Solution of ODE1 Problem 1

(1a) $e^{-rt} \to 0$ as $t \to \infty$. So $N(t) \to \frac{N_0 B}{N_0} = B$ as $t \to \infty$.

(1b),(1c) Since the logistic equation is $\frac{dN}{dt} = r(1 - \frac{N}{B})N = f(N)$,

$$\frac{d^2N}{dt^2} = \frac{d}{dt}\frac{dN}{dt} = \frac{d}{dt}f(N) = f'(N)\frac{dN}{dt} = f'(N)f(N)$$

$$\Rightarrow \frac{d^2N}{dt^2} = r^2 N\left(1 - \frac{N}{B}\right)\left(1 - \frac{2N}{B}\right).$$

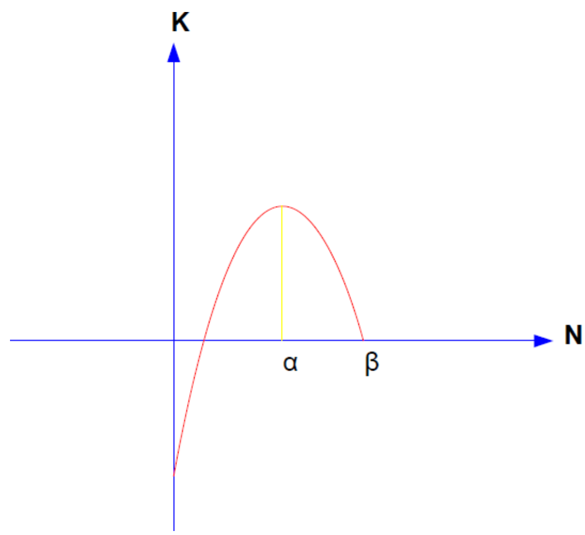
This expression is positive when $N < \frac{B}{2}$, so it follows that in this region the graph of $N(t)$ is concave up.
On the other hand, the expression is negative when $\frac{B}{2} < N < B$, and so in this region the graph of $N(t)$ is concave down.

Observe (from the explicit solution) that if $N_0 < B$, then $N(t)$ is an increasing function, so $N_0 \leq N(t) < B$ for all $t \geq 0$.

(1d) If $N > B$, then $\frac{d^2N}{dt^2} > 0$ and the graph of N is concave up.
This happens when $N_0 > B$, in which case $N(t) > B$ for all t.

Solutions of ODE1 Problems 3 a-c

(3a)



(3b) (Let us write β instead of B.) The function

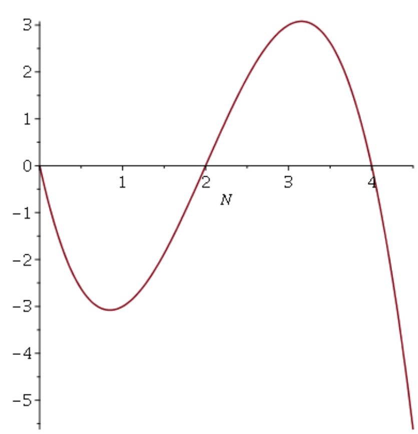
$$K(N) = -(N-\alpha)^2 + (\beta-\alpha)^2.$$

has the desired properties:

$$K(\beta) = 0,\ K'(\alpha) = 0$$

and $K'(N) > 0$ for $N < \alpha$ and $K'(N) < 0$ for $N > \alpha$. If we pick $\alpha = 3$ and $\beta = 4$, we also have $K(0) = -8 < 0$.

(3c)



Solutions of ODE1 Problem 6 (parts a,b) (the “additional problem”)

Consider the Gompertz law for tumor growth:

$$\frac{dV}{dt} = ae^{-\beta t}V$$

where β and a are positive constants. **(1)** Show (derive using separation of variables, showing details) that the solution is:

$$V(t) \;=\; V(0)e^{\frac{a}{\beta}(1-e^{-\beta t})}$$

(2) What is the limit as $t \to \infty$?

Using separation of variables,

$$\ln|V(t)| \;=\; \int \frac{dV}{V} \;=\; \int ae^{-\beta t}\,dt \;=\; -\frac{a}{\beta}e^{-\beta t} + c$$

so, taking exponentials:

$$V(t) \;=\; ke^{-\frac{a}{\beta}(e^{-\beta t})}$$

where k is a constant. Now, plugging-in $t = 0$ we have

$$V(0) = ke^{-\frac{a}{\beta}} \;\Rightarrow\; k = V(0)e^{\frac{a}{\beta}}$$

and we have: $e^{\frac{a}{\beta}}e^{\frac{a}{\beta}(-e^{-\beta t})} = e^{\frac{a}{\beta}(1-e^{-\beta t})}$

As $t \to \infty$, $e^{-\beta t} \to 0$, so $V(t) \to V(0)e^{\frac{a}{\beta}}$.