Knowledge Representation and Reasoning

Propositional Logic

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Expressivity: Complete Sets of Connectives

Recap

It can be shown that the following sets of connectives are sufficient to express whatever can be expressed with other connectives.

$\{\wedge, \neg\}$, $\{\vee, \neg\}$

$\{\supset, \neg\}$ Gottlob Frege used this set in his *Propositional Calculus*

In practice we often use the set $\{\wedge, \vee, \neg\}$ in our implementations.

Interestingly there are two singleton sets that are complete. They are,

NAND or Not–AND

α	β	$(\alpha \uparrow \beta)$
true	true	false
false	true	true
true	false	true
false	false	true

Also known as *Sheffer Stroke*

NOR or Not-OR or Joint Denial

α	β	$(\alpha \downarrow \beta)$
true	true	false
false	true	false
true	false	false
false	false	true

Also known as *Peirce's arrow*

Frege's Propositional Calculus

- Given a set of axioms / premises which set of rules suffices to derive all entailments?
- Gottlob Frege showed that just one rule Modus Ponens with the set of connectives $\{\supset, \neg\}$ is complete, given the axioms below.
- Frege's Propositional Calculus is an axiomatic system that can derive all true statements or tautologies given the following axioms (accepted as true statements):

1. *THEN-1* $\alpha \supset (\beta \supset \alpha)$ *Note that α and β are variables.*
2. *THEN-2* $(\alpha \supset (\beta \supset \gamma)) \supset ((\alpha \supset \beta) \supset (\alpha \supset \gamma))$
3. *THEN-3* $(\alpha \supset (\beta \supset \gamma)) \supset (\beta \supset (\alpha \supset \gamma))$ *Redundant*
4. *FRG-1* $(\alpha \supset \beta) \supset (\neg\beta \supset \neg\alpha)$
5. *FRG-2* $\neg\neg\alpha \rightarrow \alpha$
6. *FRG-3* $\alpha \rightarrow \neg\neg\alpha$

https://en.wikipedia.org/wiki/Frege_propositional_calculus

Completeness

- Frege proved that his Propositional Calculus is complete.
  - Completeness proofs are beyond the scope of this course.
- Frege's logic is an Axiomatic System.
  - This means that the KB is a set of axioms.
  - Axioms are statements that are accepted as true.
  - Exercise: Verify that the six axioms in Frege's calculus are tautologies.
- Therefore, Frege's calculus can prove *all* tautologies.
- Recall the Deduction Theorem:
  - $A, B, C, D \vDash E$ iff $\vDash ((A \land B \land C \land D) \supset E)$
  - IF $\vDash ((A \land B \land C \land D) \supset E)$ then $\vdash ((A \land B \land C \land D) \supset E)$

    from Completeness of Frege's Calculus

    after eliminating the $\land$ connective
  - IF $\vdash ((A \land B \land C \land D) \supset E)$ THEN $A, B, C, D \vdash E$
- This means that if $A, B, C, D \vDash E$ then $A, B, C, D \vdash E$

PROPOSITIONAL LOGIC

Propositional Logic

- *Propositional Logic* (PL) or *Propositional Calculus* is the simplest of logic languages
- The atomic sentence in PL is a *symbol*, for example P, that *stands* for a sentence.
- A symbol can stand for *any* sentence in a natural language. For example,
  - P = All men are mortal
  - P = The Moon is made of green cheese.
  - P = The Earth is round.
  - P = The Earth is flat.
  - P = The tomato is a vegetable.
  - P = The tomato is a fruit.
- The sentence is atomic or indivisible. We do not peer inside it.
- Logic itself is not concerned with meaning.
- Meaning lies in the mind of the beholder.
- A sentence or a proposition is in principle *true* or *false*.
- Truth values are externally supplied.

Propositional Logic: Syntax

Every vocabulary of any logic language has two parts.

The *constant* or *logical* part of the vocabulary

- Symbols that stand for connectives or operators
  - "$\wedge$", "$\vee$", "$\neg$", and "$\supset$"…
- Brackets "(", ")", "{", "}"…
- The constant symbols "$\bot$" and "$\top$"
  - Called *Bottom* and *Top* respectively
  - Sentences whose truth value is constant
  - Respectively *false* and *true*

The *variable* or *domain specific* part of the vocabulary

- A countable set of propositional symbols $\mathbb{P} = \{P_1, P_2, P_3, \ldots\}$
  - commonly used symbols $\{P, Q, R, P_1, Q_1, R_1, \ldots\}$

Sentences of Propositional Logic

Sentences of a PL are the *well formed formulas* or simply *formulas*.

The set $\mathbb{F}$ of formulas is defined by structural recursion as follows.

Base: Every proposition symbol is a formula. These are the *atomic formulas*.

If $\boldsymbol{\alpha} \in \mathbb{P}$ then $\boldsymbol{\alpha} \in \mathbb{F}$

Unary connective:

If $\alpha \in \mathbb{F}$ then $\neg\alpha \in \mathbb{F}$

Binary connectives: Let $\circ$ be a binary connective

If $\alpha \in \mathbb{F}$ and $\beta \in \mathbb{F}$ then $(\alpha \circ \beta) \in \mathbb{F}$

The set of formulas $\mathbb{F}$ is the smallest set satisfying the above.

A note of Frege's Propositional Calculus

When we say that Frege's Calculus is *Propositional*

we are *not* saying that an axiom in the calculus like

THEN-2: $(\alpha \supset (\beta \supset \gamma)) \supset ((\alpha \supset \beta) \supset (\alpha \supset \gamma))$

has constituents α, β, and γ that are from the set $\mathbb{P}$.

Rather, that the *atomic* sentences in the the logic

are sentences that we do not dissect and peer into.

α, β, and γ could well have been compound sentences

from a more expressive language,

and

$(\alpha \supset (\beta \supset \gamma)) \supset ((\alpha \supset \beta) \supset (\alpha \supset \gamma))$ would still be a tautology.

PL: Truth Functional Semantics

A valuation function $\mathbb{V} : \mathbb{P} \rightarrow$ {true, false} assigns a value to every proposition in $\mathbb{P}$

Note that this may *not* be specified by the user.

Often, the input KB has a collection of atomic and compound formulas, and the task is to determine the valuation of some other specific formulas.

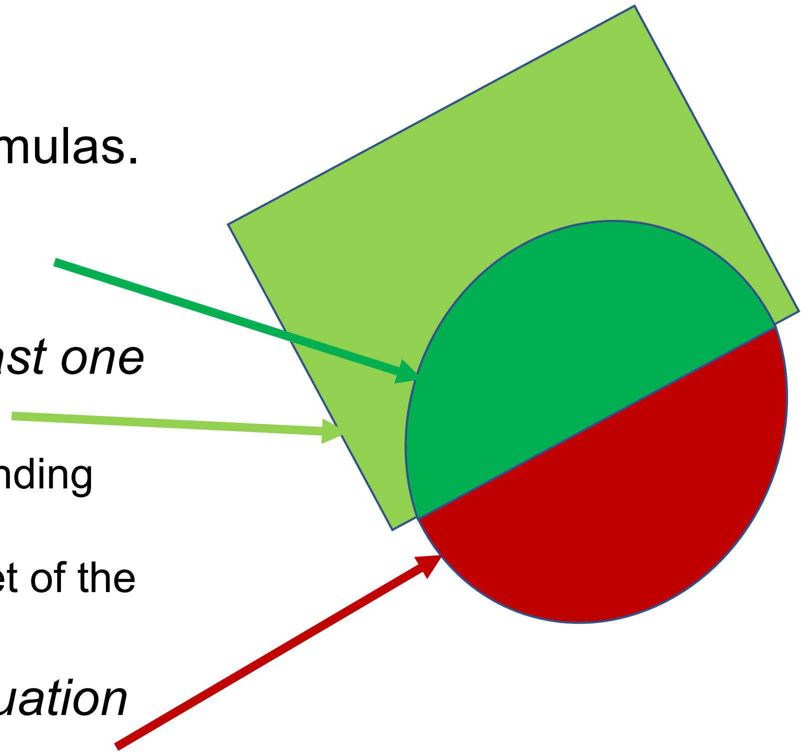
Given the valuation function we can lift it to all the formulas $\mathbb{V} : \mathbb{F} \rightarrow$ {true, false} using the semantics of the logical connectives. For example,

If $\mathbb{V}(\alpha)$ = *true* and $\mathbb{V}(\beta)$ = *true* then $\mathbb{V}(\alpha \wedge \beta)$ = *true*
else $\mathbb{V}(\alpha \wedge \beta)$ = *false*

True or Not True?

There are three categories of sentences or formulas.

- Tautologies: Formulas that are true for *every* valuation $\mathbb{V} : \mathbb{P} \rightarrow \{\text{true, false}\}$
- Satisfiable formulas: For which there is *at least one valuation* that makes it true.
  - The well known SAT problem is concerned with finding such a valuation.
  - The set of satisfiable formulas is a proper superset of the set of tautologies.
- Contradictions: Which are *false for every valuation* $\mathbb{V} : \mathbb{P} \rightarrow \{\text{true, false}\}$.
  - Also known as unsatisfiable formulas.
  - The set of unsatisfiable formulas is disjoint with the set of satisfiable formulas.



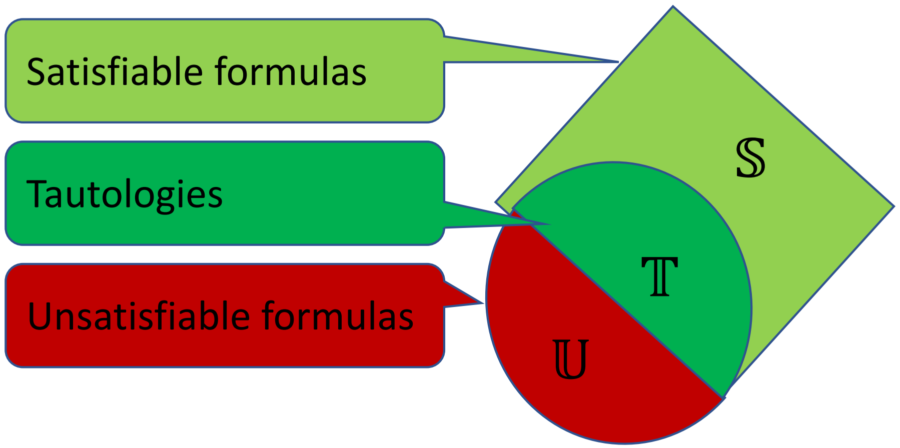
Comparing infinite sets!

Points to ponder
Given even a tiny alphabet {P, Q}…

- The set of tautologies $\mathbb{T}$ is infinite
  - $(P \lor \neg P)$
  - $(P \lor \neg P) \land (Q \lor \neg Q)$
  - $(P \supset (Q \supset P)$
  - If $\alpha, \beta \in \mathbb{T}$ then $(\alpha \lor \beta) \in \mathbb{T}$

- Let $\mathbb{U}$ be the set of unsatisfiable formulas
- For every $\alpha \in \mathbb{T}$ there is a $\neg\alpha \in \mathbb{U}$

- Let $\mathbb{S}$ be the set of satisfiable formulas
- For every $\alpha \in \mathbb{T}$ it is true that $\alpha \in \mathbb{S}$
- For every $\alpha \in \mathbb{T}$ there is a β such that
  $(\alpha \land \beta) \in \mathbb{S}$ and $(\alpha \land \beta) \notin \mathbb{T}$
- For every $\alpha \in \mathbb{S}$ it is true that $(\alpha \lor \neg\alpha) \in \mathbb{T}$



Semantics (Propositional Logic)

Atomic sentences in Propositional Logic can stand for anything. Consider,

Alice likes mathematics and she likes stories. If she likes mathematics she likes algebra. If she likes algebra and likes physics she will go to college. She does not like stories or she likes physics. She does not like chemistry and history.

Encoding: P = Alice likes mathematics. Q = Alice likes stories. R = Alice likes algebra. S = Alice likes physics. T = Alice will go to college. U = Alice likes chemistry. V = Alice likes history.

Then the given facts are, $(P \wedge Q)$

$(P \supset R)$

$((R \wedge S) \supset T)$

$(\neg Q \vee S)$

$(\neg U \wedge \neg V)$

That is, KB = $\{(P \wedge Q), (P \supset R), ((R \wedge S) \supset T), (\neg Q \vee S), (\neg U \wedge \neg V)\}$

Proofs

(in Propositional Logic)

Theorems and Proofs

- Any formula that is *true* is called a *theorem*.
- The process of establishing it to be a theorem is called *theorem proving*.
- Theorem proving is concerned with the syntactic process of constructing a proof.

There are many algorithms for theorem proving. We shall look at a few.

- Direct proof or Natural Deduction (also called Hilbert style proofs)
- Indirect proof or proof by contradiction
  - The Tableaux Method
  - The Resolution Refutation Method (covered later in First Order Logic)

We begin with Natural Deduction.

Will Alice go to college?

Encoding: P = Alice likes mathematics. Q = Alice likes stories. R = Alice likes algebra. S = Alice likes physics. T = Alice will go to college. U = Alice likes chemistry. V = Alice likes history.

Given the knowledge base

1. $(P \wedge Q)$
2. $(P \supset R)$
3. $((R \wedge S) \supset T)$
4. $(\neg Q \vee S)$
5. $(\neg U \wedge \neg V)$

We often have a query or a goal.

Is T true? Will Alice go to college?

Natural Deduction

In a direct proof we choose rules of inference and add the consequents to the KB.

Remember the geometry proofs done in school?

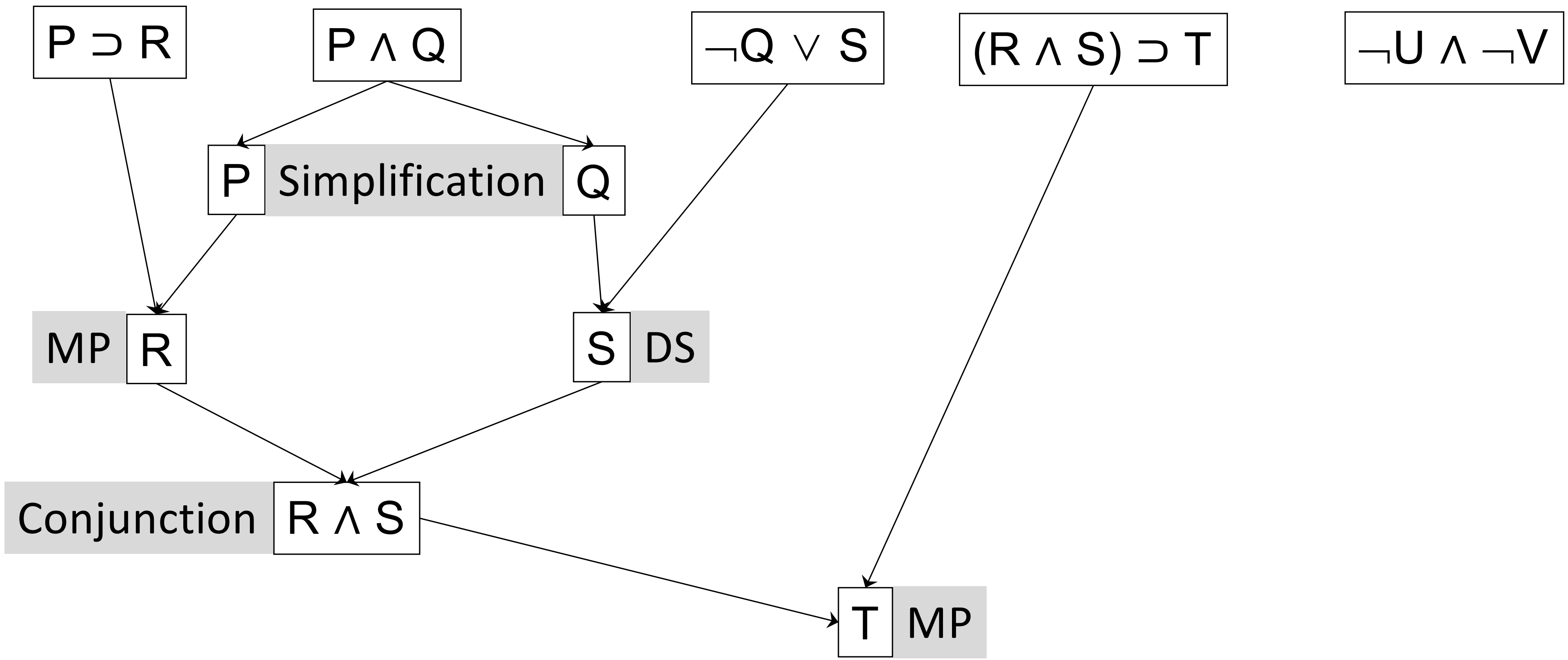
1. $(P \wedge Q)$	premise	*Alice likes mathematics and she likes stories.*
2. $(P \supset R)$	premise	*If she likes mathematics she likes algebra.*
3. $((R \wedge S) \supset T)$	premise	•
4. $(\neg Q \vee S)$	premise	•
5. P	1, simplification	
6. Q	1, simplification	Strictly speaking first replace $(P \wedge Q)$ with $(Q \wedge P)$
7. R	2, 5, modus ponens	
8. S	4, 6, disjunctive syllogism	
9. $(R \wedge S)$	7, 8, conjunction	
10. T	3, 9, modus ponens	

From $\underline{\alpha \wedge \beta\,.}$

Infer α

Simplification (S)

The Proof as a Directed Acyclic Graph (DAG)



The proof is the product that is published.
Finding the proof is a search process that logicians do not publish!

Finding Proofs

The proof of α is the end product,
that is a justification of the sentence α being true

It represents a chain of inferences linking
the given facts
to the desired goal or conclusion

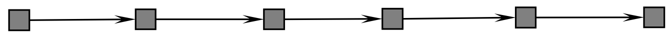
Proofs are found by a process of search
Remember the 4-colour theorem?

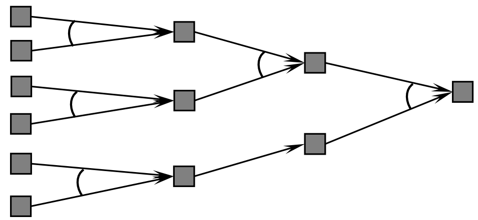
The irrelevant inferences are discarded
and only the final proof is communicated

Forward Reasoning

A data driven chain of inferences

From facts to goals





Forward Chaining

Pick a rule with matching facts

Add the consequent to the KB

Till the goal is added to the KB

The key question is

which rule and what facts?

The Deduction Theorem

One problem with natural deduction is that one has to choose a rule in each cycle. This naturally results in search.

The Deduction Theorem allows us different approaches to constructing proofs.

$\boldsymbol{\Gamma}, \boldsymbol{\alpha} \vDash \boldsymbol{\beta}$ iff $\boldsymbol{\Gamma} \vDash (\boldsymbol{\alpha} \supset \boldsymbol{\beta})$ where $\boldsymbol{\Gamma}$ is a set of formulas

For a sound and complete logic system this translates to

$\boldsymbol{\Gamma}, \boldsymbol{\alpha} \vdash \boldsymbol{\beta}$ iff $\boldsymbol{\Gamma} \vdash (\boldsymbol{\alpha} \supset \boldsymbol{\beta})$ where $\boldsymbol{\Gamma}$ is a set of formulas

Instead of guessing and adding $\boldsymbol{\alpha}$ to the premises,

we can equivalently directly prove $(\boldsymbol{\alpha} \supset \boldsymbol{\beta})$.

Show that (Premises ⊃ Conclusions) is a tautology

Given	$(P \land Q)$
	$(P \supset R)$
	$((R \land S) \supset T)$
	$(\neg Q \lor S)$
To prove	T

Is equivalent to proving the following tautology with no premises

To prove $([(P \land Q) \land (P \supset R) \land ((R \land S) \supset T) \land (\neg Q \lor S)] \supset T)$

To prove (Premises ⊃ Conclusions) assume Premises

Conversely, to prove a tautological implication ($\boldsymbol{\alpha} \supset \boldsymbol{\beta}$)

1. Create an *assumption box*
2. Add the antecedent(s) $\boldsymbol{\alpha}$
3. Prove the consequent $\boldsymbol{\beta}$
4. Exit the *assumption box*

To show $(((P \supset Q) \land (R \land S)) \supset (\neg Q \supset \neg P))$

1.	$((P \supset Q) \land (R \land S))$	Assumption
2.	$(P \supset Q)$	1, Simplification
3.	$((P \supset Q) \supset (\neg Q \supset \neg P))$	Instance of FRG-1
4.	$(\neg Q \supset \neg P)$	2,3, MP

$\therefore (((P \supset Q) \land (R \land S)) \supset (\neg Q \supset \neg P))$

A Tautological Implication

To show $((P \land Q) \supset R) \supset (P \supset (Q \supset R))$ create three assumption boxes.

1.	$((P \land Q) \supset R)$	Assumption
2.	P	Assumption
3.	Q	Assumption
4.	$(P \land Q)$	2, 3, Conjunction
5.	R	1, 4, MP
6.	$(Q \supset R)$	3, 5, Deduction theorem
7.	$(P \supset (Q \supset R))$	2, 6, Deduction theorem

$((P \land Q) \supset R) \supset (P \supset (Q \supset R))$ 1, 7, Deduction theorem

Derived Rules of Inference

- Any (long) proof can be added as a new rule of inference.
- For example, *if* you prove $(\boldsymbol{\alpha} \supset \boldsymbol{\gamma})$ *given* $(\boldsymbol{\alpha} \supset \boldsymbol{\beta})$ and $(\boldsymbol{\beta} \supset \boldsymbol{\gamma})$ then you can add $(\boldsymbol{\alpha} \supset \boldsymbol{\beta}), (\boldsymbol{\beta} \supset \boldsymbol{\gamma}) \vdash (\boldsymbol{\alpha} \supset \boldsymbol{\gamma})$ as a derived rule of inference to the original set with only M.P. defined by Frege in his *Begriffsschrift* written in 1879.

From	$\alpha \supset \beta$
and	$\underline{\beta \supset \gamma}$
Infer	$\alpha \supset \gamma$
Hypothetical Syllogism (HS)	

1.	$(\boldsymbol{\alpha} \supset \boldsymbol{\beta})$	premise
2.	$(\boldsymbol{\beta} \supset \boldsymbol{\gamma})$	premise
3.	$(\boldsymbol{\beta} \supset \boldsymbol{\gamma}) \rightarrow (\boldsymbol{\alpha} \supset (\boldsymbol{\beta} \supset \boldsymbol{\gamma}))$	THEN-1: $\alpha \supset (\beta \supset \alpha)$
4.	$(\boldsymbol{\alpha} \supset (\boldsymbol{\beta} \supset \boldsymbol{\gamma}))$	2, 3, MP
5.	$(\boldsymbol{\alpha} \supset (\boldsymbol{\beta} \supset \boldsymbol{\gamma})) \supset ((\boldsymbol{\alpha} \supset \boldsymbol{\beta}) \supset (\boldsymbol{\alpha} \supset \boldsymbol{\gamma})$	THEN-2
6.	$((\boldsymbol{\alpha} \supset \boldsymbol{\beta}) \supset (\boldsymbol{\alpha} \supset \boldsymbol{\gamma})$	4, 5, MP
7.	$(\boldsymbol{\alpha} \supset \boldsymbol{\gamma})$	1, 6, MP

Model Checking

SAT: find a *satisfying valuation* for a formula.
A valuation is also called an *interpretation*.
If the interpretation makes the formula *true* we say that it is a *model* for the formula.

If $\neg\boldsymbol{\alpha}$ *has* a model then $\boldsymbol{\alpha}$ may have a model too (for example, $\neg(P \supset Q)$).
If $\neg\boldsymbol{\alpha}$ *does not* have a model (is unsatisfiable) then $\boldsymbol{\alpha}$ *is* a tautology.

To establish that $\boldsymbol{\alpha}$ is a tautology look for a model for $\neg\boldsymbol{\alpha}$.
For example,

Look for a model for $\neg[(P \wedge (P \supset Q)) \supset Q)]$.
Q must be *false* and hence (P ∧ (P ⊃ *false*)) must be *true*.
P must be *true*, but then (*true* ⊃ *false*) cannot be *true*.
∴ There is no model for $\neg[(P \wedge (P \supset Q)) \supset Q)]$
Ergo, $[(P \wedge (P \supset Q)) \supset Q)]$ is a tautology.

Proof by Contradiction

To show that $\neg((\alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_n) \supset \beta)$ is unsatisfiable, one can add the negation of the goal to the set of premises.

$$\neg((\alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_n) \supset \beta) \equiv \neg(\neg(\alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_n) \vee \beta)$$

$$\equiv ((\alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_n) \wedge \neg\beta)$$

If the resulting set of formulas $((\alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_n) \wedge \neg\beta)$ is unsatisfiable

so is the original formula $\neg((\alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_n) \supset \beta)$.

Then $((\alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_n) \supset \beta)$ is a tautology

Model Checking: The Tableau Method

Given any $\boldsymbol{\alpha}$ the Tableau Method looks for a model for $\neg\boldsymbol{\alpha}$.

It does so by eliminating logical connectives and collecting sets of propositions or their negations (these are also called *literals*) that are a model for a given formula.

For example,

- In a model for (P ∧ Q) *both* P and Q must be present.
- In a model for (P ⊃ Q) *either* Q must be present *or* ¬P must be present.
  - There are two possibilities of finding a model.

If an interpretation has both P and ¬P then it cannot be a model.

If there is no model for $\neg\boldsymbol{\alpha}$ then $\boldsymbol{\alpha}$ is a tautology.

Tableau Rules

The following rules are used to breakdown a compound formula (for four connectives)

Connective	$\neg$	$\wedge$	$\vee$	$\supset$
Formula	$\dfrac{\neg X}{\neg X}$	$\dfrac{(X \wedge Y)}{\begin{array}{c} X \\ Y \end{array}}$	$\begin{array}{c} (X \vee Y) \\ \swarrow \quad \searrow \\ X \qquad Y \end{array}$	$\begin{array}{c} (X \supset Y) \\ \swarrow \quad \searrow \\ \neg X \qquad Y \end{array}$
Negated Formula	$\dfrac{\neg\neg X}{X}$	$\begin{array}{c} \neg(X \wedge Y) \\ \swarrow \quad \searrow \\ \neg X \qquad \neg Y \end{array}$	$\dfrac{\neg(X \vee Y)}{\begin{array}{c} \neg X \\ \neg Y \end{array}}$	$\dfrac{\neg(X \supset Y)}{\begin{array}{c} X \\ \neg Y \end{array}}$

Constructing the Tableau

- Add the negated goal to the set of premises / axioms in the tableau
- Pick an unused formula in the tableau and eliminate the main connective
- Mark the formula as used (it can be removed)
  - the constituents are added to each ”column” below it
- Each column is an interpretation
- Termination of a column :

  IF a column has both X and ¬X it cannot be a model
  - It is unsatisfiable
  - it can be closed

  ELSE IF it has no more connectives
  - it is satisfiable and is a model

Modus Ponens [(P ∧ (P ⊃ Q)) ⊃ Q)] is sound

1 ~~¬[(P ∧ (P ⊃ Q)) ⊃ Q)]~~

Eliminate the main connective ⊃

↓

2 ~~(P ∧ (P ⊃ Q))~~

¬Q

↓

P

3 ~~(P ⊃ Q))~~

¬P	Q

Closed because P and ¬P are in the column

Likewise for Q and ¬Q. No model exists.

Is Abduction $[(Q \land (P \supset Q)) \supset P)]$ valid?

1 $\cancel{\neg[(Q \land (P \supset Q)) \supset P)]}$

2 $\cancel{(Q \land (P \supset Q))}$

$\neg P$

Q

3 $\cancel{(P \supset Q))}$

$\neg P$ $\qquad\qquad$ Q

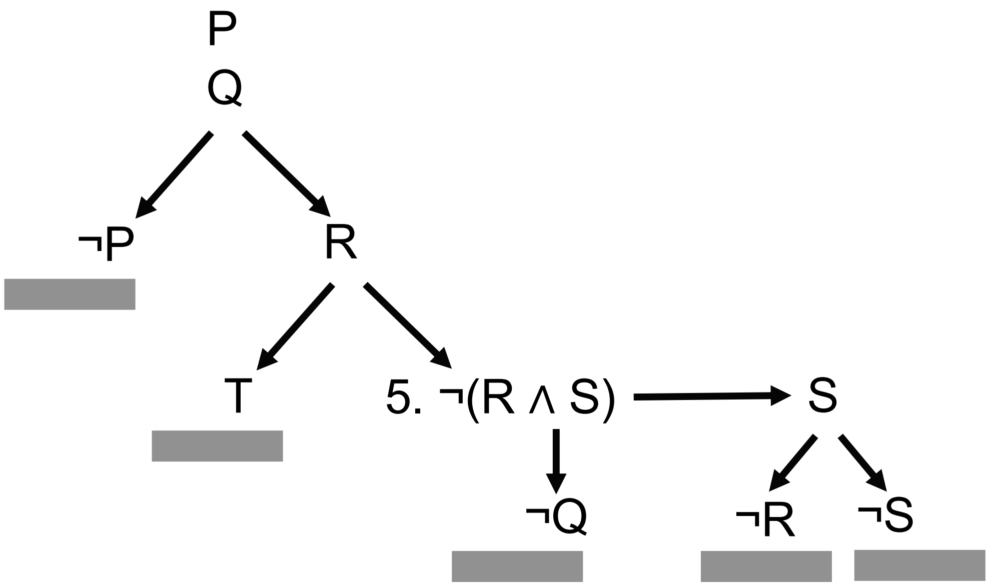
No more connectives to eliminate

$\{\neg P, Q\}$ is a model for $\neg[(Q \land (P \supset Q)) \supset P)]$

Abduction is *not* a sound rule of inference

Will Alice go to college?

1.	$(P \wedge Q)$	premise
2.	$(P \supset R)$	premise
3.	$((R \wedge S) \supset T)$	premise
4.	$(\neg Q \vee S)$	premise
	$\neg T$	negated conclusion

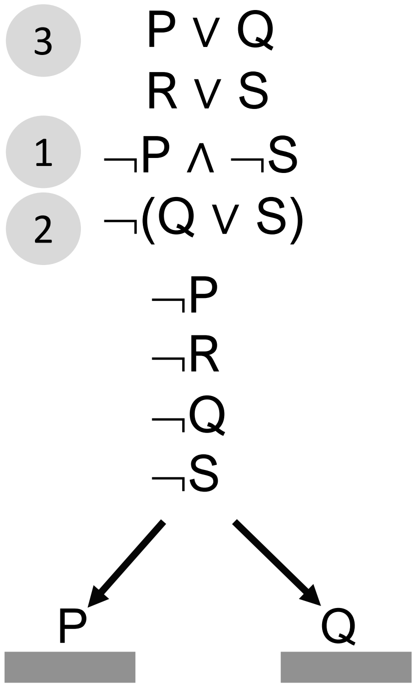


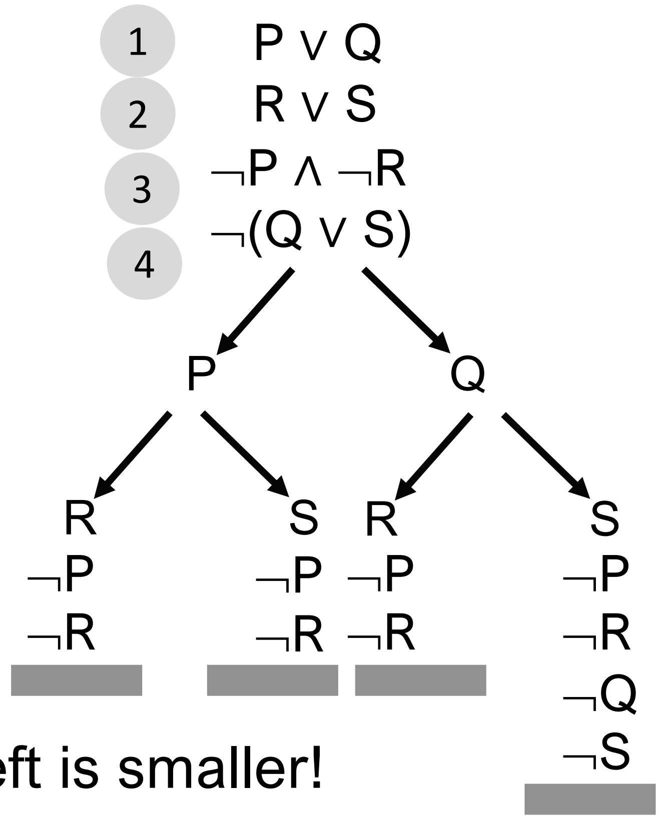
Processing the formulas in the order presented.

We are unable to find an interpretation in which the premises are true and the conclusion is false.

Order of eliminating connectives

Certain connectives introduce branching in the tableau
They are best tackled *after* the ones that do not.
Consider the following two proofs.





The tree on the left is smaller!

Proving Frege's THEN-2 Axiom

1. $\neg[(P \supset (Q \supset R))] \supset [(P \supset Q) \supset (P \supset R)]$ *Negated axiom*
2. $(P \supset (Q \supset R))$ *After 1*
3. $\neg[(P \supset Q) \supset (P \supset R)]$ *After 1*

4. $\neg P$ *After 2*
6. $(P \supset Q)$ *After 3*
7. $\neg(P \supset R)$ *After 3*

10. $\neg P$ *6*
12. P *7*

11. Q *6*
13. P *7*

5. $(Q \supset R))$ *After 2*
8. $(P \supset Q)$ *After 3*
9. $\neg(P \supset R)$ *After 3*

14. $\neg Q$ *5*

16. $\neg P$ *8*
20. P *9*

17. Q *8*

15. R *5*

18. $\neg P$ *8*
21. P *9*

19. Q *8*
22. P *9*
23. $\neg R$ *9*

from [Logical Labyrinths by Raymond Smullyan]

Proving Frege's THEN-2 Axiom

1. $\neg[(P \supset (Q \supset R))] \supset [(P \supset Q) \supset (P \supset R)]$ *Negated axiom*
2. $(P \supset (Q \supset R))$ *After 1*
3. $\neg[(P \supset Q) \supset (P \supset R)]$ *After 1*
4. $(P \supset Q)$ *After 3*
5. $\neg(P \supset R)$ *After 3*
6. P *After 5*
7. $\neg R$ *After 5*

8. $\neg P$ *After 2*

9. $(Q \supset R))$ *After 2*

10. $\neg P$ *4*

11. Q *4*

12. $\neg Q$ *9*

13. R *9*

Processing linear rules before branching rules results in a smaller tableau

Next

First Order Logic