DSP Take-Home Test

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Question 1

Consider the following discrete systems (not pictured):

List which of these are

a) Linear

B, E, F, G are linear.

A includes absolute-value => non linear.

C:

$$T\{\mathrm{az}_n + \mathrm{bu}_n\} = \frac{3(\mathrm{az}_{n-1} + \mathrm{bu}_{n-1}) + (\mathrm{az}_{n-2} + \mathrm{bu}_{n-2})}{\mathrm{az}_{n-3} + \mathrm{bu}_{n-3}}$$

$$\mathrm{aT}\{z_n\} + \mathrm{bT}\{u_n\} = a\left(\frac{3z_{n-1} + z_{n-2}}{z_{n-3}}\right) + b\left(\frac{3u_{n-1} + u_{n-2}}{u_{n-3}}\right) \neq T\{\mathrm{az}_n + \mathrm{bu}_n\}$$

D:

$$T\{\mathrm{az}_n + \mathrm{bu}_n\} = \prod_{i=0}^{8}(\mathrm{az}_{n-i} + \mathrm{bu}_{n-i})$$

$$\mathrm{aT}\{z_n\} + \mathrm{bT}\{u_n\} = a\prod_{i=0}^{8} z_{n-i} + b\prod_{i=0}^{8} u_{n-i} \neq T\{\mathrm{az}_n + \mathrm{bu}_n\}$$

H:

$$T\{\mathrm{az}_n + \mathrm{bu}_n\} = (\mathrm{az}_n + \mathrm{bu}_n) + \cos(0.7\pi n)$$

$$\mathrm{aT}\{z_n\} + \mathrm{bT}\{u_n\} = \mathrm{az}_n + \mathrm{bu}_n + 2\cos(0.7\pi n) \neq T\{\mathrm{az}_n + \mathrm{bu}_n\}$$

b) Time-Invariant

A, C, D are time invariant.

B:

$$y_{n-d} = 0.5(x_{2n-d} + x_{2n+1-d})$$

$$T\{x_{n-d}\} = 0.5(x_{2(n-d)} + x_{2(n-d)+1}) = 0.5(x_{2n-2d} + x_{2n-2d+1}) \neq y_{n-d}$$

E: $y_{n-d} = \sum_{i=-\infty}^{\infty} x_i * \delta_{i-n+d+2}$
$T\{x_{n-d}\} = \sum_{i=-\infty}^{\infty} x_{i-d} * \delta_{i-n+2} = y_n \neq y_{n-d}$

F: $y_{n-d} = x_{n-d} e^{\frac{(n-d)}{14}}$
$T\{x_{n-d}\} = x_{n-d} e^{\frac{n}{14}} \neq y_{n-d}$

G: $y_{n-d} = x_{n-d} u_{n-d}$
$T\{x_{n-d}\} = x_{n-d} u_n \neq y_{n-d}$

H: $y_{n-d} = x_{n-d} + \cos(0.7\pi(n-d))$
$T\{x_{n-d}\} = x_{n-d} + \cos(0.7\pi n) \neq y_{n-d}$

c) Causal

C, D, F, G, H are causal.

A, B, E include references to $x_k \text{ for } k > n$ => non-causal

d) Memoryless

F, G, H are memoryless.

A, B, C, D, E include references to $x_k \text{ for } k \neq n$ => use memory

Question 2

A finite-support sequence is non-zero only at a finite number of positions. If m and n are the first and last non-zero positions, respectively, then we call $n - m + 1$ the length of that sequence. What maximum length can the result of convolving two sequences of length k and l have?

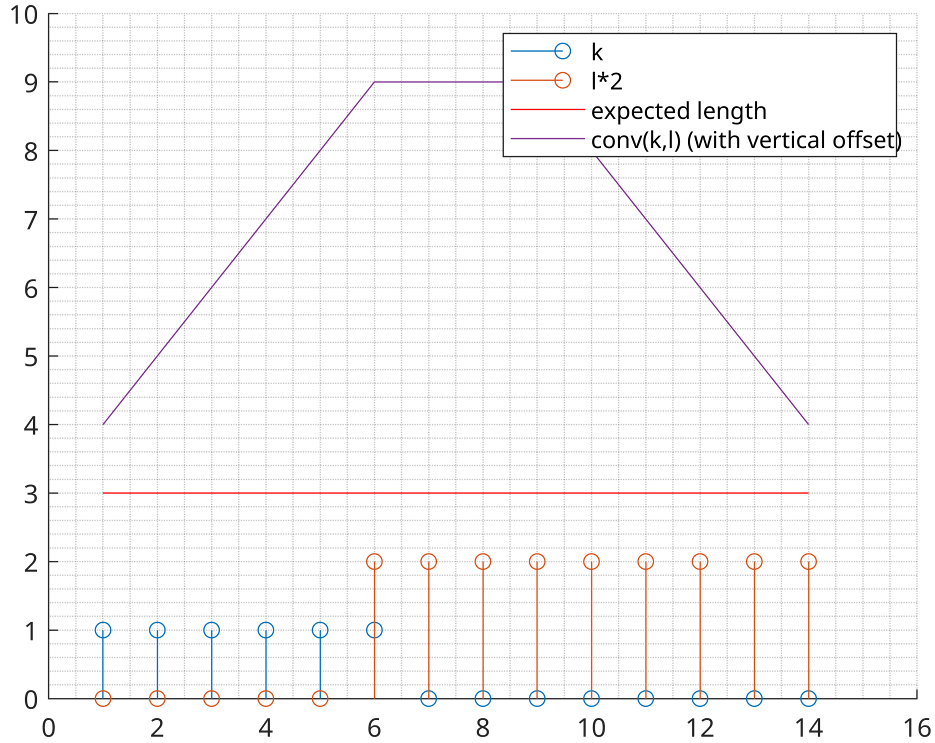
The maximum length of the convolution = the distance between the farthest two ends of both sequences when they overlap by exactly one element = $k + l - 1$

Example:

```
k = 6;
l = 9;
ks = [ones(1,k) zeros(1,l-1)];
ls = [zeros(1,k-1) ones(1,l)];
len = k + l - 1
```

```
len = 14
```

```
figure;
hold on;
stem(ks); stem(ls*2);
plot([1 len],[3 3],'r');
plot(3+conv(ones(1,k),ones(1,l)));
hold off;
legend("k","l*2","expected length","conv(k,l) (with vertical offset)")
ylim([0,10]); xlim([0,len+2]); grid minor;
```



Question 3

a) Find a pair of sequences $\{a_n\}$ and $\{b_n\}$, where each one contains at least three different values and where the convolution $\{a_n\} * \{b_n\}$ results in the all-zero sequence ...,0,0,0,....

Convolution of a,b = $c_n = \sum_{k=-\infty}^{\infty} a_k \cdot b_{n-k}$

Note: one of a, b must be infinite. PROOF

assume a,b are both finite and nonzero. without loss of generality, assume $a_n, b_n = 0 \; \forall \; n < 0$ and $a_0, b_0 \neq 0$. (e.g. that a,b both "start" at n=0.)

$$c_0 = \sum_{k=-\infty}^{\infty} a_k \cdot b_{-k} = a_0 b_0 + a_1 b_{-1} + a_2 b_{-2} \ldots = a_0 b_0 \; (\text{as } b_{-n} = 0)$$

$$a_0, b_0 \neq 0 \Rightarrow c_0 \neq 0$$

For finite nonzero a,b the convolution c will always have a nonzero tail.

Question 6's filter is equivalent to a convolution with the finite sequence [...0 1 -1 1 -1 0...] (which contains 3 different values {0,1,-1})

Question 6's filter eliminates frequencies of $\frac{f_s}{4}$ = an infinite sequence of [1 0 -1 0 1 0 -1 0]... (again, contains 3 different values {0,1,-1})

$a_n = (n \bmod 2) * (-1)^{\mathrm{floor}\left(\frac{n}{2}\right)}$ =[0 1 0 -1 0 1 0 -1]...

b_n =[0 1 -1 1 -1 0]

$\{c_n\} = \{a_n\} * \{b_n\}, c_n = 0 \,\forall\, n$

b) Let T be an LTI system with an impulse response that is not the all-zero sequence. Does every such T have a corresponding inverse LTI system T^{-1} such that $\{x_n\} = T^{-1}T\{x_n\}$ for all sequences $\{x_n\}$? Why or why not?

No.

See Question 6 for an LTI system with nonzero impulse response that can completely eliminate a specific frequency (in this case, for sampling frequency 8kHz, it can eliminate 2kHz).

For this system, all inputs consisting of a 2kHz signal at any phase and nonzero amplitude will produce a zero result.

There cannot be an LTI system that can distinguish between these all-zero results to reproduce the specific original signals

=> this LTI system is uninvertible

=> not all LTI systems with nonzero impulse response are invertible.

Question 4

a) Which of 16kHz, 32kHz, 48kHz would be suitable for distinguishing two whistle tones of 23kHz, 25kHz?

Only 32kHz could distinguish these.

Given a sample frequency f_s, you can reconstruct a signal with frequencies between $\frac{n * f_s}{2} < |f| < \frac{(n+1) * f_s}{2}$ for a specific n.

To distinguish the signals, this range must include 23kHz and 25kHz:

$f_S = 16\text{kHz} : 0 - 8;\ 8 - 16;\ 16 - 24;\ 24 - 32\ldots$ (no range with 23 and 25)

$f_S = 48\text{kHz} : 0 - 24;\ 24 - 48\ldots$ (no range with 23 and 25)

$f_S = 32\text{kHz} : 0 - 16;\ 16 - 32$ (has a range with 23 and 25 for $n = 1$)

b) For each of the two whistles: for which k will $|X_k|$ show a peak?

Each whistle emits a sine tone => when one whistle blows, $x = \sin(2\pi \mathrm{ft})$

When sampling, $x_i = \sin\left(\frac{2\pi \mathrm{fi}}{f_s}\right) = \sin\left(2\pi f * \frac{i}{n} * \frac{n}{f_s}\right)$

FFT peaks when this aligns with $e^{-2\pi j \frac{\mathrm{ik}}{n}}$ =>

$2\pi \frac{i}{n} k = 2\pi f * \frac{i}{n} * \frac{n}{f_s}$

$k = \frac{f * n}{f_s}$

$(f_s = 160\mathrm{kHz})$

Therefore for $f = 23\mathrm{kHz}, k = \frac{23 * 512}{160} = 73.6\ (\text{rounds to } 74)$. (this doesn't align with an FFT bucket perfectly => will scallop)

For $f = 25\mathrm{kHz}, k = \frac{25 * 512}{160} = 80$.

c) Which tones will cause a higher peak in the FFT result, and why?

As mentioned, the 23kHz signal does not align with an FFT bucket. This means it will not peak sharply on one bucket, but will scallop - the peak will spread to the buckets around the middle.

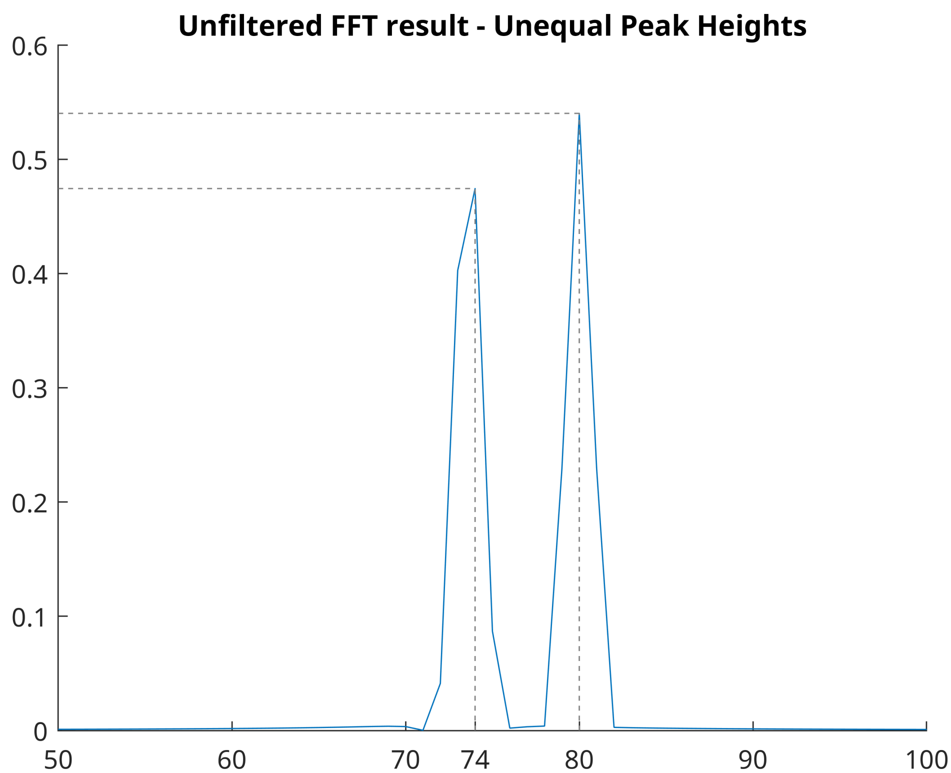
25kHz does align with an FFT bucket, so that peak will be higher than the scalloped 23kHz.

Proof:

```
f_s = 160E3;
f1 = 23E3;
f2 = 25E3;
block_size = 512;
blocks_per_s = f_s / block_size;
t_per_block = block_size / f_s;

t = (1/f_s):(1/f_s):t_per_block;
sampled_signal = sin(2*pi*f1*t) + sin(2*pi*f2*t);

% Adapted from https://uk.mathworks.com/help/matlab/ref/fft.html
% Compute the two-sided spectrum P2, with a hamming window.
% Then compute the single-sided spectrum P1 based on P2 and
% the even-valued signal length L.
Y=fft(sampled_signal .* hamming(block_size)');
L=block_size;
P2 = abs(Y/L); % Normalize impact of FFT length
% P2 is two-sided eg. includes negative frequencies
% but P2 is also symmetric
% => one-sided version is 2*(first half of P2)
```

```
P1 = P2(1:L/2);
P1(2:end) = 2*P1(2:end);
```

```
figure;
hold on;
plot(0:1:(L/2-1), P1);
% Plot lines at expected peaks
% MATLAB vectors are one-indexed, so the indices are 75 and 81
% y axis
plot([0,74],[P1(75),P1(75)],'--',"color",[0,0,0]+0.5);
plot([0,80],[P1(81),P1(81)],'--',"color",[0,0,0]+0.5);
% x axis
plot([74,74],[0,P1(75)],'--',"color",[0,0,0]+0.5);
plot([80,80],[0,P1(81)],'--',"color",[0,0,0]+0.5);
% Limit to interesting part of graph
xlim([50 100]);
xticks([50 60 70 74 80 90 100]);
title("Unfiltered FFT result - Unequal Peak Heights");
hold off;
```

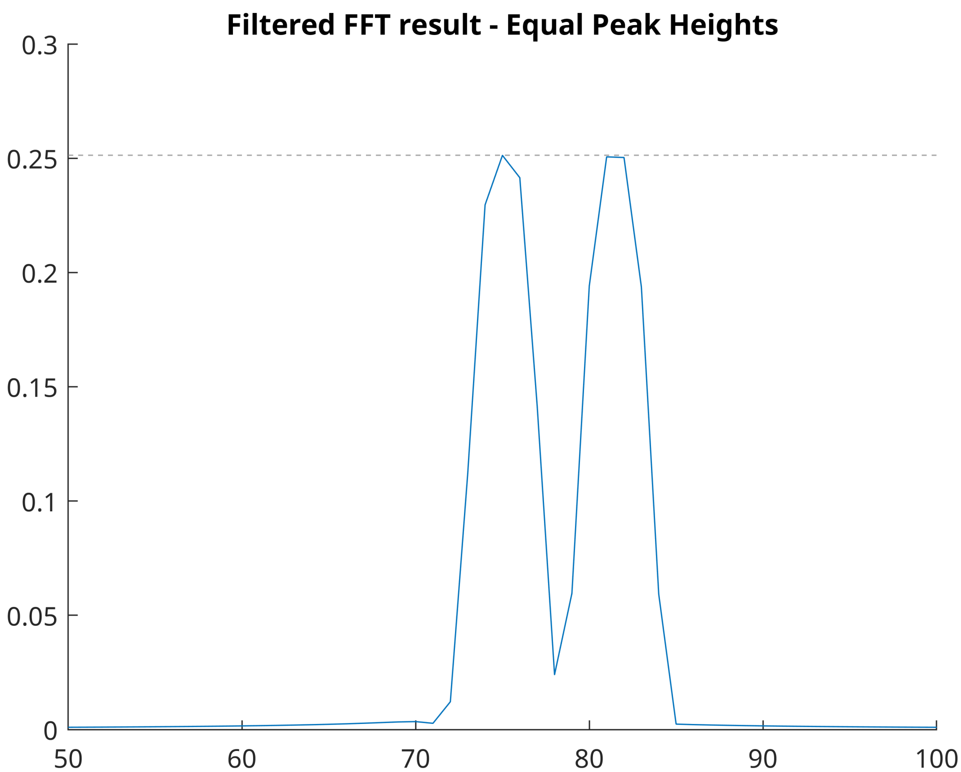


d) What additional linear processing step can you apply, and where, to reduce the relative amplitude difference between 23kHz and 25kHz?

Apply a linear "4-point moving average" filter to the FFT result.

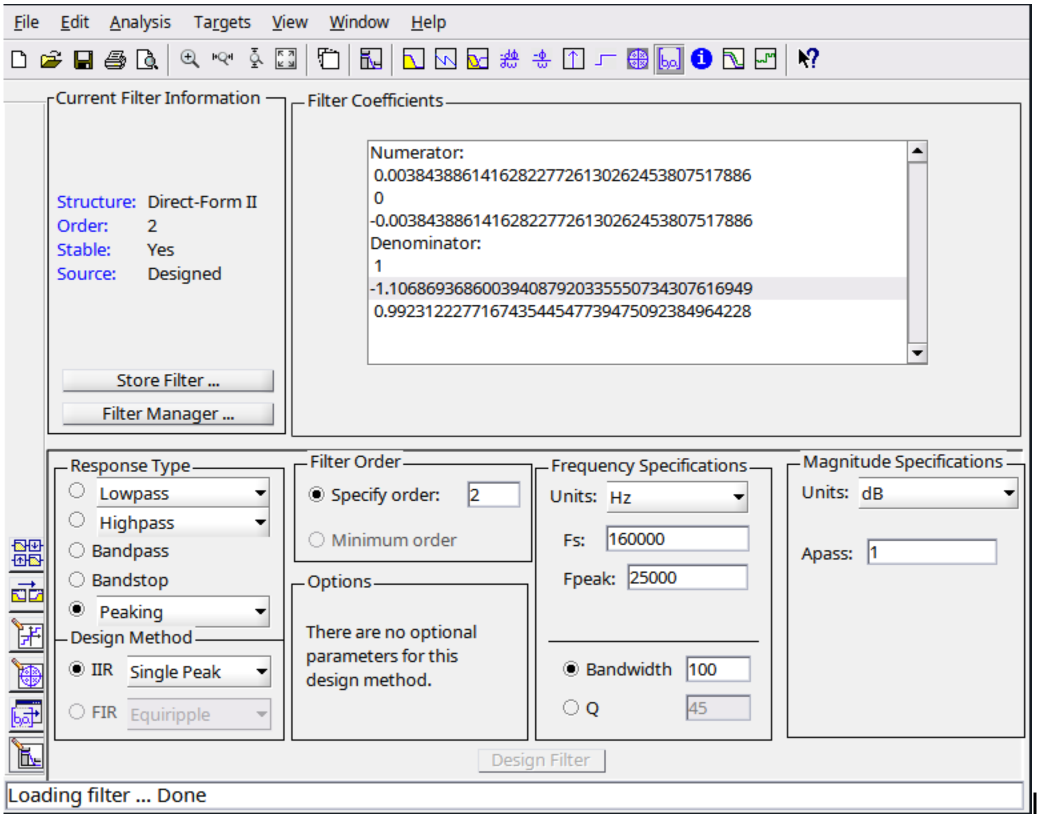
The scalloping distributes the peak among the adjacent buckets, and the moving average serves to pull them back together.

```
b=[1 1 1 1]/4;
a=[1];
P1_smoothed = filter(b, a, P1);

figure;
hold on;
plot(0:1:(L/2-1), P1_smoothed);
% The peaks won't be at exactly 74 or 80 now, though
yline(max(P1_smoothed),'--',"","color",[0,0,0]+0.5);
% Limit to interesting part of graph
xlim([50 100]);
title("Filtered FFT result - Equal Peak Heights");
hold off;
```

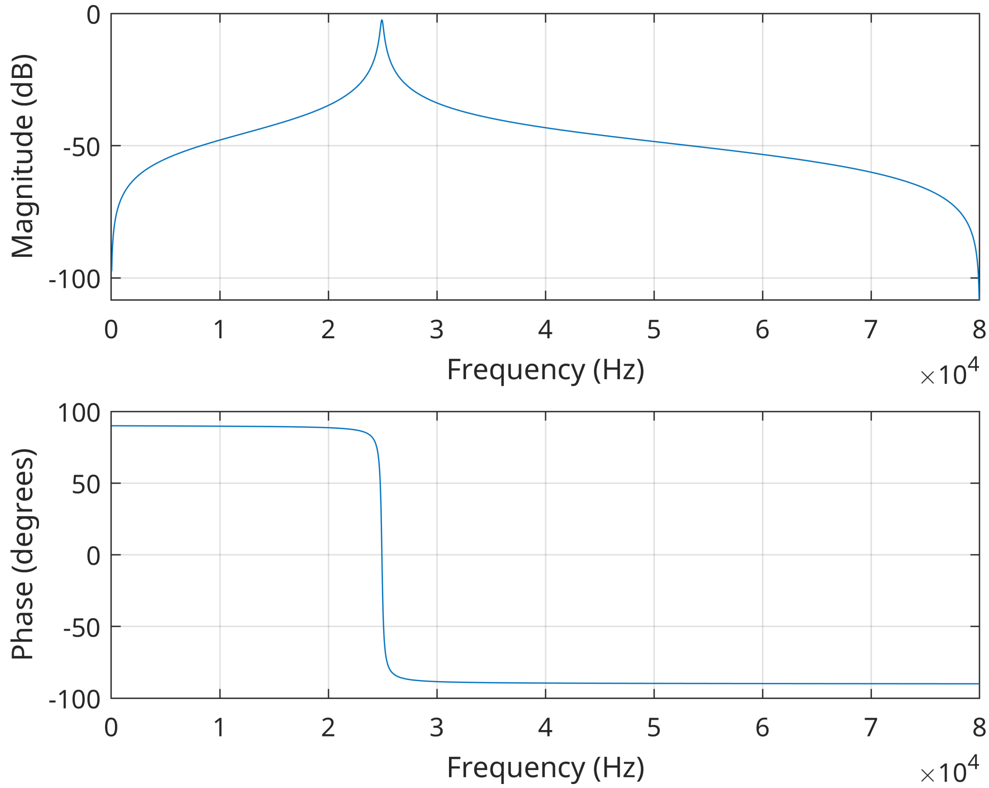


e) Give an example of a second-order peak filter, operating at f_s=160kHz with pass frequency f=25kHz, in terms of the position of poles and zeros in the z-transform of its impulse response, providing an expression for H(z)

First, I designed one with MATLAB Filter Designer:




```
f_s = 160000;
b = [3.84E-3 0 -3.84E-3];
a = [1 -1.11 0.99];
freqz(b,a,2048,f_s);
```



Then, I derived my own to get an expression for H(z):

Peak at 25kHz => Put a pole at $0.99\angle 2\pi \frac{f}{f_s}$ (and at the conjugate to cancel out the complex part).

Use 0.99 instead of 1 to dampen the response - using 1 would give infinite gain at 25kHz.

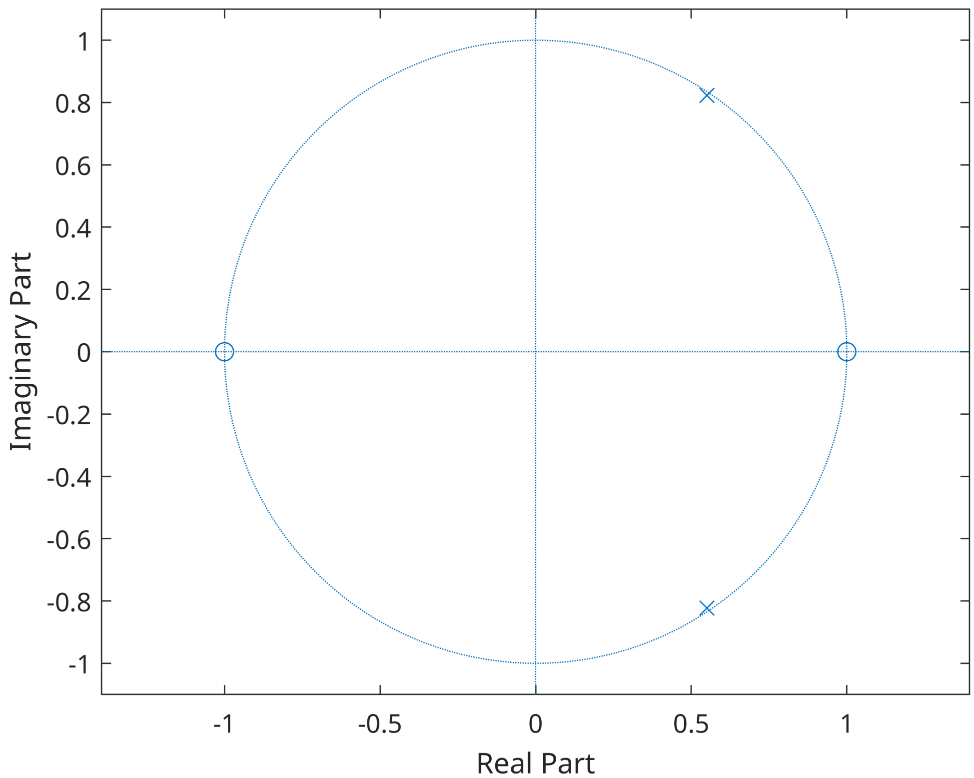
Second-order => two zeroes

Put two zeroes at -1, 1, as shown in Slide 152

```
ps = [exp(-1i*2*pi*25/160) conj(exp(-1i*2*pi*25/160))]*0.99;
zs = [1 -1];
ps, zs
```

```
ps = 1×2 complex
   0.5500 - 0.8232i   0.5500 + 0.8232i
zs = 1×2
     1    -1
```

```
zplane(zs', ps')
```



$$H(z) = \frac{b_0}{a_0} \cdot z^{2-2} \cdot \frac{(z-c_1)(z-c_2)}{(z-d_1)(z-d_2)}$$

$$= k \cdot 1 \cdot \frac{(z-1)(z+1)}{(z-(0.55-0.8232i))(z-(0.55+0.8232i))}$$

$$= k \cdot \frac{z^2-1}{z^2-1.1z+0.99^2}$$

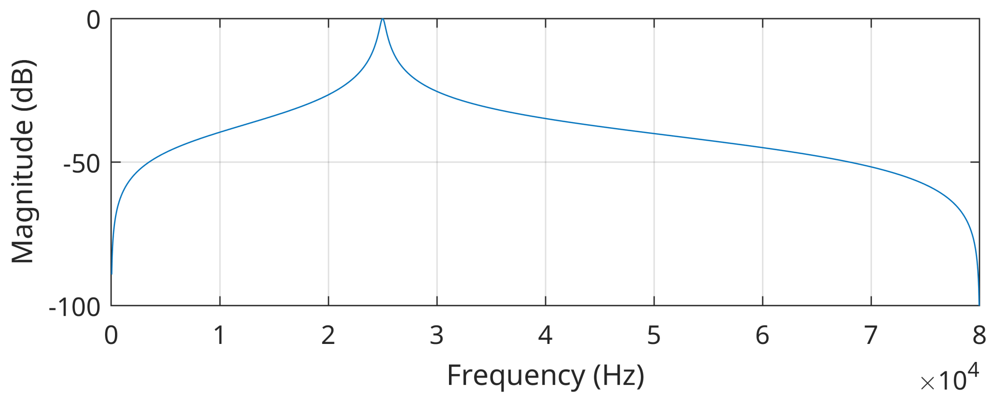
The gain k was chosen by the Filter Designer to be 0.00384, presumably to take the peak down to 0dB.

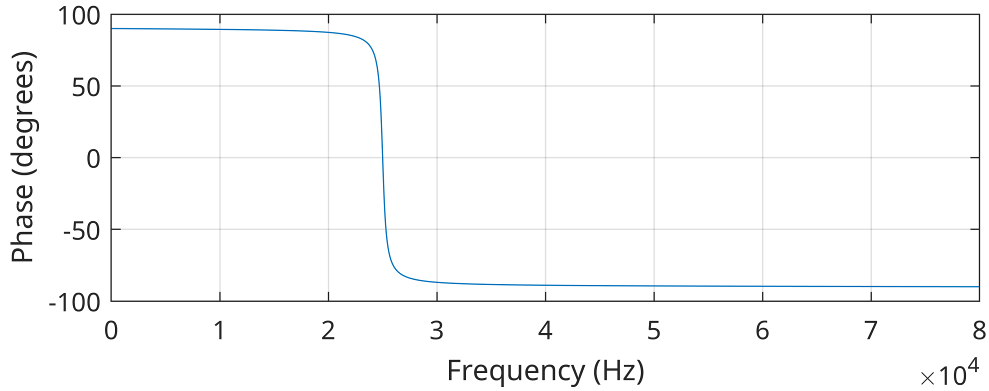
The peak for the manual calculation was 40dB, so we take k=0.01 to bring that to 0dB.

```
k = 0.01;
b = [1 0 -1] * k;
a = [1 -1.1 0.99^2];
b, a
```

```
b = 1×3
    0.0100         0   -0.0100
a = 1×3
    1.0000   -1.1000    0.9801
```

```
freqz(b,a,2048,f_s);
```





Question 5

$Y(z) = U(z) + b * X(z) * z^{-2}$

$U(z) = X(z) + c * U(z) * z^{-1} + a * X(z) * z^{-1}$

$\Rightarrow U(z)(1 - c * z^{-1}) = X(z)(1 + a * z^{-1})$

$$\Rightarrow U(z) = X(z) * \frac{1 + \mathrm{az}^{-1}}{1 - \mathrm{cz}^{-1}}$$

$$\Rightarrow Y(z) = X(z) * \frac{1 + \mathrm{az}^{-1}}{1 - \mathrm{cz}^{-1}} + X(z) * \mathrm{bz}^{-2}$$

$$\Rightarrow Y(z) = X(z) * \left(\frac{1 + \mathrm{az}^{-1}}{1 - \mathrm{cz}^{-1}} + \mathrm{bz}^{-2}\right)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + \mathrm{az}^{-1}}{1 - \mathrm{cz}^{-1}} + \mathrm{bz}^{-2}$$

Question 6

$y_n + y_{n-1} = x_n - x_{n-4}$

$\Rightarrow Y(z)(1 + z^{-1}) = X(z)(1 - z^{-4})$

$$\Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{(1 - z^{-4})}{(1 + z^{-1})} = \frac{(1 + z^{-2})(1 - z^{-2})}{(1 + z^{-1})} = \frac{(1 + z^{-2})(1 - z^{-1})(1 + z^{-1})}{(1 + z^{-1})}$$

$\Rightarrow H(z) = (1 + z^{-2})(1 - z^{-1}) = 1 + z^{-2} - z^{-1} - z^{-3} = z^0 - z^{-1} + z^{-2} - z^{-3}$

$h_0 = 1, h_1 = -1, h_2 = 1, h_3 = -1$

$y_n = x_n - x_{n-1} + x_{n-2} - x_{n-3}$

a) What is the impulse response of h?

$h\{\delta_n\} = \{1, -1, 1, -1, 0, 0...\}$

b) What is the step response of h?

for $n = 3$ onwards, $u_n, u_{n-1}, u_{n-2}, u_{n-3}$ all equal 1 => for $y_n = h\{u_n\}$, y_3 onwards = 0

similar calculations for $n = 0, 1, 2$. y is causal, so $y_n = 0 \, \forall \, n < 0$

$h\{u_n\} = \{1, 0, 1, 0, 0...\}$

c) Express as H(z)

See above for working: $H(z) = \frac{(1 - z^{-4})}{(1 + z^{-1})} = z^0 - z^{-1} + z^{-2} - z^{-3}$

d) Can you eliminate a common factor from numerator and denominator?

See above for working - eliminated common factor of $(1 + z^{-1})$

The final H(z) has no denominator => it is a Finite Impulse Response filter, not an Infinite Impulse Reponse filter.

e) Write H as a fraction using the position of its poles and zeros and draw their location in relation to the complex circle

$$H(z) = 1 - z^{-1} + z^{-2} - z^{-3} = \frac{z^3 - z^2 + z^1 - 1}{z^3} = \frac{z^0}{z^3} \cdot \frac{z^3 - z^2 + z^1 - 1}{1}$$

$\Rightarrow k = 0, m = 3$

3 zeros at nonzero locations in z

0 poles at nonzero locations in z

3 poles at z=0

Cubic function solver: $z^3 - z^2 + z^1 - 1$

```
zs = roots([1 -1 1 -1])
```

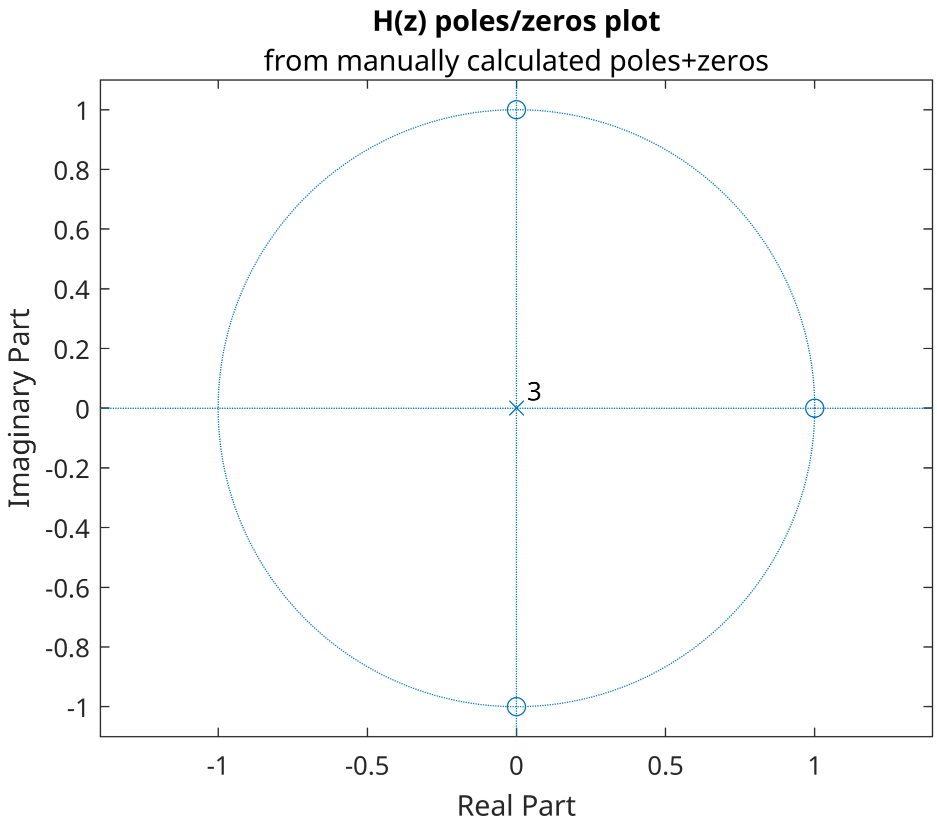
```
zs = 3×1 complex
   1.0000 + 0.0000i
   0.0000 + 1.0000i
   0.0000 - 1.0000i
```

$z = 1, j, -j \Rightarrow H(z) = 0$

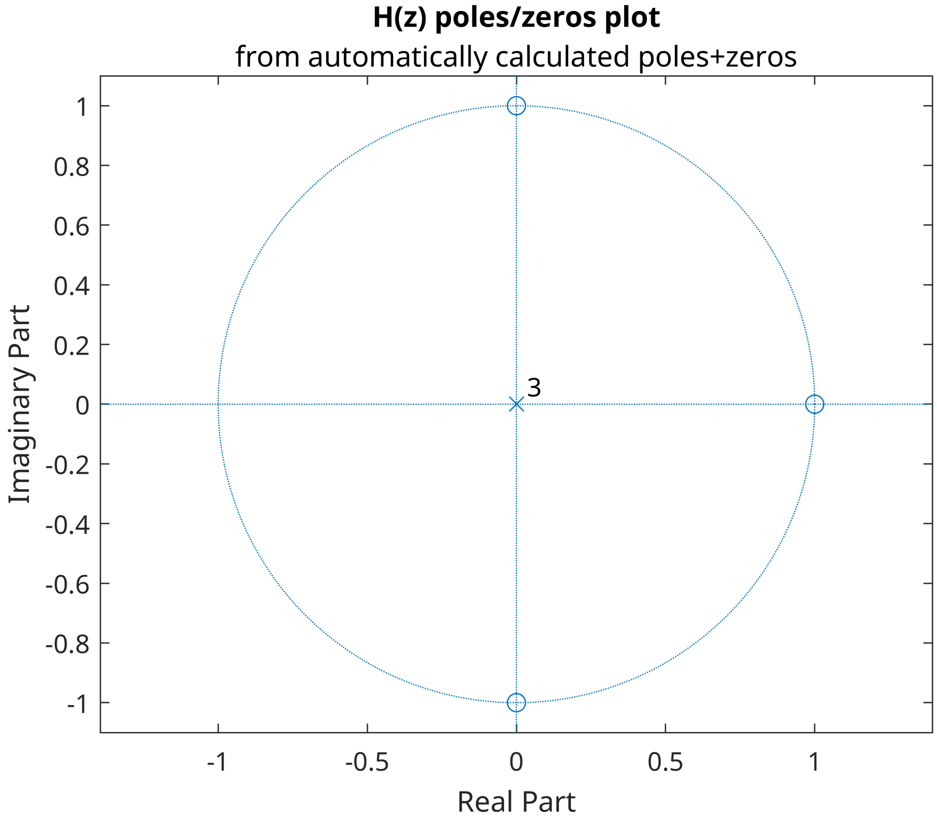
$$H(z) = \frac{b_0}{a_0} \cdot z^{-3} \cdot \frac{\prod_{l=1}^{3} (z - c_l)}{1} = \frac{(z-1)(z-j)(z+j)}{z \cdot z \cdot z}$$

```
% Plot poles, zeroes on unit circle
figure;
```

```
zplane([1 1i -1i]', [0 0 0]');
title("H(z) poles/zeros plot");
subtitle("from manually calculated poles+zeros");
```




```
% For double checking - make zplane calculate the poles, zeros from the
% transfer function.
% Should match the previous one
zplane([1 -1 1 -1], [1 0 0 0]);
title("H(z) poles/zeros plot");
subtitle("from automatically calculated poles+zeros");
```



f) If h is applied to file with sampling frequency 8000Hz, what frequency will be eliminated and what frequency will be quadrupled in amplitude?

$$A(f) = \left|H\left(e^{j2\pi\frac{f}{f_s}}\right)\right|$$

$$f_s = 8000$$

$$H(z) = (1 + z^{-2})(1 - z^{-1})$$

$$\Rightarrow |H(z)| = 0 \text{ when } z^{-1} = 1 \text{ or } z^{-2} = -1$$

$$\Rightarrow z = 1 \text{ or } z = \pm j$$

$$\Rightarrow e^{j2\pi\frac{f}{f_s}} = 1 \text{ or} \pm j$$

$$\Rightarrow f = 0 \text{ or} \pm j2\pi\frac{f}{f_s} = j\frac{\pi}{2}$$

Ignoring trivial $f = 0$ case...

$$\pm 2\frac{f}{f_s} = \frac{1}{2} \Rightarrow f = \pm\frac{f_s}{4} = \pm\frac{8000}{4} = \pm 2000$$

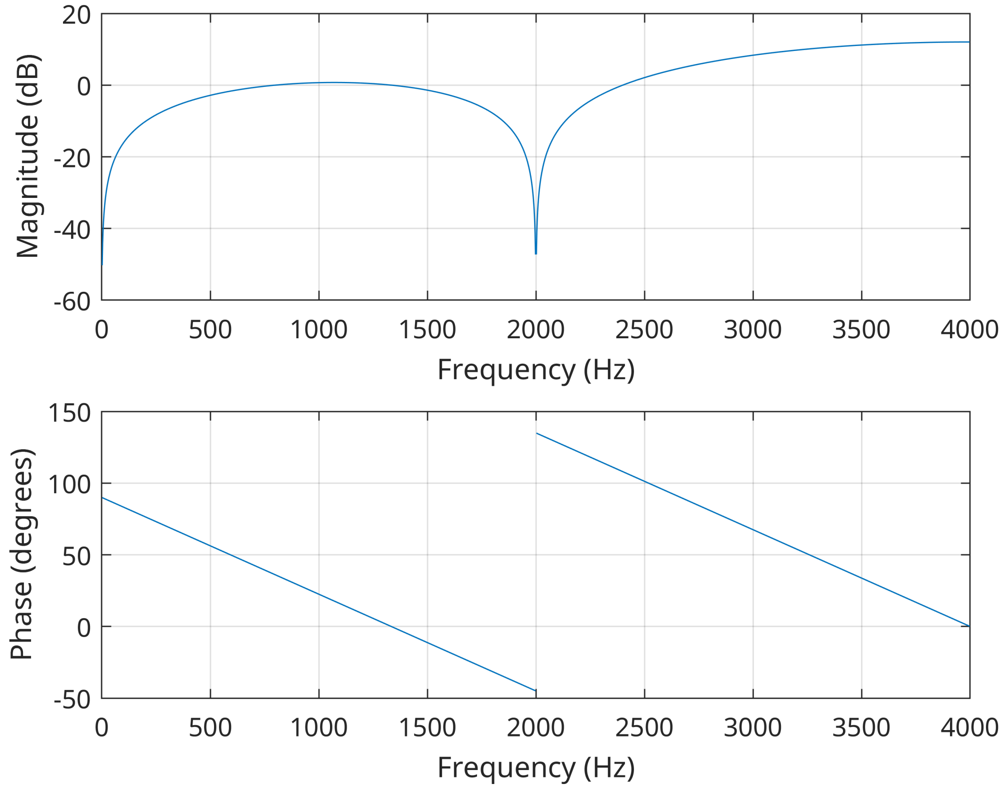
For quadrupled amplitude...

$$|H(z)| = 4 = |1 - z^{-1} + z^{-2} - z^{-3}|$$
$$\Rightarrow z^{-1} = -1 \Rightarrow z = -1$$
$$\Rightarrow e^{\mathrm{j}2\pi \frac{f}{f_s}} = -1$$
$$\Rightarrow \mathrm{j}2\pi \frac{f}{f_s} = j\pi$$
$$\Rightarrow 2\frac{f}{f_s} = 1$$
$$\Rightarrow \frac{f}{f_s} = \frac{1}{2}$$
$$\Rightarrow f = \frac{f_s}{2} = \frac{8000}{2} = 4000$$

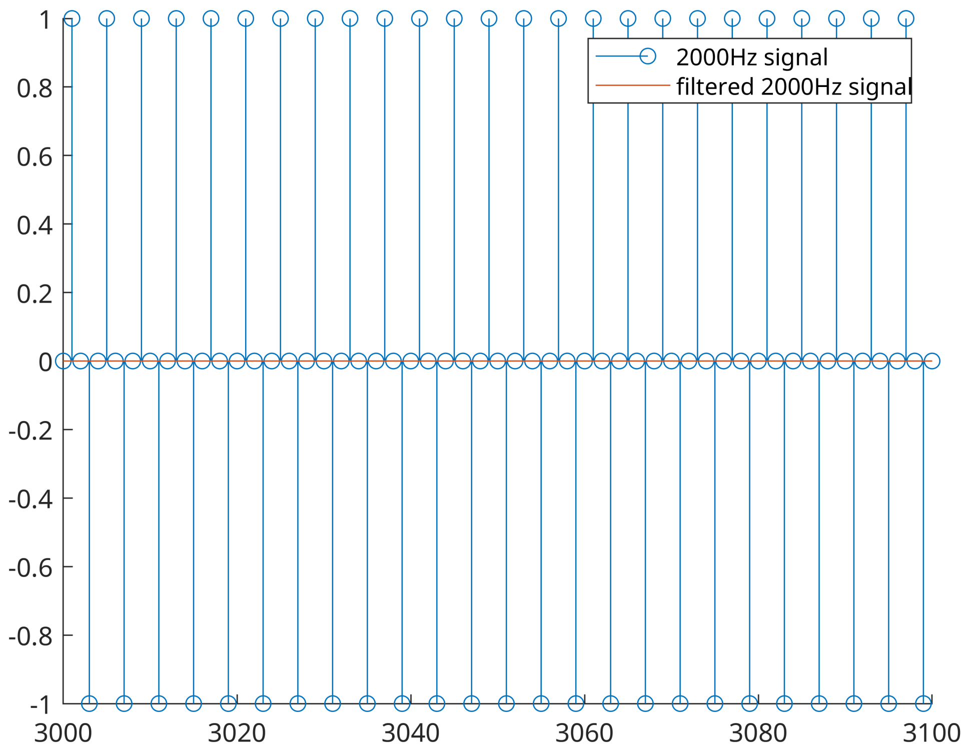
Sine waves of 2000Hz will be eliminated, sine waves of 4000Hz will be quadrupled in amplitude

Proof:

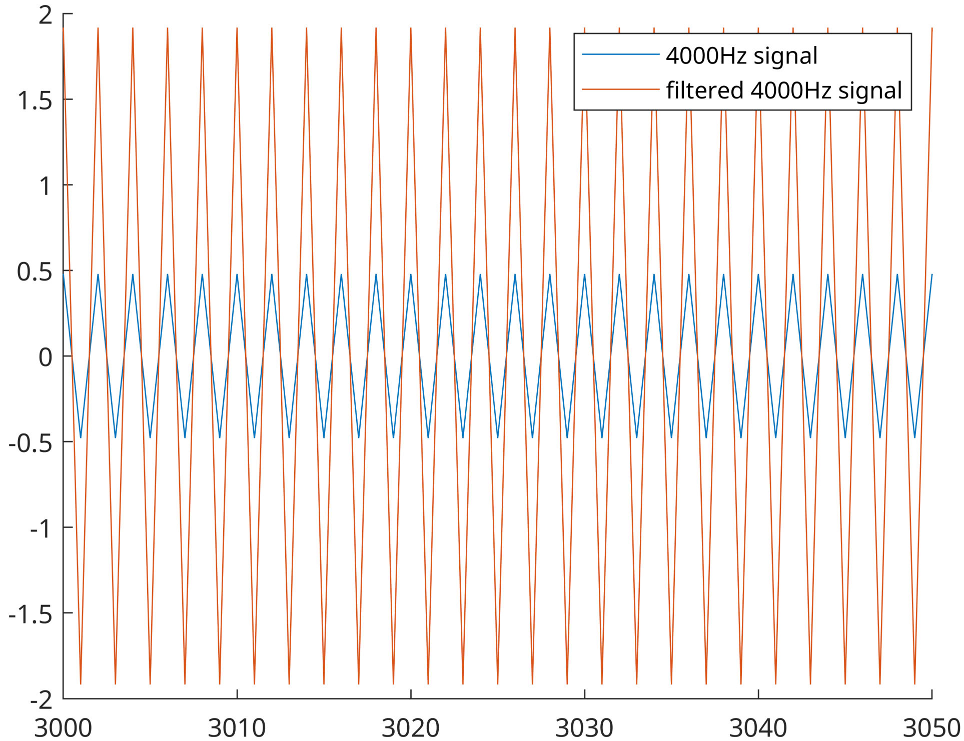
```
% From original definition
a = [1 1];%[1];
b = [1 0 0 0 -1];%[1 -1 1 -1];

f_s = 8000;
t = 1;
ts = 1/f_s:1/f_s:1;
x_2000 = sin(2*pi*2000*ts);
% Add phase offset to x_4000 so samples aren't all close to zero
x_4000 = sin(2*pi*4000*ts + 0.5);
freqz(b,a,2048,f_s);
```




```
figure;
hold on;
stem(x_2000);
plot(filter(b,a,x_2000));
xlim([3000 3100]);
legend("2000Hz signal", "filtered 2000Hz signal");
hold off;
```




```
figure;
hold on;
plot(x_4000);
plot(filter(b,a,x_4000));
xlim([3000 3050]);
legend("4000Hz signal", "filtered 4000Hz signal");
hold off;
```



4000Hz signal
filtered 4000Hz signal
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
3000
3010
3020
3030
3040
3050