Propositional Logic

General Introduction - Logic

- Its simply a science of correct reasoning – or a scientific study of the process of reasoning and the system of rules and procedures that help in the reasoning process
- Basically logic process uses some premises (inputs) and produces some conclusions (output)

Propositional Logic - propositions

- Binary-valued features on what is true about the world and what is not true
- easy to communicate
- In cases where the values of some features cannot be sensed directly, their values can be inferred from the values of other features
- They are simply statements which are either true or false

Illustration

- Motivating Example
  - Consider a robot that is able to lift a block, if that block is liftable and the robot's battery power source is adequate
  - If both are satisfied, then when the robot tries to lift a block it is holding, its arm moves.
    - x_1 (*BAT_OK*)
    - x_2 (*LIFTABLE*)
    - x_3 (*MOVES*)
  - constraint in the language of the propositional calculus

    BAT_OK $\wedge$ LIFTABLE $\supset$ MOVES

What is Logic?

- *Logic* involves
  - A *language* (with a *syntax*)
  - *Inference rule*
  - *Semantics* for associating elements of the language with elements of some subject matter
- Two logical languages
  - *propositional calculus*
  - *first-order predicate calculus* (*FOPC*)

The Language

- Elements
  - *Atoms*
    - two distinguished atoms T and F and the countably infinite set of those strings of characters that begin with a capital letter, for example, P, Q, R, …, P1, P2, ON_A_B, and so on.
  - *Connectives*
    - $\vee$, $\wedge$, $\supset$, and $\neg$, called “or”, “and”, “implies”, and “not”, respectively.
  - *Syntax* of *well-formed formula* (*wff*), also called sentences
    - Any atom is a wff.
    - If w_1 and w_2 are wffs, so are $w_1 \vee w_2$, $w_1 \wedge w_2$, $w_1 \supset w_2$, $\neg w_1$.
    - There are no other wffs.

The Language

- *Literal*
  - atoms and a $\neg$ sign in front of them
- *Antecedent* and *Consequent*
  - In $w_1 \supset w_2$, w_1 is called the antecedent of the implication.
  - w_2 is called the *consequent* of the implication.

Rule of Inference

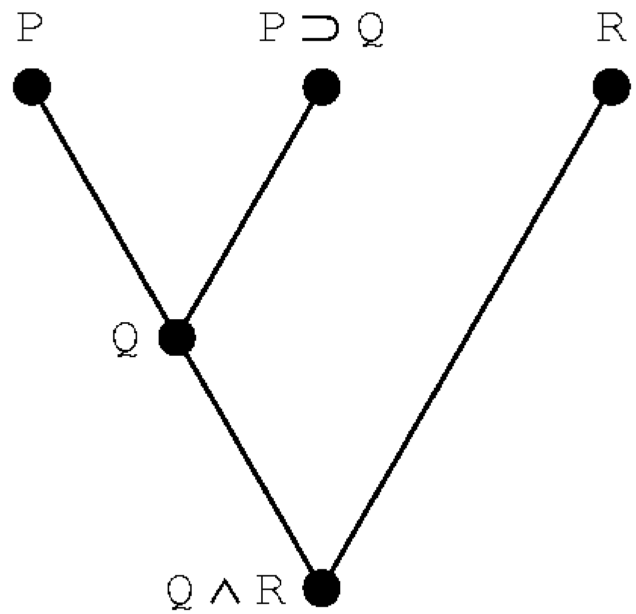
- Ways by which additional wffs can be produced from other ones
- Commonly used rules
  - ***modus ponens***: wff w_2 can be inferred from the wff w_1 and $w_1 \supset w_2$
  - ***Modus tollens:*** $\neg w_1$ can be inferred from the wff $\neg w_2$ and $w_1 \supset w_2$
  - **$\wedge$ introduction**: wff $w_1 \wedge w_2$ can be inferred from the two wffs w_1 and w_2
  - **commutativity $\wedge$**: wff $w_2 \wedge w_1$ can be inferred from the wff $w_1 \wedge w_2$
  - **$\wedge$ elimination (simplification)**: wff w_1 can be inferred from the $w_1 \wedge w_2$
  - **$\vee$ introduction (addition)**: wff $w_1 \vee w_2$ can be inferred from either from the single wff w_1 or from the single wff w_2
  - **$\neg$ elimination (Negation)**: wff w_1 can be inferred from the wff $\neg (\neg w_1)$.

Definitions of Proof

- Proof
  - The sequence of wffs $\{w_1, w_2, \ldots, w_n\}$ is called a ***proof*** of w_n from a set of wffs Δ iff each w_i is either in Δ or can be inferred from a wff earlier in the sequence by using one of the rules of inference.
- Theorem
  - If there is a **proof** of w_n from Δ, w_n is a **theorem** of the set Δ.

Example

- Given a set, Δ, of wffs: {P, R, P ⊃ Q}, {P, P ⊃ Q, Q, R, Q ∧ R} is a proof of Q ∧ R.
- The concept of proof can be based on a partial order.



Semantics

- Semantics
  - Has to do with associating elements of a logical language with elements of a domain of discourse.
  - Meaning
    - Such association
- Interpretation
  - An association of atoms with propositions
  - Denotation
    - In a given interpretation, the proposition associated with an atom

Semantics

- Under a given interpretation, atoms have *values* – *True* or *False*.
- Special Atom
  - T : always has value True
  - F : always has value False
- An interpretation by assigning values directly to the atoms in a language can be specified

Propositional Truth Table

- Given the values of atoms under some interpretation, use a truth table to compute a value for any wff under that same interpretation.
- Let w_1 and w_2 be wffs.
  - $(w_1 \wedge w_2)$ has *True* if both w_1 and w_2 have value *True*.
  - $(w_1 \vee w_2)$ has *True* if one or both w_1 or w_2 have value *True*.
  - $\neg\, w_1$ has value True if w_1 has value *False*.
  - The semantics of $\supset$ is defined in terms of $\vee$ and $\neg$.

    Specifically, $(w_1 \supset w_2)$ is an alternative and equivalent form of $(\neg\, w_1 \vee w_2)$.

Propositional Truth Table (2)

- If an agent describes its world using *n* features and these features are represented in the agent's model of the world by a corresponding set of *n* atoms, then there are 2^n different ways its world can be.
- Given values for the *n* atoms, the agent can use the truth table to find the values of any wffs.
- Suppose the values of wffs in a set of wffs are given.
  - Do those values induce a unique interpretation?
  - Usually "No."
  - Instead, there may be many interpretations that give each wff in a set of wffs the value *True* .

Satisfiability

- An interpretation *satisfies* a wff if the wff is assigned the value True under that interpretation.
- *Model*
  - An interpretation that satisfies a wff
  - In general, the more wffs that describe the world, the fewer models.
- *Inconsistent* or *Unsatisfiable*
  - When *no* interpretation satisfies a wff, the wff is inconsistent or unsatisfiable.
  - e.g. F or $P \wedge \neg P$

Validity

- A wff is said to be *valid*
  - It has value *True* under *all* interpretations of its constituent atoms.
  - e.g.
    - $P \supset P$
    - T
    - $\neg (P \wedge \neg P)$
    - $Q \vee T$
    - $[(P \supset Q) \supset P] \supset P$
    - $P \supset (Q \supset P)$
  - Use of the truth table to determine the validity of a wff takes time exponential in the number of atoms

Equivalence

- Two wffs are said to be *equivalent* iff their truth values are identical under *all* interpretations.
- DeMorgan's laws

$$\neg(w_1 \vee w_2) \equiv \neg w_1 \wedge \neg\, w_2$$

$$\neg(w_1 \wedge w_2) \equiv \neg w_1 \vee \neg\, w_2$$

- Law of the contrapositive

$$(w_1 \supset w_2) \equiv (\neg w_2 \supset \neg\, w_1)$$

- If w_1 and w_2 are equivalent, then the following formula is valid:

$$(w_1 \supset w_2) \wedge (w_2 \supset w_1)$$

Entailment

- If a wff w has value True under all of interpretations for which each of the wffs in a set Δ has value True, Δ logically entails w and w logically follows from Δ and w is a logical consequence of Δ.
- e.g.
  - $\{P\} \models P$
  - $\{P, P \supset Q\} \models Q$
  - $P \wedge Q \models P$

Soundness and Completeness

- If, for any set of wffs, Δ, and wff, w, $\Delta \vdash_R w$ implies $\Delta \models w$, the set of inference rules, *R*, is *sound*. i.e. That is, if w is derived from a set of sentences Δ using a given set of rules of inference, then *w* is entailed by Δ. Hence, inference produces only real entailments, or any sentence that follows deductively from the premises is valid.
- If, for any set of wffs, Δ, and wff, *w*, it is the case that whenever $\Delta \models w$, there exist a proof of *w* from Δ using the set of inference rules, we say that *R* is complete. That is, if w is entailed by a set of sentences Δ, then *w* can be derived from Δ using the rules of inference. Hence, inference produces all entailments, or all valid sentences can be proved from the premises.
- When inference rules are sound and complete, we can determine whether one wff follows from a set of wffs by searching for a proof.

Soundness and Completeness

- When the inference rules are sound, if we can find a proof of *w* from Δ, *w* logically follows from Δ.
- When the inference rules are complete, we will eventually be able to confirm that *w* follows from Δ by using a complete search procedure to search for a proof.
- To determine whether or not a wff logically follows from a set of wffs or can be proved from a set of wffs is, in general, an NP-hard problem.

The PSAT Problem

- Propositional satisfiability (PSAT) problem: The problem of finding a model for a formula.
- *Clause*
  - A disjunction of literals
- *Conjunctive Normal Form* (*CNF*)
  - A formula written as a conjunction of clauses
- An exhaustive procedure for solving the CNF PSAT problem is to try systematically all of the ways to assign True and False to the atoms in the formula.
  - If there are n atoms in the formula, there are 2^n different assignments.

The PSAT Problem

- Interesting Special Cases
  - 2SAT and 3SAT
  - *k*SAT problem
    - To find a model for a conjunction of clauses, the longest of which contains exactly *k* literals
  - 2SAT
    - Polynomial complexity
  - 3SAT
    - NP-complete
  - Many problems take only polynomial *expected* time.

The PSAT Problem

- GSAT
  - Nonexhaustive, greedy, hill-climbing type of search procedure
  - Begin by selecting a random set of values for all of the atoms in the formula.
    - The number of clauses having value *True* under this interpretation is noted.
  - Next, go through the list of atoms and calculate, for each one, the increase in the number of clauses whose values would be True if the value of that atom were to be changed.
    - Change the value of that atom giving the largest increase
    - Terminated after some fixed number of changes
    - May terminate at a local maximum

Metatheorems

- Important Thorems
  - Deductive theorem
    - If $\{w_1, w_2, \ldots, w_n\} \models w$, $(w_1 \wedge w_2 \wedge \ldots \wedge w_n) \supset w$ is valid.
  - Reducio ad absurdum
    - If the set Δ has a model but $\Delta \cup \{\neg w\}$ does not, then $\Delta \models w$.