Lecture 7

Example 1 (Ex. 3.2) Modeling the spread of technology

Models for the spread of technology are **very** similar to the logistic model for population growth.

Let $N(t)$ be the number of ranchers who have adopted an improved pasture technology in Uruguay. Then $N(t)$ satisfies the differential equation

$$\frac{dN}{dt} = aN\left(1 - \frac{N}{N_T}\right),$$

where N_T is the total population of ranchers. It is assumed that the rate of adoption is proportional to both the number who have adopted the technology and the fraction of the population of ranchers who have not adopted the technology.

Note: The same model can be used to describe the spread of a rumor within an organization or population.

(a) Which terms correspond to the fraction of the population who have not yet adopted the improved pasture technology?

N_T = total population of ranchers to whom a new innovation was introduced

The simplest realistic assumption that we can make concerning the spread of this innovation is that a farmer adopts the innovation only after he has been told of it by a farmer who has already adopted $= aN(t)$, and the number of farmers $N_T - N$ who are as yet unaware.

The fraction of the unawared farmers is $\boxed{\dfrac{N_T - N}{N_T} = 1 - \dfrac{N}{N_T}}$

(b) $N_T = 17{,}015$, $a = 0.490$ and $N_0 = 141$. Determine how long it takes for the improved pasture technology to spread to 80% of the population.

Solution.

We need to find t^* such that $N(t^*) = 0.8 N_T$

$$N(t) = \frac{N_T \cdot N_0}{N_0 + (N_T - N_0)e^{-at}}$$

$$\Downarrow$$

$$0.8 = \frac{141}{141 + (17{,}015 - 141)e^{-0.49t^*}}$$

$$0.8(141 + 16874)e^{-0.49t^*} = 141$$

$$t^* = -\frac{1}{0.49} \ln\left[\left(\frac{141}{0.8} - 141\right)\frac{1}{16874}\right] \approx 12.59 \approx 12.6 \text{ years.}$$

Limited growth with harvesting

The effect of harvesting a population on a regular or constant basis is extremely important to many industries. One example is the fishing industry. Will a high harvesting rate destroy the population? Will a low harvesting rate destroy the viability of the industry?

Formulating the equation

Including a constant harvesting rate in the logistic model gives

$$\left\{\begin{matrix}\text{rate of change}\\ \text{in population}\end{matrix}\right\} = \left\{\begin{matrix}\text{rate of}\\ \text{births}\end{matrix}\right\} - \left\{\begin{matrix}\text{normal rate}\\ \text{of deaths}\end{matrix}\right\} - \left\{\begin{matrix}\text{rate of deaths}\\ \text{by crowding}\end{matrix}\right\} - \underbrace{\left\{\begin{matrix}\text{rate of deaths}\\ \text{of harvesting}\end{matrix}\right\}}_{\text{constant}}$$

$$\Downarrow$$

$$\boxed{\frac{dX}{dt} = rX\left(1 - \frac{X}{K}\right) - h}$$

Here h is included as the constant rate of harvesting (total number caught per unit time, or deaths due to harvesting per unit time) and it is independent of the population size and thus could be interpreted as a quota.

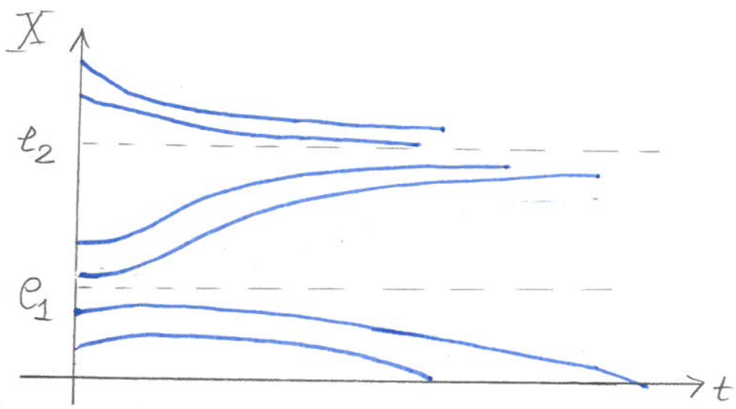
Solving the Differential Equation

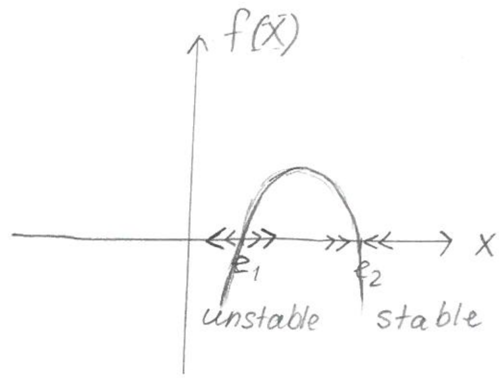
$$X' = rX - \frac{r}{K}X^2 - h = -\frac{r}{K}\left(X^2 - KX + \frac{Kh}{r}\right)$$

$$X' = -\frac{r}{K}(X - e_1)(X - e_2) \qquad e_{1,2} = \frac{K \mp \sqrt{K^2 - 4\frac{Kh}{r}}}{2}$$

e_1, e_2 - are distinct and positive $\qquad e_{1,2} = \frac{K}{2}\left(1 \mp \sqrt{1 - \frac{4h}{Kr}}\right)$

if $1 - \frac{4h}{Kr} > 0$ or $\frac{4h}{Kr} < 1$





Example Trout Extinction

$$X' = \frac{5}{2}X\left(1 - \frac{X}{10}\right) \qquad \text{units are 1000 fish}$$

We introduce harvesting at the rate of 2.25 $\left(\frac{1000 \text{ fish}}{\text{year}}\right)$

$$\Rightarrow \quad X' = \frac{5}{2}X\left(1 - \frac{X}{10}\right) - 2.25 \qquad (*)$$

a) Find the equilibrium solutions

b) Solve $(*)$ with $X(0) = 2$

c) For what values of $X(0) = x_0$ does the population become extinct?

d) Suppose $x_0 = 0.9$. How long does it take for the population to become extinct?

Solution

$$X' = \frac{5}{2}X - \frac{X^2}{4} - \frac{9}{4} = -\frac{1}{4}\left(X^2 - 10X + 9\right)$$

(a) $$X' = -\frac{1}{4}(X-1)(X-9)$$ Equilibrium $X=1$ & $X=9$

$$\frac{dX}{(X-1)(X-9)} = -\frac{1}{4}dt$$

(b)

Partial Fraction Decomposition:

$$\frac{1}{(X-1)(X-9)} = \frac{A}{X-1} + \frac{B}{X-9} = \frac{A(X-9)+B(X-1)}{(X-1)(X-9)} = \frac{X(A+B)-9A-B}{(X-1)(X-9)}$$

X^1: $A+B=0 \Rightarrow A=-B$

X^0: $-9A-B=1 \Rightarrow -9(-B)-B=1 \Rightarrow B=1/8$, $A=-1/8$

$$\int \frac{dX}{(X-1)(X-9)} = \int\left(\frac{1}{8(X-9)} - \frac{1}{8(X-1)}\right)dX = \frac{1}{8}\ln\left|\frac{X-9}{X-1}\right| = -\frac{1}{4}t + C \Rightarrow$$

$$\ln\left|\frac{X-9}{X-1}\right| = -2t + C \Rightarrow \frac{X-9}{X-1} = Ce^{-2t}$$ $$\boxed{\frac{X-9}{X-1} = \frac{(x_0-9)}{(x_0-1)}e^{-2t}}$$

$$X(0)=2 \Rightarrow \frac{2-9}{2-1} = Ce^{-2\cdot 0} \Rightarrow C=-7 \Rightarrow \frac{X-9}{X-1} = -7e^{-2t}$$

Solving for X:

$$X-9 = (X-1)(-7)e^{-2t}$$

$$X-9 = -7Xe^{-2t} + 7e^{-2t}$$

$$X(1+7e^{-2t}) = 7e^{-2t} + 9$$

$$X = \frac{9+7e^{-2t}}{1+7e^{-2t}}$$

Solving for t

$$t = -\frac{1}{2}\ln\left(-\frac{1}{7}\frac{(X-9)}{(X-1)}\right)$$

(c) If $x_0 < 1$, then $X' < 0$ and the population declines.

(d) If $x_0 = 0.9$, then $C = \frac{0.9-9}{0.9-1} = \frac{-8.1}{-0.1} = 81$, Extinct $X(t^*)=0$

$$t^* = -\frac{1}{2}\ln\left(-\frac{1}{81}\cdot 9\right)$$
$$= -\frac{1}{2}\ln\left(-\frac{1}{9}\right) \approx 1.01 \text{ year}$$

Time-delayed regulation

We discuss, briefly, how to introduce a time lag response into the logistic model of population growth

The logistic model

$$\frac{dX}{dt} = \underbrace{r\left(1 - \frac{X(t)}{K}\right)}_{R(X)} X(t)$$

r = growth rate
K = carrying capacity
$R(X) = r\left(1 - \frac{X}{K}\right)$ = density-dependent growth rate

includes an instantaneous reaction to the environment. That is, increased pressure on the resources produces an immediate response from the system (in terms of, for example, more deaths).

Often this is not realistic in that, the response usually takes affect after some time delay, or time lag. Mammals, for instance, undergo a lengthy maturation period. Vegetation needs time to recover and changed environmental conditions that may lead to increased bith rates will take time to show up in the numbers of an adult population. This leads us to a model that includes a time delay which could result from a multitude of measurable sources (maturation times, food supply, resources, crowding, etc.).

If the time lag is small compared with the natural response time $(1/r)$, then there is a tendency to overcompensate, which may produce oscillatory behavior.

In 1948 G. E. Hutchinson modified the logistic equation to incorporate a delay into the growth rate, so

$$R(X) = R(X(t-\tau)):$$

$$\frac{dX}{dt} = rX(t)\left[1 - \frac{X(t-\tau)}{K}\right] \qquad \tau = \text{delay}$$

We can write this in dimensionless form by introducing the following new variables:

$$y(\tilde{t}) = \frac{X(t)}{K}, \quad \tilde{t} = \frac{t}{\tau} \quad \Rightarrow \quad \frac{dX}{dt} = \frac{d}{dt}(Ky) = \frac{d}{d\tilde{t}}(Ky)\cdot\frac{d\tilde{t}}{dt}$$

$$= K\frac{dy}{d\tilde{t}}\cdot\frac{1}{\tau}$$

$$K\frac{dy}{d\tilde{t}}\cdot\frac{1}{\tau} = rKy(\tilde{t})[1-y(\tilde{t}-1)] \quad \Big| \div K; \times\tau$$

$$\frac{dy}{d\tilde{t}} = r\tau y(\tilde{t})[1-y(\tilde{t}-1)]. \qquad \begin{bmatrix} t := \tilde{t} \\ \alpha := r\tau \end{bmatrix}$$

$$\boxed{\frac{dy}{dt} = \alpha y(t)(1-y(t-1))}$$

Note that to specify initial conditions, one must specify the y value over the interval $[0,1]$ (or $[0,\tau]$ for the dimensional form of the equation). Often this is done by picking y to be the same value over this entire interval, so

$$y = y_0 \text{ for all } t\in[0,1].$$

The dimensionless equation has a single parameter, α, which is convenient for analysis.

There are two equilibrium solutions $y(1-y(t-1))=0$: $y_1=0$, $y_2=1$.

We are interested in if $y_2=1$ is a stable equilibrium.

The stability analysis is to assume $y=1+\delta y$ with perturbation $|\delta y|\ll 1$. If $\delta y\to 0$ as $t\to\infty$, then 1 is stable. Otherwise is unstable.

$$\frac{d}{dt}(1+\delta y) = \alpha(1+\delta y)(1-(1+\delta y(t-1)))$$

$$\Downarrow$$

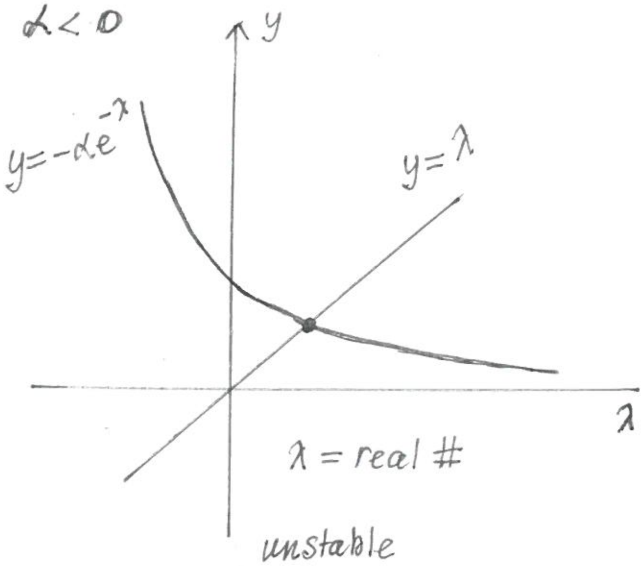
$$\frac{d\delta y}{dt} = -\alpha(1+\delta y)(\delta y(t-1)) = -\alpha\delta y(t-1) - \overbrace{\underbrace{\alpha\,\delta y\,\delta y(t-1)}_{\text{quadratic, smaller than the linear part}}}^{\text{omit}}$$

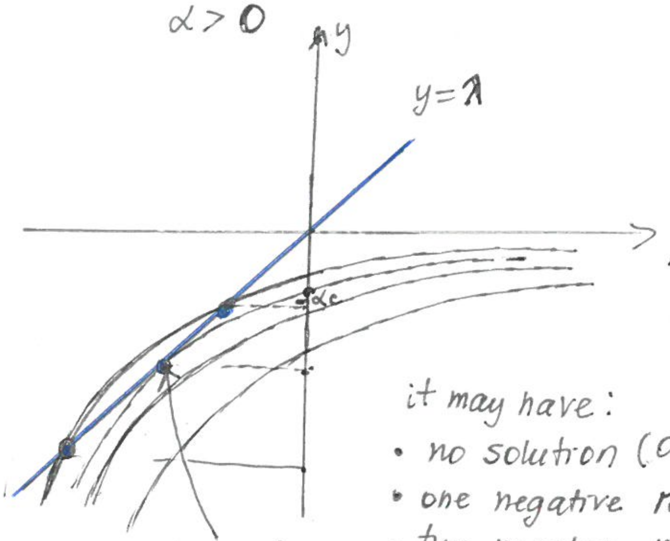
$$\Downarrow$$

$$\frac{d\delta y}{dt} = -\alpha\delta y(t-1).$$

Assume $\delta y = Ce^{\lambda t}$ $\Rightarrow$ $C\lambda e^{\lambda t} = -\alpha Ce^{\lambda(t-1)}$ $\Leftrightarrow$

$\lambda = -\alpha e^{-\lambda}$ (the transdental equation)





critical point.
$\alpha_c = \frac{1}{e}$

it may have:

- no solution ($0 < \alpha < \alpha_c$)
- one negative real eigenvalue
- two negative real eigenvalue

So for the parameter $0 < \alpha < \alpha_c$, the solution $\delta y = Ce^{\lambda t}$ will decay to zero.

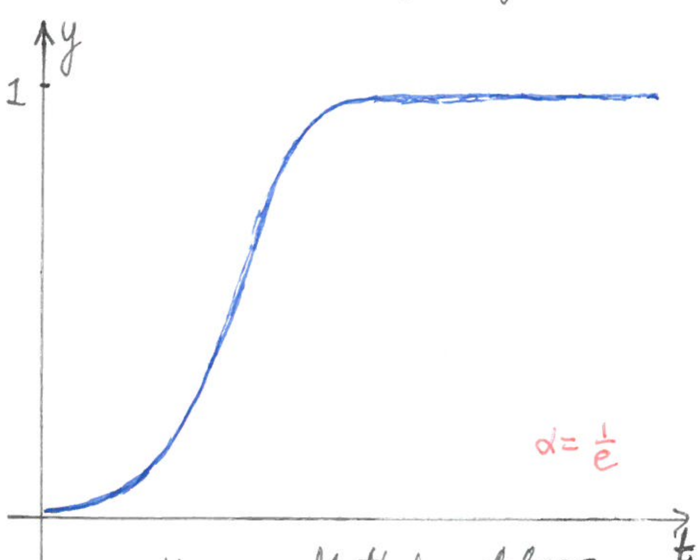
How about the case $\alpha > \alpha_c$? There is no real eigenvalue, we need to try complex eigenvalues:

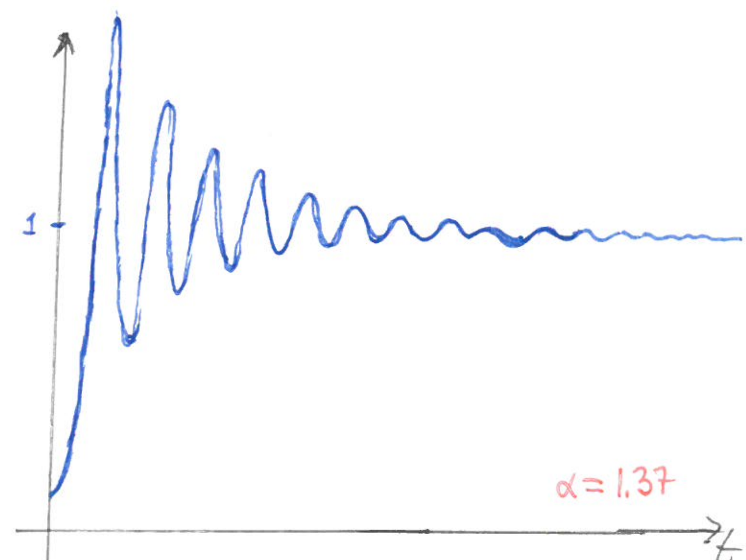
$$\lambda = \lambda_r + i\lambda_i \text{, then}$$

$$e^{(\lambda_r + i\lambda_i)t} = e^{\lambda_r t}(\cos\lambda_i t + i\sin\lambda_i t)$$

If $\lambda_r < 0$, the amplitude $e^{\lambda_r t} \xrightarrow[t\to\infty]{} 0$, and if $\lambda_r > 0$, $e^{\lambda_r t} \xrightarrow[t\to\infty]{} \infty$.

Unlike the real case, since trigonometric functions are involved, oscillation with frequency λ_i will be observed.



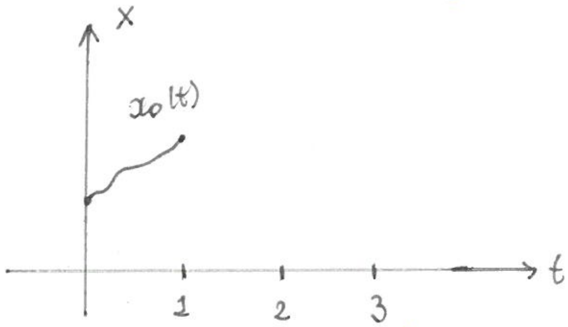


Use in Matlab dde23,

How to construct solutions of delay equations:

$$X' = f(X(t-1))$$

Let $x_0(t)$ be any continuous function on $[0,1]$



Define $x_1(t) = \int_1^t f(x_0(\xi - 1))\,d\xi + x_0(1)$ on $1 \le t \le 2$

Note $x_1(1) = x_0(1)$ Set $X(t) = \begin{cases} x_0(t) & \text{on } 0 \le t \le 1 \\ x_1(t) & \text{on } 1 \le t \le 2 \end{cases}$

Then X satisfies the delay equation on $[1,2]$.

Check: $X'(t) \underset{\substack{\uparrow \\ \text{def. of } X}}{=} x_1'(t) \underset{\substack{\uparrow \\ \text{def. of } x_1(t)}}{=} f(x_0(t-1)) \underset{\substack{\uparrow \\ \text{def. of } x_0}}{=} f(X(t-1))$

$X(t) = x_0(t)$ on $[0,1]$.

Now continue the process to get $x_2(t)$, $x_3(t)$, etc.

So any choice of $x_0(t)$ will give rise to a solution $X(t)$ on $[1,\infty)$.