FUNDAÇÃO GETÚLIO VARGAS
ESCOLA DE ECONOMIA DE SÃO PAULO

TIAGO CARDOSO BOTELHO

# CANDLE AUCTIONS

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Dissertação apresentada à Escola de Economia de São Paulo como pré-requisito à obtenção de título de mestre em Economia de Empresas.

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# Resumo

Nesta dissertação, eu desenvolvo e resolvo um modelo de *candle auctions*, ou seja, leilões dinâmicos com final aleatório. No meu modelo base, eu suponho que os participantes têm valores privados independentes, e eque eles escolhem seus lances sequencialmente, um de cada vez. Eu mostro que o segundo participante irá igualar o seu lance ao do primeiro sempre que isto for lucrativo para ele. O primeiro participante tem seu lance ótimo dado pela diferença entre seu valor e um fator de sombreamento, que depende da probabilidade de término do leilão antes que uma segunda rodada ocorra. Eu mostro que *candle auctions* podem mitigar o problema de lances tardios, mas apenas às custas da receita esperada.
**Palavras-chave**: Leilões Dinâmicos, Leilões com Final Aleatório, Licitação.

# Abstract

In this dissertation, I develop and solve a model of candle auctions, *i.e.*, dynamic auctions with random termination. In the baseline model, I assume that bidders have independent private values, and that they bid sequentially, one at a time, across two time periods. I show that the second bidder will match the bid of the first bidder whenever it is profitable for him to do so. The first bidder's optimal bid is given by the difference between his value and a shading factor, depending on the probability that the auction ends before a second round of bidding takes place. I show that although candle auctions may mitigate the issue of late bidding, they do so at the expense of expected revenue.

**Keywords:** Dynamic Auctions, Candle Auctions, Procurement.

**JEL Classification:** C73, D44, D82.

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# 1 Introduction

In his classic book, Cassady (1967) coined the term *candle auction* to refer to dynamic auctions with random termination. His apathy toward what was once a widely used auction format is not accidental. Candle auctions were gradually replaced with English auctions in the eighteenth-century, and their usage was relegated to small stakes sales in various places in Europe.

Nearly three centuries after candle auctions became a historical curiosity, some modern-day auction designers have begun to deploy it once again. As early as 2000, a patent was filed by IBM[1] to secure the rights for "a system for randomly varying the time period of an auction on the Internet". The patent was later assigned to Google LLC, and expired in 2020.

Today, there are at least two large-scale implementations of candle auctions. The London Stock Exchange[2] runs daily opening and closing (double) auction calls for market participants to buy and sell securities. These last for ten minutes, and are subject to 30-second random periods. Various other stock exchanges in Europe and Israel have adopted similar practices (see Füllbrunn and Sadrieh (2012) and the references therein). Another important application is the usage of random bidding phases in public procurement auctions[3] in Brazil. These amount to approximately 2% of yearly GDP, and therefore a good understanding of how they work is a necessary condition to ensure that taxpayer money is being well spent.

In this dissertation, I develop and solve a simple model of candle auctions. The model views a candle auction as a dynamic selling mechanism with random termination. Bidders are called to play sequentially, and in a two-period version of the model, there is a fixed probability of termination at the end of the first period.

If the auction terminates before the second bidder is given an opportunity to play, then the first bidder is the sole bidder, *i.e.*, a monopsonist. This will make him face a clear trade-off when choosing his bid: on the one hand, he wants to increase his bid, since he might face some degree of competition if the auction goes on to a second round. On the

[1] See https://patents.google.com/patent/US6665649B1/en for details.
[2] See https://www.londonstockexchange.com/discover/news-and-insights/what-auction for details.
[3] See https://www.gov.br/compras/pt-br for details (in Portuguese).

other hand, he may luck out and be the sole bidder, so he may have some room to shade his bid by a larger factor than bidders typically do in first-price auctions.

Regarding the above trade-off, the model offers two novel insights:

1. First, initial bidders shade their bids (relative to their valuations) at an even greater extent than they would do in a first-price auction. Intuitively, because the initial bidder is a monopsonist with some probability, he need not always worry about his competitors. However, he will choose positive bids in equilibrium, thereby potentially mitigating issues related to late bidding, at the expense of expected revenue.

2. Second, as a corollary of the above observation, the amount of shading (and therefore of lost revenue) is increasing in the probability of termination. The intuition here is that as the probability of being a monopsonist increases, the initial bidder needs to worry about his competitors less and less.

As robustness checks of sorts, I allow for bidders to be randomly selected to play across time periods. This does not substantially change the qualitative conclusions of the model.

## 1.1 Related Work

This dissertation is closely related to two strands of the literature on auction theory. Because this is one of the first theoretical works on the subject, I dedicate a substantial amount of space to this literature section. One can skip it with little to no loss in understanding of subsequent chapters.

### 1.1.1 Candle Auctions, Old and New

First and foremost, despite the relative neglect that candle auctions have received, several authors have shown varying degrees of interest in the subject. While Sargent and Velde (1995); Krishna (2010); Hubbard and Paarsch (2016) (as representatives of modern economic thinking) have treated candle auctions as historical footnotes, the significance of this institution has not been lost on historians.

Consider the writings of the seventeenth-century English diarist and naval administrator Pepys (2000). His pioneering observations hint at many of the issues that puzzle auction theorists to this day. He mentions *inexperienced bidders* – “backward men” – who were typically the first to bid. He also speaks of a particularly talented bidder – "one man cunninger than the rest" – who figured out that when a flame is about to go out, its smoke descends, and therefore he would always know when to place his final – and presumably winning – bid. These are no doubt early references to *late bidding* and differences in sophistication among bidders.

Of even greater historical significance was the sale by candle, in 1608, of Giovanni Vincenzo Pinelli's library. According to Grendler (1981), this man owned perhaps one of the best private libraries in Italy at the time. His extensive collection was enjoyed by various intellectuals and scholars, including Galileo Galilei. Hobson (1971) describes in minute detail the intricacies of the auction, and provides countless examples of auctions by candle dating back to 1368. More importantly, he reports what is surely one of the first known examples of late bidding, since bidders would

> (...) maliciously [...] draw out the bidding until the candle [was] much burned down, with the result that inheritances [were] hardly ever sold at their real value.

Patten (1970) gives us various examples of contemporary usages of candle auctions. In Tatworth, an eight-acre piece of land has been annually leased "by inch of candle" since at least 1832. In Chard, not far from Tatworth, market tolls are sold by candle to this day. In Chedzoy, a plot of land and a cottage are leased by candle every twenty-one years. A similarly random termination method is used to auction a piece of land in Bourne, where bidding takes place at any point in time during a boys' footrace.

One could conjure up a myriad other examples. These do not do justice to the recent interest that candle auctions have received. On the empirical side of things, Fazio and Žaldokas (2023) investigate the effects that so-called *kamikaze bidders* may have on present-day candle auctions in Brazil. Ferraz et al. (2021) exploit the random termination rule in Brazilian procurement auctions to assess the impact of winning a government contract on long-term firm growth.

There are three papers that are closely related to this dissertation. I now turn my attention to them, and make an effort to distinguish my contribution from theirs.

1. Füllbrunn and Sadrieh (2012) present the first example of a theoretical model of candle auctions. In their model, bidders choose their bids simultaneously and non-

cooperatively in each time period, and the payment rule is a dynamic version of the traditional second-price rule. In my model, bidders choose their bids sequentially, and always pay their bid in exchange for the good, *i.e.*, I adopt a dynamic variant of the first-price payment rule. Our results are qualitatively different since the second-price rule induces bidders to fully reveal their values in equilibrium.

2. Szerman (2012) closely follows Ockenfels and Roth (2006) in developing his model of candle auctions. Bidding also takes place simultaneously and non-cooperatively across time periods, and the payment rule is also a dynamic variant of the second-price rule. Again, these choices make the model easier to solve in some respects, but they do not mirror the payment rule that is used in most candle auctions.

3. Häfner and Stewart (2021) has more elements in common with my model. In particular, (two) bidders choose their bids sequentially, and the payment rule is first-price. However, in light of the underlying motivation for their paper (blockchain auctions), one of the bidders' valuation is commonly known by both bidders; and histories differ across bidders, since one of them observes all bids, whereas the other can only observe bids that increase past bids. This allows them to solve longer auctions, which is something I have yet to do.

### 1.1.2 Late Bidding and Sniping

A second strand of the literature that is important to put candle auctions in context is the literature on what is called *late bidding*. I have already alluded to this issue above. However, it was not until online auctions became popular that late bidding captured the interest of economists.

Roth and Ockenfels (2002) are among the first[4] to document various stylized facts of bidder behavior in eBay and Amazon auctions. Because eBay adopts a hard closing rule, – whereas Amazon automatically extends the bidding deadline with each new bid that arrives – one would expect there to be more late bids on eBay. This is what they find in their datasets. They conjecture that such a strategy may be a best-response to inexperienced bidders adopting an incremental bidding strategy, as if they were taking part in an English auction. Latencies in the transmission of bids, however, should partially discourage bidders from placing bids near the end of the auction.

[4] Although Bajari and Hortaçsu (2003) document similar patterns in eBay auctions at approximately the same date, providing alternative explanations for these phenomena.

In a series of related papers, Ariely et al. (2005); Ockenfels and Roth (2006) experimentally and theoretically investigate how differences in auction closing rules shape the incentives to place late bids (to snipe). Their results complement and confirm what was already evident in the field data.

In another field experiment, Ely and Hossain (2009) compare sniping (late bidding) and *squatting* (early bidding) strategies in eBay auctions. Their results indicate that although sniping has a slight advantage relative to squatting, this advantage largely depends on the presence of naïve bidders.

The key insight from this literature is that the ending rule in an auction can and often does have a major impact on how bidders behave. Although the payment rule in eBay auctions would in principle provide an incentive for bidders to bid truthfully, this is not what the data suggest (often due to inexperienced bidders).

The way I see it, at least in principle, candle auctions provide an alternative solution to the late bidding (sniping) problem. The idea is simple: the higher the risk that the auction may come to an end early on, the higher the incentive to place an early, serious bid. The qualitative predictions in my model, however, suggest that a countervailing effect prevails: conditional on having already placed a bid, a higher termination probability makes it more likely that the early bidder is the sole bidder. Thus, he has an incentive to squat with a low bid.

# 2 Candle Auctions

In this chapter, I present a two-period model of candle auctions. There are two bidders with private values. Initially, a single bidder is called to play. With probability $\lambda$, he wins the auction and pays his bid. With probability $1-\lambda$, the other bidder is also called to play. The highest bidder wins, and pays his bid in exchange for the good.

## 2.1 Model

There are two *bidders*, indexed by $i \in \{1,2\}$, vying for the purchase of a single indivisible good. I denote bidder $i$'s (private) *valuation* for the good by $v_i$. Valuations are both drawn from an absolutely continuous cumulative distribution function $F : \mathbf{R} \to [0,1]$ supported[1] on $[0,1]$ with density $f = F'$ satisfying $f\left([0,1]\right) \subset \mathbf{R}_{++}$. I assume that bidders are expected-utility maximizers with quasi-linear preferences. Formally, if $q_i$ denotes the probability that $i$ receives the good, whereas $p_i$ denotes the payment that he must make in exchange for it, his payoff when his valuation is $v_i$ will be $q_i v_i - p_i$.

The seller commits to using a *candle auction* to sell the good. This means that bidders will be called to play sequentially across (at most) two time periods, indexed by $t \in \{1,2\}$. The timing of the auction is as follows:

1. At $t=1$, bidder $i=1$ is called to place a bid $b_1 \geqslant 0$. With probability $\lambda \in [0,1]$, the auction ends, and $i=1$ gets the good in exchange for his bid. In this case, bidder $i=1$ earns a payoff of $v_1 - b_1$, whereas $i=2$ earns a payoff of 0. With probability $1-\lambda$, a second round of bidding takes place.

2. At $t=2$, bidder $i=2$ is called to place a bid $b_2 \geqslant 0$, *after observing* $b_1$. The auction then terminates with certainty. If $b_1 \leqslant b_2$, bidder $i=2$ is declared the winner, and pays his bid in exchange for the good. The payoff that accrues to him is $v_2 - b_2$, whereas $i=1$ earns a payoff of 0. If $b_1 > b_2$, bidder $i=1$ wins and pays his bid, so his payoff is $v_1 - b_1$, whereas $i=2$ earns a payoff of 0.

[1] Recall that the support of a random variable is the smallest closed set that has probability 1. The analysis admits a straightforward extension (at the expense of heavier notation) to any compactly supported distribution function.

**Remark 1.** The above description embeds the assumption that ties are broken in favor of $i = 2$, conditional on $i = 2$ placing a bid. I make this assumption to ensure that $i = 2$ has a well defined best response to any bid that $i = 1$ may choose. Alternative approaches in dynamic models of competitive bidding entail the introduction of an infinitesimal bid (see, for instance, Engelbrecht-Wiggans and Weber (1983); Hörner and Jamison (2008); Bergemann and Hörner (2018)). In the context of my model, these two approaches yield the exact same predictions.

This completes my description of candle auctions with two rounds of bidding, and two potential buyers.

# 3 Equilibrium

In this chapter, I define the strategy spaces of each player. I then define what an *equilibrium point* of the auction game is, and characterize the set of pure-strategy equilibria. I illustrate and interpret these results with a parametric example. The key idea is that in a candle auction, the probability of termination represents the probability that the initial bidder will face *no competition*. Thus, as that probability increases, his optimal bid will decrease, up to a point where it vanishes.

## 3.1 Strategies and Solutions

A candle auction induces a dynamic game of incomplete information between bidders 1 and 2. I assume (common knowledge of) sequential rationality for both bidders, as well as common knowledge of the structure of the game.

Because bidders only choose their actions at one point in time each, with no opportunity to revise their bids, the description of their strategies is straightforward.

**Definition 1.** A *(pure) strategy* for bidder 1 is a function $\tilde{b}_1 : [0,1] \to \mathbf{R}_+$, mapping 1's valuations into non-negative bids. Similarly, a *(pure) strategy* for bidder 2 is a function $\tilde{b}_2 : [0,1] \times \mathbf{R}_+ \to \mathbf{R}_+$, mapping 2's valuations and 1's potential bids into non-negative bids of his own. I denote the set of pure strategies available to player $i$ by $\Sigma_i$.

**Remark 2.** To emphasize the distinction between a strategy (a function) and a bid (a particular value of that function), I write strategies with tildes and bids without them.

**Definition 2.** An *equilibrium point* of the candle auction is a pair $\left(\tilde{b}_1, \tilde{b}_2\right) \in \Sigma_1 \times \Sigma_2$ such that:

1. Bidder 1 chooses $\tilde{b}_1$ to maximize his expected payoff, given his beliefs about his opponent's valuation distribution. Formally, for each $v_1 \in [0,1]$,

$$\tilde{b}_1(v_1) \in \operatorname{argmax} \{g(b_1|v_1) : b_1 \in \mathbf{R}_+\}$$

where $g(\cdot|v_1) : \mathbf{R}_+ \to \mathbf{R}$ is defined by

$$g(b_1|v_1) := [\lambda + (1-\lambda)\mathbf{P}(b_2 < b_1)]\,(v_1 - b_1).$$

2. Bidder 2 chooses $\tilde{b}_2$ to maximize his expected payoff, given his knowledge of 1's bid. Formally, for each $v_2 \in [0,1]$, and each $b_1 \in \mathbf{R}_+$,

$$\tilde{b}_2(v_2, b_1) \in \operatorname{argmax}\{\mathbf{1}_{[b_1,\infty)}(b_2)(v_2 - b_2) : b_2 \in \mathbf{R}_+\},$$

where $\mathbf{1}_E$ denotes the indicator function over the set $E$, *i.e.*,

$$\mathbf{1}_E(x) := \begin{cases} 1 & \text{if } x \in E; \\ 0 & \text{if } x \notin E. \end{cases}$$

## 3.2 Computing Equilibria

Armed with these definitions, I now seek to characterize the set of equilibrium points. To avoid constantly discussing an edge case, I restrict $\lambda$ to lie in the interval $[0,1)$. If $\lambda = 1$, it is clear that bidder 1 should always choose a bid equal to 0, whereas bidder 2 is never called to play with positive probability, so any strategy yields the same (vanishing) payoff to him.

**Lemma 1.** *Suppose that $F$ is twice differentiable. Moreover, assume that $f > 0$ and $f' \leqslant 0$. Fix $v_1 \in [0,1]$. Then $g'(\cdot|v_1)$ is strictly decreasing.*

*Proof.* First, note that for a given value of $b_1 \in [0, v_1]$, we have

$$g'(b_1|v_1) = -\lambda + (1-\lambda)\left[(v_1 - b_1)f(b_1) - F(b_1)\right].$$

Thus, as a linear combination of differentiable functions, $g'$ is differentiable, and

$$g''(b_1|v_1) = (1-\lambda)(v_1 - b_1)f'(b_1) - 2(1-\lambda)f(b_1).$$

Since $f' \leqslant 0$ and $f > 0$, we have $g''(b_1|v_1) < 0$. Since $b_1$ was arbitrary, the desired result ensues. □

**Remark 3.** Since $g'(v_1|v_1) = -\lambda - (1-\lambda)F(v_1) < 0$, and $g'$ is strictly decreasing, it is clear that bidder 1 will never choose a bid that exceeds $v_1$. This leads us to naturally consider another threshold value when computing equilibria. This is the role that $\overline{v}_1$ plays below.

**Theorem 1.** *Suppose that $F$ is twice differentiable and strictly increasing, with $f$ decreasing. Pose*

$$\overline{v}_1 := \min\left\{\frac{\lambda}{(1-\lambda)f(0)}, 1\right\}$$

*The set of equilibrium points of the candle auction is characterized by pairs* $\left(\tilde{b}_1, \tilde{b}_2\right) \in \Sigma_1 \times \Sigma_2$ *such that*

1. *For* $v_1 \leqslant \overline{v}_1$*, bidder* 1 *chooses* $\tilde{b}_1(v_1) = 0$*. For* $v_1 > \overline{v}_1$*, bidder* 1 *chooses his bid* $\tilde{b}_1(v_1) = b_1$ *as the unique solution to*

$$b_1 = v_1 - \frac{1}{f(b_1)}\left[F(b_1) + \frac{\lambda}{1-\lambda}\right].$$

2. *For each valuation-observed bid pair* $(v_2, b_1) \in [0,1] \times \mathbf{R}_+$*,*

$$\tilde{b}_2(v_2, b_1) \in \begin{cases} \{b_1\}, & \textit{if } b_1 < v_2; \\ [0, b_1], & \textit{if } b_1 = v_2; \\ [0, b_1), & \textit{if } b_1 > v_2. \end{cases}$$

*Proof.* Fix an arbitrary pair $(v_2, b_1) \in [0,1] \times \mathbf{R}_+$. I begin by showing that bidder 2 cannot increase his payoff by deviating from $\tilde{b}_2$ (even if bidder 1 does not behave in accordance with $\tilde{b}_1$). To do so, I will split up the analysis into three mutually exclusive cases:

1. Suppose that $b_1 < v_2$.

   If bidder 2 were to choose a bid $b_2 < b_1$, his payoff would be 0. If he were to choose a bid $b_2' = b_1$, his payoff would be $v_2 - b_2' = v_2 - b_1 > 0$. If he were to choose a bid $b_2'' > b_1$, his payoff would be $v_2 - b_2'' \in (-\infty, v_2 - b_1)$. So

$$\begin{aligned} v_2 - b_1 &> \max\{v_2 - b_2', v_2 - b_2''\} \\ &= \max\{0, v_2 - b_2''\}, \end{aligned}$$

   from whence it follows that bidder 2 cannot increase his payoff by choosing any bid different from $b_1$.

2. Suppose that $b_1 = v_2$.

   If bidder 2 were to choose a bid $b_2 > b_1$, his payoff would be $v_2 - b_2 < v_2 - b_1 = 0$, whereas if he were to choose any bid $b_2' \in [0, b_1]$, his payoff would be 0. Thus, he cannot increase his payoff by choosing any bid greater than $b_1$.

3. Suppose that $b_1 > v_2$.

If bidder 2 were to choose a bid $b_2 \geqslant b_1$, his payoff would be $v_2 - b_2 \leqslant v_2 - b_1 < 0$, whereas if he were to choose any bid $b' \in [0, h)$, his payoff would be 0. Thus, he cannot increase his payoff by choosing any bid greater than or equal to $b_1$.

The preceding analysis covers all possible $(v_2, b_1)$ pairs, and therefore shows that $\tilde{b}_2$ is always optimal for bidder 2.

Now suppose that bidder 2 behaves in accordance with $\tilde{b}_2$. I will show that bidder 1 can never increase his payoff by deviating from what $\tilde{b}_1$ prescribes that he do. Fix $v_1 \in [0, 1]$. Bidder 1's expected payoff-maximizing bid solves

$$\max_{b_1 \in \mathbf{R}_+} \left\{ \lambda(v_1 - b_1) + (1 - \lambda)\mathbf{P}(b_2 < b_1)(v_1 - b) \right\} = \max_{b_1 \in \mathbf{R}_+} g(b_1 | v_1).$$

First and foremost, since bidder 2 behaves in accordance with $\tilde{b}_2$ by assumption, and since $F$ has no atoms[1], we can replace $\mathbf{P}(b_2 < b)$ with $\mathbf{P}(v_2 < b) = F(b)$.

Lemma 1 and Remark 3 imply that we can restrict our attention without loss of generality to $[0, v_1]$ as the feasible set. Since $b_1 \mapsto g(b_1|v_1)$ is continuous, a solution to bidder 1's problem exists (by the Weierstraß[2] extreme value theorem).

Suppose that $v_1 \leqslant \overline{v}_1$. Then $g'(0|v_1) \leqslant 0$. Because $g'(\cdot|v_1)$ is strictly decreasing, either $g'(0|v_1) = 0$, or $g'(b_1|v_1) = 0$ has no solution, so that $g$ must be maximized at a boundary point. Since $g(0|v_1) > g(v_1|v_1)$, we deduce that $\tilde{b}_1(v_1) = 0$.

Now suppose that $v_1 > \overline{v}_1$. Then $g'(b_1|v_1) = 0$ has a unique solution in $(0, v_1)$. Since $g'(0|v_1) > 0$, this unique solution maximizes $g(\cdot|v_1)$ on $[0, v_1]$, and is given by

$$b_1 = v_1 - \frac{1}{f(b_1)} \left[ F(b_1) + \frac{\lambda}{1 - \lambda} \right].$$

This concludes the proof. □

## 3.3 Parametric Examples

I now provide examples to illustrate Theorem 1, *i.e.*, I specify some well-known functional forms that are widely used in auction theory for valuation distributions.

[1] Recall that if a distribution is absolutely continuous, then it must be continuous, and therefore cannot have any atoms.

[2] Let $X$ be any non-empty topological space. Let $f : X \to \mathbf{R}$ be a continuous function, and let $K$ be a compact subset of $X$. Then there exist $\underline{x}, \overline{x} \in X$ such that $f(\underline{x}) = \inf_{x \in K} f(x)$ and $f(\overline{x}) = \sup_{x \in K} f(x)$.

### 3.3.1 Uniformly Distributed Values

Suppose that $F$ is uniform on $[0, 1]$, *i.e.*, that

$$F(v) = \begin{cases} 0, & \text{if } v < 0; \\ v, & \text{if } v \in [0,1]; \\ 1 & \text{if } v > 1. \end{cases}$$

This is perhaps the example *par excellence* in auction theory, dating back to Vickrey (1961), the founding father of modern[3] auction theory.

In this special case, a closed-form expression for $\tilde{b}_1$ can be easily obtained. It is given by

$$\tilde{b}_1(v_1) = \max\left\{0, \frac{v_1}{2} - \frac{\lambda}{2(1-\lambda)}\right\}.$$

A graphical representation of how $\tilde{b}_1$ varies with $\lambda$ for various values of $v_1$ is given below:

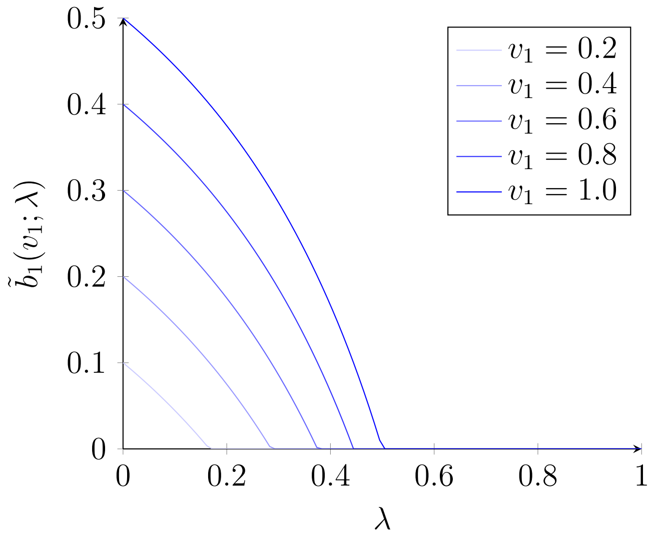


Figure 1 – Optimal first-period bids in the candle auction with valuations uniform on $[0, 1]$.

This example is instructive because it illustrates some key insights:

1. The optimal bid of bidder 1 with valuation $v_1$ is always below $\frac{1}{2}v_1$. In particular, the expected revenue that accrues to an auctioneer using a candle auction is no higher

[3] There were some papers that preceded Vickrey's seminal contribution, e.g., Friedman (1956), who had already arrived at an expression for optimal bids in first-price auctions. Stark and Rothkopf (1979) provide a comprehensive bibliography, including all works prior to Vickrey's.

than the revenue he would expect to earn in a sealed-bid, first-price auction, since bidder 2 will never bid more than what bidder 1 did.

2. Moreover, the optimal bid of bidder 1 is weakly decreasing with respect to $\lambda$.

3. Fix $v_1 \in [0, 1]$. There exists a threshold value of $\lambda$ – call it $\lambda_1$ to emphasize its dependence on $v_1$ – such that for every $\lambda \geqslant \lambda_1, \tilde{b}_1(v_1) = 0$. A uniform threshold when valuations are uniformly distributed is $\lambda = \frac{1}{2}$.

The rationale for these insights is as follows: Because $\lambda$ represents the probability that bidder 1 is a monopsonist, higher values of $\lambda$ are associated with a smaller probability of facing a competitor in the second round of bidding. Thus, as $\lambda$ increases, bidder 1 has an incentive to decrease his bid. Perhaps surprisingly, if $\lambda$ is sufficiently high, there is no belief of bidder 1 that will lead to positive first-period bids. Thus, although candle auctions may mitigate the problem of late bidding, they do so at the expense of expected revenue.

These insights still hold if we impose the assumptions of Theorem 1 on $F$:

**Proposition 1.** *Suppose that $F$ is twice differentiable, with $f > 0$ and $f' \leqslant 0$. Then $\overline{v}_1$ increases with $\lambda$. Moreover, for $v_1 > \overline{v}_1$, we have $\tilde{b}_1(v_1; \lambda)$ increasing in $v_1$ and decreasing in $\lambda$.*

*Proof.* First, note that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\lambda}{1-\lambda}\right) = \frac{1}{(1-\lambda)^2} > 0.$$

Thus, $\overline{v}_1$ is increasing in $\lambda$. It remains to show that $\tilde{b}_1(v_1; \lambda)$ is increasing in $v_1$ and decreasing in $\lambda$ for $v_1 > \overline{v}_1$.

Suppose that $v_1 > \overline{v}_1$. Then, by Theorem 1, $b_1 := \tilde{b}_1(v_1)$ solves

$$b_1 = v_1 - \frac{1}{f(b_1)}\left[F(b_1) + \frac{\lambda}{1-\lambda}\right].$$

Suppose $\lambda' > \lambda''$. Then $\frac{\lambda'}{1-\lambda'} > \frac{\lambda''}{1-\lambda''}$. Because $f > 0$ and $F > 0$, the right-hand side of the above equation increases with $\lambda$. For equality to hold, $b_1$ must decrease. A similar line of reasoning applies for $v_1' > v_1'' > \overline{v}_1$, so we are done. $\square$

# 4 Variations on a Theme

In this chapter, I address some potential shortcomings of the baseline model by allowing bidders to be randomly drawn across time periods. As one would expect, increased competition is associated with less bid shading. However, the principal qualitative result remains unchanged.

## 4.1 Bidders Called to Play at Random

In this section, I enrich the model by allowing for random orders of bidding, and for more than two bidders. Formally, instead of having bidders alternate, I assume that each bidder $i \in [n] := \{k \in \mathbf{N} : 1 \leqslant k \leqslant n\}$ is (independently) called to play at time $t$ with probability $q_i$. I further assume that there is always *someone* being called to bid, *i.e.*, $\sum_{i \in [n]} q_i = 1$, but only one bidder is called to play in each time period. The model is otherwise unchanged. In particular, ties are still broken in favor of the last bidder, regardless of his identity.

Before I characterize the set of equilibria, note that strategies are now given by *a pair of functions* for each bidder, since any one bidder may have the opportunity to play twice. This is going to make for some clumsy notation, but it won't change the qualitative conclusions of the model. Formally,

**Definition 3.** A *pure strategy* for bidder $i \in [n]$ is a pair of functions $\left(\tilde{b}_{i,1}, \tilde{b}_{i,2}\right)$ where

$$\tilde{b}_{i,1} : [0,1] \to \mathbf{R}_+$$

maps valuations into bids; and

$$\tilde{b}_{i,2} : [0,1] \times \mathbf{R}_+ \to \mathbf{R}_+$$

maps valuation-observed bids pairs into own bids. I restrict the set of pure strategies available to bidders by imposing the monotonicity constraint $\tilde{b}_{i,2} \geqslant \tilde{b}_{i,1}$ everywhere. If this restriction were not imposed, a bidder who was called to play twice in a row would lower his second-period bid to 0.

I now extend the notion of equilibrium point in a natural way.

**Definition 4.** An *equilibrium point* of the candle auction with bidders called to play at random consists of a family $\left(\left(\tilde{b}_{i,1}, \tilde{b}_{i,2}\right)\right)_{i\in[n]}$ of strategies, one for each bidder, such that

1. For each $i \in [n]$, the first-period bid is optimal given his type and his beliefs about *all* of his competitors, as well as their chances of being drawn to bid in the second period.

2. For each $i \in [n]$, the second-period bid is optimal given $i$'s type and his observation of his own (or someone else's) bid in the first period.

**Theorem 2.** *The set of equilibrium points of the candle auction game (with bidders called to play at random) is characterized by a family* $((b_{i,1}, b_{i,2}))_{i\in[n]}$ *of pairs of functions, one pair per bidder, such that*

1. *For each* $v_i \in [0,1]$*, each player i's first-period bid is given by either* 0 *or the solution to the equation*

$$b_{i,1} = v_i - \frac{q_i + \sum_{j\neq i} q_j F_j(b_{i,1})}{\sum_{j\neq i} q_j f(b_{i,1})} - \frac{\lambda}{(1-\lambda)\sum_{j\neq i} q_j f_j(b_{i,1})}.$$

2. *For each valuation-observed bid pair* $(v_i, b_1) \in [0,1] \times \mathbf{R}_+$*, player i's second-period bid must satisfy*

$$b_{i,2} \in \begin{cases} \{b_1\}, & \text{if } b_1 < v_i; \\ [0, b_1] & \text{if } b_1 = v_i; \\ [0, b_1) & \text{if } b_1 > v_i. \end{cases}$$

*Proof.* Relative to Theorem 1, the only step in the proof of this result that is mildly different is the objective function at $t = 1$, which is what I will focus on.

Define once again $g(\cdot|v_i) : \mathbf{R}_+ \to \mathbf{R}$ as follows:

$$g(b|v_i) := [v_i - b]\left(\lambda + (1-\lambda)\left(q_i + \sum_{j\neq i} q_j F_j(b)\right)\right)$$

At $t = 1$, if bidder $i$ with valuation $v_i$ is called to play, he is faced with the problem of solving

$$\max_{b\in\mathbf{R}_+}\{g(b|v_i)\}.$$

This is just

$$\max_{b \in \mathbf{R}_+} \left\{ \lambda(v_i - b) + (1-\lambda) \left[ q_i(v_i - b) + \sum_{j \neq i} q_j F_j(b)(v_i - b) \right] \right\}$$

Once again, I restrict the search for a maximizer to the compact interval $[0, v_i]$. Assuming an interior solution, $b$ must satisfy the first-order condition

$$-\lambda - (1-\lambda)q_i + (1-\lambda) \sum_{j \neq i} q_j f(b) v_i - (1-\lambda) \sum_{j \neq i} q_j (f(b)b + F(b)) = 0.$$

Thus, $b$ is implicitly determined by

$$b = v_i - \frac{q_i + \sum_{j \neq i} q_j F(b)}{\sum_{j \neq i} q_j f(b)} - \frac{\lambda}{(1-\lambda) \sum_{j \neq i} q_j f(b)}.$$

If the above equation has no solution in $\mathbf{R}_{++}$, then player $i$ should optimally choose 0 as his initial bid, since the objective function is vanishes at $b = v_i$, whereas it is positive for $b = 0$. This concludes the proof. $\square$

Notice that the optimal first-period bid now depends not only on $v_i$ and $\lambda$, but also on the probability of being faced with weak competitors on the second round of bidding (conditional on there being one). More importantly, the expression for the optimal value of $b_{i,1}$ still entails a shading factor that decreases with $\lambda$.

# 5 Discussion

In this dissertation, I developed a simple model of candle auctions, *i.e.*, of dynamic auctions with random termination. Such auctions are not only historically important, but more recently, they have been widely used in stock markets and procurement throughout the world.

In my baseline model, two bidders with private values are called to play one at a time. The initial bidder has a first-mover advantage, since with probability $\lambda$, he may be the sole bidder. The second bidder has a last-mover advantage, since he observes the bid of his adversary and will face no further competition. I show that the initial bidder will choose a bid that entails some degree of shading relative to his valuation, which is typical of first-price auctions. When the valuation distribution satisfies certain regularity conditions, the shading increases with $\lambda$, so candle auctions mitigate the sniping issue at the expense of expected revenue.

There are several fertile avenues for future research in the wake of this dissertation, given that I leave many important questions unanswered.

1. First, there is the matter of allowing for revision opportunities. I believe that the most adequate framework to do so is one of continuous-time revision games, introduced by Kamada and Kandori (2020). In their framework, players in a dynamic game have a Poisson arrival rate of revision opportunities to update their actions. Allowing for a stochastic deadline would make for an interesting model of candle auctions.

2. Second, the private values assumption is hard to justify, especially given my motivation to write this dissertation. In reality, there are likely to be both private and common components to bidder valuations. Extending the model to allow for the more general affiliated values model of Milgrom and Weber (1982) is an important next step.

3. Lastly, even in the absence of revision opportunities, it is important to understand what happens when more time periods are introduced.

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