MA106: All Propositions and Theorems

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- **Proposition**

  Let $m, n, p, q \in \mathbb{N}$. If $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times p}$ and $\mathbf{C} \in \mathbb{R}^{p\times q}$, then $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$ (which we shall write as $\mathbf{ABC}$).

- **Proposition**

  Let $m, n, p \in \mathbb{N}$. If $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{B} \in \mathbb{R}^{n\times p}$, then $(\mathbf{AB})^\mathsf{T} = \mathbf{B}^\mathsf{T}\mathbf{A}^\mathsf{T}$.

- **Proposition**

  Let $\mathbf{A} \in \mathbb{R}^{m\times n}$ be in REF with r nonzero rows. Then the linear system $\mathbf{Ax} = \mathbf{0}$ has only the zero solution if and only if $r = n$. In particular, if $m < n$, then $\mathbf{Ax} = \mathbf{0}$ has a nonzero solution.

- **Proposition**

  Let $\mathbf{A} \in \mathbb{R}^{m\times n}$. Then the linear system $\mathbf{Ax} = \mathbf{0}$ has only the zero solution if and only if any REF of $\mathbf{A}$ has n nonzero rows. In particular, if $m < n$, then $\mathbf{Ax} = \mathbf{0}$ has a nonzero solution.

- **Proposition**

  Let $\mathbf{A}$ be a square matrix. Then $\mathbf{A}$ is invertible if and only if $\mathbf{A}^\mathsf{T}$ is invertible. In this case, $(\mathbf{A}^\mathsf{T})^{-1} = (\mathbf{A}^{-1})^\mathsf{T}$.

- **Corollary**

  Let $\mathbf{A} \in \mathbb{R}^{n \times n}$. If there is $\mathbf{B} \in \mathbb{R}^{n \times n}$ such that either $\mathbf{BA} = \mathbf{I}$ or $\mathbf{AB} = \mathbf{I}$, then $\mathbf{A}$ is invertible, and $\mathbf{A}^{-1} = \mathbf{B}$.

- **Proposition**

  Let $\mathbf{A}$ and $\mathbf{B}$ be square matrices. Then $\mathbf{AB}$ is invertible if and only if $\mathbf{A}$ and $\mathbf{B}$ are invertible, and then $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

- **Proposition**

  An $n \times n$ matrix is invertible if and only if it can be transformed to the $n \times n$ identity matrix by EROs.

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  An $n \times n$ matrix is invertible if and only if it can be transformed to the $n \times n$ identity matrix by EROs.

- A set S of vectors is called **linearly dependent** if there is $m \in \mathbb{N}$, there are (distinct) vectors $\mathbf{a}_1, \ldots, \mathbf{a}_m$ in S and there are scalars $\alpha_1, \ldots, \alpha_m$, not all zero, such that

  $$\alpha_1 \mathbf{a}_1 + \cdots + \alpha_m \mathbf{a}_m = \mathbf{0}.$$

  It can be seen that S is linearly dependent $\iff$ either $\mathbf{0} \in S$ or a vector in S is a linear combination of other vectors in S.

- **Proposition**

  Let S be a set of s vectors, each of which is a linear combination of elements of a (fixed) set of r vectors. If $s > r$, then the set S is linearly dependent.

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- **Corollary**

  Let $n \in \mathbb{N}$ and S be a set of vectors of length n. If S has more than n elements, then S is linearly dependent.

- Let $\mathbf{A} \in \mathbb{R}^{m \times n}$. The **row rank** of $\mathbf{A}$ is the maximum number of linearly independent row vectors of $\mathbf{A}$. Thus the row rank of $\mathbf{A}$ is equal to r if and only if there is a linearly independent set of r rows of $\mathbf{A}$ and any set of $r+1$ rows of $\mathbf{A}$ is linearly dependent.

- **Proposition**

  If a matrix $\mathbf{A}$ is transformed to a matrix $\mathbf{A}'$ by elementary row operations, then the row ranks of $\mathbf{A}$ and $\mathbf{A}'$ are equal, that is, EROs do not alter the row rank of a matrix.

- **Proposition**

  Let a matrix $\mathbf{A}'$ be in REF. Then the nonzero rows of $\mathbf{A}'$ are linearly independent, and so the row rank of $\mathbf{A}'$ is equal to the number of nonzero rows of $\mathbf{A}'$.

- **Proposition**

  The row rank of a matrix is equal to the number of nonzero rows in any row echelon form of the matrix.

Definition

Let $\mathbf{A} \in \mathbb{R}^{m \times n}$. The column rank of $\mathbf{A}$ is the maximum number of linearly independent column vectors of $\mathbf{A}$.

Clearly, column-rank$(\mathbf{A})$ = row-rank$(\mathbf{A}^\mathsf{T})$.

Proposition

The column rank of a matrix is equal to its row rank.

- **Definition**

  A nonempty subset V of $\mathbb{R}^{n \times 1}$ is called a vector subspace, or simply a subspace of $\mathbb{R}^{n \times 1}$ if

  (i) $\mathbf{a}, \mathbf{b} \in V \implies \mathbf{a} + \mathbf{b} \in V$, and

  (ii) $\alpha \in \mathbb{R}$, $\mathbf{a} \in V \implies \alpha\,\mathbf{a} \in V$.

- Let $\mathbf{A} \in \mathbb{R}^{m \times n}$.

  Definition

  The **null space** of $\mathbf{A}$ is

  $$\mathcal{N}(\mathbf{A}) := \{\mathbf{x} \in \mathbb{R}^{n \times 1} : \mathbf{A}\mathbf{x} = \mathbf{0}\}.$$

  Definition

  The **column space** of $\mathbf{A}$ is

  $$\mathcal{C}(\mathbf{A}) := \text{ the set of all linear combinations of columns of } \mathbf{A}.$$

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Definition

A subset S of V is called a basis of V if S is linearly independent and S has maximum possible number of elements among linearly independent subsets of V.

Clearly, a basis of V has at most n elements, and any two bases of V have the same number of elements.

Definition

The dimension of V is defined as the number of elements in a basis of V. It is denoted by $\dim V$.

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Definition

Let $S \subset \mathbb{R}^{n \times 1}$. The set of all linear combinations of elements of S is denoted by $\operatorname{span} S$ and called the span of S.

Proposition

Let V be a subspace of $\mathbb{R}^{n \times 1}$, and let $S \subset V$. Then S is a basis for $V \iff S$ is linearly independent and $\operatorname{span} S = V$.

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Corollary

Let V be a subspace of $\mathbb{R}^{n \times 1}$. Every linearly independent subset of V can be enlarged to a basis for V.

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Proposition

Let $S := \{\mathbf{c}_1, \ldots, \mathbf{c}_r\}$ be a basis for a subspace V of $\mathbb{R}^{n \times 1}$, and let $\mathbf{x} \in V$. Then there are unique $\alpha_1, \ldots, \alpha_r \in \mathbb{R}$ such that $\mathbf{x} = \alpha_1 \mathbf{c}_1 + \cdots + \alpha_r \mathbf{c}_r$.

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- **Proposition**

  Let $\mathbf{A} \in \mathbb{R}^{m\times n}$, and let $\operatorname{rank}\mathbf{A} = r$. Then $\dim \mathcal{C}(\mathbf{A}) = r$ and $\dim \mathcal{N}(\mathbf{A}) = n - r$.

- **Theorem (Fundamental Theorem for Linear Systems: FTLS)**

  Let $m, n \in \mathbb{N}$ and $\mathbf{A}$ be an $m \times n$ matrix with real entries. Suppose $\boxed{\operatorname{rank}\mathbf{A} = r}$.

  (i) Homogeneous Linear System : $\mathbf{A}\mathbf{x} = \mathbf{0}$ (H)

  The solution space $\{\mathbf{x} \in \mathbb{R}^{n\times 1} : \mathbf{A}\mathbf{x} = \mathbf{0}\}$ of (H) is a subspace of $\mathbb{R}^{n\times 1}$ of dimension $\boxed{n - r}$.

  In particular, $r = n$ if and only if $\mathbf{0}$ is the only solution of (H).

  If $r < n$, then there are linearly independent solutions $\mathbf{x}_1, \dots, \mathbf{x}_{n-r}$ of (H) and every solution of (H) is a unique linear combination of these $\mathbf{x}_1, \dots, \mathbf{x}_{n-r}$.

  (ii) General Linear System: $\mathbf{A}\mathbf{x} = \mathbf{b}$ with $\mathbf{b} \in \mathbb{R}^{m\times 1}$ (G)

  (G) has a solution if and only if $\operatorname{rank}[\mathbf{A}|\mathbf{b}] = r$. In this case, let $\mathbf{x}_0$ be a particular solution of (G). If $\mathbf{x}$ is a solution of (G), then $\mathbf{x} = \mathbf{x}_0 + \mathbf{x}_h$, where $\mathbf{x}_h$ is a solution of (H) above.

- **Definition**

  Let $\mathbf{A} \in \mathbb{R}^{m\times n}$. The row space of A, denoted $\mathcal{R}(\mathbf{A})$, is defined as the subspace of $\mathbb{R}^{1\times m}$ spanned by the row vectors of $\mathbf{A}$.

- **Proposition**

  If $\mathbf{A}$ is lower triangular, then the determinant of $\mathbf{A}$ is the product of its diagonal entries.

- **Proposition**

  Let $\mathbf{A}$ be a square matrix. Then $\det \mathbf{A}^\mathsf{T} = \det \mathbf{A}$.

- **Corollary**

  If $\mathbf{A}$ is upper triangular, then the determinant of $\mathbf{A}$ is the product of its diagonal entries.

- **Proposition**

  Let $\mathbf{A}$ be a square matrix.

  (i) If two columns of $\mathbf{A}$ are interchanged, then $\det \mathbf{A}$ gets multiplied by -1.

  (ii) Addition of a multiple of a column to another column of $\mathbf{A}$ does not alter $\det \mathbf{A}$.

  (iii) Multiplication of a column of $\mathbf{A}$ by a scalar α results in the multiplication of $\det \mathbf{A}$ by α.

- **Corollary**

  Let $\mathbf{A}$ be a square matrix.

  (i) If two rows of $\mathbf{A}$ are interchanged, then $\det \mathbf{A}$ gets multiplied by -1.

  (ii) Addition of a multiple of a row to another row of $\mathbf{A}$ does not alter $\det \mathbf{A}$.

  (iii) Multiplication of a row of $\mathbf{A}$ by a scalar α results in the multiplication of $\det \mathbf{A}$ by α.

- **Proposition**

  A square matrix $\mathbf{A}$ is invertible if and only if $\det \mathbf{A} \neq 0$.