MA106

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1 Lecture 1: Introduction

- Matrices are a new universe of Numbers
- Visualizing the matrices as a column vector of row vectors or a row vector of column vectors, is an important thing
- Outer Product is called so, as its sort of doing the inner product/scalar product(or dot product!), the other way round!
- Going over the various ways to write the Product of Two Matrices
- Exercise: Proving Trivial Results like $\left(\mathbf{AB}\right)^T = \mathbf{B^T}\ \mathbf{A^T}$
- The j^{th} row of $\mathbf{AB}$ is a linear combination of the j^{th} row of $\mathbf{A}$ with coefficient of some common, and analogically in case of k^{th} column of $\mathbf{AB}$ would be
- Really Nice Question: Justifying the different cases of solutions to system of linear equations using concepts from matrices

2 Lecture 2: Linear Systems

- General Linear system will include homogeneous as well as non-homogeneous.
- **Deducing Connections:** How to relate $\mathbf{Ax} = \mathbf{b}$ to $\mathbf{Ax} = \mathbf{0}$. If $\mathbf{Ax} = \mathbf{0}$ has non-trivial solutions, than that would mean infinitely many solutions if we know just one solution exists.
- Extending the past concepts to more general cases: Using the above thing to solve any general system of m equations in n variables.

3 Lecture 3: Gaussian Elimation

Nothing as such apart from Lecture Notes introduced, Just a very nice and thoughtful question: Let $\mathbf{A} \in \mathbb{R}^{9x4}$ and $\mathbf{B} \in \mathbb{R}^{7x3}$. Is there $\mathbf{X} \in \mathbb{R}^{4x7}$ such that $\mathbf{X} \neq \mathbf{O}$ but $\mathbf{AXB} = \mathbf{O}$

4 In General Observations/Queries which people posted on Whatsapp Group

- From LEC2, where first we identify the pivot points in REF, identify the free and non-free variables and then set the non-free ones to zero, following which we also identify the basis vectors by setting each one of them to one in and getting separate solutions, so that the overall solution is a linear combination of these.
- In the REF of an inconsistent System, we can get different REFs there is no unique one, but we do have a unique Reduced REF or Row-canonical Form.

- Another way to progress, after we've identified the Free variables in REF, would be to simply substitute the Free variables $x_i = \alpha_i$, but we don't do this, Why?

- Let $\mathbf{s} := \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^\mathsf{T} \in \mathbb{R}^{n\times 1}$ be any solution of the homogeneous system, that is, $\mathbf{As} = \mathbf{0}$. Then $\mathbf{s}$ is a linear combination of the $n-r$ basic solutions $\mathbf{s}_{\ell_1}, \ldots, \mathbf{s}_{\ell_{n-r}}$. To see this, let $\mathbf{y} := \mathbf{s} - x_{\ell_1}\mathbf{s}_{\ell_1} - \cdots - x_{\ell_{n-r}}\mathbf{s}_{\ell_{n-r}}$. Then $\mathbf{Ay} = \mathbf{As} - x_{\ell_1}\mathbf{As}_{\ell_1} - \cdots - x_{\ell_{n-r}}\mathbf{As}_{\ell_{n-r}} = \mathbf{0}$, and moreover, the kth entry of $\mathbf{y}$ is 0 for each $k \in \{\ell_1, \ldots, \ell_{n-r}\}$. It then follows that $\mathbf{y} = \mathbf{0}$, that is, $\mathbf{s} = x_{\ell_1}\mathbf{s}_{\ell_1} + \cdots + x_{\ell_{n-r}}\mathbf{s}_{\ell_{n-r}}$. Thus we find that the general solution of the homogeneous system is given by

  Why is $\mathbf{y} = 0$ in the above paragraph?

- Try proving: For all $\mathbf{A} \in \mathbb{R}^{nxn}$ if $\mathbf{AE} = \mathbf{EA}$, for some $\mathbf{E} \in \mathbb{R}^{nxn}$ such that $\mathbf{E} = \mathbf{I}$.

- Let $I \subset \mathbb{R}^{nn}$ be a nonempty set closed under addition such that $\mathbf{MN}, \mathbf{NM} \in I$ whenever $\mathbf{N} \in I$ and $\mathbf{M} \in \mathbb{R}^{nxn}$. Show that either $I = 0$ or $I = \mathbb{R}^{nxn}$.

5 Lecture 4: Inverses and its usage in solving Linear Equations

- Try to prove: Let $\mathbf{A} \in \mathbb{R}^{nxn}$.iff $\mathbf{Ax} = \mathbf{b}$ has only zero solution, then $\mathbf{A}$ is invertible

- Prove that for $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{nxn}$ such that $AB = I$, then $BA = I$.

- Lwt $\mathbf{A}$ and $\mathbf{B}$ be square matrices. Then $\mathbf{AB}$ is invertible iff $\mathbf{A}$ and $\mathbf{B}$ are invertible, and then $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

- Prove that every matrix has a unique Row-canonical form.

- Row Echelon Forms are never unique.

6 Lecture 5: Inverses, Linear Dependence-Independence, Ranks

- The Gauss-Jordan Method provides us with the theoretical justification of what we used to do in 12th to find $\mathbf{A}^{-1}$ using EROs.

- If we aren't able to transform the matrix to $\mathbf{I}$, then we'll get a row of 0s and will conclude that the matrix isn't invertible, and if we don't find any such row in any step, then we'll essentially end up getting the Identity Matrix and hence, the inverse.

- A set S of vectors is called **linearly dependent** if there is $m \in \mathbb{N}$, there are (distinct) vectors $\mathbf{a}_1, \ldots, \mathbf{a}_m$ in S and there are scalars $\alpha_1, \ldots, \alpha_m$, not all zero, such that

  $$\alpha_1\mathbf{a}_1 + \cdots + \alpha_m\mathbf{a}_m = \mathbf{0}.$$

  It can be seen that S is linearly dependent $\iff$ either $\mathbf{0} \in S$ or a vector in S is a linear combination of other vectors in S.

  Investigate meaning of the last line!

- **Proposition**

  Let S be a set of s vectors, each of which is a linear combination of elements of a (fixed) set of r vectors. If $s > r$, then the set S is linearly dependent.

  Proof. Let $S := \{\mathbf{x}_1, \ldots, \mathbf{x}_s\}$, and suppose each vector in S is a linear combination of elements of the set $\{\mathbf{y}_1, \ldots, \mathbf{y}_r\}$ of r vectors and $s > r$. Then

  $$\mathbf{x}_j = \sum_{k=1}^{r} a_{jk}\mathbf{y}_k \quad \text{for } j = 1, \ldots, s, \text{ where } a_{jk} \in \mathbb{R}.$$

  Let $\mathbf{A} := [a_{jk}] \in \mathbb{R}^{s\times r}$. Then $\mathbf{A}^\mathsf{T} \in \mathbb{R}^{r\times s}$. Since $r < s$, the linear system $\mathbf{A}^\mathsf{T}\mathbf{x} = \mathbf{0}$ has a nonzero solution, that is, there are $\alpha_1, \ldots, \alpha_s$, not all zero, such that

  $$\mathbf{A}^\mathsf{T}\begin{bmatrix}\alpha_1\\ \vdots\\ \alpha_s\end{bmatrix} = \begin{bmatrix}a_{11} & \cdots & a_{s1}\\ \vdots & \vdots & \vdots\\ a_{1r} & \cdots & a_{sr}\end{bmatrix}\begin{bmatrix}\alpha_1\\ \vdots\\ \alpha_s\end{bmatrix} = \begin{bmatrix}0\\ \vdots\\ 0\end{bmatrix} \in \mathbb{R}^{r\times 1},$$

  This truly is an elegant proof, make various variations in the hypothesis condition and check why they are or aren't true.

- **Corollary**

  Let $n \in \mathbb{N}$ and S be a set of vectors of length n. If S has more than n elements, then S is linearly dependent.

  Why this is so? Proof!

- Think of vectors which can be taken as a bunch of linearly-independent vectors which can be used as basis.

- A nxn Square Matrix could be invertibe iff its rank is n.

- We can investigate linear-independence by simply taking some coefficients appropriately and see if it could possible give us non-trivial solutions, where not all of the coefficients are 0.

7 Tutorial 2: Linear Dependency, Inverses

- A rectangular matrix can only have an inverse in certain cases, (By Inverse, I mean any matrix which when multiplied with our Matrix from one side, then that gives me **I**!), Find out which ones these are, with legit mathematical arguements.

- While, we can actually consider a lot of operations on matrices as elementary or its compositions, we follow the slides and consider only three basic cases:
  1) Exchanging Two Rows
  2) Adding the scalar multiple of one of the rows to another row
  3) Multiplying a particular row with a scalar.

- Together these operations bring us the concept of Elementary Matrices, each one of which can represent an Elementary Row Operation, (As in Q2.3!)

8 Lecture 6: Rank, Subspaces, Basis and Span

- While Calculating the Row Rank, which rows shall together form a set of linearly independent row vectors is an irrelevant question, as it depends on how we do our EROs, we may get a different set of such row vectors every time, but that won't matter, only the Rank would!
- An elegant technique to prove equality: Prove $a \leq b$ and $b \leq a$.
- Also, if there's an element of symmetry in proving $a \leq b$ and $b \leq a$, we only need to prove one and then exploit symmetry to claim the other.
- Exercise: Work out the size of the Subspace, if the basis of this subspace is consisting of n vectors $\in \mathbb{R}^{nx1}$ in that space!
- Dimension of a Subspace is the same thing as the number of vectors in its basis.

9 Lecture 7: Column and Row Spaces, Determinants

- Basis is the set of maximum number of independent vectors in a subspace.
- Basis could also be seen as the smallest spanning-set.
- If we have EROs, then that may change $\mathbf{C}(A)$, similarly, ECO could change $\mathbf{R}(A)$.
- Try Proving all Properties of Determinants.
- A Scalar Multiplication of 0, at any step of any ERO has no mearning!

10 Lecture 8: Determinants and its Properties and uses(Cramer's Rule)

- Try Proving:

Lemma

Let $\mathbf{A}$ *be an* $m \times n$ *matrix, and* $r \in \mathbb{N}$. *Then*

there exists

$$\text{rank}\,\mathbf{A} \geq r \iff \exists \text{ an } r \times r \text{ submatrix } \mathbf{B} \text{ of } \mathbf{A} \text{ with } \det \mathbf{B} \neq 0$$

- Try Proving:

Theorem

Let $\mathbf{A}$ be a square matrix. Then $\mathbf{C}^\top\mathbf{A} = (\det \mathbf{A})\mathbf{I} = \mathbf{A}\mathbf{C}^\top$.
In particular, if $\det \mathbf{A} \neq 0$, then $\mathbf{A}$ is invertible and

$$\mathbf{A}^{-1} = \mathbf{C}^\top / \det \mathbf{A}.$$

- Try Proving:

Proposition

Let $\mathbf{A}$, $\mathbf{B}$ be $n \times n$ matrices. Then $\det(\mathbf{AB}) = (\det \mathbf{A})(\det \mathbf{B})$.