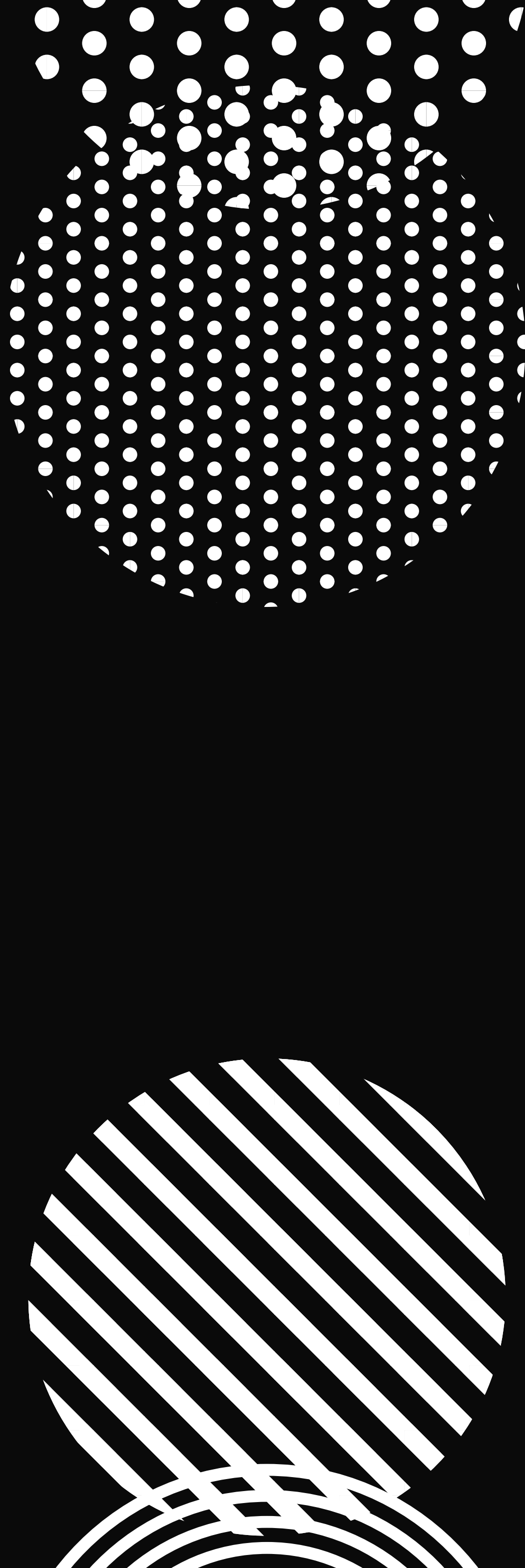


Big O, Small o

big O

$a_n = O(b_n)$, $n \to +\infty$

$\exists C > 0, n_0 \in \mathbb{N}$;

$|a_n| \leqslant C \cdot |b_n|$, $\forall n \geqslant n_0$

Prove this:

$\sqrt{n} + 2\sqrt[3]{n} + n^3 = O(n^3)$ $\exists C > 0, n_0 \in \mathbb{N}$,

$|\sqrt{n} + 2\sqrt[3]{n} + n^3| \leqslant C|n^3|$, $\forall n \geqslant n_0$

$\sqrt{n} + 2\sqrt[3]{n} + n^3 \leqslant n^3 + 2n^3 + n^3 \leqslant 4n^3$, $\forall n \geqslant 1$

?

$$\left.\begin{aligned} \sqrt{n} &< n^3 \\ \sqrt[3]{n} &< n^3 \end{aligned}\right] n \geqslant 1$$

small o,

$$a_n = o(b_n),\ n \to +\infty$$

$\forall C > 0,\ \exists n_0 \in \mathbb{N}:$

$|a_n| < C \cdot |b_n|,\ \forall n > n_0$

Prove that

$$\frac{1}{n} = o\left(\frac{2}{\sqrt[3]{n}}\right),\ \forall C > 0,\ \exists n_0$$

$$\frac{1}{n} < \frac{2}{\sqrt[3]{n}} \cdot C$$

$$\frac{n}{n^{\frac{1}{3}}} = n^{1-\frac{1}{3}} = n^{2/3}$$

$$\frac{1}{n} \overset{?}{<} C\frac{2}{\sqrt[3]{n}}$$

$$n > \frac{\sqrt[3]{n}}{2 \cdot C},\ \forall C > 0$$

$$n^{2/3} > \frac{1}{2 \cdot C} \Leftrightarrow n > \frac{1}{(2C)^{3/2}} = n_0$$

$$\frac{1}{n} < \frac{2}{\sqrt[3]{n}} C$$

3.2. c)

$$a_n = \frac{1}{2n-5}$$ Is monotonic?

Upper / Lower bound

$a_{n+1} > a_n$? $\forall n > k$

$$\frac{1}{2n+2-5} - \frac{1}{2n-5} > 0 \ ?$$

$$\frac{1}{2n-3} - \frac{1}{2n-5} > 0$$

$$\frac{\cancel{2n} - 5 - \cancel{2n} + 3}{(2n-3)(2n-5)} > 0$$

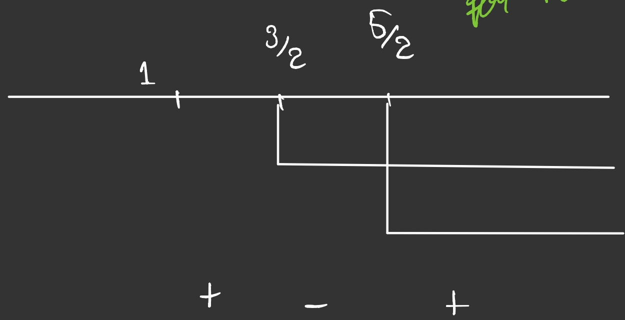
$$\frac{-2}{(2n-3)(2n-5)} > 0 \ ?$$

$$\frac{2}{(2n-3)(2n-5)} < 0$$

$2n-3>0 \Leftrightarrow n>3/2$

$2n-5>0 \Leftrightarrow n>5/2$

for $n>\frac{5}{2}$, the seq is decreasing



$\therefore$ The seq is eventually decreasing

Upper bound

$a_{n+1} < a_n \quad \forall n \geq 3$

lower bound, upper bound

$\underset{\substack{\uparrow \\ -1}}{M_1} \leq a_n \leq \underset{\substack{\uparrow \\ 1}}{M_2}$

$a_3 > a_4 > a_5 > \dots$

$a_1 = -\frac{1}{3}, \; a_2 = -1, \; a_3 = 1$

lower bound

$$\lim_{n\to+\infty} \frac{1}{\underbrace{2n-5}_{+\infty}} = \frac{1}{+\infty} = 0$$

$$\lim_{n \to +\infty} a_n = c \in \mathbb{R}$$

$\forall \varepsilon > 0, \exists n_0$ such that

$$a_n \in (c-\varepsilon, c+\varepsilon), \forall n > n_0$$

$$\Updownarrow$$

$$|a_n - c| < \varepsilon$$

$$\lim_{n \to +\infty} a_n \neq c$$

$$\forall n_0, \exists \bar{\varepsilon}, \exists n > n_0$$

$$|a_n - c| \geq \bar{\varepsilon}$$

$$\lim_{n \to +\infty} \frac{\overbrace{6n^2 - n + \sqrt{n}}^{+\infty - \infty}}{\underbrace{3n^2 + \sqrt[3]{n}}_{\to +\infty}} = \lim_{n \to +\infty} \frac{\overset{2}{\cancel{6}}\cancel{n^2}}{\cancel{3}\cancel{n^2}} = 2$$

$$= \lim_{n \to +\infty} \frac{\cancel{n^2}\left(6 - \overset{0}{\frac{1}{n}} + \overset{0}{\frac{1}{n\sqrt{n}}}\right)}{\cancel{n^2}\left(3 + \overset{0}{\frac{1}{n^{2-\frac{1}{3}}}}\right)} = 2$$

3.11. $$\lim_{n\to+\infty} \underset{+\infty}{n}\,\underbrace{\left(\underset{+\infty}{\sqrt{n^2+1}} - \underset{+\infty}{n}\right)}_{\infty - \infty}$$

$$\lim_{n\to+\infty} \left(\sqrt{n^2+1} - n \cdot \frac{(\sqrt{n^2+1}+n)}{(\sqrt{n^2+1}+n)}\right) n$$

$$\lim_{n\to+\infty} \frac{(\cancel{n^2}+1-\cancel{n^2})\overset{+\infty}{n}}{\underbrace{\sqrt{n^2+1}+n}_{+\infty}} = \lim_{n\to+\infty} \frac{\cancel{n}}{\cancel{n}\left(\sqrt{1+\underset{\to 0}{\frac{1}{n^2}}}+1\right)}$$

$$= \frac{1}{2}$$

$$\lim_{n\to+\infty} \frac{\left(\frac{1}{3}\right)^n - \left(\frac{1}{5}\right)^n}{\left(\frac{1}{3}\right)^{n^2+1} + \left(\frac{1}{3}\right)^{n^3-n}} = \frac{0}{0}$$

$\left(\frac{1}{3}\right)^n \to 0$, $\left(\frac{1}{5}\right)^n \to 0$, $n^2+1 \to \infty$, $n^3-n \to \infty$

$\underbrace{\left(\frac{1}{3}\right)^{n^2+1}}_{\to 0}$, $\underbrace{\left(\frac{1}{3}\right)^{n^3-n}}_{\to 0}$

$$\lim_{n\to+\infty} a^n = 0$$

$\uparrow$

$a < 1$

$$\left[\left(\frac{1}{3}\right)^n > \left(\frac{1}{5}\right)^n\right] \to \text{dominant term} - \left(\frac{1}{3}\right)^n$$

$$\lim_{n\to+\infty} \frac{\left(\frac{1}{3}\right)^n \left(1 - \left(\frac{3}{5}\right)^n\right)}{\left(\frac{1}{3}\right)^{n^2+1}\left(1 + \left(\frac{1}{3}\right)^{n^3-n^2-1-n}\right)}$$

$\left(\frac{3}{5}\right)^n \to 0$, $\left(\frac{1}{3}\right)^{n^3-n^2-1-n} \to 0$

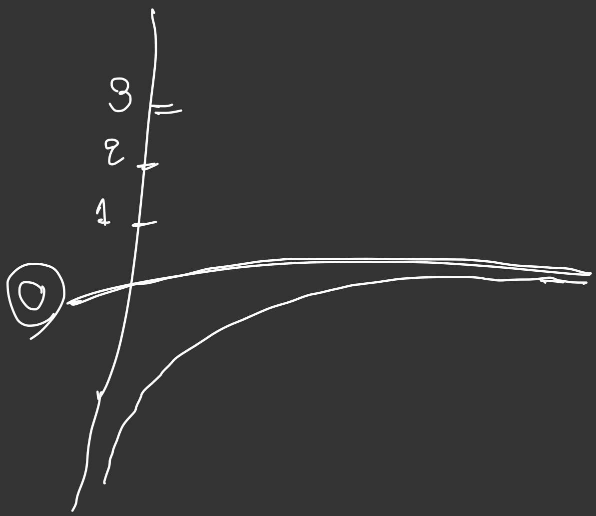
$$n^3 - n^2 - 1 - n = n^3\left(1 - \frac{1}{n} - \frac{1}{n^3} - \frac{1}{n^2}\right) \to +\infty$$

$\frac{1}{n} \to 0$, $\frac{1}{n^3} \to 0$, $\frac{1}{n^2} \to 0$

$$\lim_{n\to+\infty} \frac{\left(\frac{1}{3}\right)^n}{\left(\frac{1}{3}\right)^{n^2+1}} = \lim_{n\to+\infty} \left(\frac{1}{3}\right)^{-n^2+n-1} = +\infty$$

$-n^2+n-1 \to -\infty$

$$a < 1, \quad \frac{1}{a^{+\infty}} \to +\infty$$



3
2
1
0

$$\lim_{n\to+\infty} \frac{(-1)^n n^2 + \sqrt[4]{n} + \cos(n^{10}) + 3^{-n}}{n! + n^2}.$$

$$\lim_{n\to+\infty} \frac{(-1)^n n^2}{n!\left(1+\frac{n^2}{n!}\right)} + \frac{n^{\frac{1}{4}}}{n!\left(1+\frac{n^2}{n!}\right)} + \frac{\cos(n^{10})}{n!\left(1+\frac{n^2}{n!}\right)} + \frac{\frac{1}{3^n}}{n!\left(1+\frac{n^2}{n!}\right)}$$

$$\frac{n^2}{n!} \to 1 \qquad \frac{n^{\frac{1}{4}}}{n!} \to 0 \qquad \frac{1}{3^n} \to 0$$

$$\underset{\downarrow\atop 0}{\frac{-n^2}{n!}} \leq \underset{\downarrow\atop 0}{\frac{(-1)^n n^2}{n!}} \leq \underset{\downarrow\atop 0}{\frac{n^2}{n!}}$$

$$\underset{\downarrow\atop 0}{\frac{-1}{n!}} \leq \underset{\downarrow\atop 0}{\frac{\cos(n^{10})}{n!}} \leq \underset{\downarrow\atop 0}{\frac{1}{n!}}$$

$$= 0 + 0 + 0 + 0 = 0$$