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ASSIGNMENT — 1

1 Question 1

1.1 Derivation of Magnetic Moment in Terms of Angular Momentum

Consider a point particle of mass m, charge q revolving in a circle of radius r with a speed v. We aim to derive an expression for the magnetic moment μ in terms of the angular momentum $\mathbf{L}$.

The magnetic moment, μ, can be expressed as the cross product of current, $\mathbf{I}$, and area, $\mathbf{A}$, of the orbital circle with radius r:

$$\mu = \mathbf{I} \times \mathbf{A}$$

where $\mathbf{I} = \frac{q}{T}$, T being the time period of one revolution. Thus,

$$\mu = \frac{q}{T} \times (2\pi r^2)\hat{\mathbf{n}}$$

And,

$$T = \frac{\text{circumference}}{\text{speed}} = \frac{2\pi r}{v}$$

Hence,

$$\mu = \frac{qvr}{2}\hat{\mathbf{n}}$$

Multiplying and dividing by m,

$$\mu = \frac{q}{2m} \times (mvr)\hat{\mathbf{n}}$$

Considering $\mathbf{L} = m\mathbf{v}r$, we substitute:

$$\mu = \frac{q}{2m} \times \mathbf{L}\hat{\mathbf{n}}$$

Thus, we have derived the expression for the magnetic moment μ in terms of the angular momentum $\mathbf{L}$ as:

$$\mu = \frac{q}{2m} \times \mathbf{L}\hat{\mathbf{n}}$$

1.2 Derivation of Magnetic Moment in Terms of Angular Momentum

Consider a solid spherical ball of mass m rotating about an axis through its center with a charge q uniformly distributed on the surface of the ball. We aim to show that the magnetic moment μ is related to the angular momentum $\mathbf{L}$ by the relation $\mu = \frac{5q}{6mc}\mathbf{L}$.

The magnetic moment is given by $\mu = IA$.

Since we are dealing with a uniformly charged sphere, we will cut up the sphere into infinitely many rings that are infinitely small in height. Without loss of generality, we can cut them horizontally so that they are parallel to the x-axis.

Because the ball is rotating around the y-axis, there is an induced current. The current for each ring is given by

$$I = \frac{\sigma dA}{T}$$

where,

$$\sigma = \frac{q}{4\pi R^2}$$

is the surface charge density for a sphere of radius R and $T = \frac{2\pi}{\omega}$ is the period of the sphere's rotation.

Assuming the sphere has radius R, each ring has radius $r = R\sin\theta$ and thus

$$dA = 2\pi r R d\theta$$

is the area of each ring.

Thus

$$d\mu = dIA = dI\pi R^2 \sin^2\theta$$

Then to obtain μ of the whole sphere we need to integrate over all angles to get

$$\begin{aligned}
\mu = \int d\mu &= \int \frac{\sigma dA}{T}\pi R^2 \sin^2\theta \\
&= \int_0^\pi \frac{\sigma}{T} 2\pi R^4 \sin^3\theta d\theta \\
&= \int_0^\pi \frac{\pi q}{2T} R^2 \sin^3\theta d\theta \\
&= \frac{q}{3}\frac{2\pi}{T}R^2 \\
&= \frac{q}{3}\frac{2\pi}{T}R^2 \\
&= \frac{q}{3}\omega R^2
\end{aligned}$$

Since the angular momentum of the sphere is given by

$$\mathbf{L} = \frac{2}{5}mR^2\omega$$

we finally get, $\omega = \frac{5L}{2mR^2}$

$$\mu = \frac{5q}{6m}\mathbf{L}$$

as required.

2 Question 2

2.1 Show that $\langle +\psi | +\psi \rangle = 1, \langle -\psi | -\psi \rangle = 1, \langle -\psi | +\psi \rangle = 0, \langle +\psi | -\psi \rangle = 0.$

We know that,

$$| +\psi\rangle^\dagger = \langle +\psi|$$

$$| +\psi>^\dagger = \begin{bmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{i\phi}\sin\left(\frac{\theta}{2}\right) \end{bmatrix}^\dagger$$

$$\langle +\psi| = \begin{bmatrix} \cos\left(\frac{\theta}{2}\right) & e^{-i\phi}\sin\left(\frac{\theta}{2}\right) \end{bmatrix}$$

$$\langle +\psi| +\psi> = \begin{bmatrix} \cos\left(\frac{\theta}{2}\right) & e^{-i\phi}\sin\left(\frac{\theta}{2}\right) \end{bmatrix} \begin{bmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{i\phi}\sin\left(\frac{\theta}{2}\right) \end{bmatrix}$$

$$= \cos^2\left(\frac{\theta}{2}\right) + e^{-i\phi}e^{i\phi}\sin^2\left(\frac{\theta}{2}\right)$$

$$= 1$$

Similarly,

$$\langle -\psi| -\psi> = \begin{bmatrix} \sin\left(\frac{\theta}{2}\right) & -e^{-i\phi}\cos\left(\frac{\theta}{2}\right) \end{bmatrix} \begin{bmatrix} \sin\left(\frac{\theta}{2}\right) \\ -e^{i\phi}\cos\left(\frac{\theta}{2}\right) \end{bmatrix}$$

$$= \sin^2\left(\frac{\theta}{2}\right) + e^{-i\phi}e^{i\phi}\cos^2\left(\frac{\theta}{2}\right)$$

$$= \sin^2\left(\frac{\theta}{2}\right) + \cos^2\left(\frac{\theta}{2}\right)$$

$$= 1$$

$$\langle -\psi| +\psi> = \begin{bmatrix} \sin\left(\frac{\theta}{2}\right) & -e^{-i\phi}\cos\left(\frac{\theta}{2}\right) \end{bmatrix} \begin{bmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{i\phi}\sin\left(\frac{\theta}{2}\right) \end{bmatrix}$$

$$= \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) - e^{-i\phi}e^{i\phi}\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)$$

$$= \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) - \cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)$$

$$= 0$$

$$\langle +\psi| -\psi> = \begin{bmatrix} \cos\left(\frac{\theta}{2}\right) & e^{-i\phi}\sin\left(\frac{\theta}{2}\right) \end{bmatrix} \begin{bmatrix} \sin\left(\frac{\theta}{2}\right) \\ -e^{i\phi}\cos\left(\frac{\theta}{2}\right) \end{bmatrix}$$

$$= \cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right) - e^{-i\phi}e^{i\phi}\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)$$

$$= \cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right) - \cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)$$

$$= 0$$

2.2 Does the Quantum-NOT gate flip $|+\psi\rangle to| - \psi\rangle$? Can we design a Quantum NOT Gate that flips any Qubit to it's inverse?

The quantum NOT gate has a similar action on the input state to a NOT gate in classical logic circuits. In a classical logic circuit, if the input to a NOT gate is 0, the output is 1, and if the input is 1, the output is 0. The NOT gate 'flips' the input. The same is true of the quantum NOT gate — if the input state is $|0\rangle$, the quantum NOT gate turns this into a state of $|1\rangle$, and if the input state is $|1\rangle$, the quantum NOT gate turns this into a state of $|0\rangle$.

$$\text{QNOT} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\text{QNOT}(|+\rangle) = \text{QNOT}\left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right)$$

$$= \frac{1}{\sqrt{2}}\left(\text{QNOT}(|0\rangle) + \text{QNOT}(|1\rangle)\right)$$

Substituting $|1\rangle$ and $|0\rangle$ into the QNOT gate matrix, we get:

$$\text{QNOT}(|+\rangle) = \frac{1}{\sqrt{2}}(|1\rangle + |0\rangle) = |+\rangle$$

So, the QNOT gate does not change the state $|+\rangle$. QNOT gate alone cannot flip a qubit from any arbitrary state $|\psi\rangle$ to its orthogonal state $|\psi^{\perp}\rangle$.

3 Question 3 - Code

```python
import numpy as np

def linevo(A, p, n):
    """
    Calculate the final probabilistic state after multiple applications of the operator.

    Parameters:
        A (numpy.ndarray): Probabilistic operator representing the experiment.
        p (numpy.ndarray): Initial probabilistic state.
        n (int): Number of times the experiment is conducted.

    Returns:
        numpy.ndarray: Final probabilistic state.
    """
    # Ensure A is a numpy array
    A = np.array(A)
    # Ensure p is a column vector
    p = np.array(p).reshape(-1, 1)

    # Apply the operator A 'n' times to the initial state p
    final_state = np.linalg.matrix_power(A, n).dot(p)

    # Normalize the final state to ensure it's a valid probabilistic state
    final_state /= np.sum(final_state)

    return final_state

# Define the probabilistic operator A and initial state p
A = [[0.25, 0.25, 0.25, 0.25],
```

```
        [0.25, 0.25, 0.25, 0.25],
        [0.25, 0.25, 0.25, 0.25],
        [0.25, 0.25, 0.25, 0.25]]

p = [1, 0, 0, 0]

# Number of times the experiment is conducted
n = 3

# Calculate the final probabilistic state
final_state = linevo(A, p, n)
print("Final probabilistic state after", n, "applications of the operator A:")
print(final_state)
```

This function takes the probabilistic operator A, the initial state p, and the number of times the experiment is conducted n as input parameters, and returns the final probabilistic state after applying the operator n times to the initial state p. The final state is normalized to ensure it's a valid probabilistic state.

Submitted by Tushar Goyal on February 13, 2024.