Exercise

1 Propositional Calculus

Exercise 1.1. Check whether the following statements are tautologies.

1. $(((A \implies B) \implies B) \implies B)$
2. $(((A \implies B) \implies B) \implies A)$
3. $(((A \implies B) \implies A) \implies A)$
4. $(((B \implies C) \implies (A \implies B)) \implies (A \implies B))$
5. $((A \vee (\neg(B \wedge C))) \implies ((A \iff C) \vee B))$
6. $(A \implies (B \implies (B \implies A)))$
7. $((A \implies B) \vee (B \implies A))$
8. $((\neg(A \implies B)) \implies A)$

Exercise 1.2. Check whether the following pairs are logically equivalent.

1. $((A \implies B) \implies A)$ and A
2. $(A \iff B)$ and $((A \implies B) \wedge (B \implies A))$
3. $((\neg A) \vee B)$ and $((\neg B) \vee A)$
4. $(\neg(A \iff B))$ and $(A \iff (\neg B))$
5. $(A \vee (B \iff C))$ and $((A \vee B) \iff (A \vee C))$

2 Proofs by Contradiction

Exercise 2.1. Prove the following using proof by contradiction.

1. Prove that $\sqrt{3}$ is irrational.
2. Prove that for any prime p, $\sqrt{p}$ is irrational.
3. Let m and n be two positive integers with $n > m$. Suppose n balls are thrown into m urns in such a way that each ball will get an urn. Prove that at least one urn contains more than one ball. Suppose there are n balls and m urns with $n > m$.

4. Let V be any vector space over $\mathbb{R}$. Show that any subset (resp. superset) of a linearly independent (resp. dependent) set in V is linearly independent (resp. dependent).
5. Consider the following system of linear equations over $\mathbb{R}$:

$$A\boldsymbol{x} = \boldsymbol{y}$$

where A is an $n \times n$ matrix, $\boldsymbol{x}$ is the column vector of unknown variables and $\boldsymbol{y} \in \mathbb{R}^n$ is a fixed column vector. Suppose the rank of A is less than n. Prove that the above system has more than one solution over $\mathbb{R}$.

3 Basics of Set Theory

Exercise 3.1. Let A, B, C and $\mathcal{S}$ be three sets such that $A, B \subseteq \mathcal{S}$. Then, prove the following statements.

1. $A \subseteq B$ if and only if $A \cap \overline{B} = \emptyset$.
2. $\overline{(A \cup B)} = \overline{A} \cap \overline{B}$ and $\overline{(A \cap B)} = \overline{A} \cup \overline{B}$.
3. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
4. $S \setminus A = S \cap \overline{A}$.
5. $A \cap B = \mathcal{S} \setminus ((\mathcal{S} \setminus A) \cup (\mathcal{S} \setminus B))$ and $A \cup B = \mathcal{S} \setminus ((\mathcal{S} \setminus A) \cap (\mathcal{S} \setminus B))$.

4 Mathematical Induction

Exercise 4.1. Using induction solve the following problems.

1. Prove that $1^2 + 2^2 + \cdots + n^2 = \frac{1}{6} \cdot n(n+1)(2n+1)$.
2. Let S and T be two sets with $|S| = |T| = n$. Then, show that the number injective functions from S to T is $n!$.
3. Show that the number of arrangements of n objects is $n!$.
4. Let A be s set with $|A| = n$. Then, show that $\mathcal{P}(A) = 2^n$.
5. Prove the binomial theorem, that is,

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} \cdot a^{n-i} \cdot b^i$$

for all $i \geq 1$.

6. Let p be a prime. Show that $p|(n^p - n)$. Hint: binomial theorem.

5 Basics of Functions

Exercise 5.1. Prove the following statements.

1. A function $f : S \to T$ is injective if and only if $\forall t \in T$, $|f^{-1}(t)| \leq 1$.
2. A function $f : S \to T$ is surjective if and only if $\forall t \in T$, $|f^{-1}(t)| \geq 1$.
3. A function $f : S \to T$ is bijective if and only if $\forall t \in T$, $|f^{-1}(t)| = 1$.

Exercise 5.2. Prove or disprove the functions defined in Examples 5.1 to 5.6 are bijective.

Exercise 5.3. Let $f : S \to T$ and $g : T \to U$ be two functions. Then, prove the following statements.

1. If f and g are surjective, then $g \circ f : S \to U$ is surjective.
2. If f and g are injective, then $g \circ f : S \to U$ is injective.
3. If $g \circ f : S \to U$ is injective, then f is injective.
4. If $g \circ f : S \to U$ is surjective, then g is surjective.

Exercise 5.4. Let $f : S \to T$ and $g : T \to U$ be two functions. Then, disprove the following statements.

1. If $g \circ f : S \to U$ is surjective, then f is surjective.
2. If $g \circ f : S \to U$ is injective, then g are injective.
3. If f is injective and g is surjective, then $g \circ f : S \to U$ is bijective.

Exercise 5.5. Find the inverses of the functions defined in Examples 5.5 and 5.6.

Exercise 5.6. Prove that a function $f : S \to T$ is bijective iff it is invertible.

Exercise 5.7. Let $f : S \to T$, where S and T are nonempty finite sets. Then, prove the following statements.

1. f is one-one $\implies |S| \leq |T|$.
2. f is onto $\implies |T| \leq |S|$.
3. f is one-one and $|S| = |T| \implies f$ is bijection.
4. f is onto and $|S| = |T| \implies f$ is bijection.

Exercise 5.8. Let S be a countably infinite set and A be a finite set. Show that $S \setminus A$ is countable.

6 Combinatorics

Exercise 6.1. 1. What is the number injective functions from a set of size r to a set of size n?

2. What is the number bijective functions from a set of size n to a set of size n?
3. Let X and Y be two non-empty sets. Let the notation Y^X denote the set of all functions X to Y. Then, calculate $|Y^X|$.

4. Let V be a set of n vertices. How many graphs can be constructed using V as vertex set? Assume that self loops are not allowed.

5. Let A be a set with $|V| = n$. How many binary relations on A can be defined? How many binary relations on A are reflexive? How many binary relations on A are symmetric?

6.

Exercise 6.2. Let n and r be positive integers. Consider the following problems related to n-variables polynomial.

1. How many monomials of degree at most r is possible under following two constraints:
   (a) order of the variables matters.
   (b) order of the variable does not matter.

2. How many ways a zero polynomial in n variables can be expressed, where the degree of any monomial can be at most r?

3. How many distinct multivariate polynomials over $\{0, 1\}$ in n variables are possible, where the degree of any monomial can be at most r?