SOLUTION NOTES

MTH 7241: Fall 2022

First Practice Problems for Test 1

1). 3 balls are distributed in 3 boxes. At each step, one of the balls is selected at random, taken out of whichever box it is in, and moved at random to one of the other boxes. Let X_n be the number of balls in the first box, after n steps.

a). Find the transition matrix of the chain $X_0, X_1, \ldots$.

b). Find the stationary distribution of the chain.

$$P = \begin{array}{cc} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & \frac{2}{3} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 1 & 0 \end{pmatrix} & \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} \\ \begin{array}{cccc} 0 & \quad 1 & \quad 2 & \quad 3 \end{array} & \end{array}$$

2). Consider the following transition probability matrix for a Markov chain on 4 states:

$$P = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

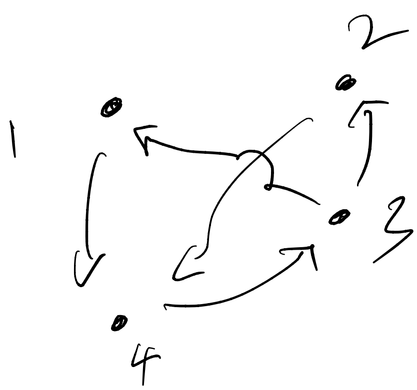
Number the states $\{1, 2, 3, 4\}$ in the order presented.

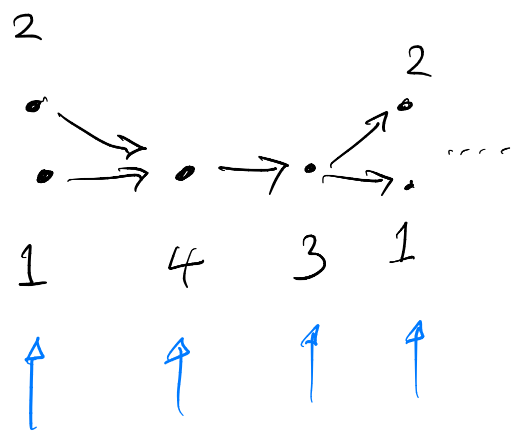
a). Find and classify the equivalence classes of the states (irreducible and transient).

b). Find a stationary distribution for the chain.

Irreducible , period = 3

$w = \frac{1}{6}(1,1,3,2)$





3). Suppose that coin 1 has probability 0.7 of coming up Heads, and coin 2 has probability 0.4 of coming up Heads. If the coin tossed today comes up Heads, then we select coin 1 to toss tomorrow, and if it comes up Tails, then we select coin 2 to toss tomorrow. If the coin initially tossed is equally likely to be coin 1 or coin 2, then what is the probability that the coin tossed on the third day after the initial toss is coin 1?

$$P = \begin{array}{cc} \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix} & \begin{array}{c} H \\ T \end{array} \\ \begin{array}{cc} H & \quad T. \end{array} & \end{array} \qquad X_n = \{H, T\}$$

$$\mathbb{P}(\text{coin } 1 \text{ on Day } 3) = \mathbb{P}(\text{Heads on Day } 2)$$

$$= \mathbb{P}(X_2 = H).$$

$$= \mathbb{P}(X_2 = H \mid \text{initial coin } 1) \cdot \tfrac{1}{2} + \mathbb{P}(X_2 = H \mid \text{initial coin } 2) \cdot \tfrac{1}{2}$$

$$\mathbb{P}(X_2 = H \mid \text{initial coin } 1) = \mathbb{P}(X_2 = H \mid X_0 = H) \cdot \mathbb{P}(X_0 = H \mid \text{initial coin } 1) + \mathbb{P}(X_2 = H \mid X_0 = T) \cdot \mathbb{P}(X_0 = T \mid \text{initial coin } 1)$$

$$= \underbrace{(0.61)}_{(P^2)_{HH}}(0.7) + \underbrace{(0.52)}_{(P^2)_{TH}}(0.3).$$

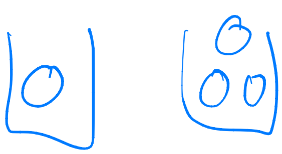
Similarly

$$\mathbb{P}(X_2 = H \mid \text{initial coin } 2) = (0.61)(0.4) + (0.52)(0.6)$$

4) Four balls are shared between box #1 and box #2. At each step a biased coin is tossed which comes up Heads with probability p. If the coin comes up Heads and box #1 is not empty, a ball is removed from box #1 and placed in box #2. If the coin comes up Heads and box #1 is empty, no balls are moved. If the coin comes up Tails and box #2 is not empty, a ball is removed from box #2 and placed in box #1. If the coin comes up Tails and box #2 is empty, no balls are moved. Let X_n be the number of balls in box #1 after n steps.

Find the transition matrix for the Markov chain $\{X_n\}$ (your answer will depend on p).

$$P = \begin{array}{cc} \begin{pmatrix} p & 1-p & 0 & 0 & 0 \\ p & 0 & 1-p & 0 & 0 \\ 0 & p & 0 & 1-p & 0 \\ 0 & 0 & p & 0 & 1-p \\ 0 & 0 & 0 & p & 1-p \end{pmatrix} & \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \\ \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} & \end{array}$$



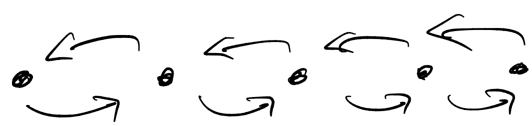
5) Consider the following transition probability matrix for a Markov chain on 5 states:

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

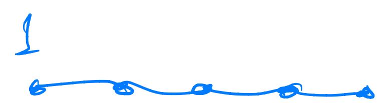
Number the states $\{1, 2, 3, 4, 5\}$ in the order presented.

Given that the chain starts in state 1, find the expected number of steps until the first return to state 1.

Irreducible period $= 2$.



$$\mu_1 = \frac{1}{W_1} = 8.$$



RW on graph.

$$W_1 = \frac{1}{1+2+2+2+1} = \frac{1}{8}$$

6) Let $\{X_n\}$ be a Markov chain, and suppose that for state i we have

$$\sum_{k=1}^{n} p_{ii}(k) = \sum_{k=1}^{n} \mathrm{P}(X_k = i \,|\, X_0 = i) = 3 - \frac{9}{\sqrt{n+8}} \quad \text{for all } n \geq 1.$$

Determine whether state i is transient or persistent (explain your reasoning).

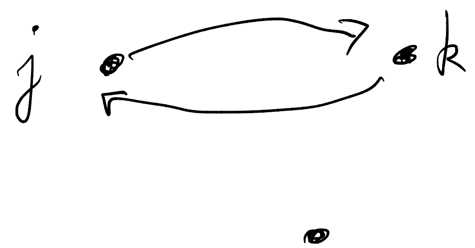
$$\lim_{n\to\infty} \sum_{k=1}^{n} p_{ii}(k) = 3 < \infty$$

$$\| $$

$$\sum_{k=1}^{\infty} p_{ii}(k)$$

$\Rightarrow$ transient.

7) Consider an irreducible chain on 3 states. **Either** prove that $p_{jj}(6) > 0$ for every state j, **or** give an example where $p_{jj}(6) = 0$ for some state j.



$P_{jj}(2) > 0 \Rightarrow P_{jj}(6) > 0.$

Irreducible $\Rightarrow$ $P_{jj}(n) > 0$ some n.

3 states $\Rightarrow$ $n \leq 3$.

$\Rightarrow$ $n \in \{1, 2, 3\}$.

all factors of 6

$\Rightarrow$ $P_{jj}(6) > 0$

$P_{jj}(3) > 0 \Rightarrow P_{jj}(6) > 0$

MTH 7241: Fall 2022

$\boxed{\text{SOLUTIONS}}$

Second Practice Problems for Test 1

1). Consider the following transition probability matrix for a Markov chain on 5 states:

$$P = \begin{pmatrix} 1/4 & 3/4 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1/3 & 2/3 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

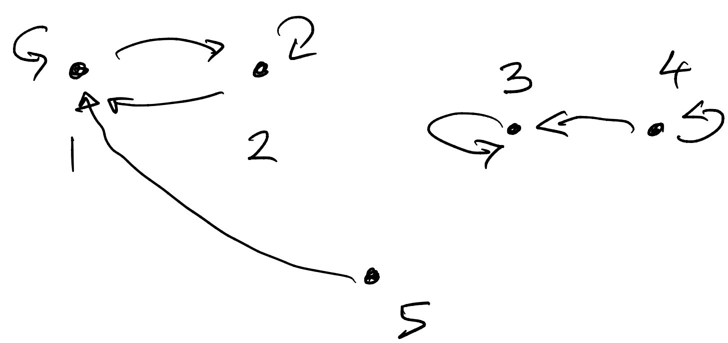
Number the states $\{1, 2, 3, 4, 5\}$ in the order presented.

a). Write down two different stationary distributions for the chain.

b). Starting from state 4, find the expected number of steps until first reaching state 3.

c). Starting from state 5, find the expected number of steps until first reaching state 2.

a)



$\{1,2\}$ closed, irreducible

$\{3\}$ absorbing

$\{4,5\}$ transient.

Different stationary distribution for each closed, irreducible class.

$\underline{\{1,2\}}$:

$$P = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$\Rightarrow \quad w_1 = \frac{1}{4} w_1 + \frac{1}{2} w_2$$

$$\Rightarrow \quad w_1 = \frac{2}{3} w_2$$

$$\Rightarrow \quad W = \left(\frac{2}{5}, \frac{3}{5}, 0, 0, 0 \right)$$

$\{\overset{3}{\cancel{5}}\}$: $\quad W = (0, 0, 1, 0, 0)$

b) $\{3, 4\}$:

$$P = \begin{array}{cc} \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ 0 & 1 \end{pmatrix} & \begin{array}{c} 4 \\ 3 \end{array} \\ \begin{array}{cc} 4 & \quad 3 \end{array} & \end{array}$$

so $\quad Q = \left(\frac{2}{3} \right)$

$$\Rightarrow \quad N = (I - Q)^{-1} = \left(\frac{1}{3} \right)^{-1} = (3)$$

$\Rightarrow$ Expected number of steps $(4 \to 3) = 3$.

c) since $5 \to 1$.

$\Rightarrow$ Expected # steps $(5 \to 2)$

$= 1 +$ Expected # steps $(1 \to 2)$.

$$\{1, 2\}: \quad P = \begin{array}{c} \\ \left(\begin{array}{cc} \frac{1}{4} & \frac{3}{4} \\ \frac{1}{2} & \frac{1}{2} \end{array}\right) \end{array} \begin{array}{c} 1 \\ 2 \end{array}$$

m_{12} = Expected # steps $(1 \to 2)$.

Condition on first step:

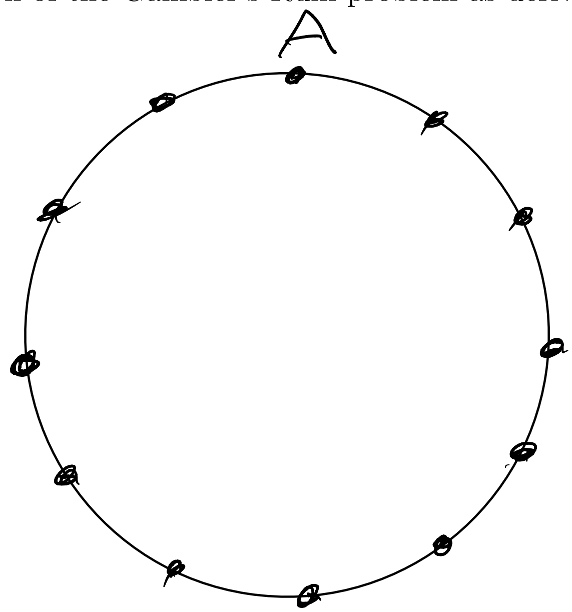
$$m_{12} = \frac{1}{4}(1 + m_{12}) + \frac{3}{4}(1)$$

$$= 1 + \frac{1}{4} m_{12}$$

$$\Rightarrow m_{12} = \frac{4}{3}.$$

$$\Rightarrow m_{52} = 1 + \frac{4}{3} = \frac{7}{3}$$

2). A particle moves between 12 points which are spaced around a circle. At each step the particle is equally likely to move one point clockwise or one point counterclockwise. Find the mean number of steps for the particle to return to its starting position. [Hint: make use of the solution of the Gambler's Ruin problem as derived in class].



This is a random walk on graph:

every vertex has degree $d = 2$.

So the Markov chain is

reversible and has stationary

distribution $$W_i = \frac{d_i}{\sum_j d_j} = \frac{2}{12(2)}$$

$$= \frac{1}{12}$$

Therefore the mean first return time is

$$\frac{1}{W_i} = 12.$$

3) A stack of m cards is shuffled as follows: at each step a random number X is chosen from $\{1, 2, \ldots, m\}$, then the card which sits at position X is removed and placed at the top of the deck. By modeling this process as a Markov chain, show that in the long-run the deck becomes randomly shuffled, so that all $m!$ orderings are equally likely.

State space = { all $m!$ orderings }

state $s = (k_1, k_2, \ldots, k_m)$

where k_i are distinct integers from $\{1, 2, \ldots, m\}$.

Transition matrix:

$$s \to s'$$

$$(k_1, \ldots, k_m) \longrightarrow (k_j, k_1, k_2, \ldots, k_{j-1}, k_{j+1}, \ldots, k_m)$$

$X = j \Rightarrow$ move card k_j to front

and $P_{ss'} = \frac{1}{m}$ (all equally likely).

The shuffling evolves as a Markov chain with this transition matrix. Clearly it is irreducible and aperiodic, so it converges to its stationary distribution. What is the stationary distribution?

Check that the transition matrix is doubly stochastic:

given s', find $s \to s'$:

say $s' = (k_1, k_2, \ldots, k_m)$

then
$$\begin{cases} s = (k_1, k_2, k_3, \ldots, k_m) \\ s = (k_2, k_1, k_3, \ldots, k_m) \\ s = (k_2, k_3, k_1, \ldots, k_m) \\ \vdots \\ s = (k_2, k_3, \ldots, k_m, k_1) \end{cases}$$

m states for $s \to s'$

all have probability $= \frac{1}{m}$

$\Rightarrow$ matrix is doubly stochastic

$\Rightarrow$ stationary distribution is uniform.

$$\Rightarrow \quad \pi_s = \frac{1}{n!}$$

$\Leftrightarrow$ all orderings equally likely.

4) Consider the following transition probability matrix for a Markov chain on 3 states:

$$P = \begin{array}{cc} \begin{pmatrix} 1/2 & 1/3 & 1/6 \\ 0 & 1/3 & 2/3 \\ 1/2 & 0 & 1/2 \end{pmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \\ \begin{array}{ccc} 1 & \quad 2 & \quad 3 \end{array} & \end{array}$$

Number the states $\{1, 2, 3\}$ in the order presented.

Find the long-run probability that the chain jumps in the following sequence of states: $1 \to 2 \to 3 \to 1 \to 2$.

$$\lim_{n\to\infty} \mathbb{P}(X_n = 1, X_{n+1} = 2, X_{n+2} = 3, X_{n+3} = 1, X_{n+4} = 2)$$

$$= P_{12} P_{23} P_{31} P_{12} \lim_{n\to\infty} \mathbb{P}(X_n = 1)$$

$$\mathbb{P}(X_n = 1, X_{n+1} = 2)$$

$$= \mathbb{P}(X_{n+1} = 2 \mid X_n = 1) \cdot \mathbb{P}(X_n = 1)$$

$$= P_{12}\, \mathbb{P}(X_n = 1)$$

$$= P_{12} P_{23} P_{31} P_{12} w_1.$$

Now $w = (w_1, w_2, w_3) = \left(\frac{2}{5}, \frac{1}{5}, \frac{2}{5}\right)$

$\Rightarrow$ get $\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)\left(\frac{2}{5}\right)$

$$= \frac{2}{135}$$

5) Let $\{X_n\}$ be a Markov chain, and suppose that for state i we have

$$\sum_{k=1}^{n} p_{ii}(k) = \sum_{k=1}^{n} \mathrm{P}(X_k = i \mid X_0 = i) \geq \frac{9}{\sqrt{n+8}} \quad \text{for all } n \geq 1.$$

Determine whether state i is transient or persistent (explain your reasoning).

Correction: it should be

$$P_{ii}(n) \geq \frac{9}{\sqrt{n+8}} \quad \text{for all } n \geq 1.$$

Then

$$\sum_{n=1}^{\infty} P_{ii}(n) \geq \sum_{n=1}^{\infty} \frac{9}{\sqrt{n+8}}$$

$$\geq 9 \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+8}}$$

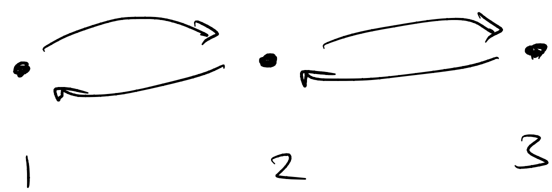
divergent harmonic series

$$= \infty$$

$\Rightarrow$ state i is persistent

6) Consider an irreducible chain on 3 states. **Either** prove that $p_{jj}(3) > 0$ for every state j, **or** give an example where $p_{jj}(3) = 0$ for some state j.

<u>False</u>: here is example



In this case $p_{jj}(3) = 0$ for every j.

7) For a branching process calculate the probability of extinction when the offspring probabilities are $p_0 = 1/4$, $p_1 = 1/2$, $p_2 = 1/8$, $p_3 = 1/8$.

Mean offspring:

$$m = 0\left(\frac{1}{4}\right) + 1\left(\frac{1}{2}\right) + 2\left(\frac{1}{8}\right) + 3\left(\frac{1}{8}\right)$$

$$= \frac{9}{8}$$

Since $m > 1 \Rightarrow \rho < 1$ (prob. of extinction)

$$\phi(s) = \frac{1}{4} + \frac{1}{2}s + \frac{1}{8}s^2 + \frac{1}{8}s^3.$$

solve

$$\phi(s) = s$$

$$\Leftrightarrow \frac{1}{8}s^3 + \frac{1}{8}s^2 - \frac{1}{2}s + \frac{1}{4} = 0.$$

$$\Leftrightarrow s^3 + s^2 - 4s + 2 = 0.$$

$$(s-1)(s^2 + 2s - 2) = 0$$

$$\Downarrow$$

$$s = \frac{-2 \pm \sqrt{4+8}}{2} = -1 \pm \sqrt{3}.$$

Solution: ρ = smallest positive solution

$$\Downarrow$$

$$\Rightarrow \rho = -1 + \sqrt{3}!$$

8) Either give an example of a finite Markov chain with no transient states and two different stationary distributions, or show that that this cannot happen.

<u>Example:</u>

FAIL → go home and study!!

$$P = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Stationary distributions:

$$w = \left(\tfrac{1}{2}, \tfrac{1}{2}, 0, 0\right)$$

$$w = \left(0, 0, \tfrac{1}{2}, \tfrac{1}{2}\right)$$

9) Either give an example of a finite Markov chain with a unique stationary distribution and one transient state, or show that that this cannot happen.

Example.

Awful!

$$P = \begin{pmatrix} 0 & 1\,\cancel{\tfrac{1}{2}} & 0 \\ 1\,\cancel{\tfrac{1}{2}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$\{3\}$ transient.

$$W = \left(\tfrac{1}{2}, \tfrac{1}{2}, 0\right)$$

unique.

10) Let X_n be an irreducible regular finite Markov chain with a unique stationary distribution $\{w_i\}$. Compute

$$\lim_{n\to\infty} \mathrm{P}\left(X_{n+1} = X_n\right)$$

$$\mathbb{P}(X_{n+1} = X_n)$$

$$= \sum_i \mathbb{P}(X_{n+1} = X_n = i \mid X_n = i)\, \mathbb{P}(X_n = i)$$

$$= \sum_i P_{ii}\, \mathbb{P}(X_n = i)$$

$$\Rightarrow \lim_{n\to\infty} \mathbb{P}(X_{n+1} = X_n) = \sum_i P_{ii}\, w_i$$

MATH 7241 Fall 2022: Problem Set #6

Due date: Friday November 4

SOLUTIONS

Reading: relevant background material for these problems can be found on Canvas 'Notes 4: Finite Markov Chains', 'FSHMC'. Also Grinstead and Snell Chapter 11.

Exercise 1 In each case below, determine whether or not the chain is reversible (note: the condition for reversibility is $w_i p_{ij} = w_j p_{ji}$ for all states i, j).

$$(a) \qquad P = \begin{pmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{pmatrix}$$

$$(b) \qquad P = \begin{pmatrix} 3/4 & 1/4 & 0 \\ 0 & 2/3 & 1/3 \\ 1/4 & 1/4 & 1/2 \end{pmatrix}$$

a) Find stationary distribution:

$$w = \left(\frac{2}{3}, \frac{1}{3}\right)$$

Check reversible equation:

$$w_1 p_{12} \overset{?}{=} w_2 p_{21}$$

$$\frac{2}{3} \cdot \frac{1}{4} \overset{?}{=} \frac{1}{3} \cdot \frac{1}{2} \quad \text{YES} \Rightarrow \text{reversible}$$

b) $p_{12} = \frac{1}{4}, \; p_{21} = 0 \Rightarrow$ impossible to satisfy $w_1 p_{12} = w_2 p_{21}$

$\Rightarrow$ not reversible

$\boxed{\textbf{Exercise 2}}$ A box contains N balls, some red and some blue. At each step, a coin is flipped with probability p of coming up Heads, and probability $1-p$ of coming up Tails. If the coin comes up Heads, a ball is chosen at random from the box and is replaced by a red ball; if the coin comes up Tails then a ball is chosen randomly from the box and replaced by a blue ball. Let X_n denote the number of red balls in the box after n steps. Find the transition matrix for the chain $\{X_n\}$, and find the stationary distribution. [Hint: is the chain reversible?] Compute $\lim_{n\to\infty} \mathrm{E}[X_n]$, and explain why you could have guessed your answer without doing the calculation.

Ex. 2 X_n = # red balls in Box 1.

State space $\{0, 1, 2, \dots, N-1\}$.

Transition matrix:

$$P_{ij} = \mathbb{P}(X_{n+1} = j \mid X_n = i) = \begin{cases} p\,\frac{N-i}{N} & \text{if } j = i+1 \\ (1-p)\,\frac{i}{N} & \text{if } j = i-1 \\ 1 - p\,\frac{N-i}{N} - (1-p)\frac{i}{N} & \text{if } j = i \end{cases}$$

Note: if $j = i+1$, then must toss Heads (prob $= p$) and must select a blue ball (prob $= \frac{N-i}{N}$). etc.

Try to solve reversible equations:

$$w_i P_{ij} = w_j P_{ji}.$$

In this case we only have the equations

$$W_i P_{i,i+1} = W_{i+1} P_{i+1,i}$$

$$\Rightarrow \quad W_{i+1} = W_i \frac{P_{i,i+1}}{P_{i+1,i}}$$

$$= W_i \frac{p}{1-p} \frac{N-i}{i+1}$$

$$= W_{i-1} \left(\frac{p}{1-p}\right)^2 \frac{N-i}{i+1} \frac{N-i+1}{i}$$

$$= \cdots\cdots$$

$$= W_0 \left(\frac{p}{1-p}\right)^{i+1} \frac{(N-i)(N-i+1)\cdots(N)}{(i+1)(i)\cdots(1)}$$

$$= W_0 \left(\frac{p}{1-p}\right)^{i+1} \binom{N}{i+1}$$

$$\Rightarrow \quad W_i = W_0 \left(\frac{p}{1-p}\right)^i \binom{N}{i} \qquad (i = 0, 1, 2, \ldots, N)$$

$$(a+b)^N = \sum_{i=0}^{N} \binom{N}{i} a^i b^{N-i}$$

Normalize to find w_0:

$$\sum_{i=0}^{N} w_i = w_0 \sum_{i=0}^{N} \left(\frac{p}{1-p}\right)^i \binom{N}{i}$$

$$= w_0 \left(1 + \frac{p}{1-p}\right)^N$$

$$= w_0 (1-p)^{-N}.$$

$$\Rightarrow \quad w_0 = (1-p)^N.$$

$$\Rightarrow \quad w_i = p^i (1-p)^{N-i} \binom{N}{i}$$

This is the binomial dist. w/ prob. p.

$$\Rightarrow \quad \lim_{N\to\infty} \mathbb{E}[X_n] = \sum_{i=0}^{N} i\, w_i$$

$$= Np. \qquad \text{b/c binomial.}$$

We could have expected this: as $N \to \infty$ the balls are randomly mixed so that the fraction of Red balls is p, and the fraction of Blue balls is $1-p$.

Exercise 3 A knight moves randomly on a standard 8×8 chessboard. At each step it chooses at random one of the possible legal moves available. Given that the knight starts in a corner of the chessboard, find the expected number of steps until its first return to its initial position. [Hint: model the knight's position using a Markov chain, and try to show that the chain is reversible]

$E[\#\text{steps first return to state } i] = w_i^{-1}$

Random walk on graph is reversible:

$$W_i = \frac{d_i}{\sum_j d_j} \qquad d_i = \#\text{ nearest neighbors of node } i.$$

8x8 chessboard

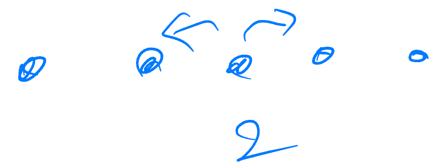
2	3	4	4	4	4	3	2
3	4	6	6	6	6	4	3
4	6	8	8	8	8	6	4
4	6	8	8	8	8	6	4
4	6	8	8	8	8	6	4
4	6	8	8	8	8	6	4
3	4	6	6	6	6	4	3
2	3	4	4	4	4	3	2

← d_i for each square

$$\sum_j d_j = 336$$

$$\Rightarrow W_{\text{corner}} = \frac{2}{336} = \frac{1}{168}$$

$\Rightarrow$ mean return time $= 168$ steps



Exercise 4 Grinstead and Snell p.423, #7.

Make 0, 4 into absorbing states

$$\Rightarrow P = \begin{array}{c} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & 0 & \frac{3}{4} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{3}{4} & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array}$$

Reorder states 1, 2, 3, 0, 4:

$$P = \begin{pmatrix} Q & R \\ 0 & I \end{pmatrix} := \begin{pmatrix} 0 & \frac{3}{4} & 0 & \frac{1}{4} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{3}{4} & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$Q = \begin{pmatrix} 0 & \frac{3}{4} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{3}{4} & 0 \end{pmatrix} \Rightarrow N = (I - Q)^{-1} = \begin{pmatrix} \frac{5}{2} & 3 & \frac{3}{2} \\ 2 & 4 & 2 \\ \frac{3}{2} & 3 & \frac{5}{2} \end{pmatrix}$$

$$NR = \begin{pmatrix} \frac{5}{2} & 3 & \frac{3}{2} \\ 2 & 4 & 2 \\ \frac{3}{2} & 3 & 5/2 \end{pmatrix} \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & 0 \\ 0 & \frac{1}{4} \end{pmatrix} = \begin{pmatrix} 5/8 & 3/8 \\ 1/2 & 1/2 \\ 3/8 & 5/8 \end{pmatrix}$$

Exercise 5 Grinstead and Snell p.423, #9.

$$P = \begin{array}{c} \\ \left(\begin{array}{ccccc} 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right) \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}$$

$\{1, 2, 3, 4\}$ transient

$$\Rightarrow \quad Q = \begin{pmatrix} 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \quad N = (I - Q)^{-1} = \begin{pmatrix} 1 & \frac{1}{4} & \frac{1}{3} & \frac{1}{2} \\ 0 & 1 & \frac{1}{3} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}$$

$$\text{Mean \# steps } (1 \to 5) = N_{11} + N_{12} + N_{13} + N_{14}$$

$$= 1 + \frac{1}{4} + \frac{1}{3} + \frac{1}{2} = \frac{25}{12}$$

Exercise 6 Grinstead and Snell p.427, #24.

Case $p = q$ done in class.

$p \neq q$: try

$$w_x = \frac{(q/p)^x - 1}{(q/p)^T - 1} = A\left(\frac{q}{p}\right)^x + B$$

$$\Rightarrow p\,w_{x+1} + q\,w_{x-1} = p\,A\left(\frac{q}{p}\right)^{x+1} + pB$$
$$+ q\,A\left(\frac{q}{p}\right)^{x-1} + qB$$
$$= q\,A\left(\frac{q}{p}\right)^{x} + p\,A\left(\frac{q}{p}\right)^{x} + A$$
$$= w_x \checkmark$$

Boundary conditions: $w_0 = 0 \Rightarrow A + B = 0$

$$w_T = 1 \Rightarrow A\left(\left(\frac{q}{p}\right)^T - 1\right) = 1.$$

Exercise 7 Recall the Gambler's Ruin Problem: a random walk on the integers $\{0, 1 \ldots, N\}$ with probability p to jump right and $q = 1 - p$ to jump left at every step, and absorbing states at 0 and N. Starting at $X_0 = k$, the probability to reach N before reaching 0 is

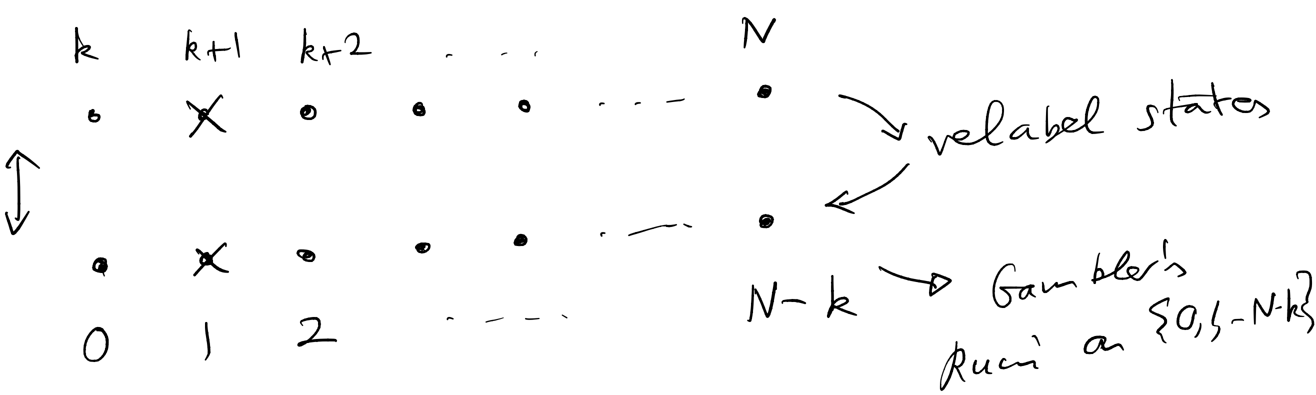
$$P_k = \frac{1-(q/p)^k}{1-(q/p)^N} \quad \text{for } p \neq \frac{1}{2}, \qquad\qquad P_k = \frac{k}{N} \quad \text{for } p = \frac{1}{2}.$$

Starting at $X_0 = k$, let R_k be the probability to reach state N without returning to state k. Use the Gambler's Ruin result to compute R_k for all $k = 0, \ldots, N$, and for all $0 < p < 1$. [Hint: condition on the first step and use the formula given above].

$$R_k = \mathbb{P}(\text{reach } N, \text{ no return to } k \mid X_0 = k)$$

Condition on Step 1:

$$R_k = \underbrace{\mathbb{P}(\text{reach } N \text{ before } k \mid X_1 = k-1)}_{0} \cdot q$$
$$+ \mathbb{P}(\text{reach } N \text{ before } k \mid X_1 = k+1) \cdot p$$



$$\Rightarrow \mathbb{P}(\text{reach } N \text{ before } k \mid X_1 = k+1)$$ original problem

$$= \mathbb{P}(\text{reach } N-k \text{ before } 0 \mid X_0 = 1)$$ Gambler's Ruin

$$\Rightarrow R_k = p \cdot \mathbb{P}(\text{reach } N-k \text{ before } 0 \mid X_0 = 1)$$

Use formula above for P_k:

$$\Rightarrow \quad R_k = \begin{cases} p \dfrac{1 - q/p}{1-(q/p)^{N-k}} & p \neq \frac{1}{2} \\[2ex] \dfrac{1}{2} \dfrac{1}{N-k} & p = \frac{1}{2} \end{cases}$$

Exercise 8 The Markov chain $X = \{X_n\}$ is defined on the state space $S = \{0, 1, 2, \dots\}$. The chain is irreducible, aperiodic and positive persistent, with stationary distribution $\{w_k\}$ $(k = 0, 1, 2, \dots)$. Let $Y = \{Y_n\}$ be an independent copy of X, and define $Z = (X, Y)$.

a). Write down the transition matrix for Z, and compute its stationary distribution (your answer will depend on w).

b). Given that the chain Z starts at the state (k, k) (so that $X_0 = Y_0 = k$), find an expression for the expected number of steps until the first return to (k, k).

a) $\mathbb{P}(Z_1 = (k,l) \mid Z_0 = (i,j))$ $\qquad Z = (X, Y)$

$= \mathbb{P}(X_1 = k \mid X_0 = i)\ \mathbb{P}(Y_1 = l \mid Y_0 = j)$

$= P_{ik} P_{jl}$

Stationary: $W_{(k,l)} = W_k W_l$.

Check: $\sum_{i,j} W_{(i,j)} P_{(i,j),(k,l)} = \sum_{i,j} W_i W_j P_{ik} P_{jl}$

$= \left(\sum_i W_i P_{ik}\right)\left(\sum_j W_j P_{jl}\right)$

$= W_k W_l$

$= W_{(k,l)}$

b) Mean return time to (k,k) is

$$\frac{1}{W_{(k,k)}} = \frac{1}{W_k^2}$$