

# Gilbert Strang's Linear Algebra 5<sup>th</sup> Edition

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## 1 Challenge Problems(1.2):

- 30. The Given scenario is very much true in case of vectors in one plane. Consider 3 vectors each pointing to a vertex of an equilateral triangle centered at origin, and that would hold it true, as in case case :  $\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \cdot \|\mathbf{v}\|} = \frac{-1}{2}$  and similarly for the other two.

Considering the same thing in 3D-xyz plane would complicate things a bit, but with benefit of hindsight we know the case of Tetrahedron, which'll hold it true for four vectors pointing to its four vertices in which case: For All pair of vectors( $\mathbf{u}, \mathbf{v} : \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \cdot \|\mathbf{v}\|} = \cos(109.47) < 0$ )

Digging Deeper into the question, Back in the 2D case, we find that we won't be able to find 4 or more vectors which hold the condition true as now, Even while considering the symmetric case i.e. The Four Vectors pointing towards the vertices of a square, we'll have  $\mathbf{u} \cdot \mathbf{v} = 0$ , and if we move anyone or more of those vectors, then if the dot product becomes negative for some of them, it'll always become positive for others.

With similar assertions in 3D, we conclude that we can't find any more than 4 such vectors, because if we try to do so in case of 5 or 6 vectors case, then we'll be unable to find any symmetric analogues which uphold these conditions.

- 31.  $x + y + z = 0 \implies x^2 + y^2 + z^2 = -2xy - 2xz - 2yz$   
 $\implies \frac{xy + yz + xz}{x^2 + y^2 + z^2} = \frac{-1}{2} = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \cdot \|\mathbf{v}\|}$

This means that the angle is always  $\frac{2\pi}{3}$

- 32. The Given Inequality is actually wrong! A Counter Example would be:

$$x = \frac{-y}{2}, z = \frac{-y}{2}, y > 0$$

$$\text{LHS: } \frac{y}{2^{\frac{2}{3}}} > (y - \frac{-y}{2} - \frac{-y}{2}) = 0$$

- 33. The total 16 possibilities brought forward by  $(\pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2})$  would be lying in the 4-dimensional hyperplane with appropriate symmetry.

Hence, We can choose one arbitrarily let it be  $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$

Clearly one of the vectors perpendicular to it will be  $(\frac{1}{2}, \frac{-1}{2}, \frac{-1}{2}, \frac{1}{2})$

and another one perpendicular to both of them would be  $(\frac{-1}{2}, \frac{-1}{2}, \frac{1}{2}, \frac{1}{2})$

and finally one perpendicular to all three of these would be  $(\frac{1}{2}, \frac{1}{2}, \frac{-1}{2}, \frac{-1}{2})$

Also observe that even if we replace the Second vector with  $(\frac{-1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{-1}{2})$ . The Quadruplet of Vectors would still be mutually-perpendicular to each other.

To Generalize, we can take any one vector(All +ve in above case), and generate a Quadruplet as:  $(\pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2})$ ,  $(\pm \frac{1}{2}, \mp \frac{1}{2}, \mp \frac{1}{2}, \pm \frac{1}{2})$  or  $(\mp \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, \mp \frac{1}{2})$ ,  $(\pm \frac{1}{2}, \pm \frac{1}{2}, \mp \frac{1}{2}, \mp \frac{1}{2})$  and  $(\mp \frac{1}{2}, \mp \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2})$

## 2 Challenge Problems(1.2):

- The Matrix C is

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

Upon some observation it becomes clear that interchanging rows  $R_3 \leftarrow R_3 + R_1$ ,  $R_5 \leftarrow R_5 + R_3$ , we'll get 0 in the last row, which means that for the system  $\mathbf{C}\mathbf{x} = \mathbf{0}$  to be consistent we must have  $b_1 + b_3 + b_5 = 0$

Also, it could be viewed as if  $R_1, R_3$  and  $R_5$  together form a linearly dependent system, whereas  $R_2$  and  $R_4$  are mutually independent, as well as independent w.r.t to the former set of dependent row vectors.

- 14) We may write  $(a, b) = (kc, kd) \implies (a, c) = (kc, c)$  and  $(b, d) = (kd, d)$ , So that clearly  $(a, c) = \frac{d}{c} \cdot (b, d)$

### 3 Challenge Problems (2.1)

- 31) *Lemma:* Exchanging two rows or two columns in a magic matrix won't change its property of being a magic matrix. (Easily Provable!)

Now, with that lemma, we can start setting one row of our matrix:

$$\begin{bmatrix} 8 & 3 & 4 \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{bmatrix}$$

Let's start guessing the next row, Let's keep 1 in the next row, and we can make sure that, we can do this, as even if it weren't so, it would've been in row below that, and by the virtue of the lemma given above, it would enable us to exchange the two rows, without any worries. Let's take it to be at  $a_{21}$ :

$$\begin{bmatrix} 8 & 3 & 4 \\ 1 & \dots & \dots \\ \dots & \dots & \dots \end{bmatrix}$$

Now, we can construct the entire matrix by just this info: as upholding the condition in column 1, we must have  $a_{31} = 6$ , From the Left-Bottom to Top-Right Diagonal  $a_{22} = 5$ , From Second Row  $a_{23} = 9$ , From the other diagonal we have, we have  $a_{33} = 2$ .

And Finally the moment of truth: If we put  $a_{32} = 7$ , which luckily!, fulfills all our requirements, hence

the Matrix  $M_3$  is: 
$$\begin{bmatrix} 8 & 3 & 4 \\ 1 & 5 & 9 \\ 6 & 7 & 2 \end{bmatrix}$$

If we multiply it by  $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$ , then we'll get a column vector with the sum i.e. 15 as all its three elements.

Coming to the question about multiplying  $M_4$  with the vector  $[1, 1, 1, 1]^T$ , it would simply give me a 4-member column vector, with the sum of the Matrices' row/column/diagonal as all of its entries, which will be 34.

- 32) The Matrix could be singular, only if the third column is a linear combination of the other two i.e. Third column  $\mathbf{w} = \alpha_1 \mathbf{u} + \alpha_2 \mathbf{v}$ .

The Typical Row Picture would be such that, there will be infinite solutions, if  $\mathbf{b}_3 = \alpha_1 \mathbf{b}_1 + \alpha_2 \mathbf{b}_3$ , and no solutions otherwise!

- 33) Clearly,  $5\mathbf{u} + 7\mathbf{v} = \mathbf{w}$   
 $\implies \mathbf{A}\mathbf{w} = 5\mathbf{A}\mathbf{u} + 7\mathbf{A}\mathbf{v}$

- 34) The Equation  $\mathbf{A}\mathbf{x} = \mathbf{b}$  for Given question would look like:

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

Applying  $R_2 \leftarrow R_2 + \frac{1}{2}R_1$

$$R_3 \leftarrow R_3 + \frac{2}{3}R_2$$

$$R_4 \leftarrow R_4 + \frac{3}{4}R_3$$

$$R_3 \leftarrow R_3 + \frac{5}{4}R_4$$

$$R_2 \leftarrow R_2 + \frac{5}{4}R_3$$

$$R_1 \leftarrow R_1 + \frac{2}{3}R_2, \text{ We get } \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{465}{96} \\ \frac{417}{48} \\ \frac{337}{32} \\ 6 \end{bmatrix}$$

- Multiplying by the given  $\mathbf{x}$  would simply mean summing up all rows, and putting those values in a column vector, hence, it would have a 9 by 1 column vector with each entry as 45.  
In order for a row/column exchange to produce another Sudoku Matrix, we must have that the numbers contained in all 1x3 row-sections for row indices that differ by a factor for 3 are the same (even if order isn't!). Hence, we may exchange any 2 rows only if their indices differ by a factor of 3 and the same holds true for columns!

## 4 Challenge Problems(2.2)

- 30) Assuming we didn't perform any Row exchanges in order to get the Pivot Matrix  $\mathbf{U}$  from  $\mathbf{A}$ , and also assuming that we didn't multiply this row by any number during the process (As, even if we did, we could view it in other way where we didn't), Then we could easily write that if  $A(5,5) = 7$ , then the same row operations as before couldn't have reduced it to 0 in the  $\mathbf{U}$  matrix, giving singularity.
- 31) The  $j^{th}$  row of  $A$  could be a linear combination of all the rows of  $A$ .  
If  $A\mathbf{x} = \mathbf{0}$  then  $U\mathbf{x} = \mathbf{0}$ , but if  $A\mathbf{x} = \mathbf{b}$  then that doesn't imply that  $U\mathbf{x} = \mathbf{b}$ , instead  $A\mathbf{x} = \mathbf{a}$ , where  $\mathbf{a}$  is obtained by some row operations of the column matrix  $\mathbf{b}$ .  
If we start from a lower-triangular  $A$ , then we would essentially get a Diagonal Matrix  $U$  as the only upper-triangular matrix reachable by EROs. (excluding row exchange!)
- 32)
  - a) 0
  - b) 0
  - c) Enumerate from 1 to 10000 as the entries of the matrix. Apply  $R_2 \leftarrow R_2 - R_1$ ,  $R_5 \leftarrow R_5 - R_4$  and then  $R_5 \leftarrow R_5 - R_2$  to have a zero row in  $R_2$  and hence a singular matrix with all non-zero, distinct entries.
  - d) Did in above answer itself! To state it formally, the  $i^{th}$  row would consist of enumeration of all natural numbers from  $100i - 99$  to  $100i$  and the  $j^{th}$  column would have the  $i^{th}$  entry in them as  $j + 100(i - 1)$ .

## 5 Challenge Problems(2.3)

- 29) Clearly, the matrix  $E$  would be obtained by the multiplication of Elementary Matrices which perform the Row-operation:  $R_4 \leftarrow R_4 - R_3$ ,  $R_3 \leftarrow R_3 - R_2$  and  $R_2 \leftarrow R_2 - R_1$ . i.e. The matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Also, Note that the Matrix obtained after it could be converted to Identity by doing another two rounds on Matrix Manipulations, first of which shall have:  $R_4 \leftarrow R_4 - R_3$  and  $R_3 \leftarrow R_3 - R_2$  and the second of which shall only have  $R_4 \leftarrow R_4 - R_3$ .

Applying the operations in the order needed and then Multiplying them, we get that

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}$$

- 30) Observe that multiplying any 2-by-2 matrix by the matrix  $A$  would be equivalent to Keeping its first row unchanged, and replacing the second row by the sum of the two rows. Similarly Multiplying by  $B$  would keep the second row as it is, and have the sum of the two rows in the first.

a) The Matrix  $E$  would be doing the inverse operation to that of matrix  $A$ , and hence would be its exact inverse i.e.  $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

b) Similarly  $F$  would be the exact inverse of matrix  $B$ , i.e.  $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

c) Clearly multiplying the matrices in the order  $E \rightarrow F \rightarrow E \rightarrow E$ , would give me the matrix  $B$ .

d) Thus,  $M = BAABA$ .

- 31) Question unclear!

## 6 Challenge Problems(2.4)

- 36) Trivial Question!

- 37) Clearly,  $BA = I = AC \implies AB = CA$ , then multiplying with  $C$  from left we get :  $C(AB) = C(CA) \implies C(AB) = C$

Then using same logic as that given we prove associativity in this context and get  $C(AB) = (CA)B \implies C(AB) = B$

Combining that with what we had above, we'll get  $B = C$

- 38)

a) In order to solve this part, we need to first appreciate the fact that whenever we write  $AB = C$ , then the  $i^{th}$  row of  $C$  is a linear combination of all the rows of  $B$ , with their coefficients in combination as the corresponding elements in the  $i^{th}$  row of  $A$ .

Now, with that in mind:

$$A^T = \begin{bmatrix} \mathbf{a}_1 & \dots & \dots & \mathbf{a}_m \end{bmatrix}$$

$$A = \begin{bmatrix} \mathbf{a}_1^T \\ \vdots \\ \mathbf{a}_m^T \end{bmatrix}$$

Now, considering  $A^T A$ , the  $i^{th}$  row of the product should be a linear combination of all the rows of  $A$ , and the contribution of the  $j^{th}$  row of  $A$  in the product shall be represented by the  $i^{th}$  component of it, which is exactly what is given (for all the entries in the product matrix) when we represent this contribution by  $\mathbf{a}_i \mathbf{a}_j^T$ .

## 7 Challenge Problems(2.5)

- 39)

$$A = \begin{bmatrix} 1 & -a & 0 & 0 \\ 0 & 1 & -b & 0 \\ 0 & 0 & 1 & -c \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\implies \begin{bmatrix} 1 & -a & 0 & 0 \\ 0 & 1 & -b & 0 \\ 0 & 0 & 1 & -c \\ 0 & 0 & 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Applying  $R_3 \leftarrow R_3 + cR_4$ ,  $R_2 \leftarrow R_2 + bR_3$  and  $R_1 \leftarrow R_1 + aR_2$ . We get:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & a & ab & abc \\ 0 & 1 & b & bc \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\implies A^{-1} = \begin{bmatrix} 1 & a & ab & abc \\ 0 & 1 & b & bc \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned}
\bullet 40) L &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ b & 0 & 1 & 0 \\ c & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & d & 1 & 0 \\ 0 & e & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & f & 1 \end{bmatrix} \\
\Rightarrow L &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ b & d & 1 & 0 \\ c & e & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & f & 1 \end{bmatrix} \\
\Rightarrow L &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ b & d & 1 & 0 \\ c & e & f & 1 \end{bmatrix}
\end{aligned}$$

The matrices  $E_1$ ,  $E_2$  and  $E_3$  are all elementary matrices and thus can be viewed as a set of EORs on Identity matrices. The Final Matrix  $L$  corresponds to the operations:

$$R_4 \leftarrow R_4 + fR_3 + eR_2 + cR_1, R_3 \leftarrow R_3 + bR_1 + dR_2 \text{ and } R_2 \leftarrow R_2 + aR_1$$

## 8 Challenge Problems(3.1)

• 30)

a) The Null Vector is a part of all subspaces, hence the vector space  $\mathbf{U} = \mathbf{S} + \mathbf{T}$  essentially contains both the subspaces  $\mathbf{S}$  and  $\mathbf{T}$ . That means the Scalar Multiplication is property already holds, as for any vector  $\mathbf{a} \in \mathbf{S}$  or  $\mathbf{T}$ , as for vectors which are neither of these but instead a sum of two vectors from each, we have for all  $\mathbf{s} \in \mathbf{S} \Rightarrow \alpha\mathbf{s} \in \mathbf{S}$  and similarly for all  $\mathbf{t} \in \mathbf{T} \Rightarrow \alpha\mathbf{t} \in \mathbf{T}$ , and hence if  $\mathbf{s} \in \mathbf{S}$  and  $\mathbf{t} \in \mathbf{T}$ , we have  $\mathbf{s} + \mathbf{t} \in \mathbf{S} + \mathbf{T}$  and so does  $\alpha(\mathbf{s} + \mathbf{t})$ , which prove the Scalar Multiplication part.

As for the addition part, then its already prove for three cases i.e. if  $\mathbf{a}$  and  $\mathbf{b}$  both come from the same subspace, or of they come from different subspaces(by the definition of the Sum!), the only case which remains is when one or both of the vectors  $\mathbf{a}$  and  $\mathbf{b}$  are part of neither subsets, but even in that case, we would easily be able to decompose the vectors  $\mathbf{a}$  and  $\mathbf{b}$  into a sum of vectors which come from the two subspaces and are hence themselves a part of them and are thus proved by the first case described above.

b)  $\mathbf{S} \cup \mathbf{T}$  won't have any vectors of the form  $\mathbf{s} + \mathbf{t}$ , where  $\mathbf{s} \in \mathbf{S}$  and  $\mathbf{t} \in \mathbf{T}$  (when the two subspaces are such that they can't have common bases, in which case, the two subspaces are essentially identical!), But  $\mathbf{S} + \mathbf{T}$  would still contain all of  $\mathbf{S} \cup \mathbf{T}$ , as The Null Vector is a part of all subspaces, hence the vector space  $\mathbf{U} = \mathbf{S} + \mathbf{T}$  essentially contains both the subspaces  $\mathbf{S}$  and  $\mathbf{T}$ .

We will also have that the bases of both  $\mathbf{S}$  and  $\mathbf{T}$  are included in  $\mathbf{S} \cup \mathbf{T}$ , and hence, I could write any vector which can be there in  $\mathbf{S} + \mathbf{T}$  using their linear Combinations so that span of  $\mathbf{S} \cup \mathbf{T}$  is  $\mathbf{S} + \mathbf{T}$ .

• 31) The Matrix  $M = \begin{bmatrix} A & B \end{bmatrix}$  would always be a Matrix whose Column space is  $\mathbf{S} + \mathbf{T}$ . It might be expressible as a matrix w