ENPM667

Final Project

SURIYA SURESH
NEHA MARNE

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LIST OF ILLUSTRATIONS

Chapter 1

INTRODUCTION

Decoding the Nature of the system

In the given system The parameters are the mass of each system are as follows

- Mass $\boldsymbol{M}$ of the cart
- Mass $\boldsymbol{m1}$, $\boldsymbol{m2}$ of the loads
- Length $\boldsymbol{l1}$, $\boldsymbol{l2}$ attaching the loads to the cart
- Length $\boldsymbol{\theta 1}$, $\boldsymbol{\theta 2}$ that the loads makes with the cart
- Total external force $\boldsymbol{F}$ on the system
- There is no *friction force* i.e. μ =0

The system parameters like $\boldsymbol{M, m1, m2, l1, l2}$ are not changing with respect to time, Making the system *Linear Time Invariant* **(LTI)**

C h a p t e r 2

EQUATION OF MOTION

Lagrangian Equation

To write the Lagrangian equation of motion let's first calculate the kinetic energy of the system. Defining variables for further context,

- x = x = Lateral position of cart
- y = y = longitudinal position of cart
- x1 = Lateral position of mass *m1*

$$x1 = x - l1 * sin(\theta 1)$$

- y1 = Longitudinal position of mass *m1*

$$y1 = -l1 * cos(\theta 1)$$

- x2 = Lateral position of mass *m2*

$$x2 = x - l2 * sin(\theta 1)$$

- y2 = Longitudinal position of mass *m2*

$$y2 = -l2 * cos(\theta 2)$$

Velocity Equations

To compute the velocities associated with each pendulum, derivatives of the above equations are computed.

$$\dot{x}1 = \dot{x} - l1 * cos(\theta 1) * \dot{\theta}1$$

$$\dot{x}2 = \dot{x} - l2 * cos(\theta 2) * \dot{\theta}2$$

$$\dot{y}1 = l1 * sin(\theta 1) * \dot{\theta}1$$

$$\dot{y}2 = l2 * sin(\theta 2) * \dot{\theta}2$$

Kinetic energy equation

The displacement of the cart is only in the positive-x direction. So, the velocity of the cart is given by Using the above equation the Kinetic Energy of the system can be written as the sum of the the kinetic energy associated with the cart as well as both the pendulums.

$$KE1 = \frac{1}{2} * m1 * (\dot{x}1^2 + \dot{y}1^2)$$

$$KE1 = \frac{1}{2} * m1 * ((\dot{x} - l1 * cos(\theta 1) * \dot{\theta}1)^2 + (l1 * sin(\theta 1) * \dot{\theta}1)^2)$$

$$KE1 = \frac{1}{2} * m1 * (\dot{x}1^2 + l1 * \dot{\theta}1^2 - 2 * l1 * cos(\theta 1) * \dot{x} * \dot{\theta}1)$$

$$KE2 = \frac{1}{2} * m2 * (\dot{x}2^2 + l2 * \dot{\theta}2^2 - 2 * l2 * cos(\theta 2) * \dot{x} * \dot{\theta}2)$$

$$KE = KE1 + KE0 + KE2$$

$$KE = \frac{1}{2} * [\dot{x}^2(M + m1 + m2) + m1 * l1 * \dot{\theta}1^2 + m2 * l2 * \dot{\theta}2^2] - m1 * l1 * cos(\theta 1) * \dot{x}\dot{\theta}1 - m2 * l2 * cos(\theta 2) * \dot{x}\dot{\theta}2$$

Potential energy equation

For computing the Potential Energy of the system, the cart height is taken as a reference. Therefore, it consists of components from the pendulums only and is given by,

$$PE = -m1 * g * l1 * cos(\theta 1) - m2 * g * l2 * cos(\theta 2)$$

Lagrangian equations Now, the Lagrange of the system can be calculated as, $\mathscr{L} = KE - PE$

$$\mathscr{L} = \frac{1}{2} * [\dot{x}^2(M + m1 + m2) + m1 * l1 * \dot{\theta}1^2 + m2 * l2 * \dot{\theta}2^2] - m1 * l1 * cos(\theta 1) * \dot{x}\dot{\theta}1 - m2 * l2 * cos(\theta 2) * \dot{x}\dot{\theta}2$$

+m1*g*l1*cos($\theta 1$) $+ m2 * g * l2 * cos(\theta 2)$

We compute the derivative of the Lagrangian with respect to

$$\frac{d}{dt}[\frac{\partial \mathcal{L}}{\partial \dot{x}}] - \frac{\partial \mathcal{L}}{\partial x} = F$$

$$\frac{d}{dt}[\frac{\partial \mathcal{L}}{\partial \dot{\theta}1}] - \frac{\partial \mathcal{L}}{\partial \theta 1} = 0$$

$$\frac{d}{dt}[\frac{\partial\mathcal{L}}{\partial\dot{\theta}2}] - \frac{\partial\mathcal{L}}{\partial\theta 1} = 0$$

$$\frac{\partial\mathcal{L}}{\partial x} = 0$$

$$\frac{d}{dt}[\frac{\partial\mathcal{L}}{\partial\dot{x}}] = (M+m1+m2)*\ddot{x}-m1*l1(the\ddot{t}a1*cos(theta1)-sin(\theta 1)*\dot{(}\theta 1)^2) = F$$

-m2*l2(*cos(theta2)-sin($\theta 2$) $* \dot{(}\theta 2)^2$)

$$\frac{\partial\mathcal{L}}{\partial\dot{\theta}1} = m1 * l1^2 * \dot{\theta}1 - m1 * l1 * cos(\theta 1)\dot{x}$$

$$\frac{d}{dt}[\frac{\partial\mathcal{L}}{\partial\dot{\theta}1}] = m1 * l1^2 * \ddot{\theta 1} - m1 * l1 * (cos(\theta 1)\ddot{x} - sin(\theta 1)\dot{x}\dot{\theta}1$$

$$\frac{d}{dt}[\frac{\partial\mathcal{L}}{\partial\dot{\theta}1}] - \frac{\partial\mathcal{L}}{\partial\dot{\theta}1} = m1 * l1^2 * \ddot{\theta 1} - m1 * l1 * cos(\theta 1)\ddot{x} + m1 * g * l1 * sin(\theta 1) = 0$$

similarly, we can write for $\theta 2$

$$\frac{d}{dt}[\frac{\partial\mathcal{L}}{\partial\dot{\theta}2}] - \frac{\partial\mathcal{L}}{\partial\dot{\theta}2} = m2 * l2^2 * \ddot{\theta 2} - m2 * l2 * cos(\theta 2)\ddot{x} + m2 * g * l2 * sin(\theta 2) = 0$$

Using the above equation we can write,

$$\ddot{x} = \frac{F + m1 * l1 * (cos(\theta 1) * \ddot{\theta 1} - sin(\theta 1)^2 * sin(\theta 1) + m2 * l2 * (cos(\theta 2) * \ddot{\theta 2} - sin(\theta 2)^2 * sin(\theta 2)}{(M + m1 + m2)}$$

$$\ddot{\theta 1} = \frac{cos(\theta 1) * \ddot{x} - g * sin(\theta 1)}{l1}$$

$$\ddot{\theta 2} = \frac{cos(\theta 2) * \ddot{x} - g * sin(\theta 2)}{l2}$$

Putting the values of $\theta 1 and \theta 2 in equation for x we get$,

Equation of motion

$$\ddot{x} = \frac{F - m1(gsin(\theta 1)cos(\theta 1 + l1sin(\theta 1)\dot{\theta}1^2 - m2(gsin(\theta 2)cos(\theta 2 + l2sin(\theta 2)\dot{\theta}2^2}{M + m1sin(\theta 1)^2 + m2sin(\theta 2)^2}$$

$$\ddot{\theta}1 = \frac{cos(\theta 1)}{l1} \frac{F - m1(gsin(\theta 1)cos(\theta 1 + l1sin(\theta 1)\dot{\theta}1^2 - m2(gsin(\theta 2)cos(\theta 2 + l2sin(\theta 2)\dot{\theta}2^2}{M + m1sin(\theta 1)^2 + m2sin(\theta 2)^2} - g\frac{sin(\theta 1)}{l1}$$

$$\ddot{\theta}2 = \frac{cos(\theta 2)}{l2} \frac{F - m1(gsin(\theta 1)cos(\theta 1 + l1sin(\theta 1)\dot{\theta}1^2 - m2(gsin(\theta 2)cos(\theta 2 + l2sin(\theta 2)\dot{\theta}2^2}{M + m1sin(\theta 1)^2 + m2sin(\theta 2)^2} - g\frac{sin(\theta 2)}{l2}$$

Chapter 3

STATE SPACE MODEL

For the state space model, we need to decide state space variable State space variables must follow the following conditions

- Can be used to compute the output of the system
- If I know X(t0) and the input from t0 until t1 then I must be able to compute X(t1)

Using the degree of freedom to pin down the state space variables we use

$$[\dot{x}, \ddot{x}, \dot{\theta}1, \ddot{\theta}2, \dot{\theta}2, \ddot{\theta}2]$$

Non=linear state space model is

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta}1 \\ \ddot{\theta}1 \\ \dot{\theta}2 \\ \ddot{\theta}2 \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \ddot{x} = \frac{F-m1(gsin(\theta1)cos(\theta1+l1sin(\theta1)\dot{\theta}1^2-m2(gsin(\theta2)cos(\theta2+l2sin(\theta2)\dot{\theta}2^2}{M+m1sin(\theta1)^2+m2sin(\theta2)^2} \\ \dot{\theta}1 \\ \frac{cos(\theta1)}{l1}\frac{F-m1(gsin(\theta1)cos(\theta1+l1sin(\theta1)\dot{\theta}1^2-m2(gsin(\theta2)cos(\theta2+l2sin(\theta2)\dot{\theta}2^2}{M+m1sin(\theta1)^2+m2sin(\theta2)^2} - g\frac{sin(\theta1)}{l1} \\ \dot{\theta}2 \\ \frac{cos(\theta2)}{l2}\frac{F-m1(gsin(\theta1)cos(\theta1+l1sin(\theta1)\dot{\theta}1^2-m2(gsin(\theta2)cos(\theta2+l2sin(\theta2)\dot{\theta}2^2}{M+m1sin(\theta1)^2+m2sin(\theta2)^2} - g\frac{sin(\theta2)}{l2} \end{bmatrix} \tag{3.1}$$

Chapter 4

LINEARISED MODEL

For linearising we use the following approximations

$$cos(\theta) = 1$$

$$sin(\theta) = \theta$$

$$\theta 1^2 = 0$$

$$\theta 2^2 = 0$$

Putting these values in equation(3.1)

$$\ddot{x} = \frac{F - m1g\theta 1 - m2g\theta 2}{M}$$

$$\ddot{\theta}1 = \frac{-g(M+m1)}{Ml1}\theta 1 - \frac{gm2}{Ml1}\theta 2 + \frac{F}{Ml1}$$

$$\ddot{\theta}1 = -\frac{gm1}{Ml2}\theta 1 - \frac{-g(M+m2)}{Ml2}\theta 2 + \frac{F}{Ml2}$$

The linearised state space model for

$$x = 0, \theta 1 = 0, \theta 2 = 0$$

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta}1 \\ \ddot{\theta}1 \\ \dot{\theta}2 \\ \ddot{\theta}2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-gm1}{M} & 0 & \frac{-gm2}{M} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-g(m1+M)}{Ml1} & 0 & \frac{-gm2}{Ml1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{-gm1}{Ml2} & 0 & \frac{-g(m2+M)}{Ml2} & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta}1 \\ \ddot{\theta}1 \\ \dot{\theta}2 \\ \ddot{\theta}2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{1}{Ml1} \\ 0 \\ \frac{1}{Ml2} \end{bmatrix} F \tag{4.1}$$

Chapter 5

CONTROLLABILITY

Since the given system is time-invariant for calculating the controllability condition the controllability matrix given below must be a full-rank matrix The dimensions of the controllability matrix ‘C’ are n x nm, hence its rank should be equal to n. Hence,

$$rank(C) = rank[B, AB, A^2B, A^3B, A^4B, A^5B] = n$$

Using the python code given in the appendix we can calculate the controllability matrix as given below, For determining the rank of the matrix we can calculate the

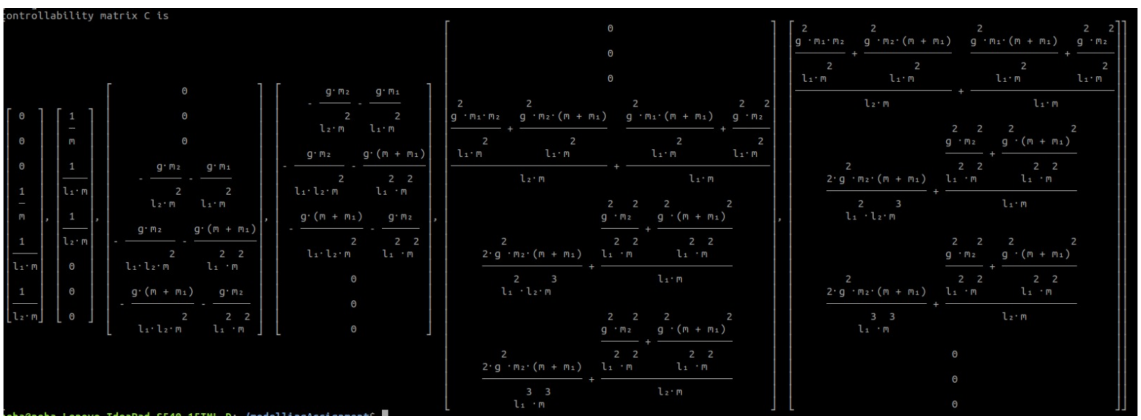

Figure 5.1: linearised Controllability matrix

determinant of the matrix

$$det(C) = \frac{-g^6(l1^2 - l2^2)}{(Ml1l2)^6}$$

The system is uncontrollable when matric "C" is full rank matrix i.e. det(C) ≠ 0 when l1 ≠ l2 system is controllable

Chapter 6

LQR CONTROLLER

putting the values of M=1000, m1=m2=100, l1=20, l2=10 in the controllability matrix and calculating its determinant. According to the above condition of determinant in the chapter on controllability, l1 ≠ l2 thus the system is **controllable**.

To obtain the LQR controller, we put the values in equation 4.1 we get,

$$\begin{bmatrix} 0 & 0 & 0 & 1.0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.0 \\ 0 & -0.98 & -0.98 & 0 & 0 & 0 \\ 0 & -0.539 & -0.049 & 0 & 0 & 0 \\ 0 & -0.098 & -1.078 & 0 & 0 & 0 \end{bmatrix}$$

Figure 6.1: A_Matrix

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.001 \\ 5.0e-5 \\ 0.0001 \end{bmatrix}$$

Figure 6.2: B_Matrix

With the given values of A and B we use tuned the Q and R as follows, R = 0.0001

```
Q =

   1.0e+03 *

    0.0100         0         0         0         0         0
         0    0.0000         0         0         0         0
         0         0    1.0000         0         0         0
         0         0         0         0         0         0
         0         0         0         0    1.0000         0
         0         0         0         0         0         0
```

Figure 6.3: Q matrix

The output of the system for given values of Q an R is, According to the lecture

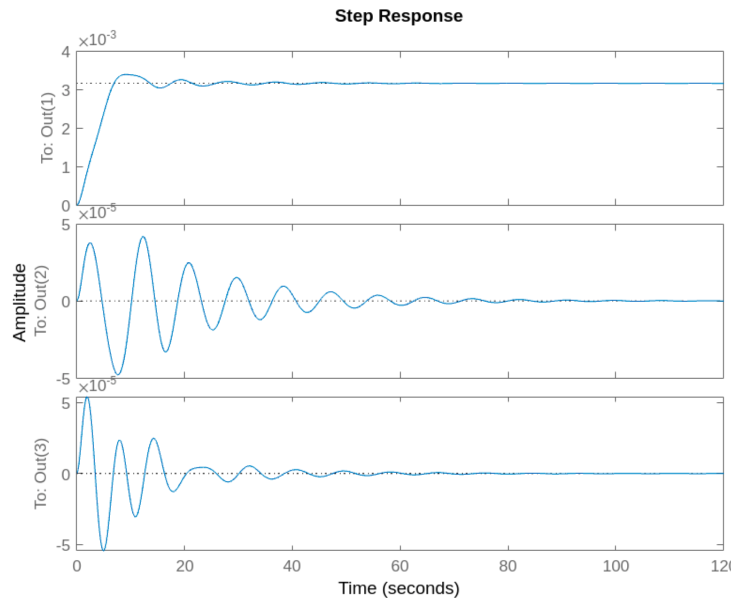

Figure 6.4: Q matrix

```
P =

  -0.0543 + 0.7183i
  -0.0543 - 0.7183i
  -0.1554 + 1.0465i
  -0.1554 - 1.0465i
  -0.3471 + 0.3755i
  -0.3471 - 0.3755i
```

Figure 6.5: eigen values

notes, Lyapunov's indirect method states that we first linearize the original system around the equilibrium point of interest and then check the eigen values as shown above. Our eigen values of A matrix have negative real part, and hence the original system is at least locally stable, around equilibrium point. In this case, a Lyapunov function for the linearized system will be valid at least locally. The initial states of the system were taken to be 10 degrees for mass 1 and 5 degrees for mass 2

Chapter 7

PART TWO

7.1 Observability

For the given system, the observability is checked for the given output vectors of: $x(t), (\theta_1(t), \theta_2(t)), (x(t), \theta_2(t))and(x(t)\theta_1(t), \theta_2(t))$

They are labeled as C1, C2, C3, and C4 respectively.The observability of the output vectors is found by the observability matrix whose rank should equal the number of rows of A which is 6.

$$\mathbf{rank}[\mathbf{C^T}(\mathbf{A^T})^\mathbf{1}\mathbf{C}...(\mathbf{A^T})^{\mathbf{n-1}}\mathbf{C}] = n$$

The C1 matrix is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The C2 Matrix is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The C3 Matrix is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

The C4 Matrix is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

The ranks of the observability matrices of C1, C3, and C4 are 6. We find that for C2 it is 4, which means that it cannot be observed and hence we do not calculate the Luenberger Observer for C2.

Chapter 8

LUENBERGER OBSERVER

Here we use the Luenberger Observer as the state observer to predict the states of the system. It is defined by the equation:

$$\dot{\hat{\vec{X}}}(t) = A\hat{\vec{X}}(t) + B_k\vec{U}_k(t) + L(\vec{Y}(t) - C\hat{\vec{X}}(t)) \tag{8.1}$$

L is the observer gain matrix and $(\vec{Y}(t) - \vec{C}\hat{\vec{X}}(t))$ is the correction term and when the time is zero, $X(0) = 0$. The estimation error is

$$\dot{\vec{X}}_e(t) = \vec{X}(t) - \hat{\vec{X}}t \tag{8.2}$$

can be represented in state space as :

$$\dot{\vec{X}}_e(t) = A\vec{X}_e(t) - L(\vec{Y}(t) - C\hat{\vec{X}}(t)) + B_d\vec{U}_d(t) \tag{8.3}$$

We assume D is zero.Therefore $\vec{Y}(t) = C\hat{\vec{X}}$.So equation 5.3 becomes:

$$\dot{\vec{X}}_e(t) = (A - LC)\vec{X}_e(t) + B_d\vec{U}_d(t) \tag{8.4}$$

We formulated the observers for the observable vectors C1, C3 and C4. The plots for the response of the system to the unit step and initial state inputs have been plotted and described below:

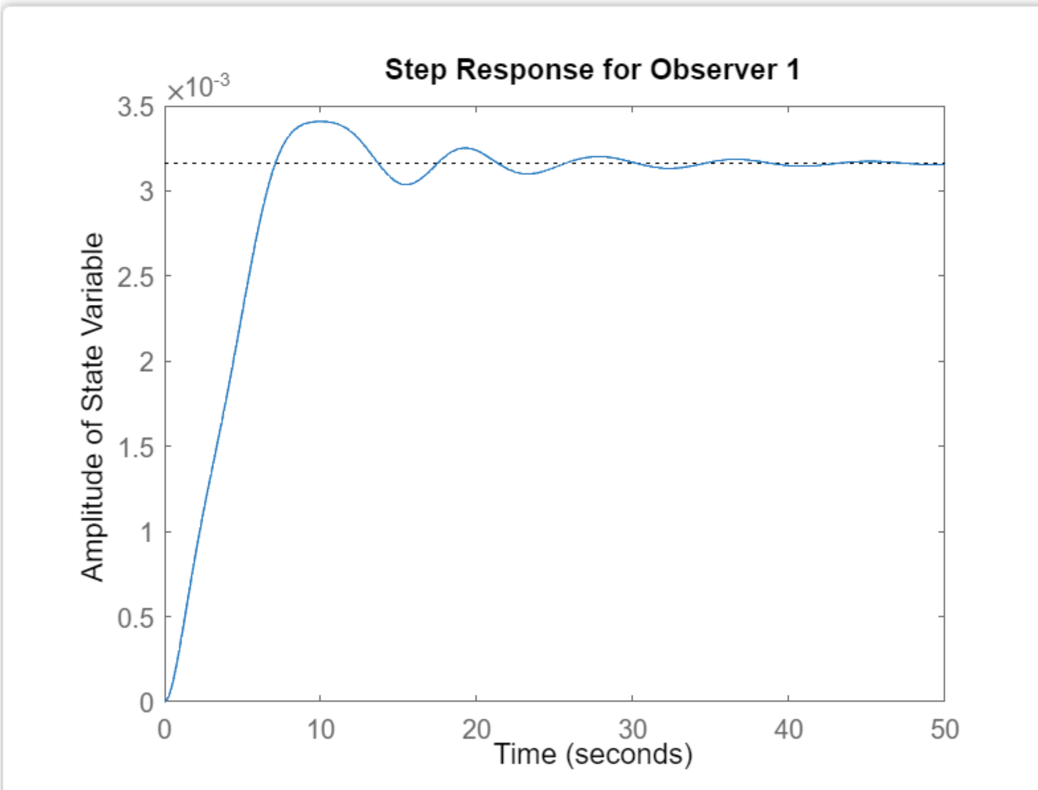

Figure 8.1: Plot of Step Response of Observer 1

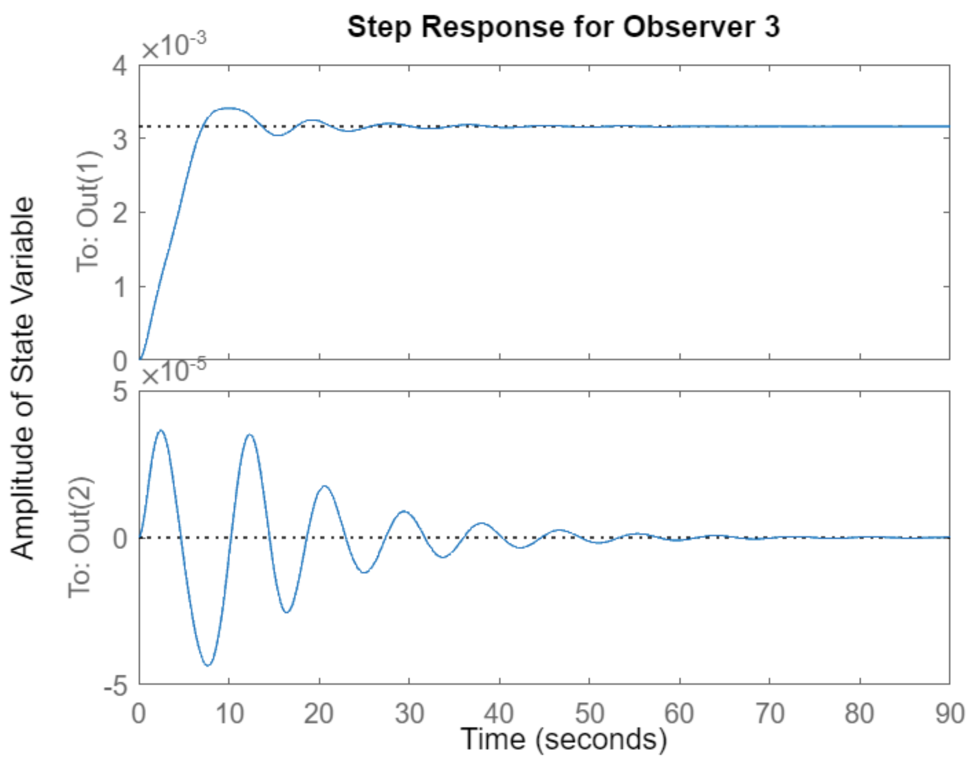

Figure 8.2: Plot of Step Response of Observer 3

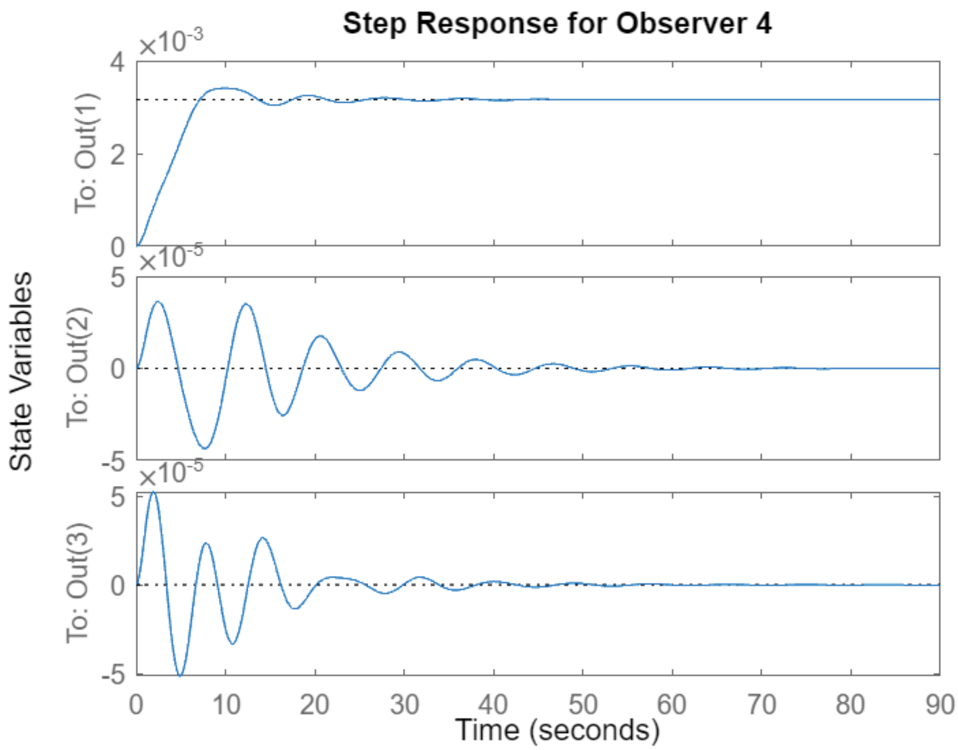

Figure 8.3: Plot of Step Response of Observer 4

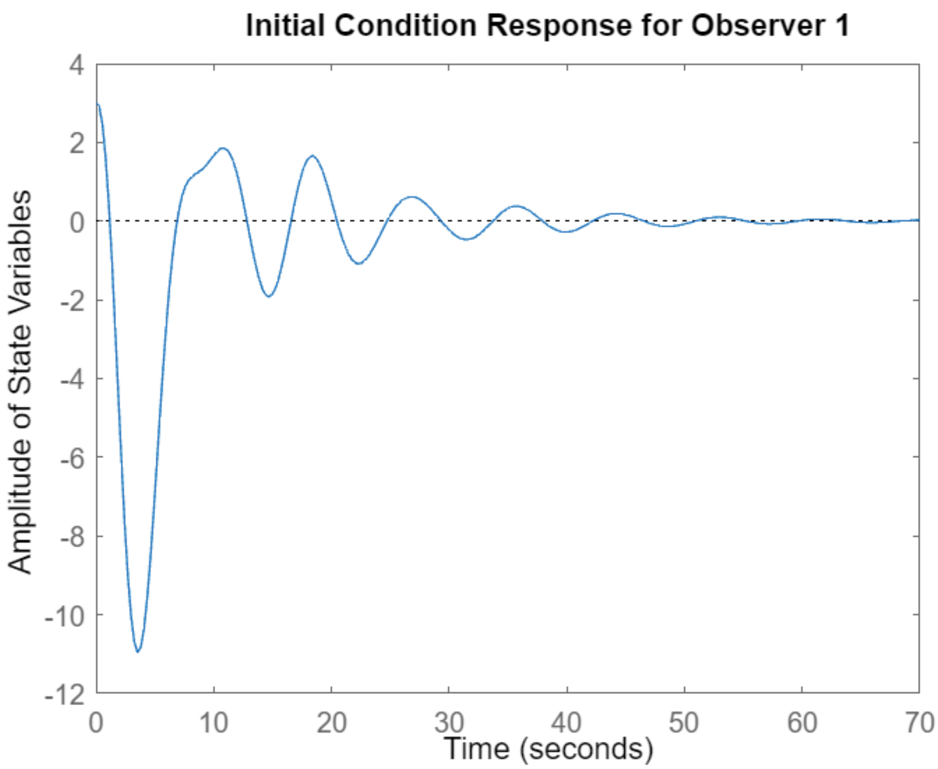

Figure 8.4: Plot of Initial State Response of Observer 1

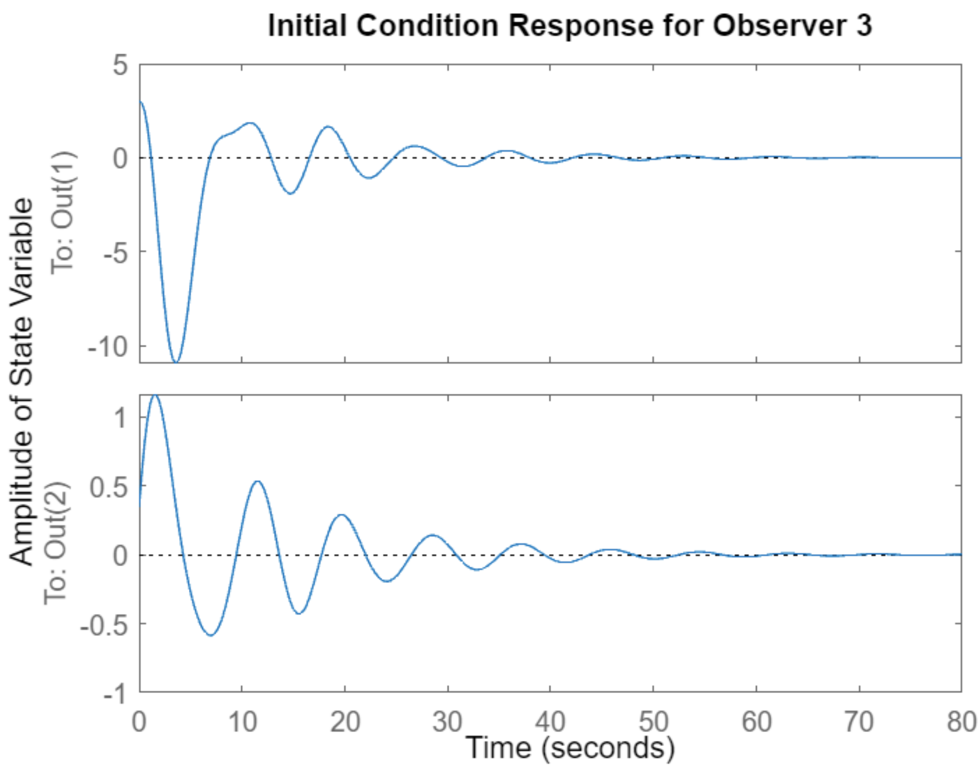

Figure 8.5: Plot of Initial State Response of Observer 3

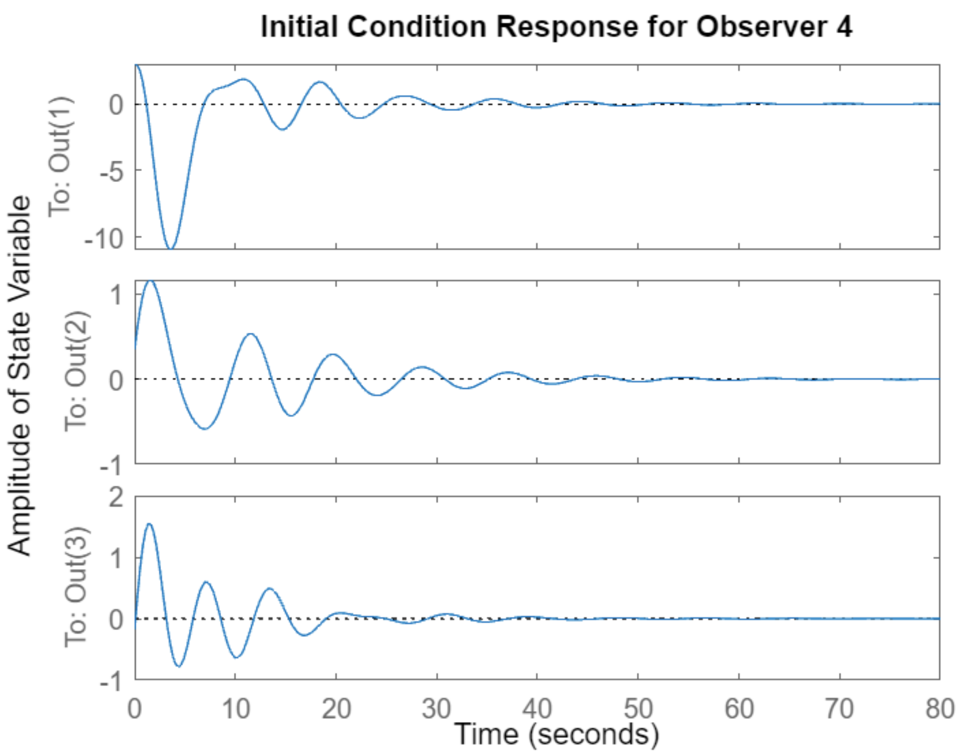

Figure 8.6: Plot of Initial State Response of Observer 4

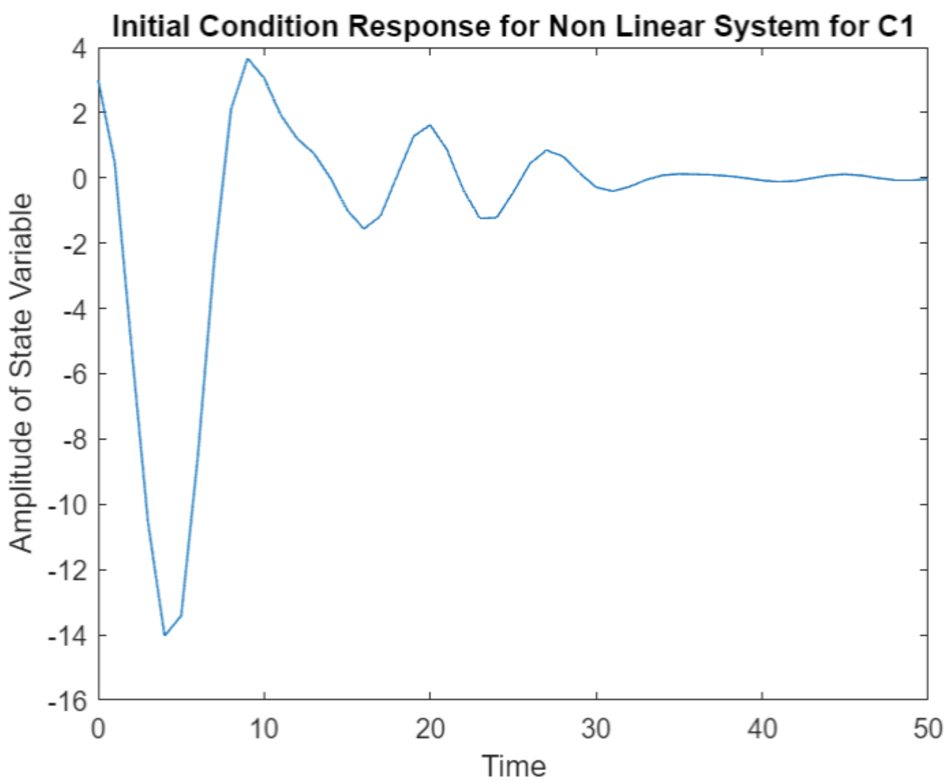

Figure 8.7: Plot of Non-Linear System Response of Observer 1

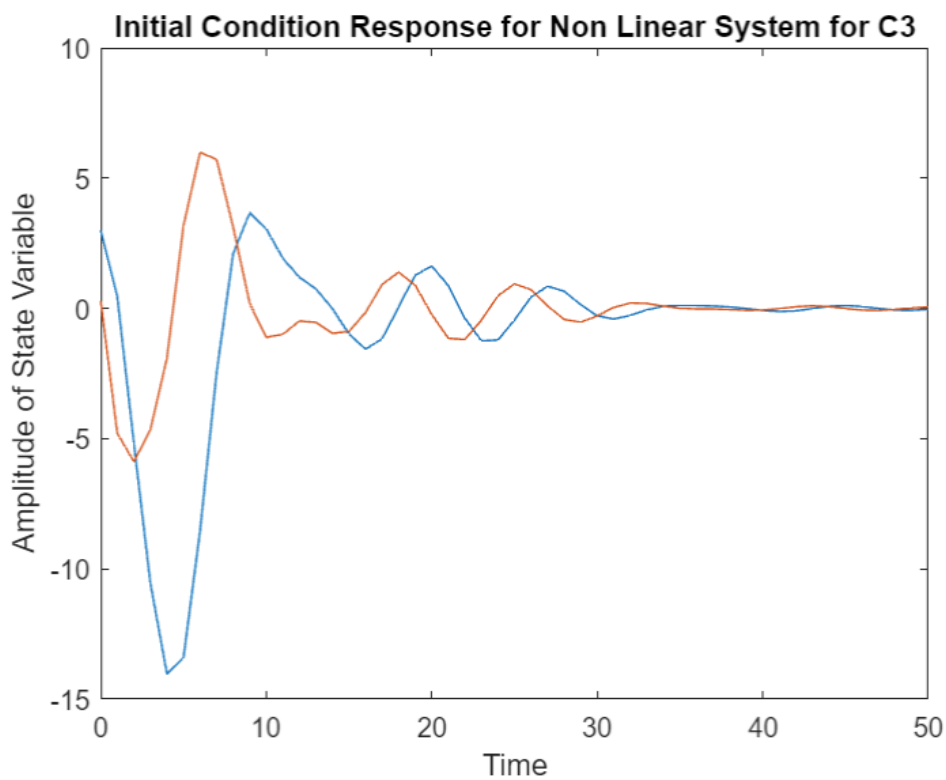

Figure 8.8: Plot of Non-Linear System Response of Observer 3

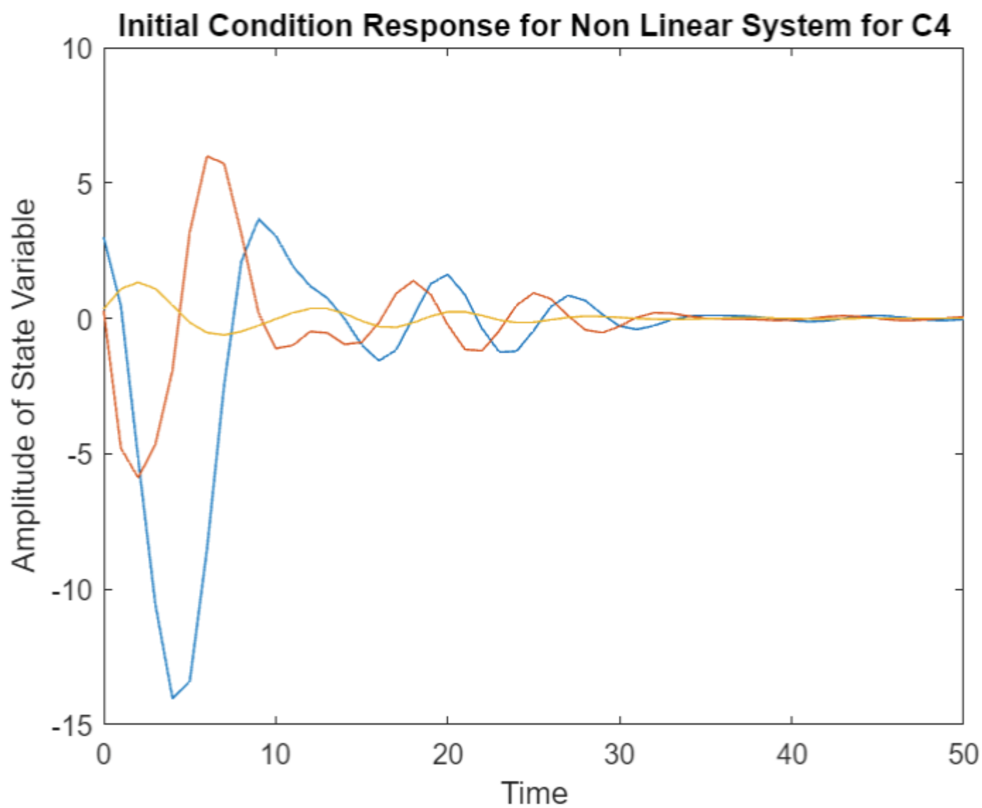

Figure 8.9: Plot of Non-Linear System Response of Observer 4

Chapter 9

LQG CONTROLLER

The LQG controller is a combination of the LQR controller and a Kalman filter. This controller takes into account process and measurement noise. The measurement and process noise are zero-mean Gaussian processes. The smallest output vector was $x(t)$, hence the C1 matrix was used for this system.

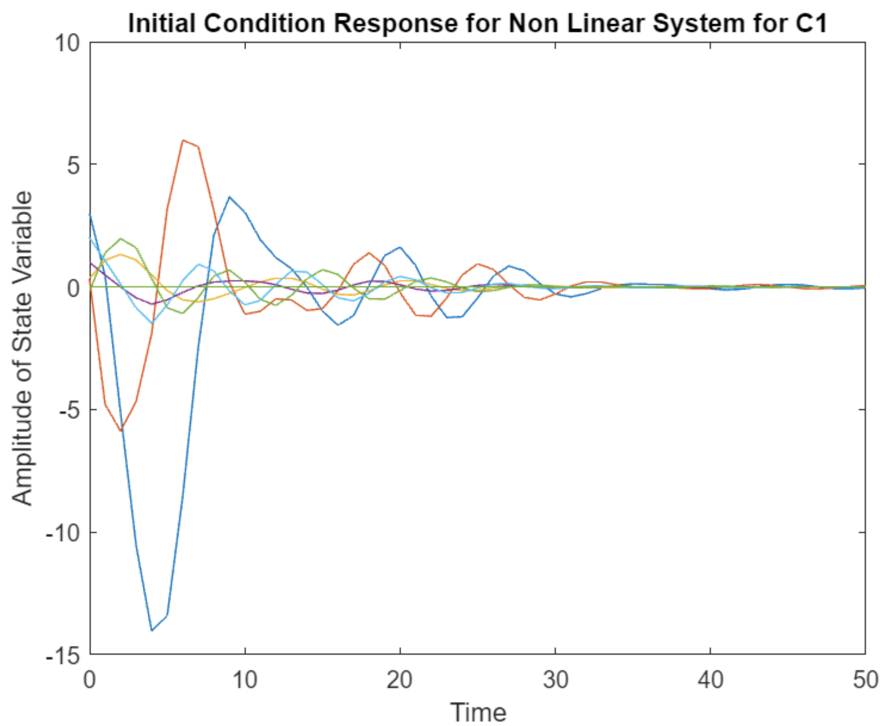

Figure 9.1: Plot of Initial State Response of C1

In order to track a constant reference on x asymptotically, the LQR controller part used in the LQG controller can be modified to

$$\int_0^\infty (\vec{X}(t) - \vec{X}_d)^T Q(\vec{X}(t) - \vec{X}_d) + (\vec{U}_k(t) - \vec{U}_\infty)^T R(\vec{U}_k(t) - \vec{U}_\infty)dt \tag{9.1}$$

.

The designed LQG controller will be able to reject constant force disturbance applied to the system. Modeling the constant force disturbance as Gaussian white noise in the controller, the system will reject the disturbance.