The Predicate Calculus

Outline

- Motivation
- The Language and Its Syntax
- Semantics
- Quantification
- Semantics of Quantifiers
- Predicate Calculus as a Language for Representing Knowledge

Motivation

- Propositional calculus
  - Expressional limitation
  - Atoms have no internal structures.
- First-order predicate calculus
  - has names for objects as well as propositions.
  - Symbols
    - Object constants
    - Relation constants
    - Function constants
  - Other constructs
  - Refer to objects in the world
  - Refer to propositions about the world

The Language and its Syntax

- Components
  - Infinite set of *object constants*
    - Aa, 125, 23B, Q, John, EiffelTower
  - Infinite set of *function constants*
    - $fatherOf^1$, $distanceBetween^2$, $times^2$
  - Infinite set of *relation constants*
    - $Parent^2$, $Large^1$, $Clear^1$
  - Propositional connectives

$$\vee,\ \wedge,\ \neg,\ \supset$$

  - Delimiters
    - (,), [,] ,(separator)

The Language and its Syntax

- Terms
  - Object constant is a term
  - Functional expression
    - fatherOf(John, Bill), times(4, plus(3, 6)), Sam
- wffs
  - Atoms
    - Relation constant of arity *n* followed by *n* terms is an *atom* (*atomic formula*)
    - An atom is a wff.
    - Greaterthan(7,2), P(A, B, C, D), Q
  - Propositional wff

$$[Greaterthan(7,2) \wedge \mathrm{Lessthan}(15,4)] \vee \neg Brother(John, Sam) \vee P$$

Semantics

- Worlds
  - Individuals
    - Objects
    - Concrete examples: Block A, Mt. Whitney, Julius Caesar, …
    - Abstract entities: 7, set of all integers, …
    - Fictional/invented entities: beauty, Santa Claus, a unicorn, honesty, …
  - Functions on individuals
    - Map *n* tuples of individuals into individuals
  - Relations over individuals
    - Property: relation of arity 1 (heavy, big, blue, …)
    - Specification of *n*-ary relation: list all the n tuples of individuals

Semantics

- Interpretations
  - Assignment: maps the followings
    - object constants into objects in the world
    - *n*-ary constants into *n*-ary functions
    - *n*-ary relation constants into *n*-ary relations
    - called *denotations* of corresponding predicate-calculus expressions
  - Domain
    - Set of objects to which object constant assignments are made
  - *True*/*False* values

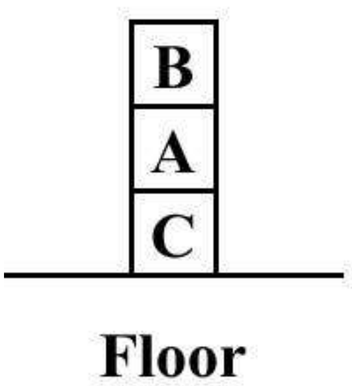

Fig.1 A Configuration of Blocks

Predicate Calculus	World
A	**A**
B	**B**
C	**C**
F1	**Floor**
On	**On**={<**B,A**>, <**A,C**>, <**C, Floor**>
Clear	**Clear={<B>}**

Table 1: A Mapping between Predicate Calculus and the World

Determination of the value of some predicate-claculus wffs

On(A,B) is *False* because <A,B> is not in the relation **On**.

Clear(B) is *True* because <B> is in the relation **Clear**.

On(C,F1) is *True* because <C,Floor> is in the relation **On**.

On(C,F1) ∧¬On(A,B) is *True* because both **On**(C,F1) and ¬ **On**(A,B) are True

Semantics

- Models and Related Notions
  - An interpretation *satisfies* a wff
    - wff has the value *True* under that interpretation
  - Model of wff
    - An interpretation that satisfies a wff
  - Valid wff
    - Any wff that has the value *True* under *all* interpretations
  - *inconsistent*/ *unsatisfiable* wff
    - Any wff that does not have a model
  - Δ *logically entails* ω ($\Delta \models \omega$)
    - A wff ω has value *True* under all of those interpretations for which each of the wffs in a set Δ has value *True*
  - *Equivalent* wffs
    - Truth values are identical under *all* interpretations

Semantics

- Knowledge
  - Predicate-calculus formulas
    - represent knowledge of an agent
  - Knowledge base of agent
    - Set of formulas

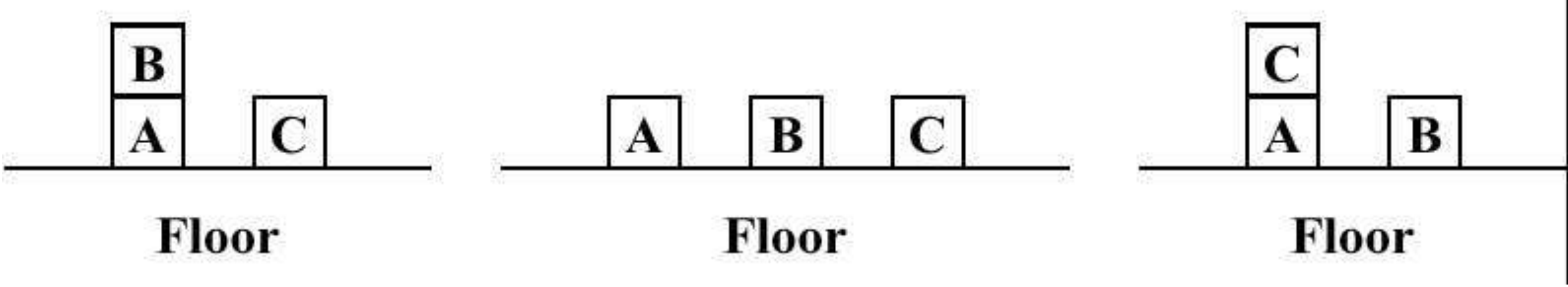

Fig. 2: Three Blocks-World Situations

Quantification

- Finite domain
  - Clear(B1) $\wedge$ Clear(B2) $\wedge$ Clear(B3) $\wedge$ Clear(B4)
  - Clear(B1) $\vee$ Clear(B2) $\vee$ Clear(B3) $\vee$ Clear(B4)
- Infinite domain
  - Problems of long conjunctions or disjunctions → impractical
- New syntactic entities
  - Variable symbols
    - consist of strings beginning with lowercase letters
    - term
  - Quantifier symbols → give expressive power to predicate-calculus
    - $\forall$: universal quantifier
    - $\exists$: existential quantifier

Quantification

- $(\forall\zeta)\omega,\ (\exists\zeta)\omega$: wff
  - ω: wff → within the scope of the quantifier
  - ζ: quantified variable
- Closed wff (closed sentence)
  - All variable symbols besides ζ in ω are quantified over in ω

$$(Ax)[P(x) \supset R(x)],\quad (\exists x)[P(x) \supset (Ey)[R(x,y) \supset S(f(x))]]$$

  - Property

$$(\forall x)[(\forall y)\omega] = (\forall y)[(\forall x)\omega] = (\forall x, y)\omega$$
$$(\forall x)[(\exists y)\omega] \neq (\exists y)[(\forall x)\omega]$$

- First-order predicate calculi
  - restrict quantification over relation and function symbols

Semantics of Quantifiers

- Universal Quantifiers
  - $(\forall\zeta)\omega(\zeta)$ = *True*
    - $\omega(\zeta)$ is *True* for *all* assignments of ζ to objects in the domain
  - Example: $(\forall x)[On(x,C) \supset \neg Clear(C)]$? in Fig. 1
    - x: A, B, C, Floor
    - investigate each of assignments in turn for each of the interpretations
- Existential Quantifiers
  - $(\exists\zeta)\omega(\zeta)$ = *True*
    - $\omega(\zeta)$ is *True* for *at least one* assignments of ζ to objects in the domain

Semantics of Quantifiers

- Useful Equivalences
  - $\neg(\forall \xi)\omega(\zeta) \equiv (\exists \zeta)\neg\omega(\zeta)$
  - $\neg(\exists \xi)\omega(\zeta) \equiv (\forall \zeta)\neg\omega(\zeta)$
  - $(\forall \xi)\omega(\zeta) \equiv (\forall \eta)\ \omega(\eta)$
- Rules of Inference
  - Propositional-calculus rules of inference → predicate calculus
    - *modus ponens*
    - Introduction and elimination of $\wedge$
    - Introduction of $\vee$
    - $\neg$ elimination
    - Resolution
  - Two important rules
    - Universal instantiation (UI)
    - Existential generalization (EG)

Semantics of Quantifiers

- Universal instantiation
  - $(\forall \xi)\omega(\zeta) \rightarrow \omega(\alpha)$
  - $\omega(\zeta)$: wff with variable ζ
  - α: constant symbol
  - $\omega(\alpha)$: $\omega(\zeta)$ with substituted for ζ throughout ω
  - Example: $(\forall x)P(x, f(x), B) \rightarrow P(A, f(A), B)$

Semantics of Quantifiers

- Existential generalization
  - $\omega(\alpha) \rightarrow (\exists\xi)\omega(\zeta)$
  - $\omega(\alpha)$: wff containing a constant symbol α
  - $\omega(\zeta)$: form with ξ replacing every occurrence of α throughout ω
  - Example: $(\forall x)Q(A, g(A), x) \rightarrow (\exists y)(\forall x)Q(y, g(y), x)$

Predicate Calculus as a Language for Representing Knowledge

- Conceptualizations
  - Predicate calculus
    - language to express and reason the knowledge about real world
    - represented knowledge: explored throughout logical deduction
  - Steps of representing knowledge about a world
    - To conceptualize a world in terms of its objects, functions, and relations
    - To invent predicate-calculus expressions with objects, functions, and relations
    - To write wffs satisfied by the world: wffs will be satisfied by other interpretations as well

Predicate Calculus as a Language for Representing Knowledge

- Usage of the predicate calculus to represent knowledge about the world in AI
  - John McCarthy (1958): first use
  - Guha & Lenat 1990, Lenat 1995, Lenat & Guha 1990
    - CYC project
    - represent millions of commonsense facts about the world
  - Nilsson 1991: discussion of the role of logic in AI
  - Genesereth & Nilsson 1987: a textbook treatment of AI based on logic

Predicate Calculus as a Language for Representing Knowledge

- Examples
  - Examples of the process of conceptualizing knowledge about a world
  - packages in an office building
    - **Package**(x): the property of something being a package
    - **Inroom**(x, y): certain object is in a certain room
    - Relation constant **Smaller**(x,y): certain object is smaller than another certain object
    - "All of the packages in room 27 are smaller than any of the packages in room 28"

$$(\forall x, y)\{[\text{Package}(x) \wedge \text{Package}(y) \wedge \text{Inroom}(x, 27) \wedge \text{Inroom}(y, 28)] \supset \text{Smaller}(x, y)\}$$

Language for Representing Knowledge

- "Every package in room 27 is smaller than one of the packages in room 28"

$$(\exists y)(\forall x)\{[\text{Package}(x) \wedge \text{Package}(y) \wedge \text{Inroom}(x,27) \wedge \text{Inroom}(y,28)] \supset \text{Smaller}(x,y)\}$$

$$(\forall x)(\exists y)\{[\text{Package}(x) \wedge \text{Package}(y) \wedge \text{Inroom}(x,27) \wedge \text{Inroom}(y,28)] \supset \text{Smaller}(x,y)\}$$

- Way of stating the arrival time of an object
  - Arrived(x,z)
  - X: arriving object
  - Z: time interval during which it arrived
  - "Package A arrived before Package B"

$$(\exists z1, z2)[\text{Arrived}(A, z1) \wedge \text{Arrived}(B, z2) \wedge \text{Before}(z1, z2)]$$

  - Temporal logic: method of dealing with time in computer science and AI

Language for Representing Knowledge

- Difficult problems in conceptualization
  - “The package in room 28 contains one quart of milk”
    - Mass nouns
    - Is milk an object having the property of being white?
    - What happens when we divide quart into two pints?
    - Does it become two objects, or does it remain as one?
  - Extensions to the predicate calculus
    - allow one agent to make statements about the knowledge of another agent
    - “Robot A knows that Package B is in room 28”