrm{COV}[N(s), N(t)] = \mathbb{E}[N(s)N(t)] - \mathbb{E}[N(s)]\,\mathbb{E}[N(t)]$$

$$\mathbb{E}[N(s)N(t)] = \mathbb{E}[N(s)(N(t) - N(s))] + \mathbb{E}[N(s)^2]$$

$$= \mathbb{E}[N(s)]\,\mathbb{E}[N(t) - N(s)]$$

$$+ \ VAR[N(s)] + \mathbb{E}[N(s)]^2$$

$$= (\lambda s)\,\lambda(t-s)$$

$$+ \ \lambda s + (\lambda s)^2$$

<u>Poisson r.v.</u> : $\mathbb{P}(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}$

$$\mathbb{E}[X(X-1)] = \sum_{k=1}^{\infty} k(k-1) \frac{\lambda^k}{k!} e^{-\lambda}$$

$$= \sum_{k=2}^{\infty} \frac{\lambda^k}{(k-2)!} e^{-\lambda}$$

$$= \lambda^2$$

$$\mathbb{E}[X^2] = \lambda^2 + \mathbb{E}[X] = \lambda^2 + \lambda$$

$$VAR[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

$$= \lambda^2 + \lambda - (\lambda)^2 = \lambda. = \mathbb{E}[X]$$

$$\mathbb{E}[N(s)\,N(t)] = \lambda^2 st - \lambda^2 s^2 + \lambda s + \lambda^2 s^2$$

$$= \lambda^2 st + \lambda s$$

$$\Rightarrow \mathrm{COV}[N(s), N(t)] = \mathbb{E}[N(s)\,N(t)] - \mathbb{E}[N(s)]\,\mathbb{E}[N(t)]$$

$$= \lambda^2 st + \lambda s - (\lambda s)(\lambda t)$$

$$\boxed{\mathrm{COV}[N(s), N(t)] = \lambda s} \qquad 0 < s < t$$

$$\mathrm{COV}[N(s), N(t)] = \lambda \min(s, t)$$

Example.

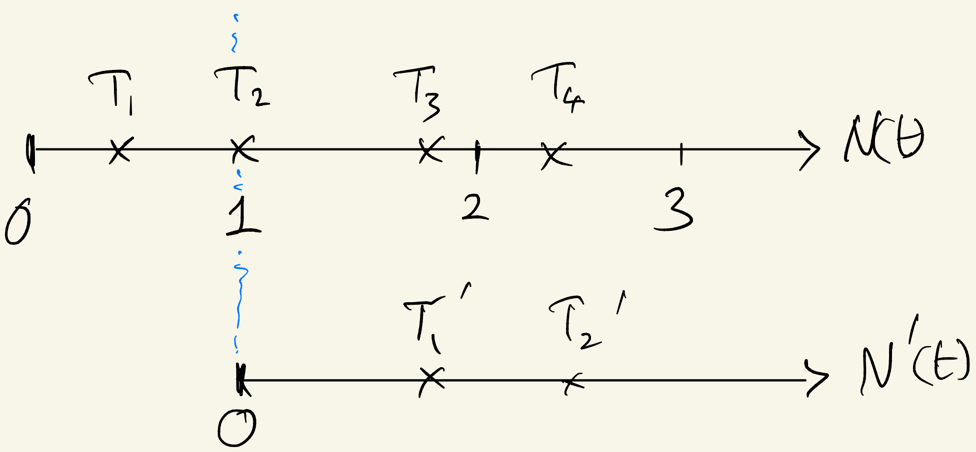
Suppose $N(t)$ is P.P. rate λ.

Compute a) $\mathbb{E}[N(3)] = 3\lambda$.

b) $\mathbb{E}[T_4] = \mathbb{E}[4^{\text{th}} \text{ arrival time}]$

$$= \frac{4}{\lambda}$$

c) $\mathbb{E}[T_4 \mid T_2 = 1]$



Markov process

$\Rightarrow$ clock starts over at $t=1$.

$\Rightarrow$ waiting for second arrival.

$$\mathbb{E}[T_4 \mid T_2 = 1] = \mathbb{E}[1 + T_4 - T_2 \mid T_2 = 1]$$

$$= 1 + \mathbb{E}[T_4 - T_2 \mid T_2 = 1]$$

independent b/c non-overlapping intervals

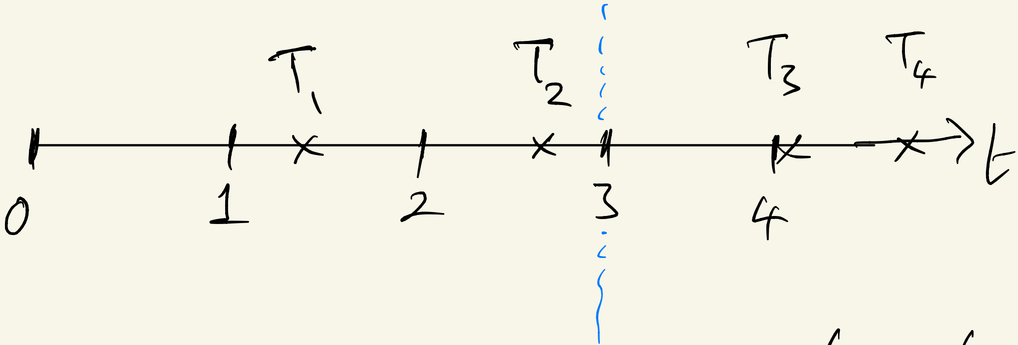
$$= 1 + \mathbb{E}[T_4 - T_2]$$

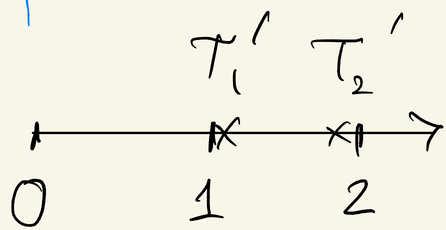
$$= 1 + \mathbb{E}[T_{4-2}]$$

$$= 1 + \mathbb{E}[T_2]$$

$$= 1 + \frac{2}{\lambda}.$$

d) $\mathbb{E}[T_4 \mid N(3) = 2]$





Clock starts over at $t=3$:

memoryless property of exponential

$\Rightarrow \quad T_1' \sim$ exponential rate λ.

$$\Rightarrow \mathbb{E}[T_4 \mid N(3)=2] = 3 + \mathbb{E}[T_2']$$

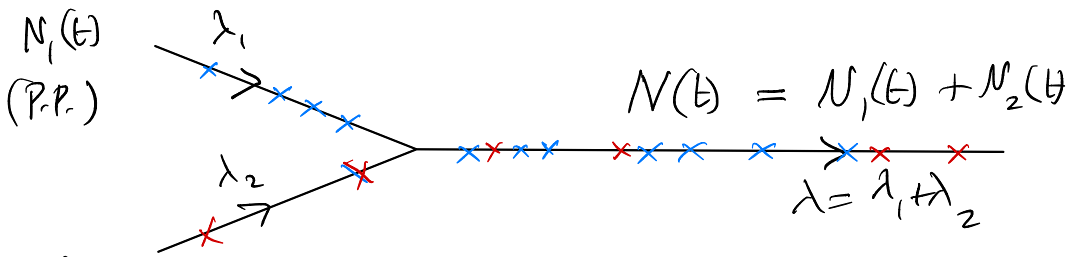
$$= 3 + \frac{2}{\lambda}.$$

The Poisson process has many remarkable properties not shared by other CTMC's, here are some examples.

Theorem 4 (The superposition theorem) *Let $N_1(t)$ and $N_2(t)$ be independent Poisson processes with rates λ_1 and λ_2 respectively. Then $N_1 + N_2$ is a Poisson process with rate $\lambda_1 + \lambda_2$.*

There is also a converse.

Theorem 5 (The thinning theorem) *Suppose that $N(t)$ is a Poisson process with rate λ, and each arrival is independently classified as one of two types: Type 1 with probability p, or Type 2 with probability $1 - p$. Let $N_1(t)$ and $N_2(t)$ be the number of arrivals of type 1 and 2 respectively. Then N_1 and N_2 are independent Poisson processes with rates λp and $\lambda(1 - p)$ respectively.*



$N_2(t)$
(P.P.)

Key fact: if $N_1(t)$, $N_2(t)$ are P.P.'s and are independent, then their sum is also a P.P. $N(t) = N_1(t) + N_2(t)$

rate $\lambda = \lambda_1 + \lambda_2$.

Mean time between arrivals for $N(t)$ is $\frac{1}{\lambda} = \frac{1}{\lambda_1 + \lambda_2}$.

Type 1 = arrivals from $N_1(t)$

Type 2 = arrivals from $N_2(t)$.

Probability that next arrival is Type 1 = $\frac{\lambda_1}{\lambda_1 + \lambda_2}$

$$P(\text{Type } 1) = \frac{\lambda_1}{\lambda_1 + \lambda_2}$$

$$\mathbb{P}(\text{Type 2}) = \frac{\lambda_2}{\lambda_1+\lambda_2}.$$

Example

Two independent PP's with $\underbrace{N_1(t),\ N_2(t)}$

rates $\lambda_1 = 5 \text{ hour}^{-1}$, $\lambda_2 = 15 \text{ hour}^{-1}$.

Merged into one process

$$N(t) = N_1(t) + N_2(t).$$

a) Prob. of at least 2 arrivals of either type in next 5 mins.

$$N(t), \quad \lambda = \lambda_1+\lambda_2 = 20 \text{ hour}^{-1}$$
$$= \frac{1}{3} \text{ min}^{-1}$$

$$\mathbb{P}(N(5) \geq 2)$$ (units are minutes)

$$= 1 - \mathbb{P}(N(5) < 2)$$

$$= 1 - \mathbb{P}(N(5) = 0) - \mathbb{P}(N(5) = 1)$$

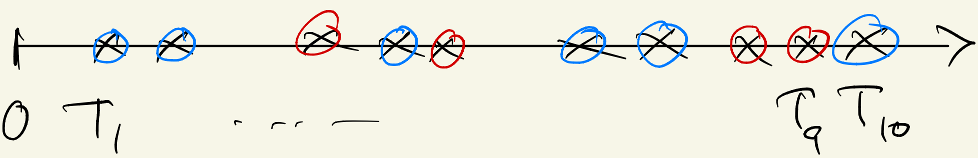
$$= 1 - e^{-5\lambda} - \frac{(5\lambda)^1}{1!} e^{-5\lambda}$$

[recall $\mathbb{P}(N(5) = n) = \frac{(\lambda(5))^n}{n!} e^{-\lambda(5)}$]

$$= 1 - e^{-5/3} - \frac{5}{3} e^{-5/3}$$

$$= 1 - \frac{8}{3} e^{-5/3}.$$

b) Find prob. that at least 2 of the next 10 arrivals are Type 2.



Each arrival is independently of Type 1 w/ prob. $p_1 = \dfrac{\lambda_1}{\lambda_1+\lambda_2}$

or Type 2 w/ prob. $p_2 = \dfrac{\lambda_2}{\lambda_1+\lambda_2}$.

$\Rightarrow$ coin tossing problems

X = number of Type 2 arrivals

among the next 10

arrivals.

$\mathbb{P}(X \geq 2) = ?$

$X \sim \text{Bin}(n, p)$

$n = 10$

$p = p_2 = \frac{\lambda_2}{\lambda_1 + \lambda_2}$

$= \frac{15}{20}$

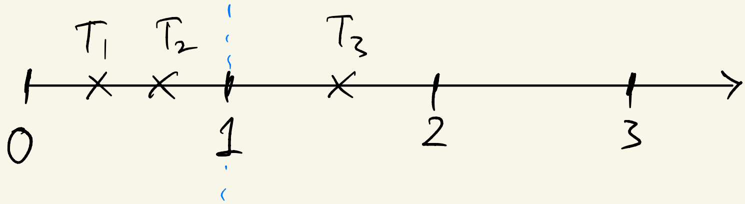
$= \frac{3}{4}.$

$\mathbb{P}(X \geq 2) = 1 - \mathbb{P}(X=0) - \mathbb{P}(X=1)$

$= 1 - (1-p)^{10} - 10(1-p)^{9} p$

$= 1 - \left(\frac{1}{4}\right)^{10} - 10\left(\frac{1}{4}\right)^{9}\left(\frac{3}{4}\right).$

Conditional distribution of arrival times.

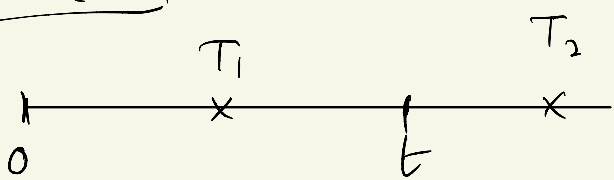


$$\mathbb{E}[T_3 \mid N(1) = 2] = 1 + \frac{1}{\lambda}$$

(starts over at $t=1$)

$$\mathbb{E}[T_2 \mid N(1) = 2] = ?$$

Simplest case:



Conditional distribution of T_1 given $N(t) = 1$.

$$\mathbb{P}(T_1 \le s \mid N(t) = 1)$$

$$= \mathbb{P}(T_1 \le s \mid T_1 \le t,\ T_2 > t)$$

Increments: $T_1 = X_1, \quad T_2 = X_1 + X_2$

$\Rightarrow \mathbb{P}(T_1 \le s \mid T_1 \le t, \quad T_2 > t)$

$$= \mathbb{P}(X_1 \le s \mid X_1 \le t, \ X_2 + X_1 > t)$$

$$= \mathbb{P}(X_1 \le s \mid X_1 \le t, \ X_2 > t - X_1)$$

$$= \frac{\mathbb{P}(X_1 \le s, \ X_1 \le t, \ X_2 > t - X_1)}{\mathbb{P}(X_1 \le t, \ X_2 > t - X_1)} \qquad (0 \le s \le t)$$

$$= \frac{\mathbb{P}(X_1 \le s, \ X_2 > t - X_1)}{\mathbb{P}(X_1 \le t, \ X_2 > t - X_1)}$$

Numerator: conditioning on X_1

$$\mathbb{P}(X_1 \le s, \ X_2 > t - X_1 \mid X_1 = u) \qquad (0 \le u \le s)$$

$$= \mathbb{P}(X_2 > t - u) \qquad \text{(independent of } X_1\text{)}$$

$$= e^{-\lambda(t-u)} = e^{-\lambda t} e^{\lambda u}.$$

Undo the conditioning:

$$\Rightarrow \mathbb{P}(X_1 \le s,\ X_2 > t - X_1)$$

$$= \int_0^s \mathbb{P}(X_1 \le s,\ X_2 > t - X_1 \mid X_1 = u)\, \lambda e^{-\lambda u}\, du$$

$$= \int_0^s e^{-\lambda t} e^{\lambda u} \lambda e^{-\lambda u}\, du$$

$$= \lambda e^{-\lambda t} s$$

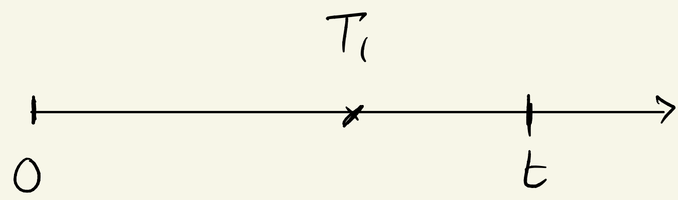
<u>Denominator</u>:

$$\mathbb{P}(X_1 \le t,\ X_2 > t - X_1) = \lambda e^{-\lambda t} \cdot t.$$

$$\Rightarrow \mathbb{P}(X_1 \le s \mid X_1 \le t,\ X_2 > t - X_1)$$

$$= \frac{\lambda s e^{-\lambda t}}{\lambda t e^{-\lambda t}}$$

$$= \frac{s}{t}.$$



$$\mathbb{P}(T_1 \le s \mid N(t) = 1) = \frac{s}{t} \quad (0 \le s \le t)$$

pdf: $$f_{T_1}(s \mid N(t) = 1) = \frac{d}{ds}\mathbb{P}(T_1 \le s \mid N(t) = 1)$$
$$= \frac{d}{ds}\left(\frac{s}{t}\right)$$
$$= \frac{1}{t} \quad (0 \le s \le t).$$

$\Rightarrow$ conditioned on $N(t) = 1$,

the first arrival time T_1 is uniform

on $[0, t]$.

Note: this does not depend on λ!

$$\mathbb{E}[T_1 \mid N(t)=1] = \frac{t}{2}$$

$$VAR[T_1 \mid N(t)=1] = \frac{t^2}{12}$$

$$\mathbb{P}(T_1 \leq \frac{t}{4} \mid N(t)=1) = \frac{1}{4}$$

$$\mathbb{P}(T_1 \leq \frac{t}{4}) = 1 - e^{-\frac{\lambda t}{4}}$$