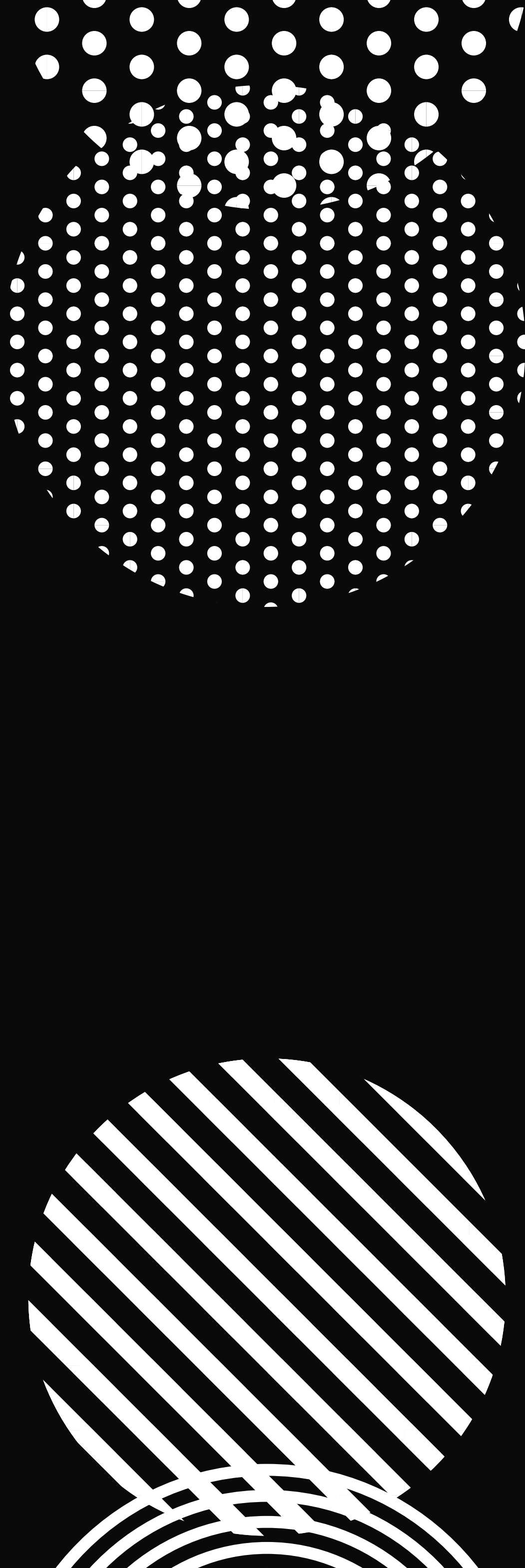

1.

a A sequence $(a_n)_{n \in \mathbb{N}}$ is strictly increasing if $\forall n \in \mathbb{N}$, $a_n < a_{n+1}$

b Yes, a monotonic sequence always has a limit.

If a_n is increasing and bounded the limit is the supremum.

— " — unbounded — " — $+\infty$.

decreasing and bounded — " — infimum.

unbounded — " — $-\infty$.

c. $a_n = n e^{-n} = \frac{n}{e^n}$

$$a_n > a_{n+1} \iff \frac{n}{e^n} > \frac{n+1}{e^{n+1}}$$

$$e \cdot n > n + 1$$

$$n(e-1) > 1$$

$\geq 1 \quad > 1$

a_n is strictly decreasing $\quad a_1 \geq a_n \quad \forall n \in \mathbb{N}$

so $a_1 = \frac{1}{e}$ is the upper bound.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{e^n} = 0$$

$$a_n = \frac{\overset{>0}{n}}{\underset{>0}{e^n}} > 0$$

so 0 is the lower bound.

d). Find b_n such that $\lim\limits_{n\to\infty} a_n \cdot b_n = -\infty$

$$a_n \cdot b_n = \underset{>0}{\frac{n}{e^n}} \cdot \underset{\substack{\text{has to be} \\ <0}}{b_n} \longrightarrow -\infty$$

$b_n = -e^n$

$$\lim_{\infty} a_n \cdot b_n = \lim_{\infty} \frac{n}{e^n}(-e^n) = \lim_{\infty} -n = -\infty$$

Pr. 2.

a). $(a_n)_{n\in\mathbb{N}}$, $(k_n)_{n\in\mathbb{N}}$ be a strictly inc. seq. of natural numbers.

Then (a_{k_n}) is a subseq of a_n.

b). If $\lim_{n\to\infty} a_n = \alpha \in \overline{\mathbb{R}}$, then for any

$(a_{k_n})_{n\in\mathbb{N}}$ $\lim_{n\to\infty} a_{k_n} = \alpha$

Corollary: If $(b_n)_{n\in\mathbb{N}}$, $(c_n)_{n\in\mathbb{N}}$ are subseqs of a_n and the limits are different, then $\lim_{n\to\infty} a_n$ does not exist.

c). $a_n = (1 + (-1)^n)(n+1)$

for odd values n,

$a_n = (1-1)(n+1) = 0$

for even n,

$a_n = (1+1)(n+1) = 2(n+1)$

$b_n = a_{2n+1} = 0$

$\lim_{n\to\infty} b_n = 0$

$$c_n = a_{2n} = 2(n+1)$$

$$\lim_{n \to \infty} c_n = +\infty$$

$$\lim b_n \neq \lim c_n \Rightarrow \lim a_n \nexists$$

d). $a_n = (1 + (-1)^n)(n+1) = O(n)$ for $n \to +\infty$

$$(\exists c > 0)(\exists N \in \mathbb{N})(\forall n \geq N)(|a_n| \leq c|n|)$$

$$\Updownarrow$$

$$a_n \leq n \cdot c$$

$$a_n \leq 2(n+1) = 2n + 2$$

$$\leq 2n + 2n$$

$$\leq 4 \cdot n$$

$$c = 4,\ N = 1$$

Pr. 3.

a. f, $x \in D_f$, x is a pt of global max if

$f(x) \geq f(y)$, $y \in D_f$

b. f, interval $J \subset D_f$

If J is a closed interval and f is continuous on J, then f has global extrema on J.

c. $f(x) = \sqrt{4 - x^2}$, find extrema of f on D_f.

$D_f = [-2, 2]$

f is continuous on D_f. $4 - x^2$ - continuous

$\sqrt{}$ - "

1. End pts.

$f(-2) = f(2) = 0$

2. $f'(x) = \dfrac{1}{2\sqrt{4-x^2}}(-2x) = \dfrac{-x}{\sqrt{4-x^2}}$

$D_{f'} = (-2, 2)$, $f'(x) = 0 \Leftrightarrow x = 0$

$f(0) = 2$

The pts of glob min are $-2, 2$.

$f(-2) = f(2) = 0$

The pt of glob max is 0.

$f(0) = 2$

d). f differentiable on (a,b), $x_0 : f(x_0) = 0$

$$(x_n) \to x_0 , \quad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Pr. 4.

a). f is defined on U_a (neighbourhood of A)

If $\lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$ exists and is finite $(\in \mathbb{R})$

then f is differentiabe at $x = a$.

b). f continuous at $x = 1$ but not differentiable at $x = 1$.

Yes, for ex, $f(x) = |x - 1|$ is continuous at $x = 1$ but not diff.

$$\lim_{x\to 1_+} \frac{|x-1| - \overset{\to 0}{|1-1|}}{x-1} = \lim_{x\to 1_+} \frac{|x-1|}{x-1} = 1$$

$$\lim_{x\to 1_-} \frac{|x-1| - |1-1|}{x-1} = \lim_{x\to 1_-} \frac{\overset{-(x-1)}{|x-1|}}{x-1} = -1$$

$$\lim_{x\to 1} \frac{|x-1| - |1-1|}{x-1} \nexists$$

f is continuous, $\lim_{x\to 1} |x-1| = 0 = |x-1|$

Pr. 4.

c). $f(x) = \dfrac{\tan x^2}{3x^a}$, $a \in \mathbb{R}$. $D_f = \mathbb{R}^+ \setminus \left\{ \sqrt{\frac{\pi}{2} + k\pi},\ k \in \mathbb{N}_0 \right\}$

x^a $x > 0$

or

$x = 0, a > 0$

f is continuous on D_f.

can we extend it by continuity to $x = 0$?

$$\lim_{x\to 0_+} \frac{\tan x^2}{3x^a} = \lim_{x\to 0_+} \frac{\sin x^2}{\cos x^2} \cdot \frac{1}{3x^a}$$

$$= \lim_{x \to 0_+} \frac{\sin x^2}{x^2} \cdot \frac{1}{3 \cos x^2 \cdot x^{a-2}}$$

$$\downarrow$$

$$1$$

$$\lim_{x \to 0_+} \underbrace{\frac{\sin x^2}{x^2}}_{\to 1} \cdot \underbrace{\frac{1}{3 \cos x^2}}_{\to \frac{1}{3}} \cdot x^{2-a}$$

$x^{2-a} \to 1,\ a = 2$

$x^{2-a} \to 0,\ a < 2$

$$\lim_{x \to 0_+} x^{\varepsilon} = \begin{cases} 0, & \varepsilon > 0 \\ 1, & \varepsilon = 0 \\ +\infty, & \varepsilon < 0 \end{cases}$$

If $a = 2$, $\lim_{x \to 0_+} f(x) = \frac{1}{3}$, $\bar{f}(0) = \frac{1}{3}$

If $a < 2$ $\lim_{x \to 0_+} f(x) = 0$, $\bar{f}(0) = 0$

$\bar{f}$ is continuous ext. of f.