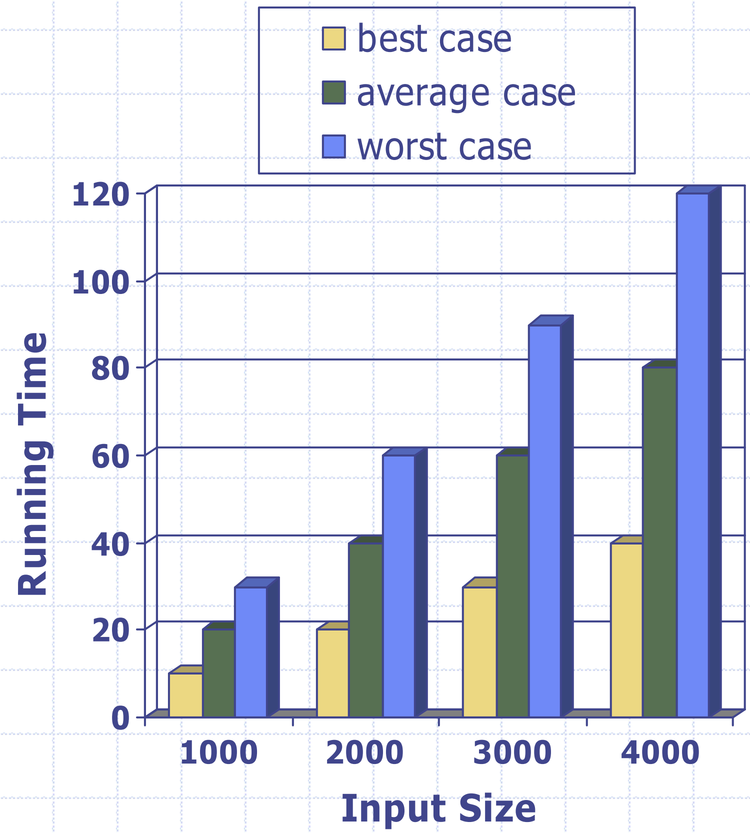
# Analysis of Algorithms

# Running Time

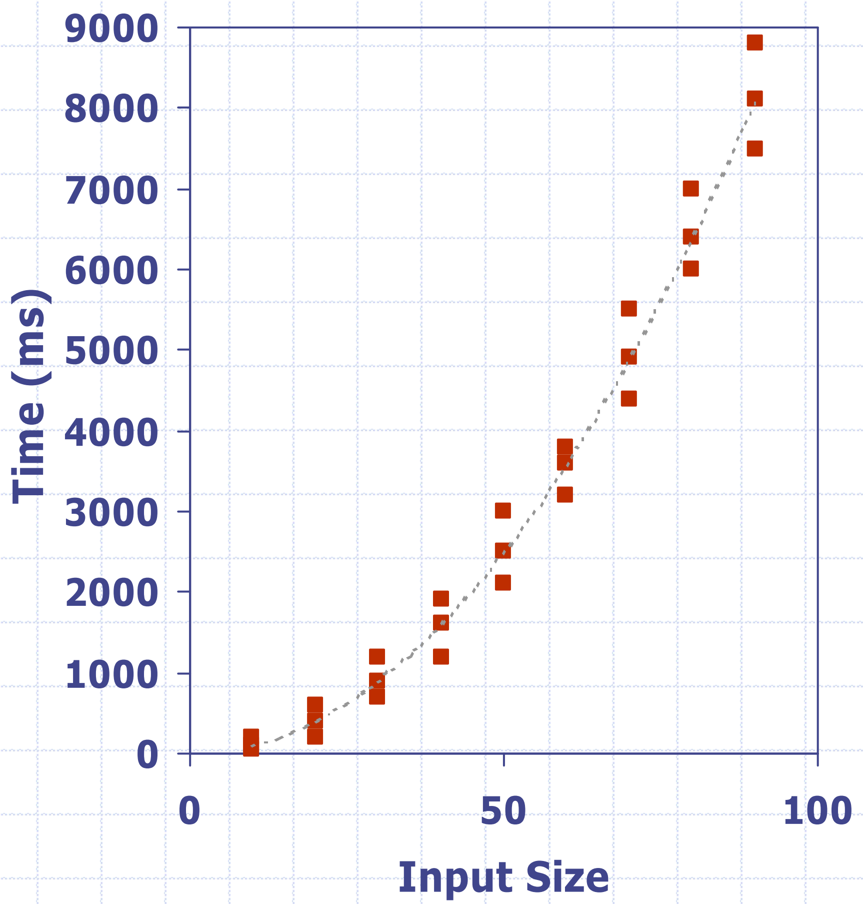
- Most algorithms transform input objects into output objects.
- The running time of an algorithm typically grows with the input size.
- Average case time is often difficult to determine.
- We focus on the worst case running time.
  - Easier to analyze
  - Crucial to applications such as games, finance and robotics



# Experimental Studies

- Write a program implementing the algorithm
- Run the program with inputs of varying size and composition, noting the time needed:

```
from time import time
start_time = time( )
run algorithm
end_time = time( )
elapsed = end_time − start_time
```




- Plot the results

# Limitations of Experiments

- It is necessary to implement the algorithm, which may be difficult
- Results may not be indicative of the running time on other inputs not included in the experiment.
- In order to compare two algorithms, the same hardware and software environments must be used

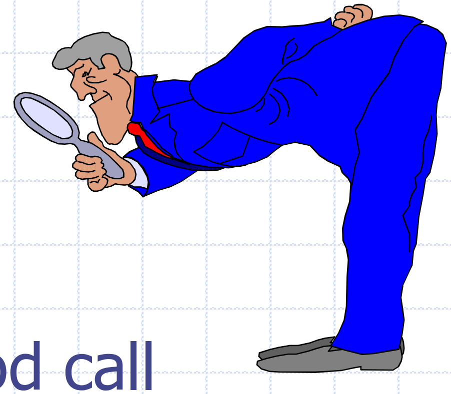
# Theoretical Analysis

- Uses a high-level description of the algorithm instead of an implementation
- Characterizes running time as a function of the input size, *n*.
- Takes into account all possible inputs
- Allows us to evaluate the speed of an algorithm independent of the hardware/software environment

# Pseudocode

- High-level description of an algorithm
- More structured than English prose
- Less detailed than a program
- Preferred notation for describing algorithms
- Hides program design issues

# Pseudocode Details



- Control flow
  - **if** … **then** … [**else** …]
  - **while** … **do** …
  - **repeat** … **until** …
  - **for** … **do** …
  - Indentation replaces braces
- Method declaration

  **Algorithm** *method* (*arg* [, *arg*…])

  **Input** …

  **Output** …
- Method call

  *method* (*arg* [, *arg*…])
- Return value

  **return** *expression*
- Expressions:

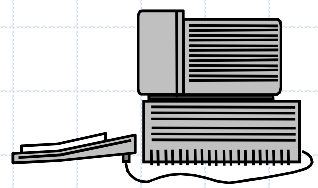
  $\leftarrow$ Assignment

  $=$ Equality testing

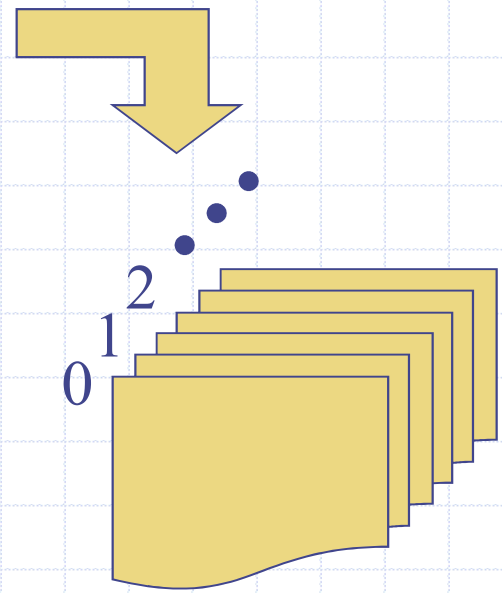
  $n^2$ Superscripts and other mathematical formatting allowed

# The Random Access Machine (RAM) Model

- A **CPU**



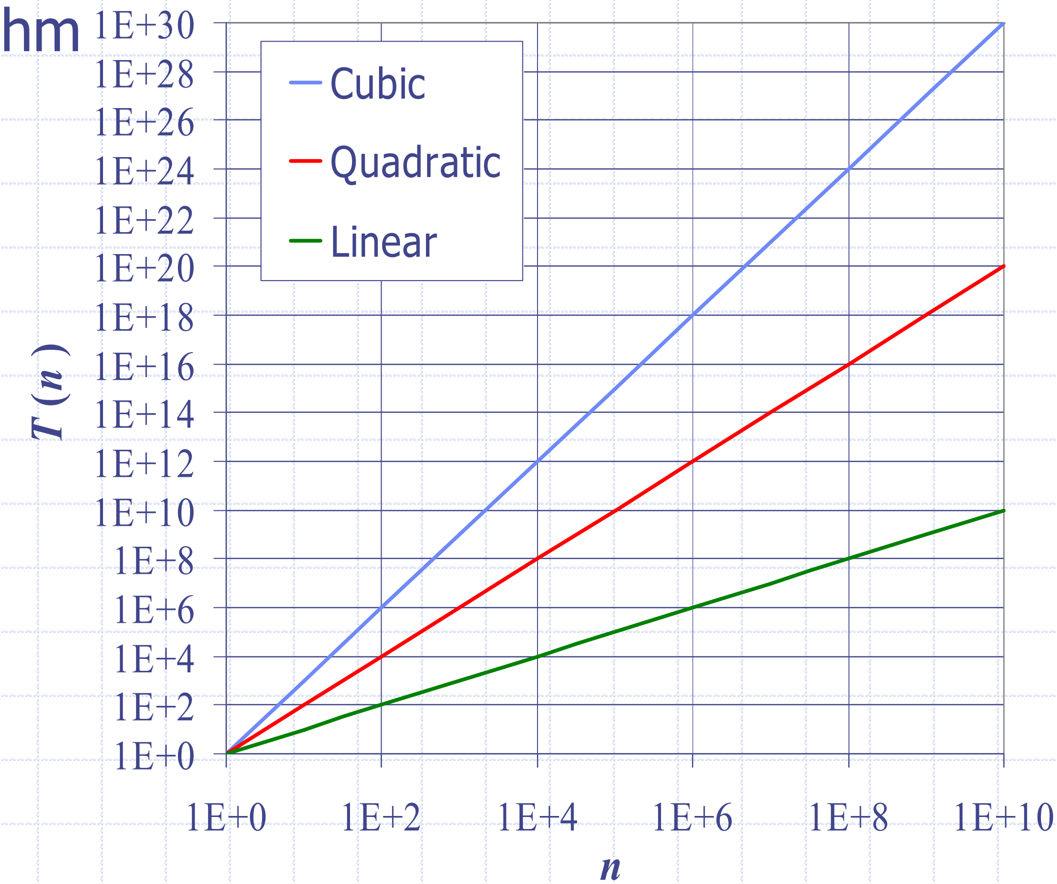
- An potentially unbounded bank of **memory** cells, each of which can hold an arbitrary number or character




- Memory cells are numbered and accessing any cell in memory takes unit time.

# Seven Important Functions

- Seven functions that often appear in algorithm analysis:
  - Constant ≈ $1$
  - Logarithmic ≈ $\log n$
  - Linear ≈ $n$
  - N-Log-N ≈ $n \log n$
  - Quadratic ≈ $n^2$
  - Cubic ≈ $n^3$
  - Exponential ≈ $2^n$
- In a log-log chart, the slope of the line corresponds to the growth rate



# Functions Graphed Using “Normal” Scale

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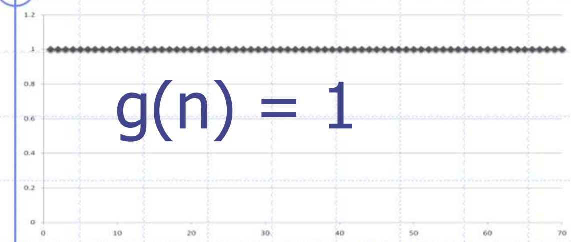
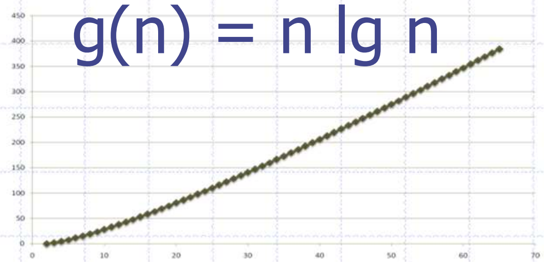
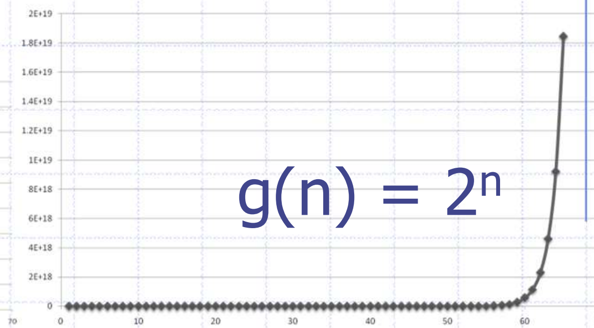
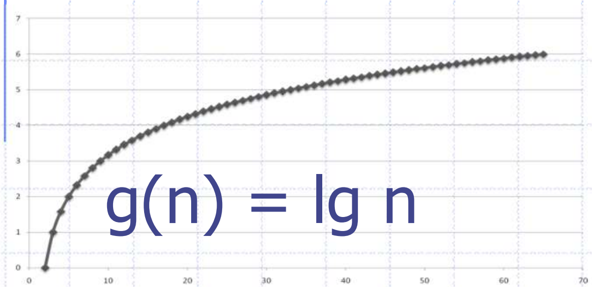
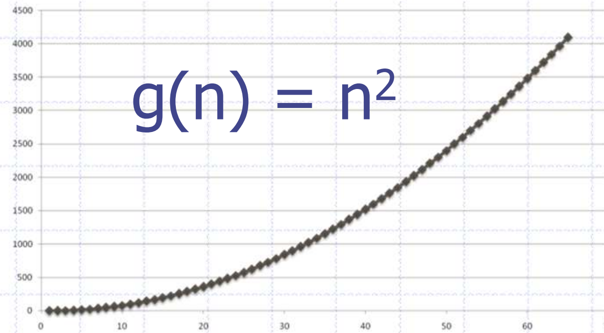
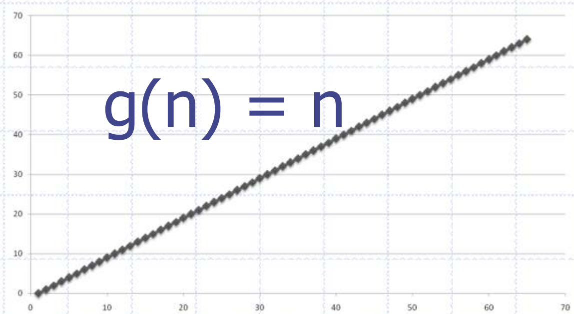


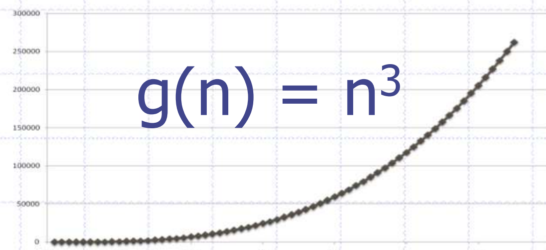



# Primitive Operations

- Basic computations performed by an algorithm
- Identifiable in pseudocode
- Largely independent from the programming language
- Exact definition not important (we will see why later)
- Assumed to take a constant amount of time in the RAM model

- Examples:
  - Evaluating an expression
  - Assigning a value to a variable
  - Indexing into an array
  - Calling a method
  - Returning from a method

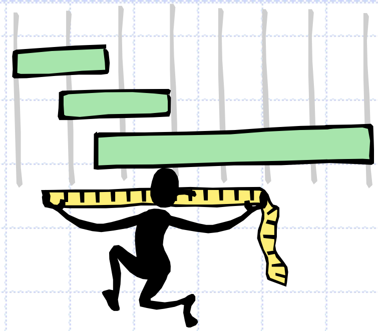
# Counting Primitive Operations

- By inspecting the pseudocode, we can determine the **maximum** number of primitive operations executed by an algorithm, as a function of the input size

```
1 def find_max(data):
2   """Return the maximum element from a nonempty Python list."""
3   biggest = data[0]            # The initial value to beat
4   for val in data:             # For each value:
5     if val > biggest           # if it is greater than the best so far,
6       biggest = val            # we have found a new best (so far)
7   return biggest               # When loop ends, biggest is the max
```

- Step 1: 2 ops, 3: 2 ops, 4: 2n ops, 5: 2n ops, 6: 0 to n ops, 7: 1 op

# Estimating Running Time



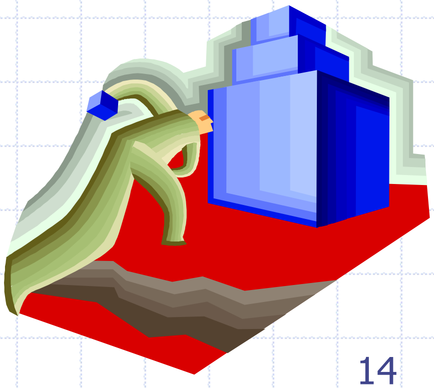
- Algorithm **find_max** executes $5n + 5$ primitive operations in the worst case, $4n + 5$ in the best case. Define:
  - $a$ = Time taken by the fastest primitive operation
  - $b$ = Time taken by the slowest primitive operation
- Let $T(n)$ be worst-case time of **find_max.** Then

$$a\,(4n + 5) \leq T(n) \leq b(5n + 5)$$

- Hence, the running time $T(n)$ is bounded by two linear functions.

# Growth Rate of Running Time

- Changing the hardware/ software environment
  - Affects $T(n)$ by a constant factor, but
  - Does not alter the growth rate of $T(n)$
- The linear growth rate of the running time $T(n)$ is an intrinsic property of algorithm **find_max**



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# Why Growth Rate Matters

| if runtime is... | time for n + 1 | time for 2 n | time for 4 n |
|---|---|---|---|
| $c \lg n$ | $c \lg (n + 1)$ | $c (\lg n + 1)$ | $c(\lg n + 2)$ |
| $c\, n$ | $c\, (n + 1)$ | $2c\, n$ | $4c\, n$ |
| $c\, n \lg n$ | $\sim c\, n \lg n + c\, n$ | $2c\, n \lg n + 2cn$ | $4c\, n \lg n + 4cn$ |
| $c\, n^2$ | $\sim c\, n^2 + 2c\, n$ | **$4c\, n^2$** | $16c\, n^2$ |
| $c\, n^3$ | $\sim c\, n^3 + 3c\, n^2$ | $8c\, n^3$ | $64c\, n^3$ |
| $c\, 2^n$ | $c\, 2^{n+1}$ | $c\, 2^{2n}$ | $c\, 2^{4n}$ |

runtime quadruples when problem size doubles

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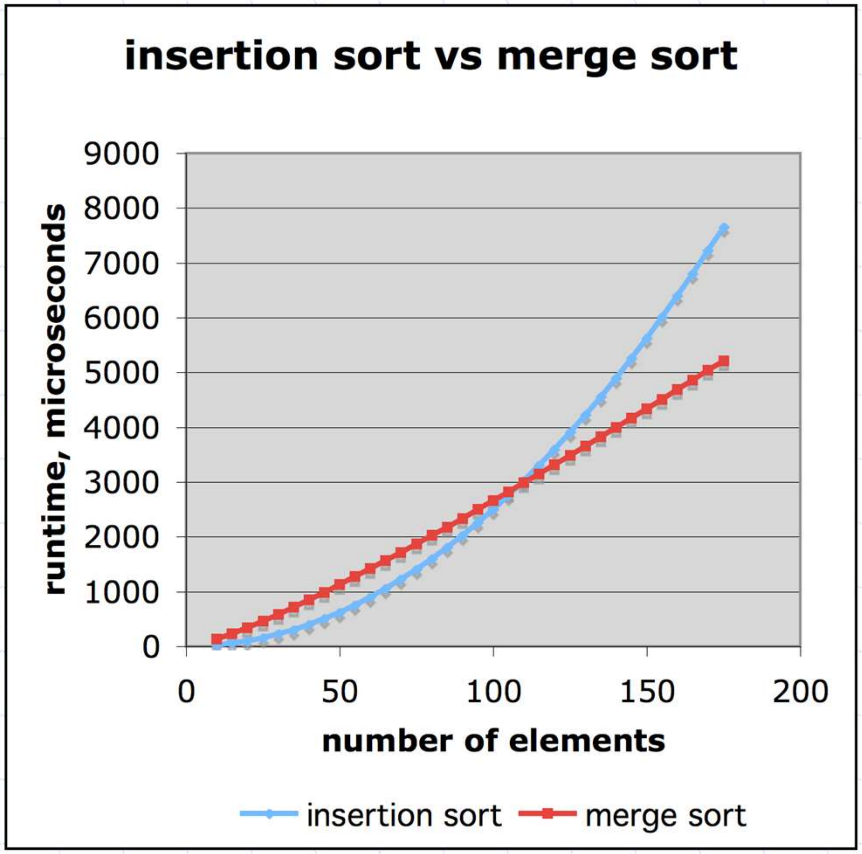
# Comparison of Two Algorithms




insertion sort is

$n^2 / 4$

merge sort is

$2 \, n \lg n$

sort a million items?

insertion sort takes roughly 70 hours

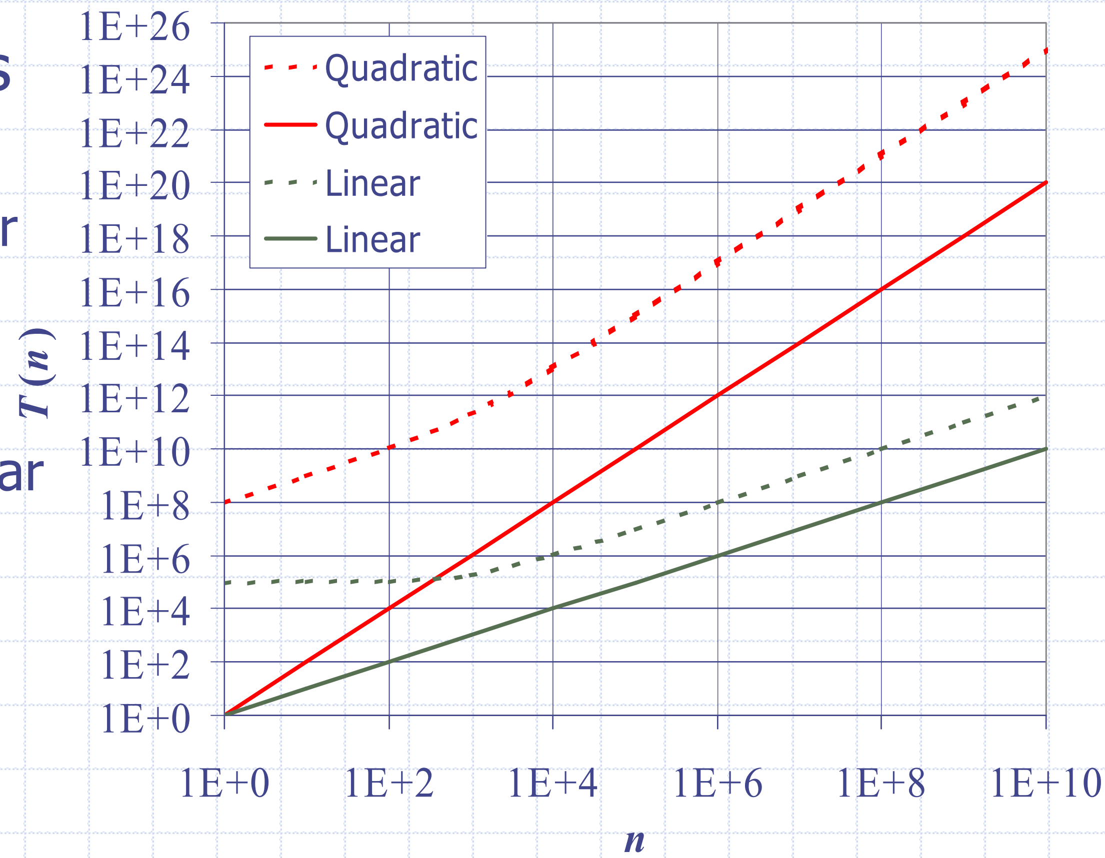
while

merge sort takes roughly 40 seconds

This is a slow machine, but if 100 x as fast then it's 40 minutes versus less than 0.5 seconds

# Constant Factors

- The growth rate is not affected by
  - constant factors or
  - lower-order terms
- Examples
  - $10^2 n + 10^5$ is a linear function
  - $10^5 n^2 + 10^8 n$ is a quadratic function

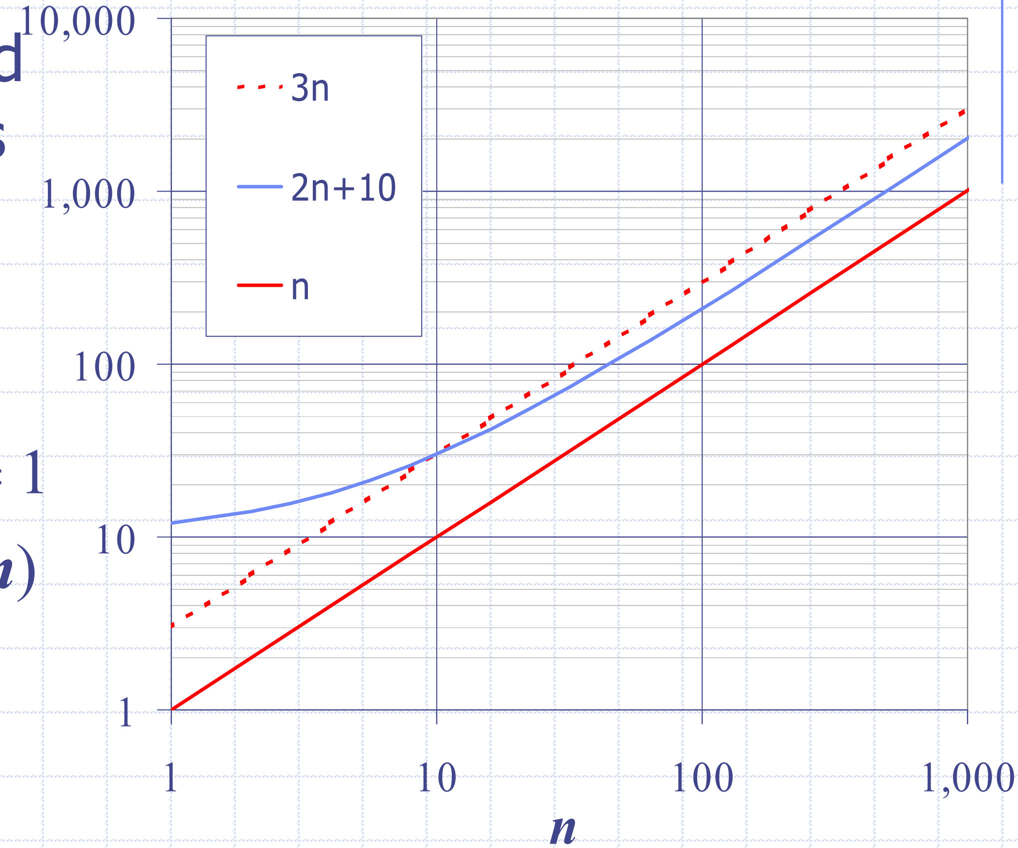


# Big-Oh Notation

- Given functions $f(n)$ and $g(n)$, we say that $f(n)$ is $O(g(n))$ if there are **positive constants** $c$ and $n_0$ such that

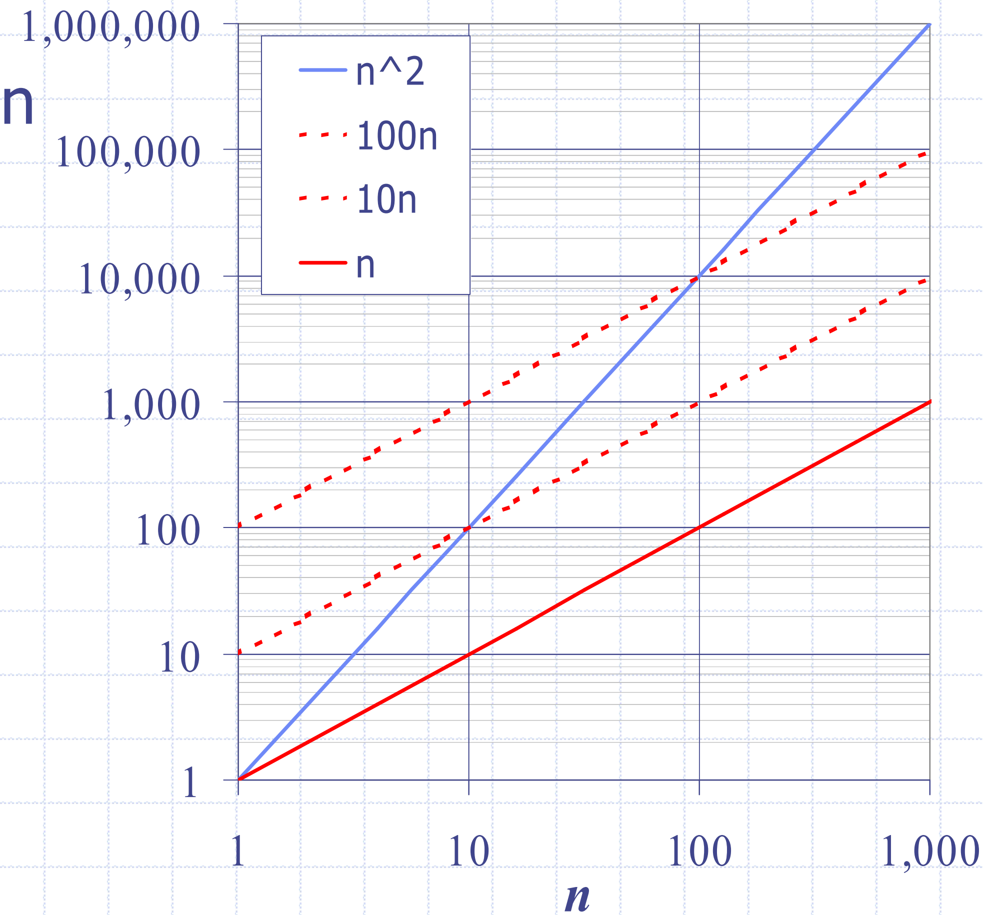
  $f(n) \leq cg(n)$ for $n \geq n_0 = 1$

- Example: $2n + 10$ is $O(n)$
  - $2n + 10 \leq cn$
  - $(c - 2)\, n \geq 10$
  - $n \geq 10/(c - 2)$
  - Pick $c = 3$ and $n_0 = 10$

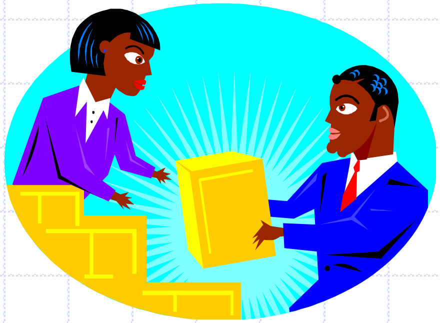


# Big-Oh Example

- Example: the function $n^2$ is not $O(n)$
  - $n^2 \leq cn$
  - $n \leq c$
  - The above inequality cannot be satisfied since $c$ must be a constant



# More Big-Oh Examples



- 7n-2

  7n-2 is O(n)

  need $c > 0$ and $n_0 \geq 1$ such that $7n-2 \leq c \bullet n$ for $n \geq n_0$

  this is true for $c = 7$ and $n_0 = 1$

- $3n^3 + 20n^2 + 5$

  $3n^3 + 20n^2 + 5$ is $O(n^3)$

  need $c > 0$ and $n_0 \geq 1$ such that $3n^3 + 20n^2 + 5 \leq c \bullet n^3$ for $n \geq n_0$

  this is true for $c = 4$ and $n_0 = 21$

- $3 \log n + 5$

  $3 \log n + 5$ is $O(\log n)$

  need $c > 0$ and $n_0 \geq 1$ such that $3\log n + 5 \leq c \bullet \log n$ for $n \geq n_0$

  this is true for $c = 8$ and $n_0 = 2$

# Big-Oh and Growth Rate

- The big-Oh notation gives an upper bound on the growth rate of a function
- The statement "$f(n)$ is $O(g(n))$" means that the growth rate of $f(n)$ is **no more than** the growth rate of $g(n)$
- We can use the big-Oh notation to rank functions according to their growth rate

| | $f(n)$ is $O(g(n))$ | $g(n)$ is $O(f(n))$ |
|---|---|---|
| $g(n)$ grows more | Yes | No |
| $f(n)$ grows more | No | Yes |
| Same growth | Yes | Yes |

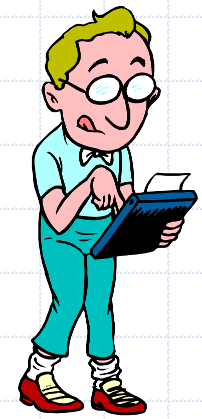
# Big-Oh Rules

- If is $f(n)$ a polynomial of degree $d$, then $f(n)$ is $O(n^d)$, i.e.,
  1. Drop lower-order terms
  2. Drop constant factors
- Use the smallest possible class of functions
  - Say "$2n$ is $O(n)$" instead of "$2n$ is $O(n^2)$"
- Use the simplest expression of the class
  - Say "$3n + 5$ is $O(n)$" instead of "$3n + 5$ is $O(3n)$"

# Asymptotic Algorithm Analysis

- The asymptotic analysis of an algorithm determines the **running time** in big-Oh notation
- To perform the asymptotic analysis
  - We find the **worst-case number** of primitive operations executed as a function of the input size
  - We express this function with big-Oh notation
- Example:
  - We say that algorithm **find_max** "runs in $O(n)$ time"
- Since constant factors and lower-order terms are eventually dropped anyhow, we can disregard them when counting primitive operations

# Math you need to Review



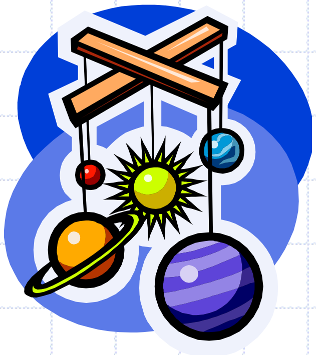
- Summations
- Logarithms and Exponents
- Proof techniques

- **properties of logarithms:**
  $\log_b(xy) = \log_b x + \log_b y$
  $\log_b (x/y) = \log_b x - \log_b y$
  $\mathrm{Log}_b x^a = a\log_b x$
  $\log_b a = \log_x a / \log_x b$
- **properties of exponentials**:
  $a^{(b+c)} = a^b a^c$
  $a^{bc} = (a^b)^c$
  $a^b / a^c = a^{(b-c)}$
  $b = a^{\log_a b}$
  $b^c = a^{c*\log_a b}$

# Relatives of Big-Oh

- **big-Omega**
  - f(n) is $\Omega(g(n))$ if there is a constant $c > 0$ and an integer constant $n_0 \geq 1$ such that $f(n) \geq c \bullet g(n)$ for $n \geq n_0$
- **big-Theta**
  - f(n) is $\Theta(g(n))$ if there are constants $c' > 0$ and $c'' > 0$ and an integer constant $n_0 \geq 1$ such that $c' \bullet g(n) \leq f(n) \leq c'' \bullet g(n)$ for $n \geq n_0$

# Example Uses of the Relatives of Big-Oh



- **$5n^2$ is $\Omega(n^2)$**

  $f(n)$ is $\Omega(g(n))$ if there is a constant $c > 0$ and an integer constant $n_0 \geq 1$ such that $f(n) \geq c \bullet g(n)$ for $n \geq n_0$

  let $c = 5$ and $n_0 = 1$

- **$5n^2$ is $\Omega(n)$**

  $f(n)$ is $\Omega(g(n))$ if there is a constant $c > 0$ and an integer constant $n_0 \geq 1$ such that $\mathrm{f}(n) \geq c \bullet g(n)$ for $n \geq n_0$

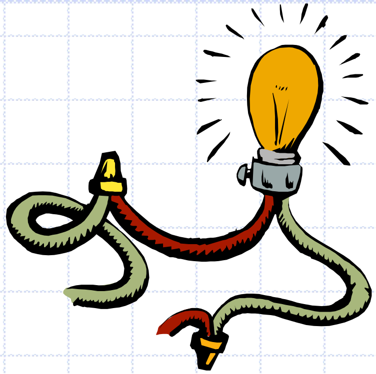
  let $c = 1$ and $n_0 = 1$

- **$5n^2$ is $\Theta(n^2)$**

  $f(n)$ is $\Theta(g(n))$ if it is $\Omega(n^2)$ and $O(n^2)$. We have already seen the former, for the latter recall that $f(n)$ is $O(g(n))$ if there is a constant $c > 0$ and an integer constant $n_0 \geq 1$ such that $\mathrm{f}(n) \leq c \bullet g(n)$ for $n \geq n_0$

  Let $c = 5$ and $n_0 = 1$

# Intuition for Asymptotic Notation



**Big-Oh**

- f(n) is O(g(n)) if f(n) is asymptotically **less than or equal** to g(n)

**big-Omega**

- f(n) is Ω(g(n)) if f(n) is asymptotically **greater than or equal** to g(n)

**big-Theta**

- f(n) is Θ(g(n)) if f(n) is asymptotically **equal** to g(n)